REALIZING INFINITE FAMILIES OF FUSION SYSTEMS OVER FINITE GROUPS

by

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ABSTRACT

In this pure mathematics thesis we study realizations of fusion systems on finite groups and determine minimal right characteristic bisets for infinite families of saturated fusion systems over 3-groups of maximal nilpotency class.

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In the Name of God, Most Gracious, Ever Merciful.

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Our Lord, bestow on us mercy from Thyself, and provide for us right guidance in our affair.(Al-Kahf, 18:11)

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CONTENTS

CHAPTER 1

INTRODUCTION

A fusion system $\mathcal{F}$ on a finite group S is a category whose objects are subgroups of S, and whose morphisms are injective group homomorphisms, which satisfy a number of other properties [1, Definition 2.1]. A saturated fusion system is a fusion system which satisfies a further set of axioms [1, Definition 2.2]. The definitions are modelled on what we call fusion categories, $\mathcal{F}_S(G)$, where G is a finite group containing S. We do not assume, in our definition of fusion categories, that S is a Sylow-p-subgroup of G. It is well known that fusion categories are indeed fusion systems. In [7], Sejong Park showed that every saturated fusion systems over a finite p-group can be realized as a fusion category. Later, Park showed in [10], that this result holds for general fusion systems. To comment on this further, we start with some definitions.

Let S be a finite group, and write $\mathcal{U}(S)$ or simply $\mathcal{U}$ to denote the universal fusion system on S. Let ϕ and ψ be two morphisms in $\mathcal{U}$. Write D_ϕ and D_ψ to denote the domains of ϕ and ψ respectively. We define

$$\Delta_{D_\phi}^{\phi} = \{(p, p\phi) \mid p \in D_\phi\},$$

to be a diagonal subgroup of $S \times S$ and write $\mathcal{O}_\phi$ to denote the right coset space $(S \times S)/\Delta_{D_\phi}^{\phi}$ in $S \times S$. This is an $S \times S$-orbit under right multiplication, and any union of these orbits constructed from a variety of morphisms gives us an $S \times S$-set. We also define

$$\mathcal{O}_\phi^\psi = ((S \times S)/\Delta_{D_\phi}^{\phi})^{\Delta_{D_\psi}^{\psi}},$$

to be the set of points in $\mathcal{O}_\phi$ fixed by $\Delta_{D_\psi}^{\psi}$ under right multiplication. For any union of these orbits, Ω, constructed from a variety of morphisms, Ω^ψ is defined similarly.

Let $\mathcal{F}$ be a fusion system on S. An $S \times S$-set Ω is called a (right) semicharacteristic biset for $\mathcal{F}$, if and only if, it satisfies the following two properties:

1. Every orbit in Ω is of the form $\mathcal{O}_\phi$ for some ϕ in $\mathcal{F}$, and

2. $|\Omega^\psi| = |\Omega^{\mathrm{Id}|_{D_\psi}}|$ for all ψ in $\mathcal{F}$.

This means that we can write our (right) semicharacteristic biset as

$$\Omega = \bigsqcup_{\phi \in \mathcal{F}} C(\phi)\mathcal{O}_\phi$$

for some integers $C(\phi) \geq 0$. Evidently, not every value of $C(\phi)$ determines that we have a (right) semicharacteristic biset. The value of the constants $C(\phi)$ come from part (2) of the definition. The key point in part (2) is that the definition has to hold for all morphisms in $\mathcal{F}$. Usually there will be too many morphisms in any given fusion system, and therefore too many equations to deal with. In order to circumvent this problem, Park introduced a definition. Let $\phi, \phi_1 \in \mathcal{U}(S)$. We say that ϕ and ϕ_1 are S-S-equivalent, and write $\phi \sim \phi_1$, if there exists $x, y \in S$ such that

$$(D_{\phi_1})^x = D_\phi \quad \text{and} \quad c_x|_{D_{\phi_1}} \circ \phi \circ c_y = \phi_1.$$

The relation is an equivalence relation, and the size of the fixed point set $|\mathcal{O}_\phi^\psi|$ is independent of the representatives of the S-S-equivalence classes. In other words, if $\phi, \phi_1, \psi, \psi_1 \in \mathcal{U}(S)$, so that $\phi \sim \phi_1$ and $\psi \sim \psi_1$, then $|\mathcal{O}_\phi^\psi| = |\mathcal{O}_{\phi_1}^{\psi_1}|$. This means we can write

$$\Omega = \bigsqcup_{\phi \in \Gamma} C(\phi)\mathcal{O}_\phi$$

where Γ denotes a set of S-S-equivalence class representatives in $\mathcal{F}$. Moreover we only need to satisfy part (2) of the definition of (right) semicharacteristic biset for S-S-equivalent class representatives. This will leave us with far fewer calculations to deal with.

Park showed that a finite group G realizing a fusion system $\mathcal{F}$ over a finite group S is constructed as the group of all permutations over a (right) semicharacteristic biset Ω, which preserves the left S-action. That is,

$$G = \{\pi \in \mathsf{Sym}(\Omega) \mid (x \circ (s_1, s_1))\pi = ((x \circ (1, s_2))\pi) \circ (s_1, 1) \text{ for all } x \in \Omega, s_1, s_2 \in S\}.$$

This group G is equal to $C_{\mathsf{Sym}(\Omega)}(S)$, when S is identified as a subgroup of G in the following way:

$$\iota : S \hookrightarrow G : (x \mapsto x \circ (1, s)).$$

Equivalently, $G \cong S \wr \mathsf{Sym}(|\Omega|/|S|)$ (See Lemma 4.3.10). A fusion system $\mathcal{F}$ on a finite p-group S is called exotic if it is not equal to $\mathcal{F}_S(G)$ for any finite G with $S \in \mathsf{Syl}_p(G)$. In [7], Park introduced the concept of the exoticity index, $e(\mathcal{F})$, of a fusion system $\mathcal{F}$ on a finite p-group S as

$$\min\{\log_p|T : S| \mid S \leq T \in \mathsf{Syl}_p(G) \text{ for some finite } G \text{ with } \mathcal{F} = \mathcal{F}_S(G)\}.$$

By definition, a fusion system $\mathcal{F}$ on a finite p group S is exotic if and only if $e(\mathcal{F}) > 0$. The group G that realizes a fusion system $\mathcal{F}$ over a finite p-group S using (right) semicharacteristic bisets gives us an upper bound on the exoticity index as

$$(|\Omega|/|S| - 1)\log_p|S| + \sum_{i=1} \left\lfloor \frac{|\Omega|/|S|}{p^i} \right\rfloor.$$

This is likely to give us a very large upper bound, and may not necessarily be anywhere near the actual value.

In [9], Park explicitly constructed a minimal right characteristic biset for saturated fusion systems $\mathcal{F}$ on extraspecial groups S of odd order p^3 and exponent p, with the property that every subgroup of S of order p^2 is $\mathcal{F}$-radical. Among these saturated fusion system are the Ruiz-Viruel exotic fusion systems, [13]. The upper bound on the exoticity index turns out to be quite large. For example, for the smallest of the three exotic fusion systems, that is when $|\mathrm{Out}_{\mathcal{F}}(S)| = 48$, the upper bound for the exoticity index is given as 425744.

Our thesis is essentially comparable to [9] in the sense that our main aim is to eventually construct fusion category realizations for a set of infinite families of saturated fusion systems over 3-groups of maximal nilpotency class, many of which are exotic. The fusion systems, which we shall call DRV-fusion systems, are described in [5, Theorem 5.9, Table 6]. We analyse these families quite thoroughly, and construct minimal right characteristic bisets for each member of these families. The following theorem is the result we obtain for what we call the DRV-9 fusion system:

Theorem. *Let S be a rank two 3-group of maximal nilpotency class, of order 3^r, $r = 2k+1 \geq 5$, and generated by s and s_1, with the presentation*

$$S = \langle s, s_1, s_2, .., s_{r-1} \mid s_i = [s_{i-1}, s], [s_i, s_1] = s_j^3 s_{j+1}^3 s_{j+2} = s^3 = 1 \rangle,$$

where $2 \leq i \leq r-1$ and $1 \leq j \leq r-1$, assuming that $s_r = s_{r+1} = 1$. Write $t = s_1^{3^{k-1}}$ and $z = s_2^{3^{k-1}}$. Define the following morphisms:

$$\omega : S \to S : s \mapsto s^{-1}, s_1 \mapsto s_1^2 s_2 \qquad \text{(an involution of } S\text{)}$$

$$\eta : S \to S : s \mapsto s, s_1 \mapsto s_1^{-1} \qquad \text{(an involution of } S\text{)}$$

$$\theta_0 : \langle s,t \rangle \to \langle s,t \rangle : s \mapsto t, t \mapsto s^{-1} \qquad \text{(an automorphism of an extraspecial subgroup)}$$

$$\alpha_0 : \langle s,t \rangle \to \langle s,t \rangle : s \mapsto t, t \mapsto s \qquad \text{(an automorphism of an extraspecial subgroup)}$$

$$\beta_0 : \langle s,t \rangle \to \langle s,t \rangle : s \mapsto t^{-1}, t \mapsto s^{-1} \qquad \text{(an automorphism of an extraspecial subgroup)}$$

Define

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(\langle s,t \rangle), Aut_{\mathcal{F}}(S) \rangle,$$

to be a fusion system[1] *over S, such that $Out_{\mathcal{F}}(\langle s,t\rangle) \cong GL_2(3)$ and $Out_{\mathcal{F}}(S) \cong \langle \bar{\omega}, \bar{\eta}\rangle$. There exists a minimal right characteristic biset Ω for $\mathcal{F}$ given by:*

$$\Omega \cong (\mathcal{O}_{Id_S} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0}) \sqcup n_k(\mathcal{O}_{Id|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}),$$

where $n_k = 3^{2k-3} - 1$ denotes the number of copies of the corresponding orbits. Moreover, $|\Omega|/|S| = 4(m_k-1)^2$, for $m_k = 3^{2k-2}$, and the upper bound of $e(\mathcal{F})$, the exoticity index over $\mathcal{F}$, given by this construction is $2[(4k+3)m_k^2 - (8k-6)m_k + (2k+3)]$.

From [5, Table 6, Theorem 5.9] we know that the DRV-9 fusion system is not exotic. This means that its exoticity index is actually zero, and hence the upper bound given by the theorem is not very precise. This pattern is prevalent throughout all the DRV-fusion systems.

We now summarize each chapter:

- In Chapter 2 we introduce basic ideas in group theory relating to group actions, automorphisms of certain extraspecial groups and some basic definitions and results in the theory of fusion systems.

- In Chapter 3 we define S-S-equivalence over morphisms of a given fusion system. The significance of this chapter is to help us later apply results in order to reduce the number of morphisms in the construction of right characteristic bisets.

- In Chapter 4 our aim is to demonstrate a significant result of Park which shows that fusion systems over finite groups can be realized as fusion categories of certain permutation groups. We start by defining bisets, introduce characteristic bisets and finally take a look at the exoticity index of a fusion system.

- In Chapter 5 we study an infinite family of 3-groups of maximal nilpotency class, whose members we shall call S. It is this infinite family of groups upon which we shall write down, what we shall call DRV-fusion systems.

[1] As discussed in [5, Table 6, Theorem 5.9], the fusion system is in fact saturated.

- In Chapter 6 we study the automorphisms of S, and the automorphisms of certain subgroups of S. Here we also fix notation for a key set of morphisms which will be used throughout Chapters 7, 8, and Appendices A,B.

- In Chapter 7, which is perhaps the most important chapter in this thesis, the idea is to introduce, what we call, DRV-fusion systems, over infinite families of groups. Here we use results from Chapter 3 to reduce the number of morphisms used in the construction of right characteristic bisets corresponding to DRV-fusion systems. By the end of this chapter we will have a general construction of right characteristic bisets for each of the DRV-fusion systems.

- In Chapter 8 we focus on the Main Theorem of our thesis which essentially lists minimal right characteristic bisets for each of the DRV-fusion systems. We also demonstrate the proof of one of the DRV-9 fusion system in detail which then can be used as a basis for understanding the rest of the proofs found in Appendix A.

- In Appendix A we list the remaining proofs of the Main Theorem, from DRV-1 to DRV-16 (except DRV-9).

- In Appendix B we determine the fixed point sizes $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|$ for all relevant ϕ,ψ and $Q \leq D_\psi$,$P \leq D_\phi$ that are needed for the proofs of the Main Theorem.

1.1 Notation

Here we list some of the most common notation used in this thesis. Much of the notation listed here is related to the group S defined in Chapter 5. Most of them are either elements of S, subgroup of S, or morphism defined over subgroups of S. Sometimes, it may be the case that some of the notation identified below is used differently in different contexts. If that is the case, then that will be clearly stated, at least in that section, if not immediately prior to its use.

Symbol	**Description**	**Page**
<u>*Elements*</u>:		
t	$s_1^{3^{k-1}}$	64
z	$s_2^{3^{k-1}}$	64
a_k	defined by the recurrence relation $a_k \equiv_k -(a_{k-1}^2 - 3a_{k-1} + 3)$ and $a_1 \equiv_1 0$	79
b_k	defined by $\frac{1+a_k^2}{1+a_k}$	79
d_1	the element $s_1^{1+a_k} s_2^2$ which depends on k	331
d_2	the element $s_1^{5-a_k} s_2^2$ which depends on k	331
<u>*Groups*</u>:		
S	The group that we are concerned with in this thesis, with the presentation $\langle s, s_1, s_2, \ldots, s_{r-1} \mid s_i = [s_{i-1}, s], [s_i, s_1] = s_j^3 s_{j+1}^3 s_{j+2} = s^3 = 1 \rangle$, where $2 \leq i \leq r-1$, $1 \leq j \leq r-1$, assuming that $s_r = s_{r+1} = 1$, $r \geq 5$ and odd.	58
V_λ	$\langle s s_1^\lambda, z \rangle$	85
E_λ	$\langle s s_1^\lambda, t \rangle$	85
A_0	$\langle t, z \rangle$	64
A	$\langle s_1, s_2 \rangle$	58
$\Omega_n(G)$	$\langle g \in G \mid g^{p^n} = 1 \rangle$ for any finite group G	25
<u>*Morphisms*</u>:		
ω	automorphism of S, defined by $s \mapsto s^{-1}, s_1 \mapsto s_1^2 s_2$	73

η	automorphism of S, defined by $s \mapsto s, s_1 \mapsto s_1^{-1}$	73
θ_T	automorphism of T, defined by $x_T \mapsto y_T, y_T \mapsto x_T^{-1}$, for $T = V_\lambda, E_\lambda, A$	85
θ_T^{-1}	automorphism of T, defined by $x_T \mapsto y_T^{-1}, y_T \mapsto x_T$, for $T = V_\lambda, E_\lambda, A$	85
α_T	automorphism of T, defined by $x_T \mapsto y_T, y_T \mapsto x_T$, for $T = V_\lambda, E_\lambda, A$	85
β_T	automorphism of T, defined by $x_T \mapsto y_T^{-1}, y_T \mapsto x_T^{-1}$, for $T = V_\lambda, E_\lambda, A$	85
$\hat{\theta}_T$	automorphism of T, defined by $x_T \mapsto x_T, y_T \mapsto y_T^{-1}$, for $T = V_\lambda, E_\lambda, A$	86
$\hat{\theta}_T^{-1}$	automorphism of T, defined by $x_T \mapsto x_T^{-1}, y_T \mapsto y_T$, for $T = V_\lambda, E_\lambda, A$	86
$\hat{\alpha}_T^{-1}$	automorphism of T, defined by $x_T \mapsto x_T, y_T \mapsto y_T$, for $T = V_\lambda, E_\lambda, A$	86
$\hat{\beta}_T^{-1}$	automorphism of T, defined by $x_T \mapsto x_T^{-1}, y_T \mapsto y_T^{-1}$, for $T = V_\lambda, E_\lambda, A$	86
$\Theta_{1,\nu}$	morphism from V_ν to A_0, defined by $ss_1^\nu \mapsto z, z \mapsto t^{-1}$	87
$\Theta_{2,\nu}$	morphism from V_ν to A_0, defined by $ss_1^\nu \mapsto z^{-1}, z \mapsto t$	87
$\Theta_{3,\nu}$	morphism from V_ν to A_0, defined by $ss_1^\nu \mapsto z, z \mapsto t$	87
$\Theta_{4,\nu}$	morphism from V_ν to A_0, defined by $ss_1^\nu \mapsto z^{-1}, z \mapsto t^{-1}$	87
$\vartheta_{1,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto z^{-1}, z \mapsto (ss_1^{-\nu})^{-1}$	87
$\vartheta_{2,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto z, z \mapsto ss_1^{-\nu}$	87
$\vartheta_{3,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto z^{-1}, z \mapsto ss_1^{-\nu}$	87
$\vartheta_{4,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto z, z \mapsto (ss_1^{-\nu})^{-1}$	87
$\hat{\vartheta}_{1,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto ss_1^{-\nu}, z \mapsto z$	88
$\hat{\vartheta}_{2,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto (ss_1^{-\nu})^{-1}, z \mapsto z^{-1}$	88
$\hat{\vartheta}_{3,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto ss_1^{-\nu}, z \mapsto z^{-1}$	88
$\hat{\vartheta}_{4,\nu}$	morphism from V_ν to $V_{-\nu}$, defined by $ss_1^\nu \mapsto (ss_1^{-\nu})^{-1}, z \mapsto z$	88
$\xi_{1,\nu}$	morphism from V_ν to V_0, defined by $ss_1^\nu \mapsto s^{-1}, z \mapsto z^{-1}$	88
$\xi_{2,\nu}$	morphism from V_ν to V_0, defined by $ss_1^\nu \mapsto s, z \mapsto z$	88

$\xi_{3,v}$	morphism from V_v to V_0, defined by $ss_1^v \mapsto s^{-1}, z \mapsto z$	88
$\xi_{4,v}$	morphism from V_v to V_0, defined by $ss_1^v \mapsto s, z \mapsto z^{-1}$	88

$\xi_{5,v}$	morphism from V_v to V_0, defined by $ss_1^v \mapsto z, z \mapsto s$	88
$\xi_{6,v}$	morphism from V_v to V_0, defined by $ss_1^v \mapsto z^{-1}, z \mapsto s^{-1}$	88
$\xi_{7,v}$	morphism from V_v to V_0, defined by $ss_1^v \mapsto z, z \mapsto s^{-1}$	88
$\xi_{8,v}$	morphism from V_v to V_0, defined by $ss_1^v \mapsto z^{-1}, z \mapsto s$	88

Sets:

D_ϕ	the domain of a morphism ϕ	29
I_ϕ	the image of a morphism ϕ	29
Δ_P^ϕ	the $S \times S$ subgroup $\{(p, p\phi) \mid p \in P\}$ whenever $P \leq S$ and $\phi \in \mathrm{Hom}(P, S)$	29
$\mathcal{O}_\phi$	$(S \times S)/\Delta_P^\phi$, the set of right cosets of Δ_P^ϕ in $S \times S$	29
$\mathcal{T}_{full}$	the set $\{V_\lambda, E_\lambda, A \mid \lambda \in \{-1,0,1\}\}$	85
$\mathcal{T}$	a subset of $\mathcal{T}_{full}$ associated with a particular fusion system	89
$\mathcal{D}$	Set of all nontrivial S conjugacy class representatives in S, chosen so that $\mathcal{D} \supseteq \{Q \mid Q \leq \langle s_1 \rangle, \langle s_2 \rangle, \langle s_1 s_2^{-1} \rangle\}$	108
$\mathcal{D}_{V_\lambda}$	The set $\{\langle ss_1^\lambda \rangle, \langle z \rangle, \langle ss_1^\lambda, z \rangle\}$ consisting of all nontrivial S-conjugacy classes in V_λ defined for $\lambda = -1, 0, 1$	108
$\mathcal{D}_\lambda$	The set $\{\langle ss_1^\lambda \rangle, \langle t \rangle, \langle ss_1^\lambda, z \rangle, \langle t, z \rangle, \langle ss_1^\lambda, t \rangle\}$ consisting of all nontrivial S-conjugacy classes in E_λ excluding $\langle z \rangle$, defined for all $\lambda = -1, 0, 1$	108
$\mathcal{D}_{E_\lambda}$	Same as $\mathcal{D}_\lambda$	108
$\mathcal{D}_A$	The set $\{Q \leq A \mid Q \neq 1, \langle tz \rangle$ or $\langle tz^{-1} \rangle\}$	108

$\mathcal{M}_T$	the set $\{\mathrm{Id}\|_S, \omega, \eta, \omega \circ \eta\} \times \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$, for $T \in \mathcal{T}_{full}$	329
$\mathcal{M}_{\hat{V}_\lambda}$	the set $\{\mathrm{Id}\|_S, \omega, \eta, \omega \circ \eta\} \times \{\theta_{\hat{V}_\lambda}, \theta_{\hat{V}_\lambda}^{-1}, \alpha_{\hat{V}_\lambda}, \beta_{\hat{V}_\lambda}\}$, for $\lambda = -1, 1$	329
$\mathcal{M}_{\vartheta_\lambda}$	the set $\{\mathrm{Id}\|_S, \omega, \eta, \omega \circ \eta\} \times \{\vartheta_{1,\lambda}, \vartheta_{2,\lambda}, \vartheta_{3,\lambda}, \vartheta_{4,\lambda}\}$, for $\lambda = -1, 1$	329
$\mathcal{M}_{\xi_\lambda}$	the set $\{\mathrm{Id}\|_S, \omega, \eta, \omega \circ \eta\} \times \{\xi_{1,\lambda}, \xi_{2,\lambda}, \xi_{3,\lambda}, \xi_{4,\lambda}\}$, for $\lambda = -1, 1$	329
$\mathcal{M}_{\xi_\lambda^{-1}}$	the set $\{\mathrm{Id}\|_S, \omega, \eta, \omega \circ \eta\} \times \{\xi_{1,\lambda}^{-1}, \xi_{2,\lambda}^{-1}, \xi_{3,\lambda}^{-1}, \xi_{4,\lambda}^{-1}\}$, for $\lambda = -1, 1$	329

$\mathcal{M}_T(Q)$	the set $\{(\psi,\phi)\in\mathcal{M}_T \mid \phi\vert_Q\circ\psi = c_g\vert_Q$ for some $g\in S\}$, for $T\in\mathfrak{T}_{full}$	329
$\mathcal{M}_{\hat{V}_\lambda}(Q)$	the set $\{(\psi,\phi)\in\mathcal{M}_{\hat{V}_\lambda} \mid \phi\vert_Q\circ\psi = c_g\vert_Q$ for some $g\in S\}$, for $\lambda=-1,1$	329
$\mathcal{M}_{\vartheta_\lambda}(Q)$	the set $\{(\psi,\phi)\in\mathcal{M}_{\vartheta_\lambda} \mid \phi\vert_Q\circ\psi = c_g\vert_Q$ for some $g\in S\}$, for $\lambda=-1,1$	329
$\mathcal{M}_{\xi_\lambda}(Q)$	the set $\{(\psi,\phi)\in\mathcal{M}_{\xi_\lambda} \mid \phi\vert_Q\circ\psi = c_g\vert_Q$ for some $g\in S\}$, for $\lambda=-1,1$	329
$\mathcal{M}_{\xi_\lambda^{-1}}(Q)$	the set $\{(\psi,\phi)\in\mathcal{M}_{\xi_\lambda^{-1}} \mid \phi\vert_Q\circ\psi = c_g\vert_Q$ for some $g\in S\}$, for $\lambda=-1,1$	329

Others:

$e(\mathcal{F})$	the exoticity index of a fusion system $\mathcal{F}$	56
$\equiv_k$	$x\equiv_k y$ if and only if $x\equiv y$ mod 3^k	78
$C_G(\phi)$	the set of elements in a group G fixed by an automorphism ϕ of G	74
$\mathcal{U}$ or $\mathcal{U}(S)$	the universal fusion system on S	12

CHAPTER 2

PRELIMINARIES

2.1 Fusion systems

We start this section by introducing the theory of fusion systems. For reference we have used [1] by Aschbacher, Kessar & Oliver; and [8] by Park. Let G be a finite group. For any $g \in G$, denote by $c_g : G \to G$, the conjugation map given by $(x)c_g = g^{-1}xg$ for $x \in G$. For any $H, K \leq G$, such that $H^g \leq K$, we define $c_{g,H,K}$ to be the restricted conjugation map $c_g : H \to K$. When the meaning is clear, we may just write c_g to denote the conjugation between the subgroups. We define

$$\mathrm{Hom}_G(H,K) = \{c_{g,H,K} \mid H^g \leq K, g \in G\}$$

to be the set of group homomorphisms (which are necessarily injective) given by conjugation by an element in G. We define $\mathrm{Iso}_G(H,K)$ to be the set of maps in $\mathrm{Hom}_G(H,K)$ that are isomorphisms of groups. Since G is finite, note that either $\mathrm{Iso}_G(H,K) = \mathrm{Hom}_G(H,K)$ or $\mathrm{Iso}_G(H,K) = \emptyset$. We denote $\mathrm{Aut}_G(H) = \mathrm{Hom}_G(H,H)$. We know that $\mathrm{Aut}_G(H) \cong N_G(H)/C_G(H)$, where $N_G(H)$ and $C_G(H)$ denote the normaliser and centraliser of H in G respectively.

We write $\mathrm{Aut}(H)$ for the full automorphism group. Note that $\mathrm{Aut}_G(H)$ is a subgroup of $\mathrm{Aut}(H)$. The inner automorphism group is defined as $\mathrm{Inn}(H) = \mathrm{Aut}_H(H)$ and the outer automorphism group is the quotient group $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$. Now, we introduce fusion systems:

Definition 2.1.1. Given a finite group S, define a fusion system $\mathcal{F}$ on S to be a category whose objects are the subgroups of S and whose morphisms $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for $P,Q \leq S$ are injective group homomorphisms such that:

(i) $\mathrm{Hom}_{\mathcal{F}}(P,Q) \supseteq \mathrm{Hom}_S(P,Q)$.

(ii) If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, then the isomorphism $\bar{\phi} : P \to P\phi$, defined by $q \mapsto q\phi$, is in $\mathrm{Hom}_{\mathcal{F}}(P,P\phi)$.

(iii) If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, then the isomorphism $\bar{\phi}^{-1} : P\phi \to P$, defined by $q \mapsto q\phi^{-1}$, is in $\mathrm{Hom}_{\mathcal{F}}(P\phi,P)$.

Definition 2.1.2. Let G be a finite group and $S \leq G$. Define a fusion category $\mathcal{F}_S(G)$ of G on S to be the category whose objects are subgroups of S and whose morphisms $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for all $P,Q \leq S$.

With the additional requirement that $S \in \mathrm{Syl}_p(G)$, with p prime, $\mathcal{F}_S(G)$ is usually the most common motivational example of a fusion system stated in the literature. Let $\mathcal{F}$ be a fusion system on a finite group S. We write $\phi \in \mathcal{F}$, if $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ for some $P \leq S$. Recall that all morphisms in $\mathcal{F}$ are injective. Therefore, if $|P| = |Q|$, for some $P,Q \leq S$, then every homomorphism in $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is a group isomorphism. Thus, in such cases, we write $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ in place of $\mathrm{Hom}_{\mathcal{F}}(P,Q)$. In particular, $\mathrm{Hom}_{\mathcal{F}}(P,P\phi)$=$\mathrm{Iso}_{\mathcal{F}}(P,P\phi)$. We also write $\mathrm{Aut}_{\mathcal{F}}(P)$ in place of $\mathrm{Hom}_{\mathcal{F}}(P,P)$.

By property (i) of the definition of fusion systems, the fusion category $\mathcal{F}_S(S)$ is contained in $\mathcal{F}$. Moreover, for every $P \leq Q$ pair of subgroups of S, the category $\mathcal{F}$ contains the inclusion maps $P \hookrightarrow Q$. By properties (i) and (ii) every morphism in $\mathcal{F}$ is a composition of an isomorphism in $\mathcal{F}$ followed by an inclusion map.

Let $\mathcal{U}(S)$, or simply, $\mathcal{U}$, be the category whose objects are subgroups of S and whose morphisms are all injective group homomorphisms between subgroups of S. Then $\mathcal{U}$ is indeed a fusion system, called the *Universal Fusion System*. Note that it is unique for a given finite group S.

Let $\mathcal{V} \subset \mathcal{U}$ denote a subset of morphisms inside $\mathcal{U} = \mathcal{U}(S)$. Then, we write $\langle \mathcal{V} \rangle$ to denote the fusion system generated by $\mathcal{V}$. Explicitly, the morphisms in $\langle V \rangle$, are the compositions of restrictions of morphisms and their inverses in $\mathcal{V} \cup \mathrm{Inn}(S)$. It can be expressed as the intersection of all fusion systems over S that contains $\mathcal{V}$.

Definition 2.1.3. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over finite groups S_1 and S_2 respectively. Then we define $\mathcal{F}_1 \times \mathcal{F}_2$ to be the fusion system on $S_1 \times S_2$, generated by the set of morphisms $(\phi_1, \phi_2) \in \mathrm{Hom}(P_1 \times P_2, Q_1 \times Q_2)$ such that $\phi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$.

Set $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $S = S_1 \times S_2$. For $P, Q \leq S$ let P_i and Q_i be the projections into S_i of P and Q respectively. Denote by $\mathcal{E}$ the category whose objects are the subgroups of S and whose morphisms are given by:

$$\mathrm{Hom}_{\mathcal{E}}(P,Q) = \{\phi|_P \mid \phi = (\phi_1, \phi_2) \text{ where } \phi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i) \text{ and } P\phi \leq Q\}$$

For every $g = (g_1, g_2) \in S$ with $P^g \leq Q$, we necessarily have $c_{g_i} \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$. Thus $c_g = (c_{g_1}, c_{g_2})|_P \in \mathrm{Mor}_{\mathcal{E}}(P,Q)$. It follows that $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Mor}_{\mathcal{E}}(P,Q)$. Moreover, every morphism is an isomorphism followed by an inclusion. Thus $\mathcal{E}$ is a fusion system on S. Every morphism in $\mathcal{E}$ is a restriction of some morphism in $\mathrm{Hom}_{\mathcal{F}_1}(P_1, Q_1) \times \mathrm{Hom}_{\mathcal{F}_2}(P_2, Q_2)$. It follows that $\mathcal{E} = \mathcal{F}$.

Definition 2.1.4. [1, Definition 2.2 & 2.4] Let $\mathcal{F}$ be a fusion system on a finite group S. Then

1. $P, Q \leq S$ are called $\mathcal{F}$-conjugate if they are isomorphic as objects of the category $\mathcal{F}$. Denote by $P^{\mathcal{F}}$, the set of all subgroups of P that are $\mathcal{F}$-conjugate to P.

2. $P \leq S$ is called fully $\mathcal{F}$-normalized if $|N_S(Q)| \leq |N_S(P)|$ for all $Q \in P^{\mathcal{F}}$.

3. $P \leq S$ is called fully $\mathcal{F}$-centralized if $|C_S(Q)| \leq |C_S(P)|$ for all $Q \in P^{\mathcal{F}}$.

4. $P \leq S$ is called fully $\mathcal{F}$-automized if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

5. $P \leq S$ is called $\mathcal{F}$-receptive if for each $Q \in P^{\mathcal{F}}$ and $\phi \in \mathrm{Iso}_{\mathcal{F}}(Q,P)$, ϕ extends to $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ where

$$N_\phi = \{g \in N_S(Q) \mid (c_g|_Q)^\phi \in \mathrm{Aut}_S(P)\}.$$

By definition, $QC_S(Q) \leq N_\phi \leq N_S(Q)$. It can be checked [8, page 4] that N_ϕ is the largest subgroup of $N_S(Q)$ to which ϕ can extend, that is, if there exists $Q \leq R \leq N_S(Q)$ with $\psi \in \mathrm{Hom}(R,S)$ such that $\psi\,|_Q = \phi$, then $R \leq N_\phi$.

We finally define saturated fusion systems:

Definition 2.1.5. [1, Definition 2.2 & Proposition 2.5] Let $\mathcal{F}$ be a fusion system on a p-group S. We say that $\mathcal{F}$ is saturated if one of the two equivalent conditions hold:

1. Every $\mathcal{F}$-conjugacy class has a subgroup of S that is $\mathcal{F}$-receptive and fully $\mathcal{F}$-automized.
2. (a) *Sylow axiom*: If $P \leq S$ is fully $\mathcal{F}$- normalized, then P is fully $\mathcal{F}$-centralized and fully $\mathcal{F}$-automized.

   (b) *Extension axiom*: If $P \leq S$ is fully $\mathcal{F}$-centralized, then P is $\mathcal{F}$-receptive.

Definition 2.1.6. Let $\mathcal{F}$ be a fusion system on a finite group S. Then we say P is $\mathcal{F}$-centric if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$.

We shall shortly see that all saturated fusion systems are generated by the $\mathcal{F}$-automorphisms of certain specified subgroups called $\mathcal{F}$-essential subgroups. To be more precise, we introduce a number of definitions:

Definition 2.1.7. Let $\mathcal{F}$ be a fusion system on a finite group S. Then we say P is $\mathcal{F}$-radical if the largest normal p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ is trivial, that is, $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.

Definition 2.1.8. Let $\mathcal{F}$ be a fusion system on a finite p-group S. We say that a subgroup P is $\mathcal{F}$-essential if P is fully normalized, $\mathcal{F}$-centric and there exists finite $Q \lneq \mathrm{Out}_{\mathcal{F}}(P)$ such that $p \mid |Q|$, but for all $\phi \in \mathrm{Out}_{\mathcal{F}}(P) \backslash Q$, $p \nmid |Q \cap Q^\phi|$.

Theorem 2.1.9. *[1, Theorem 3.5](Alperin) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then*

$$\mathcal{F} = \langle \mathit{Aut}_{\mathcal{F}}(P) \mid P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential} \rangle$$

As a direct consequence, we have the following theorem:

Theorem 2.1.10. *[1, Theorem 3.6](Alperin-Goldschmidt) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $\phi \in \mathit{Hom}_{\mathcal{F}}(P,Q)$. Then there exists $\mathcal{F}$-essential subgroups P_i*

(including the possibility that $P_i = S$), for $1 \leq i \leq n$, and automorphisms $\phi_i \in Aut_{\mathcal{F}}(P_i)$, such that

$$\phi = (\phi_1|_{Q_0}) \circ \ldots \circ (\phi_n|_{Q_{n-1}}),$$

where, $P_i \geq \langle Q_{i-1}, Q_i \rangle$ with $Q_0 = P$ and $Q_n = Q$.

2.2 Group actions

We start this section by defining a group action:

Definition 2.2.1. Let G be a group. A (right) action of G on a set Ω is a map $\Omega \times G \to \Omega$, given by $(\alpha, g) \mapsto \alpha \cdot g$, such that $\alpha \cdot 1 = \alpha$ and $(\alpha \cdot g) \cdot h = \alpha \cdot (gh)$, for all $g, h \in G$ and $\alpha \in \Omega$.

By a group action of a group G on Ω we will mean a right group action on Ω. Let $\alpha \in \Omega$. We write $G_\alpha = \mathrm{Stab}_G(\alpha)$, the subgroup of elements in G which fix α under the action on Ω. Similarly we write $\alpha G = \mathrm{Orb}_G(\alpha)$, the orbit containing α. For any $K \leq G$, write $\Omega^K = \{\alpha \in \Omega \mid \alpha \cdot k = \alpha$ for all $k \in K\}$, the set of points in Ω fixed by every element of K.

Lemma 2.2.2. *Let G be a finite group with an action on Ω. Then for any $\alpha \in \Omega$ and $g \in G$, we have $G_{\alpha \cdot g} = G_\alpha^g$.*

Proof. Note that $h \in G_{\alpha \cdot g}$ if and only if $(\alpha \cdot g) \cdot h = \alpha \cdot g$ if and only if $\alpha \cdot ghg^{-1} = \alpha$ if and only if $h \in G_\alpha^g$. $\square$

Lemma 2.2.3. *Let Ω be a transitive G-set and $H = G_\alpha$ for some $\alpha \in \Omega$. Then for $K \leq G$ we have $\Omega^K \neq \emptyset$ if and only if K is a subgroup of a G-conjugate to H.*

Proof. Let $\theta \in \Omega$. Write $\theta = \alpha \cdot g$, for some $g \in G$. Note that

$$\begin{aligned} \alpha \cdot g \in \Omega^K &\Leftrightarrow \alpha \cdot gx = \alpha \cdot g \quad \text{for all} \quad x \in K \\ &\Leftrightarrow \alpha \cdot gxg^{-1} = \alpha \quad \text{for all} \quad x \in K \\ &\Leftrightarrow gxg^{-1} \in H \quad \text{for all} \quad x \in K \\ &\Leftrightarrow K \leq H^g \end{aligned}$$

This completes the proof. $\square$

Lemma 2.2.4. *Let $\Omega = G/H$, the set of right cosets of H in G. Under the action defined by right multiplication, $\Omega^K \neq \emptyset$ if and only if K is a subgroup of a G-conjugate to H.*

Proof. Note that in the case $\Omega = G/H$, the set of right cosets of H in G, is a transitive G-set and $H = \mathrm{Stab}_G(H)$. By Lemma 2.2.3, for $K \leq G$, we have $(G/H)^K \neq \emptyset$ if and only if K is a subgroup of a G-conjugate to H. $\square$

Lemma 2.2.5. *Let Ω be a transitive G-set and $H = G_\alpha$ for some $\alpha \in \Omega$. Then $|N_G(H)/H| = |\Omega^H|$.*

Proof. Let $\theta \in \Omega$. Write $\theta = \alpha \cdot g$ for some $g \in G$. Applying the same argument as in Lemma 2.2.3, we have

$$\begin{aligned} \alpha \cdot g \in \Omega^H &\Leftrightarrow \alpha \cdot gh = \alpha \cdot g \quad \text{for all} \quad h \in H \\ &\Leftrightarrow \alpha \cdot ghg^{-1} = \alpha \quad \text{for all} \quad h \in H \\ &\Leftrightarrow H \leq H^g \Leftrightarrow g \in N_G(H). \end{aligned}$$

Note also that $\alpha \cdot g = \alpha \cdot h$, for some $g, h \in G$, if and only if $Hg = Hh$. That is, we have a bijection: $N_G(H)/H \to \Omega^H$, given by $Hg \mapsto \alpha \cdot g$. It follows that $|N_G(H)/H| = |\Omega^H|$. $\square$

Lemma 2.2.6. *Let Ω be a G-set. Let $H, K \leq G$. If K is G-conjugate to H, then $|\Omega^K| = |\Omega^H|$.*

Proof. For all $g \in G$, we have a bijection $\Omega^H \to \Omega^{H^g}$, given by $\alpha \mapsto \alpha \cdot g$ $\square$

Lemma 2.2.7. *Let G be a group with an action on Ω. Then we have an isomorphism of G-sets, given by:*

$$\pi : \bigsqcup_{\alpha \in [\Omega]} G/G_\alpha \to \Omega : G_\alpha g \mapsto \alpha \cdot g$$

where $[\Omega]$ is a fixed set of representatives of G-orbits in Ω. In particular, the orbit containing α is isomorphic to G/G_α as G-sets.

Proof. It is easily seen that π is well-defined and is an equivariant map. To see that π is an injection, suppose $(G_\alpha g)\pi = (G_\beta h)\pi$ with $\alpha, \beta \in [\Omega]$. Then $\alpha \cdot gh^{-1} = \beta$. But then

$\beta \in \mathrm{Orb}(\alpha)$ and thus $\alpha = \beta$. It follows $gh^{-1} \in G_\alpha$ from which we conclude that $G_\alpha g = G_\beta h$. To see that π is a surjection, note that for any $\beta \in \Omega$, there exists $g \in G$ such that $\alpha \cdot g = \beta$ for some $\alpha \in [\Omega]$. □

Corollary 2.2.8. *Let G be a finite group with a G-action on the set Ω. Then for all $\alpha \in \Omega$ and $g \in G$, we have $G/G_\alpha \cong G/G_\alpha^g$ as G-sets. In particular,*

$$\Omega \cong \bigsqcup_{K \in [G]} n_K \cdot (G/K) \text{ as } G\text{-sets,}$$

where $n_K \geq 0$ denotes the number of orbits in Ω which corresponds to the coset space (G/K), and where $[G]$ is any set of representatives of conjugacy classes of subgroups of G.

Proof. Note that by Lemma 2.2.7 and 2.2.2 we have $G/G_\alpha^g = G/G_{\alpha \cdot g} \cong \mathrm{Orb}_G(\alpha \cdot g) = \mathrm{Orb}_G(\alpha) \cong G/G_\alpha$. The second claim follows from Lemma 2.2.7. □

Theorem 2.2.9. *[3, Theorem 2.4.5] Let G be a finite group which acts on the sets Ω and Θ. Then the G-sets Ω and Θ are isomorphic if and only if for all $H \leq G$ we have $|\Omega^H| = |\Theta^H|$.*

Proof. If $\phi : \Omega \to \Theta$ is an isomorphism of G-sets, then ϕ induces a bijection

$$\Omega^H \to \Theta^H \text{ given by } \alpha \mapsto (\alpha)\phi.$$

Now we prove the converse. Let $[G]$ be some fixed set of representatives of conjugacy classes of subgroups of G. Let $|[G]| = r$. Order the representatives $K_1, ..., K_{|[G]|}$ in $[G]$ so that $|K_i| \leq |K_j|$ if and only if $i \leq j$ for all $i, j \in \{1, ..., r\}$. Then by 2.2.8, we have the following isomorphism of G-sets:

$$\Omega \cong \bigsqcup_{i=1}^{r} n_{K_i} \cdot (G/K_i) \text{ and } \Theta \cong \bigsqcup_{i=1}^{r} m_{K_i} \cdot (G/K_i),$$

for some n_{K_i} and m_{K_i} copies of (G/K_i). Suppose that for all $H \leq G$ we have $|\Omega^H| = |\Theta^H|$. Then for all $K_j \in [G]$,

$$\sum_{i=1}^{r} (n_{K_i} - m_{K_i}) |(G/K_i)^{K_j}| = 0.$$

In this way we obtain an $r \times r$ matrix M and a $1 \times r$ vector N with entries $M_{ij} = |(G/K_i)^{K_j}|$ and $N_{ij} = (m_{K_i} - n_{K_i})$ respectively. This implies $NM = 0$, the $1 \times r$ zero vector. By Lemma

2.2.3, $|(G/K_i)^{K_j}| \neq \emptyset$ if and only if K_j is a subgroup of a G-conjugate of K_i. So $|(G/K_i)^{K_j}| \neq \emptyset$ implies $j \leq i$. Also note that for all $i \in \{1,..,r\}$ we have $|(G/K_i)^{K_i}| = |N_G(K_i)/K_i| \geq 1$, by Lemma 2.2.5. Thus M is an lower triangular matrix with all diagonal entries nonzero. It follows that M is a non-singular matrix, and therefore $NMM^{-1} = 0M^{-1}$, that is, $N = 0$. This gives us the required solutions $n_{K_i} = m_{K_i}$ for all $i \in \{1,...,r\}$. Thus the G-sets Ω and Θ are isomorphic. □

Lemma 2.2.10. *Assume G is a finite group which acts on sets Ω and Θ. Let $\mathcal{H}$ be a set of subgroups of G, such that, if $H,K \in \mathcal{H}$, then $\langle H,K\rangle \in \mathcal{H}$. Assume that $|\Omega^H| = |\Theta^H|$ for all $H \in \mathcal{H}$. Then*

$$|\bigcup_{H\in\mathcal{H}} \Omega^H| = |\bigcup_{H\in\mathcal{H}} \Theta^H|.$$

Proof. First note that $\Omega^H \cap \Omega^K = \Omega^{\langle H,K\rangle}$ clearly holds for any two subgroups H,K of G. Listing all elements of $\mathcal{H}$ as $H_1,...,H_n$, we see that:

$$\begin{aligned}
|\bigcup_{1\leq i\leq n} \Omega^{H_i}| &= \sum_{k=1}^{n} (-1)^{k+1} (\sum_{1\leq i_1<...<i_k\leq n} |\Omega^{H_{i_1}} \cap ... \cap \Omega^{H_{i_k}}|) \\
&= \sum_{k=1}^{n} (-1)^{k+1} (\sum_{1\leq i_1<...<i_k\leq n} |\Omega^{\langle H_{i_1},...,H_{i_k}\rangle}|) = \sum_{k=1}^{n} (-1)^{k+1} (\sum_{1\leq i_1<...<i_k\leq n} |\Theta^{\langle H_{i_1},...,H_{i_k}\rangle}|) \\
&= \sum_{k=1}^{n} (-1)^{k+1} (\sum_{1\leq i_1<...<i_k\leq n} |\Theta^{H_{i_1}} \cap ... \cap \Theta^{H_{i_k}}|) = |\bigcup_{1\leq i\leq n} \Theta^{H_i}|.
\end{aligned}$$

This proves the lemma. □

Let G be a group which acts on Ω. We say that the action on Ω is free if for any $\alpha \in \Omega$, $\alpha \cdot g = \alpha$ implies $g = 1$. Otherwise we call the action non-free.

Corollary 2.2.11. *Assume G is a finite group which acts on sets Ω and Θ. Assume that $|\Omega^H| = |\Theta^H|$ for all $1 \neq H \leq G$. Then the number of elements in the non-free orbits in Ω is equal to the number of elements in non-free orbits in Θ.*

Proof. Take $\mathcal{H} = \{H \leq G \mid H \neq 1\}$ and apply Lemma 2.2.10. □

Definition 2.2.12. Let H be a finite group. Let Ω be a (right) H-set. We say that the action of H on Ω is fixed point free (or semiregular) if for any $g,h \in H, \omega \in \Omega$, such that $\omega \cdot g = \omega \cdot h$

then we have $g = h$. We say that it is transitive, if for any $\omega_1, \omega_2 \in \Omega$, there exists $g \in H$ such that $\omega_2 = \omega_1 \cdot g$. We say that it is regular if it is both fixed point free and transitive.

For the rest of this subsection, let H be any finite group, and suppose Ω is a (right) H-set. Suppose that H acts semiregularly on Ω. Define

$$G = \{\pi \in \mathsf{Sym}(\Omega) \mid (\omega \cdot x)\pi = (\omega\pi) \cdot x \text{ for all } \omega \in \Omega, x \in H\}.$$

Identify H as a subgroup of $\mathsf{Sym}(\Omega)$, in the following way:

$$\iota : H \hookrightarrow \mathsf{Sym}(\Omega) : x \mapsto (\omega \mapsto \omega \cdot x).$$

Note that

$$\begin{aligned}
\pi \in G &\Leftrightarrow (\omega \cdot x)\pi = (\omega\pi) \cdot x && \text{(for all } \omega \in \Omega, x \in H) \\
&\Leftrightarrow (\omega(x\iota))\pi = (\omega\pi)(x\iota) && \text{(for all } \omega \in \Omega, x \in H) \\
&\Leftrightarrow (x\iota)\pi = \pi(x\iota) && \text{(for all } x \in H) \\
&\Leftrightarrow \pi \in C_{\mathsf{Sym}(\Omega)}((H\iota)).
\end{aligned}$$

Thus $G = C_{\mathsf{Sym}(\Omega)}((H)\iota)$. We will write $C_{\mathsf{Sym}(\Omega)}(H)$ to denote the centralizer of $(H)\iota$ in $\mathsf{Sym}(\Omega)$. Suppose that Ω is semiregular. Let O be a orbit of Ω, and fix a $\omega_0 \in O$. Let $\omega \in O$. Then there exists a unique $x_\omega \in H$ such that $\omega = \omega_0 \cdot x_\omega$. Note that x_ω is dependent on the choice of ω_0.

Lemma 2.2.13. *Suppose that H acts regularly on Ω. Fix $\omega_0 \in \Omega$ and let $\omega \in \Omega$. For some $y \in H$, define the following map:*

$$\pi_y : \Omega \to \Omega, \textit{ given by, } \omega \mapsto (\omega_0 \cdot y) \cdot x_\omega,$$

where $\omega = \omega_0 \cdot x_\omega$. Then $\pi_y \in C_{\mathsf{Sym}(\Omega)}(H)$.

Proof. We first show that π_y is indeed a permutation. To show that π_y is an injection, let

$\omega, \omega' \in \Omega$. Then $\omega = \omega \cdot x_\omega$ and $\omega' = \omega_0 \cdot x_{\omega'}$, for some $x_\omega, x_{\omega'} \in H$. Note that

$$\begin{aligned}\omega\pi_y = \omega'\pi_y &\Leftrightarrow (\omega_0 \cdot y) \cdot x_\omega = (\omega_0 \cdot y) \cdot x_{\omega'} \\ &\Leftrightarrow x_\omega = x_{\omega'} \Leftrightarrow \omega = \omega'.\end{aligned}$$

To show that π_y is a surjection, let $\omega \in \Omega$. Then $\omega = \omega_0 \cdot x_\omega$ for some $x_\omega \in H$. Define $x_{\omega'} = y^{-1}x_\omega$. Then

$$\begin{aligned}\omega'\pi_y &= (\omega_0 \cdot y) \cdot x_{\omega'} = (\omega_0 \cdot y) \cdot (y^{-1} \cdot x_\omega) \\ &= \omega_0 \cdot x_\omega = \omega.\end{aligned}$$

It follows that π_y is a permutation. To show that π_y centralizes H, note that for any $x \in H$, we have

$$\begin{aligned}(\omega \cdot x)\pi_y &= ((\omega_0 \cdot x_\omega) \cdot x)\pi_y = (\omega_0 \cdot (x_\omega x))\pi_y \\ &= (\omega_0 \cdot y) \cdot (x_\omega x) = ((\omega_0 \cdot y) \cdot x_\omega) \cdot x \\ &= ((\omega_0 \cdot x_\omega)\pi_y) \cdot x = (\omega\pi_y) \cdot x.\end{aligned}$$

Hence $\pi_y \in C_{\mathsf{Sym}(\Omega)}(H)$. This completes the proof. □

Lemma 2.2.14. *If H acts regularly on Ω, then the map*

$$\phi : H \to C_{Sym(\Omega)}(H), \text{ defined by, } y \mapsto \pi_{y^{-1}}$$

is a group isomorphism.

Proof. Fix $\omega_0 \in \Omega$, and recall that for any $\omega \in \Omega$, we can write $\omega = \omega_0 \cdot x_\omega$ for some $x_\omega \in H$.

By recalling the definition of π_y, note that for any $y, y' \in H$, we have

$$\begin{aligned}\omega(\pi_y\pi_{y'}) = (\omega\pi_y)\pi_{y'} = ((\omega_0 \cdot y) \cdot x_\omega)\pi_{y'} \\ = (\omega_0 \cdot (yx_\omega))\pi_{y'} = (\omega_0 \cdot y') \cdot (yx_\omega) \\ = \omega_0 \cdot (y'y) \cdot x_\omega.\end{aligned}$$

Thus, $\pi_y\pi_{y'} = \pi_{y'y}$. Using this property, we see that

$$\begin{aligned}(yy')\phi = \pi_{(yy')^{-1}} = \pi_{y'^{-1}y^{-1}} \\ = \pi_{y^{-1}}\pi_{y'^{-1}} = (y\phi)(y'\phi).\end{aligned}$$

Thus, ϕ is a group homomorphism. Moreover,

$$\begin{aligned}\ker(\phi) &= \{y \in H \mid y\phi = \mathrm{Id}_\Omega\} \\ &= \{y \in H \mid \pi_{y^{-1}} = \mathrm{Id}_\Omega\} \\ &= \{y \in H \mid \omega = \omega_0 \cdot x_\omega = (\omega_0 \cdot y^{-1}) \cdot x_\omega \text{ for all } \omega \in \Omega\} \\ &= \{y \in H \mid \omega_0 = \omega_0 \cdot y^{-1}\} \\ &= \{1_H\}.\end{aligned}$$

Thus, ϕ is an injection. Let $\pi \in C_{\mathrm{Sym}(\Omega)}(H)$. Then, by regularity, there exists a unique $x_\pi \in H$ such that $\omega_0\pi = \omega_0 \cdot x_\pi$.

$$\omega\pi = (\omega_0 \cdot x_\omega)\pi = (\omega_0\pi) \cdot x_\omega = (\omega_0 \cdot x_\pi) \cdot x_\omega = \omega\pi_{x_\pi}.$$

Thus, $\pi = \pi_{x_\pi}$. This shows that ϕ is a bijection and completes the proof. □

Lemma 2.2.15. *Suppose H acts semiregularly on Ω. Then $C_{Sym(\Omega)}(H) \cong H \wr Sym(I)$, where I is the set of orbits of Ω under the action of H.*

Proof. Write $\Omega = \sqcup_{i\in I} O_i$, as the union of its orbits. Fix $\omega_i \in O_i$ for all $i \in I$. Let $\omega \in \Omega$. Then there exists $i \in I$ such that $\omega = \omega_i \cdot x_\omega$, for some $x_\omega \in H$. Let $\sigma \in \mathrm{Sym}(I)$. Define the following maps:

$$\pi_{i,y} : \Omega \to \Omega, \text{ given by, } \begin{cases} \omega \mapsto (\omega_i \cdot y) \cdot x_\omega & \text{if } \omega \in O_i \\ \omega \mapsto \omega & \text{otherwise} \end{cases}$$

$$\pi_\sigma : \Omega \to \Omega, \text{ given by, } \omega \mapsto \omega_{(i\sigma)} \cdot x_\omega.$$

Define

$$B_i = \{\pi_{i,y} \mid y \in H\}, \quad \text{and} \quad K = \{\pi_\sigma \mid \sigma \in \mathsf{Sym}(I)\}.$$

Then, by definition, note that $B_i \cap B_j = 1$ for $i \neq j$, $B_i \cap K = 1$, and $B_i^{\pi_\sigma} = B_{(i\sigma)}$. If

$$B = \prod_{i \in I} B_i, \quad \text{then we have} \quad B \rtimes K \cong H \wr \mathsf{Sym}(I).$$

It suffices to show that $B \rtimes K = C_{\mathsf{Sym}(\Omega)}(H)$.

<u>Claim 1</u>: $B \rtimes K \leq C_{\mathsf{Sym}(\Omega)}(H)$.

We show that $\pi_{i,y}, \pi_\sigma \in C_{\mathsf{Sym}(\Omega)}(H)$ for all $i \in I, y \in H$ and $\sigma \in \mathsf{Sym}(\Omega)$. The map defined by $\pi_{i,y}$ is the same map as the one in Lemma 2.2.13, when restricted to the action of H to O_i. By Lemma 2.2.13, it follows that $\pi_{i,y}$ is indeed a permutation and centralizes H, that is, $\pi_{i,y} \in C_{\mathsf{Sym}(\Omega)}(H)$.

Let $\omega, \omega' \in \Omega$. Then $\omega = \omega_i \cdot x_\omega$ and $\omega' = \omega_j \cdot x_{\omega'}$ for some $x_\omega, x_{\omega'} \in \Omega$ and $i, j \in I$. Let

$\sigma \in \mathsf{Sym}(I)$. Then

$$\begin{aligned}
\omega\pi_\sigma = \omega'\pi_\sigma &\Leftrightarrow (\omega_i \cdot x_\omega)\pi_\sigma = (\omega_j \cdot x_{\omega'})\pi_\sigma \\
&\Leftrightarrow \omega_{(i\sigma)} \cdot x_\omega = \omega_{(j\sigma)} \cdot x_{\omega'} \\
&\Leftrightarrow \omega_{(j\sigma)} \in O_{(i\sigma)} \\
&\Leftrightarrow i\sigma = j\sigma \\
&\Leftrightarrow i = j.
\end{aligned}$$

Since H acts semiregularly on Ω, $\omega_{(i\sigma)} \cdot x_\omega = \omega_{(j\sigma)} \cdot x_{\omega'}$, if and only if, $x_\omega = x_{\omega'}$. Thus $\omega\pi_\sigma = \omega'\pi_\sigma$, if and only if, $\omega = \omega'$. It follows that π_σ is a permutation. Moreover, for any $x \in H$, we have

$$\begin{aligned}
(\omega \cdot x)\pi_\sigma &= ((\omega_i \cdot x_\omega) \cdot x)\pi_\sigma = (\omega_i \cdot (x_\omega x))\pi_\sigma \\
&= \omega_{(i\sigma)} \cdot (x_\omega x) = (\omega_{(i\sigma)} \cdot x_\omega) \cdot x \\
&= (\omega\pi_\sigma) \cdot x.
\end{aligned}$$

Thus $\pi_\sigma \in C_{\mathsf{Sym}(\Omega)}(H)$. We have shown that $\pi_{i,y}, \pi_\sigma \in C_{\mathsf{Sym}(\Omega)}(H)$ for all $i \in I, y \in H$ and $\sigma \in \mathsf{Sym}(I)$. Thus $B \rtimes K \leq C_{\mathsf{Sym}(\Omega)}(H)$.

<u>Claim 2</u>: $C_{\mathsf{Sym}(\Omega)}(H) \leq B \rtimes K$.

Let $\pi \in C_{\mathsf{Sym}(\Omega)}(H)$. For any $\omega \in \Omega$, we have $\omega = \omega_i \cdot x_\omega$ for some $i \in I$ and $x_\omega \in H$, in which case we may write $\omega_i\pi = \omega_{(i\sigma)} \cdot x_i$ for some $\sigma \in \mathsf{Sym}(I)$ and $x_i \in H$. Thus

$$\begin{aligned}
\omega\pi &= (\omega_i \cdot x_\omega)\pi = (\omega_i\pi) \cdot x_\omega \\
&= (\omega_{(i\sigma)} \cdot x_i) \cdot x_\omega = (\omega_{(i\sigma)} \cdot (x_i x_\omega)) \\
&= (\omega_i \cdot (x_i x_\omega))\pi_\sigma = ((\omega_i \cdot x_i) \cdot x_\omega)\pi_\sigma \\
&= (\omega\pi_{i,x_i})\pi_\sigma.
\end{aligned}$$

It follows that

$$\pi = [\prod_{i \in I} \pi_{i,x_i}] \cdot \pi_\sigma \in B \rtimes K.$$

This completes the proof of the claim, and hence the proof of the Lemma. $\square$

2.3 Homocyclic p-groups

For any group G, we define

$$\mathrm{rank}(G) = \min\{|X| \mid \langle X \rangle = G, X \subset G\},$$

the cardinality of the smallest generating set of a group G.

Let p be a prime. A homocyclic p-group P of rank r, is defined to be a finite direct product of r cyclic groups of order p^k. That is, P is isomorphic to the additive group $\mathbb{Z}/p^k\mathbb{Z} \times ... \times \mathbb{Z}/p^k\mathbb{Z}$. When $k = 1$, P is called elementary abelian. In that case, $\mathbb{Z}/p\mathbb{Z}$ is a finite field of p elements, and the group P itself can be considered as the r-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$. It follows that $\mathrm{Aut}(P)$ is isomorphic to $\mathrm{GL}_r(\mathbb{Z}/p\mathbb{Z})$. For general k, $\mathbb{Z}/p^k\mathbb{Z}$ is the ring of integers, modulo p^k. Therefore, $\mathrm{Aut}(P)$ is isomorphic to $\mathrm{GL}_r(\mathbb{Z}/p^k\mathbb{Z})$, the group of invertible $r \times r$ matrices over the ring $\mathbb{Z}/p^k\mathbb{Z}$. Write $P = \langle g_1, ..., g_r \rangle$. Explicitly, automorphisms can be represented by

$$\begin{bmatrix} g_1 \mapsto g_1^{a_{11}} \cdot .. \cdot g_r^{a_{1r}} \\ \vdots \\ g_r \mapsto g_1^{a_{r1}} \cdot .. \cdot g_r^{a_{rr}} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} \dots & a_{1,r} \\ \vdots & \vdots \\ a_{r1} \dots & a_{rr} \end{bmatrix}$$

Note that we have an automorphism if and only if the determinant is invertible in $\mathbb{Z}/p^k\mathbb{Z}$.

Let G be a finite p-group. For $n \in \mathbb{N}$, we define

$$\Omega_n(G) = \langle g \in G \mid g^{p^n} = 1 \rangle.$$

We also define $\Phi(G)$, the Frattini subgroup of G, to be the intersection of all maximal subgroups in G. We say that $g \in G$ is a non-generator if whenever I is a generating set of G such that $g \in I$, then $I \backslash \{g\}$ is also a generating set of G. By [6, Chapter 5, Theorem 1.1], $\Phi(G)$ is the set of all non-generators of G.

Lemma 2.3.1. *Let P be a finite group. Then*

1. *$\Phi(P)$ is characteristic, and therefore, a normal subgroup of G.*
2. *If P is a p-group, then $\Phi(P)$ contains the commutator subgroup $[P,P]$, and is the smallest normal subgroup of P such that $P/\Phi(P)$ is elementary abelian.*

Proof. Let M be a maximal subgroup of P and let $\phi \in \mathrm{Aut}(M)$. Then $(M)\phi$ is also a maximal subgroup. Thus, $(\Phi(P))\alpha = (\cap_{M \text{ maximal}} M)\alpha = \cap_{M \text{ maximal}}(M\alpha) = \Phi(P)$, that is, $\Phi(P)$ is characteristic, and therefore, normal in P. By [6, Chapter 6, Theorem 1.6], we also have $[P,P] \leq \Phi(P)$. $\square$

For any finite group G, we define exponent of G to be the least common multiple of the orders of elements of G.

Lemma 2.3.2. *Let P be a homocyclic group. Define $Q = C_{Aut(P)}(P/\Phi(P))$. Then*

1. *$P/\Phi(P) \cong \Omega_1(P)$.*
2. *The map $Aut(P) \to Aut(\Omega_1(P))$, defined by $\psi \mapsto \psi|_{\Omega_1(P)}$, is a homomorphism with kernel Q. In particular, Q is normal in $Aut(P)$.*
3. *Q is a p-group.*

Proof. P is a p-group of some rank r and exponent p^k. Say, $P = \langle g_1, ..., g_r \rangle$.

1. Note that $\Omega_1(P) = \langle g_1^{p^{k-1}}, .., g_r^{p^{k-1}} \rangle$. Consider the map: $\phi : P \to \Omega_1(P), g \mapsto g^{p^{k-1}}$. This is a surjective homomorphism, with kernel, $\ker(\phi) = \{g \in P \mid g^{p^{k-1}} = 1\}$. We

show $\ker(\phi) = \Phi(P)$. Write $P_i = \langle g_1, .., g_i^p, .., g_r \rangle$. Then P_i is maximal. Thus, $\Phi(P) \leq \bigcap_{i=1,..,r} P_i = \langle g_1^p, .., g_r^p \rangle$. Suppose $\Phi(P) \lneq \langle g_1^p, .., g_r^p \rangle$. Then $g_i^p \notin M$, for some maximal $M < P$. Thus, $M\langle g_i^p \rangle = P$. This means $g_i = g_i^a g_i^{bp}$ for some $g_i^a \in M$ and $p \nmid a$. But then $g_i \in M$, a contradiction. It follows that $\ker(\phi) = \Phi(P)$, and by the first isomorphism theorem, the map $\bar{\phi} : P/\Phi(P) \to \Omega_1(P)$, defined by $g\Phi(P) \mapsto g^{p^{k-1}}$ is an isomorphism.

2. Define the map: $\gamma : \mathrm{Aut}(P) \to \mathrm{Aut}(\Omega_1(P))$ by $\psi \mapsto \psi|_{\Omega_1(P)}$. Since $\Omega_1(P)$ is a characteristic subgroup, the map is well-defined. It is clearly a homomorphism. We show $\ker(\gamma) = Q$. Suppose $\psi \in Q$. Let $g \in \Omega_1(P)$. Then $g = h_1^{p^{k-1}}$ for some $h_1 \in P$. Now, $g\psi = (h_1^{p^{k-1}})\psi = (h_1\psi)^{p^{k-1}}$. Since $\psi \in Q$, $h_1\psi = h_1h_2$, for some $h_2 \in \Phi(P)$. Thus, $g\psi = (h_1h_2)^{p^{k-1}} = h_1^{p^{k-1}} = g$, that is, $\psi \in \ker(\gamma)$. Now suppose $\psi \notin Q$. This means there exists $h \in P$ such that $(h\Phi(P))\psi \neq h\Phi(P)$. Under the isomorphism $\bar{\phi}$, we have $(h(\Phi(P))\psi)\bar{\phi} = ((h\psi)\Phi(P))\bar{\phi} = (h\psi)^{p^{k-1}} = (h^{p^{k-1}})\psi$ and $(h\Phi(P))\bar{\phi} = h^{p^{k-1}}$. It follows that $(h^{p^{k-1}})\psi \neq h^{p^{k-1}}$, and since $h^{p^{k-1}} \in \Omega_1(P)$, we have, $\psi \notin \ker(\gamma)$. Thus, $\ker(\gamma) = Q$.

3. By a Theorem of Burnside [6, Chapter 5, Theorem 1.4] no p'-automorphism of P, other than the identity map, acts trivially on $P/\Phi(P)$. It follows that Q is a p-group.

□

Lemma 2.3.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Define $Q = C_{Aut(P)}(P/\Phi(P))$. If $P \leq S$ is homocyclic and $\mathcal{F}$-essential, then*

1. *$Aut_{\mathcal{F}}(P) \cap Q = Inn(P) = 1$.*

2. *$Aut_{\mathcal{F}}(P)$ is isomorphic to a subgroup of $Aut(P/\Phi(P))$.*

Proof. We show $\mathrm{Aut}_{\mathcal{F}}(P) \cap Q = \mathrm{Inn}(P)$: Let $g \in P$. Then, by Lemma 2.3.1, for all $h\Phi(P) \in P/\Phi(P)$, we have $(h\phi(P))^g = h^g\Phi(P) = h[h,g]\Phi(P) = h\Phi(P)$. Hence $c_g \in Q$. This shows us $\mathrm{Inn}(P) \leq \mathrm{Aut}_{\mathcal{F}}(P) \cap Q$. By Lemma 2.3.2, Q is a normal p-subgroup of $\mathrm{Aut}(P)$. This means $\mathrm{Aut}_{\mathcal{F}}(P) \cap Q \leq O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Since P is $\mathcal{F}$-essential, P is $\mathcal{F}$-radical, that is, $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$. It follows that $\mathrm{Aut}_{\mathcal{F}}(P) \cap Q = \mathrm{Inn}(P)$. Since P is abelian, $\mathrm{Inn}(P) = 1$, proving part 1.

Using the second isomorphism theorem, we have

$$\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_{\mathcal{F}}(P) \cap Q \cong \mathrm{Aut}_{\mathcal{F}}(P)Q/Q \leq \mathrm{Aut}(P)/Q.$$

By Lemma 2.3.2, $\mathrm{Aut}(P)/Q$ is isomorphic to a subgroup of $\mathrm{Aut}(\Omega_1(P))$. This proves part 2. □

2.4 Extraspecial groups

Let p be an odd prime. Write $E = p_+^{1+2}$, the extraspecial groups of order p^3 and exponent p. As given in [13], we have the following presentation

$$E \cong \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = [x, y]z^{-1} = 1 \rangle$$

and every element of E can be written uniquely in the form $x^a y^b z^c$ for some $a, b, c \in \{0, 1, ..., p-1\}$. Every automorphism of E can be uniquely defined by the extension to a homomorphism of the maps

$$\begin{array}{rcl} x & \mapsto & x^a y^b z^i \\ y & \mapsto & x^c y^d z^j \end{array}$$

where $ad - bc \neq 0 \pmod{p}$ and $a, b, c, i, j \in \{0, 1, ..., p-1\}$. It can be checked, that in this case z maps to z^{ad-bc}. Moreover, every inner automorphism can be given by the extension to a homomorphism of the maps

$$\begin{array}{rcl} x & \mapsto & xz^i \\ y & \mapsto & yz^j \end{array}$$

with z, being central in E, mapping to itself. It follows that the elements of the outer automorphism group of E can be identified as elements in $\mathsf{GL}_2(p)$ in the following way

$$\begin{bmatrix} x \mapsto x^a y^b \\ y \mapsto x^c y^d \end{bmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $\begin{bmatrix} x \mapsto x^a y^b \\ y \mapsto x^c y^d \end{bmatrix}$ denotes the coset of inner automorphisms in E, containing the automorphism generated by the maps $x \mapsto x^a y^b$ and $y \mapsto x^c y^d$. Note that, if the image of z is z^j, the determinant of the corresponding matrix is j. The centre of E is of order p, and thus $Z(E) = \langle z \rangle$. The group E contains $p+1$ elementary abelian groups of order p^2, all of which contain the centre of E, and which we denote by $V_k = \langle v_k, z \rangle$, for $k = 0, .., p$. Explicitly, we can write

$$v_k = \begin{cases} xy^k & \text{if } 0 \leq k \leq p-1 \\ y & \text{if } k = p \end{cases}$$

Considering V_k as a vector space over a finite field $\mathbb{F}_p$ of order p, with its basis elements the two given generators of V_k, we see that

$$\mathrm{Aut}_E V_k \cong \left\{ \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \Big| j \in \mathbb{F}_p \right\}.$$

Let F be the subgroup of $\mathrm{Out}(E)$ with $F \cong \mathsf{SL}_2(p)$. Write F_1 to be the set of maps in $\mathrm{Aut}(E)$ which correspond to F, that is, $F_1/\mathrm{Inn}(E) \cong F$. Since every map in F_1 fixes z, we see that

$$\mathrm{Aut}_{F_1} V_k \cong \left\{ \begin{pmatrix} i & j \\ 0 & 1 \end{pmatrix} \Big| i \in \mathbb{F}_p^*, j \in \mathbb{F}_p \right\}.$$

CHAPTER 3

EQUIVALENT MORPHISMS IN FUSION SYSTEMS

Let S be a finite group, and let $\phi \in \mathrm{Hom}(P,S)$ for some $P \leq S$. Recall that, if $\mathcal{F}$ is a fusion system on S, we say that ϕ is a homomorphism (or either a morphism, or a map) in $\mathcal{F}$, and write $\phi \in \mathcal{F}$, if $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Recall that $\mathcal{U}(S)$, or simply $\mathcal{U}$, is the universal fusion system on S. We now introduce some notation which will be useful throughout the thesis.

Let $\phi \in \mathcal{U}$. Then $\phi \in \mathrm{Hom}(P,S)$ for some $P \leq S$. Define

$$D_\phi = \mathrm{Dom}(\phi) = P \text{ and } I_\phi = \mathrm{Im}(\phi),$$

the domain and image of ϕ in S respectively. We define

$$\Delta_P^\phi = \{(u,(u)\phi) \mid u \in P\},$$

a diagonal subgroup of $S \times S$ determined by ϕ. This can be interpreted as the graph of ϕ. By $\mathcal{O}_\phi$ we denote the right coset space $(S \times S)/\Delta_P^\phi$ in the direct product $S \times S$. Let $\psi \in \mathcal{U}$. Then $\psi \in \mathrm{Hom}(Q,S)$ for some $Q \leq S$. We also define

$$\mathcal{O}_\phi^\psi = ((S \times S)/\Delta_P^\phi)^{\Delta_Q^\psi},$$

the set of points in $\mathcal{O}_\phi$ fixed by Δ_Q^ψ via right multiplication. We will use the following facts throughout the upcoming results:

Lemma 3.0.1. *Let $\phi \in \mathcal{U}(S)$. Then for any $g,h \in S$, we have $(\Delta^{\phi}_{D_\phi})^{(g,h)} = \Delta^{c_{g^{-1}} \circ \phi \circ c_h}_{(D_\phi)c_g}$.*

Proof. By definition

$$\begin{aligned}(\Delta^{\phi}_{D_\phi})^{(g,h)} &= (g^{-1},h^{-1})\Delta^{\phi}_{D_\phi}(g,h) \\ &= \{(g^{-1}pg, h^{-1}(p\phi)h) \mid p \in D_\phi\} \\ &= \Delta^{c_{g^{-1}} \circ \phi \circ c_h}_{(D_\phi)c_g}.\end{aligned}$$

This completes the proof. □

Lemma 3.0.2. *Let $\phi \in \mathcal{U}(S)$. Then $H \leq \Delta^{\phi}_{D_\phi}$, if and only if, $H = \Delta^{\psi}_{D_\psi}$ and $\psi = \phi|_{D_\psi}$.*

Proof. Suppose $H \leq \Delta^{\phi}_{D_\phi}$. Define $X = \{x \mid (x, x\phi) \in H\}$. Then $H = \Delta^{\phi|_X}_{X}$, as required. On the other hand, if $H = \Delta^{\psi}_{D_\psi}$ for some $\psi = \phi|_{D_\psi}$, then by definition $H \leq \Delta^{\phi}_{D_\phi}$. This completes the proof. □

Lemma 3.0.3. *Let $\phi \in \mathcal{U}$ and $x \in D_\phi$. Then $c_x|_{D_\phi} \circ \phi = \phi \circ c_{(x\phi)}$.*

Proof. For any $g \in D_\phi$, we have $(gc_x)\phi = (x^{-1}gx)\phi = (x\phi)^{-1}(g\phi)(x\phi) = (g\phi)c_{(x\phi)}$. □

3.1 Definition and results

Definition 3.1.1. Let $\phi, \psi \in \mathcal{U}(S)$. Then we say that ϕ is S-S-equivalent to ψ, and write $\phi \sim \psi$, if there exists $x, y \in S$ such that

$$(D_\psi)^x = D_\phi \quad \text{and} \quad c_x|_{D_\psi} \circ \phi \circ c_y = \psi.$$

Writing $\bar{\psi}$ and $\bar{\phi}$ to denote the restriction of ψ and ϕ to their domains and images respectively, the following diagram illustrates the condition:

$$\begin{array}{ccc} D_\psi & \xrightarrow{\bar{\psi}} & I_\psi \\ {\scriptstyle c_x|_{D_\psi}}\downarrow & & \uparrow{\scriptstyle c_y|_{I_\phi}} \\ D_\phi & \xrightarrow{\bar{\phi}} & I_\phi \end{array}$$

Given two morphisms ϕ and ψ in $\mathcal{U}$, we define

$$N_{\psi,\phi} = \{x \in S \mid \exists y \in S \text{ with } (D_\psi)^x \leq D_\phi, \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi\}.$$

This is not a subgroup in general, and $N_{\psi,\phi} \neq \emptyset$ if and only if $\phi|_R \sim \psi$ for some $R \leq D_\phi$. Moreover,

$$\begin{aligned} N_{\phi,\phi} &= \{x \in S \mid \exists y \in S \text{ with } (D_\phi)^x = D_\phi, \text{ and } c_x|_{D_\phi} \circ \phi \circ c_y = \phi\} \\ &= \{x \in N_S(D_\phi) \mid \exists y \in N_S(I_\phi) \text{ with } c_x|_{D_\phi} \circ \phi \circ c_y = \phi\} \\ &= \{x \in N_S(D_\phi) \mid (c_x)^\phi \in \mathrm{Aut}_S(I_\phi)\} \\ &= N_\phi, \end{aligned}$$

where N_ϕ is the subgroup defined in Section 2.1.

Lemma 3.1.2. *Let $\phi, \psi \in \mathcal{U}(S)$. Then $\Delta^\phi_{D_\phi}$ and $\Delta^\psi_{D_\psi}$ are $S \times S$-conjugate, if and only if, ϕ and ψ are S-S-equivalent.*

Proof. Suppose that ϕ and ψ are S-S-equivalent. Then, there exists $x, y \in S$ such that $(D_\psi)^x = D_\phi$ and $c_x|_{D_\psi} \circ \phi \circ c_y = \psi$. Thus,

$$\begin{aligned} (\Delta^\phi_{D_\phi})^{(x^{-1},y)} &= \{(p^{x^{-1}}, (p\phi)^y) \mid p \in D_\phi\} \\ &= \{(p^{x^{-1}}, (((p^{x^{-1}})^x)\phi)^y) \mid p \in D_\phi\} \\ &= \{(q, ((q^x)\phi)^y \mid q \in (D_\phi)^{x-1}\} \\ &= \{(q, q\psi) \mid q \in D_\psi\} \\ &= \Delta^\psi_{D_\psi}. \end{aligned}$$

Now suppose that $\Delta^\phi_{D_\phi}$ and $\Delta^\psi_{D_\psi}$ are $S \times S$-conjugate. This means, there exists $x, y \in S$ such that $(\Delta^\phi_{D_\phi})^{(x^{-1},y)} = \Delta^\psi_{D_\psi}$, that is, $\{(p^{x^{-1}}, (p\phi)^y) \mid p \in D_\phi\} = \{(q, q\psi) \mid q \in D_\psi\}$. This means $p = q^x$ for some $p \in D_\phi$, and $q\psi = ((q^x)\phi)^y$ for all $q \in D_\psi$. Thus, $(D_\psi)^x = D_\phi$ and $c_x|_{D_\psi} \circ \phi \circ c_y = \psi$, that is, ϕ and ψ are S-S-equivalent. □

Lemma 3.1.3. *The relation defined in 3.1.1 on morphisms in $\mathcal{U}(S)$ is an equivalence relation.*

Proof. This follows from Lemma 3.1.2 and the fact that conjugation maps between subgroups of a group is an equivalence relation in the group. $\square$

Let $\phi \in \mathcal{F}$, where $\mathcal{F}$ is a fusion system on S. We write $[\phi]$ to denote the S-S-equivalence class of morphisms in $\mathcal{F}$ containing ϕ.

Lemma 3.1.4. *Let $\phi, \psi \in \mathcal{U}(S)$ be two maps induced by conjugation by elements in S (this means $\phi = c_g|_{D_\phi}$ and $\psi = c_h|_{D_\psi}$ for some $g, h \in S$). Suppose D_ϕ is S-conjugate to D_ψ. Then $\phi \sim \psi$.*

Proof. Let $P = D_\phi$ and $Q = D_\psi$. Say $P^x = Q$. Define $y = h^{-1}x^{-1}g$. Then $c_x|_P \circ c_h \circ c_y = c_g|_P$, as required. $\square$

Lemma 3.1.5. *Let $Q \leq S$ and $\phi, \psi \in Aut(Q)$. If ϕ and ψ are representatives of the same element in $Out(Q)$, then $\phi \sim \psi$.*

Proof. There exists $x \in Q$ such that $\phi = c_x|_Q \circ \psi$. It follows that $\phi \sim \psi$. $\square$

Let $Q \leq S$ and $\psi \in \mathrm{Aut}(Q)$. Note that 3.1.5 is not saying that the equivalence class of S-S-equivalent homomorphisms, $[\psi]$, containing ψ is a subset of $\mathrm{Aut}(Q)$. It usually never is. There may be homomorphisms $\phi \in \mathcal{U} \backslash \mathrm{Aut}(Q)$ such that $\phi \sim \psi$. This will be the case when D_ϕ or I_ϕ is distinct from Q. For example, if $\psi = c_g|_Q$ with $g \in N_S(Q)$ and $\phi = c_h|_Q$ with $h \notin N_S(Q)$, then 3.1.4 implies $\psi \sim \phi$, but we have $\phi \notin \mathrm{Aut}(Q)$.

Lemma 3.1.6. *Let $\phi_1, \phi_2 \in \mathcal{U}(S)$ be two isomorphisms, and suppose that $\psi \in Aut(S)$. Then*

1. $\phi_1 \sim \phi_2 \Leftrightarrow \phi_1^{-1} \sim \phi_2^{-1}$,
2. $\phi_1 \sim \phi_2 \Leftrightarrow \phi_1 \circ \psi \sim \phi_2 \circ \psi$.

Proof. Let $P_1 = D_{\phi_1}$ and $P_2 = D_{\phi_2}$. Similarly, let $Q_1 = I_{\phi_1}$ and $Q_2 = I_{\phi_2}$. We have

$$\begin{aligned}
\phi_1 \sim \phi_2 &\Leftrightarrow P_2^x = P_1 \text{ and } c_x|_{P_2} \circ \phi_1 \circ c_y = \phi_2 && \text{(for some } x, y \in S) \\
&\Leftrightarrow P_2^x = P_1 \text{ and } c_y|_{Q_2} \circ \phi_1^{-1} \circ c_x = \phi_2^{-1} && \text{(for some } x, y \in S) \\
&\Leftrightarrow \phi_1^{-1} \sim \phi_2^{-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\phi_1 \sim \phi_2 &\Leftrightarrow P_2^x = P_1 \text{ and } c_x|_{P_2} \circ \phi_1 \circ c_y = \phi_2 && \text{(for some } x, y \in S\text{)} \\
&\Leftrightarrow P_2^x = P_1 \text{ and } c_x|_{P_2} \circ \phi_1 \circ c_y \circ \psi = \phi_2 \circ \psi && \text{(for some } x, y \in S\text{)} \\
&\Leftrightarrow P_2^x = P_1 \text{ and } c_x|_{P_2} \circ (\phi_1 \circ \psi) \circ c_{(y\psi)} = \phi_2 \circ \psi && \text{(for some } x, y \in S\text{)} \\
&\Leftrightarrow \phi_1 \circ \psi \sim \phi_2 \circ \psi.
\end{aligned}$$

This completes the proof. □

Lemma 3.1.7. *[12, Lemma 3.10] Let ϕ and ψ be homomorphism in $\mathcal{U}(S)$. Then*

$$|\mathcal{O}_\phi^\psi| = \frac{|N_{\psi,\phi}|}{|D_\phi|}|C_S(I_\psi)|.$$

Proof. Let $(x,y) \in S \times S$. Note that

$$\begin{aligned}
\Delta_{D_\phi}^\phi(x^{-1}, y) \in \mathcal{O}_\phi^\psi &\Leftrightarrow \Delta_{D_\phi}^\phi(x^{-1}, y)(q, q\psi) = \Delta_{D_\phi}^\phi(x^{-1}, y) \quad \text{for all } q \in D_\psi \\
&\Leftrightarrow (q, q\psi) \in (x, y^{-1})\Delta_{D_\phi}^\phi(x^{-1}, y) \quad \text{for all } q \in D_\psi \\
&\Leftrightarrow \Delta_{D_\psi}^\psi \leq (x, y^{-1})\Delta_{D_\phi}^\phi(x^{-1}, y) \\
&\Leftrightarrow \Delta_{D_\psi}^\psi \leq \Delta_{(D_\phi)c_{x^{-1}}}^{c_x \circ \phi \circ c_y} && \text{(by Lemma 3.0.1)} \\
&\Leftrightarrow (D_\psi)^x \leq D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi \\
&\Leftrightarrow x \in N_{\psi,\phi} \quad \text{for some } y \in S
\end{aligned}$$

Fix $x \in N_{\psi,\phi}$. Then there exists a corresponding $y \in S$ such that $c_x|_{D_\psi} \circ \phi \circ c_y = \psi$. Now,

$$\begin{aligned}
y' \in S \text{ with } c_x|_{D_\psi} \circ \phi \circ c_{y'} = \psi \implies (q\psi)^{y^{-1}} &= (q\psi)^{y'^{-1}} \quad \text{for all } q \in D_\psi \\
&\Leftrightarrow y^{-1}y' \in C_S(I_\psi) \\
&\Leftrightarrow y' \in yC_S(I_\psi).
\end{aligned}$$

Thus, the number of elements in $S \times S$ whose corresponding right coset in $\mathcal{O}_\phi$ is fixed by $\Delta_{D_\psi}^\psi$

is $|N_{\psi,\phi}||C_S(I_\psi)|$. This means,

$$|\mathcal{O}_\phi^\psi| = \frac{|N_{\psi,\phi}||C_S(I_\psi)|}{|\Delta_{D_\phi}^\phi|} = \frac{|N_{\psi,\phi}||C_S(I_\psi)|}{|D_\phi|},$$

as desired. □

Lemma 3.1.8. *Let ϕ and ψ be morphisms in $\mathcal{U}(S)$. Then the size of the fixed point set $|\mathcal{O}_\phi^\psi|$ is independent of the representatives of the S-S-equivalent classes. In other words, let ϕ_1, $\psi_1 \in \mathcal{U}$ so that $\phi \sim \phi_1$ and $\psi \sim \psi_1$. Then*

$$|N_{\psi,\phi}| = |N_{\psi_1,\phi_1}|.$$

In particular, $|\mathcal{O}_\phi^\psi| = |\mathcal{O}_{\phi_1}^{\psi_1}|$.

Proof. Let $P = D_\phi$, $P_1 = D_{\phi_1}$, $Q = D_\psi$ and $Q_1 = D_{\psi_1}$. There exists $a,b,u,v \in S$ such that $P_1 = P^u$, $Q_1 = Q^a$ and $\phi_1 = c_{u^{-1}} \circ \phi \circ c_v$, $\psi_1 = c_{a^{-1}} \circ \psi \circ c_b$. Note that $|C_S(Q_1\psi_1)| = |C_S(Q^a\psi_1)| = |C_S((Q\psi)^b)| = |C_S(Q\psi)|$ and $|P_1| = |P|$. This means $|C_S(I_{\psi_1})| = |C_S(I_\psi)|$ and $|D_{\phi_1}| = |D_\phi|$. Also,

$$\begin{aligned}
|N_{\psi_1,\phi_1}| &= |\{x \in S \mid \exists y \in S \text{ with } Q_1^x \leq P_1, \text{ and } c_x|_{Q_1} \circ \phi_1 \circ c_y = \psi_1\}| \\
&= |\{x \in S \mid \exists y \in S \text{ with } Q_1^x \leq P_1, \text{ and } c_x|_{Q_1} \circ c_{u^{-1}} \circ \phi \circ c_v \circ c_y = c_a^{-1}|_{Q_1} \circ \psi \circ c_b\}| \\
&= |\{x \in S \mid \exists y \in S \text{ with } Q^{axu^{-1}} \leq P, \text{ and } c_{axu^{-1}}|_Q \circ \phi \circ c_{vyb^{-1}} = \psi\}| \\
&= |\{x \in S \mid \exists y \in S \text{ with } Q^x \leq P, \text{ and } c_x|_Q \circ \phi \circ c_y = \psi\}| \\
&= |N_{\psi,\phi}|.
\end{aligned}$$

The claim $|\mathcal{O}_\phi^\psi| = |\mathcal{O}_{\phi_1}^{\psi_1}|$ now follows from 3.1.7. □

Lemma 3.1.9. *Let $\phi, \psi \in \mathcal{U}(S)$. Suppose that ϕ extends to some $\bar{\phi} \in Hom(N_\phi, S)$. Then, for all $x \in S$, we have*

$$x \in N_{\psi,\phi} \Leftrightarrow C_S(D_\psi)xN_\phi \leq N_{\psi,\phi}.$$

In particular, if $\bar{\phi} \in Hom(N_S(D_\phi), S)$, then $N_{\psi,\phi} \neq \emptyset \Leftrightarrow |N_{\psi,\phi}| \geq |N_S(D_\phi)|$.

Proof. Let $P = D_\phi$ and $Q = D_\psi$. Note that $N_{\psi,\phi} \neq \emptyset \Leftrightarrow$ there exists $x, y \in S$ such that $Q^x \leq P$ and $c_x|_Q \circ \phi \circ c_y = \psi$. Let $a \in C_S(Q)$ and $b \in N_\phi \leq N_S(P)$. Then $Q^{axb} = Q^{xb} \leq P^b = P$ and

$$\begin{aligned} c_{axb}|_Q \circ \phi \circ c_{(b\bar{\phi})^{-1}y} &= c_{xb}|_Q \circ \phi \circ c_{(b\bar{\phi})^{-1}y} \\ &= c_x|_Q \circ \phi \circ c_{(b\bar{\phi})} \circ c_{(b\bar{\phi})^{-1}y} \\ &= c_x|_Q \circ \phi \circ c_y \\ &= \psi. \end{aligned}$$

It follows from above that $x \in N_{\psi,\phi} \Leftrightarrow C_S(Q)xN_\phi \leq N_{\psi,\phi}$. □

3.2 Equivalence classes

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Let ϕ be a homomorphism in $\mathcal{F}$. Recall that we write $[\phi]$ to denote the S-S-equivalence class of morphisms in $\mathcal{F}$ containing ϕ. We also write Γ to denote the set of all S-S-equivalence classes in $\mathcal{F}$. Note that any two morphisms that share the same class will have domains of equal order. Thus we can define Γ_i to be a set of S-S-equivalence classes of morphisms in $\mathcal{F}$ with domains of index p^i. We can therefore write,

$$\Gamma = \bigsqcup_{i=0}^{\infty} \Gamma_i.$$

Recall that $\mathrm{Out}_{\mathcal{F}}(S)$ is the set of cosets of $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$.

Lemma 3.2.1. *Γ_0 can be identified with $Out_{\mathcal{F}}(S)$.*

Proof. By definition, the elements of Γ_0 are the S-S equivalence classes of morphisms in $\mathcal{F}$ with domains equal to S. Let $[\phi], [\psi] \in \Gamma_0$. Then

$$\begin{aligned} \phi \sim \psi &\Leftrightarrow \psi = c_x \circ \phi \circ c_y \text{ for some } x, y \in S \\ &\Leftrightarrow \psi = \phi \circ c_{(x\phi)} \circ c_y && \text{(Since } \phi \in \mathrm{Aut}_{\mathcal{F}}(S)) \\ &\Leftrightarrow \phi \text{ and } \psi \text{ are in the same coset of } \mathrm{Inn}(S). \end{aligned}$$

So ϕ and ψ correspond to the same element of $\mathrm{Out}_{\mathcal{F}}(S)$. $\square$

Definition 3.2.2. Let $\mathcal{F}$ be a fusion system on a finite p-group S. Let $\phi \in \mathcal{F}$. We say that ϕ is *(properly) extendable* if there exists a morphism $\bar{\phi}$ in $\mathcal{F}$ such that $D_{\bar{\phi}} > D_\phi$ and $\bar{\phi}|_{D_\phi} = \phi$. If there is no such $\bar{\phi}$, then we say that ϕ is *non-extendable*. We say that ϕ is *fully extendable* if we can choose $D_{\bar{\phi}}$ to be equal to S.

Lemma 3.2.3. *Let $\phi \in \mathcal{F}$. Then ϕ is extendable to $R > D_\phi$ if and only if every morphism inside $[\phi]$ is extendable to some S-conjugate of R.*

Proof. Let $\varphi \in [\phi]$. Let $Q = D_\phi$ and $P = D_\varphi$. Then there exist $x, y \in S$ such that $\varphi = c_x|_P \circ \phi \circ c_y$ with $P^x = Q$. Suppose that ϕ is extendable to $R > Q$. Then there exist $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(R, S)$ such that $\phi = \bar{\phi}|_Q$. Define $T = R^{x^{-1}} \supset Q^{x^{-1}}$ and $\bar{\varphi} = c_x|_T \circ \bar{\phi} \circ c_y$. Then $T > P$ and $\bar{\varphi}|_P = c_x|_P \circ \bar{\phi} \circ c_y = c_x|_P \circ \phi \circ c_y = \varphi$ as required. $\square$

Thus we can write $\Gamma_i = \Gamma_i^e \sqcup \Gamma_i^n$, where

$$\Gamma_i^e = \{[\phi] \in \Gamma_i \mid \phi \text{ is extendable } \}$$

$$\Gamma_i^n = \{[\phi] \in \Gamma_i \mid \phi \text{ is non-extendable } \}.$$

Note that $\Gamma_0 = \Gamma_0^n$.

CHAPTER 4

PERMUTATION GROUPS WHICH REALIZE FUSION SYSTEMS

4.1 Bisets

We introduce a number of basic results on bisets, many of which have been implicitly stated by Sejong Park in [7], though not always proved.

Definition 4.1.1. Let S_1 and S_2 be two finite groups. Let Ω be a right S_2-set via the operation $\cdot$ and a left S_1-set via the operation $*$. Then Ω is an S_1-S_2 biset if and only if

$$(s_1 * x) \cdot s_2 = s_1 * (x \cdot s_2),$$

for all $x \in \Omega$ and $s_1 \in S_1, s_2 \in S_2$. If $S = S_1 = S_2$, then Ω is an S-S biset.

Where there is no ambiguity, we will remove the brackets and the actions and just write $s_1 x s_2$.

Lemma 4.1.2. *For any finite group S, we have the following equivalence between S-S-bisets and $S \times S$-sets:*

1. *If Ω is a S-S-biset, via right S action $\cdot$ and a left S-action $*$, then Ω is an $S \times S$-set under the (right) action:*

$$\Omega \times (S \times S) \to \Omega, \textit{ given by, } (x, (s_1, s_2)) \mapsto x \circ (s_1, s_2) = s_1^{-1} * (x \cdot s_2).$$

2. *If Ω is a (right) $S \times S$-set via $\circ$, then Ω is an S-S-biset under the following right and left S-actions:*

$$\Omega \times S \to \Omega, \textit{ given by, } (x,s) \mapsto x \cdot s = x \circ (1,s).$$

$$S \times \Omega \to \Omega, \textit{ given by, } (s,x) \mapsto s * x = x \circ (s^{-1},1).$$

Proof. 1. If Ω is an S-S biset, under the left and right actions defined above, then for all $s_1, s_2, t_1, t_2 \in S$ and $x \in \Omega$, we have

$$\begin{aligned}
(x \circ (s_1,s_2)) \circ (t_1,t_2) &= t_1^{-1} * ((s_1^{-1} * (x \cdot s_2)) \cdot t_2) \\
&= t_1^{-1} * (((s_1^{-1} * x) \cdot s_2) \cdot t_2) = t_1^{-1} * ((s_1^{-1} * x) \cdot (s_2 t_2)) \\
&= (t_1^{-1} * (s_1^{-1} * x)) \cdot (s_2 t_2) = ((s_1 t_1)^{-1} * x) \cdot (s_2 t_2) \\
&= (s_1 t_1)^{-1} * (x \cdot (s_2 t_2)) = x \circ (s_1 t_1, s_2 t_2) \\
&= x \circ ((s_1,s_2)(t_1,t_2)).
\end{aligned}$$

Since $x \circ (1,1) = 1 * (x \cdot 1) = 1 * x = x$, the claim that Ω is an $S \times S$-set now follows.

2. If Ω is an $(S \times S)$-set under the action defined above, then for all $s_1, s_2 \in S$ and $x \in \Omega$, we have

$$\begin{aligned}
(s_1 * x) \cdot s_2 = (s_1 * x) \circ (1,s_2) &= (x \circ (s_1^{-1},1)) \circ (1,s_2) = x \circ (s_1^{-1},s_2) \\
&= (x \circ (1,s_2)) \circ (s_1^{-1},1) = (x \cdot s_2) \circ (s_1^{-1},1) = s_1 * (x \cdot s_2).
\end{aligned}$$

We also have

$$\begin{aligned}
(x \cdot s_1) \cdot s_2 &= (x \circ (1,s_1)) \circ (1,s_2) = x \circ ((1,s_1)(1,s_2)) \\
&= x \circ (1,s_1 s_2) = x \cdot (s_1 s_2) \\
s_2 * (s_1 * x) &= (x \circ (s_1^{-1},1)) \circ (s_2^{-1},1) = x \circ ((s_1^{-1},1)(s_2^{-1},1)) \\
&= x \circ ((s_2 s_1)^{-1},1) = (s_2 s_1) * x.
\end{aligned}$$

Since $x \cdot 1 = x \circ (1,1) = x$ and $1 * x = x \circ (1,1) = x$, we have shown that $\cdot$ and $*$ are indeed right and left S actions respectively. It follows that Ω is an S-S-biset.

□

Lemma 4.1.2 tells us that we can think of S-S-bisets as $S \times S$-sets, and vice versa. We look at a specific example of the correspondence given in 4.1.2. Given $P \leq S$ and $\phi \in \mathrm{Hom}(P,S)$, define

$$S \times_{(P,\phi)} S = (S \times S)/\sim \text{ where } (s_1,s_2) \sim (s_1 u^{-1}, (u\phi)s_2) \text{ for } s_1,s_2 \in S, u \in P$$

Note that $\sim$ is indeed an equivalence relation: It is clearly reflexive. For the symmetric property, suppose $(s_1,s_2) \sim (s_3,s_4)$. Then $s_3 = s_1 u_1^{-1}$ and $s_4 = (u_1\phi)s_2$ for some $u_1 \in P$. Let $u_2 = u_1^{-1}$. Then, $s_1 = s_3 u_2^{-1}$ and $s_2 = ((u_2)\phi)s_4$, that is $(s_3,s_4) \sim (s_1,s_2)$. For the transitivity property, suppose that $(s_1,s_2) \sim (s_3,s_4)$ and $(s_3,s_4) \sim (s_5,s_6)$. Then, there exists $u_1,u_2 \in P$ such that $s_3 = s_1 u_1^{-1}$, $s_4 = (u_1\phi)s_2$, and also $s_5 = s_3 u_2^{-1}$, $s_6 = (u_2\phi)s_4$. Let $u_3 = u_2 u_1$. Then $s_5 = s_3 u_2^{-1} = s_1 u_1^{-1} u_2^{-1} = s_1 u_3^{-1}$, and $s_6 = (u_2\phi)s_4 = (u_2\phi)(u_1\phi)s_2 = ((u_2u_1)\phi)s_2 = (u_3\phi)s_2$, that is, $(s_1,s_2) \sim (s_5,s_6)$.

Let $\langle\langle s_1,s_2\rangle\rangle$ denote the $\sim$-equivalence class containing (s_1,s_2). Note that $S \times_{(P,\phi)} S$ is an S-S-biset under the well-defined left and right actions $t \cdot \langle\langle s_1,s_2\rangle\rangle = \langle\langle ts_1,s_2\rangle\rangle$ and $\langle\langle s_1,s_2\rangle\rangle \cdot t = \langle\langle s_1,s_2t\rangle\rangle$, where $t \in S$. We show this explicitly for the left action: It's well-defined because, for any different representative of the equivalence class containing (s_1,s_2), namely $(s_1u^{-1},(u\phi)s_2)$ for some $u \in P$, we have $t \cdot \langle\langle s_1u^{-1},(u\phi)s_2\rangle\rangle = \langle\langle ts_1u^{-1},(u\phi)s_2\rangle\rangle = \langle\langle ts_1,s_2\rangle\rangle$. Also, for any $t_1 \in S$, we have $(tt_1) \cdot \langle\langle s_1,s_2\rangle\rangle = \langle\langle (tt_1)s_1,s_2\rangle\rangle = \langle\langle t(t_1s_1),s_2\rangle\rangle = t \cdot \langle\langle t_1s_1,s_2\rangle\rangle = t \cdot (t_1 \cdot \langle\langle s_1,s_2\rangle\rangle)$, and $1 \cdot \langle\langle s_1,s_2\rangle\rangle = \langle\langle s_1,s_2\rangle\rangle$.

Equivalently, by Lemma 4.1.2, the S-S-biset under the given left and right action, is an $S \times S$-set under the action $\langle\langle s_1,s_2\rangle\rangle(t_1,t_2) = \langle\langle t_1^{-1}s_1,s_2t_2\rangle\rangle$. Recall

$$\Delta_P^\phi = \{(u,(u)\phi) \mid u \in P\}.$$

Δ_P^{ϕ} is a subgroup of $S \times S$. Thus $S \times S$ acts on the right cosets of Δ_P^{ϕ} under right multiplication.

Lemma 4.1.3. *The S-S-biset $S \times_{(P,\phi)} S$ under the action*

$$t \cdot \langle\langle s_1, s_2 \rangle\rangle = \langle\langle ts_1, s_2 \rangle\rangle \text{ and } \langle\langle s_1, s_2 \rangle\rangle \cdot t = \langle\langle s_1, s_2 t \rangle\rangle$$

is equivalent to the $(S \times S)$-set $(S \times S)/\Delta_P^{\phi}$, defined by right multiplication.

Proof. Take $\pi : S \times_{(P,\phi)} S \to (S \times S)/\Delta_P^{\phi} : \langle\langle s_1, s_2 \rangle\rangle \mapsto \Delta_P^{\phi}(s_1^{-1}, s_2)$.
First we show that π is well defined. Taking a different representative of the equivalence class containing (s_1, s_2), note that

$$\begin{aligned}
(s_1 u^{-1}, (u)\phi s_2)\pi &= \Delta_P^{\phi}(u s_1^{-1}, (u)\phi s_2) \\
&= \Delta_P^{\phi}(u, (u)\phi)(s_1^{-1}, s_2) \\
&= \Delta_P^{\phi}(s_1^{-1}, s_2) = (s_1, s_2)\pi.
\end{aligned}$$

Now we check that π is an equivariant map:

$$\begin{aligned}
(\langle\langle s_1, s_2 \rangle\rangle(t_1, t_2))\pi &= \langle\langle t_1^{-1} s_1, s_2 t_2 \rangle\rangle\pi \\
&= \Delta_P^{\phi}(s_1^{-1} t_1, s_2 t_2) \\
&= \Delta_p^{\phi}(s_1^{-1}, s_2)(t_1, t_2) \\
&= (\langle\langle s_1, s_2 \rangle\rangle)\pi(t_1, t_2).
\end{aligned}$$

It is left to show that it is a bijection. To see that π is an injection, note that

$$\begin{aligned}
(\langle\langle s_1, s_2 \rangle\rangle)\pi = (\langle\langle s_3, s_4 \rangle\rangle)\pi &\implies \Delta_P^{\phi}(s_1^{-1}, s_2) = \Delta_P^{\phi}(s_3^{-1}, s_4) \\
&\implies (s_3^{-1}, s_4) = (u s_1^{-1}, (u)\phi s_2) \quad \text{for some } u \in P \\
&\implies (s_3, s_4) = (s_1 u^{-1}, (u)\phi s_2) \\
&\implies \langle\langle s_1, s_2 \rangle\rangle = \langle\langle s_3, s_4 \rangle\rangle.
\end{aligned}$$

Surjectivity holds since for any $\Delta_P^{\phi}(s_1, s_2) \in (S \times S)/\Delta_P^{\phi}$, we have $(\langle\langle s_1^{-1}, s_2 \rangle\rangle)\pi = \Delta_P^{\phi}(s_1, s_2)$. $\square$

Note that the S-S-biset in Lemma 4.1.3 is transitive and is free on the right and also free on the left if ϕ is injective.

4.2 Characteristic bisets

Let S be a finite p-group. For an S-S biset Ω, $Q \leq S$ and $\phi : Q \to S$ an injective group homomorphism, denote by ${}_Q\Omega$, the left Q-set, the restriction of the left S-action to Q and ${}_\phi\Omega$, the restriction of the left S-action to Q induced by ϕ. Explicitly, if $*$ denotes the left S-action on Ω, then the restrictions of the left S-actions to Q are as follows:

$$Q \times {}_Q\Omega \to {}_Q\Omega, \text{ given by, } (q,x) \mapsto q * x,$$
$$Q \times {}_\phi\Omega \to {}_\phi\Omega, \text{ given by, } (q,x) \mapsto q \cdot x = (q\phi) * x.$$

Then ${}_Q\Omega$ and ${}_\phi\Omega$ are Q-S bisets. From the perspective of $S \times S$-sets, if $\circ$ denotes the $S \times S$-action on Ω, then the restrictions of $S \times S$-actions to $Q \times S$ are as follows:

$${}_Q\Omega \times (Q \times S) \to {}_Q\Omega, \text{ given by, } (x,(q,s)) \mapsto x \circ (q,s),$$
$${}_\phi\Omega \times (Q \times S) \to {}_\phi\Omega, \text{ given by, } (x,(q,s)) \mapsto x \bullet (q,s) = x \circ (q\phi, s).$$

Similarly, we define Ω_Q and Ω_ϕ: Denote by Ω_Q, the restriction of the right S-action to Q and Ω_ϕ, the restriction of the right S-action to Q induced by ϕ. Explicitly, if $*$ denotes the right S-action on Ω, then the restrictions of the right S-actions to Q are as follows:

$$\Omega_Q \times Q \to \Omega_Q, \text{ given by, } (x,q) \mapsto x * q,$$
$$\Omega_\phi \times Q \to \Omega_\phi, \text{ given by, } (x,q) \mapsto x \cdot q = x * (q\phi).$$

Then Ω_Q and Ω_ϕ are S-Q bisets. From the perspective of $S\times S$-sets, if $\circ$ denotes the $S\times S$-action on Ω, then the restrictions of the $S\times S$-actions to $S\times Q$ are as follows:

$$\Omega_Q \times (Q\times S) \to \Omega_Q, \text{ given by, } (x,(s,q)) \mapsto x\circ(s,q),$$
$$\Omega_\phi \times (Q\times S) \to \Omega_\phi, \text{ given by, } (x,(s,q)) \mapsto x\bullet(s,q) = x\circ(s,q\phi).$$

Thus, equivalently, Ω_Q and Ω_Q are $S\times Q$-sets.

Recall that we write $\mathcal{O}_\phi = (S\times S)/\Delta^{\phi}_{D_\phi}$. We may equivalently write $\mathcal{O}_\phi = S\times_{(Q,\phi)} S$, by Lemma 4.1.3. For any $\mathcal{V}\subseteq \mathcal{U}(S)$, if

$$\Omega = \bigsqcup_{\varphi\in\mathcal{V}} \mathcal{O}_\varphi, \tag{1}$$

then, Ω is a union of $S\times S$-orbits, as seen in Section 4.1. Note that Ω is itself an S-S-biset, or, equivalently, an $S\times S$-set. In the construction of Ω here, we define

$$\Omega^\phi = \Omega^{\Delta^\phi_Q},$$

whenever it is clear. As another example, if $P = Q\phi$, we write $\Omega^{\phi^{-1}} = \Omega^{\Delta_P^{\phi^{-1}}}$. We use a similar notation for $Q\times S$ and $S\times Q$-sets.

Lemma 4.2.1. *Let Ω be a union of $S\times S$-orbits, of the form $\mathcal{O}_\varphi$, for some $\varphi\in\mathcal{U}(S)$, as described in (1). Let ϕ and $\psi\in\mathcal{U}(S)$. Write $Q = D_\phi$.*

1. *If $D_\psi \leq Q$, then $\Omega^\psi = ({}_Q\Omega)^\psi$ and $({}_\phi\Omega)^\psi = \Omega^{(\phi^{-1}\psi)|_P}$ (where $P = (D_\psi)\phi$).*
2. *If $I_\psi \leq Q$, then $\Omega^\psi = (\Omega_Q)^\psi$ and $(\Omega_\phi)^\psi = \Omega^{\psi\phi}$.*

In particular, $({}_\phi\Omega)^\phi = \Omega^{Id|_{I_\phi}}$ and $(\Omega_\phi)^{Id|_{D_\phi}} = \Omega^\phi$.

Proof. Denote by $\circ$ the $S\times S$ action, and by $\bullet$ the $Q\times S$ action via ϕ. We use a similar notation for $S\times Q$ actions.

1. Suppose $D_\psi\leq Q$. Since the domain of ψ is contained in Q, we have $\Omega^\psi = ({}_Q\Omega)^\psi$ by

definition. Write $P = (D_\psi)\phi$. Now

$$
\begin{aligned}
\Omega^{(\phi^{-1}\psi)|_P} &= \Omega^{\Delta_P^{(\phi^{-1}\psi)|_P}} \\
&= \{\omega \in \Omega \mid \omega \circ (p, p\phi^{-1}\psi) = \omega \text{ for all } (p, p\phi^{-1}\psi) \in \Delta_P^{(\phi^{-1}\psi)|_P} \text{ where } p \in P\} \\
&= \{\omega \in \Omega \mid \omega \circ (q\phi, q\psi) = \omega \text{ for all } (q\phi, q\psi) \in \Delta_P^{(\phi^{-1}\psi)|_P} \text{ where } q \in D_\psi\} \\
&= \{\omega \in \Omega \mid \omega \bullet (q, q\psi) = \omega \text{ for all } (q, q\psi) \in \Delta_{D_\psi}^{\psi} \text{ where } q \in D_\psi\} \\
&= ({}_\phi\Omega)^{\Delta_{D_\psi}^{\psi}} \\
&= ({}_\phi\Omega)^{\psi}.
\end{aligned}
$$

2. Suppose $I_\psi \leq Q$. Since the image of ψ is contained in Q, we have $\Omega^\psi = (\Omega_Q)^\psi$ by definition. Now

$$
\begin{aligned}
\Omega^{\psi\phi} &= \Omega^{\Delta_{D_\psi}^{\psi\phi}} \\
&= \{\omega \in \Omega \mid \omega \circ (p, (p\psi)\phi) = \omega \text{ for all } (p, (p\psi)\phi) \in \Delta_{D_\psi}^{\psi\phi} \text{ where } p \in D_\psi\} \\
&= \{\omega \in \Omega \mid \omega \bullet (p, p\psi) = \omega \text{ for all } (p, p\psi) \in \Delta_{D_\psi}^{\psi} \text{ where } p \in D_\psi\} \\
&= (\Omega_\phi)^{\Delta_{D_\psi}^{\psi}} \\
&= (\Omega_\phi)^{\psi}.
\end{aligned}
$$

as required. The final claim follows by taking $\psi = \phi$ in the first case, and $\psi = \mathrm{Id}|_Q$ in the second case. □

Lemma 4.2.2. *[9, Lemma 3.2] Let Ω be a union of $(S \times S)$-orbits of the form $\mathcal{O}_\phi$, for some $\phi \in \mathcal{U}(S)$. Then $\Omega^H \neq \emptyset$, for some $H \leq S \times S$, if and only if H is $S \times S$-conjugate to a subgroup of $\Delta_{D_\phi}^{\phi}$, for some $\Delta_{D_\phi}^{\phi}$ in Ω. In particular, H is of the form $\Delta_{D_\varphi}^{\varphi}$ for some $\varphi \in \mathcal{U}(S)$.*

Proof. $\Omega^H \neq \emptyset$ if and only if $\mathcal{O}_\phi^H \neq \emptyset$ for some orbit $\mathcal{O}_\phi$ in Ω. By Lemma 2.2.4, $\mathcal{O}_\phi^H \neq \emptyset$ if and only if H is a subgroup of some $S \times S$-conjugate of $\Delta_{D_\phi}^{\phi}$, that is, for some $g, h \in S$, we have $H \leq (\Delta_{D_\phi}^{\phi})^{(g,h)} = \Delta_{D_\varphi}^{\varphi}$, where $D_\varphi = D_\phi^g$ and $\varphi = c_{g^{-1}}|_{D_\varphi} \circ \phi \circ c_h$. The claim now follows from Lemma 3.0.2. □

By definition, if Ω is a union of $(S \times S)$-orbits of the form $\mathcal{O}_\phi$ for some $\phi \in \mathcal{U}(S)$, then we can write

$$\Omega = \bigsqcup_{i=0}^{\infty} \Omega_i$$

where Ω_i consists of those orbits $\mathcal{O}_\phi$ in Ω with $|S : D_\phi| = p^i$.

Lemma 4.2.3. *[9, Lemma 3.2] Let Ω be a union of $(S \times S)$-sets of the form $\mathcal{O}_\phi$ for some $\phi \in \mathcal{U}(S)$. Let $\phi \in \mathcal{U}(S)$ and write $p^i = |S : D_\phi|$. Then*

$$\Omega^\phi = \bigsqcup_{0 \le j \le i} \Omega_j^\phi.$$

Proof. By Lemma 4.2.2, if $\Omega_j^\phi \neq \emptyset$ for some integer $j \geq 0$, then $\Delta_{D_\phi}^\phi$ is a subgroup of some $S \times S$-conjugate of $\Delta_{D_\psi}^\psi$, where $\Delta_{D_\psi}^\psi$ is in Ω_j. This means $|D_\phi| = |\Delta_{D_\phi}^\phi| \leq |\Delta_{D_\psi}^\psi| = |D_\psi|$, and therefore $p^i = |S : D_\phi| \geq |S : D_\psi| = p^j$, that is, $j \leq i$. □

Definition 4.2.4. [10, Definition 2] Let $\mathcal{F}$ be a fusion system on a finite group S. Then an S-S biset Ω is called a *right semicharacteristic biset* for $\mathcal{F}$ if

1. Every $(S \times S)$-orbit in Ω is of the form $\mathcal{O}_\phi$ for some $\phi \in \mathcal{F}$,
2. $\Omega_{D_\phi} \cong \Omega_\phi$ as S-D_ϕ-bisets for every $\phi \in \mathcal{F}$.

Ω is called a *left semicharacteristic biset* if instead for the latter condition, we have ${}_{D_\phi}\Omega \cong {}_\phi\Omega$ as D_ϕ-S-bisets for every $\phi \in \mathcal{F}$. The set Ω is called a *semicharacteristic biset* if it is both a right semicharacteristic biset and a left semicharacteristic biset.

Definition 4.2.5. Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then an S-S biset Ω is called a *right characteristic biset* for $\mathcal{F}$ if it is a right semicharacteristic biset with the property that $|\Omega|/|S| \not\equiv 0$ mod p.

Note that the definition makes sense since the size of a right semicharacteristic biset Ω is always a multiple of $|S|$.
A *left characteristic biset* is defined similarly. Ω is called a *characteristic biset* if it is both a left and right characteristic biset.

Lemma 4.2.6. *Let $\mathcal{F}$ be a fusion system on a finite p-group S and $\phi, \psi \in \mathcal{F}$.*

1. *If Ω is a right semicharacteristic biset, and $I_\psi \leq D_\phi$, we have $|\Omega^\psi| = |\Omega^{\psi\phi}|$. In particular, by taking $\psi = Id|_{D_\phi}$, we have $|\Omega^\phi| = |\Omega^{Id|_{D_\phi}}|$.*

2. *If Ω is a left semicharacteristic biset, and $D_\psi \leq D_\phi$, we have $|\Omega^\psi| = |\Omega^{(\phi^{-1}\psi)|_P}|$, where $P = (D_\psi)\phi$. In particular, by taking $\psi = \phi$, we have $|\Omega^\phi| = |\Omega^{Id|_{I_\phi}}|$.*

Proof. Write $Q = D_\phi$. Suppose that Ω is a right semicharacteristic biset, and $I_\psi \leq Q$. Then by definition $\Omega_Q \cong \Omega_\phi$. Using Lemmas 2.2.9 and 4.2.1 we have

$$|\Omega^\psi| = |(\Omega_Q)^\psi| = |(\Omega_\phi)^\psi| = |\Omega^{\psi\phi}|.$$

Similarly, suppose that Ω is a left semicharacteristic biset, and $D_\phi \leq D_\psi$. Then by definition ${}_Q\Omega \cong {}_\phi\Omega$. By applying Lemmas 2.2.9 and 4.2.1 again, we see that

$$|\Omega^\psi| = |({}_Q\Omega)^\psi| = |({}_\phi\Omega)^\psi| = |\Omega^{(\phi^{-1}\psi)|_P}|$$

where $P = (D_\psi)\phi$. □

Lemma 4.2.7. *Let $\mathcal{F}$ be a fusion system on a finite group S.*

1. *An S-S biset Ω is a right semicharacteristic biset if and only if*

   (a) *Every orbit in Ω is of the form $\mathcal{O}_\phi$ for some $\phi \in \mathcal{F}$*

   (b) *$|\Omega^\phi| = |\Omega^{Id|_{D_\phi}}|$ for all $\phi \in \mathcal{F}$.*

2. *An S-S biset Ω is a left semicharacteristic biset if and only if*

   (a) *Every orbit in Ω is of the form $\mathcal{O}_\phi$ for some $\phi \in \mathcal{F}$*

   (b) *$|\Omega^\phi| = |\Omega^{Id|_{I_\phi}}|$ for all $\phi \in \mathcal{F}$.*

Proof. 1. If Ω is a right semicharacteristic biset, then by Lemma 4.2.6 the two properties hold. Conversely, suppose that the two properties holds. Let $\psi \in \mathcal{F}$ with $I_\psi \leq D_\phi$. Then,

$$|(\Omega_{D_\phi})^\psi| = |\Omega^\psi| = |\Omega^{\mathrm{Id}|_{D_\psi}}| = |\Omega^{\psi\phi}| = |(\Omega_\phi)^\psi|,$$

where the first and last equality is due to Lemma 4.2.1, and where the second and third equality is a consequence of the second part of the assumption. By Lemma 4.2.2, $\Omega^H_{D_\phi}, \Omega^H_\phi \neq \emptyset$ for some $H \leq S \times D_\phi$, if and only if H is of the form $\Delta^{\psi}_{D_\psi}$, for some $\psi \in \mathcal{F}$ and $I_\psi \leq D_\phi$. It follows that $|\Omega^H_{D_\phi}| = |\Omega^H_\phi|$ for all $H \leq S \times D_\phi$. By Burnside's Lemma 2.2.9, we have $\Omega_{D_\phi} \cong \Omega_\phi$. Thus, Ω is a right semicharacteristic biset.

2. If Ω is a left semicharacteristic biset, then by Lemma 4.2.6 the two properties hold. Conversely, suppose that the two properties holds. Let $\psi \in \mathcal{F}$ with $D_\psi \leq D_\phi$. Then, by defining $P = (D_\psi)\phi$, we have

$$|({}_{D_\phi}\Omega)^\psi| = |\Omega^\psi| = |\Omega^{\mathrm{Id}|_{I_\psi}}| = |\Omega^{(\phi^{-1}\psi)|_P}| = |({}_\phi\Omega)^\psi|,$$

where the first and last equality is due to Lemma 4.2.1, and where the second and third equality is a consequence of the second part of the assumption. By Lemma 4.2.2, ${}_{D_\phi}\Omega^H, {}_\phi\Omega^H \neq \emptyset$ for some $H \leq D_\phi \times S$, if and only if H is of the form $\Delta^{\psi}_{D_\psi}$, for some $\psi \in \mathcal{F}$ and $D_\psi \leq D_\phi$. It follows that $|{}_{D_\phi}\Omega^H| = |{}_\phi\Omega^H|$ for all $H \leq D_\phi \times S$. By Burnside's Lemma 2.2.9, we have ${}_{D_\phi}\Omega \cong {}_\phi\Omega$. Thus, Ω is a left semicharacteristic biset.

□

Lemma 4.2.8. *[9, Proposition 4.1] Let $\mathcal{F}$ be a fusion system on a finite p-group S. Let Ω be a right (or left) semicharacteristic biset corresponding to $\mathcal{F}$. Then, there is an integer $C_0 > 0$ such that*

$$\Omega_0 \cong C_0 \bigsqcup_{[\phi]\in\Gamma_0} \mathcal{O}_\phi.$$

If Ω is right (or left) characteristic biset for $\mathcal{F}$, then $p \nmid C_0$.

Proof. Suppose that Ω is a right (or left) semicharacteristic biset for $\mathcal{F}$. By 4.2.7, Ω_0 is a disjoint union of orbits of the form $\mathcal{O}_\phi$ for some $\phi \in \mathrm{Aut}_{\mathcal{F}}(S)$. By 4.2.7 and 3.1.8, we may assume that every orbit $\mathcal{O}_\phi$ in Ω_0 is induced by an S-S-equivalence class representative $\phi \in \mathrm{Aut}_{\mathcal{F}}(S)$. Thus,

$$\Omega_0 = \bigsqcup_{[\phi]\in\Gamma_0} C(\phi)\mathcal{O}_\phi,$$

where $C(\phi) \geq 0$ are integers to be determined. Note that, for any other $[\phi] \in \Gamma_0$,

$$|\Omega^{\phi}| = |\Omega_0^{\phi}| = C(\phi)|\mathcal{O}_{\phi}^{\phi}| = C(\phi)|Z(S)|$$

where the first, second and third equalities are due to Lemmas 4.2.3, 4.2.2 and 3.1.7. Note that for any other $[\psi] \in \Gamma_0$, we have $|\Omega^{\psi}| = |\Omega^{\mathrm{Id}|_S}| = |\Omega^{\phi}|$ by Lemma 4.2.6. Thus, $C(\psi)|Z(S)| = C(\phi)|Z(S)|$, that is, $C(\psi) = C(\phi)$. It follows that we can fix a constant $C_0 = C(\phi)$ for every $[\phi] \in \Gamma_0$. Now suppose that Ω is a right (or left) characteristic biset for $\mathcal{F}$. Note that for any $j \geq 1$, $|\Omega_j|/|S|$ is divisible by p. Thus,

$$|\Omega|/|S| = \sum_{i=0}^{\infty} |\Omega_j|/|S| \equiv |\Omega_0|/|S| \equiv \sum_{[\phi]\in\Gamma_0} C(\phi) \equiv |\Gamma_0|C_0 \not\equiv 0 \pmod{p}$$

Thus $p \nmid C_0$, and we are done. □

4.3 Construction of finite groups realizing fusion systems

Definition 4.3.1. A fusion system $\mathcal{F}$ on a finite group S is called realizable if there exists a finite group G and a subgroup S_0 of G isomorphic to S, with S_0 not necessarily Sylow in G, such that $\mathcal{F} \cong \mathcal{F}_{S_0}(G)$.

Definition 4.3.2. A saturated fusion system $\mathcal{F}$ on a finite p-group S is called exotic if $\mathcal{F}$ is not isomorphic to $\mathcal{F}_{S_0}(G)$ for any finite group G with $S_0 \in \mathrm{Syl}_p(G)$.

Under these definitions, an exotic fusion system $\mathcal{F}$ over a finite p-group S may be realizable. The definition of a realizable fusion system is different to the one found in literature, which additionally require S_0 to be a Sylow p-subgroup in G when defining saturated fusion systems over p-groups. Under this literature definition many examples of saturated fusion systems exist which are in fact far from realizable. See, for example, the exotic fusion systems discovered by Ruiz and Viruel [13]. The aim of this section is to show that every fusion system is realizable under the definition we have taken on board. The majority of what follows is due to [4] and [7]. When working with fusion systems over p-groups, this shows that the Sylow condition is quite a strong condition to impose on saturated fusion systems. We state and prove a number of technical results due to Carles Broto, Ran Levi and Bob Oliver. By way of an example, however, we begin by showing that every universal fusion system is realizable.

Lemma 4.3.3. *Let S be a finite group. Then the universal fusion system $\mathcal{U}$ over S is realizable.*

Proof. Let $G = \mathrm{Sym}(S)$. We identify S in G via $s \mapsto (\alpha \mapsto \alpha s)$ for all $s \in S$. We show that G realises the universal fusion system on S. By definition, we have $\mathcal{F}_S(G) \subseteq \mathcal{U}$. It suffices to show the converse statement. Let $P \leq S$ and $\phi : P \to S$ a monomorphism . Let $1, t_2, ..., t_l$ and $1, t_2', ..., t_l'$ be (left) coset representatives of P and $P\phi$ in S respectively. Given $\alpha \in S$, there exists $p, p' \in P$ such that

$$\alpha = t_i p = t_j'(p'\phi).$$

Define a permutation π, and its inverse, by the following:

$$\pi : S \to S, \quad \text{given by} \quad \alpha \mapsto t_i'(p\phi)$$
$$\pi^{-1} : S \to S, \quad \text{given by} \quad \alpha \mapsto t_j p'.$$

We show π is a permutation. Indeed, if $\alpha\pi = \alpha'\pi$ where $\alpha = t_i p, \alpha' = t_k q$, for $p, q \in P$, then $t_i'(p\phi) = t_k'(q\phi)$. This gives $t_k' \in t_i'(P\phi)$. So $t_i' = t_k'$, and thus $t_i = t_k$ and $p = q$, as ϕ is a monomorphism. It follows that $\alpha = \alpha'$ and π is a permutation. Since $\alpha\pi\pi^{-1} = (t_i'(p\phi))\pi^{-1} = t_i p = \alpha$ and $\alpha\pi^{-1}\pi = (t_j p')\pi = t_j'(p'\phi) = \alpha$, we have π^{-1} is indeed the inverse of π. Note that for any $q \in P$ we have

$$\alpha(qc_\pi) = t_j'(p'\phi)\pi^{-1}q\pi = t_j p' q\pi = t_j'(p'q)\phi = t_j'(p'\phi)(q\phi) = \alpha(q\phi)$$

Thus $\phi = c_\pi|_P$. It follows that $\mathcal{U} = \mathcal{F}_S(G)$. $\square$

Lemma 4.3.4. *Let $\mathcal{F}$ be a fusion system on a finite group S. Let $\mathcal{H}$ be a set of subgroups of S closed under $\mathcal{F}$-conjugacy and taking subgroups. Let Ω_0 be an S-set such that for all*

$$P \leq S \text{ with } P \notin \mathcal{H} \text{ and } P' \in P^{\mathcal{F}}, \text{ we have } |\Omega_0{}^P| = |\Omega_0{}^{P'}|. \qquad (\star)$$

Then there exist S-sets $\Omega \supseteq \Omega_1 \supseteq \Omega_0$ such that:

1. Ω_1 consists of multiple copies of Ω_0;

2. $\Omega^P = \Omega_1^P$ for all $P \notin \mathcal{H}$; and

3. $|\Omega^P| = |\Omega^{P'}|$ for all pairs $P, P' \leq S$, $P' \in P^{\mathcal{F}}$.

Moreover, for all $P \leq S$ and for all $\alpha \in Hom_{\mathcal{F}}(P,S)$, Ω considered as a P-set via restriction is isomorphic to Ω as a P-set via α.

Proof. We prove the first three claims by induction on $|\mathcal{H}|$ for all S-sets Ω_0 satisfying property $(\star)$. If $\mathcal{H} = \emptyset$, then we can take $\Omega = \Omega_0$ and there is nothing to prove. So assume $\mathcal{H} \neq \emptyset$. Let $P \in \mathcal{H}$ be maximal with respect to inclusion and being in $\mathcal{H}$. Define $\mathcal{H}' = \mathcal{H} \setminus P^{\mathcal{F}}$. Then $\mathcal{H}'$ is closed under $\mathcal{F}$-conjugacy and taking subgroups. Since $|\mathcal{H}'| > |\mathcal{H}|$, we may assume that the result holds for $\mathcal{H}'$. That is, if there exists an S-set Ω_2 such that condition $(\star)$ holds for $\mathcal{H}'$, then we can find $\Omega \supseteq \Omega_3 \supseteq \Omega_2$ and a positive integer n such that $\Omega_3 = n \cdot \Omega_2$; and $\Omega^T = \Omega_3^T$ for all $T \leq S$ with $T \notin \mathcal{H}'$; and also $|\Omega^T| = |\Omega^{T'}|$ for all $T, T' \leq S$ with $T' \in T^{\mathcal{F}}$. We will construct an Ω_2 that satisfies condition $(\star)$ for $\mathcal{H}'$ which will then be used to give us the required Ω and Ω_1 for $\mathcal{H}$, by induction.
Assume P is chosen so that $|\Omega_0^P| = \max\{|\Omega_0^R| \mid R \in P^{\mathcal{F}}\}$. For $R \in P^{\mathcal{F}}$ define

$$n_R = \frac{m_1|R|(|\Omega_0^P| - |\Omega_0^R|)}{|N_S(R)|}$$

where m_1 is a constant independent of R chosen to be the smallest positive integer so that n_R is an integer itself. Alternatively, we can define

$$m_1 = \prod_{T \in [P^{\mathcal{F}}]} |N_S(T)|$$

where $[P^{\mathcal{F}}]$ denotes a fixed set of representative of S-conjugacy classes in $P^{\mathcal{F}}$. In either case n_R is an integer. As $|\Omega_0^P| \geq |\Omega_0^R|$, n_R is a non-negative integer. Define

$$\Omega_2 = m_1 \cdot \Omega_0 \sqcup \bigsqcup_{R \in [P^{\mathcal{F}}]} n_R \cdot (S/R).$$

Now let $R' \in P^{\mathcal{F}}$. By Lemma 2.2.4,

$$
\begin{aligned}
|\Omega_2^{R'}| &= m_1|\Omega_0^{R'}| + n_R|(S/R)^{R'}| && \text{(for some } R \in [P^{\mathcal{F}}] \text{ such that } R' \text{ is } S\text{-conjugate to } R) \\
&= m_1|\Omega_0^{R}| + n_R|(S/R)^{R'}| && \text{(by Lemma 2.2.6)} \\
&= m_1|\Omega_0^{R}| + n_R|(S/R)^{R}| && \text{(by Lemma 2.2.6)} \\
&= m_1|\Omega_0^{R}| + n_R|N_S(R)/R| && \text{(by Lemma 2.2.5)} \\
&= m_1|\Omega_0^{R}| + m_1(|\Omega_0^{P}| - |\Omega_0^{R}|) && \text{(by construction of } n_R) \\
&= m_1|\Omega_0^{P}| \\
&= |\Omega_2^{P}|. && \text{(since } n_P = 0\text{, and by applying Lemma 2.2.4)}
\end{aligned}
$$

Moreover, for $Q \notin \mathcal{H}$,

1. $\Omega_2^Q = m_1 \cdot \Omega_0^Q$ (by Lemma 2.2.3).

2. $|\Omega_2^Q| = m_1|\Omega_0^Q| = m_1|\Omega_0^{Q'}| = |\Omega_2^{Q'}|$ for $Q' \in Q^{\mathcal{F}}$ (using condition $(\star)$ for $\mathcal{H}$).

It follows that for $Q \notin \mathcal{H}'$ and for all $Q' \in Q^{\mathcal{F}}$ we have $|\Omega_2^{Q'}| = |\Omega_2^Q|$. We have satisfied condition $(\star)$ for $\mathcal{H}'$. By the induction hypothesis, there exist S-sets $\Omega \supseteq \Omega_3 \supseteq \Omega_2$ such that:

1. $\Omega_3 = m_2 \cdot \Omega_2$ for some positive integer m_2.

2. $\Omega^Q = \Omega_3^Q$ for all $Q \leq S, Q \notin \mathcal{H}'$.

3. $|\Omega^Q| = |\Omega^{Q'}|$ for all $Q, Q' \leq S, Q' \in Q^{\mathcal{F}}$.

Set $\Omega_1 = m_1 m_2 \cdot \Omega_0$. Note that $\Omega \supseteq \Omega_1 \supseteq \Omega_0$. For $Q \notin \mathcal{H}$, we have

$$\Omega^Q = \Omega_3^Q = m_2 \cdot \Omega_2^Q = m_2 m_1 \cdot \Omega_0^Q = \Omega_1^Q$$

The result now follows for $\mathcal{H}$ as desired. The second claim follows by Theorem 2.2.9 and (3) of the claim that we have just proved. □

Recall that for an S-S-biset Ω, $Q \leq S$ and $\phi \in \mathrm{Hom}(Q,S)$, we denote by ${}_Q\Omega$, the restriction of the left S-action to Q and ${}_\phi\Omega$, the restriction of the left S-action to Q induced by ϕ. Then ${}_Q\Omega$ and ${}_\phi\Omega$ are $Q \times S$ bisets.

Lemma 4.3.5. *Let $\mathcal{F}$ be a fusion system on a finite group S. Then there exists a semicharacteristic biset for $\mathcal{F}$ (containing $\mathcal{O}_{Id|_S}$ as an $S\times S$-orbit), that is, there is a finite $S\times S$-set Ω such that:*

1. *Every orbit in Ω is isomorphic to $\mathcal{O}_\phi$ for some $Q\leq S$ and $\phi\in \mathrm{Hom}_{\mathcal{F}}(Q,S)$, one of which is isomorphic to $\mathcal{O}_{Id|_S}$.*

2. *The Q-S-bisets ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic for all $Q\leq S$ and $\phi\in \mathrm{Hom}_{\mathcal{F}}(Q,S)$.*

3. *The S-Q-bisets Ω_Q and Ω_ϕ are isomorphic for all $Q\leq S$ and $\phi\in Hom_{\mathcal{F}}(Q,S)$.*

Proof. Define:

$$\Omega_0=\bigsqcup_{[\phi]\in\Gamma_0}\mathcal{O}_\phi.$$

where Γ_0 is defined as in Section 3.2. Given $g\in S$ and $\phi\in \mathrm{Aut}_{\mathcal{F}}(S)$,

$$(S\times S)/\Delta_S^\phi\to(S\times S)/\Delta_S^{\phi c_g},\text{ given by }\Delta_S^\phi(u,v)\mapsto\Delta_S^{\phi c_g}(u,v)$$

is an isomorphism of $(S\times S)$-sets. It follows that Ω_0 is a well-defined $S\times S$-set. Define:

$$\mathcal{H}=\{\Delta_P^\phi\mid P\lneq S,\phi\in \mathrm{Hom}_{\mathcal{F}}(P,S)\}.$$

Note that $\mathcal{F}\times\mathcal{F}$ as defined in Definition 2.1.3 is a fusion system on $S\times S$. Moreover, $\mathcal{H}$ is closed under subgroups and $\mathcal{F}\times\mathcal{F}$-conjugacy: The former case is clear whereas for the latter case, if $\alpha=(\alpha_1,\alpha_2)|_{\Delta_P^\phi}$ is a homomorphism in $\mathcal{F}\times\mathcal{F}$, then $(\Delta_P^\phi)\alpha=\Delta_{P\alpha_1}^{\alpha_1^{-1}\phi\alpha_2}\in\mathcal{H}$.

Let $Q\leq S\times S$. Suppose $x=\Delta_S^\phi(u,v)\in\Omega_0^Q$ for some $\phi\in \mathrm{Aut}_{\mathcal{F}}(S)$. Then $Q\subseteq \mathrm{Stab}_{S\times S}(x)=(u^{-1},v^{-1})\Delta_S^\phi(u,v)=\Delta_S^{c_{u^{-1}}\phi c_v}$. We show that for all $Q,Q'\leq S\times S$ which are $\mathcal{F}\times\mathcal{F}$-conjugate and not in $\mathcal{H}$, $|\Omega_0^Q|=|\Omega_0^{Q'}|$. Suppose that Q is not in $\mathcal{H}$ and not of the form Δ_S^ϕ. If $\Omega_0^Q\neq\emptyset$, then $Q\lneq\Delta_S^\phi$, for some $\phi\in \mathrm{Aut}_{\mathcal{F}}(S)$, that is $Q\in\mathcal{H}$, a contradiction. It follows that $\Omega_0^Q=\emptyset$. If Q' is $(\mathcal{F}\times\mathcal{F})$-conjugate to Q, then by Lemma 2.2.6, $|\Omega_0^Q|=\emptyset=|\Omega_0^{Q'}|$. Now suppose $Q=\Delta_S^\phi$ and is $(\mathcal{F}\times\mathcal{F})$-conjugate to some Q', say, under $\alpha=(\alpha_1,\alpha_2)|_Q$ where $\alpha_i\in \mathrm{Aut}_{\mathcal{F}}(S)$. Then $\Delta_S^\psi(u,v)\in\Omega_0^Q$ for some $\psi\in \mathrm{Out}_{\mathcal{F}}(S)$, if and only if, $\Delta_S^{\alpha_1^{-1}\psi\alpha_2}(u\alpha_1,v\alpha_2)\in\Omega_0^{Q'}$. Thus

$|\Omega_0^Q| = |\Omega_0^{Q'}|$.

We have satisfied condition $(\star)$ of Lemma 4.3.4. Therefore, there exists $(S\times S)$-sets $\Omega \supseteq \Omega_1 \supseteq \Omega_0$ and a positive integer n such that $\Omega_1 = n\cdot\Omega_0$; and $|\Omega^Q| = |\Omega^{Q'}|$ for any $\mathcal{F}\times\mathcal{F}$-conjugate subgroups $Q, Q' \leq S\times S$; and also $\Omega^Q = \Omega_1^Q$ for any subgroup $Q \leq S\times S$ not in $\mathcal{H}$.

Let $\alpha \in \Omega\backslash\Omega_1$ and $Q = \mathsf{Stab}_{S\times S}(\alpha)$. If $Q \notin \mathcal{H}$, then $\alpha \in \Omega^Q = \Omega_1^Q$, a contradiction. So $Q \in \mathcal{H}$. Say $Q = \Delta_R^\psi$ for some $R \lneq S$ and $\psi \in \mathsf{Hom}_{\mathcal{F}}(R,S)$. By the orbit-stabilizer theorem $|\mathsf{Orb}_{S\times S}(\alpha)| = |S\times S|/|\Delta_R^\psi| = |(S\times S)/\Delta_R^\psi|$. It follows that the map:

$$\mathsf{Orb}_{S\times S}(\alpha) \to (S\times S)/\Delta_R^\psi : \alpha\cdot(g_1,g_2) \mapsto \Delta_R^\psi(g_1,g_2)$$

is an isomorphism of $(S\times S)$-sets. Thus, every orbit inside Ω is isomorphic to $\mathcal{O}_\phi$ for some $Q \leq S$ and $\phi \in \mathsf{Hom}_{\mathcal{F}}(Q,S)$.

For part (2), define $\bar{\phi} = (\phi, \mathsf{Id})$ for some $\phi \in \mathsf{Hom}_{\mathcal{F}}(Q,S)$. Then $\bar{\phi} \in \mathsf{Hom}_{\mathcal{F}\times\mathcal{F}}(Q\times S, S\times S)$. By Lemma 4.3.4, Ω as a $(Q\times S)$-set via restriction is isomorphic to Ω as a $(Q\times S)$-set via $\bar{\phi}$, i.e. the $(Q\times S)$-sets ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic.

For part (3), define $\hat{\phi} = (\mathsf{Id}, \phi)$ for some $\phi \in \mathsf{Hom}_{\mathcal{F}}(Q,S)$. Then $\hat{\phi} \in \mathsf{Hom}_{\mathcal{F}\times\mathcal{F}}(S\times Q, S\times S)$. By Lemma 4.3.4, Ω as a $(S\times Q)$-set via restriction is isomorphic to Ω as a $(S\times Q)$-set via $\hat{\phi}$, i.e. the $(S\times Q)$-sets Ω_Q and Ω_ϕ are isomorphic. $\square$

Lemma 4.3.6. *Let $\mathcal{F}$ be a fusion system on a finite group S and $\phi \in \mathsf{Hom}_{\mathcal{F}}(Q,S)$, a morphism in $\mathcal{F}$. Let Ω be an $S\times S$-set and G the group of all permutations of Ω that preserve the right S-action. That is,*

$$G = \{\pi \in Sym(\Omega) \mid (x\circ(s_1,s_2))\pi = ((x\circ(s_1,1))\pi)\circ(1,s_2) \text{ for all } x\in\Omega, s_1,s_2\in S\}.$$

Identify S as a subgroup of G in the following way:

$$\iota : S \hookrightarrow G : s \mapsto (x \mapsto x\circ(s,1)).$$

Then ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic as $Q\times S$-sets if and only if ϕ is a morphism in $\mathcal{F}_S(G)$.

Proof. Write $\circ$ to denote the $Q\times S$ action, and $\bullet$ to denote $Q\times S$ action via ϕ. If ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic, then there exists $\pi\in \mathrm{Sym}(\Omega)$ such that for all $s_1,s_2\in S$ we have

$$\begin{aligned}(x\circ(s_1,s_2))\pi &= (x\circ(s_1,1)(1,s_2))\pi\\ &= (x\circ(s_1,1))\pi\bullet(1,s_2)\\ &= (x\circ(s_1,1))\pi\circ(1,s_2), && \text{(since } 1\phi=1)\end{aligned}$$

that is, $\pi\in G$. For any $s\in S$, we write $s\iota$ to denote the image of s in G. If $\pi=s\iota$ for some $s\in S$, we write $\pi\iota^{-1}$ to denote the preimage of π in S. Thus, $\iota^{-1}|_{(Q\iota)}\phi\iota$ translates the map ϕ into G, as illustrated below:

$$\begin{array}{ccc} G & \xrightarrow{\iota^{-1}|_{(Q\iota)}\phi\iota} & G\\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota}\\ Q & \xrightarrow{\phi} & Q\phi\end{array}$$

Thus,

$$\begin{aligned}&{}_Q\Omega \text{ and } {}_\phi\Omega \text{ are isomorphic as } Q\times S\text{-sets}\\ \iff& (x\circ(q,s))\pi=(x\pi)\bullet(q,s) \text{ for all } x\in\Omega, q\in Q, s\in S, \text{ and some } \pi\in G\\ \iff& (x\circ(q,s))\pi=(x\pi)\circ(q\phi,s) \text{ for all } x\in\Omega, q\in Q, s\in S, \text{ and some } \pi\in G\\ \iff& ((x\circ(q,1))\pi)\circ(1,s)=((x\pi)\circ(q\phi,1))\circ(1,s) \text{ for all } x\in\Omega, q\in Q, s\in S, \text{ and some } \pi\in G\\ \iff& (x\circ(q,1))\pi=(x\pi)\circ(q\phi,1) \text{ for all } x\in\Omega, q\in Q, \text{ and some } \pi\in G\\ \iff& ((y\pi^{-1})\circ(q,1))\pi=y\circ(q\phi,1) \text{ for all } y\in\Omega, q\in Q, \text{ and some } \pi\in G\\ \iff& ((y\pi^{-1})(q\iota))\pi=(y)((q\phi)\iota) \text{ for all } y\in\Omega, q\in Q, \text{ and some } \pi\in G\\ \iff& (q\iota)c_\pi=(q\iota)\iota^{-1}\phi\iota \text{ for all } q\in Q, \text{ and some } \pi\in G\\ \iff& \iota^{-1}|_{(Q\iota)}\phi\iota\in \mathrm{Hom}_{\mathcal{F}_S(G)}((Q)\iota,(Q)\phi\iota).\end{aligned}$$

It follows that $\iota^{-1}|_{(Q\iota)}\phi\iota\in \mathrm{Hom}_{\mathcal{F}_S(G)}((Q)\iota,(Q)\phi\iota)$ if and only if ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic under some $\pi\in G$, as $Q\times S$-sets. □

We have a similar Lemma for the case when we are dealing with isomorphism of $S\times Q$-sets:

Lemma 4.3.7. *Let $\mathcal{F}$ be a fusion system on a finite group S and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$, a morphism in $\mathcal{F}$. Let Ω be an $S \times S$-set and G the group of all permutations of Ω that preserve the right S-action. That is,*

$$G = \{\pi \in \mathit{Sym}(\Omega) \mid (x \circ (s_1,s_2))\pi = ((x \circ (1,s_2))\pi) \circ (s_1,1) \text{ for all } x \in \Omega, s_1, s_2 \in S\}.$$

Identify S as a subgroup of G in the following way:

$$\iota : S \hookrightarrow G : s \mapsto (x \mapsto x \circ (1,s)).$$

Then Ω_Q and Ω_ϕ are isomorphic as $S \times Q$-sets if and only if ϕ is a morphism in $\mathcal{F}_S(G)$.

Proof. The proof is almost identical to the proof of Lemma 4.3.6. □

Theorem 4.3.8. *[7, Theorem 3](Park) Every fusion system $\mathcal{F}$ on a finite group S is realizable.*

Proof. Let $\mathcal{F}$ be a fusion system over a finite group S. By Lemma 4.3.5 there exists a left semicharacteristic biset Ω (containing $\mathcal{O}_{\mathrm{Id}|_S}$ as an $S \times S$-orbit) for $\mathcal{F}$. Then for all $Q \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$, the $Q \times S$-sets ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic. Thus, by Lemma 4.3.6, under the identification given in Lemma 4.3.6, ϕ is a homomorphism in $\mathcal{F}_S(G)$, where

$$G = \{\pi \in \mathrm{Sym}(\Omega) \mid (x \circ (s_1,s_2))\pi = ((x \circ (s_1,1))\pi) \circ (1,s_2) \text{ for all } x \in \Omega, s_1, s_2 \in S\}.$$

This means $\mathcal{F} \subseteq \mathcal{F}_S(G)$. Conversely, suppose that $\phi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q,S)$. We show that $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$. Note that $\Delta_S^{\mathrm{Id}|_S} \in \Omega^{\mathrm{Id}|_{I_\phi}}$. Thus $\Omega^{\mathrm{Id}|_{I_\phi}} \neq \emptyset$. Since ${}_Q\Omega$ and ${}_\phi\Omega$ are isomorphic as $Q \times S$-sets, we have

$$|\Omega^\phi| = |({}_Q\Omega)^\phi| = |({}_\phi\Omega)^\phi| = |\Omega^{\mathrm{Id}|_{I_\phi}}| \neq 0.$$

where the first and third equalities are due to Lemma 4.2.1, and where the second equality is due to Theorem 2.2.9. Hence, there exists $\Delta_P^\psi(u,v) \in \Omega^\phi$ such that $\Delta_P^\psi(u,v)(q,q\phi) = \Delta_P^\psi(u,v)$ for all $q \in Q$. For some $p \in P$ this implies $(uq, v(q\phi)) = (p, p\psi)(u,v)$ and therefore

$$q\phi = v^{-1}((uqu^{-1})\psi)v = qc_{u^{-1}}\psi c_v,$$

for all $q \in Q$. Since $\psi \in \mathcal{F}$ by construction of Ω, we have $\phi \in \mathcal{F}$. Hence $\mathcal{F} = \mathcal{F}_S(G)$. $\square$

Similarly, we can use Lemmas 4.3.5 and 4.3.7 on right semicharacteristic bisets (instead of left) to prove that every fusion system $\mathcal{F}$ on a finite group S is realizable.

Using Lemmas 4.3.5 and 4.3.6, we can deduce that $G \cong S \wr \mathrm{Sym}(|\Omega|/|S|)$, the wreath product of S with the symmetric group $\mathrm{Sym}(|\Omega|/|S|)$, as follows:

Lemma 4.3.9. *Let $\Omega = \bigsqcup_{i \in I} \mathcal{O}_{\phi_i}$ be a union of $S \times S$-orbits, for some $\phi_i \in \mathcal{U}(S)$, and some index set I. Let G be the group of all permutations of Ω that preserve the right S-action, that is,*

$$G = \{\pi \in \mathit{Sym}(\Omega) \mid (x \circ (s_1, s_2))\pi = ((x \circ (s_1, 1))\pi) \circ (1, s_2) \text{ for all } x \in \Omega, s_1, s_2 \in S\}.$$

Then $G \cong S \wr \mathit{Sym}(|\Omega|/|S|)$.

Proof. Define the following action of S on Ω, $\Omega \times S \to \Omega$, given by $(x, s) \mapsto (x \circ (1, s))$, where $\circ$ denotes the $S \times S$-action. We show that S acts semiregularly on Ω. (Note that the $S \times S$-action need not be semiregular). Let $s \in S$ and $\Delta_Q^\phi(g, h) \in \Omega$. Then

$$\begin{aligned}
\Delta_Q^\phi(g,h)(1,s) = \Delta_Q^\phi(g,h) &\Leftrightarrow \Delta_Q^\phi(g,hs) = \Delta_Q^\phi(g,h) \\
&\Leftrightarrow (qg, (q\phi)hs) = (g,h) && \text{(for some } q \in Q) \\
&\Leftrightarrow qg = g \text{ and } (q\phi)hs = h \\
&\Leftrightarrow s = 1.
\end{aligned}$$

It follows that S is semiregular. We can also identify S as a subgroup of $\mathrm{Sym}(\Omega)$ in the following way:

$$\iota : S \hookrightarrow \mathrm{Sym}(\Omega) : s \mapsto (\omega \mapsto (\omega \circ (1, s))).$$

Note that

$$\begin{aligned}
G &= \{\pi \in \mathsf{Sym}(\Omega) \mid (x \circ (s_1, s_2))\pi = ((x \circ (s_1, 1))\pi) \circ (1, s_2) \text{ for all } x \in \Omega, s_1, s_2 \in S\} \\
&= \{\pi \in \mathsf{Sym}(\Omega) \mid (x \circ (1, s))\pi = (x\pi) \circ (1, s) \text{ for all } x \in \Omega, s \in S\} \\
&= \{\pi \in \mathsf{Sym}(\Omega) \mid (x(s\iota))\pi = (x\pi)(s\iota) \text{ for all } x \in \Omega, s \in S\} \\
&= \{\pi \in \mathsf{Sym}(\Omega) \mid (s\iota)\pi = \pi(s\iota) \text{ for all } s \in S\} \\
&= C_{\mathsf{Sym}(\Omega)}((S)\iota).
\end{aligned}$$

By semiregularity, the number of orbits under the action of S is $|\Omega|/|S|$. Thus, by an application of Lemma 2.2.15, we have $G \cong S \wr \mathsf{Sym}(|\Omega|/|S|)$. □

Lemma 4.3.10. *Let $\mathcal{F}$ be a fusion system on a finite group S. Let G be the finite group in Theorem* 4.3.8 *that realizes $\mathcal{F}$, constructed as a permutation group of a left semicharacteristic biset Ω for $\mathcal{F}$. Then $G \cong S \wr Sym(|\Omega|/|S|)$.*

Proof. By Lemma 4.3.5, there exists a left semicharacteristic biset for the fusion system $\mathcal{F}$, which is an $S \times S$-set, and which we call Ω. We can write $\Omega = \bigsqcup_{i \in I} \mathcal{O}_{\phi_i}$ for some index set I. By Lemma 4.3.9, we can conclude $G \cong S \wr \mathsf{Sym}(|\Omega|/|S|)$. □

Definition 4.3.11. We define the exoticity index $e(\mathcal{F})$ as

$$\min\{\log_p |T : S| \mid S \leq T \in \mathsf{Syl}_p(G) \text{ for some finite } G \text{ with } \mathcal{F} \cong \mathcal{F}_S(G)\},$$

for a fusion system $\mathcal{F}$ over a finite p-group S.

Lemma 4.3.12. *Let $T \in Syl_p(G)$, where G is a finite group in Theorem* 4.3.8 *that realizes the fusion system $\mathcal{F}$ on a finite p-group S. Then*

$$log_p |T : S| = (|\Omega|/|S| - 1) log_p |S| + \sum_{i=1} \left\lfloor \frac{|\Omega|/|S|}{p^i} \right\rfloor,$$

where Ω denotes a left semicharacteristic biset for $\mathcal{F}$.

Proof. By Lemma 4.3.10, $G \cong S \wr \mathsf{Sym}(|\Omega|/|S|)$. This means $|G| = |S|^{|\Omega|/S|} |\mathsf{Sym}(|\Omega|/|S|)|$.

We have

$$\begin{aligned}
\log_p|T:S| &= \log_p\left(\frac{|S|^{|\Omega|/|S|}p^{\nu_p(|\mathsf{Sym}(|\Omega|/|S|)|)}}{|S|}\right) \\
&= (|\Omega|/|S|-1)\log_p|S| + \nu_p(|\mathsf{Sym}(|\Omega|/|S|)|) \\
&= (|\Omega|/|S|-1)\log_p|S| + \sum_{i=1}\left\lfloor \frac{|\Omega|/|S|}{p^i}\right\rfloor.
\end{aligned}$$

This completes the proof. □

CHAPTER 5

A FAMILY OF 3-GROUPS OF MAXIMAL NILPOTENCY CLASS

5.1 Properties

The 3-groups of maximal nilpotency class and order 3^i, for $i \geq 4$, were classified by Blackburn in [2, p.88]. We consider special cases of these groups. Let $r \geq 5$ be odd. Define S_r to be a rank two group of maximal nilpotency class, of order 3^r, with the following presentation:

$$S_r = \langle s, s_1, s_2, \dots, s_{r-1} \mid s_i = [s_{i-1}, s], [s_i, s_1] = s_j^3 s_{j+1}^3 s_{j+2} = s^3 = 1 \rangle.$$

where $2 \leq i \leq r-1$, $1 \leq j \leq r-1$, assuming that $s_r = s_{r+1} = 1$. Since there will be no ambiguity we will just write S to denote the group S_r. Since r is odd, we write

$$r = 2k+1,$$

for some $k \in \mathbb{N}$. We state a number of results arising from the structure of S. Write

$$A_i = \langle s_i, s_{i+1}, \dots, s_{r-1} \rangle$$

and define $A = A_1$.

Lemma 5.1.1. *[5, Lemma A.9, Lemma A.11] Assuming the notation in the definition of S, we have for all $i = 1, \dots, r-1$:*

1. $A_i = \langle s_i, s_{i+1} \rangle$ *and is abelian.*

2. A_i *is characteristic of order* 3^{r-i} *in* S.

3. s_i *has order* $3^{k-\lfloor \frac{i-1}{2} \rfloor}$, *or equivalently,* s_{r-i} *has order* $3^{\lceil \frac{i}{2} \rceil}$. *In particular,* s_1 *and* s_2 *each have order* 3^k.

Proof. 1. Fix $1 \le i \le r-1$. For all $1 \le j \le r-1$, we have $s_j^3 s_{j+1}^3 s_{j+2} = 1$. Thus, for all $i+2 \le j \le r-1$, s_j can be written as a product of s_i and s_{i+1}. This means $A_i = \langle s_i, s_{i+1} \rangle$. Since $A_i \le A$, it suffices to prove that A is abelian. Choosing $i = 1$, we see that for all $1 \le j \le r-1$, s_j can be written as a product of s_1 and s_2. Since $[s_2, s_1] = 1$, it follows that A is abelian.

2. Let $a_i \in A_i$. Then $a_i = s_i^m a_{i+1}$ for some $a_{i+1} \in A_{i+1}$. Then

$$[a_i, s] = [s_i^m a_{i+1}, s] = a_{i+1}^{-1} s_i^{-m} s^{-1} s_i^m a_{i+1} s = a_{i+1}^{-1} [s_i^m, s] a_{i+1} = a_{i+1}^{-1} s_{i+1}^m a_{i+1} \in A_{i+1}.$$

Note also that $s_{i+1} = [s_i, s] \in [A_i, S]$ and similarly we can see that $s_{i+2} \in [A_i, S]$. Thus $A_{i+1} = [A_i, S]$. We show by induction that for all $1 \le i \le r-1$, A_i is characteristic in S. A is the unique abelian subgroup of index 3, because s does not commute with s_1 or s_2. Thus A is characteristic in S. Suppose that for some $1 \le i-1 \le r-1$, A_i is characteristic. We show that A_{i+1} is characteristic. Let $\phi \in \mathrm{Aut}(S)$. Note that $A_{i+1}\phi = ([A_i, S])\phi = [A_i\phi, S\phi] = [A_i, S] = A_{i+1}$, that is, A_{i+1} is characteristic.

Let $1 \le i \le r-1$. We show that A_i is of order 3^{r-i}. The subgroup A is of index 3 since $s^3 = 1$. Suppose that for some $1 \le i \le r-1$, A_i is of order 3^{r-i}. The quotient group A_i/A_{i+1} is of order 3 because $s_i^3 = s_{i+1}^{-3} s_{i+2}^{-1} \in A_{i+1}$. Thus $|A_{i+1}| = |A_i|/|A_i/A_{i+1}| = 3^{r-i}/3 = 3^{r-(i+1)}$.

3. We prove the claim that $s_{r-i} = 3^{\lceil \frac{i}{2} \rceil}$ for all $1 \le i \le r-1$. Note that $|s_{r-1}| = 3$, and since $s_{r-2}^3 s_{r-1}^3 = 1$, we also have $|s_{r-2}| = 3$, as required. Suppose that, for some $1 \le i \le r-3$, we already know the orders of s_{r-i} and $s_{r-(i+1)}$. Write $m = \lceil \frac{i}{2} \rceil$. Then $|s_{r-i}| = 3^m$ and

$|s_{r-(i+1)}| = 3^{m+\delta}$, where

$$\delta = \begin{cases} 1 & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

Note that $m + (m+\delta) = \lceil \frac{i}{2} \rceil + \lceil \frac{i+1}{2} \rceil = i+1$, and from part (1) and (2), $|A_{r-(i+1)}| = |\langle s_{r-(i+1)}, s_{r-i} \rangle| = 3^{2k-(r-(i+1))+1} = 3^{i+1}$. This means that no nontrivial power of $s_{r-(i+1)}$ is equal to a nontrivial power of s_{r-i}. Note that $s^3_{r-(i+2)} = s^{-3}_{r-(i+1)} s^{-1}_{r-i}$. Thus $s^{3^m}_{r-(i+2)} = s^{-3^m}_{r-(i+1)} s^{-3^{m-1}}_{r-i} \neq 1$, and therefore $|s^{3^m}_{r-(i+2)}| = 3$. If $|s_{r-(i+2)}| = 3^u$ for some $u \leq m$, then by writing $v = m-u$, we see that $s^{3^m}_{r-(i+2)} = (s^{3^u}_{r-(i+2)})^{3^v} = 1$, a contradiction. It follows that $|s_{r-(i+2)}| = 3^{m+1} = 3^{\lceil \frac{i+2}{2} \rceil}$, as required.

This completes the proof of the lemma. $\square$

By 5.1.1, we see that $A_i \cong \mathbb{Z}_{3^u} \times \mathbb{Z}_{3^v}$, where $u = k - \lfloor \frac{i-1}{2} \rfloor$ and $v = k - \lfloor \frac{i}{2} \rfloor$. In particular, $A \cong \mathbb{Z}_{3^k} \times \mathbb{Z}_{3^k}$.

Fix $1 \leq i \leq r-1$. Write $\bar{s} = s$ and $\bar{s}_{i'} = s_{(i-1)+i'}$ for all $1 \leq i' \leq r-i$. By 5.1.1 we have $[\bar{s}_{i'}, \bar{s}_{1'}] = 1$. It follows that $\bar{s}_{i'} = [\bar{s}_{i'-1}, \bar{s}]$ and $[\bar{s}_{i'}, \bar{s}_{1'}] = \bar{s}^3_{j'} \bar{s}^3_{j'+1} \bar{s}_{j'+2} = \bar{s}^3 = 1$, for all $2 \leq i' \leq r-i$ and $1 \leq j' \leq r-i$. Thus, we can write

$$S_{(r+1)-i} = \langle s, A_i \rangle \leq S_r = S,$$

for all $1 \leq i \leq r-1$. Note that

$$S \cong A \rtimes \langle s \rangle.$$

This holds because $\langle s \rangle$ has order 3, $A\langle s \rangle = S$, and by 5.1.1, A is normal in S. Define $z = s_{r-1}$. We have $Z(S) = \langle z \rangle$.

Let G be a group. Set $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. Then $\gamma_i(G)$ are the terms of the lower central series of G. As stated earlier, S is a group of maximal nilpotency class.

Explicitly, the lower central series is given by:

$$S > A_2 > A_3 > ... > A_{r-1} = \langle z \rangle > 1.$$

In groups of maximal class, it is known that the upper central series coincides with the lower central series, but in the opposite direction.

Lemma 5.1.2. *Let G be a finite p-group of maximal class. Let $H \trianglelefteq G$ of index at least p^2. Then $H = \gamma_i(G)$ and $|G : H| = p^i$ for some $i \geq 2$.*

Proof. If H is of index p^2, then G/H is abelian. This implies $H \geq G'$. Since G is of maximal class, $H = G'$, i.e. $H = \gamma_2(G)$. Now suppose $|G : H| = p^i$ for some $i > 2$ and that the result holds for all normal subgroups of S of index p^j for $2 \leq j < i$. Define

$$K = \{x \in G \mid [x,y] \in H \text{ for all } y \in G\}.$$

Translated, K is the preimage of $Z(G/H)$ in G. Since G is nilpotent, $Z(G/H) \neq 1$. Thus, $H < K$. As $K \trianglelefteq G$ and $H < K$, by the induction hypothesis we have $K = \gamma_j(G)$ and therefore $H \geq [\gamma_j(G), G] = \gamma_{j+1}(G)$ for some $j \geq 2$. Since $|\gamma_j(G)/\gamma_{j+1}(G)| = p$, we actually have $H = \gamma_{j+1}(G)$. Since $|G : \gamma_{j+1}(G)| = p^{j+1}$, we can conclude $j+1 = i$, that is, $H = \gamma_i(G)$. □

Lemma 5.1.3. *Let $P \leq S$ such that $P \nleq A$. Then $|P : P \cap A| = 3$. In particular, if $P \cap A = 1$, then $|P| = 3$.*

Proof. Since $s^3 = 1$, we have $|S : A| = 3$. Since $A \triangleleft S$, we can apply the Second Isomorphism Theorem: $|P : P \cap A| = |PA : A| = |S : A| = 3$. □

Lemma 5.1.4. *Let $P \leq S$. Then rank$(P) \leq 2$.*

Proof. If $P \leq A$, then rank$(P) \leq 2$, since $A = \langle s_1, s_2 \rangle$ is a direct product of two cyclic groups. So suppose that $P \nleq A$. Let $g \in P \backslash A$. Then, by 5.1.3, $P = (A \cap P)\langle g \rangle$. Now $A \cap P \trianglelefteq A\langle g \rangle = S$. By 5.1.2, either $A \cap P$ is of index p in S or $A \cap P = \gamma_i(S)$ where $|A : A \cap P| = p^i$. If the former case holds, then $A = S$, a contradiction. So the latter case holds. Then

$$P' \geq [A \cap P, P] = [A \cap P, A\langle g \rangle] = [A \cap P, S] = \gamma_{i+1}(S).$$

By 5.1.3, $|P : A\cap P| = 3$. Since S is of maximal class, $|A\cap P : [A\cap P,S]| = [\gamma_i(S) : \gamma_{i+1}(S)] = 3$. Therefore $|P : [A\cap P,S]| = 3^2$. It follows that $|P : P'| \leq |P : [A\cap P,S]| = 3^2$. By Lemma 2.3.1, the Frattini subgroup $\mathrm{Frat}(P) \geq P'$. Thus $|P : \mathrm{Frat}(P)| \leq 3^2$ and, by [11, Theorem 5.3.2], $\mathrm{rank}(P) \leq 2$. □

Lemma 5.1.5. *Let $x \in A$ be an element of order 3^n for some $1 \leq n \leq k$. Then $x = s_1^{a\cdot 3^{k-n}} s_2^{b\cdot 3^{k-n}}$ for some integers $0 \leq a,b \leq 3^n - 1$ with one of them coprime to 3.*

Proof. Write $x = s_1^c s_2^d$ for some $0 \leq c,d \leq 3^k - 1$. Now $|x| = 3^n$ implies that $c\cdot 3^n \equiv 0 \mod 3^k$ and $d\cdot 3^n \equiv 0 \mod 3^k$. That is, $c = a\cdot 3^{k-n}$ and $d = b\cdot 3^{k-n}$, where $0 \leq a,b \leq 3^n - 1$. If a and b were both divisible by 3, then $x^{3^{n-1}} = 1$, giving us a contradiction. □

Recall, that for $n \in \mathbb{N}$, we write

$$\Omega_n(A) = \langle g \in A \mid g^{3^n} = 1\rangle.$$

Lemma 5.1.6. *$\Omega_n(A)$ is characteristic in A and normal in S.*

Proof. Group automorphisms preserve the orders of elements. Thus $\Omega_n(A)$ is characteristic in A. Since $A \trianglelefteq S$, we have $\Omega_n(A) \trianglelefteq S$. □

Lemma 5.1.7. *For all $n \in \{1,..,k\}$, we have $\Omega_n(A) = \langle s_1^{3^{k-n}}, s_2^{3^{k-n}}\rangle = A_{r-2n} = A_{2(k-n)+1}$, isomorphic to $\mathbb{Z}_{3^n}\times\mathbb{Z}_{3^n}$, containing the centre of S. In particular, $\Omega_k(A) = A$.*

Proof. Let $g \in A$. By 5.1.5, $|g| = 3^n \Leftrightarrow g = s_1^{a\cdot 3^{k-n}} s_2^{b\cdot 3^{k-n}}$ for $0 \leq a,b \leq 3^n - 1$ with one coprime to 3. It follows that $\Omega_n(A) \leq \langle s_1^{3^{k-n}}, s_2^{3^{k-n}}\rangle$. But $|s_1^{3^{k-n}}| = |s_2^{3^{k-n}}| = 3^n$. Hence $\Omega_n(A) = \langle s_1^{3^{k-n}}, s_2^{3^{k-n}}\rangle$. By 5.1.1, $|s_{r-2n}| = |s_{r-2n+1}| = 3^n$. It follows that $A_{r-2n} \leq \Omega_n(A)$. Since also by 5.1.1, $|A_{r-2n}| = 3^{2n}$, we have the desired equality. □

For the calculations in this thesis, we need to determine how elements of S interact with each other.

Lemma 5.1.8. *[5, Lemma A.12] Assuming the notation in the definition of S we have:*

1. *$s_1^s = s_1 s_2$ and $s_2^s = s_1^{-3} s_2^{-2}$; and*

2. $s_1^{s^{-1}} = s_1^{-2}s_2^{-1}$ *and* $s_2^{s^{-1}} = s_1^3 s_2$.

Thus every element of S *is either of the form* sa *or* $s^{-1}a$ *or* a *where* $a \in A$ *a product of elements of* s_1 *and* s_2.

Proof. Note that $s_1^s = s_1[s_1,s] = s_1s_2$ and $s_2^s = s_2[s_2,s] = s_2s_3 = s_2(s_1^{-3}s_2^{-3}) = s_1^{-3}s_2^{-2}$. Similarly, we have $s_1^{s^{-1}} = (s_1^s)^s = (s_1s_2)^s = s_1^s s_2^s = (s_1s_2)(s_1^{-3}s_2^{-2}) = s_1^{-2}s_2^{-1}$ and $s_2^{s^{-1}} = (s_2^s)^s = (s_1^{-3}s_2^{-2})^s = (s_1^s)^{-3}(s_2^s)^{-2} = (s_1s_2)^{-3}(s_1^{-3}s_2^{-2})^{-2} = s_1^3s_2$, as required. □

We can write $S = A \cup sA \cup s^{-1}A$.

Lemma 5.1.9. *Every element in* $S \backslash A$ *has order* 3.

Proof. By 5.1.8, every element in $S\backslash A$ is of the form sa or $s^{-1}a$ for some $a \in A$. Note first that

$$(s^{-1}a)^3 = (s^{-1}a)(ssa)(s^{-1}a) = a^s a^{s^{-1}} a = a^{s^{-1}} a^s a = (sa)(s^{-1}s^{-1}a)(sa) = (sa)^3.$$

Thus, it suffices to show that $(sa)^3 = 1$. Write $a = s_1^i s_2^j$ for some $0 \le i, j \le 3^k - 1$. Then $a^s = (s_1^s)^i(s_2^s)^j = (s_1s_2)^i(s_1^{-3}s_2^{-2})^j = s_1^{i-3j}s_2^{i-2j}$ and $a^{s^{-1}} = (s_1^{s^{-1}})^i(s_2^{s^{-1}})^j = (s_1^{-2}s_2^{-1})^i(s_1^3s_2)^j = s_1^{-2i+3j}s_2^{-i+j}$. Thus $(sa)^3 = (s_1^{-2i+3j}s_2^{-i+j})(s_1^{i-3j}s_2^{i-2j})(s_1^i s_2^j) = 1$ as required. □

Lemma 5.1.10. *Suppose* $P \le S$ *is abelian such that* $P \not\le A$. *Then* P *is elementary abelian of order at most* 9. *In particular,* $P \le \langle g, z\rangle$ *for some* $g \in S\backslash A$, *i.e.* $P = 1, \langle g\rangle$, *or* $\langle g, z\rangle$.

Proof. Since P is abelian, $P \cap A \le C_S(\langle A, P\rangle) = C_S(S) = Z(S)$. Therefore, $|P \cap A| \le 3$. By 5.1.3, it follows that $|P| \le 3^2$. Therefore, P is elementary abelian of order at most 9, containing the centre of S. □

5.2 Conjugacy classes of subgroups

If $\mathcal{F}$ is a fusion system on S, and ψ a morphism in $\mathcal{F}$, then over its characteristic biset Ω, 4.2.7 tells us that the number points fixed by ψ is equal to the number of points in Ω fixed by the identity map restricted to D_ψ. Since we want a right characteristic biset of minimal size, 3.1.8

implies that it suffices to select orbits $\mathcal{O}_\phi$ for Ω with ϕ chosen up to S-S-equivalence. We will therefore need to classify morphisms up to S-S-equivalence. By definition, if two morphisms are S-S-equivalent, then their domains are S-conjugate. In particular, this means that we need to determine conjugacy classes of subgroups of S.

Define $t = s_1^{3^{k-1}}$ and $z = s_2^{3^{k-1}}$ is the generator of $Z(S)$. Recall that $\Omega_1(A) = \langle t, z\rangle$ from Lemma 5.1.7. We also write $A_0 = \langle t, z\rangle$.

Lemma 5.2.1. *Let $P \leq A$ be a cyclic group of order 3^i for some $1 \leq i \leq k$. Write $x = s_1^{3^{k-i}}$ and $y = s_2^{3^{k-i}}$. If $i \geq 2$, then up to S-conjugacy*

$$P \in \{\langle x^{9a}y\rangle, \langle xy^{3b}\rangle \mid 0 \leq a \leq 3^{i-2}-1, 0 \leq b \leq 3^{i-1}-1\}.$$

On the other hand, if $i = 1$, then up to S-conjugacy, $P \in \{\langle t\rangle, \langle z\rangle\}$.

Proof. First suppose that $i = 1$. By 5.1.7, $P \leq \Omega_1(A) = \langle t, z\rangle$. So $P \in \{\langle t\rangle, \langle tz\rangle, \langle tz^2\rangle, \langle z\rangle\}$. Since $t^s = tz$ and $t^{s^{-1}} = tz^2$, we may assume up to S-conjugacy that $P \in \{\langle t\rangle, \langle z\rangle\}$. We now consider the case when $i \geq 2$:

<u>claim 1</u>: If $i \geq 2$, then $C_S(P) = N_S(P) = A$. In particular, the conjugacy class containing P is $P^S = \{P, P^s, P^{s^{-1}}\}$.

Proof of claim. Since $P \neq Z(S)$, $C_S(P) < S$. Assume for contradiction that $C_S(P) < N_S(P)$. Since $A \leq C_S(P)$, this implies, $A = C_S(P)$ and $P \trianglelefteq S$. Since P is cyclic, we have $|S : P| \geq 3^{k+1} \geq 3^2$. By 5.1.2, we have $P = \gamma_j(S)$ for some $2 \leq j \leq r$. If $j \leq r-2$, then $P \geq \gamma_{r-2}(S) = A_{r-2} = \langle t, z\rangle$, by 5.1.7. This is an elementary abelian subgroup of order 9, which gives us a contradiction. So $j \in \{r-1, r\}$, that is $P = Z(S)$ or $P = 1$, contradicting that $i \geq 2$. So $C_S(P) = N_S(P) = A$.

<u>claim 2</u>: Let

$$B_1 = \{\langle x^{3a}y\rangle \mid 0 \leq a \leq 3^{i-1}-1\} \text{ and } B_2 = \{\langle xy^b\rangle \mid 0 \leq b \leq 3^i - 1\},$$

and put $B = B_1 \cup B_2$. Then, $P \in B$.

Proof of claim. Let $P = \langle g \rangle$. By 5.1.5 we can write $g = x^a y^b$ for $0 \leq a, b \leq 3^i - 1$ with one of the two integers coprime to 3. First suppose that a is coprime to 3. Then there exists an n coprime to 3 such that $an \equiv 1 \bmod 3^k$. Thus $P = \langle g^n \rangle = \langle xy^{bn} \rangle$. That is, $P \in B_2$. It is left to consider the case when a is not coprime to 3 but b is coprime to 3. Write $g = x^{3a'} y^b$. By the same argument, there exists an n coprime to 3 such that $bn \equiv 1 \bmod 3^k$. Thus $P = \langle g^n \rangle = \langle x^{3na'} y \rangle$. That is, $P \in B_1$.

<u>claim 3</u>: Let

$$C_1 = \{\langle x^{9a} y \rangle \mid 0 \leq a \leq 3^{i-2} - 1\} \text{ and } C_2 = \{\langle xy^{3b} \rangle \mid 0 \leq b \leq 3^{i-1} - 1\},$$

and put $C = C_1 \cup C_2$. Then up to S-conjugacy $P \in C$.

Proof of claim. Note that $C_1 \subset B_1$, $C_2 \subset B_2$ and $C \subset B$. We show that every element in C is in a distinct conjugacy class. Suppose $P, Q \in C$ are distinct subgroups S-conjugate by some $h \in S$. Since $A \leq C_S(P)$, we may assume that $h \in \{s, s^{-1}\}$. By redefining Q if necessary, we may write $h = s$. Note that $P \in C_1$ if and only if $Q \in C_1$. First assume $P \in C_1$. Write $P = \langle x^{9a} y \rangle$ and $Q = \langle x^{9a'} y \rangle$. Then there exists an n such that $(x^{9a}y)^s = (x^{9a'}y)^n$, that is, $x^{9a-3}y^{9a-2} = x^{9na'}y^n$. Thus,

$$9a - 3 \equiv 9na' \bmod 3^i \quad \text{and} \quad 9a - 2 \equiv n \bmod 3^i.$$

This implies $9a - 3 \equiv 9a'(9a - 2) \bmod 3^i$, that is, $3(a + 2a' - 9aa') \equiv 1 \bmod 3^i$. This reveals a contradiction as $3 \nmid 1$. Now assume $P \in C_2$. Write $P = \langle xy^{3b} \rangle$ and $Q = \langle xy^{3b'} \rangle$. We arrive at a similar contradiction: There exists an n such $(xy^{3b})^s = (xy^{3b'})^n$, that is, $x^{1-9b}y^{1-6b} = x^n y^{3nb'}$. Thus,

$$1 - 9b \equiv n \bmod 3^i \quad \text{and} \quad 1 - 6b \equiv 3nb' \bmod 3^i.$$

This implies that $1 - 6b \equiv 3b'(1 - 9b) \bmod 3^i$, that is, $3(2b + b' - 18bb') \equiv 1 \bmod 3^i$, again a contradiction. Thus, C is a collection of groups that lie in distinct S-conjugacy classes.

We claim that these are all of them. By claim 1, each conjugacy class contains 3 subgroups. Thus, $|C| = 3^{i-2}+3^{i-1} = (3^{i-1}+3^{i})/3 = |B|/3$. Thus, the subgroups in C are all the conjugacy class representatives of subgroups of S of order 3^{i}. $\square$

Lemma 5.2.2. *Let $P \leq A$ be group of rank 2. Write $P \cong \mathbb{Z}/3^{i}\mathbb{Z} \times \mathbb{Z}/3^{j}\mathbb{Z}$ for some $1 \leq j \leq i$. Write $x = s_1^{3^{k-i}}$ and $y = s_2^{3^{k-i}}$. If $i-j \geq 2$, then, up to S-conjugacy,*

$$P \in \{\langle x^{9a}y, x^{3^{i-j}}\rangle, \langle xy^{3b}, y^{3^{i-j}}\rangle \mid 0 \leq a \leq 3^{i-j-2}-1, 0 \leq b \leq 3^{i-j-1}-1\}.$$

If on the other hand, $i-j=1$, then up to S-conjugacy $P \in \{\langle x, y^3\rangle, \langle x^3, y\rangle\}$. Finally, if $i-j=0$, then $P = \Omega_i(A) = \langle x, y\rangle$.

Proof. First consider the case when $i-j=0$. By 5.1.7, $P = \Omega_i(A) = \langle x, y\rangle$.

Now suppose that $i-j \geq 1$. In particular, note that $i \geq 2$. Write $P = \langle g, h\rangle$ with $|g| = 3^{i}$ and $|h| = 3^{j}$. We may assume one of the two elements, say g, is the generator of an S-conjugacy class representative of a cyclic subgroup of order 3^{i}. By 5.2.1, $g \in \{x^{9a}y, xy^{3b} \mid 0 \leq a \leq 3^{i-2}-1, 0 \leq b \leq 3^{i-1}-1\}$. Write $h = (x^{a'}y^{b'})^{3^{i-j}}$, for $0 \leq a', b' \leq 3^{j}-1$. First suppose that $g = x^{9a}y$ for some a. Then $hg^{-b'\cdot 3^{i-j}} = x^{(a'-9ab')\cdot 3^{i-j}}$. We may therefore write $h = x^{a'\cdot 3^{i-j}}$ for some a'. Now suppose that $g = xy^{3b}$ for some b. Then $hg^{-a'\cdot 3^{i-j}} = y^{(b'-3ba')\cdot 3^{i-j}}$. In this case, we may write $h = y^{b'\cdot 3^{i-j}}$. It follows that

$$P = \langle g, h\rangle \in \{\langle x^{9a}y, x^{a'\cdot 3^{i-j}}\rangle, \langle xy^{3b}, y^{b'\cdot 3^{i-j}}\rangle \mid 0 \leq a \leq 3^{i-2}-1, 0 \leq b \leq 3^{i-1}-1, 0 \leq a', b' \leq 3^{j}-1\}.$$

Note that $3 \nmid a'$ and $3 \nmid b'$, otherwise $|h| < 3^{j}$. Since there exists n and m, both coprime to 3, such that $a'n \equiv b'm \equiv 1 \bmod 3^{j}$, we can choose $a' = b' = 1$. If $i-j=1$, then we can also choose $a = b = 0$. In that case,

$$P \in \{\langle x, y^3\rangle, \langle y, x^3\rangle\}.$$

Note that these two subgroups are not equal and are not S-conjugate to each other. It follows that these two are all the S-conjugacy class representatives of subgroups isomorphic to $\mathbb{Z}/3^i\mathbb{Z} \times \mathbb{Z}/3^j\mathbb{Z}$, when $i-j=1$.

Now suppose that $i-j \geq 2$. In this case we can choose $0 \leq a \leq 3^{i-j-2}-1$ and $0 \leq b \leq 3^{i-j-1}-1$. Thus,

$$P = \langle g,h\rangle \in \{\langle x^{9a}y, x^{3^{i-j}}\rangle, \langle xy^{3b}, y^{3^{i-j}}\rangle \mid 0 \leq a \leq 3^{i-j-2}-1, 0 \leq b \leq 3^{i-j-1}-1\}.$$

It suffices to show that these are all the S-conjugacy class representatives of subgroups isomorphic to $\mathbb{Z}/3^i\mathbb{Z} \times \mathbb{Z}/3^j\mathbb{Z}$. Write

$$C_1 = \{\langle x^{9a}y, x^{3^{i-j}}\rangle \mid 0 \leq a \leq 3^{i-j-2}-1\} \text{ and } C_2 = \{\langle xy^{3b}, y^{3^{i-j}}\rangle \mid 0 \leq b \leq 3^{i-j-1}-1\},$$

and define $C = C_1 \cup C_2$. Suppose $P = \langle g_1,h_1\rangle, Q = \langle g_2,h_2\rangle \in C$ are S-conjugate by some $h \in S$. Then there exists integers c and d such that

$$g_2^h = g_1^c h_1^d.$$

Since $A \leq C_S(P)$, we may assume $h \in \{1,s,s^{-1}\}$. If $h = s^{-1}$, then $P^s = Q^{s^{-1}}$. Therefore, it suffices to assume $h \in \{1,s\}$.

<u>case 1</u>: $h=1$.
First suppose that $P,Q \in C_1$. Write $P = \langle x^{9a}y, x^{3^{i-j}}\rangle$ and $Q = \langle x^{9a'}y, x^{3^{i-j}}\rangle$ for some $a \neq a'$. Now $x^{9a'}y = (x^{9a}y)^c(x^{3^{i-j}})^d = x^{9ac+d\cdot 3^{i-j}}y^c$. Thus, $c \equiv 1 \bmod 3^i$ and $9ac + d\cdot 3^{i-j} \equiv 9a' \bmod 3^i$. This implies $a'-a \equiv d\cdot 3^{i-j-2} \bmod 3^{i-2}$, i.e. $a=a'$, a contradiction.
Now suppose that $P,Q \in C_2$. Write $P = \langle xy^{3b}, y^{3^{i-j}}\rangle$ and $Q = \langle xy^{3b'}, y^{3^{i-j}}\rangle$ for some $b \neq b'$. This implies $xy^{3b'} = (xy^{3b})^c(y^{3^{i-j}})^d = x^c y^{3bc+d\cdot 3^{i-j}}$. Thus, $c \equiv 1 \bmod 3^i$ and $3bc + d\cdot 3^{i-j} \equiv 3b' \bmod 3^i$. This implies that $b-b' \equiv d\cdot 3^{i-j-1} \bmod 3^{i-1}$, i.e. $b=b'$, a contradiction.
It is left to consider the case when $P \in C_1$ and $Q \in C_2$. Write $P = \langle x^{9a}y, x^{3^{i-j}}\rangle$ and $Q = \langle xy^{3b}, y^{3^{i-j}}\rangle$ for some a,b. This implies $xy^{3b} = (x^{9a})^c(x^{3^{i-j}})^d = x^{9ac+d\cdot 3^{i-j}}y^c$. $1 \equiv 9ac + d\cdot$

3^{i-j} mod 3^i. This reveals a contradiction, as $3 \nmid 1$.

case 2 $h = s$.

First suppose $P, Q \in C_1$. Write $P = \langle x^{9a}y, x^{3^{i-j}} \rangle$ and $Q = \langle x^{9a'}y, x^{3^{i-j}} \rangle$ for some a, a'. Now $(x^{9a'}y)^s = (x^{9a}y)^c(x^{3^{i-j}})^d$, that is, $x^{9a'-3}y^{9a'-2} = x^{9ac+d\cdot 3^{i-j}}y^c$. Thus, $9a'-3 \equiv 9ac + d \cdot 3^{i-j}$ mod 3^i and $9a'-2 \equiv c$ mod 3^i. It follows that, $3a' - 3a(9a'-2) - d \cdot 3^{i-j} \equiv 1$ mod 3^i, a contradiction.

Now suppose $P, Q \in C_2$. Write $P = \langle xy^{3b}, y^{3^{i-j}} \rangle$ and $Q = \langle xy^{3b'}, y^{3^{i-j}} \rangle$. Now $(xy^{3b'})^s = (xy^{3b})^c(y^{3^{i-j}})^d$, that is, $x^{1-9b'}y^{1-6b'} = x^c y^{3bc+d\cdot 3^{i-j}}$. Thus, $1 - 9b' \equiv c$ mod 3^i and $1 - 6b' \equiv 3bc + d \cdot 3^{i-j}$ mod 3^i. This implies $6b' + 3b(1-9b') + d \cdot 3^{i-j} \equiv 1$ mod 3^i, a contradiction.

It is left to consider the case when $P \in C_1$ and $Q \in C_2$. Write $P = \langle x^{9a}y, x^{3^{i-j}} \rangle$ and $Q = \langle xy^{3b}, y^{3^{i-j}} \rangle$. Now $(xy^{3b})^s = (x^{9a}y)^c(x^{3^{i-j}})^d$, that is, $x^{1-9b}y^{1-6b} = x^{9ac+d\cdot 3^{i-j}}y^c$. Thus, $1-6b \equiv c$ mod 3^i and $1-9b \equiv 9ac + d \cdot 3^{i-j}$ mod 3^i. This implies, $9b + 9a(1-6b) + d \cdot 3^{i-j} \equiv 1$ mod 3^i, a contradiction. $\square$

Lemma 5.2.3. *Let $P \leq S$ be a cyclic subgroup such that $P \nleq A$. Then $|P| = 3$ and up to S-conjugacy*

$$P \in \{\langle s \rangle, \langle ss_1 \rangle, \langle ss_1^{-1} \rangle\}.$$

Proof. Let $C = \{s, ss_1, ss_1^{-1}, s^{-1}, s^{-1}s_1, s^{-1}s_1^{-1}\}$.

claim 1: No two elements in C are S-conjugate.

Proof of claim. By definition, $A_2 = \langle s_2, s_3 \rangle$. Since $s_1^3 \in A_2$, the quotient group $\bar{S} = S/A_2$ is elementary abelian of order 9, generated by $\bar{s}$ and $\bar{s_1}$. It follows that no two elements in $\bar{S}$ are $\bar{S}$-conjugate. Let $g_1, g_2 \in C$ such that $g_1 \neq g_2$. Then $\bar{g_1} \neq \bar{g_2}$. If $g_1^h = g_2$ for some $h \in S$, then $\bar{g_1}^{\bar{h}} = \bar{g_2}$, a contradiction. Thus, no two elements in C are S-conjugate.

claim 2: Let $g \in C$. Then $N_S(\langle g \rangle) = C_S(\langle g \rangle) = C_S(g) = \langle g, z \rangle$.

Proof of claim. We know that $\langle g, z \rangle \leq C_S(g) = C_S(\langle g \rangle) \leq N_S(\langle g \rangle)$. Note that $C_A(g) \leq C_S(A, g) = C_S(S) = Z(S)$. Therefore, by 5.1.3, $|C_S(g)| = 3|C_S(g) \cap A| = 3|C_A(g)| \leq 3|Z(S)| =$

3^2. It follows that $\langle g,z\rangle = C_S(g)$. By claim 1, g is not S-conjugate to its inverse. Therefore, $N_S(\langle g\rangle) = C_S(\langle g\rangle)$, as required.

Proof of Lemma. By claim 1, the subgroups $\langle s\rangle, \langle ss_1\rangle$ and $\langle ss_1^{-1}\rangle$, all of which are of order 3 by 5.1.9, are not S-conjugate to each other. Let $g \in \{s, ss_1, ss_1^{-1}\}$. Then the conjugacy class containing $\langle g\rangle$, denoted $\langle g\rangle^S$, has order $|\langle g\rangle^S| = |S/N_S(\langle g\rangle)| = |S/\langle g,z\rangle| = 3^{2k-1}$, by claim 2. So $|\langle s\rangle^S| + |\langle ss_1\rangle^S| + |\langle ss_1^{-1}\rangle^S| = 3^{2k}$. The total number of elements of order 3 in these three conjugacy classes is therefore $2\cdot 3^{2k}$. On the other hand, $|S\backslash A| = |S| - |A| = 3^{2k+1} - 3^{2k} = 2\cdot 3^{2k}$. It follows that up to S-conjugacy, $\langle s\rangle, \langle ss_1\rangle$ and $\langle ss_1^{-1}\rangle$ are the only three subgroups of order 3 not in A. $\square$

Lemma 5.2.4. *We have the following S-conjugacy classes for subgroups of order 3 not contained in A:*

$$\langle s\rangle^S = \{\langle ss_1^{3i}s_2^j\rangle \mid 0 \le i \le 3^{k-1}, 0 \le j \le 3^k\}$$
$$\langle ss_1\rangle^S = \{\langle ss_1^{3i+1}s_2^j\rangle \mid 0 \le i \le 3^{k-1}, 0 \le j \le 3^k\}$$
$$\langle ss_1^{-1}\rangle^S = \{\langle ss_1^{3i-1}s_2^j\rangle \mid 0 \le i \le 3^{k-1}, 0 \le j \le 3^k\}.$$

Proof. Write $g = ss_1^l$ for some $l \in \{-1,0,1\}$. Write $h = s^n s_1^a s_2^b$ for some n,a,b and $n \in \{-1,0,1\}$. Explicitly,

$$g^h = \begin{cases} ss_1^{3b+l}s_2^{3b-a+l} & \text{if } n = 1 \\ ss_1^{3b-2l}s_2^{3b-a-l} & \text{if } n = -1 \\ ss_1^{3b+l}s_2^{3b-a} & \text{if } n = 0 \end{cases}$$

Write $g^h = ss_1^{l'}s_2^{m'}$. We immediately see that $l' \equiv l \bmod 3$. Note that we can vary our choices for a and b to obtain all possible values of l' and m' which satisfy the congruence relation. $\square$

Lemma 5.2.5. *Let $P \le S$ with rank$(P) = 2$ and $P \not\le A$. Write $x_i = s_1^{3^{k-i}}$ and $y_i = s_2^{3^{k-i}}$. Up to S-conjugacy*

$$P \in \{\langle s,v\rangle, \langle ss_1,v\rangle, \langle ss_1^{-1},v\rangle \mid v = x_i \text{ or } y_i \text{ for some } 1 \le i \le k\}.$$

If $v = x_i$, then $P \cong (\mathbb{Z}_{3^i} \times \mathbb{Z}_{3^i}) \rtimes \mathbb{Z}_3$ and $|P| = 3^{2i+1}$. Otherwise $P \cong (\mathbb{Z}_{3^i} \times \mathbb{Z}_{3^{i-1}}) \rtimes \mathbb{Z}_3$ and $|P| = 3^{2i}$.

Proof. claim 1: Fix an element $g \in \{s, ss_1, ss_1^{-1}\}$. Write $B = \langle g, x_i \rangle$ and $C = \langle g, y_i \rangle$ for some $1 \le i \le k$. Then $y_i \in B$, $x_i^3 \in C$ and $\Omega_{i-1}(A) \le B, C$. Moreover, $B \cong (\mathbb{Z}_{3^i} \times \mathbb{Z}_{3^i}) \rtimes \mathbb{Z}_3$ and $C \cong (\mathbb{Z}_{3^i} \times \mathbb{Z}_{3^{i-1}}) \rtimes \mathbb{Z}_3$.

Proof of claim. Note that $x_i^{-1} x_i^g = y_i \in B$ and $(y_i^2 y_i^g)^{-1} = x_i^3 \in C$. It follows that $x_{i-1} = x_i^3$ and $y_{i-1} = y_i^3$ are elements of both B and C. By 5.1.7, we therefore have $\Omega_{i-1}(A) = \langle x_{i-1}, y_{i-1} \rangle \le B, C$. For the latter claim, it suffices to note that for $P \in \{B, C\}$ we have $P \cap A \trianglelefteq \langle g, A \rangle = S$, and that every element in B can be written as $g^l x_i^a y_i^b$, for some $0 \le a, b \le 3^i - 1$, and $-1 \le l \le 1$, and every element in C can be written as $g^l x_i^{3a} y_i^b$ for some $0 \le a \le 3^{i-1} - 1$, $0 \le b \le 3^i - 1$ and $-1 \le l \le 1$.

claim 2: Up to S-conjugacy $P \in \{\langle s, v \rangle, \langle ss_1, v \rangle, \langle ss_1^{-1}, v \rangle \mid v = x_i \text{ or } y_i \text{ for some } 1 \le i \le k\}$.

Proof of claim. If $P = S$, we have $P = \langle s, s_1 \rangle = \langle s, x_k \rangle$ as required. So assume $P < S$. Let $g \in P \backslash A$. Then 5.1.3 implies that $P = \langle g, P \cap A \rangle$. Note that $P \cap A \trianglelefteq \langle g, A \rangle = S$. Since P is to be chosen up to S-conjugacy, we can assume $g \in \{s, ss_1, ss_1^{-1}\}$ using 5.2.3. Since P is of index at least 3, $P \cap A$ is of index at least 3^2. Therefore, by 5.1.2, we have $P \cap A = \gamma_j(S) = A_j$ for some $j \ge 2$.

First suppose that j is odd. Let $i = (r - j)/2$. Then 5.1.7 implies that $P \cap A = A_j = \Omega_i(A) = \langle x_i, y_i \rangle$. In this case $|P \cap A| = 3^{2i}$. Therefore, by 5.1.3, $|P| = 3^{2i+1}$. Now suppose j is even. Note that $A_{j-1} < P \cap A = A_j < A_{j+1}$. Let $i = (r - (j+1))/2$. By 5.1.7, we have $\langle x_{i-1}, y_{i-1} \rangle = \Omega_{i-1}(A) < P \cap A < \Omega_i(A) = \langle x_i, y_i \rangle$. Now $\overline{\Omega_i(A)} = \Omega_i(A)/\Omega_{i-1}(A) = \langle \overline{x_i}, \overline{y_i} \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, $\overline{P \cap A} \in \{\langle \overline{x_i} \rangle, \langle \overline{y_i} \rangle, \langle \overline{x_i y_i} \rangle, \langle \overline{x_i}^2 \overline{y_i} \rangle\}$, that is, $P \cap A \in \{\langle \Omega_{i-1}, v \rangle \mid v \in \{x_i, y_i, x_i y_i x_i^2 y_i\}\}$. So $P \in \{\langle g, \Omega_{i-1}, v \rangle \mid v \in \{x_i, y_i, x_i y_i x_i^2 y_i\}\}$. Since $(x_i y_i)^{g^{-1}} = x_i$ and $(x_i^2 y_i)^g = x_i^{-1}$, it follows that $P \in \{\langle g, \Omega_{i-1}, v \rangle \mid v \in \{x_i, y_i\}\}$. Now claim 1 implies $P = \langle g, x_i \rangle$ or $P = \langle g, y_i \rangle$, as required. $\square$

Example 5.2.6. We write down a number of S-conjugacy class representatives for small subgroups of S. We use 5.2.1, 5.2.2, 5.2.3 and 5.2.5. Write $x_i = s_1^{3^{k-i}}$ and $y_i = s_2^{3^{k-i}}$. Then S-conjugacy class representatives of S of order

- Five classes of groups of order 3 are:

$$\langle s\rangle, \langle ss_1\rangle, \langle ss_1^{-1}\rangle, \langle t\rangle, \langle z\rangle.$$

- Eight classes of groups of order 9 are:

$$\langle s,z\rangle, \langle ss_1,z\rangle, \langle ss_1^{-1},z\rangle, \langle x_2\rangle, \langle y_2\rangle, \langle x_2y_2^3\rangle, \langle x_2y_2^6\rangle, \langle t,z\rangle.$$

- Seventeen classes of groups of order 27 are:

$$\begin{aligned}
&\langle s,t\rangle, \langle ss_1,t\rangle, \langle ss_1^{-1},t\rangle,\\
&\langle y_3\rangle, \langle x_3^9y_3\rangle, \langle x_3^{18}y_3\rangle, \langle x_3\rangle, \langle x_3y_3^3\rangle, \langle x_3y_3^6\rangle,\\
&\langle x_3y_3^9\rangle, \langle x_3y_3^{12}\rangle, \langle x_3y_3^{15}\rangle, \langle x_3y_3^{18}\rangle, \langle x_3y_3^{21}\rangle, \langle x_3y_3^{24}\rangle,\\
&\langle x_2,z\rangle, \langle t,y_2\rangle.
\end{aligned}$$

The following example looks at the case where $|S|$ is small:

Example 5.2.7. We restrict ourselves to the case $r=2$, that is, the smallest case in our family of groups. In this case $|S|=3^5$. The S-conjugacy class representatives of S of order

- Five classes of groups of order 3 are:

$$\langle s\rangle, \langle ss_1\rangle, \langle ss_1^{-1}\rangle, \langle t\rangle, \langle z\rangle.$$

- Eight classes of groups of order 9 are:

$$\langle s,z\rangle, \langle ss_1,z\rangle, \langle ss_1^{-1},z\rangle, \langle s_1\rangle, \langle s_2\rangle, \langle s_1z\rangle, \langle s_1z^2\rangle, \langle t,z\rangle.$$

- Five classes of groups of order 27 are:

$$\langle s,t\rangle, \langle ss_1,t\rangle, \langle ss_1^{-1},t\rangle, \langle s_1,z\rangle, \langle s_2,t\rangle.$$

Note that this is different to the example given in $5.2.6$ because here $k = 2$, and thus by definition, x_i, y_i must satisfy $0 \leq i \leq 2$.

- Four classes of groups of order 81 are

$$\langle s, s_2 \rangle, \langle ss_1, s_2 \rangle, \langle ss_1^{-1}, s_2 \rangle, \langle s_1, s_2 \rangle.$$

CHAPTER 6

AUTOMORPHISM GROUPS

6.1 Automorphisms of the full group S

In Chapter 7, we shall define fusion systems whose right characteristic biset we intend to compute. This calls for the study of morphisms that generate the fusion systems. In this section, we study properties of certain involutions in $\mathrm{Aut}(S)$. The purpose of this section is to check under which restrictions, to subgroups of S, can these involutions be identified as conjugation maps. We start by restating a result in [5], where our group S is denoted by $B(3,r;0,0,0)$ in the lemma.

Lemma 6.1.1. *[5, Lemma A.14] Every automorphism of S can be uniquely determined by sending*

$$s \mapsto s^e s_1^{e'} s_2^{e''}$$
$$s_1 \mapsto s_1^{f'} s_2^{f''}$$

where $e = \pm 1, 0 \leq e', e'', f', f'' \leq 3^k - 1$ and such that $3 \nmid f'$.

Define $\omega, \eta \in \mathrm{Aut}(S)$ to be group automorphisms of S given by:

$$\omega: \ s \mapsto s^{-1}; \quad s_1 \mapsto s_1^2 s_2$$

$$\eta: \ s \mapsto s; \quad s_1 \mapsto s_1^{-1}$$

We determine the image of s_2:

$$\begin{aligned}s_2\omega &= [s_1\omega, s\omega] = [s_1^2s_2, s^{-1}] = (s_1^{-2}s_2^{-1})s(s_1^2s_2)s^{-1}\\ &= (s_1^{-2}s_2^{-1})(s_1^{s^{-1}})^2(s_2^{s^{-1}}) = (s_1^{-2}s_2^{-1})(s_1^{-2}s_2^{-1})^2(s_1^3s_2) = s_1^{-3}s_2^{-2}.\end{aligned}$$

and,

$$\begin{aligned}s_2\eta &= [s_1\eta, s\eta] = [s_1^{-1}, s] = s_1s^{-1}s_1^{-1}s\\ &= s_1(s_1^s)^{-1} = s_1(s_1s_2)^{-1} = s_2^{-1}.\end{aligned}$$

Note also that η is an involution in $\mathrm{Aut}(S)$ and inverts every element of A. Since

$$(s_1)\omega^2 = (s_1^2s_2)\omega = (s_1\omega)^2(s_2)\omega = (s_1^2s_2)^2(s_1^{-3}s_2^{-2}) = s_1,$$

the map ω is also an involution in $\mathrm{Aut}(S)$. Let G be a group and ϕ an automorphism of G. Denote by $C_G(\phi)$, the set of elements in G fixed by ϕ.

Lemma 6.1.2. *The following holds:*

1. $C_S(\omega) = \langle s_1^3s_2\rangle$.
2. $C_S(\eta) = \langle s\rangle$.
3. $C_S(\omega\circ\eta) = \langle s_1s_2\rangle$.

In particular, $C_S(\omega)$ and $C_S(\omega\circ\eta)$ contain the centre of S.

Proof. 1. Suppose for contradiction that $C_S(\omega) \not\leq A$. Say $q \in C_S(\omega)\backslash A$. Write $q = s^ia$ for some $i \in \{-1, 1\}$ and $a \in A$. Then

$$(qA)\omega = (s^iA)\omega = s^{-i}A = q^{-1}A,$$

a contradiction. It follows that $C_S(\omega) \leq A$. Write $g = s_1^3s_2$. Then

$$g\omega = (s_1^2s_2)^3(s_1^{-3}s_2^{-2}) = g,$$

that is, $\langle g\rangle \in C_S(\omega)$. Suppose $C_S(\omega) \gneq \langle g\rangle$. Since $|g| = 3^k$, $C_S(\omega)$ is non-cyclic, and therefore contains an elementary abelian subgroup of order 3^2. Since $\Omega_1(A)$ is the unique elementary abelian subgroup of order 3^2 in A by 5.1.7, $C_S(\omega) \gneq \Omega_1(A) = \langle t, z\rangle$. But $t\omega = t^{-1}z$, a contradiction. We conclude that $C_S(\omega) = \langle s_1^3 s_2\rangle$.

2. If $C_S(\eta) \gneq \langle s\rangle$, then by Lemma 5.1.3, $C_S(\eta) \cap A \neq 1$, a contradiction, as η inverts every element of A. It follows that $C_S(\eta) = \langle s\rangle$.

3. Suppose for contradiction that $C_S(\omega\circ\eta) \nleq A$. Let $q \in C_S(\omega\circ\eta)\backslash A$. Write $q = s^i a$ for some $i \in \{-1, 1\}$ and $a \in A$. Then

$$(qA)\omega\circ\eta = (s^i A)\omega\circ\eta = s^{-i}A = q^{-1}A,$$

a contradiction. It follows that $C_S(\omega\circ\eta) \leq A$. Write $g = s_1 s_2$. Then $g\omega\circ\eta = ((s_1^2 s_2)(s_1^{-3} s_2^{-2}))\eta = g$, that is $\langle g\rangle \in C_S(\omega\circ\eta)$. By supposing $C_S(\omega\circ\eta) \gneq \langle g\rangle$, and by following the same method as in part (1), we achieve $C_S(\omega\circ\eta) \gneq \Omega_1(A)$. But $t\omega\circ\eta = tz^{-1}$, a contradiction. Thus, $C_S(\omega\circ\eta) = \langle s_1 s_2\rangle$.

□

Lemma 6.1.3. *Suppose that $\eta|_Q = c_g|_Q$ for some $g \in S$ and $Q \leq S$. If $Q \neq 1$, then Q is S-conjugate to $\langle s\rangle$.*

Proof. Suppose $1 \neq Q \leq S$.

<u>Claim 1</u> $Q \cap A = 1$ and $|Q| = 3$.

Proof of claim. Let $q \in Q \cap A$. Then $q\eta = q^{-1}$, that is η normalizes $Q \cap A$. Since η is an involution, this implies g^2 centralizes $Q \cap A$. Since g is 3-element, g centralizes $Q \cap A$. It follows that η centralizes $Q \cap A$, and therefore $Q \cap A = 1$. By 5.1.3, we have $|Q| = 3$.

Proof of Lemma. Write $Q = \langle q\rangle$ and $q = ss_1^i s_2^j$ for some $0 \leq i, j \leq 3^k - 1$. There exists

$0 \leq m \leq 3^{k-1}-1$, $0 \leq n \leq 3^k-1$ such that $s^g = ss_1^{3m}s_2^n$. It follows that

$$q\eta = q^g \Leftrightarrow ss_1^{-i}s_2^{-j} = \begin{cases} ss_1^{3m}s_2^n s_1^i s_2^j & \text{if } g \in A \\ ss_1^{3m}s_2^n s_1^{i-3j}s_2^{i-2j} & \text{if } g \in sA \\ ss_1^{3m}s_2^n s_1^{-2i+3j}s_2^{-i+j} & \text{if } g \in s^{-1}A \end{cases}$$

$$\Rightarrow \begin{cases} 3m+2i \equiv 0 \bmod 3^k & \text{if } g \in A \\ 3m+2i-3j \equiv 0 \bmod 3^k & \text{if } g \in sA \\ 3m-i+3j \equiv 0 \bmod 3^k & \text{if } g \in s^{-1}A \end{cases}$$

In all cases we see that $3 \mid i$, that is, $q = ss_1^{3i'}s_2^j$ where $i = 3i'$ for some $0 \leq i' \leq 3^{k-1}-1$. That is, Q is S-conjugate to $\langle s \rangle$, by 5.2.4. $\square$

Lemma 6.1.4. *1. $\omega|_Q = c_g|_Q$ for some $g \in S$ and some $Q \leq S$ if and only if one of the following holds:*

(a) $Q = Z(S) = \langle s_2^{3^{k-1}} \rangle$ and $g \in S$.

(b) $Q \leq \langle s_1^3 s_2 \rangle$ and $g \in A$.

(c) $Q \leq \langle s_2 \rangle$ and $g \in sA$.

(d) $Q \leq \langle s_1^3 s_2^2 \rangle$ and $g \in s^{-1}A$.

2. $(\omega \circ \eta)|_Q = c_g|_Q$ for some $g \in S$ and $Q \leq S$ if and only if one of the following holds:

(a) $Q \leq \langle s_1 s_2 \rangle$ and $g \in A$.

(b) $Q \leq \langle s_1^2 s_2 \rangle$ and $g \in sA$.

(c) $Q \leq \langle s_1 \rangle$ and $g \in s^{-1}A$.

In particular, when $\omega|_Q$ or $(\omega \circ \eta)|_Q$ is a conjugation map, for some $Q \leq S$, then $Q \leq A$.

Proof. We first verify that $\omega|_Q$ and $(\omega \circ \eta)|_Q$ are conjugation maps for the groups Q and elements g specified in the lemma. Let $g \in S$. Note that $\omega|_{Z(S)} = c_g|_{Z(S)}$ by 6.1.2 since $Z(S) \leq C_S(\omega)$. Thus $1(a)$ holds. Again, by 6.1.2, if $g \in A$, then $\omega|_{\langle s_1^3 s_2 \rangle} = c_g|_{\langle s_1^3 s_2 \rangle}$ and $\omega \circ \eta|_{\langle s_1 s_2 \rangle} = c_g|_{\langle s_1 s_2 \rangle}$. Thus $1(b)$ and $2(a)$ holds. Since $(s_2)\omega = s_2^s = s_1^{-3}s_2^{-2}$ and $(s_1^2 s_2)\omega \circ \eta =$

$(s_1^2 s_2)^s = s_1^{-1}$, we have $\omega|_{\langle s_2\rangle} = c_g|_{\langle s_2\rangle}$ and $\omega\circ\eta|_{\langle s_1^2 s_2\rangle} = c_g|_{\langle s_1^2 s_2\rangle}$ for all $g\in sA$. Thus $1(c)$ and $(2b)$ holds. Since $(s_1^3 s_2^2)\omega = (s_1^3 s_2^2)^{s^{-1}} = s_2^{-1}$ and $(s_1)\omega\circ\eta = s_1^{s^{-1}} = s_1^{-2}s_2^{-1}$, we also have $\omega|_{\langle s_1^3 s_2^2\rangle} = c_g|_{\langle s_1^3 s_2^2\rangle}$ and $\omega\circ\eta|_{\langle s_1\rangle} = c_g|_{\langle s_1\rangle}$ for all $g\in s^{-1}A$. Thus $1(d)$ and $2(c)$ holds.

Now suppose that $\phi|_Q = c_g|_Q$ for some $\phi\in\{\omega,\omega\circ\eta\}$, $g\in S$ and $Q\le S$. If $\phi=\omega$, we may assume $Q\ne Z(S)$.

Claim 1 $Q\le A$.

Proof of claim. Suppose that $Q\not\le A$. Let $q\in Q\backslash A$ and write $q = s^i a$ for some $i\in\{-1,1\}$ and $a\in A$. Also write $g = s^j b$ for some $j\in\{-1,0,1\}$ and $b\in A$. Then

$$\begin{aligned} &(qA)\phi = (s^iA)\phi = s^{-i}A \\ \text{and}\quad &(qA)^g = (s^iA)^g = (s^g)^iA = (s^b)^iA = (s(b^s)^{-1}b)^iA = s^iA \end{aligned}$$

which is a contradiction. It follows that $Q\le A$.

Claim 2 Q is cyclic of order at most 3^k containing the centre of S.

Proof of claim. It suffices to show that Q is cyclic. Suppose not. Then $|Q|\ge 3^2$ and Q contains an elementary abelian subgroup of A of order 3^2. By 5.1.7, $\Omega_1(A)$ is the unique elementary abelian subgroup of A of order 3^2. Therefore $\Omega_1(A)\le Q$. Note however $t\omega = t^{-1}z$ and $z\omega\circ\eta = z^{-1}$, whereas $t^g = t^{s^i} = tz^i$ and $z^g = z$ for some $i\in\{-1,0,1\}$. That is, $t\omega\ne t^g$ and $z\omega\circ\eta\ne z^g$, revealing a contradiction. We conclude that Q is cyclic.

Proof of Lemma. Since $Q\le A$ is abelian we see that $c_g|_Q$ can be identified as one of $\mathrm{Id}|_Q$, $c_s|_Q$ or $c_{s^{-1}}|_Q$. First suppose that $\phi=\omega$. Since $(s_1^3 s_2)\omega = (s_1^3 s_2)^{\mathrm{Id}}$, $(s_2)\omega = s_2^s$ and

$(s_1^3 s_2^2)\omega = (s_1^3 s_2^2)^{s^{-1}}$, we have

$$g = 1 \implies \omega|_{Q\langle s_1^3 s_2\rangle} = \mathrm{Id}|_{Q\langle s_1^3 s_2\rangle} \implies Q \leq \langle s_1^3 s_2\rangle$$
$$g = s \implies \omega|_{Q\langle s_2\rangle} = c_s|_{Q\langle s_2\rangle} \implies Q \leq \langle s_2\rangle$$
$$g = s^{-1} \implies \omega|_{Q\langle s_1^3 s_2^2\rangle} = c_{s^{-1}}|_{Q\langle s_1^3 s_2^2\rangle} \implies Q \leq \langle s_1^3 s_2^2\rangle$$

where the latter implications hold by the application of Claim 2 to $Q\langle s_1^3 s_2\rangle$, $Q\langle s_2\rangle$ and $Q\langle s_1^3 s_2^2\rangle$ respectively. Similarly, when $\phi = \omega \circ \eta$, we have $(s_1 s_2)\omega \circ \eta = (s_1 s_2)^{\mathrm{Id}}$, $(s_1^2 s_2)\omega \circ \eta = (s_1^2 s_2)^s$ and $(s_1)\omega \circ \eta = (s_1)^{s^{-1}}$ which reveal that $Q \leq \langle s_1 s_2\rangle$, $\langle s_1^2 s_2\rangle$ and $\langle s_1\rangle$ when $g = 1, s$ and s^{-1} respectively. $\square$

6.2 Automorphisms of an abelian subgroup A

We mentioned in Section 6.1 that we needed to study involutions of $\mathrm{Aut}(S)$. In Chapter 7, the fusion systems that we'll define will include those that will be generated by $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ automorphisms of A. In this section, we study automorphisms of A, the unique maximal abelian subgroup of S. We construct a number of morphisms and show that they satisfy the necessary properties, which will be needed to generate our fusion systems. Thus, we explicitly determine the generators of $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ automorphisms of A. Recall from Section 2.3 that $\mathrm{Aut}(A) \cong \mathrm{GL}_2(\mathbb{Z}/3^k\mathbb{Z})$, the group of invertible 2×2 matrices over the ring $\mathbb{Z}/3^k\mathbb{Z}$. The maps can be given by,

$$\begin{bmatrix} s_1 \mapsto s_1^{a_{11}} s_2^{a_{12}} \\ s_2 \mapsto s_1^{a_{21}} s_2^{a_{22}} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

For integers x, y, define $x \equiv_k y \Leftrightarrow x \equiv y \bmod 3^k$. Under the identification of automorphisms of A above, we define the following automorphisms of A:

$$\theta_A = \begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix}, \alpha_A = \begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix} \quad \text{and} \quad \beta_A = \begin{bmatrix} -a_k & -b_k \\ 2a_k-1 & a_k \end{bmatrix}$$

where $a_1 \equiv_1 0$, and for $k \geq 2$, we have

$$a_k \equiv_k -(a_{k-1}^2 - 3a_{k-1} + 3), \quad \text{and} \quad b_k \equiv_k \frac{1+a_k^2}{1+a_k}.$$

Since $3 \mid a_k$, b_k is well-defined. Defining $a'_{k-1} = a_{k-1} + n3^{k-1}$, for some fixed a_{k-1} and $k \geq 1$, we see that

$$\begin{aligned} -({a'_{k-1}}^2 - 3a'_{k-1} + 3) &= -((a_{k-1} + n3^{k-1})^2 - 3(a_{k-1} + n3^{k-1}) + 3) \\ &\equiv_k -(a_{k-1}^2 - 3a_{k-1} + 3). \\ &\equiv_k -a_k \end{aligned}$$

Thus, a_k modulo 3^k, is independent of our choice of our value of a_{k-1}. Note that

$$\begin{aligned} \begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} \begin{bmatrix} -a_k & -b_k \\ a_k+1 & a_k \end{bmatrix} &= \begin{bmatrix} b_k(a_k+1) - a_k^2 & a_kb_k - a_kb_k \\ a_k(a_k+1) - a_k(a_k+1) & b_k(a_k+1) - a_k^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where the latter equality holds as a consequence of the application of the definition of b_k. Thus, we have

$$\theta_A^{-1} = \begin{bmatrix} -a_k & -b_k \\ a_k+1 & a_k \end{bmatrix}.$$

The four automorphisms can be written as

$$\phi = \delta(\phi) \begin{bmatrix} a_k & b_k \\ -c_k(\phi) & -a_k \end{bmatrix}$$

where $\delta(\phi) \in \{-1,1\}$ and

$$c_k(\phi) = \begin{cases} a_k+1 & \text{if } \phi \in \{\theta_A, \theta_A^{-1}\} \\ 2a_k-1 & \text{if } \phi \in \{\alpha_A, \beta_A\}. \end{cases}$$

Let $g \in S$. Since $|S:A| = 3$, the conjugation map $c_g|_A$, restricted to A, can be identified as $c_{s^{m_g}}|_A$ for some $m_g \in \{-1,0,1\}$. Let $q = s_1^i s_2^j \in A$ for some $0 \leq i,j \leq 3^k - 1$. Then $q^g = s_1^{a_g i - 3m_g j} s_2^{m_g i + b_g j}$, where

$$a_g = \begin{cases} 1 & \text{if } m_g \neq -1 \\ -2 & \text{if } m_g = -1 \end{cases} \quad \text{and} \quad b_g \begin{cases} 1 & \text{if } m_g \neq 1 \\ -2 & \text{if } m_g = 1. \end{cases}$$

Under the identification,

$$c_g|_A = \begin{bmatrix} a_g & m_g \\ -3m_g & b_g \end{bmatrix}.$$

Lemma 6.2.1. *We have $a_k^2 \equiv_k 2a_k - 3$.*

Proof. We prove this by induction on k. When $k = 1$, we have $a_1^2 \equiv_1 2a_1 - 3 \equiv_1 0$. Suppose that $k > 1$ and that the result holds for the case $k-1$, that is, $a_{k-1}^2 \equiv_{k-1} 2a_{k-1} - 3$. Writing, $a_{k-1}^2 = 2a_{k-1} - 3 + u \cdot 3^{k-1}$ for some $u \in \mathbb{Z}$, we have

$$\begin{aligned}
& a_{k-1}^2 \equiv_{k-1} 2a_{k-1} - 3 \\
\Leftrightarrow \quad & 3a_{k-1}^2 \equiv_k 6a_{k-1} - 9 \\
\Leftrightarrow \quad & a_{k-1}^2 \equiv_k -2(a_{k-1}^2 - 3a_{k-1} + 3) - 3 \\
\Leftrightarrow \quad & (u \cdot 3^{k-1} - a_{k-1})^2 \equiv_k -2(a_{k-1}^2 - 3a_{k-1} + 3) - 3 \\
\Leftrightarrow \quad & [(2a_{k-1} - 3 + u \cdot 3^{k-1}) - 3a_{k-1} + 3]^2 \equiv_k 2a_k - 3 \\
\Leftrightarrow \quad & [a_{k-1}^2 - 3a_{k-1} + 3]^2 \equiv_k 2a_k - 3 \\
\Leftrightarrow \quad & a_k^2 \equiv_k 2a_k - 3
\end{aligned}$$

It follows that $a_k^2 \equiv_k 2a_k - 3$ for all $k \in \mathbb{Z}$. $\square$

The fusion systems that we'll define will include those that will be generated by $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ outer automorphisms of A. As mentioned in the introduction of this chapter, in order to compute characteristic bisets we will need to study the morphisms that generate our fusion systems. Thus, in Lemma 6.2.2, we explicitly write the generators of $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ outer automorphisms of A.

Lemma 6.2.2. *We have*

$$\langle Aut_{\mathcal{F}}(A) \rangle = \begin{cases} \langle \theta_A, c_s|_A \rangle & \text{if } Out_{\mathcal{F}}(A) \cong SL_2(3) \\ \langle \alpha_A, c_s|_A \rangle = \langle \theta_A, \alpha_A, \beta_A, c_s|_A \rangle & \text{if } Out_{\mathcal{F}}(A) \cong GL_2(3) \end{cases}$$

where $|\theta_A| = 4$ *and* $|\alpha_A| = |\beta_A| = 2$.

Proof. Since $|S : A| = 3$, $\mathrm{Aut}_S(A)$ is generated by the conjugation map $c_s|_A$. Define $H_1 = \langle \theta_A, \theta_A^{c_s} \rangle$, $H_2 = \langle \theta_A, c_s|_A \rangle$, and $H_3 = \langle \alpha_A, c_s|_A \rangle$. We show that $H_1 < H_2 < H_3$ and $H_1 \cong \mathbb{Q}_8$, $H_2 \cong \mathrm{SL}_2(3)$, $H_3 \cong \mathrm{GL}_2(3)$. In the proof we will make use of a number of easy-to-check identities, including

$$3b_k \equiv_k 2a_k.$$

Claim 1. $\theta_A^2 = -\mathrm{Id}$. Moreover, $|\theta_A| = 4$ and $|\alpha_A| = |\beta_A| = 2$.

Proof of claim. By an application of the definition of b_k, we have

$$\theta_A^2 = \begin{bmatrix} a_k^2 - b_k(1+a_k) & 0 \\ 0 & a_k^2 - b_k(1+a_k) \end{bmatrix} = -\mathrm{Id}_2$$
$$\alpha_A^2 = \begin{bmatrix} a_k^2 + b_k(1-2a_k) & 0 \\ 0 & a_k^2 + b_k(1-2a_k) \end{bmatrix} = \mathrm{Id}_2$$

It follows that $|\theta_A| = 4$ and $|\alpha_A| = 2$. Since $\beta_A = -\alpha_A$, we also have $|\beta_A| = 2$.

Claim 2 $H_1 \cong \mathbb{Q}_8$.

Proof of claim. Note that

$$\theta_A^{c_s} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_k-3b_k & a_k-2b_k \\ 2a_k-1 & a_k-1 \end{bmatrix}$$
$$= \begin{bmatrix} -4a_k+6b_k+1 & -3a_k+4b_k+1 \\ 5a_k-9b_k-1 & 4a_k-6b_k-1 \end{bmatrix} = \begin{bmatrix} 1 & -a_k+b_k+1 \\ -(a_k+1) & -1 \end{bmatrix}.$$

Thus,

$$\theta_A \circ \theta_A^{c_s} = \begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} \begin{bmatrix} 1 & b_k-a_k+1 \\ -(a_k+1) & -1 \end{bmatrix}$$
$$= \begin{bmatrix} a_k-b_k(a_k+1) & a_k(1-a_k+b_k)-b_k \\ -(a_k+1)+a_k(a_k+1) & -(a_k+1)(1-a_k+b_k)+a_k \end{bmatrix}$$
$$= \begin{bmatrix} 2-a_k & b_k-a_k+1 \\ 2a_k-4 & a_k-2 \end{bmatrix}$$
$$= \begin{bmatrix} -a_k+(a_k+1)(1-a_k+b_k) & -b_k+a_k(1-a_k+b_k) \\ a_k(a_k+1)-(a_k+1) & b_k(a_k+1)-a_k \end{bmatrix}$$
$$= \begin{bmatrix} 1 & b_k-a_k+1 \\ -(a_k+1) & -1 \end{bmatrix} \begin{bmatrix} -a_k & -b_k \\ a_k+1 & a_k \end{bmatrix} = \theta_A^{c_s} \circ \theta_A \circ (-\mathrm{Id}).$$

We have shown that $\theta_A \circ \theta_A^{c_s} = \theta_A^{c_s} \circ \theta_A \circ (-\mathrm{Id})$, that is, θ_A and $\theta_A^{c_s}$ commute modulo $\langle -\mathrm{Id} \rangle$. Since θ_A and $\theta_A^{c_s}$ are distinct, and by the last claim, each of order 4, we have, $H_1 \cong \mathbb{Q}_8$.

<u>Claim 3</u> c_s normalizes H_1, whereas α_A normalizes H_2.

Proof of claim. To show that c_s normalizes H_1, it suffices to show that $\theta_A^{c_s^{-1}} \in H_1$. Using the calculation in the last claim, we have,

$$\theta_A^{c_s^{-1}} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & b_k - a_k \\ 2-a_k & 0 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2-a_k & b_k - a_k + 1 \\ 2a_k - 4 & a_k - 2 \end{bmatrix} = \theta_A \circ \theta_A^{c_s} \in H_1.$$

Now to show that α_A normalizes H_2, it suffices to show that $\theta_A^{\alpha_A} \in H_2$ and $c_s^{\alpha_A} \in H_2$. We have

$$\theta_A^{\alpha_A} = \begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix} \begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} \begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix}$$
$$= \begin{bmatrix} a_k^2 - b_k(a_k+1) & 0 \\ 3 & a_k^2 + b_k(1-2a_k) \end{bmatrix} \begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix} = \begin{bmatrix} -a_k & -b_k \\ a_k+1 & a_k \end{bmatrix} = \theta_A^{-1} \in H_2.$$

Again using the calculation in the last claim, we also have

$$c_s^{\alpha_A} c_s = \left(\begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \right)^2 = \left(\begin{bmatrix} -a_k & a_k - 2b_k \\ a_k+1 & 1 \end{bmatrix} \right)^2$$
$$= \begin{bmatrix} a_k^2 + (a_k+1)(a_k-2b_k) & (1-a_k)(a_k-2b_k) \\ (1-a_k)(a_k+1) & (a_k+1)(a_k-2b_k)+1 \end{bmatrix} = \begin{bmatrix} a_k - 2 & (1-a_k)(a_k-2b_k) \\ 4-2a_k & 2-a_k \end{bmatrix}$$
$$= \begin{bmatrix} -a_k + (a_k+1)b_k & -a_k(-a_k+b_k+1)+b_k \\ 4-2a_k & (a_k+1)(-a_k+b_k+1)-a_k \end{bmatrix}$$
$$= \begin{bmatrix} -a_k & -b_k \\ a_k+1 & a_k \end{bmatrix} \begin{bmatrix} 1 & -a_k+b_k+1 \\ -a_k-1 & -1 \end{bmatrix} = \theta_A^{-1} \circ \theta_A^{c_s} \in H_2.$$

It follows that $c_s^{\alpha_A} = \theta_A^{-1} \circ \theta_A^{c_s} \circ c_{s^{-1}} \in H_2$. Thus α_A normalizes H_2.

<u>Claim 4</u> $\theta_A \in H_3$.

Proof of claim. Using the calculation in the last claim, note that

$$
\begin{aligned}
c_{s^{-1}} \circ (c_s^{\alpha_A}) \circ c_{s^{-1}} &= c_{s^{-1}} \circ [(c_s^{\alpha_A}) \circ c_s] \circ c_s \\
&= \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_k - 2 & (1-a_k)(a_k - 2b_k) \\ 4 - 2a_k & 2 - a_k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (a_k - 2) + b_k(1 - a_k) \\ a_k - 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -a_k & -b_k \\ a_k + 1 & a_k \end{bmatrix} = \theta_A^{-1}.
\end{aligned}
$$

It follows that θ_A^{-1}, and therefore θ_A, is in H_3.

Proof of Lemma. By claim 2, $H_1 \cong \mathbb{Q}_8$. By claim 3, c_s normalizes H_1. It follows that, $H_2 = \langle \theta_A, c_s \rangle \cong H_1 \rtimes \langle c_s \rangle \cong \mathsf{SL}_2(3)$. By claim 3 and 4, the involution α_A normalizes, but does not centralize H_2. Thus, $H_3 = \langle \alpha_A, c_s \rangle \cong H_2 \rtimes \langle \alpha_A \rangle \cong \mathsf{GL}_2(3)$. By claim 4, $H_3 = \langle \theta_A, \theta_A^{-1}, \alpha_A, \beta_A, c_s \rangle$ and we are done. $\square$

6.3 Other automorphisms

In the previous sections we studied selected automorphisms of S, and of its abelian subgroup, A. Here, we describe automorphisms of extraspecial and elementary abelian subgroups of S, which, as we'll see in Chapter 7, will be used to generate fusion systems. As we shall see in this section, the automorphisms that we introduce are related to those that we discussed in Section 6.2. Hence, we move some of the discussion for automorphisms of A to this section. Recall that we defined $t = s_1^{3^{k-1}}$ and $z = s_2^{3^{k-1}}$.

For $\lambda = -1, 0, 1$, define $E_\lambda = \langle s s_1^\lambda, t \rangle$, the extraspecial subgroup of S of exponent 3 and order 27. Note that

$$
\begin{aligned}
(s s_1^\lambda)\omega = (s\omega)(s_1\omega)^\lambda &= s^{-1} s_1^{2\lambda} s_2^\lambda = (s_1^s)^{2\lambda} (s_2^s)^\lambda s^{-1} \\
&= (s_1 s_2)^{2\lambda} (s_1^{-3} s_2^{-2})^\lambda s^{-1} = s s_1^\lambda.
\end{aligned}
$$

Thus, ω normalizes E_λ, and since η inverts every element of A, η fixes E_0 but swaps E_1 with E_{-1}. Let $V_\lambda = \langle ss_1^\lambda, z\rangle$. Then V_λ is an elementary abelian subgroup of order 9 in E_λ. Similarly, ω fixes V_λ, and η fixes V_0, but swaps V_1 with V_{-1}. Define

$$\mathcal{T}_{full} = \{V_\lambda, E_\lambda, A \mid \lambda \in \{-1,0,1\}\},$$

a set of seven subgroups of S. The subscript *full* is used to denote the fact that it is the collection of all possible subgroups of S which will be used in the construction of our fusion systems. Let $T \in \mathcal{T}_{full}$. Let $x_T, y_T \in T = \langle x_T, y_T\rangle$ be defined as follows:

T	x_T	y_T
V_λ	ss_1^λ	z
E_λ	ss_1^λ	t
A	s_1	$s_1^{a_k} s_2^{b_k}$

where a_k and b_k are defined as in Section 6.2. We define θ_T, θ_T^{-1}, α_T, $\beta_T \in \mathrm{Aut}(T)$ for all $T \in \mathcal{T}_{full}$, to be automorphisms of T as:

$$\begin{aligned}
\theta_T :&\ x_T \mapsto y_T; \quad y_T \mapsto x_T^{-1}\\
\theta_T^{-1} :&\ x_T \mapsto y_T^{-1}; \quad y_T \mapsto x_T\\
\alpha_T :&\ x_T \mapsto y_T; \quad y_T \mapsto x_T\\
\beta_T :&\ x_T \mapsto y_T^{-1}; \quad y_T \mapsto x_T^{-1}.
\end{aligned}$$

Note that θ_T^{-1} is indeed the inverse of θ_T. We check that θ_A, α_A and β_A as defined in Section 6.2 satisfy the definitions here. Indeed, under the identification $\mathrm{Aut}(A) \to \mathrm{GL}_2(\mathbb{Z}/3^k\mathbb{Z})$, we

have

$$s_1\theta_A = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} = \begin{bmatrix} a_k & b_k \end{bmatrix} = s_1^{a_k}s_2^{b_k}$$

$$s_1\alpha_A = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix} = \begin{bmatrix} a_k & b_k \end{bmatrix} = s_1^{a_k}s_2^{b_k}$$

$$s_1\beta_A = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} -a_k & -b_k \\ 2a_k-1 & a_k \end{bmatrix} = \begin{bmatrix} -a_k & -b_k \end{bmatrix} = (s_1^{a_k}s_2^{b_k})^{-1}.$$

and

$$s_1^{a_k}s_2^{b_k}\theta_A = \begin{bmatrix}a_k & b_k\end{bmatrix}\begin{bmatrix} a_k & b_k \\ -(a_k+1) & -a_k \end{bmatrix} = \begin{bmatrix} a_k^2 - b_k(a_k+1) & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix} = s_1^{-1}$$

$$s_1^{a_k}s_2^{b_k}\alpha_A = \begin{bmatrix}a_k & b_k\end{bmatrix}\begin{bmatrix} a_k & b_k \\ 1-2a_k & -a_k \end{bmatrix} = \begin{bmatrix} a_k^2 + b_k(1-2a_k) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} = s_1$$

$$s_1^{a_k}s_2^{b_k}\beta_A = \begin{bmatrix}a_k & b_k\end{bmatrix}\begin{bmatrix} -a_k & -b_k \\ 2a_k-1 & a_k \end{bmatrix} = \begin{bmatrix} -a_k^2 + b_k(2a_k-1) & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix} = s_1^{-1}$$

Thus the two definitions of θ_A, α_A and β_A agree.

We also define $\hat{\theta}_T, \hat{\theta}_T^{-1}, \hat{\alpha}_T, \hat{\beta}_T \in \mathrm{Aut}(T)$ for all $T \in \mathcal{T}_{full}$, to be automorphisms of T as follows:

$$\hat{\theta}_T: \ x_T \mapsto x_T; \quad y_T \mapsto y_T^{-1}$$

$$\hat{\theta}_T^{-1}: \ x_T \mapsto x_T^{-1}; \quad y_T \mapsto y_T$$

$$\hat{\alpha}_T: \ x_T \mapsto x_T; \quad y_T \mapsto y_T$$

$$\hat{\beta}_T: \ x_T \mapsto x_T^{-1}; \quad y_T \mapsto y_T^{-1}.$$

Evidently, $\hat{\alpha}_T = \mathrm{Id}|_T$, which we add for completeness.

We make further definitions. We define a number of morphisms between distinct subgroups of V_λ and E_λ for $\lambda = -1, 0, 1$. The reason for this is that in certain fusion systems, as we shall see in Chapter 7, not every morphism is a composition of restriction of maps in $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T)\rangle$ for some $T \in \mathcal{T}$. It may be the case that there exists $T_1 \in \mathcal{T}$ with $T_1 \neq T$, such that a morphism can only be written as a restriction of a composition of restrictions of automorphisms of T and T_1, other than that of S.

Define $\Theta_{\mu,\lambda} : \langle ss_1^\lambda, z\rangle \to \langle t, z\rangle$, for $\mu = 1, .., 4$ and $\lambda = -1, 0, 1$ to be a set of maps, as follows:

$$\Theta_{1,\lambda} = \theta_{V_\lambda} \circ \theta_\lambda : \ ss_1^\lambda \mapsto z; \quad z \mapsto t^{-1}$$

$$\Theta_{2,\lambda} = \theta_{V_\lambda}^{-1} \circ \theta_\lambda : \ ss_1^\lambda \mapsto z^{-1}; \quad z \mapsto t$$

$$\Theta_{3,\lambda} = \alpha_{V_\lambda} \circ \theta_\lambda : \ ss_1^\lambda \mapsto z; \quad z \mapsto t$$

$$\Theta_{4,\lambda} = \beta_{V_\lambda} \circ \theta_\lambda : \ ss_1^\lambda \mapsto z^{-1}; \quad z \mapsto t^{-1}.$$

Define $\vartheta_{\mu,\lambda} : \langle ss_1^\lambda, z\rangle \to \langle ss_1^{-\lambda}, z\rangle$, for $\mu = 1, .., 4$ and $\lambda = -1, 1$, to be a set of maps, as follows:

$$\vartheta_{1,\lambda} = \theta_{V_\lambda} \circ \eta : \ ss_1^\lambda \mapsto z^{-1}; \quad z \mapsto (ss_1^{-\lambda})^{-1}$$

$$\vartheta_{2,\lambda} = \theta_{V_\lambda}^{-1} \circ \eta : \ ss_1^\lambda \mapsto z; \quad z \mapsto ss_1^{-\lambda}$$

$$\vartheta_{3,\lambda} = \alpha_{V_\lambda} \circ \eta : \ ss_1^\lambda \mapsto z^{-1}; \quad z \mapsto ss_1^{-\lambda}$$

$$\vartheta_{4,\lambda} = \beta_{V_\lambda} \circ \eta : \ ss_1^\lambda \mapsto z; \quad z \mapsto (ss_1^{-\lambda})^{-1}.$$

Define $\hat{\vartheta}_{\mu,\lambda} : \langle ss_1^{\lambda}, z\rangle \to \langle ss_1^{-\lambda}, z\rangle$, for $\mu = 1,..,4$ and $\lambda = -1, 1$, to be a set of maps, as follows:

$$\hat{\vartheta}_{1,\lambda} = \hat{\theta}_{V_\lambda} \circ \eta : \ ss_1^{\lambda} \mapsto ss_1^{-\lambda}; \quad z \mapsto z$$

$$\hat{\vartheta}_{2,\lambda} = \hat{\theta}_{V_\lambda}^{-1} \circ \eta : \ ss_1^{\lambda} \mapsto (ss_1^{-\lambda})^{-1}; \quad z \mapsto z^{-1}$$

$$\hat{\vartheta}_{3,\lambda} = \hat{\alpha}_{V_\lambda} \circ \eta : \ ss_1^{\lambda} \mapsto ss_1^{-\lambda}; \quad z \mapsto z^{-1}$$

$$\hat{\vartheta}_{4,\lambda} = \hat{\beta}_{V_\lambda} \circ \eta : \ ss_1^{\lambda} \mapsto (ss_1^{-\lambda})^{-1}; \quad z \mapsto z.$$

Define $\xi_{\mu,\lambda} : \langle ss_1^{\lambda}, z\rangle \to \langle s, z\rangle$, for $\mu = 1,..,4$ and $\lambda = -1, 1$ to be the set of maps, as follows:

$$\xi_{1,\lambda} = \hat{\theta}_{V_\lambda} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto s^{-1}; \quad z \mapsto z^{-1}$$

$$\xi_{2,\lambda} = \hat{\theta}_{V_\lambda}^{-1} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto s; \quad z \mapsto z$$

$$\xi_{3,\lambda} = \hat{\alpha}_{V_\lambda} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto s^{-1}; \quad z \mapsto z$$

$$\xi_{4,\lambda} = \hat{\beta}_{V_\lambda} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto s; \quad z \mapsto z^{-1}.$$

Define $\xi_{\mu,\lambda} : \langle ss_1^{\lambda}, z\rangle \to \langle s, z\rangle$, for $\mu = 5,..,8$ and $\lambda = -1, 1$ to be the set of maps, as follows:

$$\xi_{5,\lambda} = \theta_{V_\lambda} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto z; \quad z \mapsto s$$

$$\xi_{6,\lambda} = \theta_{V_\lambda}^{-1} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto z^{-1}; \quad z \mapsto s^{-1}$$

$$\xi_{7,\lambda} = \alpha_{V_\lambda} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto z; \quad z \mapsto s^{-1}$$

$$\xi_{8,\lambda} = \beta_{V_\lambda} \circ \theta_\lambda \circ \theta_0 : \ ss_1^{\lambda} \mapsto z^{-1}; \quad z \mapsto s.$$

CHAPTER 7

DRV-FUSION SYSTEMS

In Chapter 6 we studied properties of certain automorphisms of S, and automorphisms of specified subgroups of S. The reason for this is that we are going to eventually compute right characteristic bisets for a number of saturated fusion systems on S, generated by these automorphisms. In this chapter, we list these fusion systems, as given in [5, Theorem 5.9, Table 6], and, for convenience, we call them DRV-fusion systems. If Ω is a right (or left) characteristic biset for any saturated fusion system $\mathcal{F}$ over S, then recall, by definition, Ω is a disjoint union of orbits of the form $\mathcal{O}_\phi$ for some morphism $\phi \in \mathcal{F}$. However, by Lemma 3.1.8 and Lemma 4.2.7, it suffices to assume that every orbit $\mathcal{O}_\phi$ in Ω is induced by an S-S-equivalence class representative $\phi \in \mathcal{F}$. Thus, after this motivation, the aim of this chapter is to determine morphisms up to S-S equivalence, for each DRV-fusion system.

Recall that $\mathcal{T}_{full} = \{V_\lambda, E_\lambda, A \mid \lambda \in \{-1,0,1\}\}$. For any DRV-fusion system $\mathcal{F}$, there is a corresponding $\mathcal{T} \subseteq \mathcal{T}_{full}$, the set of $\mathcal{F}$-essential proper subgroups of S, which, along with $\mathrm{Aut}_{\mathcal{F}}(S)$, generate the fusion system, that is,

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T) \mid T \in \mathcal{T} \rangle.$$

Here is the theorem which lists our DRV-fusion systems:

Theorem 7.0.1. *[5, Table 6, Theorem 5.9] Let $\mathcal{F}$ be a saturated fusion system on S, with at least one $\mathcal{F}$-essential proper subgroup. Then the outer automorphisms of S and the $\mathcal{F}$-essential subgroups are as those listed in the table below:*

Name	S	V_0	E_0	E_1	E_{-1}	A	$\mathcal{T}$
DRV-1	$\langle\bar{\omega}\rangle$		$SL_2(3)$				$\{E_0\}$
DRV-2	$\langle\bar{\omega}\rangle$			$SL_2(3)$	$SL_2(3)$		$\{E_{\pm 1}\}$
DRV-3	$\langle\bar{\omega}\rangle$		$SL_2(3)$	$SL_2(3)$	$SL_2(3)$		$\{E_0, E_{\pm 1}\}$
DRV-4	$\langle\bar{\eta}\rangle$					$SL_2(3)$	$\{A\}$
DRV-5	$\langle\bar{\eta}\circ\bar{\omega}\rangle$	$SL_2(3)$					$\{V_0\}$
DRV-6	$\langle\bar{\eta},\bar{\omega}\rangle$					$GL_2(3)$	$\{A\}$
DRV-7	$\langle\bar{\eta},\bar{\omega}\rangle$			$SL_2(3)$	$SL_2(3)$		$\{E_{\pm 1}\}$
DRV-8	$\langle\bar{\eta},\bar{\omega}\rangle$			$SL_2(3)$	$SL_2(3)$	$GL_2(3)$	$\{E_{\pm 1}, A\}$
DRV-9	$\langle\bar{\eta},\bar{\omega}\rangle$		$GL_2(3)$				$\{E_0\}$
DRV-10	$\langle\bar{\eta},\bar{\omega}\rangle$		$GL_2(3)$			$GL_2(3)$	$\{E_0, A\}$
DRV-11	$\langle\bar{\eta},\bar{\omega}\rangle$		$GL_2(3)$	$SL_2(3)$	$SL_2(3)$		$\{E_0, E_{\pm 1}\}$
DRV-12	$\langle\bar{\eta},\bar{\omega}\rangle$		$GL_2(3)$	$SL_2(3)$	$SL_2(3)$	$GL_2(3)$	$\{E_0, E_{\pm 1}, A\}$
DRV-13	$\langle\bar{\eta},\bar{\omega}\rangle$	$GL_2(3)$					$\{V_0\}$
DRV-14	$\langle\bar{\eta},\bar{\omega}\rangle$	$GL_2(3)$				$GL_2(3)$	$\{V_0, A\}$
DRV-15	$\langle\bar{\eta},\bar{\omega}\rangle$	$GL_2(3)$		$SL_2(3)$	$SL_2(3)$		$\{V_0, E_{\pm 1}\}$
DRV-16	$\langle\bar{\eta},\bar{\omega}\rangle$	$GL_2(3)$		$SL_2(3)$	$SL_2(3)$	$GL_2(3)$	$\{V_0, E_{\pm 1}, A\}$

Table 7.1: List of saturated fusion systems on S

We make a few remarks. From the table, we can see that $V_{\pm 1} \notin \mathcal{T}$ for any saturated fusion system $\mathcal{F}$. The reason for accommodating $V_{\pm 1}$ will become apparent later on in the chapter. We also note that, if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ for any saturated fusion system $\mathcal{F}$, then $E_1 \in \mathcal{T} \Leftrightarrow E_{-1} \in \mathcal{T}$. As we will mention later, η swaps E_1 with E_{-1}. Thus, in theory, it suffices to have only one of the two $\mathcal{F}$-essential subgroups in the construction of such fusion systems.

It is easy to determine, up to S-S-equivalence, morphisms that are restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(S)$ in any DRV-fusion system $\mathcal{F}$. We prove this first, and before we move on to Section 7.1, we show that for any morphism that is not a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(S)$, its domain (and similarly its image), up to S-conjugacy, is a subgroup of T for some $T \in \mathcal{T}$. This is slightly helpful in the sense that just by looking at the size of the domain of any morphism we can consider whether the morphism is generated by a proper $\mathcal{F}$-essential subgroups of S. In Section 7.1, we determine, up to S-S-equivalence, morphisms that are generated by automorphisms of a single proper subgroup of S, other than S itself, where the bulk of our work for this chapter lies.

Let $\mathcal{F}$ be a DRV-fusion system. Let $P,Q \leq S$. By construction, for every $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ we can write

$$\phi = (\phi_1|_{P_0}) \circ \ldots \circ (\phi_n|_{P_{n-1}}) \tag{I}$$

with $P_0 = P$, $P_n = Q$, and $(P_{i-1})\phi_i \leq P_i$ and either $\phi_i \in \bigsqcup_{T\in\mathcal{T}} \mathrm{Aut}_{\mathcal{F}}(T)$, in which case $T \geq \langle P_{i-1}, P_i\rangle$, or $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(S)$. We may assume that $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(S)$ whenever i is odd, and $\phi_i \in \bigsqcup_{T\in\mathcal{T}} \mathrm{Aut}_{\mathcal{F}}(T)$ otherwise.

Lemma 7.0.2. *Let $\mathcal{F}$ be a DRV-fusion system. Let $\phi : P \to S$ be a map induced by restriction of maps in $Aut_{\mathcal{F}}(S)$. Let $Q \leq S$ be any S-conjugate to P. Then up to S-S equivalence*

$$\phi = \begin{cases} Id|_Q, \omega|_Q & \text{if } \mathcal{F} = \text{DRV-1,2,3} \\ Id|_Q, \eta|_Q & \text{if } \mathcal{F} = \text{DRV-4} \\ Id|_Q, (\omega\circ\eta)|_Q & \text{if } \mathcal{F} = \text{DRV-5} \\ Id|_Q, \omega|_Q, \eta|_Q \text{ or } (\omega\circ\eta)|_Q & \text{otherwise} \end{cases}$$

Proof. Since ω and η commute, and depending on the DRV-fusion system $\mathcal{F}$, we may write $\phi = c_g|_P \circ \omega^i \circ \eta^j$ for some $i, j \in \{0,1\}$ such that

$$\mathcal{F} = \begin{cases} \text{DRV-1,2,3} \implies j = 0 \\ \text{DRV-4} \implies i = 0 \\ \text{DRV-5} \implies i = j. \end{cases}$$

Write $\psi = \omega^i \circ \eta^j$ and let $h \in S$ such that $P^{gh} = Q$. Then $c_{gh}|_P \circ \psi = c_g|_P \circ \psi \circ c_{(h\psi)} = \phi \circ c_{(h\psi)}$.

This means $\psi|_Q = c_{h^{-1}g^{-1}}|_Q \circ \phi \circ c_{(h\psi)}$, and the result follows. $\square$

Lemma 7.0.3. *Let $\psi \in \{Id_S, \omega, \eta, \omega \circ \eta\}$ and $T \in \mathfrak{T}_{full}$. Then, either ψ fixes T, or, if $\psi \in \{\eta, \omega \circ \eta\}$, then ψ permutes E_1 with E_{-1}, and V_1 with V_{-1}.*

Proof. By Lemma 5.1.1, A is characteristic in S. So, we may suppose $T \in \mathfrak{T}_{full} \backslash \{A\}$. Upon calculating, we see that $t\omega = t^{-1}z$ and $(ss_1^{\lambda})\omega = (ss_1^{\lambda})^{-1}$ for any $\lambda \in \{-1,0,1\}$. Thus, $T\omega = T$. We also know that η inverts every element of A, and centralizes s. Thus, whenever $\psi \in \{\eta, \eta \circ \omega\}$, then ψ fixes E_0 and V_0, but permutes E_1 with E_{-1}, and V_1 with V_{-1}. $\square$

The following, Lemma 7.0.4 is a standard fact.

Lemma 7.0.4. *Let $\mathcal{F}$ be a DRV-fusion system. If $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$, is not a restriction of some element in $Aut_{\mathcal{F}}(S)$, then P is a subgroup of some S-conjugate of T, for some $T \in \mathfrak{T}$.*

Proof. By (I) we may write $\phi = (\phi_1|_{P_0}) \circ \ldots \circ (\phi_n|_{P_{n-1}})$ with $n \geq 2$, $P = P_0$, $Q = P_{n-1}\phi_n$, $\phi_1 \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\phi_2 \in \mathrm{Aut}_{\mathcal{F}}(T)$, for some $T \in \mathfrak{T}$. Then $\phi_1 = \psi \circ c_g$ for some $g \in S$ and $\psi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$. Since $P\phi_1 = P_0\phi_1 \leq P_1 \leq \langle P_1, P_2 \rangle \leq T$, we have,

$$
\begin{aligned}
P\phi_1 = P(\psi \circ c_g) \leq T &\Leftrightarrow P(\psi \circ c_g \circ \psi^{-1}) \leq T\psi^{-1} \\
&\Leftrightarrow P^{(g\psi^{-1})} \leq T\psi^{-1}
\end{aligned}
$$

Note that $V_1, V_{-1} \notin \mathfrak{T}$ by Table 7.1. By Lemma 7.0.3, either ψ preserves elements of $\mathfrak{T}$, or, if additionally $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $E_{\pm 1} \in \mathfrak{T}$, ψ may permute E_1 with E_{-1}. Setting $h = g\psi^{-1}$, there exists $T_1 \in \mathfrak{T}$ such that,

$$
T_1 = \begin{cases} E_{\pm 1} & \text{if } \eta \in \mathrm{Aut}_{\mathcal{F}}(S), T = E_{\pm 1} \\ T & \text{otherwise} \end{cases}
$$

with the property $P^h \leq T_1$. This completes the proof of the lemma. $\square$

Corollary 7.0.5. *Let $\mathcal{F}$ be a DRV-fusion system. If $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$, is not a restriction of some element in $Aut_{\mathcal{F}}(S)$, then $P\phi$ is a subgroup of some S-conjugate of some element of T, for some $T \in \mathfrak{T}$.*

Proof. We may assume that ϕ is an isomorphism. Define $\varphi = \phi^{-1}$. Then by Lemma 7.0.4 $D_\varphi = P\phi$ is a subgroup of some S-conjugate of T, for some $T \in \mathfrak{T}$. □

Corollary 7.0.6. *Let $\mathcal{F}$ be a DRV-fusion system. If $\phi \in Hom_{\mathcal{F}}(P,Q)$ where $|P| > max_{T\in\mathfrak{T}}|T|$, then ϕ is a restriction of some element in $Aut_{\mathcal{F}}(S)$.*

Proof. If ϕ was not a restriction of some element in $\mathrm{Aut}_{\mathcal{F}}(S)$, then by 7.0.4, P would be a subgroup of some S-conjugate to $T \in \mathfrak{T}$, or if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $E_{\pm 1} \in \mathfrak{T}$, P would be a subgroup of some S-conjugate of $E_{\pm 1}$. Thus, in either case, this contradicts our assumption on the order of P. □

7.1 $\mathcal{F}$-morphisms generated by automorphisms of S and a single proper subgroup

Let $\mathcal{F}$ be a DRV-fusion system. The aim of this section is to explicitly list the morphisms, up to S-S-equivalence, that are generated by automorphisms of S and exactly one proper subgroup T, for some $T \in \mathfrak{T}$, that is, the fusion subsystem,

$$\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T) \rangle.$$

We do this in two general steps. First we show that for any morphism in the fusion subsystem, that is not a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(S)$, can be chosen, up to S-S-equivalence, to be restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(T)$ (with some modifications in the case where $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$). Second, we explicitly list those morphisms that are restrictions of elements $\mathrm{Aut}_{\mathcal{F}}(T)$, as we shall see in Lemma 7.1.8 and thereafter.

Let $\phi \in \mathcal{F}$. In Lemma 7.0.4 we showed that, up to S-conjugacy, D_ϕ, the domain of ϕ, (and similarly, its image I_ϕ) is a subgroup of T for some $T \in \mathfrak{T}$, provided that ϕ is not a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(S)$. We start this section by making a stronger claim. We show that, by decomposing the morphism by its generators, we can restrict this T to be the first one in the series of components in the decomposition of ϕ. The situation is similar when we look at I_ϕ, the image of ϕ.

Let $\mathcal{F}$ be a DRV-fusion system. If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, then recall that we can write

$$\phi = (\phi_1|_{P_0}) \circ .. \circ (\phi_n|_{P_{n-1}}) \tag{I}$$

with the property that $P_0 = P, P_n = Q$, and $(P_{i-1})\phi_i \leq P_i$ and, if i is even, $\phi_i \in \bigsqcup_{T\in\mathcal{T}} \mathrm{Aut}_{\mathcal{F}}(T)$, in which case $T \geq \langle P_{i-1}, P_i\rangle$ or, if i is odd, $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(S)$.

Lemma 7.1.1. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$, is not a restriction of $Aut_{\mathcal{F}}(S)$. We know that in the decomposition of ϕ in the form given in (I), one of the following holds:*

1. *The integer $n \geq 2$ is even and $\phi_n \in Aut_{\mathcal{F}}(T)$, or*
2. *The integer $n \geq 3$ is odd and $\phi_{n-1} \in Aut_{\mathcal{F}}(T)$,*

for some $T \in \mathcal{T}$. Then one of the following holds:

1. *$P\phi$ is a subgroup of an S-conjugate of T, or*
2. *If $\eta \in Aut_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$, then $P\phi$ is a subgroup of an S-conjugate of $E_{\pm 1}$.*

Proof. If n is even and $\phi_n \in \mathrm{Aut}_{\mathcal{F}}(T)$, then $P\phi \leq P_{n-1}\phi_n \leq T$, and we are done. It is left to consider the case when n is odd. Then $\phi_{n-1} \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\phi_n \in \mathrm{Aut}_{\mathcal{F}}(S)$. Write $R = P_{n-2}\phi_{n-1}$ and $\varphi = \phi_n|_R$. We can write $\varphi = \psi|_R \circ c_g$ for some $g \in S$ and $\psi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$. Note that $R \leq T$. Note also that $T \neq V_1, V_{-1}$ by Table 7.1. Thus, by Lemma 7.0.3, either $R\varphi = R^g \leq T^g$, or, if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$, then $R\varphi \leq E_{\pm 1}^g$. That is, either $R\varphi$ is a subgroup of an S-conjugate of T, or, if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$, $R\varphi$ is a subgroup of some S-conjugate of $E_{\pm 1}$. Since $P\phi \leq P_{n-2}\phi_{n-1}\phi_n = R\varphi$, the same consequence holds for $P\phi$. □

Corollary 7.1.2. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$, is not a restriction of $Aut_{\mathcal{F}}(S)$. We know that in the decomposition of ϕ in the form given in (I), $n \geq 2$, and $\phi_2 \in Aut_{\mathcal{F}}(T)$ for some $T \in \mathcal{T}$. Then*

1. *P is a subgroup of some S-conjugate of T, or*

2. *If $\eta \in Aut_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$, then P is a subgroup of some S-conjugate of $E_{\pm 1}$.*

Proof. We may assume that ϕ, along with its components, ϕ_i for $1 \leq i \leq n$ are isomorphisms. We may assume n is odd, for otherwise, we may define $\phi_{n+1} = \mathrm{Id}|_{P_n}$. Define $\varphi = \phi^{-1}$, $\varphi_{n+1-i} = \phi_i^{-1}$ for each $1 \leq i \leq n$ and $Q_{n-i} = P_{i-1}\phi_i$. Write $Q = P\phi$. Then $\varphi = (\varphi_1|_{Q_0}) \circ .. \circ (\varphi_n|_{Q_{n-1}})$. The second assumption holds in Lemma 7.1.1 with respect to φ. It follows that the conclusion of that lemma holds for $Q\varphi = P$. $\square$

Corollary 7.1.3. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$, is not a restriction of $Aut_{\mathcal{F}}(S)$. Let $T \in \mathcal{T}$ and suppose that $\phi \in \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T)\rangle$. Then*

1. *P and $P\phi$ are subgroups of some S-conjugates of T, or*

2. *If $\eta \in Aut_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$, then P and $P\phi$ are subgroups of some S-conjugates of $E_{\pm 1}$.*

Proof. This follows from Lemma 7.1.1 and Lemma 7.1.2. $\square$

Note that it is possible to prove Corollary 7.1.3 directly from Lemma 7.0.4.

From Table 7.1, for any DRV-fusion system $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(E_\lambda)$ for $\lambda = -1, 0, 1$, is either $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ whenever $E_\lambda \in \mathcal{T}$. It is possible that there may be certain morphisms between subgroups of E_λ that are not restrictions of automorphisms of E_λ. Lemma 7.1.4 attempts to address this for certain DRV-fusion systems:

Lemma 7.1.4. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $E_{\pm 1} \in \mathcal{T}$ and $\eta \in Aut_{\mathcal{F}}(S)$. Then*

$$\{\varphi|_P \mid \varphi \in Aut(E_\lambda), P \lneq E_\lambda, \lambda = -1, 1\} \subseteq \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(E_{\pm 1})\rangle.$$

Proof. Let $\lambda = -1, 1$ and define $u = s_2^{3^{k-2}}$. We begin by mentioning that for any integer i, $(ss_1^\lambda)^{u^i} = (ss_1^\lambda)t^i z^i$ and $(ss_1^\lambda)^{t^i} = ss_1^\lambda z^{-i}$. Thus, under the identification $\mathrm{Out}_{\mathcal{F}}(E_\lambda) \to \mathrm{GL}_2(3)$,

we have

$$\overline{c_u|_{E_\lambda}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and, by definition} \quad \overline{\alpha_\lambda} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\mathrm{GL}_2(3) \cong \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$, we have $\mathrm{Out}(E_\lambda) = \langle \overline{c_u|_{E_\lambda}}, \overline{\alpha_\lambda} \rangle$. Thus, for the proof of the lemma, it suffices to show that $\alpha_\lambda|_P \in \mathcal{F}$ for every $P \lneq E_\lambda$ (note that $\alpha_\lambda \notin \mathrm{Aut}_{\mathcal{F}}(E_\lambda)$ since $\mathrm{Out}_{\mathcal{F}}(E_\lambda) \cong \mathrm{SL}_2(3)$ by Table 7.1 and the fact that $\det(\alpha_\lambda) = -1$). We may assume that $P \in \{\langle ss_1^\lambda t^i, z \rangle, \langle t, z \rangle \mid i = -1, 0, 1\}$, the set of elementary abelian subgroups of E_λ of order 3^2 (since every subgroup of order 3 is inside one of these). For $i = -1, 0, 1$, define

$$\varphi_i = c_{u^{-i}t^{-i}}|_{\langle ss_1^\lambda t^i, z \rangle} \circ \theta_\lambda^{-1} \circ \eta \circ \theta_\lambda^{-1} \circ c_{u^{-i}t^{-i}} \circ \theta_\lambda.$$

Then $\alpha_\lambda|_{\langle ss_1^\lambda t^i, z \rangle} = \varphi_i$ and $\alpha_\lambda|_{\langle t, z \rangle} = (\varphi_0)^{-1}$. This completes the proof. □

Lemma 7.1.5. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $\phi \in Hom_{\mathcal{F}}(P, Q)$, for some $P, Q \leq S$, is not a restriction of $Aut_{\mathcal{F}}(S)$. Suppose that $\phi \in \mathcal{F}_1 = \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T) \rangle$ for some $T \in \mathcal{T}$. Then $\phi \sim \psi|_R$ for*

1. *$\psi \in Aut_{\mathcal{F}}(T)$ and $R \leq T$ for some $T \in \mathcal{T}$, or*

2. *If $T = E_{\pm 1}$ and $\eta \in Aut_{\mathcal{F}}(S)$, then*

$$\psi \in \{\varphi|_P, (\varphi \circ \eta)|_P \mid \varphi \in Aut(E_\lambda), P \lneq E_\lambda\} \sqcup \{\varphi, \varphi \circ \eta \mid \varphi \in Aut_{\mathcal{F}}(E_\lambda)\},$$

   and $R \leq E_\lambda$ for some $\lambda = -1, 1$.

where $\sim$ denotes the S-S-equivalence relation.

Proof. By 7.1.3 there exists $g, h \in S$ such that, either

1. $P \leq T^g$ and $P\phi \leq T^h$, or,

2. If $T = E_{\pm 1}$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$, then $P \leq E_{\pm 1}^g$ and $P\phi \leq E_{\pm 1}^h$.

Assume the first case. Then $c_g|_{P^{g^{-1}}} \circ \phi \circ c_{h^{-1}}$ is a restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(T)$. It is left to consider the second case. Assume that $T = E_{\pm 1}$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$. Write $\gamma = c_g|_{P^{g^{-1}}} \circ \phi \circ c_{h^{-1}}$, $Q = P^{g^{-1}}$ and $R = (P\phi)^{h^{-1}}$. Every morphism in $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$ maps $\langle z \rangle$ to itself. This means that any morphism between subgroups of E_λ for some $\lambda = -1, 1$, is a restriction of a morphism in the full automorphism group $\mathrm{Aut}(E_\lambda)$. Thus, if $Q, R \leq E_\lambda$ for some $\lambda = -1, 1$, then by Lemma 7.1.4 we have

$$\begin{aligned}
\gamma \in &\{\varphi|_P \mid \varphi \in \mathrm{Aut}(E_\lambda), P \leq E_\lambda\} \cap \mathcal{F} \\
= &\{\varphi|_P \mid \varphi \in \mathrm{Aut}(E_\lambda), P \lneq E_\lambda\} \cap \mathcal{F} \sqcup \{\varphi \mid \varphi \in \mathrm{Aut}(E_\lambda)\} \cap \mathcal{F} \\
= &\{\varphi|_P \mid \varphi \in \mathrm{Aut}(E_\lambda), P \lneq E_\lambda\} \sqcup \{\varphi \mid \varphi \in \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\}.
\end{aligned}$$

If $Q \leq E_\lambda$ and $R \leq E_{-\lambda}$ for some $\lambda = -1, 1$, then $\gamma \circ \eta$ brings us into the previous situation. This means that

$$\gamma \in \{\varphi|_P \circ \eta \mid \varphi \in \mathrm{Aut}(E_\lambda), P \lneq E_\lambda\} \sqcup \{\varphi \circ \eta \mid \varphi \in \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\}.$$

This completes the proof. $\square$

At the start of this section we mentioned two steps in achieving the aim of this section. Lemma 7.1.5 completes the first step by showing that for any DRV-fusion system $\mathcal{F}$, if $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T)\rangle$, for some $T \in \mathcal{T}$, and ϕ is not a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(S)$, then ϕ can be chosen, up to S-S-equivalence, to be a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(T)$ (with some modifications in the case when $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$). In the next set of results, we explicitly list those morphisms that are restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(T)$, up to S-S-equivalence.

Let $\mathcal{F}$ be a DRV-fusion system. Let $\phi \in \mathrm{Aut}_{\mathcal{F}}(T)$ for some $T \in \mathcal{T}$. It turns out that, if $H = \mathrm{Aut}_S(T)$, then a set of double coset representatives of $(\bar{H}, \bar{H})$ in $\mathrm{Out}_{\mathcal{F}}(T)$, where $\bar{H}$ is the quotient of H by $\mathrm{Inn}(T)$, gives us a small set containing all the morphisms up to S-S-equivalence. This is not to say that the set itself is necessarily a set of S-S-equivalence class representatives - it may be the case that, in general, two morphisms in the set are S-S-equivalent. Nevertheless, when $\mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{SL}_2(3)$ we get just two morphisms, as opposed to 24, and when $\mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{GL}_2(3)$, we get just four morphisms, as opposed to 48. This is proved in Lemma 7.1.8.

When ϕ is a strict restriction of some morphism in $\mathrm{Aut}_{\mathcal{F}}(T)$, then, as seen in Corollary 7.1.9, it suffices to take restrictions of morphisms found in Lemma 7.1.8. However, we attempt to do better than this. We cannot use the previous method involving double cosets, fundamentally because the set $\mathrm{Aut}_{\mathcal{F}}(T)$ is a group, but restrictions of morphisms do not form a group. Hence, in such cases we resort to more direct methods, as seen in Lemmas 7.1.10, 7.1.11 and 7.1.12, for $T = E_\lambda$, $T = V_\lambda$ and $T = A$, respectively. Finally, in Lemma 7.1.13 we collect the results, and it turns out that we can just choose ϕ to be restrictions of those automorphisms of T that were already chosen up to S-S-equivalence, but with domains chosen up to S-conjugacy (other than for the case $T = A$).

Finally, Corollary 7.1.15 concludes this section by returning to Lemma 7.1.5 and explicitly listing morphisms up to S-S-equivalence in the more general case when $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T) \rangle$ for some $T \in \mathcal{T}$.

Naturally, there will be subgroups $T \in \mathcal{T}_{full} \backslash \mathcal{T}$, not $\mathcal{F}$-essential in $\mathcal{F}$, which also generate nontrivial automorphism groups. We do not need to explicitly calculate this for all DRV-fusion systems. However, as we shall see in Section 7.2, we will need this for some DRV-fusion systems. Thus, alongside the rest of the calculations, we determine, up to S-S-equivalence, morphisms that are restrictions of $\mathrm{Aut}_{\mathcal{F}}(T)$ for selected $T \in \mathcal{T}_{full} \backslash \mathcal{T}$. We begin by recalling some details.

Let $T \in \mathcal{T}_{full}$. Recall from Section 6.3 that we defined $x_T, y_T \in T$ as follows:

T	x_T	y_T
V_λ	ss_1^λ	z
E_λ	ss_1^λ	t
A	s_1	$s_1^{a_k}s_2^{b_k}$

where $\lambda = -1, 0, 1$, and where a_k, b_k are defined in Section 6.2. We also defined θ_T, θ_T^{-1}, $\alpha_T, \beta_T \in \mathrm{Aut}(T)$ as automorphisms of T as follows:

$$\theta_T: \; x_T \mapsto y_T; \quad y_T \mapsto x_T^{-1} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\theta_T^{-1}: \; x_T \mapsto y_T^{-1}; \quad y_T \mapsto x_T \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\alpha_T: \; x_T \mapsto y_T; \quad y_T \mapsto x_T \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\beta_T: \; x_T \mapsto y_T^{-1}; \quad y_T \mapsto x_T^{-1} \qquad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In addition to the above, we also defined $\hat{\theta}_T$, $\hat{\theta}_T^{-1}$, $\hat{\alpha}_T, \hat{\beta}_T \in \mathrm{Aut}(T)$ to be automorphisms of T as follows:

$$\hat{\theta}_T: \; x_T \mapsto x_T; \quad y_T \mapsto y_T^{-1} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\theta}_T^{-1}: \; x_T \mapsto x_T^{-1}; \quad y_T \mapsto y_T \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\alpha}_T: \; x_T \mapsto x_T; \quad y_T \mapsto y_T \qquad \mathbb{1}$$

$$\hat{\beta}_T: \; x_T \mapsto x_T^{-1}; \quad y_T \mapsto y_T^{-1}. \qquad -\mathbb{1}.$$

Evidently, $\hat{\alpha}_T = \mathrm{Id}|_T$, which we added for completeness. It turns out that for most of the rest of the morphisms, namely $\hat{\theta}_T, \hat{\theta}_T^{-1}$ and $\hat{\beta}_T$, are also restrictions (or S-S-equivalent to restrictions) of automorphisms of S, as shown in the table below:

T	$\hat{\theta}_T$	$\hat{\theta}_T^{-1}$	$\hat{\alpha}_T$	$\hat{\beta}_T$
V_0	$\eta\|_{V_0}$	$\omega\|_{V_0}$	$\mathsf{Id}\|_{V_0}$	$\omega\circ\eta\|_{V_0}$
$V_{\pm1}$		$\omega\|_{V_0}$	$\mathsf{Id}\|_{V_{\pm1}}$	
E_0	$\eta\|_{E_0}$	$\omega\circ\eta\|_{E_0}$	$\mathsf{Id}\|_{E_0}$	$\sim\omega\|_{E_0}$
$E_{\pm1}$			$\mathsf{Id}\|_{E_{\pm1}}$	$\sim\omega\|_{E_{\pm1}}$
A	$\sim\omega\circ\eta\|_A$	$\sim\omega\|_A$	$\mathsf{Id}\|_A$	$\eta\|_A$

The notation $\sim$ denotes that the corresponding maps are S-S-equivalent, which we will discuss further in Lemma 7.1.8. However, it turns out that when $\lambda=-1,1$, then $\hat{\theta}_{V_\lambda}$ and $\hat{\beta}_{V_\lambda}$ are not restrictions (or S-S-equivalent to restrictions) of elements in $\mathsf{Aut}_{\mathcal{F}}(S)$. This is because the only generator in $\mathsf{Aut}_{\mathcal{F}}(S)$ that could invert z is η. But the problem is that η takes ss_1^{λ} to $ss_1^{-\lambda}$. In the case $\hat{\theta}_\lambda$ and $\hat{\theta}_\lambda^{-1}$ for $\lambda=-1,1$, we will not need these maps, as it turns out the that determinant for these maps, when identified as elements of $\mathsf{GL}_2(3)$, is -1, but $\mathsf{Out}_{\mathcal{F}}(E_\lambda)$ is always an $\mathsf{SL}_2(3)$ in every DRV-fusion system.

Recall from Section 2.3 the definitions of $\Omega_n(G)$ and $\Phi(G)$, for any finite group G. Recall also, from Lemma 5.1.7 that $\Omega_1(A)=A_0=\langle t,z\rangle$.

Lemma 7.1.6. *Let $\mathcal{F}$ be a DRV-fusion system. Then $Aut_{\mathcal{F}}(A)\cong Aut_{\mathcal{F}}(\Omega_1(A))$.*

Proof. Write $Q=C_{\mathsf{Aut}(A)}(A/\Phi(A))$. By Lemma 2.3.2, and by restricting to $\mathcal{F}$-automorphisms, the map $\gamma\colon \mathsf{Aut}_{\mathcal{F}}(A)\to\mathsf{Aut}_{\mathcal{F}}(\Omega_1(A))$, via $\psi\mapsto\psi|_{\Omega_1(A)}$, is a group homomorphism with kernel $Q\cap\mathsf{Aut}_{\mathcal{F}}(A)$. Since A is abelian, Lemma 2.3.3 implies $Q\cap\mathsf{Aut}_{\mathcal{F}}(A)=1$, that is, γ is injective. Let $\phi\in\mathsf{Aut}_{\mathcal{F}}(\Omega_1(A))$. $\Omega_1(A)$ is fully $\mathcal{F}$-centralized, and therefore, by 2.1.5, is $\mathcal{F}$-receptive. Therefore, ϕ extends to $N_\phi\geq C_S(\Omega_1(A))=A$, that is, γ is surjective. □

Lemma 7.1.7. *Let $\mathcal{F}$ be a DRV-fusion system. If $E_0,A\in\mathcal{T}$, then $Out_{\mathcal{F}}(V_0)\cong GL_2(3)$. If $E_{\pm1},A\in\mathcal{T}$, then $Out_{\mathcal{F}}(V_\lambda)\cong GL_2(3)$ for all $\lambda=-1,1$.*

Proof. By definition of $\mathcal{F}$, looking at Table 7.1, we have $\eta\in\mathsf{Aut}_{\mathcal{F}}(S)$, and by Lemma 7.1.6, we have $\mathsf{Aut}_{\mathcal{F}}(A)\cong\mathsf{Aut}_{\mathcal{F}}(\langle t,z\rangle)\cong\mathsf{GL}_2(3)$. Note that for any $\phi\in\mathsf{Aut}_{\mathcal{F}}(\langle t,z\rangle)$ and any $\lambda=-1,0,1$, the map defined by $\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}\circ\phi\circ\theta_\lambda^{-1}\in\mathsf{Aut}_{\mathcal{F}}(V_\lambda)$. Thus, there is a bijection of sets between elements $\mathsf{Aut}_{\mathcal{F}}(\langle t,z\rangle)$ and elements in $\mathsf{Aut}_{\mathcal{F}}(V_\lambda)$. □

Define the following subgroup of $\mathsf{GL}_2(3)$:

$$H = \left\{ \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \middle| \delta \in \{-1,0,1\} \right\} \leq \mathsf{SL}_2(3).$$

Then the double cosets, and their representatives, corresponding to (H,H) in $\mathsf{GL}_2(3)$ are as those given in the table below:

x	HxH	Size	Contained in
$\mathbb{1}$	$\left\{ \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \middle\| \delta \in \{-1,0,1\} \right\}$	3	$\mathsf{SL}_2(3)$
$-\mathbb{1}$	$\left\{ \begin{pmatrix} -1 & \delta \\ 0 & -1 \end{pmatrix} \middle\| \delta \in \{-1,0,1\} \right\}$	3	$\mathsf{SL}_2(3)$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\left\{ \begin{pmatrix} \delta & 1-\delta\varepsilon \\ -1 & \varepsilon \end{pmatrix} \middle\| \delta,\varepsilon \in \{-1,0,1\} \right\}$	9	$\mathsf{SL}_2(3)$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\left\{ \begin{pmatrix} \delta & \delta\varepsilon-1 \\ 1 & \varepsilon \end{pmatrix} \middle\| \delta,\varepsilon \in \{-1,0,1\} \right\}$	9	$\mathsf{SL}_2(3)$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\left\{ \begin{pmatrix} 1 & \delta \\ 0 & -1 \end{pmatrix} \middle\| \delta \in \{-1,0,1\} \right\}$	3	$\mathsf{GL}_2(3)\backslash\mathsf{SL}_2(3)$
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\left\{ \begin{pmatrix} -1 & \delta \\ 0 & 1 \end{pmatrix} \middle\| \delta \in \{-1,0,1\} \right\}$	3	$\mathsf{GL}_2(3)\backslash\mathsf{SL}_2(3)$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\left\{ \begin{pmatrix} \delta & \delta\varepsilon+1 \\ 1 & \varepsilon \end{pmatrix} \middle\| \delta,\varepsilon \in \{-1,0,1\} \right\}$	9	$\mathsf{GL}_2(3)\backslash\mathsf{SL}_2(3)$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\left\{ \begin{pmatrix} \delta & -\delta\varepsilon-1 \\ -1 & \varepsilon \end{pmatrix} \middle\| \delta,\varepsilon \in \{-1,0,1\} \right\}$	9	$\mathsf{GL}_2(3)\backslash\mathsf{SL}_2(3)$

Note that while the eight elements are representatives of the double coset (H,H) in $\mathsf{GL}_2(3)$, the first four are representatives of (H,H) in $\mathsf{SL}_2(3)$.

Recall from Table 7.1, if $E_0, A \in \mathcal{T}$, then $V_0 \notin \mathcal{T}$. Similarly, if $E_{\pm 1}, A \in \mathcal{T}$, then $V_{\pm 1} \notin \mathcal{T}$. Thus, there is no (plausible) contradiction between the two statements in Lemma 7.1.8, below.

Lemma 7.1.8. *Let $\mathcal{F}$ be a DRV-fusion system, and let $T \in \mathcal{T}_{full}$. Suppose that $\phi \in Aut_{\mathcal{F}}(T)$ is not a restriction of $Aut_{\mathcal{F}}(S)$.*

1. *If $T \in \mathcal{T}$, then up to S-S equivalence*

$$\phi \in \begin{cases} \{\theta_T, \theta_T^{-1}\} & \text{if } Out_{\mathcal{F}}(T) \cong SL_2(3) \\ \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\} & \text{if } Out_{\mathcal{F}}(T) \cong GL_2(3). \end{cases}$$

2. *If $E_0, A \in \mathcal{T}$ and $T = V_0$, or, if $E_{\pm 1}, A \in \mathcal{T}$ and $T = V_\lambda$, for some $\lambda = -1, 1$, then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_{V_0}, \theta_{V_0}^{-1}, \alpha_{V_0}, \beta_{V_0}\} & \text{if } E_0, A \in \mathcal{T} \\ \{\theta_{V_\lambda}, \theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}, \beta_{V_\lambda} \mid \lambda = -1, 1\} \sqcup \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda} \mid \lambda = -1, 1\} & \text{if } E_{\pm 1}, A \in \mathcal{T}. \end{cases}$$

Proof. Let $\phi, \psi \in \mathrm{Aut}_{\mathcal{F}}(T)$. Write $H = \mathrm{Aut}_S(T)$ and $\bar{H} = \mathrm{Out}_S(T)$. Then,

$$\bar{\psi} \in \bar{H}\bar{\phi}\bar{H} \Leftrightarrow \psi \in H\phi H \Rightarrow \phi \sim \psi.$$

Since double cosets corresponding to $(\bar{H}, \bar{H})$ in $\mathrm{Out}_{\mathcal{F}}(T)$ partition the group, it suffices to find a set of double coset representatives of $(\bar{H}, \bar{H})$ in $\mathrm{Out}_{\mathcal{F}}(T)$. Under the identification $\mathrm{Out}_{\mathcal{F}}(T) \to \mathrm{GL}_2(3)$, we have:

$$\bar{H} = \left\{ \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \middle| \delta \in \{-1, 0, 1\} \right\} \leq \mathrm{SL}_2(3).$$

Suppose $T \in \mathcal{T}$. Then note that $\mathrm{Out}_{\mathcal{F}}(T)$ is isomorphic to one of $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. If $\mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{SL}_2(3)$, then for each double coset representative of $(\bar{H}, \bar{H})$ in $\mathrm{SL}_2(3)$, we have the following identifications:

T	$\mathbb{1}$	$-\mathbb{1}$	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$
V_0	$\overline{\mathsf{Id}\vert_{V_0}}$	$\overline{\omega \circ \eta\vert_{V_0}}$	$\overline{\theta_{V_0}}$	$\overline{\theta_{V_0}^{-1}}$
E_λ	$\overline{\mathsf{Id}\vert_{E_\lambda}}$	$\overline{\omega\vert_{E_\lambda}}$	$\overline{\theta_{E_\lambda}}$	$\overline{\theta_{E_\lambda}^{-1}}$
A	$\overline{\mathsf{Id}\vert_{A}}$	$\overline{\eta\vert_{A}}$	$\overline{\theta_{A}}$	$\overline{\theta_{A}^{-1}}$

It follows that θ_T and θ_T^{-1} are all the morphisms up to S-S-equivalence in $\mathsf{Aut}_{\mathcal{F}}(T)$ that are not restrictions of elements in $\mathsf{Aut}_{\mathcal{F}}(S)$.

If on the other hand $\mathsf{Out}_{\mathcal{F}}(T) \cong \mathsf{GL}_2(3)$, then for each double coset representative of $(\bar{H},\bar{H})$ in $\mathsf{GL}_2(3)$, we have the following additional identifications:

T	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$
V_0	$\overline{\eta\vert_{V_0}}$	$\overline{\omega\vert_{V_0}}$	$\overline{\alpha_{V_0}}$	$\overline{\beta_{V_0}}$
E_0	$\overline{\eta\vert_{E_0}}$	$\overline{\omega \circ \eta\vert_{E_0}}$	$\overline{\alpha_{E_0}}$	$\overline{\beta_{E_0}^{-1}}$
A	$\sim \overline{\omega \circ \eta\vert_{A}}$	$\sim \overline{\omega\vert_{A}}$	$\overline{\theta_{A}}$	$\overline{\theta_{A}^{-1}}$

We remark that $\overline{\omega|_A} = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\overline{\omega \circ \eta|_A} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. However, the two maps are in the same double coset as to the matrices listed in the table above. It follows that θ_T, θ_T^{-1}, α_T and β_T are all the morphisms up to S-S-equivalence in $\mathsf{Aut}_{\mathcal{F}}(T)$ that are not restrictions of elements in $\mathsf{Aut}_{\mathcal{F}}(S)$.

Now suppose that $E_0, A \in \mathfrak{T}$ and $T = V_0$, or $E_{\pm 1}, A \in \mathfrak{T}$ and $T = V_\lambda$, for some $\lambda = -1, 1$. By Lemma 7.1.7, $\mathsf{Out}_{\mathcal{F}}(V_0) \cong \mathsf{GL}_2(3)$ if $E_0, A \in \mathfrak{T}$, and $\mathsf{Out}_{\mathcal{F}}(V_\lambda) \cong \mathsf{GL}_2(3)$ if $E_{\pm 1}, A \in \mathfrak{T}$. For each double coset representative of $(\bar{H},\bar{H})$ in $\mathsf{GL}_2(3)$, where $\bar{H} = \mathsf{Out}_S(V_\lambda)$, we have the following identifications:

T	$\mathbb{1}$	$-\mathbb{1}$	$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$
V_0	$\overline{\mathsf{Id}\vert_{V_0}}$	$\overline{\omega \circ \eta\vert_{V_0}}$	$\overline{\theta_{V_0}}$	$\overline{\theta_{V_0}^{-1}}$	$\overline{\eta\vert_{V_0}}$	$\overline{\omega\vert_{V_0}}$	$\overline{\alpha_{V_0}}$	$\overline{\beta_{V_0}}$
$V_\lambda, \lambda = -1, 1$	$\overline{\mathsf{Id}\vert_{V_\lambda}}$	$\overline{\hat{\beta}_{V_\lambda}}$	$\overline{\theta_{V_\lambda}}$	$\overline{\theta_{V_\lambda}^{-1}}$	$\overline{\hat{\theta}_{V_\lambda}}$	$\overline{\omega\vert_{V_\lambda}}$	$\overline{\alpha_{V_\lambda}}$	$\overline{\beta_{V_\lambda}^{-1}}$

Thus, when $\lambda = 0$, then $\theta_{V_0}, \theta_{V_0}^{-1}, \alpha_{V_0}, \beta_{V_0}$ are all the morphisms up to S-S-equivalence in $\mathrm{Aut}_{\mathcal{F}}(V_0)$ that are not restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(S)$, as was the case when V_0 was an element of $\mathcal{T}$ in the proof of part (1). More notably, however, when $\lambda = -1, 1$, then $\theta_{V_\lambda}, \theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}, \beta_{V_\lambda}$, and additionally $\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}$ are the morphisms up to S-S-equivalence in $\mathrm{Aut}_{\mathcal{F}}(V_\lambda)$ that are not restrictions of $\mathrm{Aut}_{\mathcal{F}}(S)$. $\square$

Corollary 7.1.9. *Let $\mathcal{F}$ be a DRV-fusion system and let $T \in \mathcal{T}_{full}$. Suppose that ϕ is a restriction of an element in $Aut_{\mathcal{F}}(T)$, and not a restriction of $Aut_{\mathcal{F}}(S)$.*

1. *If $T \in \mathcal{T}$, then up to S-S-equivalence,*

$$\phi \in \begin{cases} \{\theta_T|_Q, \theta_T^{-1}|_Q \mid Q \leq T\} & \text{if } Out_{\mathcal{F}}(T) \cong SL_2(3) \\ \{\theta_T|_Q, \theta_T^{-1}|_Q, \alpha_T|_Q, \beta_T|_Q \mid Q \leq T\} & \text{if } Out_{\mathcal{F}}(T) \cong GL_2(3). \end{cases}$$

2. *If $E_0, A \in \mathcal{T}$ and $T = V_0$, or, if $E_{\pm 1}, A \in \mathcal{T}$ and $T = V_\lambda$, for some $\lambda = -1, 1$, then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_{V_0}|_Q, \theta_{V_0}^{-1}|_Q, \alpha_{V_0}|_Q, \beta_{V_0}|_Q \mid Q \leq V_0\} & \text{if } E_0, A \in \mathcal{T} \\ \{\theta_{V_\lambda}|_Q, \theta_{V_\lambda}^{-1}|_Q, \alpha_{V_\lambda}|_Q, \beta_{V_\lambda}|_Q \mid Q \leq V_\lambda, \lambda = -1, 1\} & \\ \sqcup \{\hat{\theta}_{V_\lambda}|_Q, \hat{\beta}_{V_\lambda}|_Q \mid Q \leq V_\lambda, \lambda = -1, 1\} & \text{if } E_{\pm 1}, A \in \mathcal{T}. \end{cases}$$

Proof. Write $P = D_\phi$ and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\phi = \varphi|_P$. If $T \in \mathcal{T}$, then by 7.1.8, there exists

$$\psi \in \begin{cases} \{\theta_T, \theta_T^{-1}\} & \text{if } \mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{SL}_2(3) \\ \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\} & \text{if } \mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{GL}_2(3). \end{cases}$$

such that $\varphi \sim \psi$. If $E_0, A \in \mathcal{T}$ and $T = V_0$, or, if $E_{\pm 1}, A \in \mathcal{T}$ and $T = V_\lambda$, for some $\lambda = -1, 1$, then by Lemma 7.1.8, there exists

$$\psi \in \begin{cases} \{\theta_{V_0}, \theta_{V_0}^{-1}, \alpha_{V_0}, \beta_{V_0}\} & \text{if } E_0, A \in \mathcal{T} \\ \{\theta_{V_\lambda}, \theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}, \beta_{V_\lambda} \mid \lambda = -1, 1\} \sqcup \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda} \mid \lambda = -1, 1\} & \text{if } E_{\pm 1}, A \in \mathcal{T}. \end{cases}$$

such that $\varphi \sim \psi$. Under restriction, for either of the two cases, we have $\varphi|_P \sim \psi|_Q$ for some S-conjugate $Q \leq T$ of P. The result now follows. □

Lemma 7.1.10. *Let $\mathcal{F}$ be a DRV-fusion system and let $T = E_\lambda$ for some $\lambda = -1, 0, 1$. Suppose that one of the following holds:*

1. *$Out_{\mathcal{F}}(T) \cong SL_2(3)$ for $\lambda = -1, 1$, whenever $\eta \in Aut_{\mathcal{F}}(S)$ and $E_{\pm 1} \in \mathcal{T}$, or $Out_{\mathcal{F}}(T) \cong GL_2(3)$ for $\lambda = 0$.*

2. *$Out_{\mathcal{F}}(T) \cong SL_2(3)$ whenever the first case does not hold.*

Let

$$\{V, W\} \in \Big\{\{\langle z \rangle\}, \{\langle ss_1^\lambda \rangle\}, \{\langle t \rangle\}, \{\langle ss_1^\lambda, z \rangle\}, \{\langle t, z \rangle\}, \\ \{\langle ss_1^\lambda \rangle, \langle t \rangle\}, \{\langle ss_1^\lambda, z \rangle, \langle t, z \rangle\}\Big\}.$$

If $\{V, W\} \neq \{\langle z \rangle\}$, define $v \in V \cap \{ss_1^\lambda, t\}$ and $w \in W \cap \{ss_1^\lambda, t\}$. Then, up to S-S-equivalence, every non-trivial proper restriction of a morphism in $Aut_{\mathcal{F}}(T)$ is a morphism $\phi \in Hom_{\mathcal{F}}(V, W)$, defined by

$$\phi : \begin{cases} v \mapsto w^i & \text{if } z \notin V \\ v \mapsto w^i;\ z \mapsto z^j & \text{if } z \in V, V \neq \langle z \rangle \\ z \mapsto z^j & \text{if } V = \langle z \rangle \end{cases}$$

where $i, j \in \{-1, 1\}$, whenever case (1) holds, and where $i \in \{-1, 1\}, j = 1$, whenever case (2) holds.

Proof. By Lemma 7.1.4, we remark that whenever $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $E_{\pm 1} \in \mathcal{T}$, every proper restriction of a morphism that exists in the full automorphism group $\mathrm{Aut}(E_\lambda)$, for $\lambda = -1, 1$, exists in $\mathcal{F}$. This is the reason why this case is set alongside the case when $\mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{GL}_2(3)$.

Let ψ be a non-trivial proper restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(T)$. Write $P = D_\psi$ and $Q = I_\psi$. if $P = \langle z \rangle$, then $z\psi = z^j$ where $j \in \{-1,1\}$ whenever case (1) holds, and where $j = 1$ whenever case (2) holds. So suppose $P \neq \langle z \rangle$. Then there exists $x, y \in T$, $i \in \{-1,1\}$ and $l \in \{-1,0,1\}$ such that

$$P = \begin{cases} \langle x \rangle & \text{if } z \notin P \\ \langle x, z \rangle & \text{if } z \in P \end{cases} \quad Q = \begin{cases} \langle y \rangle & \text{if } z \notin Q \\ \langle y, z \rangle & \text{if } z \in Q \end{cases}$$

$$(x)\psi = \begin{cases} y^i & \text{if } z \notin P \\ y^i z^l & \text{if } z \in P. \end{cases} \quad \text{and} \quad (z)\psi = z^j \text{ if } z \in P$$

where $j \in \{-1,1\}$ whenever case (1) holds, and where $j = 1$ whenever case (2) holds. If $z \notin P$, then P and Q are S-conjugate to $\langle ss_1^\lambda \rangle$ or $\langle t \rangle$. If on the other hand, $z \in P$, then P and Q are S-conjugate to $\langle ss_1^\lambda, z \rangle$ or $\langle t, z \rangle$. It follows that there exists $g, h \in S$ such that $P^g = V$ and $Q^h = W$ for some unordered pair $\{V, W\}$ specified in the Lemma. Upon redefining generators, if necessary, we have $h \in N_S(y, w)$ and

$$g \in \begin{cases} N_S(x, v) & \text{if } z \notin P \\ N_S(x, vz^m) & \text{if } z \in P \end{cases}$$

where m is chosen such that $mj \equiv l \bmod 3$. It follows that $\psi = c_g|_P \circ \phi \circ c_{h^{-1}}$, where ϕ is the map described in the lemma. □

Lemma 7.1.11. *Let $\mathcal{F}$ be a DRV-fusion system and let $T = V_\lambda$ for some $\lambda = -1, 0, 1$. Suppose that $\mathrm{Aut}_{\mathcal{F}}(T) \cong SL_2(3)$ or $GL_2(3)$. Let*

$$\{V, W\} \in \Big\{ \{\langle z \rangle\}, \{\langle ss_1^\lambda \rangle\}, \{\langle ss_1^\lambda \rangle, \langle z \rangle\} \Big\}.$$

Define $v \in V \cap \{ss_1^{\lambda}, z\}$ and $w \in W \cap \{ss_1^{\lambda}, z\}$. Then, up to S-S-equivalence, every non-trivial proper restriction of a morphism in $Aut_{\mathcal{F}}(T)$ is a morphism $\phi \in Hom_{\mathcal{F}}(V, W)$, defined by

$$\phi : v \mapsto w^i$$

where $i \in \{-1, 1\}$.

Proof. Let ψ be a non-trivial proper restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(T)$. Write $P = D_{\psi}$ and $Q = I_{\psi}$. Then $|P| = |Q| = 3$. There exists $x, y \in T$ such that

$$P = \langle x \rangle, \quad Q = \langle y \rangle \quad \text{and} \quad x\psi = y^i$$

for some $i \in \{-1, 1\}$. Note that P and Q are S-conjugate to one of $\langle ss_1^{\lambda} \rangle$ and $\langle z \rangle$. Thus, there exists $g, h \in S$ such that $P^g = V$ and $Q^h = W$ for some unordered pair $\{V, W\}$ specified in the Lemma. Upon redefining generators, if necessary, we have $g \in N_S(x, v)$ and $h \in N_S(y, w)$. It follows that $\psi = c_g|_P \circ \phi \circ c_{h^{-1}}$, where ϕ is the map described in the lemma. $\square$

Lemma 7.1.12. *Let $\mathcal{F}$ be a DRV-fusion system, and suppose $Out_{\mathcal{F}}(A) \cong SL_2(3)$ or $GL_2(3)$. Let*

$$\{V, W\} \in \Big\{ \{\langle z \rangle\}, \{\langle t \rangle\}, \{\langle t \rangle, \langle z \rangle\} \Big\}.$$

Define $v \in V \cap \{t, z\}$ and $w \in W \cap \{t, z\}$. Then, up to S-S-equivalence, every non-trivial restriction of a morphism in $Aut_{\mathcal{F}}(A)$ with domain of order 3, is a morphism $\phi \in Hom_{\mathcal{F}}(V, W)$, defined by

$$\phi : v \mapsto w^i$$

where $i \in \{-1, 1\}$.

Proof. When a subgroup of A has order 3, then it is inside $\Omega_1(A) = \langle t, z \rangle$. In the proof of Lemma 7.1.11, it suffices to replace ss_1^{λ} with t, and V_{λ} with A. The claim now follows. $\square$

Let $T \in \mathcal{T}_{full}$. Define

$$\mathcal{D}_T = \begin{cases} \text{set of nontrivial } S\text{-conjugacy class representatives of subgroups of } T & \text{if } T \neq A \\ \text{set of all nontrivial subgroups in } A \text{ not including } \langle tz \rangle \text{ or } \langle tz^{-1} \rangle & \text{if } T = A. \end{cases}$$

For our convenience, however, we shall fix $\mathcal{D}_T$ in the following way:

$$\mathcal{D}_T = \begin{cases} \{\langle ss_1^{\lambda} \rangle, \langle z \rangle, \langle ss_1^{\lambda}, z \rangle\} & \text{if } T = V_\lambda \\ \{\langle ss_1^{\lambda} \rangle, \langle t \rangle, \langle ss_1^{\lambda}, z \rangle, \langle t, z \rangle, \langle ss_1^{\lambda}, t \rangle\} & \text{if } T = E_\lambda \\ \{Q \leq A \mid Q \neq 1, \langle tz \rangle \text{ or } \langle tz^{-1} \rangle\} & \text{if } T = A. \end{cases}$$

where $\lambda = -1, 0, 1$. For simplicity, we also write $\mathcal{D}_\lambda = \mathcal{D}_{E_\lambda}$. We also define

$$\mathcal{D}_{A_0} = \{\langle z \rangle, \langle t \rangle, \langle t, z \rangle\}.$$

We also define $\mathcal{D}$ to be the set of S-conjugacy class representatives in S (not including the identity). For $\mathcal{D}$, the S-conjugacy class representatives that are not contained in A are easy to write down, by Lemma 5.2.3 and Lemma 5.2.5. Thus we may fix,

$$\mathcal{D} \supset \{\langle s \rangle, \langle ss_1 \rangle, \langle ss_1^{-1} \rangle, \langle s, v \rangle, \langle ss_1, v \rangle, \langle ss_1^{-1}, v \rangle \mid v \in \{s_1^{3^{k-i}}, s_2^{3^{k-i}}\} \text{ where } i = 1, \ldots, k\}.$$

For convenience, we may also choose $\langle z \rangle, \langle t \rangle$, $\langle t, z \rangle \in \mathcal{D}$. This ensures $\mathcal{D}_T \subset \mathcal{D}$ for all $T \neq A$.

Note that $\langle s_2 \rangle^s = \langle s_1^3 s_2^2 \rangle$, $\langle s_2 \rangle^{s^{-1}} = \langle s_1^3 s_2 \rangle$ and $|S|/|N_S(\langle s_2 \rangle)| = 3$. This means that the conjugacy class containing $\langle s_2 \rangle$ is $\langle s_2 \rangle^S = \{\langle s_1^3 s_2 \rangle, \langle s_2 \rangle, \langle s_1^3 s_2^2 \rangle\}$. Similarly, the conjugacy class containing $\langle s_1 \rangle$ and $\langle s_1 s_2^{-1} \rangle$ is $\{\langle s_1 s_2 \rangle, \langle s_1 \rangle, \langle s_1^2 s_2 \rangle\}$ and $\{\langle s_1^4 s_2^3 \rangle, \langle s_1 s_2^{-1} \rangle, \langle s_1^5 s_2^2 \rangle\}$, respectively. We may reasonably choose,

$$\mathcal{D} \supset \{Q \mid Q \leq \langle s_1 \rangle, \langle s_2 \rangle, \langle s_1 s_2^{-1} \rangle\}.$$

This means that no subgroup of $\langle s_1^3 s_2 \rangle, \langle s_1^3 s_2^2 \rangle, \langle s_1 s_2 \rangle$ or $\langle s_1^2 s_2 \rangle$ is in $\mathcal{D}$. For the rest of the subgroups of A in $\mathcal{D}$ we leave them as arbitrary.

Lemma 7.1.13. *Let $\mathcal{F}$ be a DRV-fusion system and let $T \in \mathcal{T}_{full}$. Suppose that ϕ is a restriction of an element in $Aut_{\mathcal{F}}(T)$, and is not a restriction of an element in $Aut_{\mathcal{F}}(S)$.*

1. *If $T \in \mathcal{T}$, then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_T|_Q, \theta_T^{-1}|_Q \mid Q \in \mathcal{D}_T\} & \text{if } Out_{\mathcal{F}}(T) \cong SL_2(3) \\ \{\theta_T|_Q, \theta_T^{-1}|_Q, \alpha_T|_Q, \beta_T|_Q \mid Q \in \mathcal{D}_T\} & \text{if } Out_{\mathcal{F}}(T) \cong GL_2(3). \end{cases}$$

2. *If $T = V_0$ and $\{E_0, A\} \subseteq \mathcal{T}$, or, if $T = V_{\pm 1}$ and $\{E_{\pm 1}, A\} \subseteq \mathcal{T}$, then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_{V_0}|_Q, \theta_{V_0}^{-1}|_Q, \alpha_{V_0}|_Q, \beta_{V_0}|_Q \mid Q \in \mathcal{D}_{V_0}\} & \text{if } E_0, A \in \mathcal{T} \\ \{\theta_{V_\lambda}|_Q, \theta_{V_\lambda}^{-1}|_Q, \alpha_{V_\lambda}|_Q, \beta_{V_\lambda}|_Q \mid Q \in \mathcal{D}_{V_\lambda}, \lambda = -1, 1\} \sqcup \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}\} & \text{if } E_{\pm 1}, A \in \mathcal{T}. \end{cases}$$

Proof. We split the two claims into three cases: $T = A$, $T = V_\lambda$ or $T = E_\lambda$, for $\lambda = -1, 0, 1$. It is in the second case, that is, $T = V_\lambda$, that we prove the second claim of the Lemma.

Case 1 $T = A$.

In this case $\mathrm{Out}_{\mathcal{F}}(A) \cong \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. If $|D_\phi| \geq 3^2$, then the result holds by definition of $\mathcal{D}_A$. So it suffices to consider the case $|D_\phi| = 3$. Lemma 7.1.12 gives us the following morphisms, up to S-S equivalence:

Morphisms		Equal to
$\langle t \rangle \to \langle t \rangle$	$t \mapsto t$	$\mathrm{Id}\|_{\langle t \rangle}$
	$t \mapsto t^{-1}$	$\eta\|_{\langle t \rangle}$
$\langle z \rangle \to \langle z \rangle$	$z \mapsto z$	$\mathrm{Id}\|_{\langle z \rangle}$
	$z \mapsto z^{-1}$	$\eta\|_{\langle z \rangle}$
$\langle t \rangle \to \langle z \rangle$	$t \mapsto z$	$\theta_A\|_{\langle t \rangle}$
	$t \mapsto z^{-1}$	$\theta_A^{-1}\|_{\langle t \rangle}$
$\langle z \rangle \to \langle t \rangle$	$z \mapsto t$	$\theta_A^{-1}\|_{\langle z \rangle}$
	$z \mapsto t^{-1}$	$\theta_A\|_{\langle z \rangle}$

This completes the proof for this case.

Case 2: $T = V_\lambda$.

In this case, once again, $\mathrm{Out}_{\mathcal{F}}(V_\lambda) \cong \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. When $|D_\phi| = 3$, Lemma 7.1.11 gives the following morphisms, up to S-S equivalence:

Morphisms		Equal to
$\langle ss_1^\lambda \rangle \to \langle ss_1^\lambda \rangle$	$ss_1^\lambda \mapsto ss_1^\lambda$	$\mathrm{Id}\|_{\langle ss_1^\lambda \rangle}$
	$ss_1^\lambda \mapsto (ss_1^\lambda)^{-1}$	$\omega\|_{\langle ss_1^\lambda \rangle}$
$\langle z \rangle \to \langle z \rangle$	$z \mapsto z$	$\mathrm{Id}\|_{\langle z \rangle}$
	$z \mapsto z^{-1}$	$\eta\|_{\langle z \rangle}$
$\langle ss_1^\lambda \rangle \to \langle z \rangle$	$ss_1^\lambda \mapsto z$	$\theta_{V_\lambda}\|_{\langle ss_1^\lambda \rangle}$
	$ss_1^\lambda \mapsto z^{-1}$	$\theta_{V_\lambda}^{-1}\|_{\langle ss_1^\lambda \rangle}$
$\langle z \rangle \to \langle ss_1^\lambda \rangle$	$z \mapsto ss_1^\lambda$	$\theta_{V_\lambda}^{-1}\|_{\langle z \rangle}$
	$z \mapsto (ss_1^\lambda)^{-1}$	$\theta_{V_\lambda}\|_{\langle z \rangle}$

By using Lemma 7.1.8 for the situation where $D_\phi = V_\lambda$, this completes the proof for this case. It is here that we have proved the second claim of this lemma, through the application of Lemma 7.1.8.

Case 3: $T = E_\lambda$.

Suppose that $\mathrm{Out}_{\mathcal{F}}(E_\lambda) \cong \mathrm{SL}_2(3)$ for $\lambda = -1, 0, 1$. If $|D_\phi| = 3^2$, then Lemma 7.1.10 gives us the following 8 morphisms, up to S-S equivalence:

Morphisms		Equal to
$\langle ss_1^\lambda, z\rangle \to \langle ss_1^\lambda, z\rangle$	$ss_1^\lambda \mapsto ss_1^\lambda,\ z \mapsto z$	$\mathrm{Id}\|_{\langle ss_1^\lambda, z\rangle}$
	$ss_1^\lambda \mapsto (ss_1^\lambda)^{-1},\ z \mapsto z$	$\omega\|_{\langle ss_1^\lambda, z\rangle}$
$\langle t, z\rangle \to \langle t, z\rangle$	$t \mapsto t,\ z \mapsto z$	$\mathrm{Id}\|_{\langle t, z\rangle}$
	$t \mapsto t^{-1},\ z \mapsto z$	$c_{s^{-1}}\|_{\langle t, z\rangle} \circ \omega$
$\langle ss_1^\lambda, z\rangle \to \langle t, z\rangle$	$ss_1^\lambda \mapsto t,\ z \mapsto z$	$\theta_\lambda\|_{\langle ss_1^\lambda, z\rangle}$
	$ss_1^\lambda \mapsto t^{-1},\ z \mapsto z$	$\theta_\lambda^{-1}\|_{\langle ss_1^\lambda, z\rangle}$
$\langle t, z\rangle \to \langle ss_1^\lambda, z\rangle$	$t \mapsto ss_1^\lambda,\ z \mapsto z$	$\theta_\lambda^{-1}\|_{\langle t, z\rangle}$
	$t \mapsto (ss_1^\lambda)^{-1},\ z \mapsto z$	$\theta_\lambda\|_{\langle t, z\rangle}$

When $|D_\phi| = 3$ Lemma 7.1.10 also gives the following 9 morphisms, up to S-S-equivalence:

Morphisms		Equal to
$\langle ss_1^\lambda\rangle \to \langle ss_1^\lambda\rangle$	$ss_1^\lambda \mapsto ss_1^\lambda$	$\mathrm{Id}\|_{\langle ss_1^\lambda\rangle}$
	$ss_1^\lambda \mapsto (ss_1^\lambda)^{-1}$	$\omega\|_{\langle ss_1^\lambda\rangle}$
$\langle t\rangle \to \langle t\rangle$	$t \mapsto t$	$\mathrm{Id}\|_{\langle t\rangle}$
	$t \mapsto t^{-1}$	$c_{s^{-1}}\|_{\langle t\rangle} \circ \omega$
$\langle z\rangle \to \langle z\rangle$	$z \mapsto z$	$\mathrm{Id}\|_{\langle z\rangle}$
$\langle ss_1^\lambda\rangle \to \langle t\rangle$	$ss_1^\lambda \mapsto t$	$\theta_\lambda\|_{\langle ss_1^\lambda\rangle}$
	$ss_1^\lambda \mapsto t^{-1}$	$\theta_\lambda^{-1}\|_{\langle ss_1^\lambda\rangle}$
$\langle t\rangle \to \langle s\rangle$	$t \mapsto ss_1^\lambda$	$\theta_\lambda^{-1}\|_{\langle t\rangle}$
	$t \mapsto (ss_1^\lambda)^{-1}$	$\theta_\lambda\|_{\langle t\rangle}$

By using Lemma 7.1.8 for the situation when the domain of ϕ is E_λ, this completes the proof when $\mathrm{Out}_{\mathcal{F}}(E_\lambda) \cong \mathrm{SL}_2(3)$.

Now suppose that $\mathrm{Out}_{\mathcal{F}}(E_0) \cong \mathrm{GL}_2(3)$.[1] If $|D_\phi| = 3^2$ then Lemma 7.1.10 gives us the following additional 8 morphisms, up to S-S-equivalence:

[1]Note that for $\lambda = -1, 1$, $\mathrm{Out}_{\mathcal{F}}(E_\lambda) \ncong \mathrm{GL}_2(3)$ for any DRV-fusion system.

Morphisms		Equal to
$\langle s,z\rangle \to \langle s,z\rangle$	$s \mapsto s,\ z \mapsto z^{-1}$	$\eta\|_{\langle s,z\rangle}$
	$s \mapsto s^{-1},\ z \mapsto z^{-1}$	$\omega\|_{\langle s,z\rangle} \circ \eta$
$\langle t,z\rangle \to \langle t,z\rangle$	$t \mapsto t,\ z \mapsto z^{-1}$	$c_{s^{-1}}\|_{\langle t,z\rangle} \circ \omega \circ \eta$
	$t \mapsto t^{-1},\ z \mapsto z^{-1}$	$\eta\|_{\langle t,z\rangle}$
$\langle s,z\rangle \to \langle t,z\rangle$	$s \mapsto t,\ z \mapsto z^{-1}$	$\alpha_0\|_{\langle s,z\rangle}$
	$s \mapsto t^{-1},\ z \mapsto z^{-1}$	$\beta_0\|_{\langle s,z\rangle}$
$\langle t,z\rangle \to \langle s,z\rangle$	$t \mapsto s,\ z \mapsto z^{-1}$	$\alpha_0\|_{\langle t,z\rangle}$
	$t \mapsto s^{-1},\ z \mapsto z^{-1}$	$\beta_0\|_{\langle t,z\rangle}$

When $\mathrm{Out}_{\mathcal{F}}(E_0) \cong \mathrm{GL}_2(3)$ and $|D_\phi| = 3$, we additionally have the following morphism:

Morphisms		Equal to
$\langle z\rangle \to \langle z\rangle$	$z \mapsto z^{-1}$	$\eta\|_{\langle z\rangle}$

By using Lemma 7.1.8 for the situation when the domain of ϕ is E_λ, this completes the proof for this case. $\square$

Lemma 7.1.14. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $E_{\pm 1} \in \mathcal{T}$ and $\eta \in Aut_{\mathcal{F}}(S)$. Suppose that $\phi \in \mathcal{F}$ is a restriction of an element of the full automorphism group $Aut(E_\lambda)$, for some $\lambda = -1, 1$, but not a restriction of an element of $Aut_{\mathcal{F}}(S)$. Then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_\lambda, \theta_\lambda^{-1} \mid \lambda = -1, 1\} & \text{if } |D_\phi| = |E_\lambda| \\ \{\theta_\lambda|_Q, \theta_\lambda^{-1}|_Q, \alpha_\lambda|_Q, \beta_\lambda|_Q \mid Q \in \mathcal{D}_\lambda, Q \neq E_\lambda\} & \\ \sqcup \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}\} & \text{if } |D_\phi| \lneq |E_\lambda|. \end{cases}$$

Proof. If $D_\phi = E_\lambda$, then by definition $\phi \in \mathrm{Aut}_{\mathcal{F}}(E_\lambda)$. Now, by applying Lemma 7.1.13 we get the two listed morphisms up to S-S-equivalence. If $|D_\phi| = 3^2$, then Lemma 7.1.10 gives us, in addition to the $\mathrm{SL}_2(3)$ ones listed in Case 3 in the proof of Lemma 7.1.13, the following 8 morphisms, up to S-S-equivalence:

Morphisms		Equal to
$\langle ss_1^\lambda, z\rangle \to \langle ss_1^\lambda, z\rangle$	$ss_1^\lambda \mapsto ss_1^\lambda$, $z \mapsto z^{-1}$	$\hat{\theta}_{V_\lambda}$
	$ss_1^\lambda \mapsto (ss_1^\lambda)^{-1}$, $z \mapsto z^{-1}$	$\hat{\beta}_{V_\lambda}$
$\langle t, z\rangle \to \langle t, z\rangle$	$t \mapsto t$, $z \mapsto z^{-1}$	$(\omega \circ \eta \circ c_s)\|_{\langle t,z\rangle}$
	$t \mapsto t^{-1}$, $z \mapsto z^{-1}$	$\eta\|_{\langle t,z\rangle}$
$\langle ss_1^\lambda, z\rangle \to \langle t, z\rangle$	$ss_1^\lambda \mapsto t$, $z \mapsto z^{-1}$	$\alpha_\lambda\|_{\langle ss_1^\lambda,z\rangle}$
	$ss_1^\lambda \mapsto t^{-1}$, $z \mapsto z^{-1}$	$\beta_\lambda\|_{\langle ss_1^\lambda,z\rangle}$
$\langle t, z\rangle \to \langle ss_1^\lambda, z\rangle$	$t \mapsto ss_1^\lambda$, $z \mapsto z^{-1}$	$\alpha_\lambda\|_{\langle t,z\rangle}$
	$t \mapsto (ss_1^\lambda)^{-1}$, $z \mapsto z^{-1}$	$\beta_\lambda\|_{\langle t,z\rangle}$

When $|D_\phi| = 3$, Lemma 7.1.10 gives, in addition to the $\mathrm{SL}_2(3)$ ones listed in Case 3 in the proof of Lemma 7.1.13, the following morphism:

Morphisms		Equal to
$\langle z\rangle \to \langle z\rangle$	$z \mapsto z^{-1}$	$\eta\|_{\langle z\rangle}$

This completes the proof. $\square$

Before we move on, we remark that there is no contradiction between Lemma 7.1.13 and Lemma 7.1.14. In the case where $T = E_{\pm 1}$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$, the statement in Lemma 7.1.13 is talking of those morphisms that are restrictions of those in $\mathrm{Aut}_{\mathcal{F}}(T)$. However, as Lemma 7.1.14 demonstrates, not every morphism is a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(T)$.

Corollary 7.1.15. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$, is not a restriction of $Aut_{\mathcal{F}}(S)$. Suppose that $\phi \in \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T)\rangle$ for some $T \in \mathcal{T}$. Then, up to S-S-equivalence, either*

$$\phi \in \begin{cases} \{\theta_T|_Q, \theta_T^{-1}|_Q \mid Q \in \mathcal{D}_T\} & \text{if } Out_{\mathcal{F}}(T) \cong SL_2(3) \\ \{\theta_T|_Q, \theta_T^{-1}|_Q, \alpha_T|_Q, \beta_T|_Q \mid Q \in \mathcal{D}_T\} & \text{if } Out_{\mathcal{F}}(T) \cong GL_2(3). \end{cases}$$

or, if $T = E_{\pm 1}$, and $\eta \in Aut_{\mathcal{F}}(S)$, then

$$\phi \in \begin{cases} \{\theta_\lambda|_Q, \theta_\lambda^{-1}|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\} \\ \{(\theta_\lambda \circ \eta)|_Q, (\theta_\lambda^{-1} \circ \eta)|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\} \\ \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda}\}. \end{cases}$$

Proof. By Lemma 7.1.5,

1. Either ϕ is S-S-equivalent to a restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(T)$, or
2. if additionally, $T = E_{\pm 1}$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$, then ϕ is S-S-equivalent to an element in the set

$$\{\varphi|_P, (\varphi \circ \eta)|_P \mid \varphi \in \mathrm{Aut}(E_\lambda), P \lneq E_\lambda, \lambda = -1, 1\} \sqcup \{\varphi, \varphi \circ \eta \mid \varphi \in \mathrm{Aut}_{\mathcal{F}}(E_\lambda), \lambda = -1, 1\}.$$

In the first case it suffices to impose Lemma 7.1.13 and the result follows. For the second case, we note that, by Lemma 3.1.6, for any $\psi_1, \psi_2 \in \mathrm{Aut}(E_\lambda)$, we have $\psi_1 \sim \psi_2 \Leftrightarrow \psi_1 \circ \eta \sim \psi_2 \circ \eta$. Thus, by imposing Lemma 7.1.14, we get the following set of morphisms:

$$\phi \in \begin{cases} \{\theta_\lambda \circ \eta^i, \theta_\lambda^{-1} \circ \eta^i \mid \lambda = -1, 1, i = 0, 1\} & \text{if } |D_\phi| = E_\lambda \\ \{\theta_\lambda|_Q \circ \eta^i, \theta_\lambda^{-1}|_Q \circ \eta^i, \alpha_\lambda|_Q \circ \eta^i, \beta_\lambda|_Q \circ \eta^i \mid Q \in \mathcal{D}_\lambda, Q \neq E_\lambda, i = 0, 1\} & \\ \cup\{\hat{\theta}_{V_\lambda} \circ \eta^i, \hat{\beta}_{V_\lambda} \circ \eta^i \mid i = 0, 1\} & \text{if } |D_\phi| \lneq E_\lambda. \end{cases}$$

However, it turns out that

$$\begin{aligned} (\theta_\lambda \circ \eta)|_{\langle ss_1^\lambda, z\rangle} &= \beta_\lambda|_{\langle ss_1^\lambda, z\rangle}, \quad (\theta_\lambda \circ \eta)|_{\langle t,z\rangle} = \beta_{-\lambda}|_{\langle t,z\rangle} \\ (\theta_\lambda^{-1} \circ \eta)|_{\langle ss_1^\lambda, z\rangle} &= \alpha_\lambda|_{\langle ss_1^\lambda, z\rangle}, \quad (\theta_\lambda^{-1} \circ \eta)|_{\langle t,z\rangle} = \alpha_{-\lambda}|_{\langle t,z\rangle} \\ \theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle} &= (\alpha_\lambda \circ \eta)|_{\langle ss_1^\lambda, z\rangle}, \quad \theta_{-\lambda}^{-1}|_{\langle t,z\rangle} = (\alpha_\lambda \circ \eta)|_{\langle t,z\rangle} \\ \theta_\lambda|_{\langle ss_1^\lambda, z\rangle} &= (\beta_\lambda \circ \eta)|_{\langle ss_1^\lambda, z\rangle}, \quad \theta_{-\lambda}|_{\langle t,z\rangle} = (\beta_\lambda \circ \eta)|_{\langle t,z\rangle} \end{aligned}$$

Moreover, by definition, $\hat{\theta}_{V_\lambda} \circ \eta = \hat{\vartheta}_{1,\lambda}$ and $\hat{\beta}_{V_\lambda} \circ \eta = \hat{\vartheta}_{4,\lambda}$. This completes the proof. $\square$

7.2 $\mathcal{F}$-morphisms generated by automorphisms of S and multiple proper subgroups

Let $\mathcal{F}$ be a DRV-fusion system and let $T \in \mathcal{T}$. In Section 7.1, we showed that when $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T) \rangle$, and is not a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(S)$, then, up to S-S-equivalence, ϕ can be chosen to be restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(T)$ (with some modifications in the case when $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T = E_{\pm 1}$). However, it may be the case that

$$\{\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T) \rangle \mid T \in \mathcal{T}\} \subsetneq \mathcal{F},$$

that is, there may be morphisms that are generated by products of automorphisms of multiple $\mathcal{F}$-essential subgroups of S. In particular this means that for such a DRV-fusion system, $|\mathcal{T}| \geq 2$. In this section we address this problem. Let $\phi \in \mathcal{F}$. Lemma 7.2.1 gives us a condition on D_ϕ under which $\phi \notin \bigsqcup_{T \in \mathcal{T}} \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T) \rangle$. Intuitively, this is only interesting when D_ϕ conjugates into several $T \in \mathcal{T}$. Hence D_ϕ is expected to be very small. We use this condition to determine morphisms, up to S-S-equivalence, for various DRV-fusion systems with the property that $|\mathcal{T}| \geq 2$, as seen in Lemma 7.2.3 and Lemma 7.2.4.

Let $\mathcal{F}$ be a DRV-fusion system. If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, then recall that we can write

$$\phi = (\phi_1|_{P_0}) \circ .. \circ (\phi_n|_{P_{n-1}}) \tag{I}$$

with the property that $P_0 = P, P_n = Q$, and $(P_{i-1})\phi_i \leq P_i$ and, if i is even, $\phi_i \in \bigsqcup_{T \in \mathcal{T}} \mathrm{Aut}_{\mathcal{F}}(T)$, in which case $T \geq \langle P_{i-1}, P_i \rangle$ or, if i is odd, $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(S)$.

Lemma 7.2.1. *Let $\mathcal{F}$ be a DRV-fusion system. Suppose that $\phi \in Hom_{\mathcal{F}}(P, Q)$, for some $P, Q \leq S$, is not a restriction of an element in $Aut_{\mathcal{F}}(S)$. Suppose that either $|\mathcal{T}| = 1$, or, if $|\mathcal{T}| > 1$, then $P \not\leq (T_1 \cap T_2)\psi$ for every $\psi \in \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T_1) \rangle \cup \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T_2) \rangle$ and $T_1 \neq T_2$, $T_1, T_2 \in \mathcal{T}$. Then, for some $T \in \mathcal{T}$, we have $\phi \in \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T) \rangle$.*

Proof. By (I) we may write $\phi = (\phi_1|_{P_0}) \circ .. \circ (\phi_n|_{P_{n-1}})$ with $n \geq 1$, $P = P_0$ and $Q = P_{n-1}\phi_n$. If $|\mathcal{T}| = 1$, the claim holds by definition. If $\mathcal{T} = \{E_{\pm 1}\}$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$, then, since $(E_1)\eta =$

E_{-1}, the claim also holds. So we assume $|\mathcal{T}| > 1$ with $P \nleq (T_1 \cap T_2)\psi$ for every $\psi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T_1)\rangle \cup \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T_2)\rangle$, $T_1 \neq T_2$ and $T_1, T_2 \in \mathcal{T}$. Let $T_1 \in \mathcal{T}$ be defined such that $\phi_2 \in \mathrm{Aut}_{\mathcal{F}}(T_1)$. Suppose for contradiction that there exists $4 \leq i \leq n$, which we may assume to be the least, such that $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(T_2)$ for some $T_2 \in \mathcal{T}\backslash\{T_1\}$.[1] Then $\phi = \varphi_1 \circ \varphi_2$ where $\varphi_1 = (\phi_1|_{P_0}) \circ .. \circ (\phi_{i-1}|_{P_{i-2}})$ and $\varphi_2 = (\phi_i|_{P_{i-1}}) \circ .. \circ (\phi_n|_{P_{n-1}})$. By construction, $\varphi_1 \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T_1)\rangle$. By 7.1.3, we have $P\varphi_1 \leq T_1^g$, or, if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T_1 = E_{\pm 1}$, then $P\varphi_1 \leq E_{\pm 1}^g$, for some $g \in S$. By construction, we also have $P\varphi_1 \leq T_2$. Thus $P\varphi_1 \leq T_1^g \cap T_2$ or $P\varphi_1 \leq E_{\pm 1}^g \cap T_2$. By inspection, $T_1^g \cap T_2 = T_1 \cap T_2 \in \{\langle t, z\rangle, \langle z\rangle\}$ depending on the subgroup T_1 and T_2. Similarly, in the case when $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T_1 = E_{\pm 1}$, we have $E_{\pm 1}^g \cap T_2 = E_{-1} \cap T_2 = E_1 \cap T_2 \in \{\langle t, z\rangle, \langle z\rangle\}$. It follows that $P \leq (T_1 \cap T_2)\varphi_1^{-1}$, contradicting our assumption. □

Lemma 7.2.2. *Let $\mathcal{F}$ be a DRV-fusion system. Let $\phi, \psi \in Hom_{\mathcal{F}}(P,Q)$, for some $P, Q \leq S$. Fix a $\lambda, \mu = -1, 0, 1$. We have the following facts:*

1. *If $P, Q \leq \langle ss_1^{\lambda}, z\rangle$, then $\phi \sim \psi$, if and only if, $\phi \circ \theta_\lambda \sim \psi \circ \theta_\lambda$.*
2. *If $P, Q \leq \langle ss_1^{\lambda}, z\rangle$, then $\phi \sim \psi$, if and only if, $\phi \circ \theta_\lambda \circ \theta_\mu \sim \psi \circ \theta_\lambda \circ \theta_\mu$.*
3. *If $P \leq \langle ss_1^{\lambda}, z\rangle$ and $Q \leq \langle t, z\rangle$, then $\phi \sim \psi$, if and only if, $\phi \circ \theta_0 \sim \psi \circ \theta_0$.*

Proof. 1. Note that $\mathrm{Aut}_S(V_\lambda) = \langle c_t|_{V_\lambda}\rangle$. Thus, when normalizing $\langle ss_1^{\lambda}, z\rangle$ by an element $y \in S$, we may write $y = t^i$. Similarly, note that $\mathrm{Aut}_S(\langle t, z\rangle) = \langle c_s|_{\langle t,z\rangle}\rangle$. Thus, when normalizing $\langle t, z\rangle$ by an element of $y' \in S$, we may write $y' = s^j$. By writing $j = -i$, it

[1] We can say more. If $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T_1 = E_1$, then $T_2 \neq E_{-1}$. Otherwise that would not be a contradiction, because, as we know $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_1)\rangle = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_{-1})\rangle$. Similarly, if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $T_1 = E_{-1}$, then $T_2 \neq E_1$

follows that $c_y|_{V_\lambda} \circ \theta_\lambda = c_{t^i}|_{V_\lambda} \circ \theta_\lambda = \theta_\lambda|_{\langle ss_1^\lambda, z\rangle} \circ c_{s^{-i}} = \theta_\lambda|_{\langle ss_1^\lambda, z\rangle} \circ c_{y'}$. Thus,

$$\begin{aligned}
\phi \sim \psi &\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi && \text{(for some } x, y \in S) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y \circ \theta_\lambda = \psi \circ \theta_\lambda && \text{(for some } x, y \in S) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_{t^i} \circ \theta_\lambda = \psi \circ \theta_\lambda && \text{(for some } x \in S, i = -1, 0, 1) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ (\phi \circ \theta_\lambda) \circ c_{s^{-i}} = \psi \circ \theta_\lambda && \text{(for some } x, \in S, i = -1, 0, 1) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ (\phi \circ \theta_\lambda) \circ c_{y'} = \psi \circ \theta_\lambda && \text{(for some } x, y' \in S) \\
&\Leftrightarrow \phi \circ \theta_\lambda \sim \psi \circ \theta_\lambda.
\end{aligned}$$

2. Here, when normalizing $\langle ss_1^\lambda, z\rangle$ by an element of $y \in S$, we may write $y = t^i$. Similarly, when normalizing $\langle ss_1^\mu, z\rangle$ by an element of $y' \in S$, we may write $y' = t^j$. Writing $j = -i$, it follows that $c_y|_{V_\lambda} \circ \theta_\lambda \circ \theta_\mu = c_{t^i}|_{V_\lambda} \circ \theta_\lambda \circ \theta_\mu = \theta_\lambda|_{\langle ss_1^\lambda, z\rangle} \circ \theta_\mu \circ c_{t^{-i}} = \theta_\lambda|_{\langle ss_1^\lambda, z\rangle} \circ \theta_\mu \circ c_{y'}$. Thus,

$$\begin{aligned}
\phi \sim \psi &\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi && \text{(for some } x, y \in S) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y \circ \theta_\lambda \circ \theta_\mu = \psi \circ \theta_\lambda \circ \theta_\mu && (x, y \in S) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_{t^i} \circ \theta_\lambda \circ \theta_\mu = \psi \circ \theta_\lambda \circ \theta_\mu && (x \in S, i = -1, 0, 1) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ (\phi \circ \theta_\lambda \circ \theta_\mu) \circ c_{t^{-i}} = \psi \circ \theta_\lambda \circ \theta_\mu && (x, \in S, i = -1, 0, 1) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ (\phi \circ \theta_\lambda \circ \theta_\mu) \circ c_{y'} = \psi \circ \theta_\lambda \circ \theta_\mu && (x, y' \in S) \\
&\Leftrightarrow \phi \circ \theta_\lambda \sim \psi \circ \theta_\lambda.
\end{aligned}$$

3. This case is similar to the previous ones. In this case we let $y = s^i$ and $y' = t^i$, and we

see that $c_y|_{\langle t,z\rangle} \circ \theta_0 = c_{s^i}|_{\langle t,z\rangle} \circ \theta_0 = \theta_0|_{\langle t,z\rangle} \circ c_{t^i} = \theta_0|_{\langle t,z\rangle} \circ c_{y'}$. Thus,

$$\begin{aligned}
\phi \sim \psi &\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi && \text{(for some } x,y \in S) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y \circ \theta_0 = \psi \circ \theta_0 && \text{(for some } x,y \in S) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ \phi \circ c_{s^i} \circ \theta_0 = \psi \circ \theta_0 && \text{(for some } x \in S, i = -1,0,1) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ (\phi \circ \theta_0) \circ c_{t^i} = \psi \circ \theta_0 && \text{(for some } x, \in S, i = -1,0,1) \\
&\Leftrightarrow D_\psi^x = D_\phi \text{ and } c_x|_{D_\psi} \circ (\phi \circ \theta_0) \circ c_{y'} = \psi \circ \theta_0 && \text{(for some } x,y' \in S) \\
&\Leftrightarrow \phi \circ \theta_0 \sim \psi \circ \theta_0.
\end{aligned}$$

This completes the proof. $\square$

For the next lemma recall morphisms defined in Section 6.3.

Lemma 7.2.3. *Let $\mathcal{F}$ be a DRV-fusion system. Let $\phi \in Hom_{\mathcal{F}}(P,Q)$, for some $P,Q \leq S$. Suppose that $\phi \notin \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T)\rangle$ for all $T \in \mathcal{T}$. We have following facts:*

1. *Suppose that $\{E_0, A\} \subseteq \mathcal{T}$. If P,Q are subgroups of one of $\langle s,z\rangle$ or $\langle t,z\rangle$, then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_{V_0}|_R, \theta_{V_0}^{-1}|_R, \alpha_{V_0}|_R, \beta_{V_0}|_R \mid R \in \mathcal{D}_{V_0}\} & \text{if } P,Q \leq \langle s,z\rangle \\ \{\Theta_{1,0}|_R, \Theta_{2,0}|_R, \Theta_{3,0}|_R, \Theta_{4,0}|_R \mid R \in \mathcal{D}_{V_0}\} & \text{if } P \leq \langle s,z\rangle, Q \leq \langle t,z\rangle \\ \{\Theta_{1,0}^{-1}|_R, \Theta_{2,0}^{-1}|_R, \Theta_{3,0}^{-1}|_R, \Theta_{4,0}^{-1}|_R \mid R \in \mathcal{D}_{A_0}\} & \text{if } P \leq \langle t,z\rangle, Q \leq \langle s,z\rangle \end{cases}$$

2. *Suppose that $\{E_{\pm 1}, A\} \subseteq \mathcal{T}$. If P,Q are subgroups of one of $\langle ss_1^\lambda, z\rangle$ or $\langle t,z\rangle$, for some $\lambda = -1,1$, then up to S-S-equivalence*

$$\phi \in \begin{cases} \{\theta_{V_\lambda}|_R, \theta_{V_\lambda}^{-1}|_R, \alpha_{V_\lambda}|_R, \beta_{V_\lambda}|_R \mid R \in \mathcal{D}_{V_\lambda}\} & \text{if } P,Q \leq \langle ss_1^\lambda, z\rangle \\ \{\Theta_{1,\lambda}|_R, \Theta_{2,\lambda}|_R, \Theta_{3,\lambda}|_R, \Theta_{4,\lambda}|_R \mid R \in \mathcal{D}_{V_\lambda}\} & \text{if } P \leq \langle ss_1^\lambda, z\rangle, Q \leq \langle t,z\rangle \\ \{\Theta_{1,\lambda}^{-1}|_R, \Theta_{2,\lambda}^{-1}|_R, \Theta_{3,\lambda}^{-1}|_R, \Theta_{4,\lambda}^{-1}|_R \mid R \in \mathcal{D}_{A_0}\} & \text{if } P \leq \langle t,z\rangle, Q \leq \langle ss_1^\lambda, z\rangle \\ \{\vartheta_{1,\lambda}|_R, \vartheta_{2,\lambda}|_R, \vartheta_{3,\lambda}|_R, \vartheta_{4,\lambda}|_R \mid R \in \mathcal{D}_{V_\lambda}\} & \text{if } P \leq \langle ss_1^\lambda, z\rangle, Q \leq \langle ss_1^{-\lambda}, z\rangle. \end{cases}$$

3. *Suppose that $\mathcal{T} = \{E_0, E_{\pm 1}\}$.*[1] *If P,Q are subgroups of one of $\langle ss_1^{\lambda}, z\rangle$ or $\langle t, z\rangle$, for some $\lambda = -1,0,1$, then up to S-S-equivalence, $\lambda = -1,1$ and*

$$\phi \in \begin{cases} \{\xi_{1,\lambda}|_R, \xi_{2,\lambda}|_R, \xi_{3,\lambda}|_R, \xi_{4,\lambda}|_R \mid R = \langle ss_1^{\lambda}\rangle, \langle ss_1^{\lambda}, z\rangle\} & \text{if } P \leq \langle ss_1^{\lambda}, z\rangle, Q \leq \langle s, z\rangle \\ \{\xi_{1,\lambda}^{-1}|_R, \xi_{2,\lambda}^{-1}|_R, \xi_{3,\lambda}^{-1}|_R, \xi_{4,\lambda}^{-1}|_R \mid R = \langle s\rangle, \langle s, z\rangle\} & \text{if } P \leq \langle s, z\rangle, Q \leq \langle ss_1^{\lambda}, z\rangle. \end{cases}$$

Proof. We prove cases 1 and 2 together. By Lemma 7.1.6, first note that, since $A \in \mathcal{T}$ and $\mathrm{Aut}_{\mathcal{F}}(A) \cong \mathrm{GL}_2(3)$, we have $\mathrm{Aut}_{\mathcal{F}}(A) \cong \mathrm{Aut}_{\mathcal{F}}(\langle t,z\rangle) \cong \mathrm{Aut}(\langle t,z\rangle) \cong \mathrm{GL}_2(3)$. Thus, if $P,Q \leq \langle t,z\rangle$, then $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(A)\rangle$, a contradiction. By Lemma 7.1.7, if $E_0, A \in \mathcal{T}$, then $\mathrm{Aut}_{\mathcal{F}}(V_0) \cong \mathrm{GL}_2(3)$, and, if $E_{\pm 1}, A \in \mathcal{T}$, then $\mathrm{Aut}_{\mathcal{F}}(V_{\pm 1}) \cong \mathrm{GL}_2(3)$. Thus, if $P,Q \leq \langle s,z\rangle$, then ϕ is a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(V_0)$. Similarly, if $P,Q \leq \langle ss_1^{\lambda}, z\rangle$, for $\lambda = -1,1$, then ϕ is a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(V_{\lambda})$. By Lemma 7.1.13, if $E_0, A \in \mathcal{T}$ and $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(V_0)\rangle$, or, if $E_{\pm 1}, A \in \mathcal{T}$ and $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(V_{\lambda})\rangle$, for $\lambda = -1,1$, then up to S-S-equivalence

$$\phi \in \begin{cases} \{\theta_{V_0}|_R, \theta_{V_0}^{-1}|_R, \alpha_{V_0}|_R, \beta_{V_0}|_R \mid R \in \mathcal{D}_{V_0}\} & \text{if } E_0, A \in \mathcal{T} \\ \{\theta_{V_\lambda}|_R, \theta_{V_\lambda}^{-1}|_R, \alpha_{V_\lambda}|_R, \beta_{V_\lambda}|_R \mid R \in \mathcal{D}_{V_\lambda}, \lambda = -1,1\} & \text{if } E_{\pm 1}, A \in \mathcal{T} \end{cases}$$

where we have excluded $\hat{\theta}_{V_\lambda}$ and $\hat{\beta}_{V_\lambda}$ because they lie in $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$ for $\lambda = -1,1$, by Lemma 7.1.15.

Now, if $P \leq \langle ss_1^{\lambda}, z\rangle$ and $Q \leq \langle t,z\rangle$ for some $\lambda = -1,0,1$, then $\phi = \varphi \circ \theta_\lambda$ for some $\varphi \in \langle \mathrm{Aut}_{\mathcal{F}}(V_\lambda)\rangle$. Applying Lemma 7.2.2 (1) and by recalling the definitions we have the required morphisms:

$$\phi \in \begin{cases} \{\Theta_{1,0}|_R, \Theta_{2,0}|_R, \Theta_{3,0}|_R, \Theta_{4,0}|_R \mid R \in \mathcal{D}_{V_0}\} & \text{if } E_0, A \in \mathcal{T} \\ \{\Theta_{1,\lambda}|_R, \Theta_{2,\lambda}|_R, \Theta_{3,\lambda}|_R, \Theta_{4,\lambda}|_R \mid R \in \mathcal{D}_{V_\lambda}, \lambda = -1,1\} & \text{if } E_{\pm 1}, A \in \mathcal{T}. \end{cases}$$

If $P \leq \langle t,z\rangle$ and $Q \leq \langle ss_1^{\lambda}, z\rangle$ for some $\lambda = -1,0,1$, then an application of Lemma 3.1.6 gives

[1] This is the DRV-11 fusion system.

us the required result:

$$\phi \in \begin{cases} \{\Theta_{1,0}^{-1}|_R, \Theta_{2,0}^{-1}|_R, \Theta_{3,0}^{-1}|_R, \Theta_{4,0}^{-1}|_R \mid R \in \mathcal{D}_{V_0}\} & \text{if } E_0, A \in \mathcal{T} \\ \{\Theta_{1,\lambda}^{-1}|_R, \Theta_{2,\lambda}^{-1}|_R, \Theta_{3,\lambda}^{-1}|_R, \Theta_{4,\lambda}^{-1}|_R \mid R \in \mathcal{D}_{V_\lambda}, \lambda = -1, 1\} & \text{if } E_{\pm 1}, A \in \mathcal{T}. \end{cases}$$

If $P \leq \langle ss_1^{\lambda}, z\rangle$ and $Q \leq \langle ss_1^{-\lambda}, z\rangle$ for some $\lambda = -1, 1$, then, $\phi = \varphi \circ \eta$ for some $\varphi \in \mathrm{Aut}_{\mathcal{F}}(V_\lambda)$. Once again, an application of Lemma 3.1.6 gives the required morphisms:

$$\phi \in \{\vartheta_{1,\lambda}|_R, \vartheta_{2,\lambda}|_R, \vartheta_{3,\lambda}|_R, \vartheta_{4,\lambda}|_R \mid R \in \mathcal{D}_{V_\lambda}, \lambda = -1, 1\} \quad \text{if } E_{\pm 1}, A \in \mathcal{T}.$$

Now, we prove case 3. By Lemma 7.1.4, we note that

$$\{\varphi|_R \mid \varphi \in \mathrm{Aut}(E_\lambda), P \lneq E_\lambda, \lambda = -1, 1\} \subseteq \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_{\pm 1})\rangle.$$

We also note that $\mathrm{Aut}_{\mathcal{F}}(E_0) \cong \mathrm{GL}_2(3)$. Thus, for $\lambda = -1, 0, 1$, under the identification $\mathrm{Aut}_{\mathcal{F}}(\langle ss_1^{\lambda}, z\rangle) \to \mathrm{GL}_2(3)$, we have

$$\theta_\lambda|_{\langle ss_1^{\lambda}, z\rangle} \circ c_s \circ \theta_\lambda = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \alpha_\lambda|_{\langle ss_1^{\lambda}, z\rangle} \circ c_s \circ \theta_\lambda = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } c_{t^{-1}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly, under the identification $\mathrm{Aut}_{\mathcal{F}}(\langle t, z\rangle) \to \mathrm{GL}_2(3)$, we have

$$\theta_0|_{\langle t, z\rangle} \circ c_t \circ \theta_0 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \alpha_0|_{\langle t, z\rangle} \circ c_{t^{-1}} \circ \theta_0^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } c_s|_{\langle t, z\rangle} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle \cong D_{12}$, a dihedral group of order 12. Let $R \in \{\langle ss_1^{\lambda}, z\rangle, \langle t, z\rangle \mid \lambda = -1, 0, 1\}$. If $\phi \in \mathrm{Aut}_{\mathcal{F}}(R)$, then $z\phi \in \langle z\rangle$. Thus, $D_{12} \leq \mathrm{Aut}_{\mathcal{F}}(R) < \mathrm{GL}_2(3)$. This implies $\mathrm{Aut}_{\mathcal{F}}(R) \cong D_{12}$. We use this to make a number of conclusions.

Suppose $\lambda = -1, 0, 1$. Then, if $P, Q \leq \langle ss_1^{\lambda}, z\rangle$, then $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$, a contradiction. If $P, Q \leq \langle t, z\rangle$, then $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_0)\rangle$, a contradiction. If $P \leq \langle ss_1^{\lambda}, z\rangle$ and $Q \leq \langle t, z\rangle$, then $\phi = \varphi \circ \theta_\lambda$ for some $\varphi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$, that is, $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$, a contradiction. The case is similar if $P \leq \langle t, z\rangle$ and $Q \leq \langle ss_1^\lambda, z\rangle$.

Suppose $\lambda = -1, 1$. If $P \leq \langle ss_1^\lambda, z\rangle$ and $Q \leq \langle ss_1^{-\lambda}, z\rangle$, then $\phi = \varphi \circ \eta$ for some $\varphi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$, that is, $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$, a contradiction. If $P \leq \langle ss_1^\lambda, z\rangle$ and $Q \leq \langle s, z\rangle$, then $\phi = \varphi \circ \theta_0$ for some $\varphi \in \langle \mathrm{Hom}_{\mathcal{F}}(\langle ss_1^\lambda, z\rangle, \langle t, z\rangle)\rangle$. It has already been discussed that $\langle \mathrm{Hom}_{\mathcal{F}}(\langle ss_1^\lambda, z\rangle, \langle t, z\rangle)\rangle \subseteq \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$. By Lemma 7.1.15, the morphisms, up to S-S-equivalence, in $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E_\lambda)\rangle$ that could possibly be S-S-equivalent to morphisms in $\langle \mathrm{Hom}_{\mathcal{F}}(\langle ss_1^\lambda, z\rangle, \langle t, z\rangle)\rangle$, and which are not restrictions of elements of $\mathrm{Aut}_{\mathcal{F}}(S)$, are as follows:

$$\begin{aligned}
&\{\theta_\lambda|_R, \theta_\lambda^{-1}|_R, (\theta_\lambda \circ \eta)|_R, (\theta_\lambda^{-1} \circ \eta)|_R \mid R = \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda\rangle, \lambda = -1, 1\} \\
=&\{\theta_\lambda|_R, \theta_\lambda^{-1}|_R, \alpha_\lambda|_R, \beta_\lambda|_R \mid R = \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda\rangle\ \lambda = -1, 1\}.
\end{aligned}$$

Now, by definition we see that

$$\theta_\lambda|_{\langle ss_1^\lambda, z\rangle} \circ \theta_0 = \xi_{4,\lambda}, \quad \theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle} \circ \theta_0 = \xi_{3,\lambda}$$
$$\alpha_\lambda \circ \theta_0 = \xi_{2,\lambda}, \quad \beta_\lambda \circ \theta_0 = \xi_{1,\lambda}.$$

Applying Lemma 7.2.2 (3) gives us the required morphisms in this case, namely

$$\{\xi_{1,\lambda}|_R, \xi_{2,\lambda}|_R, \xi_{3,\lambda}|_R, \xi_{4,\lambda}|_R \mid R = \langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle, \lambda = -1, 1\}.$$

If $P \leq \langle s, z\rangle$ and $Q \leq \langle ss_1^\lambda, z\rangle$, for some $\lambda = -1, 1$, then an application of Lemma 3.1.6 gives us the required morphisms, namely

$$\{\xi_{1,\lambda}^{-1}|_R, \xi_{2,\lambda}^{-1}|_R, \xi_{3,\lambda}^{-1}|_R, \xi_{4,\lambda}^{-1}|_R \mid R = \langle s\rangle, \langle s, z\rangle, \lambda = -1, 1\}.$$

This completes the proof. $\square$

Lemma 7.2.4. *Let $\phi \in Hom_{\mathcal{F}}(P, Q)$ be not a restriction of $Aut_{\mathcal{F}}(S)$. Suppose that for every $T \in \mathcal{T}$, $\phi \notin \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(T)\rangle$. Then, up to S-S-equivalence, ϕ is one of the following listed* *in the table below:*

$\mathcal{T}$	*Maps up to* S-S*-conjugacy*
E_0, A (DRV-10)	$\{\theta_{V_0}\|_Q, \theta_{V_0}^{-1}, \alpha_{V_0}\|_Q, \beta_{V_0}\|_Q \mid Q \in \mathcal{D}_{V_0}\}$ $\{\Theta_{1,0}\|_Q, \Theta_{2,0}\|_Q, \Theta_{3,0}\|_Q, \Theta_{4,0}\|_Q \mid Q \in \mathcal{D}_{V_0}\}$ $\{\Theta_{1,0}^{-1}\|_Q, \Theta_{2,0}^{-1}\|_Q, \Theta_{3,0}^{-1}\|_Q, \Theta_{4,0}^{-1}\|_Q \mid Q \in \mathcal{D}_{V_0}\}$
V_0, A (DRV-14)	$\{\theta_0\|_Q, \theta_0\|_Q^{-1} \mid Q = \langle s\rangle, \langle t\rangle\}$
$E_{\pm 1}, A$ (DRV-8)	$\{\theta_{V_\lambda}\|_Q, \theta_{V_\lambda}^{-1}\|_Q, \alpha_{V_\lambda}\|_Q, \beta_{V_\lambda}\|_Q \mid Q \in \lambda \in \{-1,1\}, \mathcal{D}_{V_\lambda}\}$ $\{\vartheta_{1,\lambda}\|_Q, \vartheta_{2,\lambda}\|_Q, \vartheta_{3,\lambda}\|_Q, \vartheta_{4,\lambda}\|_Q \mid \lambda \in \{-1,1\}, Q \in \mathcal{D}_{V_\lambda}\}$ $\{\Theta_{1,\lambda}\|_Q, \Theta_{2,\lambda}\|_Q, \Theta_{3,\lambda}\|_Q, \Theta_{4,\lambda}\|_Q \mid \lambda \in \{-1,1\}, Q \in \mathcal{D}_{V_\lambda}\}$ $\{\Theta_{1,\lambda}^{-1}\|_Q, \Theta_{2,\lambda}^{-1}\|_Q, \Theta_{3,\lambda}^{-1}\|_Q, \Theta_{4,\lambda}^{-1}\|_Q \mid \lambda \in \{-1,1\}, Q \in \mathcal{D}_{V_\lambda}\}$
$E_0, E_{\pm 1}$ (DRV-11)	$\{\xi_{1,\lambda}\|_Q, \xi_{2,\lambda}\|_Q, \xi_{3,\lambda}\|_Q, \xi_{4,\lambda}\|_Q \mid \lambda = -1,1, Q \in \mathcal{D}_{V_\lambda}\}$ $\{\xi_{1,\lambda}\|_Q, \xi_{1,\lambda}^{-1}\|_Q, \xi_{2,\lambda}^{-1}\|_Q, \xi_{3,\lambda}^{-1}\|_Q, \xi_{4,\lambda}^{-1}\|_Q \mid \lambda = -1,1, Q \in \mathcal{D}_{V_0}\}$
$V_0, E_{\pm 1}$ (DRV-15)	*None*
$V_0, E_{\pm 1}, A$ (DRV-16)	*All the maps in the cases* $\mathcal{T} = \{V_0, A\}$, $\mathcal{T} = \{E_{\pm 1}, A\}$, *and as well as the following:* $\{\xi_{1,\lambda}\|_{\langle ss_1^\lambda\rangle}, \xi_{2,\lambda}\|_{\langle ss_1^\lambda\rangle}, \xi_{1,\lambda}^{-1}\|_{\langle s\rangle}, \xi_{2,\lambda}^{-1}\|_{\langle s\rangle} \mid \lambda = -1,1\}$
$E_0, E_{\pm 1}, A$ (DRV-12)	*All the maps in the cases* $\mathcal{T} = \{E_0, A\}$ *and* $\mathcal{T} = \{E_{\pm 1}, A\}$ *as well as the following:* $\{\xi_{1,\lambda}\|_Q, \xi_{2,\lambda}\|_Q, \xi_{3,\lambda}\|_Q, \xi_{4,\lambda}, \xi_{5,\lambda}\|_Q, \xi_{6,\lambda}\|_Q, \xi_{7,\lambda}\|_Q, \xi_{8,\lambda}\|_Q \mid \lambda = -1,1, Q \in \mathcal{D}_{V_\lambda}\}$ $\{\xi_{1,\lambda}^{-1}\|_Q, \xi_{2,\lambda}^{-1}\|_Q, \xi_{3,\lambda}^{-1}\|_Q, \xi_{4,\lambda}^{-1}\|_Q, \xi_{5,\lambda}^{-1}\|_Q, \xi_{6,\lambda}^{-1}\|_Q, \xi_{7,\lambda}^{-1}\|_Q, \xi_{8,\lambda}^{-1}\|_Q \mid \lambda = -1,1, Q \in \mathcal{D}_{V_0}\}$

Proof. By 7.2.1 we necessarily have $P \leq (T_1 \cap T_2)\psi$ for some $\psi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T_1)\rangle \cup \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T_2)\rangle$, $T_1 \neq T_2$ and $T_1, T_2 \in \mathcal{T}$. In particular, this means $|\mathcal{T}| > 1$. We may assume that $Q = P\phi$, otherwise ϕ is just an isomorphism, followed by an inclusion map. P and Q can also be chosen up to S-conjugacy, since ϕ is being defined up to S-S-equivalence. We prove the Lemma case by case. Note that, by Lemma 7.1.6 and by Table 7.1, whenever $A \in \mathcal{T}$, then $\mathrm{Aut}_{\mathcal{F}}(A) \cong \mathrm{Aut}_{\mathcal{F}}(\langle t, z\rangle) \cong \mathrm{Aut}(\langle t, z\rangle) \cong \mathrm{GL}_2(3)$, and therefore whenever $P, Q \leq$

$\langle t,z\rangle$, then $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(A)\rangle$.

Case 1: $\mathcal{T} = \{E_0, A\}$.

In this case $E_0 \cap A = \langle t,z\rangle$. Then P,Q are subgroups of one of $\langle s,z\rangle$ or $\langle t,z\rangle$. Applying Lemma 7.2.3 gives us the required morphisms.

Case 2: $\mathcal{T} = \{V_0, A\}$.

In this case $V_0 \cap A = \langle z\rangle$. Then up to S-conjugacy $P,Q \in \{\langle s\rangle, \langle t\rangle, \langle z\rangle\}$. Since $\phi \notin \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(V_0)\rangle \cup \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(A)\rangle$, we have the following possibilities:

Morphisms		Equal to
$\langle s\rangle \to \langle t\rangle$	$s \mapsto t$	$\theta_{V_0}\vert_{\langle s\rangle} \circ \theta_A^{-1} = \theta_0\vert_{\langle s\rangle}$
	$s \mapsto t^{-1}$	$\theta_{V_0}\vert_{\langle s\rangle} \circ \theta_A = \theta_0^{-1}\vert_{\langle s\rangle}$
$\langle t\rangle \to \langle s\rangle$	$t \mapsto s$	$\theta_A\vert_{\langle t\rangle} \circ \theta_{V_0}^{-1} = \theta_0^{-1}\vert_{\langle t\rangle}$
	$t \mapsto s^{-1}$	$\theta_A^{-1}\vert_{\langle t\rangle} \circ \theta_{V_0}^{-1} = \theta_0\vert_{\langle t\rangle}$

Case 3: $\mathcal{T} = \{E_{\pm 1}, A\}$.

In this case $A \cap E_{\pm 1} = E_1 \cap E_{-1} = \langle t,z\rangle$. Then P,Q are subgroups of one $\langle ss_1^{\lambda}, z\rangle$ or $\langle t,z\rangle$, for some $\lambda = -1,1$. Applying Lemma 7.2.3 gives us the required morphisms.

Case 4: $\mathcal{T} = \{E_0, E_{\pm 1}\}$.

In this case $E_0 \cap E_{\pm 1} = E_1 \cap E_{-1} = \langle t,z\rangle$. Then P,Q are subgroups of one of $\langle ss_1^{\lambda}, z\rangle$ or $\langle t,z\rangle$ for some $\lambda = -1,0,1$. Applying Lemma 7.2.3 gives us the required morphisms.

Case 5: $\mathcal{T} = \{V_0, E_{\pm 1}\}$.

In this case $V_0 \cap E_{\pm 1} = \langle z\rangle$. Then $P,Q \in \{\langle s\rangle, \langle z\rangle\}$. In this case $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(V_0)\rangle = \langle \mathrm{Aut}(V_0)\rangle$, a contradiction. So there are no morphisms in this case.

Case 6: $\mathcal{T} = \{V_0, E_{\pm 1}, A\}$.

In this case $A \cap E_{\pm 1} = E_1 \cap E_{-1} = \langle t,z\rangle$ and $V_0 \cap A = V_0 \cap E_{\pm 1} = \langle z\rangle$. Then P,Q are either subgroups of one of $\langle ss_1^{\lambda}, z\rangle$ or $\langle t,z\rangle$ for some $\lambda = -1,1$, or $\{P,Q\} \in \{\{\langle s\rangle, \langle ss_1^{\lambda}\rangle\}, \{\langle s\rangle, \langle t\rangle\}, \{\langle s\rangle, \langle z\rangle\} \mid$

$\lambda = -1, 1\}$. In the first case, with the application of Lemma 7.2.3, we get the same morphisms as in Case 3. It is left to consider the latter case. If $\{P, Q\} = \{\langle s\rangle, \langle z\rangle\}$, then $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(V_0)\rangle$, a contradiction. If $\{P, Q\} = \{\langle s\rangle, \langle t\rangle\}$, then $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(V_0), \mathrm{Aut}_{\mathcal{F}}(A)\rangle$, which gives us the relevant morphisms from Case 2. If $\{P, Q\} = \{\langle s\rangle, \langle ss_1^\lambda\rangle\}$ for some $\lambda = -1, 1$, then we get the following possibilities:

Morphisms		Equal to
$\langle s\rangle \to \langle ss_1^\lambda\rangle$	$s \mapsto ss_1^\lambda$	$\theta_{V_0}\|_{\langle s\rangle} \circ \theta_A^{-1} \circ \theta_\lambda^{-1} = \xi_{1,\lambda}\|_{\langle s\rangle}^{-1}$
	$s \mapsto (ss_1^\lambda)^{-1}$	$\theta_{V_0}^{-1}\|_{\langle s\rangle} \circ \theta_A^{-1} \circ \theta_\lambda^{-1} = \xi_{2,\lambda}\|_{\langle s\rangle}^{-1}$
$\langle ss_1^\lambda\rangle \to \langle s\rangle$	$ss_1^\lambda \mapsto s$	$\theta_\lambda\|_{\langle ss_1^\lambda\rangle} \circ \theta_A \circ \theta_{V_0}^{-1} = \xi_{1,\lambda}\|_{\langle ss_1^\lambda\rangle}$
	$ss_1^\lambda \mapsto s^{-1}$	$\theta_\lambda\|_{\langle ss_1^\lambda\rangle} \circ \theta_A \circ \theta_{V_0} = \xi_{2,\lambda}\|_{\langle ss_1^\lambda\rangle}$

Case 7: $\mathcal{T} = \{E_0, E_{\pm 1}, A\}$.

In this case $E_0 \cap E_{\pm 1} = E_0 \cap A = E_{\pm 1} \cap A = \langle t, z\rangle$. Then P, Q are subgroups of one of $\langle ss_1^\lambda, z\rangle$ or $\langle t, z\rangle$ for $\lambda = -1, 0, 1$. If P, Q are subgroups of $\langle s, z\rangle$ or $\langle t, z\rangle$, then we apply Lemma 7.2.3 to give us the morphisms as in Case 1. If P, Q are subgroups of one of $\langle ss_1^\lambda, z\rangle$ or $\langle t, z\rangle$ for $\lambda = -1, 1$, then again we apply Lemma 7.2.3 to give us the morphisms as in Case 3. It is left to consider the case where, for $\lambda = -1, 1$, either $P \leq \langle ss_1^\lambda, z\rangle$ and $Q \leq \langle s, z\rangle$, or, $P \leq \langle s, z\rangle$ and $Q \leq \langle ss_1^\lambda, z\rangle$. In the first case, $\phi = \varphi \circ \theta_\lambda \circ \theta_0$ for some $\varphi \in \langle \mathrm{Aut}_{\mathcal{F}}(V_\lambda)\rangle$. Applying Lemma 7.2.2 (2) gives us the required morphisms. For the second case, it suffices to apply Lemma 3.1.6. $\square$

7.3 General construction of Ω

In this section, we collect the results derived from previous sections, and give a general construction of a right characteristic biset Ω, for each DRV-fusion system. We first make a number of definitions. For each $\lambda = -1, 0, 1$, define

$$\begin{aligned}\Pi_\lambda = & \bigsqcup_{Q \in \mathcal{D}_{V_\lambda}} C(\theta_{V_\lambda}|_Q)\mathcal{O}_{\theta_{V_\lambda}|_Q} \sqcup C(\theta_{V_\lambda}^{-1}|_Q)\mathcal{O}_{\theta_{V_\lambda}^{-1}|_Q} \sqcup C(\alpha_{V_\lambda})\mathcal{O}_{\alpha_{V_\lambda}|_Q} \sqcup C(\beta_{V_\lambda})\mathcal{O}_{\beta_{V_\lambda}|_Q} \\ & \sqcup \bigsqcup_{Q \in \mathcal{D}_{V_\lambda}} C(\Theta_{1,\lambda})\mathcal{O}_{\Theta_{1,\lambda}|_Q} \sqcup C(\Theta_{2,\lambda})\mathcal{O}_{\Theta_{2,\lambda}|_Q} \sqcup C(\Theta_{3,\lambda})\mathcal{O}_{\Theta_{3,\lambda}|_Q} \sqcup C(\Theta_{4,\lambda})\mathcal{O}_{\Theta_{4,\lambda}|_Q} \\ & \sqcup \bigsqcup_{Q \in \mathcal{D}_{A_0}} C(\Theta_{1,\lambda}^{-1})\mathcal{O}_{\Theta_{1,\lambda}^{-1}|_Q} \sqcup C(\Theta_{2,\lambda}^{-1})\mathcal{O}_{\Theta_{2,\lambda}^{-1}|_Q} \sqcup C(\Theta_{3,\lambda}^{-1})\mathcal{O}_{\Theta_{3,\lambda}^{-1}|_Q} \sqcup C(\Theta_{4,\lambda}^{-1})\mathcal{O}_{\Theta_{4,\lambda}^{-1}|_Q}\end{aligned}$$

where $C(\star) \geq 0$ are some undetermined integer constants. Let $\mathcal{F}$ be a DRV-fusion system. Recall that for each such fusion system, there is a corresponding $\mathcal{T}$, the set of $\mathcal{F}$-essential subgroups of S which generate the fusion system. We define $\Omega_\mathcal{T} = \emptyset$, if $|\mathcal{T}| = 1$ or, if $\mathrm{Out}_\mathcal{F}(S) \not\cong \langle \bar{\omega}, \bar{\eta} \rangle$. Otherwise, we make the following definitions for each of the remaining DRV-fusion systems, where $C(\star) \geq 0$ are again some undetermined integer constants:

<u>DRV-8</u>: $\mathcal{T} = \{E_{\pm 1}, A\}$, then

$$\begin{aligned}\Omega_\mathcal{T} = & \Pi_1 \sqcup \Pi_{-1} \\ & \sqcup \bigsqcup_{Q \in \mathcal{D}_{V_\lambda}} C(\vartheta_{1,\lambda}|_Q)\mathcal{O}_{\vartheta_{1,\lambda}|_Q} \sqcup C(\vartheta_{2,\lambda}|_Q)\mathcal{O}_{\vartheta_{2,\lambda}|_Q} \sqcup C(\vartheta_{3,\lambda}|_Q)\mathcal{O}_{\vartheta_{3,\lambda}|_Q} \sqcup C(\vartheta_{4,\lambda}|_Q)\mathcal{O}_{\vartheta_{4,\lambda}|_Q}.\end{aligned}$$

<u>DRV-10</u>: $\mathcal{T} = \{E_0, A\}$, then $\Omega_\mathcal{T} = \Pi_0$.

<u>DRV-11</u>: $\mathcal{T} = \{E_0, E_{\pm 1}\}$, then

$$\begin{aligned}\Omega_\mathcal{T} = & \bigsqcup_{\substack{\lambda=-1,1 \\ Q \in \mathcal{D}_{V_\lambda}}} C(\xi_{1,\lambda}|_Q)\mathcal{O}_{\xi_{1,\lambda}|_Q} \sqcup C(\xi_{2,\lambda}|_Q)\mathcal{O}_{\xi_{2,\lambda}|_Q} \sqcup C(\xi_{3,\lambda}|_Q)\mathcal{O}_{\xi_{3,\lambda}|_Q} \sqcup C(\xi_{4,\lambda}|_Q)\mathcal{O}_{\xi_{4,\lambda}|_Q} \\ & \sqcup \bigsqcup_{\substack{\lambda=-1,1 \\ Q \in \mathcal{D}_{V_0}}} C(\xi_{1,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{1,\lambda}^{-1}|_Q} \sqcup C(\xi_{2,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{2,\lambda}^{-1}|_Q} \sqcup C(\xi_{3,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{3,\lambda}^{-1}|_Q} \sqcup C(\xi_{4,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{4,\lambda}^{-1}|_Q}.\end{aligned}$$

DRV-12: $\mathcal{T} = \{E_0, E_{\pm 1}, A\}$, then

$$\begin{aligned}
\Omega_{\mathcal{T}} = {} & \Omega_{E_0,A} \sqcup \Omega_{E_{\pm 1},A} \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1 \\ Q \in \mathcal{D}_{V_\lambda}}} C(\xi_{1,\lambda}|_Q)\mathcal{O}_{\xi_{1,\lambda}|_Q} \sqcup C(\xi_{2,\lambda}|_Q)\mathcal{O}_{\xi_{2,\lambda}|_Q} \sqcup C(\xi_{3,\lambda}|_Q)\mathcal{O}_{\xi_{3,\lambda}|_Q} \sqcup C(\xi_{4,\lambda}|_Q)\mathcal{O}_{\xi_{4,\lambda}|_Q} \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1 \\ Q \in \mathcal{D}_{V_\lambda}}} C(\xi_{5,\lambda}|_Q)\mathcal{O}_{\xi_{5,\lambda}|_Q} \sqcup C(\xi_{6,\lambda}|_Q)\mathcal{O}_{\xi_{6,\lambda}|_Q} \sqcup C(\xi_{7,\lambda}|_Q)\mathcal{O}_{\xi_{7,\lambda}|_Q} \sqcup C(\xi_{8,\lambda}|_Q)\mathcal{O}_{\xi_{8,\lambda}|_Q} \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1 \\ Q \in \mathcal{D}_{V_0}}} C(\xi_{1,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{1,\lambda}^{-1}|_Q} \sqcup C(\xi_{2,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{2,\lambda}^{-1}|_Q} \sqcup C(\xi_{3,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{3,\lambda}^{-1}|_Q} \sqcup C(\xi_{4,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{4,\lambda}^{-1}|_Q} \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1 \\ Q \in \mathcal{D}_{V_0}}} C(\xi_{5,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{5,\lambda}^{-1}|_Q} \sqcup C(\xi_{6,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{6,\lambda}^{-1}|_Q} \sqcup C(\xi_{7,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{7,\lambda}^{-1}|_Q} \sqcup C(\xi_{8,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{8,\lambda}^{-1}|_Q}.
\end{aligned}$$

DRV-14: $\mathcal{T} = \{V_0, A\}$, then

$$\Omega_{\mathcal{T}} = \bigsqcup_{Q = \langle s \rangle, \langle t \rangle} C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0^{-1}|_Q}.$$

DRV-15: $\mathcal{T} = \{V_0, E_{\pm 1}\}$, then $\Omega_{\mathcal{T}} = \emptyset$.

DRV-16: $\mathcal{T} = \{V_0, E_{\pm 1}, A\}$, then

$$\Omega_{\mathcal{T}} = \Omega_{V_0,A} \sqcup \Omega_{E_{\pm 1},A} \sqcup \left[\bigsqcup_{\mu=1,2} C(\xi_{\mu,\lambda}|_{\langle s s_1^\lambda \rangle})\mathcal{O}_{\xi_{\mu,\lambda}|_{\langle s s_1^\lambda \rangle}} \sqcup C(\xi_{\mu,\lambda}^{-1}|_{\langle s \rangle})\mathcal{O}_{\xi_{\mu,\lambda}^{-1}|_{\langle s \rangle}} \right].$$

Finally, if $E_{\pm 1} \notin \mathcal{T}$ or $\eta \notin \mathrm{Aut}_{\mathcal{F}}(S)$, we define $\Omega_{E_{\pm 1},\eta} = \emptyset$. Otherwise, if $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $E_{\pm 1} \in \mathcal{T}$, then we define

$$\begin{aligned}
\Omega_{E_{\pm 1},\eta} = {} & \bigsqcup_{\substack{Q \in \mathcal{D}_\lambda \\ \lambda=-1,1}} C(\theta_\lambda \circ \eta|_Q)\mathcal{O}_{\theta_\lambda \circ \eta|_Q} \sqcup C(\theta_\lambda^{-1} \circ \eta|_Q)\mathcal{O}_{\theta_\lambda^{-1} \circ \eta|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\vartheta_{4,\lambda}}.
\end{aligned}$$

Theorem 7.3.1. *Let $\mathcal{F}$ be a DRV-fusion system. Let Ω be a right characteristic biset for $\mathcal{F}$.*

Then

$$\Omega \cong \bigsqcup_{Q\in\mathcal{D}} \left[C(Id|_Q)\mathcal{O}_{Id|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|Q}\right]$$
$$\sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_T\\ T\in\mathcal{T}}} \left[C(\theta_T|_Q)\mathcal{O}_{\theta_T|_Q} \sqcup C(\theta_T^{-1}|_Q)\mathcal{O}_{\theta_T^{-1}|_Q} \sqcup C(\alpha_T|_Q)\mathcal{O}_{\alpha_T|_Q} \sqcup C(\beta_T|_Q)\mathcal{O}_{\beta_T|_Q}\right]$$
$$\sqcup\Omega_{E_{\pm1},\eta} \sqcup \Omega_{\mathcal{T}}.$$

under the conditions that

1. $C(\star) \geq 0$ *are integers,*

2. (a) *If* $Out_{\mathcal{F}}(S) \cong \langle\overline{\omega}\rangle$*, then* $C(\eta|_Q) = C(\omega\circ\eta|_Q) = 0$ *for all* $Q \in \mathcal{D}$*, and* $C(Id) = C(\omega) > 0$*.*

   (b) *If* $Out_{\mathcal{F}}(S) \cong \langle\overline{\eta}\rangle$*, then* $C(\omega|_Q) = C(\omega\circ\eta|_Q) = 0$ *for all* $Q \in \mathcal{D}$*, and* $C(Id) = C(\eta) > 0$*.*

   (c) *If* $Out_{\mathcal{F}}(S) \cong \langle\overline{\omega\circ\eta}\rangle$*, then* $C(\omega|_Q) = C(\eta|_Q) = 0$ *for all* $Q \in \mathcal{D}$*, and* $C(Id) = C(\omega\circ\eta) > 0$*.*

   (d) *If* $Out_{\mathcal{F}}(S) \cong \langle\overline{\omega},\overline{\eta}\rangle$*, then* $C(Id) = C(\omega) = C(\eta) = C(\omega\circ\eta) > 0$*.*

3. *If* $Out_{\mathcal{F}}(T) \cong SL_2(3)$*, for some* $T \in \mathcal{T}$*, then* $C(\alpha_T|_Q) = C(\beta_T|_Q) = 0$ *for all* $Q \in \mathcal{D}_T$*.*

Proof. By Lemma 4.2.7, Ω is a disjoint union of orbits of the form $\mathcal{O}_\phi$ for some morphisms $\phi \in \mathcal{F}$. By Lemma 4.2.7 and Lemma 3.1.8, we may assume that every orbit $\mathcal{O}_\phi$ in Ω is induced by an S-S-equivalence class representative $\phi \in \mathcal{F}$. Write $Q = D_\phi$. If ϕ is a restriction of an element in $\mathrm{Aut}_{\mathcal{F}}(S)$, then Lemma 7.0.2 implies that we can choose Q to be an S-conjugacy class representative and

$$\phi \in \begin{cases} \mathrm{Id}|_P, \omega|_P & \text{if } \mathcal{F} = \text{DRV-1,2,3} \\ \mathrm{Id}|_P, \eta|_P & \text{if } \mathcal{F} = \text{DRV-4} \\ \mathrm{Id}|_P, (\omega\circ\eta)|_P & \text{if } \mathcal{F} = \text{DRV-5} \\ \mathrm{Id}|_Q, \omega|_Q, \eta|_Q, \omega|_Q\circ\eta & \text{otherwise} \end{cases}$$

If on that other hand, ϕ is not a restriction of $\mathrm{Aut}_{\mathcal{F}}(S)$, and $\phi \in \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T)\rangle$, for some $T \in \mathcal{T}$, then 7.1.15 implies that either

$$\phi \in \begin{cases} \{\theta_T|_Q, \theta_T^{-1}|_Q \mid Q \in \mathcal{D}_T\} & \text{if } \mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{SL}_2(3) \\ \{\theta_T|_Q, \theta_T^{-1}|_Q, \alpha_T|_Q, \beta_T|_Q \mid Q \in \mathcal{D}_T\} & \text{if } \mathrm{Out}_{\mathcal{F}}(T) \cong \mathrm{GL}_2(3) \end{cases}$$

or, if $T = E_{\pm 1}$ and $\eta \in \mathrm{Aut}_{\mathcal{F}}(S)$, then

$$\phi \in \begin{cases} \{\theta_\lambda|_Q, \theta_\lambda^{-1}|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\} \\ \{(\theta_\lambda \circ \eta)|_Q, (\theta_\lambda^{-1} \circ \eta)|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\} \\ \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda}\}. \end{cases}$$

If, finally, ϕ is not a restriction of $\mathrm{Aut}_{\mathcal{F}}(S)$, and $\phi \notin \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(T)\rangle$, for every $T \in \mathcal{T}$, then Lemma 7.2.4 lists in the table the possible maps for ϕ. The latter bits of (2) (for example $C(\mathrm{Id}) = C(\omega) > 0$ in 2(a)) holds by Lemma 4.2.8. This completes the proof. $\square$

CHAPTER 8

MAIN THEOREM

Theorem 8.0.2. *Let S be a rank two 3-group of maximal nilpotency class, of order 3^{2k+1} and generated by s and s_1, with the presentation*

$$S = \langle s, s_1, s_2, s_3, .., s_{2k} \mid s_i = [s_{i-1}, s], [s_i, s_1] = s_j^3 s_{j+1}^3 s_{j+2} = s^3 = 1 \rangle,$$

where $2 \le i \le 2k$ and $1 \le j \le 2k$, assuming that $s_{2k+1} = s_{2k+2} = 1$. Define $\omega, \eta \in Aut(S)$ to be involutions given by the maps $s \mapsto s^{-1}$, $s_1 \mapsto s_1^2 s_2$ and $s \mapsto s$, $s_1 \mapsto s_1^{-1}$. Let $\mathcal{F}$ be one of the following sixteen fusion systems generated by the outer automorphisms of S and $\mathcal{F}$-essential subgroups of S:

Name	S	V_0	E_0	E_1	E_{-1}	A	$\mathcal{T}$
DRV-1	$\langle\bar{\omega}\rangle$		$SL_2(3)$				$\{E_0\}$
DRV-2	$\langle\bar{\omega}\rangle$			$SL_2(3)$	$SL_2(3)$		$\{E_1, E_{-1}\}$
DRV-3	$\langle\bar{\omega}\rangle$		$SL_2(3)$	$SL_2(3)$	$SL_2(3)$		$\{E_0, E_1, E_{-1}\}$
DRV-4	$\langle\bar{\eta}\rangle$					$SL_2(3)$	$\{A\}$
DRV-5	$\langle\bar{\eta}\circ\bar{\omega}\rangle$	$SL_2(3)$					$\{V_0\}$
DRV-6	$\langle\bar{\eta}, \bar{\omega}\rangle$					$GL_2(3)$	$\{A\}$
DRV-7	$\langle\bar{\eta}, \bar{\omega}\rangle$			$SL_2(3)$	$SL_2(3)$		$\{E_{\pm 1}\}$
DRV-8	$\langle\bar{\eta}, \bar{\omega}\rangle$			$SL_2(3)$	$SL_2(3)$	$GL_2(3)$	$\{E_{\pm 1}, A\}$
DRV-9	$\langle\bar{\eta}, \bar{\omega}\rangle$		$GL_2(3)$				$\{E_0\}$
DRV-10	$\langle\bar{\eta}, \bar{\omega}\rangle$		$GL_2(3)$			$GL_2(3)$	$\{E_0, A\}$
DRV-11	$\langle\bar{\eta}, \bar{\omega}\rangle$		$GL_2(3)$	$SL_2(3)$	$SL_2(3)$		$\{E_0, E_{\pm 1}\}$
DRV-12	$\langle\bar{\eta}, \bar{\omega}\rangle$		$GL_2(3)$	$SL_2(3)$	$SL_2(3)$	$GL_2(3)$	$\{E_0, E_{\pm 1}, A\}$
DRV-13	$\langle\bar{\eta}, \bar{\omega}\rangle$	$GL_2(3)$					$\{V_0\}$
DRV-14	$\langle\bar{\eta}, \bar{\omega}\rangle$	$GL_2(3)$				$GL_2(3)$	$\{V_0, A\}$
DRV-15	$\langle\bar{\eta}, \bar{\omega}\rangle$	$GL_2(3)$		$SL_2(3)$	$SL_2(3)$		$\{V_0, E_{\pm 1}\}$
DRV-16	$\langle\bar{\eta}, \bar{\omega}\rangle$	$GL_2(3)$		$SL_2(3)$	$SL_2(3)$	$GL_2(3)$	$\{V_0, E_{\pm 1}, A\}$

Then for each of the 16 (saturated) fusion systems $\mathcal{F}$, we have the following list of minimal right characteristic bisets, denoted by Ω. Alongside these constructions we write down the sizes of these right characteristic bisets and the upper bound on the exoticity index of the fusion systems given by these constructions.

<u>DRV-1</u>: In this case

$$\Omega \cong (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega}) \sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}}) \sqcup (3^{2k-3}-1)(\mathcal{O}_{Id|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s,z\rangle}}),$$

Moreover, $|\Omega|/|S| = 2(3^{2k-2}-1)^2$, with

$$e(\mathcal{F}) \leq (3^{2k-2}-1)^2(4k+3) - 4k.$$

<u>DRV-2</u>: In this case

$$\Omega \cong (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega}) \sqcup \bigsqcup_{\lambda=-1,1} \left[(\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}) \sqcup (3^{2k-3}-1)(\mathcal{O}_{Id|_{\langle ss_1^\lambda,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^\lambda,z\rangle}})\right],$$

Moreover, $|\Omega|/|S| = 2[2\cdot 3^{4k-4} - 2^2\cdot 3^{2k-2}+1]$, with

$$e(\mathcal{F}) \leq 2[3^{4k-4} - 2\cdot 3^{2k-2}](4k+3)+4.$$

<u>DRV-3</u>: In this case

$$\Omega \cong (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega}) \sqcup \bigsqcup_{\lambda=-1,0,1} \left[(\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}) \sqcup (3^{2k-3}-1)(\mathcal{O}_{Id|_{\langle ss_1^\lambda,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^\lambda,z\rangle}})\right].$$

Moreover, $|\Omega|/|S| = 2[3^{4k-3} - 2\cdot 3^{2k-1}+1]$, with

$$e(\mathcal{F}) \leq [3^{4k-3} - 2\cdot 3^{2k-1}](4k+3)+3.$$

<u>DRV-4</u>: In this case

$$\Omega \cong [\mathcal{O}_{Id_S} \sqcup \mathcal{O}_{\eta}] \sqcup [\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}}].$$

Moreover, $|\Omega|/|S| = 2^3$, which is independent of k, with

$$e(\mathcal{F}) \leq 14k+9.$$

<u>DRV-5</u>: In this case

$$\Omega \cong [\mathcal{O}_{Id_S} \sqcup \mathcal{O}_{\omega\circ\eta}] \sqcup [\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}}] \sqcup (3^{2k-2}-1)[\mathcal{O}_{Id|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}].$$

Moreover, $|\Omega|/|S| = 2(3^{2k-1}-1)^2$*, with*

$$e(\mathcal{F}) \leq (3^{2k-1}-1)^2(4k+3) + 2\cdot 3^{2k-2}(3^{2k}-1) - (4k+1).$$

DRV-6: In this case

$$\Omega = (\mathcal{O}_{Id|_S} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}).$$

Moreover, $|\Omega|/|S| = 2^4$*, which is independent of* k*, with*

$$e(\mathcal{F}) \leq 30k + 21.$$

DRV-7: In this case

$$\begin{aligned}
\Omega \cong\; & (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)[\mathcal{O}_{Id|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}] \\
& \sqcup \bigsqcup_{\lambda=-1,1} [\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}] \sqcup [\mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}] \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}].
\end{aligned}$$

Moreover, $|\Omega|/|S| = 4[2\cdot 3^{2k-2}-1]^2$*, with*

$$e(\mathcal{F}) \leq \begin{cases} 6349 & \text{if } k=2 \\ 8\cdot 3^{2k-2}[3^{2k-2}-1][4k+3] + (4k+6) & \text{if } k\geq 3. \end{cases}$$

DRV-8: *In this case*

$$\begin{aligned}
\Omega \cong & (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)(\mathcal{O}_{Id|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2^2 \cdot 3^{4k-4}(\mathcal{O}_{Id|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup 2(\mathcal{O}_{Id|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_A|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\theta_{V_\lambda}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}} \sqcup \mathcal{O}_{\beta_{V_\lambda}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_\lambda}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}(\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1\\ \ell=-1,1}} (\mathcal{O}_{\Theta^{\ell}_{1,\lambda}} \sqcup \mathcal{O}_{\Theta^{\ell}_{2,\lambda}} \sqcup \mathcal{O}_{\Theta^{\ell}_{3,\lambda}} \sqcup \mathcal{O}_{\Theta^{\ell}_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\vartheta_{2,\lambda}} \sqcup \mathcal{O}_{\vartheta_{3,\lambda}} \sqcup \mathcal{O}_{\vartheta_{4,\lambda}}).
\end{aligned}$$

Moreover, $|\Omega|/|S| = 2^3[2^4\cdot 3^{6k-5}+38\cdot 3^{4k-4}-5\cdot 3^{2k-2}+2]$, *with*

$$e(\mathcal{F}) \leq \begin{cases} 1673177 & \text{if } k=2 \\ 1545484964 & \text{if } k=3 \\ 2^2[2^4\cdot 3^{6k-5}+38\cdot 3^{4k-4}-5\cdot 3^{2k-2}](4k+3)+(28k+19) & \text{if } k\geq 4 \end{cases}$$

DRV-9: *In this case*

$$\Omega \cong (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0}) \sqcup n_k \cdot (\mathcal{O}_{Id|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}),$$

where $n_k = 3^{2k-3}-1$. *Moreover,* $|\Omega|/|S| = 4(3^{2k-2}-1)^2$, *with*

$$e(\mathcal{F}) \leq 2[(4k+3)3^{4k-4}-(8k-6)3^{2k-2}+(2k+3)].$$

<u>DRV-10</u>: In this case

$$\begin{aligned}
\Omega \cong\ & (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup [3^{2k-3}-1](\mathcal{O}_{Id|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}})\\
&\sqcup 3^{4k-4}(\mathcal{O}_{Id|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}) \sqcup (\mathcal{O}_{Id|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}})\\
&\sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A})\\
&\sqcup 3^{2k-2}(\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}})\\
&\sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0})\\
&\sqcup 3^{2k-2}(\theta_0|_{\langle s\rangle} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\beta_0|_{\langle s\rangle}}) \sqcup 3^{2k-2}(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\beta_0|_{\langle t\rangle}})\\
&\sqcup (\mathcal{O}_{\Theta_{1,0}} \sqcup \mathcal{O}_{\Theta_{2,0}} \sqcup \mathcal{O}_{\Theta_{3,0}} \sqcup \mathcal{O}_{\Theta_{4,0}}) \sqcup (\mathcal{O}_{\Theta_{1,0}^{-1}} \sqcup \mathcal{O}_{\Theta_{2,0}^{-1}} \sqcup \mathcal{O}_{\Theta_{3,0}^{-1}} \sqcup \mathcal{O}_{\Theta_{4,0}^{-1}}).
\end{aligned}$$

Moreover, $|\Omega|/|S| = 3^{6k-4}+22\cdot 3^{4k-4}+13\cdot 3^{2k-2}+4$, *with*

$$e(\mathcal{F}) \leq \frac{1}{2}[3^{6k-4}+22\cdot 3^{4k-4}+13\cdot 3^{2k-2}](4k+3)+6k.$$

<u>DRV-11</u>: In this case

$$\begin{aligned}
\Omega \cong & (\mathcal{O}_{Id_S} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta})\\
&\sqcup \bigsqcup_{\lambda=-1,0,1} (3^{2k-3}-1)[\mathcal{O}_{Id|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}]\\
&\sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0})\\
&\sqcup \bigsqcup_{\lambda=-1,1} [\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}] \sqcup [\mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}]\\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}]\\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\xi_{1,\lambda}} \sqcup \mathcal{O}_{\xi_{2,\lambda}} \sqcup \mathcal{O}_{\xi_{3,\lambda}} \sqcup \mathcal{O}_{\xi_{4,\lambda}}]\\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\xi_{1,\lambda}^{-1}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{-1}} \sqcup \mathcal{O}_{\xi_{3,\lambda}^{-1}} \sqcup \mathcal{O}_{\xi_{4,\lambda}^{-1}}].
\end{aligned}$$

Moreover, $|\Omega|/|S| = 4[3^{4k-2}-2\cdot 3^{2k-1}+1]$, *with*

$$e(\mathcal{F}) \leq 2[(3^{4k-2}-2\cdot 3^{2k-1})(4k+3)+(2k+3)].$$

DRV-12: *In this case*

$$
\begin{aligned}
\Omega \cong & (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} (3^{2k-3}-1)(\mathcal{O}_{Id|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}) \\
& \sqcup 3^{4k-4}(\mathcal{O}_{Id|_{\langle s\rangle}} \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2^2\cdot 3^{4k-4}(\mathcal{O}_{Id|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup 3(\mathcal{O}_{Id|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}) \\
& \sqcup (17\cdot 3^{2k-3}-3)(\mathcal{O}_{\theta_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_A|_{\langle z\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0}) \\
& \sqcup 3^{2k-1}(\mathcal{O}_{\theta_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\beta_0}|_{\langle s\rangle}) \\
& \sqcup 3^{2k-1}(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\beta_0|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-1}(\mathcal{O}_{\theta_\lambda|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-1}(\mathcal{O}_{\theta_\lambda|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}(\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} 3^{2k-2}(\mathcal{O}_{\theta_{V_\lambda}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}} \sqcup \mathcal{O}_{\beta_{V_\lambda}}) \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} (17\cdot 3^{4k-5}-3^{2k-1})(\mathcal{O}_{\theta_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_\lambda}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,0,1\\ \ell=-1,1}} (\mathcal{O}_{\Theta_{1,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{2,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{3,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{4,\lambda}^{\ell}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\vartheta_{2,\lambda}} \sqcup \mathcal{O}_{\vartheta_{3,\lambda}} \sqcup \mathcal{O}_{\vartheta_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\lambda,\ell=-1,1} 3^{2k-3}(\mathcal{O}_{\xi_{1,\lambda}^{\ell}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{\ell}} \sqcup \mathcal{O}_{\xi_{3,\lambda}^{\ell}} \sqcup \mathcal{O}_{\xi_{4,\lambda}^{\ell}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{4k-4}+2\cdot 3^{2k-3})(\mathcal{O}_{\xi_{1,\lambda}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\xi_{3,\lambda}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\xi_{4,\lambda}|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{4k-4}+2\cdot 3^{2k-3})(\mathcal{O}_{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{3,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{4,\lambda}^{-1}|_{\langle s\rangle}}) \\
& \sqcup \bigsqcup_{\lambda,\ell=-1,1} 3^{2k-2}(\mathcal{O}_{\xi_{5,\lambda}^{\ell}} \sqcup \mathcal{O}_{\xi_{5,\lambda}^{\ell}} \sqcup \mathcal{O}_{\xi_{5,\lambda}^{\ell}} \sqcup \mathcal{O}_{\xi_{5,\lambda}^{\ell}}).
\end{aligned}
$$

Moreover, $|\Omega|/|S| = 2^3[5\cdot 3^{6k-3} + 2^5\cdot 3^{4k-3} + 2\cdot 3^{2k-1} + 2]$, *with*

$$e(\mathcal{F}) \leq 2^2[5\cdot 3^{6k-3} + 2^5\cdot 3^{4k-3} + 2\cdot 3^{2k-1}](4k+3) + 15(2k+1).$$

DRV-13: In this case

$$\Omega \cong (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}) \sqcup n_k\cdot(\mathcal{O}_{Id|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}),$$

where $n_k = 3^{2k-2} - 1$. *Moreover,* $|\Omega|/|S| = 4(3^{2k-1}-1)^2$, *with*

$$e(\mathcal{F}) \leq 2(3^{2k-1}-1)^2(4k+3) - 4k.$$

DRV-14: In this case

$$\begin{aligned}\Omega \cong &(\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (3^{2k-2}-1)(\mathcal{O}_{Id|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}})\\ &\sqcup(\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}) \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A})\\ &\sqcup 2(\mathcal{O}_{\theta_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}}) \sqcup 2(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle t\rangle}}).\end{aligned}$$

Moreover, $|\Omega|/|S| = 4[3^{4k-2} + 4\cdot 3^{2k-1} + 4]$, *with*

$$e(\mathcal{F}) \leq 2[3^{4k-2} + 4\cdot 3^{2k-1}](4k+3) + 6(5k+3).$$

DRV-15: In this case

$$\begin{aligned}\Omega \cong\ &(\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta})\\ &\sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)[\mathcal{O}_{Id|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}]\\ &\sqcup (3^{2k-2}-1)[\mathcal{O}_{Id|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}]\\ &\sqcup \bigsqcup_{\lambda=-1,1} [\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}] \sqcup [\mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}]\\ &\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}]\\ &\sqcup [\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}]\\ &\sqcup [\mathcal{O}_{\theta_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_0}|_{\langle z\rangle}}].\end{aligned}$$

Moreover, $|\Omega|/|S| = 4[13\cdot 3^{4k-4} - 3^{2k-2} + 1]$, *with*

$$e(\mathcal{F}) \leq 2[13\cdot 3^{4k-4} - 3^{2k-2}](4k+3) + 4(k+1).$$

DRV-16: *In this case*

$$
\begin{aligned}
\Omega \cong & (\mathcal{O}_{Id} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)(\mathcal{O}_{Id|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}) \\
& \sqcup (3^{2k-2}-1)(\mathcal{O}_{Id|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2^2\cdot 3^{4k-4}(\mathcal{O}_{Id|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup 2(\mathcal{O}_{Id|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_A|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\theta_{V_\lambda}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}} \sqcup \mathcal{O}_{\beta_{V_\lambda}}) \\
& \sqcup (3^{4k-3}-2^2\cdot 3^{2k-2}+1)(\mathcal{O}_{\theta_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_0}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_\lambda}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}(\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1\\ \ell=-1,1}} (\mathcal{O}_{\Theta_{1,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{2,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{3,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{4,\lambda}^{\ell}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\vartheta_{2,\lambda}} \sqcup \mathcal{O}_{\vartheta_{3,\lambda}} \sqcup \mathcal{O}_{\vartheta_{4,\lambda}}) \\
& \sqcup 2(\mathcal{O}_{\theta_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}}) \sqcup 2(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(2\cdot 3^{2k-2}+1)(\mathcal{O}_{\xi_{1,\lambda}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}})
\end{aligned}
$$

Moreover, $|\Omega|/|S| = 2^2[47\cdot 3^{6k-5} + 76\cdot 3^{4k-4} + 11\cdot 3^{2k-2} + 4]$, *with*

$$e(\mathcal{F}) \leq \begin{cases} 2399046 & \text{if } k = 2 \\ 2262981344 & \text{if } k = 3 \\ 2[47\cdot 3^{6k-5} + 76\cdot 3^{4k-4} + 11\cdot 3^{2k-2}](4k+3) + (30k+11) & \text{if } k \geq 4. \end{cases}$$

8.1 Proof Example

For each fusion system $\mathcal{F}$, we know that Ω, a corresponding mininal right characteristic biset, is isomorphic to a union of $(S\times S)$-sets of the form $\mathcal{O}_\phi$, for some $\phi \in \mathcal{F}$, for some unspecified constants, $C(\phi) \geq 0$, as given in Theorem 7.3.1. Not every value of the constant determines that we have a right characteristic biset. Ω is a right characteristic biset for $\mathcal{F}$ if and only if it satisfies the conditions in Lemma 4.2.7. For the proof, the idea is to determine under which values of the constants we have $|\Omega^\psi| = |\Omega^{\mathrm{Id}|_{D_\psi}}|$ for every $\psi \in \mathcal{F}$. We will have a number of simultaneous equations to solve. The constants that we derive, will give us a candidate minimal solution for Ω, which we shall show is minimal.

We use our solution to determine an upper bound on the exoticity index using Lemma 4.3.12. Note that this is not the actual value of the exoticity index, because our solution is only minimal in the sense that it is the minimal solution among all possible right characteristic bisets. In other words, there may be valid solutions that are derived from other routes than those involving right (or left) characteristic bisets.

We now give a full proof for a particular example, namely DRV-9. In the proof that follows we call this fusion system $\mathcal{F}$. The proofs for the rest of the examples will be found in Appendix A.

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\Omega \cong \bigsqcup_{Q\in\mathcal{D}} \left[C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q}\right]$$
$$\sqcup \bigsqcup_{Q\in\mathcal{D}_0} \left[C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0^{-1}|_Q} \sqcup C(\alpha_0|_Q)\mathcal{O}_{\alpha_0|_Q} \sqcup C(\beta_0|_Q)\mathcal{O}_{\beta_0|_Q}\right]$$

where the constants are non-negative integers and are yet to be determined. Not every choice of the constant will give us a right characteristic biset.

Equations:

Now, Ω is a right characteristic biset, if and only if, by Definition 4.2.5 and Lemma 4.2.7, we have

1. Every orbit in Ω is of the form $\mathcal{O}_\phi$ for some $\phi \in \mathcal{F}$,
2. $|\Omega^\psi| = |\Omega^{\mathsf{Id}|_{D_\psi}}|$ for every $\psi \in \mathcal{F}$.
3. $|\Omega|/|S|$ is not a multiple of 3.

With the construction of the general form of Ω, the first condition has already been satisfied, irrespective of what the values of the constants corresponding to each orbit are. By Lemma 4.2.8, the third condition is satisfied, if and only if

$$3 \nmid C(\mathsf{Id})$$

where, by the same lemma, we already know that $C(\mathsf{Id}) = C(\omega) = C(\eta) = C(\omega\circ\eta) > 0$. Thus, once we have determined the values of each constant, we will be in a position to say whether the resulting Ω satisfies the third condition. Now, a large chunk of the remainder of the proof is focused in determining $C(\phi)$ for every $\phi \in \mathcal{F}$, so that we can satisfy the second condition. Denote by $\mathcal{V}$, the set of morphisms $\phi \in \mathcal{F}$ such that $\mathcal{O}_\phi$ is an orbit in the general form of Ω. By definition, for any $\psi \in \mathcal{F}$, we have

$$|\Omega^\psi| = \sum_{\phi\in\mathcal{V}} C(\phi)|\mathcal{O}_\phi^\psi|.$$

By Lemma 3.1.8, it suffices to satisfy the second condition for morphisms ψ chosen up to S-S-equivalence. But we already know what the representatives of these classes are - they are those in the set $\mathcal{V}$ itself. Explicitly, we can say that $\psi \in \mathcal{V}$, where $\mathcal{V}$ is the union of the following two sets:

$$\begin{cases} \{\mathsf{Id}|_Q, \omega|_Q, \eta|_Q, \omega \circ \eta|_Q \mid Q \in \mathcal{D}\} \\ \{\theta_0|_Q, \theta_0^{-1}|_Q, \alpha_0|_Q, \beta_0|_Q \mid Q \in \mathcal{D}_0\} \end{cases}$$

We shall write down the equations for $|\Omega^{\psi}|$, and also $|\Omega^{\mathsf{Id}|_{D_\psi}}|$, for each $\psi \in \mathcal{V}$. However, we shall only justify them for $\psi \in \hat{\mathcal{V}}$, where $\hat{\mathcal{V}}$ is the union of the following two sets:

$$\begin{cases} \{\mathsf{Id}|_Q \mid Q \in \mathcal{D}\} \\ \{\theta_0|_Q \mid Q \in \mathcal{D}_0\} \end{cases}$$

that is, the first column of the presented list $\mathcal{V}$. We have chosen to justify equations for a smaller but carefully selected set of morphisms $\hat{\mathcal{V}}$, for the reason that the justifications for the rest of the equations in $\mathcal{V}$ are similar to the ones we have in $\hat{\mathcal{V}}$.[1]

We justify the equation in $\hat{\mathcal{V}}$ by making use of the *Calculations Locator* on page 296 in Appendix B, in order to help us find the relevant fixed point sizes of orbits that we need for this DRV-9 fusion system. Note that for these equations, calculations are found in the same page/area in Appendix B, due to the simplicity of this case.[2]

Here are the equations, which will be very helpful in demonstrating that our candidate solution is indeed a right characteristic biset (see section on *Candidate solution*) and that it is minimal (see section on *Minimality*). We have split it into two parts:

[1] This pattern occurs for the remaining DRV-fusion systems as well. It turns out that morphisms occurs in quadruples just like here. Although for DRV-9 fusion system we shall write the equations for all $\psi \in \mathcal{V}$, for the remaining DRV-fusion systems, we shall only write equations for a smaller subset, akin to $\hat{\mathcal{V}}$.

[2] As we shall see in the calculations of the remaining DRV-fusion systems in Appendix A, the fixed point sizes calculations can be found all over the place in Appendix B.

$\underline{\psi \in \{\mathsf{Id}|_Q, \omega|_Q, \eta|_Q, \omega\circ\eta|_Q \mid Q \in \mathcal{D}_0\}}$:

From page 298: $|\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\omega|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\eta|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$. Moreover, $|\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\mathsf{Id}|_P}|$, for $P \in \mathcal{D}$, is dependent on P, so we leave it as it is.[1]
From page 298: $|\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\theta_0|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\alpha_0|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\mathsf{Id}|_{\langle s,t\rangle}}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\mathsf{Id}|_P}|. \tag{ⓐ$_1$}$$

Similarly,

$$|\Omega^{\omega|_{\langle s,t\rangle}}| = \sum_{P\in\mathcal{D}} C(\omega|_P)|\mathcal{O}^{\omega|_{\langle s,t\rangle}}_{\omega|_P}| \tag{ⓐ$_2$}$$

$$|\Omega^{\eta|_{\langle s,t\rangle}}| = \sum_{P\in\mathcal{D}} C(\eta|_P)|\mathcal{O}^{\eta|_{\langle s,t\rangle}}_{\eta|_P}| \tag{ⓐ$_3$}$$

$$|\Omega^{\omega\circ\eta|_{\langle s,t\rangle}}| = \sum_{P\in\mathcal{D}} C(\omega\circ\eta|_P)|\mathcal{O}^{\omega\circ\eta|_{\langle s,t\rangle}}_{\omega\circ\eta|_P}|. \tag{ⓐ$_4$}$$

From page 298: $|\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\omega|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\eta|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$. Moreover, $|\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\mathsf{Id}|_P}|$, for $P \in \mathcal{D}$, is dependent on P, so we leave it as it is.
From page 298: $|\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\theta_0}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\theta_0^{-1}}| = 3^2$, and for all $P \in \mathcal{D}_0\backslash\{\langle s,t\rangle\}$, we have $|\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\theta_0|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\theta_0^{-1}|_P}| = 0$. Moreover, $|\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\theta_{\alpha_0|_P}}| = |\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\mathsf{Id}|_{\langle s,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle s,z\rangle}}_{\mathsf{Id}|_P}| + 3^2[C(\theta_0) + C(\theta_0^{-1})]. \tag{ⓑ$_1$}$$

Similarly,

$$|\Omega^{\omega|_{\langle s,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\omega|_P)|\mathcal{O}^{\omega|_{\langle s,z\rangle}}_{\omega|_P}| + 3^2[C(\theta_0) + C(\theta_0^{-1})] \tag{ⓑ$_2$}$$

$$|\Omega^{\eta|_{\langle s,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\eta|_P)|\mathcal{O}^{\eta|_{\langle s,z\rangle}}_{\eta|_P}| + 3^2[C(\alpha_0) + C(\beta_0)] \tag{ⓑ$_3$}$$

$$|\Omega^{\omega\circ\eta|_{\langle s,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\omega\circ\eta|_P)|\mathcal{O}^{\omega\circ\eta|_{\langle s,z\rangle}}_{\omega\circ\eta|_P}| + 3^2[C(\alpha_0) + C(\beta_0)]. \tag{ⓑ$_4$}$$

---

[1] When we need to know the exact value of $|\mathcal{O}^{\mathsf{Id}|_{\langle s,t\rangle}}_{\mathsf{Id}|_P}|$ for a particular P, we may refer back to Appendix B.

From page 298: $|\mathcal{O}_{\omega|_P}^{\mathsf{Id}_{\langle t,z\rangle}}| = |\mathcal{O}_{\eta|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| = |\mathcal{O}_{\omega\circ\eta|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| = 0$ for all $P \in \mathcal{D}$. Moreover, $|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}|$ is dependent on P, so we leave it as it is.

From page 298: $|\mathcal{O}_{\theta_0|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| = |\mathcal{O}_{\theta_0^{-1}|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| = |\mathcal{O}_{\alpha_0|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| = |\mathcal{O}_{\beta_0|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\mathsf{Id}|_{\langle t,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}|. \qquad \text{ⓒ}_1$$

Similarly,

$$|\Omega^{\omega|_{\langle t,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\omega|_{\langle t,z\rangle}}| \qquad \text{ⓒ}_2$$

$$|\Omega^{\eta|_{\langle t,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\eta|_{\langle t,z\rangle}}| \qquad \text{ⓒ}_3$$

$$|\Omega^{\omega\circ\eta|_{\langle t,z\rangle}}| = \sum_{P\in\mathcal{D}} C(\omega\circ\eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\omega\circ\eta|_{\langle t,z\rangle}}|. \qquad \text{ⓒ}_4$$

From page 298: $|\mathcal{O}_{\omega|_P}^{\mathsf{Id}_{\langle s\rangle}}| = |\mathcal{O}_{\omega\circ\eta|_P}^{\mathsf{Id}|_{\langle s\rangle}}| = 0$ for all $P \in \mathcal{D}$. Moreover, $|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle s\rangle}}|$ and $|\mathcal{O}_{\eta|_P}^{\mathsf{Id}|_{\langle s\rangle}}|$, for $P \in \mathcal{D}$, is dependent on P, so we leave them as they are.

From page 298: $|\mathcal{O}_{\theta_0}^{\mathsf{Id}|_{\langle s\rangle}}| = |\mathcal{O}_{\theta_0^{-1}}^{\mathsf{Id}|_{\langle s\rangle}}| = |\mathcal{O}_{\alpha_0}^{\mathsf{Id}|_{\langle s\rangle}}| = |\mathcal{O}_{\beta_0}^{\mathsf{Id}|_{\langle s\rangle}}| = 3^2$, and for all $P \in \mathcal{D}_0 \backslash \{\langle s,t\rangle\}$, we have $|\mathcal{O}_{\theta_0|_P}^{\mathsf{Id}|_{\langle s\rangle}}| = |\mathcal{O}_{\theta_0^{-1}|_P}^{\mathsf{Id}|_{\langle s\rangle}}| = |\mathcal{O}_{\alpha_0|_P}^{\mathsf{Id}|_{\langle s\rangle}}| = |\mathcal{O}_{\beta_0|_P}^{\mathsf{Id}|_{\langle s\rangle}}| = 0$. Thus,

$$|\Omega^{\mathsf{Id}|_{\langle s\rangle}}| = \sum_{P\in\mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle s\rangle}}| + C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\eta|_{\langle s\rangle}}|] + 3^2[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)]. \quad \text{ⓓ}_1$$

Similarly,

$$|\Omega^{\omega|_{\langle s\rangle}}| = \sum_{P\in\mathcal{D}} [C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\omega|_{\langle s\rangle}}| + C(\omega\circ\eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\omega\circ\eta|_{\langle s\rangle}}|] + 3^2[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \qquad \text{ⓓ}_2$$

$$|\Omega^{\eta|_{\langle s\rangle}}| = \sum_{P\in\mathcal{D}} [C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\eta|_{\langle s\rangle}}| + C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\eta|_{\langle s\rangle}}|] + 3^2[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \quad \text{ⓓ}_3$$

$$|\Omega^{\omega\circ\eta|_{\langle s\rangle}}| = \sum_{P\in\mathcal{D}} [C(\omega\circ\eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\omega\circ\eta|_{\langle s\rangle}}| + C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\omega\circ\eta|_{\langle s\rangle}}|] + 3^2[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)]. \qquad \text{ⓓ}_4$$

From page 298: $|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\omega|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\eta|_P}| = 0$ for all $P \in \mathcal{D}$. Moreover, $|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\mathsf{Id}|_P}|$ and $|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\omega\circ\eta|_P}|$, for $P \in \mathcal{D}$, is dependent on P, so we leave it as it is.

From page 298: $|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\theta_0|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\alpha_0|_P}| = |\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\mathsf{Id}|_{\langle t\rangle}}| = \sum_{P\in\mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\mathsf{Id}|_P}| + C(\omega\circ\eta|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\omega\circ\eta|_P}|]. \qquad ⓔ_1$$

Similarly,

$$|\Omega^{\omega|_{\langle t\rangle}}| = \sum_{P\in\mathcal{D}} [C(\omega|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\omega|_P}| + C(\eta|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\eta|_P}|] \qquad ⓔ_2$$

$$|\Omega^{\eta|_{\langle t\rangle}}| = \sum_{P\in\mathcal{D}} [C(\omega|_P)|\mathcal{O}^{\eta|_{\langle t\rangle}}_{\omega|_P}| + C(\eta|_P)|\mathcal{O}^{\eta|_{\langle t\rangle}}_{\eta|_P}|] \qquad ⓔ_3$$

$$|\Omega^{\omega\circ\eta|_{\langle t\rangle}}| = \sum_{P\in\mathcal{D}} [C(\omega\circ\eta|_P)|\mathcal{O}^{\omega\circ\eta|_{\langle t\rangle}}_{\omega\circ\eta|_P}| + C(\mathsf{Id}|_P)|\mathcal{O}^{\omega\circ\eta|_{\langle t\rangle}}_{\mathsf{Id}|_P}|]. \qquad ⓔ_4$$

$\underline{\psi \in \{\theta_0|_Q, \theta_0^{-1}|_Q, \alpha_0|_Q, \beta_0|_Q \mid Q \in \mathcal{D}_0\}}$:

From page 301: $|\mathcal{O}^{\theta_0}_{\mathsf{Id}|_P}| = |\mathcal{O}^{\theta_0}_{\omega|_P}| = |\mathcal{O}^{\theta_0}_{\eta|_P}| = |\mathcal{O}^{\theta_0}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$.

From page 301: $|\mathcal{O}^{\theta_0}_{\theta_0}| = 3$ and for all $P \in \mathcal{D}_0\backslash\{\langle s,t\rangle\}$, $|\mathcal{O}^{\theta_0}_{\theta_0|_P}| = 0$. Moreover, $|\mathcal{O}^{\theta_0}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\theta_0}_{\alpha_0|_P}| = |\mathcal{O}^{\theta_0}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\theta_0}| = 3C(\theta_0). \qquad ⓕ_1$$

Similarly,

$$|\Omega^{\theta_0^{-1}}| = 3C(\theta_0^{-1}) \qquad ⓕ_2$$

$$|\Omega^{\alpha_0}| = 3C(\alpha_0) \qquad ⓕ_3$$

$$|\Omega^{\beta_0}| = 3C(\beta_0). \qquad ⓕ_4$$

From page 301: $|\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\mathsf{Id}|_P}| = |\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\omega|_P}| = |\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\eta|_P}| = |\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$.

From page 301: $|\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\theta_0}| = 3^{2k}$ and $|\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\theta_0|_{\langle s,z\rangle}}| = 3^{2k+1}$, and for all $P \in \mathcal{D}_0\backslash\{\langle s,t\rangle, \langle s,z\rangle\}$, we have $|\mathcal{O}^{\theta_0}_{\theta_0|_P}| = 0$. Moreover, $|\mathcal{O}^{\theta_0}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\theta_0}_{\alpha_0|_P}| = |\mathcal{O}^{\theta_0|_{\langle s,z\rangle}}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\theta_0|_{\langle s,z\rangle}}| = 3^{2k}C(\theta_0) + 3^{2k+1}C(\theta_0|_{\langle s,z\rangle}). \qquad \text{ⓖ}_1$$

Similarly,

$$|\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}| = 3^{2k}C(\theta_0^{-1}) + 3^{2k+1}C(\theta_0^{-1}|_{\langle s,z\rangle}) \qquad \text{ⓖ}_2$$

$$|\Omega^{\alpha_0|_{\langle s,z\rangle}}| = 3^{2k}C(\alpha_0) + 3^{2k+1}C(\alpha_0|_{\langle s,z\rangle}) \qquad \text{ⓖ}_3$$

$$|\Omega^{\beta_0|_{\langle s,z\rangle}}| = 3^{2k}C(\beta_0) + 3^{2k+1}C(\beta_0|_{\langle s,z\rangle}). \qquad \text{ⓖ}_4$$

From page 301: $|\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\mathsf{Id}|_P}| = |\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\omega|_P}| = |\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\eta|_P}| = |\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$.
From page 301: $|\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\theta_0}| = 3^{2k}$ and $|\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\theta_0|_{\langle t,z\rangle}}| = 3^{2k+1}$, and for all $P \in \mathcal{D}_0 \backslash \{\langle s,t\rangle, \langle t,z\rangle\}$ we have $|\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\theta_0|_P}| = 0$. Moreover, $|\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\alpha_0|_P}| = |\mathcal{O}^{\theta_0|_{\langle t,z\rangle}}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\theta_0|_{\langle t,z\rangle}}| = 3^{2k}C(\theta_0) + 3^{2k+1}C(\theta_0|_{\langle t,z\rangle}). \qquad \text{ⓗ}_1$$

Similarly,

$$|\Omega^{\theta_0^{-1}|_{\langle t,z\rangle}}| = 3^{2k}C(\theta_0^{-1}) + 3^{2k+1}C(\theta_0^{-1}|_{\langle t,z\rangle}) \qquad \text{ⓗ}_2$$

$$|\Omega^{\alpha_0|_{\langle t,z\rangle}}| = 3^{2k}C(\alpha_0) + 3^{2k+1}C(\alpha_0|_{\langle t,z\rangle}) \qquad \text{ⓗ}_3$$

$$|\Omega^{\beta_0|_{\langle t,z\rangle}}| = 3^{2k}C(\beta_0) + 3^{2k+1}C(\beta_0|_{\langle t,z\rangle}). \qquad \text{ⓗ}_4$$

From page 301: $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\mathsf{Id}|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\omega|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\eta|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$.
From page 301: $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\alpha_0}| = 3^{2k}$, $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0|_{\langle s,z\rangle}}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\alpha_0|_{\langle s,z\rangle}}| = 3^{2k+1}$, $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0|_{\langle s\rangle}}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\alpha_0|_{\langle s\rangle}}| = 3^{2k+1}$ and for all $P \in \mathcal{D}_0 \backslash \{\langle s,t\rangle, \langle s,z\rangle, \langle s\rangle\}$, $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\alpha_0|_P}| = 0$. Moreover, $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\beta_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\theta_0|_{\langle s\rangle}}| = 3^{2k}[C(\theta_0) + C(\alpha_0)] + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle})] + 3^{2k+1}[C(\theta_0|_{\langle s\rangle}) + C(\alpha_0|_{\langle s\rangle})]. \qquad \text{ⓘ}_1$$

Similarly,

$$|\Omega^{\theta_0^{-1}|_{\langle s\rangle}}| = 3^{2k}[C(\theta_0^{-1}) + C(\beta_0)] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle s\rangle}) + C(\beta_0|_{\langle s\rangle})] \qquad ⓘ_2$$

$$|\Omega^{\alpha_0|_{\langle s\rangle}}| = 3^{2k}[C(\theta_0) + C(\alpha_0)] + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle})] + 3^{2k+1}[C(\theta_0|_{\langle s\rangle}) + C(\alpha_0|_{\langle s\rangle})] \qquad ⓘ_3$$

$$|\Omega^{\beta_0|_{\langle s\rangle}}| = 3^{2k}[C(\theta_0^{-1}) + C(\beta_0)] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle s\rangle}) + C(\beta_0|_{\langle s\rangle})]. \qquad ⓘ_4$$

From page 301: $|\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\mathsf{Id}|_P}| = |\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\omega|_P}| = |\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\eta|_P}| = |\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\omega\circ\eta|_P}| = 0$ for all $P \in \mathcal{D}$.
From page 301: $|\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\theta_0}| = |\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\beta_0}| = 3^{2k}$, $|\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\theta_0|_{\langle t,z\rangle}}| = |\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\beta_0|_{\langle t,z\rangle}}| = 3^{2k+1}$, $|\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\theta_0|_{\langle t\rangle}}| = |\mathcal{O}^{\theta_0|_{\langle t\rangle}}_{\beta_0|_{\langle t\rangle}}| = 3^{2k+1}$ and for all $P \in \mathcal{D}_0 \backslash \{\langle s,t\rangle, \langle t,z\rangle, \langle t\rangle\}$, $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\beta_0|_P}| = 0$. Moreover, $|\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\theta_0^{-1}|_P}| = |\mathcal{O}^{\theta_0|_{\langle s\rangle}}_{\alpha_0|_P}| = 0$ for all $P \in \mathcal{D}_0$. Thus,

$$|\Omega^{\theta_0|_{\langle t\rangle}}| = 3^{2k}[C(\theta_0) + C(\beta_0)] + 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})] + 3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})]. \qquad ⓙ_1$$

Similarly,

$$|\Omega^{\theta_0^{-1}|_{\langle t\rangle}}| = 3^{2k}[C(\theta_0^{-1}) + C(\alpha_0)] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle})] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle})] \qquad ⓙ_2$$

$$|\Omega^{\alpha_0|_{\langle t\rangle}}| = 3^{2k}[C(\theta_0^{-1}) + C(\alpha_0)] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle})] + 3^{2k+1}[C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle})] \qquad ⓙ_3$$

$$|\Omega^{\beta_0|_{\langle t\rangle}}| = 3^{2k}[C(\theta_0) + C(\beta_0)] + 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})] + 3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})]. \qquad ⓙ_4$$

Candidate solution:

We have the following designated values for the constants:

$$C(\mathsf{Id}) = C(\omega) = C(\eta) = C(\omega\circ\eta) = 1$$
$$C(\theta_0) = C(\theta_0^{-1}) = C(\alpha_0) = C(\beta_0) = 1$$
$$C(\mathsf{Id}|_{\langle s,z\rangle}) = C(\omega|_{\langle s,z\rangle}) = C(\eta|_{\langle s,z\rangle}) = C(\omega\circ\eta|_{\langle s,z\rangle}) = 3^{2k-3}-1,$$

with the rest of the constants set to zero. Writing Ω^* to denote this solution, we have

$$\Omega^* \cong (\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0}) \sqcup n_k \cdot (\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}})$$

where $n_k = 3^{2k-3}-1$. This is a right characteristic biset for the DRV-9 fusion system. To check that this is indeed the case it requires us to check that the following equalities hold:[1]

$$|(\Omega^*)^{\mathsf{Id}|_Q}| = |(\Omega^*)^{\omega|_Q}| = |(\Omega^*)^{\eta|_Q}| = |(\Omega^*)^{\omega\circ\eta|_Q}| \qquad \text{(for all } Q \in \mathcal{D})$$
$$|(\Omega^*)^{\mathsf{Id}|_Q}| = |(\Omega^*)^{\theta_0|_Q}| = |(\Omega^*)^{\theta_0^{-1}|_Q}| = |(\Omega^*)^{\alpha_0|_Q}| = |(\Omega^*)^{\beta_0|_Q}|. \qquad \text{(for all } Q \in \mathcal{D}_0)$$

For this we can use *Equations* section above or the *Calculations Locator* on page 296, in Appendix B. Because the vast majority of the constants in our candidate solution are zero, this is an easy check. We now demonstrate that this indeed is a right characteristic biset:

When $Q \in \mathcal{D}$ with $|Q| > 3^3$, we have

$$|(\Omega^*)^{\mathsf{Id}|_Q}| = |\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_Q}| = |C_S(Q)|, \quad |(\Omega^*)^{\omega|_Q}| = |\mathcal{O}_{\omega}^{\omega}| = |C_S(Q)|,$$
$$|(\Omega^*)^{\eta|_Q}| = |\mathcal{O}_{\eta}^{\eta}| = |C_S(Q)|, \quad |(\Omega^*)^{\omega\circ\eta|_Q}| = |\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta}| = |C_S(Q)|.$$

[1]For the rest of the DRV-fusion systems found in Appendix A, the candidate solutions will also have the vast majority of the constants to be zero. Unlike here, in those cases we shall not demonstrate that the candidate solution is a solution.

Thus, if $|Q| > 3^3$, then the sizes are all equal. When $Q = \langle s,t \rangle$, then by ⓐ$_1$,..,ⓐ$_4$ and ⓕ$_1$,..,ⓕ$_4$, we have

$$
\begin{aligned}
&|(\Omega^*)^{\mathsf{Id}|_{\langle s,t\rangle}}| = |\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_{\langle s,t\rangle}}| = 3, \quad |(\Omega^*)^{\omega|_{\langle s,t\rangle}}| = |\mathcal{O}_{\mathsf{Id}}^{\omega|_{\langle s,t\rangle}}| = 3 \\
&|(\Omega^*)^{\eta|_{\langle s,t\rangle}}| = |\mathcal{O}_{\eta}^{\eta|_{\langle s,t\rangle}}| = 3, \quad |(\Omega^*)^{\omega\circ\eta|_{\langle s,t\rangle}}| = |\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta|_{\langle s,t\rangle}}| = 3 \\
&|(\Omega^*)^{\theta_0}| = |\mathcal{O}_{\theta_0}^{\theta_0}| = 3, \quad |(\Omega^*)^{\theta_0^{-1}}| = |\mathcal{O}_{\theta_0^{-1}}^{\theta_0^{-1}}| = 3 \\
&|(\Omega^*)^{\alpha_0}| = |\mathcal{O}_{\alpha_0}^{\alpha_0}| = 3, \quad |(\Omega^*)^{\beta_0}| = |\mathcal{O}_{\beta_0}^{\beta_0}| = 3,
\end{aligned}
$$

Thus, if $Q = \langle s,t \rangle$, the sizes are all equal. When $Q = \langle s,z \rangle$, then, by ⓑ$_1$,..,ⓑ$_4$ and ⓖ$_1$,..,ⓖ$_4$, we have

$$
\begin{aligned}
&|(\Omega^*)^{\mathsf{Id}|_{\langle s,z\rangle}}| = |\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_{\langle s,z\rangle}}| + n_k|\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}}^{\mathsf{Id}|_{\langle s,z\rangle}}| + [|\mathcal{O}_{\theta_0}^{\mathsf{Id}|_{\langle s,z\rangle}}| + |\mathcal{O}_{\theta_0^{-1}}^{\mathsf{Id}|_{\langle s,z\rangle}}|] = 3^2 + n_k \cdot 3^3 + 2 \cdot 3^2 = 3^{2k} \\
&|(\Omega^*)^{\omega|_{\langle s,z\rangle}}| = |\mathcal{O}_{\omega}^{\omega|_{\langle s,z\rangle}}| + n_k|\mathcal{O}_{\omega|_{\langle s,z\rangle}}^{\omega|_{\langle s,z\rangle}}| + [|\mathcal{O}_{\theta_0}^{\omega|_{\langle s,z\rangle}}| + |\mathcal{O}_{\theta_0^{-1}}^{\omega|_{\langle s,z\rangle}}|] = 3^2 + n_k \cdot 3^3 + 2 \cdot 3^2 = 3^{2k} \\
&|(\Omega^*)^{\eta|_{\langle s,z\rangle}}| = |\mathcal{O}_{\eta}^{\eta|_{\langle s,z\rangle}}| + n_k|\mathcal{O}_{\eta|_{\langle s,z\rangle}}^{\eta|_{\langle s,z\rangle}}| + [|\mathcal{O}_{\alpha_0}^{\eta|_{\langle s,z\rangle}}| + |\mathcal{O}_{\beta_0}^{\eta|_{\langle s,z\rangle}}|] = 3^2 + n_k \cdot 3^3 + 2 \cdot 3^2 = 3^{2k} \\
&|(\Omega^*)^{\omega\circ\eta|_{\langle s,z\rangle}}| = |\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta|_{\langle s,t\rangle}}| + n_k|\mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}^{\omega\circ\eta|_{\langle s,z\rangle}}| + [|\mathcal{O}_{\alpha_0}^{\omega\circ\eta|_{\langle s,z\rangle}}| + |\mathcal{O}_{\beta_0}^{\omega\circ\eta|_{\langle s,z\rangle}}|] = 3^2 + n_k \cdot 3^3 + 2 \cdot 3^2 = 3^{2k} \\
&|(\Omega^*)^{\theta_0|_{\langle s,z\rangle}}| = |\mathcal{O}_{\theta_0}^{\theta_0|_{\langle s,z\rangle}}| = 3^{2k}, \quad |(\Omega^*)^{\theta_0^{-1}|_{\langle s,z\rangle}}| = |\mathcal{O}_{\theta_0^{-1}}^{\theta_0^{-1}|_{\langle s,z\rangle}}| = 3^{2k} \\
&|(\Omega^*)^{\alpha_0|_{\langle s,z\rangle}}| = |\mathcal{O}_{\alpha_0}^{\alpha_0|_{\langle s,z\rangle}}| = 3^{2k}, \quad |(\Omega^*)^{\beta_0|_{\langle s,z\rangle}}| = |\mathcal{O}_{\beta_0}^{\beta_0|_{\langle s,z\rangle}}| = 3^{2k},
\end{aligned}
$$

Thus, if $Q = \langle s,z \rangle$, then the sizes are all equal. When $Q = \langle t,z \rangle$, then, by ⓒ$_1$,..,ⓒ$_4$ and ⓗ$_1$,..,ⓗ$_4$, we have

$$
\begin{aligned}
&|(\Omega^*)^{\mathsf{Id}|_{\langle t,z\rangle}}| = |\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_{\langle t,z\rangle}}| = 3^{2k}, \quad |(\Omega^*)^{\omega|_{\langle t,z\rangle}}| = |\mathcal{O}_{\mathsf{Id}}^{\omega|_{\langle t,z\rangle}}| = 3^{2k} \\
&|(\Omega^*)^{\eta|_{\langle t,z\rangle}}| = |\mathcal{O}_{\eta}^{\eta|_{\langle t,z\rangle}}| = 3^{2k}, \quad |(\Omega^*)^{\omega\circ\eta|_{\langle t,z\rangle}}| = |\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta|_{\langle t,z\rangle}}| = 3^{2k} \\
&|(\Omega^*)^{\theta_0|_{\langle t,z\rangle}}| = |\mathcal{O}_{\theta_0}^{\theta_0|_{\langle t,z\rangle}}| = 3^{2k}, \quad |(\Omega^*)^{\theta_0^{-1}|_{\langle t,z\rangle}}| = |\mathcal{O}_{\theta_0^{-1}}^{\theta_0^{-1}|_{\langle t,z\rangle}}| = 3^{2k} \\
&|(\Omega^*)^{\alpha_0|_{\langle t,z\rangle}}| = |\mathcal{O}_{\alpha_0}^{\alpha_0|_{\langle t,z\rangle}}| = 3^{2k}, \quad |(\Omega^*)^{\beta_0|_{\langle t,z\rangle}}| = |\mathcal{O}_{\beta_0}^{\beta_0|_{\langle t,z\rangle}}| = 3^{2k}.
\end{aligned}
$$

Thus, if $Q=\langle t,z\rangle$, then the sizes are equal. When $Q=\langle s\rangle$, then, by ⓓ$_1$,..,ⓓ$_4$ and ⓘ$_1$,..,ⓘ$_4$, we have[1]

$$
\begin{aligned}
|(\Omega^*)^{\mathsf{Id}|_{\langle s\rangle}}| &= [|\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_{\langle s\rangle}}|+|\mathcal{O}_{\eta}^{\mathsf{Id}|_{\langle s\rangle}}|]+n_k[|\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}}^{\mathsf{Id}|_{\langle s\rangle}}|+|\mathcal{O}_{\eta|_{\langle s,z\rangle}}^{\mathsf{Id}|_{\langle s\rangle}}|]+[|\mathcal{O}_{\theta_0}^{\mathsf{Id}|_{\langle s\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\mathsf{Id}|_{\langle s\rangle}}|+|\mathcal{O}_{\alpha_0}^{\mathsf{Id}|_{\langle s\rangle}}|+|\mathcal{O}_{\beta_0}^{\mathsf{Id}|_{\langle s\rangle}}|]\\
&= 2\cdot 3^2+n_k\cdot 2\cdot 3^3+2^2\cdot 3^2=2\cdot 3^{2k}\\
|(\Omega^*)^{\omega|_{\langle s\rangle}}| &= [|\mathcal{O}_{\omega}^{\omega|_{\langle s\rangle}}|+|\mathcal{O}_{\omega\circ\eta}^{\omega|_{\langle s\rangle}}]+n_k[|\mathcal{O}_{\omega|_{\langle s,z\rangle}}^{\omega|_{\langle s,z\rangle}}|+|\mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}^{\omega|_{\langle s\rangle}}|]+[|\mathcal{O}_{\theta_0}^{\omega|_{\langle s\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\omega|_{\langle s\rangle}}|+|\mathcal{O}_{\alpha_0}^{\omega|_{\langle s\rangle}}|+|\mathcal{O}_{\beta_0}^{\omega|_{\langle s\rangle}}|]\\
&= 2\cdot 3^2+n_k\cdot 2\cdot 3^3+2^2\cdot 3^2=2\cdot 3^{2k}\\
|(\Omega^*)^{\eta|_{\langle s\rangle}}| &= [|\mathcal{O}_{\eta}^{\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\mathsf{Id}}^{\eta|_{\langle s\rangle}}|]+n_k[|\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}}^{\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}}^{\eta|_{\langle s\rangle}}|]+[|\mathcal{O}_{\theta_0}^{\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\alpha_0}^{\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\beta_0}^{\eta|_{\langle s\rangle}}|]\\
&= 2\cdot 3^2+n_k\cdot 2\cdot 3^3+2^2\cdot 3^2=2\cdot 3^{2k}\\
|(\Omega^*)^{\omega\circ\eta|_{\langle s\rangle}}| &= [|\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\omega}^{\omega\circ\eta|_{\langle s\rangle}}|]+n_k[|\mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}^{\omega\circ\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\omega|_{\langle s,z\rangle}}^{\omega\circ\eta|_{\langle s\rangle}}|]\\
&\quad +[|\mathcal{O}_{\theta_0}^{\omega\circ\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\omega\circ\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\alpha_0}^{\omega\circ\eta|_{\langle s\rangle}}|+|\mathcal{O}_{\beta_0}^{\omega\circ\eta|_{\langle s\rangle}}|]=2\cdot 3^2+n_k\cdot 2\cdot 3^3+2^2\cdot 3^2=2\cdot 3^{2k}\\
|(\Omega^*)^{\theta_0|_{\langle s\rangle}}| &= |\mathcal{O}_{\theta_0}^{\theta_0|_{\langle s\rangle}}|+|\mathcal{O}_{\alpha_0}^{\theta_0|_{\langle s\rangle}}|=2\cdot 3^{2k},\quad |(\Omega^*)^{\theta_0^{-1}|_{\langle s\rangle}}|=|\mathcal{O}_{\theta_0^{-1}}^{\theta_0^{-1}|_{\langle s\rangle}}|+|\mathcal{O}_{\beta_0}^{\theta_0^{-1}|_{\langle s\rangle}}|=2\cdot 3^{2k}\\
|(\Omega^*)^{\alpha_0|_{\langle s\rangle}}| &= |\mathcal{O}_{\alpha_0}^{\alpha_0|_{\langle s\rangle}}|+|\mathcal{O}_{\theta_0}^{\alpha_0|_{\langle s\rangle}}|=2\cdot 3^{2k},\quad |(\Omega^*)^{\beta_0|_{\langle s\rangle}}|=|\mathcal{O}_{\beta_0}^{\beta_0|_{\langle s\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\beta_0|_{\langle s\rangle}}|=2\cdot 3^{2k}.
\end{aligned}
$$

Thus, if $Q=\langle s\rangle$, then the sizes are all equal. When $Q=\langle t\rangle$, then, by ⓔ$_1$,..,ⓔ$_4$ and ⓙ$_1$,..,ⓙ$_4$, we have[2]

$$
\begin{aligned}
&|(\Omega^*)^{\mathsf{Id}|_{\langle t\rangle}}|=|\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_{\langle t\rangle}}|+|\mathcal{O}_{\omega\circ\eta}^{\mathsf{Id}|_{\langle t\rangle}}|=2\cdot 3^{2k},\quad |(\Omega^*)^{\omega|_{\langle t\rangle}}|=|\mathcal{O}_{\mathsf{Id}}^{\omega|_{\langle t\rangle}}|+|\mathcal{O}_{\eta}^{\omega|_{\langle t\rangle}}|=2\cdot 3^{2k}\\
&|(\Omega^*)^{\eta|_{\langle t\rangle}}|=|\mathcal{O}_{\eta}^{\eta|_{\langle t\rangle}}|+|\mathcal{O}_{\omega}^{\eta|_{\langle t\rangle}}|=2\cdot 3^{2k},\quad |(\Omega^*)^{\omega\circ\eta|_{\langle t\rangle}}|=|\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta|_{\langle t\rangle}}|+|\mathcal{O}_{\mathsf{Id}}^{\omega\circ\eta|_{\langle t\rangle}}|=2\cdot 3^{2k}\\
&|(\Omega^*)^{\theta_0|_{\langle t\rangle}}|=|\mathcal{O}_{\theta_0}^{\theta_0|_{\langle t\rangle}}|+|\mathcal{O}_{\beta_0}^{\theta_0|_{\langle t\rangle}}|=2\cdot 3^{2k},\quad |(\Omega^*)^{\theta_0^{-1}|_{\langle t\rangle}}|=|\mathcal{O}_{\theta_0^{-1}}^{\theta_0^{-1}|_{\langle t\rangle}}|+|\mathcal{O}_{\alpha_0}^{\theta_0^{-1}|_{\langle t\rangle}}|=2\cdot 3^{2k}\\
&|(\Omega^*)^{\alpha_0|_{\langle t\rangle}}|=|\mathcal{O}_{\alpha_0}^{\alpha_0|_{\langle t\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\alpha_0|_{\langle t\rangle}}|=2\cdot 3^{2k},\quad |(\Omega^*)^{\beta_0|_{\langle t\rangle}}|=|\mathcal{O}_{\beta_0}^{\beta_0|_{\langle t\rangle}}|+|\mathcal{O}_{\theta_0}^{\beta_0|_{\langle t\rangle}}|=2\cdot 3^{2k}.
\end{aligned}
$$

[1]Since $\mathsf{Id}|_{\langle s\rangle}=\omega|_{\langle s\rangle}$, $\eta|_{\langle s\rangle}=\omega\circ\eta|_{\langle s\rangle}$, $\theta_0|_{\langle s\rangle}=\alpha_0|_{\langle s\rangle}$ and $\theta_0^{-1}|_{\langle s\rangle}=\beta_0|_{\langle s\rangle}$, it suffices to calculate for $|(\Omega^*)^{\mathsf{Id}|_{\langle s\rangle}}|$, $|(\Omega^*)^{\eta|_{\langle s\rangle}}|$, $|(\Omega^*)^{\theta_0|_{\langle t\rangle}}|$ and $|(\Omega^*)^{\theta_0^{-1}|_{\langle t\rangle}}|$. However we demonstrate it for all of them.

[2]Since $\theta_0|_{\langle t\rangle}=\beta_0|_{\langle t\rangle}$ and $\theta_0^{-1}|_{\langle t\rangle}=\alpha_0|_{\langle t\rangle}$, it suffices to calculate for $|(\Omega^*)^{\theta_0|_{\langle t\rangle}}|$ and $|(\Omega^*)^{\theta_0^{-1}|_{\langle t\rangle}}|$, other than those coming from restrictions of automorphisms of S. However we demonstrate it for all four of them.

Thus, when $Q=\langle t\rangle$, the sizes are all equal. When $Q=\langle z\rangle$, we have[1]

$$|(\Omega^*)^{\mathsf{Id}|_{\langle z\rangle}}| = [|\mathcal{O}_{\mathsf{Id}}^{\mathsf{Id}|_{\langle z\rangle}}|+|\mathcal{O}_{\omega}^{\mathsf{Id}|_{\langle z\rangle}}|]+n_k[|\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}}^{\mathsf{Id}|_{\langle z\rangle}}|+|\mathcal{O}_{\omega|_{\langle s,z\rangle}}^{\mathsf{Id}|_{\langle z\rangle}}|]+[|\mathcal{O}_{\theta_0}^{\mathsf{Id}|_{\langle z\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\mathsf{Id}|_{\langle z\rangle}}|] = 2\cdot 3^{2k+1}[3^{2k-2}-1]^2$$

$$|(\Omega^*)^{\omega|_{\langle z\rangle}}| = [|\mathcal{O}_{\omega}^{\omega|_{\langle z\rangle}}|+|\mathcal{O}_{\mathsf{Id}}^{\omega|_{\langle z\rangle}}|]+n_k[|\mathcal{O}_{\omega|_{\langle s,z\rangle}}^{\omega|_{\langle z\rangle}}|+|\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}}^{\omega|_{\langle z\rangle}}|]+[|\mathcal{O}_{\theta_0}^{\omega|_{\langle z\rangle}}|+|\mathcal{O}_{\theta_0^{-1}}^{\omega|_{\langle z\rangle}}|] = 2\cdot 3^{2k+1}[3^{2k-2}-1]^2$$

$$|(\Omega^*)^{\eta|_{\langle z\rangle}}| = [|\mathcal{O}_{\eta}^{\eta|_{\langle z\rangle}}|+|\mathcal{O}_{\omega\circ\eta}^{\eta|_{\langle z\rangle}}|]+n_k[|\mathcal{O}_{\omega|_{\langle s,z\rangle}}^{\eta|_{\langle z\rangle}}|+|\mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}^{\eta|_{\langle z\rangle}}|]+[|\mathcal{O}_{\alpha_0}^{\eta|_{\langle z\rangle}}|+|\mathcal{O}_{\beta_0}^{\eta|_{\langle z\rangle}}|] = 2\cdot 3^{2k+1}[3^{2k-2}-1]^2$$

$$\begin{aligned}|(\Omega^*)^{\omega\circ\eta|_{\langle z\rangle}}| &= [|\mathcal{O}_{\omega\circ\eta}^{\omega\circ\eta|_{\langle z\rangle}}|+|\mathcal{O}_{\eta}^{\omega\circ\eta|_{\langle z\rangle}}|]+n_k[|\mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}}^{\omega\circ\eta|_{\langle z\rangle}}|+|\mathcal{O}_{\eta|_{\langle s,z\rangle}}^{\omega\circ\eta|_{\langle z\rangle}}|]+[|\mathcal{O}_{\alpha_0}^{\omega\circ\eta|_{\langle z\rangle}}|+|\mathcal{O}_{\beta_0}^{\omega\circ\eta|_{\langle z\rangle}}|]\\ &= 2\cdot 3^{2k+1}[3^{2k-2}-1]^2.\end{aligned}$$

Thus, when $Q=\langle z\rangle$, then the sizes are all equal. It follows that Ω^* is a right characteristic biset. Now, we show that our candidate solution is minimal.

Minimality:

We now show that our candidate example is of minimal size for all possible constants in the general form Ω. First we define a few numbers which will be used later on. Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. Define

$$\sigma_S^{\psi|_{\langle s,t\rangle}} = \sum_{P\in\mathcal{D}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,t\rangle}}| \text{ and } \sigma_S^{\psi|_{\langle s,z\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq\langle s,z\rangle}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}|.$$

Now we determine the minimal size of the general form Ω, using the section on *Equations*. In order to determine minimality, we do not need to determine the minimal value of each constant. By Lemma $A.0.1$,[2] it suffices to ascertain the minimal value of a sum of a number of constants which correspond to morphisms with domains of equal size.

By Lemma 4.2.8, we begin by noting that $C(\mathsf{Id}),C(\omega),C(\eta),C(\omega\circ\eta)\geq 1$.

[1]Since $\omega|_{\langle z\rangle}=\mathsf{Id}|_{\langle z\rangle}$ and $\omega\circ\eta|_{\langle z\rangle}=\eta|_{\langle z\rangle}$, it suffices to check the cases for $|(\Omega^*)^{\mathsf{Id}|_{\langle z\rangle}}|$ and $|(\Omega^*)^{\eta|_{\langle z\rangle}}|$. However, we demonstrate it for all four of them.

[2]We justify this for each of the two upcoming examples.

$\underline{C(\theta_0), C(\theta_0^{-1}), C(\alpha_0), C(\beta_0)}$:

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle s,t\rangle}}| + |\Omega^{\omega|_{\langle s,t\rangle}}| + |\Omega^{\eta|_{\langle s,t\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle s,t\rangle}}| = |\Omega^{\theta_0}| + |\Omega^{\theta_0^{-1}}| + |\Omega^{\alpha_0}| + |\Omega^{\beta_0}| \\
\Longrightarrow\ & 3[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}] \\
&= 3[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)],
\end{aligned}
$$

where we have used ⓐ$_1$,..,ⓐ$_4$ and ⓕ$_1$,..,ⓕ$_4$.

This implies $[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \geq 2^2$. ①$_9$

Note that $|D_{\theta_0}| = |D_{\theta_0^{-1}}| = |D_{\alpha_0}| = |D_{\beta_0}| = 3^3$. Thus, by an application of Lemma $A.0.1$, the assignment of values $C(\theta_0) = C(\theta_0^{-1}) = C(\alpha_0) = C(\beta_0) = 1$, will give us a minimal right characteristic biset.

$\underline{C(\mathsf{Id}|_{\langle s,z\rangle}), C(\omega|_{\langle s,z\rangle}), C(\eta|_{\langle s,z\rangle}), C(\omega\circ\eta|_{\langle s,z\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle s,z\rangle}}| + |\Omega^{\omega|_{\langle s,z\rangle}}| + |\Omega^{\eta|_{\langle s,z\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle s,z\rangle}}| = |\Omega^{\theta_0|_{\langle s,z\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}| + |\Omega^{\alpha_0|_{\langle s,z\rangle}}| + |\Omega^{\beta_0|_{\langle s,z\rangle}}| \\
\Longrightarrow\ & 3^2[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + 3^3[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})] \\
&\quad + [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] + 2\cdot 3^2[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&= 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \\
\Longrightarrow\ & 3^3[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})] \\
&= (3^{2k} - 2\cdot 3^2)[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&\quad + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \\
&\quad - 3^2[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] - [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]
\end{aligned}
$$

where we have used ⓑ$_1$,..,ⓑ$_4$ and ⓖ$_1$,..,ⓖ$_4$.

Let $\psi \in \{\mathsf{Id}_S, \omega, \eta, \omega\circ\eta\}$. Then for all $P \in \mathcal{D}\backslash\{\langle s,z\rangle\}$, we have

$$|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| = \frac{|N_S(\langle s,z\rangle,P)||C_S(\langle s,z\rangle)}{|P|} = \frac{3|N_S(\langle s,t\rangle,P)||C_S(\langle s,t\rangle)}{|P|} = |\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,t\rangle}}|.$$

Thus, we have $[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] = 3[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]$. By using this fact and the substitution from ①$_9$, we have

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})]\\
=&(3^{2k-1} - 2\cdot 3)\left[3[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\right]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})]\\
&-3^2[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] - [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]\\
=&(3^{2k} - 3^3)[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + (3^{2k-1} - 3^2)[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})]\\
\geq&(3^{2k} - 3^3)[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)].
\end{aligned}$$

This gives us $[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})] \geq 2^2\cdot(3^{2k-3} - 1)$. ②$_9$

Note that $|D_{\mathsf{Id}|_{\langle s,z\rangle}}| = |D_{\omega|_{\langle s,z\rangle}}| = |D_{\eta|_{\langle s,z\rangle}}| = |D_{\omega\circ\eta|_{\langle s,z\rangle}}| = 3^2$. Thus, by an application of Lemma $A.0.1$, the assignment of values $C(\mathsf{Id}|_{\langle s,z\rangle}) = C(\omega|_{\langle s,z\rangle}) = C(\eta|_{\langle s,z\rangle}) = C(\omega\circ\eta|_{\langle s,z\rangle}) = 3^{2k-3} - 1$, will give us a minimal right characteristic biset.

We have therefore shown that our candidate solution is minimal by size. We now use our candidate solution to determine an upper bound on the exoticity index, and show that it is minimal by size.

Exoticity:

Note that $|\Omega^*|/|S| = 4(3^{2k-2} - 1)^2$. We use Lemma 4.3.12 to determine an upper bound on the exoticity index for our fusion system. We have,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{4(3^{2k-2}-1)^2}{3^i} \right\rfloor.$$

Let $n_i = \left\lfloor \frac{4(3^{2k-2}-1)^2}{3^i} \right\rfloor$. Now,

$$\begin{aligned} i = 1 &\implies n_i = 4 \cdot 3^{2k-3}(3^{2k-2}-2)+1 \\ 2 \leq i \leq 2k-2 &\implies n_i = 4 \cdot 3^{2k-2-i}[3^{2k-2}-2] \\ i = 2k-1 &\implies n_i = 4 \cdot 3^{2k-3} - 3 \\ 2k \leq i \leq 4k-4 &\implies n_i = 4 \cdot 3^{4k-4-i} - 1. \end{aligned}$$

Thus,

$$\begin{aligned} v_3(\mathsf{Sym}(|\Omega^*|/|S|)) =& [4 \cdot 3^{2k-3}(3^{2k-2}-2)+1] + \left[\sum_{i=2}^{2k-2} 4 \cdot 3^{2k-2-i}[3^{2k-2}-2]\right] \\ &+ [4 \cdot 3^{2k-3} - 3] + \left[\sum_{i=2k}^{4k-4} 4\dot{3}^{4k-4-i} - 1\right] \\ =& 2 \cdot 3^{4k-4} - 4 \cdot 3^{2k-2} - (2k-3). \end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω, realizing our fusion system. Then

$$\log_3|T:S| = 2[(4k+3)3^{4k-4} - (8k-6)3^{2k-2} + (2k+3)].$$

Thus, we have an upper bound for our exoticity index.

CHAPTER 9

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Appendices

APPENDIX A

PROOFS OF THE MAIN THEOREM

For each of the DRV-fusion systems, we will split the proofs into five parts. The last two parts, namely, *Minimality* and *Exoticity*, will be presented as separate subsections. In this appendix the proofs will be presented relatively briefly. We refer the reader to Section 8.1 for a detailed run through of a particular DRV-9 fusion system, as an example. We structure our proofs in the following way:

General form:
Here we will use Theorem 7.3.1 to state the general form of the right characteristic biset Ω, that is,

$$\Omega \cong \bigsqcup_{\phi \in \mathcal{V}} C(\phi)\mathcal{O}_{\phi}$$

for some $\mathcal{V}$, a set of morphisms in $\mathcal{F}$ containing a set of S-S-equivalence class representatives Γ for $\mathcal{F}$. Here $C(\phi) \geq 0$ are integer constants yet to be determined.

Equations:
Here we will use Appendix B extensively to determine a series of equations. For Ω to be a right characteristic biset we need to ensure that Lemma 4.2.7 holds. This means we need to ensure

$$|\Omega^{\mathrm{Id}|_{D_{\psi}}}| = |\Omega^{\psi}|$$

holds for all $\psi \in \mathcal{F}$. However, by Lemma 3.1.8, is suffices to ensure that the above equation holds for a set of morphisms that contain S-S-equivalence class representatives, and they are those in the set $\mathcal{V}$. Thus the idea is to write down the equations for

$$|\Omega^{\mathrm{Id}|_{D_{\psi}}}| \quad \text{and} \quad |\Omega^{\psi}|,$$

for all $\psi \in \mathcal{V}$. However, we shall write down the equations for about one fourth of them because it turns out that many of these morphisms occur in quadruples, and their equations are similar (though not the same) and the reader can provide the others easily. For example, if $\theta_0, \theta_0^{-1}, \alpha_0$ and β_0 are all in $\mathcal{V}$, we shall only write the equation for $|\Omega^{\theta_0}|$.

The procedure for calculating equations is as follows. Note that

$$|\Omega^{\psi}| = \sum_{\phi \in \mathcal{V}} C(\phi)|\mathcal{O}_{\phi}^{\psi}|.$$

This gives us a series of pairs $\{(\psi, \phi) \mid \phi \in \mathcal{V}\}$. For each pair (ψ, ϕ) we use the *Calculations Locator* on Page 296, in Appendix B, to determine the exact page number in which the calculation for $|\mathcal{O}_{\phi}^{\psi}|$ is to be found. Once that has been found we (mostly) replace $|\mathcal{O}_{\phi}^{\psi}|$ in the equation for $|\Omega^{\psi}|$, with the exact value given in Appendix B. Fortunately, in the majority of the cases, we have $|\mathcal{O}_{\phi}^{\psi}| = 0$. This makes the equation for $|\Omega^{\psi}|$ significantly simpler.

As has already been mentioned we shall only write down the equations for about one fourth of the maps. Readers may wish to confirm the calculation for the rest of them if they so wish. However, in some ways, for many of these equations, their calculations will be embedded in a series of simultaneous equations found under the section on *Minimality*. More on that below.

Candidate solution:

Here we will present our candidate minimal right characteristic biset which we shall write as Ω^*. Checking to see that a candidate solution is a right characteristic biset is a lot easier than finding one. To check Ω^* is a right characteristic biset we have to ensure that

$$|(\Omega^*)^{\mathrm{Id}|_{D\psi}}| = |(\Omega^*)^{\psi}|$$

holds. Checking takes a lot of paper space, so we leave it to the reader to satisfy for themselves that the equations do indeed hold. The reader may use the *Equations* section, or more fully, the *Calculations Locator* on page 296.

Minimality:

Here we will show that our candidate solution Ω^* is minimal with respect to size of right characteristic bisets. We will not show that it is uniquely minimal (up to isomorphism), though that may be the case. We use the *Equations* section extensively (or more fully the *Calculations Locator* on page 296). We will start with Lemma 4.3.5 from which we know that the constants corresponding to S-S-equivalent class representative of automorphisms of S are non zero. For example, if $\omega, \eta, \omega \circ \eta \in \mathrm{Aut}_{\mathcal{F}}(S)$, we will have

$$C(\mathrm{Id}_S), C(\omega), C(\eta), C(\omega \circ \eta) > 0.$$

Then for a set of morphisms, say, $\psi_1, \psi_2, \psi_3, \psi_4$, with $Q = D_{\psi_1} = D_{\psi_2} = D_{\psi_3} = D_{\psi_4}$, we shall use the following equality

$$|\Omega^{\mathrm{Id}|_Q}| + |\Omega^{\omega|_Q}| + |\Omega^{\eta|_Q}| + |\Omega^{\omega\circ\eta|_Q}| = |\Omega^{\psi_1}| + |\Omega^{\psi_2}| + |\Omega^{\psi_3}| + |\Omega^{\psi_4}|,$$

which comes from Lemma 4.2.7, and where Ω is the *General form* of the right characteristic biset for the fusion system. This equation will give us minimal values for a sum of constants, say, $C(\psi_1) + C(\psi_2) + C(\psi_3) + C(\psi_4)$. Then, if this sum is equal to the sum of the constants coming from the candidate solution Ω^*, then Lemma $A.0.1$ (2) tells us that we do not need to determine the minimal values of $C(\psi_1), C(\psi_2), C(\psi_3), C(\psi_4)$ individually. Based on this approach, step by step, we calculate minimal values for the rest of the constants.

Throughout the *Minimality* section we shall use Lemma $A.0.1$ (2) extensively, which suffices to ascertain the minimal values of a sum of a number of constants (corresponding to morphisms with domains of equal size). As mentioned earlier, the *Equations* section only calculates equations for about one fourth of them, as a guide. In order to determine minimality, we need to know the equations for all four of them. Though they are not all mentioned in the *Equations* section, they can be easily derived from the *Calculations Locator* on page 296, in Appendix B.

Exoticity:
Here we will determine an upper bound of the exoticity index derived from our candidate minimal solution Ω^*. We first determine $|\Omega^*|/|S|$. We then directly use Lemma 4.3.12 to determine an upper bound on the exoticity index coming from Ω^*.

Note that this will be an upper bound, and not necessarily the exact exoticity index, since there may be other methods of constructing finite groups that realize DRV-fusion systems. In fact, many of the DRV-fusion systems are not exotic,[1] meaning that their exoticity indices will be zero, although our constructions of minimal right characteristic bisets will give us large upper bounds on exoticity indices.

This completes the structure of the proofs. We now prove Lemma $A.0.1$:

Lemma A.0.1. *Let $\mathcal{F}$ be a fusion system. Let $\mathcal{C}$ be a set of (or a set that contains) representatives of $S \times S$-classes of morphisms in $\mathcal{F}$. Let $\mathcal{P}$ be a partition of $\mathcal{C}$. Let Ω be a right*

[1]See [5, Table 6, Theorem 5.9], which actually lists the groups which realize the non-exotic fusion systems.

semicharacteristic biset, and write $\Omega = \sqcup_{\phi \in \mathcal{C}} C_\Omega(\phi)\mathcal{O}_\phi$*, for constants* $C_\Omega(\phi) \geq 0$ *which depend on* Ω*. For each* $I \in \mathcal{P}$*, write* $\Omega_I = \sqcup_{\phi \in I} C_\Omega(\phi)\mathcal{O}_\phi$*.*

1. *Suppose that for all* $I \in \mathcal{P}$*,* $|\Omega_I| \geq M_I \geq 0$*, where* M_I *is a constant independent of* Ω*. Then*

$$|\Omega| \geq \sum_{I \in \mathcal{P}} M_I.$$

   In particular, any right semicharacteristic biset Ω *that satisfies* $|\Omega_I| = M_I$*, for all* $I \in \mathcal{P}$*, is minimal.*

2. *Suppose that for every* $I \in \mathcal{P}$*, we have* $|D_{\phi_1}| = |D_{\phi_2}|$ *whenever* $\phi_1, \phi_2 \in I$*. Set* $D_I = |D_\phi|$ *for all* $\phi \in I$*. Suppose that* $\sum_{\phi \in I} C_\Omega(\phi) \geq N_I$*, where* N_I *is a constant independent of* Ω*. Then*

$$\frac{|\Omega|}{|S|} \geq \sum_{I \in \mathcal{P}} \frac{N_I |S|}{D_I}.$$

   In particular, any right semicharacteristic biset Ω *that satisfies* $\sum_{\phi \in I} C_\Omega(\phi) = N_I$*, for all* $I \in \mathcal{P}$*, is minimal.*

Proof. 1. Trivially,

$$|\Omega| = \sum_{I \in \mathcal{P}} |\Omega_I| \geq \sum_{I \in \mathcal{P}} M_I.$$

Suppose that for some right semicharacteristic biset Ω, we have $|\Omega_I| = M_I$ for all $I \in \mathcal{P}$. Then Ω is minimal because it achieves the minimality of the inequality just proved.

2. Note that

$$\begin{aligned}
\frac{|\Omega|}{|S|} &= \frac{1}{|S|} \sum_{\phi \in I, I \in \mathcal{P}} C_\Omega(\phi)|\mathcal{O}_\phi| = \frac{1}{|S|} \sum_{\phi \in I, I \in \mathcal{P}} C_\Omega(\phi) \frac{|S|^2}{|\Delta^\phi_{D_\phi}|} \\
&= \sum_{\phi \in I, I \in \mathcal{P}} C_\Omega(\phi) \frac{|S|}{|D_\phi|} = \sum_{\phi \in I, I \in \mathcal{P}} C_\Omega(\phi) \frac{|S|}{D_I} \\
&= \sum_{I \in \mathcal{P}} \left(\frac{|S|}{D_I} \sum_{\phi \in I} C_\Omega(\phi) \right) \geq \sum_{I \in \mathcal{P}} \frac{N_I |S|}{D_I}.
\end{aligned}$$

Suppose that for some right semicharacteristic biset Ω, we have $\sum_{\phi \in I} C_\Omega(\phi) = N_I$ for all $I \in \mathcal{P}$. Then Ω is minimal because it achieves the minimality of the inequality just proved.

□

A.1 DRV-1,2,3

General form:
By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\Omega \cong \bigsqcup_{Q\in\mathcal{D}} \left[C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \right] \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_\lambda \\ E_\lambda\in\mathcal{T}}} \left[C(\theta_\lambda|_Q)\mathcal{O}_{\theta_\lambda|_Q} \sqcup C(\theta_\lambda^{-1})\mathcal{O}_{\theta_\lambda^{-1}} \right]$$

Equations:
We list the following equations:

$\psi \in \{\theta_\lambda|_Q \mid Q\in\mathcal{D}_\lambda, \lambda=-1,0,1\}$:

Provided $E_\lambda\in\mathcal{T}$, we have

$$\begin{aligned}
|\Omega^{\theta_\lambda}| &= 3C(\theta_\lambda) \\
|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}}| &= 3^{2k}C(\theta_\lambda) + 3^{2k+1}C(\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}) \\
|\Omega^{\theta_\lambda|_{\langle t,z\rangle}}| &= 3^{2k}C(\theta_\lambda) + 3^{2k+1}C(\theta_\lambda|_{\langle t,z\rangle}) \\
|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda\rangle}}| &= 3^{2k}C(\theta_\lambda) + 3^{2k+1}C(\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}) + 3^{2k+1}C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}) \\
|\Omega^{\theta_\lambda|_{\langle t\rangle}}| &= 3^{2k}C(\theta_\lambda) + 3^{2k+1}C(\theta_\lambda|_{\langle t,z\rangle}) + 3^{2k+1}C(\theta_\lambda|_{\langle t\rangle})
\end{aligned}$$

$\psi \in \{\mathsf{Id}|_Q \mid Q\in\mathcal{D}_\lambda, \lambda=-1,0,1\}$:

Provided $E_\lambda\in\mathcal{T}$, we have

$$\begin{aligned}
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda, t\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda, t\rangle}}| \\
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}} &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}}| + 3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})] \\
|\Omega^{\mathsf{Id}|_{\langle t,z\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| \\
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}| + 3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})] \\
|\Omega^{\mathsf{Id}|_{\langle t\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t\rangle}}|
\end{aligned}$$

Candidate solution:

We have the following designated values for the constants:

$$
\begin{aligned}
&C(\mathsf{Id}_S) = C(\omega) = 1 && \\
&C(\theta_\lambda) = C(\theta_\lambda^{-1}) = 1 && \text{(if } E_\lambda \in \mathcal{T}) \\
&C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) = C(\omega|_{\langle ss_1^\lambda, z\rangle}) = 3^{2k-3} - 1, && \text{(if } E_\lambda \in \mathcal{T})
\end{aligned}
$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$\Omega^* \cong (\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_\omega) \sqcup \bigsqcup_{E_\lambda \in \mathcal{T}} \left[(\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}) \sqcup n_k(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^\lambda, z\rangle}}) \right]$$

where $n_k = 3^{2k-3} - 1$. It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.1.1 Minimality

We write down a few definitions. Let $\psi \in \{\mathsf{Id}_S, \omega\}$. Define:

$$\sigma_S^{\psi|_{\langle ss_1^\lambda, t\rangle}} = \sum_{P \in \mathcal{D}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda, t\rangle}}| \text{ and } \sigma_S^{\psi|_{\langle ss_1^\lambda, z\rangle}} = \sum_{\substack{P \in \mathcal{D} \\ P \neq \langle ss_1^\lambda, z\rangle}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda, z\rangle}}|$$

Now we determine minimum values of the constants:

By Lemma 4.2.8, we already know that $C(\mathsf{Id}_S), C(\omega) \geq 1$.

<u>$C(\theta_\lambda), C(\theta_\lambda^{-1})$ for $\lambda = -1, 0, 1$:</u>

Provided $E_\lambda \in \mathcal{T}$, we have

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda, t\rangle}}| + |\Omega^{\omega|_{\langle ss_1^\lambda, t\rangle}}| = |\Omega^{\theta_\lambda}| + |\Omega^{\theta_\lambda^{-1}}| \\
\implies &3[C(\mathsf{Id}) + C(\omega)] + [\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda, t\rangle}} + \sigma_S^{\omega|_{\langle ss_1^\lambda, t\rangle}}] = 3[C(\theta_\lambda) + C(\theta_\lambda^{-1})] \qquad ①_{1,2,3}
\end{aligned}
$$

This gives us $[C(\theta_\lambda) + C(\theta_\lambda)] \geq 2$, as required.

<u>$C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}), C(\omega|_{\langle ss_1^\lambda, z\rangle})$ for $\lambda = -1, 0, 1$:</u>

Provided $E_\lambda \in \mathcal{T}$, we have

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\omega|_{\langle ss_1^\lambda,z\rangle}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}}| \\
\implies & 3^2[C(\mathsf{Id})+C(\omega)]+3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}] \\
&+2\cdot 3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})] \\
&=3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})]+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})] \\
\implies & 3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})] \\
&=(3^{2k}-2\cdot 3^2)[C(\theta_\lambda)+C(\theta_\lambda^{-1})]+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})] \\
&-3^2[C(\mathsf{Id})+C(\omega)]-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}]
\end{aligned}
$$

Let $\psi \in \{\mathsf{Id}_S, \omega\}$. Then for all $P \in \mathcal{D}\backslash\{\langle ss_1^\lambda, z\rangle\}$, we have

$$
|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda,z\rangle}}| = \frac{|N_S(\langle ss_1^\lambda,z\rangle,P)||C_S(\langle ss_1^\lambda,z\rangle)|}{|P|} = \frac{3|N_S(\langle ss_1^\lambda,t\rangle,P)||C_S(\langle ss_1^\lambda,t\rangle)|}{|P|} = 3|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda,t\rangle}}|.
$$

Thus, $[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}] = 3[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}]$. By using this fact and the substitution from ①, we have

$$
\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})] \\
=&(3^{2k-1}-2\cdot 3)\left[3[C(\mathsf{Id}_S)+C(\omega)]+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}]\right] \\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})]-3^2[C(\mathsf{Id})+C(\omega)]-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}] \\
=&(3^{2k}-3^3)[C(\mathsf{Id}_S)+C(\omega)]+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})] \\
&+(3^{2k-1}-3^2)[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}] \\
\geq&(3^{2k}-3^3)[C(\mathsf{Id}_S)+C(\omega)]
\end{aligned}
$$

It follows that $[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})] \geq 2\cdot(3^{2k-3}-1)$, as required. ②$_{1,2,3}$

We have therefore shown that our candidate solution is of minimal size.

A.1.2 Exoticity

Note that $|\Omega^*|/|S| = 2[m(3^{4k-4}-2\cdot 3^{2k-2})+1]$, where $m = |\mathcal{T} \cap \{E_\lambda \mid \lambda = -1,0,1\}|$.

We first consider the DRV-1 fusion system. In that case $|\Omega^*|/|S| = 2(3^{2k-2}-1)^2$. Thus,

$$\begin{aligned}
v_3(\mathsf{Sym}(|\Omega^*|/|S|)) &= \sum_{i=1} \left\lfloor \frac{2(3^{2k-2}-1)^2}{3^i} \right\rfloor \\
&= \sum_{i=1} \left\lfloor 2(3^{4k-4-i} + \frac{1-2\cdot 3^{2k-2}}{3^i}) \right\rfloor
\end{aligned}$$

Write $n_i = \left\lfloor 2(3^{4k-4-i} + \frac{1-2\cdot 3^{2k-2}}{3^i}) \right\rfloor$. Now,

$$\begin{aligned}
1 \le i \le 2k-2 &\implies n_i = 2\cdot 3^{2k-2-i}[3^{2k-2}-2] \\
i = 2k-1 &\implies n_{2k-1} = 2[3^{2k-3}-1] \\
2k \le i \le 4k-4 &\implies n_i = 2\cdot 3^{4k-4-i} - 1
\end{aligned}$$

Thus,

$$\begin{aligned}
v_3(\mathsf{Sym}(|\Omega^*|/|S|)) &= \left[\sum_{i=1}^{2k-2} 2\cdot 3^{2k-2-i}[3^{2k-2}-2]\right] + \left[\sum_{2k}^{4k-4} 2\cdot 3^{4k-4-i} - 1\right] \\
&+ 2[3^{2k-3}-1] \\
&= [(3^{2k-2}-2)(3^{2k-2}-1)] + [(3^{2k-3}-1) - (2k-3)] \\
&+ 2[3^{2k-3}-1] \\
&= (3^{2k-2}-1)^2 - 2k + 1
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3|T:S| &= [2(3^{2k-2}-1)^2 - 1](2k+1) + (3^{2k-2}-1)^2 - 2k + 1 \\
&= (3^{2k-2}-1)^2(4k+3) - 4k
\end{aligned}$$

This is our upper bound for the exoticity index of DRV-1 fusion system.

We now consider the DRV-2 fusion system. In that case $|\Omega^*|/|S| = 2[2\cdot 3^{4k-4} - 2^2\cdot 3^{2k-2} + 1]$. Thus,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{2[2\cdot 3^{4k-4} - 2^2\cdot 3^{2k-2} + 1]}{3^i} \right\rfloor$$

Write $n_i = \left\lfloor \frac{2[2\cdot 3^{4k-4} - 2^2\cdot 3^{2k-2}+1]}{3^i} \right\rfloor$. Now,

$$\begin{aligned} 1 \le i \le 2k-2 &\implies n_i = 4\cdot 3^{4k-4} - i - 8\cdot 3^{2k-2-i} \\ i = 2k-1 &\implies n_i = 4\cdot 3^{2k-3} - 3 \\ 2k \le i \le 4k-4 &\implies n_i = 4\cdot 3^{4k-4-i} - 1 \\ i = 4k-3 &\implies n_i = 1 \end{aligned}$$

Thus,

$$\begin{aligned} v_3(\mathsf{Sym}(|\Omega^*|/|S|)) &= \left[\sum_{i=1}^{2k-2} 4\cdot 3^{4k-4} - i - 8\cdot 3^{2k-2-i}\right] + [4\cdot 3^{2k-3} - 3] + \left[\sum_{2k}^{4k-4} 4\cdot 3^{4k-4-i} - 1\right] + 1 \\ &= 2[3^{4k-4} - 2\cdot 3^{2k-2}] - (2k-3) \end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing the fusion system. Then

$$\begin{aligned} \log_3|T:S| &= (2[2\cdot 3^{4k-4} - 2^2\cdot 3^{2k-2}+1]-1)(2k+1) + 2[3^{4k-4} - 2\cdot 3^{2k-2}] - (2k-3) \\ &= 2[3^{4k-4} - 2\cdot 3^{2k-2}](4k+3) + 4 \end{aligned}$$

This is our upper bound for the exoticity index for the DRV-2 fusion system.

We now consider the DRV-3 fusion system. In that case $|\Omega^*|/|S| = 2[3^{4k-3} - 2\cdot 3^{2k-1} + 1]$. Thus,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{2[3^{4k-3} - 2\cdot 3^{2k-1}+1]}{3^i} \right\rfloor$$

Write $n_i = \left\lfloor \frac{2[3^{4k-3} - 2\cdot 3^{2k-1}+1]}{3^i} \right\rfloor$. Now,

$$\begin{aligned} 1 \le i \le 2k-1 &\implies n_i = 2\cdot 3^{4k-3-i} - 4\cdot 3^{2k-1-i} \\ i = 2k &\implies n_i = 2\cdot 3^{2k-3} - 2 \\ 2k+1 \le i \le 4k-3 &\implies n_i = 2\cdot 3^{4k-3-i} - 1 \end{aligned}$$

Thus,

$$\begin{aligned} v_3(\mathsf{Sym}(|\Omega^*|/|S|)) &= \left[\sum_{i=1}^{2k-1} 2\cdot 3^{4k-3-i} - 4\cdot 3^{2k-1-i}\right] + [2\cdot 3^{2k-3} - 2] + \left[\sum_{2k+1}^{4k-3} 2\cdot 3^{4k-3-i} - 1\right] \\ &= 3^{4k-3} - 2\cdot 3^{2k-1} - 2(k-1). \end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}\log_3|T:S| &= (2[3^{4k-3}-2\cdot 3^{2k-1}+1]-1)(2k+1)+3^{4k-3}-2\cdot 3^{2k-1}-2(k-1)\\ &= [3^{4k-3}-2\cdot 3^{2k-1}](4k+3)+3.\end{aligned}$$

This is our upper bound for the exoticity index of DRV-3 fusion system.

A.4 DRV-4

General form:
By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\Omega \cong \bigsqcup_{Q\in\mathcal{D}} \left[C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q}\right] \sqcup \bigsqcup_{Q\in\mathcal{D}_A} \left[C(\theta_A|_Q)\mathcal{O}_{\theta_A} \sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q}\right]$$

Equations:
We list the following equations:

$\psi \in \{\mathsf{Id}|_Q \mid Q \in \mathcal{D}_A\}$:

If $Q = \langle t\rangle$,

$$\begin{aligned} |\Omega^{\mathsf{Id}|_Q}| = &\sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}|^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}| \\ &+ \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\le P}} C(\theta_A|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\theta_A|_P}| + \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-2}\rangle\le P}} C(\theta_A^{-1}|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\theta_A^{-1}|_P}| \end{aligned}$$

If $|Q| \neq \langle t\rangle$,

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}|^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}|$$

$\psi \in \{\theta_A|_Q \mid Q \in \mathcal{D}_A\}$:

In any case we have

$$|\Omega^{\theta_A|_Q}| = \sum_{\substack{P\in\mathcal{D}_A\\ Q\le P}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}|$$

Candidate Solution:
We have the following designated values for the solution:

$$C(\mathsf{Id}_S) = C(\eta) = 1$$
$$C(\theta_A) = C(\theta_A^{-1}) = 1$$

Writing Ω^* to denote this solution, we have

$$\Omega^* \cong [\mathcal{O}_{\mathsf{Id}_S} \sqcup \mathcal{O}_\eta] \sqcup [\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}}].$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.4.1 Minimality

We show that our solution is minimal. Note that, by Lemma 4.2.8, we have $C(\mathsf{Id}), C(\eta) \geq 1$. Now,

$$\begin{aligned} & |\Omega^{\mathsf{Id}|_A}| + |\Omega^{\eta|_A}| = |\Omega^{\theta_A}| + |\Omega^{\theta_A^{-1}}| \\ \Longrightarrow\ & 3^{2k}[C(\mathsf{Id}_S) + C(\eta)] + 3^{2k+1}[C(\mathsf{Id}|_A) + C(\eta|_A)] = 3^{2k}[C(\theta_A) + C(\theta_A^{-1})] \end{aligned}$$

This gives $[C(\theta_A) + C(\theta_A^{-1})] \geq 2$, as required. ①$_4$

It follows that our candidate solution is a minimal one.

A.4.2 Exoticity

Note that $|\Omega^*|/|S| = 2^3$. Thus,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{2^3}{3^i} \right\rfloor = 2$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\log_3 |T : S| = (2^3 - 1)(2k+1) + 2 = 14k + 9$$

Thus, we have an upper bound for our exoticity index.

A.5 DRV-5

General form:
By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\Omega \cong \bigsqcup_{Q\in\mathcal{D}} \left[C(\mathrm{Id}|_Q)\mathcal{O}_{\mathrm{Id}|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \right] \sqcup \bigsqcup_{Q\in\mathcal{D}_{V_0}} \left[C(\theta_{V_0}|_Q)\mathcal{O}_{\theta_{V_0}} \sqcup C(\theta_{V_0}^{-1}|_Q)\mathcal{O}_{\theta_{V_0}^{-1}|_Q} \right]$$

Equations:
We list the following equations:

$\psi \in \{\mathrm{Id}|_Q \mid Q \in \mathcal{D}_{V_0}\}$:

$$\begin{aligned}
|\Omega^{\mathrm{Id}|_{\langle s,z\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathrm{Id}|_P)|\mathcal{O}_{\mathrm{Id}|_P}^{\mathrm{Id}|_{\langle s,z\rangle}}| \\
|\Omega^{\mathrm{Id}|_{\langle s\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathrm{Id}|_P)|\mathcal{O}_{\mathrm{Id}|_P}^{\mathrm{Id}|_{\langle s\rangle}}| + 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1})] \\
|\Omega^{\mathrm{Id}|_{\langle z\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathrm{Id}|_P)|\mathcal{O}_{\mathrm{Id}|_P}^{\mathrm{Id}|_{\langle z\rangle}}|
\end{aligned}$$

$\psi \in \{\theta_{V_0}|_Q \mid Q \in \mathcal{D}_{V_0}\}$:

$$\begin{aligned}
|\Omega^{\theta_{V_0}}| &= 3^2 C(\theta_{V_0}) \\
|\Omega^{\theta_{V_0}|_{\langle s\rangle}}| &= 3^{2k+1}C(\theta_{V_0}) + 3^{2k+2}C(\theta_{V_0}|_{\langle s\rangle}) \\
|\Omega^{\theta_{V_0}|_{\langle z\rangle}}| &= 3^{2k+1}C(\theta_V 0) + 3^{2k+2}C(\theta_{V_0}|_{\langle z\rangle})
\end{aligned}$$

Candidate solution:
We have the following values for the constants:

$$\begin{aligned}
&C(\mathrm{Id}) = C(\omega\circ\eta) = 1 \\
&C(\theta_{V_0}) = C(\theta_{V_0}^{-1}) = 1 \\
&C(\mathrm{Id}|_{\langle s\rangle}) = C(\omega\circ\eta|_{\langle s\rangle}) = 3^{2k-2} - 1,
\end{aligned}$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$\Omega^* \cong [\mathcal{O}_{\mathrm{Id}_S} \sqcup \mathcal{O}_{\omega\circ\eta}] \sqcup [\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}}] \sqcup n_k \cdot [\mathcal{O}_{\mathrm{Id}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}],$$

where $n_k = 3^{2k-2} - 1$. It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.5.1 Minimality

We start by defining a few things. Let $\psi \in \{\mathsf{Id}_S, \omega \circ \eta\}$.

$$\sigma_S^{\psi|_{\langle s,z\rangle}} = \sum_{P \in \mathcal{D}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| \text{ and } \sigma_S^{\psi|_{\langle s\rangle}} = \sum_{\substack{P \in \mathcal{D} \\ P \neq \langle s\rangle}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s\rangle}}|$$

Now, we determine the minimal values of the constants:

By Lemma 4.2.8, we already know that $C(\mathsf{Id}_S), C(\omega), C(\eta), C(\omega \circ \eta) \geq 1$.

$\underline{C(\theta_{V_0}), C(\theta_{V_0}^{-1}), C(\alpha_{V_0}), C(\beta_{V_0})}$:

$$\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle s,z\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle s,z\rangle}}| = |\Omega^{\theta_{V_0}}| + |\Omega^{\theta_{V_0}^{-1}}| \\
\Longrightarrow\ & 3^2[C(\mathsf{Id}_S) + C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] = 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1})]
\end{aligned}$$

This gives us $[C(\theta_{V_0}) + C(\theta_{V_0}^{-1})] \geq 2^2$, as required. ①$_5$

$\underline{C(\mathsf{Id}|_{\langle s\rangle}), C(\omega|_{\langle s\rangle}), C(\eta|_{\langle s\rangle}), C(\omega\circ\eta|_{\langle s\rangle})}$:

$$\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle s\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle s\rangle}}| = |\Omega^{\theta_{V_0}|_{\langle s\rangle}}| + |\Omega^{\theta_{V_0}^{-1}|_{\langle s\rangle}}| \\
\Longrightarrow\ & 3^2[C(\mathsf{Id}_S) + C(\omega\circ\eta)] + 3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}] + 2\cdot 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1})] \\
&= 3^{2k+1}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1})] + 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle})] \\
\Longrightarrow\ & 3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
&= (3^{2k+1} - 2\cdot 3^2)[C(\theta_{V_0}) + C(\theta_{V_0}^{-1})] + 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle})] \\
&\quad - 3^2[C(\mathsf{Id}_S) + C(\omega\circ\eta)] - [\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}]
\end{aligned}$$

Let $\psi \in \{\mathsf{Id}_S, \omega\circ\eta\}$. Then, for all $P \in \mathcal{D}\backslash\{\langle s\rangle\}$, we have

$$|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| = \frac{|N_S(\langle s,z\rangle, P)||C_S(\langle s,z\rangle)|}{|P|} = \frac{|N_S(\langle s\rangle, P)||C_S(\langle s\rangle)|}{|P|} = |\mathcal{O}_{\psi|_P}^{\psi|_{\langle s\rangle}}|.$$

Thus, by definition, we have $\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}} = \sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}$. By using this property, and by substituting in ①, we see that

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
=&(3^{2k-1}-2)\left[3^2[C(\mathsf{Id}_S)+C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]\right] \\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle})] - 3^2[C(\mathsf{Id}_S)+C(\omega\circ\eta)] - [\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}] \\
=&(3^{2k+1}-3^3)[C(\mathsf{Id}_S)+C(\omega\circ\eta)] + 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle})] + (3^{2k-1}-3)[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] \\
\geq&(3^{2k+1}-3^3)[C(\mathsf{Id}_S)+C(\omega\circ\eta)]
\end{aligned}$$

It follows that $[C(\mathsf{Id}_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \geq 2\cdot(3^{2k-2}-1)$, as required. ②$_5$

A.5.2 Exoticity

We have $|\Omega^*|/|S| = 2(3^{2k-1}-1)^2$. Thus,

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) &= \sum_{i=1} \left\lfloor \frac{2(3^{2k-1}-1)^2}{3^i} \right\rfloor \\
&= \sum_{i=1} \left\lfloor 2(3^{4k-2-i} + \frac{1-2\cdot 3^{2k-1}}{3^i}) \right\rfloor
\end{aligned}$$

Write $n_i = \left\lfloor 2(3^{4k-2-i} + \frac{1-2\cdot 3^{2k-1}}{3^i}) \right\rfloor$. Now,

$$\begin{aligned}
1 \leq i \leq 2k-1 &\implies n_i = 2\cdot 3^{2k-1-i}[3^{2k-1}-2] \\
i = 2k &\implies n_{2k} = 2\cdot 3^{2k-2}-2 \\
2k+1 \leq i \leq 4k-2 &\implies n_i = 2\cdot 3^{4k-2-i}-1
\end{aligned}$$

Thus

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = &\left[\sum_{i=1}^{2k-1} 2\cdot 3^{2k-1-i}[3^{2k-1}-2]\right] + \left[\sum_{i=2k+1}^{4k-2} 2\cdot 3^{4k-2-i}-1\right] \\
&+ (2\cdot 3^{4k-2}-2) \\
&= [(3^{2k-1}-2)(3^{2k-1}-1)] + [(3^{2k-2}-1)-(2k-2)] \\
&+ (2\cdot 3^{4k-2}-2) \\
&= 3^{2k-2}[3^{2k+1}-8]-(2k-1)
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}\log_3|T:S| &= [2(3^{2k-1}-1)^2-1](2k+1)+3^{2k-2}[3^{2k+1}-8]-(2k-1)\\ &= (3^{2k-1}-1)^2(4k+3)+2\cdot 3^{2k-2}(3^{2k}-1)-(4k+1).\end{aligned}$$

This is our upper bound for our exoticity index on our DRV-5 fusion system.

A.6 DRV-6

General form:
By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}\Omega \cong & \bigsqcup_{Q\in\mathcal{D}}\left[C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q}\sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q}\sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q}\sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q}\right]\\ &\sqcup\bigsqcup_{Q\in\mathcal{D}_A}\left[C(\theta_A|_Q)\mathcal{O}_{\theta_A|_Q}\sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q}\sqcup C(\alpha_A|_Q)\mathcal{O}_{\alpha_A|_Q}\sqcup C(\beta_A|_Q)\mathcal{O}_{\beta_A|_Q}\right]\end{aligned}$$

Equations:
We list the following equations:

$\psi\in\{\mathsf{Id}|_Q \mid Q\in\mathcal{D}_A\}$:

If $Q=\langle t\rangle$,

$$\begin{aligned}|\Omega^{\mathsf{Id}|_Q}| = & \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}|^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}| + C(\omega\circ\eta|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\omega\circ\eta|_P}|\\ &+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\le P}} C(\theta_A|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\theta_A|_P}|+C(\beta_A|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\beta_A|_P}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-2}\rangle\le P}} C(\theta_A^{-1}|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\theta_A^{-1}|_P}|+C(\alpha_A|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\alpha_A|_P}|\end{aligned}$$

If $|Q|>3$ and $Q\le\langle s_1\rangle,\langle s_1^s\rangle,\langle s_1^{s^{-1}}\rangle$,

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}|^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}| + C(\omega\circ\eta|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\omega\circ\eta|_P}|$$

If either $Q=\langle z\rangle$, or $|Q|>3$ with $Q\le\langle s_2\rangle,\langle s_2^s\rangle,\langle s_2^{s^{-1}}\rangle$,

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}|^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}| + C(\omega|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\omega|_P}|$$

If $|Q|>3$ and $Q\le\langle d_1\rangle,\langle d_1^s\rangle,\langle d_1^{s^{-1}}\rangle$,

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}| + \sum_{\substack{P\in\mathcal{D}_A\\ Q^g\le P\cap\langle d_1\rangle}} C(\beta_A|_P)|\mathcal{O}^{\mathsf{Id}|_P}_{\beta_A|_P}|$$

If $|Q|>3$ and $Q\le\langle d_2\rangle,\langle d_2^s\rangle,\langle d_2^{s^{-1}}\rangle$,

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}| + \sum_{\substack{P\in\mathcal{D}_A\\ Q^g\le P\cap\langle d_2\rangle}} C(\alpha_A|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\alpha_A|_P}|$$

If Q satisfies none of the above properties, then

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P \in \mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_Q}_{\mathsf{Id}|_P}|$$

<u>$\psi \in \{\theta_A|_Q \mid Q \in \mathcal{D}_A\}$</u>:

Define

$$\mathbb{D}_1 = \{(Q,P) \in \mathcal{D}_A \times \mathcal{D}_A \mid |Q| > 3 \text{ and } Q \leq \langle s_1\rangle \cap P, \langle d_1\rangle^s \cap P, \langle d_2\rangle^{s^{-1}} \cap P, \langle s_1\rangle^s \cap P^{s^{-1}}, \langle s_1\rangle^{s^{-1}} \cap P\}$$
$$\mathbb{D}_2 = \{(Q,P) \in \mathcal{D}_A \times \mathcal{D}_A \mid |Q| > 3 \text{ and } Q \leq \langle s_2\rangle^s \cap P, \langle d_1\rangle^{s^{-1}} \cap P, \langle d_2\rangle^s \cap P, \langle s_2\rangle^{s^{-1}} \cap P^{s^{-1}}, \langle s_2\rangle \cap P^s\}$$

If $Q = \langle t\rangle$,

$$|\Omega^{\theta_A|_Q}| = \sum_{\substack{P \in \mathcal{D}_A \\ Q \leq P}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}| + C(\alpha_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\alpha_A|_P}|$$

If $Q = \langle z\rangle$,

$$|\Omega^{\theta_A|_Q}| = \sum_{\substack{P \in \mathcal{D}_A \\ Q \leq P}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}| + C(\beta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\beta_A|_P}|$$

If $|Q| > 3$ and $Q \leq \langle s_1\rangle, \langle s_1^s\rangle, \langle s_1^{s^{-1}}\rangle, \langle d_1^s\rangle, \langle d_2^{s^{-1}}\rangle$,

$$|\Omega^{\theta_A|_Q}| = \sum_{\substack{P \in \mathcal{D}_A \\ (Q,P) \in \mathbb{D}_1}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}| + C(\alpha_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\alpha_A|_P}|$$

If $|Q| > 3$ and $Q \leq \langle s_2\rangle, \langle s_2^s\rangle, \langle s_2^{s^{-1}}\rangle, \langle d_1^{s^{-1}}\rangle, \langle d_2^s\rangle$,

$$|\Omega^{\theta_A|_Q}| = \sum_{\substack{P \in \mathcal{D}_A \\ (Q,P) \in \mathbb{D}_2}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}| + C(\beta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\beta_A|_P}|$$

If $Q = \langle d_1\rangle$,

$$|\Omega^{\theta_A|_Q}| = \sum_{P \in \mathcal{D}} C(P)|\mathcal{O}^{\theta_A|_Q}_{\omega\circ\eta|_P}| + \sum_{\substack{P \in \mathcal{D}_A \\ Q \leq P}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}|$$

If $Q = \langle d_2\rangle$,

$$|\Omega^{\theta_A|_Q}| = \sum_{P \in \mathcal{D}} C(P)|\mathcal{O}^{\theta_A|_Q}_{\omega|_P}| + \sum_{\substack{P \in \mathcal{D}_A \\ Q \leq P}} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_Q}_{\theta_A|_P}|$$

If Q is none of the above,

$$|\Omega^{\theta_A|_Q}| = \sum_{\substack{P \in \mathcal{D}_A \\ Q \leq P}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_Q}|$$

<u>**Candidate solution**</u>:
We have the following designated values for the constants:

$$C(\mathsf{Id}_S) = C(\omega) = C(\eta) = C(\omega \circ \eta) = 1$$
$$C(\theta_A) = C(\theta_A^{-1}) = C(\alpha_A) = C(\beta_A) = 1,$$

with every other constant equated to 0. Writing Ω^* to denote this solution, we have

$$\Omega^* = (\mathcal{O}_{\mathsf{Id}|_S} \sqcup \mathcal{O}_\omega \sqcup \mathcal{O}_\eta \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}).$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.6.1 Minimality

We determine the minimal values of the constants. Note that $C(\mathsf{Id}), C(\omega), C(\eta), C(\omega \circ \eta) \geq 1$. Now,

$$\begin{aligned} &|\Omega^{\mathsf{Id}|_A}| + |\Omega^{\omega|_A}| + |\Omega^{\eta|_A}| + |\Omega^{\omega\circ\eta|_A}| = |\Omega^{\theta_A}| + |\Omega^{\theta_A^{-1}}| + |\Omega^{\alpha_A}| + |\Omega^{\beta_A}| \\ \Longrightarrow\ & 3^{2k}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega \circ \eta)] + 3^{2k+1}[C(\mathsf{Id}|_A) + C(\omega|_A) + C(\eta|_A) + C(\omega \circ \eta|_A)] \\ &= 3^{2k}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \end{aligned}$$

This gives $[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \geq 2^2$, as required. ①$_6$

It follows that our candidate solution is a minimal one.

A.6.2 Exoticity

Note that $|\Omega^*|/|S| = 2^4$. Thus,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{2^4}{3^i} \right\rfloor = 6$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\log_3|T:S| = (2^4-1)(2k+1)+6 = 30k+21$$

Thus, we have an upper bound for our exoticity index.

A.7 DRV-7

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}
\Omega \cong & \bigsqcup_{Q \in \mathcal{D}} C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega \circ \eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\
& \sqcup \bigsqcup_{\substack{Q \in \mathcal{D}_\lambda \\ \lambda=-1,1}} C(\theta_\lambda|_Q)\mathcal{O}_{\theta_\lambda|_Q} \sqcup C(\theta_\lambda^{-1}|_Q)\mathcal{O}_{\theta_\lambda^{-1}|_Q} \\
& \sqcup \bigsqcup_{\substack{Q \in \mathcal{D}_\lambda \\ \lambda=-1,1}} C(\theta_\lambda \circ \eta|_Q)\mathcal{O}_{\theta_\lambda\circ\eta|_Q} \sqcup C(\theta_\lambda^{-1} \circ \eta|_Q)\mathcal{O}_{\theta_\lambda\circ\eta|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\hat{\vartheta}_{4,\lambda}}
\end{aligned}$$

Equations:

We list the following equations:

$\underline{\psi \in \{\mathsf{Id}|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}}$:

$$\begin{aligned}
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda, t\rangle}}| &= \bigsqcup_{P \in \mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda, t\rangle}}| \\
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}}| &= \bigsqcup_{P \in \mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}}| + 3^2[C(\theta_\lambda) + C(\theta_\lambda^{-1})] \\
|\Omega^{\mathsf{Id}|_{\langle t, z\rangle}}| &= \bigsqcup_{P \in \mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t, z\rangle}}| \\
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}| &= \bigsqcup_{P \in \mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}| + 3^2[C(\theta_\lambda) + C(\theta_\lambda^{-1})] + 3^3 C(\hat{\theta}_{V_\lambda}) \\
|\Omega^{\mathsf{Id}|_{\langle t\rangle}}| &= \bigsqcup_{P \in \mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t\rangle}}| + C(\omega \circ \eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\mathsf{Id}|_{\langle t\rangle}}|]
\end{aligned}$$

$\underline{\psi \in \{\theta_\lambda|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}}$:

$$
\begin{aligned}
|\Omega^{\theta_\lambda}| &= 3C(\theta_\lambda)\\
|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}}| &= 3^{2k}C(\theta_\lambda) + 3^{2k+1}C(\theta_\lambda|_{\langle ss_1^\lambda, z\rangle})\\
|\Omega^{\theta_\lambda|_{\langle t, z\rangle}}| &= 3^{2k}C(\theta_\lambda) + 3^{2k+1}C(\theta_\lambda|_{\langle t, z\rangle})\\
|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda\rangle}}| &= 3^{2k}[C(\theta_\lambda) + C(\theta_\lambda^{-1}\circ\eta)] + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda, z\rangle})]\\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}) + C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
|\Omega^{\theta_\lambda|_{\langle t\rangle}}| &= 3^{2k}[C(\theta_\lambda) + C(\theta_{-\lambda}\circ\eta)] + 3^{2k+1}[C(\theta_\lambda|_{\langle t, z\rangle}) + C(\theta_{-\lambda}\circ\eta|_{\langle t, z\rangle})]\\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle t\rangle}) + C(\theta_{-\lambda}\circ\eta|_{\langle t\rangle})]
\end{aligned}
$$

$\underline{\psi \in \{\theta_\lambda\circ\eta|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}}$:

$$
\begin{aligned}
|\Omega^{\theta_\lambda\circ\eta}| &= 3C(\theta_\lambda\circ\eta)\\
|\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda, z\rangle}}| &= 3^{2k}C(\theta_\lambda\circ\eta) + 3^{2k+1}C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda, z\rangle})\\
|\Omega^{\theta_\lambda\circ\eta|_{\langle t, z\rangle}}| &= 3^{2k}C(\theta_\lambda\circ\eta) + 3^{2k+1}C(\theta_\lambda\circ\eta|_{\langle t, z\rangle})\\
|\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}}| &= 3^{2k}[C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1})] + 3^{2k+1}[C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle})]\\
&\quad + 3^{2k+1}[C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})]\\
|\Omega^{\theta_\lambda\circ\eta|_{\langle t\rangle}}| &= 3^{2k}[C(\theta_\lambda\circ\eta) + C(\theta_{-\lambda})] + 3^{2k+1}[C(\theta_\lambda\circ\eta|_{\langle t, z\rangle}) + C(\theta_{-\lambda}|_{\langle t, z\rangle})]\\
&\quad + 3^{2k+1}[C(\theta_\lambda\circ\eta|_{\langle t\rangle}) + C(\theta_{-\lambda}|_{\langle t\rangle})]
\end{aligned}
$$

$\underline{\psi \in \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda} \mid \lambda = -1, 1\}}$:

$$
\begin{aligned}
|\Omega^{\hat{\theta}_{V_\lambda}}| &= 3^3C(\hat{\theta}_{V_\lambda})\\
|\Omega^{\hat{\beta}_{V_\lambda}}| &= 3^3C(\hat{\beta}_{V_\lambda})\\
|\Omega^{\hat{\vartheta}_{1,\lambda}}| &= 3^3C(\hat{\vartheta}_{1,\lambda})\\
|\Omega^{\hat{\vartheta}_{4,\lambda}}| &= 3^3C(\hat{\vartheta}_{4,\lambda})
\end{aligned}
$$

Candidate solution:

We have the following designated values for the constants:

$$
\begin{aligned}
&C(\mathsf{Id}_S) = C(\omega) = C(\eta) = C(\omega\circ\eta) = 1\\
&C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) = C(\omega|_{\langle ss_1^\lambda, z\rangle}) = C(\eta|_{\langle ss_1^\lambda, z\rangle}) = C(\omega\circ\eta|_{\langle s_1^\lambda, z\rangle}) = 3^{2k-3} - 1\\
&C(\theta_\lambda) = C(\theta_\lambda^{-1}) = C(\theta_\lambda\circ\eta) = C(\theta_\lambda^{-1}\circ\eta) = 1\\
&C(\hat{\theta}_{V_\lambda}) = C(\hat{\beta}_{V_\lambda}) = C(\hat{\vartheta}_{1,\lambda}) = C(\hat{\vartheta}_{4,\lambda}) = 3^{2k-3},
\end{aligned}
$$

for all $\lambda = -1, 1$. We equate the rest of the constants to be zero. Writing Ω^* to denote this solution, we have

$$\begin{aligned}\Omega^* \cong\ & (\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\ & \sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)[\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}] \\ & \sqcup \bigsqcup_{\lambda=-1,1} [\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}] \sqcup [\mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}] \\ & \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}].\end{aligned}$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.7.1 Minimality

We start by defining a few things. Let $\psi \in \{\mathsf{Id}_S, \omega, \eta, \omega\circ\eta\}$. Then, we define

$$\sigma_S^{\psi|_{\langle ss_1^{\lambda},t\rangle}} = \sum_{P\in\mathcal{D}, P\neq S} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^{\lambda},t\rangle}}| \quad \text{and} \quad \sigma_S^{\psi|_{\langle ss_1^{\lambda},z\rangle}} = \sum_{\substack{P\in\mathcal{D} \\ P\neq S, \langle ss_1^{\lambda},z\rangle}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^{\lambda},z\rangle}}|$$

Now we determine the minimal values of the constants:

We already know by Lemma 4.2.8 that $C(\mathsf{Id}_S), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$.

<u>$C(\theta_\lambda), C(\theta_\lambda^{-1}), C(\theta_\lambda\circ\eta), C(\theta_\lambda^{-1}\circ\eta)$ for $\lambda = -1, 1$:</u>

$$\begin{aligned} & |\Omega^{\theta_\lambda}| + |\Omega^{\theta_\lambda^{-1}}| + |\Omega^{\theta_\lambda\circ\eta}| + |\Omega^{\theta_\lambda^{-1}\circ\eta}| = |\Omega^{\mathsf{Id}|_{\langle ss_1^{\lambda},t\rangle}}| + |\Omega^{\omega|_{\langle ss_1^{\lambda},t\rangle}}| + |\Omega^{\eta|_{\langle ss_1^{\lambda},t\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle ss_1^{\lambda},t\rangle}}| \\ \Longrightarrow\ & 3[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] \\ & = 3[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle ss_1^{\lambda},t\rangle}} + \sigma_S^{\omega|_{\langle ss_1^{\lambda},t\rangle}} + \sigma_S^{\eta|_{\langle ss_1^{\lambda},t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle ss_1^{\lambda},t\rangle}}]\end{aligned}$$

This gives us $[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] \geq 2^2$, as required. ①$_7$

$\underline{C(\hat{\theta}_{V_\lambda}), C(\hat{\beta}_{V_\lambda}), C(\hat{\vartheta}_{1,\lambda}), C(\hat{\vartheta}_{4,\lambda}) \text{ for } \lambda=-1,1:}$

$$
\begin{aligned}
&|\Omega^{\hat{\theta}_{V_\lambda}}|+|\Omega^{\hat{\beta}_{V_\lambda}}|+|\Omega^{\hat{\vartheta}_{1,\lambda}}|+|\Omega^{\hat{\vartheta}_{4,\lambda}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|\\
\implies & 3^3[C(\hat{\theta}_{V_\lambda})+C(\hat{\theta}_{V_\lambda}^{-1})+C(\hat{\vartheta}_{1,\lambda})+C(\hat{\vartheta}_{4,\lambda})]\\
&=3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&\quad+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]
\end{aligned}
$$

Thus, $[C(\hat{\theta}_{V_\lambda})+C(\hat{\theta}_{V_\lambda}^{-1})+C(\hat{\vartheta}_{1,\lambda})+C(\hat{\vartheta}_{4,\lambda})] \geq 2^2\cdot 3^{2k-3}$, as required. ②$_7$

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}), C(\omega|_{\langle ss_1^\lambda,z\rangle}), C(\eta|_{\langle ss_1^\lambda,z\rangle}), C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}) \text{ for } \lambda=-1,1:}$

Using substitution from ①$_7$ we have

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\omega|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\eta|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|\\
&=|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|\\
\implies & 3^2[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&\quad+3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&\quad+2\cdot 3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&\quad+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}]\\
&=3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&\quad+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
\implies & 3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&=(3^{2k-1}-2\cdot 3)\Big[3[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\Big]\\
&\quad+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&\quad-3^2[(C\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}]\\
&=(3^{2k}-3^3)[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&\quad+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&\quad+(3^{2k-1}-2\cdot 3)[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\\
&\quad-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}]
\end{aligned}
$$

Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. Then for all $P\in\mathcal{D}\backslash\{\langle ss_1^\lambda\rangle,\langle ss_1^\lambda,z\rangle\}$, we have

$$
3|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda,t\rangle}}|=\frac{3|N_S(\langle ss_1^\lambda,t\rangle,P)||C_S(\langle ss_1^\lambda,t\rangle)|}{|P|}=\frac{|N_S(\langle ss_1^\lambda,z\rangle,P)||C_S(\langle ss_1^\lambda,z\rangle)|}{|P|}=|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}|
$$

This implies $\sigma_S^{\psi|_{\langle ss_1^\lambda, z\rangle}} = 3\sigma_S^{\psi|_{\langle ss_1^\lambda, t\rangle}}$. Thus,

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
=&(3^{2k}-3^3)[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+(3^{2k-1}-3^2)[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\\
\geq&(3^{2k}-3^3)[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]
\end{aligned}$$

Thus, $[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \geq 2^2\cdot(3^{2k-3}-1)$. ⑦$_3$

Thus, our candidate solution that we have chosen is minimal.

A.7.2 Exoticity

Note that $|\Omega^*|/|S| = 4[2\cdot 3^{2k-2}-1]^2$. Thus,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1}\left\lfloor \frac{4(2\cdot 3^{2k-2}-1)^2}{3^i}\right\rfloor$$

Write $n_i = \left\lfloor \frac{4(2\cdot 3^{2k-2}-1)^2}{3^i}\right\rfloor$. Now,

$$\begin{aligned}
i=1 &\implies n_i = 16\cdot 3^{2k-3}(3^{2k-2}-1)+1\\
2\leq i\leq 2k-2 &\implies n_i = 16\cdot 3^{2k-2-i}(3^{2k-2}-1)\\
i=2k-1 &\implies n_i = 16\cdot 3^{2k-3}-6\\
i=2k &\implies n_i = 16\cdot 3^{2k-4}-2\\
2k+1\leq i\leq 4k-4 &\implies n_i = 16\cdot 3^{4k-4-i}-1 && (k\geq 3)\\
i=4k-3 &\implies n_i = \begin{cases}4 & \text{if } k=2\\ 5 & \text{if } k\geq 3\end{cases}\\
i=4k-2 &\implies n_i = 1
\end{aligned}$$

Then, when $k \geq 3$,

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) =& [16 \cdot 3^{2k-3}(3^{2k-2}-1)+1] + \left[\sum_{i=2}^{2k-2} 16 \cdot 3^{2k-2-i}[3^{2k-2}-]\right] \\
& + [16 \cdot 3^{2k-3}-6] + [16 \cdot 3^{2k-4}-2] \\
& + \left[\sum_{i=2k+1}^{4k-4} 16 \cdot 3^{4k-4-i}-1\right] + 6 \\
=& 8 \cdot 3^{2k-2}(3^{2k-2}-1)-(2k-3)
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3|T:S| &= \begin{cases} [4\cdot(3^{2k-2}-1)^2-1](2k+1)+[8\cdot 3^{2k-2}[3^{2k-2}-1]-(2k-2)] & \text{if } k=2 \\ [4\cdot(3^{2k-2}-1)^2-1](2k+1)+[8\cdot 3^{2k-2}[3^{2k-2}-1]-(2k-3)] & \text{if } k\geq 3 \end{cases} \\
&= \begin{cases} 6349 & \text{if } k=2 \\ 8\cdot 3^{2k-2}[3^{2k-2}-1][4k+3]+(4k+6) & \text{if } k\geq 3. \end{cases}
\end{aligned}$$

Thus, we have an upper bound on our exoticity index.

A.8 DRV-8

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}
\Omega \cong & \bigsqcup_{Q\in\mathcal{D}} C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\
& \sqcup \bigsqcup_{Q\in\mathcal{D}_A} C(\theta_A|_Q)\mathcal{O}_{\theta_A|_Q} \sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q} \sqcup C(\alpha_A|_Q)\mathcal{O}_{\alpha_A|_Q} \sqcup C(\beta_A|_Q)\mathcal{O}_{\beta_A|_Q} \\
& \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_\lambda \\ \lambda=-1,1 \\ \ell=0,1}} C(\theta_\lambda\circ\eta^\ell|_Q)\mathcal{O}_{\theta_\lambda\circ\eta^\ell|_Q} \sqcup C(\theta_\lambda^{-1}\circ\eta^\ell|_Q)\mathcal{O}_{\theta_\lambda^{-1}\circ\eta^\ell|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\hat{\vartheta}_{4,\lambda}} \\
& \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_{V_\lambda} \\ \lambda=-1,1}} C(\theta_{V_\lambda}|_Q)\mathcal{O}_{\theta_{V_\lambda}} \sqcup C(\theta_{V_\lambda}^{-1}|_Q)\mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup C(\alpha_{V_\lambda}|_Q)\mathcal{O}_{\alpha_{V_\lambda}|_Q} \sqcup C(\beta_{V_\lambda}|_Q)\mathcal{O}_{\beta_{V_\lambda}|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\Theta_{1,\lambda})\mathcal{O}_{\Theta_{1,\lambda}} \sqcup C(\Theta_{2,\lambda})\mathcal{O}_{\Theta_{2,\lambda}} \sqcup C(\Theta_{3,\lambda})\mathcal{O}_{\Theta_{3,\lambda}} \sqcup C(\Theta_{4,\lambda})\mathcal{O}_{\Theta_{4,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\Theta_{1,\lambda}^{-1})\mathcal{O}_{\Theta_{1,\lambda}} \sqcup C(\Theta_{2,\lambda}^{-1})\mathcal{O}_{\Theta_{2,\lambda}} \sqcup C(\Theta_{3,\lambda}^{-1})\mathcal{O}_{\Theta_{3,\lambda}} \sqcup C(\Theta_{4,\lambda}^{-1})\mathcal{O}_{\Theta_{4,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\vartheta_{1,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{2,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{3,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{4,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}}
\end{aligned}$$

Note that there is a difference between the construction in Theorem 7.3.1 and the one above. The difference is that we have excluded orbits induced by morphisms $\Theta_{\mu,\lambda}|_Q$, $\vartheta_{\mu,\lambda}|_Q$ for $Q \lneq \langle ss_1^\lambda, z\rangle$, and $\Theta_{\mu,\lambda}^{-1}|_Q$ for $Q \lneq \langle t,z\rangle$, for all $\mu = 1,..,4$ and $\lambda = -1,1$. This is because all of these morphisms are restrictions of those from the set

$$\{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T \mid T = V_\lambda, A, \lambda = -1,1\},$$

which are already included in the the general construction.

Equations:

We list the following equations:

$\psi \in \{\mathsf{Id}|_Q \mid Q \in \mathcal{D}_A \cup \mathcal{D}_\lambda, \lambda = -1,1\}$:

When $Q \in \mathcal{D}_A \backslash \{\langle z\rangle\}$, then the equations are the same as those for DRV-6. When $Q \in \mathcal{D}_\lambda \backslash \{\langle ss_1^\lambda\rangle, \langle t\rangle, \langle z\rangle\}$, for $\lambda = -1,1$, then the equations are the same as those for DRV-7. So, it suffices to consider the case $Q \in \{\langle ss_1^\lambda \rangle, \langle z \rangle \mid \lambda = -1, 1\}$:

$$
\begin{aligned}
|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda \rangle}}| &= \sum_{P \in \mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda \rangle}}| + 3^2[C(\theta_\lambda) + C(\theta_\lambda^{-1})] \\
&\quad + 3^2[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda}^{-1}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] + 3^3 C(\hat{\theta}_{V_\lambda}) \\
|\Omega^{\mathsf{Id}_{\langle z \rangle}}| &= \sum_{P \in \mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle z \rangle}}| + C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\mathsf{Id}|_{\langle z \rangle}}|] + \sum_{\substack{P \in \mathcal{D}_\lambda, \langle z \rangle \leq P \\ \lambda = -1, 1}} \frac{3^{4k+2}}{|P|}[C(\theta_\lambda|_P) + C(\theta_\lambda^{-1}|_P)] \\
&\quad + \sum_{\lambda = -1, 1} 3^{4k}[C(\hat{\vartheta}_{1,\lambda}) + C(\hat{\vartheta}_{4,\lambda})]
\end{aligned}
$$

<u>$\psi \in \{\theta_A|_Q \mid Q \in \mathcal{D}_A\}$</u>:

When $Q \in \mathcal{D}_A \backslash \{\langle t \rangle, \langle z \rangle\}$, then the equations are the same as those for DRV-6. So, we consider the cases when $Q \in \{\langle t \rangle, \langle z \rangle\}$:

$$
\begin{aligned}
|\Omega^{\theta_A|_{\langle t \rangle}}| &= \sum_{P \in \mathcal{D}_A} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_{\langle t \rangle}}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\theta_A|_{\langle t \rangle}}| \\
&\quad + 3^{4k-1}[C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1})] + 3^{4k-1}[C(\Theta_{2,-\lambda}^{-1}) + C(\Theta_{3,-\lambda}^{-1})] \\
|\Omega^{\theta_A|_{\langle z \rangle}}| &= \sum_{P \in \mathcal{D}_A} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_{\langle z \rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\theta_A|_{\langle z \rangle}}| \\
&\quad + 3^{4k-1}[C(\Theta_{1,\lambda}) + C(\Theta_{4,\lambda})] + 3^{4k-1}[C(\Theta_{1,-\lambda}) + C(\Theta_{4,-\lambda})]
\end{aligned}
$$

<u>$\psi \in \{\theta_\lambda|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}$</u>:

When $Q \in \mathcal{D}_\lambda \backslash \{\langle ss_1^\lambda \rangle, \langle t \rangle, \langle z \rangle\}$, for $\lambda = -1, 1$, then the equations are the same as those for DRV-7. So, we consider the cases when $Q \in \{\langle ss_1^\lambda \rangle, \langle t \rangle \mid \lambda = -1, 1\}$:

$$
\begin{aligned}
|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda \rangle}}| &= 3^{2k}[C(\theta_\lambda) + C(\theta_\lambda^{-1} \circ \eta)] + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda, z \rangle}) + C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda, z \rangle})] \\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda \rangle}) + C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda \rangle})] \\
&\quad + 3^{2k}[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \\
|\Omega^{\theta_\lambda|_{\langle t \rangle}}| &= 3^{2k}[C(\theta_\lambda) + C(\theta_{-\lambda}^{-1} \circ \eta)] + 3^{2k+1}[C(\theta_\lambda|_{\langle t, z \rangle}) + C(\theta_{-\lambda}^{-1} \circ \eta|_{\langle t, z \rangle})] \\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle t \rangle}) + C(\theta_{-\lambda}^{-1} \circ \eta|_{\langle t \rangle})] \\
&\quad + 3^{2k}[C(\Theta_{1,\lambda}^{-1}) + C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})]
\end{aligned}
$$

<u>$\psi \in \{\theta_\lambda \circ \eta|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}$</u>:

When $Q \in \mathcal{D}_\lambda \backslash \{\langle ss_1^\lambda \rangle, \langle t \rangle, \langle z \rangle\}$, for $\lambda = -1, 1$, then the equations are the same as those for DRV-7. So, we consider the cases when $Q \in \{\langle ss_1^\lambda \rangle, \langle t \rangle \mid \lambda = -1, 1\}$:

$$\begin{aligned}
|\Omega^{\theta_\lambda \circ \eta|_{\langle ss_1^\lambda \rangle}}| &= 3^{2k}[C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1})] + 3^{2k+1}[C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda, z \rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda, z \rangle})] \\
&+ 3^{2k+1}[C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda \rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda \rangle})] \\
&+ 3^{2k}[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \\
|\Omega^{\theta_\lambda \circ \eta|_{\langle t \rangle}}| &= 3^{2k}[C(\theta_\lambda \circ \eta) + C(\theta_{-\lambda})] + 3^{2k+1}[C(\theta_\lambda \circ \eta|_{\langle t, z \rangle}) + C(\theta_{-\lambda}|_{\langle t, z \rangle})] \\
&+ 3^{2k+1}[C(\theta_\lambda \circ \eta|_{\langle t \rangle}) + C(\theta_{-\lambda}|_{\langle t \rangle})] \\
&+ 3^{2k}[C(\Theta_{1,-\lambda}^{-1}) + C(\Theta_{2,-\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})]
\end{aligned}$$

<u>$\psi \in \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda} \mid \lambda = -1, 1\}$</u>:

In this case the equations are the same as those for DRV-7.

<u>$\psi \in \{\theta_{V_\lambda}|_Q \mid Q \in \mathcal{D}_{V_\lambda}, \lambda = -1, 1\}$</u>:

$$\begin{aligned}
|\Omega^{\theta_{V_\lambda}}| &= 3^2 C(\theta_{V_\lambda}) \\
|\Omega^{\theta_{V_\lambda}|_{\langle ss_1^\lambda \rangle}}| &= 3^{2k+1}[C(\theta_{V_\lambda}) + C(\alpha_{V_\lambda})] + 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda \rangle}) + C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda \rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,\lambda}) + C(\Theta_{3,\lambda})] \\
&+ 3^{2k+1}[C(\vartheta_{2,\lambda}) + C(\vartheta_{4,\lambda})] \\
|\Omega^{\theta_{V_\lambda}|_{\langle z \rangle}}| &= 3^{2k+1}[C(\theta_{V_\lambda}) + C(\beta_{V_\lambda})] + 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z \rangle}) + C(\beta_{V_\lambda}|_{\langle z \rangle})] \\
&+ 3^{2k+1}[C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \\
&+ 3^{2k+1}[C(\vartheta_{2,-\lambda}) + C(\vartheta_{3,-\lambda})]
\end{aligned}$$

<u>$\psi \in \{\Theta_{1,\lambda} \mid \lambda = -1, 1\}$</u>:

$$|\Omega^{\Theta_{1,\lambda}}| = 3^{2k} C(\Theta_{1,\lambda})$$

<u>$\psi \in \{\Theta_{1,\lambda}^{-1} \mid \lambda = -1, 1\}$</u>:

$$|\Omega^{\Theta_{1,\lambda}^{-1}}| = 3^{2k} C(\Theta_{1,\lambda}^{-1})$$

<u>$\psi \in \{\vartheta_{1,\lambda} \mid \lambda = -1, 1\}$</u>:

$$|\Omega^{\vartheta_{1\lambda}}| = 3^2 C(\vartheta_{1,\lambda})$$

<u>**Candidate solution**</u>:

We have the following designated valued for the constants:

$$
\begin{aligned}
&C(\mathsf{Id}_S) = C(\omega) = C(\eta) = C(\omega\circ\eta) = 1 \\
&C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) = C(\omega|_{\langle ss_1^\lambda, z\rangle}) = C(\eta|_{\langle ss_1^\lambda, z\rangle}) = C(\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle}) = 3^{2k-3}-1 && (\text{for } \lambda=-1,1) \\
&C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle}) = C(\omega|_{\langle ss_1^\lambda\rangle}) = C(\eta|_{\langle ss_1^\lambda\rangle}) = C(\omega\circ\eta|_{\langle t\rangle}) = 2^2\cdot 3^{4k-4} && (\text{for } \lambda=-1,1) \\
&C(\mathsf{Id}|_{\langle t\rangle}) = C(\omega|_{\langle t\rangle}) = C(\eta|_{\langle ss_1^\lambda\rangle}) = C(\omega\circ\eta|_{\langle t\rangle}) = 2 \\
&C(\theta_A) = C(\theta_A^{-1}) = C(\alpha_A) = C(\beta_A) = 1 \\
&C(\theta_A|_{\langle z\rangle}) = C(\theta_A^{-1}|_{\langle z\rangle}) = C(\alpha_A|_{\langle z\rangle}) = C(\beta_A|_{\langle z\rangle}) = 2(3^{2k-3}-1) \\
&C(\theta_\lambda) = C(\theta_\lambda^{-1}) = C(\theta_\lambda\circ\eta) = C(\theta_\lambda^{-1}\circ\eta) = 1 && (\text{for } \lambda=-1,1) \\
&C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}) = C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}) = C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}) = C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle}) = 2\cdot 3^{2k-2} && (\text{for } \lambda=-1,1) \\
&C(\theta_\lambda|_{\langle t\rangle}) = C(\theta_\lambda^{-1}|_{\langle t\rangle}) = C(\theta_\lambda\circ\eta|_{\langle t\rangle}) = C(\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}) = 2\cdot 3^{2k-2} && (\text{for } \lambda=-1,1) \\
&C(\theta_{V_\lambda}) = C(\theta_{V_\lambda}) = C(\alpha_{V_\lambda}) = C(\beta_{V_\lambda}) = 3^{2k-2} && (\text{for } \lambda=-1,1) \\
&C(\theta_{V_\lambda}|_{\langle z\rangle}) = C(\theta_{V_\lambda}|_{\langle z\rangle}) = C(\alpha_{V_\lambda}|_{\langle z\rangle}) = C(\beta_{V_\lambda}|_{\langle z\rangle}) = 3^{2k-2}(3^{2k-2}-2) && (\text{for } \lambda=-1,1) \\
&C(\Theta_{1,\lambda}) = C(\Theta_{2,\lambda}) = C(\Theta_{3,\lambda}) = C(\Theta_{4,\lambda}) = 1 && (\text{for } \lambda=-1,1) \\
&C(\hat\theta_{V_\lambda}) = C(\hat\beta_{V_\lambda}) = C(\hat\vartheta_{1,\lambda}) = C(\hat\vartheta_{4,\lambda}) = 3^{2k-3} && (\text{for } \lambda=-1,1) \\
&C(\Theta_{1,\lambda}^{-1}) = C(\Theta_{2,\lambda}^{-1}) = C(\Theta_{3,\lambda}^{-1}) = C(\Theta_{4,\lambda}^{-1}) = 1 && (\text{for } \lambda=-1,1) \\
&C(\vartheta_{1,\lambda}) = C(\Theta_{2,\lambda}) = C(\vartheta_{3,\lambda}) = C(\vartheta_{4,\lambda}) = 3^{2k-2}, && (\text{for } \lambda=-1,1)
\end{aligned}
$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$
\begin{aligned}
\Omega^* \cong &(\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 2^2\cdot 3^{4k-4}(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
&\sqcup 2(\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}) \\
&\sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 2(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_A|_{\langle z\rangle}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} (\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\theta_{V_\lambda}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}} \sqcup \mathcal{O}_{\beta_{V_\lambda}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_\lambda}|_{\langle z\rangle}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}(\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}) \\
&\sqcup \bigsqcup_{\substack{\lambda=-1,1\\ \ell=-1,1}} (\mathcal{O}_{\Theta^{\ell}_{1,\lambda}} \sqcup \mathcal{O}_{\Theta^{\ell}_{2,\lambda}} \sqcup \mathcal{O}_{\Theta^{\ell}_{3,\lambda}} \sqcup \mathcal{O}_{\Theta^{\ell}_{4,\lambda}}) \\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\vartheta_{2,\lambda}} \sqcup \mathcal{O}_{\vartheta_{3,\lambda}} \sqcup \mathcal{O}_{\vartheta_{4,\lambda}})
\end{aligned}
$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.8.1 Minimality

We write down a few definitions. Let $\psi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$. For $\lambda = -1, 1$, define

$$\begin{aligned}
\sigma_S^{\psi|_{\langle ss_1^\lambda\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq\langle ss_1^\lambda\rangle,\langle ss_1^\lambda,z\rangle,S}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda\rangle}}| \\
\sigma_S^{\psi|_{\langle z\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq\langle ss_1^\mu,z\rangle,S,\mu=-1,1}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}| + |C(\psi\circ\omega|_P)|\mathcal{O}_{\psi\circ\omega|_P}^{\psi|_{\langle z\rangle}}| \\
\sigma_A^{\psi|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P}} C(\phi_1|_P)|\mathcal{O}_{\phi_1|_P}^{\psi|_{\langle t\rangle}}| + C(\phi_2|_P)|\mathcal{O}_{\phi_2|_P}^{\psi|_{\langle t\rangle}}| + \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P}} C(\phi_3|_P)|\mathcal{O}_{\phi_3|_P}^{\psi|_{\langle t\rangle}}| + C(\phi_4|_P)|\mathcal{O}_{\phi_4|_P}^{\psi|_{\langle t\rangle}}|,
\end{aligned}$$

where

$$(\phi_1,\phi_2,\phi_3,\phi_4) = \begin{cases} (\theta_A, \beta_A, \alpha_A, \theta_A^{-1}) & \text{if } \psi = \mathrm{Id}_S \\ (\alpha_A, \theta_A^{-1}, \beta_A, \theta_A) & \text{if } \psi = \omega \\ (\theta_A^{-1}, \alpha_A, \theta_A, \beta_A) & \text{if } \psi = \eta \\ (\beta_A, \theta_A, \alpha_A, \theta_A^{-1}) & \text{if } \psi = \omega\circ\eta \end{cases}$$

Similarly, for $\psi \in \{\theta_A, \theta_A^{-1}, \alpha_A, \beta_A\}$, define

$$\begin{aligned}
\sigma_A^{\psi|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ P\neq A}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle t\rangle}}| + C(\psi_1|_P)|\mathcal{O}_{\psi_1|_P}^{\psi|_{\langle t\rangle}}| \\
\sigma_A^{\psi|_{\langle z\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ P\neq\langle z\rangle,A}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}| + C(\psi_2|_P)|\mathcal{O}_{\psi_2|_P}^{\psi|_P}|
\end{aligned}$$

where

$$\psi_1 = \begin{cases} \alpha_A & \text{if } \phi = \theta_A \\ \beta_A & \text{if } \phi = \theta_A^{-1} \\ \theta_A & \text{if } \phi = \alpha_A \\ \theta_A^{-1} & \text{if } \phi = \beta_A \end{cases} \text{ and } \psi_2 = \begin{cases} \beta_A & \text{if } \phi = \theta_A \\ \alpha_A & \text{if } \phi = \theta_A^{-1} \\ \theta_A & \text{if } \phi = \beta_A \\ \theta_A^{-1} & \text{if } \phi = \alpha_A \end{cases}$$

Now we determine minimal values of the constants. Note that, by Lemma 4.2.8, we have $C(\mathrm{Id}), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$.

$\underline{C(\theta_A), C(\theta_A^{-1}), C(\alpha_A), C(\beta_A)}$:

By ①$_6$, we have $[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \geq 2^2$. ①$_8$

$\underline{C(\theta_\lambda), C(\theta_\lambda^{-1}), C(\theta_\lambda \circ \eta), C(\theta_\lambda^{-1} \circ \eta) \text{ for } \lambda = -1, 1:}$

By ①$_7$, we have $[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1} \circ \eta)] \geq 2^2$. ②$_8$

$\underline{C(\hat{\theta}_{V_\lambda}), C(\hat{\beta}_{V_\lambda}), C(\hat{\vartheta}_{1,\lambda}), C(\hat{\vartheta}_{4,\lambda}) \text{ for } \lambda = -1, 1:}$

By ②$_7$, we have $[C(\hat{\theta}_{V_\lambda}) + C(\hat{\beta}_{V_\lambda}) + C(\hat{\vartheta}_{1,\lambda}) + C(\hat{\vartheta}_{4,\lambda})] \geq 2^2 \cdot 3^{2k-3}$. ③$_8$

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}), C(\omega|_{\langle ss_1^\lambda, z\rangle}), C(\eta|_{\langle ss_1^\lambda, z\rangle}), C(\beta|_{\langle ss_1^\lambda, z\rangle}) \text{ for } \lambda = -1, 1:}$

By ③$_7$, we have $[C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}), C(\omega|_{\langle ss_1^\lambda, z\rangle}), C(\eta|_{\langle ss_1^\lambda, z\rangle}), C(\beta|_{\langle ss_1^\lambda, z\rangle})] \geq 2^2 \cdot (3^{2k-3} - 1)$. ④$_8$

$\underline{C(\theta_{V_\lambda}), C(\theta_{V_\lambda}^{-1}), C(\alpha_{V_\lambda}), C(\beta_{V_\lambda}) \text{ for } \lambda = -1, 1:}$

$$\begin{aligned}
&|\Omega^{\theta_{V_\lambda}}| + |\Omega^{\theta_{V_\lambda}^{-1}}| + |\Omega^{\alpha_{V_\lambda}}| + |\Omega^{\beta_{V_\lambda}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}}| + |\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle}}| + |\Omega^{\theta_\lambda \circ \eta|_{\langle ss_1^\lambda, z\rangle}}| + |\Omega^{\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda, z\rangle}}| \\
\Longrightarrow\ & 3^2[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda}^{-1}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \\
&= 3^{2k}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1} \circ \eta)] \\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda, z\rangle})] \qquad ⑤_8
\end{aligned}$$

Thus, by using ①$_7$, we have $[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda}^{-1}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \geq 2^2 \cdot 3^{2k-2}$, as required.

$\underline{C(\Theta_{1,\lambda}), C(\Theta_{2,\lambda}), C(\Theta_{3,\lambda}), C(\Theta_{4,\lambda}) \text{ for } \lambda = -1, 1:}$

$$\begin{aligned}
&|\Omega^{\Theta_{1,\lambda}}| + |\Omega^{\Theta_{2,\lambda}}| + |\Omega^{\Theta_{3,\lambda}}| + |\Omega^{\Theta_{4,\lambda}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}}| + |\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle}}| + |\Omega^{\theta_\lambda \circ \eta|_{\langle ss_1^\lambda, z\rangle}}| + |\Omega^{\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda, z\rangle}}| \\
\Longrightarrow\ & 3^{2k}[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \\
&= 3^{2k}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1} \circ \eta)] \\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda, z\rangle}) + C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda, z\rangle})] \qquad ⑥_8
\end{aligned}$$

Thus, by using ①$_7$, we have $[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \geq 2^2$, as required.

$\underline{C(\Theta_{1,\lambda}^{-1}), C(\Theta_{2,\lambda}^{-1}), C(\Theta_{3,\lambda}^{-1}), C(\Theta_{4,\lambda}^{-1}) \text{ for } \lambda = -1, 1:}$

$$\begin{aligned}
&|\Omega^{\Theta_{1,\lambda}^{-1}}| + |\Omega^{\Theta_{2,\lambda}^{-1}}| + |\Omega^{\Theta_{3,\lambda}^{-1}}| + |\Omega^{\Theta_{4,\lambda}^{-1}}| = |\Omega^{\theta_\lambda|_{\langle t,z\rangle}}| + |\Omega^{\theta_\lambda^{-1}|_{\langle t,z\rangle}}| + |\Omega^{\theta_\lambda\circ\eta|_{\langle t,z\rangle}}| + |\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle}}| \\
\Longrightarrow\ & 3^{2k}[C(\Theta_{1,\lambda}^{-1}) + C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \\
&= 3^{2k}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1} \circ \eta)] \\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle t,z\rangle}) + C(\theta_\lambda^{-1}|_{\langle t,z\rangle}) + C(\theta_\lambda \circ \eta|_{\langle t,z\rangle}) + C(\theta_\lambda^{-1} \circ \eta|_{\langle t,z\rangle})] \qquad ⑦_8
\end{aligned}$$

Thus, by using $①_7$, we have $[C(\Theta_{1,\lambda}^{-1}) + C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \geq 2^2$, as required.

$\underline{C(\vartheta_{1,\lambda}), C(\vartheta_{2,\lambda}), C(\vartheta_{3,\lambda}), C(\vartheta_{4,\lambda}) \text{ for } \lambda = -1, 1:}$

$$\begin{aligned}
&|\Omega^{\vartheta_{1,\lambda}}| + |\Omega^{\vartheta_{2,\lambda}}| + |\Omega^{\vartheta_{3,\lambda}}| + |\Omega^{\vartheta_{4,\lambda}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}}| + |\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle}}| + |\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle}}| + |\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle}}| \\
\Longrightarrow\ & 3^{2}[C(\vartheta_{1,\lambda}) + C(\vartheta_{2,\lambda}) + C(\vartheta_{3,\lambda}) + C(\vartheta_{4,\lambda})] \\
&= 3^{2k}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1} \circ \eta)] \\
&\quad + 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}) + C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle}) + C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda,z\rangle}) + C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda,z\rangle})] \qquad ⑧_8
\end{aligned}$$

Thus, by using $①_7$, we have $[C(\vartheta_{1,\lambda}) + C(\vartheta_{2,\lambda}) + C(\vartheta_{3,\lambda}) + C(\vartheta_{4,\lambda})] \geq 2^2 \cdot 3^{2k-2}$, as required.

$\underline{C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})}$ for $\lambda=-1,1$:

$$
\begin{aligned}
&|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}}|=|\Omega^{\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle}}|\\
\Longrightarrow\ &3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+2\cdot3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
=&3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
\Longrightarrow\ &3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=&3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&-3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&-3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]
\end{aligned}
$$

Thus, by using substitution from ⑤$_8$, we have

$$
\begin{aligned}
&3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=&(3^{2k+1}-3^2)[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
\geq&(3^{2k+1}-3^2)[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
\geq&2^2[(3^{2k+1}-3^2)\cdot3^{2k-2}+3^{2k}+3^{2k+1}\cdot3^{2k-2}]=2^3\cdot3^{4k-1} \qquad ⑨_8
\end{aligned}
$$

It follows that $[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta)]\geq 2^3\cdot3^{2k-2}$, as required.

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle}), C(\omega|_{\langle ss_1^\lambda\rangle}), C(\eta|_{\langle ss_1^\lambda\rangle}), C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})\text{ for }\lambda=-1,1\text{:}}$

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\omega|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\eta|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}|\\
=&|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle}}|\\
\Longrightarrow\ & 3^2[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle})+C(\omega|_{\langle ss_1^\lambda\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}]\\
&+2\cdot3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+2\cdot3^2[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&2\cdot3^2[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^3[C(\hat\theta_{V_\lambda})+C(\hat\beta_{V_\lambda})+C(\hat\vartheta_{1,\lambda})+C(\hat\vartheta_{4,\lambda})]\\
=&2\cdot3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+2\cdot3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+2\cdot3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+2^2\cdot3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
\Longrightarrow\ & 3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle})+C(\omega|_{\langle ss_1^\lambda\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=&2\cdot3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+2\cdot3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+2\cdot3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+2^2\cdot3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&-3^2[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&-3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}]\\
&-2\cdot3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&-2\cdot3^2[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&-2\cdot3^2[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&-3^3[C(\hat\theta_{V_\lambda})+C(\hat\beta_{V_\lambda})+C(\hat\vartheta_{1,\lambda})+C(\hat\vartheta_{4,\lambda})]
\end{aligned}
$$

By using substitution from ②$_7$, ⑤$_8$, ⑥$_8$ and ⑧$_8$, we have

$$
\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle})+C(\omega|_{\langle ss_1^\lambda\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=&2\cdot 3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+2\cdot 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+2\cdot 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+2^2\cdot\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
&-3^2[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&-3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}]\\
&-2\cdot 3^2[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&-2\cdot\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
&-2\cdot\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
&-\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
=&(3^{2k}-2\cdot 3^2)[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+2\cdot 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&-3^2[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&-3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}]
\end{aligned}
$$

Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. Then for all $P\in\mathcal{D}\backslash\{\langle ss_1^\lambda\rangle\}$, we have

$$|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda\rangle}}|=\frac{|N_S(\langle ss_1^\lambda\rangle,P)||C_S(\langle ss_1^\lambda\rangle)|}{|P|}=\frac{|N_S(\langle ss_1^\lambda,z\rangle,P)||C_S(\langle ss_1^\lambda,z\rangle)|}{|P|}=|\mathcal{O}_{\psi|_P}^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}|$$

This implies $\sigma_S^{\psi|_{\langle ss_1^\lambda, z\rangle}} = \sigma_S^{\psi|_{\langle ss_1^\lambda\rangle}}$. Using substitution from ①$_7$ and ③$_7$, and then by applying the equality, we have

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle})+C(\omega|_{\langle ss_1^\lambda\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=&(3^{2k-1}-2\cdot 3)\Big[3[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\Big]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+2\cdot 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&-3^2[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&-\Big[(3^{2k}-3^3)[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&(3^{2k-1}-2\cdot 3)[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}]\Big]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}]\\
=&2\cdot 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}]\\
=&2\cdot 3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]
\end{aligned}$$

Thus, by ⑨$_8$, we have $[C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\geq 2^4\cdot 3^{4k-4}$, as required. ⑩$_8$

$\underline{C(\theta_{V_\lambda}|_{\langle z\rangle}), C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle}), C(\alpha_{V_\lambda}|_{\langle z\rangle}), C(\beta_{V_\lambda}|_{\langle z\rangle}) \text{ for } \lambda = -1,1:}$

$$
\begin{aligned}
&|\Omega^{\theta_{V_\lambda}|_{\langle z\rangle}}| + |\Omega^{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}}| = |\Omega^{\mathsf{Id}|_{\langle z\rangle}}| + |\Omega^{\eta|_{\langle z\rangle}}| \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda}^{-1}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \\
&+ 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle}) + C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_\lambda}|_{\langle z\rangle}) + C(\beta_{V_\lambda}|_{\langle z\rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,\lambda}^{-1}) + C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \\
&+ 3^{2k+1}[C(\vartheta_{1,-\lambda}) + C(\vartheta_{2,-\lambda}) + C(\vartheta_{3,-\lambda}) + C(\vartheta_{4,-\lambda})] \\
=& 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ \sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) + C(\omega|_{\langle ss_1^\lambda, z\rangle}) + C(\eta|_{\langle ss_1^\lambda, z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle})] \\
&+ [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
&+ \sum_{\mu=-1,1} \Big[3^{4k-1}[C(\theta_\mu) + C(\theta_\mu^{-1}) + C(\theta_\mu\circ\eta) + C(\theta_\mu^{-1}\circ\eta)] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu, z\rangle})] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}|_{\langle t, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle t, z\rangle})] \\
&+ 3^{4k+1}[C(\theta_\mu|_{\langle z\rangle}) + C(\theta_\mu^{-1}|_{\langle z\rangle}) + C(\theta_\mu\circ\eta|_{\langle z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat\theta_{V_\mu}) + C(\hat\beta_{V_\mu}) + C(\hat\vartheta_{1,\mu}) + C(\hat\vartheta_{4,\mu})] \\
\Longrightarrow\ & 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle}) + C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_\lambda}|_{\langle z\rangle}) + C(\beta_{V_\lambda}|_{\langle z\rangle})] \\
=& 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ \sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) + C(\omega|_{\langle ss_1^\lambda, z\rangle}) + C(\eta|_{\langle ss_1^\lambda, z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle})] \\
&+ [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
&+ \sum_{\mu=-1,1} \Big[3^{4k-1}[C(\theta_\mu) + C(\theta_\mu^{-1}) + C(\theta_\mu\circ\eta) + C(\theta_\mu^{-1}\circ\eta)] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu, z\rangle})] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}|_{\langle t, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle t, z\rangle})] \\
&+ 3^{4k+1}[C(\theta_\mu|_{\langle z\rangle}) + C(\theta_\mu^{-1}|_{\langle z\rangle}) + C(\theta_\mu\circ\eta|_{\langle z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat\theta_{V_\mu}) + C(\hat\beta_{V_\mu}) + C(\hat\vartheta_{1,\mu}) + C(\hat\vartheta_{4,\mu})] \\
&- 3^{2k+1}[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda}^{-1}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \\
&- 3^{2k+1}[C(\Theta_{1,\lambda}^{-1}) + C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \\
&- 3^{2k+1}[C(\vartheta_{1,-\lambda}) + C(\vartheta_{2,-\lambda}) + C(\vartheta_{3,-\lambda}) + C(\vartheta_{4,-\lambda})]
\end{aligned}
$$

By using substitution from ⑤$_8$, ⑦$_8$ and ⑧$_8$, note that

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda}|_{\langle z\rangle})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&-3^{2k-1}\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
&-3\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})]\Big]\\
&-3^{2k-1}\Big[3^{2k}[C(\theta_{-\lambda})+C(\theta_{-\lambda}^{-1})+C(\theta_{-\lambda}\circ\eta)+C(\theta_{-\lambda}^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_{-\lambda}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_{-\lambda}^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_{-\lambda}\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_{-\lambda}^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&-3^{2k+1}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&-3^{2k+2}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]
\end{aligned}
$$

By using substitution from ①$_7$, note that

$$
\begin{aligned}
& 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda|_{\langle z\rangle}})+C(\beta_{V_\lambda}|_{\langle z\rangle})] \\
=& 3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)] \\
& +\sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \\
& +[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}] \\
& +\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})] \\
& +3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
& -\Big[3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^{2k}[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\Big] \\
& -3^{2k+2}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})] \\
& +\sum_{\mu=-1,1} 3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})] \\
& =\sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \\
& +[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]-3^{2k+1}[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}] \\
& +\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})] \\
& +3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
& -3^{2k+2}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})] \\
& +\sum_{\mu=-1,1} 3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})] \\
\geq & \sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \\
& +[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]-3^{2k}[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}] \\
& +\sum_{\mu=-1,1} 3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]
\end{aligned}
$$

Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. For all $P\in\mathcal{D}$, note that

$$
\begin{aligned}
|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}|+|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle z\rangle}}| &= \frac{2|N_S(\langle z\rangle,P)||C_S(\langle z\rangle)|}{|P|} \\
&\geq \frac{2\cdot 3^{2k}|N_S(\langle ss_1^\lambda,t\rangle,P)||C_S(\langle ss_1^\lambda,t\rangle)|}{|P|}=3^{2k}[|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda,t\rangle}}|+|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle ss_1^\lambda,t\rangle}}|]
\end{aligned}
$$

It follows that $\sigma_S^{\psi|_{\langle z\rangle}} \geq \sigma_S^{\psi|_{\langle ss_1^\lambda,t\rangle}} + \sigma_S^{\psi\circ\omega|_{\langle ss_1^\lambda,t\rangle}}$. Thus

$$\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle}) + C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_\lambda|_{\langle z\rangle}}) + C(\beta_{V_\lambda}|_{\langle z\rangle})] \\
\geq &\sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}) + C(\omega|_{\langle ss_1^\lambda,z\rangle}) + C(\eta|_{\langle ss_1^\lambda,z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat{\theta}_{V_\mu}) + C(\hat{\beta}_{V_\mu}) + C(\hat{\vartheta}_{1,\mu}) + C(\hat{\vartheta}_{4,\mu})] \\
\geq &2^3\cdot 3^{4k}(3^{2k-3}-1) + 2^3\cdot 3^{4k}\cdot 3^{2k-3} = 2^3[2\cdot 3^{6k-3} - 3^{4k}]
\end{aligned}$$

Thus, $[C(\theta_{V_\lambda}|_{\langle z\rangle}) + C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_\lambda|_{\langle z\rangle}}) + C(\beta_{V_\lambda}|_{\langle z\rangle})] \geq 2^3[2\cdot 3^{4k-5} - 3^{2k-2}]$. 11_8

$\underline{C(\theta_A|_{\langle z\rangle}), C(\theta_A^{-1}|_{\langle z\rangle}), C(\alpha_A|_{\langle z\rangle}), C(\beta_A|_{\langle z\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\theta_A|_{\langle z\rangle}}| + |\Omega^{\theta_A^{-1}|_{\langle z\rangle}}| = |\Omega^{\mathsf{Id}|_{\langle z\rangle}}| + |\Omega^{\eta|_{\langle z\rangle}}| \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \\
&+3^{4k}[C(\theta_A|_{\langle z\rangle}) + C(\theta_A^{-1}|_{\langle z\rangle}) + C(\alpha_A|_{\langle z\rangle}) + C(\beta_A|_{\langle z\rangle})] + [\sigma_A^{\theta_A|_{\langle z\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}] \\
&+ \sum_{\mu=-1,1} \left[3^{4k-1}[C(\Theta_{1,\mu}) + C(\Theta_{2,\mu}) + C(\Theta_{3,\mu}) + C(\Theta_{4,\mu})]\right] \\
&= 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ \sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda, z\rangle}) + C(\omega|_{\langle ss_1^\lambda, z\rangle}) + C(\eta|_{\langle ss_1^\lambda, z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
&+ \sum_{\mu=-1,1} \Big[3^{4k-1}[C(\theta_\mu) + C(\theta_\mu^{-1}) + C(\theta_\mu\circ\eta) + C(\theta_\mu^{-1}\circ\eta)] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu, z\rangle})] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}|_{\langle t, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle t, z\rangle})] \\
&+ 3^{4k+1}[C(\theta_\mu|_{\langle z\rangle}) + C(\theta_\mu^{-1}|_{\langle z\rangle}) + C(\theta_\mu\circ\eta|_{\langle z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat{\theta}_{V_\mu}) + C(\hat{\beta}_{V_\mu}) + C(\hat{\vartheta}_{1,\mu}) + C(\hat{\vartheta}_{1,\mu})] \\
\Longrightarrow\ & 3^{4k}[C(\theta_A|_{\langle z\rangle}) + C(\theta_A^{-1}|_{\langle z\rangle}) + C(\alpha_A|_{\langle z\rangle}) + C(\beta_A|_{\langle z\rangle})] + [\sigma_A^{\theta_A|_{\langle z\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}] \\
&= 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ \sum_{\lambda=-1,1} 3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda, z\rangle}) + C(\omega|_{\langle ss_1^\lambda, z\rangle}) + C(\eta|_{\langle ss_1^\lambda, z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
&+ \sum_{\mu=-1,1} \Big[3^{4k-1}[C(\theta_\mu) + C(\theta_\mu^{-1}) + C(\theta_\mu\circ\eta) + C(\theta_\mu^{-1}\circ\eta)] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle ss_1^\mu, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu, z\rangle})] \\
&+ 3^{4k}[C(\theta_\mu|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}|_{\langle t, z\rangle}) + C(\theta_\mu\circ\eta|_{\langle t, z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle t, z\rangle})] \\
&+ 3^{4k+1}[C(\theta_\mu|_{\langle z\rangle}) + C(\theta_\mu^{-1}|_{\langle z\rangle}) + C(\theta_\mu\circ\eta|_{\langle z\rangle}) + C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat{\theta}_{V_\mu}) + C(\hat{\beta}_{V_\mu}) + C(\hat{\vartheta}_{1,\mu}) + C(\hat{\vartheta}_{1,\mu})] \\
&- 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \\
&- \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}) + C(\Theta_{2,\mu}) + C(\Theta_{3,\mu}) + C(\Theta_{4,\mu})]
\end{aligned}
$$

By using substitution from ①$_6$ and ⑥$_8$, we have

$$\begin{aligned}
&3^{4k}[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]+[\sigma_A^{\theta_A|_{\langle z\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{1,\mu})]\\
&-\Big[3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^{2k+2}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\Big]\\
&-\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\Big]\\
=&\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{1,\mu})]\\
&-3^{2k+2}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\\
\geq&\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{1,\mu})]\\
\geq&2^3\cdot 3^{4k}(2\cdot 3^{2k-3}-1)
\end{aligned}$$

Thus,

$$3^{4k}[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]+[\sigma_A^{\theta_A|_{\langle z\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}]\geq 2^3\cdot 3^{4k}(2\cdot 3^{2k-3}-1). \tag{$\star$}$$

Recall that

$$\sigma_A^{\theta_A|_{\langle z\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ P\neq\langle z\rangle,A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_{\langle z\rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\theta_A|_{\langle z\rangle}}|.$$

$$\sigma_A^{\theta_A|_{\langle z\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ P\neq\langle z\rangle,A}} C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\theta_A^{-1}|_{\langle z\rangle}}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\theta_A^{-1}|_{\langle z\rangle}}|.$$

Write

$$N = \{\theta_A|_P, \theta_A^{-1}|_P, \alpha_A|_P, \beta_A|_P \mid P\in\mathcal{D}_A, P\neq\langle z\rangle, A\},$$
$$\hat{N} = N\sqcup\{\theta_A|_{\langle z\rangle}, \theta_A^{-1}|_{\langle z\rangle}, \alpha_A|_{\langle z\rangle}, \beta_A|_{\langle z\rangle}\}.$$

Note that N is defined to be those morphisms that come from the definitions of $\sigma_A^{\theta_A|_{\langle z\rangle}}$ and $\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}$. Let Ω^* be our candidate characteristic biset. This means

$$C_{\Omega^*}(\theta_A|_{\langle z\rangle}) = C_{\Omega^*}(\theta_A^{-1}|_{\langle z\rangle}) = C_{\Omega^*}C(\alpha_A|_{\langle z\rangle}) = C_{\Omega^*}(\beta_A|_{\langle z\rangle}) = 2(2\cdot 3^{2k-3}-1).$$

and for all $\phi\in N$, $C_{\Omega^*}(\phi)=0$. That is,

$$\sum_{\phi\in\hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| = 2^3\cdot 3^{4k}(2\cdot 3^{2k-3}-1).$$

Let Ω' be any minimal characteristic biset with the condition that

$$C_{\Omega'}(\theta_A|_{\langle z\rangle}) + C_{\Omega'}(\theta_A^{-1}|_{\langle z\rangle}) + C_{\Omega'}(\alpha_A|_{\langle z\rangle}) + C_{\Omega'}(\beta_A|_{\langle z\rangle}) = n \quad \text{where} \quad 0\le n < 2^3(2\cdot 3^{2k-3}-1). \tag{$\star\star$}$$

We show that Ω^* is the optimal choice. From the rest of the calculations in this subsection and from Lemma $A.0.1$, whenever $\phi\notin\hat{N}$, we may assume

$$C_{\Omega'}(\phi)\ge C_{\Omega^*}(\phi). \tag{$\star\star\star$}$$

Moreover, there exists $\emptyset\neq I\subseteq N$ such that for all $\phi\in I$ we have $C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}|>0$ and by ($\star$) and ($\star\star$) we have

$$\sum_{\phi\in I} C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| \ge [2^3(2\cdot 3^{2k-3}-1)-n]\cdot 3^{4k}, \tag{$\star\star\star\star$}$$

where $\psi \in \{\theta_A, \theta_A^{-1}\}$. From calculations in Appendix B, for each $\phi \in I$, we have

$$|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| = \frac{3^{4k+1}}{|D_\phi|}.$$

Thus, by ($\star\star\star\star$) we have

$$\begin{aligned}
&\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| \geq [2^3(2\cdot 3^{2k-3}-1)-n]\cdot 3^{4k} \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+1}}{|D_\phi|} + n\cdot 3^{4k} \geq [2^3(2\cdot 3^{2k-3}-1)]\cdot 3^{4k} && \text{(by definition of } N\text{)} \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+2}}{|D_\phi|} + n\cdot 3^{4k+1} \geq [2^3(2\cdot 3^{2k-3}-1)]\cdot 3^{4k+1} \\
\Longrightarrow &[C_{\Omega'}(\theta_A|_{\langle z\rangle})|\mathcal{O}_{\theta_A|_{\langle z\rangle}}| + C_{\Omega'}(\theta_A^{-1}|_{\langle z\rangle})|\mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}}| + C_{\Omega'}(\alpha_A|_{\langle z\rangle})|\mathcal{O}_{\alpha_A|_{\langle z\rangle}}| + C_{\Omega'}(\beta_A|_{\langle z\rangle})|\mathcal{O}_{\beta_A|_{\langle z\rangle}}|] \\
&+\sum_{\phi \in N} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq [2^3(2\cdot 3^{2k-3}-1)]\cdot 3^{4k+1} && \text{(by } (\star\star) \text{ and definition of } |\mathcal{O}_\phi|\text{)} \\
\Longrightarrow &\sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq [2^3(2\cdot 3^{2k-3}-1)]\cdot 3^{4k+1}. && \text{(by definition of } \hat{N}\text{)}
\end{aligned}$$

Using this property, we have

$$\begin{aligned}
|\Omega'| - |\Omega^*| &\geq \sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| - \sum_{\phi \in \hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| \\
&\geq [2^3(2\cdot 3^{2k-3}-1)]\cdot 3^{4k+1} - [2^3(2\cdot 3^{2k-3}-1)]\cdot 3^{4k} > 0,
\end{aligned}$$

where the first inequality is due to ($\star\star\star$). It follows that Ω^* is indeed minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

Thus, we can choose $[C(\theta_A|_{\langle z\rangle}) = C(\theta_A^{-1}|_{\langle z\rangle}) = C(\alpha_A|_{\langle z\rangle}) = C(\beta_A|_{\langle z\rangle})] = 2(2\cdot 3^{2k-3}-1)$. 12_8

$\underline{C(\theta_\lambda|_{\langle t\rangle}), C(\theta_\lambda^{-1}|_{\langle t\rangle}), C(\theta_\lambda\circ\eta|_{\langle t\rangle}), C(\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}) \text{ for } \lambda=-1,1}$:

$$
\begin{aligned}
&|\Omega^{\theta_\lambda|_{\langle t\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle t\rangle}}|+|\Omega^{\theta_\lambda|_{\langle t\rangle}\circ\eta}|+|\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}|=|\Omega^{\theta_A|_{\langle t\rangle}}|+|\Omega^{\theta_A^{-1}|_{\langle t\rangle}}|+|\Omega^{\alpha_A|_{\langle t\rangle}}|+|\Omega^{\beta_A^{-1}|_{\langle t\rangle}}|\\
\Longrightarrow &\sum_{\mu=-1,1} 3^{2k}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\\
&+\sum_{\mu=-1,1} 2\cdot 3^{2k}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&=2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,1} 2\cdot 3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
\Longrightarrow &\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\\
&=2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,1} 2(3^{4k-1}-3^{2k})[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&-\sum_{\mu=-1,1} 3^{2k}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&-\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]
\end{aligned}
$$

By using substitution from ⑦$_8$, we have

$$
\begin{aligned}
&\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\\
&=2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,1} 2(3^{4k-1}-3^{2k})[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&-\sum_{\mu=-1,1} 3^{2k}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&=2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,1} (2\cdot 3^{4k-1}-3^{2k+1})[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&\geq 2^3\cdot 3^{2k+1}+2^4\cdot 3^{4k-1}-2^3\cdot 3^{2k+1}=2^4\cdot 3^{4k-1}
\end{aligned}
$$

Thus, $\sum_{\mu=-1,1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\geq 2^4\cdot 3^{2k-2}$. ⑬$_8$

$\underline{C(\mathsf{Id}|_{\langle t\rangle}), C(\omega|_{\langle t\rangle}), C(\eta|_{\langle t\rangle}), C(\omega\circ\eta|_{\langle t\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\theta_A|_{\langle t\rangle}}| + |\Omega^{\theta_A^{-1}|_{\langle t\rangle}}| = |\Omega^{\mathsf{Id}|_{\langle t\rangle}}| + |\Omega^{\omega|_{\langle t\rangle}}| \\
\Longrightarrow\ & 3^{2k}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ 3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle}) + C(\omega|_{\langle t\rangle}) + C(\eta|_{\langle t\rangle}) + C(\omega\circ\eta|_{\langle t\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_S^{\omega|_{\langle t\rangle}}] \\
&+ 2\cdot 3^{2k}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_A^{\omega|_{\langle t\rangle}}] \\
=& 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
&+ \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
\Longrightarrow\ & 3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle}) + C(\omega|_{\langle t\rangle}) + C(\eta|_{\langle t\rangle}) + C(\omega\circ\eta|_{\langle t\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_S^{\omega|_{\langle t\rangle}} + \sigma_A^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_A^{\omega|_{\langle t\rangle}}] \\
=& 3^{2k}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
&+ \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
&- 3^{2k}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)]
\end{aligned}
$$

By using substitution from ①$_6$, we have

$$
\begin{aligned}
& 3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle}) + C(\omega|_{\langle t\rangle}) + C(\eta|_{\langle t\rangle}) + C(\omega\circ\eta|_{\langle t\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_S^{\omega|_{\langle t\rangle}} + \sigma_A^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_A^{\omega|_{\langle t\rangle}}] \\
=& 3^{2k}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
&+ \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
&- \Big[3^{2k}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] - 3^{2k+1}[C(\mathsf{Id}|_A) + C(\omega|_A) + C(\eta|_A) + C(\omega\circ\eta|_A)]\Big] \\
=& 3^{2k+1}[C(\mathsf{Id}|_A) + C(\omega|_A) + C(\eta|_A) + C(\omega\circ\eta|_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
&+ \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
\geq & \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
\geq & 2^3\cdot 3^{4k-1}.
\end{aligned}
$$

Thus,

$$
3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle}) + C(\omega|_{\langle t\rangle}) + C(\eta|_{\langle t\rangle}) + C(\omega\circ\eta|_{\langle t\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_S^{\omega|_{\langle t\rangle}} + \sigma_A^{\mathsf{Id}|_{\langle t\rangle}} + \sigma_A^{\omega|_{\langle t\rangle}}] \geq 2^3\cdot 3^{4k-1}, \tag{$\star$}
$$

We demonstrate that $[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]=2^3$ is the optimum choice. Recall the following definitions:

$$\begin{aligned}
\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\mathsf{Id}|_{\langle t\rangle}}|\\
\sigma_S^{\omega|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\omega|_{\langle t\rangle}}|+C(\omega\circ\eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\omega|_{\langle t\rangle}}|\\
\sigma_A^{\mathsf{Id}|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P,P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P,P\neq A}} C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\mathsf{Id}|_{\langle t\rangle}}|\\
\sigma_A^{\omega|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P,P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\omega|_{\langle t\rangle}}|+C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\omega|_{\langle t\rangle}}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P,P\neq A}} C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\omega|_{\langle t\rangle}}|+C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\omega|_{\langle t\rangle}}|
\end{aligned}$$

Write

$$\begin{aligned}
N_1 &= \{\mathsf{Id}|_P,\omega|_P,\eta|_P,\omega\circ\eta|_P \mid P\in\mathcal{D},P\neq S,\langle t\rangle,\langle ss_1^{\lambda}\rangle,\langle ss_1^{\lambda},z\rangle,\lambda=-1,1\},\\
N_2 &= \{\theta_A|_P,\theta_A^{-1}|_P,\alpha_A|_P,\beta_A|_P \mid P\in\mathcal{D}_A,P\neq A,\text{ and } tz^{-1}\in P \text{ or } tz\in P\},\\
N_3 &= \{\mathsf{Id}|_P,\omega|_P,\eta|_P,\omega\circ\eta|_P \mid P=\langle ss_1^{\lambda}\rangle,\langle ss_1^{\lambda},z\rangle,\lambda=-1,1\},\\
N &= N_1\sqcup N_2\\
\hat{N} &= N\sqcup\{\mathsf{Id}|_{\langle t\rangle},\omega|_{\langle t\rangle},\eta|_{\langle t\rangle},\omega\circ\eta|_{\langle t\rangle}\}
\end{aligned}$$

Note that $N\sqcup N_3$ is the set of morphisms which come from the definitions of $\sigma_S^{\mathsf{Id}|_{\langle t\rangle}},\sigma_S^{\omega|_{\langle t\rangle}},\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}$ and $\sigma_A^{\omega|_{\langle t\rangle}}$. Let Ω^* be our candidate characteristic biset. Under Ω^* we have

$$C_{\Omega^*}(\mathsf{Id}|_{\langle t\rangle})=C_{\Omega^*}(\omega|_{\langle t\rangle})=C_{\Omega^*}(\eta|_{\langle t\rangle})=C_{\Omega^*}(\omega\circ\eta|_{\langle t\rangle})=2,$$

and for all $\phi\in N$, we have $C_{\Omega^*}(\phi)=0$. When $\phi\in N_3$, $C_{\Omega^*}(\phi)>0$, and is the reason why we have stopped short of including N_3 in N. Thus,

$$\sum_{\phi\in\hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_{\phi}|=8\cdot 3^{4k+1}.$$

Suppose that Ω' is a minimal right characteristic biset with the condition that

$$C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})+C_{\Omega'}(\omega|_{\langle t\rangle})+C_{\Omega'}(\eta|_{\langle t\rangle})+C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})=n \quad\text{where}\quad 0\leq n<8. \tag{$\star\star$}$$

From the rest of the calculations in this subsection and from Lemma $A.0.1$, we may assume for all $\phi\notin\hat{N}$ that

$$C_{\Omega'}(\phi)\geq C_{\Omega^*}(\phi). \tag{$\star\star\star$}$$

Moreover, there exists $\emptyset \neq I \subseteq N$ such that for all $\phi \in I$, we have $C_{\Omega'}(\phi)|\mathcal{O}_{\phi}^{\psi|_{\langle t\rangle}}| > 0$ and by $(\star)$ and $(\star\star)$ we have

$$\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_{\phi}^{\psi|_{\langle t\rangle}}| \geq (8-n)\cdot 3^{4k-1}, \qquad (\star\star\star\star)$$

where $\psi \in \{\mathsf{Id}, \omega\}$. We show that Ω^* is minimal by size. In other words we have to show that $|\Omega'| \geq |\Omega^*|$. From results in Appendix B, it can be checked that for each $\phi \in I$ and $\psi \in \{\mathsf{Id}, \omega\}$, we have

$$|\mathcal{O}_{\phi}^{\psi|_{\langle t\rangle}}| = \frac{3^{4k}}{|D_\phi|} \text{ or } \frac{3^{4k+1}}{|D_\phi|}.$$

Suppose that for every $\phi \in I$, $|\mathcal{O}_{\phi}^{\psi|_{\langle t\rangle}}| = \frac{3^{4k}}{|D_\phi|}$. Then, by $(\star\star\star\star)$,

$$\begin{aligned}
&\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_{\phi}^{\psi|_{\langle t\rangle}}| \geq (8-n)\cdot 3^{4k-1} \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k}}{|D_\phi|} + n\cdot 3^{4k-1} \geq 8\cdot 3^{4k-1} && \text{(by definition of } I \text{ and } N\text{)} \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+2}}{|D_\phi|} + n\cdot 3^{4k+1} \geq 8\cdot 3^{4k+1} \\
\Longrightarrow &C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega'}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega'}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}| \\
&\quad + \sum_{\phi \in N} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 8\cdot 3^{4k+1} && \text{(by } (\star\star) \text{ and definition of } |\mathcal{O}_\phi|\text{)} \\
\Longrightarrow &\sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 8\cdot 3^{4k+1}, && \text{(by definition of } \hat{N}\text{)}
\end{aligned}$$

Thus

$$|\Omega'| - |\Omega^*| \geq \sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| - \sum_{\phi \in \hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| \geq 8\cdot 3^{4k+1} - 8\cdot 3^{4k+1} = 0,$$

where the first inequality is due to $(\star\star\star)$. It follows that Ω^* is minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

It is left to consider the case when there exists $\phi \in I$ such that $|\mathcal{O}_{\phi}^{\psi|_{\langle t\rangle}}| = \frac{3^{4k+1}}{|D_\phi|}$. From results in Appendix B, this only occurs when $\phi \in N_1$ and $\langle t, z\rangle \leq D_\phi$. Thus, for all $\mu = 1,..,4$ and

$\lambda = -1, 1$, we have

$$
\begin{aligned}
&|\Omega^{\Theta^{-1}_{\mu,\lambda}}| = |\Omega^{\psi|_{\langle t,z\rangle}}| \\
\implies & 3^{2k} C(\Theta^{-1}_{\mu,\lambda}) = 3^{2k} C(\psi) + \frac{3^{4k+1}}{|D_\phi|} C(\phi) + [\text{extra..}] \\
\implies & C(\Theta^{-1}_{\mu,\lambda}) \geq C(\psi) + \frac{3^{2k+1}}{|D_\phi|} C(\phi) \geq C(\psi) + 3C(\phi) \geq 4 \qquad \text{(as } \phi \in I\text{)}
\end{aligned}
$$

where the second last inequality is because $|D_\phi| \leq 3^{2k}$. Note that, $C_{\Omega^*}(\Theta^{-1}_{\mu,\lambda}) = 1$. However, $C_{\Omega'}(\Theta_{\mu,\lambda}) \geq 4$. Now

$$
\begin{aligned}
|\Omega'| - |\Omega^*| \geq & [C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega'}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega'}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}|] \\
& + \sum_{\lambda=-1,1} [C_{\Omega'}(\Theta^{-1}_{1,\lambda})|\mathcal{O}_{\Theta^{-1}_{1,\lambda}}| + C_{\Omega'}(\Theta^{-1}_{2,\lambda})|\mathcal{O}_{\Theta^{-1}_{2,\lambda}}| + C_{\Omega'}(\Theta^{-1}_{3,\lambda})|\mathcal{O}_{\Theta^{-1}_{3,\lambda}}| + C_{\Omega'}(\Theta^{-1}_{4,\lambda})|\mathcal{O}_{\Theta^{-1}_{4,\lambda}}|] \\
& - [C_{\Omega^*}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega^*}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega^*}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega^*}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}|] \\
& - \sum_{\lambda=-1,1} [C_{\Omega^*}(\Theta^{-1}_{1,\lambda})|\mathcal{O}_{\Theta^{-1}_{1,\lambda}}| + C_{\Omega^*}(\Theta^{-1}_{2,\lambda})|\mathcal{O}_{\Theta^{-1}_{2,\lambda}}| + C_{\Omega^*}(\Theta^{-1}_{3,\lambda})|\mathcal{O}_{\Theta^{-1}_{3,\lambda}}| + C_{\Omega^*}(\Theta^{-1}_{4,\lambda})|\mathcal{O}_{\Theta^{-1}_{4,\lambda}}|] \\
\geq & n \cdot 3^{4k+1} + 8 \cdot 4 \cdot 3^{4k} - 8 \cdot 3^{4k+1} - 8 \cdot 3^{4k} \\
= & n \cdot 3^{4k+1} \geq 0
\end{aligned}
$$

It follows that Ω^* is minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

So we may assume $C(\mathsf{Id}|_{\langle t\rangle}) = C(\omega|_{\langle t\rangle}) = C(\eta|_{\langle t\rangle}) = C(\omega\circ\eta|_{\langle t\rangle}) = 2$. 14$_8$

With the minimality of each of the non-zero constants of Ω^* demonstrated, we have therefore shown that it is minimal by size.

A.8.2 Exoticity

Note that $|\Omega^*|/|S| = 2^3[2^4 \cdot 3^{6k-5} + 38 \cdot 3^{4k-4} - 5 \cdot 3^{2k-2} + 2]$. Thus,

$$
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{2^3[2^4 \cdot 3^{6k-5} + 38 \cdot 3^{4k-4} - 5 \cdot 3^{2k-2} + 2]}{3^i} \right\rfloor
$$

Write $n_i = \left\lfloor \frac{2^3[2^4\cdot 3^{6k-5}+38\cdot 3^{4k-4}-5\cdot 3^{2k-2}+2]}{3^i} \right\rfloor$. Now,

$$
\begin{aligned}
i=1 &\implies n_i = 128\cdot 3^{6k-6}+304\cdot 3^{4k-5}-40\cdot 3^{2k-3}+5 \\
i=2 &\implies n_i = 128\cdot 3^{6k-7}+304\cdot 3^{4k-6}-40\cdot 3^{2k-4}+1 \\
3\le i\le 2k-2 &\implies n_i = 128\cdot 3^{6k-5-i}+304\cdot 3^{4k-4-i}-40\cdot 3^{2k-2-i} && (k\ge 3) \\
i=2k-1 &\implies n_i = 128\cdot 3^{4k-4}+304\cdot 3^{2k-3}-14 \\
i=2k &\implies n_i = 128\cdot 3^{4k-5}+304\cdot 3^{2k-4}-5 \\
i=2k+1 &\implies n_i = \begin{cases} 1251 & \text{if } k=2 \\ 128\cdot 3^{4k-6}+304\cdot 3^{2k-5}-2 & \text{if } k\ge 3 \end{cases} \\
2k+2\le i\le 4k-4 &\implies n_i = 128\cdot 3^{6k-5-i}+304\cdot 3^{4k-4-i}-1 && (k\ge 3) \\
i=4k-3 &\implies n_i = 128\cdot 3^{2k-2}+101 \\
i=4k-2 &\implies n_i = 128\cdot 3^{2k-3}+33 \\
i=4k-1 &\implies n_i = 128\cdot 3^{2k-4}+11 \\
i=4k &\implies n_i = \begin{cases} 46 & \text{if } k=2 \\ 128\cdot 3^{2k-5}+3 & \text{if } k\ge 3 \end{cases} \\
i=4k+1 &\implies n_i = \begin{cases} 15 & \text{if } k=2 \\ 128\cdot 3^{2k-6}+1 & \text{if } k\ge 3 \end{cases} \\
4k+2\le i\le 6k-5 &\implies n_i = 128\cdot 3^{6k-5-i} && (k\ge 4) \\
i=6k-4 &\implies n_i = \begin{cases} 46 & \text{if } k=2 \\ 43 & \text{if } k=3 \\ 42 & \text{if } k\ge 4 \end{cases} \\
i=6k-3 &\implies n_i = \begin{cases} 15 & \text{if } k=2 \\ 14 & \text{if } k\ge 3 \end{cases} \\
i=6k-2 &\implies n_i = \begin{cases} 5 & \text{if } k=2 \\ 4 & \text{if } k\ge 3 \end{cases} \\
i=6k-1 &\implies n_i = 1
\end{aligned}
$$

Thus, by summing over the range $1\le i\le 6k-1$, we have

$$
\nu_3(\mathrm{Sym}(|\Omega^*|/|S|)) = \begin{cases} 152102 & \text{if } k=2 \\ 103032323 & \text{if } k=3 \\ 2^2[2^4\cdot 3^{6k-5}+38\cdot 3^{4k-4}-5\cdot 3^{2k-2}+1]-2k & \text{if } k\ge 3 \end{cases}
$$

Let T be Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\log_3|T:S| = \begin{cases} [|\Omega^*|/|S|-1]\cdot 5 + 152102 & \text{if } k=2 \\ [|\Omega^*|/|S|-1]\cdot 7 + 103032323 & \text{if } k=3 \\ [|\Omega^*|/|S|-1](2k+1) + [2^2[2^4\cdot 3^{6k-5} + 38\cdot 3^{4k-4} - 5\cdot 3^{2k-2}+1]-2k] & \text{if } k\geq 4 \end{cases}$$

$$= \begin{cases} 1673177 & \text{if } k=2 \\ 1545484964 & \text{if } k=3 \\ 2^2[2^4\cdot 3^{6k-5} + 38\cdot 3^{4k-4} - 5\cdot 3^{2k-2}](4k+3) + (28k+19) & \text{if } k\geq 4 \end{cases}$$

Thus, we have an upper bound on our exoticity index.

A.9 DRV-9

For this refer to the proof given in Section 8.1.

A.10 DRV-10

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}\Omega \cong & \bigsqcup_{Q\in\mathcal{D}} C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\ & \sqcup \bigsqcup_{Q\in\mathcal{D}_A} C(\theta_A|_Q)\mathcal{O}_{\theta_A|_Q} \sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q} \sqcup C(\alpha_A|_Q)\mathcal{O}_{\alpha_A|_Q} \sqcup C(\beta_A|_Q)\mathcal{O}_{\beta_A|_Q} \\ & \sqcup \bigsqcup_{Q\in\mathcal{D}_{E_0}} C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0^{-1}|_Q} \sqcup C(\alpha_0|_Q)\mathcal{O}_{\alpha_0|_Q} \sqcup C(\beta_0|_Q)\mathcal{O}_{\beta_0|_Q} \\ & \sqcup \bigsqcup_{Q\in\mathcal{D}_{V_0}} C(\theta_{V_0}|_Q)\mathcal{O}_{\theta_{V_0}|_Q} \sqcup C(\theta_{V_0}^{-1}|_Q)\mathcal{O}_{\theta_{V_0}^{-1}|_Q} \sqcup C(\alpha_{V_0}|_Q)\mathcal{O}_{\alpha_{V_0}|_Q} \sqcup C(\beta_{V_0}|_Q)\mathcal{O}_{\beta_{V_0}|_Q} \\ & \sqcup [C(\Theta_{1,0})\mathcal{O}_{\Theta_{1,0}} \sqcup C(\Theta_{2,0})\mathcal{O}_{\Theta_{2,0}} \sqcup C(\Theta_{3,0})\mathcal{O}_{\Theta_{3,0}} \sqcup C(\Theta_{4,0})\mathcal{O}_{\Theta_{4,0}}] \\ & \sqcup [C(\Theta_{1,0}^{-1})\mathcal{O}_{\Theta_{1,0}^{-1}} \sqcup C(\Theta_{2,0}^{-1})\mathcal{O}_{\Theta_{2,0}^{-1}} \sqcup C(\Theta_{3,0}^{-1})\mathcal{O}_{\Theta_{3,0}^{-1}} \sqcup C(\Theta_{4,0})\mathcal{O}_{\Theta_{4,0}^{-1}}]\end{aligned}$$

Note that there is a difference between the construction in Theorem 7.3.1 and the one above. The difference is that we have excluded orbits induced by morphisms $\Theta_{\mu,0}|_Q$, for $Q \lneq \langle s,z\rangle$, and $\Theta_{\mu,0}^{-1}|_Q$ for $Q \lneq \langle t,z\rangle$, for all $\mu = 1,..,4$. This is because all of these morphisms are restrictions of those from the set

$$\{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T \mid T = V_0, A\},$$

which are already included in the general construction.

Equations:

We list the following equations:

$\psi \in \{\mathsf{Id}|_Q \mid Q \in \mathcal{D}\cup\mathcal{D}_A\}$:

If $Q \in \mathcal{D}_A \backslash \mathcal{D}_{A_0}$, then the equations are the same as those for DRV-6.

If $Q \in \mathcal{D}\backslash(\mathcal{D}_A \cup \mathcal{D}_0)$, then

$$|\Omega^{\mathsf{Id}|_Q}| = \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_Q}|$$

Now for the rest of Q:

$$
\begin{aligned}
|\Omega^{\mathsf{Id}|_{\langle s,t\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle s,t\rangle}}| \\
|\Omega^{\mathsf{Id}|_{\langle s,z\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle s,z\rangle}}| + 3^2[C(\theta_0)+C(\theta_0^{-1})] \\
|\Omega^{\mathsf{Id}|_{\langle t,z\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t,z\rangle}}| \\
|\Omega^{\mathsf{Id}|_{\langle s\rangle}}| &= \sum_{P\in\mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle s\rangle}}| + C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\mathsf{Id}|_{\langle s\rangle}}|] + 3^2[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)] \\
&\quad + 3^2[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})] \\
|\Omega^{\mathsf{Id}|_{\langle t\rangle}}| &= \sum_{P\in\mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t\rangle}}| + C(\omega\circ\eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\mathsf{Id}|_{\langle t\rangle}}|] \\
&\quad + \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\le P}} [C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|] + \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\le P}} [C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\mathsf{Id}|_{\langle t\rangle}}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|] \\
|\Omega^{\mathsf{Id}|_{\langle z\rangle}}| &= \sum_{P\in\mathcal{D}} [C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle z\rangle}}| + C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\mathsf{Id}|_{\langle z\rangle}}|] + \sum_{\substack{P\in\mathcal{D}_0\\ P\ge\langle z\rangle}} \frac{3^{4k+2}}{|P|}[C(\theta_0|_P)+C(\theta_0^{-1}|_P)]
\end{aligned}
$$

$\underline{\psi \in \{\theta_0|_Q \mid Q\in\mathcal{D}_0\}}$:

$$
\begin{aligned}
|\Omega^{\theta_0}| &= 3C(\theta_0) \\
|\Omega^{\theta_0|_{\langle s,z\rangle}}| &= 3^{2k}C(\theta_0) + 3^{2k+1}C(\theta_0|_{\langle s,z\rangle}) \\
|\Omega^{\theta_0|_{\langle t,z\rangle}}| &= 3^{2k}C(\theta_0) + 3^{2k+1}C(\theta_0|_{\langle t,z\rangle}) \\
|\Omega^{\theta_{\langle s\rangle}}| &= 3^{2k}[C(\theta_0)+C(\alpha_0)] + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})] + 3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})] \\
&\quad + 3^{2k}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})] \\
|\Omega^{\theta_0|_{\langle t\rangle}}| &= 3^{2k}[C(\theta_0)+C(\beta_0)] + 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,\rangle})] + 3^{2k+1}[C(\theta_0|_{\langle t\rangle})+C(\beta_0|_{\langle t\rangle})] \\
&\quad + 3^{2k}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})]
\end{aligned}
$$

$\underline{\psi \in \{\theta_A|_Q \mid Q\in\mathcal{D}_A\}}$:

If $Q\in\mathcal{D}_A\backslash\mathcal{D}_{A_0}$, then the equations are the same as those for DRV-6. If $Q\in\mathcal{D}_{A_0}$, we determine the cases below:

$$
\begin{aligned}
|\Omega^{\theta_A|_{\langle t,z\rangle}}| &= \sum_{P\in\mathcal{D}_A} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_{\langle t,z\rangle}}_{\theta_A|_P}| \\
|\Omega^{\theta_A|_{\langle t\rangle}}| &= \sum_{P\in\mathcal{D}_A} [C(\theta_A|_P)|\mathcal{O}^{\theta_A|_{\langle t\rangle}}_{\theta_A|_P}| + C(\alpha_A|_P)|\mathcal{O}^{\theta_A|_{\langle t\rangle}}_{\alpha_A|_P}|] \\
&\quad + 3^{4k-1}[C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1})] \\
|\Omega^{\theta_A|_{\langle z\rangle}}| &= \sum_{P\in\mathcal{D}_A} C(\theta_A|_P)|\mathcal{O}^{\theta_A|_{\langle z\rangle}}_{\theta_A|_P}| + C(\beta_A|_P)|\mathcal{O}^{\theta_A|_{\langle z\rangle}}_{\beta_A|_P}| \\
&\quad + 3^{4k-1}[C(\Theta_{1,0}) + C(\Theta_{4,0})]
\end{aligned}
$$

$\underline{\psi \in \{\theta_{V_0}|_Q \mid Q \in \mathcal{D}_{V_0}\}}$:

$$
\begin{aligned}
|\Omega^{\theta_{V_0}}| &= 3^2 C(\theta_{V_0}) \\
|\Omega^{\theta_{V_0}|_{\langle s\rangle}}| &= 3^{2k+1}[C(\theta_{V_0}) + C(\alpha_{V_0})] + 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\alpha_{V_0}|_{\langle s\rangle})] \\
&\quad + 3^{2k+1}[C(\Theta_{1,0}) + C(\Theta_{3,0})] \\
|\Omega^{\theta_{V_0}|_{\langle z\rangle}}| &= 3^{2k+1}[C(\theta_{V_0}) + C(\beta_{V_0})] + 3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \\
&\quad + 3^{2k+1}[C(\Theta_{2,0}^{-1}) + C(\Theta_{4,0}^{-1})]
\end{aligned}
$$

$\underline{\psi = \Theta_{1,0}}$:

$$|\Omega^{\Theta_{1,0}|_{\langle s,z\rangle}}| = 3^{2k} C(\Theta_{1,0})$$

$\underline{\psi = \Theta_{1,0}^{-1}}$:

$$|\Omega^{\Theta_{1,0}^{-1}}| = 3^{2k} C(\Theta_{1,0}^{-1})$$

Candidate solution:

We have the following designated values for the constants:

$$
\begin{aligned}
&C(\mathsf{Id}) = C(\omega) = C(\eta) = C(\omega \circ \eta) = 1\\
&C(\mathsf{Id}|_{\langle s,z\rangle}) = C(\omega|_{\langle s,z\rangle}) = C(\eta|_{\langle s,z\rangle}) = C(\omega \circ \eta|_{\langle s,z\rangle}) = 3^{2k-3} - 1\\
&C(\mathsf{Id}|_{\langle s\rangle}) = C(\omega|_{\langle s\rangle}) = C(\eta|_{\langle s\rangle}) = C(\omega \circ \eta|_{\langle s\rangle}) = 3^{4k-4}\\
&C(\mathsf{Id}|_{\langle t\rangle}) = C(\omega|_{\langle t\rangle}) = C(\eta|_{\langle t\rangle}) = C(\omega \circ \eta|_{\langle t\rangle}) = 1\\
&C(\theta_A) = C(\theta_A^{-1}) = C(\alpha_A) = C(\beta_A) = 1\\
&C(\theta_{V_0}) = C(\theta_{V_0}^{-1}) = C(\alpha_{V_0}) = C(\beta_{V_0}) = 3^{2k-2}\\
&C(\theta_0) = C(\theta_0^{-1}) = C(\alpha_0) = C(\beta_0) = 1\\
&C(\theta_0|_{\langle s\rangle}) = C(\theta_0^{-1}|_{\langle s\rangle}) = C(\alpha_0|_{\langle s\rangle}) = C(\beta_0|_{\langle s\rangle}) = 3^{2k-2}\\
&C(\theta_0|_{\langle t\rangle}) = C(\theta_0|_{\langle t\rangle}) = C(\alpha_0|_{\langle t\rangle}) = C(\beta_0|_{\langle t\rangle}) = 3^{2k-2}\\
&C(\Theta_{1,0}) = C(\Theta_{2,0}) = C(\Theta_{3,0}) = C(\Theta_{4,0}) = 1\\
&C(\Theta_{1,0}^{-1}) = C(\Theta_{2,0}^{-1}) = C(\Theta_{3,0}^{-1}) = C(\Theta_{4,0}^{-1}) = 1,
\end{aligned}
$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$
\begin{aligned}
\Omega^* \cong\; &(\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup [3^{2k-3} - 1](\mathcal{O}_{\mathsf{Id}|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s,z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s,z\rangle}})\\
&\sqcup 3^{4k-4}(\mathcal{O}_{\mathsf{Id}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}) \sqcup (\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}})\\
&\sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A})\\
&\sqcup 3^{2k-2}(\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}})\\
&\sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0})\\
&\sqcup 3^{2k-2}(\theta_0|_{\langle s\rangle} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\beta_0|_{\langle s\rangle}}) \sqcup 3^{2k-2}(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\beta_0|_{\langle t\rangle}})\\
&\sqcup (\mathcal{O}_{\Theta_{1,0}} \sqcup \mathcal{O}_{\Theta_{2,0}} \sqcup \mathcal{O}_{\Theta_{3,0}} \sqcup \mathcal{O}_{\Theta_{4,0}}) \sqcup (\mathcal{O}_{\Theta_{1,0}^{-1}} \sqcup \mathcal{O}_{\Theta_{2,0}^{-1}} \sqcup \mathcal{O}_{\Theta_{3,0}^{-1}} \sqcup \mathcal{O}_{\Theta_{4,0}^{-1}}).
\end{aligned}
$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.10.1 Minimality

We now show that our candidate example is of minimal size for all possible constants in the general construction Ω. First we define a few things which will be used later on. Let

$\psi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$. Then, we define

$$\sigma_S^{\psi|_{\langle s,t\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq S}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,t\rangle}}|$$
$$\sigma_S^{\psi|_{\langle s,z\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle s,z\rangle}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}|$$
$$\sigma_S^{\psi|_{\langle s\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle s,z\rangle,\langle s\rangle}} [C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s\rangle}}| + C(\phi_1|_P)|\mathcal{O}_{\phi_1|_P}^{\psi|_P}|]$$
$$\sigma_S^{\psi|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} [C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle t\rangle}}| + C(\phi_2|_P)|\mathcal{O}_{\phi_2|_P}^{\psi|_{\langle t\rangle}}|]$$

Where ϕ_1, ϕ_2 are chosen such that $\psi\circ\phi_1 = \eta$ and $\psi\circ\phi_2 = \omega\circ\eta$. We also define:

$$\sigma_A^{\mathrm{Id}|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P\\ P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\mathrm{Id}|_{\langle t\rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\mathrm{Id}|_{\langle t\rangle}}| + \sum_{\substack{P\in\mathcal{D}_A\\ \langle t,z\rangle\leq P\\ P\neq A}} C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\mathrm{Id}|_{\langle t\rangle}}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\mathrm{Id}|_{\langle t\rangle}}|$$
$$\sigma_A^{\omega|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P\\ P\neq A}} C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\omega|_{\langle t\rangle}}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\omega|_{\langle t\rangle}}| + \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P\\ P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\omega|_{\langle t\rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\omega|_{\langle t\rangle}}|$$
$$\sigma_A^{\theta_A|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_{\langle t\rangle}}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\theta_A|_{\langle t\rangle}}|$$
$$\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ P\neq A}} C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\theta_A^{-1}|_{\langle t\rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\theta_A^{-1}|_{\langle t\rangle}}|$$

Now we determine the minimal size of Ω.

We begin by noting that $C(\mathrm{Id}), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$ from Lemma 4.2.8.

$\underline{C(\theta_0), C(\theta_0^{-1}), C(\alpha_0), C(\beta_0)}$:

$$\begin{aligned} &|\Omega^{\theta_0}| + |\Omega^{\theta_0^{-1}}| + |\Omega^{\alpha_0}| + |\Omega^{\beta_0}| = |\Omega^{\mathrm{Id}|_{\langle s,t\rangle}}| + |\Omega^{\omega|_{\langle s,t\rangle}}| + |\Omega^{\eta|_{\langle s,t\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle s,t\rangle}}| \\ \Longrightarrow\ & 3[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\ &= 3[C(\mathrm{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathrm{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}] \qquad ①_{10} \end{aligned}$$

This gives us $[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \geq 2^2$, as required.

$\underline{C(\theta_A), C(\theta_A^{-1}), C(\alpha_A), C(\beta_A)}$:

$$\begin{aligned}
&|\Omega^{\theta_A}| + |\Omega^{\theta_A^{-1}}| + |\Omega^{\alpha_A}| + |\Omega^{\beta_A}| = |\Omega^{\mathsf{Id}|_A}| + |\Omega^{\omega|_A}| + |\Omega^{\eta|_A}| + |\Omega^{\omega\circ\eta|_A}| \\
\implies & 3^{2k}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \\
&= 3^{2k}[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&\quad + 3^{2k+1}[C(\mathsf{Id}|_A) + C(\omega|_A) + C(\eta|_A) + C(\omega\circ\eta|_A)] \qquad ②_{10}
\end{aligned}$$

This gives us that $[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \geq 2^2$, as required.

$\underline{C(\theta_{V_0}), C(\theta_{V_0}^{-1}), C(\alpha_{V_0}), C(\beta_{V_0})}$:

$$\begin{aligned}
&|\Omega^{\theta_{V_0}}| + |\Omega^{\theta_{V_0}^{-1}}| + |\Omega^{\alpha_{V_0}}| + |\Omega^{\beta_{V_0}}| = |\Omega^{\theta_0|_{\langle s,z\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}| + |\Omega^{\alpha_0|_{\langle s,z\rangle}}| + |\Omega^{\beta_0|_{\langle s,z\rangle}}| \\
\implies & 3^{2}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
&= 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&\quad + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \qquad ③_{10}
\end{aligned}$$

Using the result derived from $①_{10}$, this gives us $[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \geq 2^2 \cdot 3^{2k-2}$, as required.

$\underline{C(\Theta_{1,0}), C(\Theta_{2,0}), C(\Theta_{3,0}), C(\Theta_{4,0})}$:

$$\begin{aligned}
&|\Omega^{\Theta_{1,0}}| + |\Omega^{\Theta_{2,0}}| + |\Omega^{\Theta_{3,0}}| + |\Omega^{\Theta_{4,0}}| = |\Omega^{\theta_0|_{\langle s,z\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}| + |\Omega^{\alpha_0|_{\langle s,z\rangle}}| + |\Omega^{\beta_0|_{\langle s,z\rangle}}| \\
\implies & 3^{2k}[C(\Theta_{1,0}) + C(\Theta_{2,0}) + C(\Theta_{3,0}) + C(\Theta_{4,0})] \\
&= 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&\quad + 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \qquad ④_{10}
\end{aligned}$$

Using the result derived from ①, this gives us $[C(\Theta_{1,0}) + C(\Theta_{2,0}) + C(\Theta_{3,0}) + C(\Theta_{4,0})] \geq 2^2$, as required.

$\underline{C(\Theta_{1,0}^{-1}), C(\Theta_{2,0}^{-1}), C(\Theta_{3,0}^{-1}), C(\Theta_{4,0}^{-1})}$:

$$\begin{aligned}
&|\Omega^{\Theta_{1,0}^{-1}}| + |\Omega^{\Theta_{2,0}^{-1}}| + |\Omega^{\Theta_{3,0}^{-1}}| + |\Omega^{\Theta_{4,0}^{-1}}| = |\Omega^{\theta_0|_{\langle t,z\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle t,z\rangle}}| + |\Omega^{\alpha_0|_{\langle t,z\rangle}}| + |\Omega^{\beta_0|_{\langle t,z\rangle}}| \\
\implies & 3^{2k}[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
&= 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&\quad + 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})] \qquad ⑤_{10}
\end{aligned}$$

Using the result derived from ①$_{10}$, this gives us $[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})] \geq 2^2$, as required.

<u>$C(\mathsf{Id}|_{\langle s,z\rangle}), C(\omega|_{\langle s,z\rangle}), C(\eta|_{\langle s,z\rangle}), C(\omega\circ\eta|_{\langle s,z\rangle})$:</u>

Using substitution from ①$_{10}$, we have

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle s,z\rangle}}|+|\Omega^{\omega|_{\langle s,z\rangle}}|+|\Omega^{\eta|_{\langle s,z\rangle}}|+|\Omega^{\omega\circ\eta|_{\langle s,z\rangle}}| = |\Omega^{\theta_0|_{\langle s,z\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}|+|\Omega^{\alpha_0|_{\langle s,z\rangle}}|+|\Omega^{\beta_0|_{\langle s,z\rangle}}|\\
\Longrightarrow\ & 3^2[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^3[C(\mathsf{Id}|_{\langle s,z\rangle})+C(\omega|_{\langle s,z\rangle})+C(\eta|_{\langle s,z\rangle})+C(\omega\circ\eta|_{\langle s,z\rangle})]\\
&+2\cdot 3^2[(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]+\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}}+\sigma_S^{\omega|_{\langle s,z\rangle}}+\sigma_S^{\eta|_{\langle s,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}\\
=&3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
\Longrightarrow\ & 3^3[C(\mathsf{Id}|_{\langle s,z\rangle})+C(\omega|_{\langle s,z\rangle})+C(\eta|_{\langle s,z\rangle})+C(\omega\circ\eta|_{\langle s,z\rangle})]\\
=&-3^2[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+(3^{2k-1}-2\cdot 3)\Big[3[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\Big]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}}+\sigma_S^{\omega|_{\langle s,z\rangle}}+\sigma_S^{\eta|_{\langle s,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]\\
=&(3^{2k}-3^3)[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+(3^{2k-1}-2\cdot 3)[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&-[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}}+\sigma_S^{\omega|_{\langle s,z\rangle}}+\sigma_S^{\eta|_{\langle s,z\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]
\end{aligned}
$$

Let $\psi \in \{\mathsf{Id}_S, \omega, \eta, \omega\circ\eta\}$. Then for all $P \in \mathcal{D}\backslash\{\langle s\rangle, \langle s,z\rangle, S\}$, we have

$$
|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| = \frac{|N_S(\langle s,z\rangle,P)||C_S(\langle s,z\rangle)|}{|P|} = \frac{3|N_S(\langle s,t\rangle,P)||C_S(\langle s,t\rangle)|}{|P|} = 3|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,t\rangle}}|
$$

This implies $\sigma_S^{\psi|_{\langle s,z\rangle}} = 3\sigma_S^{\psi|_{\langle s,t\rangle}}$. Thus,

$$
\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle s,z\rangle})+C(\omega|_{\langle s,z\rangle})+C(\eta|_{\langle s,z\rangle})+C(\omega\circ\eta|_{\langle s,z\rangle})]\\
=&(3^{2k}-3^3)[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+(3^{2k-2}-3^2)[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\\
\geq&(3^{2k}-3^3)[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)] \qquad ⑥_{10}
\end{aligned}
$$

Thus, $[C(\mathrm{Id}|_{\langle s,z\rangle})+C(\omega|_{\langle s,z\rangle})+C(\eta|_{\langle s,z\rangle})+C(\omega\circ\eta|_{\langle s,z\rangle})] \geq 2^2\cdot(3^{2k-3}-1)$, as required.

$\underline{C(\theta_0|_{\langle s\rangle}), C(\theta_0^{-1}|_{\langle s\rangle}), C(\alpha_0|_{\langle s\rangle}), C(\beta_0|_{\langle s\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\theta_{V_0}|_{\langle s\rangle}}|+|\Omega^{\theta_{V_0}^{-1}|_{\langle s\rangle}}| = |\Omega^{\theta_0|_{\langle s\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle s\rangle}}|\\
\implies &3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&=3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\\
&+2\cdot 3^{2k}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
\implies &3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\\
&=3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+3^{2k}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&-3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&-3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]
\end{aligned}
$$

Note that,

$$
\begin{aligned}
&|\Omega^{\Theta_{1,0}}|+|\Omega^{\Theta_{2,0}}|+|\Omega^{\Theta_{3,0}}|+|\Omega^{\Theta_{4,0}}| = |\Omega^{\theta_0|_{\langle s,z\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}|+|\Omega^{\alpha_0|_{\langle s,z\rangle}}|+|\Omega^{\beta_0|_{\langle s,z\rangle}}|\\
\implies &3^{2k}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&=3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})] \qquad ⑦_{10}
\end{aligned}
$$

Thus, by applying ⑦$_{10}$, we have

$$\begin{aligned}
&3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
=&3^{2k-1}\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\Big]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
=&3^{4k-1}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})] \qquad ⑧_{10}\\
\geq&3^{4k-1}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]
\end{aligned}$$

Thus, $[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\geq 2^2\cdot 3^{2k-2}$, as required.

$\underline{C(\mathsf{Id}|_{\langle s\rangle}), C(\omega|_{\langle s\rangle}), C(\eta|_{\langle s\rangle}), C(\omega\circ\eta|_{\langle s\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle s\rangle}}| + |\Omega^{\omega|_{\langle s\rangle}}| = |\Omega^{\theta_0|_{\langle s\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle s\rangle}}| \\
\Longrightarrow\ & 3^2[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ 3^3[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})] \\
&+ 3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
&+ 2\cdot 3^2[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&+ 2\cdot 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] + \sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega|_{\langle s\rangle}} \\
=&3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&+ 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \\
&+ 3^{2k+1}[C(\theta_0|_{\langle s\rangle}) + C(\theta_0^{-1}|_{\langle s\rangle}) + C(\alpha_0|_{\langle s\rangle}) + C(\beta_0|_{\langle s\rangle})] \\
&+ 2\cdot 3^{2k}[C(\Theta_{1,0}) + C(\Theta_{2,0}) + C(\Theta_{3,0}) + C(\Theta_{4,0})] \\
\Longrightarrow\ & 3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
=&(3^{2k} - 2\cdot 3^2)[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&+ 3^{2k+1}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \\
&+ 3^{2k+1}[C(\theta_0|_{\langle s\rangle}) + C(\theta_0^{-1}|_{\langle s\rangle}) + C(\alpha_0|_{\langle s\rangle}) + C(\beta_0|_{\langle s\rangle})] \\
&+ 2\cdot 3^{2k}[C(\Theta_{1,0}) + C(\Theta_{2,0}) + C(\Theta_{3,0}) + C(\Theta_{4,0})] \\
&- 3^2[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&- 3^3[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})] \\
&- 2\cdot 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] - \sigma_S^{\mathsf{Id}|_{\langle s\rangle}} - \sigma_S^{\omega|_{\langle s\rangle}}
\end{aligned}
$$

Applying ③$_{10}$,⑥$_{10}$, ⑦$_{10}$ and ⑧$_{10}$, we have

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle s\rangle})+C(\omega|_{\langle s\rangle})+C(\eta|_{\langle s\rangle})+C(\omega\circ\eta|_{\langle s\rangle})]\\
=&(3^{2k}-2\cdot 3^2)[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+\Big[3^{4k-1}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\Big]\\
&+2\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\Big]\\
&-3^2[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&-\Big[(3^{2k}-3^3)[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+(3^{2k-2}-3^2)[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\Big]\\
&-2\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\Big]-\sigma_S^{\mathsf{Id}|_{\langle s\rangle}}-\sigma_S^{\omega|_{\langle s\rangle}}\\
=&(2\cdot 3^2-3^{2k})[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&(3^{4k-1}+3^{2k}-2\cdot 3^2)[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+(3^2-3^{2k-2})[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]-\sigma_S^{\mathsf{Id}|_{\langle s\rangle}}-\sigma_S^{\omega|_{\langle s\rangle}}
\end{aligned}$$

Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. Let ϕ_1 be defined such that $\psi\circ\phi_1=\eta$. Then for all $P\in\mathcal{D}\backslash\{\langle s\rangle,\langle s,z\rangle,S\}$, we have

$$|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s\rangle}}|=|\mathcal{O}_{\phi_1|_P}^{\psi|_{\langle s\rangle}}|=\frac{|N_S(\langle s\rangle,P)||C_S(\langle s\rangle)|}{|P|}=\frac{3|N_S(\langle s,t\rangle,P)||C_S(\langle s,t\rangle)|}{|P|}=3|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,t\rangle}}|$$

Thus, $\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega|_{\langle s\rangle}} = \sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}$. Thus, by using this fact, and by substituting in ①$_{10}$, we have

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
\geq&(2\cdot 3^2 - 3^{2k})[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&(3^{4k-1} + 3^{2k} - 2\cdot 3^2)[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&+(3^2 - 3^{2k-2})[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}] - \sigma_S^{\mathsf{Id}|_{\langle s\rangle}} - \sigma_S^{\omega|_{\langle s\rangle}} \\
=&(2\cdot 3^2 - 3^{2k})[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&(3^{4k-2} + 3^{2k-1} - 2\cdot 3)\Big[3[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\Big] \\
&+(3^2 - 3^{2k-2})[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}] - \sigma_S^{\mathsf{Id}|_{\langle s\rangle}} - \sigma_S^{\omega|_{\langle s\rangle}} \\
=&3^{4k-1}[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+(3^{4k-2} + 2\cdot 3^{2k-2} + 3)[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}} + \sigma_S^{\omega|_{\langle s,t\rangle}} + \sigma_S^{\eta|_{\langle s,t\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}] - \sigma_S^{\mathsf{Id}|_{\langle s\rangle}} - \sigma_S^{\omega|_{\langle s\rangle}} \\
\geq&3^{4k-1}[C(\mathsf{Id}) + C(\omega) + C(\eta) + C(\omega\circ\eta)]
\end{aligned}$$

Thus, we have $[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \geq 2\cdot 3^{4k-4}$, as required. ⑨$_{10}$

$\underline{C(\theta_0|_{\langle t\rangle}), C(\theta_0^{-1}|_{\langle t\rangle}), C(\alpha_0|_{\langle t\rangle}), C(\beta_0|_{\langle t\rangle})}$:

$$\begin{aligned}
&|\Omega^{\theta_0|_{\langle t\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle t\rangle}}| = |\Omega^{\theta_A|_{\langle t\rangle}}| + |\Omega^{\theta_A^{-1}|_{\langle t\rangle}}| \\
\Longrightarrow\ & 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] + 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})] \\
&+3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})] \\
&+2\cdot 3^{2k}[C(\Theta_{1,0}^{-1}|) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
=&3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + 3^{4k-1}[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
&+\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})] \\
=&3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + (3^{4k-1} - 2\cdot 3^{2k})[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
&+\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
&-3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] - 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})]
\end{aligned}$$

Note that

$$
\begin{aligned}
&|\Omega^{\Theta_{1,0}^{-1}}|+|\Omega^{\Theta_{2,0}^{-1}}|+|\Omega^{\Theta_{3,0}^{-1}}|+|\Omega^{\Theta_{4,0}^{-1}}|=|\Omega^{\theta_0|_{\langle t,z\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle t,z\rangle}}|+|\Omega^{\alpha_0|_{\langle t,z\rangle}}|+|\Omega^{\beta_0|_{\langle t,z\rangle}}|\\
\Longrightarrow\ &3^{2k}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})]\\
=&3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})] \qquad (10)_{10}
\end{aligned}
$$

Thus, by substituting in $(10)_{10}$, we have

$$
\begin{aligned}
&3^{2k+1}[C(\theta_0|_{\langle t\rangle})+C(\theta_0^{-1}|_{\langle t\rangle})+C(\alpha_0|_{\langle t\rangle})+C(\beta_0|_{\langle t\rangle})]\\
=&3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+(3^{2k-1}-2)\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\Big]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}\\
&-3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]-3^{2k+1}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
=&3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+(2^2\cdot 3^{4k-1}-2^2\cdot 3^{2k+1})[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+(3^{4k}-3^{2k+2})[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}
\end{aligned}
$$

Substituting in $(1)_{10}$ and $(2)_{10}$, we have

$$
\begin{aligned}
&3^{2k+1}[C(\theta_0|_{\langle t\rangle})+C(\theta_0^{-1}|_{\langle t\rangle})+C(\alpha_0|_{\langle t\rangle})+C(\beta_0|_{\langle t\rangle})]\\
=&3^{2k+1}\Big[[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\Big]\\
&+(2^2\cdot 3^{4k-2}-2^2\cdot 3^{2k})\Big[3[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\Big]\\
&+(3^{4k}-3^{2k+2})[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}\\
=&3^{4k-1}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^{2k+2}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\\
&+(3^{4k}-3^{2k+2})[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}\\
\geq&3^{4k-1}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]
\end{aligned}
$$

It follows that $[C(\theta_0|_{\langle t\rangle})+C(\theta_0^{-1}|_{\langle t\rangle})+C(\alpha_0|_{\langle t\rangle})+C(\beta_0|_{\langle t\rangle})]\geq 2^2\cdot 3^{2k-2}$, as required. $(11)_{10}$

$\underline{C(\mathsf{Id}|_{\langle t\rangle}),C(\omega|_{\langle t\rangle}),C(\eta|_{\langle t\rangle}),C(\omega\circ\eta|_{\langle t\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\mathsf{Id}|_{\langle t\rangle}}|+|\Omega^{\omega|_{\langle t\rangle}}|=|\Omega^{\theta_A|_{\langle t\rangle}}|+|\Omega^{\theta_A^{-1}|_{\langle t\rangle}}| \\
\Longrightarrow\ & 3^{2k}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+2\cdot 3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)] \\
&+3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}} \\
=&3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+3^{4k-1}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})] \\
&+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
\Longrightarrow\ & 3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}} \\
=&3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+3^{4k-1}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})] \\
&+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}-3^{2k}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]
\end{aligned}
$$

Thus, by applying ①$_{10}$, ②$_{10}$ and ⑩$_{10}$, we have

$$
\begin{aligned}
&3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}} \\
=&3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+3^{2k-1}\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)] \\
&+3^{2k+1}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\Big]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
&-3^{2k}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)] \\
=&3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+3^{4k-1}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)] \\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
&-3^{2k}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)] \\
=&3^{2k}\Big[[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta)]\Big] \\
&+3^{4k-2}\Big[3[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\Big] \\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
&-3^{2k}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)] \\
=&3^{4k-1}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^{2k+1}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)] \\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]+\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}} \\
\geq&3^{4k-1}[C(\mathsf{Id})+C(\omega)+C(\eta)+C(\omega\circ\eta)]
\end{aligned}
$$

Thus,

$$
3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}}]\geq 2^2\cdot 3^{4k-1}, \tag{$\star$}
$$

We demonstrate that $[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]=2^2$ is the optimum choice. Recall the following definitions:

$$
\begin{aligned}
\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} C(\mathsf{Id}|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\mathsf{Id}|_P}|+C(\eta|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\eta|_P}| \\
\sigma_S^{\omega|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} C(\omega|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\omega|_P}|+C(\omega\circ\eta|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\omega\circ\eta|_P}| \\
\sigma_A^{\mathsf{Id}|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P,P\neq A}} C(\theta_A|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\theta_A|_P}|+C(\beta_A|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\beta_A|_P}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P,P\neq A}} C(\alpha_A|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\alpha_A|_P}|+C(\theta_A^{-1}|_P)|\mathcal{O}^{\mathsf{Id}|_{\langle t\rangle}}_{\theta_A^{-1}|_P}| \\
\sigma_A^{\omega|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P,P\neq A}} C(\theta_A|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\theta_A|_P}|+C(\beta_A|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\beta_A|_P}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P,P\neq A}} C(\alpha_A|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\alpha_A|_P}|+C(\theta_A^{-1}|_P)|\mathcal{O}^{\omega|_{\langle t\rangle}}_{\theta_A^{-1}|_P}|
\end{aligned}
$$

Write

$$
\begin{aligned}
N_1 &= \{\mathsf{Id}|_P,\omega|_P,\eta|_P,\omega\circ\eta|_P \mid P\in\mathcal{D},P\neq S,\langle t\rangle,\langle s\rangle,\langle s,z\rangle\},\\
N_2 &= \{\theta_A|_P,\theta_A^{-1}|_P,\alpha_A|_P,\beta_A|_P \mid P\in\mathcal{D}_A,P\neq A,\ \text{and } tz^{-1}\in P \text{ or } tz\in P\},\\
N_3 &= \{\mathsf{Id}|_P,\omega|_P,\eta|_P,\omega\circ\eta|_P \mid P=\langle s\rangle,\langle s,z\rangle\},\\
N &= N_1\sqcup N_2\\
\hat{N} &= N\sqcup\{\mathsf{Id}|_{\langle t\rangle},\omega|_{\langle t\rangle},\eta|_{\langle t\rangle},\omega\circ\eta|_{\langle t\rangle}\}.
\end{aligned}
$$

Note that $N\sqcup N_3$ is the set of morphisms which come from the definitions of $\sigma_S^{\mathsf{Id}|_{\langle t\rangle}},\sigma_S^{\omega|_{\langle t\rangle}},\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}$ and $\sigma_A^{\omega|_{\langle t\rangle}}$. Let Ω^* be our candidate characteristic biset. Under Ω^* we have

$$C_{\Omega^*}(\mathsf{Id}|_{\langle t\rangle})=C_{\Omega^*}(\omega|_{\langle t\rangle})=C_{\Omega^*}(\eta|_{\langle t\rangle})=C_{\Omega^*}(\omega\circ\eta|_{\langle t\rangle})=1,$$

and for all $\phi\in N$, we have $C_{\Omega^*}(\phi)=0$. When $\phi\in N_3$, $C_{\Omega^*}(\phi)>0$, and is the reason why we have stopped short of including N_3 in N. Thus,

$$\sum_{\phi\in\hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi|=4\cdot 3^{4k+1}.$$

Suppose that Ω' is a minimal right characteristic biset with the condition that

$$C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})+C_{\Omega'}(\omega|_{\langle t\rangle})+C_{\Omega'}(\eta|_{\langle t\rangle})+C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})=n \quad \text{where} \quad 0\leq n<4. \tag{$\star\star$}$$

From the rest of the calculations in this subsection and from Lemma $A.0.1$, we may assume for all $\phi\notin\hat{N}$ that

$$C_{\Omega'}(\phi)\geq C_{\Omega^*}(\phi). \tag{$\star\star\star$}$$

Moreover, there exists $\emptyset \neq I \subseteq N$ such that for all $\phi \in I$, we have $C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle t\rangle}}| > 0$ and by $(\star)$ and $(\star\star)$ we have

$$\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle t\rangle}}| \geq (4-n)\cdot 3^{4k-1}, \tag{$\star\star\star\star$}$$

where $\psi \in \{\mathsf{Id}, \omega\}$. We show that Ω^* is minimal by size. In other words we have to show that $|\Omega'| \geq |\Omega^*|$. From results in Appendix B, it can be checked that for each $\phi \in I$ and $\psi \in \{\mathsf{Id}, \omega\}$, we have

$$|\mathcal{O}_\phi^{\psi|_{\langle t\rangle}}| = \frac{3^{4k}}{|D_\phi|} \text{ or } \frac{3^{4k+1}}{|D_\phi|}.$$

Suppose that for every $\phi \in I$, $|\mathcal{O}_\phi^{\psi|_{\langle t\rangle}}| = \frac{3^{4k}}{|D_\phi|}$. Then, by $(\star\star\star\star)$,

$$\begin{aligned}
&\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle t\rangle}}| \geq (4-n)\cdot 3^{4k-1} \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k}}{|D_\phi|} + n\cdot 3^{4k-1} \geq 4\cdot 3^{4k-1} && \text{(by definition of } I \text{ and } N) \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+2}}{|D_\phi|} + n\cdot 3^{4k+1} \geq 4\cdot 3^{4k+1} \\
\Longrightarrow\, &C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega'}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega'}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}| \\
&\quad + \sum_{\phi \in N} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 4\cdot 3^{4k+1} && \text{(by } (\star\star) \text{ and definition of } |\mathcal{O}_\phi|) \\
\Longrightarrow &\sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 4\cdot 3^{4k+1}, && \text{(by definition of } \hat{N})
\end{aligned}$$

Thus

$$|\Omega'| - |\Omega^*| \geq \sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| - \sum_{\phi \in \hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| \geq 4\cdot 3^{4k+1} - 4\cdot 3^{4k+1} = 0,$$

where the first inequality is due to $(\star\star\star)$. It follows that Ω^* is minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

It is left to consider the case when there exists $\phi \in I$ such that $|\mathcal{O}_\phi^{\psi|_{\langle t\rangle}}| = \frac{3^{4k+1}}{|D_\phi|}$. From results in

Appendix B, this only occurs when $\phi \in N_1$ and $\langle t,z\rangle \leq D_\phi$. Thus, for all $\mu = 1,..,4$, we have

$$
\begin{aligned}
&|\Omega^{\Theta_{\mu,0}^{-1}}| = |\Omega^{\psi|_{\langle t,z\rangle}}| \\
\implies & 3^{2k}C(\Theta_{\mu,0}^{-1}) = 3^{2k}C(\psi) + \frac{3^{4k+1}}{|D_\phi|}C(\phi) + [\text{extra..}] \\
\implies & C(\Theta_{\mu,0}^{-1}) \geq C(\psi) + \frac{3^{2k+1}}{|D_\phi|}C(\phi) \geq C(\psi) + 3C(\phi) \geq 4 \qquad\qquad (\text{as } \phi \in I)
\end{aligned}
$$

where the second last inequality is because $|D_\phi| \leq 3^{2k}$. Note that, $C_{\Omega^*}(\Theta_{\mu,\lambda}^{-1}) = 1$. However, $C_{\Omega'}(\Theta_{\mu,0}) \geq 4$. Now

$$
\begin{aligned}
|\Omega'| - |\Omega^*| \geq & [C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega'}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega'}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}|] \\
& + [C_{\Omega'}(\Theta_{1,0}^{-1})|\mathcal{O}_{\Theta_{1,0}^{-1}}| + C_{\Omega'}(\Theta_{2,0}^{-1})|\mathcal{O}_{\Theta_{2,0}^{-1}}| + C_{\Omega'}(\Theta_{3,0}^{-1})|\mathcal{O}_{\Theta_{3,0}^{-1}}| + C_{\Omega'}(\Theta_{4,0}^{-1})|\mathcal{O}_{\Theta_{4,0}^{-1}}|] \\
& - [C_{\Omega^*}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega^*}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega^*}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega^*}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}|] \\
& - [C_{\Omega^*}(\Theta_{1,0}^{-1})|\mathcal{O}_{\Theta_{1,0}^{-1}}| + C_{\Omega^*}(\Theta_{2,0}^{-1})|\mathcal{O}_{\Theta_{2,0}^{-1}}| + C_{\Omega^*}(\Theta_{3,0}^{-1})|\mathcal{O}_{\Theta_{3,0}^{-1}}| + C_{\Omega^*}(\Theta_{4,0}^{-1})|\mathcal{O}_{\Theta_{4,0}^{-1}}|] \\
\geq & n \cdot 3^{4k+1} + 4\cdot 4\cdot 3^{4k} - 4\cdot 3^{4k+1} - 4\cdot 3^{4k} \\
= & n \cdot 3^{4k+1} \geq 0
\end{aligned}
$$

It follows that Ω^* is minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

So we may assume $C(\mathsf{Id}|_{\langle t\rangle}) = C(\omega|_{\langle t\rangle}) = C(\eta|_{\langle t\rangle}) = C(\omega\circ\eta|_{\langle t\rangle}) = 1$. $\quad$ 12$_{10}$

Thus, our candidate solution that we have chosen is minimal.

A.10.2 Exoticity

Note that $|\Omega^*|/|S| = 3^{6k-4} + 22\cdot 3^{4k-4} + 13\cdot 3^{2k-2} + 4$. We use Lemma 4.3.12 to determine an upper bound on the exoticity index for our fusion system. We have

$$
v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{3^{6k-4} + 22\cdot 3^{4k-4} + 13\cdot 3^{2k-2} + 4}{3^i} \right\rfloor
$$

Write $n_i = \left\lfloor \frac{3^{6k-4}+22\cdot 3^{4k-4}+13\cdot 3^{2k-2}+4}{3^i} \right\rfloor$. Now,

$$\begin{aligned}
i = 1 &\implies n_i = 3^{6k-5} + 22\cdot 3^{4k-5} + 13\cdot 3^{2k-3} + 1 \\
2 \le i \le 2k-2 &\implies n_i = 3^{6k-4-i} + 22\cdot 3^{4k-4-i} + 13\cdot 3^{2k-2-i} \\
i = 2k-1 &\implies n_i = 3^{4k-3} + 22\cdot 3^{2k-3} + 4 \\
i = 2k &\implies n_i = 3^{4k-4} + 22\cdot 3^{2k-4} + 1 \\
2k+1 \le i \le 4k-4 &\implies n_i = 3^{6k-4-i} + 22\cdot 3^{4k-4-i} \\
i = 4k-3 &\implies n_i = 3^{2k-1} + 7 \\
i = 4k-2 &\implies n_i = 3^{2k-2} + 2 \\
4k-1 \le i \le 6k-4 &\implies n_i = 3^{6k-4-i}
\end{aligned}$$

Thus,

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) =& [3^{6k-5} + 22\cdot 3^{4k-5} + 13\cdot 3^{2k-3} + 1] + \left[\sum_{i=2}^{2k-2} 3^{6k-4-i} + 22\cdot 3^{4k-4-i} + 13\cdot 3^{2k-2} - i\right] \\
&+ [3^{4k-4} + 22\cdot 3^{2k-4} + 1] + [3^{4k-3} + 22\cdot 3^{2k-3} + 4] \\
&+ \left[\sum_{i=2k+1}^{4k-4} 3^{6k-4-i} + 22\cdot 3^{4k-4-i}\right] + [3^{2k-1} + 7] \\
&+ [3^{6k-2} + 2] + \left[\sum_{4k-1}^{6k-4} 3^{6k-4-i}\right] \\
=& \frac{1}{2}[3^{6k-4} + 22\cdot 3^{4k-4} + 13\cdot 3^{2k-2} - 6]
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\log_3 |T:S| = \frac{1}{2}[3^{6k-4} + 22\cdot 3^{4k-4} + 13\cdot 3^{2k-2}](4k+3) + 6k.$$

This is our upper bound for our exoticity index.

A.11 DRV-11

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}
\Omega \cong & \bigsqcup_{Q \in \mathcal{D}} C(\mathrm{Id}|_Q)\mathcal{O}_{\mathrm{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega \circ \eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\
& \sqcup \bigsqcup_{Q \in \mathcal{D}_0} C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0^{-1}|_Q} \sqcup C(\alpha_0|_Q)\mathcal{O}_{\alpha_0|_Q} \sqcup C(\beta_0|_Q)\mathcal{O}_{\beta_0|_Q} \\
& \sqcup \bigsqcup_{\substack{Q \in \mathcal{D}_\lambda \\ \lambda=-1,1 \\ \ell=0,1}} C(\theta_\lambda \circ \eta^\ell|_Q)\mathcal{O}_{\theta_\lambda\circ\eta^\ell|_Q} \sqcup C(\theta_\lambda^{-1} \circ \eta^\ell|_Q)\mathcal{O}_{\theta_\lambda\circ\eta^\ell} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\hat{\vartheta}_{4,\lambda}} \\
& \sqcup \bigsqcup_{\substack{Q \in \mathcal{D}_{V_\lambda} \\ \lambda=-1,1}} C(\xi_{1,\lambda}|_Q)\mathcal{O}_{\xi_{1,\lambda}|_Q} \sqcup C(\xi_{2,\lambda}|_Q)\mathcal{O}_{\xi_{2,\lambda}|_Q} \sqcup C(\xi_{3,\lambda}|_Q)\mathcal{O}_{\xi_{3,\lambda}|_Q} \sqcup C(\xi_{4,\lambda}|_Q)\mathcal{O}_{\xi_{4,\lambda}|_Q} \\
& \sqcup \bigsqcup_{\substack{Q \in \mathcal{D}_{V_0} \\ \lambda=-1,1}} C(\xi_{1,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{1,\lambda}^{-1}|_Q} \sqcup C(\xi_{2,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{2,\lambda}^{-1}|_Q} \sqcup C(\xi_{3,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{3,\lambda}^{-1}|_Q} \sqcup C(\xi_{4,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{4,\lambda}^{-1}|_Q}
\end{aligned}$$

Equations:

We list the following equations:

$\psi \in \{\mathrm{Id}|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1,0,1\}$:

If $Q \in \mathcal{D}_\lambda \backslash \{\langle z \rangle\}$ for $\lambda = -1,1$, then the equations are the same as those for DRV-7. If $Q \in \mathcal{D}_0 \backslash \{\langle z \rangle\}$, then the equations are the same as those for DRV-9.

$\psi \in \{\theta_\lambda|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1,1\}$:

If $Q \in \mathcal{D}_\lambda \backslash \{\langle z \rangle\}$, for $\lambda = -1,1$, then the equations are the same as those for DRV-7.

$\psi \in \{\theta_0|_Q \mid Q \in \mathcal{D}_0\}$:

If $Q \in \mathcal{D}_0 \backslash \{\langle z \rangle\}$, then the equations are the same as those for DRV-9.

$\psi \in \{\theta_\lambda \circ \eta|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1,1\}$:

If $Q \in \mathcal{D}_\lambda \backslash \{\langle z \rangle\}$, for $\lambda = -1,1$, then the equations are the same as those for DRV-7.

$\underline{\psi \in \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda} \mid \lambda = -1,1\}}$:

In these cases, the equations are the same as those for DRV-7.

$\underline{\psi \in \{\xi_{1,\lambda}|_Q \mid Q \in \mathcal{D}_{V_\lambda}, \lambda = -1,1\}}$:

$$|\Omega^{\xi_{1,\lambda}}| = 3^3 C(\xi_{1,\lambda})$$
$$|\Omega^{\xi_{1,\lambda}|_{\langle ss_1^\lambda \rangle}}| = 3^3[C(\xi_{1,\lambda}) + C(\xi_{3,\lambda})] + 3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda \rangle}) + C(\xi_{3,\lambda}|_{\langle ss_1^\lambda \rangle})]$$

$\underline{\psi \in \{\xi_{1,\lambda}^{-1}|_Q \mid Q \in \mathcal{D}_{V_0}, \lambda = -1,1\}}$:

$$|\Omega^{\xi_{1,\lambda}^{-1}}| = 3^3 C(\xi_{1,\lambda}^{-1})$$
$$|\Omega^{\xi_{1,\lambda}^{-1}}| = 3^3[C(\xi_{1,\lambda}) + C(\xi_{3,\lambda})] + 3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda \rangle}) + C(\xi_{3,\lambda}|_{\langle ss_1^\lambda \rangle})]$$

Candidate solution:

We have the following designated values for the constants:

$$C(\mathsf{Id}_S) = C(\omega) = C(\eta) = C(\omega \circ \eta) = 1$$
$$C(\mathsf{Id}|_{\langle ss_1^\lambda, z \rangle}) = C(\omega|_{\langle ss_1^\lambda, z \rangle}) = C(\eta|_{\langle ss_1^\lambda, z \rangle}) = C(\omega \circ \eta|_{\langle s_1^\lambda, z \rangle}) = 3^{2k-3} - 1 \quad (\text{for } \lambda = -1,0,1)$$
$$C(\theta_0) = C(\theta_0^{-1}) = C(\alpha_0) = C(\beta_0) = 1$$
$$C(\theta_\lambda) = C(\theta_\lambda^{-1}) = C(\theta_\lambda \circ \eta) = C(\theta_\lambda^{-1} \circ \eta) = 1 \quad (\text{for } \lambda = -1,1)$$
$$C(\hat{\theta}_{V_\lambda}) = C(\hat{\beta}_{V_\lambda}) = C(\hat{\vartheta}_{1,\lambda}) = C(\hat{\vartheta}_{4,\lambda}) = 3^{2k-3} \quad (\text{for } \lambda = -1,1)$$
$$C(\xi_{1,\lambda}) = C(\xi_{2,\lambda}) = C(\xi_{3,\lambda}) = C(\xi_{4,\lambda}) = 3^{2k-3} \quad (\text{for } \lambda = -1,1)$$
$$C(\xi_{1,\lambda}^{-1}) = C(\xi_{2,\lambda}^{-1}) = C(\xi_{3,\lambda}^{-1}) = C(\xi_{4,\lambda}^{-1}) = 3^{2k-3} \quad (\text{for } \lambda = -1,1)$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$\begin{aligned}
\Omega^* \cong & \mathcal{O}_{\mathsf{Id}_S} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta} \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} (3^{2k-3}-1)[\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}] \\
& \sqcup \mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0} \\
& \sqcup \bigsqcup_{\lambda=-1,1} [\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}] \sqcup [\mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}] \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}] \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\xi_{1,\lambda}} \sqcup \mathcal{O}_{\xi_{2,\lambda}} \sqcup \mathcal{O}_{\xi_{3,\lambda}} \sqcup \mathcal{O}_{\xi_{4,\lambda}}] \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\xi_{1,\lambda}^{-1}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{-1}} \sqcup \mathcal{O}_{\xi_{3,\lambda}^{-1}} \sqcup \mathcal{O}_{\xi_{4,\lambda}^{-1}}]
\end{aligned}$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.11.1 Minimality

We now determine minimal values of the constants:

We already know, by Lemma 4.2.8, that $C(\mathsf{Id}_S), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$.

$C(\theta_\lambda), C(\theta_\lambda^{-1}), C(\theta_\lambda\circ\eta), C(\theta_\lambda^{-1}\circ\eta)$ for $\lambda = -1,1$:

By ①$_7$, we get $[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] \geq 2^2$. ①$_{11}$

$C(\theta_0), C(\theta_0^{-1}), C(\alpha_0), C(\beta_0)$:

By ①$_9$, we get $[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \geq 2^2$. ②$_{11}$

$C(\mathsf{Id}|_{\langle ss_1^{\lambda},z\rangle}), C(\omega|_{\langle ss_1^{\lambda},z\rangle}), C(\eta|_{\langle ss_1^{\lambda},z\rangle}), C(\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle})$ for $\lambda = -1,0,1$:

By ③$_7$, we get $[C(\mathsf{Id}|_{\langle ss_1^{\lambda},z\rangle}) + C(\omega|_{\langle ss_1^{\lambda},z\rangle}) + C(\eta|_{\langle ss_1^{\lambda},z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle})] \geq 2^2\cdot(3^{2k-3}-1)$ for $\lambda=-1,1$. Similarly, by ②$_9$, we get $[C(\mathsf{Id}|_{\langle s,z\rangle}) + C(\omega|_{\langle s,z\rangle}) + C(\eta|_{\langle s,z\rangle}) + C(\omega\circ\eta|_{\langle s,z\rangle})] \geq 2^2\cdot(3^{2k-3}-1)$. ③$_{11}$

$C(\hat{\theta}_{V_\lambda}), C(\hat{\beta}_{V_\lambda}), C(\vartheta_{1,\lambda}), C(\vartheta_{4,\lambda})$ for $\lambda = -1,1$:

By ②$_7$, we get $[C(\hat{\theta}_{V_\lambda})+C(\hat{\beta}_{V_\lambda})+C(\hat{\vartheta}_{1,\lambda})+C(\hat{\vartheta}_{4,\lambda})] \geq 2^2\cdot 3^{2k-3}$. ④$_{11}$

$\underline{C(\xi_{1,\lambda}), C(\xi_{2,\lambda}), C(\xi_{3,\lambda}), C(\xi_{4,\lambda}) \text{ for } \lambda=-1,1\text{:}}$

$$\begin{aligned}
&|\Omega^{\xi_{1,\lambda}}|+|\Omega^{\xi_{2,\lambda}}|+|\Omega^{\xi_{3,\lambda}}|+|\Omega^{\xi_{4,\lambda}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|+|\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle}}|\\
\implies & 3^3[C(\xi_{1,\lambda})C(\xi_{2,\lambda})+C(\xi_{3,\lambda})+C(\xi_{4,\lambda})]\\
&=3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&\quad+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]
\end{aligned}$$

Thus, by using ①$_7$, $[C(\xi_{1,\lambda})+C(\xi_{2,\lambda})+C(\xi_{3,\lambda})+C(\xi_{4,\lambda})] \geq 2^2\cdot 3^{2k-3}$, as required. ⑤$_{11}$

$\underline{C(\xi_{1,\lambda}^{-1}), C(\xi_{2,\lambda}^{-1}), C(\xi_{3,\lambda}^{-1}), C(\xi_{4,\lambda}^{-1}) \text{ for } \lambda=-1,1\text{:}}$

$$\begin{aligned}
&|\Omega^{\xi_{1,\lambda}^{-1}}|+|\Omega^{\xi_{2,\lambda}^{-1}}|+|\Omega^{\xi_{3,\lambda}^{-1}}|+|\Omega^{\xi_{4,\lambda}^{-1}}| = |\Omega^{\theta_0|_{\langle s,z\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle s,z\rangle}}|+|\Omega^{\alpha_0|_{\langle s,z\rangle}}|+|\Omega^{\beta_0|_{\langle s,z\rangle}}|\\
\implies & 3^3[C(\xi_{1,\lambda}^{-1})+C(\xi_{2,\lambda}^{-1})+C(\xi_{3,\lambda}^{-1})+C(\xi_{4,\lambda}^{-1})]\\
&=3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]
\end{aligned}$$

Thus, by using ①$_9$, we have $[C(\xi_{1,\lambda}^{-1})+C(\xi_{2,\lambda}^{-1})+C(\xi_{3,\lambda}^{-1})+C(\xi_{4\lambda}^{-1})] \geq 2^2\cdot 3^{2k-3}$. ⑥$_{11}$

A.11.2 Exoticity

Note that $|\Omega^*|/|S| = 4[3^{4k-2}-2\cdot 3^{2k-1}+1]$. Thus,

$$v_3(\text{Sym}(|\Omega^*|/|S|)) = \sum_{i=1}\left\lfloor \frac{4[3^{4k-2}-2\cdot 3^{2k-1}+1]}{3^i}\right\rfloor$$

Write $n_i = \left\lfloor \frac{4[3^{4k-2}-2\cdot 3^{2k-1}+1]}{3^i}\right\rfloor$. Now,

$$\begin{aligned}
i=1 &\implies n_i = 4\cdot 3^{4k-3}-8\cdot 3^{2k-2}+1\\
2\leq i\leq 2k-1 &\implies n_i = 4\cdot 3^{2k-1-i}[3^{2k-1}-2]\\
i=2k &\implies n_i = 4\cdot 3^{2k-2}-3\\
2k+1\leq i\leq 4k-2 &\implies n_i = 4\cdot 3^{4k-2-i}-1\\
i=4k-1 &\implies n_i = 1
\end{aligned}$$

Thus,

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) =& [4\cdot 3^{4k-3} - 8\cdot 3^{2k-2} + 1] + \left[\sum_{i=2}^{2k-1} 4\cdot(3^{2k-1}-2)\cdot 3^{2k-1-i}\right] \\
&+ [4\cdot 3^{2k-2} - 3] + \left[\sum_{i=2k+1}^{4k-2} 4\cdot 3^{4k-2-i} - 1\right] + 1 \\
=& 2\cdot 3^{4k-2} - 2^2\cdot 3^{4k-1} - (2k-3)
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3|T:S| &= (|\Omega^*|/|S|-1)(2k+1) + [2\cdot 3^{4k-2} - 2^2\cdot 3^{4k-1} - (2k-3)] \\
&= 2[(3^{4k-2} - 2\cdot 3^{2k-1})(4k+3) + (2k+3)].
\end{aligned}$$

Thus, we have an upper bound for our exoticity index.

A.12 DRV-12

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$
\begin{aligned}
\Omega \cong & \bigsqcup_{Q\in\mathcal{D}} C(\mathrm{Id}|_Q)\mathcal{O}_{\mathrm{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\
& \sqcup \bigsqcup_{Q\in\mathcal{D}_A} C(\theta_A|_Q)\mathcal{O}_{\theta_A|_Q} \sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q} \sqcup C(\alpha_A|_Q)\mathcal{O}_{\alpha_A|_Q} \sqcup C(\beta_A|_Q)\mathcal{O}_{\beta_A|_Q} \\
& \sqcup \bigsqcup_{Q\in\mathcal{D}_0} C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\alpha_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\beta_0|_Q)\mathcal{O}_{\theta_0|_Q} \\
& \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_\lambda \\ \lambda=-1,1 \\ \ell=0,1}} C(\theta_\lambda\circ\eta^\ell|_Q)\mathcal{O}_{\theta_\lambda\circ\eta^\ell|_Q} \sqcup C(\theta_\lambda^{-1}\circ\eta^\ell|_Q)\mathcal{O}_{\theta_\lambda^{-1}\circ\eta^\ell|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\hat{\vartheta}_{4,\lambda}} \\
& \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_{V_\lambda} \\ \lambda=-1,0,1}} C(\theta_{V_\lambda}|_Q)\mathcal{O}_{\theta_{V_\lambda}} \sqcup C(\theta_{V_\lambda}^{-1}|_Q)\mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup C(\alpha_{V_\lambda}|_Q)\mathcal{O}_{\alpha_{V_\lambda}|_Q} \sqcup C(\beta_{V_\lambda}|_Q)\mathcal{O}_{\beta_{V_\lambda}|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} C(\Theta_{1,\lambda})\mathcal{O}_{\Theta_{1,\lambda}} \sqcup C(\Theta_{2,\lambda})\mathcal{O}_{\Theta_{2,\lambda}} \sqcup C(\Theta_{3,\lambda})\mathcal{O}_{\Theta_{3,\lambda}} \sqcup C(\Theta_{4,\lambda})\mathcal{O}_{\Theta_{4,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} C(\Theta_{1,\lambda}^{-1})\mathcal{O}_{\Theta_{1,\lambda}} \sqcup C(\Theta_{2,\lambda}^{-1})\mathcal{O}_{\Theta_{2,\lambda}} \sqcup C(\Theta_{3,\lambda}^{-1})\mathcal{O}_{\Theta_{3,\lambda}} \sqcup C(\Theta_{4,\lambda}^{-1})\mathcal{O}_{\Theta_{4,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\vartheta_{1,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{2,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{3,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{4,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \\
& \sqcup \bigsqcup_{\substack{Q=\langle ss_1^\lambda\rangle,\langle ss_1^\lambda,z\rangle \\ \lambda=-1,1}} C(\xi_{1,\lambda}|_Q)\mathcal{O}_{\xi_{1,\lambda}|_Q} \sqcup C(\xi_{2,\lambda}|_Q)\mathcal{O}_{\xi_{2,\lambda}|_Q} \sqcup C(\xi_{3,\lambda}|_Q)\mathcal{O}_{\xi_{3,\lambda}|_Q} \sqcup C(\xi_{4,\lambda}|_Q)\mathcal{O}_{\xi_{4,\lambda}|_Q} \\
& \sqcup \bigsqcup_{\substack{Q=\langle s\rangle,\langle s,z\rangle \\ \lambda=-1,1}} C(\xi_{1,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{1,\lambda}^{-1}|_Q} \sqcup C(\xi_{2,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{2,\lambda}^{-1}|_Q} \sqcup C(\xi_{3,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{3,\lambda}^{-1}|_Q} \sqcup C(\xi_{4,\lambda}^{-1}|_Q)\mathcal{O}_{\xi_{4,\lambda}^{-1}|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\xi_{5,\lambda})\mathcal{O}_{\xi_{5,\lambda}} \sqcup C(\xi_{6,\lambda})\mathcal{O}_{\xi_{6,\lambda}} \sqcup C(\xi_{7,\lambda})\mathcal{O}_{\xi_{7,\lambda}} \sqcup C(\xi_{8,\lambda})\mathcal{O}_{\xi_{8,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\xi_{5,\lambda}^{-1})\mathcal{O}_{\xi_{5,\lambda}^{-1}} \sqcup C(\xi_{6,\lambda}^{-1})\mathcal{O}_{\xi_{6,\lambda}^{-1}} \sqcup C(\xi_{7,\lambda}^{-1})\mathcal{O}_{\xi_{7,\lambda}^{-1}} \sqcup C(\xi_{8,\lambda}^{-1})\mathcal{O}_{\xi_{8,\lambda}^{-1}}
\end{aligned}
$$

As already explained in the case for DRV-8 and DRV-10, there is a difference between the construction in Theorem 7.3.1 and the one above. The difference here is that we have excluded orbits induced by the following morphisms:

1. $\Theta_{\mu,\lambda}|_Q$, for $Q \lneq \langle ss_1^\lambda, z\rangle$, $\lambda = -1,0,1$, and $\mu = 1,..,4$.
2. $\Theta_{\mu,\lambda}^{-1}|_Q$ for $Q \lneq \langle t,z\rangle$, $\lambda = -1,0,1$ and $\mu = 1,..,4$.

3. $\vartheta_{\mu,\lambda}|_Q$ for $Q \lneq \langle ss_1^\lambda, z\rangle$, $\lambda=-1,1$, and $\mu=1,..,4$.

4. $\xi_{\mu,\lambda}|_Q$ for $Q \lneq \langle ss_1^\lambda, z\rangle$, $\lambda=-1,1$ and $\mu=1,..,8$ (excluding $\xi_{\mu,\lambda}|_{\langle ss_1^\lambda\rangle}$ for $\mu=1,2,3,4$).

5. $\xi_{\mu,\lambda}^{-1}|_Q$ for $Q \lneq \langle s, z\rangle$, $\lambda=-1,1$ and $\mu=1,..,8$ (excluding $\xi_{\mu,\lambda}^{-1}|_{\langle s\rangle}$ for $\mu=1,2,3,4$)

This is because all of these morphisms are restrictions of those from the set

$$\{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T \mid T = V_\lambda, A, \lambda=-1,0,1\},$$

which are already included in the the general construction.

Equations:
We list the following equations:

$\psi \in \{\mathsf{Id}|_Q \mid Q \in \mathcal{D}_A \cup \mathcal{D}_\lambda, \lambda=-1,0,1\}$:

If $Q \in \mathcal{D}_A \backslash \{\langle z\rangle\}$, then the equations are the same as those for DRV-8. If $Q \in \mathcal{D}_\lambda \backslash \{\langle z\rangle\}$ for $\lambda=-1,1$, then the equations are the same as those for DRV-8. If $Q \in \mathcal{D}_0 \backslash \{\langle z\rangle\}$, then the equations are the same as those for DRV-10. So suppose $Q=\langle z\rangle$. Then

$$\begin{aligned}
|\Omega^{\mathsf{Id}|_{\langle z\rangle}}| = &\sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle z\rangle}}| + C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\mathsf{Id}|_{\langle z\rangle}}| + \sum_{\substack{P\in\mathcal{D}_\lambda, \langle z\rangle\le P\\ \lambda=-1,0,1}} \frac{3^{4k+2}}{|P|}[C(\theta_\lambda|_P) + C(\theta_\lambda^{-1}|_P)] \\
&+ \sum_{\lambda=-1,1} 3^{4k}[C(\hat{\vartheta}_{1,\lambda}) + C(\hat{\vartheta}_{4,\lambda})] \\
&+ \sum_{\lambda=-1,1} 3^{4k}[C(\xi_{3,\lambda}) + C(\xi_{4,\lambda})] + \sum_{\lambda=-1,1} 3^{4k}[C(\xi_{3,\lambda}^{-1}) + C(\xi_{4,\lambda}^{-1})]
\end{aligned}$$

$\psi \in \{\theta_A|_Q \mid Q \in \mathcal{D}_A\}$:

If $Q \in \mathcal{D}_A \backslash \{\langle t\rangle, \langle z\rangle\}$, then the equations are the same as those for DRV-8 (or equivalently DRV-6 or DRV-10). So suppose $Q \in \{\langle t\rangle, \langle z\rangle\}$. Then

$$\begin{aligned}
|\Omega^{\theta_A|_{\langle t\rangle}}| &= \sum_{P\in\mathcal{D}_A} [C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_P}| + C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\theta_A|_P}|] \\
&\quad + \sum_{\lambda=-1,0,1} 3^{4k-1}[C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,0}^{-1})] \\
|\Omega^{\theta_A|_{\langle z\rangle}}| &= \sum_{P\in\mathcal{D}_A} [C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_{\langle z\rangle}}| + C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\theta_A|_{\langle z\rangle}}|] \\
&\quad + \sum_{\lambda=-1,0,1} 3^{4k-1}[C(\Theta_{1,\lambda}) + C(\Theta_{4,\lambda})]
\end{aligned}$$

$\underline{\psi \in \{\theta_\lambda|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1,0,1\}}$:

If $Q \in \mathcal{D}_\lambda \backslash \{\langle z \rangle\}$, for $\lambda = -1,1$, then the equations are the same as DRV-8. If $Q \in \mathcal{D}_0 \backslash \{\langle z \rangle\}$, then the equations are the same as those for DRV-10. If $Q = \langle z \rangle$, then $\theta_\lambda|_Q = \mathsf{Id}|_Q$, for all $\lambda = -1,0,1$.

$\underline{\psi \in \{\theta_\lambda \circ \eta \mid Q \in \mathcal{D}_\lambda, \lambda = -1,1\}}$:

If $Q \in \mathcal{D}_\lambda \backslash \{\langle z \rangle\}$, for $\lambda = -1,1$, then the equations are the same as DRV-8. If $Q = \langle z \rangle$, then $\theta_\lambda \circ \eta|_Q = \eta|_Q$, for all $\lambda = -1,1$.

$\underline{\psi\{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda} \mid \lambda = -1,1\}}$:

In this case, the equations are the same as those for DRV-8.

$\underline{\psi \in \{\theta_{V_\lambda}|_Q \mid Q \in \mathcal{D}_{V_\lambda}, \lambda = -1,0,1\}}$

When $Q = \langle ss_1^\lambda, z \rangle$ for $\lambda = -1,1$, then the equations are the same as those for DRV-8. If $Q = \langle s, z \rangle$, then the equation is the same as that for DRV-12. First suppose that $Q \in \{\langle ss_1^\lambda \rangle, \langle z \rangle\}$, for $\lambda - 1, 1$. Then

$$\begin{aligned}
|\Omega^{\theta_{V_\lambda}|_{\langle ss_1^\lambda \rangle}}| &= 3^{2k+1}[C(\theta_{V_\lambda}) + C(\alpha_{V_\lambda})] + 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda \rangle}) + C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda \rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,\lambda}) + C(\Theta_{3,\lambda})] \\
&+ 3^{2k+1}[C(\vartheta_{2,\lambda}) + C(\vartheta_{4,\lambda})] \\
&+ 3^{2k+1}[C(\xi_{5,\lambda}) + C(\xi_{7,\lambda})] \\
|\Omega^{\theta_{V_\lambda}|_{\langle z \rangle}}| &= 3^{2k+1}[C(\theta_{V_\lambda}) + C(\beta_{V_\lambda})] + 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z \rangle}) + C(\beta_{V_\lambda}|_{\langle z \rangle})] \\
&+ 3^{2k+1}[C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \\
&+ 3^{2k+1}[C(\vartheta_{2,-\lambda}) + C(\vartheta_{3,-\lambda})] \\
&+ 3^{2k+1}[C(\xi_{6,\lambda}^{-1}) + C(\xi_{8,\lambda}^{-1})]
\end{aligned}$$

Now, suppose $Q \in \{\langle s\rangle, \langle z\rangle\}$ and $\lambda = 0$. Then

$$
\begin{aligned}
|\Omega^{\theta_{V_0}|_{\langle s\rangle}}| &= 3^{2k+1}[C(\theta_{V_0}) + C(\alpha_{V_0})] + 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\alpha_{V_0}|_{\langle s\rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,0}) + C(\Theta_{3,0})] \\
&+ \sum_{\lambda=-1,1} 3^{2k+1}[C(\xi^{-1}_{5,\lambda}) + C(\xi^{-1}_{8,\lambda})] \\
|\Omega^{\theta_{V_0}|_{\langle z\rangle}}| &= 3^{2k+1}[C(\theta_{V_0}) + C(\beta_{V_0})] + 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\beta_{V_0}|_{\langle s\rangle})] \\
&+ 3^{2k+1}[C(\Theta^{-1}_{2,0}) + C(\Theta^{-1}_{4,0})] \\
&+ \sum_{\lambda=-1,1} 3^{2k+1}[C(\xi_{6,\lambda}) + C(\xi_{7,\lambda})]
\end{aligned}
$$

$\underline{\psi \in \{\Theta_{1,\lambda} \mid \lambda = -1,0,1\}}$:

If $\lambda = -1,1$, then the equations are the same as those for DRV-8. If $\lambda = 0$, then the equations is the same as that for DRV-10.

$\underline{\psi \in \{\Theta^{-1}_{1,\lambda} \mid \lambda = -1,0,1\}}$:

If $\lambda = -1,1$, then the equations are the same as those for DRV-8. If $\lambda = 0$, then the equations is the same as that for DRV-10.

$\underline{\psi \in \{\vartheta_{1,\lambda} \mid \lambda = -1,1\}}$:

In this case the equations are the same as those for DRV-8.

$\underline{\psi \in \{\xi_{1,\lambda}|_Q \mid Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \lambda = -1,1\}}$:

$$
\begin{aligned}
|\Omega^{\xi_{1,\lambda}}| &= 3^3 C(\xi_{1,\lambda}) \\
|\Omega^{\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}}| &= 3^3[C(\xi_{1,\lambda}) + C(\xi_{3,\lambda})] + 3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle})]
\end{aligned}
$$

$\underline{\psi \in \{\xi^{-1}_{1,\lambda}|_Q \mid Q \in \{\langle s\rangle, \langle s,z\rangle\}, \lambda = -1,1\}}$:

$$
\begin{aligned}
|\Omega^{\xi^{-1}_{1,\lambda}}| &= 3^3 C(\xi^{-1}_{1,\lambda}) \\
|\Omega^{\xi_{1,\lambda}|_{\langle s\rangle}}| &= 3^3[C(\xi^{-1}_{1,\lambda}) + C(\xi^{-1}_{3,\lambda})] + 3^3[C(\xi^{-1}_{1,\lambda}|_{\langle s\rangle}) + C(\xi^{-1}_{3,\lambda}|_{\langle s\rangle})]
\end{aligned}
$$

$\underline{\psi \in \{\xi_{5,\lambda} \mid \lambda = -1,1\}}$:

$$|\Omega^{\xi_{5,\lambda}}| = 3^2 C(\xi_{5,\lambda})$$

$\underline{\psi \in \{\xi_{5,\lambda}^{-1} \mid \lambda = -1,1\}}$:

$$|\Omega^{\xi_{5,\lambda}}| = 3^2 C(\xi_{5,\lambda}^{-1})$$

Candidate solution:

We have the following designated valued for the constants:

$$\begin{aligned}
&C(\mathsf{Id}_S) = C(\omega) = C(\eta) = C(\omega \circ \eta) = 1 \\
&C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) = C(\omega|_{\langle ss_1^\lambda, z\rangle}) = C(\eta|_{\langle ss_1^\lambda, z\rangle}) = C(\omega \circ \eta|_{\langle ss_1^\lambda, z\rangle}) = 3^{2k-3} - 1 \quad (\text{for } \lambda = -1,0,1) \\
&C(\mathsf{Id}|_{\langle s\rangle}) = C(\omega|_{\langle s\rangle}) = C(\eta|_{\langle s\rangle}) = C(\omega \circ \eta|_{\langle s\rangle}) = 3^{4k-4} \\
&C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle}) = C(\omega|_{\langle ss_1^\lambda\rangle}) = C(\eta|_{\langle ss_1^\lambda\rangle}) = C(\omega \circ \eta|_{\langle ss_1^\lambda\rangle}) = 2^2 \cdot 3^{4k-4} \quad (\text{for } \lambda = -1,1) \\
&C(\mathsf{Id}|_{\langle t\rangle}) = C(\omega|_{\langle t\rangle}) = C(\eta|_{\langle ss_1^\lambda\rangle}) = C(\omega \circ \eta|_{\langle t\rangle}) = 3 \\
&C(\theta_A) = C(\theta_A^{-1}) = C(\alpha_A) = C(\beta_A) = 1 \\
&C(\theta_A|_{\langle z\rangle}) = C(\theta_A^{-1}|_{\langle z\rangle}) = C(\alpha_A|_{\langle z\rangle}) = C(\beta_A|_{\langle z\rangle}) = 17 \cdot 3^{2k-3} - 3 \\
&C(\theta_0) = C(\theta_0^{-1}) = C(\alpha_0) = C(\beta_0) = 1 \\
&C(\theta_0|_{\langle s\rangle}) = C(\theta_0^{-1}|_{\langle s\rangle}) = C(\alpha_0|_{\langle s\rangle}) = C(\beta_0|_{\langle s\rangle}) = 3^{2k-1} \\
&C(\theta_0|_{\langle t\rangle}) = C(\theta_0^{-1}|_{\langle t\rangle}) = C(\alpha_0|_{\langle t\rangle}) = C(\beta_0|_{\langle t\rangle}) = 3^{2k-1} \\
&C(\theta_\lambda) = C(\theta_\lambda^{-1}) = C(\theta_\lambda \circ \eta) = C(\theta_\lambda^{-1} \circ \eta) = 1 \quad (\text{for } \lambda = -1,1) \\
&C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}) = C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}) = C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda\rangle}) = C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda\rangle}) = 3^{2k-1} \quad (\text{for } \lambda = -1,1) \\
&C(\theta_\lambda|_{\langle t\rangle}) = C(\theta_\lambda^{-1}|_{\langle t\rangle}) = C(\theta_\lambda \circ \eta|_{\langle t\rangle}) = C(\theta_\lambda^{-1} \circ \eta|_{\langle t\rangle}) = 3^{2k-1} \quad (\text{for } \lambda = -1,1) \\
&C(\hat{\theta}_{V_\lambda}) = C(\hat{\beta}_{V_\lambda}) = C(\hat{\vartheta}_{1,\lambda}) = C(\hat{\vartheta}_{4,\lambda}) = 3^{2k-3} \quad (\text{for } \lambda = -1,1) \\
&C(\theta_{V_\lambda}) = C(\theta_{V_\lambda}) = C(\alpha_{V_\lambda}) = C(\beta_{V_\lambda}) = 3^{2k-2} \quad (\text{for } \lambda = -1,0,1) \\
&C(\theta_{V_\lambda}|_{\langle z\rangle}) = C(\theta_{V_\lambda}|_{\langle z\rangle}) = C(\alpha_{V_\lambda}|_{\langle z\rangle}) = C(\beta_{V_\lambda}|_{\langle z\rangle}) = 17 \cdot 3^{4k-5} - 3^{2k-1} \quad (\text{for } \lambda = -1,0,1) \\
&C(\Theta_{1,\lambda}) = C(\Theta_{2,\lambda}) = C(\Theta_{3,\lambda}) = C(\Theta_{4,\lambda}) = 1 \quad (\text{for } \lambda = -1,1) \\
&C(\Theta_{1,\lambda}^{-1}) = C(\Theta_{2,\lambda}^{-1}) = C(\Theta_{3,\lambda}^{-1}) = C(\Theta_{4,\lambda}^{-1}) = 1 \quad (\text{for } \lambda = -1,1) \\
&C(\vartheta_{1,\lambda}) = C(\Theta_{2,\lambda}) = C(\vartheta_{3,\lambda}) = C(\vartheta_{4,\lambda}) = 3^{2k-2}, \quad (\text{for } \lambda = -1,1) \\
&C(\xi_{1,\lambda}) = C(\xi_{2,\lambda}) = C(\xi_{3,\lambda}) = C(\xi_{4,\lambda}) = 3^{2k-3} \quad (\text{for } \lambda = -1,1) \\
&C(\xi_{1,\lambda}^{-1}) = C(\xi_{2,\lambda}^{-1}) = C(\xi_{3,\lambda}^{-1}) = C(\xi_{4,\lambda}^{-1}) = 3^{2k-3} \quad (\text{for } \lambda = -1,1) \\
&C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}) = C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}) = C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle}) = C(\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle}) = 3^{4k-4} + 2 \cdot 3^{2k-3} \\
&\qquad (\text{for } \lambda = -1,1) \\
&C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle}) = C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle}) = C(\xi_{3,\lambda}^{-1}|_{\langle s\rangle}) = C(\xi_{4,\lambda}^{-1}|_{\langle s\rangle}) = 3^{4k-4} + 2 \cdot 3^{2k-3} \quad (\text{for } \lambda = -1,1) \\
&C(\xi_{5,\lambda}) = C(\xi_{6,\lambda}) = C(\xi_{7,\lambda}) = C(\xi_{8,\lambda}) = 3^{2k-2} \quad (\text{for } \lambda = -1,1) \\
&C(\xi_{5,\lambda}^{-1}) = C(\xi_{6,\lambda}^{-1}) = C(\xi_{7,\lambda}^{-1}) = C(\xi_{8,\lambda}^{-1}) = 3^{2k-2}, \quad (\text{for } \lambda = -1,1)
\end{aligned}$$

with the rest of constants equated to 0. Writing Ω^* to denote this solution, we have

$$
\begin{aligned}
\Omega^* \cong & (\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} (3^{2k-3}-1)(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^\lambda, z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^\lambda, z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle}}) \\
& \sqcup 3^{4k-4}(\mathcal{O}_{\mathsf{Id}|_{\langle s\rangle}} \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2^2\cdot 3^{4k-4}(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^\lambda\rangle}}) \\
& \sqcup 3(\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}) \\
& \sqcup (17\cdot 3^{2k-3}-3)(\mathcal{O}_{\theta_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_A|_{\langle z\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_0} \sqcup \mathcal{O}_{\theta_0^{-1}} \sqcup \mathcal{O}_{\alpha_0} \sqcup \mathcal{O}_{\beta_0}) \\
& \sqcup 3^{2k-1}(\mathcal{O}_{\theta_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\beta_0}|\langle s\rangle) \\
& \sqcup 3^{2k-1}(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\alpha_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\beta_0|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-1}(\mathcal{O}_{\theta_\lambda|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-1}(\mathcal{O}_{\theta_\lambda|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}(\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} 3^{2k-2}(\mathcal{O}_{\theta_{V_\lambda}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}} \sqcup \mathcal{O}_{\beta_{V_\lambda}}) \\
& \sqcup \bigsqcup_{\lambda=-1,0,1} (17\cdot 3^{4k-5}-3^{2k-1})(\mathcal{O}_{\theta_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_\lambda}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,0,1\\ \ell=-1,1}} (\mathcal{O}_{\Theta_{1,\lambda}^\ell} \sqcup \mathcal{O}_{\Theta_{2,\lambda}^\ell} \sqcup \mathcal{O}_{\Theta_{3,\lambda}^\ell} \sqcup \mathcal{O}_{\Theta_{4,\lambda}^\ell}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\vartheta_{2,\lambda}} \sqcup \mathcal{O}_{\vartheta_{3,\lambda}} \sqcup \mathcal{O}_{\vartheta_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\lambda,\ell=-1,1} 3^{2k-3}(\mathcal{O}_{\xi_{1,\lambda}^\ell} \sqcup \mathcal{O}_{\xi_{2,\lambda}^\ell} \sqcup \mathcal{O}_{\xi_{3,\lambda}^\ell} \sqcup \mathcal{O}_{\xi_{4,\lambda}^\ell}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{4k-4}+2\cdot 3^{2k-3})(\mathcal{O}_{\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle}} \sqcup \mathcal{O}_{\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{4k-4}+2\cdot 3^{2k-3})(\mathcal{O}_{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{3,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{4,\lambda}^{-1}|_{\langle s\rangle}}) \\
& \sqcup \bigsqcup_{\lambda,\ell=-1,1} 3^{2k-2}(\mathcal{O}_{\xi_{5,\lambda}^\ell} \sqcup \mathcal{O}_{\xi_{5,\lambda}^\ell} \sqcup \mathcal{O}_{\xi_{5,\lambda}^\ell} \sqcup \mathcal{O}_{\xi_{5,\lambda}^\ell})
\end{aligned}
$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.12.1 Minimality

We write down a few definitions. Let $\psi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$. For $\lambda = -1, 1$, define

$$\begin{aligned}
\sigma_S^{\psi|_{\langle ss_1^\lambda\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq\langle ss_1^\lambda\rangle,\langle ss_1^\lambda,z\rangle,S}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda\rangle}}| \\
\sigma_S^{\psi|_{\langle z\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq\langle ss_1^\mu,z\rangle,S,\mu=-1,0,1}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}| + |C(\psi\circ\omega|_P)|\mathcal{O}_{\psi\circ\omega|_P}^{\psi|_{\langle z\rangle}}| \\
\sigma_A^{\psi|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P}} C(\phi_1|_P)|\mathcal{O}_{\phi_1|_P}^{\psi|_{\langle t\rangle}}| + C(\phi_2|_P)|\mathcal{O}_{\phi_2|_P}^{\psi|_{\langle t\rangle}}| + \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P}} C(\phi_3|_P)|\mathcal{O}_{\phi_3|_P}^{\psi|_{\langle t\rangle}}| + C(\phi_4|_P)|\mathcal{O}_{\phi_4|_P}^{\psi|_{\langle t\rangle}}|,
\end{aligned}$$

where

$$(\phi_1,\phi_2,\phi_3,\phi_4) = \begin{cases} (\theta_A,\beta_A,\alpha_A,\theta_A^{-1}) & \text{if } \psi = \mathrm{Id}_S \\ (\alpha_A,\theta_A^{-1},\beta_A,\theta_A) & \text{if } \psi = \omega \\ (\theta_A^{-1},\alpha_A,\theta_A,\beta_A) & \text{if } \psi = \eta \\ (\beta_A,\theta_A,\alpha_A,\theta_A^{-1}) & \text{if } \psi = \omega\circ\eta \end{cases}$$

Similarly, for $\psi \in \{\theta_A, \theta_A^{-1}, \alpha_A, \beta_A\}$, define

$$\begin{aligned}
\sigma_A^{\psi|_{\langle t\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ P\neq A}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle t\rangle}}| + C(\psi_1|_P)|\mathcal{O}_{\psi_1|_P}^{\psi|_{\langle t\rangle}}| \\
\sigma_A^{\psi|_{\langle z\rangle}} &= \sum_{\substack{P\in\mathcal{D}_A\\ P\neq\langle z\rangle,A}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}| + C(\psi_2|_P)|\mathcal{O}_{\psi_2|_P}^{\psi|_P}|
\end{aligned}$$

where

$$\psi_1 = \begin{cases} \alpha_A & \text{if } \phi = \theta_A \\ \beta_A & \text{if } \phi = \theta_A^{-1} \\ \theta_A & \text{if } \phi = \alpha_A \\ \theta_A^{-1} & \text{if } \phi = \beta_A \end{cases} \text{ and } \psi_2 = \begin{cases} \beta_A & \text{if } \phi = \theta_A \\ \alpha_A & \text{if } \phi = \theta_A^{-1} \\ \theta_A & \text{if } \phi = \beta_A \\ \theta_A^{-1} & \text{if } \phi = \alpha_A \end{cases}$$

Now we determine minimal values of the constants. By Lemma 4.2.8, we already know that $C(\mathrm{Id}_S), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$.

$\underline{C(\theta_A), C(\theta_A^{-1}), C(\alpha_A), C(\beta_A)}$:

By ①$_8$, we have $[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)] \geq 2^2$. ①$_{12}$

$\underline{C(\theta_0), C(\theta_0^{-1}), C(\alpha_0), C(\beta_0)}$

By ①$_{10}$, we have $[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)] \geq 2^2$. ②$_{12}$

$\underline{C(\theta_\lambda), C(\theta_\lambda^{-1}), C(\theta_\lambda \circ \eta), C(\theta_\lambda^{-1} \circ \eta) \text{ for } \lambda = -1, 1}$:

By ②$_8$, we have $[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda \circ \eta)+C(\theta_\lambda^{-1} \circ \eta)] \geq 2^2$. ③$_{12}$

$\underline{C(\hat{\theta}_{V_\lambda}), C(\hat{\beta}_{V_\lambda}), C(\hat{\vartheta}_{1,\lambda}), C(\hat{\vartheta}_{4,\lambda}) \text{ for } \lambda = -1, 1}$:

By ③$_8$, we have $[C(\hat{\theta}_{V_\lambda})+C(\hat{\beta}_{V_\lambda})+C(\hat{\vartheta}_{1,\lambda})+C(\hat{\vartheta}_{4,\lambda})] \geq 2^2 \cdot 3^{2k-3}$. ④$_{12}$

$\underline{C(\mathsf{Id}|_{\langle s,z\rangle}), C(\omega|_{\langle s,z\rangle}), C(\eta|_{\langle s,z\rangle}), C(\omega \circ \eta|_{\langle s,z\rangle})}$:

By ⑥$_{10}$, we have $[C(\mathsf{Id}|_{\langle s,z\rangle})+C(\omega|_{\langle s,z\rangle})+C(\eta|_{\langle s,z\rangle})+C(\omega \circ \eta|_{\langle s,z\rangle})] \geq 2^2 \cdot [3^{2k-3}-1]$. ⑤$_{12}$

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}), C(\omega|_{\langle ss_1^\lambda,z\rangle}), C(\eta|_{\langle ss_1^\lambda,z\rangle}), C(\omega \circ \eta|_{\langle ss_1^\lambda,z\rangle}) \text{ for } \lambda = -1, 1}$:

By ④$_8$, $[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega \circ \eta|_{\langle ss_1^\lambda,z\rangle})] \geq 2^2 \cdot [3^{2k-3}-1]$. ⑥$_{12}$

$\underline{C(\theta_{V_0}), C(\theta_{V_0}^{-1}), C(\alpha_{V_0}), C(\beta_{V_0})}$:

By ③$_{10}$, we have $[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})] \geq 2^2 \cdot 3^{2k-2}$. ⑦$_{12}$

$\underline{C(\theta_{V_\lambda}), C(\theta_{V_\lambda}^{-1}), C(\alpha_{V_\lambda}), C(\beta_{V_\lambda}) \text{ for } \lambda = -1, 1}$:

By ⑤$_8$, we have $[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})] \geq 2^2 \cdot 3^{2k-2}$. ⑧$_{12}$

$\underline{C(\Theta_{1,0}), C(\Theta_{2,0}), C(\Theta_{3,0}), C(\Theta_{4,0})}$:

By ④$_{10}$, we have $[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})] \geq 2^2$. ⑨$_{12}$

$\underline{C(\Theta_{1,\lambda}), C(\Theta_{2,\lambda}), C(\Theta_{3,\lambda}), C(\Theta_{4,\lambda}) \text{ for } \lambda = -1, 1}$:

By ⑥$_8$, we have $[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})] \geq 2^2$. ⑩$_{12}$

$C(\Theta_{1,0}^{-1}),C(\Theta_{2,0}^{-1}),C(\Theta_{3,0}^{-1}),C(\Theta_{4,0}^{-1})$:

By ⑤$_{10}$, we have $[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})] \geq 2^2$. ⑪$_{12}$

$C(\Theta_{1,\lambda}^{-1}),C(\Theta_{2,\lambda}^{-1}),C(\Theta_{3,\lambda}^{-1}),C(\Theta_{4,\lambda}^{-1})$ for $\lambda=-1,1$:

By ⑦$_8$, we have $[C(\Theta_{1,\lambda}^{-1})+C(\Theta_{2,\lambda}^{-1})+C(\Theta_{3,\lambda}^{-1})+C(\Theta_{4,\lambda}^{-1})] \geq 2^2$. ⑫$_{12}$

$C(\vartheta_{1,\lambda}),C(\vartheta_{2,\lambda}),C(\vartheta_{3,\lambda}),C(\vartheta_{4,\lambda})$ for $\lambda=-1,1$:

By ⑧$_8$ we have $[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})] \geq 2^2\cdot 3^{2k-2}$. ⑬$_{12}$

$C(\xi_{1,\lambda}),C(\xi_{2,\lambda}),C(\xi_{2,\lambda}),C(\xi_{4,\lambda})$ for $\lambda=-1,1$:

$$|\Omega^{\xi_{1,\lambda}}|+|\Omega^{\xi_{2,\lambda}}|+|\Omega^{\xi_{3,\lambda}}|+|\Omega^{\xi_{4,\lambda}}| = |\Omega^{\theta_{V_\lambda}}|+|\Omega^{\theta_{V_\lambda^{-1}}}|+|\Omega^{\alpha_{V_\lambda}}|+|\Omega^{\beta_{V_\lambda}}|$$
$$\Longrightarrow 3^3[C(\xi_{1,\lambda})+C(\xi_{2,\lambda})+C(\xi_{3,\lambda})+C(\xi_{4,\lambda})] = 3^2[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]$$

Thus, by ⑧$_{12}$, we have $[C(\xi_{1,\lambda})+C(\xi_{2,\lambda})+C(\xi_{3,\lambda})+C(\xi_{4,\lambda})] \geq 2^2\cdot 3^{2k-3}$. ⑭$_{12}$

$C(\xi_{1,\lambda}^{-1}),C(\xi_{2,\lambda}^{-1}),C(\xi_{2,\lambda}^{-1}),C(\xi_{4,\lambda}^{-1})$ for $\lambda=-1,1$:

$$|\Omega^{\xi_{1,\lambda}^{-1}}|+|\Omega^{\xi_{2,\lambda}^{-1}}|+|\Omega^{\xi_{3,\lambda}^{-1}}|+|\Omega^{\xi_{4,\lambda}^{-1}}| = |\Omega^{\theta_{V_0}}|+|\Omega^{\theta_{V_0^{-1}}}|+|\Omega^{\alpha_{V_0}}|+|\Omega^{\beta_{V_0}}|$$
$$\Longrightarrow 3^3[C(\xi_{1,\lambda}^{-1})+C(\xi_{2,\lambda}^{-1})+C(\xi_{3,\lambda}^{-1})+C(\xi_{4,\lambda}^{-1})] = 3^2[C(\theta_{V_0})+C(\theta_{V_0^{-1}})+C(\alpha_{V_0})+C(\beta_{V_0})]$$

Thus, by ⑦$_{12}$, we have $[C(\xi_{1,\lambda}^{-1})+C(\xi_{2,\lambda}^{-1})+C(\xi_{3,\lambda}^{-1})+C(\xi_{4,\lambda}^{-1})] \geq 2^2\cdot 3^{2k-3}$. ⑮$_{12}$

$C(\xi_{5,\lambda}),C(\xi_{6,\lambda}),C(\xi_{7,\lambda}),C(\xi_{8,\lambda})$ for $\lambda=-1,1$:

$$|\Omega^{\xi_{5,\lambda}}|+|\Omega^{\xi_{6,\lambda}}|+|\Omega^{\xi_{7,\lambda}}|+|\Omega^{\xi_{8,\lambda}}| = |\Omega^{\theta_{V_\lambda}}|+|\Omega^{\theta_{V_\lambda^{-1}}}|+|\Omega^{\alpha_{V_\lambda}}|+|\Omega^{\beta_{V_\lambda}}|$$
$$\Longrightarrow 3^2[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})] = 3^2[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]$$

Thus, by ⑧$_{12}$, we have $[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})] \geq 2^2\cdot 3^{2k-2}$. ⑯$_{12}$

$\underline{C(\xi_{5,\lambda}^{-1}),C(\xi_{6,\lambda}^{-1}),C(\xi_{7,\lambda}^{-1}),C(\xi_{8,\lambda}^{-1}) \text{ for } \lambda=-1,1}$:

$$|\Omega^{\xi_{5,\lambda}^{-1}}|+|\Omega^{\xi_{6,\lambda}^{-1}}|+|\Omega^{\xi_{7,\lambda}^{-1}}|+|\Omega^{\xi_{8,\lambda}^{-1}}| = |\Omega^{\theta_{V_0}}|+|\Omega^{\theta_{V_0^{-1}}}|+|\Omega^{\alpha_{V_0}}|+|\Omega^{\beta_{V_0}}|$$
$$\Longrightarrow 3^2[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})] = 3^2[C(\theta_{V_0})+C(\theta_{V_0^{-1}})+C(\alpha_{V_0})+C(\beta_{V_0})]$$

Thus, by ⑦$_{12}$, we have $[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})] \geq 2^2\cdot 3^{2k-2}$. ⑰$_{12}$

$\underline{C(\mathrm{Id}|_{\langle s\rangle}),C(\omega|_{\langle s\rangle}),C(\eta|_{\langle s\rangle}),C(\omega\circ\eta|_{\langle s\rangle})}$:

By ⑧$_{10}$, we have $[C(\mathrm{Id}|_{\langle s\rangle})C(\omega|_{\langle s\rangle})+C(\eta|_{\langle s\rangle})+C(\omega\circ\eta|_{\langle s\rangle})] \geq 2^2\cdot 3^{4k-4}$. ⑱$_{12}$

$\underline{C(\mathrm{Id}|_{\langle ss_1^\lambda\rangle}),C(\omega|_{\langle ss_1^\lambda\rangle}),C(\eta|_{\langle ss_1^\lambda\rangle}),C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle}) \text{ for } \lambda=-1,1}$:

By ⑩$_8$, we have $[C(\mathrm{Id}|_{\langle ss_1^\lambda\rangle})C(\omega|_{\langle ss_1^\lambda\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})] \geq 2^4\cdot 3^{4k-4}$. ⑲$_{12}$

$\underline{C(\theta_0|_{\langle s\rangle}),C(\theta_0^{-1}|_{\langle s\rangle}),C(\alpha_0|_{\langle s\rangle}),C(\beta_0|_{\langle s\rangle})}$:

By using ④$_{10}$, we have

$$
\begin{aligned}
&|\Omega^{\theta_0|_{\langle s\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle s\rangle}}|=|\Omega^{\theta_{V_0}|_{\langle s\rangle}}|+|\Omega^{\theta_{V_0}^{-1}|_{\langle s\rangle}}|\\
\Longrightarrow\ & 3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\theta_0|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\\
&+2\cdot 3^{2k}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&+\sum_{\lambda=-1,1}3^{2k+1}[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})]\\
\Longrightarrow\ & 3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&+\sum_{\lambda=-1,1}3^{2k+1}[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})]\\
&-3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&-3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&-2\cdot 3^{2k}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+\sum_{\lambda=-1,1}3^{2k+1}[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})]\\
\geq&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+\sum_{\lambda=-1,1}3^{2k+1}[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})]\\
\geq&2^2\cdot 3^{2k+1}\cdot 3^{2k-2}+2^3\cdot 3^{2k+1}\cdot 3^{2k-2}=2^2\cdot 3^{4k}
\end{aligned}
$$

Thus, $[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})+C(\alpha_0|_{\langle s\rangle})+C(\beta_0|_{\langle s\rangle})]\geq 2^2\cdot 3^{2k-1}$, as required. ⑳$_{12}$

$C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})$ for $\lambda=-1,1$:

$$
\begin{aligned}
&|\Omega^{\theta_\lambda|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}}|=|\Omega^{\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle}}|\\
\Longrightarrow\ &3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
&+2\cdot 3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
=\ &3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
\Longrightarrow\ &3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=\ &3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
&-3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&-3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]
\end{aligned}
$$

Thus, by using substitution from ⑤$_8$, we have

$$
\begin{aligned}
&3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\\
=&(3^{2k+1}-3^2)[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
\geq&(3^{2k+1}-3^2)[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
\geq&2^2[(3^{2k+1}-3^2)\cdot 3^{2k-2}+3^{2k}+3^{2k+1}\cdot 3^{2k-2}+3^{2k+1}\cdot 3^{2k-2}]=2^2\cdot 3^{4k}
\end{aligned}
$$

It follows that $[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\geq 2^2\cdot 3^{2k-1}$, as required. ㉑$_{12}$

$\underline{C(\theta_{V_0}|_{\langle z\rangle}), C(\theta_{V_0}^{-1}|_{\langle z\rangle}), C(\alpha_{V_0}|_{\langle z\rangle}), C(\beta_{V_0}|_{\langle z\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\theta_{V_0}|_{\langle z\rangle}}| + |\Omega^{\theta_{V_0}^{-1}|_{\langle z\rangle}}| = |\Omega^{\mathsf{Id}|_{\langle z\rangle}}| + |\Omega^{\eta|_{\langle z\rangle}}| \\
\implies & 3^{2k+1}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
&+ 3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
&+ \sum_{\mu=-1,1} 3^{2k+1}[C(\xi_{5,\mu}) + C(\xi_{6,\mu}) + C(\xi_{7,\mu}) + C(\xi_{8,\mu})] \\
=& 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+ \sum_{\mu=-1,0,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^{\mu},z\rangle}) + C(\omega|_{\langle ss_1^{\mu},z\rangle}) + C(\eta|_{\langle ss_1^{\mu},z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^{\mu},z\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
&+ \Big[3^{4k-1}[C(\theta_0) + C(\theta_{\mu}^{-1}) + C(\alpha_0) + C(\beta_0)] \\
&+ 3^{4k}[C(\theta_0|_{\langle s,z\rangle}) + C(\theta_0^{-1}|_{\langle s,z\rangle}) + C(\alpha_0|_{\langle s,z\rangle}) + C(\beta_0|_{\langle s,z\rangle})] \\
&+ 3^{4k}[C(\theta_0|_{\langle t,z\rangle}) + C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})] \\
&+ 3^{4k+1}[C(\theta_0|_{\langle z\rangle}) + C(\theta_0^{-1}|_{\langle z\rangle}) + C(\alpha_0|_{\langle z\rangle}) + C(\beta_0|_{\langle z\rangle})]\Big] \\
&+ \sum_{\mu=-1,1} \Big[3^{4k-1}[C(\theta_{\mu}) + C(\theta_{\mu}^{-1}) + C(\theta_{\mu}\circ\eta) + C(\theta_{\mu}^{-1}\circ\eta)] \\
&+ 3^{4k}[C(\theta_{\mu}|_{\langle ss_1^{\mu},z\rangle}) + C(\theta_{\mu}^{-1}|_{\langle ss_1^{\mu},z\rangle}) + C(\theta_{\mu}\circ\eta|_{\langle ss_1^{\mu},z\rangle}) + C(\theta_{\mu}^{-1}\circ\eta|_{\langle ss_1^{\mu},z\rangle})] \\
&+ 3^{4k}[C(\theta_{\lambda}|_{\langle t,z\rangle}) + C(\theta_{\mu}^{-1}|_{\langle t,z\rangle}) + C(\theta_{\mu}\circ\eta|_{\langle t,z\rangle}) + C(\theta_{\mu}^{-1}\circ\eta|_{\langle t,z\rangle})] \\
&+ 3^{4k+1}[C(\theta_{\mu}|_{\langle z\rangle}) + C(\theta_{\mu}^{-1}|_{\langle z\rangle}) + C(\theta_{\mu}\circ\eta|_{\langle z\rangle}) + C(\theta_{\mu}^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat{\theta}_{V_{\mu}}) + C(\hat{\beta}_{V_{\mu}}) + C(\hat{\vartheta}_{1,\mu}) + C(\hat{\vartheta}_{4,\mu})] \\
&+ \sum_{\mu,\ell=-1,1} 3^{4k}[C(\xi_{1,\mu}^{\ell}) + C(\xi_{2,\mu}^{\ell}) + C(\xi_{3,\mu}^{\ell}) + C(\xi_{4,\mu}^{\ell})]
\end{aligned}
$$

This means,

$$
\begin{aligned}
\implies & 3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle})+C(\theta_{V_0}^{-1}|_{\langle z\rangle})+C(\alpha_{V_0}|_{\langle z\rangle})+C(\beta_{V_0}|_{\langle z\rangle})] \\
=& 3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)] \\
&+\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^{\mu},z\rangle})+C(\omega|_{\langle ss_1^{\mu},z\rangle})+C(\eta|_{\langle ss_1^{\mu},z\rangle})+C(\omega\circ\eta|_{\langle ss_1^{\mu},z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}] \\
&+\Big[3^{4k-1}[C(\theta_0)+C(\theta_{\mu}^{-1})+C(\alpha_0)+C(\beta_0)] \\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})] \\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})] \\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big] \\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_{\mu})+C(\theta_{\mu}^{-1})+C(\theta_{\mu}\circ\eta)+C(\theta_{\mu}^{-1}\circ\eta)] \\
&+3^{4k}[C(\theta_{\mu}|_{\langle ss_1^{\mu},z\rangle})+C(\theta_{\mu}^{-1}|_{\langle ss_1^{\mu},z\rangle})+C(\theta_{\mu}\circ\eta|_{\langle ss_1^{\mu},z\rangle})+C(\theta_{\mu}^{-1}\circ\eta|_{\langle ss_1^{\mu},z\rangle})] \\
&+3^{4k}[C(\theta_{\lambda}|_{\langle t,z\rangle})+C(\theta_{\mu}^{-1}|_{\langle t,z\rangle})+C(\theta_{\mu}\circ\eta|_{\langle t,z\rangle})+C(\theta_{\mu}^{-1}\circ\eta|_{\langle t,z\rangle})] \\
&+3^{4k+1}[C(\theta_{\mu}|_{\langle z\rangle})+C(\theta_{\mu}^{-1}|_{\langle z\rangle})+C(\theta_{\mu}\circ\eta|_{\langle z\rangle})+C(\theta_{\mu}^{-1}\circ\eta|_{\langle z\rangle})]\Big] \\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_{\mu}})+C(\hat{\beta}_{V_{\mu}})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})] \\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^{\ell})+C(\xi_{2,\mu}^{\ell})+C(\xi_{3,\mu}^{\ell})+C(\xi_{4,\mu}^{\ell})] \\
&-3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})] \\
&-3^{2k+1}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})] \\
&-\sum_{\mu=-1,1}3^{2k+1}[C(\xi_{5,\mu})+C(\xi_{6,\mu})+C(\xi_{7,\mu})+C(\xi_{8,\mu})]
\end{aligned}
$$

Using substitution from ③$_{10}$, ⑤$_{10}$, and by using ⑯$_{12}$ and ⑤$_8$ together, we have

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle})+C(\theta_{V_0}^{-1}|_{\langle z\rangle})+C(\alpha_{V_0}|_{\langle z\rangle})+C(\beta_{V_0}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\mu,z\rangle})+C(\omega|_{\langle ss_1^\mu,z\rangle})+C(\eta|_{\langle ss_1^\mu,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\mu,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k-1}[C(\theta_0)+C(\theta_\mu^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&-3^{2k-1}\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\Big]\\
&-3\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\Big]\\
&-\sum_{\mu=-1,1}3^{2k-1}\Big[3^{2k}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\Big]
\end{aligned}
$$

By Simplifying, we have

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle})+C(\theta_{V_0}^{-1}|_{\langle z\rangle})+C(\alpha_{V_0}|_{\langle z\rangle})+C(\beta_{V_0}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^{\mu},z\rangle})+C(\omega|_{\langle ss_1^{\mu},z\rangle})+C(\eta|_{\langle ss_1^{\mu},z\rangle})+C(\omega\circ\eta|_{\langle ss_1^{\mu},z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^{\ell})+C(\xi_{2,\mu}^{\ell})+C(\xi_{3,\mu}^{\ell})+C(\xi_{4,\mu}^{\ell})]\\
&-3^{2k+1}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&-3^{2k+2}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]
\end{aligned}
$$

By using substitution from ①$_{10}$, we have

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle})+C(\theta_{V_0}^{-1}|_{\langle z\rangle})+C(\alpha_{V_0}|_{\langle z\rangle})+C(\beta_{V_0}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^{\mu},z\rangle})+C(\omega|_{\langle ss_1^{\mu},z\rangle})+C(\eta|_{\langle ss_1^{\mu},z\rangle})+C(\omega\circ\eta|_{\langle ss_1^{\mu},z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^{\ell})+C(\xi_{2,\mu}^{\ell})+C(\xi_{3,\mu}^{\ell})+C(\xi_{4,\mu}^{\ell})]\\
&-3^{2k}\Big[3[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\Big]\\
&-3^{2k+2}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]
\end{aligned}
$$

Simplifying, we get,

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle})+C(\theta_{V_0}^{-1}|_{\langle z\rangle})+C(\alpha_{V_0}|_{\langle z\rangle})+C(\beta_{V_0}|_{\langle z\rangle})]\\
=&\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\mu,z\rangle})+C(\omega|_{\langle ss_1^\mu,z\rangle})+C(\eta|_{\langle ss_1^\mu,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+\Big[3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]-3^{2k}[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]\\
&-3^{2k+2}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
\geq&\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\mu,z\rangle})+C(\omega|_{\langle ss_1^\mu,z\rangle})+C(\eta|_{\langle ss_1^\mu,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]-3^{2k}[\sigma_S^{\mathsf{Id}|_{\langle s,t\rangle}}+\sigma_S^{\omega|_{\langle s,t\rangle}}+\sigma_S^{\eta|_{\langle s,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle s,t\rangle}}]
\end{aligned}
$$

Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. For all $P\in\mathcal{D}$, note that

$$
\begin{aligned}
|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}|+|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle z\rangle}}|&=\frac{2|N_S(\langle z\rangle,P)||C_S(\langle z\rangle)|}{|P|}\\
&\geq\frac{2\cdot 3^{2k}|N_S(\langle s,t\rangle,P)||C_S(\langle s,t\rangle)|}{|P|}=3^{2k}[|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,t\rangle}}|+|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle s,t\rangle}}|]
\end{aligned}
$$

It follows that $\sigma_S^{\psi|_{\langle z\rangle}} = \sigma_S^{\psi|_{\langle s,t\rangle}} + \sigma_S^{\psi\circ\omega|_{\langle s,t\rangle}}$. Thus

$$\begin{aligned}
&3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \\
\geq &\sum_{\mu=-1,0,1} 3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\mu,z\rangle}) + C(\omega|_{\langle ss_1^\mu,z\rangle}) + C(\eta|_{\langle ss_1^\mu,z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\mu,z\rangle})] \\
&+ \sum_{\mu=-1,1} 3^{4k}[C(\hat{\theta}_{V_\mu}) + C(\hat{\beta}_{V_\mu}) + C(\hat{\vartheta}_{1,\mu}) + C(\hat{\vartheta}_{4,\mu})] \\
&+ \sum_{\mu,\ell=-1,1} 3^{4k}[C(\xi_{1,\mu}^\ell) + C(\xi_{2,\mu}^\ell) + C(\xi_{3,\mu}^\ell) + C(\xi_{4,\mu}^\ell)] \\
\geq &3\cdot 2^2\cdot 3^{4k}(3^{2k-3}-1) + 2^3\cdot 3^{4k}\cdot 3^{2k-3} + 2^4\cdot 3^{4k}\cdot 3^{2k-2} \\
= &2^2[17\cdot 3^{6k-3} - 3^{4k+1}]
\end{aligned}$$

Thus, $[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \geq 2^2[17\cdot 3^{4k-5} - 3^{2k-1}]$. 22_{12}

$\underline{C(\theta_{V_\lambda}|_{\langle z\rangle}), C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle}), C(\alpha_{V_\lambda}|_{\langle z\rangle}), C(\beta_{V_\lambda}|_{\langle z\rangle}) \text{ for } \lambda=-1,1:}$

$$
\begin{aligned}
&|\Omega^{\theta_{V_\lambda}|_{\langle z\rangle}}|+|\Omega^{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}}|=|\Omega^{\mathsf{Id}|_{\langle z\rangle}}|+|\Omega^{\eta|_{\langle z\rangle}}|\\
\Longrightarrow\ & 3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda|_{\langle z\rangle}})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda}|_{\langle z\rangle})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,\lambda}^{-1})+C(\Theta_{2,\lambda}^{-1})+C(\Theta_{3,\lambda}^{-1})+C(\Theta_{4,\lambda}^{-1})]\\
&+3^{2k+1}[C(\vartheta_{1,-\lambda})+C(\vartheta_{2,-\lambda})+C(\vartheta_{3,-\lambda})+C(\vartheta_{4,-\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\mu=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\mu,z\rangle})+C(\omega|_{\langle ss_1^\mu,z\rangle})+C(\eta|_{\langle ss_1^\mu,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\mu,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k-1}[C(\theta_0)+C(\theta_\mu^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]
\end{aligned}
$$

This means,

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda}|_{\langle z\rangle})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\mu,z\rangle})+C(\omega|_{\langle ss_1^\mu,z\rangle})+C(\eta|_{\langle ss_1^\mu,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\mu,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k-1}[C(\theta_0)+C(\theta_\mu^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&-3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&-3^{2k+1}[C(\Theta_{1,\lambda}^{-1})+C(\Theta_{2,\lambda}^{-1})+C(\Theta_{3,\lambda}^{-1})+C(\Theta_{4,\lambda}^{-1})]\\
&-3^{2k+1}[C(\vartheta_{1,-\lambda})+C(\vartheta_{2,-\lambda})+C(\vartheta_{3,-\lambda})+C(\vartheta_{4,-\lambda})]\\
&-3^{2k+1}[C(\xi_{5,\lambda}^{-1})+C(\xi_{6,\lambda}^{-1})+C(\xi_{7,\lambda}^{-1})+C(\xi_{8,\lambda}^{-1})]
\end{aligned}
$$

By using substitution from ⑤$_8$, ⑦$_8$, ⑧$_8$, and also by using ③$_{10}$ and ⑰$_{12}$ together, note that

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda}|_{\langle z\rangle})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k-1}[C(\theta_0)+C(\theta_\mu^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{4k}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\\
&+3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&-3^{2k-1}\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
&-3\Big[3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})]\Big]\\
&-3^{2k-1}\Big[3^{2k}[C(\theta_{-\lambda})+C(\theta_{-\lambda}^{-1})+C(\theta_{-\lambda}\circ\eta)+C(\theta_{-\lambda}^{-1}\circ\eta)]\\
&+3^{2k+1}[C(\theta_{-\lambda}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_{-\lambda}^{-1}|_{\langle ss_1^\lambda,z\rangle})+C(\theta_{-\lambda}\circ\eta|_{\langle ss_1^\lambda,z\rangle})+C(\theta_{-\lambda}^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\Big]\\
&-3^{2k-1}\Big[3^{2k}[C(\theta_0)+C(\theta_0^{-1})+C(\alpha_0)+C(\beta_0)]\\
&+3^{2k+1}[C(\theta_0|_{\langle s,z\rangle})+C(\theta_0^{-1}|_{\langle s,z\rangle})+C(\alpha_0|_{\langle s,z\rangle})+C(\beta_0|_{\langle s,z\rangle})]\Big]
\end{aligned}
$$

By simplifying, we get,

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&-3^{2k+1}[C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta)]\\
&-3^{2k+2}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]
\end{aligned}
$$

By using substitution from ①$_7$, note that

$$
\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda|_{\langle z\rangle}})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\Big[3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&-\Big[3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^{2k}[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\Big]\\
&-3^{2k+2}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
=&\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]-3^{2k+1}[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\\
&+\Big[3^{4k}[C(\theta_0|_{\langle t,z\rangle})+C(\theta_0^{-1}|_{\langle t,z\rangle})+C(\alpha_0|_{\langle t,z\rangle})+C(\beta_0|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_0|_{\langle z\rangle})+C(\theta_0^{-1}|_{\langle z\rangle})+C(\alpha_0|_{\langle z\rangle})+C(\beta_0|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&-3^{2k+2}[C(\theta_\lambda|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}|_{\langle t,z\rangle})+C(\theta_\lambda\circ\eta|_{\langle t,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]
\end{aligned}
$$

Thus,

$$\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda^{-1}|_{\langle z\rangle}})+C(\alpha_{V_\lambda|_{\langle z\rangle}})+C(\beta_{V_\lambda|_{\langle z\rangle}})]\\
\geq&\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]-3^{2k}[\sigma_S^{\mathsf{Id}|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\eta|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\omega\circ\eta|_{\langle ss_1^\lambda,t\rangle}}]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]
\end{aligned}$$

Let $\psi\in\{\mathsf{Id}_S,\omega,\eta,\omega\circ\eta\}$. For all $P\in\mathcal{D}$, note that

$$\begin{aligned}
|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}|+|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle z\rangle}}|&=\frac{2|N_S(\langle z\rangle,P)||C_S(\langle z\rangle)|}{|P|}\\
&\geq\frac{2\cdot 3^{2k}|N_S(\langle ss_1^\lambda,t\rangle,P)||C_S(\langle ss_1^\lambda,t\rangle)|}{|P|}=3^{2k}[|\mathcal{O}_{\psi|_P}^{\psi|_{\langle ss_1^\lambda,t\rangle}}|+|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle ss_1^\lambda,t\rangle}}|]
\end{aligned}$$

It follows that $\sigma_S^{\psi|_{\langle z\rangle}}=\sigma_S^{\psi|_{\langle ss_1^\lambda,t\rangle}}+\sigma_S^{\psi\circ\omega|_{\langle ss_1^\lambda,t\rangle}}$. Thus

$$\begin{aligned}
&3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda|_{\langle z\rangle}})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\\
\geq&\sum_{\lambda=-1,0,1}3^{4k}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}3^{4k}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
\geq&3\cdot 2^2\cdot 3^{4k}(3^{2k-3}-1)+2^3\cdot 3^{4k}\cdot 3^{2k-3}+2^4\cdot 3^{4k}\cdot 3^{2k-2}=2^2[17\cdot 3^{6k-3}-3^{4k+1}]
\end{aligned}$$

Thus, $[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda|_{\langle z\rangle}})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\geq 2^2[17\cdot 3^{4k-5}-3^{2k-1}]$. 23$_{12}$

$\underline{C(\theta_A|_{\langle z\rangle}),C(\theta_A^{-1}|_{\langle z\rangle}),C(\alpha_A|_{\langle z\rangle}),C(\beta_A|_{\langle z\rangle})}$:

$$
\begin{aligned}
&|\Omega^{\theta_A|_{\langle z\rangle}}|+|\Omega^{\theta_A^{-1}|_{\langle z\rangle}}|=|\Omega^{\mathsf{Id}|_{\langle z\rangle}}|+|\Omega^{\eta|_{\langle z\rangle}}|\\
\Longrightarrow\ & 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]\\
&+3^{4k}[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]+[\sigma_A^{\theta_A|_{\langle z\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}]\\
&\quad+\sum_{\mu=-1,0,1}\Big[3^{4k-1}[C(\Theta_{1,\mu})+C(\Theta_{2,\mu})+C(\Theta_{3,\mu})+C(\Theta_{4,\mu})]\Big]\\
&=3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&\quad+\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&\quad+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&\quad+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&\quad+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&\quad+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&\quad+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&\quad+\sum_{\mu,\ell=-1,1}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
\Longrightarrow\ & 3^{4k}[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]+[\sigma_A^{\theta_A|_{\langle z\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}]\\
&=3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&\quad+\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&\quad+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&\quad+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&\quad+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&\quad+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&\quad+\sum_{\mu=-1,1}3^{4k}[C(\hat{\theta}_{V_\mu})+C(\hat{\beta}_{V_\mu})+C(\hat{\vartheta}_{1,\mu})+C(\hat{\vartheta}_{4,\mu})]\\
&\quad+\sum_{\mu,\ell=-1,1}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&\quad-3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]\\
&\quad-\sum_{\mu=-1,1}3^{4k-1}[C(\Theta_{1,\mu})+C(\Theta_{2,\mu})+C(\Theta_{3,\mu})+C(\Theta_{4,\mu})]
\end{aligned}
$$

By using substitution from ①$_6$ and ⑥$_8$, we have

$$\begin{aligned}
&3^{4k}[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]+[\sigma_A^{\theta_A|_{\langle z\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}]\\
=&3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\\
&+3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&-\Big[3^{2k+1}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]+3^{2k+2}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\Big]\\
&-\sum_{\mu=-1,1}\Big[3^{4k-1}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+3^{4k}[C(\theta_\mu|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu\circ\eta|_{\langle ss_1^\mu,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle ss_1^\mu,z\rangle})]\Big]\\
=&\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle z\rangle}}+\sigma_S^{\eta|_{\langle z\rangle}}]\\
&+\sum_{\mu=-1,1}\Big[3^{4k}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+3^{4k+1}[C(\theta_\mu|_{\langle z\rangle})+C(\theta_\mu^{-1}|_{\langle z\rangle})+C(\theta_\mu\circ\eta|_{\langle z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle z\rangle})]\Big]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
&-3^{2k+2}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\\
\geq&\sum_{\lambda=-1,1}3^{4k}[C(\mathsf{Id}_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})]\\
&+\sum_{\mu=-1,1}3^{4k}[C(\hat\theta_{V_\mu})+C(\hat\beta_{V_\mu})+C(\hat\vartheta_{1,\mu})+C(\hat\vartheta_{4,\mu})]\\
&+\sum_{\mu,\ell=-1,1}[C(\xi_{1,\mu}^\ell)+C(\xi_{2,\mu}^\ell)+C(\xi_{3,\mu}^\ell)+C(\xi_{4,\mu}^\ell)]\\
\geq&3\cdot2^2\cdot3^{4k}\cdot(3^{2k-3}-1)+2^3\cdot3^{4k}\cdot3^{2k-3}+2^4\cdot3^{4k}\cdot3^{2k-2}=2^2[17\cdot3^{6k-3}-3^{4k+1}].
\end{aligned}$$

Thus,

$$3^{4k}[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]+[\sigma_A^{\theta_A|_{\langle z\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}]\geq 2^2[17\cdot 3^{6k-3}-3^{4k+1}]. \tag{$\star$}$$

Recall that

$$\sigma_A^{\theta_A|_{\langle z\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ P\neq\langle z\rangle,A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\theta_A|_{\langle z\rangle}}|+C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\theta_A|_{\langle z\rangle}}|.$$

$$\sigma_A^{\theta_A|_{\langle z\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ P\neq\langle z\rangle,A}} C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\theta_A^{-1}|_{\langle z\rangle}}|+C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\theta_A^{-1}|_{\langle z\rangle}}|.$$

Write

$$N=\{\theta_A|_P,\theta_A^{-1}|_P,\alpha_A|_P,\beta_A|_P \mid P\in\mathcal{D}_A, P\neq\langle z\rangle, A\},$$
$$\hat{N}=N\sqcup\{\theta_A|_{\langle z\rangle},\theta_A^{-1}|_{\langle z\rangle},\alpha_A|_{\langle z\rangle},\beta_A|_{\langle z\rangle}\}.$$

Note that N is defined to be those morphisms that come from the definitions of $\sigma_A^{\theta_A|_{\langle z\rangle}}$ and $\sigma_A^{\theta_A^{-1}|_{\langle z\rangle}}$. Let Ω^* be our candidate characteristic biset. This means

$$C_{\Omega^*}(\theta_A|_{\langle z\rangle})=C_{\Omega^*}(\theta_A^{-1}|_{\langle z\rangle})=C_{\Omega^*}C(\alpha_A|_{\langle z\rangle})=C_{\Omega^*}(\beta_A|_{\langle z\rangle})=[17\cdot 3^{2k-3}-3].$$

and for all $\phi\in N$, $C_{\Omega^*}(\phi)=0$. That is,

$$\sum_{\phi\in\hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| = 2^2[17\cdot 3^{6k-3}-3^{4k+1}].$$

Let Ω' be any minimal characteristic biset with the condition that

$$C_{\Omega'}(\theta_A|_{\langle z\rangle})+C_{\Omega'}(\theta_A^{-1}|_{\langle z\rangle})+C_{\Omega'}(\alpha_A|_{\langle z\rangle})+C_{\Omega'}(\beta_A|_{\langle z\rangle})=n \quad \text{where} \quad 0\leq n<2^2[17\cdot 3^{2k-3}-3]. \tag{$\star\star$}$$

From the rest of the calculations in this subsection and from Lemma $A.0.1$, whenever $\phi\notin\hat{N}$, we may assume

$$C_{\Omega'}(\phi)\geq C_{\Omega^*}(\phi). \tag{$\star\star\star$}$$

Moreover, there exists $\emptyset \neq I \subseteq N$ such that for all $\phi \in I$ we have $C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| > 0$ and by $(\star)$ and $(\star\star)$ we have

$$\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| \geq [2^2[17 \cdot 3^{2k-3} - 3] - n] \cdot 3^{4k}, \qquad (\star\star\star\star)$$

where $\psi \in \{\theta_A, \theta_A^{-1}\}$. From calculations in Appendix B, for each $\phi \in I$, we have

$$|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| = \frac{3^{4k+1}}{|D_\phi|}.$$

Thus, by $(\star\star\star\star)$ we have

$$\begin{aligned}
&\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_\phi^{\psi|_{\langle z\rangle}}| \geq [2^2[17 \cdot 3^{2k-3} - 3] - n] \cdot 3^{4k} \\
\implies &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+1}}{|D_\phi|} + n \cdot 3^{4k} \geq 2^2[17 \cdot 3^{2k-3} - 3] \cdot 3^{4k} && \text{(by definition of } N) \\
\implies &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+2}}{|D_\phi|} + n \cdot 3^{4k+1} \geq 2^2[17 \cdot 3^{2k-3} - 3] \cdot 3^{4k+1} \\
\implies &[C_{\Omega'}(\theta_A|_{\langle z\rangle})|\mathcal{O}_{\theta_A|_{\langle z\rangle}}| + C_{\Omega'}(\theta_A^{-1}|_{\langle z\rangle})|\mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}}| + C_{\Omega'}(\alpha_A|_{\langle z\rangle})|\mathcal{O}_{\alpha_A|_{\langle z\rangle}}| + C_{\Omega'}(\beta_A|_{\langle z\rangle})|\mathcal{O}_{\beta_A|_{\langle z\rangle}}|] \\
&+ \sum_{\phi \in N} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 2^2[17 \cdot 3^{2k-3} - 3] \cdot 3^{4k+1} && \text{(by } (\star\star) \text{ and definition of } |\mathcal{O}_\phi|) \\
\implies &\sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 2^2[17 \cdot 3^{2k-3} - 3] \cdot 3^{4k+1}. && \text{(by definition of } \hat{N})
\end{aligned}$$

Using this property, we have

$$\begin{aligned}
|\Omega'| - |\Omega^*| \geq &\sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| - \sum_{\phi \in \hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| \\
\geq &2^2[17 \cdot 3^{2k-3} - 3] \cdot 3^{4k+1} - 2^2[17 \cdot 3^{2k-3} - 3] \cdot 3^{4k} > 0,
\end{aligned}$$

where the first inequality is due to $(\star\star\star)$. It follows that Ω^* is indeed minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

Thus, we can choose $[C(\theta_A|_{\langle z\rangle}) = C(\theta_A^{-1}|_{\langle z\rangle}) = C(\alpha_A|_{\langle z\rangle}) = C(\beta_A|_{\langle z\rangle})] = [17 \cdot 3^{2k-3} - 3]$. 24_{12}

$\underline{C(\theta_0|_{\langle t\rangle}), C(\theta_0^{-1}|_{\langle t\rangle}), C(\alpha_0|_{\langle t\rangle}), C(\beta_0|_{\langle t\rangle})}$:

$$\begin{aligned}
&|\Omega^{\theta_0|_{\langle t\rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle t\rangle}}| = |\Omega^{\theta_A|_{\langle t\rangle}}| + |\Omega^{\theta_A^{-1}|_{\langle t\rangle}}| \\
\Longrightarrow\ & 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
& + 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})] \\
& + 3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})] \\
& + 2\cdot 3^{2k}[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
=& 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
& + \sum_{\mu=-1,0,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})] \\
=& 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
& + (3^{4k-1} - 2\cdot 3^{2k})[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
& + \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
& - 3^{2k}[C(\theta_0) + C(\theta_0^{-1}) + C(\alpha_0) + C(\beta_0)] \\
& - 3^{2k+1}[C(\theta_0|_{\langle t,z\rangle}) + C(\theta_0^{-1}|_{\langle t,z\rangle}) + C(\alpha_0|_{\langle t,z\rangle}) + C(\beta_0|_{\langle t,z\rangle})]
\end{aligned}$$

Applying ⑤$_{10}$, we have

$$\begin{aligned}
& 3^{2k+1}[C(\theta_0|_{\langle t\rangle}) + C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})] \\
=& 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
& + (3^{4k-1} - 2\cdot 3^{2k})[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
& + \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
& - 3^{2k}[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
=& 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t\rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}] \\
& + (3^{4k-1} - 3^{2k+1})[C(\Theta_{1,0}^{-1}) + C(\Theta_{2,0}^{-1}) + C(\Theta_{3,0}^{-1}) + C(\Theta_{4,0}^{-1})] \\
& + \sum_{\mu=-1,1} 3^{4k-1}[C(\Theta_{1,\mu}^{-1}) + C(\Theta_{2,\mu}^{-1}) + C(\Theta_{3,\mu}^{-1}) + C(\Theta_{4,\mu}^{-1})] \\
\geq& 2^2\cdot 3^{2k+1} + 2^2\cdot(3^{4k-1} - 3^{2k+1}) + 2^3\cdot 3^{4k-1} = 2^2\cdot 3^{4k}
\end{aligned}$$

Thus, $[C(\theta_0|_{\langle t\rangle}) + C(\theta_0^{-1}|_{\langle t\rangle}) + C(\alpha_0|_{\langle t\rangle}) + C(\beta_0|_{\langle t\rangle})] \geq 2^2\cdot 3^{2k-1}$, as required.

$\underline{C(\theta_\lambda|_{\langle t\rangle}), C(\theta_\lambda^{-1}|_{\langle t\rangle}), C(\theta_\lambda\circ\eta|_{\langle t\rangle}), C(\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}) \text{ for } \lambda=-1,1:}$

$$
\begin{aligned}
&|\Omega^{\theta_\lambda|_{\langle t\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle t\rangle}}|+|\Omega^{\theta_\lambda|_{\langle t\rangle}\circ\eta}|+|\Omega^{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}|=|\Omega^{\theta_A|_{\langle t\rangle}}|+|\Omega^{\theta_A^{-1}|_{\langle t\rangle}}|+|\Omega^{\alpha_A|_{\langle t\rangle}}|+|\Omega^{\beta_A^{-1}|_{\langle t\rangle}}|\\
\implies &\sum_{\mu=-1,1} 3^{2k}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&+\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]\\
&+\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\\
&+\sum_{\mu=-1,1} 2\cdot 3^{2k}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
=&2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,0,1} 2\cdot 3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
\implies &\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\\
=&2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+2\cdot 3^{4k-1}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})]\\
&+\sum_{\mu=-1,1} 2(3^{4k-1}-3^{2k})[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&-\sum_{\mu=-1,1} 3^{2k}[C(\theta_\mu)+C(\theta_\mu^{-1})+C(\theta_\mu\circ\eta)+C(\theta_\mu^{-1}\circ\eta)]\\
&-\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t,z\rangle})+C(\theta_\mu^{-1}|_{\langle t,z\rangle})+C(\theta_\mu\circ\eta|_{\langle t,z\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t,z\rangle})]
\end{aligned}
$$

By using substitution from ⑦$_8$, we have

$$
\begin{aligned}
&\sum_{\mu=-1,1} 3^{2k+1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\\
=&2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+2\cdot 3^{4k-1}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})]\\
&+\sum_{\mu=-1,1} 2(3^{4k-1}-3^{2k})[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&-\sum_{\mu=-1,1} 3^{2k}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
=&2\cdot 3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}+\sigma_A^{\alpha_A|_{\langle t\rangle}}+\sigma_A^{\beta_A^{-1}|_{\langle t\rangle}}]\\
&+2\cdot 3^{4k-1}[C(\Theta_{1,0}^{-1})+C(\Theta_{2,0}^{-1})+C(\Theta_{3,0}^{-1})+C(\Theta_{4,0}^{-1})]\\
&+\sum_{\mu=-1,1} (2\cdot 3^{4k-1}-3^{2k+1})[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
\geq&2^3\cdot 3^{2k+1}+2^3\cdot 3^{4k-1}+2^4\cdot 3^{4k-1}-2^3\cdot 3^{2k+1}=2^3\cdot 3^{4k}
\end{aligned}
$$

Thus, $\sum_{\mu=-1,1}[C(\theta_\mu|_{\langle t\rangle})+C(\theta_\mu^{-1}|_{\langle t\rangle})+C(\theta_\mu\circ\eta|_{\langle t\rangle})+C(\theta_\mu^{-1}\circ\eta|_{\langle t\rangle})]\geq 2^3\cdot 3^{2k-1}$. ㉕$_{12}$

<u>$C(\mathsf{Id}|_{\langle t\rangle}),C(\omega|_{\langle t\rangle}),C(\eta|_{\langle t\rangle}),C(\omega\circ\eta|_{\langle t\rangle})$</u>:

$$
\begin{aligned}
&|\Omega^{\theta_A|_{\langle t\rangle}}|+|\Omega^{\theta_A^{-1}|_{\langle t\rangle}}|=|\Omega^{\mathsf{Id}|_{\langle t\rangle}}|+|\Omega^{\omega|_{\langle t\rangle}}|\\
\Longrightarrow\, & 3^{2k}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]\\
&+3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}]\\
&+2\cdot 3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}}]\\
=&3^{2k+1}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,0,1}3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
\Longrightarrow\, & 3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}}]\\
=&3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,0,1}3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&-3^{2k}[C(\mathsf{Id}_S)+C(\omega)+C(\eta)+C(\omega\circ\eta)]
\end{aligned}
$$

By using substitution from ①$_6$, we have

$$
\begin{aligned}
&3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}}]\\
=&3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,0,1}3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
&-\Big[3^{2k}[C(\theta_A)+C(\theta_A^{-1})+C(\alpha_A)+C(\beta_A)]-3^{2k+1}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]\Big]\\
=&3^{2k+1}[C(\mathsf{Id}|_A)+C(\omega|_A)+C(\eta|_A)+C(\omega\circ\eta|_A)]+[\sigma_A^{\theta_A|_{\langle t\rangle}}+\sigma_A^{\theta_A^{-1}|_{\langle t\rangle}}]\\
&+\sum_{\mu=-1,0,1}3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
\geq&\sum_{\mu=-1,0,1}3^{4k-1}[C(\Theta_{1,\mu}^{-1})+C(\Theta_{2,\mu}^{-1})+C(\Theta_{3,\mu}^{-1})+C(\Theta_{4,\mu}^{-1})]\\
\geq&3\cdot 2^2\cdot 3^{4k-1}=2^2\cdot 3^{4k}
\end{aligned}
$$

Thus,

$$3^{4k-1}[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]+[\sigma_S^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_S^{\omega|_{\langle t\rangle}}+\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}+\sigma_A^{\omega|_{\langle t\rangle}}]\geq 2^2\cdot 3^{4k}, \quad (\star)$$

We demonstrate that $[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]=2^2\cdot 3$ is an optimum choice. Recall the following definitions:

$$\sigma_S^{\mathsf{Id}|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\mathsf{Id}|_{\langle t\rangle}}|$$

$$\sigma_S^{\omega|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle t\rangle}} C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\omega|_{\langle t\rangle}}|+C(\omega\circ\eta|_P)|\mathcal{O}_{\omega\circ\eta|_P}^{\omega|_{\langle t\rangle}}|$$

$$\sigma_A^{\mathsf{Id}|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P,P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P,P\neq A}} C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\mathsf{Id}|_{\langle t\rangle}}|+C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\mathsf{Id}|_{\langle t\rangle}}|$$

$$\sigma_A^{\omega|_{\langle t\rangle}} = \sum_{\substack{P\in\mathcal{D}_A\\ \langle tz\rangle\leq P,P\neq A}} C(\theta_A|_P)|\mathcal{O}_{\theta_A|_P}^{\omega|_{\langle t\rangle}}|+C(\beta_A|_P)|\mathcal{O}_{\beta_A|_P}^{\omega|_{\langle t\rangle}}|+\sum_{\substack{P\in\mathcal{D}_A\\ \langle tz^{-1}\rangle\leq P,P\neq A}} C(\alpha_A|_P)|\mathcal{O}_{\alpha_A|_P}^{\omega|_{\langle t\rangle}}|+C(\theta_A^{-1}|_P)|\mathcal{O}_{\theta_A^{-1}|_P}^{\omega|_{\langle t\rangle}}|$$

Write

$$\begin{aligned}
N_1 &= \{\mathsf{Id}|_P,\omega|_P,\eta|_P,\omega\circ\eta|_P \mid P\in\mathcal{D}, P\neq S,\langle t\rangle,\langle ss_1^{\lambda}\rangle,\langle ss_1^{\lambda},z\rangle,\lambda=-1,0,1\},\\
N_2 &= \{\theta_A|_P,\theta_A^{-1}|_P,\alpha_A|_P,\beta_A|_P \mid P\in\mathcal{D}_A,P\neq A,\text{ and } tz^{-1}\in P \text{ or } tz\in P\},\\
N_3 &= \{\mathsf{Id}|_P,\omega|_P,\eta|_P,\omega\circ\eta|_P \mid P=\langle ss_1^{\lambda}\rangle,\langle ss_1^{\lambda},z\rangle,\lambda=-1,0,1\},\\
N &= N_1\sqcup N_2\\
\hat{N} &= N\sqcup\{\mathsf{Id}|_{\langle t\rangle},\omega|_{\langle t\rangle},\eta|_{\langle t\rangle},\omega\circ\eta|_{\langle t\rangle}\}
\end{aligned}$$

Note that $N\sqcup N_3$ is the set of morphisms which come from the definitions of $\sigma_S^{\mathsf{Id}|_{\langle t\rangle}},\sigma_S^{\omega|_{\langle t\rangle}},\sigma_A^{\mathsf{Id}|_{\langle t\rangle}}$ and $\sigma_A^{\omega|_{\langle t\rangle}}$. Let Ω^* be our candidate characteristic biset. This means

$$C_{\Omega^*}(\mathsf{Id}|_{\langle t\rangle})=C_{\Omega^*}(\omega|_{\langle t\rangle})=C_{\Omega^*}(\eta|_{\langle t\rangle})=C_{\Omega^*}(\omega\circ\eta|_{\langle t\rangle})=3,$$

and for all $\phi\in N$, we have $C_{\Omega^*}(\phi)=0$. When $\phi\in N_3$, $C_{\Omega^*}(\phi)>0$, and is the reason why we have stopped short of including N_3 in N. Thus,

$$\sum_{\phi\in\hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| = 4\cdot 3^{4k+2}.$$

Suppose that Ω' is a minimal right characteristic biset with the condition that

$$C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})+C_{\Omega'}(\omega|_{\langle t\rangle})+C_{\Omega'}(\eta|_{\langle t\rangle})+C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})=n \quad \text{where} \quad 0\leq n<12. \qquad (\star\star)$$

From the rest of the calculations in this subsection and from Lemma $A.0.1$, we may assume for all $\phi\notin\hat{N}$ that

$$C_{\Omega'}(\phi)\geq C_{\Omega^*}(\phi). \qquad (\star\star\star)$$

Moreover, there exists $\emptyset \neq I \subseteq N$ such that for all $\phi \in I$, we have $C_{\Omega'}(\phi)|\mathcal{O}_{\phi}^{\psi|_{\langle t \rangle}}| > 0$ and by $(\star)$ and $(\star\star)$ we have

$$\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_{\phi}^{\psi|_{\langle t \rangle}}| \geq (12-n) \cdot 3^{4k-1}, \tag{$\star\star\star\star$}$$

where $\psi \in \{\mathsf{Id}, \omega\}$. We show that Ω^* is minimal by size. From results in Appendix B, it can be checked that for each $\phi \in I$ and $\psi \in \{\mathsf{Id}, \omega\}$, we have

$$|\mathcal{O}_{\phi}^{\psi|_{\langle t \rangle}}| = \frac{3^{4k}}{|D_\phi|} \text{ or } \frac{3^{4k+1}}{|D_\phi|}.$$

Suppose that for every $\phi \in I$, $|\mathcal{O}_{\phi}^{\psi|_{\langle t \rangle}}| = \frac{3^{4k}}{|D_\phi|}$. Then, by $(\star\star\star\star)$,

$$\begin{aligned}
&\sum_{\phi \in I} C_{\Omega'}(\phi)|\mathcal{O}_{\phi}^{\psi|_{\langle t \rangle}}| \geq (12-n) \cdot 3^{4k-1} \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k}}{|D_\phi|} + n \cdot 3^{4k-1} \geq 4 \cdot 3^{4k} && \text{(by definition of } I \text{ and } N) \\
\Longrightarrow &\sum_{\phi \in N} C_{\Omega'}(\phi)\frac{3^{4k+2}}{|D_\phi|} + n \cdot 3^{4k+1} \geq 4 \cdot 3^{4k+2} \\
\Longrightarrow &C_{\Omega'}(\mathsf{Id}|_{\langle t \rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t \rangle}}| + C_{\Omega'}(\omega|_{\langle t \rangle})|\mathcal{O}_{\omega|_{\langle t \rangle}}| + C_{\Omega'}(\eta|_{\langle t \rangle})|\mathcal{O}_{\eta|_{\langle t \rangle}}| + C_{\Omega'}(\omega \circ \eta|_{\langle t \rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t \rangle}}| \\
&\quad + \sum_{\phi \in N} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 4 \cdot 3^{4k+2} && \text{(by } (\star\star) \text{ and definition of } |\mathcal{O}_\phi|) \\
\Longrightarrow &\sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| \geq 4 \cdot 3^{4k+2}, && \text{(by definition of } \hat{N})
\end{aligned}$$

Thus

$$|\Omega'| - |\Omega^*| \geq \sum_{\phi \in \hat{N}} C_{\Omega'}(\phi)|\mathcal{O}_\phi| - \sum_{\phi \in \hat{N}} C_{\Omega^*}(\phi)|\mathcal{O}_\phi| \geq 4 \cdot 3^{4k+2} - 4 \cdot 3^{4k+2} = 0,$$

where the first inequality is due to $(\star\star\star)$. It follows that Ω^* is minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

It is left to consider the case when there exists $\phi \in I$ such that $|\mathcal{O}_{\phi}^{\psi|_{\langle t \rangle}}| = \frac{3^{4k+1}}{|D_\phi|}$. From results in Appendix B, this only occurs when $\phi \in N_1$ and $\langle t, z \rangle \leq D_\phi$. Thus, for all $\mu = 1, .., 4$ and

$\lambda = -1, 0, 1$, we have

$$
\begin{aligned}
&|\Omega^{\Theta^{-1}_{\mu,\lambda}}| = |\Omega^{\psi|_{\langle t,z\rangle}}| \\
\Longrightarrow\ & 3^{2k} C(\Theta^{-1}_{\mu,\lambda}) = 3^{2k} C(\psi) + \frac{3^{4k+1}}{|D_\phi|} C(\phi) + [\text{extra..}] \\
\Longrightarrow\ & C(\Theta^{-1}_{\mu,\lambda}) \geq C(\psi) + \frac{3^{2k+1}}{|D_\phi|} C(\phi) \geq C(\psi) + 3C(\phi) \geq 4 \qquad\qquad (\text{as } \phi \in I)
\end{aligned}
$$

where the second last inequality is because $|D_\phi| \leq 3^{2k}$. Note that, $C_{\Omega^*}(\Theta^{-1}_{\mu,\lambda}) = 1$. However, $C_{\Omega'}(\Theta_{\mu,\lambda}) \geq 4$. Now

$$
\begin{aligned}
|\Omega'| - |\Omega^*| \geq & [C_{\Omega'}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega'}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega'}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega'}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}|] \\
& + \sum_{\lambda=-1,0,1} [C_{\Omega'}(\Theta^{-1}_{1,\lambda})|\mathcal{O}_{\Theta^{-1}_{1,\lambda}}| + C_{\Omega'}(\Theta^{-1}_{2,\lambda})|\mathcal{O}_{\Theta^{-1}_{2,\lambda}}| + C_{\Omega'}(\Theta^{-1}_{3,\lambda})|\mathcal{O}_{\Theta^{-1}_{3,\lambda}}| + C_{\Omega'}(\Theta^{-1}_{4,\lambda})|\mathcal{O}_{\Theta^{-1}_{4,\lambda}}|] \\
& - [C_{\Omega^*}(\mathsf{Id}|_{\langle t\rangle})|\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}}| + C_{\Omega^*}(\omega|_{\langle t\rangle})|\mathcal{O}_{\omega|_{\langle t\rangle}}| + C_{\Omega^*}(\eta|_{\langle t\rangle})|\mathcal{O}_{\eta|_{\langle t\rangle}}| + C_{\Omega^*}(\omega\circ\eta|_{\langle t\rangle})|\mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}|] \\
& - \sum_{\lambda=-1,0,1} [C_{\Omega^*}(\Theta^{-1}_{1,\lambda})|\mathcal{O}_{\Theta^{-1}_{1,\lambda}}| + C_{\Omega^*}(\Theta^{-1}_{2,\lambda})|\mathcal{O}_{\Theta^{-1}_{2,\lambda}}| + C_{\Omega^*}(\Theta^{-1}_{3,\lambda})|\mathcal{O}_{\Theta^{-1}_{3,\lambda}}| + C_{\Omega^*}(\Theta^{-1}_{4,\lambda})|\mathcal{O}_{\Theta^{-1}_{4,\lambda}}|] \\
\geq & n \cdot 3^{4k+1} + 12 \cdot 4 \cdot 3^{4k} - 12 \cdot 3^{4k+1} - 12 \cdot 3^{4k} \\
= & n \cdot 3^{4k+1} \geq 0.
\end{aligned}
$$

It follows that Ω^* is minimal (if the minimality of the rest of the constants in this subsection are demonstrated).

So we may assume $C(\mathsf{Id}|_{\langle t\rangle}) = C(\omega|_{\langle t\rangle}) = C(\eta|_{\langle t\rangle}) = C(\omega\circ\eta|_{\langle t\rangle}) = 3$. 26$_{12}$

$\underline{C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle}), C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle}), C(\xi_{3,\lambda}^{-1}|_{\langle s\rangle}), C(\xi_{4,\lambda}^{-1}|_{\langle s\rangle})\text{ for }\lambda=-1,1:}$

$$
\begin{aligned}
&|\Omega^{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}}|+|\Omega^{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}}| = |\Omega^{\theta_{V_0}|_{\langle s\rangle}}|+|\Omega^{\theta_{V_0}^{-1}|_{\langle s\rangle}}| \\
\Longrightarrow\ & 3^3[C(\xi_{1,\lambda}^{-1})+C(\xi_{2,\lambda}^{-1})+C(\xi_{3,\lambda}^{-1})+C(\xi_{4,\lambda}^{-1})] \\
&+3^3[C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{3,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{4,\lambda}^{-1}|_{\langle s\rangle})] \\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})] \\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})] \\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})] \\
&+\sum_{\mu=-1,1}3^{2k+1}[C(\xi_{5,\mu}^{-1})+C(\xi_{6,\mu}^{-1})+C(\xi_{7,\mu}^{-1})+C(\xi_{8,\mu}^{-1})] \\
\Longrightarrow\ & 3^3[C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{3,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{4,\lambda}^{-1}|_{\langle s\rangle})] \\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})] \\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})] \\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})] \\
&+\sum_{\mu=-1,1}3^{2k+1}[C(\xi_{5,\mu}^{-1})+C(\xi_{6,\mu}^{-1})+C(\xi_{7,\mu}^{-1})+C(\xi_{8,\mu}^{-1})] \\
&-3^3[C(\xi_{1,\lambda}^{-1})+C(\xi_{2,\lambda}^{-1})+C(\xi_{3,\lambda}^{-1})+C(\xi_{4,\lambda}^{-1})]
\end{aligned}
$$

By using substitution from (15)$_{12}$, we have

$$
\begin{aligned}
&3^3[C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{3,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{4,\lambda}^{-1}|_{\langle s\rangle})]\\
=&3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&+\sum_{\mu=-1,1}3^{2k+1}[C(\xi_{5,\mu}^{-1})+C(\xi_{6,\mu}^{-1})+C(\xi_{7,\mu}^{-1})+C(\xi_{8,\mu}^{-1})]\\
&-3^2[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
=&(3^{2k+1}-3^2)[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\theta_{V_0}^{-1}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&+\sum_{\mu=-1,1}3^{2k+1}[C(\xi_{5,\mu}^{-1})+C(\xi_{6,\mu}^{-1})+C(\xi_{7,\mu}^{-1})+C(\xi_{8,\mu}^{-1})]\\
\geq&(3^{2k+1}-3^2)[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\
&+3^{2k+1}[C(\Theta_{1,0})+C(\Theta_{2,0})+C(\Theta_{3,0})+C(\Theta_{4,0})]\\
&+\sum_{\mu=-1,1}3^{2k+1}[C(\xi_{5,\mu}^{-1})+C(\xi_{6,\mu}^{-1})+C(\xi_{7,\mu}^{-1})+C(\xi_{8,\mu}^{-1})]\\
\geq&2^2\cdot(3^{2k+1}-3^2)\cdot 3^{2k-2}+2^2\cdot 3^{2k+1}+2^3\cdot 3^{2k+1}\cdot 3^{2k-2}\\
=&2^3\cdot 3^{2k}+2^2\cdot 3^{4k}
\end{aligned}
$$

Thus, $[C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{3,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{4,\lambda}^{-1}|_{\langle s\rangle})]\geq 2^2[3^{4k-3}+2\cdot 3^{2k-3}]$. (27)$_{12}$

$\underline{C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}), C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}), C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle}), C(\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle})}$ for $\lambda = -1, 1$:

$$
\begin{aligned}
&|\Omega^{\xi_{1,\lambda}}| + |\Omega^{\xi_{2,\lambda}}| = |\Omega^{\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle}}| + |\Omega^{\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle}}| \\
\Longrightarrow\ & 3^3[C(\xi_{1,\lambda}) + C(\xi_{2,\lambda}) + C(\xi_{3,\lambda}) + C(\xi_{4,\lambda})] \\
&+ 3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle})] \\
=& 3^{2k+1}[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda^{-1}}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \\
&+ 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle}) + C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \\
&+ 3^{2k+1}[C(\vartheta_{1,\lambda}) + C(\vartheta_{2,\lambda}) + C(\vartheta_{3,\lambda}) + C(\vartheta_{4,\lambda})] \\
&+ 3^{2k+1}[C(\xi_{5,\lambda}) + C(\xi_{6,\lambda}) + C(\xi_{7,\lambda}) + C(\xi_{8,\lambda})] \\
\Longrightarrow\ & 3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle})] \\
=& 3^{2k+1}[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda^{-1}}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \\
&+ 3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle}) + C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle}) + C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})] \\
&+ 3^{2k+1}[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \\
&+ 3^{2k+1}[C(\vartheta_{1,\lambda}) + C(\vartheta_{2,\lambda}) + C(\vartheta_{3,\lambda}) + C(\vartheta_{4,\lambda})] \\
&+ 3^{2k+1}[C(\xi_{5,\lambda}) + C(\xi_{6,\lambda}) + C(\xi_{7,\lambda}) + C(\xi_{8,\lambda})] \\
&- 3^3[C(\xi_{1,\lambda}) + C(\xi_{2,\lambda}) + C(\xi_{3,\lambda}) + C(\xi_{4,\lambda})]
\end{aligned}
$$

Thus, by using substitution from (14)$_{12}$, we have

$$\begin{aligned}
&3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle})]\\
=&3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
&-3^2[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
=&(3^{2k+1}-3^2)[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle ss_1^\lambda\rangle})+C(\alpha_{V_\lambda}|_{\langle ss_1^\lambda\rangle})+C(\beta_{V_\lambda}|_{\langle ss_1^\lambda\rangle})]\\
&+3^{2k+1}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
\geq&(3^{2k+1}-3^2)[C(\theta_{V_\lambda})+C(\theta_{V_\lambda^{-1}})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})]\\
&+3^{2k+1}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})]\\
&+3^{2k+1}[C(\vartheta_{1,\lambda})+C(\vartheta_{2,\lambda})+C(\vartheta_{3,\lambda})+C(\vartheta_{4,\lambda})]\\
&+3^{2k+1}[C(\xi_{5,\lambda})+C(\xi_{6,\lambda})+C(\xi_{7,\lambda})+C(\xi_{8,\lambda})]\\
\geq&2^2\cdot(3^{2k+1}-3^2)\cdot 3^{2k-2}+2^2\cdot 3^{2k+1}+2^2\cdot 3^{2k+1}\cdot 3^{2k-2}+2^2\cdot 3^{2k+1}\cdot 3^{2k-2}\\
=&2^2[3^{4k}+2\cdot 3^{2k}]
\end{aligned}$$

Thus, $[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{3,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{4,\lambda}|_{\langle ss_1^\lambda\rangle})]\geq 2^2[3^{4k-3}+2\cdot 3^{2k-3}]$, as required. (28)$_{12}$

Thus, our candidate solution that we have chosen is minimal.

A.12.2 Exoticity

Note that $|\Omega^*|/|S| = 2^3[5\cdot 3^{6k-3}+2^5\cdot 3^{4k-3}+2\cdot 3^{2k-1}+2]$. Thus,

$$\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1}\left\lfloor \frac{2^3[5\cdot 3^{6k-3}+2^5\cdot 3^{4k-3}+2\cdot 3^{2k-1}+2]}{3^i}\right\rfloor$$

Write $n_i = \left\lfloor \frac{2^3[5\cdot 3^{6k-3}+2^5\cdot 3^{4k-3}+2\cdot 3^{2k-1}+2]}{3^i} \right\rfloor$. Now,

$$\begin{aligned}
i=1 &\implies n_i = 40\cdot 3^{6k-4}+256\cdot 3^{4k-4}+16\cdot 3^{2k-2}+5 \\
i=2 &\implies n_i = 40\cdot 3^{6k-5}+256\cdot 3^{4k-5}+16\cdot 3^{2k-3}+1 \\
3\le i\le 2k-1 &\implies n_i = 40\cdot 3^{6k-3-i}+256\cdot 3^{4k-3-i}+16\cdot 3^{2k-1-i} \\
i=2k &\implies n_i = 40\cdot 3^{4k-3}+256\cdot 3^{2k-3}+5 \\
i=2k+1 &\implies n_i = 40\cdot 3^{4k-4}+256\cdot 3^{2k-4}+1 \\
2k+2\le i\le 4k-3 &\implies n_i = 40\cdot 3^{6k-3-i}+256\cdot 3^{4k-3-i} && (k\ge 3) \\
i=4k-2 &\implies n_i = 40\cdot 3^{2k-1}+85 \\
i=4k-1 &\implies n_i = 40\cdot 3^{2k-2}+28 \\
i=4k &\implies n_i = 40\cdot 3^{2k-3}+9 \\
i=4k+1 &\implies n_i = 40\cdot 3^{2k-4}+3 \\
i=4k+2 &\implies n_i = \begin{cases} 14 & \text{if } k=2 \\ 40\cdot 3^{2k-5}+1 & \text{if } k\ge 3 \end{cases} \\
4k+3\le i\le 6k-3 &\implies n_i = 40\cdot 3^{6k-3-i} && (k\ge 3) \\
i=6k-2 &\implies n_i = 13 \\
i=6k-1 &\implies n_i = 4 \\
i=6k &\implies n_i = 1
\end{aligned}$$

Thus, by summing over the range $1\le i\le 6k$, we have

$$\nu_3(\mathrm{Sym}(|\Omega^*|/|S|)) = 2^2[5\cdot 3^{6k-3}+2^5\cdot 3^{4k-3}+2\cdot 3^{2k-1}].$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3|T:S| &= [|\Omega^*|/|S|-1](2k+1)+2^2[5\cdot 3^{6k-3}+2^5\cdot 3^{4k-3}+2\cdot 3^{2k-1}] \\
&= 2^2[5\cdot 3^{6k-3}+2^5\cdot 3^{4k-3}+2\cdot 3^{2k-1}](4k+3)+15(2k+1).
\end{aligned}$$

Thus, we have an upper bound on our exoticity index.

A.13 DRV-13

General form:
By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}\Omega \cong &\bigsqcup_{Q\in\mathcal{D}}\left[C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q}\sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q}\sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q}\sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q}\right]\\ &\sqcup\bigsqcup_{Q\in\mathcal{D}_{V_0}}\left[C(\theta_{V_0}|_Q)\mathcal{O}_{\theta_{V_0}|_Q}\sqcup C(\theta_{V_0}^{-1}|_Q)\mathcal{O}_{\theta_{V_0}^{-1}|_Q}\sqcup C(\alpha_{V_0}|_Q)\mathcal{O}_{\alpha_{V_0}|_Q}\sqcup C(\beta_{V_0}|_Q)\mathcal{O}_{\beta_{V_0}|_Q}\right]\end{aligned}$$

Equations:
We list the following equations:

$\psi\in\{\mathsf{Id}|_Q\mid Q\in\mathcal{D}_{V_0}\}$:

$$\begin{aligned}|\Omega^{\mathsf{Id}|_{\langle s,z\rangle}}| &= \sum_{P\in\mathcal{D}} C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}_{\langle s,z\rangle}}|\\ |\Omega^{\mathsf{Id}_{\langle s\rangle}}| &= \sum_{P\in\mathcal{D}}[C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}_{\langle s\rangle}}|+C(\eta|_P)|\mathcal{O}_{\eta|_P}^{\mathsf{Id}|_{\langle s\rangle}}|]+3^2[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\\ |\Omega^{\mathsf{Id}|_{\langle z\rangle}}| &= \sum_{P\in\mathcal{D}}[C(\mathsf{Id}|_P)|\mathcal{O}_{\mathsf{Id}|_P}^{\mathsf{Id}|_{\langle z\rangle}}|+C(\omega|_P)|\mathcal{O}_{\omega|_P}^{\mathsf{Id}|_{\langle z\rangle}}|]\end{aligned}$$

$\psi\in\{\theta_{V_0}|_Q\mid Q\in\mathcal{D}_{V_0}\}$:

$$\begin{aligned}|\Omega^{\theta_{V_0}}| &= 3^2C(\theta_{V_0})\\ |\Omega^{\theta_{V_0}|_{\langle s\rangle}}| &= 3^{2k+1}[C(\theta_{V_0})+C(\alpha_{V_0})]+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\alpha_{V_0}|_{\langle s\rangle})]\\ |\Omega^{\theta_{V_0}|_{\langle z\rangle}}| &= 3^{2k+1}[C(\theta_{V_0})+C(\beta_{V_0})]+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle})+C(\beta_{V_0}|_{\langle s\rangle})]\end{aligned}$$

Candidate solution:
We have the following designated values for the constants:

$$\begin{aligned}&C(\mathsf{Id}_S)=C(\omega)=C(\eta)=C(\omega\circ\eta)=1\\ &C(\theta_{V_0})=C(\theta_{V_0}^{-1})=C(\alpha_{V_0})=C(\beta_{V_0})=1\\ &C(\mathsf{Id}|_{\langle s\rangle})=C(\eta|_{\langle s\rangle})=C(\omega|_{\langle s\rangle})=C(\omega\circ\eta|_{\langle s\rangle})=3^{2k-2}-1\end{aligned}$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$\Omega^*\cong(\mathcal{O}_{\mathsf{Id}}\sqcup\mathcal{O}_\omega\sqcup\mathcal{O}_\eta\sqcup\mathcal{O}_{\omega\circ\eta})\sqcup(\mathcal{O}_{\theta_{V_0}}\sqcup\mathcal{O}_{\theta_{V_0}^{-1}}\sqcup\mathcal{O}_{\alpha_{V_0}}\sqcup\mathcal{O}_{\beta_{V_0}})\sqcup n_k\cdot(\mathcal{O}_{\mathsf{Id}|_{\langle s\rangle}}\sqcup\mathcal{O}_{\omega|_{\langle s\rangle}}\sqcup\mathcal{O}_{\eta|_{\langle s\rangle}}\sqcup\mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}),$$

where $n_k = 3^{2k-2} - 1$. It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.13.1 Minimality

We start by defining a few things. Let $\psi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$. Then define

$$\begin{aligned}
\sigma_S^{\psi|_{\langle s,z\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| \\
\sigma_S^{\psi|_{\langle s\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle s\rangle}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s\rangle}}| + C(\phi|_P)|\mathcal{O}_{\phi|_P}^{\psi|_{\langle s\rangle}}|.
\end{aligned}$$

where $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ is defined so that $\psi \circ \phi = \eta$. Now, we determine the minimal size of Ω, by looking at the smallest possible values of the constants.

By Lemma 4.2.8, note that $C(\mathrm{Id}), C(\omega), C(\eta), C(\omega \circ \eta) \geq 1$.

<u>$C(\theta_{V_0}), C(\theta_{V_0}^{-1}), C(\alpha_{V_0}), C(\beta_{V_0})$</u>:

$$\begin{aligned}
&|\Omega^{\mathrm{Id}|_{\langle s,z\rangle}}| + |\Omega^{\omega|_{\langle s,z\rangle}}| + |\Omega^{\eta|_{\langle s,z\rangle}}| + |\Omega^{\omega\circ\eta|_{\langle s,z\rangle}}| = |\Omega^{\theta_{V_0}}| + |\Omega^{\theta_{V_0}^{-1}}| + |\Omega^{\alpha_{V_0}}| + |\Omega^{\beta_{V_0}}| \\
\implies& 3^2[C(\mathrm{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathrm{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] \\
&= 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})]
\end{aligned}$$

This gives us $[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \geq 2^2$, as required. ①$_{13}$

<u>$C(\mathrm{Id}|_{\langle s\rangle}), C(\omega|_{\langle s\rangle}), C(\eta|_{\langle s\rangle}), C(\omega\circ\eta|_{\langle s\rangle})$</u>:

$$\begin{aligned}
&|\Omega^{\mathrm{Id}|_{\langle s\rangle}}| + |\Omega^{\omega|_{\langle s\rangle}}| = |\Omega^{\theta_{V_0}|_{\langle s\rangle}}| + |\Omega^{\theta_{V_0}^{-1}|_{\langle s\rangle}}| \\
\implies& 3^2[C(\mathrm{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + 3^3[C(\mathrm{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
&+ [\sigma_S^{\mathrm{Id}|_{\langle s\rangle}} + \sigma_S^{\omega|_{\langle s\rangle}} + \sigma_S^{\eta|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}] + 2\cdot 3^2[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
&= 3^{2k+1}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
&+ 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle}) + C(\alpha_{V_0}|_{\langle s\rangle}) + C(\beta_{V_0}|_{\langle s\rangle})] \\
\implies& 3^3[C(\mathrm{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
&= (3^{2k+1} - 2\cdot 3^2)[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
&+ 3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle}) + C(\alpha_{V_0}|_{\langle s\rangle}) + C(\beta_{V_0}|_{\langle s\rangle}) \\
&- 3^2[C(\mathrm{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] - [\sigma_S^{\mathrm{Id}|_{\langle s\rangle}} + \sigma_S^{\omega|_{\langle s\rangle}} + \sigma_S^{\eta|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}]
\end{aligned}$$

Let $\psi \in \{\mathsf{Id}_S, \omega, \eta, \omega \circ \eta\}$. Then, for all $P \in \mathcal{D} \backslash \{\langle s \rangle\}$, we have

$$|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| = \frac{|N_S(\langle s,z\rangle, P)||C_S(\langle s,z\rangle)|}{|P|} = \frac{|N_S(\langle s\rangle, P)||C_S(\langle s\rangle)|}{|P|} = |\mathcal{O}_{\psi|_P}^{\psi|_{\langle s\rangle}}| = |\mathcal{O}_{\phi|_P}^{\psi|_{\langle s\rangle}}|,$$

where $\phi \in \{\mathsf{Id}_S, \omega, \eta, \omega \circ \eta\}$ is defined such that $\psi \circ \phi = \eta$. Thus, by definition, we have $\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega|_{\langle s\rangle}} = \sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}$. By using this property, and by substituting in ①$_{13}$, we see that

$$\begin{aligned}
&3^3[C(\mathsf{Id}|_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \\
=&(3^{2k-1}-2)\left[3^2[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]\right] \\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle}) + C(\alpha_{V_0}|_{\langle s\rangle}) + C(\beta_{V_0}|_{\langle s\rangle}) \\
&-3^2[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&-[\sigma_S^{\mathsf{Id}|_{\langle s\rangle}} + \sigma_S^{\omega|_{\langle s\rangle}} + \sigma_S^{\eta|_{\langle s\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s\rangle}}] \\
=&(3^{2k+1}-3^3)[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
&+3^{2k+2}[C(\theta_{V_0}|_{\langle s\rangle}) + C(\theta_{V_0}|_{\langle s\rangle}) + C(\alpha_{V_0}|_{\langle s\rangle}) + C(\beta_{V_0}|_{\langle s\rangle}) \\
&+(3^{2k-1}-3)[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] \\
\geq&(3^{2k+1}-3^3)[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)]
\end{aligned}$$

It follows that $[C(\mathsf{Id}_{\langle s\rangle}) + C(\omega|_{\langle s\rangle}) + C(\eta|_{\langle s\rangle}) + C(\omega\circ\eta|_{\langle s\rangle})] \geq 2^2 \cdot (3^{2k-2}-1)$. ②$_{13}$

We have shown that our candidate solution is of minimal size.

A.13.2 Exoticity

Note that $|\Omega^*|/|S| = 4(3^{2k-1}-1)^2$. Thus,

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) &= \sum_{i=1} \left\lfloor \frac{4(3^{2k-1}-1)^2}{3^i} \right\rfloor \\
&= \sum_{i=1} \left\lfloor 4(3^{4k-2-i} + \frac{1-2\cdot 3^{2k-1}}{3^i}) \right\rfloor
\end{aligned}$$

Write $n_i = \left\lfloor 4(3^{4k-2-i} + \frac{1-2\cdot 3^{2k-1}}{3^i}) \right\rfloor$. Now,

$$\begin{aligned}
i = 1 &\implies n_1 = 4\cdot 3^{4k-3} - 8\cdot 3^{2k-2} + 1\\
2 \le i \le 2k-1 &\implies n_i = 4\cdot 3^{2k-1-i}[3^{2k-1} - 2]\\
i = 2k &\implies n_{2k} = 4\cdot 3^{2k-2} - 3\\
2k+1 \le i \le 4k-2 &\implies n_i = 4\cdot 3^{4k-2-i} - 1\\
i = 4k-1 &\implies n_{4k-1} = 1
\end{aligned}$$

Thus

$$\begin{aligned}
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = &\left[\sum_{i=2}^{2k-1} 4\cdot 3^{2k-1-i}[3^{2k-1}-2]\right] + \left[\sum_{i=2k+1}^{4k-2} 4\cdot 3^{4k-2-i} - 1\right]\\
&+ (4\cdot 3^{4k-3} - 8\cdot 3^{2k-2} + 1) + (4\cdot 3^{4k-2} - 3) + 1\\
&= [2(3^{2k-1}-2)(3^{2k-2}-1)] + [2(3^{2k-2}-1) - (2k-2)]\\
&+ (4\cdot 3^{4k-3} - 4\cdot 3^{2k-2} - 1)\\
&= 2(3^{2k-1}-1)^2 - 2k + 1
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3|T : S| &= [4(3^{2k-1}-1)^2 - 1](2k+1) + 2(3^{2k-1}-1)^2 - 2k + 1\\
&= 2(3^{2k-1}-1)^2(4k+3) - 4k.
\end{aligned}$$

This is our upper bound for our exoticity index on our DRV-13 fusion system.

A.14 DRV-14

General form:
By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$
\begin{aligned}
\Omega \cong & \bigsqcup_{Q \in \mathcal{D}} C(\mathrm{Id}|_Q)\mathcal{O}_{\mathrm{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega \circ \eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\
& \sqcup \bigsqcup_{Q \in \mathcal{D}_A} C(\theta_A|_Q)\mathcal{O}_{\theta_A|_Q} \sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q} \sqcup C(\alpha_A|_Q)\mathcal{O}_{\alpha_A|_Q} \sqcup C(\beta_A|_Q)\mathcal{O}_{\beta_A|_Q} \\
& \sqcup \bigsqcup_{Q \in \mathcal{D}_{V_0}} C(\theta_{V_0}|_Q)\mathcal{O}_{\theta_{V_0}} \sqcup C(\theta_{V_0}^{-1}|_Q)\mathcal{O}_{\theta_{V_0}^{-1}|_Q} \sqcup C(\alpha_{V_0}|_Q)\mathcal{O}_{\alpha_{V_0}|_Q} \sqcup C(\beta_{V_0}|_Q)\mathcal{O}_{\beta_{V_0}} \\
& \sqcup \bigsqcup_{Q \in \{\langle s\rangle, \langle t\rangle\}} C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0^{-1}|_Q}
\end{aligned}
$$

Equations:
We list the following equations:

$\psi \in \{\mathrm{Id}|_Q \mid Q \in \mathcal{D}_A \cup \mathcal{D}_{V_0}\}$:

When $Q \in \mathcal{D}_A$, then the equations are the same as those for DRV-6. Similarly, when $Q \in \mathcal{D}_{V_0}$, then the equations are the same as those for DRV-13.

$\psi \in \{\theta_A|_Q \mid Q \in \mathcal{D}_A\}$:

In this case, the equations are the same as those for DRV-6.

$\psi \in \{\theta_{V_0}|_Q \mid Q \in \mathcal{D}_{V_0}\}$:

In this case, the equations are the same as those for DRV-13.

$\psi \in \{\theta_0|_Q \mid Q = \langle s\rangle, \langle t\rangle\}$:

$$
\begin{aligned}
|\Omega^{\theta_0|_{\langle s\rangle}}| &= 3^{2k+1} C(\theta_0|_{\langle s\rangle}) \\
|\Omega^{\theta_0|_{\langle t\rangle}}| &= 3^{2k+1} C(\theta_0|_{\langle t\rangle})
\end{aligned}
$$

Candidate solution:

We have the following designated values for the constants:

$$
\begin{aligned}
&C(\mathrm{Id}_S) = C(\omega) = C(\eta) = C(\omega \circ \eta) = 1\\
&C(\mathrm{Id}|_{\langle s\rangle}) = C(\omega|_{\langle s\rangle}) = C(\eta|_{\langle s\rangle}) = C(\omega \circ \eta|_{\langle s\rangle}) = 3^{2k-2} - 1\\
&C(\theta_A) = C(\theta_A^{-1}) = C(\alpha_A) = C(\beta_A) = 1\\
&C(\theta_{V_0}) = C(\theta_{V_0}^{-1}) = C(\alpha_{V_0}) = C(\beta_{V_0}) = 1\\
&C(\theta_0|_{\langle s\rangle}) = C(\theta_0^{-1}|_{\langle s\rangle}) = 2\\
&C(\theta_0|_{\langle t\rangle}) = C(\theta_0^{-1}|_{\langle t\rangle}) = 2,
\end{aligned}
$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$
\begin{aligned}
\Omega^* \cong &(\mathcal{O}_{\mathrm{Id}} \sqcup \mathcal{O}_\omega \sqcup \mathcal{O}_\eta \sqcup \mathcal{O}_{\omega\circ\eta}) \sqcup (3^{2k-2}-1)(\mathcal{O}_{\mathrm{Id}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}})\\
&\sqcup(\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}) \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A})\\
&\sqcup 2(\mathcal{O}_{\theta_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}}) \sqcup 2(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle t\rangle}})
\end{aligned}
$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.14.1 Minimality

We start by first making a definition. Let $\psi \in \{\theta_A, \theta_A^{-1}\}$. Then define

$$
\sigma_A^{\psi|_{\langle t\rangle}} = \sum_{\substack{P \in \mathcal{D}_A\\ \langle t\rangle \leq P, P \neq A}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle t\rangle}}| + C(\psi_1|_P)|\mathcal{O}_{\psi_1|_P}^{\psi|_{\langle t\rangle}}|
$$

where

$$
\psi_1 = \begin{cases} \alpha_A & \text{if } \psi = \theta_A\\ \beta_A & \text{if } \psi = \theta_A^{-1} \end{cases}
$$

Now we determine the minimal values of the constants.

By Lemma 4.2.8, we already know that $C(\mathrm{Id}_S), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$.

<u>$C(\theta_A), C(\theta_A^{-1}), C(\alpha_A), C(\beta_A)$</u>:

By ①$_6$, we have $[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \geq 2^2$. ①$_{14}$

$\underline{C(\theta_{V_0}), C(\theta_{V_0}^{-1}), C(\alpha_{V_0}), C(\beta_{V_0})}$:

By ①$_{13}$, we have $[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \geq 2^2$. ②$_{14}$

$\underline{C(\mathsf{Id}|_{\langle s \rangle}), C(\omega|_{\langle s \rangle}), C(\eta|_{\langle s \rangle}), C(\omega \circ \eta|_{\langle s \rangle})}$:

By ②$_{13}$, we have $[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \geq 2^2 \cdot (3^{2k-2} - 1)$. ③$_{14}$

$\underline{C(\theta_0|_{\langle s \rangle}), C(\theta_0^{-1}|_{\langle s \rangle})}$:

$$
\begin{aligned}
& |\Omega^{\theta_0|_{\langle s \rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle s \rangle}}| = |\Omega^{\theta_{V_0}|_{\langle s \rangle}}| + |\Omega^{\theta_{V_0}^{-1}|_{\langle s \rangle}}| \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_0|_{\langle s \rangle}) + C(\theta_0^{-1}|_{\langle s \rangle})] \\
& = 3^{2k+1}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
& \quad + 3^{2k+2}[C(\theta_{V_0}|_{\langle s \rangle}) + C(\theta_{V_0}^{-1}|_{\langle s \rangle}) + C(\alpha_{V_0}|_{\langle s \rangle}) + C(\beta_{V_0}|_{\langle s \rangle})] \\
& \geq 3^{2k+1}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
& \geq 2^2 \cdot 3^{2k+1}
\end{aligned}
$$

Thus, $[C(\theta_0|_{\langle s \rangle}) + C(\theta_0^{-1}|_{\langle s \rangle})] \geq 2^2$, as required. ④$_{14}$

$\underline{C(\theta_0|_{\langle t \rangle}), C(\theta_0^{-1}|_{\langle t \rangle})}$:

$$
\begin{aligned}
& |\Omega^{\theta_0|_{\langle t \rangle}}| + |\Omega^{\theta_0^{-1}|_{\langle t \rangle}}| = |\Omega^{\theta_A|_{\langle t \rangle}}| + |\Omega^{\theta_A^{-1}|_{\langle t \rangle}}| \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_0|_{\langle t \rangle}) + C(\theta_0^{-1}|_{\langle t \rangle})] \\
& = 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] + [\sigma_A^{\theta_A|_{\langle t \rangle}} + \sigma_A^{\theta_A^{-1}|_{\langle t \rangle}}] \\
& \geq 3^{2k+1}[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \\
& \geq 2^2 \cdot 3^{2k+1}
\end{aligned}
$$

Thus, $[C(\theta_0|_{\langle t \rangle}) + C(\theta_0^{-1}|_{\langle t \rangle})] \geq 2^2$, as required. ⑤$_{14}$

It follows that our candidate solution is minimal.

A.14.2 Exoticity

Note that $|\Omega^*|/|S| = 4[3^{4k-2}+4\cdot 3^{2k-1}+4]$. Thus,

$$v_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{4[3^{4k-2}+4\cdot 3^{2k-1}+4]}{3^i} \right\rfloor$$

Write $n_i = \left\lfloor \frac{4[3^{4k-2}+4\cdot 3^{2k-1}+4]}{3^i} \right\rfloor$. Now,

$$\begin{aligned}
i=1 &\implies n_i = 4\cdot 3^{4k-3}+16\cdot 3^{2k-2}+5\\
i=2 &\implies n_i = 4\cdot 3^{4k-4}+16\cdot 3^{2k-3}+1\\
3\le i\le 2k-1 &\implies n_i = 4\cdot 3^{4k-2-i}+16\cdot 3^{2k-1-i}\\
i=2k &\implies n_i = 4\cdot 3^{2k-2}+5\\
i=2k+1 &\implies n_i = 4\cdot 3^{2k-3}+1\\
2k+2\le i\le 4k-2 &\implies n_i = 4\cdot 3^{4k-2-i}\\
i=4k-1 &\implies n_i = 1
\end{aligned}$$

Thus,

$$\begin{aligned}
v_3(\mathsf{Sym}(|\Omega^*|/|S|)) =& [4\cdot 3^{4k-3}+16\cdot 3^{2k-2}+5]+[4\cdot 3^{4k-4}+16\cdot 3^{2k-3}+1]\\
&+\left[\sum_{i=3}^{2k-1} 4\cdot 3^{4k-2-i}+16\cdot 3^{2k-1-i}\right]+[4\cdot 3^{2k-2}+5]\\
&+[4\cdot 3^{2k-3}+1]+\left[\sum_{i=2k+2}^{4k-2} 4\cdot 3^{4k-2-i}\right]+1\\
=& 2\cdot 3^{4k-2}+8\cdot 3^{2k-1}+3
\end{aligned}$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3|T:S| =& [4(3^{4k-2}+4\cdot 3^{2k-1}+4)-1][2k+1]+[3^{4k-2}+8\cdot 3^{2k-1}+3]\\
=& 2[3^{4k-2}+4\cdot 3^{2k-1}](4k+3)+6(5k+3).
\end{aligned}$$

Thus, we have our upper bound on our exoticity index.

A.15 DRV-15

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$\begin{aligned}\Omega \cong & \bigsqcup_{P\in\mathcal{D}} C(\mathsf{Id}|_Q)\mathcal{O}_{\mathsf{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\ & \sqcup \bigsqcup_{Q\in\mathcal{D}_{V_0}} C(\theta_{V_0}|_Q)\mathcal{O}_{\theta_{V_0}|_Q} \sqcup C(\theta_{V_0}^{-1}|_Q)\mathcal{O}_{\theta_{V_0}^{-1}} \sqcup C(\alpha_{V_0}|_Q)\mathcal{O}_{\alpha_{V_0}|_Q} \sqcup C(\beta_{V_0}|_Q)\mathcal{O}_{\beta_{V_0}|_Q} \\ & \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_\lambda\\ \lambda=-1,1\\ \ell=0,1}} C(\theta_\lambda\circ\eta^\ell|_Q)\mathcal{O}_{\theta_\lambda\circ\eta^\ell|_Q} \sqcup C(\theta_\lambda^{-1}\circ\eta^\ell|_Q)\mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_Q} \\ & \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\theta_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\theta_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\theta_{V_\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\theta_{V_\lambda}}\end{aligned}$$

Equations:

We list the following equations:

$\psi\in\{\mathsf{Id}|_Q \mid Q\in\mathcal{D}_\lambda\cup\mathcal{D}_{V_0}, \lambda=-1,1\}$:

If $Q\in\mathcal{D}_\lambda$ for $\lambda=-1,1$, then the equations are the same as those for DRV-7. If $Q\in\mathcal{D}_{V_0}$, then the equations are the same as those for DRV-13.

$\psi\in\{\theta_\lambda|_Q \mid Q\in\mathcal{D}_\lambda, \lambda=-1,1\}$:

In this case the equations are the same as those for DRV-7.

$\psi\in\{\theta_\lambda\circ\eta|_Q \mid Q\in\mathcal{D}_\lambda, \lambda=-1,1\}$:

In this case the equations are the same as those for DRV-7.

$\psi\in\{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda}\}$:

In this case the equations are the same as those for DRV-7.

$\psi\in\{\theta_{V_0}|_Q \mid Q\in\mathcal{D}_{V_0}\}$:

In this case the equations are the same as those for DRV-13.

Candidate solution:

We have the following designated values for the constants:

$$\begin{aligned}
&C(\mathsf{Id}_S) = C(\omega) = C(\eta) = C(\omega\circ\eta) = 1\\
&C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}) = C(\omega|_{\langle ss_1^\lambda, z\rangle}) = C(\eta|_{\langle ss_1^\lambda, z\rangle}) = C(\omega\circ\eta|_{\langle ss_1^\lambda, z\rangle}) = 3^{2k-3}-1\\
&C(\mathsf{Id}|_{\langle s\rangle}) = C(\omega|_{\langle s\rangle}) = C(\eta|_{\langle s\rangle}) = C(\omega\circ\eta|_{\langle s\rangle}) = 3^{2k-2}-1\\
&C(\theta_\lambda) = C(\theta_\lambda^{-1}) = C(\theta_\lambda\circ\eta) = C(\theta_\lambda^{-1}\circ\eta) = 1\\
&C(\hat\theta_{V_\lambda}) = C(\hat\beta_{V_\lambda}) = C(\hat\vartheta_{1,\lambda}) = C(\hat\vartheta_{4,\lambda}) = 3^{2k-3}\\
&C(\theta_{V_0}) = C(\theta_{V_0}^{-1}) = C(\alpha_{V_0}) = C(\beta_{V_0}) = 1\\
&C(\theta_{V_0}|_{\langle z\rangle}) = C(\theta_{V_0}^{-1}|_{\langle z\rangle}) = C(\alpha_{V_0}|_{\langle z\rangle}) = C(\beta_{V_0}|_{\langle z\rangle}) = 2\cdot 3^{2k-3}(3^{2k-2}-2),
\end{aligned}$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$\begin{aligned}
\Omega^* \cong\ &(\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_\omega \sqcup \mathcal{O}_\eta \sqcup \mathcal{O}_{\omega\circ\eta})\\
&\sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)[\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^\lambda,z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^\lambda,z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}}]\\
&\sqcup (3^{2k-2}-1)[\mathcal{O}_{\mathsf{Id}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}]\\
&\sqcup \bigsqcup_{\lambda=-1,1} [\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}}] \sqcup [\mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}]\\
&\sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}[\mathcal{O}_{\hat\theta_{V_\lambda}} \sqcup \mathcal{O}_{\hat\beta_{V_\lambda}} \sqcup \mathcal{O}_{\hat\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\hat\vartheta_{4,\lambda}}]\\
&\sqcup [\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}]\\
&\sqcup [\mathcal{O}_{\theta_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_0}|_{\langle z\rangle}}].
\end{aligned}$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.15.1 Minimality

We start by defining a few things. Let $\psi \in \{\mathsf{Id}_S, \omega, \eta, \omega\circ\eta\}$. Then, we define

$$\begin{aligned}
\sigma_S^{\psi|_{\langle s,z\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S}} C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| \quad \text{and}\\
\sigma_S^{\psi|_{\langle z\rangle}} &= \sum_{\substack{P\in\mathcal{D}\\ P\neq S,\langle ss_1^\lambda,z\rangle,\lambda=-1,1}} [C(\psi|_P)|\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}| + C(\psi\circ\omega|_P)|\mathcal{O}_{\psi\circ\omega|_P}^{\psi\circ\omega|_{\langle z\rangle}}|]
\end{aligned}$$

We also define

$$\sigma_\lambda^{\mathsf{Id}|_{\langle z\rangle}} = \sum_{\substack{P\in\mathcal{D}_\lambda \\ P\neq\langle ss_1^\lambda,t\rangle}} C(\theta_\lambda|_P)|\mathcal{O}_{\theta_\lambda|_P}^{\mathsf{Id}|_{\langle z\rangle}}| + C(\theta_\lambda^{-1}|_P)|\mathcal{O}_{\theta_\lambda^{-1}|_P}^{\mathsf{Id}|_{\langle z\rangle}}|$$

$$\sigma_\lambda^{\eta|_{\langle z\rangle}} = \sum_{\substack{P\in\mathcal{D}_\lambda \\ P\neq\langle ss_1^\lambda,t\rangle}} C(\theta_\lambda\circ\eta|_P)|\mathcal{O}_{\theta_\lambda\circ\eta|_P}^{\eta|_{\langle z\rangle}}| + C(\theta_\lambda^{-1}\circ\eta|_P)|\mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_P}^{\eta|_{\langle z\rangle}}|$$

Now, we determine the minimal values of the constants:

By Lemma 4.2.8, note that $C(\mathsf{Id}_S), C(\omega), C(\eta), C(\omega\circ\eta) \geq 1$.

$\underline{C(\theta_\lambda), C(\theta_\lambda^{-1}), C(\theta_\lambda\circ\eta), C(\theta_\lambda^{-1}\circ\eta)}$:

By ①$_7$, we get $C(\theta_\lambda)+C(\theta_\lambda^{-1})+C(\theta_\lambda\circ\eta)+C(\theta_\lambda^{-1}\circ\eta) \geq 2^2$. ①$_{15}$

$\underline{C(\hat{\theta}_{V_\lambda}), C(\hat{\beta}_{V_\lambda}), C(\hat{\vartheta}_{1,\lambda}), C(\hat{\vartheta}_{4,\lambda})}$:

By ②$_7$, we get $C(\hat{\theta}_{V_\lambda})+C(\hat{\beta}_{V_\lambda})+C(\hat{\vartheta}_{1,\lambda})+C(\hat{\vartheta}_{4,\lambda}) \geq 2^2\cdot 3^{2k-3}$. ②$_{15}$

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}), C(\omega|_{\langle ss_1^\lambda,z\rangle}), C(\eta|_{\langle ss_1^\lambda,z\rangle}), C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})}$:

By ③$_7$ we get $C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle})+C(\omega|_{\langle ss_1^\lambda,z\rangle})+C(\eta|_{\langle ss_1^\lambda,z\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle}) \geq 2^2\cdot(3^{2k-3}-1)$, as required. ③$_{15}$

$\underline{C(\theta_{V_0}), C(\theta_{V_0}^{-1}), C(\alpha_{V_0}), C(\beta_{V_0})}$:

By ①$_{13}$, we get $C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0}) \geq 2^2$. ④$_{15}$

$\underline{C(\mathsf{Id}|_{\langle s\rangle}), C(\omega|_{\langle s\rangle}), C(\eta|_{\langle s\rangle}), C(\beta|_{\langle s\rangle})}$:

By ②$_{13}$, we get $C(\mathsf{Id}|_{\langle s\rangle})+C(\omega|_{\langle s\rangle})+C(\eta|_{\langle s\rangle})+C(\omega\circ\eta|_{\langle s\rangle}) \geq 2^2\cdot(3^{2k-2}-1)$. ⑤$_{15}$

$\underline{C(\theta_{V_0}|_{\langle z\rangle}), C(\theta_{V_0}^{-1}|_{\langle z\rangle}), C(\alpha_{V_0}|_{\langle z\rangle}), C(\beta_{V_0}|_{\langle z\rangle})}$:

Using substitution from ①$_{13}$, we have

$$
\begin{aligned}
&|\Omega^{\theta_{V_0}|_{\langle z\rangle}}| + |\Omega_{V_0}^{\theta^{-1}|_{\langle z\rangle}}| = |\Omega^{\mathsf{Id}|_{\langle z\rangle}}| + |\Omega^{\eta|_{\langle z\rangle}}| \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_{V_0}) + C(\theta_{V_0}^{-1}) + C(\alpha_{V_0}) + C(\beta_{V_0})] \\
& + 3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \\
&= 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
& + 3^{4k}\sum_{\lambda=-1,1}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}) + C(\omega|_{\langle ss_1^\lambda,z\rangle}) + C(\eta|_{\langle ss_1^\lambda,z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
& + 3^{4k-1}\sum_{\lambda=-1,1}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] + \sum_{\lambda=-1,1}[\sigma_\lambda^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_\lambda^{\eta|_{\langle z\rangle}}] \\
\Longrightarrow\ & 3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \\
&= 3^{2k+1}[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] \\
& + 3^{4k}\sum_{\lambda=-1,1}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}) + C(\omega|_{\langle ss_1^\lambda,z\rangle}) + C(\eta|_{\langle ss_1^\lambda,z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
& + 3^{4k-1}\sum_{\lambda=-1,1}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] + \sum_{\lambda=-1,1}[\sigma_\lambda^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_\lambda^{\eta|_{\langle z\rangle}}] \\
& - 3^{2k-1}\left[3^2[C(\mathsf{Id}_S) + C(\omega) + C(\eta) + C(\omega\circ\eta)] + [\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]\right] \\
&= 3^{4k}\sum_{\lambda=-1,1}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}) + C(\omega|_{\langle ss_1^\lambda,z\rangle}) + C(\eta|_{\langle ss_1^\lambda,z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] + [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}] \\
& + 3^{4k-1}\sum_{\lambda=-1,1}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] + \sum_{\lambda=-1,1}[\sigma_\lambda^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_\lambda^{\eta|_{\langle z\rangle}}] \\
& - 3^{2k-1}[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}]
\end{aligned}
$$

Let $\psi \in \{\mathsf{Id}_S, \omega, \eta, \omega\circ\eta\}$. Then for any $P \in \mathcal{D}$, we have

$$3^{2k-1}|\mathcal{O}_{\psi|_P}^{\psi|_{\langle s,z\rangle}}| = \frac{3^{2k-1}|N_S(\langle s,z\rangle,P)||C_S(\langle s,z\rangle)|}{|P|} \le \frac{|N_S(\langle z\rangle,P)||C_S(\langle z\rangle)|}{|P|} = |\mathcal{O}_{\psi|_P}^{\psi|_{\langle z\rangle}}|.$$

It follows that $3^{2k-1}[\sigma_S^{\mathsf{Id}|_{\langle s,z\rangle}} + \sigma_S^{\omega|_{\langle s,z\rangle}} + \sigma_S^{\eta|_{\langle s,z\rangle}} + \sigma_S^{\omega\circ\eta|_{\langle s,z\rangle}}] \le [\sigma_S^{\mathsf{Id}|_{\langle z\rangle}} + \sigma_S^{\eta|_{\langle z\rangle}}]$. Thus,

$$
\begin{aligned}
& 3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \\
\ge & 3^{4k}\sum_{\lambda=-1,1}[C(\mathsf{Id}|_{\langle ss_1^\lambda,z\rangle}) + C(\omega|_{\langle ss_1^\lambda,z\rangle}) + C(\eta|_{\langle ss_1^\lambda,z\rangle}) + C(\omega\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \\
& + 3^{4k-1}\sum_{\lambda=-1,1}[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda\circ\eta) + C(\theta_\lambda^{-1}\circ\eta)] \\
\ge & 2^3\cdot 3^{4k}(3^{2k-3}-1) + 2^3\cdot 3^{4k-1} = 2^3\cdot 3^{4k-1}(3^{2k-2}-1).
\end{aligned}
$$

This gives us $[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \ge 2^3\cdot 3^{2k-3}(3^{2k-2}-2)$, as required. ⑥$_{15}$

A.15.2 Exoticity

Note that $|\Omega^*|/|S| = 4[13 \cdot 3^{4k-4} - 3^{2k-2} + 1]$. Thus,

$$\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{4[13 \cdot 3^{4k-4} - 3^{2k-2} + 1]}{3^i} \right\rfloor$$

Write $n_i = \left\lfloor \frac{4[13 \cdot 3^{4k-4} - 3^{2k-2} + 1]}{3^i} \right\rfloor$. Now,

$$\begin{aligned}
i = 1 &\implies n_i = 52 \cdot 3^{4k-5} - 4 \cdot 3^{2k-3} + 1 \\
2 \leq i \leq 2k-2 &\implies n_i = 52 \cdot 3^{4k-4-i} - 4 \cdot 3^{2k-2-i} \\
i = 2k-1 &\implies n_i = 52 \cdot 3^{2k-3} - 2 \\
2k \leq i \leq 4k-4 &\implies n_i = 52 \cdot 3^{4k-4-i} - 1 \\
i = 4k-3 &\implies n_i = 17 \\
i = 4k-2 &\implies n_i = 5 \\
i = 4k-1 &\implies n_i = 1
\end{aligned}$$

Then, by summing over $1 \leq i \leq 4k-1$, we have

$$\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = 26 \cdot 3^{4k-4} - 2 \cdot 3^{2k-2} - (2k-1).$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$\begin{aligned}
\log_3 |T : S| &= [|\Omega^*|/|S| - 1](2k+1) + \nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) \\
&= 2[13 \cdot 3^{4k-4} - 3^{2k-2}](4k+3) + 4(k+1).
\end{aligned}$$

Thus, we have an upper bound on our exoticity index.

A.16 DRV-16

General form:

By Theorem 7.3.1, we know that any potential right characteristic biset Ω has the following form:

$$
\begin{aligned}
\Omega \cong & \bigsqcup_{Q\in\mathcal{D}} C(\mathrm{Id}|_Q)\mathcal{O}_{\mathrm{Id}|_Q} \sqcup C(\omega|_Q)\mathcal{O}_{\omega|_Q} \sqcup C(\eta|_Q)\mathcal{O}_{\eta|_Q} \sqcup C(\omega\circ\eta|_Q)\mathcal{O}_{\omega\circ\eta|_Q} \\
& \sqcup \bigsqcup_{Q\in\mathcal{D}_A} C(\theta_A|_Q)\mathcal{O}_{\theta_A|_Q} \sqcup C(\theta_A^{-1}|_Q)\mathcal{O}_{\theta_A^{-1}|_Q} \sqcup C(\alpha_A|_Q)\mathcal{O}_{\alpha_A|_Q} \sqcup C(\beta_A|_Q)\mathcal{O}_{\beta_A|_Q} \\
& \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_\lambda \\ \lambda=-1,1 \\ \ell=0,1}} C(\theta_\lambda\circ\eta^\ell)\mathcal{O}_{\theta_\lambda\circ\eta^\ell} \sqcup C(\theta_\lambda^{-1}\circ\eta^\ell)\mathcal{O}_{\theta_\lambda^{-1}\circ\eta^\ell} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\hat{\theta}_{V_\lambda})\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup C(\hat{\beta}_{V_\lambda})\mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup C(\hat{\vartheta}_{1,\lambda})\mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup C(\hat{\vartheta}_{4,\lambda})\mathcal{O}_{\hat{\vartheta}_{4,\lambda}} \\
& \sqcup \bigsqcup_{\substack{Q\in\mathcal{D}_{V_\lambda} \\ \lambda=-1,0,1}} C(\theta_{V_\lambda}|_Q)\mathcal{O}_{\theta_{V_\lambda}} \sqcup C(\theta_{V_\lambda}^{-1}|_Q)\mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup C(\alpha_{V_\lambda}|_Q)\mathcal{O}_{\alpha_{V_\lambda}|_Q} \sqcup C(\beta_{V_\lambda}|_Q)\mathcal{O}_{\beta_{V_\lambda}|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\Theta_{1,\lambda})\mathcal{O}_{\Theta_{1,\lambda}} \sqcup C(\Theta_{2,\lambda})\mathcal{O}_{\Theta_{2,\lambda}} \sqcup C(\Theta_{3,\lambda})\mathcal{O}_{\Theta_{3,\lambda}} \sqcup C(\Theta_{4,\lambda})\mathcal{O}_{\Theta_{4,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\Theta_{1,\lambda}^{-1})\mathcal{O}_{\Theta_{1,\lambda}} \sqcup C(\Theta_{2,\lambda}^{-1})\mathcal{O}_{\Theta_{2,\lambda}} \sqcup C(\Theta_{3,\lambda}^{-1})\mathcal{O}_{\Theta_{3,\lambda}} \sqcup C(\Theta_{4,\lambda}^{-1})\mathcal{O}_{\Theta_{4,\lambda}} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\vartheta_{1,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{2,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{3,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup C(\vartheta_{4,\lambda})\mathcal{O}_{\vartheta_{1,\lambda}} \\
& \sqcup \bigsqcup_{Q\in\{\langle s\rangle,\langle t\rangle\}} C(\theta_0|_Q)\mathcal{O}_{\theta_0|_Q} \sqcup C(\theta_0^{-1}|_Q)\mathcal{O}_{\theta_0^{-1}|_Q} \\
& \sqcup \bigsqcup_{\lambda=-1,1} C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle})\mathcal{O}_{\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}} \sqcup C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle})\mathcal{O}_{\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}} \sqcup C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})\mathcal{O}_{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}} \sqcup C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})\mathcal{O}_{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}}
\end{aligned}
$$

As already explained in the case for DRV-8, DRV-10, and DRV-12, there is a difference between the construction in Theorem 7.3.1 and the one above. The difference is that we have excluded orbits induced by morphisms $\Theta_{\mu,\lambda}|_Q$, $\vartheta_{\mu,\lambda}|_Q$ for $Q \lneq \langle ss_1^\lambda, z\rangle$, and $\Theta_{\mu,\lambda}^{-1}|_Q$ for $Q \lneq \langle t,z\rangle$, for all $\mu = 1,..,4$ and $\lambda = -1,1$. This is because all of these morphisms are restrictions of those from the set

$$\{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T \mid T = V_\lambda, A, \lambda = -1,1\},$$

which are already included in the the general construction.

Equations:

We list the following equations:

$\{\mathsf{Id}|_Q \mid Q \in \mathcal{D}_A \cup \mathcal{D}_\lambda \cup \mathcal{D}_{V_0}, \lambda = -1, 1\}$:

When $Q \in \mathcal{D}_A$, then equations are the same as those for DRV-8. When $Q \in \mathcal{D}_{V_0} \backslash \{\langle z \rangle\}$, then the equations are the same as those for DRV-14. Similarly, when $Q \in \mathcal{D}_\lambda$ for $\lambda = -1, 1$, then the equations are the same as those for DRV-8.

$\psi \in \{\theta_A|_Q \mid Q \in \mathcal{D}_A\}$:

In this case the equations are the same as those for DRV-8.

$\psi \in \{\theta_\lambda|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}$:

In this case the equations are the same as those for DRV-8.

$\psi \in \{\theta_\lambda \circ \eta|_Q \mid Q \in \mathcal{D}_\lambda, \lambda = -1, 1\}$:

In this case the equations are the same as those for DRV-8.

$\psi \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}, \hat{\vartheta}_{1,\lambda}, \hat{\vartheta}_{4,\lambda} \mid \lambda = -1, 1\}$:

In this case the equations are the same as those for DRV-8.

$\psi \in \{\theta_{V_\lambda}|_Q \mid Q \in \mathcal{D}_{V_\lambda} \mid \lambda = -1, 0, 1\}$:

If $\lambda = -1, 1$, then the equations are the same as those for DRV-8. If $\lambda = 0$, then the equations are the same as those for DRV-14.

$\psi \in \{\Theta_{1,\lambda} \mid \lambda = -1, 1\}$:

In this case the equations are the same as those for DRV-8.

$\psi \in \{\Theta_{1,\lambda}^{-1} \mid \lambda = -1, 1\}$:

In this case the equations are the same as those for DRV-8.

$\psi \in \{\vartheta_{1,\lambda} \mid \lambda = -1, 1\}$:

In this case the equations are the same as those for DRV-8.

$\underline{\psi \in \{\theta_0|_Q \mid Q = \langle s \rangle, \langle t \rangle\}}$:

In this case the equations are the same as those for DRV-14.

$\underline{\psi \in \{\xi_{1,\lambda}|_{\langle ss_1^{\lambda} \rangle} \mid \lambda = -1, 1\}}$:

$$|\Omega^{\xi_{1,\lambda}|_{\langle ss_1^{\lambda} \rangle}}| = 3^3 C(\xi_{1,\lambda}|_{\langle ss_1^{\lambda} \rangle})$$

$\underline{\psi \in \{\xi_{1,\lambda}^{-1}|_{\langle s \rangle} \mid \lambda = -1, 1\}}$:

$$|\Omega^{\xi_{1,\lambda}^{-1}|_{\langle s \rangle}}| = 3^3 C(\xi_{1,\lambda}^{-1}|_{\langle s \rangle})$$

Candidate solution:
We have the following desgnated values for the constants: We have the same non-zero values as those for DRV-8 and the same non-zero values as those for DRV-14. We have the following additional values:

$$C(\xi_{1,\lambda}|_{\langle ss_1^{\lambda} \rangle}) = C(\xi_{2,\lambda}|_{\langle ss_1^{\lambda} \rangle}) = 3^{2k-2}(2 \cdot 3^{2k-2} + 1) \qquad \text{(for } \lambda = -1, 1)$$
$$C(\xi_{1,\lambda}^{-1}|_{\langle s \rangle}) = C(\xi_{2,\lambda}^{-1}|_{\langle s \rangle}) = 2 \cdot 3^{2k-2} \qquad \text{(for } \lambda = -1, 1)$$
$$C(\theta_{V_0}|_{\langle z \rangle}) = C(\theta_{V_0}^{-1}|_{\langle z \rangle}) = C(\alpha_{V_0}|_{\langle z \rangle}) = C(\beta_{V_0}|_{\langle z \rangle}) = 3^{4k-3} - 2^2 \cdot 3^{2k-2} + 1,$$

with the rest of the constants equated to 0. Writing Ω^* to denote this solution, we have

$$
\begin{aligned}
\Omega^* \cong & (\mathcal{O}_{\mathsf{Id}} \sqcup \mathcal{O}_{\omega} \sqcup \mathcal{O}_{\eta} \sqcup \mathcal{O}_{\omega\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (3^{2k-3}-1)(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda},z\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda},z\rangle}}) \\
& \sqcup (3^{2k-2}-1)(\mathcal{O}_{\mathsf{Id}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle s\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle s\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle s\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2^2\cdot 3^{4k-4}(\mathcal{O}_{\mathsf{Id}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup 2(\mathcal{O}_{\mathsf{Id}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega|_{\langle t\rangle}} \sqcup \mathcal{O}_{\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\omega\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_A} \sqcup \mathcal{O}_{\theta_A^{-1}} \sqcup \mathcal{O}_{\alpha_A} \sqcup \mathcal{O}_{\beta_A}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_A^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_A|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_A|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} (\mathcal{O}_{\theta_\lambda} \sqcup \mathcal{O}_{\theta_\lambda^{-1}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\theta_\lambda|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda\circ\eta|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle}}) \\
& \sqcup (\mathcal{O}_{\theta_{V_0}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_0}} \sqcup \mathcal{O}_{\beta_{V_0}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\theta_{V_\lambda}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}} \sqcup \mathcal{O}_{\beta_{V_\lambda}}) \\
& \sqcup (3^{4k-3}-2^2\cdot 3^{2k-2}+1)(\mathcal{O}_{\theta_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_0}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_0}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_0}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(2\cdot 3^{2k-3}-1)(\mathcal{O}_{\theta_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\theta_{V_\lambda}^{-1}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\alpha_{V_\lambda}|_{\langle z\rangle}} \sqcup \mathcal{O}_{\beta_{V_\lambda}|_{\langle z\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-3}(\mathcal{O}_{\hat{\theta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\beta}_{V_\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{1,\lambda}} \sqcup \mathcal{O}_{\hat{\vartheta}_{4,\lambda}}) \\
& \sqcup \bigsqcup_{\substack{\lambda=-1,1\\ \ell=-1,1}} (\mathcal{O}_{\Theta_{1,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{2,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{3,\lambda}^{\ell}} \sqcup \mathcal{O}_{\Theta_{4,\lambda}^{\ell}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(\mathcal{O}_{\vartheta_{1,\lambda}} \sqcup \mathcal{O}_{\vartheta_{2,\lambda}} \sqcup \mathcal{O}_{\vartheta_{3,\lambda}} \sqcup \mathcal{O}_{\vartheta_{4,\lambda}}) \\
& \sqcup 2(\mathcal{O}_{\theta_0|_{\langle s\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle s\rangle}}) \sqcup 2(\mathcal{O}_{\theta_0|_{\langle t\rangle}} \sqcup \mathcal{O}_{\theta_0^{-1}|_{\langle t\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 3^{2k-2}(2\cdot 3^{2k-2}+1)(\mathcal{O}_{\xi_{1,\lambda}|_{\langle ss_1^{\lambda}\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}|_{\langle ss_1^{\lambda}\rangle}}) \\
& \sqcup \bigsqcup_{\lambda=-1,1} 2\cdot 3^{2k-2}(\mathcal{O}_{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}} \sqcup \mathcal{O}_{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}})
\end{aligned}
$$

It is left to show that this solution is minimal, and to determine an upper bound on the exoticity index.

A.16.1 Minimality

Now we determine the minimal values of the constants.

We already know, by Lemma 4.2.8, that $C(\mathsf{Id}_S), C(\omega), C(\eta), C(\omega \circ \eta) \geq 1$.

$\underline{C(\theta_A), C(\theta_A^{-1}), C(\alpha_A), C(\beta_A)}$:

By ①$_8$, we have $[C(\theta_A) + C(\theta_A^{-1}) + C(\alpha_A) + C(\beta_A)] \geq 2^2$. ①$_{16}$

$\underline{C(\theta_\lambda), C(\theta_\lambda^{-1}), C(\theta_\lambda \circ \eta), C(\theta_\lambda^{-1} \circ \eta) \text{ for } \lambda = -1, 1}$:

By ②$_8$, we have $[C(\theta_\lambda) + C(\theta_\lambda^{-1}) + C(\theta_\lambda \circ \eta) + C(\theta_\lambda^{-1} \circ \eta)] \geq 2^2$. ②$_{16}$

$\underline{C(\hat{\theta}_{V_\lambda}), C(\hat{\beta}_{V_\lambda}), C(\hat{\vartheta}_{1,\lambda}), C(\hat{\vartheta}_{4,\lambda}) \text{ for } \lambda = -1, 1}$:

By ③$_8$, we have $[C(\hat{\theta}_{V_\lambda}) + C(\hat{\beta}_{V_\lambda}) + C(\hat{\vartheta}_{1,\lambda}) + C(\hat{\vartheta}_{4,\lambda})] \geq 2^2 \cdot 3^{2k-3}$. ③$_{16}$

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}), C(\omega|_{\langle ss_1^\lambda, z\rangle}), C(\eta|_{\langle ss_1^\lambda, z\rangle}), C(\omega \circ \eta|_{\langle ss_1^\lambda, z\rangle}) \text{ for } \lambda = -1, 1}$:

By ④$_8$, we have $[C(\mathsf{Id}|_{\langle ss_1^\lambda, z\rangle}), C(\omega|_{\langle ss_1^\lambda, z\rangle}), C(\eta|_{\langle ss_1^\lambda, z\rangle}), C(\omega \circ \eta|_{\langle ss_1^\lambda, z\rangle})] \geq 2^2 \cdot (3^{2k-3} - 1)$. ④$_{16}$

$\underline{C(\theta_{V_\lambda}), C(\theta_{V_\lambda}^{-1}), C(\alpha_{V_\lambda}), C(\beta_{V_\lambda}) \text{ for } \lambda = -1, 1}$:

By ⑤$_8$, we have $[C(\theta_{V_\lambda}) + C(\theta_{V_\lambda}^{-1}) + C(\alpha_{V_\lambda}) + C(\beta_{V_\lambda})] \geq 2^2 \cdot 3^{2k-2}$. ⑤$_{16}$

$\underline{C(\Theta_{1,\lambda}), C(\Theta_{2,\lambda}), C(\Theta_{3,\lambda}), C(\Theta_{4,\lambda}) \text{ for } \lambda = -1, 1}$:

By ⑥$_8$, we have $[C(\Theta_{1,\lambda}) + C(\Theta_{2,\lambda}) + C(\Theta_{3,\lambda}) + C(\Theta_{4,\lambda})] \geq 2^2$. ⑥$_{16}$

$\underline{C(\Theta_{1,\lambda}^{-1}), C(\Theta_{2,\lambda}^{-1}), C(\Theta_{3,\lambda}^{-1}), C(\Theta_{4,\lambda}^{-1}) \text{ for } \lambda = -1, 1}$:

By ⑦$_8$, we have $[C(\Theta_{1,\lambda}^{-1}) + C(\Theta_{2,\lambda}^{-1}) + C(\Theta_{3,\lambda}^{-1}) + C(\Theta_{4,\lambda}^{-1})] \geq 2^2$. ⑦$_{16}$

$\underline{C(\vartheta_{1,\lambda}), C(\vartheta_{2,\lambda}), C(\vartheta_{3,\lambda}), C(\vartheta_{4,\lambda}) \text{ for } \lambda = -1, 1}$:

By ⑧$_8$, we have $[C(\vartheta_{1,\lambda}) + C(\vartheta_{2,\lambda}) + C(\vartheta_{3,\lambda}) + C(\vartheta_{4,\lambda})] \geq 2^2 \cdot 3^{2k-2}$. ⑧$_{16}$

$\underline{C(\theta_\lambda|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda \circ \eta|_{\langle ss_1^\lambda\rangle}), C(\theta_\lambda^{-1} \circ \eta|_{\langle ss_1^\lambda\rangle}) \text{ for } \lambda = -1, 1}$:

By ⑨$_8$, $[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda\circ\eta|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})]\geq 2^3\cdot 3^{2k-2}$. ⑨$_{16}$

$\underline{C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle}),C(\omega|_{\langle ss_1^\lambda\rangle}),C(\eta|_{\langle ss_1^\lambda\rangle}),C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})\text{ for }\lambda=-1,1:}$

By ⑩$_8$, we have $[C(\mathsf{Id}|_{\langle ss_1^\lambda\rangle})+C(\omega|_{\langle ss_1^\lambda\rangle})+C(\eta|_{\langle ss_1^\lambda\rangle})+C(\omega\circ\eta|_{\langle ss_1^\lambda\rangle})]\geq 2^4\cdot 3^{4k-4}$. ⑩$_{16}$

$\underline{C(\theta_{V_\lambda}|_{\langle z\rangle}),C(\theta_{V_\lambda^{-1}}|_{\langle z\rangle}),C(\alpha_{V_\lambda}|_{\langle z\rangle}),C(\beta_{V_\lambda}|_{\langle z\rangle})\text{ for }\lambda=-1,1:}$

By ⑪$_8$, $[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda^{-1}}|_{\langle z\rangle})+C(\alpha_{V_\lambda}|_{\langle z\rangle})+C(\beta_{V_\lambda}|_{\langle z\rangle})]\geq 2^3(2\cdot 3^{4k-5}-3^{2k-2})$. ⑪$_{16}$

$\underline{C(\theta_A|_{\langle z\rangle}),C(\theta_A^{-1}|_{\langle z\rangle}),C(\alpha_A|_{\langle z\rangle}),C(\beta_A|_{\langle z\rangle}):}$

By ⑫$_8$, we have $[C(\theta_A|_{\langle z\rangle})+C(\theta_A^{-1}|_{\langle z\rangle})+C(\alpha_A|_{\langle z\rangle})+C(\beta_A|_{\langle z\rangle})]\geq 2^3(2\cdot 3^{2k-3}-1)$. ⑫$_{16}$

$\underline{C(\theta_\lambda|_{\langle t\rangle}),C(\theta_\lambda^{-1}|_{\langle t\rangle}),C(\theta_\lambda\circ\eta|_{\langle t\rangle}),C(\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle})\text{ for }\lambda=-1,1:}$

By ⑬$_8$, we have $\sum_{\lambda=-1,1}[C(\theta_\lambda|_{\langle t\rangle})+C(\theta_\lambda^{-1}|_{\langle t\rangle})+C(\theta_\lambda\circ\eta|_{\langle t\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle t\rangle})]\geq 2^4\cdot 3^{2k-2}$.

$\underline{C(\mathsf{Id}|_{\langle t\rangle}),C(\omega|_{\langle t\rangle}),C(\eta|_{\langle t\rangle}),C(\omega\circ\eta|_{\langle t\rangle}):}$

By ⑭$_8$, we can choose so that $[C(\mathsf{Id}|_{\langle t\rangle})+C(\omega|_{\langle t\rangle})+C(\eta|_{\langle t\rangle})+C(\omega\circ\eta|_{\langle t\rangle})]\geq 2^3$. ⑭$_{16}$

$\underline{C(\theta_{V_0}),C(\theta_{V_0}^{-1}),C(\alpha_{V_0}),C(\beta_{V_0}):}$

By ②$_{14}$, we have $[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})]\geq 2^2$. ⑮$_{16}$

$\underline{C(\mathsf{Id}|_{\langle s\rangle}),C(\omega|_{\langle s\rangle}),C(\eta|_{\langle s\rangle}),C(\omega\circ\eta|_{\langle s\rangle})}$

By ③$_{14}$, we have $[C(\mathsf{Id}|_{\langle s\rangle})+C(\omega|_{\langle s\rangle})+C(\eta|_{\langle s\rangle})+C(\omega\circ\eta|_{\langle s\rangle})]\geq 2^2(3^{2k-2}-1)$. ⑯$_{16}$

$\underline{C(\theta_0|_{\langle s\rangle}),C(\theta_0^{-1}|_{\langle s\rangle}):}$

By ④$_{14}$, we have $[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})]\geq 2^2$. ⑰$_{16}$

$\underline{C(\theta_0|_{\langle t\rangle}),C(\theta_0^{-1}|_{\langle t\rangle}):}$

By ⑤$_{14}$, we have $[C(\theta_0|_{\langle t\rangle})+C(\theta_0^{-1}|_{\langle t\rangle})]\geq 2^2$. ⑱$_{16}$

$\underline{C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}), C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}) \text{ for } \lambda=-1,1:}$

$$\begin{aligned}
&|\Omega^{\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle}}| = |\Omega^{\theta_\lambda|_{\langle ss_1^\lambda\rangle}}|+|\Omega^{\theta_\lambda^{-1}|_{\langle ss_1^\lambda\rangle}}| \\
\Longrightarrow\ & 3^3[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle})] \\
&=3^{2k}[C(\theta_\lambda)+C(\theta_\lambda^{-1}\circ\eta)]+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda,z\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda,z\rangle})] \\
&\quad+3^{2k+1}[C(\theta_\lambda|_{\langle ss_1^\lambda\rangle})+C(\theta_\lambda^{-1}\circ\eta|_{\langle ss_1^\lambda\rangle})] \\
&\quad+3^{2k}[C(\Theta_{1,\lambda})+C(\Theta_{2,\lambda})+C(\Theta_{3,\lambda})+C(\Theta_{4,\lambda})] \\
&\geq 2\cdot 3^{2k}+2^2\cdot 3^{2k-2}\cdot 3^{2k+1}+2^2\cdot 3^{2k}=2\cdot 3^{2k+1}+2^2\cdot 3^{4k-1}
\end{aligned}$$

Thus, $[C(\xi_{1,\lambda}|_{\langle ss_1^\lambda\rangle})+C(\xi_{2,\lambda}|_{\langle ss_1^\lambda\rangle})]\geq 2\cdot 3^{2k-2}+2^2\cdot 3^{4k-4}$, as required. 19_{16}

$\underline{C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle}), C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle}) \text{ for } \lambda=-1,1:}$

$$\begin{aligned}
&|\Omega^{\xi_{1,\lambda}^{-1}|_{\langle s\rangle}}|+|\Omega^{\xi_{2,\lambda}^{-1}|_{\langle s\rangle}}| = |\Omega^{\theta_0|_{\langle s\rangle}}|+|\Omega^{\theta_0^{-1}|_{\langle s\rangle}}| \\
\Longrightarrow\ & 3^3[C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})] \\
&=3^{2k+1}[C(\theta_0|_{\langle s\rangle})+C(\theta_0^{-1}|_{\langle s\rangle})] \\
&\geq 2^2\cdot 3^{2k+1}
\end{aligned}$$

Thus, $[C(\xi_{1,\lambda}^{-1}|_{\langle s\rangle})+C(\xi_{2,\lambda}^{-1}|_{\langle s\rangle})]\geq 2^2\cdot 3^{2k-2}$, as required. 20_{16}

$\underline{C(\theta_{V_0}|_{\langle z\rangle}), C(\theta_{V_0}^{-1}|_{\langle z\rangle}), C(\alpha_{V_0}|_{\langle z\rangle}), C(\beta_{V_0}|_{\langle z\rangle}):}$

Let $\lambda=-1,1$. Then

$$\begin{aligned}
&|\Omega^{\theta_{V_0}|_{\langle z\rangle}}|+|\Omega^{\theta_{V_0}|_{\langle z\rangle}}| = |\Omega^{\theta_{V_\lambda}|_{\langle z\rangle}}|+|\Omega^{\theta_{V_\lambda^{-1}}|_{\langle z\rangle}}| \\
\Longrightarrow\ & 3^{2k+1}[C(\theta_{V_0})+C(\theta_{V_0}^{-1})+C(\alpha_{V_0})+C(\beta_{V_0})] \\
&\quad+3^{2k+2}[C(\theta_{V_0}|_{\langle z\rangle})+C(\theta_{V_0}^{-1}|_{\langle z\rangle})+C(\alpha_{V_0}|_{\langle z\rangle})+C(\beta_{V_0}|_{\langle z\rangle})] \\
&=3^{2k+1}[C(\theta_{V_\lambda})+C(\theta_{V_\lambda}^{-1})+C(\alpha_{V_\lambda})+C(\beta_{V_\lambda})] \\
&\quad+3^{2k+2}[C(\theta_{V_\lambda}|_{\langle z\rangle})+C(\theta_{V_\lambda}^{-1}|_{\langle z\rangle})+C(\alpha_{V_\lambda}|_{\langle z\rangle})+C(\beta_{V_\lambda}|_{\langle z\rangle})] \\
&\quad+3^{2k+1}[C(\Theta_{1,\lambda}^{-1})+C(\Theta_{2,\lambda}^{-1})+C(\Theta_{3,\lambda}^{-1})+C(\Theta_{4,\lambda}^{-1})] \\
&\quad+3^{2k+1}[C(\vartheta_{1,-\lambda})+C(\vartheta_{2,-\lambda})+C(\vartheta_{3,-\lambda})+C(\vartheta_{4,-\lambda})] \\
&\geq 2^2\cdot 3^{2k-2}\cdot 3^{2k+1}+2^2\cdot 3^{2k-2}(3^{2k-2}-2)\cdot 3^{2k+2}+2^2\cdot 3^{2k+1}+2^2\cdot 3^{2k-2}\cdot 3^{2k+1} \\
&=2^2\cdot 3^{2k+1}[3^{4k-3}-2^2\cdot 3^{2k-2}+1]
\end{aligned}$$

Thus, $[C(\theta_{V_0}|_{\langle z\rangle}) + C(\theta_{V_0}^{-1}|_{\langle z\rangle}) + C(\alpha_{V_0}|_{\langle z\rangle}) + C(\beta_{V_0}|_{\langle z\rangle})] \geq 2^2 \cdot [3^{4k-3} - 2^2 \cdot 3^{2k-2} + 1]$, as required. 21_{16}

We have shown that our candidate solution is of minimal size.

A.16.2 Exoticity

Note that $|\Omega^*|/|S| = 2^2[47 \cdot 3^{6k-5} + 76 \cdot 3^{4k-4} + 11 \cdot 3^{2k-2} + 4]$. Thus,

$$\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \sum_{i=1} \left\lfloor \frac{2^2[47 \cdot 3^{6k-5} + 76 \cdot 3^{4k-4} + 11 \cdot 3^{2k-2} + 4]}{3^i} \right\rfloor$$

Write $n_i = \left\lfloor \frac{2^2[47 \cdot 3^{6k-5} + 76 \cdot 3^{4k-4} + 11 \cdot 3^{2k-2} + 4]}{3^i} \right\rfloor$. Now,

$$\begin{aligned}
i = 1 &\implies n_i = 188 \cdot 3^{6k-6} + 304 \cdot 3^{4k-5} + 44 \cdot 3^{2k-3} + 5 \\
i = 2 &\implies n_i = 188 \cdot 3^{6k-7} + 304 \cdot 3^{4k-6} + 44 \cdot 3^{2k-4} + 1 \\
3 \leq i \leq 2k-2 &\implies n_i = 188 \cdot 3^{6k-5-i} + 304 \cdot 3^{4k-4-i} + 44 \cdot 3^{2k-2-i} \qquad (k \geq 3) \\
i = 2k-1 &\implies n_i = \begin{cases} 16154 & \text{if } k = 2 \\ 188 \cdot 3^{4k-4} + 304 \cdot 3^{2k-3} + 14 & \text{if } k \geq 3 \end{cases} \\
i = 2k &\implies n_i = \begin{cases} 5384 & \text{if } k = 2 \\ 188 \cdot 3^{4k-5} + 304 \cdot 3^{2k-4} + 5 & \text{if } k \geq 3 \end{cases} \\
i = 2k+1 &\implies n_i = \begin{cases} 1795 & \text{if } k = 2 \\ 188 \cdot 3^{4k-6} + 304 \cdot 3^{2k-5} + 5 & \text{if } k \geq 3 \end{cases} \\
2k+2 \leq i \leq 4k-4 &\implies n_i = 188 \cdot 3^{6k-5-i} + 304 \cdot 3^{4k-4-i}
\end{aligned}$$

Moreover,

$$
\begin{aligned}
i = 4k-3 &\implies n_i = \begin{cases} 1795 & \text{if } k=2 \\ 188 \cdot 3^{2k-2} + 101 & \text{if } k \geq 3 \end{cases} \\
i = 4k-2 &\implies n_i = \begin{cases} 597 & \text{if } k=2 \\ 188 \cdot 3^{2k-3} + 34 & \text{if } k \geq 3 \end{cases} \\
i = 4k-1 &\implies n_i = 188 \cdot 3^{2k-4} + 11 \\
i = 4k &\implies n_i = \begin{cases} 66 & \text{if } k=2 \\ 188 \cdot 3^{2k-5} + 3 & \text{if } k \geq 3 \end{cases} \\
i = 4k+1 &\implies n_i = \begin{cases} 22 & \text{if } k=2 \\ 188 \cdot 3^{2k-6} + 1 & \text{if } k \geq 3 \end{cases} \\
4k+2 \leq i \leq 6k-5 &\implies n_i = 188 \cdot 3^{6k-5-i} && (k \geq 4) \\
i = 6k-4 &\implies n_i = 62 \\
i = 6k-3 &\implies n_i = 20 \\
i = 6k-2 &\implies n_i = 6 \\
i = 6k-1 &\implies n_i = 2
\end{aligned}
$$

Thus, by summing over the range $1 \leq i \leq 6k-1$, we have

$$
\nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) = \begin{cases} 218091 & \text{if } k=2 \\ 150865415 & \text{if } k=3 \\ 2[47 \cdot 3^{6k-5} + 76 \cdot 3^{4k-4} + 11 \cdot 3^{2k-2} - 2] & \text{if } k \geq 4 \end{cases}
$$

Let T be a Sylow 3-subgroup inside the group G constructed from Ω^*, realizing our fusion system. Then

$$
\begin{aligned}
\log_3 |T:S| &= [|\Omega^*|/|S| - 1](2k+1) + \nu_3(\mathsf{Sym}(|\Omega^*|/|S|)) \\
&= \begin{cases} 2399046 & \text{if } k=2 \\ 2262981344 & \text{if } k=3 \\ 2[47 \cdot 3^{6k-5} + 76 \cdot 3^{4k-4} + 11 \cdot 3^{2k-2}](4k+3) + (30k+11) & \text{if } k \geq 4 \end{cases}
\end{aligned}
$$

Thus, we have an upper bound on our exoticity index.

APPENDIX B

CALCULATIONS FOR THE FIXED POINT SIZES, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|$

If Ω is a right characteristic biset for any saturated fusion system $\mathcal{F}$ over S, then by definition, Ω is a disjoint union of orbits of the form $\mathcal{O}_\phi$ for some morphism $\phi \in \mathcal{F}$. Thus, we can write

$$\Omega = \bigsqcup_{\phi \in \mathcal{F}} C(\phi)\mathcal{O}_\phi$$

for some $C(\phi) \geq 0$. Not every value of $C(\phi)$ determines that we have a right characteristic biset. This Ω, by Lemma 4.2.7, is a right characteristic biset if and only if satisfies

$$|\Omega^\psi| = |\Omega^{\mathrm{Id}|_{D_\psi}}| \text{ for all } \psi \in \mathcal{F},$$

other than the additional property that $p \nmid (|\Omega|/|S|)$. This condition will help us to determine possible values of $C(\phi)$. Since

$$|\Omega^\psi| = \sum_{\phi \in \mathcal{F}} C(\phi)|\mathcal{O}_\phi^\psi|,$$

the aim of this appendix is to determine the fixed point sizes $|\mathcal{O}_\phi^\psi|$ for each relevant pair (ψ, ϕ). Fortunately, by Lemma 3.1.8, we only need to determine these sizes for morphisms ϕ and ψ up to S-S-equivalence in $\mathcal{F}$. For each DRV-fusion system, we have already listed all such morphims, which can be seen in Lemma 7.2.4, or more fully, in Theorem 7.3.1. In order to calculate $|\mathcal{O}_\phi^\psi|$, we use Lemma 3.1.7, which states that

$$|\mathcal{O}_\phi^\psi| = \frac{|N_{\psi,\phi}|}{|D_\phi|}|C_S(I_\psi)|,$$

where

$$N_{\psi,\phi} = \{x \in S \mid \exists y \in S \text{ with } (D_\psi)^x \leq D_\phi, \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi\}.$$

Usually, $|C_S(I_\psi)|$ will be easy to determine, and $|D_\phi|$ will be evident from the definition of ϕ itself. Thus, the focus will be primarily on calculating $|N_{\psi,\phi}|$. In order to determine this, we will generally suppose $N_{\psi,\phi} \neq \emptyset$, in which case there exists $x \in N_{\psi,\phi} \subseteq N_S(D_\psi, D_\phi)$. This means that

$$q \in D_\psi \implies \exists y \in S \text{ with } (q^x)\phi)^y = q\psi, \tag{$\star$}$$

for all $q \in D_\psi$ with $q^x \in D_\phi$. From this we will either derive a contradiction, or determine some conditions under which the above holds. We will usually do so in the following way. Write $p = q^x$. We will be able to write $(\star)$ as

$$(p_1^{\delta_p(\phi)})^y = q_1^{\delta_q(\psi)},$$

where, unless specified, we will have

$$\delta_p(\phi) = \begin{cases} 1 & \text{if } p\phi = p_1 \\ -1 & \text{if } p\phi = p_1^{-1} \end{cases} \quad \text{and} \quad \delta_q(\phi) = \begin{cases} 1 & \text{if } q\phi = q_1 \\ -1 & \text{if } q\phi = q_1^{-1}. \end{cases}$$

This will tell us under which conditions is $p_1^{\delta_p(\phi)}$ S-conjugate to $q_1^{\delta_q(\psi)}$. The condition will usually involve the vaules of $\delta_p(\phi)$ and $\delta_q(\psi)$.

Nevertheless, at times we will use different methods.

Calculations Locator:

The following two tables are there to help us locate a page of a particular result, and its corresponding proof.

Label name	Set of Morphisms	Set of Domains	Conditions
M1	$\mathrm{Id}_S, \omega, \eta, \omega \circ \eta$	$\mathcal{D}$	None
M2	$\theta_T, \theta_T^{-1}, \alpha_T, \beta_T$	$\mathcal{D}_T$	$T \in \mathcal{T}_{full}$
M3	$\Theta_{\mu,\nu}$	$\mathcal{D}_{V_\nu}$	$\mu = 1,..,4, \nu = -1,0,1$
M4	$\Theta_{\mu,\nu}^{-1}$	$\mathcal{D}_{A_0}$	$\mu = 1,..,4, \nu = -1,0,1$
M5	$\vartheta_{\mu,\nu}$	$\mathcal{D}_{V_\nu}$	$\mu = 1,..,4, \nu = -1,1$
M6	$\xi_{\mu,\nu}$	$\mathcal{D}_{V_\nu}$	$\mu = 1,..,8, \nu = -1,1$
M7	$\xi_{\mu,\nu}^{-1}$	$\mathcal{D}_{V_0}$	$\mu = 1,..,8, \nu = -1,1$
M8	$\theta_\nu \circ \eta, \theta_\nu^{-1} \circ \eta$	$\mathcal{D}_\nu$	$\nu = -1,1$
M9	$\hat{\theta}_{V_\nu}, \hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}, \hat{\beta}_{V_\nu}$	$\mathcal{D}_{V_\nu}$	$\nu = -1,1$
M10	$\hat{\vartheta}_{\mu,\nu}$	$\mathcal{D}_{V_\nu}$	$\mu = 1,..,4, \nu = -1,1$

Table B.1: Calculations Locator. Table 1 of 2.

$\lvert\mathcal{O}_{\phi\vert_P}^{\psi\vert_Q}\rvert$		ψ									
Page refs.		M1	M2	M3	M4	M5	M6	M7	M8	M9	M10
ϕ	M1	298,344	301,352	308,381	310,388	312,394	314,400	318,412	321,423	324,429	326,436
	M2	298,345	301,353	308,381	310,388	312,394	314,400	318,412	321,423	324,430	326,436
	M3	299,348	303,367	308,383	310,389	312,395	315,402	318,414	322,424	324,431	326,438
	M4	299,348	304,369	309,384	310,390	312,396	315,403	318,415	322,425	324,432	326,438
	M5	299,348	304,371	309,384	311,390	313,396	315,404	319,416	322,426	324,432	326,438
	M6	299,349	305,372	309,385	311,391	313,397	315,405	319,417	322,426	325,432	327,439
	M7	299,350	305,375	309,386	311,392	313,398	317,408	319,418	323,427	325,433	327,440
	M8	299,350	306,377	310,387	312,393	314,399	317,409	321,421	323,428	325,434	327,440
	M9	300,350	307,379	310,387	312,393	314,399	317,410	321,422	323,428	325,435	327,441
	M10	300,352	307,380	310,387	312,393	314,399	317,411	321,422	323,428	325,436	327,441

Table B.2: Calculations Locator. Table 2 of 2.

Search example: Finding the value of $\lvert\mathcal{O}_{\Theta_{0,1}}^{\theta_0\vert_{\langle t\rangle}}\rvert$.

In this case $\psi = \theta_0$ and $\phi = \Theta_{0,1}$. The first table tells us that we need to look at the pair (M2,M3) in the second table. This second table give us the pair (303,367). This means that the result is located in page 303, and proof starts in page 367. From the page 303, we see that $\lvert\mathcal{O}_{\Theta_{0,1}}^{\theta_0\vert_{\langle t\rangle}}\rvert = 3^{2k}$.

B.1 Results

B.1.1 $\psi \in \{\mathbf{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $Q \in \mathcal{D}$

1. Let $\phi \in \{\mathrm{Id}|_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{|N_S(Q,P)||C_S(Q)|}{|P|} & \text{if } Q \leq S; \psi \circ \phi = \mathrm{Id}|_S \\ & \text{or } Q \leq \langle s_2 \rangle; \psi \circ \phi = \omega \\ & \text{or } Q \leq \langle s \rangle; \psi \circ \phi = \eta \\ & \text{or } Q \leq \langle s_1 \rangle; \psi \circ \phi = \omega \circ \eta \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T \mid T \in \mathcal{T}_{full}\}$ and $P \in \mathcal{D}_T$.

   Suppose $T = V_\lambda$ for $\lambda = -1, 0, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or

$$(\psi, \phi) \in \{\mathrm{Id}_S, \omega\} \times \{\theta_{V_\lambda}, \theta_{V_\lambda^{-1}}, \alpha_{V_\lambda}, \beta_{V_\lambda}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\lambda \rangle, P = \langle ss_1^\lambda, z \rangle \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\eta, \omega \circ \eta\} \times \{\theta_{V_0}, \theta_{V_0}, \alpha_{V_0}, \beta_{V_0}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle s \rangle, P = \langle s, z \rangle, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose $T = E_\lambda$, for $\lambda = -1, 0, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\mathrm{Id}_S, \omega\} \times \{\theta_\lambda, \theta_\lambda^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q \in \{\langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z \rangle\}, P = \langle ss_1^\lambda, t \rangle \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^\lambda, z \rangle, \langle ss_1^\lambda, t \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\mathrm{Id}_S, \omega\} \times \{\alpha_\lambda, \beta_\lambda\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\lambda \rangle, P = \langle ss_1^\lambda, t \rangle \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\eta, \omega \circ \eta\} \times \{\theta_\lambda, \theta_\lambda^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q \in \{\langle s \rangle, \langle s, z \rangle\}, P = \langle s, z \rangle, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\eta, \omega \circ \eta\} \times \{\alpha_\lambda, \beta_\lambda\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle s \rangle, P = \langle s, z \rangle, \lambda = 0 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^\lambda, z \rangle, \langle ss_1^\lambda, t \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = A$. Then

$$(\psi,\phi) \in \{(\mathrm{Id}_S,\theta_A),(\omega,\alpha_A),(\eta,\theta_A^{-1}),(\omega\circ\eta,\beta_A)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k}}{|P|} & \text{if } Q = \langle t\rangle, \langle tz^{-1}\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\mathrm{Id}_S,\theta_A^{-1}),(\omega,\beta_A),(\eta,\theta_A),(\omega\circ\eta,\alpha_A)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k}}{|P|} & \text{if } Q = \langle t\rangle, \langle tz\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\mathrm{Id}_S,\alpha_A),(\omega,\theta_A),(\eta,\beta_A),(\omega\circ\eta,\theta_A^{-1})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k}}{|P|} & \text{if } Q^g \le P\cap\langle d_2\rangle, g\in S \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\mathrm{Id}_S,\beta_A),(\omega,\theta_A^{-1}),(\eta,\alpha_A),(\omega\circ\eta,\theta_A)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k}}{|P|} & \text{if } Q^g \le P\cap\langle d_1\rangle, g\in S \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\phi = \Theta_{\mu,\nu}$ and $P \in \mathcal{D}_{V_\nu}$ for $\mu = 1,..,4$ and $\nu = -1,0,1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

4. Let $\phi = \Theta_{\mu,\nu}^{-1}$ and $P \in \mathcal{D}_{V_\nu}$ for $\mu = 1,..,4$ and $\nu = -1,0,1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

5. Let $\phi = \vartheta_{\mu,\nu}$ and $P \in \mathcal{D}_{V_\nu}$ for $\mu = 1,..,4$ and $\nu = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\eta,\omega\circ\eta\}\times\{\vartheta_{1,\nu},\vartheta_{2,\nu},\vartheta_{3,\nu},\vartheta_{4,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\nu\rangle, P = \langle ss_1^\nu, z\rangle \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi = \xi_{\mu,\nu}$ and $P \in \mathcal{D}_{V_\nu}$ for $\mu = 1,..,8$ and $\nu = -1,1$. Then either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned}(\psi,\phi) &\in \{\mathrm{Id}_S,\omega\}\times\{\xi_{3,\nu},\xi_{4,\nu}\} \\ &\sqcup\{\eta,\omega\circ\eta\}\times\{\xi_{1,\nu},\xi_{2,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

7. Let $\phi = \xi_{\mu,\nu}^{-1}$ and $P \in \mathcal{D}_{V_0}$ for $\mu = 1,..,8$ and $\nu = -1,1$. Then either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned}(\psi,\phi) &\in \{\mathrm{Id}_S,\omega\}\times\{\xi_{3,\nu}^{-1},\xi_{4,\nu}^{-1}\} \\ &\sqcup\{\eta,\omega\circ\eta\}\times\{\xi_{1,\nu}^{-1},\xi_{2,\nu}^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

8. Let $\phi = \theta_\lambda^{\pm 1} \circ \eta$ for $\lambda = -1, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\eta, \omega \circ \eta\} \times \{\theta_\lambda^{\pm 1} \circ \eta\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q \in \{\langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z\rangle\}, P = \langle ss_1^\lambda, t\rangle \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\phi \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}^{-1}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\}$ for $\lambda = -1, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{(\mathsf{Id}, \hat{\theta}_{V_\lambda}), (\omega, \hat{\beta}_{V_\lambda})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{(\mathsf{Id}, \hat{\theta}_{V_\lambda}^{-1}), (\omega, \hat{\alpha}_{V_\lambda})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ & \text{or } Q = P = \langle ss_1^\lambda, z\rangle \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &(\psi, \phi) \in \{\eta, \omega \circ \eta\} \times \{\hat{\theta}_{V_\lambda}, \hat{\beta}_{V_\lambda}\} \\ &\sqcup \{(\mathsf{Id}_S, \hat{\theta}_{V_\lambda}^{-1}), (\omega, \hat{\alpha}_{V_\lambda})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

10. Let $\phi = \hat{\vartheta}_{\mu,\lambda}$ for $\mu = 1, .., 4$ and $\lambda = -1, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{(\eta, \hat{\vartheta}_{1,V_\lambda}), (\omega \circ \eta, \hat{\vartheta}_{4,V_\lambda})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{(\eta, \hat{\vartheta}_{2,V_\lambda}), (\omega \circ \eta, \hat{\vartheta}_{3,V_\lambda})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ & \text{or } Q = P = \langle ss_1^\lambda, z\rangle \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &(\psi, \phi) \in \{\mathsf{Id}_S, \omega\} \times \{\hat{\vartheta}_{1,V_\lambda}, \hat{\vartheta}_{4,V_\lambda}\} \\ &\sqcup \{(\eta, \hat{\vartheta}_{2,V_\lambda}), (\omega \circ \eta, \hat{\vartheta}_{3,V_\lambda})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

B.1.2 $\psi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $Q \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$

1. Let $\phi \in \{\mathsf{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   Suppose $T = V_\lambda$ for $\lambda = -1, 0, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

   Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_\lambda, \theta_\lambda^{-1}\} \times \{\mathsf{Id}_S, \omega\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\alpha_0, \beta_0\} \times \{\eta, \omega \circ \eta\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose $T = A$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{(\alpha_A, \mathsf{Id}_S), (\theta_A, \omega), (\beta_A, \eta), (\theta_A^{-1}, \omega \circ \eta)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k}|N_S(Q,P)|}{|P|} & \text{if } \langle d_2 \rangle \leq Q \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{(\beta_A, \mathsf{Id}_S), (\theta_A^{-1}, \omega), (\alpha_A, \eta), (\theta_A, \omega \circ \eta)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k}|N_S(Q,P)|}{|P|} & \text{if } \langle d_1 \rangle \leq Q \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_{T_1}, \theta_{T_1}^{-1}, \alpha_{T_1}, \beta_{T_1}\}$ and $T_1 \in \mathcal{T}_{full}$.

   Suppose that $T_1 \neq T$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned}(\psi, \phi) \in &\{\theta_\lambda, \theta_\lambda^{-1}\} \times \{\theta_\mu, \theta_\mu^{-1}\} \\ &\sqcup \{\alpha_\lambda, \beta_\lambda\} \times \{\alpha_\mu, \beta_\mu\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^\mu, z \rangle, \langle ss_1^\mu, t \rangle\}, \\ & \quad \lambda \neq \mu \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Suppose that $T_1 = T = V_\lambda$ for some $\lambda = -1,0,1$. Then

$$\{\psi,\phi\} \in \left\{\{\theta_{V_\lambda},\alpha_{V_\lambda}\},\{\theta_{V_\lambda}^{-1},\beta_{V_\lambda}\}\right\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\{\psi,\phi\} \in \left\{\{\theta_{V_\lambda},\beta_{V_\lambda}\},\{\theta_{V_\lambda}^{-1},\alpha_{V_\lambda}\}\right\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\{\psi,\phi\} \in \left\{\{\theta_{V_\lambda},\theta_{V_\lambda}^{-1}\},\{\alpha_{V_\lambda},\beta_{V_\lambda}\}\right\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$$

$$\psi = \phi \implies |\mathcal{O}_{\phi|_Q}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ & \text{or } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ 3^2 & \text{if } Q = P = \langle ss_1^\lambda, z\rangle \\ \emptyset & \text{otherwise} \end{cases}$$

Suppose $T_1 = T = E_\lambda$, for $\lambda = -1,0,1$. Then

$$\{\psi,\phi\} \in \left\{\{\theta_\lambda,\alpha_\lambda\},\{\theta_\lambda^{-1},\beta_\lambda\}\right\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle s\rangle\} \times \{\langle s, z\rangle, \langle s, t\rangle\} \\ 3^{2k+1} & \text{if } Q = P = \langle s\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\{\psi,\phi\} \in \left\{\{\theta_\lambda,\beta_\lambda\},\{\theta_\lambda^{-1},\alpha_\lambda\}\right\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle t\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ 3^{2k+1} & \text{if } Q = P = \langle t\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\{\psi,\phi\} \in \left\{\{\theta_\lambda,\theta_\lambda^{-1}\},\{\alpha_\lambda,\beta_\lambda\}\right\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t, z\rangle, \\ & \quad \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$\psi = \phi \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle s\rangle, \langle s, z\rangle\} \times \{\langle s, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ & \text{or } (Q,P) \in \{\langle t\rangle, \langle t, z\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ 3^{2k+1} & \text{if } Q = P \in \{\langle ss_1^\lambda, t\rangle, \langle t\rangle\} \\ \frac{3^{4k+2}}{|P|} & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t, z\rangle, \\ & \quad \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ 3 & \text{if } Q = P = \langle ss_1^\lambda, t\rangle \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T_1 = T = A$. Then

$$\{\psi,\phi\} \in \left\{\{\theta_T,\alpha_T\}\{\theta_T^{-1},\beta_T\}\right\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \dfrac{3^{4k}}{|P|} & \text{if } Q \in \{\langle s_1\rangle \cap P, \langle d_1^s\rangle \cap P, \langle d_2^{s^{-1}}\rangle \cap P\} \\ & \text{or } Q \in \{\langle s_1^s\rangle \cap P^{s^{-1}}, \langle s_1^{s^{-1}}\rangle \cap P^s\}, \\ & \quad Q \neq \langle t\rangle \\ \dfrac{3^{4k+1}}{|P|} & \text{if } Q = \langle t\rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$\{\psi,\phi\} \in \left\{\{\theta_T,\beta_T\},\{\theta_T^{-1},\alpha_T\}\right\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \dfrac{3^{4k}}{|P|} & \text{if } Q \in \{\langle s_2^s\rangle \cap P, \langle d_1^{s^{-1}}\rangle \cap P, \langle d_2^s\rangle \cap P\} \\ & \text{or } Q \in \{\langle s_2\rangle \cap P^s, \langle s_2^{s^{-1}}\rangle \cap P^{s^{-1}}\}, \\ & \quad Q \neq \langle z\rangle \\ \dfrac{3^{4k+1}}{|P|} & \text{if } Q = \langle z\rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$\{\psi,\phi\} \in \left\{\{\theta_T,\theta_T^{-1}\},\{\alpha_T,\beta_T\}\right\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$$

$$\psi = \phi \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \dfrac{3^{4k}}{|P|} & \text{if } Q \notin \{\langle t\rangle,\langle z\rangle\}, Q \leq P \\ \dfrac{3^{4k+1}}{|P|} & \text{if } Q \in \{\langle t\rangle,\langle z\rangle\} \cap P \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\phi = \Theta_{\mu,\nu}$, for some $\mu = 1,..,4$ and $\nu = -1,0,1$.

   If $T = V_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

   $$(\psi,\phi) \in \{\theta_{V_\lambda},\alpha_{V_\lambda}\} \times \{\Theta_{1,\nu},\Theta_{3,\nu}\}$$
   $$\sqcup\{\theta_{V_\lambda}^{-1}\beta_{V_\lambda}\} \times \{\Theta_{2,\nu},\Theta_{4,\nu}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\nu\rangle, \langle ss_1^\nu, z\rangle\}, \\ & \quad \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

   If $T = E_\lambda$ for $\lambda = -1,0,1$, then

   $$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^{2k} & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\nu, z\rangle, \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

If $T = A$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_A, \beta_A\} \times \{\Theta_{1,\nu}, \Theta_{4,\nu}\}$$

$$\sqcup \{\theta_A^{-1}, \alpha_A\} \times \{\Theta_{2,\nu}, \Theta_{3,\nu}\} \implies |\mathcal{O}^{|\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

4. Let $\phi = \Theta^{-1}_{\mu,\nu}$, for some $\mu = 1,..,4$ and $\nu = -1,0,1$.

   If $T = V_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_{V_\lambda}, \beta_{V_\lambda}\} \times \{\Theta^{-1}_{2,\nu}, \Theta^{-1}_{4,\nu}\}$$

$$\sqcup \{\theta^{-1}_{V_\lambda}, \alpha_{V_\lambda}\} \times \{\Theta^{-1}_{1,\nu}, \Theta^{-1}_{3,\nu}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$

   If $T = E_\lambda$ for $\lambda = -1,0,1$, then

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^{2k} & Q = \langle t\rangle, P = \langle t,z\rangle, \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

   If $T = A$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_A, \alpha_A\} \times \{\Theta^{-1}_{2,\nu}, \Theta^{-1}_{3,\nu}\}$$

$$\sqcup \{\theta_A^{-1}, \beta_A\} \times \{\Theta^{-1}_{1,\nu}, \Theta^{-1}_{4,\nu}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle t\rangle, P \in \{\langle t\rangle, \langle t,z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

5. Let $\phi = \vartheta_{\mu,\nu}$ for some $\mu = 1,..,4$ and $\nu = -1,1$.

If $T = V_\lambda$ for $\lambda = -1, 0, 1$, then, either, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_{V_\lambda}, \alpha_{V_\lambda}\} \times \{\vartheta_{2,\nu}, \vartheta_{4,\nu}\}$$

$$\sqcup \{\theta_{V_\lambda}^{-1}, \beta_{V_\lambda}\} \times \{\vartheta_{1,\nu}, \vartheta_{3,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda \rangle, P \in \{\langle ss_1^\nu \rangle, \langle ss_1^\nu, z \rangle\}, \\ & \quad \lambda = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\theta_{V_\lambda}, \beta_{V_\lambda}\} \times \{\vartheta_{2,\nu}, \vartheta_{3,\nu}\}$$

$$\sqcup \{\theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}\} \times \{\vartheta_{1,\nu}, \vartheta_{4,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\}, \\ & \quad \lambda = -\nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

If$T = E_\lambda$ for $\lambda = -1, 0, 1$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

6. Let $\phi = \xi_{\mu,\nu}$ for $\mu = 1, .., 8$ and $\nu = -1, 1$.

If $T = V_\lambda$, for $\lambda = -1, 0, 1$, then, either, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_{V_\lambda}, \alpha_{V_\lambda}\} \times \{\xi_{5,\nu}, \xi_{7,\nu}\}$$

$$\sqcup \{\theta_{V_\lambda}^{-1}, \beta_{V_\lambda}\} \times \{\xi_{6,\nu}, \xi_{8,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda \rangle, P \in \{\langle ss_1^\nu \rangle, \langle ss_1^\nu, z \rangle\}, \\ & \quad \lambda = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\theta_{V_0}^{-1}, \alpha_{V_0}\} \times \{\xi_{5,\nu}, \xi_{8,\nu}\}$$

$$\sqcup \{\theta_{V_0}, \beta_{V_0}\} \times \{\xi_{6,\nu}, \xi_{7,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\}, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $T = E_\lambda$ for some $\lambda = -1, 0, 1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_\lambda, \theta_\lambda^{-1}\} \times \{\xi_{3,\nu}, \xi_{4,\nu}\}$$

$$\sqcup \{\alpha_\lambda, \beta_\lambda\} \times \{\xi_{1,\nu}, \xi_{2,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Let $\phi = \xi_{\mu,\nu}^{-1}$ for $\mu = 1, .., 8$ and $\nu = -1, 1$.

If $T = V_\lambda$ for $\lambda = -1, 0, 1$, then, either, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_{V_0}, \alpha_{V_0}\} \times \{\xi_{5,\nu}^{-1}, \xi_{8,\nu}^{-1}\}$$
$$\sqcup \{\theta_{V_0}^{-1}, \beta_{V_0}\} \times \{\xi_{6,\nu}^{-1}, \xi_{7,\nu}^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s, z \rangle\}, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$(\psi, \phi) \in \{\theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}\} \times \{\xi_{5,\nu}^{-1}, \xi_{7,\nu}^{-1}\}$$
$$\sqcup \{\theta_{V_\lambda}, \beta_{V_\lambda}\} \times \{\xi_{6,\nu}^{-1}, \xi_{8,\nu}^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s, z \rangle\}, \lambda = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

If $T = E_\lambda$ for $\lambda = -1, 0, 1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_\lambda, \theta_\lambda^{-1}\} \times \{\xi_{3,\nu}^{-1}, \xi_{4,\nu}^{-1}\}$$
$$\sqcup \{\alpha_\lambda, \beta_\lambda\} \times \{\xi_{1,\nu}^{-1}, \xi_{2,\nu}^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s s_1^\lambda, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_\nu^{\pm 1} \circ \eta$, for some $\nu = -1, 1$.

   If $T = V_\lambda$ for $\lambda = -1, 0, 1$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

If $T = E_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{(\alpha_\lambda, \theta_\nu^{-1}\circ\eta), (\beta_\lambda, \theta_\nu\circ\eta)\}$$
$$\implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = \nu \neq 0 \\ & \text{or } Q \in \{\langle t\rangle, \langle t,z\rangle\}, P \in \{\langle t,z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = -\nu \neq 0 \\ 3^{2k+1} & \text{if } Q = P = \langle ss_1^\lambda\rangle, \lambda = \nu \neq 0 \\ & \text{or } Q = P = \langle t\rangle, \lambda = -\nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\theta_\lambda, \theta_\nu^{-1}\circ\eta), (\theta_\lambda^{-1}, (\theta_\nu^{-1}\circ\eta))\}$$
$$\implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = \nu \neq 0 \\ 3^{2k+1} & \text{if } Q = P = \langle ss_1^\lambda\rangle, \lambda = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\theta_\lambda, \theta_\nu\circ\eta), (\theta_\lambda^{-1}, \theta_\nu^{-1}\circ\eta)\}$$
$$\implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle t\rangle, P \in \{\langle t,z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = -\nu \neq 0 \\ 3^{2k+1} & \text{if } Q = P = \langle t\rangle, \lambda = -\nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

If $T = A$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

9. Let $\phi \in \{\hat\theta_{V_\nu}, \hat\theta_{V_\nu}^{-1}, \hat\alpha_{V_\nu}, \hat\beta_{V_\nu}\}$ for $\nu = -1,1$.

   If $T = V_\lambda$ for $\lambda = -1,0,1$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

   If $T = E_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_\lambda, \theta_\lambda^{-1}\} \times \{\hat\theta_{V_\nu}^{-1}, \hat\alpha_{V_\nu}\}$$
$$\sqcup \{\alpha_\lambda, \beta_\lambda\} \times \{\hat\theta_{V_\nu}, \hat\beta_{V_\nu}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\nu, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

   If $T = A$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

10. Let $\phi = \hat\vartheta_{V_{\mu,\nu}}$ for $\mu = 1,..,4$ and $\nu = -1,1$.

    If $T = V_\lambda$ for $\lambda = -1,0,1$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

If $T = E_\lambda$ for $\lambda = -1, 0, 1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_\lambda, \theta_\lambda^{-1}\} \times \{\hat{\vartheta}_{1,\nu}, \hat{\vartheta}_{4,\nu}\}$$

$$\sqcup \{\alpha_\lambda, \beta_\lambda\} \times \{\hat{\vartheta}_{2,\nu}, \hat{\vartheta}_{3,\nu}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.1.3 $\psi = \Theta_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1, .., 4$ and $\nu = -1, 0, 1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.
2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

If $T = V_\lambda$ for $\lambda = -1, 0, 1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\Theta_{1,\nu}, \Theta_{3,\nu}\} \times \{\theta_{V_\lambda}, \alpha_{V_\lambda}\}$$

$$\sqcup \{\Theta_{2,\nu}, \Theta_{4,\nu}\} \times \{\theta_{V_\lambda}^{-1}, \beta_{V_\lambda}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu \rangle, P \in \{\langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z \rangle\}, \\ & \quad \nu = \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

If $T = E_\lambda$ for $\lambda = -1, 0, 1$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

If $T = A$, then, either $\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if,

$$(\psi, \phi) \in \{\Theta_{2,\nu}, \Theta_{3,\nu}\} \times \{\theta_A^{-1}, \alpha_A\}$$

$$\sqcup \{\Theta_{1,\nu}, \Theta_{4,\nu}\} \times \{\theta_A, \beta_A\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle z \rangle, \langle z \rangle \leq P, P \in \mathcal{D}_A \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$, $\nu_1 = -1,0,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \Big\{(\Theta_{1,\nu},\Theta_{4,\nu_1}),(\Theta_{2,\nu},\Theta_{3,\nu_1}),$$

$$(\Theta_{4,\nu},\Theta_{1,\nu_1}),(\Theta_{3,\nu},\Theta_{2,\nu_1})\Big\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \Big\{(\Theta_{1,\nu},\Theta_{3,\nu_1}),(\Theta_{2,\nu},\Theta_{4,\nu_1}),$$

$$(\Theta_{3,\nu},\Theta_{1,\nu_1}),(\Theta_{4,\nu},\Theta_{2,\nu_1})\Big\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu_1}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi = \phi \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}\rangle \\ \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle; P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ 3^{2k} & \text{if } Q = P = \langle ss_1^{\nu}, z\rangle, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,4$, $\nu_1 = -1,0,1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$, $\nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\Theta_{1,\nu},\Theta_{3,\nu}\} \times \{\vartheta_{2,\nu_1},\vartheta_{4,\nu_1}\}$$

$$\sqcup\{\Theta_{2,\nu},\Theta_{4,\nu}\} \times \{\vartheta_{1,\nu_1},\vartheta_{3,\nu_1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$, $\nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\Theta_{1,\nu},\Theta_{3,\nu}\} \times \{\xi_{5,\nu_1},\xi_{7,\nu_1}\}$$

$$\sqcup\{\Theta_{2,\nu},\Theta_{4,\nu}\} \times \{\xi_{6,\nu_1},\xi_{8,\nu_1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$, $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$
\begin{aligned}
(\psi,\phi) \in & \{\Theta_{1,0},\Theta_{3,0}\} \times \{\xi^{-1}_{5,\nu_1},\xi^{-1}_{7,\nu_1}\} \\
& \sqcup \{\Theta_{2,0},\Theta_{4,0}\} \times \{\xi^{-1}_{6,\nu_1},\xi^{-1}_{8,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

8. Let $\phi = \theta^{\pm 1}_{\nu_1} \circ \eta$ for $\nu_1 = -1,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}^{-1}_{V_{\nu_1}}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$, for $\nu_1 = -1,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu_1\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

B.1.4 $\psi = \Theta^{-1}_{\mu,\nu}$ and $Q \in \mathcal{D}_{A_0}$, for $\mu = 1,..,4$ and $\nu = -1,0,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   If $T = V_\lambda$ for $\lambda = -1,0,1$, then, either, $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$
\begin{aligned}
(\psi,\phi) \in & \{\Theta^{-1}_{1,\nu},\Theta^{-1}_{3,\nu}\} \times \{\theta^{-1}_{V_\lambda},\alpha_{V_\lambda}\} \\
& \sqcup \{\Theta^{-1}_{2,\nu},\Theta^{-1}_{4,\nu}\} \times \{\theta_{V_\lambda},\beta_{V_\lambda}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\lambda}, z\rangle\}, \nu = \lambda \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

   If $T = E_\lambda$, for $\lambda = -1,0,1$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

   If $T = A$, then, either, $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$
\begin{aligned}
(\psi,\phi) \in & \{\Theta^{-1}_{2,\nu},\Theta^{-1}_{3,\nu}\} \times \{\theta_A,\alpha_A\} \\
& \sqcup \{\Theta^{-1}_{1,\nu},\Theta^{-1}_{4,\nu}\} \times \{\theta_A^{-1},\beta_A\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle t\rangle, \langle t\rangle \leq P, P \in \mathcal{D}_A \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$, $\nu_1 = -1,0,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

4. Let $\phi = \Theta^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$, $\nu_1 = -1,0,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \Big\{(\Theta^{-1}_{1,\nu},\Theta^{-1}_{4,\nu_1}),(\Theta^{-1}_{2,\nu},\Theta^{-1}_{3,\nu_1}), (\Theta^{-1}_{4,\nu},\Theta^{-1}_{1,\nu_1}),(\Theta^{-1}_{3,\nu},\Theta^{-1}_{1,\nu_1})\Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q=\langle t\rangle, P\in\{\langle t\rangle,\langle t,z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \Big\{(\Theta^{-1}_{1,\nu},\Theta^{-1}_{3,\nu_1}),(\Theta^{-1}_{2,\nu},\Theta^{-1}_{4,\nu_1}), (\Theta^{-1}_{3,\nu},\Theta^{-1}_{1,\nu_1}),(\Theta^{-1}_{4,\nu},\Theta^{-1}_{2,\nu_1})\Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle t,z\rangle\}, \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi=\phi \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q=\langle t\rangle, P\in\{\langle t\rangle,\langle t,z\rangle\} \\ \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle t,z\rangle\}, \nu=\nu_1 \\ 3^{2k} & \text{if } Q=P=\langle t,z\rangle, \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$. Then

$$(\psi,\phi) \in \{\Theta^{-1}_{1,\nu},\Theta^{-1}_{3,\nu}\}\times\{\vartheta_{2,\nu_1},\vartheta_{3,\nu_1}\} \sqcup \{\Theta^{-1}_{2,\nu},\Theta^{-1}_{4,\nu}\}\times\{\vartheta_{1,\nu_1},\vartheta_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^{\nu_1},z\rangle, \\ & \quad \nu=-\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$, $\nu_1 = -1,1$. Then

$$(\phi,\psi) \in \{\Theta^{-1}_{1,0}\Theta^{-1}_{3,0},\}\times\{\xi_{5,\nu_1},\xi_{8,\nu_1}\} \sqcup \{\Theta^{-1}_{2,0}\Theta^{-1}_{4,0},\}\times\{\xi_{6,\nu_1},\xi_{7,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^{\nu_1},z\rangle\}, \\ & \quad \nu_1=0 \\ 0 & \text{otherwise} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$, $\nu_1 = -1,1$. Then

$$(\phi,\psi) \in \{\Theta^{-1}_{1,\nu}\Theta^{-1}_{3,\nu},\}\times\{\xi^{-1}_{5,\nu_1},\xi^{-1}_{7,\nu_1}\} \sqcup \{\Theta^{-1}_{2,\nu}\Theta^{-1}_{4,\nu},\}\times\{\xi^{-1}_{6,\nu_1},\xi^{-1}_{8,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle s,z\rangle\}, \\ & \quad \nu=\nu_1\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$, for $\nu_1 = -1, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_2 = 1,..,4$ and $\nu_1 = -1, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.1.5 $\psi = \vartheta_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..,4$ and $\nu = -1, 1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   If $T = V_\lambda$ for $\lambda = -1, 0, 1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\vartheta_{2,\nu}, \vartheta_{4,\nu}\} \times \{\theta_{V_\lambda}, \alpha_{V_\lambda}\}$$

$$\sqcup \{\vartheta_{1,\nu}, \vartheta_{3,\nu}\} \times \{\theta_{V_\lambda}^{-1}, \beta_{V_\lambda}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu \rangle, P \in \{\langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z \rangle\}, \\ & \quad \nu = \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi, \phi) \in \{\vartheta_{2,\nu}, \vartheta_{3,\nu}\} \times \{\theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}\}$$

$$\sqcup \{\vartheta_{1,\nu}, \vartheta_{4,\nu}\} \times \{\theta_{V_\lambda}, \beta_{V_\lambda}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\lambda, z \rangle\}, \\ & \quad \nu = -\lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

   If $T = E_\lambda$ for $\lambda = -1, 0, 1$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

   If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4, \nu_1 = -1, 0, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\vartheta_{1,\nu}, \vartheta_{3,\nu}\} \times \{\Theta_{2,\nu_1}, \Theta_{4,\nu_1}\}$$

$$\sqcup \{\vartheta_{2,\nu}, \vartheta_{4,\nu}\} \times \{\Theta_{1,\nu_1}, \Theta_{3,\nu_1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu \rangle, P \in \{\langle ss_1^{\nu_1} \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

4. Let $\phi = \Theta^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4, \nu_1 = -1,0,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$
\begin{aligned}
&(\psi,\phi) \in \{\vartheta_{2,\nu}, \vartheta_{3,\nu}\} \times \{\Theta^{-1}_{1,\nu_1}, \Theta^{-1}_{3,\nu_1}\} \\
&\qquad \sqcup \{\vartheta_{1,\nu}, \vartheta_{4,\nu}\} \times \{\Theta^{-1}_{2,\nu_1}, \Theta^{-1}_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle\}, \\ & \quad \nu = -\nu_1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$, $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$
\begin{aligned}
&(\psi,\phi) \in \Big\{ (\vartheta_{1,\nu}, \vartheta_{3,\nu_1}), (\vartheta_{2,\nu}, \vartheta_{4,\nu_1}), \\
&\qquad (\vartheta_{3,\nu}, \vartheta_{1,\nu_1}), (\vartheta_{4,\nu}, \vartheta_{2,\nu_1}) \Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s s_1^{\nu} \rangle, P \in \{\langle s s_1^{\nu_1} \rangle, \langle s s_1^{\nu_1}, z \rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases} \\
&(\psi,\phi) \in \Big\{ (\vartheta_{1,\nu}, \vartheta_{4,\nu_1}), (\vartheta_{2,\nu}, \vartheta_{3,\nu_1}), \\
&\qquad (\vartheta_{4,\nu}, \vartheta_{1,\nu_1}), (\vartheta_{3,\nu}, \vartheta_{2,\nu_1}) \Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s s_1^{\nu_1}, z \rangle\}, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases} \\
&\qquad\qquad\qquad \psi = \phi \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s s_1^{\nu} \rangle, P \in \{\langle s s_1^{\nu_1} \rangle, \langle s s_1^{\nu_1}, z \rangle\}, \nu = \nu_1 \\ & \text{or } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s s_1^{\nu_1}, z \rangle\}, \nu = \nu_1 \\ 3^2 & \text{if } Q = P = \langle s s_1^{\nu}, z \rangle, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$, $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$
\begin{aligned}
&(\psi,\phi) \in \{\vartheta_{2,\nu}, \vartheta_{4,\nu}\} \times \{\xi_{5,\nu_1}, \xi_{7,\nu_1}\} \\
&\qquad \sqcup \{\vartheta_{1,\nu}, \vartheta_{3,\nu}\} \times \{\xi_{6,\nu_1}, \xi_{8,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s s_1^{\nu} \rangle, P \in \{\langle s s_1^{\nu_1} \rangle, \langle s s_1^{\nu_1}, z \rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$, $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\vartheta_{2,\nu},\vartheta_{3,\nu}\} \times \{\xi_{5,\nu_1},\xi_{7,\nu_1}\}$$
$$\sqcup\{\vartheta_{1,\nu},\vartheta_{4,\nu}\} \times \{\xi_{6,\nu_1},\xi_{8,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \\ & \quad \nu = -\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

8. Let $\phi = \theta^{\pm 1}_{\nu_1} \circ \eta$ for $\nu_1 = -1,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}^{-1}_{V_{\nu_1}}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$, for $\nu_1 = -1,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

B.1.6 $\psi = \xi_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..,8$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$ and $P \in \mathcal{D}$. Then either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\xi_{3,\nu},\xi_{4,\nu}\} \times \{\mathrm{Id}_S,\omega\}$$
$$\sqcup\{\xi_{1,\nu},\xi_{2,\nu}\} \times \{\eta,\omega\circ\eta\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta^{-1}_T, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$

If $T = V_\lambda$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in Q\{\xi_{5,\nu},\xi_7\} \times \{\theta_{V_\lambda},\alpha_{V_\lambda}\}$$
$$\sqcup\{\xi_{6,\nu},\xi_8\} \times \{\theta^{-1}_{V_\lambda},\beta_{V_\lambda}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \nu = \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in Q\{\xi_{5,\nu},\xi_8\} \times \{\theta^{-1}_{V_0},\alpha_{V_0}\}$$
$$\sqcup\{\xi_{6,\nu},\xi_7\} \times \{\theta_{V_0},\beta_{V_0}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $T=E_\lambda$ for $\lambda=-1,0,1$, then, either, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$, or, if

$$(\psi,\phi)\in\{\xi_{3,\nu},\xi_{4,\nu}\}\times\{\theta_\lambda,\theta_\lambda^{-1}\}$$
$$\sqcup\{\xi_{1,\nu},\xi_{2,\nu}\}\times\{\alpha_\lambda,\beta_\lambda\}\implies|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases}\frac{3^{4k+2}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^\lambda,z\rangle,\langle t,z\rangle,\langle ss_1^\lambda,t\rangle\}\\ 0 & \text{otherwise}\end{cases}$$

If $T=A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

3. Let $\phi=\Theta_{\mu_1,\nu_1}$ for $\mu_1=1,..,4,\nu_1=-1,0,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$, or, if

$$(\psi,\phi)\in\{\xi_{5,\nu},\xi_{7,\nu}\}\times\{\Theta_{1,\nu_1},\Theta_{3,\nu_1}\}$$
$$\sqcup\{\xi_{6,\nu},\xi_{8,\nu}\}\times\{\Theta_{2,\nu_1},\Theta_{4,\nu_1}\}\implies\quad|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases}\frac{3^{2k+3}}{|P|} & \text{if } Q=\langle ss_1^\nu\rangle, P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\},\\ & \quad \nu=\nu_1\\ 0 & \text{otherwise}\end{cases}$$

4. Let $\phi=\Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1=1,..,4,\nu_1=-1,0,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$, or, if

$$(\psi,\phi)\in\{\xi_{5,\nu},\xi_{8,\nu}\}\times\{\Theta_{1,0}^{-1},\Theta_{3,0}^{-1}\}$$
$$\sqcup\{\xi_{6,\nu},\xi_{7,\nu}\}\times\{\Theta_{2,0}^{-1},\Theta_{4,0}^{-1}\}\implies\quad|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases}\frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle t,z\rangle\},\nu_1=0\\ 0 & \text{otherwise}\end{cases}$$

5. Let $\phi=\vartheta_{\mu_1,\nu_1}$ for $\mu_1=1,..,4,\nu_1=-1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$, or, if

$$(\psi,\phi)\in\{\xi_{5,\nu},\xi_{7,\nu}\}\times\{\vartheta_{2,\nu_1},\vartheta_{4,\nu_1}\}$$
$$\sqcup\{\xi_{6,\nu},\xi_{8,\nu}\}\times\{\vartheta_{1,\nu_1},\vartheta_{3,\nu_1}\}\implies\quad|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases}\frac{3^{2k+3}}{|P|} & \text{if } Q=\langle ss_1^\nu\rangle, P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}\\ & \quad \nu=\nu_1\\ 0 & \text{otherwise}\end{cases}$$

6. Let $\phi=\xi_{\mu_1,\nu_1}$ for $\mu_1=1,..,8,\nu_1=-1,1$.

Suppose $1 \le \mu, \mu_1 \le 4$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \Big\{(\xi_{1,\nu},\xi_{3,\nu_1}),(\xi_{2,\nu},\xi_{4,\nu_1}),$$

$$(\xi_{3,\nu},\xi_{1,\nu_1}),(\xi_{4,\nu},\xi_{2,\nu_1})\Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_2^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \Big\{(\xi_{1,\nu},\xi_{2,\nu_1}),(\xi_{2,\nu},\xi_{1,\nu_1}),$$

$$(\xi_{3,\nu},\xi_{4,\nu_1}),(\xi_{4,\nu},\xi_{3,\nu_1})\Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$\psi = \phi \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ & \text{or } Q = P = \langle ss_1^{\nu}, z\rangle, \nu = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \le \mu, \mu_1 \le 8$. Then, either, $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \Big\{(\xi_{5,\nu},\xi_{7,\nu}),(\xi_{6,\nu_1},\xi_{8,\nu_1}),$$

$$(\xi_{7,\nu},\xi_{5,\nu}),(\xi_{8,\nu_1},\xi_{6,\nu_1})\Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \Big\{(\xi_{5,\nu},\xi_{8,\nu}),(\xi_{6,\nu_1},\xi_{7,\nu_1}),$$

$$(\xi_{8,\nu},\xi_{5,\nu}),(\xi_{7,\nu_1},\xi_{6,\nu_1})\Big\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$\psi = \phi \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = \nu_1 \\ & \text{or } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 3^2 & \text{if } Q = P = \langle ss_1^{\nu}, z\rangle, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $1 \leq \mu \leq 4$ and $5 \leq \mu_1 \leq 8$. Then,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^{\nu} \rangle, P = \langle ss_1^{\nu_1}, z \rangle, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \leq \mu \leq 8$ and $1 \leq \mu_1 \leq 4$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Let $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8, \nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned} &(\psi, \phi) \in \{\xi_{1,\nu}, \xi_{2,\nu}\} \times \{\xi_{1,\nu_1}^{-1}, \xi_{2,\nu_1}^{-1}\} \\ &\quad \sqcup \{\xi_{3,\nu}, \xi_{4,\nu}\} \times \{\xi_{3,\nu_1}^{-1}, \xi_{4,\nu_1}^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s, z \rangle\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\xi_{1,\nu}, \xi_{2,\nu}\} \times \{\theta_{\nu_1} \circ \eta, \theta_{\nu_1}^{-1} \circ \eta\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \\ & \quad \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned} &(\psi, \phi) \in \{\xi_{1,\nu}, \xi_{2,\nu}\} \times \{\hat{\theta}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\} \\ &\sqcup \{\xi_{3,\nu}, \xi_{4,\nu}\} \times \{\hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

10. Let $\phi = \hat{\vartheta}_{\mu_2,\nu_1}$ for $\mu_2 = 1,..,4$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned} &(\psi, \phi) \in \{\xi_{1,\nu}, \xi_{2,\nu}\} \times \{\hat{\vartheta}_{2,\nu_1}, \hat{\vartheta}_{3,\nu_1}\} \\ &\sqcup \{\xi_{3,\nu}, \xi_{4,\nu}\} \times \{\hat{\vartheta}_{1,\nu_1}, \hat{\vartheta}_{4,\nu_1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

B.1.7 $\psi = \xi_{\mu,\nu}^{-1}$ and $Q \in \mathcal{D}_{V_0}$, for $\mu = 1,..,8$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\xi_{3,\nu}^{-1}, \xi_{4,\nu}^{-1}\} \times \{\mathrm{Id}_S, \omega\} \sqcup \{\xi_{1,\nu}^{-1}, \xi_{2,\nu}^{-1}\} \times \{\eta, \omega \circ \eta\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

If $T = V_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\xi_{5,\nu}^{-1}, \xi_{8,\nu}^{-1}\} \times \{\theta_{V_0}, \alpha_{V_0}\} \sqcup \{\xi_{6,\nu}^{-1}, \xi_{7,\nu}^{-1}\} \times \{\theta_{V_0}^{-1}, \beta_{V_0}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{\xi_{5,\nu}^{-1}, \xi_{7,\nu}^{-1}\} \times \{\theta_{V_\lambda}^{-1}, \alpha_{V_\lambda}\} \sqcup \{\xi_{6,\nu}^{-1}, \xi_{8,\nu}^{-1}\} \times \{\theta_{V_\lambda}, \beta_{V_\lambda}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\lambda, z \rangle\}, \nu = \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

If $T = E_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\xi_{3,\nu}^{-1}, \xi_{4,\nu}^{-1}\} \times \{\theta_\lambda, \theta_\lambda^{-1}\} \sqcup \{\xi_{1,\nu}^{-1}, \xi_{2,\nu}^{-1}\} \times \{\alpha_\lambda, \beta_\lambda\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\lambda, z \rangle, \langle t,z \rangle, \langle ss_1^\lambda, t \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4, \nu_1 = -1,0,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{\xi_{5,\nu}^{-1}, \xi_{8,\nu}^{-1}\} \times \{\Theta_{1,0}, \Theta_{3,0}\} \sqcup \{\xi_{6,\nu}^{-1}, \xi_{7,\nu}^{-1}\} \times \{\Theta_{2,0}, \Theta_{4,0}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \nu_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $\phi = \Theta^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4, \nu_1 = -1,0,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\xi^{-1}_{5,\nu}, \xi^{-1}_{7,\nu}\} \times \{\Theta^{-1}_{1,\nu_1}, \Theta^{-1}_{3,\nu_1}\}$$
$$\sqcup \{\xi^{-1}_{6,\nu}, \xi^{-1}_{8,\nu}\} \times \{\Theta^{-1}_{2,\nu_1}, \Theta^{-1}_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \\ & \quad \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4, \nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\xi^{-1}_{5,\nu}, \xi^{-1}_{7,\nu}\} \times \{\vartheta_{2,\nu_1}, \vartheta_{3,\nu_1}\}$$
$$\sqcup \{\xi^{-1}_{6,\nu}, \xi^{-1}_{8,\nu}\} \times \{\vartheta_{1,\nu_1}, \vartheta_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \nu = -\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8, \nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\xi^{-1}_{1,\nu}, \xi^{-1}_{2,\nu}\} \times \{\xi_{1,\nu_1}, \xi_{2,\nu_1}\}$$
$$\sqcup \{\xi^{-1}_{3,\nu}, \xi^{-1}_{4,\nu}\} \times \{\xi_{3,\nu_1}, \xi_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise.} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8, \nu_1 = -1,1$.

Suppose $1 \leq \mu, \mu_1 \leq 4$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \Big\{(\xi_{1,\nu}^{-1},\xi_{3,\nu_1}^{-1}),(\xi_{2,\nu}^{-1},\xi_{4,\nu_1}^{-1}),$$

$$(\xi_{3,\nu}^{-1},\xi_{1,\nu_1}^{-1}),(\xi_{4,\nu}^{-1},\xi_{2,\nu_1}^{-1})\Big\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s,z\rangle\}, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \Big\{(\xi_{1,\nu}^{-1},\xi_{2,\nu_1}^{-1}),(\xi_{2,\nu}^{-1},\xi_{1,\nu_1}^{-1}),$$

$$(\xi_{3,\nu}^{-1},\xi_{4,\nu_1}^{-1}),(\xi_{4,\nu}^{-1},\xi_{3,\nu_1}^{-1})\Big\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$\psi = \phi \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s,z\rangle\}, \nu = \nu_1 \\ & \text{or } Q = P = \langle s,z\rangle, \nu = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \leq \mu, \mu_1 \leq 8$. Then, either, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \Big\{(\xi_{5,\nu}^{-1},\xi_{8,\nu_1}^{-1}),(\xi_{6,\nu}^{-1},\xi_{7,\nu_1}^{-1}),$$

$$(\xi_{8,\nu}^{-1},\xi_{5,\nu_1}^{-1}),(\xi_{7,\nu}^{-1},\xi_{6,\nu_1}^{-1})\Big\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s,z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \Big\{(\xi_{5,\nu}^{-1},\xi_{7,\nu_1}^{-1}),(\xi_{6,\nu}^{-1},\xi_{8,\nu_1}^{-1}),$$

$$(\xi_{7,\nu}^{-1},\xi_{5,\nu_1}^{-1}),(\xi_{8,\nu}^{-1},\xi_{6,\nu_1}^{-1})\Big\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi = \phi \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s,z\rangle\} \\ & \text{or } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \nu = \nu_1 \\ 3^2 & \text{if } Q = P = \langle s,z\rangle, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $1 \leq \mu \leq 4$ and $5 \leq \mu_1 \leq 8$. Then,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle s\rangle, P = \langle s,z\rangle, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \leq \mu \leq 8$ and $1 \leq \mu_1 \leq 4$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\xi_{1,\nu}^{-1}, \xi_{2,\nu}^{-1}\} \times \{\theta_{\nu_1} \circ \eta, \theta_{\nu_1}^{-1} \circ \eta\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \\ & \quad \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned} &(\psi, \phi) \in \{\xi_{1,\nu}^{-1}, \xi_{2,\nu}^{-1}\} \times \{\hat{\theta}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\} \\ &\sqcup \{\xi_{3,\nu}^{-1}, \xi_{4,\nu}^{-1}\} \times \{\hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1, \nu_1}$ for $\mu_2 = 1, .., 4$ and $\nu_1 = -1, 1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned} &(\psi, \phi) \in \{\xi_{1,\nu}^{-1}, \xi_{2,\nu}^{-1}\} \times \{\hat{\vartheta}_{2,\nu_1}, \hat{\vartheta}_{3,\nu_1}\} \\ &\sqcup \{\xi_{3,\nu}^{-1}, \xi_{4,\nu}^{-1}\} \times \{\hat{\vartheta}_{1,\nu_1}, \hat{\vartheta}_{4,\nu_1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

B.1.8 $\psi = \theta_{\nu}^{\pm 1} \circ \eta$ and $Q \in \mathcal{D}_{\nu}$, for $\nu = -1, 1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi, \phi) \in \{\theta_\nu \circ \eta, \theta_\nu^{-1} \circ \eta\} \times \{\eta, \omega \circ \eta\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

If $T = V_\lambda$ for $\lambda = -1, 0, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

If $T = E_\lambda$ then

$$
\begin{aligned}
&(\psi,\phi) \in \{(\theta_\nu \circ \eta, \beta_\lambda), (\theta_\nu^{-1} \circ \eta, \alpha_\lambda)\} \\
\implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = &\begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = \nu \neq 0 \\ & \text{or } Q \in \{\langle t\rangle, \langle t, z\rangle\}, P \in \{\langle t, z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = -\nu \neq 0 \\ 3^{2k+1} & \text{if } Q = P = \langle ss_1^\lambda\rangle, \lambda = \nu \neq 0 \\ & \text{or } Q = P = \langle t\rangle, \lambda = -\nu \neq 0 \\ 0 & \text{otherwise} \end{cases} \\
&(\psi,\phi) \in \{(\theta_\nu^{-1} \circ \eta, \theta_\lambda), (\theta_\nu \circ \eta, \theta_\lambda^{-1})\} \\
\implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = &\begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = \nu \neq 0 \\ 3^{2k+1} & \text{if } Q = P = \langle ss_1^\lambda\rangle, \lambda = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases} \\
&(\psi,\phi) \in \{(\theta_\nu \circ \eta, \theta_\lambda), (\theta_\nu^{-1} \circ \eta, \theta_\lambda^{-1})\} \\
\implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = &\begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle t\rangle, P \in \{\langle t, z\rangle, \langle ss_1^\nu, t\rangle\}, \lambda = -\nu \neq 0 \\ 3^{2k+1} & \text{if } Q = P = \langle t\rangle, \lambda = -\nu \neq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1.,,4$ and $\nu_1 = -1,0,1$. Then

$$
|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^{2k} & \text{if } Q = \langle ss_1^\nu\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \nu = \nu_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}
$$

4. Let $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1.,,4$ and $\nu_1 = -1,0,1$. Then

$$
|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^{2k} & \text{if } Q = \langle t\rangle, P\langle t, z\rangle, \nu = -\nu_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}
$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_\nu \circ \eta, \theta_\nu^{-1} \circ \eta\} \times \{\xi_{1,\nu_1}, \xi_{2,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, P \in \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherewise} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_\nu \circ \eta, \theta_\nu^{-1} \circ \eta\} \times \{\xi^{-1}_{1,\nu_1}, \xi^{-1}_{2,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, P \in \langle s, z\rangle\} \\ 0 & \text{otherewise} \end{cases}$$

8. Let $\phi = \theta^{\pm 1}_{\nu_1} \circ \eta$ for $\nu_1 = -1,1$. Then

$$\begin{aligned} \{\psi,\phi\} \in &\{(\theta_\nu \circ \eta, \theta_{\nu_1}^{-1} \circ \eta), (\theta_\nu^{-1} \circ \eta, \theta_{\nu_1} \circ \eta)\} \\ &\implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t,z\rangle, \\ & \qquad \langle ss_1^{\nu_1}, z\rangle, \langle ss_1^{\nu_1}, t\rangle\}, \nu = \nu_1 \neq 0 \\ 0 & \text{otherwise} \end{cases} \\ \{\psi,\phi\} \in &\{(\theta_\nu \circ \eta, \theta_{\nu_1} \circ \eta), (\theta_\nu^{-1} \circ \eta, \theta_{\nu_1}^{-1} \circ \eta)\} \\ &\implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^{\nu}\rangle, \langle ss_1^{\nu}, z\rangle\} \times \{\langle ss_1^{\nu_1}, z\rangle, \langle ss_1^{\nu_1}, t\rangle\}, \nu = \nu_1 \neq 0 \\ & \text{or } (Q,P) \in \{\langle t\rangle, \langle t,z\rangle\} \times \{\langle t,z\rangle, \langle ss_1^{\nu_1}, t\rangle\}, \nu = \nu_1 \neq 0 \\ 3^{2k+1} & \text{if } Q = P \in \{\langle ss_1^{\nu}, t\rangle, \langle t\rangle\}, \nu = \nu_1 \neq 0 \\ \frac{3^{4k+2}}{|P|} & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t,z\rangle, \langle ss_1^{\nu_1}, z\rangle, \langle ss_1^{\nu_1}, t\rangle\}, \nu = \nu_1 \neq 0 \\ 3 & \text{if } Q = P = \langle ss_1^{\nu}, t\rangle, \nu = \nu_1 \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}^{-1}_{V_{\nu_1}}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or if,

$$(\psi,\phi) \in \{\theta_\nu \circ \eta, \theta_\nu^{-1} \circ \eta\} \times \{\hat{\theta}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\theta_\nu \circ \eta, \theta_\nu^{-1} \circ \eta\} \times \{\hat{\vartheta}_{2,\nu_1}, \hat{\vartheta}_{3,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

B.1.9 $\psi \in \{\hat{\theta}_{V_\nu}, \hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}, \hat{\beta}_{V_\nu}\}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\nu = -1, 1$

1. Let $\phi \in \{\mathsf{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{(\hat{\theta}_{V_\nu}, \mathsf{Id}_S), (\hat{\beta}_{V_\nu}, \omega)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\nu \rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\hat{\alpha}_{V_\nu}, \mathsf{Id}_S), (\hat{\theta}_{V_\nu}^{-1}, \omega)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q \in \{\langle ss_1^\nu \rangle, \langle ss_1^\nu, z \rangle\}, Q \leq P \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &(\psi,\phi) \in \{\hat{\theta}_{V_\nu}\hat{\beta}_{V_\nu}\} \times \{\eta, \omega \circ \eta\} \\ &\sqcup \{(\hat{\theta}_{V_\nu}^{-1}, \mathsf{Id}_S), (\hat{\alpha}_{V_\nu}, \omega)\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   If $T = V_\lambda$ for $\lambda = -1, 0, 1$, then

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\nu \rangle, P = \langle ss_1^\lambda, z \rangle, \lambda = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

   If $T = E_\lambda$ for $\lambda = -1, 0, 1$, then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$\begin{aligned} &(\psi,\phi) \in \{\hat{\theta}_{V_\nu}, \hat{\beta}_{V_\nu}\} \times \{\alpha_\lambda, \beta_\lambda\} \\ &\sqcup \{\hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}\} \times \{\theta_\lambda, \theta_\lambda^{-1}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\nu \rangle, P = \langle ss_1^\lambda, t \rangle, \nu = \lambda \neq 0 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^\lambda, z \rangle, \langle ss_1^\lambda, t \rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} &(\psi,\phi) \in \{\hat{\theta}_{V_\nu}, \hat{\beta}_{V_\nu}\} \times \{\theta_\lambda, \theta_\lambda^{-1}\} \\ &\sqcup \{\hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}\} \times \{\alpha_\lambda, \beta_\lambda\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\nu \rangle, P = \langle ss_1^\lambda, t \rangle, \nu = \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

   If $T = A$, then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 0, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

4. Let $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 0, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 1$. Then $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu = 1,..,8$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\hat{\theta}_{V_\nu}, \hat{\beta}_{V_\nu}\} \times \{\xi_{1,\nu_1}, \xi_{2,\nu_1}\}$$
$$\sqcup \{\hat{\theta}^{-1}_{V_\nu}, \hat{\alpha}_{V_\nu}\} \times \{\xi_{3,\nu_1}, \xi_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\hat{\theta}_{V_\nu}, \hat{\beta}_{V_\nu}\} \times \{\xi^{-1}_{1,\nu_1}, \xi^{-1}_{2,\nu_1}\}$$
$$\sqcup \{\hat{\theta}^{-1}_{V_\nu}, \hat{\alpha}_{V_\nu}\} \times \{\xi^{-1}_{3,\nu_1}, \xi^{-1}_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

8. Let $\phi = \theta^{\pm 1}_{\nu_1} \circ \eta$ for $\nu_1 = -1,1$. Then

$$(\psi,\phi) \in \{\hat{\theta}^{-1}_{V_\nu}, \hat{\beta}_{V_\nu}\} \times \{\theta_{\nu_1} \circ \eta, \theta^{-1}_{\nu_1} \circ \eta\}$$
$$\implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}^{-1}_{V_{\nu_1}}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{(\hat{\theta}_{V_\nu}, \hat{\alpha}_{V_{\nu_1}}), (\hat{\alpha}_{V_\nu}, \hat{\theta}_{V_{\nu_1}}),$$
$$(\hat{\theta}^{-1}_{V_\nu}, \hat{\beta}_{V_{\nu_1}}), (\hat{\beta}_{V_\nu}, \hat{\theta}^{-1}_{V_{\nu_1}})\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^{\nu} \rangle, P \in \{\langle ss_1^{\nu_1} \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\hat{\theta}_{V_\nu}, \hat{\beta}_{V_{\nu_1}}), (\hat{\beta}_{V_\nu}, \hat{\theta}_{V_{\nu_1}}),$$
$$(\hat{\theta}^{-1}_{V_\nu}, \hat{\alpha}_{V_{\nu_1}}), (\hat{\alpha}_{V_\nu}, \hat{\theta}^{-1}_{V_{\nu_1}})\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\hat{\theta}_{V_\nu}, \hat{\theta}_{V_{\nu_1}}), (\hat{\theta}^{-1}_{V_\nu}, \hat{\theta}^{-1}_{V_{\nu_1}}),$$
$$(\hat{\alpha}_{V_\nu}, \hat{\alpha}_{V_{\nu_1}}), (\hat{\beta}_{V_\nu}, \hat{\beta}_{V_{\nu_1}})\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^3 & \text{if } Q = P = \langle ss_1^{\nu} \rangle, \nu = \nu_1 \\ & \text{or } Q \in \{\langle ss_1^{\nu} \rangle, \langle ss_1^{\nu}, z \rangle\}, P = \langle ss_1^{\nu_1}, z \rangle, \lambda = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$\begin{aligned} &(\psi,\phi) \in \{\hat{\theta}_{V_\nu}, \hat{\beta}_{V_\nu}\} \times \{\hat{\vartheta}_{2,\nu_1}, \hat{\vartheta}_{3,\nu_1}\} \\ \sqcup &\{\hat{\theta}^{-1}_{V_\nu}, \hat{\alpha}_{V_\nu}\} \times \{\hat{\vartheta}_{1,\nu_1}, \hat{\vartheta}_{4,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

B.1.10 $\psi = \hat{\vartheta}_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..,4$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$ and $P \in \mathcal{D}$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{(\hat{\vartheta}_{1,V_\nu}, \eta), (\hat{\vartheta}_{4,V_\nu}, \omega\circ\eta)\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\nu\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\hat{\vartheta}_{3,V_\nu}, \eta), (\hat{\vartheta}_{2,V_\nu}, \omega\circ\eta)\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^3 & \text{if } Q \in \{\langle ss_1^\nu\rangle, \langle ss_1^\nu, z\rangle\}, Q \le P \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &(\psi,\phi) \in \{\hat{\vartheta}_{1,V_\nu} \hat{\vartheta}_{4,V_\nu}\} \times \{\mathrm{Id}_S, \omega\} \\ \sqcup &\{(\hat{\vartheta}_{2,V_\nu}, \eta), (\hat{\vartheta}_{3,V_\nu}, \omega\circ\eta)\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   If $T = V_\lambda$, for $\lambda = -1,0,1$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

   If $T = E_\lambda$ for $\lambda = -1,0,1$, then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$\begin{aligned} &(\psi,\phi) \in \{\hat{\vartheta}_{1,\nu}, \hat{\vartheta}_{4,\nu}\} \times \{\theta_\lambda, \theta_\lambda^{-1}\} \\ \sqcup &\{\hat{\vartheta}_{2,\nu}, \hat{\vartheta}_{3,\nu}\} \times \{\alpha_\lambda, \beta_\lambda\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

   If $T = A$, then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

3. Let $\phi = \theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

4. Let $\phi = \theta^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$. Then $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$.

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^{\nu}\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \nu = \nu_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\hat{\vartheta}_{1,\nu}, \hat{\vartheta}_{4,\nu}\} \times \{\xi_{3,\nu_1}, \xi_{4,\nu_1}\}$$
$$\sqcup \{\hat{\vartheta}_{2,\nu}, \hat{\vartheta}_{3,\nu}\} \times \{\xi_{1,\nu_1}, \xi_{2,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\hat{\vartheta}_{1,\nu}, \hat{\vartheta}_{4,\nu}\} \times \{\xi^{-1}_{3,\nu_1}, \xi^{-1}_{4,\nu_1}\}$$
$$\sqcup \{\hat{\vartheta}_{2,\nu}, \hat{\vartheta}_{3,\nu}\} \times \{\xi^{-1}_{1,\nu_1}, \xi^{-1}_{2,\nu_1}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

8. Let $\phi = \theta^{\pm 1}_{\nu_1} \circ \eta$ for $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\hat{\vartheta}_{2,\nu}, \hat{\vartheta}_{3,\nu}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^{\nu_1}, z\rangle, \langle ss_1^{\nu_1}, t\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}^{-1}_{V_{\nu_1}}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1,1$. Then, either $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$, or, if

$$(\psi,\phi) \in \{\hat{\vartheta}_{1,\nu}, \hat{\vartheta}_{4,\nu}\} \times \{\hat{\theta}^{-1}_{V_{\nu_1}}, \hat{\alpha}_{V_{\nu_1}}\}$$
$$\sqcup \{\hat{\vartheta}_{2,\nu}, \hat{\vartheta}_{3,\nu}\} \times \{\hat{\theta}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\} \implies |\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$. Then, either $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$, or, if

$$(\psi,\phi) \in \{(\hat{\vartheta}_{1,\nu},\hat{\vartheta}_{3,\nu_1}),(\hat{\vartheta}_{3,\nu},\hat{\vartheta}_{1,\nu_1}),$$
$$(\hat{\vartheta}_{2,\nu},\hat{\vartheta}_{4,\nu_1}),(\hat{\vartheta}_{4,\nu},\hat{\vartheta}_{2,\nu_1})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\hat{\vartheta}_{1,\nu},\hat{\vartheta}_{4,\nu_1}),(\hat{\vartheta}_{4,\nu},\hat{\vartheta}_{1,\nu_1}),$$
$$(\hat{\vartheta}_{2,\nu},\hat{\vartheta}_{3,\nu_1}),(\hat{\vartheta}_{3,\nu},\hat{\vartheta}_{2,\nu_1})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\psi,\phi) \in \{(\hat{\vartheta}_{1,\nu},\hat{\vartheta}_{1,\nu_1}),(\hat{\vartheta}_{2,\nu},\hat{\vartheta}_{2,\nu_1}),$$
$$(\hat{\vartheta}_{3,\nu},\hat{\vartheta}_{3,\nu_1}),(\hat{\vartheta}_{4,\nu},\hat{\vartheta}_{4,\nu_1})\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = P = \langle ss_1^{\nu}\rangle, \nu = \nu_1 \\ & \text{or } Q \in \{\langle ss_1^{\nu}\rangle, \langle ss_1^{\nu}, z\rangle\}, P = \langle ss_1^{\nu_1}, z\rangle, \nu = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

B.2 Proofs

Before we prove the results in Section $B.1$, we state, and prove a few lemmas. We first start with a series of definitions.

For any $T \in \mathcal{T}_{full}$, we write $\mathcal{M}_T = \{\mathsf{Id}_S, \omega, \eta, \omega \circ \eta\} \times \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$, and for any $Q \leq S$ define

$$\mathcal{M}_T(Q) = \{(\psi, \phi) \in \mathcal{M}_T \mid \phi|_Q \circ \psi = c_g \text{ for some } g \in S\}.$$

We also write $\mathcal{M}_{\hat{V}_\lambda} = \{\mathsf{Id}|_S, \omega, \eta, \omega \circ \eta\} \times \{\theta_{\hat{V}_\lambda}, \theta_{\hat{V}_\lambda}^{-1}, \alpha_{\hat{V}_\lambda}, \beta_{\hat{V}_\lambda}\}$, and for any $Q \leq \langle ss_1^\lambda, z\rangle$, define

$$\mathcal{M}_{\hat{V}_\lambda}(Q) = \{(\psi, \phi) \in \mathcal{M}_{\hat{V}_\lambda} \mid \phi|_Q \circ \psi = c_g|_Q \text{ for some } g \in S\}.$$

We also write $\mathcal{M}_{\vartheta_\lambda} = \{\mathsf{Id}|_S, \omega, \eta, \omega \circ \eta\} \times \{\vartheta_{1,\lambda}, \vartheta_{2,\lambda}, \vartheta_{3,\lambda}, \vartheta_{4,\lambda}\}$, and for any $Q \leq \langle ss_1^\lambda, z\rangle$, define

$$\mathcal{M}_{\vartheta_\lambda}(Q) = \{(\psi, \phi) \in \mathcal{M}_{\vartheta_\lambda} \mid \phi|_Q \circ \psi = c_g|_Q \text{ for some } g \in S\}.$$

We also write $\mathcal{M}_{\xi_\lambda} = \{\mathsf{Id}|_S, \omega, \eta, \omega \circ \eta\} \times \{\xi_{1,\lambda}, \xi_{2,\lambda}, \xi_{3,\lambda}, \xi_{4,\lambda}\}$, and for any $Q \leq \langle ss_1^\lambda, z\rangle$, define

$$\mathcal{M}_{\xi_\lambda}(Q) = \{(\psi, \phi) \in \mathcal{M}_{\xi_\lambda} \mid \phi|_Q \circ \psi = c_g|_Q \text{ for some } g \in S\}.$$

Finally, we also write $\mathcal{M}_{\xi_\lambda^{-1}} = \{\mathsf{Id}|_S, \omega, \eta, \omega \circ \eta\} \times \{\xi_{1,\lambda}^{-1}, \xi_{2,\lambda}^{-1}, \xi_{3,\lambda}^{-1}, \xi_{4,\lambda}^{-1}\}$, and for $Q \leq \langle s, z\rangle$, we define

$$\mathcal{M}_{\xi_\lambda^{-1}}(Q) = \{(\psi, \phi) \in \mathcal{M}_{\xi_\lambda^{-1}} \mid \phi|_Q \circ \psi = c_g|_Q \text{ for some } g \in S\}.$$

We use these definitions in a number of proofs.

Lemma B.2.1. *Let $\psi \in \{Id|_S, \omega, \eta, \omega \circ \eta\}$ and $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ for some $T \in \mathcal{T}_{full}$. Then $\phi|_Q \circ \psi = c_g|_Q$ for some $1 \neq Q \leq S$ and $g \in S$ if and only if $Q \leq P$ and $g \in C$, where $(P; C)$ is the pair listed in the table corresponding to maps (ϕ, ψ).*

$\underline{T = V_\lambda}$:
When $\underline{\lambda = 0}$:

ϕ \ ψ	$Id\|_S$	ω	η	$\omega\circ\eta$
θ_{V_0}	$\langle sz^{-1}\rangle;\langle s,z\rangle t$	$\langle sz\rangle;\langle s,z\rangle$	$\langle sz^{-1}\rangle;\langle s,z\rangle$	$\langle sz\rangle;\langle s,z\rangle t^{-1}$
$\theta_{V_0}^{-1}$	$\langle sz\rangle;\langle s,z\rangle t^{-1}$	$\langle sz^{-1}\rangle;\langle s,z\rangle$	$\langle sz\rangle;\langle s,z\rangle$	$\langle sz^{-1}\rangle;\langle s,z\rangle t$
α_{V_0}	$\langle sz\rangle;\langle s,z\rangle$	$\langle sz^{-1}\rangle;\langle s,z\rangle t$	$\langle sz\rangle;\langle s,z\rangle t^{-1}$	$\langle sz^{-1}\rangle;\langle s,z\rangle$
β_{V_0}	$\langle sz^{-1}\rangle;\langle s,z\rangle$	$\langle sz\rangle;\langle s,z\rangle t^{-1}$	$\langle sz^{-1}\rangle;\langle s,z\rangle t$	$\langle sz\rangle;\langle s,z\rangle$

When $\underline{\lambda=-1,1}$:

ϕ \ ψ	$Id\|_S$	ω	η	$\omega\circ\eta$
θ_{V_λ}	$\langle ss_1^\lambda z^{-1}\rangle;\langle ss_1^\lambda,z\rangle t$	$\langle ss_1^\lambda z\rangle;\langle ss_1^\lambda,z\rangle$	$\emptyset$	$\emptyset$
$\theta_{V_\lambda}^{-1}$	$\langle ss_1^\lambda z\rangle;\langle ss_1^\lambda,z\rangle t^{-1}$	$\langle ss_1^\lambda z^{-1}\rangle;\langle ss_1^\lambda,z\rangle$	$\emptyset$	$\emptyset$
α_{V_λ}	$\langle ss_1^\lambda z\rangle;\langle ss_1^\lambda,z\rangle$	$\langle ss_1^\lambda z^{-1}\rangle;\langle ss_1^\lambda,z\rangle t$	$\emptyset$	$\emptyset$
β_{V_λ}	$\langle ss_1^\lambda z^{-1}\rangle;\langle ss_1^\lambda,z\rangle$	$\langle ss_1^\lambda z\rangle;\langle ss_1^\lambda,z\rangle t^{-1}$	$\emptyset$	$\emptyset$

$\underline{T=E_\lambda}$:
When $\underline{\lambda=0}$. *Let* $u=s_2^{3^{k-2}}$. *In this case, for all* $\ell\in\{-1,0,1\}$, *we have:*

ϕ \ ψ	$Id\|_S$	ω	η	$\omega\circ\eta$
θ_0	$\langle st^{-1},z\rangle;\langle s,t\rangle u^{-1}t$ *or* $\langle z\rangle;S$	$\langle st,z\rangle;\langle s,t\rangle ut$ *or* $\langle z\rangle;S$	$\langle st^{-1}z^\ell\rangle;\langle st^{-1},z\rangle t^{1-\ell}$	$\langle stz^\ell\rangle;\langle st,z\rangle t^{-\ell}$
θ_0^{-1}	$\langle st,z\rangle;\langle s,t\rangle ut^{-1}$ *or* $\langle z\rangle;S$	$\langle st^{-1},z\rangle;\langle s,t\rangle u^{-1}t^{-1}$ *or* $\langle z\rangle;S$	$\langle stz^\ell\rangle;\langle st,z\rangle t^{-1-\ell}$	$\langle st^{-1}z^\ell\rangle;\langle st^{-1},z\rangle t^{-\ell}$
α_0	$\langle stz^\ell\rangle;\langle st,z\rangle t^{-1-\ell}$	$\langle st^{-1}z^\ell\rangle;\langle st^{-1},z\rangle t^{-\ell}$	$\langle st,z\rangle;\langle s,t\rangle ut^{-1}$ *or* $\langle z\rangle;S$	$\langle st^{-1},z\rangle;\langle s,t\rangle u^{-1}t^{-1}$ *or* $\langle z\rangle;S$
β_0	$\langle st^{-1}z^\ell\rangle;\langle st^{-1},z\rangle t^{-1-\ell}$	$\langle stz^\ell\rangle;\langle st,z\rangle t^{-\ell}$	$\langle st^{-1},z\rangle;\langle s,t\rangle u^{-1}t$ *or* $\langle z\rangle;S$	$\langle st,z\rangle;\langle s,t\rangle u$ *or* $\langle z\rangle;S$

When $\underline{\lambda=-1,1}$: *As before, let* $u=s_2^{3^{k-2}}$ *and* $\ell\in\{-1,0,1\}$. *Then:*

$\phi \backslash \psi$	$Id\|_S$	ω	η	$\omega\circ\eta$
θ_λ	$\langle ss_1^\lambda t^{-1}, z\rangle; \langle ss_1^\lambda, t\rangle u^{-1}t$ *or* $\langle z\rangle; S$	$\langle ss_1^\lambda t, z\rangle; \langle ss_1^\lambda, t\rangle ut$ *or* $\langle z\rangle; S$	$\emptyset$	$\emptyset$
θ_λ^{-1}	$\langle ss_1^\lambda t, z\rangle; \langle ss_1^\lambda, t\rangle ut^{-1}$ *or* $\langle z\rangle; S$	$\langle ss_1^\lambda t^{-1}, z\rangle; \langle ss_1^\lambda, t\rangle u^{-1}t^{-1}$ *or* $\langle z\rangle; S$	$\emptyset$	$\emptyset$
α_λ	$\langle ss_1^\lambda tz^\ell\rangle; \langle ss_1^\lambda t, z\rangle t^{-1-\ell}$	$\langle ss_1^\lambda t^{-1}z^\ell\rangle; \langle ss_1^\lambda t^{-1}, z\rangle t^{-\ell}$	$\langle z\rangle; S$	$\langle z\rangle; S$
β_λ	$\langle ss_1^\lambda t^{-1}z^\ell\rangle; \langle ss_1^\lambda t^{-1}, z\rangle t^{-1-\ell}$	$\langle ss_1^\lambda tz^\ell\rangle; \langle ss_1^\lambda t, z\rangle t^{-\ell}$	$\langle z\rangle; S$	$\langle z\rangle; S$

$\underline{T = A}$:

$\phi \backslash \psi$	$Id\|_S$	ω	η	$\omega\circ\eta$
θ_A	$\langle tz^{-1}\rangle; s^{-1}A$	$\langle d_2\rangle; s^{-1}A$	$\langle tz\rangle; sA$	$\langle d_1\rangle; s^{-1}A$
θ_A^{-1}	$\langle tz\rangle; sA$	$\langle d_1\rangle; s^{-1}A$	$\langle tz^{-1}\rangle; s^{-1}A$	$\langle d_2\rangle; s^{-1}A$
α_A	$\langle d_2\rangle; A$	$\langle tz^{-1}\rangle; sA$	$\langle d_1\rangle; A$	$\langle tz\rangle; A$
β_A	$\langle d_1\rangle; A$	$\langle tz\rangle; A$	$\langle d_2\rangle; A$	$\langle tz^{-1}\rangle; sA$

where $d_1 = s_1^{1+a_k}s_2^2$, $d_2 = s_1^{5-a_k}s_2^2$ *and where* a_k *is defined recursively as follows:*

$$a_1 \equiv 0 \ \mathit{mod}\ 3 \ \mathit{and}\ a_k \equiv -(a_{k-1}^2 - 3a_{k-1} + 3) \ \mathit{mod}\ 3^k.$$

Proof. We prove the lemma case by case.

Case 1: $T = V_\lambda$.
Let $Q \leq S$ and $g \in S$. We have $(ss_1^\lambda)\phi \circ \psi = z^{\pm 1}$ and $z\phi \circ \psi = (ss_1^\lambda)^{\pm 1}$. However, $(ss_1^\lambda)^{\pm 1}$ is not S-conjugate to $z^{\pm 1}$. It follows that $ss_1^\lambda, z \notin Q$. Thus, $Q = \langle (ss_1^\lambda)^i z^j\rangle$ for some $i, j \in \{-1, 1\}$. Since $((ss_1^\lambda)^i z^j)^2 = (ss_1^\lambda)^{-i}z^{-j}$, we may write $Q = \langle ss_1^\lambda z^j\rangle$. Define

$$\delta(\psi) = \begin{cases} 1 & \text{if } (ss_1^\lambda)\psi \in sA \\ -1 & \text{if } (ss_1^\lambda)\psi \in s^{-1}A \end{cases} \quad \text{and} \quad \delta(\phi) = \begin{cases} 1 & \text{if } z\phi = ss_1^\lambda \\ -1 & \text{if } z\phi = (ss_1^\lambda)^{-1} \end{cases}$$

Note that

$$(ss_1^\lambda z^j)\phi \circ \psi \in \begin{cases} s^j A & \text{if } \delta(\phi)\delta(\psi) = 1 \\ s^{-j}A & \text{if } \delta(\phi)\delta(\psi) = -1 \end{cases}$$

Since no element of sA is S-conjugate to an element of $s^{-1}A$, we have

$$j = \begin{cases} 1 & \text{if } \delta(\phi)\delta(\psi) = 1 \\ -1 & \text{if } \delta(\phi)\delta(\psi) = -1 \end{cases}$$

That is,

$$Q \leq \begin{cases} \langle ss_1^{\lambda} z \rangle & \text{if } \delta(\phi)\delta(\psi) = 1 \\ \langle ss_1^{\lambda} z^{-1} \rangle & \text{if } \delta(\phi)\delta(\psi) = -1. \end{cases}$$

Note that, if $\lambda \neq 0$, and $\psi \in \{\eta, \omega \circ \eta\}$, then $(ss_1^{\lambda} z^j)\phi \circ \psi \in V_{-\lambda} \cap (S \backslash A)$. However, no element of $V_{\lambda} \cap (S \backslash A)$ is S-conjugate to an element of $V_{-\lambda} \cap (S \backslash A)$. Thus, in that case, there is no such subgroup Q. After determining the values of $\delta(\phi)$ and $\delta(\psi)$ for all maps ϕ and ψ we can fill in the values of P in the table. Choose a P from the table and write $N = N_S(P, P\phi \circ \psi)$. Note that, for all P, $N_S(P) = \langle s, z \rangle$. In order to determine C, it suffices to find one element $g \in N$, since $N = N_S(P)g = \langle s, z \rangle g$, and we are done.

Case 2: $T = E_{\lambda}$.
Let $Q \leq S$ and $g \in S$. We have $(ss_1^{\lambda})\phi \circ \psi \in t^{\pm 1}\langle z \rangle$ and $t\phi \circ \psi \in s^{\pm 1}A$. However, no element of A is S-conjugate to an element lying outside of A. It follows that $ss_1^{\lambda}, t \notin Q$. Thus $Q \leq \langle (ss_1^{\lambda})^i t^j, z \rangle$ for some $i, j \in \{-1, 1\}$. Since

$$((ss_1^{\lambda})^i t^j)^2 = (ss_1^{\lambda})^{-i} (t^{(ss_1^{\lambda})^i})^j t^j \in (ss_1^{\lambda})^{-i} (t\langle z \rangle)^j t^j = (ss_1^{\lambda})^{-i} t^{-j} \langle z \rangle,$$

it suffices to write $Q \leq \langle ss_1^{\lambda} t^j, z \rangle$. Define

$$\delta_1(\psi) = \begin{cases} 1 & \text{if } (ss_1^{\lambda})\psi \in sA \\ -1 & \text{if } (ss_1^{\lambda})\psi \in s^{-1}A \end{cases} \qquad \delta_1(\phi) = \begin{cases} 1 & \text{if } t\phi = ss_1^{\lambda} \\ -1 & \text{if } t\phi = (ss_1^{\lambda})^{-1} \end{cases}$$

and

$$\delta_2(\psi) = \begin{cases} 1 & \text{if } z\psi = z \\ -1 & \text{if } z\psi = z^{-1} \end{cases} \qquad \delta_2(\phi) = \begin{cases} 1 & \text{if } z\phi = z \\ -1 & \text{if } z\phi = z^{-1} \end{cases}$$

Then

$$(ss_1^{\lambda} t^j)\phi \circ \psi \in \begin{cases} s^j A & \text{if } \delta_1(\phi)\delta_1(\psi) = 1 \\ s^{-j} A & \text{if } \delta_1(\phi)\delta_1(\psi) = -1 \end{cases} \quad \text{and} \quad z\phi \circ \psi = \begin{cases} z & \text{if } \delta_2(\phi)\delta_2(\psi) = 1 \\ z^{-1} & \text{if } \delta_2(\phi)\delta_2(\psi) = -1 \end{cases}$$

Since no element of sA is S-conjugate to an element of $s^{-1}A$, we have

$$j = \begin{cases} 1 & \text{if } \delta_1(\phi)\delta_1(\psi) = 1 \\ -1 & \text{if } \delta_1(\phi)\delta_1(\psi) = -1 \end{cases}$$

Similarly, z is not S-conjugate to z^{-1}. We conclude,

$$Q \leq \begin{cases} \langle ss_1^{\lambda}t, z\rangle & \text{if } \delta_1(\phi)\delta_1(\psi) = \delta_2(\phi)\delta_2(\psi) = 1 \\ \langle ss_1^{\lambda}t^{-1}, z\rangle & \text{if } -\delta_1(\phi)\delta_1(\psi) = \delta_2(\phi)\delta_2(\psi) = 1 \\ \langle ss_1^{\lambda}tz^{\ell}\rangle, \ell \in \{-1,0,1\} & \text{if } \delta_1(\phi)\delta_1(\psi) = -\delta_2(\phi)\delta_2(\psi) = 1 \\ \langle ss_1^{\lambda}t^{-1}z^{\ell}\rangle, \ell \in \{-1,0,1\} & \text{if } -\delta_1(\phi)\delta_1(\psi) = -\delta_2(\phi)\delta_2(\psi) = 1. \end{cases}$$

Note also that, if $\lambda \neq 0$ and $\psi \in \{\eta, \omega \circ \eta\}$, then $(ss_1^{\lambda}t^jz^{\ell})\phi \circ \psi \in E_{-\lambda} \cap (S\backslash A)$. However, by 5.2.3 and 5.2.4, no element of $E_{\lambda} \cap (S\backslash A)$ is S-conjugate to an element in $E_{-\lambda} \cap (S\backslash A)$. Thus, in that case there is no such subgroup Q, other than $\langle z\rangle$. After determining the values of $\delta_1(\phi), \delta_1(\psi), \delta_2(\phi)$ and $\delta_2(\psi)$ for all maps ϕ and ψ we can fill in the values for P in the table as required. Finally, choose a P from the table and write $N = N_S(P, P\phi \circ \psi)$. In order to determine C, it suffices to find $N_S(P)$ and one element $g \in N$.

<u>Case 3</u>: $T = A$.
For this case we will need to recall definitions and results from Section 6.2. Note that for $n = 0, 1$, we have

$$\begin{aligned} \theta_A \circ \omega^n \circ \eta &= \theta_A^{-1} \circ \omega^n \\ \theta_A^{-1} \circ \omega^n \circ \eta &= \theta_A \circ \omega^n \\ \alpha_A \circ \omega^n \circ \eta &= \beta_A \circ \omega^n \\ \beta_A \circ \omega^n \circ \eta &= \alpha_A \circ \omega^n. \end{aligned}$$

Thus, for this case, it suffices to determine Q and g for the first two columns of the table. Since A is abelian of index 3 it suffices to assume that $g = s^{m_g}$ for some $m_g \in \{-1,0,1\}$. Suppose $q = s_1^{i_k}s_2^{j_k} \in Q$, where, as we recall, $k \geq 2$ is the fixed integer which determines the size of S. If there is no ambiguity, we'll write $i_k = i$ and $j_k = j$. If $\psi = \mathrm{Id}_S$, then under the identification $\mathrm{Aut}(A) \to \mathrm{GL}_2(\mathbb{Z}/3^{2k}\mathbb{Z})$, we have

$$q\phi = q^g \Leftrightarrow \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} \delta(\phi)a_k & \delta(\phi)b_k \\ -\delta(\phi)c_k(\phi) & -\delta(\phi)a_k \end{bmatrix} = \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} a_g & m_g \\ -3m_g & b_g \end{bmatrix}$$

where a_k, b_k, $\delta(\phi), c_k(\phi), a_g$ and b_g are as defined in Section 6.2. This equality holds if and only if the following two congruences are satisfied:

$$\begin{aligned} \delta(\phi)(a_k i - c_k(\phi) j) &\equiv_k a_g i - 3m_g j \\ \text{and} \quad \delta(\phi)(b_k i - a_k j) &\equiv_k m_g i + b_g j. \end{aligned}$$

Similarly, if $\psi = \omega$, then

$$(q\phi)\omega = q^g \Leftrightarrow \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} \delta(\phi)a_k & \delta(\phi)b_k \\ -\delta(\phi)c_k(\phi) & -\delta(\phi)a_k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} a_g & m_g \\ -3m_g & b_g \end{bmatrix}$$

which holds if and only if the two congruences are satisfied:

$$\begin{aligned} \delta(\phi)[(2a_k - 3b_k)i - (2c_k(\phi) - 3a_k)j] &\equiv_k a_g i - 3m_g \\ \text{and} \quad \delta(\phi)[(a_k - 2b_k)i - (c_k(\phi) - 2a_k)j] &\equiv_k m_g i + b_g j. \end{aligned}$$

For simplicity, we write $a = a_g, b = b_g$, and $m = m_g$, where there is no ambiguity. Explicitly, we have the following table:

(ϕ, ψ)	$ai - 3mj \equiv_k$	$mi + bj \equiv_k$
$(\theta_A, \mathsf{Id}_S)$	$a_k i - (a_k + 1)j$	$b_k i - a_k j$
$(\theta_A^{-1}, \mathsf{Id}_S)$	$-a_k i + (a_k + 1)j$	$-b_k i + a_k j$
$(\alpha_A, \mathsf{Id}_S)$	$a_k i - (2a_k - 1)j$	$b_k i - a_k j$
(β_A, Id_S)	$-a_k i + (2a_k - 1)j$	$-b_k i + a_k j$
(θ_A, ω)	$(2a_k - 3b_k)i + (a_k - 2)j$	$(a_k - 2b_k)i + (a_k - 1)j$
(θ_A^{-1}, ω)	$-(2a_k - 3b_k)i - (a_k - 2)j$	$-(a_k - 2b_k)i - (a_k - 1)j$
(α_A, ω)	$(2a_k - 3b_k)i - (a_k - 2)j$	$(a_k - 2b_k)i + j$
(β_A, ω)	$-(2a_k - 3b_k)i + (a_k - 2)j$	$-(a_k - 2b_k)i - j$

Define $\alpha_k \equiv_k a_{k-1}^2 - 2a_{k-1} + 3$. By Lemma 6.2.1, we have $\alpha_k \equiv_{k-1} 0$. We also have $a_k \equiv_k -(\alpha_k - a_{k-1})$. It follows that

$$a_k \equiv_{k-1} a_{k-1} \text{ and } b_k \equiv_{k-1} b_{k-1}.$$

Thus, the equations in the above table satisfy

$$i_k \equiv_{k-1} i_{k-1} \quad \text{and} \quad j_k \equiv_{k-1} j_{k-1}.$$

We solve the equations in the table by dividing them to various cases:

<u>Case 3.1</u>:

Suppose that for each pair (ϕ,ψ), and for two fixed elements $m \in \{-1,0,1\}$ (which, as we'll see, is chosen to depend on the pair (ϕ,ψ)), we have $i_{k-1} \equiv_{k-1} j_{k-1} \equiv_{k-1} 0$. That is $i_k \equiv_k 3^{k-1}u$ and $j_k \equiv_k 3^{k-1}v$ for some integers u,v. We show $i_k \equiv_k j_k \equiv_k 0$. Substituting in these congruences into those in the table above gives us the following ones,[1] reduced to modulo 3:

(ϕ,ψ)	$au \equiv_1$	$mu+bv \equiv_1$
$(\theta_A, \mathsf{Id}_S)$	$-v$	u
$(\theta_A^{-1}, \mathsf{Id}_S)$	v	$-u$
$(\alpha_A, \mathsf{Id}_S)$	v	u
(β_A, Id_S)	$-v$	$-u$
(θ_A, ω)	$-2v$	$-2u-v$
(θ_A^{-1}, ω)	$2v$	$2u+v$
(α_A, ω)	$2v$	$-2u+v$
(β_A, ω)	$-2v$	$2u-v$

We substitute in two values of m for each pair (ϕ,ψ), including their corresponding values for a and b.[2] We do not substitute in the corresponding third value of m for the pair of maps, which we'll discuss in the later cases, for the reason that they will not provide us with any meaningful result.

[1]For example, if $(\phi,\psi) = (\alpha_A, \mathsf{Id}_S)$, then $ai_k - 3mj_k \equiv_k a_k i_k - (2a_k - 1)j_k$ implies $3^{k-1}au - 3^k mv \equiv_k 3^{k-1}a_k u - 3^{k-1}(2a_k-1)v$, that is, $au - 3mv \equiv_1 a_k u - (2a_k - 1)v$. Since $3 \mid a_k$, this is equivalent to saying $au \equiv_1 v$. Similarly, $mi_k + bj_k \equiv_k b_k i_k - a_k j_k$ gives $mu + bv \equiv_1 u$.

[2]For example, if $(\phi,\psi) = (\alpha_A, \mathsf{Id}_S)$ and $m = -1$, then $a = -2$ and $b = 1$. Thus, $au \equiv_1 v$ gives $-2u \equiv_1 v$, and $mu + bv \equiv_1 u$ gives $-u + v \equiv_1 u$, as seen in the table.

(ϕ,ψ)	$m=-1$	$m=0$	$m=1$
(θ_A,Id_S)	-	$u\equiv_1 -v$; $v\equiv_1 u$	$u\equiv_1 -v$; $u-2v\equiv_1 u$
$(\theta_A^{-1},\mathsf{Id}_S)$	$-2u\equiv_1 v$; $-u+v\equiv_1 -u$	$u\equiv_1 v$; $v\equiv_1 -u$	-
(α_A,Id_S)	$-2u\equiv_1 v$; $-u+v\equiv_1 u$	-	$u\equiv_1 v$; $u-2v\equiv_1 u$
(β_A,Id_S)	$-2u\equiv_1 -v$; $-u+v\equiv_1 -u$	-	$u\equiv_1 -v$; $u-2v\equiv_1 -u$
(θ_A,ω)	-	$u\equiv_1 -2v$; $v\equiv_1 -2u-v$	$u\equiv_1 -2v$; $u-2v\equiv_1 -2u-v$
(θ_A^{-1},ω)	-	$u\equiv_1 2v$; $v\equiv_1 2u+v$	$u\equiv_1 2v$; $u-2v\equiv_1 2u+v$
(α_A,ω)	$-2u\equiv_1 2v$; $-u+v\equiv_1 2u-v$	$u\equiv_1 2v$; $v\equiv_1 -2u+v$	-
(β_A,ω)	$-2u\equiv_1 -2v$; $-u+v\equiv_1 2u-v$	-	$u\equiv_1 -2v$; $u-2v\equiv_1 2u-v$

Each corresponding pair of congruences deduce $u\equiv_1 v\equiv_1 0$. It follows that $i_k\equiv_k j_k\equiv_k 0$. Now, when we force $k=1$, the equations for pairs of maps, are as exactly those in the table above (resulting in without the reduction). It follows, by induction, that for all $k\geq 1$, we have $i_k\equiv_k j_k\equiv_k 0$, for the selected cases of m. This means, for the cases considered here, $Q=1$. It is left to consider the cases of m not considered here.

<u>Case 3.2</u>:
Suppose that for each of the following pairs $(\phi,\psi)\in\{(\theta_A,\mathsf{Id}_S),(\theta_A^{-1},\mathsf{Id}_S),(\alpha_A,\omega),(\beta_A,\omega)\}$, and that for the third m not considered in the previous case, we have $3i_{k-1}\equiv_{k-1}3j_{k-1}\equiv_{k-1}0$. It follows that there exists integers u,v such that $i_k\equiv_k 3^{k-2}u$ and $j_k\equiv_k 3^{k-2}v$. We show $3i_k\equiv_k 3j_k\equiv_k 0$. Substituting in these congruences into the table on page 334 gives us the following ones, reduced to modulo 3:

(ϕ,ψ)	m			
(θ_A,Id_S)	-1	$6u-7v\equiv_2 -2u+3v$ $4u-6v\equiv_2 -u+v$	$\Longrightarrow$	$u\equiv_2 -v$ $3u\equiv_2 3v\equiv_2 0$
$(\theta_A^{-1},\mathrm{Id}_S)$	1	$-6u+7v\equiv_2 u-3v$ $-4u+6v\equiv_2 u-2v$	$\Longrightarrow$	$u\equiv_2 v$ $3u\equiv_2 3v\equiv_2 0$
(α_A,ω)	1	$-4v\equiv_2 u-3v$ $-2u+v\equiv_2 u-2v$	$\Longrightarrow$	$u\equiv_2 -v$ $3u\equiv_2 3v\equiv_2 0$
(β_A,ω)	0	$4u\equiv_2 u$ $2u-v\equiv_2 v$	$\Longrightarrow$	$u\equiv_2 v$ $3u\equiv_2 3v\equiv_2 0$

Each corresponding pair gives us $3u\equiv_2 3v\equiv_2 0$. This means $3i_k\equiv_k 3j_k\equiv_k 0$. If $(\phi,\psi)\in\{(\theta_A,\mathrm{Id}_S),(\alpha_A,\omega)\}$, we also have $i_k\equiv_k -j_k$, that is, $Q\leq\langle tz^{-1}\rangle$. If $(\phi,\psi)\in\{(\theta_A^{-1},\mathrm{Id}_S),(\beta_A,\omega)\}$, then $i_k\equiv_k j_k$, that is, $Q\leq\langle tz\rangle$. Now, when $k=2$, the equations for pairs of maps are exactly as those in the table above, without the reduction. It follows, by induction, that for all $k\geq 2$, we have $Q\leq\langle tz^{-1}\rangle$ whenever $(\phi,\psi)\in\{(\theta_A,\mathrm{Id}_S),(\alpha_A,\omega)\}$, and $Q\leq\langle tz\rangle$ whenever $(\phi,\psi)\in\{(\theta_A^{-1},\mathrm{Id}_S),(\beta_A,\omega)\}$.

Case 3.3:

We have a small number of cases left to consider. In this case, we do not use induction, but direct calculation. Given a pair (ϕ,ψ) and $m\in\{-1,0,1\}$, we label the two corresponding congruences, given in the table on page 334, as π_1 and π_2, after rearranging them to make one side equivalent to 0 mod 3^k. It turns out that it suffices to consider only one of them, as shown in the table below. We attempt to spell out the first example as an illustration:
When $(\phi,\psi)=(\alpha_A,\mathrm{Id}_S)$ and $m=0$, then we have the two congruence relations:

$$\begin{aligned} & i\equiv_k a_k i-(2a_k-1)j \quad \text{and} \quad j\equiv b_k i-a_k j \\ \Longrightarrow\ & (a_k-1)i-(2a_k-1)j\equiv_k 0 \quad \text{and} \quad b_k i-(a_k+1)j\equiv_k 0 \end{aligned}$$

Let $\pi_1=(a_k-1)i-(2a_k-1)j$ and $\pi_2=b_k i-(a_k+1)j$. Now, since $\pi_1\equiv_k 0$, we have

$$\begin{aligned} & (a_k^2+1)\pi_1\equiv_k 0 && \text{(which we can multiply since } 3\mid a_k) \\ \Leftrightarrow & (a_k-1)(a_k^2+1)i-(2a_k-1)(a_k^2+1)\equiv_k 0 \\ \Leftrightarrow & (a_k-1)(2a_k-2)i-(2a_k-1)(2a_k-2)j\equiv_k 0 && \text{(since } a_k^2\equiv_k 2a_k-3) \\ \Leftrightarrow & (2a_k^2-4a_k+2)i-2(2a_k^2-3a_k+1)j\equiv_k 0 \\ \Leftrightarrow & -4i-2(a_k-5)j\equiv_k 0 && \text{(since } a_k^2\equiv_k 2a_k-3) \\ \Leftrightarrow & 2i+(a_k-5)j\equiv_k 0. \end{aligned}$$

Similarly, $\pi_2\equiv_k 0\Leftrightarrow(a_k+1)(a_k-1)\pi_2\equiv_k 0\Leftrightarrow 2i+(a_k-5)j\equiv_k 0$, as shown in the table below.

Here are the full set of examples which were left to consider:

(ϕ,ψ)	m	
(α_A,Id_S)	0	$(a_k+1)(a_k-1)\pi_2 \equiv_k 0$ $\Leftrightarrow (a_k+1)(a_k-1)[b_k i-(a_k+1)j] \equiv_k 0$ $\Leftrightarrow (a_k-1)(a_k^2+1)i-(a_k-1)(a_k+1)^2 j \equiv_k 0$ $\Leftrightarrow 2i+(a_k-5)j \equiv_k 0.$ Similarly, $(a_k^2+1)\pi_1 \equiv_k 0 \Leftrightarrow 2i+(a_k-5)j \equiv_k 0.$
(β_A,Id_S)	0	$(a_k+1)(a_k-1)\pi_2 \equiv_k 0$ $\Leftrightarrow (a_k+1)(a_k-1)[-b_k i+(a_k-1)j] \equiv_k 0$ $\Leftrightarrow -(a_k-1)(a_k^2+1)i+(a_k-1)^2(a_k+1)j \equiv_k 0$ $\Leftrightarrow 2i-(a_k+1)j \equiv_k 0.$ Similarly, $b_k(a_k-1)\pi_1 \equiv_k 0 \Leftrightarrow 2i-(a_k+1)j \equiv_k 0.$
(θ_A,ω)	-1	$(a_k+1)\pi_2 \equiv_k 0$ $\Leftrightarrow (a_k+1)[(a_k-2b_k+1)i+(a_k-2)j] \equiv_k 0$ $\Leftrightarrow [(a_k+1)^2-2(a_k^2+1)]i+(a_k-2)(a_k+1)j \equiv_k 0$ $\Leftrightarrow 2i+(a_k-5)j \equiv_k 0.$ Similarly, $\pi_1 \equiv_k 0 \Leftrightarrow 2i+(a_k-5)j \equiv_k 0.$
(θ_A^{-1},ω)	-1	$(a_k+1)\pi_1 \equiv_k 0$ $\Leftrightarrow (a_k+1)[(-2a_k+3b_k+2)i-(a_k+1)j] \equiv_k 0$ $\Leftrightarrow [2(1-a_k)(a_k+1)+3(a_k^2+1)]i-(a_k+1)^2 j \equiv_k 0$ $\Leftrightarrow (2a_k+2)i-(4a_k-2)j \equiv_k 0$ $\Leftrightarrow 2(a_k+1)i-(a_k+1)^2 j \equiv_k 0$ $\Leftrightarrow 2i-(a_k+1)j \equiv_k 0.$ Similarly, $\pi_1 \equiv_k 0 \Leftrightarrow 2i-(a_k+1)j \equiv_k 0.$

This means that

$$(\phi,\psi) \in \{(\alpha_A,\mathrm{Id}_S),(\theta_A,\omega)\} \implies 2i \equiv_k (5-a_k)j$$
$$(\phi,\psi) \in \{(\beta_A,\mathrm{Id}_S),(\theta_A^{-1},\omega)\} \implies 2i \equiv_k (a_k+1)j.$$

This completes the proof. □

Lemma B.2.2. *Let $\psi \in \{Id_S,\omega,\eta,\omega\circ\eta\}$.*

1. *If $\phi = \Theta^{\pm 1}_{\mu,\lambda}$ for $\mu = 1,..4$ and $\lambda = -1,0,1$, then for all $1 \neq Q \leq S$, $\phi|_Q \circ \psi$ is never a conjugation map.*

2. *If $\phi = \vartheta_{\mu,\lambda}$ for $\mu = 1,..,4$ and $\lambda = -1,1$, then $\phi|_Q \circ \psi = c_g|_Q$ for some $1 \neq Q \leq S$ and $g \in S$ if and only if $Q \leq P$ and $g \in C$, where $(P;C)$ is the pair listed in the table below, corresponding to the maps (ϕ,ψ).*

ϕ \ ψ	Id_S	ω	η	$\omega\circ\eta$
$\vartheta_{1,\lambda}$	$\emptyset$	$\emptyset$	$\langle ss_1^{\lambda}z^{-1}\rangle;\langle ss_1^{\lambda},z\rangle t$	$\langle ss_1^{\lambda}z\rangle;\langle ss_1^{\lambda},z\rangle$
$\vartheta_{2,\lambda}$	$\emptyset$	$\emptyset$	$\langle ss_1^{\lambda}z\rangle;\langle ss_1^{\lambda},z\rangle t^{-1}$	$\langle ss_1^{\lambda}z^{-1}\rangle;\langle ss_1^{\lambda},z\rangle$
$\vartheta_{3,\lambda}$	$\emptyset$	$\emptyset$	$\langle ss_1^{\lambda}z\rangle;\langle ss_1^{\lambda},z\rangle$	$\langle ss_1^{\lambda}z^{-1}\rangle;\langle ss_1^{\lambda},z\rangle t$
$\vartheta_{4,\lambda}$	$\emptyset$	$\emptyset$	$\langle ss_1^{\lambda}z^{-1}\rangle;\langle ss_1^{\lambda},z\rangle$	$\langle ss_1^{\lambda}z\rangle;\langle ss_1^{\lambda},z\rangle t^{-1}$

3. *Let $\phi=\xi^{\pm1}_{\mu,\lambda}$ for $\mu=1,..,8$ and $\lambda=-1,1$. If $\mu=5,..,8$, then for all $1\neq Q\leq S$, $\phi|_Q\circ\psi$ is never a conjugation map. If $\mu=1,..,4$, then $\phi|_Q\circ\psi=c_g|_Q$ for some $1\neq Q\leq S$ and $g\in S$ if and only if $Q\leq P$ and $g\in C$, where $(P;C)$ is the pair listed in the table below, corresponding to the maps (ϕ,ψ).*

ϕ \ ψ	Id_S	ω	η	$\omega\circ\eta$
$\xi^{\pm1}_{1,\lambda}$	$\emptyset$	$\emptyset$	$\langle z\rangle;S$	$\langle z\rangle;S$
$\xi^{\pm1}_{2,\lambda}$	$\emptyset$	$\emptyset$	$\langle z\rangle;S$	$\langle z\rangle;S$
$\xi^{\pm1}_{3,\lambda}$	$\langle z\rangle;S$	$\langle z\rangle;S$	$\emptyset$	$\emptyset$
$\xi^{\pm1}_{4,\lambda}$	$\langle z\rangle;S$	$\langle z\rangle;S$	$\emptyset$	$\emptyset$

Proof. We prove the lemma case by case:
<u>Case 1</u> Suppose $\phi=\Theta^{\pm1}_{\mu,\lambda}$ for $\mu=1,..,4$ and $\lambda=-1,0,1$.
First suppose $\phi=\Theta_{\mu,\lambda}$. Let $Q\leq S$ and $g\in S$. We have $(ss_1^{\lambda})\phi\circ\psi=z^{\pm1}\psi=z^{\pm1}$ and $z\phi\circ\psi=t^{\pm1}\psi\in t^{\pm1}\langle z\rangle$. However ss_1^{λ} is not S-conjugate to $z^{\pm1}$, nor does z conjugate into $t^{\pm1}\langle z\rangle$. So $ss_1^{\lambda},z\notin Q$. We may assume $Q=\langle ss_1^{\lambda}z^i\rangle$ for $i=-1,1$. However, $(ss_1^{\lambda}z^i)\phi\circ\psi=(z^{\pm1}t^{\pm1})\psi=t^{\pm1}z^{\pm1}$. Since $ss_1^{\lambda}z^i$ is not S-conjugate to any element of $\langle t,z\rangle$, $Q=1$.

Now suppose $\phi=\Theta^{-1}_{\mu,\lambda}$. Let $Q\leq S$ and $g\in S$. We have $(t\phi)\psi=z^{\pm1}$ and $(z\phi)\psi=(ss_1^{\pm\lambda})^{\pm1}$. Since z is central in S, we have $t,z\notin Q$. We may assume $Q=\langle tz^i\rangle$ for $i=-1,1$. However, $(tz^i)\phi\circ\psi=z^{\pm1}(ss_1^{\pm\lambda})^{\pm1}$. Since $\langle t,z\rangle$ is normal in S, we have $tz^i\notin Q$. So $Q=1$.

<u>Case 2</u> Suppose $\phi=\vartheta_{\mu,\lambda}$ for $\mu=1,..,4$ and $\lambda=-1,1$.
Let $Q\leq S$ and $g\in S$. We have $ss_1^{\lambda}\phi\circ\psi=z^{\pm1}\psi=z^{\pm1}$ and $z\phi\circ\psi=(ss_1^{\lambda})^{\pm1}\psi=(ss_1^{-\lambda})^{\pm1}\psi\notin A$. Thus $ss_1^{\lambda},z\notin Q$. So assume $Q=\langle ss_1^{\lambda}z^i\rangle$ for $i=-1,1$. Define

$$\delta(\psi)=\begin{cases}1 & s\psi=s\\ -1 & s\psi=s^{-1}\end{cases}\qquad \delta(\phi)=\begin{cases}1 & z\phi=ss_1^{-\lambda}\\ -1 & z\phi=(ss_1^{-\lambda})^{-1}\end{cases}$$

We have

$$(ss_1^{\lambda}z^i)\phi\circ\psi = (z^{\pm1}(ss_1^{-\lambda})^{\delta(\phi)i})\psi = \begin{cases} z^{\pm1}(ss_1^{\lambda})^{\delta(\phi)\delta(\psi)i} & \text{if } \psi = \eta, \omega\circ\eta \\ z^{\pm1}(ss_1^{-\lambda})^{\delta(\phi)\delta(\psi)i} & \text{if } \psi = \mathrm{Id}_S, \omega \end{cases}$$

Thus, if $\eta = Id_S, \omega$, then $ss_1^{\lambda}z^i \notin Q$. If, on the other hand $\psi = \eta, \omega\circ\eta$, then $\delta(\psi)\delta(\phi) = i$, thereby giving us the results in the table. The set C can be calculated easily.

<u>Case 3</u> Suppose $\phi = \xi_{\mu,\lambda}^{\pm1}$ for $\mu = 1,..,4$ and $\lambda = -1,1$.
First suppose that $\phi = \xi_{\mu,\lambda}$. Let $Q \leq S$ and $g \in S$. For $i = -1,0,1$, we have $(ss_1^{\lambda}z^i)\phi\circ\psi \in \langle s,z\rangle$ for $i = -1,0,1$. However $ss_1^{\lambda}z^i$ is not S-conjugate to any element of $\langle s,z\rangle$. So $ss_1^{\lambda}z^i \notin Q$. It follows that $Q = \langle z\rangle$. If $\mu = 5,..,8$, then $z\phi\circ\psi = s^{\pm1}$. Thus, in this case $z \notin Q$. We are left with the case $\mu = 1,..,4$ and $Q = \langle z\rangle$. Define

$$\delta(\psi) = \begin{cases} 1 & \text{if } z\psi = z \\ -1 & \text{if } z\psi = z^{-1} \end{cases} \qquad \delta(\phi) = \begin{cases} 1 & \text{if } z\phi = z \\ -1 & \text{if } z\phi = z^{-1} \end{cases}$$

Then

$$z\phi\circ\psi = \begin{cases} z & \text{if } \delta(\phi)\delta(\psi) = 1 \\ z^{-1} & \text{if } \delta(\phi)\delta(\psi) = -1 \end{cases}$$

Note that z is not S-cojugate to its inverse. Thus, $\mu = 1,..,4$, $Q = \langle z\rangle$ and $\delta(\phi)\delta(\psi) = 1$. After determining the values of $\delta(\psi)$ and $\delta(\phi)$ we can fill in the values of P in the table. Since z is central in S, we have $C = S$.

When $\phi = \xi_{\mu,\lambda}^{-1}$, then again, we have $sz^i \notin Q$ for all $i = -1,0,1$, since in that case, $(sz^i)\phi\circ\psi$ is not S-conjugate of sz^i. We therefore have $Q = \langle z\rangle$ which occurs only when $\delta(\phi)\delta(\psi) = 1$. Hence the values of P and C are the same as in the case for $\phi = \xi_{\mu,\lambda}$.

□

Lemma B.2.3. *Let $\psi \in \{Id_S, \omega, \eta, \omega\circ\eta\}$. Let $\phi \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\}$ for $\lambda = -1,1$. Then $\phi\circ\psi = c_g|_Q$ for some $1 \neq Q \leq S$ and $g \in S$ if and only if $Q \leq P$ and $g \in C$, where $(P;C)$ is the pair listed in the table below, corresponding to the maps (ϕ,ψ).*

ϕ \\ ψ	$Id\|_S$	ω	η	$\omega\circ\eta$
$\hat{\theta}_{V_\lambda}$	$\langle ss_1^\lambda z^i\rangle;\langle ss_1^\lambda,z\rangle t^{-i}$ $i=-1,0,1$	$\emptyset$	$\langle z\rangle;S$	$\langle z\rangle;S$
$\hat{\theta}_{V_\lambda}^{-1}$	$\langle z\rangle;S$	$\langle ss_1^\lambda,z\rangle;\langle ss_1^\lambda,z\rangle$ $\langle z\rangle;S$	$\emptyset$	$\emptyset$
$\hat{\alpha}_{V_\lambda}$	$\langle ss_1^\lambda,z\rangle;\langle ss_1^\lambda,z\rangle$ $\langle z\rangle;S$	$\langle z\rangle;S$	$\emptyset$	$\emptyset$
$\hat{\beta}_{V_\lambda}$	$\emptyset$	$\langle ss_1^\lambda z^i\rangle;\langle ss_1^\lambda,z\rangle t^{-i}$ $i=-1,0,1$	$\langle z\rangle;S$	$\langle z\rangle;S$

Proof. Define

$$\delta_\lambda(\psi)=\begin{cases}1 & \text{if } (ss_1^\lambda)\psi=ss_1^{\pm1}\\ -1 & \text{if } (ss_1^\lambda)\psi=(ss_1^{\pm1})^{-1}\end{cases} \quad\text{and}\quad \delta_\lambda(\phi)=\begin{cases}1 & \text{if } (ss_1^\lambda)\phi=ss_1^\lambda\\ -1 & \text{if } (ss_1^\lambda)\phi=(ss_1^\lambda)^{-1}\end{cases}$$

$$\delta'_\lambda(\psi)=\begin{cases}1 & \text{if } (ss_1^\lambda)\psi=(ss_1^\lambda)^{\pm1}\\ -1 & \text{if } (ss_1^\lambda)\psi=(ss_1^{-\lambda})^{\pm1}\end{cases}$$

$$\delta_z(\psi)=\begin{cases}1 & \text{if } z\psi=z\\ -1 & \text{if } z\psi=z^{-1}\end{cases} \quad\text{and}\quad \delta_z(\phi)=\begin{cases}1 & \text{if } z\phi=z\\ -1 & \text{if } z\phi=z^{-1}\end{cases}$$

Note that

$$(ss_1^\lambda z^i)\phi\circ\psi=\begin{cases} ss_1^\lambda z^{\pm1} & \text{if } \delta'_\lambda(\psi)=1,\delta_\lambda(\psi)\delta_\lambda(\phi)=1\\ (ss_1^\lambda)^{-1}z^{\pm1} & \text{if } \delta'_\lambda(\psi)=1,\delta_\lambda(\psi)\delta_\lambda(\phi)=-1\\ ss_1^{-\lambda}z^{\pm1} & \text{if } \delta'_\lambda(\psi)=-1,\delta_\lambda(\psi)\delta_\lambda(\phi)=1\\ (ss_1^{-\lambda})^{-1}z^{\pm1} & \text{if } \delta'_\lambda(\psi)=-1,\delta_\lambda(\psi)\delta_\lambda(\phi)=-1\end{cases}$$

and

$$z\phi\circ\psi=\begin{cases} z & \text{if } \delta_z(\psi)\delta_z(\phi)=1\\ z^{-1} & \text{otherwise}\end{cases}$$

It follows that

$$Q\le\begin{cases}\langle ss_1^\lambda,z\rangle & \text{if } \delta'_\lambda(\psi)=1,\delta_\lambda(\psi)\delta_\lambda(\phi)=1,\delta_z(\psi)\delta_z(\phi)=1\\ \langle ss_1^\lambda z^j\rangle, j=-1,0,1 & \text{if } \delta'_\lambda(\psi)=1,\delta_\lambda(\psi)\delta_\lambda(\phi)=1,\delta_z(\psi)\delta_z(\phi)=-1\end{cases}$$

After determining the values of $\delta_\lambda(\psi),\delta_\lambda(\phi),\delta'_\lambda(\psi),\delta_z(\psi)$ and $\delta_z(\phi)$ we can fill in the values for all P. The set C can be calculated easily. □

Lemma B.2.4. *Let $Q \leq S$. Define $u = s_2^{3^{k-2}}$ and let $\lambda = -1, 0, 1, i \in \{1, 2\}$. Then*

1. *If $Q \in \mathcal{D}$, then* $N_S(Q, \langle ss_1^\lambda z^i \rangle) = \begin{cases} t^{-i}\langle ss_1^\lambda, z\rangle & \textit{if } Q = \langle ss_1^\lambda \rangle \\ \emptyset & \textit{otherwise} \end{cases}$

2. *If $Q \in \mathcal{D}$ and $j \in \{-1, 0, 1\}$, then* $N_S(Q, \langle ss_1^\lambda t^i z^j \rangle) = \begin{cases} u^i t^{i-j}\langle ss_1^\lambda t^i, z\rangle & \textit{if } Q = \langle ss_1^\lambda \rangle \\ \emptyset & \textit{otherwise} \end{cases}$

3. *If $Q \in \mathcal{D}$, then* $N_S(Q, \langle ss_1^\lambda t^i, z \rangle) = \begin{cases} \cup_{j \in \{-1,0,1\}} u^i t^j \langle ss_1^\lambda t^i, z\rangle & \textit{if } Q \in \{\langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z\rangle\} \\ S & \textit{if } Q = \langle z \rangle \\ \emptyset & \textit{otherwise} \end{cases}$

4. *If $Q \in \mathcal{D}_A$ and $j \in \{-1, 0, 1\}$, then* $N_S(Q, \langle d_i \rangle) = \begin{cases} s^j A & \textit{if } Q^{s^j} \leq \langle d_i \rangle \\ \emptyset & \textit{otherwise} \end{cases}$

5. *If $Q \in \mathcal{D}_A$, then* $N_S(Q, \langle tz^i \rangle) = \begin{cases} s^i A & \textit{if } Q = \langle t \rangle \\ \emptyset & \textit{otherwise} \end{cases}$

Proof. 1. Write $N = N_S(Q, \langle ss_1^\lambda z^i \rangle)$. If $N \neq \emptyset$, then it is immediate that $|Q| \leq 3$ and $Q \not\leq A$, since A is normal in S. Moreover ss_1^λ is not S-conjugate to ss_1^μ for $\mu \neq \lambda$ and $\mu = -1, 0, 1$. Thus $Q = \langle ss_1^\lambda \rangle$. Now we calculate $N = N_S(\langle ss_1^\lambda \rangle, \langle ss_1^\lambda z^i \rangle)$. We have $t^{-i} \in N$. Thus $N = N_S(ss_1^\lambda, ss_1^\lambda z^i) = t^{-i} C_S(ss_1^\lambda z^i) = t^{-i} C_S(ss_1^\lambda) = t^{-i}\langle ss_1^\lambda, z\rangle$.

2. Write $N = N_S(Q, \langle ss_1^\lambda t^i z^j \rangle)$. If $N \neq \emptyset$, then it is immediate that $|Q| \leq 3$ and $Q \not\leq A$ since A is normal in S. Moreover $ss_1^\lambda t^i z^j$ is not S-conjugate to ss_1^μ for $\mu \neq \lambda$ and $\mu \in \{-1, 0, 1\}$. Thus $Q = \langle ss_1^\lambda \rangle$. We calculate $N = N_S(\langle ss_1^\lambda \rangle, \langle ss_1^\lambda t^i z^j \rangle)$ for this case. It is easily verified that $u^i t^{i-j} \in N$. Thus $N = u^i t^{i-j} C_S(\langle ss_1^\lambda t^i z^j \rangle) = u^i t^{i-j} C_S(ss_1^\lambda t^i) = u^i t^{i-j}\langle ss_1^\lambda t^i, z\rangle$.

3. Write $N = N_S(Q, \langle ss_1^\lambda t^j, z \rangle)$. Suppose $N \neq \emptyset$. Then it is immediate that $|Q| \leq 3^2$ and $Q \not\leq A$ (other than $\langle z \rangle$) since A is normal in S. Moreover, $ss_1^\lambda t^j$ is not S-conjugate to ss_1^μ for $\mu \neq \lambda$ and $\mu \in \{-1, 0, 1\}$. It follows that $Q \in \{\langle z \rangle, \langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z \rangle\}$. We calculate N for these cases. If $Q = \langle z \rangle$, then $N = S$. So suppose $Q \in \{\langle ss_1^\lambda \rangle, \langle ss_1^\lambda, z \rangle\}$. Rewrite $N = N_S(\langle ss_1^\lambda \rangle, \langle ss_1^\lambda t^j, z \rangle)$. Since z is central in S, we have $N_S(\langle ss_1^\lambda, z \rangle, \langle ss_1^\lambda t^j, z \rangle) = N$. Thus it suffices to calculate N. Since

$$\begin{aligned} N_S(\langle ss_1^\lambda \rangle, \langle ss_1^\lambda t^i, z \rangle) &= \bigsqcup_{j=-1,0,1} N_S(\langle ss_1^\lambda \rangle, \langle ss_1^\lambda t^i z^j \rangle) \\ &= \bigsqcup_{j=-1,0,1} u^i t^{i-j}\langle ss_1^\lambda t^i, z \rangle \end{aligned}$$

by the previous result.

4. Write $N = N_S(Q, \langle d_i \rangle)$. Suppose $g \in N$. We may assume $g = s^j$ for some $j \in \{-1, 0, 1\}$. Thus $s^j A \leq N_S(Q, \langle s_1 s_2^i \rangle)$. If also $s^n A \leq N_S(Q, \langle d_i \rangle)$ for some $j \neq n \in \{-1, 0, 1\}$, then $Q \leq \langle d_i \rangle^{s^{-j}} \cap \langle d_i \rangle^{s^{-n}} = 1$, by definition of d_i, a contradiction. It follows that

$$N_S(Q, \langle d_i \rangle) = \begin{cases} s^j A & \text{if } Q^{s^j} \leq \langle d_i \rangle \\ \emptyset & \text{otherwise} \end{cases}$$

5. Write $N = N_S(Q, \langle tz^i \rangle)$. If $N \neq \emptyset$, then $|Q| \leq 3$ and $Q \leq A$. This leaves $\langle t \rangle$ and $\langle z \rangle$ as possible cases for Q. However, only $\langle t \rangle$ is S-conjugate to $\langle tz^i \rangle$. Now we calculate $N = N_S(\langle t \rangle, \langle tz^i \rangle)$. We have $s^i \in N$. Thus, $N = N_S(t, tz^i) = s^i C_S(tz^i) = s^i C_S(t) = s^i A$.

□

If ψ is a restriction of some morphism $\bar{\psi}$ in $\mathrm{Aut}(S)$, with $Q \leq D_\psi$, then note that $C_S(Q\psi) = C_S(Q\bar{\psi}) = (C_S(Q))\bar{\psi}$, where the latter equality holds because

$$\begin{aligned} g \in C_S(Q\bar{\psi}) &\Leftrightarrow (q\bar{\psi})^g = (q\bar{\psi}) && \text{(for all } q \in Q) \\ &\Leftrightarrow (q^{g\bar{\psi}^{-1}})\bar{\psi} = q\bar{\psi} && \text{(for all } q \in Q) \\ &\Leftrightarrow q^{g\bar{\psi}^{-1}} = q && \text{(for all } q \in Q) \\ &\Leftrightarrow g \in (C_S(Q))\bar{\psi} \end{aligned}$$

This means $|C_S(Q\psi)| = |C_S(Q)|$. Thus, in this case,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \frac{|N_{\psi|_Q, \phi|_P}||C_S(Q\psi)|}{|P|} = \frac{|N_{\psi|_Q, \phi|_P}||C_S(Q)|}{|P|},$$

for any morphism ϕ such that $P \leq D_\phi$. Thus, when ψ is a restriction of an element in $\mathrm{Aut}(S)$, then in determining $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|$, it suffices to determine $|N_{\psi|_Q, \phi|_P}|$ and $|C_S(Q)|$.

Throughout the following set of proofs, we may use the following equivalence. Given $x \in N_{\psi|_Q, \phi|_P}$ for some morphism ϕ and domain P, and some automorphisms $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$, there exists $y \in S$ such that

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow \phi|_{Q^x} = c_{x^{-1}}|_{Q^x} \circ \psi \circ c_{y^{-1}} \\ &\Leftrightarrow \phi|_{Q^x} = c_{x^{-1}(y\psi)^{-1}}|_{Q^x} \circ \psi \\ &\Leftrightarrow \phi|_{Q^x} \circ \psi = c_{x^{-1}(y\psi)^{-1}}|_{Q^x} \end{aligned}$$

where the last equivalence is because ψ is either the identity map or an involution. By Lemma $B.2.1$ and $B.2.2$ we already know under which circumstances will $\phi|_{Q^x} \circ \psi$ will be a conjugation map. Using this, and Lemma $B.2.4$, we can determine $N_{\psi|_Q, \phi|_P}$ and hence $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|$.

We may in some cases define the following: Suppose that for some morphism ψ with $x \in D_\psi \leq S$, we have $(x\psi) = y$ or y^{-1}. Then define

$$\delta_x(\psi) = \begin{cases} 1 & \text{if } (x\psi) = y \\ -1 & \text{if } (x\psi) = y^{-1} \end{cases}$$

The proof will have the following structure. Let ψ and ϕ be two maps with $Q \leq D_\psi \leq S$ and $P \leq D_\phi \leq S$. We suppose that $N_{\psi|_Q,\phi|_P} \neq \emptyset$, in which case there exists $x \in N_{\psi|_Q,\phi|_P} \subseteq N_S(Q,P)$. This means that

$$q \in Q \implies \exists y \in S \text{ with } ((q^x)\phi)^y = q\psi$$

for all $q \in Q$ with $q^x \in P$. From this we will either derive a contradiction, or determine some conditions under which the above holds. We now list our proofs:

B.2.1 $\psi \in \{\mathbf{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $Q \in \mathcal{D}$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Then, for some $y \in S$, we have

   $$\begin{aligned} c_x|_Q \circ \phi \circ c_y = \psi|_Q &\Leftrightarrow c_{x(y\phi)}|_Q \circ \phi = \psi|_Q \\ &\Leftrightarrow c_{x(y\phi)}|_Q = \psi|_Q \circ \phi \end{aligned}$$

   where the latter equivalence holds since ϕ is an automorphism of order at most 2. Recall that $\mathcal{D}$ is the set of S-conjugacy class representatives in S, containing $\langle s \rangle, \langle s_1 \rangle$ and $\langle s_2 \rangle$. Thus, if $Q \not\leq \langle s_2 \rangle$ and $\psi \circ \phi = \omega$, or if $Q \not\leq \langle s \rangle$ and $\psi \circ \phi = \eta$, or if $Q \not\leq \langle s_1 \rangle$ and $\psi \circ \phi = \omega \circ \eta$, then by 6.1.3 and 6.1.4, $\psi|_Q \circ \phi$ cannot be equal to a restriction of a conjugation map. Therefore, in these cases, $|N_{\psi|_Q,\phi|_P}| = 0$. In the rest of the cases, if we define y, which are permitted to do, so that

   $$x(y\phi) = \begin{cases} 1 & \text{if } \psi \circ \phi = \mathrm{Id}_S \\ s & \text{if } Q \leq \langle s_2 \rangle; \psi \circ \phi = \omega \\ s & \text{if } Q \leq \langle s \rangle; \psi \circ \phi = \eta \\ s^{-1} & \text{if } Q \leq \langle s_1 \rangle; \psi \circ \phi = \omega \circ \eta \end{cases}$$

then 6.1.3 and 6.1.4 implies $\psi|_Q \circ \phi$ is indeed a conjugation map. This means $|N_{\psi|_P,\phi|_Q}| = |N_S(Q,P)|$. Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{|N_S(Q,P)||C_S(Q)|}{|P|} & \text{if } Q \leq S, \psi\circ = \mathrm{Id}_S \\ & \text{or } Q \leq \langle s_2\rangle, \psi\circ\phi = \omega \\ & \text{or } Q \leq \langle s\rangle, \psi\circ\phi = \eta \\ & \text{or } Q \leq \langle s_1\rangle, \psi\circ\phi = \omega\circ\eta \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Suppose $T = V_\lambda$ for $\lambda = -1, 0, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. By Lemma $B.2.1$, we have

$$x \in N_S(Q, \langle ss_1^\lambda z^i\rangle) \quad \text{and} \quad (\psi,\phi) \in \mathcal{M}_{V_\lambda}(\langle ss_1^\lambda z^i\rangle).$$

   for some $i = -1, 1$. By Lemma $B.2.4$, if $Q \neq \langle ss_1^\lambda\rangle$, then $N_S(Q, \langle ss_1^\lambda z^i\rangle) = \emptyset$, and thus $N_{\psi|_Q,\phi|_P} = \emptyset$. So suppose $Q = \langle ss_1^\lambda\rangle$. We have

$$N_S(Q,P) = \begin{cases} \langle ss_1^\lambda, t\rangle & \text{if } P = \langle ss_1^\lambda, z\rangle \\ \langle ss_1^\lambda, z\rangle & \text{if } P = \langle ss_1^\lambda\rangle \\ \emptyset & \text{if } P = \langle z\rangle \end{cases}$$

   By applying $B.2.4$ again, we see that

$$N_S(Q,P) \cap N_S(Q, \langle ss_1^\lambda z^i\rangle) = \begin{cases} t^{-i}\langle ss_1^\lambda, z\rangle & \text{if } P = \langle ss_1^\lambda, z\rangle, Q = \langle ss_1^\lambda\rangle \\ \emptyset & \text{otherwise} \end{cases}$$

   Collecting the data,

$$|N_{\psi|_Q,\phi|_P}| = \begin{cases} 3^2 & \text{if } P = \langle ss_1^\lambda, z\rangle, Q = \langle ss_1^\lambda\rangle, (\psi,\phi) \in \mathcal{M}_{V_\lambda}(\langle ss_1^\lambda z^i\rangle \text{ for } i = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

   Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^2 & \text{if } P = \langle ss_1^\lambda, z\rangle, Q = \langle ss_1^\lambda\rangle, (\psi,\phi) \in \mathcal{M}_{V_\lambda}(\langle ss_1^\lambda z^i\rangle \text{ for } i = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$.

Let $x \in N_{\psi|_Q, \phi|_P}$. By Lemma $B.2.1$, we have

$$\begin{aligned} x \in & N_S(Q, \langle ss_1^\lambda t^i, z\rangle) \quad \text{and} \quad (\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle), \quad \text{or} \\ x \in & N_S(Q, \langle ss_1^\lambda t^i z^j\rangle) \quad \text{and} \quad (\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i z^j\rangle) \end{aligned}$$

for some $i = -1, 1$ and $j = -1, 0, 1$. By $B.2.4$, if $Q \notin \{\langle z\rangle, \langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}$ and $(\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle)$, then $N_S(Q, \langle ss_1^\lambda t^i, z\rangle) = \emptyset$, that is, $|N_{\psi|_Q, \phi|_P}| = 0$. Similarly, if $Q \neq \langle ss_1^\lambda\rangle$ and $(\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i z^j\rangle)$, then $N_S(Q, \langle ss_1^\lambda t^i z^j\rangle) = \emptyset$, that is, $|N_{\psi|_Q, \phi|_P}| = 0$. We study both cases separately. First suppose that $Q \in \{\langle z\rangle, \langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}$ and $(\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle)$. Define $u = s_2^{3^{k-2}}$. We have

$$N_S(Q, P) = \begin{cases} \langle ss_1^\lambda, z\rangle & \text{if } Q = P = \langle ss_1^\lambda\rangle \\ \langle ss_1^\lambda, t\rangle & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P = \langle ss_1^\lambda, z\rangle \\ \langle ss_1^\lambda, u\rangle & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P = \langle ss_1^\lambda, t\rangle \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ \emptyset & \text{otherwise} \end{cases}$$

By $B.2.4$, we see that

$$|N_S(Q, \langle ss_1^\lambda t^i, z\rangle) \cap N_S(Q, P)| = \begin{cases} 3^3 & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P = \langle ss_1^\lambda, t\rangle \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\} \\ \emptyset & \text{otherwise} \end{cases}$$

Collecting the data,

$$|N_{\psi|_Q, \phi|_P}| = \begin{cases} 3^3 & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P = \langle ss_1^\lambda, t\rangle, (\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle) \text{ for } i = -1, 1 \\ 3^{2k+1} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, (\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle), i = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, P = \langle ss_1^\lambda, t\rangle, (\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle) \text{ for } i = -1, 1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, (\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z\rangle)), i = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

Now suppose that $Q = \langle ss_1^\lambda\rangle$ and $(\psi, \phi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i z^j\rangle)$ for $i = -1, 1$ and $j = -1, 0, 1$.

We have

$$N_S(Q,P) = \begin{cases} \langle ss_1^{\lambda}, z\rangle & \text{if } P = \langle ss_1^{\lambda}\rangle \\ \langle ss_1^{\lambda}, t\rangle & \text{if } P = \langle ss_1^{\lambda}, z\rangle \\ \langle ss_1^{\lambda}, u\rangle & \text{if } P = \langle ss_1^{\lambda}, t\rangle \\ \emptyset & \text{otherwise} \end{cases}$$

which is similar to the previous situation. By $B.2.4$, we see that

$$|N_S(Q, \langle ss_1^{\lambda} t^i z^j\rangle) \cap N_S(Q,P)| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^{\lambda}\rangle, P = \langle ss_1^{\lambda}, t\rangle \\ 0 & \text{otherwise} \end{cases}$$

By Lemma $B.2.1$ we have $M_{\lambda}(\langle ss_1^{\lambda} t^i z^j\rangle) = M_{\lambda}(\langle ss_1^{\lambda} t^i z^{\ell}\rangle)$ for all $\ell = -1,0,1$. By collecting the data,

$$\begin{aligned} |N_{\psi|_Q,\phi|_P}| &= \begin{cases} \sum_{\ell\in\{-1,0,1\}} |N_S(Q, \langle ss_1^{\lambda} t^i z^{\ell}\rangle) \cap N_S(Q,P)| & \text{if } Q = \langle ss_1^{\lambda}\rangle, P = \langle ss_1^{\lambda}, t\rangle, \\ & \quad (\psi,\phi) \in \mathcal{M}_{\lambda}(\langle ss_1^{\lambda} t^i z^j\rangle), \\ & \quad \text{for } i = -1,1, j = -1,0,1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 3^3 & \text{if } Q = \langle ss_1^{\lambda}\rangle, P = \langle ss_1^{\lambda}, t\rangle, (\psi,\phi) \in \mathcal{M}_{\lambda}(\langle ss_1^{\lambda} t^i z^j\rangle) \text{ for } i = -1,1, j = -1,0,1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^{\lambda}\rangle, P = \langle ss_1^{\lambda}, t\rangle, (\psi,\phi) \in \mathcal{M}_{\lambda}(\langle ss_1^{\lambda} t^i z^j\rangle) \text{ for } i = -1,1, j = -1,0,1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = A$.

Let $x \in N_{\psi|_Q,\phi|_P}$. By Lemma $B.2.1$, we have

$$\begin{aligned} x \in & N_S(Q, \langle d_i\rangle) \quad \text{and} \quad (\psi,\phi) \in \mathcal{M}_A(\langle d_i\rangle), \quad \text{or} \\ x \in & N_S(Q, \langle tz^j\rangle) \quad \text{and} \quad (\psi,\phi) \in \mathcal{M}_A(\langle tz^j\rangle). \end{aligned}$$

for some $i = 1,2$ and $j = -1,1$. We have

$$N_S(Q,P) \geq \begin{cases} s^m A & \text{if } Q^{s^m} \leq P \text{ for some } m = -1,0,1 \\ \emptyset & \text{otherwise} \end{cases}$$

First suppose that $x \in N_S(Q,\langle d_i\rangle)$ and $(\psi,\phi) \in \mathcal{M}_A(\langle d_i\rangle)$ for some $i=1,2$. By $B.2.4$, we have

$$N_S(Q,P)\cap N_S(Q,\langle d_i\rangle) = \begin{cases} s^mA & \text{if } Q^{s^m} \leq P\cap\langle d_i\rangle \text{ for some } m=-1,0,1 \\ \emptyset & \text{otherwise} \end{cases}$$

Similarly, if we suppose that $x \in N_S(Q,\langle tz^i\rangle)$ and $(\psi,\phi)\in\mathcal{M}_A(\langle tz^j\rangle)$, for some $j=-1,1$, then by $B.2.4$, we have

$$N_S(Q,P)\cap N_S(Q,\langle tz^j\rangle) = \begin{cases} s^mA & \text{if } Q^{s^m} \leq P\cap\langle tz^j\rangle \text{ for some } m=-1,0,1 \\ \emptyset & \text{otherwise} \end{cases}$$

Collecting the data,

$$|N_{\psi|_Q,\phi|_P}| = \begin{cases} 3^{2k} & \text{if } (\psi,\phi)\in\mathcal{M}_A(\langle d_i\rangle), Q^g\leq P\cap\langle d_i\rangle \text{ for some } g\in S \\ & \text{or } (\psi,\phi)\in\mathcal{M}_A(\langle tz^i\rangle), Q^g\leq P\cap\langle tz^i\rangle \text{ for some } g\in S \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k}}{|P|} & \text{if } (\psi,\phi)\in\mathcal{M}_A(\langle d_i\rangle), Q^g\leq P\cap\langle d_i\rangle \text{ for some } g\in S, i=1,2 \\ & \text{or } (\psi,\phi)\in\mathcal{M}_A(\langle tz^j\rangle), Q^g\leq P\cap\langle tz^j\rangle \text{ for some } g\in S, j=-1,1 \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\phi=\Theta_{\mu,\nu}$ and $P\in\mathcal{D}_{V_\nu}$for $\mu=1,..,4$, $\nu=-1,0,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. By Lemma $B.2.2$, $\phi|_{Q^x}\circ\psi$ is never a conjugation map, that is, $|N_{\psi|_Q,\phi|_P}|=0$.

4. Let $\phi=\Theta^{-1}_{\mu,\nu}$ and $P\in\mathcal{D}_{V_\nu}$for $\mu=1,..,4$, $\nu=-1,0,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. By Lemma $B.2.2$, $\phi|_{Q^x}\circ\psi$ is never a conjugation map, that is, $|N_{\psi|_Q,\phi|_P}|=0$.

5. Let $\phi=\vartheta_{\mu,\nu}$ and $P\in\mathcal{D}_{V_\nu}$ for $\mu=1,..,4$ and $\nu=-1,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. By Lemma $B.2.2$

   $$x\in N_S(Q,\langle ss_1^\nu z^i\rangle) \quad \text{and} \quad (\psi,\phi)\in\mathcal{M}_\vartheta(\langle ss_1^\nu z^i\rangle)$$

   for $i=-1,1$. By Lemma $B.2.4$, if $Q\neq\langle ss_1^\nu\rangle$, then $N_S(Q,\langle ss_1^\nu z^i\rangle)=\emptyset$, that is, $|N_{\psi|_Q,\phi|_P}|=$

0. So suppose $Q = \langle ss_1^\nu \rangle$. We have

$$N_S(Q,P) = \begin{cases} \langle ss_1^\nu, t\rangle & \text{if } P = \langle ss_1^\nu, z\rangle \\ \langle ss_1^\nu, z\rangle & \text{if } P = \langle ss_1^\nu \rangle \\ \emptyset & P = \langle z\rangle \end{cases}$$

By applying Lemma $B.2.4$ again, we see that

$$N_S(Q,P) \cap N_S(Q, \langle ss_1^\lambda z^i\rangle) = \begin{cases} t^i \langle ss_1^\nu, z\rangle & \text{if } P = \langle ss_1^\nu, z\rangle, Q = \langle ss_1^\nu\rangle \\ \emptyset & \text{otherwise} \end{cases}$$

Collecting the data,

$$|N_{\psi|_Q, \phi|_P}| = \begin{cases} 3^2 & \text{if } P = \langle ss_1^\nu, z\rangle, Q = \langle ss_1^\nu\rangle, (\psi,\phi) \in \mathcal{M}_\vartheta(\langle ss_1^\nu z^i\rangle) \text{ for } i = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } P = \langle ss_1^\nu, z\rangle, Q = \langle ss_1^\nu\rangle, (\psi,\phi) \in \mathcal{M}_\vartheta(\langle ss_1^\nu z^i\rangle) \text{ for } i = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi = \xi_{\mu,\nu}$ and $P \in \mathcal{D}_\nu$ for $\mu = 1, .., 8$, $\nu = -1, 1$.

   Let $x \in N_{\psi|_Q, \phi|_P}$. By Lemma $B.2.2$, we have

$$x \in N_S(Q, \langle z\rangle) \quad \text{and} \quad (\psi, \phi) \in \mathcal{M}_\xi(\langle z\rangle).$$

   Thus, if $Q \neq \langle z\rangle$, then $N_S(Q, \langle z\rangle) = \emptyset$, that is, $N_{\psi|_Q,\phi|_P} = \emptyset$. So suppose $Q = \langle z\rangle$. Note that

$$N_S(Q,P) = \begin{cases} S & \text{if } Q = \langle z\rangle \leq P \\ \emptyset & \text{otherwise} \end{cases}$$

   Since $N_S(Q,P) \cap N_S(Q, \langle z\rangle) = N_S(Q,P)$, we can conclude

$$N_{\psi|_Q,\phi|_P} = \begin{cases} S & \text{if } Q = \langle z\rangle \leq P, (\psi,\phi) \in \mathcal{M}_\xi(\langle z\rangle) \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P, (\psi,\phi) \in \mathcal{M}_\xi(\langle z\rangle) \\ 0 & \text{otherwise} \end{cases}$$

7. Let $\phi = \xi^{-1}_{\mu,\nu}$ and $P \in \mathcal{D}_{V_0}$ for $\mu = 1,..,8$ and $\nu = -1, 1$.

   Since, in this case $(\psi,\phi) \in \mathcal{M}_{\xi^{-1}}(R) \Leftrightarrow (\psi^{-1},\phi) \in \mathcal{M}_\xi(R)$ for all $1 \neq R \le S$, the argument is exactly the same as the previous case. Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P, (\psi,\phi) \in \mathcal{M}_{\xi^{-1}}(\langle z\rangle) \\ 0 & \text{otherwise} \end{cases}$$

8. Let $\phi = \theta^{\pm 1}_\lambda \circ \eta$ for $\lambda = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. For any $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi|_Q \quad \text{for } \phi_1 \in \{\theta_\lambda, \theta^{-1}_\lambda\} \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi|_Q \circ \eta \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi|_Q \circ \eta \end{aligned}$$

   Thus, it suffices to look at the case when $\psi \circ \eta \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $\phi_1 \in \{\theta_\lambda, \theta^{-1}_\lambda\}$.

9. Let $\phi \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}^{-1}_{V_\lambda}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\}$ for $\lambda = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. By Lemma B.2.3, we have

$$\begin{aligned} x \in & N_S(Q, \langle ss_1^\lambda, z\rangle) \quad \text{and} \quad (\psi,\phi) \in M_{\hat{V}_\lambda}(\langle ss_1^\lambda, z\rangle) \\ x \in & N_S(Q, \langle ss_1^\lambda z^i\rangle) \quad \text{and} \quad (\psi,\phi) \in M_{\hat{V}_\lambda}(\langle ss_1^\lambda z^i\rangle) \\ x \in & N_S(Q, \langle z\rangle) \quad \text{and} \quad (\psi,\phi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle z\rangle) \end{aligned}$$

   for some $i = -1, 0, 1$. When $(\psi,\phi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle ss_1^\lambda, z\rangle)$, then $(\psi,\phi) \in \{(\mathrm{Id}_S, \hat{\alpha}_{V_\lambda}), (\omega, \hat{\theta}^{-1}_{V_\lambda})\}$, and we know by definition that $\hat{\alpha}_{V_\lambda} = \mathrm{Id}|_{\langle ss_1^\lambda, z\rangle}$ and $\hat{\theta}^{-1}_{V_\lambda} = \omega|_{\langle ss_1^\lambda, z\rangle}$. So it suffices to use corresponding results, from which we get

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ & \text{or } Q = P = \langle ss_1^\lambda, z\rangle \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

By Lemma $B.2.4$, if $Q \neq \langle ss_1^\lambda\rangle$ and $(\psi,\phi) \in M_{\hat{V}_\lambda}(\langle ss_1^\lambda z^i\rangle)$, then $N_S(Q, \langle ss_1^\lambda z^i\rangle) = \emptyset$, that is, $|N_{\psi|_Q,\phi|_P}| = 0$. So suppose $Q = \langle ss_1^\lambda\rangle$ and $(\psi,\phi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle ss_1^\lambda z^i\rangle)$. Note that

$$N_S(Q,P) = \begin{cases} \langle ss_1^\lambda, z\rangle & \text{if } P = \langle ss_1^\lambda\rangle \\ \langle ss_1^\lambda, t\rangle & \text{if } P = \langle ss_1^\lambda, z\rangle \\ \emptyset & \text{otherwise} \end{cases}$$

Note that

$$N_S(Q, \langle ss_1^\lambda z^i\rangle) \cap N_S(Q,P) = \begin{cases} t^{-i}\langle ss_1^\lambda, z\rangle & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\lambda, z\rangle \\ & \text{or } Q = P = \langle ss_1^\lambda\rangle, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Collecting the data,

$$|N_{\psi|_Q,\phi|_P}| = \begin{cases} \sum_{i=-1,0,1} |N_S(Q, \langle ss_1^\lambda z^i\rangle) \cap N_S(Q,P)| & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\lambda, z\rangle \\ |N_S(Q, \langle ss_1^\lambda\rangle) \cap N_S(Q,P)| & \text{if } Q = P = \langle ss_1^\lambda\rangle \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 3^2 & \text{if } Q = P = \langle ss_1^\lambda\rangle \\ 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\lambda, z\rangle \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\} \\ 0 & \text{otherwise} \end{cases}$$

Now suppose $Q = \langle z \rangle$ and $(\psi, \phi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle z \rangle)$. Note that

$$N_S(Q,P) = \begin{cases} S & \text{if } P \in \{\langle z \rangle, \langle ss_1^\lambda, z \rangle\} \\ \emptyset & \text{otherwise} \end{cases}$$

Since $N_S(Q,P) \leq N_S(Q, \langle z \rangle)$, it follows that

$$|N_{\psi|_Q,\phi|_P}| = \begin{cases} 3^{2k+1} & \text{if } P \in \{\langle z \rangle, \langle ss_1^\lambda, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\lambda, z \rangle\} \\ 0 & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu,\lambda}$ for $\mu = 1,..,4$ and $\lambda = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. For any $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi|_Q \quad \text{for } \phi_1 \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}^{-1}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\} \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi|_Q \circ \eta \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi|_Q \circ \eta \end{aligned}$$

Thus, it suffices to look at the case when $\psi \circ \eta \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $\phi_1 \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}^{-1}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\}$.

B.2.2 $\psi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $Q \in \mathcal{D}_T$ for $T \in \mathcal{T}_{full}$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

Suppose $T = V_\lambda$ for $\lambda = -1, 0, 1$.

By Lemma $B.2.1$, the only time when $\psi|_Q \circ \phi$ is a conjugation map is when $Q \leq \langle ss_1^\lambda z^i \rangle$ for some $i = -1, 1$. However $\langle ss_1^\lambda z^i \rangle \notin \mathcal{D}_{V_\lambda}$. Thus, $|N_{\psi|_Q,\phi|_P}| = 0$.

Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$.

By Lemma $B.2.1$, $\psi|_Q \circ \phi$ is a conjugation map when $Q \leq \langle ss_1^\lambda t^i, z \rangle$ and $(\phi, \psi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i, z \rangle)$ for $i = -1, 1$, or when $Q \leq \langle ss_1^\lambda t^i z^j \rangle$ and $(\phi, \psi) \in \mathcal{M}_\lambda(\langle ss_1^\lambda t^i z^j \rangle)$ for

$i=-1,1, j=-1,0,1$. In the latter case, $\langle ss_1^{\lambda}t^iz^j\rangle \notin \mathcal{D}_{\lambda}$, which implies $|N_{\psi|_Q,\phi|_P}|=0$. In the former case, the only subgroup Q in $\langle ss_1^{\lambda}t^i, z\rangle$ that lies in $\mathcal{D}_{\lambda}$ is $\langle z\rangle$.

Let $Q=\langle z\rangle$ and $(\phi,\psi)\in\mathcal{M}_{\lambda}(\langle z\rangle)$. Let $x\in N_{\psi|_Q,\phi|_P}$. Then, by choosing any value of $y\in S$, we see that $c_{x(y\phi)}|_Q=\psi|_Q\circ\phi$. Collecting the data,

$$N_{\psi|_Q,\phi|_P}=\begin{cases} N_S(Q,P) & \text{if } Q=\langle z\rangle, (\phi,\psi)\in\mathcal{M}_{\lambda}(\langle z\rangle) \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q=\langle z\rangle\leq P, (\phi,\psi)\in\mathcal{M}_{\lambda}(\langle z\rangle) \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T=A$.

By Lemma $B.2.1$, $\psi|_Q\circ\phi$ is a conjugation map when $Q\leq\langle d_i\rangle$ and $(\phi,\psi)\in\mathcal{M}_A(\langle d_i\rangle)$ for $i=1,2$, or when $Q\leq\langle tz^j\rangle$ and $(\phi,\psi)\in\mathcal{M}_A(\langle tz^j\rangle)$ for $j=-1,1$. In the latter case $\langle tz^j\rangle\notin\mathcal{D}_A$, which implies $|N_{\psi|_Q,\phi|_P}|=0$. We consider the former case.

Let $Q\leq\langle d_i\rangle$ and $(\phi,\psi)\in\mathcal{M}_A(\langle d_i\rangle)$. Suppose $x\in N_{\psi|_Q,\phi|_P}$. By choosing a suitable element $y\in S$ such that $x(y\phi)=1$, we see that $c_{x(y\phi)}|_Q=\psi|_Q\circ\phi$. Collecting the data,

$$N_{\psi|_Q,\phi|_P}=\begin{cases} N_S(Q,P) & \text{if } \langle d_i\rangle\leq Q, (\phi,\psi)\in\mathcal{M}_A(\langle d_i\rangle) \text{ for } i=1,2 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases} \frac{3^{2k}|N_S(Q,P)|}{|P|} & \text{if } \langle d_i\rangle\leq Q, (\phi,\psi)\in\mathcal{M}_A(\langle d_i\rangle) \text{ for } i=1,2 \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi\in\{\theta_{T_1},\theta_{T_1}^{-1},\alpha_{T_1},\beta_{T_1}\}$ and $P\in\mathcal{D}_{T_1}$, for $T_1\in\mathcal{T}_{full}$.

First suppose that $T_1\neq T$.

Suppose $(T,T_1)=(V_{\lambda},V_{\mu})$ for some $\lambda\neq\mu$. let $x\in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\lambda}\notin Q$ since

ss_1^λ is not S-conjugate to any element of P. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\mu)^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)} \\ &\implies \lambda = \mu, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $(T,T_1) = (E_\lambda, E_\mu)$ for some $\lambda \neq \mu$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\lambda \notin Q$ since ss_1^λ is not S-conjugate to any element of P. If $t \in Q$, then $t^x = tz^i$ for some $i = -1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies ((ss_1^\mu)^{\delta_t(\phi)} z^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_t(\psi)} \\ &\implies \lambda = \mu, \text{ a contradiction.} \\ z \in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle, \langle ss_1^\mu, z\rangle, \langle ss_1^\mu, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle, \langle ss_1^\mu, z\rangle, \langle ss_1^\mu, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $(T,T_1) = (V_\lambda, A)$ and let $x \in N_{\psi|_Q,\phi|_P}$. Note that $s \notin Q$, since ss_1^λ is not S-conjugate to any element of A. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $(T,T_1)=(A,V_\lambda)$ and let $x\in N_{\psi|_Q,\phi|_P}$. Since A is normal in S, the only elements that could S-conjugate into P are in $A\cap V_\lambda=\langle z\rangle$. For some $y\in S$, we have

$$\begin{aligned} z\in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

Suppose $(T,T_1)=(E_\lambda,A)$ and let $x\in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\lambda\notin Q$, since ss_1^λ is not S-conjugate to any element of A. If $t\in Q$, then $t^x=tz^i$ for some $i=-1,0,1$. For some $y\in S$, we have

$$\begin{aligned} t\in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies (z^{\delta_t(\phi)}t^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_t(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

Suppose $(T,T_1)=(A,E_\lambda)$ and let $x\in N_{\psi|_Q,\phi|_P}$. Since A is normal in S, the only elements that could S-conjugate into P are in $A\cap E_\lambda=\langle t,z\rangle$. If $t\in Q$, then $t^x=tz^i$ for some $i=-1,0,1$. For some $y\in S$, we have

$$\begin{aligned} t\in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies ((ss_1^\lambda)^{\delta_t(\phi)}z^{i\delta_z(\phi)})^y = z^{\delta_t(\psi)} \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

Suppose $(T,T_1)=(V_\mu,E_\lambda)$ and let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\mu\in Q$, then $\lambda=\mu$ and $(ss_1^\mu)^x= ss_1^\mu t^iz^j$ for some $i,j=-1,0,1$. Thus, for some $y\in S$, we have

$$\begin{aligned} ss_1^\mu\in Q &\implies ((ss_1^\lambda t^iz^j)\phi)^y = (ss_1^\mu)\psi \\ &\implies (t^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_t(\phi)}z^{j\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z^x\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^\mu)^{\delta_z(\psi)} \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $(T, T_1) = (E_\lambda, V_\mu)$ and let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$, since it is not S-conjugate to any element of P. If $ss_1^\lambda \in Q$, then $\lambda = \mu$ and $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for some $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q &\implies ((ss_1^\mu z^i)\phi)^y = (ss_1^\lambda)\psi \\ &\implies (z^{\delta_\mu(\phi)}(ss_1^\mu)^{i\delta_z(\phi)})^y = t^{\delta_t(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\mu)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

<u>Suppose $T = T_1 = V_\lambda$ for $\lambda = -1, 0, 1$.</u>

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\lambda \in Q$, then $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for some $i = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q \implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\lambda)\psi &\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_z(\phi)})^y = z^{\delta_\lambda(\psi)} \\ &\implies i = 0 \text{ and } \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ z \in Q \implies (z\phi)^y = z\psi &\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^\lambda, z\rangle & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } Q = \langle ss_1^\lambda, z\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle ss_1^\lambda, z\rangle & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } Q = P = \langle ss_1^\lambda, z\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ S & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{|\langle ss_1^\lambda, z\rangle||C_S((Q)\psi)|}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } Q = P = \langle ss_1^\lambda, z\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ \dfrac{|S||C_S((Q)\psi)|}{|P|} & \text{if } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 3^2 & \text{if } Q = P = \langle ss_1^\lambda, z\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

<u>Suppose $T = T_1 = E_\lambda$ for $\lambda = -1, 0, 1$.</u>

Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^\lambda \in Q$, then $(ss_1^\lambda)^x = ss_1^\lambda t^i z^j$, for some $i, j = -1, 0, 1$, and if $t \in Q$, then $t^x = tz^l$ for some $i = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q \implies ((ss_1^\lambda t^i z^j)\phi)^y = (ss_1^\lambda)\psi &\implies (t^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_t(\phi)} z^{\delta_z(\phi)})^y = t^{\delta_\lambda(\psi)} \\ &\implies i = 0 \text{ and } \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ t \in Q \implies ((tz^i)\phi)^y = t\psi &\implies ((ss_1^\lambda)^{\delta_t(\phi)} z^{i\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_t(\psi)} \\ &\implies \delta_t(\psi)\delta_t(\phi) = 1 \\ z \in Q \implies ((z)\phi)^y = z\psi &\implies z^{\delta_z(\phi)} = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1 \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q, \phi|_P} = \begin{cases} \langle ss_1^\lambda, t\rangle \cap N_S(Q,P) & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } Q = \langle ss_1^\lambda, z\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = \langle ss_1^\lambda, t\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_t(\psi)\delta_t(\phi) = 1 \\ N_S(Q,P) & \text{if } Q = \langle t\rangle, \delta_t(\psi)\delta_t(\phi) = 1 \\ & \text{or } Q = \langle t, z\rangle, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

That is,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} \langle ss_1^\lambda, t\rangle & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle ss_1^\lambda, z\rangle\} \times \{\langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = P = \langle ss_1^\lambda, t\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ \langle ss_1^\lambda, z\rangle & \text{if } Q = P = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ S & \text{if } (Q,P) \in \{\langle t\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle t, z\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ A & \text{if } Q = P = \langle t\rangle, \delta_t(\psi)\delta_t(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{|\langle ss_1^\lambda, t\rangle||C_S((Q)\psi)}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle ss_1^\lambda, z\rangle\} \times \{\langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \\ & \quad \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = P = \langle ss_1^\lambda, t\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ \frac{|\langle ss_1^\lambda, z\rangle||C_S((Q)\psi)|}{|P|} & \text{if } Q = P = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ \frac{|S||C_S((Q)\psi)|}{|P|} & \text{if } (Q,P) \in \{\langle t\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle t, z\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \frac{|A||C_S((Q)\psi)|}{|P|} & \text{if } Q = P = \langle t\rangle, \delta_t(\psi)\delta_t(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } (Q,P) \in \{\langle ss_1^\lambda\rangle\} \times \{\langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle ss_1^\lambda, z\rangle\} \times \{\langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle t\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = 1 \\ & \text{or } (Q,P) \in \{\langle t, z\rangle\} \times \{\langle t, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ 3 & \text{or } Q = P = \langle ss_1^\lambda, t\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ 3^{2k+1} & \text{if } Q = P = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\lambda(\phi) = 1 \\ & \text{or } Q = P = \langle t\rangle, \delta_t(\psi)\delta_t(\phi) = 1 \\ \frac{3^{4k+2}}{|P|} & \text{or } (Q,P) \in \{\langle z\rangle\} \times \{\langle z\rangle, \langle t, z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = T_1 = A$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Then there exists $y \in S$ such that $c_x|_Q \circ \phi \circ c_y = \psi|_Q$. Since $P,Q \leq A$, we may assume $x = s^{m_x}$ and $y = s^{m_y}$ for some $m_x, m_y = -1,0,1$. Under the identification $\mathrm{Aut}(A) \to \mathrm{GL}_2(\mathbb{Z}/3^k\mathbb{Z})$, recall, from Section 6.2, that

$$c_x|_A = \begin{bmatrix} a_x & m_x \\ -3m_x & b_x \end{bmatrix} \quad \text{and} \quad \phi = \begin{bmatrix} \delta_1(\phi)a_k & \delta_1(\phi)b_k \\ -\delta_1(\phi)c_k(\phi) & -\delta_1(\phi)a_k \end{bmatrix}$$

for some $\delta_1(\phi) \in \{-1,1\}$ and

$$c_k(\phi) = \begin{cases} a_k+1 & \text{if } \phi \in \{\theta_A, \theta_A^{-1}\} \\ 2a_k-1 & \text{if } \phi \in \{\alpha_A, \beta_A\} \end{cases}$$

where, for $g = x,y$, we have

$$a_g = \begin{cases} 1 & \text{if } m_g \neq -1 \\ -2 & \text{if } m_g = -1 \end{cases} \quad \text{and} \quad b_g = \begin{cases} 1 & \text{if } m_g \neq 1 \\ -2 & \text{if } m_g = 1. \end{cases}$$

Define

$$\delta_2(\phi) = \begin{cases} -\delta_1(\phi) & \text{if } \phi \in \{\theta_A, \theta_A^{-1}\} \\ \delta_1(\phi) & \text{if } \phi \in \{\alpha_A, \beta_A\} \end{cases}$$

Explicitly, from the definitions of the morphisms, we get

$$\delta_1(\phi) = \begin{cases} 1 & \text{if } \phi \in \{\theta_A, \alpha_A\} \\ -1 & \text{if } \phi \in \{\theta_A^{-1}, \beta_A\} \end{cases} \quad \text{and} \quad \delta_2(\phi) = \begin{cases} 1 & \text{if } \in \{\theta_A^{-1}, \alpha_A\} \\ -1 & \text{if } \phi \in \{\theta_A, \beta_A\} \end{cases}$$

We also define

$$\delta = \{i \in \{1,2\} \mid \delta_i(\psi) = \delta_i(\phi)\}.$$

This means,

$$\delta = \begin{cases} \emptyset & \text{if } (\phi,\psi) \in \{\{\theta_A, \theta_A^{-1}\}, \{\alpha_A, \beta_A\}\} \\ \{1\} & \text{if } (\phi,\psi) \in \{\{\theta_A, \beta_A\}, \{\theta_A^{-1}, \alpha_A\}\} \\ \{2\} & \text{if } (\phi,\psi) \in \{\{\theta_A, \alpha_A\}, \{\theta_A^{-1}, \beta_A\}\} \\ \{1,2\} & \text{if } \phi = \psi. \end{cases}$$

In particular, note that we have the following properties of the morphisms:

ϕ	θ_A	θ_A^{-1}	α_A	β_A
$(\delta_1(\phi), \delta_2(\phi))$	$(1,-1)$	$(-1,1)$	$(1,1)$	$(-1,-1)$

Now that we have set the stage, by making a number of definitions, we begin our calculations. Let $q = s_1^{i_k} s_2^{j_k} \in Q$ for some $i_k, j_k \in \{1,..,3^k - 1\}$. If there is no ambiguity, we'll write $i_k = i$ and $j_k = j$. Note that, $(q)c_x \circ \phi \circ c_y = (q)\psi$ if and only if

$$\begin{aligned}
&\begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} a_x & m_x \\ -3m_x & b_x \end{bmatrix} \begin{bmatrix} \delta_1(\phi)a_k & \delta_1(\phi)b_k \\ -\delta_1(\phi)c_k(\phi) & -\delta_1(\phi)a_k \end{bmatrix} \begin{bmatrix} a_y & m_y \\ -3m_y & b_y \end{bmatrix} \\
&= \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} \delta_1(\psi)a_k & \delta_1(\psi)b_k \\ -\delta_1(\psi)c_k(\psi) & -\delta_1(\psi)a_k \end{bmatrix}
\end{aligned}$$

which holds, if and only if, the following two congruences are satisfied:

$$\begin{aligned}
&a_y[\delta_1(\phi)a_k(a_x i - 3m_x j) - \delta_1(\phi)c_k(\phi)(m_x i + b_x j)] \\
&-3m_y[\delta_1(\phi)b_k(a_x i - 3m_x j) - \delta_1(\phi)a_k(m_x i + b_x j)] \equiv_k \delta_1(\psi)a_k i - \delta_1(\psi)c_k(\psi)j \\
&m_y[\delta_1(\phi)a_k(a_x i - 3m_x j) - \delta_1(\phi)c_k(\phi)(m_x i + b_x j)] \\
&\quad +b_y[\delta_1(\phi)b_k(a_x i - 3m_x j) - \delta_1(\phi)a_k(m_x i + b_x j)] \equiv_k \delta_1(\psi)b_k i - \delta_1(\psi)a_k j
\end{aligned}$$

where $\equiv_k$ denotes modulo 3^k. We split our calculations into three cases as follows, depending on the values of m_x, m_y and δ. The reason for this will become apparent later on.

$m_x m_y$ \ δ	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
0	Case 2	Case 1	Case 1	Case 3
$\neq 0$	Case 3	Case 2	Case 2	Case 2

<u>Case 1</u>

In this case we have $(c_k(\phi), c_k(\psi)) \in \{(a_k+1, 2a_k-1), (2a_k-1, a_k+1)\}$. First suppose that $(c_k(\phi), c_k(\psi)) = (a_k+1, 2a_k-1)$, that is, $(\phi, \psi) \in \{\theta_A, \theta_A^{-1}\} \times \{\alpha_A, \beta_A\}$. The following table solves the two congruences for i_k and j_k and thus determines Q for different values of (m_x, m_y).

(m_x, m_y)	$(c_k(\phi), c_k(\psi)) = (1 + a_k, 1 - 2a_k)$
$(0,0)$	$\delta_1(\phi)a_k i+\delta_2(\phi)(a_k+1)j\equiv_k\delta_1(\psi)a_k i+\delta_2(\psi)(1-2a_k)j$ $\delta_1(\phi)b_k i-\delta_1(\phi)a_k j\equiv_k\delta_1(\psi)b_k i-\delta_1(\psi)a_k j$ If $\delta=\{1\}$, then $j\equiv_k 0$. If follows that $Q\leq\langle s_1\rangle$ If $\delta=\{2\}$, then $2i\equiv_k 3j$. It follows that $Q\leq\langle s_1^3 s_2^2\rangle=\langle s_2\rangle^s$
$(0,1)$	$[a_k\delta_1(\phi)i+\delta_2(\phi)(a_k+1)j]-3[\delta_1(\phi)b_k i-\delta_1(\phi)a_k j]\equiv_k\delta_1(\psi)a_k i+\delta_2(\psi)(1-2a_k)j$ $[\delta_1(\phi)a_k i+\delta_2(\phi)(a_k+1)j]-2[\delta_1(\phi)b_k i-\delta_1(\phi)a_k j]\equiv_k\delta_1(\psi)b_k i-\delta_1(\psi)a_k j$ If $\delta=\{1\}$, then $a_k i\equiv_k(1-2a_k)j$ Since $i\equiv_k a_k-5$ and $j\equiv_k a_k-3$ satisfy this congruence, we have $Q\leq\langle s_1^{a_k-5}s_2^{a_k-3}\rangle=\langle d_1\rangle^s$ If $\delta=\{2\}$, then $(a_k-b_k)i\equiv_k j$, that is, $(a_k-1)i\equiv_k(a_k+1)j$ Since $i\equiv_k-(1+a_k)$ and $j\equiv_k(1-a_k)$ satisfy this congruence, we have $Q\leq\langle s_1^{-1-ak}s_2^{1-a_k}\rangle=\langle d_2\rangle^s$
$(0,-1)$	$-2[\delta_1(\phi)a_k i+\delta_2(\phi)(a_k+1)j]+3[\delta_1(\phi)b_k i-\delta_1(\phi)a_k j]\equiv_k\delta_1(\psi)a_k i+\delta_2(\psi)(1-2a_k)j$ $-[\delta_1(\phi)a_k i+\delta_2(\phi)(a_k+1)j]+[\delta_1(\phi)b_k i-\delta_1(\phi)a_k j]\equiv_k\delta_1(\psi)b_k i-\delta_1(\psi)a_k j$ If $\delta=\{1\}$, then $a_k i\equiv_k(1+a_k)j$ Since $i\equiv_k 2a_k-4$ and $j\equiv_k a_k-3$ satisfies this congruence, we have $Q\leq\langle s_1^{2a_k-4}s_2^{a_k-3}\rangle=\langle d_2\rangle^{s^{-1}}$. If $\delta=\{2\}$, then $(-a_k+3b_k)i\equiv_k 3(a_k-1)j$ and $(-a_k+2b_k)i\equiv_k(a_k-1)j$ $\Longrightarrow a_k i\equiv_k 3(a_k-1)j$ and $(a_k-1)i\equiv_k 2(a_k-2)j$ Since $i\equiv_k 4-2a_k$ and $j\equiv_k a_k-1$ satisfy these congruences, we have $Q\leq\langle s_1^{4-2a_k}s_2^{1-a_k}\rangle=\langle d_1\rangle^{s^{-1}}$
$(1,0)$	$\delta_1(\phi)a_k(i-3j)+\delta_2(\phi)(a_k+1)(i-2j)\equiv_k\delta_1(\psi)a_k i+\delta_2(\psi)(1-2a_k)j$ $\delta_1(\phi)(i-3j)b_k-\delta_1(\phi)a_k(i-2j)\equiv_k\delta_1(\psi)b_k i-\delta_1(\psi)a_k j$ If $\delta=\{1\}$, then $i\equiv_k j$. It follows that $Q\leq\langle s_1 s_2\rangle=\langle s_1\rangle^s$ If $\delta=\{2\}$, then $i\equiv_k 3j$. It follows that $Q\leq\langle s_1^3 s_2\rangle=\langle s_2\rangle^{s^{-1}}$
$(-1,0)$	$\delta_1(\phi)a_k(-2i+3j)+\delta_2(\phi)(a_k+1)(-i+j)\equiv_k\delta_1(\psi)a_k i+\delta_2(\psi)(1-2a_k)j$ $\delta_1(\phi)(-2+3j)b_k-\delta_1(\phi)a_k(-i+j)\equiv_k\delta_1(\psi)b_k i-\delta_1(\psi)a_k j$ If $\delta=\{1\}$, then $i\equiv_k 2j$. It follows that $Q\leq\langle s_1^2 s_2\rangle=\langle s_1\rangle^{s^{-1}}$ If $\delta=\{2\}$, then $i\equiv_k 0$. It follows that $Q\leq\langle s_2\rangle$

In the case when $(c_k(\phi), c_k(\psi)) = (2a_k - 1, a_k + 1)$, we have $(\phi, \psi) \in \{\alpha_A, \beta_A\} \times \{\theta_A, \theta_A^{-1}\}$. Write $P = Q^x$ and $p = q^x$ for some $q \in Q$ and $x \in N_{\psi|_Q, \phi|_P}$. Then, there exists $y \in S$ such that $(q)c_x \circ \phi \circ c_y = (q)\psi$, which holds if and only if, $(p)\phi = (p)c_{x^{-1}} \circ \psi \circ c_{y^{-1}}$. Note that $(\psi, \phi) \in \{\theta_A, \theta_A^{-1}\} \times \{\alpha_A, \beta_A\}$ and $(c_k(\psi), c_k(\phi)) = (1 + a_k, 2a_k - 1)$. Thus, with ψ and ϕ switched, i_k and j_k satisfy the congruences listed in the table above, corresponding to the pair $(m_{x^{-1}}, m_{y^{-1}}) = (-m_x, -m_y)$. This means $P \leq R$, and therefore $Q \leq R^{x^{-1}}$, for some subgroup R, as described in the table below:

(m_{x-1},m_{y-1})	(m_x,m_y)	$c_k(\phi),c_k(\psi)=(1-2a_k,1+a_k)$
(0,0)	(0,0)	If $\delta=\{1\}$, then $P\leq\langle s_1\rangle$. So $Q\leq\langle s_1\rangle$. If $\delta=\{2\}$ then $P\leq\langle\langle s_2\rangle^s$. So $Q\leq\langle s_2\rangle^s$
(0,1)	(0,1)	If $\delta=\{1\}$, then $P\leq\langle d_1\rangle^s$. So $Q\leq\langle d_1\rangle^s$. If $\delta=\{2\}$, then $P\leq\langle d_2\rangle^s$. So $Q\leq\langle d_2\rangle^s$
$(0,-1)$	$(0,-1)$	If $\delta=\{1\}$, then $P\leq\langle d_2\rangle^{s^{-1}}$. So $Q\leq\langle d_2\rangle^{s^{-1}}$. If $\delta=\{2\}$, then $P\leq\langle d_1\rangle^{s^{-1}}$. So $Q\leq\langle d_1\rangle^{s^{-1}}$.
(1,0)	$(-1,0)$	If $\delta=\{1\}$, then $P\leq\langle s_1\rangle^s$. So $Q\leq\langle s_1\rangle^{s^{-1}}$. If $\delta=\{2\}$, then $P\leq\langle s_2\rangle^{s^{-1}}$. So $Q\leq\langle s_2\rangle^s$
$(-1,0)$	(1,0)	If $\delta=\{1\}$, then $P\leq\langle s_1\rangle^{s^{-1}}$. So $Q\leq\langle s_1\rangle^s$. If $\delta=\{2\}$, then $P\leq\langle s_2\rangle$. So $Q\leq\langle s_2\rangle^{s^{-1}}$

Case 2

Unlike the previous case, we prove this case by induction on k. Suppose that $k=1$. Then the two congruences simplify to

$$a_y\delta_2(\phi)(m_xi+b_xj)\equiv_1\delta_2(\psi)j$$
$$m_y\delta_2(\phi)(m_xi+b_xj)+a_xb_y\delta_1(\phi)i\equiv_1\delta_1(\psi)i$$

We choose $k=1$, instead of $k=2$ as our base case for induction, because the corresponding congruences are simpler. This is despite the fact that we do not need to find the solution to the congruences when $k=1$, as $k\geq 2$, in the construction of our DRV-fusion systems. The following table lists the congruences for i_1 and j_1, identified as u and v respectively. The blank cells represent those instances from cases 1 and 3.

(m,n)	$\delta=\emptyset$	$\delta=\{1\}$	$\delta=\{2\}$	$\delta=\{1,2\}$
(0,0)	$-v\equiv_1 v$ $-u\equiv_1 u$			
(0,1)	$-v\equiv_1 v$ $\pm v+2u\equiv_1 u$			
$(0,-1)$	$2v\equiv_1 v$ $\pm 2v-u\equiv_1 u$			
(1,0)	$-(u-2v)\equiv_1 v$ $-u\equiv_1 u$			
(1,1)		$u-2v\equiv_1 v$ $\pm(-u+v)+2u\equiv_1 u$	$-(u-2v)\equiv_1 v$ $\pm(u-2v)-2u\equiv_1 u$	$(u-2v)\equiv_1 v$ $\pm(u-2v)-2u\equiv_1 u$
$(1,-1)$		$-2(u-2v)\equiv_1 v$ $\pm(u-2v)-u\equiv_1 u$	$2(-u+v)\equiv_1 v$ $\pm(u-2v)+u\equiv_1 u$	$-2(u-2v)\equiv_1 v$ $\pm(u-2v)-2u\equiv_1 u$
$(-1,0)$	$-(-u+v)\equiv_1 v$ $2u\equiv_1 u$			
$(-1,1)$		$(-u+v)\equiv_1 v$ $\pm(-u+v)-4u\equiv_1 u$	$-(-u+v)\equiv_1 v$ $\pm(-u+v)+4u\equiv_1 u$	$(-u+v)\equiv_1 v$ $\pm(-u+v)+4u\equiv_1 u$
$(-1,-1)$		$-2(-u+v)\equiv_1 v$ $\pm(-u+v)+2u\equiv_1 u$	$2(-u+v)\equiv_1 v$ $\pm(-u+v)-2u\equiv_1 u$	$-2(-u+v)\equiv_1 v$ $\pm(-u+v)-2u\equiv_1 u$

The unique property of this case is that each pair of the simplified congruences listed in the table above evaluate to $i_1 \equiv_1 j_1 \equiv_1 0$. Now suppose that $k \geq 2$ and assume for induction that $i_{k-1} \equiv_{k-1} j_{k-1} \equiv_{k-1} 0$. Since $a_k \equiv_{k-1} a_{k-1}$ and $b_k \equiv_{k-1} b_{k-1}$, we have $i_k \equiv_{k-1} i_{k-1}$ and $j_k \equiv_{k-1} j_{k-1}$. By induction, $i_k \equiv_k 3^{k-1}u$ and $j_k \equiv_k 3^{k-1}v$ for some $u, v \in \mathbb{Z}$. We can now reduce the congruences to modulo 3, as follows:

$$\begin{aligned} a_y\delta_2(\phi)(m_xu + b_xv) &\equiv_1 \delta_2(\psi)v \\ m_y\delta_2(\phi)(m_xu + b_xv) + a_xb_y\delta_1(\phi)u &\equiv_1 \delta_1(\psi)u \end{aligned}$$

Switching u and v with i_k and j_k respectively, these congruences are those in the calculation for $k = 1$. It follows that $u \equiv_1 v \equiv_1 0$, that is, $i_k \equiv_k j_k \equiv_k 0$, as required.

Case 3

We prove this case also by induction on k. Suppose that $k = 2$. Since $7\delta_2(\phi) \equiv_2 -\delta_1(\phi)c_k(\phi)$, we have the congruences simplified to

$$\begin{aligned} a_y[6\delta_1(\phi)(a_xi - 3m_xj) + 7\delta_2(\phi)(m_xi + b_xj)] & \\ -3m_y[4\delta_1(\phi)a_xi] &\equiv_2 6\delta_1(\psi)i + 7\delta_2(\psi)j \\ m_y[6\delta_1(\phi)(a_xi - 3m_xj) + 7\delta_2(\phi)(m_xi + b_xj)] & \\ +b_y[4\delta_1(\phi)(a_xi - 3m_xj) - 6\delta_1(\phi)(m_xi + b_xj)] &\equiv_2 4\delta_1(\psi)i - 6\delta_1(\psi)j \end{aligned}$$

The following tables solves the congruences for i_2 and j_2 and thus determine Q for different values of (m_x, m_y). First suppose that $m_xm_y \neq 0$ and $\delta = \emptyset$:

(m_x, m_y)	$\delta = \emptyset$
$(1,1)$	$[6\delta_1(\phi)(u-3v)+7\delta_2(\phi)(u-2v)]-12\delta_1(\phi)u \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)v$ $[6\delta_1(\phi)(u-3v)+7\delta_2(\phi)(u-2v)]-2[4\delta_1(\phi)(u-3v)-6\delta_2(\phi)(u-2v)] \equiv_2 4\delta_1(\psi)-6\delta_1(\psi)$ If $\delta_2(\phi) \neq \delta_1(\phi)$, then $u \equiv_2 v \equiv_2 0$ If $\delta_2(\phi) \neq \delta_1(\phi)$ then $3u \equiv_2 3v \equiv_2 0$ and $v \equiv_2 u$. It follows that $Q \leq \langle tz \rangle$.
$(1,-1)$	$-2[6\delta_1(\phi)(u-3v)+7\delta_2(\phi)(u-2v)]+12\delta_1(\phi)u \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)$ $-[6\delta_1(\phi)(u-3v)+7\delta_2(\phi)(u-2v)]+[4\delta_1(\phi)(u-3v)-6\delta_1(\phi)(u-2v)] \equiv_2 4\delta_1(\psi)-6\delta_1(\psi)$ If $\delta_2(\phi)=\delta_1(\phi)$ then $3u \equiv_2 3v \equiv_2 0$ and $v \equiv_2 u$. It follows that $Q \leq \langle tz \rangle$. If $\delta_2(\phi) \neq \delta_1(\phi)$ then $u \equiv_2 v \equiv_2 0$
$(-1,1)$	$[6\delta_1(\phi)(-2u+3v)+7\delta_2(\phi)(-u+v)]+24\delta_1(\phi)u \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)v$ $[6\delta_1(\phi)(-2u+3v)+7\delta_2(\phi)(-u+v)]-2[4\delta_1(\phi)(-2u+3v)-6\delta_1(\phi)(-u+v)] \equiv_2 4\delta_1(\psi)u-6\delta_1(\psi)v$ If $\delta_2(\phi)=\delta_1(\phi)$ then $3u \equiv_2 3v \equiv_2 0$ and $v \equiv_2 -u$. It follows that $Q \leq \langle tz^{-1} \rangle$ $\delta_2(\phi) \neq \delta_1(\phi)$, then $u \equiv_2 v \equiv_2 0$.
$(-1,-1)$	$-2[6\delta_1(\phi)(-2u+3v)+7\delta_2(\phi)(-u+v)]-24\delta_1(\phi)u \equiv_2 6\delta_2(\psi)u+7\delta_2(\psi)v$ $-[6\delta_1(\phi)(-2u+3v)+7\delta_2(\phi)(-u+v)]+[4\delta_1(\phi)(-2u+3v)-6\delta_1(\phi)(-u+v)] \equiv_2 4\delta_1(\psi)u-6\delta_1(\psi)v$ If $\delta_2(\phi) \neq \delta_1(\phi)$, then $u \equiv_2 v \equiv_2 0$. If $\delta_2(\phi)=\delta_1(\phi)$, then $3u \equiv_2 3v \equiv_2 0$ and $v \equiv_2 -u$. It follows that $Q \leq \langle tz^{-1} \rangle$

Now suppose that $m_x m_y = 0$ and $\delta = \{1,2\}$:

(m_x, m_y)	$\delta = \{1,2\}$
$(0,1)$	$[6\delta_1(\phi)u+7\delta_2(\phi)v]-12\delta_1(\phi)u \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)$ $[6\delta_1(\phi)u+7\delta_2(\phi)v]-2[4\delta_1(\phi)-6\delta_1(\phi)v] \equiv_2 4\delta_1(\psi)u-6\delta_1(\psi)v$ $\Longrightarrow 3u \equiv_2 v \equiv_2 0$. It follows that $Q \leq \langle t \rangle$
$(0,-1)$	$-2[6\delta_1(\phi)u+7\delta_2(\phi)v]+12\delta_1(\phi)u \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)v$ $-[6\delta_1(\phi)u+7\delta_2(\phi)v]+[4\delta_1(\phi)u-6\delta_1(\phi)v] \equiv_2 4\delta_1(\psi)u-6\delta_1(\psi)$ $\Longrightarrow 3u \equiv_2 v \equiv_2 0$. It follows that $Q \leq \langle t \rangle$.
$(1,0)$	$[6\delta_1(\phi)(u-3v)+7\delta_2(\phi)(u-2v)] \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)v$ $[4\delta_1(\phi)(u-3v)-6\delta_1(\phi)(u-2v)] \equiv_2 4\delta_1(\psi)u-6\delta_1(\psi)v$ $\Longrightarrow u \equiv_2 3v \equiv_2 0$. It follows that $Q \leq \langle z \rangle$.
$(-1,0)$	$[6\delta_1(\phi)(-2u+3v)+7\delta_2(\phi)(-u+v)] \equiv_2 6\delta_1(\psi)u+7\delta_2(\psi)v$ $[4\delta_1(\phi)(-2u+3v)-6\delta_1(\phi)(-u+v)] \equiv_2 4\delta_1(\psi)u-6\delta_1(\psi)v$ $\Longrightarrow u \equiv_2 3v \equiv_2 0$. It follows that $Q \leq \langle z \rangle$.

Now fix (m_x, m_y) and $\delta \in \{\emptyset, \{1,2\}\}$ and suppose $k \geq 3$. We may assume for induction

$3i_{k-1} \equiv_{k-1} 3j_{k-1} \equiv_{k-1} 0$ and one of the following:

$i_{k-1} \equiv_{k-1} j_{k-1}$	if	$i_2 \equiv_2 j_2$
$i_{k-1} \equiv_{k-1} -j_{k-1}$	if	$i_2 \equiv_2 j_2$
$i_{k-1} \equiv_{k-1} j_k-1 \equiv_{k-1} 0$	if	$i_2 \equiv_2 j_2 \equiv_2 0$
$i_k-1 \equiv_{k-1} 0$	if	$i_2 \equiv_2 0$
$j_{k-1} \equiv_{k-1} 0$	if	$j_2 \equiv_2 0$

Since $a_k \equiv_{k-1} a_{k-1}$ and $b_k \equiv_{k-1} b_{k-1}$, we have $i_k \equiv_{k-1} i_{k-1}$ and $j_k \equiv_{k-1} j_{k-1}$. By the induction assumption $i_{k-1} = 3^{k-2}u'$ and $j_{k-1} = 3^{k-2}v'$ for some $u', v' \in \mathbb{Z}$. We can write $i_k = 3^{k-2}u$ and $j_k = 3^{k-2}v$ for some $u, v \in \mathbb{Z}$ satisfying $u = u' + 3u''$ and $v = v' + 3v''$. We can now reduce the congruences to modulo 9, as follows

$$
\begin{aligned}
&a_y[6\delta_1(\phi)(a_xu - 3m_xv) + 7\delta_2(\phi)(m_xu + b_xv)] \\
&\qquad\qquad -3m_y[4\delta_1(\phi)a_xu] \equiv_2 6\delta_1(\psi)u + 7\delta_2(\psi)v \\
&m_y[6\delta_1(\phi)(a_xu - 3m_xv) + 7\delta_2(\phi)(m_xu + b_xv)] \\
&+b_y[4\delta_1(\phi)(a_xu - 3m_xv) - 6\delta_1(\phi)(m_xu + b_xv)] \equiv_2 4\delta_1(\psi)i - 6\delta_1(\psi)v
\end{aligned}
$$

Switching u and v with i_2 and j_2 respectively, these congruences are the same as those in the calculation for $k = 2$. Note $3i_k \equiv_k 3^{k-1}u$ and $3j_k \equiv_k 3^{k-1}v$ and

$i_k \equiv_k j_k + 3^{k-2}(u - v)$	if	$i_{k-1} \equiv_{k-1} j_{k-1}$
$i_k \equiv_k -j_k + 3^{k-2}(u + v)$	if	$i_{k-1} \equiv_{k-1} -j_{k-1}$
$i_k \equiv_k 3^{k-2}u$ and $j_k \equiv_k 3^{k-2}v$	if	$i_{k-1} \equiv_{k-1} j_{k-1} \equiv_{k-1} 0$
$i_k \equiv_k 3^{k-2}u$	if	$i_{k-1} \equiv_{k-1} 0$
$j_k \equiv_k 3^{k-2}u$	if	$j_{k-1} \equiv_{k-1} 0$

Using these facts, the following tables list the solutions to the congruences and determine Q for varying values of (m_x, m_y).

(m_x,m_y)	$\delta=\emptyset$
(1,1)	If $\delta_2(\phi)=\delta_1(\phi)$, then $u\equiv_2 v\equiv_2 0$. It follows that $i_k\equiv_k j_k\equiv_k 0$. If $\delta_2(\phi)\neq\delta_1(\phi)$, then $3u\equiv_2 3\equiv_2 0$ and $v\equiv_2 u$. It follows that $i_k\equiv_k j_k$ and $Q\leq\langle tz\rangle$.
(1,−1)	If $\delta_2(\phi)=\delta_1(\phi)$, then $3u\equiv_2 3v\equiv_2 0$ and $v\equiv_2 u$. It follows that $i_k\equiv_k j_k$ and $Q\leq\langle tz\rangle$ If $\delta_2(\phi)\neq\delta_1(\phi)$ then $u\equiv_2 v\equiv_2 0$. It follows that $i_k\equiv_k j_k\equiv_k 0$
(−1,1)	If $\delta_2(\phi)=\delta_1(\phi)$ then $3u\equiv_2 3v\equiv_2 0$ and $v\equiv_2 -u$. It follows that $i_k\equiv_k -j_k$ and $Q\leq\langle tz^{-1}\rangle$. If $\delta_2(\phi)\neq\delta_1(\phi)$, then $u\equiv_2 v\equiv_2 0$. It follows that $i_k\equiv_k j_k\equiv_k 0$
(−1,−1)	If $\delta_2(\phi)\neq\delta_1(\phi)$, then $u\equiv_2 v\equiv_2 0$. It follows that $i_k\equiv_2 j_k\equiv_k 0$ If $\delta_2(\phi)=\delta_1(\phi)$, then $3u\equiv_2 3v\equiv_2 0$ and $v\equiv_2 -u$. It follows that $i_k\equiv_k -j_k$ and $Q\leq\langle tz^{-1}\rangle$.

(m_x,m_y)	$\delta=\{1,2\}$
(0,1)	$3u\equiv_2 v\equiv_2 0$. It follows that $3i_k\equiv_k j_k\equiv_k 0$ and $Q\leq\langle t\rangle$
(0,−1)	$3u\equiv_2 v\equiv_2 0$. It follows that $3i_k\equiv_k j_k\equiv_k 0$ and $Q\leq\langle t\rangle$
(1,0)	$u\equiv_2 3v\equiv_2 0$. It follows that $3i_k\equiv_k j_k\equiv_k 0$ and $Q\leq\langle z\rangle$
(−1,0)	$u\equiv_2 3v\equiv_2 0$. It follows that $i_k\equiv_k 3j_k\equiv_k 0$ and $Q\leq\langle z\rangle$

Collecting the data from the three cases,

$$\{\psi,\phi\}\in\{\{\theta_A,\alpha_A\},\{\theta_A^{-1},\beta_A\}\implies|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases}\dfrac{3^{2k}|A\cap N_S(Q,P)|}{|P|} & \text{if } Q\leq\langle s_1\rangle,\langle d_1\rangle^s,\langle d_2\rangle^{s^{-1}}\\ \dfrac{3^{2k}|sA\cap N_S(Q,P)|}{|P|} & \text{if } Q\leq\langle s_1\rangle^s\\ \dfrac{3^{2k}|s^{-1}A\cap N_S(Q,P)|}{|P|} & \text{if } Q\leq\langle s_1\rangle^{s^{-1}}\\ \dfrac{3^{2k+1}|A\cap N_S(Q,P)}{|P|} & \text{if } Q=\langle t\rangle\end{cases}$$

$$=\begin{cases}\dfrac{3^{4k}}{|P|} & \text{if } Q\leq\langle s_1\rangle\cap P,\langle d_1\rangle^s\cap P,\langle d_2\rangle^{s^{-1}}\cap P\\ & \text{or } Q\leq\langle s_1\rangle^s\cap P^{s^{-1}},\langle s_1\rangle^{s^{-1}}\cap P^s\\ \dfrac{3^{4k+1}}{|P|} & \text{if } Q=\langle t\rangle\leq P\end{cases}$$

$$\{\psi,\phi\}\in\{\{\theta_A,\beta_A\},\{\theta_A^{-1},\alpha_A\}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \dfrac{3^{2k}|A\cap N_S(Q,P)|}{|P|} & \text{if } Q\le \langle s_2\rangle^s, \langle d_1\rangle^{s^{-1}}, \langle d_2\rangle^s \\ \dfrac{3^{2k}|sA\cap N_S(Q,P)|}{|P|} & \text{if } Q\le \langle s_2\rangle^{s^{-1}} \\ \dfrac{3^{2k}|s^{-1}A\cap N_S(Q,P)|}{|P|} & \text{if } Q\le \langle s_2\rangle \\ \dfrac{3^{2k}|N_S(Q,P)}{|P|} & \text{if } Q=\langle z\rangle \end{cases}$$

$$= \begin{cases} \dfrac{3^{4k}}{|P|} & \text{if } Q\le \langle s_2\rangle^s\cap P, \langle d_1\rangle^{s^{-1}}\cap P, \langle d_2\rangle^s\cap P \\ & \text{or } Q\le \langle s_2\rangle^{s^{-1}}\cap P^{s^{-1}}, \langle s_2\rangle\cap P^s \\ \dfrac{3^{4k+1}}{|P|} & \text{if } Q=\langle z\rangle\le P \end{cases}$$

$$\{\psi,\phi\}\in\{\{\theta_A,\theta_A^{-1}\},\{\alpha_A,\beta_A\}\} \implies |\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$$

$$\psi=\phi \implies |\mathcal{O}_{\phi|_P}^{\psi_Q}| = \begin{cases} \dfrac{3^{2k}|A\cap N_S(Q,P)|}{|P|} & \text{if } Q\notin\{\langle t\rangle,\langle z\rangle\} \\ \dfrac{3^{2k+1}|A\cap N_S(Q,P)|}{|P|} & \text{if } Q=\langle t\rangle \\ \dfrac{3^{2k}|N_S(Q,P)|}{|P|} & \text{if } Q=\langle z\rangle \end{cases}$$

$$= \begin{cases} \dfrac{3^{4k}}{|P|} & \text{if } Q\notin\{\langle t\rangle,\langle z\rangle\}, Q\le P \\ \dfrac{3^{4k+1}}{|P|} & \text{if } Q\in\{\langle t\rangle,\langle z\rangle\}\le P \end{cases}$$

3. Suppose $\phi=\Theta_{\mu,\nu}$ for $\mu=1,..,4$ and $\nu=-1,0,1$.

Suppose $T=V_\lambda$ for $\lambda=-1,0,1$. Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\lambda\in Q$, then $\nu=\lambda$ and $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for some $i=-1,0,1$. For some $y\in S$, we have

$$\begin{aligned} ss_1^\lambda\in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \implies (z^{\delta_\nu(\phi)}t^{\delta_z(\phi)i})^y = z^{\delta_\lambda(\psi)} \\ &\qquad\qquad\qquad\qquad\qquad\quad \implies i=0 \quad \text{and} \quad \delta_\lambda(\psi)=\delta_\nu(\phi) \\ z\in Q &\implies (z\phi)^y = z\psi \implies t^{\delta_z(\phi)} = (ss_1^\nu)^{\delta_\lambda(\psi)}, \quad \text{a contradiction.} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} \langle ss_1^\lambda, z\rangle \cap N_S(Q,P) & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi) = \delta_\nu(\phi), \nu = \lambda \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle ss_1^\lambda, z\rangle & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\lambda(\psi) = \delta_\nu(\phi), \nu = \lambda \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi_P}^{\psi|_Q}| = \begin{cases} \frac{|\langle ss_1^\lambda, z\rangle||C_S(Q\psi)|}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\lambda(\psi) = \delta_\nu(\phi), \nu = \lambda \\ 0 & \text{othersiwise} \end{cases}$$
$$= \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\lambda(\psi) = \delta_\nu(\phi), \nu = \lambda \\ 0 & \text{othersiwise} \end{cases}$$

Suppose that $T = E_\lambda$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\lambda \in Q$, then $\lambda = \nu$, and $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for some $i = -1, 0, 1$. Since t is not S-conjugate to any element of P, $t \notin Q$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \implies (z^{\delta_\nu(\phi)} t^{\delta_z(\phi) i})^y = t^{\delta_\lambda(\psi)} \\ &\qquad\qquad\qquad\qquad\qquad\quad \implies i\delta_z(\phi)\delta_\lambda(\psi) = 1 \\ z \in Q &\implies ((z^x)\phi)^y = z\psi \implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Note that in the case when $ss_1^\lambda \in Q$, if $i = 1$, we have $x \in N_S(\langle ss_1^\lambda\rangle, \langle ss_1^\lambda z\rangle) = \langle ss_1^\lambda, z\rangle t^{-1}$. Otherwise, if $i = -1$, we have $x \in N_S(\langle ss_1^\lambda\rangle, \langle ss_1^\lambda z^{-1}\rangle) = \langle ss_1^\lambda, z\rangle t$. Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} \langle ss_1^\lambda, z\rangle t^{-1} \cap N_S(Q,P) & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_z(\phi)\delta_\lambda(\psi) = 1, \lambda = \nu \\ \langle ss_1^\lambda, z\rangle t \cap N_S(Q,P) & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_z(\phi)\delta_\lambda(\psi) = -1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle ss_1^\lambda, z\rangle t^{-1} & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\nu, z\rangle, \delta_z(\phi)\delta_\lambda(\psi) = 1, \lambda = \nu \\ \langle ss_1^\lambda, z\rangle t & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\nu, z\rangle, \delta_z(\phi)\delta_\lambda(\psi) = -1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^2|C_S(Q\psi)|}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\nu, z\rangle, \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 3^{2k} & \text{if } Q = \langle ss_1^\lambda\rangle, P = \langle ss_1^\nu, z\rangle, \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $T = A$. Let $x \in N_{\psi|_P,\phi|_Q}$. Since A is normal in S, the only elements that could S-conjugate into P are in $A \cap \langle ss_1^\lambda, z\rangle = \langle z\rangle$. For some $y \in S$, we have

$$z \in Q \implies ((z^x)\phi)^y = z\psi \implies (t^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}$$
$$\implies \delta_z(\psi)\delta_z(\phi) = 1.$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+1}|C_S(Q\psi)|}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $\phi = \Theta_{\mu,\nu}^{-1}$ for $\mu = 1,..,4$ and $\nu = -1,0,1$.

   Suppose that $T = V_\lambda$ for some $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\lambda \notin Q$. For some $y \in S$, we have

$$z \in Q \implies ((z^x)\phi)^y = z\psi \implies (ss_1^\nu)^{\delta_z(\phi)} = (ss_1^\lambda)^{\delta_z(\psi)}$$
$$\implies \lambda = \nu \text{ and } \delta_z(\phi)\delta_z(\psi) = 1$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\phi)\delta_z(\psi) = 1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \lambda = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that E_λ. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\lambda \notin Q$. If $t \in Q$, then $t^x = tz^i$ for some $i = -1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies (z^{\delta_t(\phi)}(ss_1^\nu)^{\delta_z(\phi)i})^y = (ss_1^\lambda)^{\delta_t(\psi)} \\ &\implies \lambda = \nu \text{ and } i\delta_z(\phi)\delta_t(\psi) = 1, \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\nu)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} sA \cap N_S(Q,P) & \text{if } Q = \langle t\rangle, \delta_z(\phi)\delta_t(\psi) = 1, \lambda = \nu \\ s^{-1}A \cap N_S(Q,P) & \text{if } Q = \langle t\rangle, \delta_z(\phi)\delta_t(\psi) = -1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} sA & \text{if } Q = \langle t\rangle, P = \langle t,z\rangle, \delta_z(\phi)\delta_t(\psi) = 1, \lambda = \nu \\ s^{-1}A & \text{if } Q = \langle t\rangle, P = \langle t,z\rangle, \delta_z(\phi)\delta_t(\psi) = -1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^{2k} & \text{if } Q = \langle t\rangle, P = \langle t,z\rangle, \lambda = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Let $T = A$. Let $x \in N_{\psi|_Q,\phi|_P}$. The only elements that could conjugate into P are in $A \cap P \leq \langle t,z\rangle$. If $t \in Q$, then $t^x = tz^i$ for some $i = -1,0,1$. Thus, for some $y \in S$, we

have

$$\begin{aligned}
t\in Q &\implies ((tz^i)\phi)^y = t\psi \\
&\implies (z^{\delta_t(\phi)}(ss_1^{\nu})^{\delta_z(\phi)i})^y = z^{\delta_t(\psi)} \\
&\implies i=0 \text{ and } \delta_t(\phi)\delta_t(\psi)=1 \\
z\in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^{\nu})^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} A\cap N_S(Q,P) & \text{if } Q=\langle t\rangle, \delta_t(\phi)\delta_t(\psi)=1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} A & \text{if } Q=\langle t\rangle, P\in\{\langle t\rangle,\langle t,z\rangle\}, \delta_t(\phi)\delta_t(\psi)=1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q=\langle t\rangle, P\in\{\langle t\rangle,\langle t,z\rangle\}, \delta_t(\phi)\delta_t(\psi)=1 \\ 0 & \text{otherwise.} \end{cases}$$

5. Let $\phi=\vartheta_{\mu,\nu}$ for $\mu=1,..,4$ and $\nu=-1,1$.

   Suppose that $T=V_\lambda$ for $\lambda=-1,0,1$. Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\lambda\in Q$ then $\lambda=\nu\neq 0$. For some $y\in S$, we have

$$\begin{aligned}
ss_1^\lambda\in Q \implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi &\implies z^{\delta_\nu(\phi)}(ss_1^{-\nu})^{i\delta_z(\phi)} = z^{\delta_\lambda(\psi)} \\
&\implies i=0 \text{ and } \delta_\lambda(\psi)\delta_\nu(\phi)=1 \\
z\in Q \implies ((z^x)\phi)^y = z\psi &\implies ((ss_1^{-\nu})^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)} \\
&\implies \nu=-\lambda\neq 0 \text{ and } \delta_z(\psi)\delta_z(\phi)=1.
\end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P)\cap\langle ss_1^{\lambda},z\rangle & \text{if } Q=\langle ss_1^{\lambda}\rangle, \delta_{\lambda}(\psi)\delta_{\nu}(\phi)=1, \lambda=\nu\neq 0 \\ N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1, \lambda=-\nu\neq 0 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\lambda},z\rangle & \text{if } Q=\langle ss_1^{\lambda}\rangle, P\in\{\langle ss_1^{\nu}\rangle,\langle ss_1^{\nu},z\rangle\}, \delta_{\lambda}(\psi)\delta_{\nu}(\phi)=1, \lambda=\nu\neq 0 \\ S & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^{\nu},z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1, \lambda=-\nu\neq 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle ss_1^{\lambda}\rangle, P\in\{\langle ss_1^{\nu}\rangle,\langle ss_1^{\nu},z\rangle\}, \delta_{\lambda}(\psi)\delta_{\nu}(\phi)=1, \lambda=\nu\neq 0 \\ & \text{or } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^{\nu},z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1, \lambda=-\nu\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $T=E_{\lambda}$ for $\lambda=-1,0,1$. Let $x\in N_{\psi|_Q,\phi|_P}$. Note that t is not S-conjugate to any element of P. Thus, $t\notin Q$. If $ss_1^{\lambda}\in Q$, then $(ss_1^{\lambda})^x = ss_1^{\lambda}z^i$ for some $i=-1,0,1$. Thus, for some $y\in S$, we have

$$ss_1^{\lambda}\in Q \implies ((ss_1^{\lambda}z^i)\phi)^y = (ss_1^{\lambda})\psi \implies (z^{\delta_{\nu}(\phi)}(ss_1^{-\nu})^{\delta_z(\phi)})^y = t^{\delta_{\lambda}(\psi)}, \text{ a contradiction.}$$
$$z\in Q \implies ((z^x)\phi)^y = z\psi \implies ((ss_1^{-\nu})^{\delta_z(\phi)})^y = z^{\delta_{\lambda}(\psi)} \text{ a contradiction.}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

Suppose that $T=A$. Let $x\in N_{\psi|_Q,\phi|_P}$. Since A is normal in S, every element that conjugates into P is in $A\cap\langle ss_1^{\lambda},z\rangle = \langle z\rangle$. For some $y\in S$, we have

$$z\in Q \implies ((z^x)\phi)^y = z\psi \implies ((ss_1^{-\nu})^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

6. Let $\phi=\xi_{\mu,\nu}$ for $\mu=1,..,8$ and $\nu=-1,1$.

Suppose that $T=V_{\lambda}$ for $\lambda=-1,0,1$. If $ss_1^{\lambda}\in Q$ then $\lambda=\nu\neq 0$ and $(ss_1^{\lambda})^x=(ss_1^{\lambda})^x = ss_1^{\lambda}z^i$ for some $i=-1,0,1$.

First suppose that $1 \leq \mu \leq 4$. For some $y \in S$,

$$\begin{aligned}
ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \\
&\implies (s^{\delta_\nu(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_\lambda(\psi)}, \text{ a contradiction} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now, suppose that $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \\
&\implies (z^{\delta_\nu(\phi)} s^{i\delta_z(\phi)})^y = z^{\delta_\lambda(\psi)} \\
&\implies i = 0 \text{ and } \delta_\lambda(\psi)\delta_\nu(\phi) = 1 \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (s^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)} \\
&\implies \lambda = 0 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^\lambda, z\rangle & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \neq 0 \\ N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = 0 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle ss_1^\lambda, z\rangle & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\nu\rangle, \langle ss_1^\nu, z\rangle\}, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \neq 0 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\nu, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = 0 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\nu\rangle, \langle ss_1^\nu, z\rangle\}, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \neq 0 \\ & \text{or } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\nu, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $T = E_\lambda$ for $\lambda = -1, 0, 1$. If $ss_1^\lambda \in Q$, then $\lambda = \nu \neq 0$ and $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for some $i = -1, 0, 1$. Note also that $t \notin Q$.

First suppose that $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \\ &\implies (s^{\delta_\nu(\phi)} z^{i\delta_z(\phi)})^y = t^{\delta_\lambda(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose $5 \leq \mu \leq 8$. For some $y \in S$,

$$\begin{aligned} ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \\ &\implies (z^{\delta_\nu(\phi)} s^{i\delta_z(\phi)})^y = t^{\delta_\lambda(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose that $T = A$. In this cases, the only elements of Q that could conjugate into P are in $A \cap \langle ss_1^\nu, z \rangle = \langle z \rangle$.

First suppose that $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now, suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Let $\phi = \xi_{\mu,\nu}^{-1}$ for $\mu = 1,..,8$ and $\nu = -1,1$.

Suppose that $T = V_\lambda$ for some $\lambda = -1,0,1$. If $ss_1^\lambda \in Q$, then $\lambda = 0$ and $s^x = sz^i$ for $i = -1,0,1$.

First suppose $1 \leq \mu \leq 4$. For some $y \in Q$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies ((ss_1^\nu)^{\delta_s(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_s(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now, suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_s(\phi)}(ss_1^\nu)^{i\delta_z(\phi)})^y = z^{\delta_s(\psi)} \\ &\implies i = 0 \text{ and } \delta_s(\psi)\delta_s(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\nu)^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)} \\ &\implies \lambda = \nu \neq 0 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle s,z \rangle & \text{if } Q = \langle s \rangle, \delta_s(\psi)\delta_s(\phi) = 1, \lambda = 0 \\ N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = \nu \neq 0 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle s,z \rangle & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \delta_s(\psi)\delta_s(\phi) = 1, \lambda = 0 \\ S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s,z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = \nu \neq 0 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle s\rangle, P\in\{\langle s\rangle,\langle s,z\rangle\}, \delta_s(\psi)\delta_s(\phi)=1, \lambda=0 \\ & \text{or } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1, \lambda=\nu\neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $T = E_\lambda$ for $\lambda=-1,0,1$. If $ss_1^\lambda \in Q$ then $\lambda=0$ and $s^x = sz^i$ for some $i=-1,0,1$. Note also that $t\notin Q$.

First suppose $1\le\mu\le 4$. For some $y\in S$, we have

$$\begin{aligned} s\in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies ((ss_1^\nu)^{\delta_s(\phi)} z^{i\delta_z(\phi)})^y = t^{\delta_s(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi)=1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ 0 & \text{otherwise} \end{cases}$$

Now suppose $5\le\mu\le 8$. For some $y\in S$, we have

$$\begin{aligned} s\in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_z(\phi)}(ss_1^\nu)^{i\delta_z(\phi)})^y = t^{\delta_s(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\nu)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = A$. Since A is normal in S, the only elements in Q that could conjugate into S are in $A \cap \langle s, z \rangle = \langle z \rangle$.

First suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now suppose that $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu})^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_\nu^{\pm 1} \circ \eta$, for some $\nu = -1, 1$.

   Suppose that $T = V_\lambda$ for $\lambda = -1, 0, 1$. Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^\lambda \in Q$, then $\lambda = \nu$ and $(ss_1^\lambda)^x = ss_1^\lambda t^i z^j$ for $i, j = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q &\implies ((ss_1^\nu t^i z^j)\phi)^y = (ss_1^\lambda)\psi \\ &\implies (t^{\delta_\nu(\phi)}(ss_1^{-\nu})^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = z^{\delta_\lambda(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

   Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$. Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^\lambda \in Q$, then $\lambda = \nu$ and $(ss_1^\lambda)^x = ss_1^\lambda t^i z^j$ for some $i, j = -1, 0, 1$. If $t \in Q$, then $t^x = tz^\ell$ for some $\ell = -1, 0, 1$.

For some $y \in S$, we have

$$\begin{aligned}
ss_1^\lambda \in Q &\implies ((ss_1^\nu t^i z^j)\phi)^y = (ss_1^\lambda)\psi \\
&\implies (t^{\delta_\nu(\phi)}(ss_1^{-\nu})^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = t^{\delta_\lambda(\psi)} \\
&\implies i = 0 \text{ and } \delta_\lambda(\psi)\delta_\nu(\phi) = 1 \\
t \in Q &\implies ((tz^\ell)\phi)^y = t\psi \\
&\implies ((ss_1^{-\nu})^{\delta_t(\phi)} z^{\ell\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_t(\psi)} \\
&\implies \lambda = -\nu \text{ and } \delta_t(\psi)\delta_t(\phi) = 1 \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases}
\langle ss_1^\lambda, t\rangle \cap N_S(Q,P) & \text{if } Q = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \\
& \text{or } Q = \langle ss_1^\lambda, z\rangle, \delta_\lambda(\psi)\delta_\nu(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \lambda = \nu \\
N_S(Q,P) & \text{if } Q = \langle t\rangle, \delta_t(\psi)\delta_t(\phi) = 1, \lambda = -\nu \\
& \text{or } Q = \langle t,z\rangle, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \lambda = -\nu \\
\emptyset & \text{otherwise}
\end{cases} \\
&= \begin{cases}
\langle ss_1^\lambda, z\rangle & \text{if } Q = P = \langle ss_1^\lambda\rangle, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \\
\langle ss_1^\lambda, t\rangle & \text{if } Q = \langle ss_1^\lambda\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \\
& \text{or } Q = \langle ss_1^\lambda, z\rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_\lambda(\psi)\delta_\nu(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \\
& \quad \text{and } \lambda = \nu \neq 0 \\
A & \text{if } Q = P = \langle t\rangle, \delta_t(\psi)\delta_t(\phi) = 1, \lambda = -\nu \\
S & \text{if } Q = \langle t\rangle, P \in \{\langle t,z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = 1, \lambda = -\nu \\
& \text{or } Q = \langle t,z\rangle, P \in \{\langle t,z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \lambda = -\nu \\
\emptyset & \text{otherwise}
\end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\lambda \rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \neq 0 \\ & \text{or } Q = \langle ss_1^\lambda, z \rangle, P \in \{\langle ss_1^\nu, z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_\lambda(\psi)\delta_\nu(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \\ & \quad \text{and } \lambda = \nu \neq 0 \\ & \text{or } Q = \langle t \rangle, P \in \{\langle t, z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = 1, \lambda = -\nu \\ & \text{or } Q = \langle t, z \rangle, P \in \{\langle t, z\rangle, \langle ss_1^\nu, t\rangle\}, \delta_t(\psi)\delta_t(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \lambda = -\nu \\ 3^{2k+1} & \text{if } Q = P = \langle ss_1^\lambda \rangle, \delta_\lambda(\psi)\delta_\nu(\phi) = 1, \lambda = \nu \neq 0 \\ & \text{or } Q = P = \langle t \rangle, \delta_t(\psi)\delta_t(\phi) = 1, \lambda = -\nu \\ 0 & \text{otherwise} \end{cases}$$

Let $T = A$. Let $x \in N_{\psi|_Q, \phi|_P}$. Note that the only elements that could be in Q are in $A \cap \langle ss_1^\nu, z \rangle = \langle t, z \rangle$. If $t \in Q$, then $t^x = tz^i$ for some $i = -1, 0, 1$. For some $y \in S$,

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies ((ss_1^\nu)^{\delta_t(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_\nu}, \hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}, \hat{\beta}_{V_\nu}\}$ for $\nu = -1, 1$.

Suppose $T = V_\lambda$ for $\lambda = -1, 0, 1$. Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^\lambda \in Q$ then $\lambda = \nu$ and $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for $i = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \\ &\implies ((ss_1^\nu)^{\delta_\nu(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_\lambda(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^\lambda)^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$. Let $x \in N_{\psi|_Q, \phi|_P}$. Note that $t \notin Q$. If $ss_1^\lambda \in Q$, then $\lambda = \nu$ and $(ss_1^{\lambda})^x = ss_1^{\lambda} z^i$ for $i = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\lambda} \in Q &\implies ((ss_1^{\nu} z^i)\phi)^y = (ss_1^{\lambda})\psi \\
&\implies ((ss_1^{\nu})^{\delta_{\nu}(\phi)} z^{i\delta_z(\phi)})^y = t^{\delta_{\lambda}(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q, \phi|_P} &= \begin{cases} N_S(Q,P) & Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} S & Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = A$. Since A is normal, the only elements that lie Q are in $A \cap \langle ss_1^{\nu}, z \rangle = \langle z \rangle$. Thus, for some $y \in S$,

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu,\nu}$ for $\mu = 1, .., 4$ and $\nu = -1, 1$.

Suppose $T = V_{\lambda}$ for $\lambda = -1, 0, 1$. Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^{\lambda} \in Q$ then $\lambda = \nu$ and $(ss_1^{\lambda})^x = ss_1^{\lambda} z^i$ for $i = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\lambda} \in Q &\implies ((ss_1^{\nu} z^i)\phi)^y = (ss_1^{\lambda})\psi \\
&\implies ((ss_1^{-\nu})^{\delta_{\nu}(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_{\lambda}(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = (ss_1^{\lambda})^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = E_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. If $ss_1^\lambda \in Q$, then $\lambda = \nu$ and $(ss_1^\lambda)^x = ss_1^\lambda z^i$ for $i = -1,0,1$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^\lambda \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\lambda)\psi \\
&\implies ((ss_1^{-\nu})^{\delta_\nu(\phi)} z^{i\delta_z(\phi)})^y = t^{\delta_\lambda(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} S & Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^\nu, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = A$. Since A is normal, the only elements that lie Q are in $A \cap \langle ss_1^\nu, z \rangle = \langle z \rangle$. Thus, for some $y \in S$,

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.2.3 $\psi = \Theta_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..,4$ and $\nu = -1,0,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   By Lemma $B.2.2$, $\psi|_Q \circ \phi$ is never a conjugation map. It follows that $|N_{\psi|_Q,\phi|_P}| = 0$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Suppose that $T = V_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \lambda$ and

$(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q \implies ((ss_1^{\lambda} z^i)\phi)^y = (ss_1^{\nu})\psi &\implies (z^{\delta_\lambda(\phi)}(ss_1^{\lambda})^{\delta_z(\phi)i})^y = z^{\delta_\nu(\psi)} \\ &\implies i = 0 \text{ and } \delta_\nu(\psi)\delta_\lambda(\phi) = 1. \\ z \in Q \implies (z\phi)^y = z\psi &\implies (ss_1^{\lambda})^{\delta_z(\phi)} = t^{\delta_z(\psi)}, \text{a contradiction.} \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\lambda}\rangle, \langle ss_1^{\lambda}, z\rangle\}, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\lambda}\rangle, \langle ss_1^{\lambda}, z\rangle\}, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then, $\lambda = \nu$ and $(ss_1^{\nu})^x = ss_1^{\nu} t^i z^j$ for some $i, j = -1, 0, 1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\lambda} t^i z^j)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (t^{\delta_\lambda(\phi)}(ss_1^{\lambda})^{\delta_t(\phi)i} z^{\delta_z(\phi)j})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = A$. Let $x \in N_{\psi|_Q,\phi|_P}$. In this case, the only elements in Q that could S-conjugate into P are in $\langle z\rangle$. Thus, for some $y \in S$, we have

$$\begin{aligned} z \in Q \implies ((z^x)\phi)^y = z\psi &\implies (t^{\delta_z(\phi)})^y = t^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Thus,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\phi)\delta_z(\psi)=1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q=\langle z\rangle, \langle z\rangle \subseteq P, P\in \mathcal{D}_A, \delta_z(\phi)\delta_z(\psi)=1 \\ \emptyset & \text{otherwise} \end{cases}$$

It follows that

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+1}|C_S(Q\psi)|}{|P|} & \text{if } Q=\langle z\rangle, \langle z\rangle \subseteq P, P\in \mathcal{D}_A, \delta_z(\psi)\delta_z(\phi)=1 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q=\langle z\rangle, \langle z\rangle \subseteq P, P\in \mathcal{D}_A, \delta_z(\psi)\delta_z(\phi)=1 \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\phi=\Theta_{\mu_1,\nu_1}$ for $\mu_1=1,..,4$ and $\nu_1=-1,0,1$.

Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu=\nu_1$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for some $i=-1,0,1$. Thus, for some $y\in S$, we have

$$\begin{aligned} ss_1^\nu \in Q &\implies ((ss_1^\nu z^i)\phi)^y = (ss_1^\nu)\psi \implies (z^{\delta_{\nu_1}(\phi)}t^{\delta_z(\phi)i})^y = z^{\delta_\nu(\psi)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \implies i=0 \text{ and } \delta_\nu(\psi)=\delta_{\nu_1}(\phi) \\ z\in Q &\implies ((z^x)\phi)^y = z\psi \implies (t^{\delta_z(\phi)})^y = t^{\delta_z(\psi)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \implies \delta_z(\psi)=\delta_z(\phi) \end{aligned}$$

Collecting the data, we have

$$N_{\psi|_Q,\phi|_P} = \begin{cases} \langle ss_1^\nu, z\rangle \cap N_S(Q,P) & \text{if } Q=\langle ss_1^\nu\rangle; \delta_\nu(\psi)=\delta_{\nu_1}(\phi); \nu=\nu_1 \\ N_S(Q,P) & \text{if } Q=\langle z\rangle; \delta_z(\psi)=\delta_z(\phi) \\ \langle ss_1^\nu, z\rangle \cap N_S(Q,P) & \text{if } Q=\langle ss_1^\nu, z\rangle; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=\delta_z(\psi)\delta_z(\phi)=1; \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle ss_1^\nu, z\rangle & \text{if } Q=\langle ss_1^\nu\rangle; P\in\{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ S & \text{if } Q=\langle z\rangle; P\in\{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}; \delta_z(\psi)=\delta_z(\phi) \\ \langle ss_1^\nu, z\rangle & \text{if } Q=P=\langle ss_1^\nu, z\rangle; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=\delta_z(\psi)\delta_z(\phi)=1; \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{|\langle ss_1^{\nu},z\rangle||C_S(Q\psi)|}{|P|} & \text{if } Q=\langle ss_1^{\nu}\rangle; P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ \frac{|S||C_S(Q\psi)|}{|P|} & \text{if } Q=\langle z\rangle; P\in\{\langle z\rangle,\langle ss_1^{\nu_1},z\rangle\}; \delta_z(\psi)=\delta_z(\phi) \\ \frac{|\langle ss_1^{\nu},z\rangle||C_S(Q\psi)|}{|P|} & \text{if } Q=P=\langle ss_1^{\nu},z\rangle; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=\delta_z(\psi)\delta_z(\phi)=1; \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle ss_1^{\nu}\rangle; P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ \frac{3^{4k+1}}{|P|} & \text{if } Q=\langle z\rangle; P\in\{\langle z\rangle,\langle ss_1^{\nu_1},z\rangle\}; \delta_z(\psi)=\delta_z(\phi) \\ 3^{2k} & \text{if } Q=P=\langle ss_1^{\nu},z\rangle; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=\delta_z(\psi)\delta_z(\phi)=1; \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $\phi=\Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1=1,..,4$ and $\nu_1=-1,0,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu}\notin Q$. For some $y\in S$,

$$\begin{aligned} z\in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \quad \text{a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

5. Let $\phi=\vartheta_{\mu_1,\nu_1}$ for $\mu_1=1,..,4$ and $\nu_1=-1,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu}\in Q$, then $\nu=\nu_1$ and $(ss_1^{\nu})^x=ss_1^{\nu}z^i$ for some $i=-1,0,1$. Thus, for some $y\in S$, we have

$$\begin{aligned} ss_1^{\nu}\in Q &\implies ((ss_1^{\nu}z^i)\phi)^y=(ss_1^{\nu})\psi \implies (z^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{\delta_z(\phi)i})^y = z^{\delta_\nu(\psi)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \implies i=0 \text{ and } \delta_\nu(\phi)=\delta_{\nu_1}(\psi) \\ z\in Q &\implies ((z^x)\phi)^y = z\psi \implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \quad \text{a contradiction.} \end{aligned}$$

   Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} \langle ss_1^{\nu},z\rangle\cap N_S(Q,P) & \text{if } Q=\langle ss_1^{\nu}\rangle; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\nu},z\rangle & \text{if } Q=\langle ss_1^{\nu}\rangle; P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}; \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{|\langle ss_1^{\nu},z\rangle||C_S(Q\psi)|}{|P|} & \text{if } Q=\langle ss_1^{\nu}\rangle; P\in\{\langle ss_1^{\nu}\rangle,\langle ss_1^{\nu},z\rangle\}; \delta_{\nu}(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle ss_1^{\nu}\rangle; P\in\{\langle ss_1^{\nu}\rangle,\langle ss_1^{\nu},z\rangle\}; \delta_{\nu}(\psi)\delta_{\nu_1}(\phi)=1; \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Suppose $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for some $i = -1,0,1$.

Suppose $1 \le \mu_1 \le 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (s^{\delta_{\nu}(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_{\nu}(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \le \mu \le 8$. For some $y \in S$,we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu}(\phi)} s^{i\delta_z(\phi)})^y = z^{\delta_{\nu}(\psi)} \\ &\implies i=0 \text{ and } \delta_{\nu}(\psi)\delta_{\nu_1}(\phi)=1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P)\cap\langle ss_1^{\nu},z\rangle & \text{if } Q=\langle ss_1^{\nu}\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi)=1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle ss_1^{\nu},z\rangle & \text{if } Q=\langle ss_1^{\nu}\rangle, P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}, \\ & \quad \delta_{\nu}(\psi)\delta_{\nu_1}(\phi)=1, \nu=\nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

7. Suppose $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = 0$ and $s^x = sz^i$ for $i = -1, 0, 1$.

Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_s(\phi)} z^{i\delta_z(\psi)})^y = z^{\delta_s(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_s(\phi)}(ss_1^{\nu_1})^{i\delta_z(\psi)})^y = z^{\delta_s(\psi)} \\ &\implies i = 0 \text{ and } \delta_s(\psi)\delta_s(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle s, z\rangle & \text{if } Q = \langle s\rangle, \delta_s(\psi)\delta_s(\phi) = 1, \nu = 0 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle s, z\rangle & \text{if } Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s, z\rangle\}, \\ & \quad \delta_s(\psi)\delta_s(\phi) = 1, \nu = 0 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_Q}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s,z\rangle\}, \\ & \quad \delta_s(\psi)\delta_s(\phi) = 1, \nu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^{\nu} \in Q$ then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu_1} t^i z^j$ for $i, j = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} t^i z^j)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (t^{\delta_{\nu_1}}(ss_1^{-\nu_1})^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$. Then, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu_1, \nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$. Then, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = t^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.2.4 $\psi = \Theta_{\mu,\nu}^{-1}$ and $Q \in \mathcal{D}_{A_0}$, for $\mu = 1,..4$ and $\nu = -1,0,1$.

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   By Lemma $B.2.2$, $\psi|_Q \circ \phi$ is never a conjugation map. It follows that $|N_{\psi|_Q,\phi|_P}| = 0$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Suppose $T = V_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q \implies ((z^x)\phi)^y = z\psi &\implies ((ss_1^{\lambda})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\ &\implies \nu = \lambda \text{ and } \delta_z(\phi)\delta_z(\psi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \lambda \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\lambda}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \lambda \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\lambda}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

   Suppose $T = E_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $t \in Q$, then $t^x = tz^i$ for some

$i=-1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies ((ss_1^{\lambda})^{\delta_t(\phi)} z^{\delta_z(\phi)i})^y = z^{\delta_t(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = A$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $t \in Q$ then $t^x = tz^i$ for some $i=-1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies (z^{\delta_t(\phi)} t^{\delta_z(\phi)i})^y = z^{\delta_t(\psi)} \\ &\implies i = 0 \text{ and } \delta_t(\phi)\delta_t(\psi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap A & \text{if } Q = \langle t \rangle, \delta_t(\psi)\delta_t(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} A & \text{if } Q = \langle t \rangle, \langle t \rangle \leq P, P \in \mathcal{D}_A, \delta_t(\psi)\delta_t(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle t \rangle, \langle t \rangle \leq P, P \in \mathcal{D}_A, \delta_t(\psi)\delta_t(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = (ss_1^{\nu_1})^{\delta_z(\psi)} \quad \text{a contradiction} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

4. Let $\phi = \Theta^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $t \in Q$, then $t^x = tz^i$ for some $i = -1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned}
t \in Q &\implies ((tz^i)\phi)^y = t\psi \\
&\implies (z^{\delta_t(\phi)}(ss_1^{\nu_1})^{\delta_z(\phi)i})^y = z^{\delta_t(\psi)} \\
&\implies i = 0 \text{ and } \delta_t(\phi) = \delta_t(\psi) \\
z \in Q &\implies ((z^x)\phi)^y = z\psi \\
&\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\
&\implies \nu = \nu_1 \text{ and } \delta_z(\phi) = \delta_z(\psi)
\end{aligned}$$

   Thus,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} A \cap N_S(Q,P) & \text{if } Q = \langle t\rangle, \delta_t(\phi)\delta_t(\psi) = 1 \\ N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\phi)\delta_z(\psi) = 1, \nu = \nu_1 \\ A \cap N_S(Q,P) & \text{if } Q = \langle t,z\rangle, \delta_z(\phi)\delta_z(\psi) = \delta_t(\phi)\delta_t(\psi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} A & \text{if } Q = \langle t\rangle, P \in \{\langle t\rangle, \langle t,z\rangle\}, \delta_t(\phi)\delta_t(\psi) = 1 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \nu = \nu_1 \\ A & \text{if } Q = P = \langle t,z\rangle, \delta_z(\phi)\delta_z(\psi) = \delta_t(\phi)\delta_t(\psi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

   It follows that

$$\begin{aligned}
|\mathcal{O}^{\psi|_Q}_{\phi|_P}| &= \begin{cases} \frac{3^{2k}|C_S(((\langle z\rangle)\psi)|}{|P|} & \text{if } Q = \langle t\rangle, P \in \{\langle t\rangle, \langle t,z\rangle\}, \delta_t(\phi)\delta_t(\psi) = 1 \\ \frac{3^{2k+1}|C_S(((\langle z\rangle)\psi)|}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \nu = \nu_1 \\ \frac{3^{2k}|C_S(((\langle t,z\rangle)\psi)|}{|P|} & \text{if } Q = P = \langle t,z\rangle, \delta_z(\phi)\delta_z(\psi) = \delta_t(\phi)\delta_t(\psi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{3^{4k+1}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_t(\phi)\delta_t(\psi) = 1 \\ \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \nu = \nu_1 \\ 3^{2k} & \text{if } Q = P = \langle t,z\rangle, \delta_z(\phi)\delta_z(\psi) = \delta_t(\phi)\delta_t(\psi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q \implies ((z^x)\phi)^y = z\psi \implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y &= (ss_1^{\nu})^{\delta_z(\psi)} \\ \implies \nu = -\nu_1 \text{ and } \delta_z(\phi) &= \delta_z(\psi) \end{aligned}$$

Thus,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\phi) = \delta_z(\psi), \nu = -\nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\phi) = \delta_z(\psi), \nu = -\nu_1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

It follows that,

$$\begin{aligned} |\mathcal{O}_{\phi|_P}^{\psi|_Q}| &= \begin{cases} \frac{3^{2k+1}|C_S((\langle z \rangle)\psi)|}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\phi) = \delta_z(\psi), \nu = -\nu_1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\phi) = \delta_z(\psi), \nu = -\nu_1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

6. Suppose $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$.

   Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\ &\implies \nu = 0 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1 \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = 0 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

7. Suppose $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$.

   Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\ &\implies \nu = \nu_1 \text{ and } \delta_z(\psi)\delta_z(\psi) = 1. \end{aligned}$$

   Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \neq 0 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $t \in Q$ then $t^x = tz^i$ for some $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_t(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_t(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_t(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

9. $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

10. $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.2.5 $\psi = \vartheta_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..4$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   By $B.2.2$, the only time when $\psi|_Q \circ \phi$ is a conjugation map is when $Q \leq \langle ss_1^\lambda z^i \rangle$ for some $i = -1,1$. However, $\langle ss_1^\lambda z^i \rangle \notin \mathcal{D}_{V_\nu}$. Thus, $|N_{\psi|_Q,\phi|_P}| = 0$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Suppose $T = V_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \lambda \neq 0$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for $i = -1,0,1$. For some $y \in S$, we have

$$
\begin{aligned}
ss_1^\nu \in Q &\implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\nu)\psi \\
&\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{\delta_z(\phi)i})^y = z^{\delta_\nu(\psi)} \\
&\implies i = 0 \text{ and } \delta_\nu(\psi)\delta_\lambda(\phi) = 1 \\
z \in Q &\implies ((z^x)\phi)^y = z\psi \\
&\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\lambda
\end{aligned}
$$

   Collecting the data,

$$
\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\lambda \neq 0 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\lambda \neq 0 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}
$$

   Thus,

$$
|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ & \text{or } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\lambda \neq 0 \\ 0 & \text{otherwise.} \end{cases}
$$

   Suppose $T = E_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \lambda \neq 0$ and

$(ss_1^\nu)^x = ss_1^\nu t^i z^j$ for some $i, j = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^\nu \in Q &\implies ((ss_1^\lambda t^i z^j)\phi)^y = (ss_1^\nu)\psi \\ &\implies (t^{\delta_\lambda(\phi)}(ss_1^\lambda)^{\delta_t(\phi)i} z^{\delta_z(\phi)j})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies ((z^x)\phi)^y = z\psi \\ &\implies z^{\delta_z(\phi)} = (ss_1^{-\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = A$. Since A is normal in S, the elements in Q that S-conjugate into P are in $\langle z \rangle$. Thus, for some $y \in S$, we have

$$z \in Q \implies ((z^x)\phi)^y = z\psi \implies (t^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)}, \text{ a contradiction.}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \nu_1$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for some $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^\nu \in Q \implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^\nu)\psi &\implies (z^{\delta_{\nu_1}(\phi)} t^{\delta_z(\phi)i})^y = z^{\delta_\nu(\phi)} \\ &\implies i = 0 \text{ and } \delta_\nu(\psi) = \delta_{\nu_1}(\phi) \\ z \in Q \implies ((z^x)\phi)^y = z\psi &\implies (t^{\delta_z(\phi)})^y = (ss_1^{-\nu_1})^{\delta_z(\psi)}, \quad \text{a contradiction.} \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^\nu, z \rangle & \text{if } Q = \langle ss_1^\nu \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle ss_1^\nu, z \rangle & \text{if } Q = \langle ss_1^\nu \rangle, P \in \{\langle ss_1^{\nu_1} \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \\ & \quad \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu \rangle, P \in \{\langle ss_1^{\nu_1} \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \\ & \quad \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

4. Let $\phi = \Theta^{-1}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q \implies ((z^x)\phi)^y = z\psi &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)} \\ &\implies \nu = -\nu_1 \text{ and } \delta_z(\phi)\delta_z(\psi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \nu = -\nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \nu = -\nu_1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle\}, \delta_z(\phi)\delta_z(\psi) = 1, \nu = -\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Suppose that $ss_1^{\nu} \in Q$. Then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu_1}z^i$ for some $i = -1,0,1$. Then for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q \implies ((ss_1^{\nu}z^i)\phi)^y = (ss_1^{\nu})\psi &\implies (z^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{\delta_z(\phi)i})^y = z^{\delta_\nu(\psi)} \\ &\implies i = 0 \text{ and } \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1 \\ z \in Q \implies ((z^x)\phi)^y = z\psi &\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)} \\ &\implies \nu = \nu_1 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} \langle ss_1^{\nu}, z\rangle \cap N_S(Q,P) & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle ss_1^{\nu}, z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = P = \langle ss_1^{\nu}, z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{|\langle ss_1^{\nu}, z\rangle||C_S(Q\psi)|}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = P = \langle ss_1^{\nu}, z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \frac{|S||C_S(Q\psi)|}{|P|} & \text{if } Q = \langle z\rangle; P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ 3^2 & \text{if } Q = P = \langle ss_1^{\nu}, z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Suppose $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$.

Suppose $1 \le \mu_1 \le 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (s^{\delta_{\nu_1}(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies (z^{\delta_{\nu_1}(\phi)}s^{i\delta_z(\phi)})^y = z^{\delta_{\nu}(\psi)} \\
&\implies i = 0 \text{ and } \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1 \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (s^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)} \\
&\implies \nu = 0, \text{ a contradiction.}
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

7. Suppose $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$.

   Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have,

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)} \\
&\implies \nu = -\nu_1 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1, \nu=-\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle s,z\rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi)=1, \nu=-\nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle s,z\rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi)=1, \nu=-\nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

8. Let $\phi=\theta_{\nu_1}^{\pm1}\circ\eta$ for $\mu_1=-1,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu\in Q$ then $\nu=\nu_1$ and $(ss_1^\nu)^x=ss_1^\nu t^i z^j$ for $i,j=-1,0,1$. Thus, for some $y\in S$, we have

$$\begin{aligned} ss_1^\nu\in Q &\implies ((ss_1^{\nu_1}t^iz^j)\phi)^y=(ss_1^\nu)\psi \\ &\implies (t^{\delta_\lambda(\phi)}(ss_1^{-\nu})^{i\delta_t(\phi)}z^{j\delta_z(\phi)})^y=z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y=z\psi \\ &\implies (z^{\delta_z(\phi)})^y=(ss_1^{-\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

9. Let $\phi\in\{\hat{\theta}_{V_{\nu_1}},\hat{\theta}_{V_{\nu_1}}^{-1},\hat{\alpha}_{V_{\nu_1}},\hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1=-1,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu\in Q$ then $\nu=\nu_1$ and $(ss_1^\nu)^x=ss_1^\nu z^i$ for $i=-1,0,1$. Thus, for some $y\in S$, we have

$$\begin{aligned} ss_1^\nu\in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y=(ss_1^\nu)\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y=z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y=z\psi \\ &\implies (z^{\delta_z(\phi)})^y=(ss_1^{-\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

10. Let $\phi=\hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1=1,..,4$ and $\nu_1=-1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Then for some $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi_1|_Q \circ \eta \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

where $\psi_1 \in \{\theta_{V_\nu}, \theta_{V_\nu}^{-1}, \alpha_{V_\nu}, \beta_{V_\nu}\}$ and $\phi_1 \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$. It follows that we can apply this case.

B.2.6 $\psi = \xi_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..8$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   By Lemma $B.2.2$, $\psi|_Q \circ \phi$ is a conjugation map only when $Q = \langle z \rangle$ and $(\phi, \psi) \in \mathcal{M}_\xi(\langle z \rangle)$. Let $Q = \langle z \rangle$. Suppose $x \in N_{\psi|_Q,\phi|_P}$. Then, by choosing any $y \in S$, we see that $c_{x(y\phi)}|_Q = \psi|_Q \circ \phi$. Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, (\phi,\psi) \in \mathcal{M}_\xi(\langle z \rangle) \\ \emptyset & \text{otherwise} \end{cases}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \le P, (\phi,\psi) \in \mathcal{M}_\xi(\langle z \rangle) \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Suppose $T = V_\lambda$ for some $\lambda = -1,0,1$. If $ss_1^\nu \in Q$, then $\nu = \lambda \neq 0$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for some $i = -1,0,1$.

   First suppose $1 \le \mu \le 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\nu \in Q &\implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\nu)\psi \\ &\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_z(\phi)})^y = s^{\delta_\nu(\psi)} \\ &\implies \lambda = 0 \implies \nu = 0, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now, suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^\nu \in Q &\implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\nu)\psi \\
&\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)} \\
&\implies i = 0 \text{ and } \delta_\nu(\psi)\delta_\lambda(\phi) = 1. \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = s^{\delta_z(\psi)} \\
&\implies \lambda = 0 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = 0 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = 0 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^\nu\rangle, P \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_\nu(\psi)\delta_\lambda(\phi) = 1, \nu = \lambda \neq 0 \\ & \text{or } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^\lambda, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = E_\lambda$ for $\lambda = -1, 0, 1$. If $ss_1^\nu \in Q$, then $\nu = \lambda \neq 0$ and $(ss_1^\nu)^x = ss_1^\nu t^i z^j$ for some $i, j = -1, 0, 1$.

First suppose $1 \le \mu \le 4$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^\nu \in Q &\implies ((ss_1^\lambda t^i z^j)\phi)^y = (ss_1^\nu)\psi \\
&\implies (t^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_t(\phi)} z^{i\delta_z(\phi)})^y = s^{\delta_\nu(\psi)} \\
&\implies \lambda = 0, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1
\end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\lambda}, z\rangle, \langle t, z\rangle, \langle ss_1^{\lambda}, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\lambda}, z\rangle, \langle t, z\rangle, \langle ss_1^{\lambda}, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\lambda} t^i z^j)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (t^{\delta_\lambda(\phi)} (ss_1^{\lambda})^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = (z\psi) \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = A$. Since A is normal, the only elements in Q that S-conjugate into P are in $A \cap \langle ss_1^{\nu}, z\rangle = \langle z\rangle$.

Suppose $1 \le \mu \le 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = s^{\delta_z(\psi)} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Suppose $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4, \nu_1 = -1,0,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1,0,1$.

Suppose that $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)} t^{i\delta_z(\phi)})^y = s^{\delta_{\nu}(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose that $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)} t^{i\delta_z(\phi)})^y = z^{\delta_{\nu}(\psi)} \\ &\implies i = 0 \text{ and } \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = s^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

4. Suppose $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,4, \nu_1 = -1,0,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$.

   Suppose that $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose that $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = s^{\delta_z(\psi)} \\ &\implies \nu_1 = 0 \text{ and } \delta_z(\psi)\delta_z(\phi_) = 1 \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu_1 = 0 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu_1 = 0 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

5. Suppose $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1 \neq 0$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$.

   Suppose $1 \le \mu \le 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{i\delta_z(\phi)})^y = s^{\delta_{\nu}(\psi)} \\ &\implies \nu = \nu_1 = 0, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (ss_1^{-\nu_1})^{\delta_z(\phi)} = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)} \\ &\implies i = 0 \text{ and } \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (ss_1^{-\nu_1})^{\delta_z(\phi)} = s^{\delta_z(\psi)} \\ &\implies \nu_1 = 0, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) \cap \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\nu}, z\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Suppose $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for $i = -1, 0, 1$.

   Suppose $1 \le \mu, \mu_1 \le 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (s^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = s^{\delta_\nu(\psi)} \\ &\implies \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = \langle ss_1^{\nu}, z\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\psi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\nu}, z\rangle & \text{if } Q = P = \langle ss_1^{\nu}\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \langle ss_1^{\nu}, t\rangle & \text{if } Q = \langle ss_1^{\nu}\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = P = \langle ss_1^{\nu}, z\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\psi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle ss_1^{\nu}\rangle, P \in \{\langle ss_1^{\nu_1}\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = P = \langle ss_1^{\nu}, z\rangle, \delta_{\nu}(\psi)\delta_{\nu_1}(\psi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu, \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)} s^{i\delta_z(\phi)})^y = z^{\delta_{\nu}(\psi)} \\ &\implies i = 0 \text{ and } \delta_{\nu}(\psi)\delta_{\nu_1}(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = s^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P)\cap\langle ss_1^{\nu},z\rangle & \text{if } Q=\langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1, \nu=\nu_1 \\ & \text{or } Q=\langle ss_1^{\nu},z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\psi)=\delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\nu},z\rangle & \text{if } Q=\langle ss_1^{\nu}\rangle, P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}, \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1, \nu=\nu_1 \\ & \text{or } Q=P=\langle ss_1^{\nu},z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\psi)=\delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ S & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^{\nu_1},z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle ss_1^{\nu}\rangle, P\in\{\langle ss_1^{\nu_1}\rangle,\langle ss_1^{\nu_1},z\rangle\}, \delta_\nu(\psi)\delta_{\nu_1}(\phi)=1, \nu=\nu_1 \\ & \text{or } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle ss_1^{\nu_1},z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ 3^2 & \text{if } Q=P=\langle ss_1^{\nu},z\rangle, \delta_\nu(\psi)\delta_{\nu_1}(\psi)=\delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Suppose $1\le\mu\le4$ and $5\le\mu_1\le8$. For some $y\in S$, we have

$$\begin{aligned} ss_1^{\nu}\in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y=(ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}s^{i\delta_z(\phi)})^y=s^{\delta_\nu(\psi)} \\ &\implies i\neq0 \text{ and } i\delta_\nu(\psi)\delta_z(\phi)=1 \\ z\in Q &\implies (z\phi)^y=z\psi \\ &\implies (s^{\delta_z(\phi)})^y=z^{\delta_z(\psi)}, \text{ a contradiction} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P)\cap\langle ss_1^{\nu},z\rangle t^{-1} & \text{if } Q=\langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_z(\phi)=i=1, \nu=\nu_1 \\ N_S(Q,P)\cap\langle ss_1^{\nu},z\rangle t & \text{if } Q=\langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_z(\phi)=i=-1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle ss_1^{\nu},z\rangle t^{-1} & \text{if } Q=\langle ss_1^{\nu}\rangle, P=\langle ss_1^{\nu_1},z\rangle, \delta_\nu(\psi)\delta_z(\phi)=i=1, \nu=\nu_1 \\ \langle ss_1^{\nu},z\rangle t & \text{if } Q=\langle ss_1^{\nu}\rangle, P=\langle ss_1^{\nu_1},z\rangle, \delta_\nu(\psi)\delta_z(\phi)=i=-1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^{\nu}\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \delta_\nu(\psi)\delta_z(\phi) \neq 0, \nu = \nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose $5 \leq \mu \leq 8$ and $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (s^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = s^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Suppose $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$.

Suppose $1 \leq \mu, \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \leq \mu, \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = s^{\delta_z(\phi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $1 \leq \mu \leq 4$ and $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have,

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu \leq 8$ and $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have,

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = s^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} t^i z^j$ for $i, j = -1, 0, 1$.

Suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} t^i z^j)\phi)^y = (ss_1^{\nu_1})\psi \\ &\implies (t^{\delta_{\nu_1}(\phi)} (ss_1^{-\nu_1})^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = s^{\delta_{\nu}(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q, \phi|_P} &= \begin{cases} N_S(Q, P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} t^i z^j)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (t^{\delta_{\nu_1}(\phi)} (ss_1^{-\nu_1})^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = s^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$.

Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)} z^{i\delta_z(\phi)})^y = s^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q, \phi|_P} &= \begin{cases} N_S(Q, P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = z^{\delta_{\nu}(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = s^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for $i = -1,0,1$.

Suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies ((ss_1^{-\nu_1})^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = s^{\delta_{\nu}(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies ((ss_1^{-\nu_1})^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = z^{\delta_\nu(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = s^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.2.7 $\psi = \xi_{\mu,\nu}^{-1}$ and $Q \in \mathcal{D}_{V_0}$, for $\mu = 1,..8$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   By Lemma $B.2.2$, $\psi|_Q \circ \phi$ is a conjugation map when $Q = \langle z \rangle$ and $(\phi, \psi) \in \mathcal{M}_{\xi^{-1}}(\langle z \rangle)$. Let $Q = \langle z \rangle$. Suppose $x \in N_{\psi|_Q, \phi|_P}$. Then, by choosing any $y \in S$, we see that $c_{x(y\phi)}|_Q = \psi|_Q \circ \phi$. Collecting the data,

$$N_{\psi|Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, (\phi, \psi) \in \mathcal{M}_{\xi^{-1}}(\langle z \rangle) \\ \emptyset & \text{otherwise} \end{cases}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle \leq P, (\phi, \psi) \in \mathcal{M}_{\xi^{-1}}(\langle z \rangle) \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Suppose $T = V_\lambda$. If $s \in Q$, then $\lambda = 0$ and $s^x = sz^i$ for $i = -1, 0, 1$.

   First suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned}
s \in Q &\implies ((sz^i)\phi)^y = s\psi \\
&\implies (z^{\delta_s(\phi)}s^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_s(\psi)} \\
&\implies \nu = 0, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^{\lambda})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now, suppose $5 \leq \mu \leq 8$. Thus, for some $y \in S$, we have

$$\begin{aligned}
s \in Q &\implies ((sz^i)\phi)^y = s\psi \\
&\implies (z^{\delta_s(\phi)} s^{i\delta_z(\phi)})^y = z^{\delta_s(\psi)} \\
&\implies i = 0 \text{ and } \delta_s(\psi)\delta_s(\phi) = 1. \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^{\lambda})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\
&\implies \nu = \lambda \neq 0 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle s,z \rangle & \text{if } Q = \langle s \rangle, \delta_s(\psi)\delta_s(\phi) = 1, \lambda = 0 \\ N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \lambda \neq 0 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle s,z \rangle & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \delta_s(\psi)\delta_s(\phi) = 1, \lambda = 0 \\ S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\lambda}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \lambda \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \delta_s(\psi)\delta_s(\phi) = 1, \lambda = 0 \\ & \text{or } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\lambda}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1, \nu = \lambda \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose $T = E_{\lambda}$ for $\lambda = -1, 0, 1$. If $s \in Q$, then $\lambda = 0$ and $s^x = sz^i$ for $i = -1, 0, 1$.

First suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned}
s \in Q &\implies ((sz^i)\phi)^y = s\psi \\
&\implies (t^{\delta_s(\phi)} z^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_s(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\lambda}, z\rangle, \langle t, z\rangle, \langle ss_1^{\lambda}, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\lambda}, z\rangle, \langle t, z\rangle, \langle ss_1^{\lambda}, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Now, suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (t^{\delta_s(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_s(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\phi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = A$. Since A is normal in S, the only elements in Q that could conjugate in P are in $A \cap \langle s, z\rangle = \langle z\rangle$.

First suppose $1 \le \mu \le 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Now, suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Suppose $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $s \in Q$, then $\nu_1 = 0$ and $s^x = sz^i$ for $i = -1,0,1$.

Suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_s(\phi)} t^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_s(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_s(\phi)} t^{i\delta_z(\phi)})^y = z^{\delta_s(\psi)} \\ &\implies i = 0 \text{ and } \delta_s(\psi)\delta_s(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle s,z \rangle & \text{if } Q = \langle s \rangle, \delta_s(\psi)\delta_s(\phi) = 1, \nu_1 = 0 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle s,z \rangle & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \\ & \quad \delta_s(\psi)\delta_s(\phi) = 1, \nu_1 = 0 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle s \rangle, P \in \{\langle s \rangle, \langle s,z \rangle\}, \\ & \quad \delta_s(\psi)\delta_s(\phi) = 1, \nu_1 = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

4. Suppose $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $s \notin Q$.

Suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose that $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\ &\implies \nu = \nu_1 \text{ and } \delta_z(\psi) = \delta_z(\phi). \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu_1 = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $s \notin Q$.

   Suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = 0$. Suppose $5 \leq \mu \leq 8$. For some $y \in S$,

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\ &\implies \nu = -\nu_1 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1 \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\nu_1 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \\ & \quad \delta_z(\psi)\delta_z(\phi) = 1, \nu = -\nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

6. Suppose $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $s \notin Q$.

   Suppose $1 \leq \mu, \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle ss_1^{\nu_1}, z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \leq \mu, \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $1 \leq \mu \leq 4$ and $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have,

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$. Suppose $5 \leq \mu \leq 8$ and $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have,

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Suppose $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $s \in Q$, then $s^x = sz^i$ for $i = -1, 0, 1$.

Suppose $1 \leq \mu, \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_s(\phi)} z^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_s(\psi)} \\ &\implies \nu = \nu_1 \text{ and } \delta_s(\psi)\delta_s(\phi) = 1 \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle s\rangle, \delta_s(\psi)\delta_s(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = \langle s,z\rangle, \delta_s(\psi)\delta_s(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} \langle s,z\rangle & \text{if } Q = P = \langle s\rangle, \delta_s(\psi)\delta_s(\phi) = 1, \nu = \nu_1 \\ \langle s,t\rangle & \text{if } Q = \langle s\rangle, P = \langle s,z\rangle, \delta_s(\psi)\delta_s(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = P = \langle s,z\rangle, \delta_s(\psi)\delta_s(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle s\rangle, P \in \{\langle s\rangle, \langle s,z\rangle\}, \delta_s(\psi)\delta_s(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = P = \langle s,z\rangle, \delta_s(\psi)\delta_s(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle s,z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $5 \le \mu, \mu_1 \le 8$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_s(\phi)}(ss_1^{\nu_1})^{i\delta_z(\phi)})^y = z^{\delta_s(\psi)} \\ &\implies i = 0 \text{ and } \delta_s(\psi)\delta_s(\phi) = 1 \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)} \\ &\implies \nu = \nu_1 \text{ and } \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) \cap \langle s,z\rangle & \text{if } Q=\langle s\rangle, \delta_s(\psi)\delta_s(\phi)=1 \\ & \text{or } Q=\langle s,z\rangle, \delta_s(\phi)\delta_s(\psi)=\delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle s,z\rangle & \text{if } Q=\langle s\rangle, \delta_s(\psi)\delta_s(\phi)=1 \\ & \text{or } Q=\langle s,z\rangle, \delta_s(\phi)\delta_s(\psi)=\delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ S & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{2k+3}}{|P|} & \text{if } Q=\langle s\rangle, \delta_s(\psi)\delta_s(\phi)=1 \\ & \text{or } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ 3^2 & \text{or } Q=\langle s,z\rangle, \delta_s(\phi)\delta_s(\psi)=\delta_z(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $1 \le \mu \le 4$ and $5 \le \mu_1 \le 8$. For some $y \in S$, we have,

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies (z^{\delta_s(\phi)}(ss_1^{\nu_1})^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_s(\psi)} \\ &\implies \nu=\nu_1 \text{ and } i \neq 0 \text{ and } i\delta_z(\phi)\delta_s(\psi)=1 \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) \cap \langle s,z\rangle t^{-1} & \text{if } Q=\langle s\rangle, \delta_s(\psi)\delta_z(\phi)=i=1, \nu=\nu_1 \\ N_S(Q,P) \cap \langle s,z\rangle t & \text{if } Q=\langle s\rangle, \delta_s(\psi)\delta_z(\phi)=i=-1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle s,z\rangle t^{-1} & \text{if } Q=\langle s\rangle, P=\langle s,z\rangle, \delta_s(\psi)\delta_z(\phi)=i=1, \nu=\nu_1 \\ \langle s,z\rangle t & \text{if } Q=\langle s\rangle, P=\langle s,z\rangle, \delta_s(\psi)\delta_z(\phi)=i=-1, \nu=\nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle s \rangle, P = \langle s, z \rangle, \delta_s(\psi)\delta_z(\phi) \neq 0, \nu = \nu_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose $5 \le \mu \le 8$ and $1 \le \mu_1 \le 4$. For some $y \in S$, we have

$$\begin{aligned} s \in Q &\implies ((sz^i)\phi)^y = s\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_s(\phi)} z^{i\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q, \phi|_P}$. Note that $s \notin Q$.

Suppose $1 \le \mu \le 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q, \phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle t, z \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle ss_1^{\nu_1}, t \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \le \mu \le 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q, \phi|_P}$. Note that $s \notin Q$.

   Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q, \phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle ss_1^{\nu_1}, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle ss_1^{\nu_1}, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

10. Let $\phi = \hat{\vartheta}_{\mu_1, \nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q, \phi|_P}$. Note that $s \notin Q$.

Suppose $1 \leq \mu \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle ss_1^{\nu_1}, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle ss_1^{\nu_1}, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

B.2.8 $\psi = \theta_\nu^{\pm 1} \circ \eta$ and $Q \in \mathcal{D}_\nu$, for $\nu = -1, 1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. For any $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi|_P \circ c_y = \psi_1|_Q \circ \eta \quad \text{for } \psi_1 \in \{\theta_\lambda, \theta_\lambda^{-1}\} \\ &\Leftrightarrow c_x|_Q \circ \phi|_P \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ \phi|_P \circ \eta \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

   Thus, it suffices to look at the case when $\phi \circ \eta \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $\psi_1 \in \{\theta_\lambda, \theta_\lambda^{-1}\}$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

Suppose that $T = V_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. If $ss_1^\nu \in Q$, then $\nu = \lambda$ and $(ss_1^\nu)^x = ss_1^\lambda z^i$ for $i = -1,0,1$. For some $y \in S$, we have

$$\begin{aligned} ss_1^\nu \in Q &\implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\nu)\psi \\ &\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_z(\phi)})^y = t^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = E_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. Then for some $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi|_P \circ c_y \circ \eta = \psi_1|_Q \circ \eta \text{ (for } \psi_1 = \theta_\nu^{\pm 1}) \\ &\Leftrightarrow c_x|_Q \circ \phi|_P \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ (\phi|_P \circ \eta) \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

Thus $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = |\mathcal{O}_{(\phi\circ\eta)|_P}^{\psi_1|_Q}|$, which we have already calculated in another section. Although in the calculation of $|\mathcal{O}_{(\phi\circ\eta)|_P}^{\psi_1|_Q}|$, $\phi\circ\eta$ is limited to $\theta_\lambda^{\pm1}\circ\eta$ for $\lambda = -1,1$, the proof is the same when $\phi\circ\eta \in \{\theta_\lambda\circ\eta, \theta_\lambda^{-1}\circ\eta, \alpha_\lambda\circ\eta, \beta_\lambda\circ\eta \mid \lambda = -1,0,1\}$, though the proof gives us slightly different results.

Suppose $T = A$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\nu \notin Q$. If $t \in Q$ then $t^x = tz^i$ for $i = -1,0,1$. For some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)\phi = t\psi \\ &\implies (z^{\delta_t(\phi)}t^{i\delta_z(\phi)})^y = (ss_1^\nu)^{\delta_t(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. If $ss_1^\nu \in Q$, then $\nu = \nu_1$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for

$i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)} t^{i\delta_z(\phi)})^y = t^{\delta_\nu}(\psi) \\ &\implies i\delta_\nu(\psi)\delta_z(\phi) = 1 \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap \langle ss_1^{\nu}, z\rangle t^{-1} & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_z(\phi) = 1, i = 1, \nu = \nu_1 \\ N_S(Q,P) \cap \langle ss_1^{\nu}, z\rangle t & \text{if } Q = \langle ss_1^{\nu}\rangle, \delta_\nu(\psi)\delta_z(\phi) = -1, i = -1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle ss_1^{\nu}, z\rangle t^{-1} & \text{if } Q = \langle ss_1^{\nu}\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \delta_\nu(\psi)\delta_z(\phi) = 1, i = 1, \nu = \nu_1 \\ \langle ss_1^{\nu}, z\rangle t & \text{if } Q = \langle ss_1^{\nu}\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \delta_\nu(\psi)\delta_z(\phi) = -1, i = -1, \nu = \nu_1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^{2k} & \text{if } Q = \langle ss_1^{\nu}\rangle, P = \langle ss_1^{\nu_1}, z\rangle, \nu = \nu_1 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 0, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$. If $t \in Q$, then $t^x = tz^i$ for $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} t \in Q &\implies ((tz^i)\phi)^y = t\psi \\ &\implies (z^{\delta_t(\phi)} (ss_1^{\nu_1})^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_t(\psi)} \\ &\implies \nu = -\nu_1 \neq 0 \text{ and } i\delta_t(\psi)\delta_z(\phi) = 1 \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data, we have

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P)\cap sA & \text{if } Q=\langle t\rangle, \delta_t(\psi)\delta_z(\phi)=1, i=1, \nu=-\nu_1\neq 0 \\ N_S(Q,P)\cap s^{-1}A & \text{if } Q=\langle t\rangle, \delta_t(\psi)\delta_z(\phi)=-1, i=-1, \nu=-\nu_1\neq 0 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} sA & \text{if } Q=\langle t\rangle, P=\langle t,z\rangle, \delta_t(\psi)\delta_z(\phi)=1, i=1, \nu=-\nu_1\neq 0 \\ s^{-1}A & \text{if } Q=\langle t\rangle, P=\langle t,z\rangle, \delta_t(\psi)\delta_z(\phi)=-1, i=-1, \nu=-\nu_1\neq 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}^{\psi|_Q}_{\phi|_P}| = \begin{cases} 3^{2k} & \text{if } Q=\langle t\rangle, P=\langle t,z\rangle, \nu=-\nu_1\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

5. Let $\phi=\vartheta_{\mu_1,\nu_1}$ for $\mu_1=1,..,4$ and $\nu_1=-1,0,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. Note that $t\notin Q$. If $ss_1^{\nu}\in Q$, then $\nu=\nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for some $i=-1,0,1$. Then, for some $y\in S$, we have

$$\begin{aligned} ss_1^{\nu}\in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{i\delta_z(\phi)})^y = t^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}^{\psi|_Q}_{\phi|_P}|=0$.

6. Let $\phi=\xi_{\mu_1,\nu_1}$ for $\mu_1=1,..,8$ and $\nu_1=-1,1$.

   Let $x\in N_{\psi|_Q,\phi|_P}$. Note that $t\notin Q$. If $ss_1^{\nu}\in Q$, then $\nu=\nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for some $i=-1,0,1$.

   Suppose $1\le\mu_1\le 4$. For some $y\in S$, we have

$$\begin{aligned} ss_1^{\nu}\in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (s^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = t^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi)=1. \end{aligned}$$

Collecting the data, we have

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^{\nu_1}, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^{\nu_1}, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Suppose that $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i))^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}s^{i\delta_z(\phi)})^y = t^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Let $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1,1$.

   Suppose $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$ and $ss_1^{\nu} \notin Q$.

   Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$,

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle s, z\rangle, \langle z\rangle\}, \delta_Z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle s,z\rangle, \langle z\rangle\}, \delta_Z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Then for some $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q \Leftrightarrow & \ c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi_1|_Q \circ \eta \quad \text{for } \psi_1 = \theta_\nu^{\pm 1}, \phi_1 = \theta_{\nu_1}^{\pm 1} \\ \Leftrightarrow & \ c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi_1|_Q \\ \Leftrightarrow & \ c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

   It follows that we can apply the case when $(\psi_1, \phi_1) = (\theta_\nu^{\pm 1}, \theta_{\nu_1}^{\pm 1})$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $t \notin Q$. If $ss_1^\nu \in Q$, then $\nu = \nu_1$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for some $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^\nu \in Q \implies & ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^\nu)\psi \\ \implies & ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)} z^{i\delta_z(\phi)})^y = t^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q \implies & (z\phi)^y = z\psi \\ \implies & (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ \implies & \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^{\nu_1}, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle ss_1^{\nu_1}, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Then for some $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi_1|_Q \circ \eta \quad \text{for } \psi_1 = \theta_\nu^{\pm 1}, \phi_1 \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\} \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

It follows that we can apply the case when $\psi_1 = \theta_\nu^{\pm 1}$ and $\phi_1 \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$.

B.2.9 $\psi \in \{\hat{\theta}_{V_\nu}, \hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}, \hat{\beta}_{V_\nu}\}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\nu = -1, 1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega \circ \eta\}$ and $P \in \mathcal{D}$.

   By Lemma $B.2.3$, $\psi|_Q \circ \phi$ is a conjugation map when

   $$\begin{aligned} &Q = \langle s s_1^\lambda z^i \rangle \quad \text{and} \quad (\phi, \psi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle z \rangle) \text{ for } i = -1, 0, 1, \text{ or} \\ &Q = \langle z \rangle \quad \text{and} \quad (\phi, \psi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle z \rangle), \text{ or} \\ &Q \le \langle s s_1^\lambda, z \rangle \quad \text{and} \quad (\phi, \psi) \in \mathcal{M}_{\hat{V}_\lambda}(\langle s s_1^\lambda, z \rangle). \end{aligned}$$

   In the first case, $\langle s s_1^\lambda z^i \rangle \notin \mathcal{D}_{V_\lambda}$ for $i = -1, 1$. So suppose $i = 0$. Let $x \in N_{\psi|_Q,\phi|_P}$. Then by defining $y = x^{-1}$, we see that $c_{x(y\phi)}|_Q = \psi|_Q \circ \phi$. It follows that

   $$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle s s_1^\lambda \rangle \\ \emptyset & \text{otherwise} \end{cases}$$

   Thus,

   $$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = \langle s s_1^\lambda \rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

   For the second case, let $x \in N_{\psi|_Q,\phi|_P}$. Then for any $y \in S$, we see that $c_{x(y\phi)}|_Q = \psi|_Q \circ \phi$. It follows that

   $$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

For the final case, note that $(\phi,\psi) \in \{(\mathrm{Id}_S, \hat{\alpha}_{V_\lambda}), (\omega, \hat{\theta}_{V_\lambda}^{-1})\}$. Since $\hat{\alpha}_{V_\lambda} = \mathrm{Id}|_{\langle ss_1^\lambda, z\rangle}$ and $\hat{\theta}_{V_\lambda}^{-1} = \omega|_{\langle ss_1^\lambda, z\rangle}$, we apply results from another section to get

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q \in \{\langle ss_1^\lambda\rangle, \langle ss_1^\lambda, z\rangle\}, Q \le P \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle \le P \\ 0 & \text{otherwise} \end{cases}$$

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $P \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

   Let $T = V_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \lambda$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for $i = -1,0,1$. Thus, for some $y \in S$,

$$\begin{aligned} ss_1^\nu \in Q &\implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\nu)\psi \\ &\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_z(\phi)})^y = (ss_1^\nu)^{\delta_\nu(\psi)} \\ &\implies i\delta_\nu(\psi)\delta_z(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) \cap t^{-1}\langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, i = 1, \delta_\nu(\psi)\delta_z(\phi) = 1, \lambda = \nu \\ N_S(Q,P) \cap t\langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, i = -1, \delta_\nu(\psi)\delta_z(\phi) = -1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} t^{-1}\langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, P = \langle ss_1^\lambda, z\rangle, i = 1, \delta_\nu(\psi)\delta_z(\phi) = 1, \lambda = \nu \\ t\langle ss_1^\nu, z\rangle & \text{if } Q = \langle ss_1^\nu\rangle, P = \langle ss_1^\lambda, z\rangle, i = -1, \delta_\nu(\psi)\delta_z(\phi) = -1, \lambda = \nu \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q = \langle ss_1^\nu\rangle, P = \langle ss_1^\lambda, z\rangle, \lambda = \nu \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T=E_\lambda$ for $\lambda=-1,0,1$. Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu\in Q$, then $\nu=\lambda$ and $(ss_1^\nu)^x=ss_1^\nu t^i z^j$ for $i,j=-1,0,1$. For some $y\in S$, we have

$$\begin{aligned}
ss_1^\nu\in Q &\implies ((ss_1^\lambda t^i z^j)\phi)^y=(ss_1^\nu)\psi\\
&\implies (t^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_t(\phi)}z^{j\delta_z(\phi)})^y=(ss_1^\nu)^{\delta_\nu(\psi)}\\
&\implies i\delta_\nu(\psi)\delta_t(\phi)=1.\\
z\in Q &\implies (z\phi)^y=z\psi\\
&\implies (z^{\delta_z(\phi)})^y=z^{\delta_z(\psi)}\\
&\implies \delta_z(\psi)\delta_z(\phi)=1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases}
N_S(Q,P)\cap u\langle ss_1^\nu,t\rangle & \text{if } Q=\langle ss_1^\nu\rangle, i=1, \delta_\nu(\psi)\delta_t(\phi)=1\\
N_S(Q,P)\cap u^{-1}\langle ss_1^\nu,t\rangle & \text{if } Q=\langle ss_1^\nu\rangle, i=-1, \delta_\nu(\psi)\delta_t(\phi)=-1\\
N_S(Q,P) & \text{if } Q=\langle z\rangle, \delta_z(\psi)\delta_z(\phi)=1\\
\emptyset & \text{otherwise}
\end{cases}\\
&= \begin{cases}
u\langle ss_1^\nu,t\rangle & \text{if } Q=\langle ss_1^\nu\rangle, P=\langle ss_1^\lambda,t\rangle, i=1, \delta_\nu(\psi)\delta_t(\phi)=1, \lambda=\nu\\
u^{-1}\langle ss_1^\nu,t\rangle & \text{if } Q=\langle ss_1^\nu\rangle, P=\langle ss_1^\lambda,t\rangle, i=-1, \delta_\nu(\psi)\delta_t(\phi)=-1, \lambda=\nu\\
S & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle t,z\rangle,\langle ss_1^\lambda,z\rangle,\langle ss_1^\lambda,t\rangle\}, \delta_z(\psi)\delta_z(\phi)=1\\
\emptyset & \text{otherwise}
\end{cases}
\end{aligned}$$

where $u=s_2^{3^{k-2}}$. Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=\begin{cases}
3^2 & \text{if } Q=\langle ss_1^\nu\rangle, P=\langle ss_1^\lambda,t\rangle, \lambda=\nu\\
\frac{3^{4k+2}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle z\rangle,\langle t,z\rangle,\langle ss_1^\lambda,z\rangle,\langle ss_1^\lambda,t\rangle\}, \delta_z(\psi)\delta_z(\phi)=1\\
0 & \text{otherwise}
\end{cases}$$

Suppose $T=A$. Let $x\in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\nu\notin Q$. For some $y\in S$, we have

$$\begin{aligned}
z\in Q &\implies (z\phi)^y=z\psi\\
&\implies (t^{\delta_z(\phi)})^y=z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}|=0$.

3. Let $\phi=\Theta_{\mu_1,\nu_1}$ for $\mu_1=1,..,4$ and $\nu_1=-1,0,1$.

Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu\in Q$, then $(ss_1^\nu)^x=ss_1^\nu z^i$ and $\nu=\nu_1$. Thus, for some $y\in S$, we

have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies (z^{\delta_{\nu_1}(\phi)} t^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_\nu(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

4. $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$. Note that $ss_1^{\nu} \notin Q$. Thus for some $y \in S$, we have

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^{\nu})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

5. Let $\phi = \vartheta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$ then $(ss_1^{\nu})^x = ss_1^{\nu_1} z^i$ and $\nu = \nu_1$. Thus, for some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies (z^{\delta_z(\phi)} (ss_1^{-\nu_1})^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_\nu(\psi)} \\
&\implies \nu = -\nu_1, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

6. Let $\phi = \xi_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,8$ and $\nu = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu_1} z^i$ for some $i = -1,0,1$.

Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies (s^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_\nu(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\
&\implies (z^{\delta_{\nu_1}(\phi)}s^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_\nu(\psi)}, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (s^{\delta_z(\psi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Let $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$.

Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1
\end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle t, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle t, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\lambda})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, the $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}t^iz^j$ for $i, j = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}t^iz^j)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (t^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{i\delta_t(\phi)}z^{j\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_\nu(\psi)} \\ &\implies \nu = -\nu_1, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle ss_1^{\nu_1}, t \rangle, \langle ss_1^{\nu_1}, z \rangle, \langle t, z \rangle, \langle z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle ss_1^{\nu_1}, t\rangle, \langle ss_1^{\nu_1}, z\rangle, \langle t, z\rangle, \langle z\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{\nu_1}\}$ for $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q, \phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \nu_1$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^\nu \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^\nu)\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)} z^{i\delta_z(\phi)})^y = (ss_1^\nu)^{\delta_\nu(\psi)} \\ &\implies \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1 \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q, \phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle ss_1^\nu \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ & \text{or } Q = \langle ss_1^\nu, z \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} \langle ss_1^\nu, z \rangle & \text{if } Q = P = \langle ss_1^\nu \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ \langle ss_1^\nu, t \rangle & \text{if } Q = \langle ss_1^\nu \rangle, P = \langle ss_1^{\nu_1}, z \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle ss_1^\nu, z \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ S & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^3 & \text{if } Q = P = \langle ss_1^\nu \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle ss_1^\nu \rangle, P = \langle ss_1^{\nu_1}, z \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = 1, \nu = \nu_1 \\ & \text{or } Q = \langle ss_1^\nu, z \rangle, \delta_\nu(\psi)\delta_{\nu_1}(\phi) = \delta_z(\psi)\delta_z(\phi) = 1, \nu = \nu_1 \\ \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$ then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for $i = -1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)}z^{i\delta_z(\phi)})^y = (ss_1^{\nu})^{\delta_\nu(\psi)} \\ &\implies \nu = -\nu_1, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

B.2.10 $\psi = \hat{\vartheta}_{\mu,\nu}$ and $Q \in \mathcal{D}_{V_\nu}$, for $\mu = 1,..,4$ and $\nu = -1,1$

1. Let $\phi \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$ and $P \in \mathcal{D}$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. For any $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi|_P \circ c_y = \psi_1|_Q \circ \eta \quad \text{for } \psi_1 \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}^{-1}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\} \\ &\Leftrightarrow c_x|_Q \circ \phi \circ \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ \phi|_P \circ \eta \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

   Thus, it suffices to look at the case when $\phi\circ\eta \in \{\mathrm{Id}_S, \omega, \eta, \omega\circ\eta\}$ and $\psi_1 \in \{\hat{\theta}_{V_\lambda}, \hat{\theta}_{V_\lambda}^{-1}, \hat{\alpha}_{V_\lambda}, \hat{\beta}_{V_\lambda}\}$.

2. Let $\phi \in \{\theta_T, \theta_T^{-1}, \alpha_T, \beta_T\}$ and $T \in \mathcal{D}_T$, for $T \in \mathcal{T}_{full}$.

Suppose $T = V_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \lambda$ and $(ss_1^\nu)^x = ss_1^\nu z^i$ for $i = -1,0,1$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^\nu \in Q &\implies ((ss_1^\lambda z^i)\phi)^y = (ss_1^\nu)\psi \\
&\implies (z^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_\nu(\psi)} \\
&\implies \nu = -\lambda, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies ((ss_1^\lambda)^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

Suppose $T = E_\lambda$ for $\lambda = -1,0,1$. Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^\nu \in Q$, then $\nu = \lambda$ and $(ss_1^\nu)^x = ss_1^\nu t^i z^j$ for $i,j = -1,0,1$. For some $y \in S$, we have

$$\begin{aligned}
ss_1^\nu \in Q &\implies ((ss_1^\lambda t^i z^j)\phi)^y = (ss_1^\nu)\psi \\
&\implies (t^{\delta_\lambda(\phi)}(ss_1^\lambda)^{i\delta_t(\phi)} z^{j\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_\nu(\psi)} \\
&\implies \nu = -\lambda, \text{ a contradiction.} \\
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (z^{\delta_Z(\phi)})^y = z^{\delta_z(\psi)} \\
&\implies \delta_z(\psi)\delta_z(\phi) = 1.
\end{aligned}$$

Collecting the data,

$$\begin{aligned}
N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z\rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} S & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z\rangle, P \in \{\langle z\rangle, \langle t,z\rangle, \langle ss_1^\lambda, z\rangle, \langle ss_1^\lambda, t\rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $T = A$. Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^\nu \notin Q$. For some $y \in S$, we have

$$\begin{aligned}
z \in Q &\implies (z\phi)^y = z\psi \\
&\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.}
\end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

3. Let $\phi = \Theta_{\mu_1,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,0,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for $i = -1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}t^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (t^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

4. Let $\phi = \Theta_{\mu_1,\nu_1}^{-1}$ for $\mu = 1,..,4$ and $\nu_1 = -1,0,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Note that $ss_1^{\nu} \notin Q$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

5. Let $\phi = \vartheta_{\mu,\nu_1}$ for $\mu_1 = 1,..,4$ and $\nu_1 = -1,1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu}z^i$ for $i = -1,0,1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)}(ss_1^{-\nu_1})^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_\nu(\psi)} \\ &\implies i\delta_\nu(\psi)\delta_z(\phi) = 1. \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{-\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P)\cap\langle ss_1^{\nu},z\rangle t^{-1} & \text{if } Q=\langle ss_1^{\nu}\rangle, i=1, \delta_\nu(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ N_S(Q,P)\cap\langle ss_1^{\nu},z\rangle t & \text{if } Q=\langle ss_1^{\nu}\rangle, i=-1, \delta_\nu(\psi)\delta_z(\phi)=-1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} \langle ss_1^{\nu},z\rangle t^{-1} & \text{if } Q=\langle ss_1^{\nu}\rangle, P=\langle ss_1^{\nu},z\rangle, i=1, \delta_\nu(\psi)\delta_z(\phi)=1, \nu=\nu_1 \\ \langle ss_1^{\nu},z\rangle t & \text{if } Q=\langle ss_1^{\nu}\rangle, P=\langle ss_1^{\nu},z\rangle, i=-1, \delta_\nu(\psi)\delta_z(\phi)=-1, \nu=\nu_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} 3^2 & \text{if } Q=\langle ss_1^{\nu}\rangle, P=\langle ss_1^{\nu_1},z\rangle, \nu=\nu_1 \\ 0 & \text{otherwise} \end{cases}$$

6. Let $\phi=\xi_{\mu_1,\nu_1}$ for $\mu_1=1,..,8$ and $\nu_1=-1,1$.

Let $x\in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu}\in Q$ then $\nu=\nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu_1}z^i$ for $i=-1,0,1$.

Suppose $1\le\mu_1\le 4$. For some $y\in S$, we have

$$\begin{aligned} ss_1^{\nu}\in Q &\implies ((ss_1^{\nu_1}z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (s^{\delta_{\nu_1}(\phi)}z^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z\in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}. \end{aligned}$$

Collecting the data,

$$N_{\psi|_Q,\phi|_P} = \begin{cases} N_S(Q,P) & \text{if } Q=\langle z\rangle, P\in\{\langle ss_1^{\nu_1},z\rangle,\langle z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ \emptyset & \text{otherwise} \end{cases}$$
$$= \begin{cases} S & \text{if } Q=\langle z\rangle, P\in\{\langle ss_1^{\nu_1},z\rangle,\langle z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ \emptyset & \text{otherwise} \end{cases}$$

Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q=\langle z\rangle, P\in\{\langle ss_1^{\nu_1},z\rangle,\langle z\rangle\}, \delta_z(\psi)\delta_z(\phi)=1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies (z^{\delta_{\nu_1}(\phi)} s^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_\nu(\psi)}, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (s^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

7. Let $\phi = \xi_{\mu_1,\nu_1}^{-1}$ for $\mu_1 = 1,..,8$ and $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. Since $\nu \neq 0$, note that $ss_1^{\nu} \notin Q$.

   Suppose $1 \leq \mu_1 \leq 4$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s,z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle s,z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose $5 \leq \mu_1 \leq 8$. For some $y \in S$, we have

$$\begin{aligned} z \in Q &\implies (z\phi)^y = z\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_z(\phi)})^y = z^{\delta_z(\psi)}, \text{ a contradiction.} \end{aligned}$$

   Thus, $|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = 0$.

8. Let $\phi = \theta_{\nu_1}^{\pm 1} \circ \eta$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Then for some $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi_1|_Q \circ \eta \quad \text{for } \phi_1 = \theta_{\nu_1}^{\pm 1}, \psi_1 \in \{\hat{\theta}_{V_\nu}, \hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}, \hat{\beta}_{V_\nu}\} \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

It follows that we can apply the case when $\phi_1 = \theta_{\nu_1}^{\pm 1}$ and $\psi_1 \in \{\hat{\theta}_{V_\nu}, \hat{\theta}_{V_\nu}^{-1}, \hat{\alpha}_{V_\nu}, \hat{\beta}_{V_\nu}\}$.

9. Let $\phi \in \{\hat{\theta}_{V_{\nu_1}}, \hat{\theta}_{V_{\nu_1}}^{-1}, \hat{\alpha}_{V_{\nu_1}}, \hat{\beta}_{V_{\nu_1}}\}$ for $\nu_1 = -1, 1$.

   Let $x \in N_{\psi|_Q,\phi|_P}$. If $ss_1^{\nu} \in Q$, then $\nu = \nu_1$ and $(ss_1^{\nu})^x = ss_1^{\nu} z^i$ for $i = -1, 0, 1$. Thus, for some $y \in S$, we have

$$\begin{aligned} ss_1^{\nu} \in Q &\implies ((ss_1^{\nu_1} z^i)\phi)^y = (ss_1^{\nu})\psi \\ &\implies ((ss_1^{\nu_1})^{\delta_{\nu_1}(\phi)} z^{i\delta_z(\phi)})^y = (ss_1^{-\nu})^{\delta_z(\psi)} \\ &\implies \nu = -\nu_1, \text{ a contradiction.} \\ z \in Q &\implies (z\phi)^y = z\psi \\ &\implies (z^{\delta_z(\phi)})^y = z^{\delta_z(\psi)} \\ &\implies \delta_z(\psi)\delta_z(\phi) = 1. \end{aligned}$$

   Collecting the data,

$$\begin{aligned} |N_{\psi|_Q,\phi|_P} &= \begin{cases} N_S(Q,P) & \text{if } Q = \langle z \rangle, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \\ &= \begin{cases} S & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

   Thus,

$$|\mathcal{O}_{\phi|_P}^{\psi|_Q}| = \begin{cases} \frac{3^{4k+2}}{|P|} & \text{if } Q = \langle z \rangle, P \in \{\langle z \rangle, \langle ss_1^{\nu_1}, z \rangle\}, \delta_z(\psi)\delta_z(\phi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

10. Let $\phi = \hat{\vartheta}_{\mu_1,\nu_1}$ for $\mu_1 = 1, .., 4$ and $\nu_1 = -1, 1$.

Let $x \in N_{\psi|_Q,\phi|_P}$. Then for some $y \in S$, we have

$$\begin{aligned} c_x|_Q \circ \phi|_P \circ c_y = \psi|_Q &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y = \psi_1|_Q \circ \eta \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ \eta \circ c_y \circ \eta = \psi_1|_Q \\ &\Leftrightarrow c_x|_Q \circ \phi_1|_P \circ c_{(y\eta)} = \psi_1|_Q \end{aligned}$$

where $\psi_1 \in \{\hat{\theta}_{V_v}, \hat{\theta}_{V_v}^{-1}, \hat{\alpha}_{V_v}, \hat{\beta}_{V_v}\}$ and $\phi_1 \in \{\hat{\theta}_{V_{v_1}}, \hat{\theta}_{V_{v_1}}^{-1}, \hat{\alpha}_{V_{v_1}}, \hat{\beta}_{V_{v_1}}\}$. It follows that we can apply these cases.