Partial Groups

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Abstract

In this thesis, we seek to extend some results of group theory to a new structure in algebra, called partial groups. Initially, we will prove a number of basic results of partial groups, introducing the elementary concepts of partial groups as abelian, nilpotent, homomorphism partial groups and Coprime Action on partial groups and some other ideas. After that, we are going to prove some results of characteristic p members in partial groups. These results are two uniqueness theorems of characteristic p members and further uniqueness theorems in partial groups. The principle result of this work is an extension of the Solvable Signalizer Functor Theorem to partial groups.

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Contents

Chapter 1

Preface

Very often, group theoretic proofs leave one with the impression that they may apply to a more general class of objects - for example, instances where they are only involved in the consideration of proper subgroups of a group and their normalizers. We will explore this stance by defining a class of objects termed Partial Groups, which is intended to be a more natural setting for many group theoretic arguments. This reveals the exciting possibility that many non-existence results may translate into classification theorems and that interesting algebraic objects may emerge. In this research our objective is to introduce partial groups as algebraic objects that inherit local group theoretic properties of groups and provided a framework through which local group theoretic arguments can be forwarded. We then generalize some group theoretic results to this more general setting.

This exposition comprises six chapters: Chapter 2 contains preliminary notions, basic definitions and some important well-known results which will be needed to develop the subject in the subsequent chapters. Initially, we start with some background information, introducing the concept of primitive pairs, Coprime Action and some other ideas. Finally, we present proofs of two special cases of Thompson's Fixed-point-free Automorphism Theorem, Sylow's Theorem for Coprime Action and Burnside's $p^{\alpha}q^{\beta}$-Theorem. This chapter aims at making the present research as self-contained as possible. However, basic knowledge of group theory has been presumed.

Chapter 3 is devoted to the basic study of partial groups. We start this chapter by defining a partial group as a finite set Γ such that

1- Each member of Γ is a finite group.

2- If $X,\ Y \in \Gamma$, then $X \cap Y \in \Gamma$ and the binary operation on $X \cap Y$ induced by the binary operation on X coincides with the binary operation induced by the binary operation on Y.

3- If $X \in \Gamma$ with X not the trivial group, then there exists $N \in \Gamma$ with the property that whenever $H \in \Gamma$ with $X \leqslant H$, then $N \cap H = N_H(X)$.

4- If $H \in \Gamma$ and K char H, then $K \in \Gamma$.

For example, let G be a finite group and Γ the collection of all subgroups of G. We notice that all the conditions from 1 to 4 are satisfied. Hence Γ is a partial group. We denote Γ by $\Gamma(G)$. After that, we give some basic definitions and results for partial groups. A partial group Γ is called abelian, nilpotent or solvable partial group if every member of Γ is abelian, nilpotent or

solvable respectively. We have shown that a connected partial group (see Definition 3.2.13) that is abelian or nilpotent has a unique maximal member. However, a connected solvable partial group may not have a unique maximal member (see Proposition 3.2.24). The set of all maximal members in a partial group Γ, denoted by Γ^*. This set plays an important role in a partial group Γ. For instance, if Γ^* has only one member G, then $\Gamma = \Gamma(G)$. In Chapter 3 we also present a number of results on subpartial groups, homomorphisms and primitive pairs of partial groups. A member of Γ, M is primitive if for $1 \neq K$ char M, then $N_\Gamma(K) = M$. The set of all primitive members in a partial group Γ, denoted by Γ^{**}. Since $\Gamma^* \subseteq \Gamma^{**}$, Γ^{**} has a great influence on a partial group Γ as shown in the next result, called Bender's Maximal Subgroup Theorem for Partial groups. This result is a close approximation to Bender's original Theorem and concludes that any two distinct members of Γ^{**} which are linked in Bender's sense (see Definition 2.5.8) are either equal or both have characteristic p for some prime p. Then we introduce automorphisms of partial groups, and we begin with a definition of a group action on a partial group Γ as follows:
let Γ be a partial group and A a group. We say that A acts on Γ if the following hold

(1) A acts on each member of Γ.

(2) If $X,\ Y \in \Gamma$ with $X \leqslant Y$, then action of A on X coincides with the action of A on Y restricted to X.

(3) For each $B \trianglelefteq A$ with $B \neq 1$, there exists $C \in \Gamma$ with the property:
whenever $X \in \Gamma$ then $C \cap X = C_X(B)$.

(4) For each $B \trianglelefteq A$ and for each $X \in \Gamma$, $[X,\ B] \in \Gamma$.

If in addition each member of Γ has order coprime to the order of A, then we say that A acts coprimely on Γ. The next result (Thompson's Fixed-point-free Automorphism Theorem for partial groups) is about a fixed point free action on a partial group Γ that is if A acts fixed point freely on every member of Γ. Our result is a consequence of Thompson's Fixed-point-free Automorphism Theorem and one observation on nilpotent partial groups. Next, we will extend some results on Sylow's Theorem of Coprime Action to partial groups, and one of these results is Transitivity Theorem for Partial Groups. This result shows that if we let A be an elementary abelian group that acts coprimely on a partial group Γ, and we let $\mathrm{Syl}_p\Gamma$ be denoted to the set of all maximal p-groups in Γ, so under certain assumptions on Γ such that $P_1,\ P_2 \in \mathrm{Syl}_p\Gamma$, then $C_A(P_1) = C_A(P_2)$. We also will extend a result of Solvable Signalizer Functor Theorem to nilpotent partial groups which tell us that under certain assumptions on a nilpotent partial group Γ such that if the group A that is elementary abelian of rank $\geqslant 3$ acts coprimely on Γ, then Γ has a unique maximal member.

In the next two chapters, our aim is to generalize the result of Signalizer Functor Theorem on nilpotent partial groups to solvable partial groups which will be the substantive result of this thesis.
Chapter 4 offers insights into understanding the members that have characteristic p, for some primes p in partial groups. In this chapter, we will apply Bender's Maximal Subgroup Theorem for Partial Groups, that is an important result of studying characteristic p members of partial groups. We are going to establish two uniqueness theorems as follows:

Theorem 4.2.1. Assume the group A acts on the partial group Γ and that the following hold:

(1) p is a prime.

(2) M_1 and M_2 are p-solvable members of Γ^{**}.

(3) $O_p(M_1) \leqslant M_2$ and $O_p(M_2) \leqslant M_1$.

(4) M_1 and M_2 have characteristic p.

(5) $C_\Gamma(A) \leqslant M_1 \cap M_2$.

Then, $M_1 = M_2$.

Theorem 4.2.2. Assume the group A acts on the partial group Γ and p is a prime. Let $M \in \Gamma^{**}$ and $T \in \Gamma$. Assume that

(1) M is p-solvable with characteristic p.

(2) $O_p(M) \leqslant T \leqslant O_p(M;\, A)$.

Then, $N_\Gamma(T) \leqslant M$.

We recall that the set of A-invariant p-subgroups of G is partially ordered by inclusion. The set of maximal elements of this set is denoted by

$$\mathrm{Syl}_p(G;\, A).$$

Moreover,

$$O_p(G;\, A) = \bigcap \mathrm{Syl}_p(G;\, A),$$

At the end of this chapter, we will give further uniqueness theorems in the case partial groups are solvable.
Chapter 5 aims to establish analogue of the Solvable Signalizer Functor Theorem for partial groups which states that if we let A be an elementary abelian r-group that acts coprimely on the partial group Γ, and we suppose that $\mathrm{rank}(A) \geqslant 3$, and that every member of Γ is solvable, then Γ possesses a unique maximal member. This theorem will be the principle result of the chapter. in order to prove this result, we divide our proof into several parts. This chapter will start by describing the method followed to prove this theorem, thereafter which we will then provide a definition of certain elements of the group A that acts on the the partial group Γ, which termed exceptional elements. Next, we shall provide some result related to these exceptional elements. Furthermore, we will also prove some results of certain commutators members and the intersection of certain fitting members in Γ. Finally, we will employ these sequent results to complete the poof of the principle result of the chapter.

Chapter 6 deals with a special case of Burnside's $p^\alpha q^\beta$-Theorem which will be extended to partial groups. We will start to describe types of maximal members of these partial groups as follows:

Let $M \in \Gamma^*$, then we have
• M has type p if $\pi(F(M)) = \{p\}$.
• M has type q if $\pi(F(M)) = \{q\}$.
• M has type pq if $\pi(F(M)) = \{p,\, q\}$.

The difficulty here is that there may not be a unique maximal member of Γ so $|\Gamma^{**}| \geqslant 2$. However, we are seeking to study these maximal members of different types in Γ. We will begin with some basic results which will be used in the subsequent sections. In each type of elements in our partial group we have some results to help us understanding how these members connect to each other. Finally, we give two example theorems to indicate the types of results we obtain.

Throughout the next two theorems, we assume the following hypothesis

Hypothesis

(1) Γ is a partial group.

(2) p and q are distinct primes.

(3) For all $H \in \Gamma$, $\pi(H) \subseteq \{p,\, q\}$.

(4) Every p-group in Γ is abelian.

(5) Every q-group in Γ is abelian.

(6) For all $H \in \Gamma$ if $K \leqslant H$, then $K \in \Gamma$.

Theorem 6.6.1. (a) Suppose that $M \in \Gamma^*$ has type pq and that $F(M) \leqslant L \in \Gamma^*$. Then $M = L$.

(b) Let $M,\ L \in \Gamma^*$ and suppose that $F(M) \leqslant L$ and $M \neq L$. Then one of the following holds.

(1) M has type p, $O_p(M) \in \mathrm{Syl}_p\Gamma$, $L = O_p(M)O_q(L)$, $O_q(L) \in \mathrm{Syl}_q\Gamma$ and $L = N_\Gamma(O_q(L))$.

(2) M has type q, $O_q(M) \in \mathrm{Syl}_q\Gamma$, $L = O_q(M)O_p(L), O_p(L) \in \mathrm{Syl}_pL$ and $L = N_\Gamma(O_p(L))$.

Theorem 6.6.3. Let $M \in \Gamma^*$ and $P \in \mathrm{Syl}_pM$. Suppose that M has type q.

(a) Suppose that P is non-cyclic. Then $P \in \mathrm{Syl}_p\Gamma$.

(b) Suppose that rank $(P) \geqslant 3$.

(1) Then there exists $\mathbb{Z}_p \times \mathbb{Z}_p \cong B \leqslant P$ such that $C_\Gamma(b) \leqslant M$ for all $b \in B^\#$ and $P \in \mathrm{Syl}_p\Gamma$.

(2) If $P \leqslant L \in \Gamma^*$ with $L \neq M$, then $L = N_\Gamma(P)$.

(3) If $a \in P^\#$ then $C_\Gamma(a) \leqslant M$ or $C_\Gamma(a) \leqslant N_\Gamma(P)$.

(4) If $P_1 \in \Gamma$ is a p-group and $P_1 \cap M \neq 1$, then $P_1 \leqslant M$.

(c) Suppose rank $(P) \geqslant 3$. Let $L \in \Gamma^*$ and suppose that $M \cap L$ has non-cyclic Sylow p-subgroups. Then $M = L$ or $L = N_\Gamma(P_1)$ for some $P_1 \in \mathrm{Syl}_pM$.

Chapter 2

Preliminaries

2.1 Introduction

Let A be a group that acts on a group G. The notation $C_G(A)$ refers to the subgroup of fixed points of A in G. The operator group A acts fixed point freely on G if $C_G(A) = 1$. Often we consider the semidirect product $G \rtimes A$ so that we can apply Sylow's Theorem and Hall's Theorem.

A theorem of Frobenius states that if G is a finite Frobenius group with Frobenius complement H and Frobenius kernel K, then K is a normal subgroup of G. Hence, G is the semidirect product of H and K. An element $h \in H,\ h \neq 1$, induces an automorphism of K which leaves only the identity element fixed. Conversely, a group K is the Frobenius kernel of a Frobenius group if and only if it possesses a fixed point free automorphism of prime order. In this chapter we will discuss the Theorem of Thompson that a group admitting a fixed point free automorphism of prime order is nilpotent.

We start this chapter with background material on group theory. Presenting this information at the beginning is preferable, because it introduces notions and basic results used in the subsequent chapters. Secondly, in section 2.3 we introduce some results about Coprime Action. In the last part of this section we prove two special cases of the Theorem of Thompson on Fixed-poinit-free Automorphism. After that, in section 2.4 we review some of the basic definitions and facts in representation theory, knowledge of which will be assumed in later chapters. Finally, some important results have been presented which we shall require for the development of the subject matter in the present research. We present a simple and short proof of Burnside's Theorem that shows a group of order $p^a q^b$, where p and q are primes and $a,\ b \in \mathbb{N} \cup \{0\}$, is solvable. In the last part of this section, the concept of a primitive pair is introduced and it is shown that under certain assumptions on a group G which has a primitive pair, then the primitive pair has characteristic p.

2.2 Background Material on Group Theory

First we discuss some basic ideas from Group Theory.

Group Actions

Definition 2.2.1. Let G be a group and X a non-empty set. Then G is said to act on X if there is a function from $X \times G$ to X, usually denoted $(x, g) \mapsto xg$, such that $x1 = x$ for all $x \in X$, and for all $g, h \in G$ and $x \in X$, such that $x(gh) = (xg)h$.

Lemma 2.2.2. Let G act on X. If $x \in X$, $g \in G$ and $y = xg$, then $x = yg^{-1}$. If $x \neq x'$, then $xg \neq x'g$.

Proof. We have $y = xg$ and so $yg^{-1} = (xg)g^{-1} = x(gg^{-1}) = x1 = x$.
To show $x \neq x'$ implies $xg \neq x'g$, we show the contrapositive: if $xg = x'g$ then applying g^{-1} to both sides gives $(xg)g^{-1} = (x'g)g^{-1}$, so $x(gg^{-1}) = x'(gg^{-1})$, and so $x = x'$.

Another way to think about an action of a group on a set is that it is a certain homomorphism.

Theorem 2.2.3. An action of a group G on a set X is a group homomorphism from G to $\mathrm{Sym}(X)$, the group of permutations of X.

Proof. Suppose we have an action of G on X. We view xg as a function of x (with g fixed). That is, for each $g \in G$ we have a function $\phi_g : X \longrightarrow X$, given by $x\phi_g = xg$. The axiom

$$x = x1$$

says ϕ_1 is the identity function on X. The axiom

$$(xg_1)g_2 = x(g_1g_2)$$

says $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1g_2}$ so composition of functions on X corresponds to multiplication in G. Moreover, ϕ_g is an invertible function since $\phi_{g^{-1}}$ is an inverse: the composite of ϕ_g and $\phi_{g^{-1}}$ is ϕ_1, which is the identity function on X. Therefore $\phi_g \in \mathrm{Sym}(X)$ and $g \to \phi_g$ is a homomorphism $G \to \mathrm{Sym}(X)$.

Conversely, suppose we have a homomorphism $\theta : G \longrightarrow \mathrm{Sym}(X)$. For each $g \in G$, we have a permutation $g\theta$ on X, and $g_1\theta \circ g_2\theta = (g_1g_2)\theta$. Setting $x(g\theta) = xg$ defines a group action of G on X, since the group homomorphism properties of θ yield defining the properties of a group action.

From this viewpoint, the set of $g \in G$ that acts trivially ($xg = x$ for all $x \in X$) is simply the kernel of the homomorphism $G \longrightarrow \mathrm{Sym}(X)$ associated to the action. Therefore those g that act trivially on X are said to lie in the kernel of the action. If the kernel equals one, then G acts faithfully.

Now, let a group G act on a finite set X. Define a binary relation $\sim_G$ on X, by $x_1 \sim_G x_2$ if $x_2 = x_1g$ for some $g \in G$. Then $\sim_G$ is an equivalence relation. An orbit of G on X is an equivalence class of $\sim_G$. Thus the orbits form a partition of X. Consequently we have $|X| = \sum |O_x|$, $O_x = \{xg : g \in G\}$. Each element x has a corresponding subgroup in G, $G_x = \{g \in G : xg = x\}$. The conjugate of one subgroup is another $G_x^g = G_{xg}$. Each coset G_xg takes x to xg, and distinct cosets G_xg_1 and G_xg_2 take x to distinct values. Thus we have the Orbit- Stabililzer Formula $|G| = |O_x||G_x|$.

A finite group G acts on itself by conjugation. Then the index $[G : C_G(x)]$ is equal to the number of elements in the conjugacy class of x. If we choose a single representative element x_i from every conjugacy class, we infer from the disjointness of the conjugacy classes that $|G| = \sum |G : C_G(x_i)|$, where $C_G(x_i)$ is the centralizer of the element x_i. Observing that each element of $Z(G)$ forms a conjugacy class containing just itself gives rise to the class equation:

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where each x_i in the sum is non central.

Definition 2.2.4. An action of a group G on a set X is called transitive when the set X is non-empty and there is exactly one orbit.

Theorem 2.2.5. For $X \neq \emptyset$, an action of G on X is transitive if and only if, given x and y in X, there some $g \in G$ such that $y = xg$.

Proof. Suppose the action is transitive, so there is one orbit. Given any x in X, its orbit must fill up X, so every element of X has the form xg for some $g \in G$. Suppose that given any x and y in X, we have $y = xg$ for some $g \in G$. Fix x in X. Since every y in X has the form xg for some $g \in G$, every y is in the orbit of x. Thus X has only one orbit.

Lemma 2.2.6. (**Frattini Argument**). Suppose that G contains a normal subgroup N, which acts transitively on X. Then $G = G_x N$ for every $x \in X$.

Proof. Let $x \in X$ and $g \in G$. The transitivity of N on X gives an element $n \in N$ such that $xg = xn$. Hence, we have

$$x(gn^{-1}) = x$$

and so

$$gn^{-1} \in G_x.$$

This shows that

$$g \in G_x n \subseteq G_x N.$$

Automorphisms

Let G be a group. The set $\mathrm{Aut}(G)$ of all automorphisms of G given by

$$\theta_1\theta_2 : x \longrightarrow ((x)\theta_1)\theta_2 \ (x \in G,\ \theta_1,\ \theta_2 \in \mathrm{Aut}(G))$$

is a group, the automorphism group of G. The identity mapping is the identity of $\mathrm{Aut}(G)$, and the inverse mapping θ^{-1} the inverse of θ.
Automorphisms map finite subgroups (respectively elements) to subgroups (respectively elements) of the same order. For $a \in G$, the mapping

$$\varphi_a : G \longrightarrow G \text{ such that } g \longrightarrow g^a \ (= a^{-1}ga)$$

is bijection. Since for all $g,\ h \in G$

$$(gh)^a = a^{-1}gaa^{-1}ha = (a^{-1}ga)(a^{-1}ha) = g^a h^a,$$

φ_a is an automorphism of G, the inner automorphism induced by a. The mapping

$$\varphi : G \longrightarrow \mathrm{Aut}(G)$$

given by $a \to \varphi_a$ is a homomorphism from G to $\mathrm{Aut}(G)$, since $g^{ab} = b^{-1}a^{-1}gab = (g^a)^b$. Hence, the set of inner automorphism of G,

$$\mathrm{Inn}G := \{\varphi_a \mid a \in G\},$$

is a subgroup of $\mathrm{Aut}G$. Moreover,

$$\gamma^{-1}\varphi_a\gamma = \varphi_{(a)\gamma}\ (\gamma \in \mathrm{Aut}(G), a \in G)$$

shows that $\mathrm{Inn}G$ is a normal subgroup of $\mathrm{Aut}(G)$. Then

$$\mathrm{Ker}\varphi = \{g \in G \mid \forall a \in G,\ g^a = g\} = Z(G).$$

The homomorphism theorem then yields

$$G/Z(G) \cong \mathrm{Inn}G.$$

Lemma 2.2.7. Let G be a group. Suppose that $G/Z(G)$ is cyclic. Then G is abelian.

Proof. There exists $g \in G$ such that $G/Z(G) = \langle gZ(G)\rangle$ and thus

$$G = Z(G)\langle g\rangle.$$

Since $\langle g\rangle$ is abelian, all pairs of elements of G commute.

By defining a subgroup N of G is normal if and only if

$$N^a = N \text{ for all } a \in G.$$

Thus, a subgroup of G is normal if and only if it is mapped to itself by every inner automorphism of G.

Lemma 2.2.8. The automorphism group of a cyclic group is abelian.

Proof. We note that all automorphisms are of the form $x \to x^m$, if $\phi : x \to x^m$ and $\varphi : x \to x^n$ are two automorphisms, then

$$(x)(\phi\varphi) = ((x)\phi)\varphi = (x^m)\varphi = x^{mn} = x^{nm} = (x^n)\phi = (x)(\varphi\phi).$$

Lemma 2.2.9. [13, Lemma 2.2.4]. The automorphism group of a group of order p is cyclic.

Definition 2.2.10. A subgroup H of a group G is said to be a characteristic subgroup of G if $(H)\theta = H$ for all $\theta \in \mathrm{Aut}(G)$. We write

$$H \text{ char } G$$

to indicate that H is a characteristic subgroup of G.

Characteristic subgroups are normal in G. Moreover, 1 and G are characteristic subgroups of G. Another example of a characteristic subgroup is $Z(G)$. Indeed, for $x \in Z(G),\ g \in G$ and $\theta \in \mathrm{Aut}G$, we have

$$(x)\theta(g)\theta = (xg)\theta = (gx)\theta = (g)\theta(x)\theta,$$

and since

$$G = \{(g)\theta \mid g \in G\},$$

we get

$$(x)\theta \in Z(G).$$

Now, we show two properties of characteristic subgroups.

Lemma 2.2.11. Let N be a normal subgroup of G and K be a characteristic subgroup of N.

(1) K is normal in G.

(2) If N is characteristic in G, then also K is characteristic in G.

Proof. (1) Let $a \in G$ and ϑ_a be the inner automorphism of G induced by a. Then the restriction of ϑ_a to N is an automorphism of N since N is normal in G. Hence, K is invariant under ϑ_a for all $a \in G$. This means, K is normal in G.

(2) Since N is now characteristic subgroup of G, we can replace ϑ_a in the above argument by an arbitrary automorphism of G.

Now, let A be a group and

$$\varphi : A \longrightarrow \mathrm{Aut}(G)$$

be a homomorphism from A to $\mathrm{Aut}(G)$. Then we say that A acts on G (with respect to φ). We set

$$g^a = g^{a^\varphi}$$

and get

$$(gh)^a = g^a h^a \text{ and } (g^a)^b = g^{ab}$$

for all $g,\ h \in G$ and $a,\ b \in A$.
A subgroup H of G is A-invariant if for all $a \in A$:

$$H^a = \{h^a \mid h \in H\} = H.$$

If H is an A-invariant subgroup of G, then A acts on H with respect to the homomorphism $A \to \mathrm{Aut}(H)$ induced by φ.

Products of Groups

Products of groups are used to construct new groups from given ones (external products). On the other hand, they can be used to describe the structure of groups(internal products).

Definition 2.2.12. Let H and K be groups. The cartesian product of H and K, is defined by

$$H \times K = \{(h,\ k) \mid h \in H,\ k \in K\}.$$

We define a binary operation by declaring

$$(h_1,\ k_1)(h_2,\ k_2) = (h_1h_2,\ k_1k_2)$$

for all $(h_1,\ k_1), (h_2,\ k_2) \in H \times K$. With respect to this operation, $H \times K$ is a group.

This group is the (external) direct product of the group H and K. Obviously, the embedding

$$\varphi : H \longrightarrow H \times K$$

with

$$h \longrightarrow (h,\ 1),$$

is an isomorphism from H to $H \times 1$.
For the subgroups $G_1 = H \times 1$ and $G_2 = 1 \times K$ of $G = H \times K$, we have the following

(1) $G = G_1G_2$,

(2) $G_1,\ G_2 \trianglelefteq G$,

(3) $G_1 \cap G_2 = 1$.

Definition 2.2.13. Let G be a group. Two subgroups H and K of G are said to be complementary if $G = HK$ and $H \cap K = 1$.
We note that if H and K are complementary, then $|G| = |H||K|$.

Definition 2.2.14. Let G be a group, and let H and K be complementary in G with $H \trianglelefteq G$ and $K \trianglelefteq G$. Then G is called the (internal) direct product of the subgroups H and K.

Theorem 2.2.15. Let G be a group and let H and K be subgroups. The following statements are equivalent.

(1) H and K are complementary in G with $H \trianglelefteq G$ and $K \trianglelefteq G$.

(2) Every element of G can be expressed uniquely in the form hk, where $h \in H$ and $k \in K$; and every element of H commutes with every element of K.

Proof. Suppose that (1) holds. Since $G = HK$, every element of G can be expressed as hk, with $h \in H$ and $k \in K$. We need to show that this representation is unique. This means that, when

$h_1k_1 = h_2k_2$ with $h_1, h_2 \in H$ and $k_1, k_2 \in K$, then necessarily $h_1 = h_2$ and $k_1 = k_2$. This follows from $H \cap K = 1$. Indeed, from $h_1k_1 = h_2k_2$, we get

$$k_1k_2^{-1} = h_1^{-1}h_2 \in H \cap K = 1$$

and so

$$h_1 = h_2 \text{ and } k_1 = k_2.$$

It remains to show that any element of H commutes with any element of K. Let $h \in H$ and $k \in K$. We need to show that $hk = kh$, or, equivalently, $h^{-1}k^{-1}hk = 1$.
Now, since $h^{-1}k^{-1}h \in K$, we have

$$h^{-1}k^{-1}hk \in K.$$

Also, we have

$$h^{-1}k^{-1}hk \in H.$$

Then, we have

$$h^{-1}k^{-1}hk \in H \cap K = 1$$

and so

$$h^{-1}k^{-1}hk = 1,$$

as claimed.

Suppose that (2) holds. Then every element of G can be written in the form hk, where $h \in H$ and $k \in K$ and so $G = HK$. Now, we show that $H \cap K = 1$. Let $x \in H \cap K$. If $x \neq 1$, then $1x = x1$ are two distinct representation of $x \in G$ with $1 \in H$, $x \in K$ and $1 \in K$, $x \in H$, contrary to the hypothesis that every element of G, in particular x, can be expressed uniquely in the form hk. Thus, we get $x = 1$. This shows that $H \cap K = 1$.

Now, we show that $H \trianglelefteq G$. Let $g \in G = HK$. Then $g = hk$ for some $h \in H$ and $k \in K$. Let $h' \in H$. Thus, we have

$$\begin{aligned} g^{-1}h'g &= (hk)^{-1}h'(hk) \\ &= k^{-1}(h^{-1}h'h)k \\ &= k^{-1}k(h^{-1}h'h) \in H \end{aligned}$$

($h' \in H$, $h^{-1}h'h \in H$ and $k \in K$ commute).
Similarly, we get $K \trianglelefteq G$.

Theorem 2.2.16. Let G be a group and H, K be complementary in G. Assume that $H \trianglelefteq G$, $K \trianglelefteq G$. Then $G \cong H \times K$.

Proof. We define $\varphi : H \times K \longrightarrow G$ with

$$(h,\, k) \longrightarrow hk.$$

First, we show that φ is a homomorphism.

$$\begin{aligned} ((h_1,\, k_1)(h_2,\, k_2))\,\varphi &= (h_1h_2,\, k_1k_2)\varphi \\ &= h_1h_2k_1k_2 \end{aligned}$$

$$= (h_1, k_1)\varphi(h_2, k_2)\varphi.$$

φ is surjective because $G = HK$ by hypothesis. Now, let $(h, k) \in \mathrm{Ker}\varphi$. Then, we have

$$(h, k)\varphi = 1$$

and so

$$hk = 1.$$

Then, we get

$$h, k \in H \cap K = 1.$$

Then, we have

$$(h, k) = (1, 1)$$

and so

$$\mathrm{Ker}\varphi = \{(1, 1)\}.$$

Thus, φ is an isomorphism.

Lemma 2.2.17. Let a, b be elements of a finite group G and $ab = ba$. Let $o(a)$ be the order of a and $o(b)$ the order of b such that $(o(a), o(b)) = 1$. Then

$$\langle ab \rangle = \langle a \rangle \times \langle b \rangle,$$

and

$$o(ab) = o(a)o(b).$$

Proof. Let $m = o(a)$ and $n = o(b)$. Note that $\langle a, b \rangle$ is an abelian group, where the subgroups $\langle a \rangle$ and $\langle b \rangle$ are coprime order. Hence,

$$H = \langle a, b \rangle = \langle a \rangle \times \langle b \rangle$$

is a group of order mn.
Let $c = ab \in H$. The homomorphism

$$\varphi : \langle c \rangle \longrightarrow H/\langle a \rangle$$

with

$$c^i \longrightarrow \langle a \rangle c^i = \langle a \rangle b^i$$

is surjective.
Hence, $|\operatorname{Im} \varphi| = n$ is a divisor of $|\langle c \rangle|$. In the same way m is a divisor of $|\langle c \rangle|$.
Now, $(n, m) = 1$ implies that

$$o(c) = mn = |H|$$

and so

$$H = \langle c \rangle.$$

Lemma 2.2.18. Let $G = H \times K$. If H and K are characteristic subgroups of G, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

Proof. If α_1 and α_2 are automorphisms of H and K respectively, then

$$(h,\, k)^{\alpha} = (h^{\alpha_1},\, k^{\alpha_2})$$

defines an automorphism of $G = H \times K$,

$$\varphi : \mathrm{Aut}(H)\times\, \mathrm{Aut}(K) \longrightarrow \mathrm{Aut}(G)$$

with

$$(\alpha 1,\, \alpha_2) \longrightarrow \alpha$$

is a homomorphism and injective. Moreover, φ is surjective since H and K are characteristic subgroups of G.

Now let G be a group, and let H and K be complementary in G. Suppose that K is normal in G. Then H acts by conjugacy on K. Since the map $k \to k^h$ for $k \in K$ defines an automorphism of K, this conjugacy action my be described as a homomorphism $H \to \mathrm{Aut}(K)$. For $h \in H$ we define the automorphism ϕ_h to be the automorphism of K given by conjugation

$$\phi_h : K \longrightarrow K,$$

$$\phi_h(k) = h^{-1}kh.$$

Proposition 2.2.19. The map $H \to \phi_h$ is a homomorphism $\phi : H \longrightarrow \mathrm{Aut}(K)$.

Proof. We need to show that $\phi_{h_1}\phi_{h_2} = \phi_{h_1h_2}$ for $h_1,\ h_2 \in H$.
For $k \in K$, the left side is

$$\begin{aligned} k(\phi_{h_1}\phi_{h_2}) &= (k\phi_{h_1})\phi_{h_2} \\ &= ({h_1}^{-1}kh_1)\phi_{h_2} \\ &= {h_2}^{-1}h_1^{-1}kh_1h_2. \end{aligned}$$

The right side is

$$\begin{aligned} k\phi_{h_1h_2} &= (h_1h_2)^{-1}k(h_1h_2) \\ &= h_2^{-1}h_1^{-1}kh_1h_2. \end{aligned}$$

Definition 2.2.20. Let H and K be groups, and let $\phi : H \longrightarrow \mathrm{Aut}(K)$ be a homomorphism. The following group is called a semidirect product of K by H via ϕ, written $K \rtimes_{\phi} H$:

- The group operation, is defined as follows

$$(k_1,\, h_1)(k_2,\, h_2) = (k_1{k_2}^{h_1\phi},\, h_1h_2).$$

- The identity element is $(1_K,\, 1_H)$.

- The inverse of $(k,\, h)$ is given by $((k^{-1})^{h^{-1}\phi},\, h^{-1})$.

Theorem 2.2.21. Let G be a group and K normal subgroup with complement H in G. Let $\phi : H \longrightarrow \mathrm{Aut}(K)$ be defined by $\phi_h(k) = k^h$. Then $G \cong K \rtimes_\phi H$.

Proof. Every element of G has a unique expression of the form hk, for $h \in H$ and $k \in K$, we have

$$(k_1,\, h_1)(k_2,\, h_2) = k_1 k_2^{h_1} h_1 h_2.$$

We see that the map

$$(k,\, h) \longrightarrow kh$$

is an isomorphism from $K \rtimes_\phi H$ to G.

Proposition 2.2.22. Let G be a group and K normal subgroup with complement H in G. Let $\phi : H \longrightarrow \mathrm{Aut}(K)$ map every element of H to the identity automorphism. Then $K \rtimes_\phi H \cong K \times H$.

Proof. The hypothesis means that $k^h = k$ for all $h \in H$ and $k \in K$. Then the group operation in $K \rtimes_\phi H$ is

$$(k_1,\, h_1)(k_2,\, h_2) = (k_1 k_2,\, h_1 h_2)$$

which is the group operation in the direct product.

Sylow's Theorems

Recall Lagrange's Theorem says that for any finite group G, if $H \leqslant G$, then $|H|$ divides $|G|$. The converse of Lagrange's Theorem is false. If G is a finite group and $d \mid |G|$, where d is a positive integer, then there may not be a subgroup of G with order d. The simplest example of this is the group A_4, of order 12, which has no subgroup of order 6. Cauchy's Theorem implies that for any prime divisor p of the order of G, there is a subgroup of G whose order is p. Then Cauchy's Theorem is generalized by Sylow's First Theorem, which implies that if p^n is the maximal power of a prime p dividing the order of G, then G has a subgroup of order p^n. We are going to prove Sylow's theorems, and our proof will involve the action of a p-group G on a set X and use the fixed point congruence for such actions: $|X| \equiv |\,\mathrm{Fix}_G(X)| \bmod p$.

We start to prove Lagrange's Theorem and in order to prove it, we need the following lemma:

Lemma 2.2.23. Let H be a subgroup of a group G, and let $a,\, b \in G$. Then

(1) $aH = bH$ if and only if $b^{-1}a \in H$.

(2) If $aH \cap bH \neq \phi$, then $aH = bH$.

(3) $|aH| = |H|$ for all $a \in G$.

Proof. (1) Let $aH = bH$, then for any $h_1 \in H$ there is $h_2 \in H$ with $ah_1 = bh_2$. This gives,

$b^{-1}a = h_2h_1^{-1} \in H$. Let $b^{-1}a = h_0$. If $x \in aH$, then $x = ah,\ h \in H$. Now, we have

$$x = b(b^{-1}a)h = bh_0h \in bH$$

and so

$$aH \subseteq bH.$$

Similarly, we get $bH \subseteq aH$.

(2) Since $aH \cap bH \neq \phi$, then there exists $x \in H$ with $x \in aH \cap bH$.
This implies

$$ah_1 = x = bh_2,\ h_1,\ h_2 \in H$$

and so

$$b^{-1}a = h_2h_1^{-1}.$$

By (1), we get the result.

(3) Let $h_1,\ h_2 \in H$ such that $h_1 \neq h_2$. Then $ah_1 \neq ah_2$, since otherwise $h_1 = h_2$. Thus, we have

$$|aH| = |H|.$$

Theorem 2.2.24. (**Lagrange's Theorem**). Let H be a subgroup of a finite group G. Then $|H|$ divides $|G|$.

Proof. Let $|G| = n$ and $\{a_1H,\ a_2H, \ldots,\ a_nH\}$ be the family of all cosets of H in G. Then, we have $G = a_1H \cup a_2H \cup \ldots \cup a_nH$. By (2) of the above lemma for any two cosets a_iH and a_jH we have only two possibilities:

$$a_iH \cap a_jH = \phi \ \text{ or } \ a_iH = a_jH.$$

However, from (3) of the above lemma it follows that all cosets have exactly $|H|$ number of elements. Therefore, we have

$$|G| = |H| + \ldots + |H|,$$

and so

$$|G| = d|H|,\ d \in \mathbb{N}.$$

Corollary 2.2.25. Let G be a finite group and $a \in G$, then the order of a is a divisor of $|G|$.

Proof. Let $H = \langle a \rangle$. By Lagrange's Theorem, $|H|$ divides $|G|$.

Corollary 2.2.26. Let G be a finite group and $|G| = n$, then $g^n = 1$ for all $g \in G$.

Proof. Let d be the order of a. By Corollary 2.2.25, $d \mid n$, that is,

$$n = dk$$

for some integer k. Thus, we have

$$a^n = a^{dk} = (a^d)^k = 1.$$

Definition 2.2.27. Let G be a finite group and H a subgroup of G. Let the number of right cosets of H in G be called the index of H in G.

We denote the index of H in G by $[G:H]$.

Corollary 2.2.28. If G is a finte group, then $|G| = |H|[G:H]$.

Proof. By Lagrange's Theorem, we have

$$|G| = d|H|,\ d \in \mathbb{N}.$$

Since d is the number of cosets of H in G, $d = [G:H]$. Then, we have

$$|G| = |H|[G:H].$$

Corollary 2.2.29. If p is a prime, then every group G of order p is cyclic.

Proof. Choose $a \in G$ with $a \neq 1$, and let $|H| = \langle a \rangle$ be the cyclic subgroup generated by a. By Lagrange's Theorem, $|H|$ is a divisor of $|G| = p$. Since p is a prime and $|H| > 1$, it follows

$$|H| = p = |G|,$$

and so $H = G$, as desired.

Theorem 2.2.30. (**Cauchy's Theorem**). Let G be a group and p a prime dividing $|G|$. Then G contains an element of order p; in particular there exists a cyclic subgroup of order p in G.

Proof. Let

$$X := \{x^* := (x_1,\ x_2, \ldots,\ x_p) \mid x_1,\ x_2, \ldots,\ x_p \in G \text{ and } x_1x_2\ldots x_p = 1\}.$$

Since the components $x_1,\ x_2, \ldots x_p$ of $x^* \in X$ can be chosen independently (after which x_p uniquely determined), we get

$$|X| = |G|^{p-1} \equiv 0 \ (\text{mod } p).$$

Notice that

$$\begin{aligned} x_1x_2\ldots x_p = 1 &\Leftrightarrow x_2\ldots x_p = {x_1}^{-1} \\ &\Leftrightarrow x_2\ldots x_P x_1 = 1, \end{aligned}$$

so the cyclic group $C_p = \langle a \rangle \leqslant G$ acts on X by

$$(x_1,\ x_2, \ldots,\ x_p) \longrightarrow (x_2, \ldots x_p,\ x_1).$$

Hence,

$$|\operatorname{Fix}_{\langle a\rangle}(X)| \equiv |X| \equiv 0 \pmod{p}.$$

Since $1^* = (1, \ldots, 1) \in \operatorname{Fix}_{\langle a\rangle}(X)$, there exists $x^* = (x_1, \ldots, x_p) \neq 1^* \in \operatorname{Fix}_{\langle a\rangle}(X)$. This shows that $x_1 = x_2 = \ldots = x_p \neq 1$ and $x_1{}^p = 1$.

Theorem 2.2.31. (**The First Sylow Theorem**). Let G be a finite group and let $|G| = p^n m$ where $n \geqslant 1$ and where p does not divide m. Then G contains a subgroup of order p^i for each i where $0 \leqslant i \leqslant n$.

Proof. We will prove there is a subgroup of order p^i for $0 \leqslant i \leqslant n$. That is, if $|K| = p^i$ and $i < n$, we will show there is a p-subgroup K^* such that $K \subset K^*$ and $[K^* : K] = p$, so $|K^*| = p^{i+1}$. Then, starting with K as the trivial subgroup, we can repeat this process with K^* in place of K to create a rising tower of subgroups

$$1 = K_0 \subset K_1 \ldots$$

where $|K_i| = p^i$, and after n steps we reach K_n, which is a p-Sylow subgroup of G. Consider the left multiplication action of K on the set of the left cosets G/K, so by the fixed point congruence for actions of non-trivial p-groups, we have

$$|G/K| \equiv |\operatorname{Fix}_K(G/K)| \bmod p,$$

and so

$$\begin{aligned} kgK = gK,\ \forall k \in K &\Longleftrightarrow kg \in gK,\ \forall k \in K \\ &\Longleftrightarrow g^{-1}kg \in K,\ \forall k \in K \\ &\Longleftrightarrow g^{-1}Kg \subset K \\ &\Longleftrightarrow g^{-1}Kg = K \\ &\Longleftrightarrow g \in N_G(K). \end{aligned}$$

Thus, we get

$$\operatorname{Fix}_K(G/K) = \{gK : g \in N_G(K)\} = N_G(K)/K,$$

and so

$$[G : K] \equiv [N_G(K) : K] \bmod p.$$

When $|K| = p^i$ and $i < n$, the index $[G : K]$ is divisible by p, so $[N_G(K) : K]$ is divisible by p. Then $N_G(K)$ is a group with order divisible by p, so $N_G(K)$ has a subgroup of order p by Cauchy's Theorem. All subgroups of $N_G(K)/K$ have the form K^*/K, where K^* is a subgroup between K and $N_G(K)$. Therefore, a subgroup of order p in $N_G(K)/K$ is K^*/K such that $[K^* : K] = p$, so $|K^*| = p|K| = p^{i+1}$.

Theorem 2.2.32. (**The Second Sylow Theorem**). For each prime p, the Sylow p-subgroups of a finite group G are conjugate.

Proof. Let P_1 and P_2 be two Sylow p-subgroups of G. We will show they are conjugate. Consider the action of P_1 on G/P_2 by left multiplication. Since P_1 is a finite p-group, we have

$$|G/P_2| \equiv |\operatorname{Fix}_{P_1}(G/P_2)| \bmod p.$$

The left side $[G:P_2]$, which is non-zero modulo p since P_2 is a Sylow p-subgroup of G. Thus, $|\operatorname{Fix}_{P_1}(G/P_2)|$ can not be zero, so there is a fixed point in G/P_2, say gP_2. That is, $xgP_2 = gP_2$ for all $x \in P_2$. Then, $xg \in gP_2$ for all $x \in P_2$, so $P_1 \subset gP_1g^{-1}$. Therefore, $P_1 = gP_2g^{-1}$, since P_1 and gP_2g^{-1} have the same size.

Theorem 2.2.33. (**The Third Sylow Theorem**). For each prime p, let n_p be the number of Sylow p-subgroups of a finite group G and $|G| = p^n m$, where $n \geqslant 1$ and p does not divide m. Then

$$n_p \equiv 1 \bmod p \text{ and } n_p \mid m.$$

Proof. To show $n_p \equiv 1 \bmod p$, consider the action of the Sylow p-subgroup P_1 on the set $\mathrm{Syl}_p G$ by conjugation. The size of $\mathrm{Syl}_p G$ is n_p. Since P_1 is a finite p-group, we have

$$n_p \equiv |\{\text{ fixed points }\}| \bmod p.$$

Fixed points for P_1 acting by conjugation on $\mathrm{Syl}_p G$ are $P_2 \in \mathrm{Syl}_p G$ such that $gP_2g^{-1} = P_2$, for all $g \in P_1$. One choice for P_2 is P_1. For any such P_2, $P_1 \subset N_G(P_2)$. Also, $P_2 \subset N_G(P_2)$, so P_1 and P_2 are Sylow p-subgroups in $N_G(P_2)$. By the second Sylow Theorem, P_1 and P_2 are conjugate in $N_G(P_2)$. Since the only subgroup of $N_G(P_2)$ conjugate to P_2 is P_2, we have $P_1 = P_2$. Thus P_1 is the only fixed point when P_1 acts on $\mathrm{Syl}_p G$, so $n_p \equiv 1 \bmod p$.
To show $n_p \mid m$, consider the action of G by conjugation on $\mathrm{Syl}_p G$. Since all Sylow p-subgroups of G are conjugate, there is one orbit. A set on which a group acts with one orbit has size dividing the size of the group, so $n_p \mid |G|$. Since $n_p \equiv 1 \bmod p$, we have n_p is relatively prime to p, so $n_p \mid m$.

Now, we will give a brief explanation of Sylow p-subgroups. Let G be a finite group and p a prime. A p-subgroup P is called a Sylow p-subgroup of G if no p-subgroup of G contains P properly. Thus, the Sylow p-subgroups of G are the maximal elements of the set of p-subgroups of G. We denote the set of Sylow p-subgroups of G by $\mathrm{Syl}_p G$.
For example, $\mathrm{Syl}_p G = \{1\}$ if p is not a divisor of G (Lagrange's Theorem); and $\mathrm{Syl}_p G = \{G\}$ if G is a p-group. If G has precisely one Sylow subgroup, then this subgroup is normal. Conversely, if a Sylow p-subgroup of G is normal, then there is only one Sylow p-subgroup of G. Since automorphisms of G map Sylow p-subgroups to Sylow p-subgroups, the subgroup

$$O_p(G) := \bigcap_{P \in Syl_p G} P$$

is a characteristic p-subgroup of G. We remark that $O_p(G)$, is the largest normal subgroup in G, whose order $|O_p(G)|$ is a power of p.

Definition 2.2.34. Let p be a prime. The group G has characteristic p if $C_G(O_p(G)) \leqslant O_p(G)$.

Lemma 2.2.35. [13, Lemma 3.1.10]. Let p be a prime. Let P be a p-subgroup of a group G and p be a divisor of $|G:P|$. Then $P < N_G(P)$.

Lemma 2.2.36. Let N be a normal p-subgroup of G. Then $N \leqslant O_p(G)$.

Proof Let $P \in \mathrm{Syl}_p G$. Then NP is a p-subgroup. The maximality of P and $P \leqslant PN$ gives $N \leqslant P$.

Lemma 2.2.37. Let p be a prime. Let U be a p-subgroup but not a Sylow p-subgroup of G. Then $U < P$ for every Sylow p-subgroup of $N_G(U)$.

Proof. If $P \in \mathrm{Syl}_p G$ with $U < P$, then $U < N_P(U)$ by Lemma 2.2.35. Hence U is not a maximal p-subgroup of $N_G(U)$. On the other hand, U is contained in every Sylow p-subgroup of $N_G(U)$ since $U \trianglelefteq N_G(U)$ and so by Lemma 2.2.37 .

Lemma 2.2.38. (**Frattini Argument**). Let p be a prime. Let N be a finite normal subgroup of a group G. Let p be a prime and $P \in \mathrm{Syl}_p N$. Then $G = N_G(P)N$.

Proof. Let G act on the set $X = \mathrm{Syl}_p N$ by conjugation, and the stabilizer of P is $N_G(P)$. Moreover, by Sylow's Theorem N is transitive on X. Hence, the claim follows from Lemma 2.2.6.

Lemma 2.2.39. Let G be a group, p a prime and P a p-subgroup of G. Suppose that $N_{O_p(G)}(P) \leqslant P$. Then $O_p(G) \leqslant P$.

Proof. Since $N_{O_p(G)}(P) \leqslant P$, we have

$$N_{O_p(G)P}(P) = P.$$

Then, we get

$$O_p(G)P = P$$

and so $O_p(G) \leqslant P$.

Composition Series and Chief Series

Every finite group of order greater than one possesses a finite series of subgroups, called a composition series, such that

$$1 \lhd H_n \lhd \ldots \lhd H_1 \lhd G,$$

where H_{i+1} is a maximal normal subgroup of H_i. A composition series is therefore a normal series without repetition whose factors are all simple.
The quotient groups $G/H_1,\ H_1/H_2, \ldots,\ H_n$ are called composition factors.

A chief series of a group G is a finite collection of normal subgroups $N_i \subseteq G$,

$$1 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n = G,$$

such that each quotient group N_{i+1}/N_i, for $i = 1,\ 2, \ldots, n-1$, is a minimal normal subgroup of G/N_i.

Proposition 2.2.40. Every finite group possesses a composition series.

Proof. Let G be a finite group. We know at least one subnormal series, namely $G \geqslant 1$. Now, assume that

$$G = G_0 > G_1 > G_2 > \dots. > G_n = 1 \qquad (*)$$

be a longest possible subnormal series of G. This certainly exists since G has finitely many subgroups and so G has only finitely many subnormal series.
We want to show that $(*)$ is a composition series. Suppose that it is not. Then, we have one of the factors, say G_i/G_{i+1}, is not simple. Then, G_i/G_{i+1} possesses a non-trivial proper normal subgroup and this corresponds to a subgroup N of G with

$$G_{i+1} < N < G_i.$$

Then, we have

$$G = G_0 > G_1 > G_2 > \dots. > G_i > N > G_{i+1} > \dots > G_n = 1$$

is a subnormal series in G that is longer than $(*)$. This contradicts our assumption that $(*)$ is the longest such series. Hence, we get $(*)$ is a composition series of G.

The important thing about composition series is that the composition factors are essentially unique. This is content of the following theorem.

Theorem 2.2.41. [12, **Jordan Holder Theorem**]. Let G be a group that has a composition series. Then any two composition series for G have the same length. Moreover, if

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = 1$$

and

$$G = H_0 \triangleright H_1 \triangleright H_2 \triangleright \dots \triangleright H_n = 1$$

are two composition series for G, there exists a permutation τ of $\{0,\ 1, \dots,\ n-1\}$ such that for each $i = 0,\ 1,\ 2, \dots,\ n-1$, $G_i/G_{i+1} \cong H_{\tau(i)}/H_{\tau(i)+1}$.

Nilpotent Groups

Definition 2.2.42. A group G is nilpotent if it has a normal series

$$1 = G_1 \leqslant \cdots \leqslant G_n = G \qquad (*)$$

with

$$G_{i+1}/G_i \leqslant Z(G/G_i)$$

for all i. We call $(*)$ is a central series of G of length $n-1$. Non-trivial nilpotent groups have non-trivial center since $G_2 \leqslant Z(G)$. Conversely, the fact that p-groups have non-trivial center implies that they are nilpotent.

Theorem 2.2.43. Every finite p-group is nilpotent.

proof. We will construct a central series of a p-group G from the bottom up. The numbering will be reversed: $G_k = Z_k(G)$ which are defined as follows, $Z_0(G) = 1,\ Z_1(G) = Z(G)$. Given $Z_k(G)$, let $Z_{k+1}(G)$ be the subgroup of G which contains $Z_k(G)$ and corresponds to the center of $G/Z_k(G)$, so that $Z_{k+1}(G)/Z_k(G) = Z(G/Z_k(G))$. Since $G/Z_k(G)$ is a p-group it has a non-trivial center, making $Z_{k+1}(G) > Z_k(G)$ unless $Z_k(G) = G$. Since G is finite we must have $Z_n(G) = G$ for some n making G nilpotent of class n or less.

Theorem 2.2.44. ([15, Proposition 18.8]). Let G be a finite group. The following conditions are equivalent.

1- G is nilpotent.
2- Every Sylow subgroup of G is a normal subgroup.
3- G is isomorphic to the direct product of its Sylow subgroups.

Solvable Groups

Definition 2.2.45. A group G is solvable if it has a normal series from $1 = G_0 \leqslant G_1 \leqslant \ldots. \leqslant G_n = G$ such that G_{i+1}/G_i is abelian for $i = 0, 1, ..., n-1$.

Note that nilpotent groups are solvable since $G_{i+1}/G_i \leqslant Z(G/G_i)$ implies that G_{i+1}/G_i is abelian. It may be not obvious how to find a normal subgroup which satisfies the condition G_{i+1}/G_i is abelian. However, the commutator subgroup will satisfy that condition.

Recall the commutator is given by, $[g,\ h] = g^{-1}h^{-1}gh$, where $g,\ h \in G$.

Definition 2.2.46. The commutator subgroup G' of G is the group generated by all the commutators of the group.

The commutator subgroup $G' = [G,\ G]$ is also called the derived subgroup, and denoted $G^{(1)}$. This construction can be iterated:
$G^{(0)} := G$
$G^{(n)} := [G^{(n-1)}, G^{(n-1)}], n \in \mathbb{N}$.
The groups $G^{(2)}, G^{(3)}$....., are called the second derived subgroup, third derived subgroup, and so forth, and the descending normal series $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq$ ...., is called the derived series. The descending series of a group G, $G = G_1 \trianglerighteq G_2 \trianglerighteq \ldots \trianglerighteq G_n$, where each $G_{n+1} = [G_n, G]$, is called descending central series.

The group G is solvable if its derived series eventually reaches the trivial subgroup of G, and so $H' \neq H$ for all subgroups $1 \neq H \leqslant G$.

Lemma 2.2.47. Let θ be a homomorphism of G. Then

$$([g,\ h])\theta = [(g)\theta,\ (h)\theta]$$

for all $g,\ h \in G$.

Prof. By the definition of the commutator of G, we have

$$([g,h])\theta = (g^{-1}h^{-1}gh)\theta$$

and since θ is a homomorphism, we get the result.

We note that $(G')\theta = ((G)\theta)'$. In particular, G' is a characteristic subgroup of G. Also the commutator G'' of G' is characteristic in G.

Theorem 2.2.48. Let G be a group, and let $N \trianglelefteq G$. Then G/N is abelian if and only if G' is contained in N.

Proof. Suppose that $[h,\ k] \notin N$, where $h,\ k \in G$. Then in G/N we have

$$[hN,\ kN] = hNkNh^{-1}Nk^{-1}N = [h,\ k]N \neq N.$$

So $[hN,\ kN] \neq e_{G/N}$, and so hN and kN do not commute. Conversely, if $[h,\ k] \in N$ for all $h,\ k$ in G, then $[hN,\ kN] = e_{G/N}$ for all $h,\ k$ in G, and so G/N is abelian.

Corollary 2.2.49. G' is normal subgroup of G and G/G' is abelian.

Proof. It is clear that $G' \trianglelefteq G$.
As G' is a normal subgroup of G, G/G' is a group. Theorem 2.2.48 implies G/G' is abelian.

Lemma 2.2.50. Let G be a group. If $H,\ K \trianglelefteq G$ then $[H,\ K] \trianglelefteq G$.

Proof. Let $g \in G$, and let $h \in H,\ k \in K$. Then

$$[h,\ k]^g = (h^{-1}k^{-1}hk)^g = [h^g,\ k^g].$$

Since H and K are normal, $h^g \in H$ and $k^g \in K$, and so $[h,\ k]^g \in [H,\ K]$.
Now suppose that $l \in [H,\ K]$. Then

$$l = [h_1,\ k_1][h_2,\ k_2]\ldots[h_n,\ k_n],$$

for some $h_1,\ h_2,\ldots,\ h_n \in H$ and $k_1,\ k_2,\ldots,\ k_n \in K$. Now

$$l^g = [h_1,\ k_1]^g[h_2,\ k_2]^g\ldots[h_n,\ k_n]^g \in [H,\ K],$$

as required.

Lemma 2.2.51. [12, Lemma 1.5.4]. For $h,\ k,\ l \in G$:

$$[h,\ kl] = [h,\ l][h,\ k]^l \text{ and } [hl,\ k] = [h,\ k]^l[l,\ k].$$

Lemma 2.2.52. For subgroups H and K of G the subgroup $[H,\, K]$ is normal in $\langle H,\, K\rangle$.

Proof. For $h,\, l \in H$ and $k \in K$ Lemma 2.2.51 implies

$$[h,\, k]^l = [hl,\, k][l,\, k]^{-1} \in [H,\, K];$$

and with a similar argument $[h,\, k]^l \in [H,\, K]$ for $l \in K$.

Lemma 2.2.53. Let H and K be subgroups of a group G. Then $[H,\, K] \leqslant K$ if and only if $H \leqslant N_G(K)$.

Proof. Suppose that $H \leqslant N_G(K)$. Then, we have

$$h^{-1}k^{-1}h = k' \in K,$$

and so

$$h^{-1}k^{-1}hk = k'k \in K.$$

Thus,

$$[H,\, K] \leqslant K.$$

Conversely, suppose that $[H,\, K] \leqslant K$, but $H \nleqslant N_G(K)$. Then there exist h and k such that $h^{-1}k^{-1}h \notin K$, a contradiction.

Lemma 2.2.54. [16, **Three-Subgroups Lemma**]. Let $H,\, K,\, L$ be subgroups of G. Suppose that $[H,\, K,\, L] = [K,\, L,\, H] = 1$. Then also $[L,\, H,\, K] = 1$.

Lemma 2.2.55. Subgroups and homomorphic images of solvable groups are solvable.

Proof. For subgroups this clear by the definition of solvability. Let θ be a homomorphism of the solvable group G. Let $1 \neq K \leqslant (G)\theta$, and let $H \leqslant G$ be of minimal order such that $(H)\theta = K$. By Lemma 2.2.47, we have

$$(H')\theta = K'.$$

Since $H' < H$ the minimality of H gives $K' < K$.

Theorem 2.2.56. If G/N and N are solvable. Then G is solvable.

Proof. Let $G/N = \overline{G}$. Since $\overline{G}$ and N are solvable groups, $\overline{G}^{(a)} = 1$ and $N^{(b)} = 1$ for some positive integers $a,\, b$. Because G/N is a homomorphic image of G (under the natural homomorphism $G \to G/N$), $\overline{G}^{(n)} = G^{(n)}N/N$ for every positive integer n. Hence, $G^{(a)} \leqslant N$. Therefore, $G^{(a+b)} \leqslant N^{(b)} = 1$ and G is solvable.

Lemma 2.2.57. Let H and K be two solvable normal subgroups of the group G. Then the product HK is also a solvable subgroup of G.

Proof. Since we have

$$HK/H \cong H/H \cap K,$$

the result follows from Lemma 2.2.55 and Theorem 2.2.56.

The Fitting Subgroup

The group generated by two nilpotent normal subgroups is nilpotent and normal. The Fitting subgroup is the subgroup generated by all normal nilpotent subgroups of a group G, denoted $F(G)$. In particular, $F(G)$ is nilpotent and it is the unique largest nilpotent normal subgroup of G. $F(G)$ can be described to be the product of the normal subgroups $O_p(G)$ for all primes p.

Definition 2.2.58. Let G be a finite group, π a set of primes. We define

$$\begin{aligned} F_\pi(G) &= O_\pi(F(G)) \\ &= \langle O_p(G) \mid p \in \pi \rangle. \end{aligned}$$

Lemma 2.2.59. Suppose $G \neq 1$ is solvable. Then $F(G) \neq 1$.

Proof. Since G is solvable, $G' \neq G$.
If $G' = 1$, then G is abelian. Hence $G = F(G) \neq 1$.

If $G' \neq 1$. Then by induction $F(G') \neq 1$. Now since $F(G')$ is a characteristic subgroup of the normal subgroup G', we have $F(G') \trianglelefteq G$, and $F(G') \leqslant F(G)$. Hence, $F(G) \neq 1$.

Lemma 2.2.60. Let $N \lhd G$. Then $N \cap F(G) = F(N)$.

Proof. Certainly, $F(N)$ is nilpotent subgroup of N, and since it is characteristic in N, it is normal in G. It follows that $F(N) \subseteq N \cap F(G)$. Also, since $F(G)$ is normal subgroup of G, we know that $N \cap F(G)$ is normal subgroup of N, and so is nilpotent. Thus it is contained in $F(N)$.

Theorem 2.2.61. If G is a solvable group. Then $C_G(F(G)) \leqslant F(G)$.

Proof. Set $F = F(G)$ and $C = C_G(F(G))$ and assume by way of contradiction that $C \nsubseteq F$, so that $C \cap F \subset C$. Since C and $C \cap F$ are each normal in G, we can refine the series $1 \subseteq C \cap F \subset C \subseteq G$ to a chief series $1 = G_n \subset \ldots \subset G_2 \subset G_1 = G$. If $C = G_{s+1}$ and $C \cap F = G_{r+1}$, then $s < r$. Thus, we have $G_r \subseteq C$. Since $F \neq 1$, and F is nilpotent, we also have $C \cap F \neq 1$, whence $G_{r+1} \subset G_r$. Furthermore, G_r/G_{r+1} is abelian by Theorem 2.2.48, whence $[G_r,\ G_r] \subseteq G_{r+1} = F$. Since $G_r \subseteq C$ and C centralizes F, it follows that $[G_r,\ G_r,\ G_r] = 1$. But then G_r is nilpotent and so

$G_r \subseteq F$, by the definition of F, as $G_r \lhd G$. Thus $G_r \subseteq C \cap F = G_{r+1}$, contrary $G_{r+1} \subset G_r$.

Lemma 2.2.62. Let G be a solvable group with $\pi(G) \subseteq \{p,\ q\}$ for some primes $p,\ q$. Assume that the Sylow p-subgroups and Sylow q-subgroups of G are abelian. If $P \leqslant G$ is a p-group and $Q \leqslant G$ is a q-group of G, then $[P,\ Q] \leqslant F(G)$.

Proof. Since the Sylow p-subgroups of G are abelian, it follows that $PO_p(G)$ is abelian. Similarly, so is $QO_q(G)$. Since G is solvable, we have

$$[P,\ Q] \leqslant C_G(O_p(G)) \cap C_G(O_q(G)) \leqslant C_G(F(G)) \leqslant F(G).$$

Lemma 2.2.63. Let G be a solvable group with $\pi(G) = \{p,\ q\}$ and assume every Sylow subgroup is abelian. Let $P \in \mathrm{Syl}_p G$. If Q is a q-subgroup of G with $P \leqslant N_G(Q)$, then $Q \trianglelefteq G$. In particular, $Q \leqslant O_q(G)$.

Proof. Choose Q^* with $Q \leqslant Q^* \in \mathrm{Syl}_q G$. Then, we have $G = PQ^*$. Now, $P \leqslant N_G(Q)$ by hypothesis, and $Q^* \leqslant N_G(Q)$ because Q^* is abelian. Thus, $Q \trianglelefteq G$.

Definition 2.2.64. Let p be a prime. A group G is said to be p-solvable if the composition factors of G are either p-groups or p'-groups.

It is an immediate consequence of the definition that any solvable is p-solvable group.

Remark 2.2.65. [19, p. 845]. Suppose that G is a group and p a prime.

- If G is p-solvable and $O_p(G) = 1$, then $C_G(O_{p'}(G)) \leqslant O_{p'}(G)$.
- If G is p-solvable and $O_{p'}(G) = 1$, then G has characteristic p.

Components and the Generalized Fitting Subgroups

Definition 2.2.66. A group $K \neq 1$ is a quasisimple if K is perfect, that is $[K,\ K] = K$ and $K/Z(K)$ is simple. Clearly, for every subnormal subgroup N of a quasisimple group K either

$$N \leqslant Z(K) \ \text{ or } \ N = K.$$

Definition 2.2.67. Let G be a group. A subgroup K of G is a component of G if K is quasisimple and subnormal in G.

Lemma 2.2.68. Let K be a component of a group G and U a subnormal subgroup of G. Then $K \leqslant U$ or $[U,\ K] = 1$.

Proof. Obviously, $U = G$ implies $K \leqslant U$. Moreover, $K = G$ implies either $U = K$ or $[U,\ K] = 1$. Thus, we may assume that there exist proper normal subgroups N, M of G such that

$$K \leqslant N < G \text{ and } U \leqslant M < G.$$

In particular

$$U_1 := [U,\ K] \leqslant N \cap M,$$

and by applying Lemma 2.2.50, we get

$$K \leqslant N_N(U_1) = G_1.$$

Thus, K is a component of G_1, and U_1 is subnormal (in fact normal) in G_1. By induction on $|G|$, applied to G_1, we get

$$[U_1,\ K] = 1 \text{ or } K \leqslant U_1.$$

The first case gives

$$1 = [U,\ K,\ K] = [K,\ U,\ K],$$

and then using the Three-Subgroups Lemma, we have

$$1 = [K,\ K,\ U] = [K',\ U] = [K,\ U].$$

The second case gives $K \leqslant M$ since $U_1 = [U,\ K] \leqslant M$, and the conclusion follows by induction on $|G|$, now applied to M.

Corollary 2.2.69. Let K_1 and K_2 be components of G. Then either $K_1 = K_2$ or $[K_1,\ K_2] = 1$.

Proof. In the case $[K_1,\ K_2] \neq 1$ Lemma 2.2.68 implies $K_1 \leqslant K_2$ and by symmetry also $K_2 \leqslant K_1$.

We now define two characteristic subgroups of G.

$E(G) :=$ the subgroup generated by the components of G,
$F^*(G) := F(G)E(G)$.
$F^*(G)$ is the generalized Fitting subgroup of G. Notice that by Lemma 2.2.68,

$$[F(G),\ E(G)] = 1.$$

Theorem 2.2.70. [12, Theorem 6.5.8]. Let G be a group. Then $C_G(F^*(G)) \leqslant F^*(G)$.

p-Complements

Throughout this part of the section we assume G is a finite group and p is a prime, and we recall a group is a p'-group if it has order coprime to p.

Definition 2.2.71. The largest normal p'-group of G is denoted by

$$O_{p'}(G).$$

Definition 2.2.72. The smallest normal subgroup of G whose quotient is a p-group is denoted by

$$O^p(G).$$

Definition 2.2.73. A normal p-complement in G is a normal subgroup K of G such that, for some $P \in \mathrm{Syl}_p(G)$,

$$G = PK \text{ and } P \cap K = 1.$$

Remark 2.2.74. From the above definition, we have the following:

(1) $G = P_1K$ and $P_1 \cap K = 1$ for all $P_1 \in \mathrm{Syl}_p(G)$. This is because P_1 is conjugate to P and $K \trianglelefteq G$.

(2) K is a p'-group because $|G| = |P||K|$.

(3) Every element of G can be written uniquely in the form ab with $a \in P$ and $b \in K$.

Theorem 2.2.75. [13, **Burnside's Normal p-complement Theorem**]. Let $P \in \mathrm{Syl}_p(G)$ and suppose that

$$P \leqslant Z(N_G(P)).$$

Then G has a normal p-complement.

Corollary 2.2.76. Suppose that a sylow p-subgroup of G is contained in the center of G. Then

$$G \cong P \times O_{p'}(G).$$

Proof. Since $P \leqslant Z(G)$, we have

$$P \leqslant Z(N_G(P)).$$

Theorem 2.2.75 implies that

$$G = PO_{p'}(G).$$

Now $O_{p'}(G) \trianglelefteq G$, and $P \trianglelefteq G$ because $P \leqslant Z(G)$. Also, we have

$$P \cap O_{p'}(G) = 1,$$

and the proof is complete.

2.3 Coprime Action

Let A be a group that acts on a group G. First we introduce some notation that coincides with earlier notion if A and G are embedded in their semidirect product.
For $H \subseteq G$, $B \subseteq A$ and $a \in A$ we have

$$C_H(a) = \{h \in H \mid h^a = h\},$$

and

$$C_H(B) = \bigcap_{b \in B} C_H(b).$$

Definition 2.3.1. Let A be a group that acts on a group G. The action of A on G is coprime if

(1) $(|A|,\ |G|) = 1$,

(2) A or G is solvable.

(1) implies (2) since at least one of the groups A and B has odd order and so Feit-Thompson Theorem which states that every finite group of odd order is solvable.

Definition 2.3.2. Suppose that the group A acts on the group G. Let p be a prime. The set of A-invariant p-subgroups of G is partially ordered by inclusion. The set of maximal elements of this set is denoted by

$$\mathrm{Syl}_p(G;\ A).$$

Moreover,

$$O_p(G;\ A) = \bigcap \mathrm{Syl}_p(G;\ A),$$

which is the unique maximal $AC_G(A)$-invariant p-subgroup of G.

Now, the following two lemmas collect some results on Coprime Action. These are well known and can be found in many different books of group theory, for example in [13]. In this section, we give a proof for Sylow's Theorem for Coprime Action and two special cases of Thompson's Theorem. We will use these results in the following chapters.

Lemma 2.3.3. [20, Coprime Action page 52] Suppose that A is an elementary abelian p-group that acts on a p'-group G.

(a) $G = C_G(A)[G,\ A]$, $[G,\ A]$ is an A-invariant normal subgroup of G and $[G,\ A] = [G,\ A,\ A]$.

(b) Let N be an A-invariant normal subgroup of G and set $\overline{G} = G/N$. Then $C_{\overline{G}}(A) = \overline{C_G(A)}$. Moreover, if p is a prime then $\overline{O_p(G;A)} \leqslant O_p(\overline{G};\ A)$.

Lemma 2.3.4. [19, Coprime Action page 844] Suppose that A is a non-cyclic abelian p-group that acts on an abelian p'-group G. Then, we have the following:

(a) $G = C_G(A) \times [G,\ A]$.

(b) $G = \langle C_G(a) \mid a \in A^{\#} \rangle = \langle C_G(B) \mid B \in \mathrm{Hyp}(A) \rangle$.

(c) $[G,\ a] = \langle [C_G(b),\ a] \mid b \in A \setminus \langle a \rangle,\ a \in A^{\#} \rangle$.

Here, $\mathrm{Hyp}(A)$ denotes the set of maximal subgroups of A.

Lemma 2.3.5. Suppose that A is an elementary abelian p-group that acts on a solvable p'-group G. Then, $[G,\, A] = 1$ if and only if $[F(G),\, A] = 1$.

Proof. It is clear that if $[G,\, A] = 1$, then $[F(G),\, A] = 1$. Now we show the converse. Since $[G,\, F(G)] \leqslant F(G)$, we have

$$[G,\, F(G),\, A] \leqslant [F(G),\, A] = 1$$

Also, we have

$$[F(G),\, A,\, G] = 1.$$

By applying Lemma 2.2.54, we get

$$[A,\, G,\, F(G)] = 1.$$

Since $[A,\, G] = [G,\, A]$, we have

$$[G,\, A] \leqslant C_G(F(G)).$$

Since G is solvable, $C_G(F(G)) \leqslant F(G)$. Then

$$[G,\, A,\, A] = 1.$$

By applying Lemma 2.3.3, we get the result.

Lemma 2.3.6. [12, Lemma 8.1.4]. Let P be a p-subgroup of a group G and N a normal p'-subgroup of G. Then

$$N_G(P)N/N = N_{G/N}(PN/N).$$

Lemma 2.3.7. Let $1 \neq G$ be a nilpotent group, then $[N,\, G] < N$ for every non-trivial normal subgroup $N \leqslant G$.

Proof. Since G is nilpotent, $Z(G) \neq 1$. Let $\overline{G} = G/Z(G)$. By induction on $|G|$ that $\overline{N} = 1$ or $\left[\overline{N},\, \overline{G}\right] < \overline{N}$. The first case gives $N \leqslant Z(G)$ and so $[N,\, G] = 1 < N$. In the second case, we get $[N,\, G] < N$.

Lemma 2.3.8. (**Thompson's** $A \times B$ **Lemma**). Let A and G be p-groups, B a p'-group and let $A \times B$ be a group of operators for G. If $[B,\, C_G(A)] = 1$, then $[B,\, G] = 1$.

Proof. $C_U(A) \leqslant C_U(B)$ for all A-invariant subgroups $U \leqslant G$. Thus, we may assume by induction on $|G|$ that $[U,\, B] = 1$ for all proper A-invariant subgroups of G. As by Lemma 2.3.7, $[G,\, A]$ is a proper subgroup of G, we get $[G,\, A,\, B] = 1$. Since $[A,\, B] = 1$, we have $[A,\, B,\, G] = 1$. Lemma 2.2.54 gives $[B,\, G,\, A] = 1$. This implies that, $[B,\, G] \leqslant C_G(A) \leqslant C_G(B)$, and so $[G,\, B,\, B] = 1$. Applying Lemma 2.3.3, we get $[G,\, B] = 1$.

Lemma 2.3.9. [8, Theorem 6]. Suppose that $A \times B$ acts coprimely on the solvable group G, where A is a group of order r and B is a $\{2,\ r\}'$-group. Assume that $[C_G(A),\ B] = 1$. Then

$$[G,\ B] \leqslant F(G).$$

Lemma 2.3.10. (**Goldschmidt's Lemma**). Let G be a p-solvable group and suppose P is a p-subgroup of G. Then, $O_{p'}(N_G(P)) \leqslant O_{p'}(G)$.

Proof. By induction on $|G|$, set $K = O_{p'}(G)$, and first suppose that $K \neq 1$. Then induction yields,

$$O_{p'}(N_{G/K}(PK/K)) \leqslant O_{p'}(G/K) = 1.$$

Since $N_{G/K}(PK/K) = N_G(P)K/K$ by Lemma 2.3.6, we then obtain

$$O_{p'}(N_G(P))K/K \leqslant O_{p'}(N_G(P)K/K) = 1.$$

Hence, we get

$$O_{p'}(N_G(P)) \leqslant K,$$

as required.

We may therefore suppose that $K = 1$ and hence that $O_p(G) = F(G)$. Set $X = O_p(G)$ and $Q = O_{p'}(N_G(P))$, and observe that $C_X(P) \leqslant N_G(Q)$. It follows that $[Q,\ C_X(P)] \leqslant P \cap Q = 1$, we can apply Theorem 2.3.8 to deduce that $[Q,\ X] = 1$. Consequently $Q \leqslant C_G(X) = C_G(F(G)) \leqslant F(G)$, and since $F(G)$ is a p-group, we conclude that $Q = 1$.

Definition 2.3.11. Let $a \in A$ we define

$$[G,\ a] = \langle [g,\ a] \mid\ g \in G \rangle,$$

where $[g,\ a]$ is the commutator $g^{-1}a^{-1}ga$.

Lemma 2.3.12. Let b be an element of order r that acts coprimely on the group G. Then $[G,\ b] = [G,\ \langle b \rangle]$.

Proof. It is clear that $[G,\ b] \subseteq [G,\ \langle b \rangle]$. By the above definition, we have

$$[G,\ b] = \langle [g,\ b] \mid\ g \in G \rangle$$

and so

$$[G,\ \langle b \rangle] = \langle [g,\ b] \mid\ g \in G,\ n \in \mathbb{N} \rangle.$$

Let $[g,\ b^n] \in [G,\ \langle b \rangle]$. Then

$$\begin{aligned}[g,\ b^n] &= [g,\ b^{n-1}b] \\ &= [g,\ b][g,\ b^{n-1}][g,\ b^{n-1},\ b] \in [G,\ b].\end{aligned}$$

Thus, we get

$$[G,\ b] = [G,\ \langle b \rangle].$$

Lemma 2.3.13. Let b be an element of order r that acts coprimely on the group G. Assume that

$$[O^2([G,\ b]),\ b] = 1.$$

Then $[G,\ b] \leqslant O_2(G)$.

Proof. Let S be a $\langle b\rangle$-invariant Sylow 2-subgroup of $[G,\ b]$. Then $[G,\ b] = O^2([G,\ b])S$. Let $g \in [G,\ b]$, so $g = hs$ with $h \in O^2([G,\ b])$ and $s \in S$. Then

$$\begin{aligned} [g,\ b] &= [hs,\ b] \\ &= [h,\ b]^s[s,\ b] \\ &= [s,\ b] \\ &\leqslant S. \end{aligned}$$

Consequently

$$\begin{aligned} [G,\ b,\ b] &= \langle [g,\ b] \mid\ g \in [G,\ b]\rangle \\ &\leqslant S. \end{aligned}$$

By Coprime Action and Lemma 2.3.12, we get

$$[G,\ b] = [G,\ b,\ b] \leqslant S.$$

Since $[G,\ b] \trianglelefteq G$ we conclude that $[G,\ b] \leqslant O_2(G)$.

Lemma 2.3.14. [19, Theorem 5.2]. Let $\langle a\rangle$ be a group of prime order r that acts coprimely on the solvable group G. Let p be a prime and suppose that P is an $\langle a\rangle C_G(a)$-invariant p-subgroup of G with $P = [P,\ a]$. Assume that at least one of the following hold:

(1) $p \neq 2$.

(2) r is not Fermat.

(3) P is abelian.

(4) $C_P(a) = 1$.

Then, $P \leqslant O_p(G)$.

Corollary 2.3.15. [19, Corollary 5.3]. Let A be an elementary abelian group that acts coprimely on the solvable group G. Let p be a prime, $a \in A^{\#}$ and suppose that P is an $AC_G(a)$-invariant p-subgroup of G. Suppose also that $P = [P,\ a]$ and $C_P(A) = 1$. Then $P \leqslant O_p(G)$.

Lemma 2.3.16. [19, Theorem 5.4]. Let $\langle a\rangle$ be a group of prime order r that acts coprimely on the solvable group G. Let H be an $\langle a\rangle C_G(a)$-invariant subgroup of G. Then,

$$O_2(O^2([H,\ a])) \leqslant O_2(G).$$

Lemma 2.3.17. Suppose B is an r-group that acts on the nilpotent r'-group G. Let $H \leqslant G$ with $C_G(H) \leqslant H$. Assume that $[H, B] = 1$. Then $[G, B] = 1$.

Proof. Consider first the case that $H \trianglelefteq G$.
Let X be the semi direct product GB. Then we have,

$$H \trianglelefteq X, \; C_X(H) = BC_G(H) = BZ(H).$$

Now $[B, Z(H)] = 1$ and B has order coprime to $Z(H)$ and so B is the unique Sylow r-subgroup of $C_X(H)$. Then B is characteristic in $C_X(H) \trianglelefteq X$ and so $B \trianglelefteq X$. Then, we have

$$[G, B] \leqslant G \cap B = 1.$$

Returning to the general case, we note that $N_G(H)$ is B-invariant and apply induction to $|G : N_G(H)|$.

Sylow's Theorem for Coprime Action

Theorem 2.3.18. Let A be an elementary abelian r-group that acts on the r'-group G. Let p be a prime. Then:

(a) There exist an A-invariant Sylow p-subgroup of G.

(b) $C_G(A)$ acts transitively on the set of A-invariant Sylow p-subgroups of G.

(c) Every A-invariant p-subgroups of G is contained in an A-invariant Sylow p-subgroup of G.

Proof. (a) Let X be the semi direct product of G with A.
Choose $P \in \mathrm{Syl}_p G$. Now, $G \trianglelefteq X$ so Lemma 2.2.6 implies $X = GN_X(P)$.
Note that $A \in \mathrm{Syl}_r X$, because G is an r'-group.
Also, since $|X| = |G||N_X(P)| \diagup |G \cap N_x(P)|$ it follows that $N_X(P)$ contains a Sylow r-subgroup of X. Hence there exists $g \in G$ such that $A^g \leqslant N_X(P)$. Then, we have

$$A \leqslant N_X(P^{g^{-1}}),$$

and so, $P^{g^{-1}}$ is an A-invariant Sylow p-subgroup of G.

(b) Trivially, $C_G(A)$ acts by conjugation on the set of A-invariant Sylow p-subgroups of G. We must show this action is transitive.

Suppose that P_1 and P_2 are A-invariant Sylow p-subgroups of G. By Sylow's Theorem (2.2.32), there exists $g \in G$ such that $P_1^g = P_2$. Then, we have

$$A^g, \, A \leqslant N_X(P_2).$$

In fact, we have

$$A^g, \, A \in \mathrm{Syl}_r N_X(P_2)$$

and so,

$$N_X(P_2) = AN_G(P_2).$$

Hence there exists $c \in N_G(P_2)$ such that $A^g = A^c$.
Now $gc^{-1} \in N_G(A)$, in fact, we get

$$\left[A,\ gc^{-1}\right] \leqslant A \cap G = 1$$

and so,

$$gc^{-1} \in C_G(A).$$

Moreover, we have $P_1^{gc^{-1}} = P_2^{c^{-1}} = P_2$ because $c \in N_G(P_2)$.

(c) Let U be a maximal A-invariant p-subgroup of G. We show that U is a Sylow p-subgroup of G.
Assume that $U \notin \mathrm{Syl}_p G$. Then U is not a Sylow p-subgroup of $N_G(U)$ by Lemma 2.2.37. As $N_G(U)$ is A-invariant, there exists an A-invariant $T \in \mathrm{Syl}_p N_G(U)$ by (a). But $U < T$, which contradicts the maximality of U.

Corollary 2.3.19. Let A be an elementary abelian r-group that acts on the r'-group G. Let P_1 and P_2 be A-invariant Sylow p-subgroups of G. Then, we have

$$C_A(P_1) = C_A(P_2).$$

Proof. We work in the semidirect product AG. By part (b) of the above lemma, there exists $c \in C_G(A)$ such that $P_1^c = P_2$. Then, we have

$$\begin{aligned} C_A(P_2) &= C_A(P_1^c) \\ &= C_{A^c}(P_1^c) \\ &= (C_A(P_1))^c \\ &= C_A(P_1). \end{aligned}$$

The Solvable Signalizer Functor Theorem

Definition 2.3.20. Let r be a prime and A an elementary abelian r-group. Suppose that A acts on the group G. An A-Signalizer Functor on G is a mapping θ which assigns to each element $a \in A^\#$, an A-invariant r'-subgroup $\theta(a)$ of $C_G(a)$. The following condition must also be satisfied:

$$\theta(a) \cap C_G(b) \leqslant \theta(b),$$

for all $a,\ b \in A^\#$.

We say that θ is solvable if $\theta(a)$ is solvable for all $a \in A^\#$.

Definition 2.3.21. With the notation above, we say that θ is complete if there is an A-invariant r'-subgroup K such that

$$\theta(a) = C_K(a),$$

for all $a \in A^\#$. We say θ is solvably complete if K is solvable.

Theorem 2.3.22. [19, **The Solvable Signalizer Functor Theorem**]. With the notation in Definition 2.3.20 if θ is svable and rank $(A) \geqslant 3$ then θ is solvably complete.

Fixed Point Free Automorphism

Theorem 2.3.23. [12, **Thompson's Fixed-point-free Automorphism Theorem**]. Every group admitting a fixed point free automorphism of prime order is nilpotent.

We will prove two special cases of Thompson's Fixed-point-free Automorphism Theorem. However, we start with the following lemma:

Lemma 2.3.24. [12, Lemma 8.3.5]. Let G be a Frobenius group with Frobenius complement H and Frobenius kernel K. Suppose that G acts on a non-trivial abelian group V such that $(|V|,\ |K|) = 1$ and $C_V(K) = 1$. Then $C_V(H) \neq 1$.

Lemma 2.3.25. [13, Lemma 8.1.5]. Let A be a group that acts on the group G. Let K be an A-invariant composition factor of G that is a p-group. Then $[K,\ O_p(A)] = 1$.

Theorem 2.3.26. Let G be a solvable group and $\alpha \in \mathrm{Aut}(G)$ with prime order. Assume that $C_G(\alpha) = 1$. Then G is nilpotent.

Proof. Let $\langle\alpha\rangle$ be a group of prime order p. Let G be a group on which $\langle\alpha\rangle$ acts fixed point freely, and that is $C_G(\langle\alpha\rangle) = 1$, and $p \notin \pi(G)$. Thus we have G is p'-group. Assume the result is false, and we suppose that G is a counterexample of minimal order. Let V be a minimal α-invariant normal subgroup of G. Then there exist a prime q such that V is an elementary abelian q-group. α is fixed point free on V so V and G/V are nilpotent but G is not. Thus $V \not\subseteq Z(G)$, and $C_G(V) \neq G$. Also, α is fixed point free on $\overline{G} = G/C_G(V)$ so $\overline{G}$ is nilpotent, and there exists a prime r such that $\overline{G}_1 = O_r(\overline{G}) \neq 1$, and Lemma 2.3.25 implies that $r \neq q$. Moreover $C_V(\overline{G}_1) = 1$ since $C_V(\overline{G}_1)$ is an α-invariant normal subgroup of G and $C_G(V) \neq G$. Every non-trivial element of $\langle\alpha\rangle$ is a fixed point free on $\overline{G}_1$. Now, consider the semi direct product $\langle\alpha\rangle\overline{G}_1$. This is a Frobenius group with Frobenius complement $\langle\alpha\rangle$. But Lemma 2.3.24 shows that $1 \neq C_V(\alpha)$, a contradiction.

Theorem 2.3.27. Let G be a group with every Sylow subgroup abelian and $\alpha \in \mathrm{Aut}(G)$ with prime order r. Assume that $C_G(\alpha) = 1$. Then G is nilpotent.

Proof. Assume the result is false and let G be a minimal counterexample, then G is not nilpotent.

Claim. G does not posses an α-invariant normal subgroup N with $1 \neq N \neq G$.

Assume false. The minimality of G implies that G/N and N are nilpotent. Therefore, G is solvale and so nilpotent by Theorem 2.3.26, a contradiction.
If U is an α-invariant subgroup with $1 \neq U \neq G$, then $N_G(U) \neq G$ by our Claim. Also $N_G(U)$ is α-invariant. The minimality of G implies $N_G(U)$ is nilpotent. Let $p \in \pi(G)$ and let P be an α-invariant Sylow p-subgroup. By induction, $N_G(P)$ is nilpotent, so $N_G(P)$ has a normal

p-complement. Theorem 2.2.75 implies that G has a normal p-complement, a contradiction.

2.4 Review of Some Basic Concepts of Representation Theory

A linear representation φ of a group G is a homomorphism from G to the general linear group $GL(V)$, where V is a vector space over a field $\mathbb{F}$. If V has dimension n, then $GL(V)$ and $GL(n, \mathbb{F})$ are isomorphic.

The degree of the representation is equal to the dimension of V. If $\{0\}$ is the only invariant proper subspace of V with respect to $g\varphi$, for all $g \in G$, then the representation φ is irreducible, that is, if $W \subset V$ and $w(g\varphi) \in W$, for all $w \in W,\ g \in G$ implies $W = \{0\},\ \varphi$ is irreducible.

Two representation $\varphi_1,\ \varphi_2$ are equivalent if there exist a fixed invertible matrix A, such that $A^{-1}(g\varphi_1)A = (g)\varphi_2$ for all $g \in G$.

We shall be concerned exclusively with representations of finite degree, that is, in finite dimensional vector spaces; and these will be vector spaces over a field $\mathbb{F}$ of prime characteristic. Very often $\mathbb{F} = GF(p)$, where p is a prime.

A representation in V can be defined by a homomorphism

$$\varphi : G \longrightarrow GL(n, \mathbb{F}),$$

where n is the dimension of V.

Now, we will review some elementary results.

Definition 2.4.1. Let V be a vector space over $\mathbb{F}$ and let G be a group. Then V is an $\mathbb{F}G$-module if a multiplication $vg(v \in V,\ g \in G)$ is defined, satisfying the following conditions for all $u,\ v \in V,\ \lambda \in \mathbb{F}$ and $g,\ h \in G$:

(1) $vg \in V$,

(2) $v(gh) = (vg)h$,

(3) $v1 = v$,

(4) $(\lambda v)g = \lambda(vg)$,

(5) $(u+v)g = ug+vg$.

Note that conditions (1), (4) and (5) in the definition ensure that for all $g \in G$, the function

$$v \longrightarrow vg(v \in V)$$

is an endomorphism of V.

Definition 2.4.2. Let V be an $\mathbb{F}G$-module, and let β be a basis of V. For each $g \in G$, let $[g]_\beta$ denote the matrix of endomorphism $v \to vg$ of V, relative to the basis β.

The connection between $\mathbb{F}G$-modules and representations of G over $\mathbb{F}$ is described in the following basic result.

Theorem 2.4.3. (1) If $\varphi : G \longrightarrow GL(n,\ \mathbb{F})$ is a representation of G over $\mathbb{F}$, and $V = \mathbb{F}^n$, then V becomes an $\mathbb{F}G$-module if we define the multiplication vg by

$$vg = v(g\varphi)(v \in V,\ g \in G).$$

Moreover, there is a basis β of V such that

$$g\varphi = [g]_\beta \text{ for all } g \in G.$$

(2) Assume that V is an $\mathbb{F}G$-module and let β be a basis of V. Then the function

$$g \longrightarrow [g]_\beta \ \ (g \in G)$$

is a representation of G over $\mathbb{F}$.

Proof. We have already observation that for al $u,\ v \in \mathbb{F}^n,\ \lambda \in \mathbb{F}$ and $g,\ h \in G$, we have

$$\begin{aligned} v(g\varphi) &\in \mathbb{F}^n, \\ v((gh)\varphi) &= (v(g\varphi))(h\varphi), \\ v(1\varphi)) &= v, \\ (\lambda v)(g\varphi) &= \lambda(v(g\varphi)), \\ (u+v)(g\varphi) &= u(g\varphi) + v(g\varphi). \end{aligned}$$

Therefore, $\mathbb{F}^n$ becomes an $\mathbb{F}G$-module if we define

$$vg = v(g\varphi) \text{ for all } v \in \mathbb{F}^n,\ g \in G.$$

Moreover, if let β be the basis

$$(1\ ,\ 0,\ 0, \ldots,\ 0),\ (0,\ 1,\ 0, \ldots,\ 0), \ldots (0,\ 0,\ 0, \ldots,\ 1)$$

of $\mathbb{F}^n$, then $g\varphi = [g]_\beta$ for all $g \in G$.

(2) Let V be an $\mathbb{F}G$-module with basis β. Since $v(gh) = (vg)h$ for all $g,\ h \in G$ and all v in the basis β of V, it follows that

$$[gh]_\beta = [g]_\beta\, [h]_\beta\,.$$

In particular,

$$[1]_\beta = [g]_\beta\, [g^{-1}]_\beta$$

for all $g \in G$. Now $v1 = v$ for all $v \in V$, so $[1]_\beta$ is the identity matrix. Therefore each matrix $[g]_\beta$ is a homomorphism from G to $GL(n,\ \mathbb{F})$ (where $n = \dim V$), and hence is a representation of G over $\mathbb{F}$.

Definition 2.4.4. Let V be an $\mathbb{F}G$-module. A subset W of V is said to be an $\mathbb{F}G$-submodule of V if W is a subspace and $wg \in W$ for all $w \in W$ and all $g \in G$.
Thus an $\mathbb{F}G$-submodule of V is a subspace which is also an $\mathbb{F}G$-module.

Definition 2.4.5. Let V and W be $\mathbb{F}G$-modules. A function $\theta : V \longrightarrow W$ is said to be an $\mathbb{F}G$-homomorphism if θ is a linear transformation and

$$(vg)\theta = (v\theta)g \text{ for all } v \in V,\ g \in G.$$

Proposition 2.4.6. Let V and W be $\mathbb{F}G$-modules and let $\theta : V \longrightarrow W$ be an $\mathbb{F}G$-homomorphism. Then $\mathrm{Ker}\theta$ is an $\mathbb{F}G$-submodule of V, and $\mathrm{Im}\theta$ is an $\mathbb{F}G$-submodule of W.

Proof. Note that $\mathrm{Ker}\theta$ is a subspace of V and $\mathrm{Im}\theta$ is a subspace of W, since θ is a linear transformation. Let $v \in \mathrm{Ker}\theta$ and $g \in G$. Then, we have

$$(vg)\theta = (v\theta)g = 0g = 0$$

and so $vg \in \mathrm{Ker}\theta$. Therefore, $\mathrm{Ker}\theta$ is an $\mathbb{F}G$-submodule of V. Now let $w \in \mathrm{Im}\theta$, so that $w = v\theta$ for some $v \in V$.
For all $g \in G$, we have

$$wg = (v\theta)g = (vg)\theta \in \mathrm{Im}\theta,$$

and so $\mathrm{Im}\theta$ is an $\mathbb{F}G$-submodule of W.

Definition 2.4.7. Let V and W be $\mathbb{F}G$-modules. We call a function $\theta : V \longrightarrow W$ an $\mathbb{F}G$-isomorphism if θ is an $\mathbb{F}G$-homomorphism and θ is bijective. If there such an $\mathbb{F}G$-isomorphism, then we say that V and W are isomorphic $\mathbb{F}G$-module and write $V \cong W$.

Theorem 2.4.8. [3, **Maschke's Theorem**]. Let G be a finite group and let $\mathbb{F}$ be a field has a characteristic not dividing $|G|$. Suppose V is an $\mathbb{F}G$-module of V and that U is an $\mathbb{F}G$-submodule of V. Then there exist an $\mathbb{F}G$-submodule W of V such that $V = U \oplus W$.

Lemma 2.4.9. (**Schur's Lemma**). Let V and W be irreducible $\mathbb{F}G$-modules.

(1) If $\vartheta : V \to W$ is a $\mathbb{F}G$-homomorphism, then either ϑ is a $\mathbb{F}G$-isomorphism, or $v\vartheta = 0$ for all $v \in V$.

(2) Assume $\mathbb{F}$ is algebraically closed. If $\vartheta : V \to V$ is a $\mathbb{F}G$-isomorphism, then ϑ is a scalar multiple of the identity endomorphism 1_v.

Proof. (1) Suppose that $v\vartheta \neq 0$ for some $v \in V$. Then Im $\vartheta \neq \{0\}$. As Im ϑ is a $\mathbb{F}G$-submodule of W, and W is irreducible, we have $\mathrm{Im}\vartheta = W$. Also, $\mathrm{Ker}\vartheta$ is a $\mathbb{F}G$-submodule of V; as $\mathrm{Ker}\vartheta \neq V$ and V is irreducible, $\mathrm{Ker}\vartheta = \{0\}$. Thus ϑ is a bijection, and hence is a $\mathbb{F}G$-isomorphism.

(2) The endomorphism ϑ has an eigenvalue $\lambda \in \mathbb{F}$, and so Ker $(\vartheta - \lambda 1_v) \neq \{0\}$. Thus Ker

$(\vartheta - \lambda 1_v)$ is a non-zero $\mathbb{F}G$-submodule of V. Since V is irreducible, $\mathrm{Ker}(\vartheta - \lambda 1_v) = V$. Therefore,

$$v(\vartheta - \lambda 1_v) = 0 \text{ for all } v \in V.$$

That is, $\vartheta = \lambda 1_v$, as required.

Part (2) of Schur's Lemma has the following converse:

Proposition 2.4.10. Let G be a finite group and let $\mathbb{F}$ be a field has characteristic not dividing $|G|$. Let V be a non-zero $\mathbb{F}G$-module, and suppose that every $\mathbb{F}G$-homomorphism from V to V is a scalar multiple of 1_v. Then V is irreducible.

Proof. Suppose that V is reducible, so that V has a $\mathbb{F}G$-submodule U not equal to $\{0\}$ or V. By Maschke's Theorem, there is a $\mathbb{F}G$-submodule W of V such that

$$V = U \oplus W.$$

Then the projection $\pi : V \to V$ defined by $(u+v)\,\pi = u$ for all $u \in U,\ w \in W$ is a $\mathbb{F}G$-homomorphism, and is not a scalar multiple of 1_v, which is a contradiction. Hence, V is irreducible.

Now, we are going to present some other results which will be used in later chapters.

Definition 2.4.11. For S a finite p-group, let

$$m = \max\{|A| : A \text{ is an abelian subgroup of } S\},$$

$$\mathscr{A}(S) = \{A \leqslant S : A \text{ is abelian and } |A| = m\}$$

and

$$J(S) = \langle A : A \in \mathscr{A}(S)\rangle.$$

Lemma 2.4.12. Let p be a prime. Let G be a group that acts on the elementary abelian p-group V and suppose that

$$V = \langle C_V(S) \mid S \in \mathrm{Syl}_p G\rangle.$$

Then,

$$O_p(G/C_G(V)) = 1.$$

Proof. Let $S \in \mathrm{Syl}_p G$. Set

$$Z := C_V(S) \text{ and } C := C_G(V).$$

Since all Sylow p-subgroups of G are conjugate, we get

$$V = \langle Z^G\rangle.$$

Let $C \leqslant D \leqslant G$ such that $D/C = O_p(G/C)$. Then, we have

$$1 \neq D \cap S \in \mathrm{Syl}_p D,\ D = C(D \cap S) \text{ and } G = CN_G(D \cap S)$$

by applying Lemma 2.2.6. This gives,

$$V = \langle Z^{N_G(D \cap S)} \rangle.$$

Let $n \in N_G(D \cap S)$. Then

$$\begin{aligned}[Z^n,\ D \cap S] &= [Z,\ D \cap S]^n \\ &\leqslant [Z,\ D \cap S]^n \\ &= 1\end{aligned}$$

since $Z = C_V(S)$. We deduce that

$$[V,\ D \cap S] = 1.$$

Hence, we have

$$D \cap S \leqslant N_G(V) = C.$$

and so

$$D = C.$$

Now the group

$$\Omega_1(G) = \langle x \in G \mid x^p = 1 \rangle,$$

is a characteristic subgroup of G.

Remark 2.4.13. Let p be a prime. The group G has characteristic p (see Definition 2.2.34) if

$$F^*(G) = F(G) = O_p(G)$$

because the condition $C_G(O_p(G)) \leqslant O_p(G)$ is equivalent to the above condition.

Lemma 2.4.14. (**Thompson Lemma**). Let p be a prime and G a group with characteristic p. Let

$$V = \langle \Omega_1(Z(S)) |\ S \in \mathrm{Syl}_p G \rangle.$$

Then $V \leqslant \Omega_1(Z(O_p(G)))$, V is an elementary abelian characteristic subgroup of G and

$$O_p(G/C_G(V)) = 1.$$

Proof. Let $S \in \mathrm{Syl}_p G$. Since G has characteristic p, we have

$$\Omega_1(Z(S)) \leqslant C_G(O_p(G)) \leqslant O_p(G) \leqslant S.$$

Then V is contained in $\Omega_1(Z(O_p(G)))$ and $\Omega_1(Z(S)) = C_V(S)$, so the assertion follows from Lemma 2.4.12.

Lemma 2.4.15. [13, **Stellmacher**]. Let p be a prime and P a p-group. Let $V_1,\ V_2,\dots,\ V_n$ be elementary abelian normal subgroups of P with $V = \langle V_1,\ V_2,\dots,\ V_n\rangle$. Let $q \in \mathbb{R}$ and suppose that

$$|B/C_B(V_i)|^q \leqslant |V_i/C_{V_i}(B)|,$$

for all $1 \leqslant i \leqslant n$ and $B \leqslant P$ with $[V_i,\ B,\ B] = 1$. Then,

$$|B/C_B(V)|^q \leqslant |V/C_V(B)|,$$

for all $B \leqslant P$ with $[V,\ B,\ B] = 1$.

Proof. Let $B_0 = B$ and for $1 \leqslant k \leqslant n$ let

$$B_k = C_B(V_1) \cap \ldots \cap C_B(V_k).$$

Then for each k with $1 \leqslant k \leqslant n$, we have

$$|B_{k-1}/B_k|^q \leqslant |V_k/C_{V_k}(B_{k-1})|.$$

The map $v \longmapsto C_V(B_{k-1}) + v$ from V_k to $C_V(B_k)/C_V(B_{k-1})$ has kernal $C_{V_k}(B_{k-1})$. Hence

$$|V_k/C_{V_k}(B_{k-1})| \leqslant |C_V(B_k)/C_V(B_{k-1})|.$$

We deduce that,

$$|B_{k-1}/B_k|^q \leqslant |C_V(B_k)/C_V(B_{k-1})|.$$

Then, we have

$$\begin{aligned}&(|B_0/B_1||B_1/B_2|\ldots|B_{n-1}/B_n|)^q\\ &\leqslant |C_V(B_1)/C_V(B_0)||C_V(B_2)/C_V(B_1)|\ldots|C_V(B_n)/C_V(B_{n-1}).\end{aligned}$$

Hence,

$$|B/C_B(V)|^q \leqslant |V/C_V(B)|.$$

Definition 2.4.16. Let G be a group, p a prime, V a faithful $GF(p)G$-module and let $A \leqslant G$ be an elementary abelian p-group. Then

(1) A is quadratic on V if $[V,\ A,\ A] = 0$.

(2) A is cubic on V if $[V,\ A,\ A,\ A] = 0$.

(3) A is nearly quadratic on V if A is cubic on V and

$$[V,\ A] \leqslant [v,\ A] + C_V(A),$$

for all $v \in V \setminus ([V,\ A] + C_V(V))$.

(4) A is a $2F$-offender for G on V if $A \neq 1$ and

$$|V/C_V(A)| \leqslant |A/A \cap O_p(G)|^2.$$

We remark that if A is quadratic on V then it is nearly quadratic on V. Moreover, if A is a $2F$-offender for G on V, then $A \nleqslant O_p(G)$.

Theorem 2.4.17. Suppose that the group A acts coprimely on the group G, that p is a prime and that V is a $GF(p)AG$-module. Assume that G is faithful on V and that G is p-solvable. Then $O_p(G;\, A)$ does not contain any nearly quadratic $2F$-offender for G on V.

Proof. This proof in [19, Theorem B] under the assumption that AG acts faithfully on V. Recall that AG is the semidirect product of A and G. Let $N = C_{AG}(V)$. Then, we have

$$N \cap G = C_G(V) = 1,$$

so $|N|$ divides $|A|$.
Now, A is a Hall subgroup of AG and $AN \leqslant AG$. It follows that $AN = A$, so $N \leqslant A$. Since $N \trianglelefteq AG$, we have

$$[N,\, G] \leqslant A \cap G = 1.$$

Set $A^* = A/N$. Then A^* acts coprimely on G, V is a faithful $GF(p)A^*G$-module and

$$O_p(G;\, A^*) = O_p(G;\, A).$$

The conclusion follows from [20, Theorem B].

Theorem 2.4.18. (**Stellmacher** [20, Theorem 3.3]). Let p be a prime, let G be a p-solvable group and let V be an elementary abelian p-subgroup of G. Suppose that

(1) G has characteristic p.

(2) $O_p(G)$ normalizes V but $V \nleqslant O_p(G)$.

Then there exists $B \leqslant O_p(G)$ such that the following hold:

(a) $B/C_B(V)$ is elementary abelian and acts non-trivially and nearly quadratically on V.

(b) $|V/C_V(B)| \leqslant |B/C_B(V)|^2$.

Theorem 2.4.19. (**Thompson's Replacement Theorem** [10, Theorem 1]). Let S be a finite p-group, and let $A \in \mathscr{A}(S)$.

(a) Suppose $x \in S$ and $[x,\, A]$ is abelian. Let $M = [x,\, A]$. Then $MC_A(M) \in \mathscr{A}(S)$.

(b) Suppose B is an abelian subgroup of S that is normalized by A. If $[B,\, A,\, A] \neq 1$, there exists $A^* \in \mathscr{A}(S)$ such that $A \cap B < A^* \cap B$, $A^* \leqslant AB$ and $[A^*,\, A,\, A] = 1$.

2.5 Some Group Theoretic Concepts

Burnside's $p^\alpha q^\beta$-Theorem

We will prove the Theorem of Burnside that groups of order $p^\alpha q^\beta$, where p and q are primes and $\alpha,\ \beta$ are non-negative integers are solvable. We will use some basic results which have been presented in the previous section and the following lemma.

Lemma 2.5.1. [2, Theorem 4.23]. If a group G has a non-identity conjugacy class K such that $|K| = p^a$, where p is prime and a is a non-negative integer , then G is not simple.

Theorem 2.5.2. (Burnside's Theorem). Let G be a group of order $p^\alpha q^\beta$, where $p,\ q$ are primes and $\alpha,\ \beta$ are non-negative integers. Then G is solvable.

Proof. If $p = q$ or if either $\alpha,\ \beta$ are equals zero, then G is p-group, hence it is a solvable. We can assume that p and q are distinct primes and $\alpha,\ \beta$ are positive numbers. There must a subgroup P of order p^α. The center of the subgroup P is non-trivial because P is a p-group. Let $g \in Z(P)$ and $g \neq 1$, the order of the conjugacy class g is $|G : C_G(g)|$. As $P \subseteq C_G(g)$, $|G : C_G(g)| = q^t$, $t \neq 1$. Then, we have a non identity conjugacy class of G of order q^t. Thus, by Lemma 2.5.1, G is not simple. By the induction of the order of the group G, G has a normal subgroup N that is solvable, and G/N also solvable. Hence G is a solvable.

Primitive Pairs

We call a proper subgroup H of G primitive if $H = N_G(K)$ for every non-trivial characteristic subgroup K of H. Now we define a primitive pair of a group G as follows:

Definition 2.5.3. A primitive pair for a group G is a pair $(H_1,\ H_2)$ of distinct subgroups G that satisfy

- whenever $\{i,\ j\} = \{1,\ 2\}$ and K char H_i with $K \leqslant H_1 \cap H_2$ then $N_{H_j}(K) = H_1 \cap H_2$.

Moreover, $(H_1,\ H_2)$ has characteristic p if in addition:

- for each i, H_i has characteristic p and $O_p(H_1)O_p(H_2) \leqslant H_i$.

Now, we will present Bender's Theorem which shows under some certain assumptions that a primitive pair has characteristic p. We start with the following lemmas:

Lemma 2.5.4. Suppose that $(H_1,\ H_2)$ is a primitive pair and N is a group with N char H_1 and N char H_2. Then, $N = 1$.

Proof. Suppose that $N \neq 1$. Since H_1 and H_2 is a primitive pair and $1 \neq N \trianglelefteq H_1$, $N \leqslant H_1 \cap H_2$, implies that $H_2 = N_{H_2}(N) = H_1 \cap H_2$. Thus $H_2 \subseteq H_1$. Similarly, we get $H_1 \subseteq H_2$. Hence $H_1 = H_2$, a contradiction. Then, $N = 1$.

Lemma 2.5.5. Let H be a subgroup of a group G. Suppose that $H = \langle S \rangle$ for some $S \subseteq G$. Let $g \in G$ and suppose that $s^g \in H$ for all $s \in S$. Then $g \in N_G(H)$.

Proof. Let $h \in H$. Since $H = \langle S \rangle$, there exist $s_1,\ s_2, \dots s_n \in S \cup S^{-1}$ such that $h = s_1 s_2 \dots s_n$. Then $h^g = s_1^g s_1^g \dots s_n^g \in H$, and it follows that $g \in N_G(H)$.

Lemma 2.5.6. Let X be a group, $P \leqslant X$ a p-group and $Q \leqslant X$ a q-group, and $p,\ q$ are distinct primes. Suppose that $C_X(F(X)) \leqslant F(X),\ PF(X)$ and $QF(X)$ are nilpotent. Then $[P,\ Q] = 1$.

Proof. The group $PF(X)$ is nilpotent, so subgroups of $PF(X)$ with coprime orders commute. Note that $p \neq q$, and hence $[P,\ O_{p'}(F(X))] = 1$. Then, we have

$$P \leqslant C_X\left(O_{p'}(F(X))\right) \trianglelefteq X.$$

Similarly,

$$Q \leqslant C_X\left(O_{q'}(F(X))\right) \trianglelefteq X.$$

This implies that,

$$[P,\ Q] \leqslant C_X(O_{p'}F(X)) \cap C_X(O_{q'}F(X)) \leqslant C_X(F(X)) = Z(F(X)).$$

We claim that Q normalizes $PZ(F(X))$.
Let $g \in P$ and $h \in Q$. Then,

$$[g,\ h] \in [P,\ Q] \leqslant Z(F(X)).$$

This implies that,

$$g^{-1}h^{-1}gh \in Z(F(X))$$

and so,

$$g^{-1}g^h \in Z(F(X)).$$

Hence, we have

$$g^h \in gZ(F(X)) \subseteq PZ(F(X)).$$

Thus, the claim follows from Lemma 2.5.5. Similarly, P normalizes $QZ(F(X))$.
Since $QZ(F(X))$ is nilpotent, we have

$$Q \leqslant O_q(QZ(F(X))).$$

Now, P normalizes $O_q(QZ(F(X)))$, hence $[P,\ Q]$ is q-group.

Similarly, it is p-group. Then, we have

$$[P,\ Q] = 1.$$

Theorem 2.5.7. (Bender's Theorem). Let G be a group. Let $(H_1,\ H_2)$ be a primitive pair of G. Suppose that $F^*(H_1) \leqslant H_2$ and $F^*(H_2) \leqslant H_1$. Then there exist a prime p such that the primitive pair of $(H_1,\ H_2)$ has characteristic p.

Proof. Since $F^*(H_1) \trianglelefteq H_1$ and $F^*(H_1) \leqslant H_2$, also $F^*(H_2) \trianglelefteq H_2$ and $F^*(H_2) \leqslant H_1$, we have

$$F^*(H_1)F^*(H_2) \trianglelefteq H_1 \cap H_2$$

and,

$$F^*(H_1)F^*(H_2) \leqslant F^*(H_1 \cap H_2). \tag{1}$$

From (1), if K is a component of H_1 it is also a component of $H_1 \cap H_2$, and so

$$K \leqslant F^*(H_2) \text{ or } [F^*(H_2),\ K] = 1.$$

If $[F^*(H_2),\ K] = 1$, then $K \leqslant Z(F(H_2))$. This is a contradiction as K non-abelian. Thus, we have

$$K \leqslant F^*(H_2).$$

Note that $K \trianglelefteq \trianglelefteq H_1 \cap H_2$ and $F^*(H_2) \leqslant H_1 \cap H_2$.
So, $K \trianglelefteq \trianglelefteq F^*(H_2) \trianglelefteq H_2$. Then K is also a component of H_2. This implies that,

$$E(H_1) \leqslant E(H_2). \tag{2}$$

Similarly,

$$E(H_2) \leqslant E(H_1). \tag{3}$$

From (2) and (3), we have

$$E(H_1) = E(H_2).$$

Refer to Lemma 2.5.4. Hence,

$$E(H_1) = E(H_2) = 1.$$

Thus, we have

$$F^*(H_1) = F(H_1) \text{ and } F^*(H_2) = F(H_2).$$

Therefore,

$$C_{H_1}(F(H_1)) \leqslant F(H_1)$$

and,

$$C_{H_2}(F(H_2)) \leqslant F(H_2).$$

Also, we have

$$F(H_1) \leqslant F(H_1 \cap H_2)$$

and,

$$F(H_2) \leqslant F(H_1 \cap H_2).$$

Let $p \in \pi(F(H_1))$. Suppose that $p \notin \pi(F(H_2))$, so $F(H_2)$ is a p'-group, then we have

$$[O_p(H_1),\ F(H_2)] = 1$$

and,

$$1 \neq O_p(H_1) \leqslant C_{H_2}(F(H_2)) \leqslant F(H_2).$$

This contradicts that $F(H_2)$ is a p'-group. We deduce that,

$$p \in \pi(F(H_2)).$$

This means that,

$$\pi(F(H_1)) \subseteq \pi(F(H_2)).$$

By symmetry it follows that,

$$\pi(F(H_1)) = \pi(F(H_2)).$$

Assume the theorem false, then there exist $p,\ q \in \pi(F(H_1)) = \pi(F(H_2))$ with $p \neq q$. Since $O_p(H_1),\ O_q(H_1) \leqslant F(H_1 \cap H_2)$ also $F(H_2) \leqslant F(H_1 \cap H_2)$, we have $O_p(H_1)F(H_2)$ and $O_q(H_1)F(H_2)$ are nilpotent subgroups of H_2.

Let $h \in H_2$, then $O_p(H_1)^h F(H_2) = (O_p(H_1)F(H_2))^h$ is nilpotent. By Lemma 2.5.6 we have,

$$[O_p(H_1)^h,\ O_q(H_2)] = 1.$$

Now, $C_{H_2}(O_q(H_1)) \leqslant H_1 \cap H_2$, and then

$$O_p(H_1)^h \leqslant H_1 \cap H_2.$$

Let $I = H_1 \cap H_2$, then

$$O_p(H_1) \leqslant I^{h^{-1}},\ \forall h \in H_2.$$

Hence, we have

$$O_p(H_1) \leqslant \bigcap_{h \in H_2} I^h \trianglelefteq H_2.$$

Also,

$$I \leqslant H_1$$

and so,

$$O_p(H_1) \subseteq O_p(I).$$

Hence, we have

$$O_p(H_1) \subseteq O_p(\bigcap_{h \in H_2} I^h) \subseteq O_p(H_2).$$

This implies that,

$$O_p(H_1) \subseteq O_p(H_2).$$

By symmetry, we get

$$O_p(H_1) = O_p(H_2).$$

Then Lemma 2.5.4 implies that $O_p(H_1) = 1$, a contradiction.

Definition 2.5.8. Let M and H are two solvable subgroups of a group G we define

$$M \rightsquigarrow H$$

to mean there exists $X \leqslant F(M)$ with $XC_{F(M)}(X) \leqslant H$.

Theorem 2.5.9. [18, **Bender's Maximal Subgroup Theorem**]. Suppose that $(M,\ H)$ is a primitive pair of solvable groups for the group G and that $M \rightsquigarrow H$. Then

(a) $O_q(H) \cap M = 1$ for all $q \notin \pi(F(M))$.

(b) If $H \rightsquigarrow M$ or $O_q(H) = 1$ for all $q \notin \pi(F(M))$ the there is a prime p such that M and H have characteristic p.

Chapter 3

Basic Concepts of Partial Groups

3.1 Introduction

In this chapter we will introduce the concept of a partial group and some basic concepts. We begin with the definition of partial groups with some examples, and then we introduce the notions of abelian partial groups, nilpotent partial groups and solvable partial groups. In fact, we will prove that every connected abelian partial group and so nilpotent partial group have a unique maximal member. We give a counterexample to show that not every connected solvable partial group has a unique maximal member. In the third section of this chapter, we will introduce subpartial groups. Also, we give a definition of homomorphisms of partial groups and so kernel of homomorphism partial groups. At the end of this section, we will extend the concept of primitive pairs of groups to partial groups, and we give an analogue of Bender's Maximal Subgroup Theorem (Theorem 2.5.9) for partial groups. In the last section, we provide a notion of automorphisms of partial groups. We extend the Theorem of Thompson (Theorem 2.3.23) to partial groups. Also, we extend some results of elementary abelian groups acting on partial groups. We start to extend a result of Solvable Signalizer Functor Theorem to nilpotent partial groups. Finally, we extend Transitivity Theorem (Theorem 2.3.18) to partial groups.

3.2 The Partial Group Concept

In the present section, we start by defining partial groups and providing some basic results.

Definition 3.2.1. (**Partial Group**). A partial group is a finite set Γ such that

1- Each member of Γ is a finite group.

2- If $X,\ Y \in \Gamma$, then $X \cap Y \in \Gamma$ and the binary operation on $X \cap Y$ induced by the binary operation on X coincides with the binary operation induced by the binary operation on Y.

3- If $X \in \Gamma$ with X not the trivial group, then there exists $N \in \Gamma$ with the property: whenever $H \in \Gamma$ with $X \leqslant H$, then $N \cap H = N_H(X)$.

4- If $H \in \Gamma$ and K char H, then $K \in \Gamma$.

Remark 3.2.2. (**Definition of Partial Group**).

(a) In (2), $X \cap Y$ is a subgroup of both X and Y.

(b) It follows from (2) and (4) that Γ contains a member $\{1\}$ with the property that $1 = 1_H$ for all $H \in \Gamma$.

Definition 3.2.3. We define the identity element of a partial group Γ to be the identity of each element in Γ.

This definition makes sense because of Remark 3.2.2.

Lemma 3.2.4. Let Γ be a partial group and suppose $X \in \Gamma$. Then Γ possesses a unique maximal member N with the following property:
whenever $H \in \Gamma$ with $X \leqslant H$, then $N \cap H = N_H(X)$. Moreover, $X \trianglelefteq N$.

Proof. By the definition of a partial group Γ, there exists $N \in \Gamma$ such that whenever $H \in \Gamma$ with $1 \neq X \leqslant H$, then

$$H \cap N = N_H(X). \tag{1}$$

Now, we show that N is unique. Suppose we also have $N^* \in \Gamma$ such that whenever $H \in \Gamma$ with $1 \neq X \leqslant H$, then

$$H \cap N^* = N_H(X). \tag{2}$$

In (1) put $H = X$, then we have

$$X \cap N = N_X(X) = X.$$

Then, we have

$$X \leqslant N.$$

Now, put $H = N$ in (1), then we have

$$N = N_N(X)$$

and so $X \trianglelefteq N$.

Similarly, we get $X \trianglelefteq N^*$.

Now, put $H = N^*$ in (1), we have

$$N^* \cap N = N_{N^*}(X).$$

Since $X \trianglelefteq N^*$, we have

$$N^* \cap N = N^*.$$

Then, we get

$$N^* \leqslant N.$$

Similarly, we get

$$N \leqslant N^*.$$

Thus, we get

$$N = N^*.$$

Definition 3.2.5. Let Γ be a partial group, and suppose $X \in \Gamma$. The group N constructed in 3.2.4 is called the normalizer of X and is denoted by

$$N_\Gamma(X).$$

The centralizer of X is denoted by

$$C_\Gamma(X) = C_{N_\Gamma(X)}(X).$$

Lemma 3.2.6. Assume the notation of Definition 3.2.5. If $H \in \Gamma$ with $X \leqslant H$, then $C_\Gamma(X) \cap H = C_H(X)$.

Proof. By Lemma 3.2.4, we have

$$N_\Gamma(X) \cap H = N_H(X).$$

Then,

$$(C_\Gamma(X) \cap N_\Gamma(X)) \cap H = C_\Gamma(X) \cap N_H(X).$$

Since $C_\Gamma(X) = C_{N_\Gamma(X)}(X)$, we get

$$C_\Gamma(X) \cap H = C_H(X).$$

Example 3.2.7. (**Partial Group**).

(1) Let G be a group and Γ the set of subgroups of G.
Since each member of Γ is subgroup, their intersections are subgroups. Also, if $1 \neq X$ is a subgroup of G in Γ, there exist a subgroup of G in Γ namely $N = N_G(X)$ such that $N \cap H = N_H(X)$. The condition of characteristic subgroups is clear. Hence, Γ is a partial group.

(2) Let X and Y be two groups with $X \cap Y = 1$. Let $\Gamma = \{$subgroups of $X\} \cup \{$subgroups of $Y\}$. Then Γ is a partial group.

(3) Suppose that Γ_1 and Γ_2 are partial groups with $X \cap Y = 1, \forall X \in \Gamma_1$ and $Y \in \Gamma_2$. Then $\Gamma_1 \cup \Gamma_2$ is a partial group.

Definition 3.2.8. Let Γ be a partial group and

(a) M is a maximal member of Γ if whenever $H \in \Gamma$ with $M \leqslant H$, then $M = H$. The set of maximal members of Γ is denoted by

$$\Gamma^*.$$

(b) M is a primitive member of Γ if whenever $1 \neq X$ char M, then $M = N_\Gamma(X)$. The set of primitive members of Γ is denoted by

$$\Gamma^{**}.$$

It is clear from the definition of a partial group Γ that Γ^*, $\Gamma^{**} \subseteq \Gamma$, but Γ^* and Γ^{**} may not be partial groups.

Lemma 3.2.9. Let Γ be a partial group and A, $M \in \Gamma$. Then

(1) If $1 \neq A \trianglelefteq M$, then $M \leqslant N_\Gamma(A)$.

(2) If $1 \neq A \trianglelefteq M \in \Gamma^*$, then $M = N_\Gamma(A)$.

Proof. (1) $M = N_M(A) \leqslant N_\Gamma(A)$.

(2) we have $M \leqslant N_\Gamma(A)$ by (1). Since M is a maximal element in Γ, we get $M = N_\Gamma(A)$.

Lemma 3.2.10. Let Γ be a partial group. Then, $\Gamma^* \subseteq \Gamma^{**\cdot}$

Proof. Let $M \in \Gamma^*$ and suppose that K char M. Then $N_\Gamma(K) = M$ and so M is a primitive member of Γ. Thus, $\Gamma^* \subseteq \Gamma^{**}$.

Lemma 3.2.11. Let Γ be a partial group. Let H_1 and H_2 be two members of Γ with $H_1 \cap H_2 \neq 1$. Then $N_\Gamma(H_1) \cap N_\Gamma(H_2) \leqslant N_\Gamma(H_1 \cap H_2)$.

Proof. Since Γ is a partial group, we have $H_1 \trianglelefteq N_\Gamma(H_1) \in \Gamma$ and $H_2 \trianglelefteq N_\Gamma(H_2) \in \Gamma$. Then

$$H_1 \cap H_2 \leqslant N_\Gamma(H_1) \cap N_\Gamma(H_2) \in \Gamma.$$

Let $h \in H_1 \cap H_2$ and $n \in N_\Gamma(H_1) \cap N_\Gamma(H_2)$. Then $h \in H_1$, $n \in N_\Gamma(H_1)$ and so $h^n \in H_1$. Similarly, we get $h^n \in H_2$. Thus, we get

$$h^n \in H_1 \cap H_2$$

and so

$$H_1 \cap H_2 \trianglelefteq N_\Gamma(H_1) \cap N_\Gamma(H_2).$$

Then, we have

$$\begin{aligned} N_\Gamma(H_1) \cap N_\Gamma(H_2) &= N_{N_\Gamma(H_1) \cap N_\Gamma(H_2)}(H_1 \cap H_2) \\ &= N_\Gamma(H_1) \cap N_\Gamma(H_2) \cap N_\Gamma(H_1 \cap H_2) \\ &\subseteq N_\Gamma(H_1 \cap H_2). \end{aligned}$$

Now, we know the set of all subgroups of a group G provides a partial group. But the proper subgroups of G might not be a partial group. However, if we consider Γ is a partial group of proper subgroups of G. What does G look like?.

Proposition 3.2.12. Let G be a group, Γ the collection of proper subgroups of G and suppose Γ is a partial group. Then G has one of the following holds:

(1) G is simple.

(2) G possesses an elementary abelian normal q-subgroup K such that G/K is cyclic.

(3) G possesses a characteristically simple subgroup K with $C_G(K) = 1$ and G/K cyclic.

Proof. (1) If G is simple, then the intersection condition of Γ is satisfied. Now, let P and H are in Γ such that $1 \neq P \leqslant H$. We define $N_\Gamma(P) = N_G(P)$ which is a proper subgroup in Γ and so $N_\Gamma(P) \cap H = N_H(P)$.

We may assume for the rest of the proof that G is not simple, since otherwise (1) holds.

We claim that if $1 \neq K \trianglelefteq G$ then G/K is cyclic.
Assume false and let $N = N_\Gamma(K)$. Let $g \in G$. Since G/K is not cyclic, then $K\langle g\rangle \neq G$. Now, we have

$$\begin{aligned} K\langle g\rangle &= N_{K\langle g\rangle}(K) \cap N \\ &= K\langle g\rangle \cap N. \end{aligned}$$

This implies that $g \in N$. Then, $G = N \in \Gamma$, a contradiction. Hence the claim holds.
Let K be a minimal normal subgroup of G. Then either K is an elementary abelian q-subgroup of G for some prime q or K is the direct product of isomorphic non-abelian simple groups permuted transitively by G.
In the first case, (2) holds. Suppose the second case. Note that $Z(K) = 1$.
So, we have

$$C_G(K) \cap K = 1.$$

Suppose that $C_G(K) \neq 1$. Now $C_G(K) \trianglelefteq G$ and so $G/C_G(K)$ is cyclic. As $C_G(K) \cap K = 1$, we see that K is isomorphic to its image in $G/C_G(K)$ and so K is cyclic, a contradiction. Hence, $C_G(K) = 1$ and so (3) holds.

The graph of a partial group

A partial group Γ determines a graph as follows:

1- The vertices of the graph are the non-identity members of Γ.

2- Two distinct vertices are joined by an edge if they have non-trivial intersection.

Definition 3.2.13. (**Connected Partial Group**). We say that a partial group is connected if its graph is connected.

We note that the partial group in (1) of Example 3.2.7 is a connected partial groups. An example of a disconnected partial group is the set of proper subgroups of S_3.

Theorem 3.2.14. Every partial group is a disjoint union of connected partial groups.

Proof. Let Γ_1 be a connected component of Γ. We claim that Γ_1 is a partial group. Axioms 1, 2 and 4 hold trivially. To verify Axiom 3, suppose that $P \in \Gamma_1$ with $P \neq 1$. Since Γ is a partial group, there exists $N \in \Gamma$ such that wherever $P \leqslant H \in \Gamma$ then $N \cap H = N_H(P)$. Now, $P \leqslant N$ and $P \in \Gamma_1$ so since Γ_1 is a connected component of Γ, we have $N \in \Gamma_1$.
Recall that every graph is the disjoint union of its connected components. Hence, the partial group Γ is a disjoint union of connected partial groups.

Now, we give some important definitions and results of abelian partial groups, nilpotent partial groups and solvable partial groups as follows:

Abelian Partial Groups

Definition 3.2.15. A partial group called an abelian partial group if each member of the partial group is abelian.

Theorem 3.2.16. Every connected abelian partial group has a unique maximal member.

Proof. Let Γ be a connected partial group in which every member is abelian.
Let A and B be maximal members of Γ.
First, we suppose that $A \cap B \neq 1$. Let $N = N_\Gamma(A \cap B)$. Then we have,

$$\begin{aligned} N \cap A &= N_A(A \cap B) \\ &= A. \end{aligned}$$

Therefore, $A \leqslant N$. But A is a maximal member of Γ, so $A = N$.
Similarly, $B = N$. Hence, $A = B$.

Now, we suppose that $A \cap B = 1$. Since Γ is a connected partial group, there exist maximal members $C_1,\ C_2, \ldots, C_n$ such that

$$A \cap C_1 \neq 1,\ C_i \cap C_{i+1} \neq 1 \text{ and } C_n \cap B \neq 1,$$

for all $i < n$.

By what we have just proved, $A = C_1,\ C_i = C_{i+1}$ and $C_n = B,$ for all $i < n$. Hence, $A = B$.

Nilpotent Partial Groups

Now we will extend the definition of abelian partial groups to nilpotent partial groups as follows:

Definition 3.2.17. A partial group is called nilpotent partial group if each member of the partial group is nilpotent.

Proposition 3.2.18. Every abelian partial group is a nilpotent partial group.

Proof. Let Γ be an abelian partial group. Since every member of Γ is an abelian group, and we know that every abelian group is nilpotent, so Γ is a nilpotent partial group.

Theorem 3.2.19. Let Γ be a finite connected nilpotent partial group. Then Γ has a unique maximal element.

Proof. We claim that if A and B are maximal elements of Γ with $A \cap B \neq 1$ then $A = B$.
Assume false and choose a counter example $A,\ B$ with $|A \cap B|$ as large as possible.
Such a choice is possible since $|\Gamma|$ is finite.

Let $N = N_\Gamma(A \cap B)$. Then since A is nilpotent, we have

$$N \cap A = N_A(A \cap B) > A \cap B.$$

Choose a maximal element $M \in \Gamma$ with $N \leqslant M$. Then

$$M \cap A \geqslant\ N \cap A > A \cap B.$$

The choice of $A,\ B$ forces $M = A$.

Similarly, $M = B$, whence $A = B$, a contradiction.

Finally we conclude, as in the abelian case that Γ contains a unique maximal element.

p-Partial Groups

Definition 3.2.20. A partial group Γ is called p-partial group if every member of Γ is a p-group.

We denote the set of all p-groups of a partial group Γ by Γ_p and the set of all maximal p-groups in Γ by $\mathrm{Syl}_p\,\Gamma$.

Proposition 3.2.21. Let Γ be a partial group. Then Γ_p may not be a partial group.

Here an example.
Let Γ be the set of all subgroups of S_5. Then $\Gamma_{p=2}$ is not a partial group. Let $P = \langle(12)\rangle$, A = P x $\langle(34)\rangle$ and B = P x $\langle(45)\rangle$ are in Γ. suppose that $\Gamma_{p=2}$ is a partial group and let $N = N_\Gamma(P)$ and so N is a 2-group.
Now, we have $\langle(34)\rangle \subseteq N_A(P) \subseteq N$ and $\langle(45)\rangle \subseteq N_B(P) \subseteq N$. But $(34)(45) = (354)$, which is of order 3, a contradiction.

Corollary 3.2.22. Every connected p-partial group has a unique maximal member.

Proof. Since every p-partial group is a nilpotent partial group and so it has a unique maximal member.

Solvable Partial Groups

Definition 3.2.23. A partial group Γ is called a solvable partial group if each member of Γ is solvable.

It is clear from the above definition that abelian and nilpotent partial groups are solvable but the converse is not true.

Proposition 3.2.24. There exists a connected partial group with more than one maximal member.

Proof. Here is an example.
Let $\Gamma = \{$the set of all proper subgroups of $A_5\}$.
Since A_5 is a simple group, the collection of the proper subgroups of A_5 is a partial group by Proposition 3.2.12 as every proper subgroup of A_5 is solvable. It remains to show that A_5 is connected. A_5 has three classes of maximal subgroups which are the normalizer of Sylow 5-subgroups of order 10, the normalizer of Sylow 3-subgroups of order 6 and the group A_4 that contains Sylow 2-subgroups. Let $H = A_4 < A_5$, $|A_5 : H| = 5$. Any maximal subgroup of A_5 has non-trivial intersection with H. Let $1 \neq P \leqslant A_5$ and choose a maximal subgroup M of A_5 with $P \leqslant M$. Then there is a path between P and H and so P is in the same connected component of H. Thus, the graph of the partial group Γ is connected and so Γ is connected.

3.3 Subpartial Groups and Homomorphism of Partial Groups

Subpartial Groups

Definition 3.3.1. Let Γ be a partial group and we call $\Gamma_1 \subseteq \Gamma$ a subpartial group if Γ_1 is also a partial group.

Definition 3.3.2. We call a partial group Γ is a trivial if it contains one member only which is the identity member.

Example 3.3.3. Let G be a simple group, $\Gamma = \{\{1_G\},\ \{G\}\}$ and $\Gamma_1 = \{1_G\}$. Then Γ_1 is a trivial subpartial group.

Definition 3.3.4. We call a partial group Γ disjoint if $M_1 \cap M_2 = 1$ for all $M_1,\ M_2 \in \Gamma^*$ with $M_1 \neq M_2$.

Proposition 3.3.5. Let Γ be a non-trivial partial group. The union of two disjoint subpartial groups of Γ is also a partial group.

Proof. This follows from the definition of a disjoint partial group.

Partial Groups Homomorphism

Definition 3.3.6. Let Γ_1 and Γ_2 be partial groups. A partial group homomorphism from Γ_1 to Γ_2 is a collection $\Theta = \{\theta_A \mid A \in \Gamma_1\}$ such that

1- $\forall A \in \Gamma_1$, θ_A is a homomorphism with domain A and codomain $\theta_A(A) \in \Gamma_2$.

2- $\forall A,\ B \in \Gamma_1$, we have $\theta_A|_{A\cap B} = \theta_B|_{A\cap B}$.

3- If $A \in \Gamma_1$, then $\mathrm{Ker}\theta_A \in \Gamma_1$.

Definition 3.3.7. The kernel of a partial group homomorphism $\Theta : \Gamma_1 \longrightarrow \Gamma_2$ is defined as $\mathrm{Ker}\theta = \{\ \mathrm{Ker}\theta_A \mid A \in \Gamma_1\}$.

Theorem 3.3.8. Let $\Theta : \Gamma_1 \longrightarrow \Gamma_2$ be a partial group homomorphism. Then $\mathrm{Ker}\Theta$ is a subpartial group.

In order to prove the above theorem, we need to prove the following Lemma:

Lemma 3.3.9. Let Γ_1, and Γ_2 be two partial groups, and $\Theta : \Gamma_1 \longrightarrow \Gamma_2$ a partial group homomorphism.

(1) Let $A,\ B \in \Gamma_1$. Then $\mathrm{Ker}\theta_A \cap\ \mathrm{Ker}\theta_B = \mathrm{Ker}\theta_{A\cap B}$.

(2) Let $A,\ B \in \Gamma_1$. Then $A\cap\ \mathrm{Ker}\theta_B = \mathrm{Ker}\theta_{A\cap B} = \mathrm{Ker}\theta_A \cap B$.

(3) If $M \in \mathrm{Ker}\Theta$, then $\mathrm{Ker}\theta_M = M$.

Proof. (1) Let $\mathrm{Ker}\theta_A$, $\mathrm{Ker}\theta_B \in \mathrm{Ker}\Theta$, with $A,\ B \in \Gamma_1$. Now, consider $A\cap B \in \Gamma_1$ and the map $\theta_{A\cap B} \in \Theta$. Let $g \in \mathrm{Ker}\theta_{A\cap B}$ and so $\theta_{A\cap B}(g) = 1$. But also $g \in A$ and $\theta_{A\cap B}|_{A\cap B} = \theta_A|_{A\cap B}$. Therefore,

$$\theta_{A\cap B} = \theta_A|_{A\cap B}.$$

Then, we get

$$1 = \theta_{A\cap B}(g) = \theta_A(g),$$

and so

$$g \in \mathrm{Ker}\theta_A.$$

Similarly, we get

$$g \in \mathrm{Ker}\theta_B.$$

Thus, we have

$$\mathrm{Ker}\theta_{A\cap B} \subseteq \mathrm{Ker}\theta_A \cap\ \mathrm{Ker}\theta_B.$$

Conversely, suppose that $g \in \mathrm{Ker}\theta_A \cap\ \mathrm{Ker}\theta_B$. Now, $g \in A\cap B$ and $\theta_{A\cap B} = \theta_A|_{A\cap B}$. Then

$$\theta_{A\cap B}(g) = \theta_A(g) = 1,$$

and so,

$$g \in \mathrm{Ker}\theta_{A\cap B}.$$

Thus, we get

$$\mathrm{Ker}\theta_A \cap\ \theta_B \subseteq \mathrm{Ker}\theta_{A\cap B}.$$

Then, we have

$$\mathrm{Ker}\theta_{A\cap B} = \mathrm{Ker}\theta_A \cap\ \mathrm{Ker}\theta_B.$$

(2) By the definition of partial groups homomorphism, we have

$$\mathrm{Ker}\theta_A \cap B \subseteq \mathrm{Ker}\theta_B$$

and so

$$\mathrm{Ker}\theta_A \cap B \subseteq \mathrm{Ker}\theta_B \cap A.$$

Also, we have

$$\mathrm{Ker}\theta_B \cap A \subseteq \mathrm{Ker}\theta_A \cap B.$$

Therefore, we have

$$\mathrm{Ker}\theta_A \cap B = \mathrm{Ker}\theta_B \cap A.$$

By applying the above, we get

$$\mathrm{Ker}\theta_A \cap B \subseteq \mathrm{Ker}\theta_B \text{ and } \mathrm{Ker}\theta_A \cap B \subseteq A.$$

Then

$$\mathrm{Ker}\theta_A \cap B \subseteq \mathrm{Ker}\theta_A \cap\ \mathrm{Ker}\theta_B = \mathrm{Ker}\theta_{A\cap B}.$$

Conversely, we have

$$\begin{aligned}\mathrm{Ker}\theta_{A\cap B} &= \mathrm{Ker}\theta_{A\cap B} \cap (A\cap B)\\ &= \mathrm{Ker}\theta_{A\cap B} \cap B\\ &\subseteq \mathrm{Ker}\theta_B\\ &\subseteq B.\end{aligned}$$

Similarly, we have

$$\mathrm{Ker}\theta_{A\cap B} \subseteq \mathrm{Ker}\theta_A.$$

Therefore, we get

$$\mathrm{Ker}\theta_{A\cap B} \subseteq \mathrm{Ker}\theta_A \cap B.$$

(3) Let $M \in \mathrm{Ker}\Theta$. Then, we have

$$M = \mathrm{Ker}\theta_A$$

for some $A \in \Gamma_1$.

Now, $\mathrm{Ker}\theta_M \cap A = M\cap\ \mathrm{Ker}\theta_A = M \cap M = M$. Therefore, we get

$$M \subseteq \mathrm{Ker}\theta_M.$$

Also, we have $\mathrm{Ker}\theta_M \subseteq M$ and so we get $\mathrm{Ker}\theta_M = M$.

Proof Theorem 3.3.8. From the above lemma, we have the intersection condition of $\mathrm{Ker}\Theta$ is satisfied.

Let P and $M \in \mathrm{Ker}\Theta$ such that $P \leqslant M$, so $P = \mathrm{Ker}\theta_A$ for some $A \in \Gamma_1$. Let $N_1 = N_{\Gamma_1}(P)$. Now, $M \in \Gamma_1$ and so $N_1 \cap M = N_M(P)$. We need to construct the normalizer of P in $\mathrm{Ker}\Theta$. Let $N = \mathrm{Ker}\theta_{N_1} \in \mathrm{Ker}\Theta$. Then, we have

$$\begin{aligned} N \cap M &= \mathrm{Ker}\theta_{N_1} \cap M \\ &= N_1 \cap \mathrm{Ker}\theta_M \\ &= N_1 \cap M \\ &= N_M(P). \end{aligned}$$

Thus, we have N is the normalizer of P in $\mathrm{Ker}\Theta$.

Subpartial Groups of Normal Subgroups in Γ

Recall that given a group G, the kernels of homomorphisms with domain G coincide with the normal subgroups of G. Note that kernels are defined externally since we need another group to be the image of a homomorphism. Whereas, normal subgroups are defined internally to G. A natural question to ask is "Can we characterize the kernels of partial group homomorphisms internally to the partial group?". We have not been able to answer this question, but here is some progress.

Hypothesis:
Let Γ be a partial group such that

1- For each $A \in \Gamma$ $\exists$ N_A with $N_A \trianglelefteq A$ and $N_A \in \Gamma$.

2- For all $A,\ B \in \Gamma$, $N_A \cap B \leqslant N_B$.

Theorem 3.3.10. Assume the above hypothesis. Let $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 = \{N_A \mid A \in \Gamma\}$. Then Γ_1 is a partial group in Γ.

In order to prove the above theorem, we need to start with the following lemma:

Lemma 3.3.11. Assume the hypothesis. Suppose that $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 = \{N_A \mid A \in \Gamma\}$. Then, we have the following:

(1) $N_A \cap B = A \cap N_B$, for all $A,\ B \in \Gamma$.

(2) $N_A \cap B = N_A \cap N_B$, for all $A,\ B \in \Gamma$.

(3) $N_{A\cap B} = N_A \cap N_B$.

Proof. (1) From our hypothesis, we have

$$N_A \cap B \subseteq N_B.$$

This implies

$$A \cap (N_A \cap B) \subseteq A \cap N_B$$

and so,

$$N_A \cap B \subseteq A \cap N_B.$$

Similarly, we have

$$A \cap N_B \subseteq N_A \cap B.$$

Hence, we get

$$N_A \cap B = A \cap N_B.$$

(2) Now, we have

$$N_A \cap N_B \subseteq N_A \cap B.$$

Also,

$$N_A \cap B \subseteq N_B,$$

and so

$$N_A \cap B \subseteq N_A \cap N_B.$$

Thus, we have

$$N_A \cap B = N_A \cap N_B.$$

(3) Since A and $B \in \Gamma$, then $A \cap B \in \Gamma$. Now, we have

$$N_A \cap (A \cap B) \subseteq N_{A \cap B},$$

and so

$$N_A \cap B \subseteq N_{A \cap B}.$$

In particular,

$$N_A \cap N_B \subseteq A_{A \cap B}.$$

Also,

$$N_{A \cap B} \cap A \subseteq N_A.$$

This implies that

$$N_{A \cap B} \cap (A \cap B) \subseteq N_A \cap B,$$

and so

$$N_{A \cap B} \subseteq N_A \cap B.$$

Then, we get

$$N_{A \cap B} \subseteq N_A \cap B = N_A \cap N_B.$$

We deduce that,

$$N_{A \cap B} = N_A \cap N_B.$$

Proof Theorem 3.3.10. By the above lemma, we have the intersection of two members in Γ_1 is also in Γ_1.

Now, let $P \in \Gamma_1$ and define N^* by

$$N^* = N_{N_\Gamma(P)} \in \Gamma_1.$$

Now suppose that $M \in \Gamma_1$ and $P \leqslant M$. Choose $A \in \Gamma$ such that $M = N_A$. Then

$$\begin{aligned} M \cap N^* &= N_A \cap N_{N_\Gamma(P)} \\ &= N_A \cap N_\Gamma(P) \\ &= M \cap N_\Gamma(P) \\ &= N_M(P). \end{aligned}$$

Thus, we have the normalizer of P in Γ_1 is N^*.

Question:

Is there a partial group homomorphism Θ from a partial group Γ to another partial group Σ such that $\mathrm{Ker}\Theta = \Gamma_1 \subseteq \Gamma$?.

Primitive Pairs of Partial Groups

We will extend the concept of primitive pairs of a group to a partial group as shown in the following definition and results.

Definition 3.3.12. Let Γ be a partial group and let M_1 and M_2 be two distinct members of Γ. We call $(M_1,\ M_2)$ is a primitive pair of Γ if the following hold:

1- If $1 \neq K$ char M_1 and $K \leqslant M_1 \cap M_2$, then $N_{M_2}(K) = M_1 \cap M_2$.

2- If $1 \neq K$ char M_2 and $K \leqslant M_1 \cap M_2$, then $N_{M_1}(K) = M_1 \cap M_2$.

Lemma 3.3.13. Let M_1 and M_2 be two distinct maximal members of a partial group Γ. Then $(M_1,\ M_2)$ is a primitive pair of Γ.

Proof. Let $1 \neq A$ char M_1 and $A \leqslant M_1 \cap M_2$, we have

$$N_{M_1}(A) \leqslant N_\Gamma(A).$$

Then, we have

$$M_1 \leqslant N_\Gamma(A).$$

Since M_1 is a maximal member of Γ, we have

$$N_\Gamma(A) = M_1$$

and so

$$N_\Gamma(A) \cap M_2 = M_1 \cap M_2.$$

Then, we have

$$N_{M_2}(A) = M_1 \cap M_2.$$

Similarly, if $1 \neq A$ char M_2 and $A \leqslant M_1 \cap M_2$, then

$$N_{M_1}(A) = M_1 \cap M_2.$$

Corollary 3.3.14. Let Γ be a partial group and $M_1 \neq M_2 \in \Gamma^*$. Suppose that N char M_1 and N char M_2, where $N \in \Gamma$. Then $N = 1$.

Proof. By the above lemma, we have $(M_1,\ M_2)$ is a primitive pair and so by Lemma 2.5.4, we have $N = 1$

Corollary 3.3.15. Let Γ be a partial group and $M_1,\ M_2 \in \Gamma^*$. Suppose that N char M_1 and N char M_2, where $1 \neq N \in \Gamma$. Then $M_1 = M_2$.

Proof. Since Γ is a partial group and $N \trianglelefteq M_1,\ N \trianglelefteq M_2$, we have $M_1 = N_\Gamma(N) = M_2$.

Now we are extending Bender's Theorems (Theorem 2.5.7) for partial groups. In the following result we will apply the second part of Definition 2.5.3 for partial groups.

Theorem 3.3.16. Let Γ be a partial group and $M_1,\ M_2 \in \Gamma^*$ with $M_1 \neq M_2$. Suppose that $F^*(M_1) \leqslant M_2$ and $F^*(M_2) \leqslant M_1$. Then $(M_1,\ M_2)$ has a characteristic p for some prime p.

Proof. Since M_1 and M_2 are maximal members in Γ, $(M_1,\ M_2)$ a primitive pair in Γ. By applying Theorem 2.5.7, we get the result.

Bender's Maximal Subgroup Theorem for Partial Groups

We will establish an analogue of Theorem 2.5.9 for partial groups. We start to extend Definition 2.5.8 for partial groups as follows:

Definition 3.3.17. Let Γ be a partial group and $M,\ L \in \Gamma$. We write

$$M \rightsquigarrow L$$

and say that M is linked to L if there exists $X \in \Gamma$ with

$$X \trianglelefteq \trianglelefteq F(M) \text{ and } XC_{F^*(M)}(X) \leqslant L.$$

Note that $[E(M),\ F(M)] = 1$ (see page 29) implies that $E(M) \leqslant L$.

Theorem 3.3.18. (**Bender's Maximal Subgroup Theorem for Partial Groups**). Let Γ be a partial group and $M,\ L \in \Gamma^{**}$. Let $\pi = \pi(F(M))$. Assume that

$$M \rightsquigarrow L.$$

Then the following hold:

(a) $M \cap F_{\pi'}(L) = 1$.

(b) M has characteristic p for some prime p or $F_{\pi}(L) \leqslant M$.

(c) Suppose that

$$L \rightsquigarrow M$$

or

$$F_{\pi'}(L) = 1 \text{ and } E(L) \leqslant M,$$

then $M = L$ or there is a prime p such that M and L both have characteristic p.

Proof. (a) Set $Q = F_{\pi'}(L) \cap M$ and $P = F(M) \cap L$. Then

$$[P,\ Q] \leqslant F(M) \cap F_{\pi'}(L) = 1.$$

Now consider Q acting on $F(M)$. Since $M \rightsquigarrow L$, we have

$$C_{F(M)}(P) \leqslant P.$$

Coprime Action implies

$$[F(M),\ Q] = 1.$$

The note after the definition of $M \rightsquigarrow L$ implies that

$$E(M) \leqslant L$$

and so

$$E(M) \leqslant E(M \cap L).$$

Since $Q = F_{\pi'}(L) \cap M$, we have

$$Q \leqslant F(M \cap L).$$

Now, we have

$$[E(M \cap L),\ F(M \cap L)] = 1$$

and so

$$[E(M),\ Q] = 1.$$

We deduce that

$$[F^*(M),\ Q] = 1$$

and so

$$Q \leqslant Z(F(M)).$$

Since $F(M)$ is a π-group and Q is a π'-group, we have $Q = 1$.

(b) Assume M does not have characteristic p for any prime p. Let $p \in \pi$. Now $M \rightsquigarrow L$ so

$$E(M)Z(F(M)) \leqslant L.$$

Let $P = O_p(Z(F(M)))$ and $Q = E(M)O_{p'}(Z(F(M)))$. Now, we have

$$C_{o_p(L)}(P) \leqslant M$$

so

$$C_{O_p(M)}(P) \leqslant O_p(M \cap L).$$

Also,

$$Q \leqslant M \cap L$$

so

$$Q \leqslant E(M \cap L)O_{p'}(M \cap L).$$

Now, we have

$$[O_p(M \cap L),\ E(M \cap L)O_{p'}(M \cap L)] = 1$$

so

$$[C_{O_p(L)}(P),\ Q] = 1.$$

Since $[P,\ Q] = 1$ and since $Q = O^p(Q)$, Lemma 2.3.8 implies that

$$[O_p(L),\ Q] = 1.$$

Now M does not have characteristic p so $Q \neq 1$. Then,

$$N_\Gamma(Q) \leqslant M.$$

Hence,

$$O_p(L) \leqslant M.$$

(c) First, we claim that

$$\pi(F(L)) \subseteq \pi(F(M)). \tag{1}$$

If $F_{\pi'}(L) = 1$, then this clear. Suppose that $L \rightsquigarrow M$, then $Z(F(L)) \leqslant M$ and (1) follows from part (a).

Assume the result false, then $|\pi(F(M))| \geqslant 2$. Let $p,\ q \in \pi(F(M))$ with $p \neq q$. Set $Z = Z(F(M))$. As $M \rightsquigarrow L$, we have $Z \leqslant L$.

Now consider the action of $O_p(Z) \times O_q(Z)$ on $O_p(L)$. Since $(M,\ L)$ is a primitive pair, we have

$$N_L(O_p(Z)) \leqslant M.$$

Thus,

$$[C_{O_p(L)}(O_p(Z)),\ O_q(Z)] \leqslant O_p(L) \cap O_q(Z) = 1.$$

By Lemma 2.3.8, we get

$$[O_p(L),\ O_q(Z)] = 1$$

and so

$$O_p(L) \leqslant N_M(O_q(Z)) \leqslant M. \tag{2}$$

From (1) it follows that $F(L) \leqslant M$ and then $L \rightsquigarrow M$. In particular, there is symmetry between M and L. Hence,

$$\pi(F(L)) = \pi(F(M)).$$

Since $\pi(F(L)) = \pi(F(M))$ and $L \rightsquigarrow M$, we get $F(M) \leqslant L$. Then, we get $F^*(L) \leqslant M$ and $F^*(M) \leqslant L$. Theorem 2.5.7 implies that $M = L$.

3.4 Automorphisms of Partial Groups

Fixed Point Free Automorphism on Partial Groups

In this part of the section we start defining an action on partial groups, and we extend Thompson's Theorem (Theorem 2.3.23) to partial groups. In the second part of this section we will prove an analogue of the Solvable Signalizer Functor Theorem for nilpotent partial groups.

Now we start to extend Theorem (2.3.23 to partial groups. In order to prove this result for partial group we need to begin with the following definitions:

Definition 3.4.1. Let Γ be a partial group and A a group. We say that A acts on Γ if the following hold:

(1) A acts on each member of Γ.

(2) If $X,\ Y \in \Gamma$ with $X \leqslant Y$, then action of A on X coincides with the action of A on Y restricted to X.

(3) For each $B \trianglelefteq A$ with $B \neq 1$, there exists $C \in \Gamma$ with the property:
whenever $X \in \Gamma$ then $C \cap X = C_X(B)$.

(4) For each $B \trianglelefteq A$ and for each $X \in \Gamma$, $[X,\ B] \in \Gamma$.

If in addition each member of Γ has order coprime to the order of A, then we say that A acts coprimely on Γ.

The following example illustrates how the action of one group on another gives us a partial group.

Example 3.4.2. (Partial Group Action). Let A be a group that acts as a group of automorphisms on a simple group G. Let $\Gamma = \{H \mid H$ is a proper A-invariant subgroup of $G\}$. Then Γ is a partial group.

Proof. Let H_1 and $H_2 \in \Gamma$ and so H_1 and H_2 are A-invariant proper subgroups of G. By applying Lemma 3.2.11, we have $H_1 \cap H_2$ is also A-invariant proper subgroup of G and so $H_1 \cap H_2 \in \Gamma$. Now, let $1 \neq P$ and $H \in \Gamma$ such that $P \leqslant H$. We define $N_\Gamma(P) = N_G(P)$ and because of the normalizer of A-invariant subgroup of G is A-invariant, we have $N_\Gamma(P)$ and $N_H(P)$ are A-invariant subgroups of G. Then $N_\Gamma(P)$ and $N_H(P) \in \Gamma$ and so $N_\Gamma(P) \cap H = N_H(P)$.

Definition 3.4.3. Let Γ be a partial group and A a group. We say that A acts fixed point freely on Γ if A acts fixed point freely on every member of Γ.

Theorem 3.4.4. (Thompson's Theorem for Partial Groups). Suppose that Γ is a connected partial group that possesses a fixed point free automorphism of prime order. Then Γ has a unique maximal element.

Proof. Let A is a group acts a fixed point free automorphism of prime order on a partial group Γ. Then A acts a fixed point free automorphism on every member of Γ. By applying Theorem 2.3.23, we get Γ is a connected nilpotent partial group. Theorem 3.2.19 implies that Γ has a unique maximal member.

Lemma 3.4.5. Let Γ be a partial group and suppose that the group A acts on Γ. Let $1 \neq B \trianglelefteq A$. Then there exists a unique $C \in \Gamma$ with the property: whenever $X \in \Gamma$ then $C \cap X = C_X(B)$. Moreover, B acts trivially on C.

Proof. By Definition 3.4.1, there exists $C \in \Gamma$ such that whenever $X \in \Gamma$, then $C \cap X = C_X(B)$. Now we show that C is unique in Γ. Suppose we also have C^* such that $C^* \cap X = C_X(B)$. Then,

$$C \cap C^* = C_{C^*}(B).$$

Similarly, we get

$$C \cap C^* = C_C(B).$$

Then

$$C_C(B) = C_{C^*}(B).$$

As B acts on C, we have

$$[C,\ B] \subseteq C \cap B = 1$$

and so

$$C = C_C(B).$$

Similarly, we get

$$C^* = C_{C^*}(B).$$

Then

$$C = C^*$$

so B acts trivially on C.

Definition 3.4.6. Let Γ be a partial group and suppose that the group A acts on Γ. Let $1 \neq B \trianglelefteq A$. The centralizer of B in the group C is the group C constructed in Lemma 3.4.5 and denoted by

$$C_\Gamma(B).$$

If $B = \langle b \rangle$ for some b, then we also use the notation

$$C_\Gamma(b).$$

Lemma 3.4.7. Let Γ be a partial group and suppose that the group A acts on Γ.

(a) Suppose $1 \neq B_1,\ B_2 \trianglelefteq A$ with $B_1 \leqslant B_2$. Then

$$C_\Gamma(B_1) \supseteq C_\Gamma(B_2).$$

(b) Suppose $1 \neq B_1,\ B_2 \trianglelefteq A$. Then

$$C_\Gamma(B_1B_2) = C_\Gamma(B_1) \cap C_\Gamma(B_2).$$

Proof. (a) By Lemma 3.4.5, we have

$$C_\Gamma(B_1) \cap X = C_X(B_1),$$

whenever $X \in \Gamma$.
Put $X = C_\Gamma(B_2)$. Then, we get

$$C_\Gamma(B_1) \cap C_\Gamma(B_2) = C_{C_\Gamma(B_2)}(B_1).$$

Similarly, we get

$$C_\Gamma(B_1) \cap C_\Gamma(B_2) = C_{C_\Gamma(B_1)}(B_2).$$

Then, we have

$$C_{C_\Gamma(B_1)}(B_2) = C_{C_\Gamma(B_2)}(B_1).$$

As $B_1 \leqslant B_2$, we have

$$C_{C_\Gamma(B_2)}(B_1) = C_\Gamma(B_2)$$

and so

$$C_\Gamma(B_1) \supseteq C_\Gamma(B_2).$$

(b) By applying part (a), we have

$$C_\Gamma(B_1B_2) \leqslant C_\Gamma(B_1) \text{ and } C_\Gamma(B_1) \cap C_\Gamma(B_2).$$

Also, we have

$$C_\Gamma(B_1) \cap C_\Gamma(B_2) \leqslant C_\Gamma(B_1B_2).$$

Then, we get

$$C_\Gamma(B_1B_2) = C_\Gamma(B_1) \cap C_\Gamma(B_2).$$

Lemma 3.4.8. Let A be a non-cyclic elementary abelian group that acts coprimely on the partial group Γ. Suppose that $M \in \Gamma$ with the property

$$C_\Gamma(a) \leqslant M,$$

for all $a \in A^\#$. Then M is the unique maximal member in Γ.

Proof. Let $H \in \Gamma$. By Coprime Action, we have

$$H = \langle C_H(a) \mid a \in A^\# \rangle.$$

Then,

$$H \leqslant M.$$

Lemma 3.4.9. Let A be a non-cyclic elementary abelian group that acts coprimely on the partial group Γ. Let

$$\Gamma_0 = \{H \in \Gamma \setminus \{1\} \mid C_\Gamma(A) \leqslant N_\Gamma(H)\} \cup \{1\}.$$

Then the following hold:

(a) Γ_0 is a partial group and A acts coprimely on Γ_0

(b) $C_{\Gamma_0}(A) = C_\Gamma(A)$ and $C_{\Gamma_0}(A) \leqslant N_\Gamma(H)$ for each $H \in \Gamma \setminus \{1\}$.

(c) Γ possesses a unique maximal member if and only if Γ_0 possesses a unique maximal member.

Proof. (a) If H_1 and $H_2 \in \Gamma_0$, then H_1 and H_2 are $C_\Gamma(A)$-invariant groups in Γ. Since Γ is a partial group and the group A acts coprimely on each member of Γ, $H_1 \cap H_2 \in \Gamma$ is $C_\Gamma(A)$-invariant group and so by Lemma 3.2.11 $H_1 \cap H_2 \in \Gamma_0$.
Now, let $K \leqslant H \in \Gamma_0$. Then $K \in \Gamma$. Let $N = N_\Gamma(K)$. Since A acts coprimely on Γ and $\Gamma_0 \subseteq \Gamma$ so A acts coprimely on Γ_0. Then, A acts on N and $C_\Gamma(A) \leqslant N$. Thus, $N \in \Gamma_0$ and $N \cap H = N_H(K)$.

(b) Since $\Gamma_0 \subseteq \Gamma$, $C_{\Gamma_0}(A) \subseteq C_\Gamma(A)$. On the other hand, by the definition of Γ_0, $C_\Gamma(A) \in \Gamma$ then $C_\Gamma(A) \in \Gamma_0$. This implies that $C_\Gamma(A) \subseteq C_{\Gamma_0}(A)$. Then

$$C_\Gamma(A) = C_{\Gamma_0}(A)$$

and so $C_{\Gamma_0}(A) \leqslant N_\Gamma(H)$ for all $H \in \Gamma \setminus \{1\}$.

(c) Suppose that Γ_0 has a unique maximal member H. Then,

$$C_\Gamma(A) = C_H(A)$$

and so

$$C_\Gamma(a) \leqslant H,$$

for all $a \in A$. By applying Lemma 3.4.8, Γ has a unique maximal member.
Suppose Γ has a unique maximal member M. Since $C_{\Gamma_0}(A) = C_\Gamma(A)$, Γ has a unique maximal member of $C_{\Gamma_0}(A)$-invariant subgroup of M. As every member of Γ_0 is $C_{\Gamma_0}(A)$-invariant subgroup of M, Γ_0 has a unique maximal member.

Elementary Abelian Groups Acting on Partial Groups

Let r be a prime and A an elementary abelian r-group that acts on the r'-group G. Assume that A is non-cyclic. Then

$$G = \langle C_G(a) \mid a \in A^\# \rangle,$$

so we expect the centralizer $C_G(a)$, $a \in A^\#$ to influence the structure of G. One example is the Solvable Signalizer Functor Theorem:

Theorem 3.4.10. Let r be a prime and A an elementary abelian r-group that acts on the r'-group G. Suppose rank $(A) \geqslant 3$ and that $C_G(a)$ is solvable for all $a \in A^{\#}$. Then G is solvable.

Proof. For each $a \in A^{\#}$ we set
$$\theta(a) = C_G(a).$$
Note if $a,\ b \in A^{\#}$, then
$$\theta(a) \cap C_G(b) \leqslant C_G(b) = \theta(b).$$
Thus θ is solvable A-Signalizer Functor on G. Theorem 2.3.22 implies that θ is solvably complete and so there exists a solvable A-invariant subgroup K such that

$$\theta(a) = C_K(a),$$

for all $a \in A^{\#}$. Using Coprime Action, we have

$$G = \langle C_G(a) \mid a \in A^{\#} \rangle \leqslant K,$$

and so,
$$G = K.$$
We conclude that G is solvable.

In fact, using the Classification of Finite Simple Groups, the hypothesis can be weakened to rank $(A) \geqslant 2$. We would like to extend these results to partial groups. We will succeed for nilpotent partial groups and we shall make some progress towards extending it for solvable partial groups. As a starting point we will consider the following hypothesis:

Hypothesis $(*)$

- r is a prime and A is an elementary abelian r-group that acts on the partial group Γ.
- Each element of Γ is an r'-group.
- For each $H \in \Gamma$, every A-invariant subgroup of H is contained in Γ.

Example 3.4.11. Let r be a prime and A an elementary abelian r-group that acts on the r'-group G. Assume that G is characteristically simple and that A acts faithfully on G. Let Γ be the set of proper A-invariant subgroups of G. Then the Hypothesis $(*)$ is satisfied.

A related example as follows:

Example 3.4.12. Let r be a prime and A an elementary abelian r-group that acts on the r'-group G. Let Γ be the set of A-invariant subgroups of G. Then our hypothesis is satisfied.

Note that in the Example 3.4.12, Γ has a unique maximal member; namely G. The following is our first result attempt at generalizing Theorem 3.4.10 to partial groups.

Theorem 3.4.13. Assume Hypothesis $(*)$, that $\mathrm{rank}(A) \geqslant 3$ and that Γ is nilpotent. Then Γ has a unique maximal element.

The conclusion asserts that Γ is one of the examples constructed in Example 3.4.12.

Proof. We recall that Theorem 3.2.19 tells us

$$\text{if } M,\ L \in \Gamma^* \text{ with } M \cap L \neq 1 \text{ then } M = L. \tag{1}$$

We also need the following result about Coprime Action. Suppose that A acts on an r'-group X then

$$X = \langle C_X(B) \mid B < A;\ \text{rank }(B) = \text{rank }(A) - 1 \rangle\,. \tag{2}$$

Now choose $M \in \Gamma^*$. By (2) we may choose $B < A$ with rank $(B) =$ rank $(A) - 1$ such that $C_M(B) \neq 1$. Let $b \in B^{\#}$ and choose L with

$$C_\Gamma(b) \leqslant L \in \Gamma^*.$$

Then, we have

$$1 \neq C_M(B) \leqslant M \cap L.$$

Thus, (1) forces

$$M = L.$$

We deduce that

$$C_\Gamma(b) \leqslant M,$$

for all $b \in B^{\#}$.

Finally, let $L \in \Gamma^*$. Now rank $(A) \geqslant 3$ and so rank $(B) \geqslant 2$ and hence

$$L = \left\langle C_L(b) \mid b \in B^{\#} \right\rangle.$$

Then, we have

$$C_L(b) \leqslant C_\Gamma(b) \leqslant M.$$

We deduce that $L \leqslant M$. Thus M is the unique maximal element in Γ.

On Sylow's Theorem of Coprime Action for Partial Groups

An important result in the proof of the Solvable Signalizer Functor Theorem is the Transitivity Theorem. In what follows, we will prove an analogue of the Transitivity Theorem for partial groups, Theorem 3.4.14. Thus we suspect that we may be able to modify the proof of the Solvable Signalizer Functor Theorem to prove an analogue of Theorem 3.4.10.

We would like to extend Sylow's Theorem for Coprime Action to Hypothesis $(*)$. Since all elements of Γ are A-invariant, trivially part (a) of Theorem 2.3.18 holds. The difficulty (see Theorem 2.3.18) in extending part (b) is that there is no conjugation in a partial group. However, Corollary 2.3.19 does. Using an idea similar to the one used in Theorem 2.3.18. We can establish an analogue of Corollary 2.3.19.

Theorem 3.4.14. (**Transitivity Theorem for partial groups**). Assume Hypothesis $(*)$. Let $P_1,\ P_2 \in \mathrm{Syl}_p\Gamma$.

(a) If $P_1 \cap P_2 \neq 1$, then $C_A(P_1) = C_A(P_2)$.

(b) If rank $A \geqslant 3$, then $C_A(P_1) = C_A(P_2)$.

Proof. (a) Let $T_1 = N_{P_1}(P_1 \cap P_2)$ and $N = N_\Gamma(P_1 \cap P_2) \in \Gamma$. Then, we have

$$C_{P_1}(T_1) \leqslant C_{P_1}(P_1 \cap P_2) \leqslant T_1.$$

Lemma 2.3.17 implies that,

$$C_A(T_1) \leqslant C_A(P_1).$$

Trivially, we have

$$C_A(P_1) \leqslant C_A(T_1).$$

Then,

$$C_A(T_1) = C_A(P_1).$$

Let $T_2 = N_{P_2}(P_1 \cap P_2) \in \Gamma$. Similarly, we get

$$C_A(T_2) = C_A(P_2).$$

Now, choose Q_1 with $T_1 \leqslant Q_1 \in \mathrm{Syl}_p(N;\ A)$ and ${Q_1}^*$ with $Q_1 \leqslant {Q_1}^* \in \mathrm{Syl}_p\Gamma$. We have

$$P_1 \cap P_2 < P_1 \cap Q_1 \leqslant P_1 \cap {Q_1}^*.$$

Then by induction, we get

$$\begin{aligned} C_A(P_1) &= C_A({Q_1}^*) \\ &= C_A(Q_1). \end{aligned}$$

Next, we choose Q_2 with $T_2 \leqslant Q_2 \in \mathrm{Syl}_p(N;\ A)$ and ${Q_2}^*$ with $Q_2 \leqslant {Q_2}^* \in \mathrm{Syl}_p\Gamma$. Similarly, we have

$$C_A(P_2) = C_A(Q_2).$$

Corollary 2.3.19 implies that

$$C_A(Q_1) = C_A(Q_2).$$

Hence,

$$C_A(P_1) = C_A(P_2).$$

(b) Since rank $A \geqslant 3$ there exists $B < A$ with rank $B =$ rank $A - 1$ and $C_{P_1}(B) \neq 1$.

Let $b \in B^\#$. Choose $Q_1 \in \mathrm{Syl}_p C_\Gamma(b)$ with $C_{P_1}(B) \leqslant Q_1$ and Q_1 A-invariant.

Now, choose ${Q_1}^* \in \mathrm{Syl}_p\Gamma$ with $Q_1 \leqslant {Q_1}^*$. We have, $P_1 \cap {Q_1}^* \neq 1$ and so (a) implies that

$$C_A(P_1) = C_A({Q_1}^*) \leqslant C_A(Q_1)$$

and so Corollary 2.3.19 implies that $C_A(P_1)$ centralizes every A-invariant p-subgroup of $C_\Gamma(b)$. In particular, $C_A(P_1)$ centralizes $C_{P_2}(b)$. Now, we have

$$P_2 = \langle C_{P_2}(b) \mid b \in B^\# \rangle$$

and so, we have

$$C_A(P_1) \leqslant C_A(P_2).$$

By applying symmetry, we get

$$C_A(P_2) \leqslant C_A(P_1).$$

Thus,

$$C_A(P_1) = C_A(P_2).$$

Theorem 3.4.15. Let A be an elementary abelian group that acts coprimely on the partial group Γ. Let p be a prime, let $1 \neq P \in \Gamma$ be a p-group and let $M = N_\Gamma(P)$. Let $a \in A^\#$ and suppose that a centralizers a Sylow p-subgroup of M. Let $H \in \Gamma$ and assume that:

(a) $H \cap P \neq 1$, or

(b) $\mathrm{rank}(A) \geqslant 3$.

Then a centralizes every member of $\mathrm{Syl}_p(H;\, A)$. Moreover, if H is p-solvable then $[H,\, a] \leqslant O_{p'}(H)$.

We remark that; when $M \in \Gamma^*,\ p \in \pi(F(M))$ and $P = O_p(M)$, then $P \in \Gamma$ and $M = N_\Gamma(P)$.

Now, in order to prove the above theorem, we need the following lemma:

Lemma 3.4.16. Let A be an elementary abelian group that acts coprimely on the group M. Let p be a prime and $a \in A$. Suppose that a centralizes a Sylow p-subgroup of M. Then a centralizes every member of $\mathrm{Syl}_p(M;\, A)$.

Proof. Let P be a Sylow p-subgroup of M that is centralized by a. By Coprime Action, $C_M(a)$ acts transitively by conjugation on the set of a-invariant Sylow p-subgroups. Therefore, a centralizes every a-invariant Sylow p-subgroup of M.

Now, we start proving Theorem 3.4.15.

Proof Theorem 3.4.15. Assume (a) holds. Let $Q = H \cap P$. We proceed by induction on $|P : Q|$. Set $N = N_\Gamma(Q)$. We claim that a centralizes every member of $\mathrm{Syl}_p(N;\, A)$.
If $|P : Q| = 1$, then

$$N = M,$$

and the claim holds by hypothesis and Lemma 3.4.16.
If $|P : Q| > 1$, then

$$Q < N_P(Q)$$

and so

$$|P : N \cap Q| < |P : Q|,$$

and the claim follows from the inductive hypothesis.

Choose S with $Q \leqslant S \in \mathrm{Syl}_p(H;A)$. Then $N_S(Q)$ is an A-invariant p-subgroup of N so the claim implies

$$[N_S(Q),\, a] = 1.$$

Coprime Action forces

$$[S,\ a] = 1,$$

completing the proof in this case.

Assume that (b) holds. Let $S \in \mathrm{Syl}_p(H;\ A)$. By Corime Action there exists $B \in \mathrm{Hyp}(A)$ with

$$C_P(B) \neq 1.$$

Now, $\mathrm{rank}(A) \geqslant 3$ so $\mathrm{rank}(B) \geqslant 2$ and Coprime Action also gives

$$S = \langle C_S(b) \mid b \in B^{\#} \rangle. \tag{$*$}$$

Let $b \in B^{\#}$. We have, A is abelian so $C_S(b)$ is an A-invariant p-subgroup of $C_\Gamma(b)$. Then, we have

$$1 \neq C_P(B) \leqslant P \cap C_\Gamma(b)$$

so (a) implies that

$$[C_S(b),\ a] = 1.$$

Then $(*)$ implies that

$$[S,\ a] = 1.$$

Finally, suppose that H is p-solvable. Set $\overline{H} = H/O_{p'}(H)$. Then

$$C_{\overline{H}}(O_p(\overline{H})) \leqslant O_p(\overline{H}).$$

Now $O_p(\overline{H}) \leqslant \overline{S}$, so

$$[O_p(\overline{H}),\ a] = 1.$$

Coprime Action implies

$$[\overline{H}, a] = 1.$$

Then,

$$[H,\ a] \leqslant O_{p'}(H).$$

Chapter 4

On Characteristic p in Partial Groups

4.1 Introduction

Previously, we have proved Bender's Maximal Subgroup Theorem (Theorem 2.5.9) for partial groups, which leads to the existence of groups of characteristic p in partial groups, for some primes p. In the present chapter, we seek to study the members that have characteristic p in partial groups. In particular, we are going to establish two uniqueness theorems and further uniqueness theorems which will be used in the next chapter. We begin with two uniqueness theorems as shown in the following results:

Theorem 4.2.1. Assume the group A acts coprimely on the partial group Γ and that the following hold:

(1) p is a prime.

(2) M_1 and M_2 are p-sovable members of Γ^{**}.

(3) $O_p(M_1) \leqslant M_2$ and $O_p(M_2) \leqslant M_1$.

(4) M_1 and M_2 have characteristic p.

(5) $C_\Gamma(A) \leqslant M_1 \cap M_2$.

Then, $M_1 = M_2$.

Theorem 4.2.2. Assume the group A acts coprimely on the partial group Γ and p is a prime. Let $M \in \Gamma^{**}$ and $T \in \Gamma$. Assume that

(1) M is p-solvable with characteristic p.

(2) $O_p(M) \leqslant T \leqslant O_p(M;\, A)$.

Then, $N_\Gamma(T) \leqslant M$.

At the end of this chapter, we focus on solvable partial groups in which every member is $C_\Gamma(A)$-invariant, and we give further uniqueness theorems.

4.2 Characteristic p Uniqueness Theorems

In this section, we will prove the two uniqueness theorem. We start with the following hypothesis:

Hypothesis (A)

- The group A acts coprimely on the partial group Γ.
- p is a prime.

The following result and its proof is a minor modification of [19, Theorem A], which in turn is based on an argument of Stellmacher [22].

Theorem 4.2.1. Assume the Hypothesis (A) and that the following hold:

(1) M_1 and M_2 are p-solvable members of Γ^{**}.

(2) $O_p(M_1) \leqslant M_2$ and $O_p(M_2) \leqslant M_1$.

(3) M_1 and M_2 have characteristic p.

(4) $C_\Gamma(A) \leqslant M_1 \cap M_2$.

Then, $M_1 = M_2$.

Proof. For each i, set

$$V_i = \langle \Omega_1(Z(S)) \mid S \in \mathrm{Syl}_p(M_i) \rangle.$$

Lemma 2.4.14 implies that $V_i \leqslant \Omega_1(Z(O_p(M_i)))$, V_i is an elementary abelian characteristic subgroup of M_i, and

$$O_p(M_i/C_{M_i}(V_i)) = 1.$$

Let

$$W_1 = \langle V_2^{Aut(M_1)} \rangle \text{ char } M_1,$$

$$W_2 = \langle V_1^{Aut(M_2)} \rangle \text{ char } M_2,$$

and

$$P = O_p(M_1)O_p(M_2).$$

Note that P is a p-group because $O_p(M_1)$ and $O_p(M_2)$ are normal p-subgroups of $M_1 \cap M_2$. Moreover, P is $C_\Gamma(A)$-invariant as $C_\Gamma(A) \leqslant M_1 \cap M_2$, whence

$$P \leqslant O_p(M_1;\, A) \cap O_p(M_2;\, A)$$

by applying the second part of Definition 2.3.2.

Now, we are going to split our proof into several steps which lead to completion of the proof.

Step 1. We show that $V_1 \trianglelefteq O_p(M_2)$ and $V_2 \trianglelefteq O_p(M_1)$.

Proof. Note that $V_1 \leqslant O_p(M_1) \leqslant M_2$, $V_1 \trianglelefteq M_1$ and $O_p(M_2) \leqslant M_1$. Then $O_p(M_2)$ normalizes V_1. Suppose that $V_1 \nleqslant O_p(M_2)$. Theorem 2.4.18 implies there exists $B \leqslant O_p(M_2)$ such that $B/C_B(V_1)$ is elementary abelian and acts non-trivially and nearly quadratically on V_1. Moreover,

$$|V_1/C_{V_1}(B)| \leqslant |B/C_B(V_1)|^2.$$

Let ${M_1}^* = M_1/C_{M_1}(V_1)$. We regard V_1 as $GF(p)A{M_1}^*$-module. Since $O_p({M_1}^*) = 1$ by Lemma 2.4.14 and $|V_1/C_{V_1}(B^*)| \leqslant |B^*|^2$ it follows that B^* is a nearly quadratic $2F$-offender for ${M_1}^*$ on V_1. Now, we have

$$B \leqslant P \leqslant O_p(M_1;\, A)$$

so using Coprime Action, we have

$$B^* \leqslant O_p(M_1;\, A)^* \leqslant O_p({M_1}^*;\, A).$$

Theorem 2.4.17 supplies a contradiction. Hence, $V_1 \leqslant O_p(M_2)$. Since $O_p(M_2) \leqslant M_1$, we have

$$V_1 \trianglelefteq O_p(M_2).$$

Similarly, we have

$$V_2 \trianglelefteq O_p(M_1).$$

Step 2. Let $i \in \{1,\ 2\}$ and suppose that $B \leqslant P$ acts non-trivially and quadratically on V_i. Then,

$$|B/C_B(V_i)|^2 < |V_i/C_{V_i}(B)|.$$

Proof. Let ${M_i}^* = M_i/C_{M_i}(V_i)$. Assume the conclusion to be false, so

$$|V_i/C_{V_i}(B^*)| \leqslant |B^*|^2.$$

Since $O_p({M_i}^*) = 1$, it follows that B^* is a nearly quadratic $2F$-offender for ${M_i}^*$ on V_i. However,

$$B^* \leqslant O_p(M_i;\, A)^* \leqslant O_p({M_i}^*;\, A),$$

then Theorem 2.4.17 supplies a contradiction.

Step 3. We claim that $J(P)$ acts trivially on V_1 and V_2.

Proof. Note that $V_1 \trianglelefteq P$ so $J(P)$ does indeed act on V_1. Suppose that $[V_1,\ J(P)] \neq 1$. Now, V_1 is elementary abelian so Theorem 2.4.19 implies there exists $B \in \mathscr{A}(P)$ such that B acts non-trivially and quadratically on V_1. Since $B \in \mathscr{A}(P)$, we have

$$|B| \geqslant |C_B(V_1)V_1|$$

and so

$$|B/C_B(V_1)| \geqslant |V_1/V_1 \cap C_B(V_1)| \geqslant |V_1/C_{V_1}(B)|.$$

Step 2 supplies a contradiction. The claim follows.

Step 4. We claim that W_1 and W_2 are elementary abelian.

Proof. Suppose that $x \in \mathrm{Aut}(M_1)$ and $[V_2, V_2{}^x] \neq 1$. Since $V_1, V_2 \trianglelefteq O_p(M_1)$, we have

$$[V_2, V_2{}^x, V_2{}^x] \leqslant [V_2{}^x, V_2{}^x] = 1,$$

so $V_2{}^x$ acts quadratically on V_2.

Step 2 implies

$$|V_2{}^x/C_{V_2{}^x}(V_2)| < |V_2/C_{V_2}(V_2{}^x)|.$$

Also, we have

$$[V_2{}^{x^{-1}}, V_2] \neq 1$$

and so

$$|V_2{}^{x^{-1}}/C_{V_2{}^{x^{-1}}}(V_2)| < |V_2/C_{V_2}(V_2{}^{x^{-1}})|.$$

Conjugating the second inequality by x gives a contradiction to the first inequality, and completes the claim.

Now, we define Q to be the inverse image of $O_p(M_1/C_{M_1}(W_1))$ in M_1, so $Q \trianglelefteq M_1$.

Step 5. We show that $P \cap Q \leqslant O_p(M_1)$.

Proof. Note that $1 \neq V_2$ char M_2 so as $M_2 \in \Gamma^*$ we have $M_2 = N_\Gamma(V_2)$.
In particular,

$$C_{M_1}(V_2) \leqslant M_1 \cap M_2.$$

Now, we have

$$P = O_p(M_1)O_p(M_2) \trianglelefteq M_1 \cap M_2$$

and so $C_{M_1}(V_2)$ normalizes P.

Since $V_2 \leqslant W_1$ and W_1 is elementary abelian, we have

$$C_{M_1}(W_1) \leqslant C_{M_1}(V_2)$$

so $C_{M_1}(W_1)$ normalizes $P \cap Q$.

Choose S with $P \cap Q \leqslant S \in \mathrm{Syl}_p Q$ so that $Q = C_{M_1}(W_1)S$. Then, we have

$$\langle (P \cap Q)^Q \rangle \leqslant S$$

whence $P \cap Q \leqslant O_p(Q) \leqslant O_p(M_1)$.

Step 6. Let $i \in \{1, 2\}$ and suppose $B \leqslant O_p(M_i)$ acts quadratically on W_i. Then,

$$|B/C_B(W_i)|^2 \leqslant |W_i/C_{W_i}(B)|.$$

Proof. Without loss of generality, $i = 1$. Recall that $W_1 = \langle V_2{}^{Aut(M_1)} \rangle$. Let $x \in \mathrm{Aut}(M_1)$ and suppose $D \leqslant O_p(M_1)$ acts quadratically on $V_2{}^x$. Then $D^{x^{-1}} \leqslant O_p(M_1)$ and $D^{x^{-1}}$ acts quadratically on V_2. Step 2 implies

$$|D^{x^{-1}}/C_{D^{x^{-1}}}(V_2)|^2 \leqslant |V_2/C_{V_2}(D^{x^{-1}})|.$$

Then, we have

$$|D/C_D(V_2{}^x)|^2 \leqslant |V_2{}^x/C_{V_2{}^x}(D)|,$$

the assertion follows from Lemma 2.4.15.

Step 7. Let $B \in \mathscr{A}(P)$ and suppose $B \leqslant Q$. Then $[W_1,\, B] = 1$.

Proof. Using Step 5 we have

$$B \leqslant O_p(M_1) \leqslant P.$$

Then,

$$B \leqslant J(O_p(M_1)) \leqslant J(P).$$

Step 3 implies that

$$[V_2,\, J(O_p(M_1))] = 1.$$

Since $W_1 = \langle V_2{}^{Aut(M_1)} \rangle$, we have

$$[W_1,\, J(O_p(M_1))] = 1,$$

so

$$[W_1,\, B] = 1.$$

Step 8. We claim that $[W_1,\, J(P)] = 1$.

Proof. Assume false. Theorem 2.4.19 implies that there exists $B \in \mathscr{A}(P)$ such that B acts non-trivially and quadratically on W_1. Put

$$B_0 = B \cap Q.$$

By Step 5, $B_0 = B \cap O_p(M_1)$ and $C_B(W_1) \leqslant O_p(M_1)$, so

$$C_B(W_1) = C_{B_0}(W_1).$$

Step 6 implies that

$$|B_0/C_{B_0}(W_1)|^2 \leqslant |W_1/C_{W_1}(B_0)|. \qquad (*)$$

As $B \in \mathscr{A}(P)$, we have

$$|B| \geqslant |C_B(W_1)W_1|$$

and so

$$|B/C_B(W_1)| \geqslant |W_1/C_{W_1}(B)|. \qquad (**)$$

Squaring $(**)$, dividing by $(*)$ and using $C_B(W_1) = C_{B_0}(W_1)$ yields

$$|B/B_0|^2 \geqslant |W_1/C_{W_1}(B)||C_{W_1}(B_0)/C_{W_1}(B)|$$

$$\geqslant |W_1/C_{W_1}(B)|.$$

Let $M_1{}^* = M_1/C_{M_1}(W_1)$, so that $Q^* = O_p(M_1{}^*)$. Then,

$$|B^*/B^* \cap O_p(M_1{}^*)|^2 \geqslant |W_1/C_{W_1}(B^*)|.$$

Now B acts non-trivially on W_1 so B^* is a nearly quadratic $2F$-offender for ${M_1}^*$ on W_1. However,

$$B \leqslant P \leqslant O_p(M_1;\, A)$$

and so

$$B^* \leqslant O_p({M_1}^*;\, A),$$

Theorem 2.4.17 supplies a contradiction, and the claim follows.

We can now complete the proof by using the previous steps.

Now by Step 8 and Step 5, we have

$$J(P) \leqslant P \cap C_{M_1}(W_1) \leqslant P \cap Q \leqslant O_p(M_1) \leqslant P.$$

This forces

$$J(P) = J(O_p(M_1)).$$

Then, we have

$$J(P) \in \Gamma \text{ and } M_1 = N_\Gamma(J(P)).$$

Similarly, we get

$$M_2 = N_\Gamma(J(P)),$$

so $M_1 = M_2$.

Theorem 4.2.2. Assume Hypothesis (A). Let p be a prime, $M \in \Gamma^{**}$ and $T \in \Gamma$. Assume that

(1) M is p-solvable with characteristic p.

(2) $O_p(M) \leqslant T \leqslant O_p(M;\, A)$.

Then, $N_\Gamma(T) \leqslant M$.

Proof. Let

$$N = N_\Gamma(T) \text{ and } V = \langle \Omega_1(Z(S)) \mid S \in \mathrm{Syl}_p M \rangle$$

Thompson's Lemma implies that V is an elementary abelian p-subgroup, that V char M and that $O_p(M/C_M(V)) = 1$.

Now, we split the proof into some steps which lead to completion of the proof

Step 1. Let $B \leqslant T$ and suppose that $B/C_B(V)$ is non-trivial and elementary abelian. Suppose that B acts nearly quadratically on V. Then,

$$|B/C_B(V)|^2 < |V/C_V(B)|.$$

Proof. Let $M^* = M/C_M(V)$. Then $B^* \neq 1$, and we have

$$B^* \leqslant T^* \leqslant O_p(M;\, A)^* \leqslant O_p(M^*;\, A).$$

Theorem 2.4.17 implies that B^* is not a nearly quadratic $2F$-offender for M^* on V. Since $O_p(M^*) = 1$, we have

$$|B^*|^2 < |V/C_V(B^*)|,$$

completing proof Step 1.

In the next step we remark that the symbol nm has no meaning. We are using V^{nm} as an abbreviation for $(V^n)^m$.

Step 2. Let $n \in \mathrm{Aut}(T)$ and $m \in \mathrm{Aut}(M)$.

(a) $V^n \trianglelefteq O_p(M)$.

(b) If $B \leqslant O_p(M)$ acts non-trivially and quadratically on V^{nm}, then

$$|B/C_B(V^{nm})|^2 < |V^{nm}/C_{V^{nm}}(B)|.$$

Proof. (a) We have $V \trianglelefteq T$ so $V^n \trianglelefteq T$. In particular, $O_p(M)$ normalizes V^n. Suppose that $V^n \nleqslant O_p(M)$. Theorem 2.4.17 implies there exists $B \leqslant O_p(M)$ such that $B/C_B(V^n)$ is non-trivial, elementary abelian and acts nearly quadratically on V^n. Moreover,

$$|B/C_B(V^n)|^2 \geqslant |V^n/C_{V^n}(B)|.$$

Now $B^{n^{-1}} \leqslant T$ so Step 1 supplies a contradiction.

(b) Conjugating B by m^{-1}, and we may suppose that $m = 1$. Apply Step 1 with $B^{n^{-1}}$ in the role of B and then conjugate by n, and the proof of part (b) is complete.

Next, we define two characteristic subgroups of M as follows:
For each $n \in \mathrm{Aut}(T)$. Set

$$W_n = V^{n\mathtt{Aut}(M)} \text{ char } M$$

and

$$W = \langle W_n \mid n \in \mathrm{Aut}(T) \rangle \text{ char } M.$$

Then each W_n and W are members of Γ.

Step 3. We show the following:

(a) W is an elementary abelian normal subgroup of M.

(b) If $B \leqslant O_p(M)$ acts quadratically on W, then

$$|B/C_B(W)|^2 \leqslant |W/C_W(B)|.$$

Proof. Let $n_1, n_2 \in \mathrm{Aut}(T)$ and $m_1, m_2 \in \mathrm{Aut}(M)$. Step 2 implies that $V^{n_1m_1}, V^{n_2m_2} \trianglelefteq O_p(M)$. Assume that $[V^{n_1m_1}, V^{n_2m_2}] \neq 1$. Now, we have

$$[V^{n_1m_1}, V^{n_2m_2}, V^{n_2m_2}] \leqslant [V^{n_2m_2}, V^{n_2m_2}] = 1$$

and so Step 2 part (b) implies

$$|V^{n_2m_2}/C_{V^{n_2m_2}}(V^{n_1m_1})| < |V^{n_1m_1}/C_{V^{n_1m_1}}(V^{n_2m_2})|.$$

By symmetry, we get

$$|V^{n_1m_1}/C_{V^{n_1m_1}}(V^{n_2m_2})| < |V^{n_2m_2}/C_{V^{n_2m_2}}(V^{n_1m_1})|.$$

These inequalities are contradictory. We deduce that

$$[V^{n_1 m_1},\, V^{n_2 m_2}] = 1.$$

Hence, (a) follows and (b) follows from Step 2 and Lemma 2.4.15.

Now, we define

$$P = \langle O_p(M)^n \mid n \in N \rangle.$$

Then $P \leqslant T$ and $P \trianglelefteq N$. In particular $P \leqslant M$.

Step 4. We show that $C_M(W)$ normalizes P.

Proof. Set

$$W_0 = \langle V^n \mid n \in \mathrm{Aut}(T) \rangle.$$

Then, we have W_0 char T, $W_0 \in \Gamma$ and $W_0 \leqslant W$.
Moreover,

$$P \leqslant T \leqslant N \leqslant N_\Gamma(W_0).$$

Since $C_M(W) \leqslant C_M(W_0)$ it suffices to prove that $C_M(W_0)$ normalizes P.
Since $N \leqslant N_\Gamma(W_0)$ it follows that N normalizes $C_\Gamma(W_0)$. As V char M, we have $V \in \Gamma$ and $M = N_\Gamma(V)$. Then,

$$C_\Gamma(W_0) \leqslant N_\Gamma(V) = M$$

and so

$$C_\Gamma(W_0) = C_M(W_0).$$

Hence, N normalizes $C_M(W_0)$.

Now, since $C_M(W_0)$ normalizes $O_p(M)$ it follows that $C_M(W_0)$ normalizes $O_p(M)^n$ for each $n \in N$. The conclusion follows.

Let

$$Q = O_p(M \bmod C_M(W)) \text{ char } M.$$

Step 5. We show that $P \cap Q \leqslant O_p(M)$.

Proof. Choose $S \in \mathrm{Syl}_p Q$ with $P \cap Q \leqslant S$. Then,

$$Q = C_M(W)S.$$

By Step 4, we get

$$C_M(W) \leqslant N_M(P \cap Q)$$

and so

$$\langle (P \cap Q)^Q \rangle \leqslant S.$$

Then, we have

$$P \cap Q \leqslant O_p(Q) \leqslant O_p(M).$$

Step 6 We claim that $[W,\ J(P)] = 1$.

Proof. Assume false. Theorem 2.4.19 implies there exists $B \in \mathscr{A}(P)$ with B acting non-trivially and quadratically on W. Since $B \in \mathscr{A}(P)$ and $W \leqslant O_p(M) \leqslant P$, we have

$$|B/C_B(W)| \geqslant |W/C_W(B)|. \tag{$*$}$$

Let $B_0 = B \cap Q$. Step 5 implies $B_0 \leqslant O_p(M)$ so Step 3(b) yields

$$|B_0/C_{B_0}(W)|^2 \leqslant |W/C_W(B_0)|. \tag{$**$}$$

Note that $C_B(W) \leqslant Q$. Then,

$$C_B(W) = C_{B_0}(W).$$

Squaring $(*)$ and dividing by $(**)$ gives

$$\begin{aligned}|B/B_0|^2 &\geqslant |W/C_W(B)||C_W(B_0)/C_W(B)|\\ &\geqslant |W/C_W(B)|.\end{aligned}$$

Let $M^* = M/C_M(W)$, so $Q^* = O_p(M^*)$ and $B_0{}^* = B^* \cap Q^*$. Then,

$$|B^*/B^* \cap O_p(M^*)|^2 \geqslant |W/C_W(B^*)|.$$

Since $B^* \neq 1$, we see that B^* is a nearly quadratic $2F$-offender for M^* on W. On the other hand, $B^* \leqslant P^* \leqslant T^* \leqslant O_p(M;\ A)^* \leqslant O_p(M^*;\ A)$, so Theorem 2.4.17 supplies a contradiction, and the claim follows.

We are now in a position to complete the proof.

Step 6 and Step 5 imply $J(P) \leqslant O_p(M)$.
Now $O_p(M) \leqslant P$ whence

$$J(P) = J(O_p(M)).$$

Since $J(O_p(M))$ char M, we have $J(O_p(M))) \in \Gamma$ and $N_\Gamma(J(O_p(M))) = M$. Then, we get

$$J(P) \in \Gamma \text{ and } N_\Gamma(J(P)) = M.$$

Also, $P \leqslant T \leqslant N \in \Gamma$ and $P \trianglelefteq N$. Then, we have

$$J(P) \trianglelefteq N$$

and so

$$N \leqslant N_\Gamma(J(P)) = M.$$

The proof is complete.

4.3 Further Uniqueness Theorems

In this section, we continue to show further uniqueness theorems in the case that partial groups are solvable. We begin with the following hypothesis:

Hypothesis (B)

(1) The group A acts coprimely on the partial group Γ.

(2) Each member of Γ is solvable.

(3) $C_\Gamma(A) \leqslant N_\Gamma(X)$ for all $X \in \Gamma \setminus \{1\}$.

We remark that (3) implies $C_\Gamma(A) \leqslant M$ for all $M \in \Gamma^{**}$.

Lemma 4.3.1. Assume Hypothesis (B). Let $M,\ L \in \Gamma^{**}$ and suppose that M and L have characteristic p for some prime p. Suppose also that $O_p(M) \leqslant L$. Then $M = L$.

Proof. Let $H = M \cap L$. Since $C_\Gamma(A) \leqslant H$, we have

$$O_p(M) \leqslant O_p(H) \leqslant O_p(M;\, A).$$

Theorem 4.2.2 implies that

$$N_\Gamma(O_p(H)) \leqslant M.$$

Then, we have

$$N_{O_p(L)}(O_p(H)) \leqslant M \cap O_p(L) \leqslant O_p(H).$$

Lemma 2.2.39 forces

$$O_p(L) \leqslant O_p(H) \leqslant M.$$

By hypothesis, $O_p(M) \leqslant L$ and Theorem 4.2.1 implies $M = L$.

Lemma 4.3.2. Assume Hypothesis (B). Let $M \in \Gamma^{**}$ and suppose that M has characteristic p for some prime p. Assume further that $\mathscr{L}$ is a non-empty subset of Γ^{**} such that for each $L \in \mathscr{L}$, the following hold:

(1) L has characteristic p.

(2) $O_p(M) \cap L \neq 1$.

(3) There exists $N \in \mathscr{L}$ with $N_\Gamma(O_p(M \cap L)) \leqslant N$.

Then $\mathscr{L} = \{M\}$.

Proof. Assume false and choose $L \in \mathscr{L} \setminus \{M\}$ with $O_p(M) \cap L$ maximal. Let $H = M \cap L$, so that

$$1 \neq O_p(M) \cap L \leqslant O_p(H).$$

Choose $N \in \mathscr{L}$ with $N_\Gamma(O_p(H)) \leqslant N$. Then

$$O_p(M) \cap L \leqslant O_p(M) \cap N.$$

Suppose that $O_p(M) \cap L = O_p(M) \cap N$. Then

$$N_{O_p(M)}(O_p(H)) \leqslant O_p(M) \cap N = O_p(M) \cap L \leqslant O_p(H).$$

Lemma 2.2.39 forces

$$O_p(M) \leqslant O_p(H) \leqslant L,$$

and so Lemma 4.3.1 implies, $M = L$, a contradiction. We deduce that $O_p(M) \cap L < O_p(M) \cap N$.

The choice of L implies that $N = M$. Then,

$$N_{O_p(L)}(O_p(H)) \leqslant M \cap O_p(L) \leqslant O_p(H).$$

Lemma 2.2.39 forces

$$O_p(L) \leqslant O_p(H) \leqslant M.$$

Lemma 4.3.1 , with the role of M and L interchanged, implies $M = L$, again a contradiction.

Lemma 4.3.3. Assume Hypothesis (B). Suppose that $a \in Z(A)$ has prime order. Then there exists M with the following properties:

(a) $C_\Gamma(a) \leqslant M \in \Gamma^*$.

(b) Whenever $L \in \Gamma^{**}$ with $C_\Gamma(a) \leqslant L$ and $M \rightsquigarrow L$, then $M = L$.

Proof. Since $C_\Gamma(a) \in \Gamma$, we may choose M with $C_\Gamma(a) \leqslant M \in \Gamma^{**}$. If possible, choose M with $C_{O_p(M)}(a) = 1$, for some $p \in \pi(F(M))$. Let $\pi = \pi(F(M))$ and

$$\mathscr{L} = \{L \in \Gamma^{**} \mid C_\Gamma(a) \leqslant L \text{ and } M \rightsquigarrow L\}.$$

We claim that if $L \in \mathscr{L}$, then $F_{\pi'}(L) = 1$. Assume this to be false. Then there exists $q \in \pi(F(L)) \setminus \pi$. Theorem 2.5.9 implies that

$$M \cap O_q(L) = 1.$$

Since $C_\Gamma(a) \leqslant M$ this implies that

$$C_{O_q(L)}(a) = 1.$$

The choice of M implies there exists $p \in \pi(F(M))$ with $C_{O_p(M)}(a) = 1$. Let $Z = Z(O_p(M))$. Since $M \rightsquigarrow L$, we have

$$Z \leqslant L.$$

Moreover, since $C_L(a) \leqslant C_\Gamma(a) \leqslant M$ it follows that Z is $C_L(a)$-invariant. Since $C_{O_p(M)}(a) = 1$, we get

$$C_Z(a) = 1.$$

Coprime Action implies

$$Z = [Z,\, a].$$

Lemma 2.3.14 forces

$$Z \leqslant O_p(L).$$

Now, Z char M so $N_\Gamma(Z) = M$. As $p \neq q$, we have

$$O_q(L) \leqslant N_\Gamma(Z) \leqslant M.$$

However, $M \cap O_q(L) = 1$, a contradiction. Hence, $F_{\pi'}(L) = 1$ as claimed.

Suppose that $|\pi| \geqslant 2$. Theorem 3.3.18 implies that $\mathscr{L} = \{M\}$. Suppose that $\pi = \{p\}$ for some prime p. We claim the hypothesis of Lemma 4.3.2 are satisfied. Let $L \in \mathscr{L}$. Since $F_{\pi'}(L) = 1$ and L is solvable, we have $F^*(L) = O_p(L)$ so L has characteristic P. Since $M \rightsquigarrow L$, we have

$$1 \neq Z(O_p(M)) \leqslant O_p(M) \cap L.$$

Note that $O_p(M) \cap L \leqslant O_p(M \cap L)$ hence we may choose N with

$$N_\Gamma(O_p(M \cap L))) \leqslant N \in \Gamma^*$$

and so $N \in \Gamma^{**}$. Now we have

$$C_\Gamma(a) \leqslant M \cap L \leqslant N.$$

Since $M \rightsquigarrow L$ there exists $X \in \Gamma$ with $1 \neq X \leqslant O_p(M)$ and $XC_{O_p(M)}(X) \leqslant L$. Then

$$XC_{O_p(M)}(X) \leqslant M \cap L \leqslant N$$

and so $M \rightsquigarrow N$. Then, $N \in \mathscr{L}$. Lemma 4.3.2 implies that

$$\mathscr{L} = \{M\},$$

and hence (b) holds.

Choose M^* with $M \leqslant M^* \in \Gamma^*$. Since $\Gamma^* \subseteq \Gamma^{**}$, we have $M^* \in \mathscr{L}$ so

$$M = M^*,$$

whence $M \in \Gamma^*$ and (a) holds.

Chapter 5

An Analogue of The Solvable Signalizer Functor Theorem for Partial Groups

5.1 Introduction

The main object of this chapter is to prove the Solvable Siqnalizer Functor Theorem (Theorem 2.3.22) for partial groups which will be the main result of this chapter.

Theorem 5.1.1. (**Signalizer Functor Theorem for Partial Groups**). Let A be an elementary abelian r-group that acts coprimely on the partial group $\mathit{\Gamma}$. Suppose that $\mathrm{rank}(A) \geqslant 3$ and that every member of $\mathit{\Gamma}$ is solvable. Then $\mathit{\Gamma}$ possesses a unique maximal member.

We will use a proof by contradiction to prove this result. We start with the counterexample, and then we are going to prove some results which will be used to complete the proof of our main result.

5.2 The Counterexample

In this section we will begin the study of a counterexample to Theorem 2.3.22 for Partial Groups. Henceforth we assume Theorem 5.1.1 to be false. Thus we have a partial group $\mathit{\Gamma}$ and an elementary abelian r-group A of rank at least three that acts on $\mathit{\Gamma}$ such that the following hold:

- Each member of $\mathit{\Gamma}$ is a solvable r'-group.
- $|\mathit{\Gamma}^*| \geqslant 2$.

Using Coprime Action and Lemma 3.4.9 we may assume that

- rank $(A) = 3$.
- $C_{\mathit{\Gamma}}(A) \leqslant N_{\mathit{\Gamma}}(H)$ for all $1 \neq H \in \mathit{\Gamma}$.

Theorem 5.2.1. There exists a family $\{M_a \mid a \in A^\#\}$ such that for each $a \in A^\#$ the following hold:

(a) $C_\Gamma(a) \leqslant M_a \in \Gamma^*$.

(b) If $a' \in A$ and $\langle a' \rangle = \langle a \rangle$ then $M_{a'} = M_a$.

Let $L \in \Gamma^{**}$ and suppose that $M_a \rightsquigarrow L$. Let $\pi = \pi(F(M_a))$.

(c) If $C_\Gamma(a) \leqslant L$, then $M_a = L$. In particular, if $X \in \Gamma$ satisfies $1 \neq X \leqslant F(M_a)$ and X is $C_\Gamma(a)$-invariant, then $N_\Gamma(X) \leqslant M_a$.

(d) $M_a \cap O_{\pi'}(L) = 1$.

(e) If $|\pi| \geqslant 2$, then $F_\pi(L) \leqslant M_a$.

(f) Suppose that $|\pi| \geqslant 2$ and at least one of the following holds:

(1) $L \rightsquigarrow M_a$, or

(2) $F_{\pi'}(L) = 1$.

Then $M = L$.

Proof. Let $a \in A^\#$ and let M_a denote the subgroup M in the conclusion of Lemma 4.3.3. For each $a' \in \langle a \rangle^\#$ define $M_{a'} = M_a$. Since $\langle a' \rangle = \langle a \rangle$ we have

$$C_\Gamma(a') = C_\Gamma(a).$$

Then (a) and (b) hold. Lemma 4.3.3 implies (c) and the remaining assertions follow from Theorem 3.3.18.

Henceforth we let

$$\{M_a \mid a \in A^\#\},$$

be the family constructed in Theorem 5.2.1.

We recall that $\mathrm{Hyp}(A)$ denotes the set of maximal subgroups of A.

Lemma 5.2.2. (a) Let $B \in \mathrm{Hyp}(A)$. Then

$$|\{M_b \mid b \in B^\#\}| \geqslant 2.$$

(b) $|\{M_a \mid a \in A^\#\}| \geqslant 3$.

(c) Let $a \in A^\#$. Then $C_A(M_a)$ is cyclic.

Proof. (a) Assume that $\{M_b \mid b \in B^\#\} = \{M\}$ for some $M \in \Gamma$. Let $X \in \Gamma$. Coprime Action implies

$$X = \langle C_X(b) \mid b \in B^\# \rangle \leqslant \langle X \cap M_b \mid b \in B^\# \rangle \leqslant M.$$

Then $\Gamma^* = \{M\}$, contrary to $|\Gamma^*| \geqslant 2$. Thus (a) holds.

(b) Assume that $\{M_a \mid a \in A^{\#}\} = \{M,\ L\}$ for some $M,\ L \in \Gamma$. Now $\Gamma^* \neq \{L\}$ so there exists $X \in \Gamma$ with $X \nleqslant L$. By Coprime Action there exists $B \in \mathrm{Hyp}(A)$ with $C_X(B) \nleqslant L$. Then for each $b \in B^{\#}$, we have

$$C_X(B) \leqslant C_\Gamma(b) \leqslant M_b$$

so $M_b \neq L$ and then $M_b = M$. This contradicts (a) and completes the proof.

(c) If $b \in C_A(M_a)^{\#}$ then since $M_a \in \Gamma^*$, we have

$$M_a \leqslant C_\Gamma(b) \leqslant M_b$$

and so

$$M_a = M_b.$$

Apply (a).

Lemma 5.2.3. (a) Suppose that $P \in \Gamma$ is a non-trivial p-group for some prime p. Let $L \in \Gamma$ and suppose that $P \cap L \neq 1$. Then

$$O_{p'}(N_\Gamma(P)) \cap L \leqslant O_{p'}(L).$$

(b) Let $a \in A^{\#}$ and $L \in \Gamma$. Suppose that $O_2(M_a) \cap L \neq 1$. Then

$$O(M_a) \cap L \leqslant O(L).$$

Proof. (a) Let $D = O_{p'}(N_\Gamma(P))$. First we claim that if $Q \in \Gamma$ with $1 \neq Q \leqslant P$, then $D \leqslant O_{p'}(N_\Gamma(Q))$. Let $N = N_\Gamma(Q)$. We proceed by induction on $|P:Q|$. If $|P:Q| = 1$, then $P = Q$ and the claim follows.

Assume that $Q < P$. Let $Q_1 = N_P(Q)$ so $Q < Q_1 \in \Gamma$. By induction, $D \leqslant O_{p'}(N_\Gamma(Q_1))$. Note that $D \leqslant C_\Gamma(P) \leqslant N_\Gamma(Q) = N$ and that $Q_1 \leqslant N$. Using Lemma 2.3.10, we have

$$D \leqslant O_{p'}(N_\Gamma(Q_1)) \cap N \leqslant O_{p'}(N_N(Q_1)) \leqslant O_{p'}(N).$$

The claim is established.

Let $Q = P \cap L$. Using the claim, we have

$$D \cap L \leqslant O_{p'}(N_\Gamma(Q)) \cap L \leqslant O_{p'}(N_L(Q)).$$

The Lemma 2.3.10 implies

$$D \cap L \leqslant O_{p'}(L).$$

(b) Since $M_a \in \Gamma^*$ and $O_2(M_a) \neq 1$, we have

$$M_a = N_\Gamma(O_2(M_a)).$$

The conclusion follows from (a).

Lemma 5.2.4. Suppose $B \in \mathrm{Hyp}(A)$ and $X \in \Gamma$ satisfy

$$X \leqslant O(M_b),$$

for all $b \in B^{\#}$. Let $H \in \Gamma$. Then

$$X \cap H \leqslant O(H).$$

Prof. Let $\overline{H} = H/O(H)$. Let $b \in B^{\#}$. Then

$$[\overline{X \cap H},\ C_{O_2(\overline{H})}(b)] \leqslant \overline{O(M_b) \cap H} \cap O_2(\overline{H}) = 1.$$

Since B is non-cyclic, Coprime Action implies

$$O_2(\overline{H}) = \langle C_{O_2(\overline{H})}(b) \mid b \in B^{\#} \rangle$$

and so

$$[\overline{X \cap H},\ O_2(\overline{H})] = 1.$$

Since $O(\overline{H}) = 1$ and $\overline{H}$ is solvable, we have

$$C_{\overline{H}}(O_2(\overline{H})) \leqslant O_2(\overline{H}).$$

Now X has odd order, whence $\overline{X \cap H} = 1$. Then

$$X \cap H \leqslant O(H).$$

5.3 Exceptional Elements

We start this section with the following definition.

Definition 5.3.1. An element $a \in A^{\#}$ is exceptional if

$$[M_a,\ a] \leqslant O_2(M_a) \text{ and } O(M_a) \neq 1.$$

We let

$$A_{exc}$$

denote the set of exceptional elements of A.

We recall that for a group G that $\pi(G)$ is the set of all prime divisors of order G.

Hypothesis E

- $a \in A_{exc}$.
- When Hypothesis E holds we define $\pi := \pi(F(M_a))$, and $\pi_0 = \pi \setminus \{2\}$.

Lemma 5.3.2. Let $a \in A^{\#}$. Suppose that

$$[F_{2'}(M_a),\ a] = 1,\ O_2(O^2([M_a,\ a])) = 1 \text{ and } O(M_a) \neq 1.$$

Then $a \in A_{exc}$.

Proof. Let $F = F(O^2([M_a,\ a])) \trianglelefteq M_a$. Now $O_2(O^2([M_a,\ a]) = 1$ so F is a $2'$-group. In particular $F \leqslant F_{2'}(M_a)$ and then

$$[F,\ a] = 1.$$

Since $O^2([M_a,\ a])$ is solvable, Coprime Action implies that a centralizes $O^2([M_a,\ a])$. Le $S \in \mathrm{Syl}_2(M_a;\ a)$. Then $[M_a,\ a] = O^2([M_a,\ a])S$ so

$$[M_a,\ a] = [M_a,\ a,\ a] = [S,\ a].$$

Then $[M_a,\ a]$ is a 2-group and as $[M_a,\ a]$ is normal subgroup of M_a we have $[M_a,\ a] \leqslant O_2(M_a)$. By assumption $O(M_a) \neq 1$. Thus, $a \in A_{exc}$.

Lemma 5.3.3. Assume Hypothesis E. Let $X \in \Gamma$. Then

$$[X,\ a] \leqslant O_{\pi_0'}(X).$$

In particular, if $p \in \pi_0$ then a centralizes every A-invariant p-subgroup of X.

Proof. Let $p \in \pi_0$. We have a centralizes a Sylow p-subgroup of M_a. Since X is solvable we may apply Theorem 3.4.15 to get

$$[X,\ a] \leqslant O_{p'}(X).$$

Hence,

$$[X,\ a] \leqslant O_{\pi_0'}(X).$$

Lemma 5.3.4. Let $a,\ b \in A_{exc}$ and suppose that

$$F_{2'}(M_a) \cap F_{2'}(M_b) \neq 1.$$

Then $M_a = M_b$.

Proof. Choose $p \neq 2$ such that $P := O_p(M_a) \cap O_p(M_b) \neq 1$. Let $N = N_\Gamma(P)$ and $B = \langle a,\ b\rangle$. Lemma 5.3.3 implies that

$$[O_p(M_a),\ B] = 1.$$

Then,

$$[M_a,\ B] \leqslant C_{M_a}(O_p(M_a)) \leqslant N.$$

Also by applying Lemma 3.4.7, we have

$$C_\Gamma(B) = C_\Gamma(a) \cap C_\Gamma(B),$$

and so

$$C_\Gamma(B) \leqslant M_a \cap M_b \leqslant N.$$

By Coprime Action $M_a = C_{M_a}(B)[M_a,\ B]$, so $M_a \leqslant N$. Since $M_a \in \Gamma^*$, we have

$$M_a = N.$$

Similarly, we get

$$M_b = N,$$

so

$$M_a = M_b.$$

Lemma 5.3.5. Assume Hypothesis E, that $p \in \pi_0$, that $P \in \Gamma$ and that $1 \neq P \leqslant O_p(M_a)$. Choose N with $N_\Gamma(P) \leqslant N \in \Gamma$. Then

$$N = (M_a \cap N)F_{\pi'}(N),\ M_a \cap F_{\pi'}(N) = 1,\ C_{F_{\pi'}(N)}(a) = 1$$

and

$$[N,\ a] = [O_2(M_a) \cap N,\ a]F_{\pi'}(N).$$

Moreover, if $X \leqslant [N,\ a]$ is A-invariant then

$$X = ([O_2(M_a) \cap N,\ a] \cap X)(F_{\pi'}(N) \cap X).$$

Proof. By Coprime Action, $N = C_N(a)[N,\ a]$. Let $K = [N,\ a]$, so that K is an A-invariant normal subgroup of N. Lemma 5.3.3 implies that K is a ${\pi_0}'$-group. Also

$$C_N(a) \leqslant C_\Gamma(a) \leqslant M_a$$

so

$$N = (M_a \cap N)K. \tag{1}$$

Since $a \in A_{exc}$, we have

$$[O_p(M_a),\ a] \leqslant O_p(M_a) \cap O_2(M_a) = 1.$$

Let $Z = Z(O_p(M_a)) \leqslant N_\Gamma(P) \leqslant N$. We consider the action of $\langle a \rangle \times Z$ on K. Note that $C_K(a) \leqslant M_a$ and $Z \trianglelefteq M_a$. Then

$$[C_K(a),\ Z] \leqslant K \cap Z = 1, \tag{2}$$

since K is a ${\pi_0}'$-group and $p \in \pi_0$. Theorem 2.3.9 implies that $[K,\ Z]$ is nilpotent. Now $[K,\ Z] \trianglelefteq K$ and K is a normal ${\pi_0}'$-subgroup of N. Hence,

$$[K, Z] \leqslant F(K) \leqslant F_{{\pi_0}'}(N). \tag{3}$$

By Coprime Action, $K = C_K(Z)[K,\ Z]$. As Z char M_a, we have $N_\Gamma(Z) = M_a$ and so

$$K = (M_a \cap N)[K,\ Z].$$

Combining this with (1) and (3) yields

$$N = (M_a \cap N)F_{{\pi_0}'}(N). \tag{4}$$

Since $P \leqslant O_p(M_a)$ and $N_\Gamma(P) \leqslant N$, we have $M_a \rightsquigarrow N$. Suppose that $2 \notin \pi$. Then $\pi_0 = \pi$ so (4) implies that

$$N = (M_a \cap N)F_{\pi'}(N).$$

Suppose that $2 \in \pi$. Then $\{2,\ p\} \subseteq \pi$ so $|\pi| \geqslant 2$ as $p \in \pi_0$. Theorem 5.2.1 implies $F_\pi(N) \leqslant M_a$. Since $F_{\pi_0'}(N) = O_2(N)F_{\pi'}(N)$ and $2 \in \pi$, it follows from (4) that

$$N = (M_a \cap N)F_{\pi'}(N),$$

in this case also.
As $M_a \rightsquigarrow N$, Theorem 5.2.1 implies

$$M_a \cap F_{\pi'}(N) = 1.$$

Then $C_{F_{\pi'}(N)}(a) = 1$ because $C_\Gamma(a) \leqslant M_a$. Coprime Action implies $F_{\pi'}(N) = [F_{\pi'}(N),\ a]$ so as $F_{\pi'}(N)$ char N it follows that

$$[N,\ a] = [M_a \cap N,\ a]F_{\pi'}(N). \tag{5}$$

Since $a \in A_{exc}$, we have $[M_a \cap N,\ a] \leqslant O_2(M_a) \cap N$. Using Coprime Action, we have

$$[M_a \cap N, a] = [M_a \cap N,\ a,\ a] \leqslant [O_2(M_a) \cap N,\ a].$$

Hene,

$$[N,\ a] = [O_2(M_a) \cap N,\ a]F_{\pi'}(N). \tag{6}$$

Next, suppose that X is an A-invariant subgroup of $[N,\ a]$. Suppose that $2 \notin \pi$. Then $O_2(M_a) = 1$ so $[N,\ a] = F_{\pi'}(N)$ and $X = X \cap F_{\pi'}(N)$.

Suppose that $2 \in \pi$. Let $S = [O_2(M_a) \cap N,\ a]$. Then

$$[N,\ a] = SF_{\pi'}(N),$$

and S is an $\langle a\rangle$-invariant Sylow 2-subgroup of $[N,\ a]$. Since $C_{F_{\pi'}(N)}(a) = 1$ it follows that S is the only $\langle a\rangle$-invariant Sylow 2-subgroup of $[N,\ a]$. Consequently S contains an $\langle a\rangle$-invariant Sylow 2-subgroup of X. It follows that

$$X = (S \cap X)(F_{\pi'}(N) \cap X).$$

Lemma 5.3.6. Assume Hypothesis E, let $L \in \Gamma$ and suppose that $F_{2'}(M_a) \cap L \neq 1$. Then

$$L = (M_a \cap L)F_{\pi_0'}(L).$$

Moreover, if $X \leqslant L$ is a-invariant and has odd order then

$$[X,\ a] \leqslant F_{2'}(L).$$

Proof. By Coprime Action, $L = C_L(a)[L,\ a]$. Let $K = [L,\ a]$ so that K is an A-invariant normal subgroup of L. Lemma 5.3.3 implies that K is a $\pi_0{}'$-group. Since $C_L(a) \leqslant M_a$, we have

$$L = (M_a \cap L)K. \tag{1}$$

Choose $p \in \pi_0$ such that $P := O_p(M_a) \cap L \neq 1$. Let $N = N_\Gamma(P)$. Now $a \in A_{exc}$ so

$$[P,\ a] \leqslant P \cap O_2(M_a) = 1$$

and we may consider the action of $\langle a\rangle \times P$ on K. Since $C_K(a) \leqslant M_a$ and $P \leqslant O_p(M_a)$, we have

$$[C_K(a),\, P] \leqslant K \cap O_p(M_a) = 1. \tag{2}$$

Theorem 2.3.9 implies that $[K,\, P]$ is nilpotent so $[K,\, P] \leqslant F(K)$ and then Coprime Action implies that

$$K = C_K(P)F(K).$$

Since $K = [L,\, a]$ we have $K = [K,\, a]$ and as $F(K)$ char K it follows that

$$K = [C_K(P),\, a]F(K). \tag{3}$$

Let $Y = [C_K(P),\, a]$. Lemma 5.3.5 implies that

$$Y = (O_2(M_a) \cap Y)(F_{\pi'}(N) \cap Y). \tag{4}$$

Since $[C_K(a),\, P] = 1$ it follows that Y and $F_{\pi'}(N)$ are $C_K(a)$-invariant. Then so is $F_{\pi'}(N) \cap Y$. Lemma 5.3.5 implies that

$$C_{F_{\pi'}(N)\cap Y}(a) = 1$$

and then Lemma 2.3.14 forces $F_{\pi'}(N) \cap Y \leqslant F(K)$. Combining this with (3) and (4) gives

$$K = (O_2(M_a) \cap K)F(K). \tag{5}$$

Since K is a normal $\pi_0{}'$-subgroup of L, we have $F(K) \leqslant F_{\pi_0{}'}$. Then (1) and (5) yield

$$L = (M_a \cap L)F_{\pi_0{}'}(L), \tag{6}$$

which is the first conclusion.

Suppose that $X \leqslant L$ is a-invariant and has odd order. Now,

$$[X,\, a] \leqslant K = (O_2(M_a) \cap K)F(K)$$

whence $[X,\, a] \leqslant F_{2'}(K) \leqslant F_{2'}(L)$.

Lemma 5.3.7. The following is impossible.

(1) $B = \langle a,\, b\rangle \in \mathrm{Hyp}(A)$.

(2) $a \in A_{exc}$.

(3) $N_{M_a}(P) \leqslant M_b$ for some $1 \neq P \leqslant O_p(M_a)$ with $P \in \Gamma$ and $p \neq 2$.

(4) $M_{b'} = M_b$ for all $b' \in B \setminus \langle a\rangle$.

Proof. Assume false. Let $H \in \Gamma^*$. We shall show that $H \in \{M_a,\, M_b\}$. By Coprime Action

$$\begin{aligned}[H,\, a] &= \langle [C_H(b'),\, a] \mid b' \in B \setminus \langle a\rangle\rangle \\ &\leqslant [M_b,\, a]. \end{aligned} \tag{1}$$

Let $\pi = \pi(F(M_a))$. Lemma 5.3.5, with M_b in the role of N implies

$$M_b = (M_a \cap M_b)F_{\pi'}(M_b),\; C_{F_{\pi'}(M_b)}(a) = 1 \tag{2}$$

and that

$$[H,\ a] = (O_2(M_a) \cap [H,\ a])(F_{\pi'}(M_b) \cap [H,\ a]). \tag{3}$$

Let $V = F_{\pi'}(M_b) \cap H \in \Gamma$. Then

$$C_V(a) \leqslant C_{F_{\pi'}(M_b)}(a) = 1$$

so Coprime Action implies

$$V = [V,\ a] \leqslant [H,\ a].$$

Hence,

$$V = F_{\pi'}(M_b) \cap [H,\ a].$$

If $2 \in \pi$, then (3) implies $V = O^2([H,a])$. If $2 \notin \pi$, then $O_2(M_a) = 1$ and $V = [H,\ a]$. In both cases, $V \trianglelefteq H$.

Suppose that $V \neq 1$. Then $N_\Gamma(V) = H$. Let $I = N_{F_{\pi'}(M_b)}(V)$. As $C_{F_{\pi'}(M_b)}(a) = 1$, we have

$$I = [I,\ a] \leqslant F_{\pi'}(M_b) \cap [H,\ a] = V.$$

We deduce that $V = F_{\pi'}(M_b)$. Then

$$H = N_\Gamma(V) = N_\Gamma(F_{\pi'}(M_b)) = M_b.$$

Suppose that $V = 1$. Then (3) implies $[H,\ a] \leqslant O_2(M_a)$. Now $C_H(a) \leqslant M_a$ so Coprime Action gives

$$H = C_H(a)[H,\ a] \leqslant M_a,$$

and hence $H = M_a$.
We have shown that $\{M_c \mid c \in A^\#\} = \{M_a,\ M_b\}$. Lemma 5.2.2 supplies a contradiction.

5.4 Commutators

The aim of this section is to prove the following result and explore some of its consequences.

Theorem 5.4.1. Let $a \in A^\#$. Then

$$[O(M_a),\ a] = 1.$$

First we need some preliminary results.

Lemma 5.4.2. Let $a \in A^\#$, let $F = F_{2'}(M_a)$ and let $H \in \Gamma^{**}$. Suppose that $C_F(a) \neq 1$ and that

$$C_{M_a}(C_F(a)) \leqslant H.$$

Then, $M_a = H$ or $a \in A_{exc}$.

Proof. We may suppose that $a \notin A_{exc}$. Let

$$\pi = \pi(F(M_a)),\ \mathscr{L} = \{L \in \Gamma^{**} \mid C_{M_a}(C_F(a)) \leqslant L\},$$

$$U = O_2(O^2([M_a,\ a])) \text{ and } V = C_F(C_F(a)).$$

We must show that $\mathscr{L} = \{M_a\}$. Let $L \in \mathscr{L}$. Since $V \leqslant L$ we have $M_a \rightsquigarrow L$. Also, $C_F(a) \neq 1$ so π contains an odd prime.
Suppose that $[V,\ a] = 1$. Coprime Action implies that $[F,\ a] = 1$. Then

$$[M_a,\ a] \leqslant C_{M_a}(F) \leqslant L.$$

Now $[M_a,\ a]$ is $C_\Gamma(a)$-invariant so Lemma 2.3.16 implies that $U \leqslant O_2(L)$. As $a \notin A_{exc}$ and $[F,\ a] = 1$, Lemma 5.3.2 implies that $U \neq 1$. Then $2 \in \pi$ so $|\pi| \geqslant 2$.
Also $U \trianglelefteq M_a$ so $M_a = N_\Gamma(U)$. Then

$$N_{F(L)}(U) \leqslant M_a$$

so

$$L \rightsquigarrow M_a.$$

Since $M_a \rightsquigarrow L$, Theorem 3.3.18 implies that $M_a = L$. Hence we may assume that $[V,\ a] \neq 1$.
Since $C_\Gamma(a) \leqslant M_a$ it follows that $[V,\ a]$ is $C_\Gamma(a)$-invariant and then Theorem 5.2.1 implies that $N_\Gamma([V,\ a]) \leqslant M_a$. By Lemma 2.3.14, $[V,\ a] \leqslant F(L)$ so

$$N_{F(L)}([V,\ a]) \leqslant M_a \text{ and } L \rightsquigarrow M_a.$$

Recall that $M_a \rightsquigarrow L$. If $|\pi| \geqslant 2$, then $M_a = L$ by Theorem 3.3.18. Hence we may assume that $\pi = \{p\}$ for some prime p. In particular, M_a has characteristic p.
We will show that $\mathscr{L}$ satisfies the hypothesis of Lemma 4.3.2. Recall that L is an arbitrary member of $\mathscr{L}$. We have shown that $M_a \rightsquigarrow L$ and that $L \rightsquigarrow M_a$. Theorem 3.3.18 implies that L has characteristic p. Now

$$1 \neq [V,\ a] \leqslant O_p(M_a) \cap O_p(L) \leqslant O_p(M_a \cap L).$$

Choose N with $N_\Gamma(O_p(M_a \cap L)) \leqslant N \in \Gamma^*$. Then

$$C_{M_a}(C_F(a)) \leqslant M_a \cap L \leqslant N_\Gamma(O_p(M_a \cap L)) \leqslant N.$$

Consequently, $N \in \mathscr{L}$. Lemma 4.3.2 implies that $\mathscr{L} = \{M_a\}$. The proof is complete.

Corollary 5.4.3. Let $a \in A^\#$, let p be an odd prime and let $P \in \Gamma$. Suppose that

$$1 \neq P \leqslant C_{O_p(M_a)}(a).$$

Then $N_\Gamma(P) \leqslant M_a$ or $a \in A_{exc}$.

Proof. Choose N with $N_\Gamma(P) \leqslant N \in \Gamma^*$. Let $F = F_{2'}(M_a)$. Then

$$P \leqslant C_F(a)$$

so

$$C_{M_a}(C_F(a)) \leqslant N_\Gamma(P) \leqslant N.$$

The conclusion follows from Lemma 5.4.2.

Proof of Theorem 5.4.1. Assume false. By Coprime Action $[F(O(M_a)),\ a] \neq 1$ so there exists $p \neq 2$ with $[O_p(M_a),\ a] \neq 1$. By Coprime Action there exists $B \in \mathrm{Hyp}(A)$ with

$$P := [C_{O_p(M_a)}(B),\ a] \neq 1.$$

We claim that if $b \in B^{\#}$ then

$$P \leqslant C_{O_p(M_b)}(b).$$

Indeed $O_p(M_a) \cap M_b$ is $C_{M_b}(a)$-invariant so Lemma 2.3.14 implies that

$$[O_p(M_a) \cap M_b,\ a] \leqslant O_p(M_b).$$

Now

$$C_{O_p(M_a)}(B) \leqslant O_p(M_a) \cap M_b$$

so

$$P \leqslant O_p(M_b).$$

As

$$P \leqslant C_\Gamma(B) \leqslant C_\Gamma(b),$$

the claim follows.

Let

$$B_{exc} = B \cap A_{exc} \text{ and } B_{norm} = B^{\#} \smallsetminus B_{exc}.$$

Suppose that $b,\ c \in B_{norm}$. Then $P \leqslant C_{O_p(M_c)}(c)$ so Corollary 5.4.3 implies $N_\Gamma(P) \leqslant M_c$. Also $P \leqslant C_{O_p(M_b)}(b)$. Let $F = F_{2'}(M_b)$. Then

$$C_{M_b}(C_F(b)) \leqslant N_\Gamma(P) \leqslant M_c.$$

Lemma 5.4.2 implies that

$$M_b = M_c.$$

On the other hand, if $b,\ c \in B_{exc}$, then as $P \leqslant O_p(M_b) \cap O_p(M_c)$, Lemma 5.3.4 implies that

$$M_b = M_c.$$

Using Lemma 5.2.2 it follows that $B_{norm} \neq B^{\#}$ and that $B_{exc} \neq B^{\#}$. Choose $b \in B_{norm}$ and $e \in B_{exc}$. Again, Lemma 5.2.2 implies that $M_b \neq M_e$. Also $B = \langle b,\ e\rangle$ since $B \in \mathrm{Hyp}(A)$ and $\mathrm{rank}(A) = 3$.

We will show that the hypothesis of Lemma 5.3.7 are satisfied. Note that $P \leqslant O_p(M_e)$ by the claim and that $N_\Gamma(P) \leqslant M_b$, since $b \in B_{norm}$.
Let $b' \in B \smallsetminus \langle e\rangle$. Suppose that $b' \in B_{exc}$. Since $P \leqslant O_p(M_{b'})$ we have $O_p(M_{b'}) \neq 1$. Lemma 5.3.3 implies that

$$[O_p(M_e),\ b'] = 1.$$

Since $e \in B_{exc}$, we also have $[O_p(M_e),\ e] = 1$. Now $b \in B = \langle b',\ e\rangle$ so $[O_p(M_e),\ b] = 1$. Then

$$[M_e,\ b] \leqslant C_{M_e}(O_p(M_e)) \leqslant N_{M_e}(P) \leqslant M_b.$$

By Coprime Action, $M_e = C_{M_e}(b)[M_e,\ b] \leqslant M_b$ so $M_e = M_b$, a contradiction.
We deduce that $b' \in B_{norm}$. Consequently,

$$M_{b'} = M_b,$$

for all $b' \in B \smallsetminus \langle e\rangle$. Lemma 5.3.7 supplies a contradiction.

Corollary 5.4.4. Let $a \in A^{\#} \setminus A_{exc}$ and suppose that $O(M_a) \neq 1$. Then

$$[O_2(M_a),\ a] \neq 1.$$

Proof. Suppose $[O_2(M_a),\ a] = 1$. Then Theorem 5.4.1 implies that $[F(M_a),\ a] = 1$ and then $[M_a,\ a] = 1$ by Coprime Action. This is not the case since $a \notin A_{exc}$.

Corollary 5.4.5. Let $a \in A^{\#} \setminus A_{exc}$, let p be an odd prime and let $P \in \Gamma$. Suppose that

$$1 \neq P \leqslant O_p(M_a).$$

Then $N_\Gamma(P) \leqslant M_a$.

Proof. By Theorem 5.4.1, $[O_p(M_a),\ a] = 1$ so $P \leqslant C_{O_p(M_a)}(a)$. Apply Corollary 5.4.3.

Corollary 5.4.6. Suppose that $a,\ b \in A^{\#} \setminus A_{exc}$ and that

$$F_{2'}(M_a) \cap F_{2'}(M_b) \neq 1.$$

Then $M_a = M_b$.

Proof. Choose an odd prime p such that

$$P := O_p(M_a) \cap O_p(M_b) \neq 1.$$

Corollary 5.4.5 implies that $N_\Gamma(P) \leqslant M_b$. Then

$$M_a \rightsquigarrow M_b.$$

Similarly, we get

$$M_b \rightsquigarrow M_a.$$

Corollary 5.4.4 implies $2 \in \pi(F(M_a))$, whence $\{2,\ p\} \subseteq \pi(F(M_a))$. Theorem 5.2.1 forces $M_a = M_b$.

Corollary 5.4.7. Let $B = \langle b,\ e\rangle \in \mathrm{Hyp}(A)$ and suppose that $M_{b'} = M_b$ for all $b' \in B \setminus \langle e\rangle$. Then

$$[O(M_b),\ A] = 1,\ O_2(M_b) \neq 1 \text{ and } O(M_b) \leqslant O(M_a)$$

for all $a \in A^{\#}$.

Proof. Theorem 5.4.1 implies that

$$[O(M_b),\ b'] = 1,$$

for all $b' \in B \setminus \langle e\rangle$. Then $[O(M_b),\ B] = 1$ and in particular, $[O(M_b),\ e] = 1$, and so $O(M_b) \leqslant M_e$. If $O_2(M_b) = 1$, then $[F(M_b),\ B] = 1$ so Coprime Action implies $[M_b,\ B] = 1$ and so $B \leqslant C_A(M_b)$, contrary to Lemma 5.2.2. We deduce that $O_2(M_b) \neq 1$.
Let $c \in A\#$. By Coprime Action

$$[M_c,\ e] = \langle [C_{M_c}(b'),\ e] \mid b' \in B \smallsetminus \langle e \rangle \rangle$$
$$\leqslant [M_b,\ e].$$

Since $[O(M_b),\ e] = 1$, we have
$$[O(M_b),\ [M_b,\ e]] = 1.$$

In particular,
$$[O(M_b),\ [M_c,\ e]] = 1.$$

If $[M_c,\ e] \neq 1$ then as $[M_c,\ e] \trianglelefteq M_c$ we have $O(M_b) \leqslant N_\Gamma([M_c,\ e]) = M_c$.

If $[M_c,\ e] = 1$ then $M_c = M_e$. We have shown that $O(M_b) \leqslant M_e$ so $O(M_b) \leqslant O(M_c)$ in this case also.
Choose $D \in \mathrm{Hyp}(A)$ with $C_{O_2(M_b)}(D) \neq 1$. If $d \in D^\#$ then $O_2(M_b) \cap M_d \neq 1$ so by Lemma 5.2.3 we have
$$O(M_b) = O(M_b) \cap M_d \leqslant O(M_d).$$

Then Lemma 5.2.4 implies that $O(M_b) \leqslant O(M_a)$ for all $a \in A^\#$. Finally, Theorem 5.4.1 forces $[O(M_b), A] = 1$.

5.5 Fitting Intersections

In this section we are going to prove the following result.

Theorem 5.5.1. Let $a,\ b \in A^\#$ and suppose that
$$F_{2'}(M_a) \cap F_{2'}(M_b) \neq 1.$$

Then $M_a = M_b$.

Proof. Assume false, so that $M_a \neq M_b$. Choose an odd prime p such that
$$P := O_p(M_a) \cap O_p(M_b) \neq 1.$$

Using Lemma 5.3.4 and Corollary 5.4.6 we may suppose that
$$a \in A_{exc} \text{ and } b \notin A_{exc}.$$

Corollary 5.4.5 implies that
$$N_\Gamma(P) \leqslant M_b$$

and so $M_a \rightsquigarrow M_b$.

Let
$$P_0 = N_{O_p(M_a)}(P) \leqslant M_b.$$

Claim 1. $[P_0,\ b] \neq 1$.

Proof. Assume false. Coprime Action implies $[O_p(M_a),\ b] = 1$. Then
$$[M_a,\ b] \leqslant C_{M_a}(O_p(M_a)) \leqslant N_\Gamma(P) \leqslant M_b.$$

By Coprime Action, we have

$$M_a = C_{M_a}(b)[M_a,\ b] \leqslant M_b,$$

so $M_a = M_b$, a contradiction. We deduce that $[P_0,\ b] \neq 1$.

Let

$$\pi = \pi(F(M_a)).$$

Claim 2. The following hold.

(a) $[P_0,\ b]$ acts non-trivially on $O_2(M_b)$.

(b) $2 \notin \pi$ and $[M_a,\ a] = 1$.

(c) $[M_b,\ a] \leqslant F_{\pi'}(M_b)$.

(d) a acts fixed point freely on $O_2(M_b)$.

Proof. Lemma 5.3.5 implies that $M_b = (M_a \cap M_b)F_{\pi'}(M_b)$ and that a acts fixed point freely on $F_{\pi'}(M_b)$.
Let $X = (M_a \cap M_b)O_{F_{\pi'}}(M_b)$. Then $M_b = XO_2(M_b)$. As $P_0 \leqslant O_p(M_a)$ we have $P_0 \leqslant O(X)$.
Suppose that $[P_0,\ b]$ acts trivially on $O_2(X)$. As $M_b = XO_2(M_b)$ it follows that $[P_0,\ b] \leqslant O(M_b)$.
Using Coprime Action and Theorem 5.4.1, we have

$$[P_0,\ b] = [P_0,\ b,\ b] \leqslant [O(M_b), b] = 1,$$

contrary to Claim 1, so (a) holds.

Suppose that $2 \in \pi$. Then $\{2,\ p\} \subseteq \pi$ so as $M_a \rightsquigarrow M_b$, Theorem 5.2.1 implies that $O_2(M_b) \leqslant M_a$.
As $P_0 \leqslant O_p(M_a)$ and $P_0 \leqslant M_b$, we have

$$[P_0,\ O_2(M_b)] \leqslant O_p(M_a) \cap O_2(M_b) = 1,$$

contrary to (a). We deduce that $2 \notin \pi$, so $O_2(M_a) = 1$. As $a \in A_{exc}$ we have

$$[M_a,\ a] \leqslant O_2(M_a) = 1.$$

Thus (b) holds.

As $M_b = (M_a \cap M_b)F_{\pi'}(M_b)$ and $[M_a,\ a] = 1$ we have

$$[M_b,\ a] \leqslant F_{\pi'}(M_b),$$

so (c) holds.

Finally (d) holds because $2 \notin \pi$ and a acts fixed point freely on $F_{\pi'}(M_b)$., and this completes the proof.

Let

$$B = \langle a,\ b \rangle.$$

Claim 3. Let $c \in B \setminus \langle a \rangle$. Then $M_c = M_b$.

Proof. Theorem 5.4.1 implies that $[O_p(M_a),\ a] = 1$, so as $B = \langle a,\ c \rangle$ it follow that

$$[P_0,\ c] = [P_0,\ b] \neq 1.$$

In particular, $[P_0,\ c]$ acts non-trivially on $O_2(M_b)$. Lemma 2.3.14, with $[P_0,\ c]O_2(M_b)$ in the role of G, implies that $C_{O_2(M_b)}(c) \neq 1$. Hence

$$O_2(M_b) \cap M_c \neq 1.$$

Theorem 5.4.1 implies that

$$[P,\ B] = 1,$$

whence $P \leqslant O(M_b) \cap M_c$. Lemma 5.2.3 implies that $P \leqslant O(M_c)$. Then

$$O_2(M_c) \leqslant N_\Gamma(P) \leqslant M_b.$$

By Claim 2, $2 \notin \pi$ so Lemma 5.3.6 implies that

$$M_c = (M_a \cap M_c)F_{\pi'}(M_c).$$

As $[M_a,\ a] = 1$ this yields

$$[M_c,\ a] \leqslant F_{\pi'}(M_c). \tag{$*$}$$

Recall that $O_2(M_c) \leqslant M_b$. Let

$$V = N_{O_2(M_b)}(O_2(M_c)) \leqslant M_c.$$

Using the fact that a acts fixed point freely on $O_2(M_b)$, Coprime Action and $(*)$ we have

$$V = [V,\ a] \leqslant O_2(M_c).$$

Lemma 4.3.1 implies that

$$O_2(M_b) \leqslant O_2(M_c).$$

Since a acts fixed point freely on $O_2(M_b)$ we have

$$O_2(M_b) \leqslant [O_2(M_c),\ a].$$

By Claim (c) we have

$$[M_b,\ a] \leqslant F_{\pi'}(M_b).$$

As $O_2(M_c) \leqslant M_b$ this gives

$$[O_2(M_c),\ a] \leqslant O_2(M_b).$$

We deduce that

$$O_2(M_b) = [O_2(M_c),\ a].$$

As $M_c = (M_a \cap M_b)F_{\pi'}(M_c)$ and $[M_a,\ a] = 1$, we have $[O_2(M_c),\ a] \trianglelefteq M_c$. Then

$$M_c = N_\Gamma([O_2(M_c),\ a] = N_\Gamma(O_2(M_b)) = M_b,$$

and this completes the proof of the claim.

Lemma 5.3.7 supplies a contradiction which completes the proof.

5.6 More on Exceptional Elements (1)

This the first of three sections on exceptional elements that will almost culminate in proving their nonexistence.

Hypothesis E (1)

- $a \in A_{exc}$.
- $\pi = \pi(F(M_a))$ and $\pi_0 = \pi \setminus \{2\}$.
- $p \in \pi_0$.
- $a \in B \in \mathrm{Hyp}(A)$.
- $P := C_{O_p(M_a)}(B) \neq 1$.

The main result of this subsection is the following :

Theorem 5.6.1. Assume Hypothesis E (1) .

(a) Let $c \in A^{\#}$ and suppose that $[O(M_c),\ a] \neq 1$. Then there exists $b \in B^{\#}$ with $M_c = M_b$.

(b) Let $b \in B^{\#}$ and suppose that $[O(M_b),\ a] \neq 1$. Then $b \in A_{exc}$ and $[O(M_a),\ b] = 1$.

(c) Let $b,\ b' \in B^{\#}$ and suppose that

$$[O(M_b),\ a] \neq 1 \neq [O(M_{b'}),\ a].$$

Then $\langle b \rangle = \langle b' \rangle$.

(d) $[O(M_a),\ B] = 1$ or $O_2(M_a) = 1$.

Part (d) gives a dichotomy, each case of which will be handled in the following two sections.

Now we start with some results.

Lemma 5.6.2. Let $a \in A^{\#}$ and set

$$\begin{aligned}\mathscr{B} = \{b \in A^{\#} \mid\ & C_{O_2(M_a)}(b) = 1, \text{ or}\\ & [M_a,\ b] \leqslant O_2(M_a), \text{ or}\\ & [M_a,\ b] \leqslant O_2(M_a \bmod C_{M_a}(O_2(M_a)))\}.\end{aligned}$$

Suppose that $O(M_a) \neq 1$ and $a \in \langle \mathscr{B} \rangle$. Then $a \in A_{exc}$.

Proof. Let $M = M_a$ and $\overline{M} = M/C_M(O_2(M))$. Let $b \in \mathscr{B}$. We claim that $[\overline{M},\ b] \leqslant O_2(\overline{M})$. This is clear if b satisfies the second and third criteria in the definition of $\mathscr{B}$. Suppose that

$C_{O_2(M)}(b) = 1$. Let p be an odd prime and choose $P \in \mathrm{Syl}_p(M;\, A)$. Lemma 2.3.14, with $O_2(M)[P,\, b]$ in the role of G, implies that

$$[O_2(M),\, [P,\, b]] = 1.$$

This implies that

$$[\overline{P},\, b] = \overline{1},$$

and so $\overline{P} \leqslant C_{\overline{M}}(b)$. We deduce that $|\overline{M} : C_{\overline{M}}(b)|$ is a power of 2.
Choose $\overline{S} \in \mathrm{Syl}_2(\overline{M};\, A)$. Then $\overline{M} = C_{\overline{M}}(b)\overline{S}$ and it follows from commutator identities that $[\overline{M},\, b] \leqslant \overline{S}$. Since $[\overline{M},\, b] \trianglelefteq \overline{M}$ we deduce that $[\overline{M},\, b] \leqslant O_2(\overline{M})$. The claim is established.

The previous paragraph implies that $\langle \mathscr{B} \rangle$ centralizes $\overline{M}/O_2(\overline{M})$. Since $a \in \langle \mathscr{B} \rangle$ we have

$$[\overline{M},\, a] \leqslant O_2(\overline{M}).$$

Then

$$O^2([M,\, a]) \leqslant C_M(O_2(M)).$$

Theorem 5.4.1 implies that

$$[F_{2'}(M),\, a] = 1$$

and so

$$[M,\, a] \leqslant C_M(F_{2'}(M)).$$

Using the fact that M is solvable we have

$$O^2([M,\, a]) \leqslant C_M(F(M)) \leqslant F(M).$$

Since $O^2([M,\, a])$ is generated by elements of odd order we have

$$O^2([M,\, a]) \leqslant F_{2'}(M).$$

Then

$$[O^2([M,\, a]),\, a] = 1.$$

Let p be an odd prime and choose $P \in \mathrm{Syl}_p(M;\, A)$. By Coprime Action we have

$$[P,\, a] = [P,\, a,\, a] \leqslant [O^2([M,\, a]),\, a] = 1.$$

We deduce that $|M : C_M(a)|$ is a power of 2.
Choose $S \in \mathrm{Syl}_2(M;\, A)$. Then $M = C_M(a)S$ and so $[M,\, a] \leqslant S$. Now $[M,\, a] \trianglelefteq M$ whence $[M,\, a] \leqslant O_2(M)$. By assumption $O(M) \neq 1$ so $a \in A_{exc}$.

Lemma 5.6.3. Assume Hypothesis E (1). Let $b \in B^{\#}$. Then

$$M_b = (M_a \cap M_b)F_{\pi_0'}(M_b).$$

Moreover if $X \in \Gamma$ has odd order and $X \leqslant M_b$ then

$$[X,\, a] \leqslant F_{2'}(M_b).$$

Proof. We have

$$1 \neq P = C_{O_p(M_a)}(B) \leqslant O_p(M_a) \cap C_\Gamma(b) \leqslant F_{2'}(M_a) \cap M_b.$$

Apply Lemma 5.3.6.

Lemma 5.6.4. Assume Hypothesis E (1). Let $c \in A^{\#}$ and suppose that $[O(M_c),\ a] \neq 1$. Then there exists $b \in B^{\#}$ such that $M_c = M_b$.

Proof. Let $F = F_{2'}(M_c) = F(O(M_c))$. Since $O(M_c)$ is solvable we have $[F,\ a] \neq 1$. By Coprime Action there exists $b \in B^{\#}$ with

$$[C_F(b),\ a] \neq 1.$$

Note that $C_F(b) \leqslant M_b$. Using Lemma 5.6.3 we obtain

$$1 \neq [C_F(b),\ a] \leqslant F_{2'}(M_c) \cap F_{2'}(M_b).$$

Theorem 5.5.1 forces $M_c = M_b$.

Lemma 5.6.5. Assume Hypothesis E (1). Let $b \in B^{\#}$ and suppose that $[O(M_b),\ a] \neq 1$. Then $b \in A_{exc}$.

Proof. Let $F = F_{2'}(M_b)$. Since $O(M_b)$ is solvable we have $[F,\ a] \neq 1$. By Coprime Action there exists $D \in \mathrm{Hyp}(A)$ with $[C_F(D),\ a] \neq 1$. Now $[F,\ b] = 1$ by Theorem 5.4.1 so we may choose D with $b \in D$.
Let

$$D_0 = \{d \in D^{\#} \mid C_{O_2(M_b)}(d) = 1\}$$

and

$$D_1 = \{d \in D^{\#} \mid C_{O_2(M_b)}(d) \neq 1\}.$$

Suppose that $d \in D_1$. Then $O_2(M_b) \cap M_d \neq$ so using Lemma 5.2.3 we have

$$1 \neq [C_F(D),\ a] \leqslant O(M_b) \cap M_d \leqslant O(M_d).$$

Then $[O(M_d),\ a] \neq 1$. Lemma 5.6.4 implies

$$M_d = M_{b'}$$

for some $b' \in B^{\#}$. By Theorem 5.4.1, $[F,\ b] = 1$ and $[O(M_{b'}),\ b'] = 1$. Since $M_d = M_{b'}$ we have

$$[[C_F(D),\ a],\ b'] = 1.$$

In particular,

$$\langle b,\ b' \rangle \leqslant C_B([C_F(D),\ a]).$$

Now a is not contained in this centralizer, $a \in B^{\#}$ and $\mathrm{rank}(B) = 2$. It follows that the centralizer is cyclic and then that $\langle b \rangle = \langle b' \rangle$. We deduce that

$$M_d = M_b$$

for all $d \in D_1$. Lemma 5.2.2 implies that $D^{\#} \neq D_1$ and so $D_0 \neq \emptyset$.

Suppose that $b \notin A_{exc}$. Lemma 5.6.2 implies $b \notin \langle D_0 \rangle$ so as $b \in D$ and $\mathrm{rank}(D) = 2$ it follows that $\langle D_0 \rangle$ is cyclic. Let e be a generator for $\langle D_0 \rangle$. Then $D = \langle b,\ e \rangle$ and $M_d = M_b$ for all $d \in D \setminus \langle e \rangle$. Corollary 5.4.7 implies that $[O(M_b),\ a] = 1$, contrary to $[O(M_b),\ a] \neq 1$. We deduce that $b \in A_{exc}$.

Lemma 5.6.6. Assume Hypothesis E (1). Let $b,\ b' \in B^{\#}$ and suppose that

$$[O(M_b),\ a] \neq 1 \neq [O(M_{b'}),\ a].$$

Then $\langle b \rangle = \langle b' \rangle$.

Proof. Assume false. As rank $(B) = 2$ we have $B = \langle b,\ b' \rangle$. As $[O(M_{b'}),\ b'] = 1$ and $a \in B$ it follows that

$$[O(M_{b'}),\ b] \neq 1. \tag{$*$}$$

Lemma 5.6.5 implies that $b \in A_{exc}$. Choose D with $b \in D \in \mathrm{Hyp}(A)$ and $V := [C_{F_{2'}(M_b)}(D),\ a] \neq 1$. Note that $[F_{2'}(M_b),\ a] \neq 1$ since $[O(M_b),\ a] \neq 1$ so such choice is possible. Then Hypothesis E (1) is satisfied with $(b,\ D)$ in the role of $(a,\ B)$. Then $(*)$ and Lemma 5.6.4 imply that there exists $d \in D^{\#}$ with $M_{b'} = M_d$. Now

$$V \leqslant C_{\Gamma}(D) \leqslant M_d,$$

so $V \leqslant M_{b'}$. Lemma 5.6.3 forces

$$V \leqslant F_{2'}(M_b) \cap F_{2'}(M_{b'}).$$

Theorem 5.5.1 implies $M_b = M_{b'}$. But then

$$[O(M_{b'}),\ b] = [O(M_b),\ b] = 1,$$

a contradiction.

Lemma 5.6.7. Assume hypothesis E (1). Let $b \in B^{\#}$ and suppose that $[O(M_b),\ a] \neq 1$. Then $[O(M_a),\ b] = 1$.

Proof. Choose D with $b \in D \in \mathrm{Hyp}(A)$ and $V := [C_{F_{2'}(M_b)}(D),\ a] \neq 1$. Lemma 5.6.5 implies that $b \in A_{exc}$ so we may apply Lemma 5.6.4 with $(b,\ D)$ in the role of $(a,\ B)$. Suppose that $[O(M_a),\ b] \neq 1$. Then there exists $d \in D^{\#}$ with

$$M_a = M_d.$$

Then $V \leqslant M_a$. However, $a \in A_{exc}$ so $[M_a,\ a]$ is a 2-group and $1 \neq V = [V,\ a]$ is a $2'$-group. This is a contradiction, and we deduce that $[O(M_a),\ b] = 1$.

Lemma 5.6.8. Assume Hypothesis E (1). Then

$$[O(M_a),\ B] = 1 \text{ or } O_2(M_a) = 1.$$

Proof. Assume that $[O(M_a),\ B] \neq 1$. Let $b \in B \setminus \langle a \rangle$ so that $B = \langle a,\ b \rangle$. Since $[O(M_a),\ a] = 1$ we have

$$[O(M_a),\ b] \neq 1.$$

Lemma 5.6.7 implies that

$$[O(M_b),\ a] = 1,$$

so $O(M_b) \leqslant M_a$. As $[O(M_b),\ b] = 1$, we have

$$M_a \neq M_b.$$

Also

$$[O(M_b),\ B] = 1,$$

so as $C_A(M_b)$ is cyclic it follows that $O_2(M_b) \neq 1$. Suppose that $O_2(M_a) \cap M_b \neq 1$. Recall the subgroup P in Hypothesis E (1). Lemma 5.2.3 implies that $P \leqslant O(M_b)$. Since $O(M_b) \leqslant M_a$ and $P \leqslant O_p(M_a)$ we have

$$P \leqslant O_p(O(M_b)) = O_p(M_b)$$

and so

$$P \leqslant O_p(M_a) \cap O_p(M_b).$$

Theorem 5.5.1 forces $M_a = M_b$, a contradiction. Hence

$$O_2(M_a) \cap M_b = 1.$$

We deduce that

$$C_{O_2(M_a)}(b) = 1,$$

for all $b \in B \setminus \langle a \rangle$.
By Coprime Action, we have

$$[O_2(M_a),\ a] = \langle [C_{O_2(M_a)}(b),\ a] \mid b \in B \setminus \langle a \rangle \rangle = 1.$$

Since $[O(M_a),\ a] = 1$ we have

$$[F(M_a),\ a] = 1$$

so

$$[M_a,\ a] = 1.$$

Suppose that $O_2(M_a) \neq 1$. Let $b \in B \setminus \langle a \rangle$. Since $[M_a,\ a] = 1$, Theorem 3.4.15 implies that

$$[O_2(M_b),\ a] = 1.$$

Then

$$O_2(M_b) \leqslant M_a.$$

We have previously shown that $O_2(M_b) \neq 1$ and that $O_2(M_a) \cap M_b = 1$. However,

$$1 \neq N_{O_2(M_a)}(O_2(M_b)) \leqslant O_2(M_a) \cap M_b,$$

a contradiction. We deduce that $O_2(M_a) = 1$.

Proof of Theorem 5.6.1. (a) is Lemma 5.6.4, (b) is Lemma 5.6.5 and Lemma 5.6.7, (c) is Lemma 5.6.6 and (d) is Lemma 5.6.8.

5.7 More on Exceptional Elements (2)

The aim of this section is to prove the following:

Theorem 5.7.1. Suppose that $a \in A_{exc}$ and that $[M_a,\ a] = 1$. Then

(a) $O_2(M_a) = 1$.

(b) $[X,\, a] \leqslant O_2(X)$ for each $X \in \Gamma$.

(c) If $c \in A_{exc}$ and $[M_c,\, c] = 1$ then $\langle c \rangle = \langle a \rangle$.

Lemma 5.7.2. Let $a \in A^\#$ and suppose that $[O(M_a),\, A] = 1$. Then $O_2(M_a) \neq 1$ and $O(M_a) \leqslant O(M_c)$ for all $c \in A^\#$.

Proof. By Lemma 5.2.2, $C_A(M_a)$ is cyclic so as M_a is solvable we have $[F(M_a),\, A] \neq 1$. Now $F_{2'}(M_a) \leqslant O(M_a)$ so as $[O(M_a),\, A] = 1$ we must have $[O_2(M_a),\, A] \neq 1$. In particular, $O_2(M_a) \neq 1$. Choose $B \in \mathrm{Hyp}(A)$ such that $C_{O_2(M_a)}(B) \neq 1$. Lemma 5.2.3 implies that

$$O(M_a) \leqslant O(M_b),$$

for all $b \in B^\#$. The conclusion follows from Lemma 5.2.4.

Lemma 5.7.3. Suppose that $a \in A_{exc}$. Then

$$[O(M_a),\, A] \neq 1.$$

Proof. Assume false, so $[O(M_a),\, A] = 1$. Since $a \in A_{exc}$ we have $O(M_a) \neq 1$. Lemma 5.7.2 implies that

$$O(M_a) \leqslant O(M_c),$$

for all $c \in A^\#$.
Let $c \in A^\#$ and suppose that $[O(M_c),\, a] = 1$. Then $O(M_a) \leqslant O(M_c) \leqslant M_a$ whence $O(M_a) \trianglelefteq O(M_c)$ and $1 \neq F_{2'}(M_a) \leqslant F_{2'}(M_b)$. Theorem 5.5.1 implies that $M_a = M_c$.

Choose B with $a \in B \in \mathrm{Hyp}(A)$. Now $[F_{2'}(M_a),\, B] = 1$ so Hypothesis E (1) is satisfied. By Lemma 5.2.3 there exists $b \in B^\#$ with $M_a \neq M_b$. The previous paragraph implies that

$$[O(M_b),\, a] \neq 1.$$

Let $c \in A^\#$. If $[O(M_c),\, a] = 1$ then $M_a = M_c$. Suppose that $[O(M_c),\, a] \neq 1$. Theorem 5.6.1 (a) and (b) imply $M_c = M_b$. Then $M_c \in \{M_a,\, M_b\}$. Lemma 5.2.3 supplies a contradiction.

Proof Theorem 5.7.1. Since $a \in A_{exc}$ we have $O(M_a) \neq 1$ so there exists an odd prime p with $O_p(M_a) \neq 1$. By Coprime Action we may choose $B \in \mathrm{Hyp}(A)$ with $C_{O_p(M_a)}(B) \neq 1$. Then Hypothesis E (1) is satisfied. Recall that

$$\pi = \pi(F(M_a)) \text{ and } \pi_0 = \pi \setminus \{2\}.$$

Let

$$B_0 = \{b \in B^\# \mid [O(M_b),\, a] = 1\}$$

and

$$B_1 = \{b \in B^\# \mid [O(M_b),\, a] \neq 1\}.$$

Theorem 5.6.1 implies that $\langle B_1 \rangle$ is cyclic.

(a) Suppose that $O_2(M_a) \neq 1$. Now $[M_a,\, a] = 1$ so Theorem 3.4.15 implies that

$$[O_2(X),\, a] = 1,$$

for all $X \in \Gamma$.
Let $b \in B_0$. Then $[F_{2'}(M_b),\ a] = 1$ by the definition of B_0 and so $[F(M_b),\ a] = 1$, Coprime Action implies that

$$[M_b,\ a] = 1.$$

We deduce that

$$M_b = M_a,$$

for all $b \in B_0$. Now $B_0 = B^{\#} \setminus B_1$ and $\langle B_1 \rangle$ is cyclic. Corollary 5.4.7 implies that

$$[O(M_a),\ A] = 1,$$

contrary to Lemma 5.7.3. We deduce that $O_2(M_a) = 1$.

(b) Let $b \in B \setminus \langle a \rangle$ and suppose that $[O(M_b),\ a] \neq 1$. Theorem 5.6.1 implies that $[O(M_a),\ b] = 1$. But $F(M_a) \leqslant O(M_a)$ by (a) so $[M_a,\ b] = 1$. By Hypothesis $[M_a,\ a] = 1$ so $\langle a,\ b \rangle \leqslant C_A(M_a)$. This is a contradiction since $C_A(M_a)$ is cyclic. We deduce that

$$[O(M_b), a\,] = 1.$$

Now $M_b = (M_a \cap M_b)F_{\pi_0'}(M_b)$ by Lemma 5.6.3 so as $[M_a,\ a] = 1$ it follows that $[M_b,\ a] \leqslant O_2(M_b)$.

Now let $X \in \Gamma$. Using Coprime Action and the above, we have

$$\begin{aligned}
[X,\ a] &= \langle [C_X(b),\ a] \mid b \in B \setminus \langle a \rangle \rangle \\
&= \langle [X \cap M_b,\ a] \mid b \in B \setminus \langle a \rangle \rangle \\
&\leqslant \langle X \cap O_2(M_b) \mid b \in B \setminus \langle a \rangle \rangle \\
&\leqslant O_2(X;\ A).
\end{aligned}$$

In particular, $[X,\ a]$ is a 2-group. Since $[X,\ a] \trianglelefteq X$ we have $[X,\ a] \leqslant O_2(X)$, completing the proof.

(c) By (a), with c in the role of a, we have $O_2(M_c) = 1$. Then (b) implies that $[M_c,\ a] = 1$, so $M_c \leqslant C_\Gamma(a) \leqslant M_a$ and then $M_c = M_a$. Now, we have

$$[M_a,\ a] = [M_c,\ c] = 1$$

so

$$\langle a,\ c \rangle \leqslant C_A(M_a).$$

Since $C_A(M_a)$ is cyclic, this forces $\langle c \rangle = \langle a \rangle$.

5.8 More on Exceptional Elements (3)

The aim of this section is to prove the following result.

Theorem 5.8.1. Let $a \in A_{exc}$. Then

$$[M_a,\ a] = 1.$$

We begin with the following hypothesis.

Hypothesis E (2)

- $a \in A_{exc}$.
- $[M_a,\ a] \neq 1$.
- $B = C_A(O(M_a))$.
- $B_0 = \{b \in B^{\#} \mid [O(M_b),\ a] = 1\}$.
- $B_1 = \{b \in B^{\#} \mid [O(M_b),\ a] \neq 1\}$.

Lemma 5.8.2. Assume Hypothesis E (2).

(a) $[O_2(M_a),\ a] \neq 1$.

(b) $a \in B \in \mathrm{Hyp}(A)$.

(c) $\langle B_1 \rangle$ is cyclic.

(d) Let $b \in B_0$. Then

$$M_b = (M_a \cap M_b)O_2(M_b),\ [M_b,\ a] \leqslant O_2(M_b),\ [O(M_b),\ B] = 1 \text{ and } O_2(M_b) \neq 1.$$

(e) Let $b,\ b' \in B_0$. If $O(M_b) \cap O(M_{b'}) \neq 1$ then $M_b = M_{b'}$.

(f) Let $b,\ b' \in B_0$. If $O(M_b) \neq 1$ and $O_2(M_b) \cap M_{b'} \neq 1$ then $M_b = M_{b'}$.

Proof. (a) Since $[M_a,\ a] \neq 1$ we have $[F(M_a),\ a] \neq 1$. Now $[F_{2'}(M_a),\ a] = 1$ because $[O(M_a),\ a] = 1$. Hence $[O_2(M_a),\ a] \neq 1$.

(b) Since $a \in A_{exc}$ we have $F_{2'}(M_a) \neq 1$ so there exists an odd prime p with $O_p(M_a) \neq 1$. Choose $C \in \mathrm{Hyp}(A)$ with $C_{O_p(M_a)}C \neq 1$ and $a \in C$. Then $(a,\ C)$ satisfies Hypothesis E (1). Since $O_2(M_a) \neq 1$, Theorem 5.6.1 implies that

$$[O(M_a),\ C] = 1,$$

so $C \leqslant B$. On the other hand, $[O(M_a),\ A] \neq 1$ by Lemma 5.7.3 so as $B \neq A$. As $C \in \mathrm{Hyp}(A)$, we obtain $C = B$, whence $B \in \mathrm{Hyp}(A)$. Also, $[O(M_a),\ a] = 1$ so $a \in B$.

(c) This is Theorem 5.6.1 (c).

(d) By Lemma 5.6.3 $M_b = (M_a \cap M_b)F_{\pi'}(M_b)$. Since $[O(M_b),\ a] = 1$ we obtain

$$M_b = (M_a \cap M_b)O_2(M_b).$$

Now $a \in A_{exc}$ so $[M_a,\ a] \leqslant O_2(M_a)$. Then

$$[M_a \cap M_b,\ a] \leqslant O_2(M_a \cap M_b),$$

and it follows that $[M_b,\ a] \leqslant O_2(M_b)$.
If $\langle b \rangle = \langle a \rangle$ then $M_b = M_a$ so $[O(M_b),\ B] = 1$ by the definition of B.

Suppose that $\langle b \rangle \neq \langle a \rangle$. Then $B = \langle a,\ b \rangle$. The definition of B_0 implies that $[O(M_b),\ a] = 1$ and Theorem 5.4.1 implies that $[O(M_b),\ b] = 1$. Thus $[O(M_b),\ B] = 1$ in this case also.

Now suppose that $O_2(M_b) = 1$. Then $[M_b, a] = 1$ so $M_a = M_b$ and then $[M_a, a] = 1$, a contradiction since $[M_a, a] \neq 1$ by Hypothesis E (2).

(e) By (d), $O(M_b)$ and $O(M_{b'})$ are normal subgroups of $C_\Gamma(B)$. If $O(M_b) \cap O(M_{b'}) \neq 1$ then

$$1 \neq F(O(M_b) \cap O(M_{b'})) \leqslant F_{2'}(M_b) \cap F_{2'}(M_{b'}).$$

Theorem 5.5.1 forces $M_b = M_{b'}$.

(f) By (c), $O(M_b) \leqslant C_\Gamma(B) \leqslant M_{b'}$. Since $O_2(M_b) \cap M_{b'} \neq 1$. Lemma 5.2.3 implies

$$O(M_b) \leqslant O(M_{b'}).$$

Then (e) forces $M_b = M_{b'}$.

Lemma 5.8.3. Assume Hypothesis E (2).

(a) There exists $b \in B_0$ such that $M_a \neq M_b$.

(b) $[M_a, B] \leqslant O_2(M_a)$.

(c) $C_{O_2(M_a)}(B) = 1$.

Proof. Suppose that $M_a = M_b$ for all $b \in B_0$. Since $\langle B_1 \rangle$ is cyclic, Corollary 5.4.7 implies that

$$[O(M_a), A] = 1,$$

contrary to Lemma 5.7.3 . Hence we may choose $b \in B_0$ with

$$M_a \neq M_b.$$

This proves (a). Lemma 5.8.2 implies that $O_2(M_a) \cap M_b = 1$. In particular, we have

$$C_{O_2(M_a)}(b) = 1.$$

This proves (c). The argument in the proof of Lemma 5.6.2 forces

$$[M_a, b] \leqslant O_2(M_a \bmod C_{M_a}(O_2(M_a)).$$

As $[O(M_a), b] = 1$, we have

$$[M_a, b] \leqslant C_{M_a}(O(M_a)) \leqslant M_a$$

so

$$O^2([M_a, b]) \leqslant C_{M_a}(O(M_a)).$$

Then, we have

$$O^2([M_a, b]) \leqslant F(M_a).$$

Since $O^2([M_a, b])$ is generated by elements of odd order, we have

$$O^2([M_a, b]) \leqslant F_{2'}(M_a).$$

By using Coprime Action, we get

$$[O^2([M_a, b]), b] = 1.$$

Lemma 2.3.13 implies that
$$[M_a,\ b] \leqslant O_2(M_a).$$
Since $a \in A_{exc}$ we have $[M_a,\ a] \leqslant O_2(M_a)$ so as $B = \langle a,\ b\rangle$ we deduce that
$$[M_a,\ B] \leqslant O_2(M_a),$$
which proves (b).

Lemma 5.8.4. Assume Hypothesis E (2). Let $b \in B_0$. Then
$$M_b = C_\Gamma(B)O_2(M_b) \text{ and } C_{O_2(M_b)}(B) = 1.$$

Proof. By Lemma 5.6.3 and since $[O(M_b),\ a] = 1$ we have
$$M_b = (M_a \cap M_b)O_2(M_b).$$
Lemma 5.8.3 implies that
$$[M_a \cap M_b,\ B] \leqslant O_2(M_a \cap M_b),$$
and it follows that $[M_b, B] \leqslant O_2(M_b)$. By Coprime Action, we have
$$M_b = M_b C_{M_b}(B)[M_b,\ B] = C_{M_b}(B)O_2(M_b).$$
Now $C_\Gamma(B) \leqslant C_\Gamma(b) \leqslant M_b$ so $C_{M_b}(B) = C_\Gamma(B)$ and the first assertion holds.

Suppose that $T := C_{O_2(M_b)}(B) \neq 1$. Let $N = N_\Gamma(T)$. Since $B = C_A(O(M_a))$ we have $O(M_a) \leqslant M_b$. Also $T \leqslant C_\Gamma(a) \leqslant M_a$ so
$$[O(M_a),\ T] \leqslant O(M_a) \cap O_2(M_b) = 1.$$
Then $O(M_a) \leqslant N$. Since $T \leqslant M_a$, we have
$$1 \neq N_{O_2(M_a)}(T) \leqslant O_2(M_a) \cap N.$$
Lemma 5.2.3 implies that $O(M_a) \leqslant O(N)$. Since $1 \neq T \leqslant O_2(M_a) \cap M_b$, another application of Lemma 5.2.3 gives
$$O(M_a) \leqslant O(N) \cap M_b.$$
Lemma 5.8.2 forces $M_a = M_b$. But $C_{O_2(M_a)}(B) = 1$ by Lemma 5.8.3, a contradiction. We deduce that
$$C_{O_2(M_b)}(B) = 1,$$
and the proof is complete.

Lemma 5.8.5. Assume Hypothesis E (2). There exists $a' \in B_0 \setminus \langle a\rangle$ such that $M_{a'} = M_a$. Moreover $a' \in A_{exc}$.

Proof. Suppose not. Using Lemma 5.8.2 we have
$$C_{O(M_a)}(a') \leqslant O_2(M_a) \cap M_{a'} = 1$$

for each $a' \in B_0 \smallsetminus \langle a \rangle$.

If $B_1 = \emptyset$ then $C_{O_2(M_a)}(a') = 1$ for all $a' \in B \smallsetminus \langle a \rangle$ and Coprime Action implies $[O_2(M_a),\, a] = 1$ contrary to Lemma 5.8.2. Thus $B_1 \neq \emptyset$.

Now by Coprime Action, we have

$$[O_2(M_a),\, a] = \langle C_{O_2(M_a)}(c),\, a] \mid c \in {B_1}^{\#} \rangle.$$

By Lemma 5.8.2 (c), $\langle B_1 \rangle$ is cyclic. Let e be a generator. Then

$$[O_2(M_a),\, a] = [C_{O_2(M_a)}(e),\, a] \leqslant C_{C_{O_2(M_a)}(e)}(a) \leqslant M_e.$$

By Lemma 5.8.3, $C_{O_2(M_a)}(A) = 1$ so Corollary 2.3.15 implies that

$$[O_2(M_a),\, a] \leqslant O_2(M_e).$$

Using Theorem 5.2.1 we have

$$O(M_e) \leqslant N_\Gamma([O_2(M_a),\, a]) \leqslant M_a.$$

Then

$$1 \neq [O(M_e),\, a] \leqslant [M_a,\, a] \leqslant O_2(M_a),$$

a contradiction since $O(M_e)$ has odd order.

Lemma 5.8.6. Assume Hypothesis E (2). Suppose that $e \in B_1$. Then $[M_e,\, e] = 1,\ O_2(M_e) = 1$ and $[M_e,\, B] \leqslant F_{2'}(M_e)$.

Proof. Suppose that $O_2(M_e) \neq 1$. Since $e \in B_1$ we have $O(M_e) \neq 1$. By Lemma 5.6.5, $e \in A_{exc}$. Then Hypothesis E (2) is satisfied with e in the role of a. Let $D = C_A(O(M_e))$. Lemma 5.8.2 implies that $e \in D \in \mathrm{Hyp}(A)$. By the definition of B_1 we have $[O(M_e),\, a] \neq 1$ so $B \neq D$ and then $B \cap D = \langle e \rangle$. Lemma 5.8.5 implies there exists $e' \in D \smallsetminus \langle e \rangle$ with $M_e = M_{e'}$ and $e' \in A_{exc}$. Since $B = C_A(O(M_e))$ we have $[O(M_a),\, e'] \neq 1$.
Now $M_e = M_{e'}$ so $D = C_A(O(M_{e'})$ and Lemma 5.6.4 implies there exists $d \in D^{\#}$ with $M_a = M_d$. Then

$$O(M_e) \leqslant C_\Gamma(d) \leqslant M_d = M_a$$

and so

$$1 \neq [O(M_e),\, a] \leqslant [M_a,\, a] \leqslant O_2(M_a).$$

This is a contradiction since $O(M_e)$ has odd order. We deduce that $O_2(M_e) = 1$.
Lemma 5.6.3 implies that $M_e = (M_a \cap M_e)F_{\pi_0'}(M_e)$. Since $O_2(M_e) = 1$ we have $M_e = (M_a \cap M_e)F_{2'}(M_e)$. Let $Q = [M_a \cap M_e,\, a]$. Then $Q \leqslant O_2(M_a)$ and so $C_Q(A) = 1$ by Lemma 5.8.3. Now $C_{M_e}(a) \leqslant M_a \cap M_e$ so Q is $C_{M_e}(a)$-invariant. Corollary 2.3.15 implies that $Q \leqslant O_2(M_e)$. Then $Q = 1$ and we deduce that $[M_e,\, a] \leqslant F_{2'}(M_e)$.
By Lemma 5.8.5 there exists $a' \in B \smallsetminus \langle a \rangle$ with $M_{a'} = M_a$ and $a' \in A_{exc}$. Then $B = C_A(O(M_{a'}))$ and again $M_e = (M_{a'} \cap M_e)F_{2'}(M_e)$. Then $[M_e,\, a'] \leqslant F_{2'}(M_e)$. Since $B = \langle a,\, a' \rangle$ we have $[M_e,\, B] \leqslant F_{2'}(M_e)$.

Lemma 5.8.7. Assume Hypothesis E (2). Let $Q \in \Gamma$ and suppose that Q is a 2-group with $Q = [Q,\ B]$. Then
$$C_Q(B) = 1.$$

Proof. By Coprime Action
$$C_Q(B) = \langle C_{[C_Q(b),\ B]}(B) \mid b \in B^{\#} \rangle.$$

If $b \in B_0$ then using Lemma 5.8.4 we have $[C_Q(b),\ B] \leqslant O_2(M_b)$ and then $C_{[C_Q(b),\ B]}(B) = 1$.

If $b \in B_1$ then using Lemma 5.8.6 we have $[C_Q(b),\ B] \leqslant F_{2'}(M_b)$ so $[C_Q(b),\ B] = 1$ since Q is a 2-group. The conclusion follows.

Lemma 5.8.8. Assume Hypothesis E (2) . Suppose that $T \in \Gamma$ with $1 \neq T \leqslant O_2(M_a)$. Then
$$N_\Gamma(T) \leqslant M_a.$$

Proof. Choose N with $N_\Gamma(T) \leqslant N \in \Gamma^*$. Then $M_a \rightsquigarrow N$. Let $Z = Z(O_2(M_a))$. Then $[Z,\ a] \leqslant O_2(N)$. By Lemma 5.8.5 there exists $a' \in B_0 \setminus \langle a \rangle$ with $M_a = M_{a'}$ and $a' \in A_{exc}$. Then $[Z,\ a'] \leqslant O_2(N)$. Since $B = \langle a,\ a' \rangle$ and $C_{O_2(M_a)}(B) = 1$, we deduce that $Z \leqslant O_2(N)$. Then $N \rightsquigarrow M_a$ and Theorem 5.2.1 implies that $N = M_a$, so $N_\Gamma(T) \leqslant M_a$.

Now we are going to prove Theorem 5.8.1.

Proof of Theorem 5.8.1. Assume false. Then Hypothesis E (2) holds. By Lemma 5.8.3 there exists $b \in B_0$ with $M_a \neq M_b$. Now $O_2(M_b) \neq 1$ by Lemma 5.8.2 so Coprime Action implies there exists $D \in \mathrm{Hyp}(A)$ with $C_{O_2(M_b)}(D) \neq 1$.
Let $d \in D^{\#}$ and set
$$Q = \langle C_{O_2(M_a)}(D),\ C_{O_2(M_b)}(d) \rangle.$$
Note that Q is a 2-group since it is contained in $O_2(C_\Gamma(d);\ A)$. By Lemma 5.8.4, B has no fixed points on the generating subgroups of Q so Coprime Action implies that $Q = [Q,\ B]$.

Let $Q_0 = Q \cap M_a$. Then $1 \neq C_{O_2(M_a)}(D) \leqslant Q_0$. By Lemma 5.8.7 $C_Q(B) = 1$ so $C_{Q_0}(B) = 1$. Using Coprime Action and Lemma 5.8.3 we have
$$Q_0 = [Q_0,\ B] \leqslant O_2(M_a).$$

Using Lemma 5.8.8 we have
$$N_Q(Q_0) \leqslant Q \cap M_a = Q_0,$$
and it follows that $Q = Q_0 \leqslant O_2(M_a)$. We deduce that
$$C_{O_2(M_b)}(d) \leqslant O_2(M_a),$$
for all $d \in D^{\#}$.

Since D is non-cyclic, Coprime Action implies that $O_2(M_b) \leqslant O_2(M_a)$. Then $O_2(M_a) \cap M_b \neq 1$ and Lemma 5.8.2 forces $M_a = M_b$. This contradiction completes the proof.

5.9 Proof of the Solvable Signalizer Functor Theorem for Partial Groups

First we collect together some previous results. Recall that $\pi = \pi(F(M_a))$ and $\pi_0 = \pi \smallsetminus \{2\}$.

Lemma 5.9.1. (a) Let $a \in A^{\#}$. Then $[O(M_a),\ a] = 1$.

(b) $\langle A_{exc} \rangle$ is cyclic.

(c) If $e \in A_{exc}$ then $[M_e,\ e] = 1,\ O_2(M_e) = 1$ and $[H,\ e] \leqslant O_2(H)$ for all $H \in \Gamma$.

(d) Let $a \in A \smallsetminus A_{exc}$ and suppose that $O(M_a) \neq 1$. Then $O_2(M_a) \neq 1,\ O_2(O^2([M_a,\ a])) \neq 1$ and $M_a = N_\Gamma(O_2(O^2([M_a,\ a])))$. Moreover if $[M_a,\ a] \leqslant H \in \Gamma$ then $O_2(O^2([M_a,\ a])) \leqslant O_2(H)$.

(e) Let $a \in A \smallsetminus A_{exc}$ and let p be an odd prime. Suppose that $P \in \Gamma$ with $1 \neq P \leqslant O_p(M_a)$. Then $N_\Gamma(P) \leqslant M_a$.

Proof. (a) is Theorem 5.4.1, (b) and (c) follow from Theorem 5.7.1 and Theorem 5.8.1.

(d) Suppose that $O_2(M_a) = 1$. Now $[O(M_a),\ a] = 1$ by (a) and so $[F(M_a),\ a] = 1$. This implies that
$$[M_a,\ a] = 1 \leqslant O_2(M_a).$$
As $O_2(M_a) = 1$, we have $F_{2'}(M_a) \neq 1$ so $a \in A_{exc}$, a contradiction. Hence $O_2(M_a) \neq 1$.

Suppose that $O_2(O^2([M_a,\ a])) = 1$. Now we have
$$\begin{aligned}[O_2(M_a),\ O^2([M_a,\ a])] &\leqslant O_2(M_a) \cap O^2([M_a,\ a]) \\ &\leqslant O_2(O^2([M_a,\ a])) = 1.\end{aligned}$$
Since $[O(M_a),\ a] = 1$, $[M_a,\ a]$ centralizes $O(M_a)$. Then
$$\begin{aligned}O^2([M_a,\ a]) &\leqslant C_{M_a}(F(M_a)) \\ &\leqslant F(M_a).\end{aligned}$$
As $O^2([M_a,\ a])$ is generated by elements of odd order, we have
$$O^2([M_a,\ a]) \leqslant F_{2'}(M_a).$$
By using Coprime Action and Lemma 2.3.13, we get
$$[M_a,\ a] \leqslant O_2(M_a),$$
by applying (a). Thus $a \in A_{exc}$, a contradiction. Finally, we have $[M_a,\ a] \leqslant [H,\ a] \leqslant O_{\pi_0'}(H)$. Then
$$O_2(O^2([M_a,\ a])) \leqslant O_2(H).$$

(e) is Corollary 5.4.4, and the proof is complete.

Now, let e be a generator for $\langle A_{exc}\rangle$. Note that $e=1$ if $A_{exc}=\phi$.

Lemma 5.9.2. For all $a \in A \setminus A_{exc}$,

$$O(M_a)=1.$$

Proof. Assume false and choose a counterexample with $O(M_a)$ maximal. Let p be an odd prime with $O_p(M_a)\neq 1$. By Coprime Action, there exists $B \in \mathrm{Hyp}(A)$
with

$$P := C_{O_p(M_a)}(B) \neq 1.$$

Since $[O(M_a), a\,]=1$, we may suppose that $a \in B$.
Let

$$\mathscr{B} = \{b \in B^{\#} \mid C_{O_2(M_a)}(b)=1 \text{ or } [M_a,\, b] \leqslant O_2(M_a)\}.$$

Suppose that $b \in B \setminus \mathscr{B}$. We claim that $M_a=M_b$. Now $P \leqslant O(M_a)\cap M_b$ and the definition of $\mathscr{B}$ implies that

$$1 \neq C_{O_2(M_a)}(b) \leqslant O_2(M_a)\cap M_b.$$

Lemma 5.2.3 forces $P \leqslant O(M_b)$. Using Lemma 5.9.1 (a) and (e) we have

$$[M_b,\, b] \leqslant C_{M_b}(O(M_b)) \leqslant N_{\Gamma}(P) \leqslant M_a.$$

Since $b \notin \mathscr{B}$ we also have $[M_a,\, b] \nleqslant O_2(M_a)$ so Lemma 5.9.1 (c) and (d) implies that $b \notin A_{exc}$ and then that

$$1 \neq O_2(O^2([M_b,\, b]) \leqslant O_2(M_a).$$

Then

$$O(M_a) \leqslant N_{\Gamma}(O_2(O^2([M_b,\, b])) = M_b,$$

and then Lemma 5.2.3 implies that $O(M_a) \leqslant O(M_b)$. Since $b \in A \setminus A_{exc}$ the choice of a implies that $O(M_a)=O(M_b)$ and so $M_a=M_b$. The claim is established.
Since $a \notin A_{exc}$, Lemma 5.6.2 implies that $a \notin \langle \mathscr{B}\rangle$. Now $a \in B \in \mathrm{Hyp}(A)$ so it follows that $\langle \mathscr{B}\rangle$ is cyclic. The claim and Corollary 5.4.7 imply that

$$O(M_a) \leqslant O(M_d),$$

for all $d \in A^{\#}$.

By Lemma 5.9.1, $\langle A_{exc}\rangle$ is cyclic so we may choose $D \in \mathrm{Hyp}(A)$ with $D \cap A_{exc}=\emptyset$. Let $d \in D^{\#}$. Then $d \in A \setminus A_{exc}$ and $O(M_a) \leqslant O(M_d)$. The choice of a forces $O(M_a)=O(M_d)$ and so $M_a=M_d$. Lemma 5.2.2 supplies a contradiction.

Lemma 5.9.3. Suppose that $P \in \Gamma$ is a non-trivial 2-group. Then $O(N_{\Gamma}(P))=1$.

Proof. Assume false and let P be a maximal counterexample. Let $N=N_{\Gamma}(P)$, so that $O(N)\neq 1$. Now $P \leqslant O_2(N)$ so using Lemma 5.2.3 we have $O(N) \leqslant O(N_{\Gamma}(O_2(N)))$. The choice of P implies that

$$P = O_2(N).$$

By Coprime Action there exists $B \in \mathrm{Hyp}(A)$ with $C_P(B) \neq 1$. Let $b \in B \setminus \langle e \rangle$. Now $1 \neq C_P(B) \leqslant P \cap M_b$ so using Lemma 5.2.3 and Lemma 5.9.2 we have

$$C_{O(N)}(b) \leqslant O(N) \cap M_b \leqslant O(M_b) = 1.$$

Since $O(N) \neq 1$ and B is non-cyclic, it follows from Coprime Action that $1 \neq e \in B$ and that

$$[O(N),\ e] = 1.$$

By Coprime Action,

$$[P,\ e] = \langle [C_P(C),\ e] \mid C \in \mathrm{Hyp}(A) \rangle.$$

The previous paragraph implies that whenever $C \in \mathrm{Hyp}(A)$ and $C_P(C) \neq 1$ then $e \in C$, and hence $[C_P(C),\ e] = 1$. We deduce that $[P,\ e] = 1$. Now $P = O_2(N)$ and $[O(N),\ e] = 1$ so $[F(N),\ e] = 1$ and Coprime Action forces $[N,\ e] = 1$. In particular, e centralizes a Sylow 2-subgroup of N.

Finally choose $a \in A \setminus A_{exc}$. Lemma 5.9.2 implies that $F(M_a) = O_2(M_a) \neq 1$. The previous paragraph and Theorem 3.4.15 imply that $[O_2(M_a),\ e] = 1$. Coprime Action forces $[M_a,\ e] = 1$ and so $M_a = M_e$. However, $O_2(M_e) = 1$ by Lemma 5.9.1. This contradiction complete the proof.

Let

$$\mathscr{L}_2 = \{N_\Gamma(P) \mid P \in \Gamma \text{ is a non-trivial 2-group}\}.$$

Then $\mathscr{L}_2$ is partially ordered by inclusion and we let

$$\mathscr{L}_2{}^*$$

denote the set of maximal members of $\mathscr{L}_2$.

Lemma 5.9.4. $\mathscr{L}_2{}^* \subseteq \Gamma^{**}$.

Proof. We will show that every member of $\mathscr{L}_2{}^*$ is primitive in Γ. Let $N \in \mathscr{L}_2{}^*$, so $O(N) = 1$ by Lemma 5.9.3. Now suppose that $1 \neq X$ char N. Since $F^*(X) \leqslant F^*(N)$ and $F^* = O_2(N)$ by Lemma 5.9.3, it follow that $F^*(X) = O_2(X)$. Also $1 \neq O_2(X)$ char N so

$$N \leqslant N_\Gamma(O_2(X)) \in \mathscr{L}_2.$$

Since $N \in \mathscr{L}_2{}^*$, we get

$$N = N_\Gamma(O_2(X)).$$

As $N_\Gamma(X) \leqslant N_\Gamma(O_2(X))$, we deduce that

$$N_\Gamma(X) = N$$

and then that $N \in \Gamma^{**}$.

Lemma 5.9.5. Let $M \in \mathscr{L}_2{}^*$. If $H \in \mathscr{L}_2{}^*$ with $O_2(M) \cap H \neq 1$ then $M = H$.

Proof. We will apply Lemma 4.3.2. Let

$$\mathscr{L} = \{L \in \mathscr{L}_2{}^* \mid O_2(M) \cap L \neq 1\}.$$

Let $L \in \mathscr{L}$. Then $L \in \Gamma^{**}$ by Lemma 5.9.4, and Lemma 5.9.3 implies that L has characteristic 2. Now $N_\Gamma(O_2(M) \cap L) \in \mathscr{L}_2$ so we may choose N with

$$N_\Gamma(O_2(M) \cap L) \leqslant N \in \mathscr{L}_2{}^*.$$

Then

$$1 \neq O_2(M) \cap L \leqslant O_2(M) \cap N$$

so $N \in \mathscr{L}$. Lemma 4.3.2 implies that $\mathscr{L} = \{M\}$ and the proof is complete.

We may now complete proof of the theorem. First note that if $a \in A \setminus A_{exc}$ then $M_a \in \mathscr{L}_2{}^*$. Choose $a \in A \setminus A_{exc}$, so that $F(M_a) = O_2(M_a) \neq 1$. We claim that there exists $B \in \mathrm{Hyp}(A)$ with $C_{O_2(M_a)}(B) \neq 1$ and $B \cap A_{exc} = \emptyset$.

If $A_{exc} = \emptyset$ then this follows from Coprime Action.

Suppose that $A_{exc} \neq \emptyset$, so that $\langle A_{exc} \rangle = \langle e \rangle$. Now $O_2(M_a) \neq 1$ and $O_2(M_e) = 1$ so $M_a \neq M_e$. Then $[M_a,\, e] \neq 1$ so as $F(M_a) = O_2(M_a)$, Coprime Action implies that

$$[O_2(M_a),\, e] \neq 1.$$

By Coprime Action there exists $B \in \mathrm{Hyp}(A)$ with $[C_{O_2(M_a)}(B),\, e] \neq 1$. Then $e \notin B$ so $B \cap A_{exc} = \emptyset$.

Let $b \in B^\#$. Then

$$b \in A \setminus A_{exc}$$

so

$$M_b \subseteq \mathscr{L}_2{}^*,$$

and

$$1 \neq C_{O_2(M_a)}(B) \leqslant O_2(M_a) \cap M_b.$$

Lemma 5.9.5 implies that

$$M_a = M_b.$$

Since $B \in \mathrm{Hyp}(A)$, Lemma 5.2.2 supplies a contradiction.

Chapter 6

Special Case of Burnside's $p^\alpha q^\beta$-Theorem for Partial Groups

6.1 Introduction

Recall Burnside's $p^\alpha q^\beta$-Theorem for groups which asserts that every group of order $p^\alpha q^\beta$, where p and q are primes, is solvable. We would like to establish an analogue of Burnside's Theorem for partial groups.We do not know yet what the final answer should look like. It is quite easy to prove Burnside's Theorem for groups in which every Sylow subgroup is abelian. So a good place to start with partial groups is to impose the assumption that every group in the partial group with prime power order is abelian. Even with this additional assumption, matters are much more complex than for groups.

We will study a partial group Γ in which every element H of Γ satisfies $\pi(H) \subseteq \{p,\, q\}$ for a fixed pair of primes $p,\; q$ and in which every group in Γ with prime power order is abelian. In particular, we will study the maximal elements of Γ.

6.2 The Main Hypothesis

Throughout the remainder of this chapter, we assume the following:

Hypothesis A

(1) Γ is a partial group.

(2) p and q are distinct primes.

(3) For all $H \in \Gamma$, $\pi(H) \subseteq \{p,\, q\}$.

(4) Every p-group in Γ is abelian.

(5) Every q-group in Γ is abelian.

(6) For all $H \in \Gamma$ if $K \leqslant H$, then $K \in \Gamma$.

The elements of Γ^* divide into three types as in the following definition.

Definition 6.2.1. Let $M \in \Gamma^*$, then we have

- M has type p if $\pi(F(M)) = \{p\}$.
- M has type q if $\pi(F(M)) = \{q\}$.
- M has type pq if $\pi(F(M)) = \{p, q\}$.

Here are some examples and basic results.

Example 6.2.2. Let Γ be the set of all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_3$. Then Γ satisfies Hypothesis A.

(1) It is clear that Γ is a partial group.

(2) 2 and 3 are distinct primes.

(3) For all $H \in \Gamma$, $\pi(H) \subseteq \{2, 3\}$.

(3) Every 2-group is abelian.

(4) Every 3-group is abelian.

(5) For all $H \in \Gamma$ if $1 \neq K \leqslant H$, then $K \in \Gamma$.

Example 6.2.3. Let Γ be the set of all proper subgroups of S_3 . Then Γ satisfies Hypothesis A and is disconnected.

Example 6.2.4. Let Γ be the set of all proper subgroups of A_5. Then Γ satisfies Hypothesis A and is connected.

Proposition 6.2.5. There exists a connected partial group that satisfies Hypothesis A and has more than one maximal member.

Proof. We can use Example 6.2.4 as an example.

Now, we begin with some elementary observations. We recall that $\mathrm{Syl}_p\ \Gamma$ denotes the set of all maximal p-groups in Γ.

Lemma 6.2.6. Let $M \in \Gamma^*$ and suppose that $O_p(M) \neq 1$. If $P \in \mathrm{Syl}_p M$, then $P \in \mathrm{Syl}_p \Gamma$.

Proof. Let P_1 be a p-group with $P \leqslant P_1 \in \mathrm{Syl}_p \Gamma$. Since the p-groups in Γ are abelian, we

have
$$P_1 \leqslant C_\Gamma(P) \leqslant C_\Gamma(O_p(M)) \leqslant M.$$

Now, we have
$$P \in \mathrm{Syl}_p M$$
as
$$P \leqslant P_1 \leqslant M.$$
Then, we have $P = P_1$.

Lemma 6.2.7. Let $M \in \Gamma^*$ and suppose that $O_q(M) \neq 1$. If $Q \in \mathrm{Syl}_q M$, then $Q \in \mathrm{Syl}_q \Gamma$.

Proof. Same as the proof of the above lemma.

Proposition 6.2.8. Let M and $L \in \Gamma^*$ have type $r \in \{p,\ q\}$. Suppose that $Q \in \mathrm{Syl}_q M$ and $P \in \mathrm{Syl}_p M$. Then

(1) If $r = p$, then $M = QO_p(M)$ and $C_\Gamma(O_p(M)) = O_p(M)$.

(2) If $r = q$, then $M = PO_q(M)$ and $C_\Gamma(O_q(M)) = O_q(M)$.

(3) If $F(M) \leqslant L$ and $F(L) \leqslant M$, then $M = L$.

Proof. (1) Let $P \in \mathrm{Syl}_p M$ and $Q \in \mathrm{Syl}_q M$, so $M = PQ$ since $\pi(M) \subseteq \{p,\ q\}$. Since M has type p and P is abelian, we have
$$P \leqslant C_M(O_p(M)) \leqslant O_p(M)$$
and so,
$$P = O_p(M).$$
Thus, we have
$$M = QO_p(M).$$
Also, as $M \in \Gamma^*$ and $O_p(M) \neq 1$, we have
$$C_\Gamma(O_p(M)) \leqslant N_\Gamma(O_p(M)) = M$$
and so,
$$C_\Gamma(O_p(M)) = C_M(O_p(M)) = O_p(M).$$

(2) The proof is similar to (1).

(3) In the case $r = p$, we know $F(M) = O_p(M)$ and $F(L) = O_p(L)$. Also, we have
$$O_p(M) \in \mathrm{Syl}_p M$$
and so,
$$O_p(L) \in \mathrm{Syl}_p L.$$

This implies that

$$O_p(M) \leqslant O_p(L)$$

and

$$O_p(L) \leqslant O_p(M).$$

Then, we have

$$O_p(M) = O_p(L)$$

and so,

$$M = N_\Gamma(O_p(M)) = N_\Gamma(O_p(L)) = L$$

In the case $r = q$, we get the same result by the same argument.

Lemma 6.2.9. Suppose that $M \in \Gamma^*$ has type pq and that $F(M) \leqslant L \in \Gamma^*$. Then $M = L$.

Proof. We have $M = N_\Gamma(O_p(M))$. Then $N_L(O_p(M)) = M \cap L$. But $O_q(M) \trianglelefteq M \cap L$ and by applying Lemma 2.3.10, we get

$$1 \neq O_q(M) \leqslant O_q(N_L(O_p(M))) \leqslant O_q(L). \tag{$*$}$$

We have used the fact that if $\pi(X) \subseteq \{p,\ q\}$, then $O_{p'}(X) = O_q(X)$. The Sylow q-subgroups of L are abelian and $O_q(M)O_q(L)$ is a q-subgroup of L so

$$O_q(L) \leqslant C_\Gamma(O_q(M)) \leqslant M.$$

Similarly, we get $O_p(L) \leqslant M$. Then, we have

$$F(L) \leqslant M.$$

Interchanging M and L, $(*)$ becomes

$$1 \neq O_q(L) \leqslant O_q(M). \tag{$**$}$$

From $(*)$ and $(**)$, we get

$$O_q(L) = O_q(M)$$

and so,

$$M = N_\Gamma(O_q(M)) = N_\Gamma(O_q(L)) = L.$$

Corollary 6.2.10. Suppose that $M \in \Gamma^*$ has type pq and $1 \neq a \in F(M)$. Then $C_\Gamma(a) \leqslant M$.

Proof. Note that $\langle a \rangle \in \Gamma$ by the part (6) of Hypothesis A. Now, we have $F(M) = C_M(F(M)) \leqslant C_\Gamma(a)$. Let $L \in \Gamma^*$ with $C_\Gamma(a) \leqslant L$. Then

$$F(M) \leqslant L$$

and so by the above lemma, we get

$$M = L.$$

Lemma 6.2.11. Let $M,\ L \in \Gamma^*$ and suppose that $F(M) \leqslant L$ and $M \neq L$. Then one of the following holds.

(a) M has type p, $O_p(M) \in \mathrm{Syl}_p\Gamma$, $L = O_p(M)O_q(L)$, $O_q(L) \in \mathrm{Syl}_q\Gamma$ and $L = N_\Gamma(O_q(L))$.

(b) M has type q, $O_q(M) \in \mathrm{Syl}_q\Gamma$, $L = O_q(M)O_p(L), O_p(L) \in \mathrm{Syl}_pL$ and $L = N_\Gamma(O_p(L))$.

Proof. (a) Lemma 6.2.9 implies M does not have type pq.
Suppose that M has type p. Choose $Q_0 \in \mathrm{Syl}_q(M \cap L)$ and Q with $Q_0 \leqslant Q \in \mathrm{Syl}_qM$. By applying Proposition 6.2.8, we have

$$M = QO_p(M) \text{ and } C_M(O_p(M)) = O_p(M)$$

and so,

$$[Q_0,\ O_p(M)] \leqslant O_p(M).$$

Next we claim that $[Q_0,\ O_p(M)] \trianglelefteq M$.

Now $O_p(M)$ normalizes $[Q_0,\ O_p(M)]$. Also $Q_0 \leqslant Q$ and Q is abelian so Q normalizes Q_0. Trivially Q normalizes $O_p(M)$, hence Q normalizes $[Q_0,\ O_p(M)]$. Since $M = QO_p(M)$, the claim follows.
Also Lemma 2.2.62 implies that

$$[Q_0,\ O_p(M)] \leqslant F(L).$$

Then

$$[Q_0,\ O_p(M)] \leqslant O_p(L).$$

Suppose that $Q_0 \neq 1$. Then, since $C_M(O_p(M)) = O_p(M)$ we have,

$$1 \neq [Q_0,\ O_p(M)] \subseteq O_p(L).$$

Since $[Q_0,\ O_p(M)] \trianglelefteq M$, we have

$$N_\Gamma([Q_0,\ O_p(M)]) = M.$$

Then, we get

$$F(L) \leqslant M.$$

Proposition 6.2.8 implies that L has type p and so $F(L) = O_p(L) \in \mathrm{Syl}_pL \subseteq \mathrm{Syl}_p\Gamma$.
Also we have,

$$O_p(L) \leqslant M.$$

This implies that

$$O_p(L) \in \mathrm{Syl}_pM.$$

Now, since M has type p, we have $O_p(L) = O_p(M)$ and so $L = M$, a contradiction. We deduce that

$$Q_0 = 1 \text{ and so } M \cap L \text{ is a } p\text{-group.}$$

Now, we have $O_p(M) \leqslant M \cap L,\ O_p(M) \in \mathrm{Syl}_p M \subseteq \mathrm{Syl}_p \Gamma$ and so $M \cap L = O_p(M),\ O_p(M) \in \mathrm{Syl}_p L$ and so $N_\Gamma((O_p(M)) = M$.

Now choose $Q_1 \in \mathrm{Syl}_q L$ and so $L = Q_1 O_p(M)$. Then by Lemma 2.2.62, we have

$$[Q_1,\ O_p(M)] \leqslant F(L).$$

This implies that

$$F(L)O_p(M) \trianglelefteq L.$$

In particular,

$$O_q(L)O_p(L)O_p(M) \trianglelefteq L.$$

Since $O_p(L) \subseteq O_p(M)$, we have

$$O_q(L)O_p(M) \trianglelefteq L$$

By Lemma 2.2.6, we get

$$L = O_q(L)O_p(M)N_L(O_p(M)).$$

Also, we have

$$O_p(M)N_L(O_p(M)) \leqslant M \cap L = O_p(M).$$

Then, we get

$$L = O_q(L)O_p(M).$$

Since $M \neq L$, $O_q(L) \neq 1$, $O_q(L) \in \mathrm{Syl}_q \Gamma$ and $L = N_\Gamma(Q_q(L))$.

(b) By the same argument in part (a), we get the result.

6.3 Elements of Type pq

Throughout this part of the section we assume the following:

Hypothesis B

- $M \in \Gamma^*$ has type pq.
- $O_p(M)$ is non-cyclic.

Lemma 6.3.1. Suppose $Q \in \Gamma$ is a q-group with $O_p(M) \leqslant N_\Gamma(Q)$. Then $Q \leqslant M$.

Proof. Since $O_p(M)$ is a non-cyclic abelian group that acts on Q, by Lemma 2.3.4 (b), we have

$$Q = \langle C_Q(a) \mid 1 \neq a \in O_p(M) \rangle.$$

By Corollary 6.2.10, we get
$$Q \leqslant M.$$

Lemma 6.3.2. Let $P \in \mathrm{Syl}_p M$ and $P \leqslant L \in \Gamma^*$. Then $M = L$ or $L = N_\Gamma(P)$.

Proof. By Lemma 6.3.1, we have
$$O_q(L) \leqslant M.$$
Also by Lemma 6.2.6, we get $P \in \mathrm{Syl}_p\Gamma$. By assumption $P \leqslant L$ so $P \in \mathrm{Syl}_p L$. Then, we have
$$O_p(L) \leqslant P \leqslant M.$$
Therefore,
$$F(L) \leqslant M.$$

Suppose $\pi(F(L)) = \{p\}$ then L has type p, $P \trianglelefteq L$ and $L = N_\Gamma(P)$.

Suppose $\pi(F(L)) = \{q\}$. By Lemma 6.3.1, we have $O_q(L) \leqslant M$. By assumption $O_p(M) \leqslant P \leqslant L$, hence $[O_q(L),\, O_p(M)] \leqslant O_q(L) \cap O_p(M) = 1$. This implies that $O_p(M) \leqslant C_\Gamma(O_q(L)) = O_q(L)$, a contradiction.

If $\pi(F(L)) = \{p,\, q\}$, then Lemma 6.2.9 implies that $M = L$.

Lemma 6.3.3. Let $P \in \mathrm{Syl}_p(M)$ and $P_1 \in \Gamma$ is a p-group. Suppose that $P \cap P_1 \neq 1$. Then $P_1 \leqslant M$.

Proof. We have,
$$P_1 \leqslant C_\Gamma(P \cap P_1).$$

Also,
$$P \leqslant C_\Gamma(P \cap P_1).$$

Choose $L \in \Gamma^*$ such that $C_\Gamma(P \cap P_1) \leqslant L$ then P, $P_1 \leqslant L$. Now, we have
$$P \in \mathrm{Syl}_p(M)$$
and so,
$$P \leqslant L.$$
Then we have,
$$M = L \text{ or } N_\Gamma(P) = L.$$
Suppose that $M = L$. Then $P_1 \leqslant M$. (Because $P_1 \leqslant L$).

Suppose that $N_\Gamma(P) = L$. Now, $P \in \mathrm{Syl}_p M$ and so $P \in \mathrm{Syl}_p\Gamma$, hence $P \in \mathrm{Syl}_p L$. As $P \trianglelefteq L$, it follows that $P_1 \leqslant P$. Then $P_1 \leqslant M$.

Corollary 6.3.4. Let $P \in \mathrm{Syl}_p M$ and $x \in P^{\#}$. Then $C_\Gamma(x) \leqslant M$ or $C_\Gamma(x) \leqslant N_\Gamma(P)$.

Proof. Choose $L \in \Gamma^*$ with $C_\Gamma(x) \leqslant L$. Since $x \in P^{\#}$, we have

$$P \leqslant C_\Gamma(x) \leqslant L.$$

By Lemma 6.3.2, we get

$$L = M \text{ or } L = N_\Gamma(P).$$

If $L = M$, then $C_\Gamma(x) \leqslant M$. If $L = N_\Gamma(P)$, then $C_\Gamma(x) \leqslant N_\Gamma(P)$. This completes the proof.

Proposition 6.3.5. Let $L \in \Gamma^*$. Suppose that $M \cap L$ has order divisible by p. Then we have the following:

(1) If L has type pq, then $M = L$.

(2) If L has type q, then $M \cap L$ has cyclic Sylow p-subgroups.

(3) If L has type p, then $L = N_\Gamma(P)$ for some $P \in \mathrm{Syl}_p M$.

Proof. (1) Choose $P_0 \in \mathrm{Syl}_p(M \cap L)$. Then $P_0 \neq 1$.
Now choose P_1 and P_2 with $P_0 \leqslant P_1 \in \mathrm{Syl}_p M$ and $P_0 \leqslant P_2 \in \mathrm{Syl}_p L$. Then, we have

$$P_1 \cap P_2 \neq 1$$

and so by Lemma 6.3.3, we get

$$P_2 \leqslant M.$$

Now, we have $O_p(L) \neq 1$ so $P_2 \in \mathrm{Syl}_p \Gamma$ and so $P_2 \in \mathrm{Syl}_p M$. In particular, we have

$$O_p(M) \leqslant P_2.$$

As $P_2 \leqslant L$ and by Lemma 6.3.2, we get

$$M = L \text{ or } L = N_\Gamma(P_2).$$

If $M = L$, there is nothing further to prove.

Assume that $L = N_\Gamma(P_2)$, $P_2 = O_p(L)$. Now, we have

$$[P_2,\ O_q(L)] \leqslant P_2 \cap O_q(L) = 1.$$

As $O_p(M) \leqslant P_2$, we have

$$O_q(L) \leqslant C_\Gamma(O_p(M)) \leqslant M.$$

Then, we have

$$F(L) \leqslant M$$

and so by Lemma 6.2.9, we get

$$M = L.$$

(2) Suppose that $M \cap L$ has a non-cyclic Sylow p-subgroups. Since $|M \cap L|_p \neq 1$, there exists a $P_0 \in \mathrm{Syl}_p L$ with $P_0 \cap M \neq 1$. Then Lemma 6.3.3 implies that

$$P_0 \leqslant M.$$

Choose P with $P_0 \leqslant P \in \mathrm{Syl}_p M$. Also, since L has type q, we have $L = P_0 O_q(L)$.
Consider the action of P_0 on $O_q(L)$. By Coprime Action, we have

$$[O_q(L),\, P_0] = \langle [C_{O_q(L)}(x),\, P_0] \mid x \in P_0^{\#} \rangle.$$

Let $x \in P_0^{\#}$ and so by Corollary 6.3.4, we have

$$C_\Gamma(x) \leqslant M \text{ or } C_\Gamma(x) \leqslant N_\Gamma(P).$$

In the first case, we have $[C_{O_q(L)}(x),\, P_0] \leqslant M$.

In the second case, we have $[C_{O_q(L)}(x),\, P_0] \leqslant O_q(L) \cap P = 1 \leqslant M$. We deduce that,

$$[O_q(L),\, P_0] \leqslant M.$$

Since M has type pq and so by applying Coprime Action and Lemma 2.2.62, we have

$$[O_q(L),\, P_0] = [O_q(L),\, P_0,\, P_0] \leqslant F(M).$$

Since $[O_q(L),\, P_0]$ is a q-group, L has type q and $P_0 \neq 1$, we have

$$1 \neq [O_q(L),\, P_0] \leqslant O_q(M).$$

Recall $L = P_0 O_q(L)$ so $[P_0,\, O_q(L)] \trianglelefteq L$. Then, we have

$$O_q(M) \leqslant N_\Gamma([O_q(L),\, P_0]) = L.$$

Since $L = P_0 O_q(L)$, L has only one Sylow q-subgroup, $O_q(L)$. Hence,

$$O_q(M) \leqslant O_q(L).$$

Now, by Hypothesis B, M has type pq so

$$O_q(M) \neq 1$$

and

$$N_\Gamma(O_q(M)) = M.$$

Then, we have

$$O_q(L) \leqslant M.$$

But, $L = P_0 O_q(L) \leqslant M$ and so $M = L$. This is a contradiction because M has type pq and L has type q. We deduce that $M \cap L$ has cyclic Sylow p-subgroups.

(3) By Lemma 6.2.6, we have $O_p(L) \in \mathrm{Syl}_p \Gamma$. Since $O_p(L)$ is the only Sylow p-subgroup of L, we get

$$O_p(L) \cap M \neq 1.$$

Then by Lemma 6.3.3, we have

$$O_p(L) \leqslant M.$$

Since $O_p(L) \in \mathrm{Syl}_p \Gamma$ and $O_p(L) \leqslant M$, we get

$$O_p(L) \in \mathrm{Syl}_p M.$$

As L has type p we get $L = N_\Gamma(O_p(L))$. Put $P = O_p(L)$.

Corollary 6.3.6. Suppose $L \in \Gamma^*$ has type q and that $|M \cap L|_p \neq 1$. Then, there exists $P_0 \in \mathrm{Syl}_p(L \cap M)$ with $L = P_0 O_q(L)$, P_0 is cyclic and $M \cap O_q(L) = 1$.

Proof. By Proposition 6.3.5 (2), we can choose $P_0 \in \mathrm{Syl}_p(M \cap L)$ which is a cyclic p-subgroup of $M \cap L$. Then, we have $P_0 \leqslant M$ and P_0 is cyclic. Now, choose P_1 with $P_0 \leqslant P_1 \in \mathrm{Syl}_p L$. Then, we have

$$M \cap P_1 \neq 1,$$

and so

$$P_\cap P_1 \neq 1,$$

for some $P \in \mathrm{Syl}_p M$. By applying Lemma 6.3.3, we get

$$P_1 \leqslant M.$$

Thus, we have

$$P_0 \leqslant P_1 \leqslant M \cap L$$

so as $P_0 \in \mathrm{Syl}_p(M \cap L)$, we get

$$P_0 = P_1 \in \mathrm{Syl}_P L.$$

As L has type q, we have

$$L = P_0 O_q(L).$$

Now, suppose that $M \cap O_q(L) \neq 1$. Then, we have $O_q(L) \leqslant M$ and so $F(L) \leqslant M$. Lemma 6.2.11 implies that M has type q, a contradiction.

6.4 Elements of Type q

Every result of this part of the section can be applied to elements of type p.

Lemma 6.4.1. Let $M \in \Gamma^*$ have type q and suppose that $P \in \mathrm{Syl}_p M$ is non-cyclic. Then $P \in \mathrm{Syl}_p \Gamma$.

Proof. We have $P \in \mathrm{Syl}_p M$ and so $M = PO_q(M)$ and $O_q(M) \in \mathrm{Syl}_q \Gamma$ by Lemma 6.2.7. Since P is non-cyclic, we have

$$1 \neq [O_q(M),\, P] = \langle [C_{O_q(M)}(y),\, P] \,|\, y \in P^\# \rangle.$$

Choose $y \in P^\#$ with $V := [C_{O_q(M)}(y),\, P] \neq 1$.
Choose H with $C_\Gamma(y) \leqslant H \in \Gamma^*$. Then, we have

$$V \leqslant H.$$

Also, by Lemma 2.2.62 and Coprime Action, we have

$$V = [V,\ P] \leqslant F(H).$$

Since V is a q-group, we have

$$V = [V,\ P] \leqslant O_q(H).$$

Now, $M = PO_q(M)$ and $O_q(M)$ is abelian, so $V = [C_{O_q(M)}(y),\ P] \trianglelefteq M$. Then, we have

$$N_\Gamma(V) = M.$$

Thus, we have

$$F(H) \leqslant M.$$

Lemma 6.2.11 implies that $O_p(H) = 1$ and so H has type q. Now, $O_q(H) \in \mathrm{Syl}_q\Gamma$ and so $O_q(H) \in \mathrm{Syl}_q M$. Note that since M has type q, it has exactly one Sylow q-subgroup, $O_q(M)$. Then, we have

$$O_q(H) = O_q(M)$$

and so we get

$$H = M$$

Therefore,

$$C_\Gamma(y) \leqslant M.$$

Choose P^* with $P \leqslant P^* \in \mathrm{Syl}_p\Gamma$. This implies that

$$P^* \leqslant C_\Gamma(y) \leqslant M.$$

Thus, we have

$$P \leqslant P^* \leqslant M$$

and so,

$$P = P^*.$$

Then, we have

$$P \in \mathrm{Syl}_p\Gamma,$$

which completes the proof.

Lemma 6.4.2. Let $M \in \Gamma^*$ and suppose that M has type q. Let $P \in \mathrm{Syl}_p M$ and suppose that rank $(P) \geqslant 3$.

(1) Then there exists $\mathbb{Z}_p \times \mathbb{Z}_p \cong B \leqslant P$ such that $C_\Gamma(b) \leqslant M$ for all $b \in B^\#$ and $P \in \mathrm{Syl}_p\Gamma$.

(2) If $P \leqslant L \in \Gamma^*$ with $L \neq M$, then $L = N_\Gamma(P)$.

(3) If $a \in P^\#$ then $C_\Gamma(a) \leqslant M$ or $C_\Gamma(a) \leqslant N_\Gamma(P)$.

(4) If $P_1 \in \Gamma$ is a p-group and $P_1 \cap M \neq 1$, then $P_1 \leqslant M$.

Proof. (1) Since M has type q we have $M = PO_q(M)$ with $P \in \mathrm{Syl}_p M$ and $O_q(M) \in \mathrm{Syl}_q\Gamma$. Also by Coprime Action we have,

$$O_q(M) = \langle C_{O_q(M)}\ (B) \mid B \leqslant P,\ B \text{ elementary abelian subgroup and rank } (B) = 2\rangle.$$

Now, $[P,\ O_q(M)] \neq 1$ and so there exists $B \leqslant P$ with $B \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $1 \neq [P,\ C_{O_q(M)}(B)] \trianglelefteq M$. Note the commutator is normal of M because $M = PO_q(M)$ and $O_q(M)$ is abelian. Let $b \in B^{\#}$ and choose N with $C_\Gamma(b) \leqslant N \in \Gamma^*$. Then

$$1 \neq [P,\ C_{O_q(M)}(B)] \leqslant O_q(N).$$

This implies that

$$F(N) \leqslant N_\Gamma([P,\ C_{O_q(M)}(B)]) = M.$$

If N has type pq, then $N = M$ and so $C_\Gamma(b) \leqslant M$.

Suppose N does not have type pq. Now, $O_q(N) \neq 1$ and so N has type q. Then $O_q(N) \in \mathrm{Syl}_q\Gamma$ and so $O_q(N) \in \mathrm{Syl}_q M$. Since M has type q, we have $O_q(N) = O_q(M)$. Then, we have

$$N = M$$

and so,

$$C_\Gamma(b) \leqslant M.$$

We deduce that

$$C_\Gamma(b) \leqslant M$$

for all $b \in B^{\#}$.

(2) The result follows from part (1) and Lemma 6.3.2.

(3) Now, we have $P \leqslant L \in \Gamma^*$ with

$$L \neq M.$$

Let B denote the subgroup whose existence was proved in (1). Now, by Coprime Action we have

$$O_q(L) = \langle C_{O_q(L)}(b) \mid b \in B^{\#}\rangle \leqslant M.$$

Then, we get

$$O_q(L) \leqslant O_q(M).$$

If $O_q(L) \neq 1$, then $O_q(M) \leqslant N_\Gamma(O_q(L)) = L$, then $M = PO_q(M) \leqslant L$, a contradiction. Thus, we have $O_q(L) = 1$ and so $O_p(L) \in \mathrm{Syl}_p L$, $P = O_p(L) \in \mathrm{Syl}_p\Gamma$ and $L = N_\Gamma(P)$.

(4) Now, let $P_1 \in \Gamma$ be a p-group and suppose that $P_1 \cap M \neq 1$. With out loss of generality $P_1 \cap M \leqslant P$ and so $P_1 \leqslant C_\Gamma(P_1 \cap M)$. Choose N with $C_\Gamma(P_1 \cap M) \leqslant N \in \Gamma^*$. If $N = M$, then $P_1 \leqslant M$.

If not, then $N = N_\Gamma(P)$ by (2). But $P \in \mathrm{Syl}_p N$. Then $P_1 \leqslant P \leqslant M$. Thus, we have if $P_1 \cap M \neq 1$

with P_1 a p-subgroup, then $P_1 \leqslant M$.

Lemma 6.4.3. Let $M \in \Gamma^*$ and suppose M has type q. Let $P \in \mathrm{Syl}_p M$ and suppose rank $(P) \geqslant 3$. Let $L \in \Gamma^*$ and suppose that $M \cap L$ has non-cyclic Sylow p-subgroups. Then $M = L$ or $L = N_\Gamma(P_1)$ for some $P_1 \in \mathrm{Syl}_p M$.

Proof. Assume that $M \neq L$.
Choose $P_0 \in \mathrm{Syl}_p(M \cap L)$ and P_1 with $P_0 \leqslant P_1 \in \mathrm{Syl}_p L$. Now, $1 \neq P_0 \leqslant P_1 \cap M$ so Lemma 6.4.2 (4) implies $P_1 \leqslant M$. Hence, we have

$$P_0 \leqslant P_1 \leqslant M \cap L$$

so as $P_0 \in \mathrm{Syl}_p(M \cap L)$, we get

$$P_0 = P_1 \in \mathrm{Syl}_p L.$$

Without loss of generality $P_1 \leqslant P$.
Suppose that $O_p(L) \neq 1$. Then, we have

$$O_p(L) \leqslant P_1 \leqslant P,$$

and so

$$P \leqslant N_\Gamma(O_p(L)) = L.$$

Then Lemma 6.4.2 (2) implies

$$L = N_\Gamma(P).$$

This is the desired outcome. Hence we may assume that $O_p(L) = 1$. Then L has type q.
Since L has type q, we have

$$L = O_q(L)P_1$$

and

$$[O_q(L),\, P_1] \neq 1.$$

Since $M \cap L$ has non-cyclic Sylow p-subgroups, P_1 is non-cyclic so Coprime Action gives

$$[O_q(L),\, P_1] = \langle [C_{O_q(L)}(a),\, P_1 \mid a \in P_1^{\#}] \rangle.$$

Let $a \in P_1^{\#}$. If $C_\Gamma(a) \leqslant N_\Gamma(P)$, then

$$[C_{O_q(L)}(a),\, P_1] \leqslant P \cap O_q(L) = 1.$$

Then (3) implies

$$[O_q(L),\, P_1] \leqslant M.$$

Also $P_1 \leqslant M$ so using Coprime Action and Lemma 2.2.62, we get

$$1 \neq [O_q(L),\, P_1] = [O_q(L),\, P_1,\, P_1] \leqslant F(M).$$

Since $[O_q(L),\, P_1]$ is a q-group, we get

$$1 \neq [O_q(L),\, P_1] \leqslant O_q(M).$$

Also, as $[O_q(L),\, P_1] \trianglelefteq O_q(L)P_1 = L$, we have

$$L = N_\Gamma([O_q(L),\, P_1]).$$

As $O_q(M)$ is abelian, we get

$$O_q(M) \leqslant L.$$

As $L = O_q(L)P_1$ this implies

$$O_q(M) \leqslant O_q(L).$$

Also,

$$O_q(L) \leqslant N_\Gamma(O_q(M)) = M,$$

and again as M has type q, we have $O_q(M) \in \mathrm{Syl}_q M$ and so $O_q(L) \leqslant O_q(M)$. Then

$$O_q(L) = O_q(M)$$

and so

$$L = N_\Gamma(O_q(L)) = N_\Gamma(O_q(M)) = M.$$

We assumed $M \neq L$, so this contradiction complete the proof.

6.5 q-Groups Normalized By $O_p(M)$ When M Has Type p

Throughout this section we assume the following

Hypothesis C

- $M \in \Gamma^*$ and M has type p.
- $O_p(M)$ is elementary abelian and rank $(O_p(M)) \geqslant 3$.

In what follows we let $P = O_p(M)$. We will study q-groups in Γ that are normalized by P.

Definition 6.5.1. We define

$$\Omega(P) = \{Q \in \Gamma \mid Q \text{ is a } q\text{-group and } P \leqslant N_\Gamma(Q)\}.$$

We denote $\Omega^*(P)$ to be the set of maximal elements of $\Omega(P)$.

Now, we begin with the following elementary results.

Lemma 6.5.2. Assume Hypothesis C. Suppose that $Q \in \Omega(P)$. Then

(1) $Q \cap M = 1$.

(2) $C_Q(P) = 1$.

(3) $Q = [Q,\ P]$.

Proof. (1) We have,

$$[P,\ Q \cap M] \leqslant P \cap Q = 1.$$

Then

$$[P,\ Q \cap M] = 1.$$

Note that $P = O_p(M)$, $\pi(F(M)) = \{p\}$ so $C_M(P) = P$. Then

$$Q \cap M \leqslant C_M(P) = P.$$

Thus, we get

$$Q \cap M = 1$$

because Q is a q-group.

(2) We have,

$$C_Q(P) \leqslant Q \cap M.$$

By (1), we get

$$C_Q(P) = 1.$$

(3) Since $P \leqslant N_\Gamma(Q)$, Coprime Action implies

$$Q = [Q,\ P]C_Q(P).$$

By applying part (1), we get

$$Q = [Q,\ P].$$

Lemma 6.5.3. Assume Hypothesis C. Suppose that $Q_1,\ Q_2 \in \Omega^*(P)$ and that $Q_1 \cap Q_2 \neq 1$. Then $Q_1 = Q_2$.

Proof. Note we can prove this lemma without the hypothesis that the q-groups in Γ are abelian. Assume false and choose a counterexample $Q_1,\ Q_2$ with $|Q_1 \cap Q_2|$ as large as possible. Note that Γ is a finite partial group so this choice is possible. Let $N = N_\Gamma(Q_1 \cap Q_2)$. Let $Q_1^* = N_{Q_1}(Q_1 \cap Q_2)$. Then, we have

$$Q_1^* > Q_1 \cap Q_2.$$

Now, we have

$$C_{Q_1^*}(P) \leqslant C_{Q_1}(P) = 1.$$

Coprime Action implies that

$$Q_1^* = [Q_1^*,\ P].$$

By applying Lemma 2.2.62, we get

$$[Q_1^*,\ P] \leqslant F(N)$$

and so

$$Q_1^* \leqslant O_q(N).$$

Since $P \leqslant N$, we have

$$O_q(N) \in \Omega^*(P).$$

Choose T with

$$O_q(N) \leqslant T \in \Omega^*(P).$$

Then, we have
$$Q_1 \cap Q_2 < Q_1^* \leqslant Q_1 \cap O_q(N) \leqslant Q_1 \cap T.$$
The choice of Q_1 and Q_2 implies that
$$Q_1 = T.$$
Similarly, we get
$$Q_2 = T.$$
Hence, we have
$$Q_1 = Q_2,$$
a contradiction.

Lemma 6.5.4. Assume Hypothesis C. Then $\Omega(P)$ has a unique maximal element.

Proof. Choose $Q \in \Omega^*(P)$. Since rank $(P) \geqslant 3$, Coprime Action implies
$$Q = \{C_Q(B) \mid B < P, \text{ and rank } (B) \geqslant 2\}.$$
Hence, we may choose $B < P$ with rank $(B) \geqslant 2$ and
$$C_Q(B) \neq 1.$$
Now, choose $Q^* \in \Omega^*(P)$. Since rank $(B) \geqslant 2$, we have
$$Q^* = \langle C_{Q^*}(b) \mid b \in B^{\#} \rangle,$$
so there exists $b \in B^{\#}$ such that
$$C_{Q^*}(b) \neq 1.$$
Let $C = C_\Gamma(b)$. Then, we have
$$1 \neq C_{Q^*}(b) \leqslant C$$
and
$$1 \neq C_Q(B) \leqslant C_Q(b) \leqslant C.$$
Since $b \in P$ and P is abelian, we have
$$P \leqslant C.$$
Now, we have
$$C_{Q^*}(b) \in \Omega(P),$$
and so Lemma 6.5.2 implies that
$$C_{Q^*}(b) \leqslant O_q(C).$$
Also, we have
$$C_Q(B) \leqslant O_q(C) \in \Omega(P).$$
Choose Q^{**} with $O_q(C) \leqslant Q^{**} \in \Omega^*(P)$. Then, we have
$$1 \neq C_{Q^*}(b) \leqslant Q^* \cap Q^{**},$$
and so Lemma 6.5.3 implies that
$$Q^* = Q^{**}.$$
Also, we have
$$1 \neq C_Q(B) \leqslant Q \cap Q^{**}$$

and so Lemma 6.5.3 implies that

$$Q = Q^{**}.$$

Thus, we get

$$Q = Q^{*}.$$

Lemma 6.5.5. Assume Hypothesis C. Let $\Omega^*(P) = Q$. Then $N_\Gamma(Q) \in \Gamma^*$.

Proof. Choose L with $N_\Gamma(Q) \leqslant L \in \Gamma^*$. Then

$$P,\ Q \leqslant L.$$

By applying Lemma 6.5.2 (2) and Lemma 2.2.62, we have

$$Q = [Q,\ P] \leqslant F(L).$$

Then

$$Q \leqslant O_q(L).$$

Thus, we have

$$O_q(L) \in \Omega(P)$$

and so

$$O_q(L) \leqslant Q.$$

Then, we get

$$Q = O_q(L)$$

and so

$$L = N_\Gamma(O_q(L)) = N_\Gamma(Q).$$

Lemma 6.5.6. Assume Hypothesis C. Let $Q \in \Omega(P)$ and suppose that $P \leqslant L \in \Gamma^*$. Then either

$$L = M \text{ or } L = N_\Gamma(Q).$$

Proof. Since $P \in \mathrm{Syl}_p\Gamma$, $P \in \mathrm{Syl}_p L$.
If $O_q(L) = 1$, then L has type p and so P is the only Sylow p-subgroup of L. Then

$$L = N_\Gamma(P) = M.$$

If $O_q(L) \neq 1$, then

$$O_q(L) \leqslant Q \in \Omega(P).$$

Thus, we have

$$Q \leqslant N_\Gamma(O_q(L)) = L.$$

Also, we have

$$Q = [P,\ Q] \leqslant O_q(L) \leqslant Q.$$

Then, we get

$$O_q(L) = Q$$

and so

$$L = N_\Gamma(Q).$$

Corollary 6.5.7. Assume Hypothesis C. Let $Q \in \Omega(P)$. Let $x \in P^{\#}$. Then either

$$C_\Gamma(x) \leqslant M \text{ or } C_\Gamma(x) \leqslant N_\Gamma(Q).$$

Proof. Choose H with $C_\Gamma(x) \leqslant H \in \Gamma^*$. Then

$$P \leqslant C_\Gamma(x) \leqslant H.$$

Then the above lemma implies that

$$H = M \text{ or } H = N_\Gamma(Q).$$

Lemma 6.5.8. Assume Hypothesis C. Let $Q \in \Omega(P)$ and $L \in \Gamma^*$. Suppose that $L \cap M$ contains a p-subgroup $B \leqslant P$ of rank $\geqslant 2$. The either

$$L = N_\Gamma(Q) \text{ or } L = N_\Gamma(P_1),$$

for some $P_1 \in \mathrm{Syl}_p N_\Gamma(Q)$.

Proof. Suppose that $O_q(L) \neq 1$. Then

$$[O_q(L),\, B] = \langle [C_{O_q(L)}(b),\, B] \mid b \in B^{\#} \rangle.$$

Let $b \in B^{\#}$. Then by Corollary 6.5.7, we have

$$C_\Gamma(b) \leqslant M \text{ or } C_\Gamma(b) \leqslant N_\Gamma(Q).$$

If $C_\Gamma(b) \leqslant M$, then we have

$$[C_{O_q(L)},\, B] \leqslant O_q(L) \cap O_p(M) = 1$$

since $B \leqslant P = O_p(M)$.

If $C_\Gamma(b) \leqslant N_\Gamma(Q)$, then by Lemma 2.3.10, we have

$$[C_{O_q(L)}(b),\, B] \leqslant O_q(N_\Gamma(Q)).$$

We deduce that

$$[O_q(L),\, B] \leqslant Q.$$

Now, we claim that $[O_q(L),\, B] \trianglelefteq L$.
We have,

$$[O_q(L),\, B] \leqslant O_q(L) \leqslant Q^*,$$

for $Q^* \in \mathrm{Syl}_q L$. Then

$$Q^* \leqslant N_\Gamma\left([O_q(L),\, B]\right).$$

Now, choose P^* with $B \leqslant P^* \in \mathrm{Syl}_p L$. Since P^* is abelian, it normalize B and hence $[O_q(L),\, B]$. Since L has type pq we have

$$L = P^* Q^*.$$

Then, we have
$$[O_q(L),\ B] \trianglelefteq L.$$
Case(1). Assume that $[O_q(L),\ B] = 1$. Then, we have
$$B \leqslant O_p(L)$$
and so $\pi(F(L)) = \{p,\ q\}$. Then Lemma 6.2.9 implies that
$$N_\Gamma(B) \leqslant L$$
and so
$$P \leqslant L.$$
By applying Lemma 6.5.6, we get
$$L = M \text{ or } L = N_\Gamma(Q).$$
If $L = M$, then we get
$$L = N_\Gamma(P_1),$$
for some $P_1 \in \mathrm{Syl}_p N_\Gamma(Q)$, namely $P_1 = P$.

Case(2). Assume that $[O_q(L),\ B] \neq 1$. Then, we have
$$Q \leqslant N_\Gamma\left([O_q(L),\ B]\right) = L$$
and so
$$[Q,\ B] \leqslant O_q(L).$$
Then, we have
$$[Q,\ B] = [Q,\ B,\ B] \leqslant [O_q(L),\ B].$$
Also,
$$[O_q(L),\ B] = [O_q(L),\ B,\ B] \leqslant [Q,\ B].$$
We deduce that
$$[Q,\ B] = [O_q(L),\ B].$$
Note that $[Q,\ B] \trianglelefteq N_\Gamma(Q)$ because $Q = O_q(N_\Gamma(Q))$. Then
$$N_\Gamma(Q) = N_\Gamma([Q,\ B]).$$
Since $L = N_\Gamma\left([O_q(L),\ B]\right)$, we get
$$L = N_\Gamma(Q).$$

Now, suppose that $O_q(L) = 1$. Then L has type p so $O_p(L)$ is the unique Sylow p-subgroup of L. Then, we have
$$B \leqslant O_p(L).$$
Corollary 6.5.7 implies that
$$C_\Gamma(B) \leqslant M \text{ or } C_\Gamma(B) \leqslant N_\Gamma(Q).$$

Assume $C_\Gamma(B) \leqslant M$, then

$$O_p(L) \leqslant M$$

and so

$$O_p(L) \leqslant P.$$

Since $O_p(L) \in \mathrm{Syl}_p\Gamma$, we get

$$O_p(L) = P$$

and so

$$L = M.$$

Hence we may suppose that $L \neq M$. Then, we have

$$C_\Gamma(B) \leqslant N_\Gamma(Q).$$

Then, we get

$$O_p(L) \leqslant N_\Gamma(Q).$$

Since $O_p(L) \in \mathrm{Syl}_p\Gamma$, $O_p(L) \in \mathrm{Syl}_pN_\Gamma(Q)$. Thus, we have

$$L = N_\Gamma(P_1),$$

for some $O_p(L) = P_1 \in \mathrm{Syl}_pN_\Gamma(Q)$.

Lemma 6.5.9. Let $T \in \Gamma$ be a p-group and suppose that $T \cap P \neq 1$. Then $T \leqslant N_\Gamma(Q)$.

Proof. Since $T \cap M \neq 1$, we have

$$\langle T,\ P\rangle \leqslant C_\Gamma(T \cap P) \leqslant H \in \Gamma^*.$$

Then, we get

$$P \leqslant H.$$

By applying Lemma 6.5.6, we have

$$H = M \text{ or } H = N_\Gamma(Q).$$

Note that P is the only Sylow p-subgroup of M, hence

$$T \leqslant N_\Gamma(Q).$$

Now we study the case that $O_p(N_\Gamma(Q)) \neq 1$, and we begin with the following hypothesis.

Hypothesis D

- $M \in \Gamma^*$.
- M has type pq.
- $P \in \mathrm{Syl}_pM$.
- P is elementary abelian and rank $(P) \geqslant 3$.

Lemma 6.5.10. Assume Hypothesis D. Let $T \in \Gamma$ be a q-group and $P \leqslant N_\Gamma(T)$. Then $T \leqslant M$.

Proof. Choose $B < P$ with rank $(B) = 2$ and $C_{O_q(M)}(B) \neq 1$.
Let $V = C_{O_q(M)}(B)$. Then, we have

$$V \leqslant F(M) \text{ and } \pi(F(M)) = \{p,\ q\}.$$

Lemma 6.2.9 implies that

$$N_\Gamma(V) \leqslant M.$$

Then, we have

$$O_p(M) \leqslant O_p\left(N_\Gamma(V)\right).$$

Next, let $b \in B^{\#}$ and choose H with $C_\Gamma(b) \leqslant H \in \Gamma^*$. As $b \in P$, then we have

$$P,\ V \leqslant H.$$

Then, we get

$$1 \neq O_p(M) \leqslant O_p\left(N_H(V)\right) \leqslant O_p(H)$$

and so

$$O_p(H) \leqslant M$$

because $O_p(H)$ is abelian. Also, we have

$$O_q(H) \leqslant C_\Gamma(O_p(H)) \leqslant C_\Gamma(O_p(M)) \leqslant M.$$

Then

$$F(H) \leqslant M.$$

Suppose that $\pi(F(H)) = \{p,\ q\}$. Then by applying Lemma 6.2.9, we get

$$H = M.$$

Suppose that $\pi(F(H)) = \{p\}$. Then, we have

$$O_p(H) \in \mathrm{Syl}_p\Gamma$$

and so

$$O_p(H) \in \mathrm{Syl}_p M.$$

Also, we have

$$P \in \mathrm{Syl}_p M \text{ and } P \leqslant O_p(H).$$

Then, we get

$$P = O_p(H)$$

and so

$$H = N_\Gamma(P).$$

We deduce that

$$C_\Gamma(b) \leqslant M \text{ or } C_\Gamma(b) \leqslant N_\Gamma(P),$$

for all $b \in B^{\#}$. Now, T is a q-group. Then, we have

$$T = \langle C_T(b) |\; b \in B^{\#} \rangle$$

and so

$$T \leqslant N_{\Gamma}(P).$$

Let $b \in B^{\#}$. By the above $C_{\Gamma}(b) \leqslant M$ or $C_{\Gamma}(b) \leqslant N_{\Gamma}(P)$. Suppose that $C_{\Gamma}(b) \leqslant N_{\Gamma}(P)$. Then, we have

$$[C_T(b),\, P] \leqslant T \cap P = 1$$

and so

$$C_T(b) \leqslant C_{\Gamma}(P) \leqslant C_{\Gamma}(O_p(M)) \leqslant M.$$

Suppose that $C_{\Gamma}(b) \leqslant M$. Then, we get

$$C_T(b) \leqslant M.$$

We deduce that

$$T \leqslant M.$$

Proposition 6.5.11. Assume Hypothesis D. Suppose that $P \leqslant L \in \Gamma^*$. Then $L = M$ or $L = N_{\Gamma}(P)$ and $\pi(F(L)) = \{p\}$.

Proof. By applying the above lemma, we get

$$O_q(L) \leqslant M.$$

Suppose that $\pi(F(L)) = \{p,\, q\}$. Then, we have

$$O_p(L) \leqslant P \leqslant M.$$

Thus, we get

$$F(L) \leqslant M.$$

By applying Lemma 6.2.9, we get

$$L = M.$$

Suppose that $\pi(F(L)) = \{p\}$. Then, we get

$$L = N_{\Gamma}(P).$$

Suppose that $\pi(F(L)) = \{q\}$. Then, we have

$$O_q(L) \in \mathrm{Syl}_q\Gamma$$

and so

$$O_q(L) \in \mathrm{Syl}_q M.$$

Also, we have

$$P \in \mathrm{Syl}_p\Gamma$$

and so

$$P \in \mathrm{Syl}_p L.$$

Then, we get

$$L = PO_q(L) = M.$$

In what follows let Q be defined by

$$\Omega^*(P) = \{Q\}.$$

Again assume Hypothesis C and $P = O_p(M)$. If $O_p(N_\Gamma(Q)) \neq 1$, then we can apply the above proposition as follows.

Corollary 6.5.12. Let $L \in \Gamma^*$ with $O_p(L) \neq 1$. Let $N = N_\Gamma(Q)$ with $O_p(N) \neq 1$. Suppose that $|M \cap L|_p \neq 1$. Then $M = L$ or L has type p and $O_p(M),\ O_p(L) \in \mathrm{Syl}_p N$.

Proof. Since $|M \cap L|_p \neq 1$ and P is the unique $\mathrm{Syl}_p M$, we have

$$P \cap P_1 \neq 1,$$

for some $P_1 \in \mathrm{Syl}_p L$. By Lemma 6.5.9, we have

$$P_1 \leqslant N.$$

Since $P_1 \in \mathrm{Syl}_p \Gamma$, $P_1 \in \mathrm{Syl}_p N$. By applying proposition 6.5.11, we get the result.

Now, we assume that $O_p(N_\Gamma(Q)) = 1$. Then $N_\Gamma(Q)$ has type q with the maximal element Q and so $Q = O_q(N_\Gamma(Q)) \in \mathrm{Syl}_p \Gamma$.

Can Q normalize a non-trivial p-subgroup?. In order to answer this question, we begin with the following definition.

Definition 6.5.13. We define

$$\Omega(Q) = \{T \in \Gamma \mid T \text{ is a } p\text{-group and } Q \leqslant N_\Gamma(T)\}.$$

We denote $\Omega^*(Q)$ to be the set of maximal elements of $\Omega(Q)$.

Lemma 6.5.14. Q is elementary abelian and rank $(Q) \geqslant 3$.

Proof. Consider the semidirect product of P and Q. Then, $P \cong \mathrm{Aut}(Q)$. As P is elementary abelian of rank $\geqslant 3$, Q is non-cyclic.
Now, assume that rank $(Q) = 2$.
Since $C_Q(P) = 1$, P is faithful on $\Omega(Q)$ which is isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$. Then

$$B < P,\ B \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

and

$$C_{\Omega(Q)}(B) \neq 1.$$

Then, we have

$$\Omega(Q) = [\Omega(Q),\ B] \times C_{\Omega(Q)}(B)$$

and so

$$[\Omega(Q),\ B] \cong \mathbb{Z}_q.$$

But, B is faithful on $\Omega(Q)$, a contradiction. Thus, we get rank $(Q) \geqslant 3$.

Lemma 6.5.15. Q can not normalize a non-trivial p-subgroup in Γ.

Proof. Suppose that $T \in \Omega^*(Q)$. Let $1 \neq B < P$ with $C_Q(B) \neq 1$. Let $x \in C_Q(B^{\#})$ and suppose that $C_T(x) \neq 1$. Then, we have

$$Q,\ C_T(x),\ B \leqslant C_\Gamma(x) \leqslant L \in \Gamma^*.$$

By applying Lemma 6.5.6, we get

$$L = N_\Gamma(Q) \text{ or } L = N_\Gamma(T).$$

Suppose that $L = N_\Gamma(T)$. Then

$$[B,\ T] \leqslant T \cap Q = 1$$

and so

$$T \leqslant C_\Gamma(B).$$

By corollary 6.5.7, we get

$$C_\Gamma(B) \leqslant M \text{ or } C_\Gamma(B) \leqslant N_\Gamma(Q).$$

Then, we have

$$T \leqslant M \text{ or } T \leqslant N_\Gamma(Q).$$

If $T \leqslant M$, then $T \leqslant P$ because M has type $\{p\}$. Then

$$[T,\ Q] \leqslant Q \cap T = 1,$$

a contradiction since $N_\Gamma(Q)$ has type q.

If $T \leqslant N_\Gamma(Q)$, then we have

$$[T,\ Q] = 1,$$

a contradiction. Then

$$L = N_\Gamma(Q)$$

and so

$$C_\Gamma(x) \leqslant N_\Gamma(Q).$$

Then, we get

$$[C_T(x),\ Q] \leqslant T \cap Q = 1,$$

a contradiction.
We deduce that whenever $1 \neq B < P$ and $x \in C_Q(B^{\#})$, then $C_T(x) = 1$. Then, we get

$$C_Q(B) \text{ is cyclic.}$$

Choose $B < P$ with rank $(B) = 2$ and $C_Q(B) \neq 1$. Then

$$Q = C_Q(B)[Q,\ B].$$

Consider the action of B on $[Q,\ B]$. There exists $b \in B^{\#}$ with

$$C_{[Q,\ B]}(b) \neq 1.$$

Then, we have

$$C_Q(b) = C_Q(B) \times C_{[Q,\ B]}(b).$$

Thus, we get $C_Q(b)$ is non-cyclic. Apply the above with $\langle b\rangle$ in place of B gives a contradiction.

6.6 Conclusion

In this chapter we have proved many results of Special Case of Burnside's Theorem for Partial Groups, which we summarize in the following theorems.

Theorem 6.6.1. (a) Suppose that $M \in \Gamma^*$ has type pq and that $F(M) \leqslant L \in \Gamma^*$. Then $M = L$.

(b) Let $M,\ L \in \Gamma^*$ and suppose that $F(M) \leqslant L$ and $M \neq L$. Then one of the following holds.

(1) M has type p, $O_p(M) \in \mathrm{Syl}_p\Gamma$, $L = O_p(M)O_q(L)$, $O_q(L) \in \mathrm{Syl}_q\Gamma$ and $L = N_\Gamma(O_q(L))$.

(2) M has type q, $O_q(M) \in \mathrm{Syl}_q\Gamma$, $L = O_q(M)O_p(L), O_p(L) \in \mathrm{Syl}_pL$ and $L = N_\Gamma(O_p(L))$.

Proof. (a) is Lemma 6.2.9 and (b) is Lemma 6.2.11.

Theorem 6.6.2. Assume Hypothesis B.

(a) Suppose $Q \in \Gamma$ is a q-group with $O_p(M) \leqslant N_\Gamma(Q)$. Then $Q \leqslant M$.

(b) Let $P \in \mathrm{Syl}_pM$ and $P \leqslant L \in \Gamma^*$. Then $M = L$ or $L = N_\Gamma(P)$.

(c) Let $P \in \mathrm{Syl}_p(M)$ and $P_1 \in \Gamma$ is a p-group. Suppose that $P \cap P_1 \neq 1$. Then $P_1 \leqslant M$.

(d) Let $P \in \mathrm{Syl}_pM$ and $x \in P^{\#}$. Then $C_\Gamma(x) \leqslant M$ or $C_\Gamma(x) \leqslant N_\Gamma(P)$.

(e) Suppose that $M \cap L$ has order divisible by p. Then we have the following:

(1) If L has type pq, then $M = L$.

(2) If L has type q, then $M \cap L$ has cyclic Sylow p-subgroups.

(3) If L has type p, then $L = N_\Gamma(P)$ for some $P \in \mathrm{Syl}_pM$.

Proof. (a) is Lemma 6.3.1, (b) is Lemma 6.3.2, (c) is Lemma 6.3.3, (d) is Corollary 6.3.4 and (e) is Proposition 6.3.5.

Theorem 6.6.3. Let $M \in \Gamma^*$ and $P \in \mathrm{Syl}_p M$. Suppose that M has type q.

(a) Suppose that P is non-cyclic. Then $P \in \mathrm{Syl}_p \Gamma$.

(b) Suppose that rank $(P) \geqslant 3$.

(1) Then there exists $\mathbb{Z}_p \times \mathbb{Z}_p \cong B \leqslant P$ such that $C_\Gamma(b) \leqslant M$ for all $b \in B^\#$ and $P \in \mathrm{Syl}_p \Gamma$.

(2) If $P \leqslant L \in \Gamma^*$ with $L \neq M$, then $L = N_\Gamma(P)$.

(3) If $a \in P^\#$ then $C_\Gamma(a) \leqslant M$ or $C_\Gamma(a) \leqslant N_\Gamma(P)$.

(4) If $P_1 \in \Gamma$ is a p-group and $P_1 \cap M \neq 1$, then $P_1 \leqslant M$.

(c) Suppose rank $(P) \geqslant 3$. Let $L \in \Gamma^*$ and suppose that $M \cap L$ has non-cyclic Sylow p-subgroups. Then $M = L$ or $L = N_\Gamma(P_1)$ for some $P_1 \in \mathrm{Syl}_p M$.

Proof. (a) is Lemma 6.4.1, (b) is Lemma 6.4.2 and (c) is Lemma 6.4.3.

Theorem 6.6.4. Assume Hypothesis C. Let $P = O_p(M)$.

(a) $\Omega(P)$ has a unique maximal element.

(b) Let $Q \in \Omega(P)$. Then $N_\Gamma(P) \in \Gamma^*$.

(c) Let $Q \in \Omega(P)$. Suppose that $P \leqslant L \in \Gamma^*$. Then either

$$L = M \text{ or } L = N_\Gamma(Q).$$

(d) Let $Q \in \Omega(P)$. Let $x \in P^\#$. Then either

$$C_\Gamma(x) \leqslant M \text{ or } C_\Gamma(x) \leqslant N_\Gamma(Q).$$

(e) Let $L \in \Gamma^*$ and suppose that $L \cap M$ contains a p-subgroup $B \leqslant P$ of rank $\geqslant 2$. The either

$$L = N_\Gamma(Q) \text{ or } L = N_\Gamma(P_1),$$

for some $P_1 \in \mathrm{Syl}_p N_\Gamma(Q)$.

(f) Let $Q \in \Omega(P)$, and let T be a p-group. Suppose that $T \cap P \neq 1$. Then $T \leqslant N_\Gamma(Q)$.

Proof. (a) is Lemma 6.2.8, (b) is Lemma 6.5.5, (c) is Lemma 6.5.6, (d) is Corollary 6.5.7, (e) is Lemma 6.5.8 and (f) is Lemma 6.5.9.

Theorem 6.6.5. Assume Hypothesis D.

(a) Let $T \in \Gamma$ be a q-group and $P \leqslant N_\Gamma(T)$. Then $T \leqslant M$.

(b) Suppose that $P \leqslant L \in \Gamma^*$. Then $L = M$ or $L = N_\Gamma(P)$ and $\pi(F(L)) = \{p\}$.

Proof. (a) is Lemma 6.5.10 and (b) is Proposition 6.5.11.

REFERENCES

[1] A. Feldman, Fitting height of solvable groups admitting fixed point free automorphism groups, J of Algebra. 53 (1978) 268-295.

[2] B. Steinberg, Representation Theory of Finite Groups An Introductory Approach, Universitext, Springer-Verlag, New York, 2012.

[3] C. Delizia, H. Dietrich, P. Moravec, C. Nicotera, Group in which every non-abelian subgroup is self-centeralizing, J of Algebra, 462 (2016) 23-36.

[4] D. Gorenstein, Finite Simple Groups: An Introduction to Their Classification, Plenum Press, New York, 1982.

[5] D. M. Goldschmidt, Solvable signalizer functors on finite groups, J of Algebra. 21 (1972) 131-148.

[6] D. M. Goldschmidt, 2-signalizer functors on finite groups, J of Algebra. 21 (1972) 321-340.

[7] E. Shult, On groups admitting fixed point free operator groups, Illinois J. Math. 9 (4 (1956) 701-720.

[8] G. Glauberman, Correspondences of characters for relatively prime operator groups, Cand. J. Math. 20 (1968) 1465-1488.

[9] G. Glauberman, On solvable signalizer functors in finite groups, Proc. London Math. Soc. 33 (3) (1976) 1-27.

[10] G. Glauberman, Prime-Power Factor Groups of Finite Groups, Math. Z. 107 (1968) 159-172.

[11] G. James, M. Liebeck, Representations and Characters of Groups, 2nd ed., Cambridge University Press, Cambridge, 2004.

[12] H. Bender, On groups with abelian Sylow 2-subgroups, Math. Z. 117 (1970) 164-176.

[13] H. Kurzweil, B. Stellmacher, The Theory of Finite Groups. An Introduction, Universitext, Springer-Verlag, New York, 2004.

[14] J. Thompson, Finite group with fixed point free automorphisms of prime order, Proc. Nat. Acad. Sci. 45 (1959), 578-581.

[15] M. Aschbacher, Finite Group Theory, first ed., Cambridge Stud. Adv. Math. Vol. 10, Cambridge, 1986.

[16] M. Humphreys, A Course in Group Theory, Oxford University Press, 19996.

[17] M. Isaacs, Finite Group Theory (Graduate Studies in Math), Rhode Island, USA. Amer. Math. Soc. 92, 2008.

[18] M. Suzuki, Group Theory II, Springer-Verlag, Berlin, 1986.

[19] P. Flavell, A new proof of the solvable signalizer functor Theorem, J of Algebra. 398 (2014) 350-363.

[20] P. Flavell, Primitive pairs of p-solvable groups, J of Algebra. 324 (4) (2010) 841-859.

[21] R. Trudeau, Introduction to Graph Theory, Kent State University Press, 1996.

[22] R. Martineau, Elementary abelian fixed point free automorphism groups, 23 (2) (1972) 205-212.

[23] U. Meierfrankenfeld, B. Stellmacher, F-Stability in finite groups, Trans. Amer. Math. Soc. 361 (5) (2009) 2509-2525.

[24] W. Feit, J. Thompson, Solvability of groups of odd order, Pacific J. Math. 13 (3) (1963) 775-1029.