AN INDUCTION THEOREM INSPIRED BY BRAUER'S INDUCTION THEOREM FOR CHARACTERS OF FINITE GROUPS

by

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Abstract

Brauer's induction theorem states that every irreducible character of a finite group G can be expressed as an integral linear combination of induced characters from elementary subgroups.

The goal of this thesis is to develop our own induction theorem inspired by both Brauer's induction theorem and Global-Local conjectures. Specifically we replace the set of elementary subgroups of G by the set of subgroups of index divisible by the prime power divisors of the given character's degree.

We aim to do this by using a reduction theorem to almost simple and quasisimple groups, using the Classification of Finite Simple Groups to deal with the remaining cases.

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LIST OF NOTATION

1. $\mathrm{Syl}_p(G)$ - the set of Sylow p-subgroups of G.
2. G' - the derived subgroup of G.
3. $G \wr H$ - the wreath product of G and H.
4. $N_G(H)$ - the normalizer in G of H.
5. $C_G(H)$ - the centralizer in G of H.
6. $\mathrm{Aut}(G)$ - the group of automorphisms of G.
7. S_n - the symmetric group on n points.
8. A_n - the alternating group on n points.
9. G^n - the group $\underbrace{G \times \cdots \times G}_{n \text{ times}}$.
10. $Z^2(G, \mathbb{C}^*)$ - the group of 2-cocycles of G.
11. $B^2(G, \mathbb{C}^*)$ - the group of 2-coboundaries of G.
12. $H^2(G, \mathbb{C}^*)/M(G)$ - the Schur multiplier of G.
13. $\mathrm{El}(G)$ - the set of elementary subgroups of G.
14. $\mathrm{El}_{\max}(G)$ - the set of maximal elementary subgroups of G.
15. $\mathrm{El}_{p,a}(G)$ - the set of maximal elementary subgroups of G of index not divisible by p^a.

16. $\mathrm{PD}(G)$ - the set of p-decomposable subgroups of G.

17. $\mathrm{PD}_{\max}(G)$ - the set of maximal p-decomposable subgroups of G.

18. $\mathcal{I}_n(G)$ - the set of subgroups of index divisible by n.

19. $\mathcal{I}_n(G,H)$ - the set of subgroups L of G such that n divides $|H : L \cap H|$.

20. $v_p(\cdot)$ - the p-adic valuation function.

21. $b(E,N,H,a) = a - v_p\left(|H : EN \cap H|\right)$.

22. $\mathrm{Irr}(G)$ - the set of irreducible characters of G.

23. $\mathcal{C}(G)$ - the set of $\mathbb{Z}$-linear combinations of irreducible characters of G (generalised characters of G).

24. $\mathrm{CF}(G)$ - the set of class functions of G.

25. $\mathrm{Irr}_{p'}(G)$ - the set of irreducible characters of G of degree prime to p.

26. $\mathrm{Irr}^p(G)$ - the set of irreducible characters of G of degree divisible by p.

27. $\mathcal{C}^p(G)$ - the $\mathbb{Z}$-linear combinations of elements of $\mathrm{Irr}^p(G)$.

28. $\mathcal{C}_p(G)$ - the set of generalised characters of degree divisible by p.

29. $\mathcal{A}_n(G,H,N)/\mathcal{A}_n(A,H,N)$ - the $\mathbb{Z}$-linear combinations of induced characters from subgroups of G/subalgebras of A in $\mathcal{I}_n(G,H)$ containing N.

30. $\mathcal{A}_n(G,N)/\mathcal{A}_n(A,N)$ - $\mathcal{A}_n(G,G,N)/\mathcal{A}_n(A,G,N)$.

31. $\mathcal{A}_n(G)/\mathcal{A}_n(A)$ - $\mathcal{A}_n(G,G,1)/\mathcal{A}_n(A,G,1)$.

32. $\chi_1 \times \chi_2$ - the character of a direct product of groups.

33. 1_G - the principal character of G.

34. ρ_G - the regular character of G.

35. $\mathrm{Res}_H^G(\cdot)$ - the operator representing the restriction of characters/modules.

36. $\mathrm{Ind}_H^G(\cdot)$ - the operator representing the induction of characters/modules.

37. $\mathrm{Inf}_{G/N}^G(\cdot)$ - the operator representing the inflation of characters/modules.

38. $\mathrm{Def}_{G/N}^G(\cdot)$ - the operator representing the deflation of characters/modules.

39. $\ker(\cdot)$ - the kernel of a character/homomorphism.

40. $\langle \chi_1, \chi_2 \rangle$ - the inner product of two characters.

41. $\mathrm{Irr}(G|\theta)$ - the irreducible characters of G whose restriction has θ as an irreducible constituent.

42. $\mathrm{Irr}(G||\theta)$ - the irreducible characters of G whose restriction has θ as its only irreducible constituent.

43. $\widetilde{\pi}_\theta(\cdot)$ - the projection of a character of a direct product with θ as an irreducible in the second component.

44. $\widetilde{\pi}_L(\cdot)$ - the $\mathbb{Z}$-span of the linear irreducible constituents of a character.

45. ${}^g\chi$ - the conjugate character.

46. $I_G(\cdot)$ - the inertia subgroup of a character.

47. (G, N, θ) - a character triple.

48. (Γ, π) - a finite central extension.

49. $M^{\otimes n}$ - the module $\underbrace{M \otimes \cdots \otimes M}_{n \text{ times}}$.

50. $M^{\widetilde{\otimes} n}$ - the module of a wreath product of groups/algebras.

51. $\chi^{\widetilde{\times} n}$ - the character afforded by $M^{\widetilde{\otimes} n}$.

52. $\mathbb{C}^\alpha[G]$ - the twisted group algebra on G with associated factor set α.

CONTENTS

CHAPTER 1

INTRODUCTION

Currently, many open problems in Representation Theory and its modular counterpart take the form of global-local conjectures and are being tackled by many world renowned mathematicians. The work of Berger–Knörr [6] on the reduction of Brauer's Height Zero conjecture, Conjecture 5.58, to quasisimple groups, highlights both the long-standing nature of such conjectures and their relevance to current research in representation theory. However, the motivation for this thesis stems from a paper of Evseev [11], which discusses another major global-local conjecture, namely the McKay conjecture.

The McKay conjecture was first introduced, in the case $p = 2$, by J. McKay in [25] and states the following. We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G over $\mathbb{C}$ and $\mathrm{Irr}_{p'}(G)$ will be the elements of $\mathrm{Irr}(G)$ of degree prime to p.

Conjecture 1.1. *Let G be a finite group and p be a prime. Let $P \in \mathrm{Syl}_p(G)$. Then $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|$.*

The above conjecture clearly fulfils the criterion of a global-local conjecture relating directly the representation theory of the global group G to that of its local subgroups, namely the normaliser of a Sylow p-subgroup. However, we are not only interested in the statement of such conjectures but also the method by which such things are proved. It is becoming common practice in representation theory (and in fact group theory) to reduce problems about arbitrary groups to statements about classes of groups related to simple group. Specifically we are interested in almost simple (groups whose socle is simple) and quasisimple (perfect groups whose quotient by the centre is simple) groups. In order to achieve what is commonly referred to as a reduction theorem, we use the Classification of Finite Simple Groups. Much of the proof strategy used in this thesis is based on [19], which is a reduction theorem of the Alperin-McKay conjecture (a block-theoretic refinement of the McKay conjecture) to simple groups. However, work of Evseev [11] references the following refinement of the McKay conjecture, the Isaacs-Navarro conjecture. Following the notation of [11], we define for a prime p and $l \in \mathbb{Z}$,

$$M_l(G) := \left|\{\chi \in \operatorname{Irr}(G) : \chi(1) \equiv \pm l \bmod p\}\right|.$$

Conjecture 1.2. [20, Conjecture A] *Let G be a finite group and p be a prime. Let $P \in \operatorname{Syl}_p(G)$, then $M_l(G) = M_l(N_G(P))$ for every l prime to p.*

Evseev is however concerned with his own refinement of the Isaacs–Navarro conjecture, which we now take some time to develop. We introduce the following set of subgroups to simplify notation. As stated in [11] and found in [2, Chapter III,§11], the following set is one of the sets of subgroups which appears in the Green Correspondence, if $N_G(P) \leq H$.

Definition 1.3. [11, Definition 1.2] *Let G be a group, P be a p-subgroup of G for a prime p and $H \leq G$ such that $N_G(P) \leq H$. We define $\mathcal{S}(G, P, H)$ by*

$$\mathcal{S}(G, P, H) := \left\{Q \leq P : Q \leq {}^tP \; for\; some\; t \in G,\ t \notin H\right\}.$$

We also refer the reader to Definition 4.1 for the definition of $\mathcal{I}(G, P, \mathcal{S})$. In brief, it is a ring of induced characters for subgroups of G whose index is controlled by the Sylow p-subgroup of G. We are also interested in Definition 2.1 (v) and (vi), which gives the definition of $\mathcal{C}^p(H)$. Evseev asks whether the following property holds for a pair (G, H) where $N_G(P) \leq H \leq G$ for $P \in \operatorname{Syl}_p(G)$ where p is a prime.

(WIRC-Syl) There is a signed bijection $F : \pm\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(H)$ such that

$$F(\chi) \equiv \mathrm{Res}^G_H \chi \bmod \mathcal{C}^p(H) + \mathcal{I}\left(H, P, \mathcal{S}(G, P, H)\right),$$

for all $\chi \in \pm\mathrm{Irr}_{p'}(G)$.

By [11, Proposition 1.7], we see that if the condition (WIRC-Syl) holds for $(G, N_G(P))$ then the Isaacs-Navarro conjecture holds. It is clear that to verify (WIRC-Syl) for $(G, N_G(P))$ we must understand the character ring, $\mathcal{I}(H, P, \mathcal{S}(G, P, H))$, since the set $\mathcal{C}^p(H)$ is well understood, given that it has an explicit basis. This is the purpose of this thesis. We introduce a more general ring of induced characters denoted $\mathcal{A}_p(G)$, Definition 2.4, which contains the ring introduced by Evseev, and investigate its properties, such as which irreducible characters belong to the ring. We aim to show that any irreducible character of degree divisible by p^a can expressed as a sum of induced characters from subgroups of index divisible by p^a. The connection with this conjecture and global-local conjectures can also be seen in its method of proof. Namely, we aim to use a reduction theorem to reduce the conjecture to almost simple and quasisimple groups.

The attentive reader may have noticed the similarity between the conjecture we are aiming to prove and an existing classical result in representation theory. As eluded to in the title of this thesis, Brauer's induction theorem for characters of finite groups plays a fundamental role. Despite motivating the connection with global-local conjectures, the problem can be independently introduced using Brauer's induction theorem.

Theorem 1.4. [7, Theorem 1] *Let G be a finite group. Every character χ of G can be expressed as a linear combination with integral coefficents of characters ω^*, where ω^* is a character of G induced from a linear character of a subgroup of G.*

In his paper, Brauer observes that he is able to take the subgroups used in his proof to be the so-called elementary subgroups. These are subgroups of G which are the direct product of a cyclic group and a p-group for a prime p. Denoting this class of subgroups by El(G), we can alternatively express Brauer's Induction Theorem as follows. For all $\chi \in \mathrm{Irr}(G)$,

$$\chi = \sum_{E \in \mathrm{El}(G)} \mathrm{Ind}_E^G \phi_{(E)},$$

where $\phi_{(E)}$ is a $\mathbb{Z}$-linear combination of linear characters of E. The theorem was first conjectured by Artin, but proved by Brauer in [7] in 1947, and the theorem is now associated solely to him. For our induction theorem, we aim to replace the set of elementary subgroups by subgroups of index divisible by the prime power divisors of the character degree. Moreover, considering the proof of Lemma 4.2, we show the relationship between $\mathcal{S}(G, P, H)$ and our set of subgroups. This problem has in fact been addressed in a paper of [29] for soluble groups. His approach relies heavily on the character theory of π-separable groups, developed by Isaacs in [18], which restricts the ability to generalise the result. The reader is directed to Chapter 4 for our new proof for p-soluble groups which can be generalised more easily.

We now give an overview of each chapter and what we prove in each. In Chapter 2, we discuss the intimate details of the conjecture we make progress towards in this thesis. In particular, we aim to prove Conjecture 2.5 by way of its refinement, given by Conjecture 2.6. In Chapter 3, we give some preliminary details used throughout the thesis.

Chapter 4 introduces our ring of induced characters, $\mathcal{A}_n(G, H, N)$. We prove Conjecture 2.6 for p-soluble groups, Theorem 4.12, which as a consequence relates Conjecture 2.5 to the condition (pRes-Syl), introduced in [11]. Section 4.3 also shows how we are able to reduce Conjecture 2.5 to p-groups, which gives us a key simplification to the computational verification of Conjecture 2.11 for sporadic groups, Chapter 9. Chapter 9 provides the theory used to generate MAGMA code to verify Conjecture 2.11 for classes of almost simple and quasisimple groups relating to sporadic simple groups.

The focus of Chapter 5 is to develop further properties of the induced character ring, $\mathcal{A}_n(G, H, N)$. In particular we show how the ring behaves in symmetric groups and for wreath products, and generalise the results relating to wreath products to projective characters of twisted group algebras. We also give an overview of modular representation theory, building to Theorem 5.60, essentially excluding characters of height zero in their block from the verification of Conjecture 2.11.

In Chapter 6, we show the key results of this thesis, Theorems 6.1 and 6.3 which are reduction theorems to Brauer-good groups in the case when G has trivial centre and when G satisfies an "extensibility condition", see Condition 6.2, respectively. Since we were unable to obtain a full reduction theorem for arbitrary groups G, the reader can see that these theorems prove that groups satisfying such conditions are not minimal counterexamples to Conjecture 2.12. This can hopefully be generalised to a full reduction theorem and is one area of further research branching from this thesis.

The structure of the induced character ring for small, in terms of rank, abelian p-groups is the focus of Chapter 7. In Section 7.1.1, we show that even for rank 2 abelian p-groups, $\mathcal{A}_p(G)$ is not trivial to understand, in the sense that it is not clear from the dimension that every irreducible character of degree divisible by p lies in it. Moreover, we prove Theorem 7.1, which in some sense shows that if the abelianisation of the Sylow 2-subgroup is elementary abelian, then this is the "only" case in which Conjecture 2.5 is trivial.

Finally in Chapter 8, we prove Conjecture 2.5 for symmetric groups and in specific cases for alternating groups, see Theorem 8.1, Corollary 8.2 and Theorem 8.6.

CHAPTER 2

THE CONJECTURE

In this chapter we will formulate the conjecture we introduced in the previous chapter. Informally, we are aiming to show that every irreducible character of a finite group, G, can be decomposed as an integral linear combination of characters induced from subgroups of index divisible by the prime power divisors of the character degree. However we do this by showing a stronger statement holds. Our approach is to consider each of the highest prime power divisors of the character degrees, say p^a, and show that every irreducible character of degree divisible by p^a can be expressed as an integral linear combination of characters induced from subgroups of index divisible by p^a containing $Z(G)$. The results from Chapter 4 can then be used to prove our conjecture. We begin by introducing the relevant notation to understand the formulation of the conjecture. We remark that all groups considered will be finite.

Definition 2.1. *Let G be a group and R be a ring. We define the following sets of characters.*

(i) $\mathrm{CF}(G)$ *will denote the set of class functions of G, i.e. functions which are constant on the conjugacy classes of G;*

(ii) $\mathrm{Irr}(G)$ *will denote the set of irreducible characters of G;*

(iii) If $A \subseteq \mathrm{CF}(G)$, then $R[A]$ will denote the R-span of the characters in A, i.e. $R[A] := \{a_\chi \chi : \chi \in A \text{ and } a_\chi \in R\}$*;*

(iv) $\mathcal{C}(G) := \mathbb{Z}[\mathrm{Irr}(G)]$*;*

(v) $\mathrm{Irr}^p(G)$ *will denote the set of irreducible characters of* G *of degree divisible by* p*;*

(vi) $\mathcal{C}^p(G) := \mathbb{Z}[\mathrm{Irr}^p(G)]$*.*

(vii) $\mathcal{C}_p(G)$ *will denote the set of generalised characters of degree divisible by* p*.*

Note that $\mathcal{C}(G)$ can be given a ring structure in a natural way, defining $(\chi_1 + \chi_2)(g) := \chi_1(g) + \chi_2(g)$ and $(\chi_1\chi_2)(g) := \chi_1(g)\chi_2(g)$ for $\chi_1,\ \chi_2 \in \mathrm{Irr}(G)$ and $g \in G$. For the following definition, the reader is directed to Section 5.3 for further explanation of the notation used.

Definition 2.2. [22, §1.2]. *Let* G *be a group,* $K \leq G$*,* R *be a ring and* $\alpha \in H^2(G, \mathbb{C}^*)$ *be a factor set. Let* $A := \mathbb{C}^\alpha[G]$ *be the twisted group algebra associated to* α*. We define the following sets of projective* α*-characters of* G*.*

(i) $\mathrm{Irr}(A)$ *will denote the set of irreducible projective* α*-characters of* G*;*

(ii) If $B \subseteq \mathrm{Irr}(A)$*, then* $R[B]$ *will denote the* R*-span of the projective* α*-characters in* B*;*

(iii) $\mathcal{C}(A) := \mathbb{Z}[\mathrm{Irr}(A)]$*.*

(iv) Given the following decomposition of A*,* $A = \bigoplus_{g\in G} A_g$ *for* A*-subspaces* A_g*, we define* $A[K] = \bigoplus_{k\in K} A_k$*.*

The following definition introduces the set of subgroups that we are required to induce from.

Definition 2.3. *Let* G *be a group and* $H \trianglelefteq G$*. Let* $n \in \mathbb{N}$ *with* $n \geq 1$*.*

(i) We define $\mathcal{I}_n(G)$ *to be the set of subgroups of* G *of index divisible by* n*;*

(ii) We define $\mathcal{I}_n(G, H)$ *to be the set of subgroups* L *of* G *such that* n *divides* $|H : L \cap H|$*.*

For convenience, we introduce the following ideals of $\mathcal{C}(G)$ (see Chapter 4, for further details). One should note here, the similarities between these sets and the characters involved in Brauer's induction theorem (see Chapter 1, Theorem 1.4). Explicitly, we have replaced the elementary subgroups used by Brauer, by subgroups in $\mathcal{I}_n(G)$ (respectively $\mathcal{I}_n(G,H)$).

Definition 2.4. *Let G be a group, $H \trianglelefteq G$, $N \trianglelefteq G$ and $n \in \mathbb{N}$ with $n \geq 1$. We define the following ideals of $\mathcal{C}(G)$.*

(i)

$$\mathcal{A}_n\left(G,H,N\right) := \left\{ \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}_K^G \phi_K \ : \ \phi_K \in \mathcal{C}(K) \right\};$$

(ii) $\mathcal{A}_n(G,N) := \mathcal{A}_n(G,G,N)$;

(iii) $\mathcal{A}_n(G) := \mathcal{A}_n(G,G,1) = \mathcal{A}_n(G,1)$.

We are now in a position to formally introduce the conjecture we are aiming to prove.

Conjecture 2.5. *Let G be a group and p be a prime. Let $\chi \in \mathrm{Irr}(G)$ and $a := v_p(\chi(1)$, then $\chi \in \mathcal{A}_{p^a}(G)$.*

Where possible we prove the above conjecture directly, for example in Chapter 8, where we verify Conjecture 2.6 in the case $a = 1$ and $G = S_n$. However, our general approach is to use an induction argument and in this situation we require a stronger statement.

Conjecture 2.6. *Let G be a group and p be a prime. Let $\chi \in \mathrm{Irr}(G)$ and $a := v_p(\chi(1))$, then $\chi \in \mathcal{A}_{p^a}\left(G, Z(G)\right)$.*

Note that it is clear why Conjecture 2.6 is a stronger statement than Conjecture 2.5 and why it in fact yields Conjecture 2.5 as a consequence. It should be noted that imposing that the subgroups we induce from contain the centre is a purely technical assumption and therefore should not detract from the direct relationship between Brauer's Induction Theorem and Conjecture 2.5.

We will now discuss the approach we are aiming to use to prove Conjecture 2.6, which will give us the final reformulation of the conjecture. We use a reduction theorem, the concept of which is introduced in Chapter 1, to prove our conjecture, therefore we define a collection of subgroups, that we call "Brauer-good" groups, related to the families of almost simple and quasisimple groups that we assume the conjecture holds for.

As will become apparent in Sections 6.1 and 6.2 and the definition of Brauer-good groups, we will need to work with characters of twisted group algebras. Therefore we require analogous definitions of $\mathcal{A}_n(G, H, N)$ for projective α-characters. Once again, the reader is directed to Section 5.3 for the terminology we are using regarding twisted group algebras.

Definition 2.7. *Let G be a group, $H \trianglelefteq G$, $N \trianglelefteq G$ and $n \in \mathbb{N}$ with $n \geq 1$. Let $\alpha \in H^2(G, \mathbb{C}^*)$ and let $A := \mathbb{C}^\alpha[G]$ denote its associated twisted group algebra. We define the following ideals of $\mathcal{C}(A)$.*

(i)

$$\mathcal{A}_n(A, H, N) := \left\{ \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}_K^G \phi_K : \phi_K \in \mathcal{C}(A[K]) \right\};$$

(ii) $\mathcal{A}_n(A, N) := \mathcal{A}_n(A, G, N)$.

Contrasting Definitions 2.4 and 2.7, it will be clear whether we are considering the ideals for generalised characters or projective α-characters. The following definitions now lead up to the definition of Brauer-good groups. First we recall the definitions of almost simple and quasisimple groups.

Definition 2.8. [31, §2.5.6]. *Let S be a non-abelian simple group and G be a group such that $S \leq G \leq \mathrm{Aut}(S)$. Then G is called almost simple. We denote by $\mathcal{X}$ the collection of almost simple groups.*

Definition 2.9. [17, Chapter 9]. *Let G be a perfect group such that $G/Z(G) \cong S$ where S is non-abelian and simple. Then G is called quasisimple. We denote by $\mathcal{Y}$ the collection of quasisimple groups.*

Definition 2.10. *Let $H \in \mathcal{X} \cup \mathcal{Y}$ and let p be a prime. Let S denote the associated simple group of H. Let $\alpha \in H^2(H, \mathbb{C}^*)$ and $A := \mathbb{C}^\alpha[H]$ be the associated twisted group algebra. We say H is Brauer-good if one of the following two conditions holds.*

(i) $H \in \mathcal{X}$, *and for every* $\chi \in \mathrm{Irr}(A)$ *of degree divisible by* p^a *such that* $\mathrm{Res}^H_S \chi \in \mathrm{Irr}(A[S])$, *we have* $\chi \in \mathcal{A}_{p^a}(H, S, 1)$;

(ii) $H \in \mathcal{Y}$, *and for every* $\chi \in \mathrm{Irr}(H)$ *of degree divisible by* p^a, *we have* $\chi \in \mathcal{A}_{p^a}(H, Z(H))$.

Finally, we formulate the conjecture. Under some additional assumptions, we prove the reduction theorems in Chapter 6. The details of these additional assumptions are discussed in Sections 6.1 and 6.2.

Conjecture 2.11. *Let $H \in \mathcal{X} \cup \mathcal{Y}$, then H is Brauer-good.*

Conjecture 2.12. *Let G be a group and p be a prime. Let $\chi \in \mathrm{Irr}(G)$ and $a := v_p(\chi(1))$. If Conjecture 2.11 holds, then $\chi \in \mathcal{A}_{p^a}(G, Z(G))$.*

CHAPTER 3

BACKGROUND MATERIAL AND PRELIMINARY RESULTS

The purpose of this chapter is to combine all the important background results that we require for the project. This will also include properties of $\mathcal{I}_n(G, H)$. Since the ideal $\mathcal{A}_n(G, H, N)$ is the fundamental object of study in this thesis, we dedicate a separate chapter to its properties. The reader is directed to Chapter 4 for these details.

Moreover, since we can prove independently useful results regarding the structure of $\mathcal{A}_n(G, H, N)$ if G is a wreath product, if G is a symmetric group or if we are working over a twisted group algebra, we dedicate a separate chapter to discuss these theories and to prove these results. The same applies to an important structural result, that can be deduced from modular representation theory. The reader should see Chapter 5 and its relevant sections for these.

3.1 Finite group theory

In this section, we recall some facts, defintions and lemmas regarding the structure of finite groups that we require throughout the remaining chapters of the project. First we note that our proof strategy is to assume we have a minimal counterexample and proceed by induction, hence we will require a strict partial order on subgroups of G.

Definition 3.1. *For groups H and K, we say $H \prec K$ if one of the following holds,* *(a)* $|H : Z(H)| < |K : Z(K)|$,

(b) *Or* $|H : Z(H)| = |K : Z(K)|$ *and* $|H| < |K|$.

Throughout, we will be using left conjugation.

Definition 3.2. *Let G be a group. For $g \in G$ and $x \in G$, we define the conjugate of x by g to be ${}^{g}x = gxg^{-1}$.*

The initial case of the reduction theorem will be to prove the Conjecture 2.6 for p-soluble groups.

Definition 3.3. [16, 3D]. *Let G be a group and p be a prime. A group G is p-soluble if there exists a subnormal series,*

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = G,$$

such that N_i/N_{i-1} is either an abelian group or a p'-group.

Remark 3.4. Note that we can refine the above series to be a composition series and in this situation we yield an equivalent definition namely that a group G is p-soluble if the non-abelian composition factors of G are p'-groups.

Lemma 3.5. *Let G be a p-soluble group and H be a subgroup of G. Then H is p-soluble.*

For convenience, we also introduce the following shorthand notation.

Definition 3.6. *Let G be a group and $n \in \mathbb{N}$, $n \geq 1$. Then we define $G^n := G \times \cdots \times G$.*

We recall the statement of Brauer's Induction Theorem, from the introduction. We note the use of elementary subgroups in this theorem and emphasise their importance to both our reduction theorem and the simplification of our computations for sporadic simple groups, see Chapter 9.

Definition 3.7. *Let G be a group. Let $H \trianglelefteq G$ and $N \trianglelefteq G$.*

(i) We say that a subgroup $L \leq G$ is p-decomposable, for some prime p, if $L = P \times Q$ where P is a p-group and Q is a p'-group. We define $\mathrm{PD}(G)$ *to be the set of such subgroups.*

(ii) We say that a subgroup $E \leq G$ is l-elementary, where l is prime, if $E = P \times Q$ where P is an l-group and Q is a cyclic l'-group. We say that $E \leq G$ is elementary if it is l-elementary for some prime l. We denote by $\mathrm{El}(G)$ *the collection of elementary subgroups of G.*[†]

(iii) We define $\mathrm{PD}_{\max}(G)$ *to be the set of maximal p-decomposable subgroups of G. $E \in$* $\mathrm{PD}_{\max}(G)$ *is maximal p-decomposable if $E \leq F \leq G$ for some $F \in$* $\mathrm{PD}(G)$ *implies $E = F$ or $F = G$.*

(iv) We define $\mathrm{El}_{\max}(G)$ *to be the analagous set of elementary subgroups (c.f. (iii)).*

(v) We define $\mathrm{El}_{p,a}(G, H, N)$ *to be the maximal elementary subgroups, $E \leq G$ such that p^a does not divide $|H : EN \cap H|$.*

Let G be a group and p be a prime. Suppose that $P \in \mathrm{Syl}_p(G)$. Throughout the project we will regularly encounter the group $M := PC_G(P)$. This is because M is an element of $\mathrm{PD}_{\max}(G)$ of index prime to p (c.f. Definition 3.7 (v)). The results of Chapter 4, imply that it is enough to consider the structure of $\mathcal{A}_p(M)$ to prove Conjecture 2.5 in the case $a = 1$.

Lemma 3.8. *Let G be a group and p be a prime. If $P \in \mathrm{Syl}_p(G)$, then $PC_G(P) \in \mathrm{PD}_{\max}(G)$.*

Proof. By the Schur-Zassenhaus Theorem, Theorem 2.1 in [13], we have that $PC_G(P) = P \rtimes F$ for some p'-group F. Since $\mathrm{Aut}_{PC_G(P)}(P) \cong P/Z(P)$ is a p-group, we have that F must act as p-automorphisms, hence we have that F must act trivially on P, i.e. $PC_G(P) = P \times F$. □

[†]Note that l-elementary subgroups are indeed l-decomposable. In fact, l-elementary subgroups are p-decomposable for any prime p.

Moreover, we have the following result which shows the importance of the previous result.

Lemma 3.9. *Let G be a group and let $E \in \mathrm{El}(G)$ with $R \in \mathrm{Syl}_p(E)$, then $E \leq RC_G(R)$.*

Proof. Since $E \in \mathrm{El}(G)$, we have that $E = R \times F$ for some p'-group F. Clearly $F \leq C_G(R)$ and the result follows. $\square$

We will consider in Section 5.2, the properties of $\mathcal{A}_n(G, H, N)$ when G is a wreath product. We will use the following lemma to construct an isomorphism of wreath products.

Lemma 3.10. *Let G, N and K be groups and $\psi : K \to \mathrm{Aut}(N)$ be a homomorphism. Suppose there exist group homomorphisms $\alpha : N \to G$ and $\beta : K \to G$ such that, for all $n \in N$ and $k \in K$,*

$$\beta(k)\alpha(n)\beta(k)^{-1} = \alpha\left(\psi(k)(n)\right),$$

then the map $\phi : N \rtimes_\psi K \to G$, defined by $\phi\left((n, k)\right) = \alpha(n)\beta(k)$, is a homomorphism.

Proof. If (n, k), $(m, l) \in N \rtimes K$, then $(n, k)(m, l) = (n\psi(k)(m), kl)$, by definition. Furthermore,

$$\phi\left((n, k)(m, l)\right) = \phi\left((n\psi(k)(m), kl)\right) = \alpha\left(n\psi(k)(m)\right)\beta\left(kl\right).$$

Using the condition in the Lemma and that α is a homomorphism, we have $\alpha\left(n\psi(k)(m)\right)$ $= \alpha(n)\beta(k)\alpha(m)\beta(k)^{-1}$. Therefore,

$$\phi\left((n, k)(m, l)\right) = \alpha(n)\beta(k)\alpha(m)\beta(k)^{-1}\beta(k)\beta(l) = \alpha(n)\beta(k)\alpha(m)\beta(l),$$

and the result follows. $\square$

The problem of solving Conjecture 2.5 when G is a p-group for a prime p, can be reduced to the study of abelian p-groups through the use of its abelianisation. Here, we give appropriate definitions and facts related to the abelianisation of a group G.

Definition 3.11. *Let G be a group. We define the derived subgroup of G, denoted G', to be the subgroup of G generated by commutators of G, namely,*

$$G' := \left\langle [x, y] = x^{-1}y^{-1}xy : x, y \in G \right\rangle .$$

We use the fact that G' is normal and that G' is the smallest normal subgroup such that its corresponding quotient group is abelian, to define the following.

Definition 3.12. *Let G be a group. The quotient group G/G' is called the abelianisation of G.*

The following property of the abelianisation of semi-direct products will be used in the proof to show that the abelianisation of the Sylow 2-subgroups of the symmetric group are elementary abelian.

Lemma 3.13. (See [3, Page 54]) *Let G be a group and suppose $G = N \rtimes H$ for groups N and H. Let $L := N/N' \times H/H'$, we have that there exists $K \trianglelefteq L$ such that $G/G' \cong L/K$.*

Finally, the following lemma shows us that if we take $G = S_{pw+e}$ where $0 \leq e < p$, and a Sylow p-subgroup of G, say P, then we can view P as a subgroup of $S_p \wr S_w$ with $S_p \wr S_w$ embedded as a subgroup of G in the natural way. This lemma is used in the verification of Conjecture 2.5 for symmetric groups, namely Theorem 8.1.

Lemma 3.14. *Let p be a prime and $w, e \in \mathbb{Z}_{\geq 0}$ with $e < p$. Let $G = S_{pw+e}$ and $P \in \mathrm{Syl}_p(G)$. For any subgroup $S_p \wr S_w \times S_e$, we can choose $P \leq S_p \wr S_w$.*

Proof. It is enough to show $|S_{pw+e}|_p = |S_p \wr S_w|_p$, as we can then take P to be a subgroup of $S_p \wr S_w$, without loss of generality, and then P has the desired structure as $C_p \wr Q \in \mathrm{Syl}_p(S_p \wr S_w)$. Clearly since $e < p$ and $v_p(ab) = v_p(a) + v_p(b)$ for $a, b \in \mathbb{Z}$, we have $v_p((pw+e)!) = v_p((pw)!)$. Moreover, $v_p((pw)!) = w + v_p(w!)$ as if $x \in \mathbb{Z}$, with $1 \leq x \leq pw$ and p divides x, then $x = pi$ with $1 \leq i \leq w$. Finally, $v_p((p!)^w w!) = w\,(v_p(p!)) + v_p(w!) = w + v_p(w!)$ and we are done. $\square$

3.2 Ordinary representation theory

This section consists of preliminary definitions, conventions we follow and properties of representations and characters over a field of characteristic zero, usually taken to be $\mathbb{C}$.

Definition 3.15. [16] *Let G and H be groups and $K \leq G$.*

(i) If χ_1 and $\chi_2 \in \mathrm{CF}(G)$, we define, for $g \in G$, $\chi_1 \cdot \chi_2 \in \mathrm{CF}(G)$, by $\chi_1 \cdot \chi_2(g) := \chi_1(g)\chi_2(g)$;

(ii) If $\chi_1 \in \mathrm{CF}(G)$ and $\chi_2 \in \mathrm{CF}(H)$, we define, for $g \in G$ and $h \in H$, $\chi_1 \times \chi_2 \in \mathrm{CF}(G \times H)$ by $(\chi_1 \times \chi_2)(g, h) := \chi_1(g)\chi_2(h)$;

(iii) We denote by $1_G \in \mathrm{Irr}(G)$, the principal character of G, i.e. for all $g \in G$, $1_G(g) = 1$;

(iv) We denote by $\rho_G \in \mathcal{C}(G)$, the regular character of G, i.e. $p_G(1) = |G|$ and for all $1 \neq g \in G$, $p_G(g) = 0$;

(v) For $\chi \in \mathrm{CF}(G)$, we define $\mathrm{Res}^G_H\chi \in \mathrm{CF}(H)$ by $\mathrm{Res}^G_H\chi(h) := \chi(h)$, for $h \in H$. This is the usual restriction map of characters;

(vi) For $\chi \in \mathrm{CF}(H)$, we define, for $g \in G$, $\mathrm{Ind}^G_H\chi \in \mathrm{CF}(G)$ by

$$\mathrm{Ind}^G_H\chi(g) := \frac{1}{|H|}\sum_{x\in G}\chi^\circ\left(xgx^{-1}\right),$$

where,

$$\chi^\circ(g) := \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}.$$

This is the usual induction map of characters;

(vii) For $\chi \in \mathrm{CF}(G)$, $\ker(\chi) := \{g \in G : \chi(g) = \chi(1)\}$.

(viii) For χ_1, $\chi_2 \in \mathcal{C}(G)$,

$$\langle\chi_1, \chi_2\rangle := \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)}.$$

Given Definition 3.15, the following result is immediate.

Lemma 3.16. *Let G and H be groups with $K \leq G$, $L \leq H$. Let $\alpha \in \mathcal{C}(K)$ and $\beta \in \mathcal{C}(L)$. Then,*

$$\mathrm{Ind}_K^G \alpha \times \mathrm{Ind}_L^H \beta = \mathrm{Ind}_{K\times L}^{G\times H} (\alpha \times \beta) .$$

For the later chapters of the project, we will require more specific maps on $\mathcal{C}(G)$. The following map essentially computes a "projection" of a generalised character onto a component labelled by an irreducible. This will be useful when considering $\mathcal{A}_{p^a}(G, H, N)$ in the case when G is p-decomposable and we are solely interested in the properties of the characters of the p-group.

Definition 3.17. *Let G and H be groups. Let $\theta \in \mathrm{Irr}(H)$. We define the homomorphism $\widetilde{\pi}_\theta : \mathcal{C}(G \times H) \to \mathcal{C}(G)$, by extending $\mathbb{Z}$-linearly the following map on $\mathrm{Irr}(G \times H)$, namely for $\chi_1 \times \chi_2 \in \mathrm{Irr}(G \times H)$,*

$$\widetilde{\pi}_\theta (\chi_1 \times \chi_2) := \begin{cases} \chi_1 & \text{if } \chi_2 = \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Due to the nature of reduction theorems, we will regularly be using the representation theory of normal subgroups and quotient groups to reduce our question for all finite groups to that of finite simple groups. In particular, we need to understand how we relate this representation theory back to our original group. This area of representation theory is known as Clifford Theory. The standard results that we need to use will be given here, but a detailed explanation of the topic can be found in [16, Chapter 6]

Definition 3.18. *Let G be a group and $H \leq G$. Let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(H)$, then we say that*

(i) $\chi \in \mathrm{Irr}(G|\theta)$ *if* $\left\langle \mathrm{Res}_H^G \chi, \theta \right\rangle \neq 0$.

(ii) $\chi \in \mathrm{Irr}(G||\theta)$ *if* $\mathrm{Res}_H^G \chi = e\theta$ *for some* $e \in \mathbb{Z}$.

Definition 3.19. [16, Lemma 2.22]. *Let G be a group and let $N \trianglelefteq G$. We define* $\mathrm{Inf}^G_{G/N} : \mathcal{C}(G/N) \to \mathcal{C}(G)$ *to be the character which takes the value,*

$$\left(\mathrm{Inf}^G_{G/N}\chi\right)(g) := \chi(gN),$$

for $\chi \in \mathcal{C}(G/N)$. For $\chi \in \mathcal{C}(G)$ such that $N \leq \ker(\chi)$, we define

$$\left(\mathrm{Def}^G_{G/N}\chi\right)(gN) := \chi(g).$$

The map below takes a character to the $\mathbb{Z}$-span of its linear irreducible constituents of a generalised character. This is used along with the inflation and deflation maps defined above to reduce the structure of $\mathcal{A}_{p^a}(P)$ for a p-group P, where p is prime, to that of its abelianisation, since the derived subgroup P' of P is a subgroup of the kernel of any linear character. These properties will be discussed in Chapter 4.

Definition 3.20. *Let G be a group and $\chi \in \mathcal{C}(G)$. Suppose that $\chi = \sum_{\theta \in \mathrm{Irr}(G)} a_\theta \theta$ where $a_\theta \in \mathbb{Z}$. We define $\widetilde{\pi}_L : \mathcal{C}(G) \to \mathcal{C}(G)$ by,*

$$\widetilde{\pi}_L(\chi) := \sum_{\substack{\theta \in \mathrm{Irr}(G) \\ \theta(1)=1}} a_\theta \theta.$$

The following proposition gives us that the induction and inflation maps commute. Therefore, where applicable, we will be able to consider whether the deflation of an irreducible character to a quotient group is a sum of suitably induced characters, and if so, lift (inflate) such an expression back to our original group.

Lemma 3.21. *Let G be a group, $K \leq G$ and $N \trianglelefteq G$ with $N \leq K$. If $\chi \in \mathcal{C}(K/N)$, then*

$$\mathrm{Ind}^G_K \mathrm{Inf}^K_{K/N}\chi = \mathrm{Inf}^G_{G/N}\mathrm{Ind}^{G/N}_{K/N}\chi.$$

Proof. If $g \in G$ then,

$$\mathrm{Ind}_K^G \mathrm{Inf}_{K/N}^K \chi(g) = \frac{1}{|K|} \sum_{x \in G} \left(\mathrm{Inf}_{K/N}^K \chi\right)^{\circ} \left(xgx^{-1}\right).$$

Now it follows from the definition of χ° from Definition 3.15 (vi) that $\left(\mathrm{Inf}_{K/N}^K \chi\right)^{\circ} (xgx^{-1})$ $= \chi^{\circ} (xgx^{-1}N)$. Moreover, if $xN = yN$, for $x, y \in G$, then $\chi^{\circ} (xgx^{-1}N) = \chi^{\circ} (ygy^{-1}N)$. Therefore,

$$\mathrm{Ind}_K^G \mathrm{Inf}_{K/N}^K \chi(g) = \frac{|N|}{|K|} \sum_{xN \in G/N} \chi^{\circ}(xgx^{-1}N) = \mathrm{Ind}_{K/N}^{G/N} \chi(gN) = \mathrm{Inf}_{G/N}^G \mathrm{Ind}_{K/N}^{G/N} \chi(g).$$

□

In Chapter 4, we show that $\mathcal{A}_n(G, H, N)$ is an ideal of $\mathcal{C}(G)$. The key component of this result is the tensor product formula below.

Proposition 3.22. (See [16, Problem (5.3)]). *Let G be a group and $H \leq G$. Then for $\alpha \in \mathrm{CF}(H)$ and $\beta \in \mathrm{CF}(G)$ we have*

$$\left(\mathrm{Ind}_H^G \alpha\right) \cdot \beta = \mathrm{Ind}_H^G \left(\alpha \cdot \mathrm{Res}_H^G \beta\right).$$

We now state the key results of Clifford Theory. We begin with the notation of a conjugate character. Informally, we extend the natural group action on normal subgroups to a group action on the set of class functions of a normal subgroup.

Definition 3.23. [16, Page 78] *Let G be a group. Let $N \trianglelefteq G$, $\theta \in \mathrm{Irr}(N)$ and $g \in G$. We define ${}^g\theta$, the conjugate character to θ in G, to be the character of G defined by ${}^g\theta(n) := \theta(g^{-1}ng)$ for all $n \in N$.*

Given the above definition, it is natural to consider the stabiliser of such an action.

Definition 3.24. [16, Definition 6.10] *Let G be a group, $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. The inertia subgroup of θ in G is defined by,*

$$I_G(\theta) := \{g \in G : {}^g\theta = \theta\}.$$

Moreover, we say that θ is G-invariant if $I_G(\theta) = G$.

Clifford's Theorem, which follows, is the key result from which Clifford Theory is built.

Theorem 3.25. (See [16, Theorem 6.2]) *Let G be a group. Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Consider $\theta \in \mathrm{Irr}(N)$ such that $\chi \in \mathrm{Irr}(G|\theta)$ and let $e := \langle \mathrm{Res}^G_N \chi, \theta \rangle$. Let θ_i, for $i = 1, \ldots, t$ denote the distinct conjugates of θ in G. Then,*

$$\mathrm{Res}^G_N \chi = e \sum_{i=1}^{t} \theta_i.$$

The following definitions allow us to simplify our reduction statement considerably. Given a particular normal subgroup, N, we are able to reduce to the case that G/N is nilpotent, which, using in [16, Theorem (6.22)], yields that G is a relative M-group. In addition if our character χ is primitive, we will be able to assume that $\mathrm{Res}^G_N \chi \in \mathrm{Irr}(N)$.

Definition 3.26. [16, Definition 6.21] *Let G be a group, $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$. We say that*

(i) χ is a relative M-character (with respect to N) if there exists a subgroup H, with $N \leq H \leq G$ and $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}^G_H \psi = \chi$ and $\mathrm{Res}^H_N \psi \in \mathrm{Irr}(N)$;

(ii) G is a relative M-group (with respect to N), if every irreducible character of G is a relative M-character.

Isaacs notes in [16], following Theorem (6.22), that if G/N is nilpotent then G is a relative M-group with respect to N.

Definition 3.27. [16, Page 66] *Let G be a group. A character $\chi \in \mathrm{Irr}(G)$ is called primitive if $\chi \neq \mathrm{Ind}^G_H \theta$ for any $H < G$ and $\theta \in \mathcal{C}(H)$.*

Finally, we use the following to directly associate the representation theory of G to that of N.

Lemma 3.28. (See [16, Corollary (6.12)]) *Let G be a group and $\chi \in \mathrm{Irr}(G)$ that is primitive. Then for every $N \trianglelefteq G$, there exists $\theta_N \in \mathrm{Irr}(N)$ such that $\chi \in \mathrm{Irr}(G||\theta_N)$.*

A key result which will play a pivotal role in the proofs of this thesis is the following. We will regularly find ourselves, inflating, inducing or extending,a decomposition for a character to some overgroup, and then obtaining a decomposition for our irreducible character by restriction. Since our original decomposition is a linear combination of induced characters, we need to understand the outcome when a restriction map is applied after an induction map. The Mackey decomposition formula, or Mackey's theorem below, describes this relationship explicitly. In particular, it modifies a decomposition of restricted induced characters (which we cannot use too effectively) into a decomposition of induced characters, which is exactly the definition of our $\mathcal{A}_n(G, H, N)$.

Theorem 3.29. (See [16, Problem (5.6)]) *Let G be a group, with H, $K \leq G$. Suppose $G = \bigcup_{t \in T} HtK$ where T denotes a complete set of (H, K)-double coset representatives in G. Then for $\psi \in \mathcal{C}(H)$,*

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H (\psi) = \sum_{t \in T} \mathrm{Ind}^K_{{}^tH \cap K} \mathrm{Res}^{{}^tH}_{{}^tH \cap K} {}^t\psi.$$

Our next steps, will be to prove a series of results, which are similar in nature to those of Clifford Theory, that we use throughout the thesis.

Lemma 3.30. *Let G be a group with $K \leq H \leq G$. Suppose $\chi \in \mathrm{Irr}(G||\theta)$ for $\theta \in \mathrm{Irr}(K)$. If $\phi \in \mathrm{Irr}(H)$ such that $\chi \in \mathrm{Irr}(G|\phi)$, then $\phi \in \mathrm{Irr}(H||\theta)$.*

Proof. Suppose that $\mathrm{Res}^G_H \chi = \phi + \psi$ for some $\psi \in \mathcal{C}(H)$, where ψ is a non-negative linear combination of irreducibles of H. Then $\mathrm{Res}^G_K \chi = \mathrm{Res}^H_K \phi + \mathrm{Res}^H_K \psi$. On the other hand, $\mathrm{Res}^G_K \chi = e\theta$ for some $e \in \mathbb{Z}_{>0}$, hence $\mathrm{Res}^H_K(\phi) = f\theta$ for some $f \leq e$. □

Lemma 3.31. *Let Γ be a group and with $A \leq K \leq \Gamma$. If λ_1, $\lambda_2 \in \mathrm{Irr}(A)$ are distinct and $\chi \in \mathrm{Irr}(\Gamma||\lambda_1)$ and $\phi \in \mathrm{Irr}(K||\lambda_2)$, then*

$$\langle \chi \,,\, \mathrm{Ind}^\Gamma_K \phi \rangle = 0.$$

Proof. Suppose for a contradiction that $\langle \chi, \mathrm{Ind}^\Gamma_K \phi \rangle = \langle \mathrm{Res}^\Gamma_K \chi, \phi \rangle \neq 0$. As $\langle \mathrm{Res}^\Gamma_K \chi, \phi \rangle \neq$

0, we have that $\left\langle \mathrm{Res}^{\Gamma}_{A}\chi, \mathrm{Res}^{K}_{A}\phi \right\rangle \neq 0$ also. But,

$$\left\langle \mathrm{Res}^{\Gamma}_{A}\chi, \mathrm{Res}^{K}_{A}\phi \right\rangle = \langle e\lambda_1, f\lambda_2 \rangle = ef \langle \lambda_1, \lambda_2 \rangle = 0,$$

for $e,\ f \in \mathbb{Z}_{>0}$. □

Proposition 3.32. *Let G be a group. Suppose $\phi \in \mathrm{Irr}(H)$ where $H \leq G$ such that $\mathrm{Ind}^{G}_{H}\phi := \chi \in \mathrm{Irr}(G)$, then $Z(G) \leq H$.*

Proof. Suppose that G is a minimal counterexample and define $L := HZ(G)$. If $L < G$, then since $\mathrm{Ind}^{L}_{H}\phi \in \mathrm{Irr}(L)$ (as induction is linear), we have that $Z(L) \leq H$. However, $Z(G) \leq Z(L)$ by definition of L, so this is a contradiction and $L = G$. Moreover, $I_G(\phi) = G$ and $\mathrm{Res}^{G}_{H}\chi = |G : H|\phi$. Since $|G : H| = \left\langle \mathrm{Res}^{G}_{H}\chi, \phi \right\rangle = \left\langle \chi, \mathrm{Ind}^{G}_{H}\phi \right\rangle = 1$, we have that $H = HZ(G)$ and $Z(G) \leq H$. □

We also require the following property of $\mathcal{I}_n(G, H)$ for direct products.

Lemma 3.33. *Let G, H be groups with normal subgroups K and L respectively. If $Q \in \mathcal{I}_{p^{a_1}}(G, K)$ with $N \leq Q$ and $R \in \mathcal{I}_{p^{a_2}}(H, L)$ with $M \leq R$, then $Q \times R \in \mathcal{I}_{p^{a_1+a_2}}(G \times H, K \times L)$ with $N \times M \leq Q \times R$.*

Proof. We clearly need only to prove that $p^{a_1+a_2}$ divides the index $|K \times L : (K \times L) \cap (Q \times R)|$. However, as $(K \times L) \cap (Q \times R) = (K \cap Q) \times (L \cap R)$, p^{a_1} divides $|K : K \cap Q|$ and p^{a_2} divides $|L : L \cap R|$, we are done. □

We also rely on the theory of character triples, which can be found in [16], but we summarise here.

Definition 3.34. *Let $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ such that θ is G-invariant. We say that (G, N, θ) is a character triple.*

Definition 3.35. (See [16, Definition (11.23)]) *Let (G, N, θ) and (Γ, A, λ) be character triples and let $\tau : G/N \to \Gamma/A$ be an isomorphism. For $N \leq H \leq G$, let H^{τ} denote the inverse image in Γ of $\tau(H/N)$. For every such H, suppose there exists a map σ_H :*

$\mathcal{C}(H|\theta) \to \mathcal{C}(H^\tau|\theta)$ such that the following conditions holds for H, K with $N \leq K \leq H \leq G$ and $\chi, \psi \in \mathcal{C}(H|\theta)$.

(i) $\sigma_H(\chi + \psi) = \sigma_H(\chi) + \sigma_H(\psi)$;

(ii) $\langle \chi, \psi \rangle = \langle \sigma_H(\chi), \sigma_H(\psi) \rangle$;

(iii) $\sigma_K\left(\mathrm{Res}^H_K \chi\right) = \mathrm{Res}^{H^\tau}_{K^\tau} \sigma_H(\chi)$;

(iv) $\sigma_H(\chi\beta) = \sigma_H(\chi)\beta^\tau$ *for* $\beta \in \mathrm{Irr}(H/N)$.

Let σ denote the union of the maps σ_H. Then (τ, σ) is an isomorphism from (G, N, θ) to (Γ, A, λ).

The following theorem is useful for our purposes to be able to introduce an isomorphic character triple, essentially replacing our irreducible character (θ above) with a linear irreducible character λ.

Theorem 3.36. (See [16, Theorem (11.28)]) *Let (G, N, θ) be a character triple and let (Γ, π) be a finite central extension of G/N having the projective lifting property. Let $A := \ker(\pi)$. Then (G, N, θ) is isomorphic to (Γ, A, λ) for some $\lambda \in \mathrm{Irr}(A)$.*

CHAPTER 4

PROPERTIES OF $\mathcal{A}_N(G, H, N)$

In this chapter we prove a series of important lemmas and propositions regarding the structure of the ideal $\mathcal{A}_n(G, H, N)$. As this ideal of the character ideal is the main focus of the project, there are many technical results in this section and where possible their uses in the proofs will be described.

4.1 p-soluble groups and current research

We begin by explaining the relationship between this ideal and the underlying motivation for its study found in [11]. As seen briefly in the introduction, we start with the formal definition of $\mathcal{I}(G, P, \mathcal{S})$, where we recall the definition of $\mathcal{S}$, from Definition 1.3.

Definition 4.1. [11, Definition 1.3] *Let G be a group and p be a prime. Let $H \leq G$ and P be a p-subgroup of G. Let $\mathcal{S} := \mathcal{S}(G, P, H)$. We define $\mathcal{I}(G, P, \mathcal{S})$ to be the integral span of characters of the form $\mathrm{Ind}_L^G \phi$ where $\phi \in \mathcal{C}(L)$ where $L \leq G$ such that,*

- *$L \cap P \in \mathrm{Syl}_p(L)$;*
- *$L \cap P \in \mathcal{S}$.*

We first show directly the relationship between $\mathcal{I}(G, P, \mathcal{S})$ and our ideal $\mathcal{A}_n(G, H, N)$.

Lemma 4.2. *Let G be a group and p be a prime. Let $P \in \mathrm{Syl}_p(G)$. Suppose $H \leq G$ such that $N_G(P) \leq H$ and let $\mathcal{S} = \mathcal{S}(G, P, H)$, then $\mathcal{I}(G, P, \mathcal{S}) \subseteq \mathcal{A}_p(G)$.*

Proof. Let $\chi \in \mathcal{I}(G,P,\mathcal{S})$, then

$$\chi = \sum_{L \leq G} \mathrm{Ind}_L^G \phi_{(L)},$$

where $\phi_{(L)} \in \mathcal{C}(L)$, $L \cap P \in \mathrm{Syl}_p(L)$ and $L \cap P \in \mathcal{S}$. We show p divides $|G : L|$. It is enough to show p divides $|P : L \cap P|$. Since $L \cap P \in \mathcal{S}$, we have $L \cap P \leq P \cap {}^tP$ for some $t \in G \backslash H$. Since $t \notin N_G(P)$, $P \cap {}^tP < P$ and p divides $|P : L \cap P|$. □

Definition 4.3. [11, Section 1.1] *Let G be a group and p be a prime. Let $P \in \mathrm{Syl}_p(G)$ and $H \leq G$ such that $N_G(P) \leq H$. We say that the pair (G,H) satisfies (pRes-Syl) if,*

$$\mathrm{Res}_H^G\left(\mathcal{C}^p(G)\right) \subseteq \mathcal{C}^p(H) + \mathcal{I}\left(H,P,\mathcal{S}(G,P,H)\right).$$

Our aim is now to show that if (pRes-Syl) holds for the pair $(G, N_G(P))$, then Conjecture 2.5 holds. This requires some technical lemmas so we determine the properties of $\mathcal{A}_n(G,H,N)$ we need to prove this result. First, we see the direct influence Brauer's Induction Theorem has on our proof.

Definition 4.4. *Let $v_p : \mathbb{N} \to \mathbb{N}$ denote the p-adic valuation map, i.e. if $m \in \mathbb{N}$ such that $m = p^a m'$ where m' is prime to p, then $v_p(m) := a$.*

Definition 4.5. *Let G be a group, $H \trianglelefteq G$, $N \trianglelefteq G$ and $E \leq G$ and $a \in \mathbb{Z}_{\geq 0}$. We define,*

$$b = b(E,N,H,a) := \begin{cases} a - v_p\left(|H : EN \cap H|\right) & \textit{if } a \geq v_p\left(|H : EN \cap H|\right) \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 4.6. *Let G be a group. Let $H \trianglelefteq G$ and $N \trianglelefteq G$. Suppose that $\chi \in \mathrm{Irr}(G)$ such that p^a divides $\chi(1)$. Let $E \in \mathrm{El}_{\max}(G)$ and $b := b(E,N,H,a)$. If $\mathrm{Res}_{EN}^G \chi \in \mathcal{A}_{p^b}(EN, EN \cap H, N)$ for all $E \in \mathrm{El}_{\max}(G)$, then $\chi \in \mathcal{A}_{p^a}(G,H,N)$.*

Proof. Without loss of generality, from Brauer's Induction Theorem we may assume that,

$$1_G = \sum_{E \in \mathrm{El}_{\max}(G)} \mathrm{Ind}_E^G \phi_{(E)},$$

where $\phi_{(E)} \in \mathcal{C}(E)$. Note that this is without loss of generality since if $E \in \mathrm{El}(G)$ such that $E \leq F \in \mathrm{El}(G)$, then $\mathrm{Ind}_E^G \phi_{(E)} = \mathrm{Ind}_F^G \eta_{(F)}$ where $\eta_{(F)} \in \mathcal{C}(F)$, hence we can assume that the sum is taken over maximal elementary subgroups. By transitivity of induction, we have that there exists $\zeta_{(E)} \in \mathcal{C}(EN)$ such that,

$$1_G = \sum_{E \in \mathrm{El}_{\max}(G)} \mathrm{Ind}_{EN}^G \zeta_{(E)}.$$

Multiplying by χ, we obtain,

$$\chi = \sum_{E \in \mathrm{El}_{\max}(G)} \mathrm{Ind}_{EN}^G \left(\mathrm{Res}_{EN}^G \chi \zeta_{(E)} \right),$$

using the tensor product formula. From the assumption in the statement of the proposition, we have that for each $E \in \mathrm{El}_{\max}(G)$,

$$\mathrm{Res}_{EN}^G \chi = \sum_{\substack{K \in \mathcal{I}_{p^b}(EN, H \cap EN) \\ N \leq K}} \mathrm{Ind}_K^{EN} \theta_{(K)},$$

where $\theta_{(K)} \in \mathcal{C}(K)$. Substituting these expressions into the decomposition for χ we obtain

$$\chi = \sum_{E \in \mathrm{El}_{\max}(G)} \sum_{\substack{K \in \mathcal{I}_{p^b}(EN, H \cap EN) \\ N \leq K}} \mathrm{Ind}_K^G \left(\theta_{(K)} \mathrm{Res}_K^{EN} \zeta_{(E)} \right).$$

Now we fix an $E \in \mathrm{El}_{\max}(G)$ and $K \in \mathcal{I}_{p^b}(EN, H \cap EN)$ and consider $|H : H \cap K|$. By definition of $\mathcal{I}_{p^b}(EN, H \cap EN)$, we have that p^b divides $|EN \cap H : (EN \cap H) \cap K|$, therefore p^a divides $|H : H \cap EN||H \cap EN : (H \cap EN) \cap K|$. Since $K \leq EN$, we have p^a divides $|H : H \cap K|$, i.e. $\chi \in \mathcal{A}_{p^a}(G, H, N)$. $\square$

Corollary 4.7. *Let G be a group. Let $H \trianglelefteq G$ and $N \trianglelefteq G$. If for all $E \in \mathrm{El}_{p,a}(G,H,N)$, we have that $\mathrm{Res}^G_{EN}\chi \in \mathcal{A}_{p^b}(EN, EN \cap H, N)$, then $\chi \in \mathcal{A}_{p^a}(G,H,N)$ where b is taken as in Proposition 4.6.*

Proof. By Proposition 4.6, this statement holds if we show this for all $E \in \mathrm{El}_{\max}(G)$. However, if we take $E \in \mathrm{El}_{\max}(G)$ with $E \notin \mathrm{El}_{p,a}(G,H,N)$, then p^a divides $|H : EN \cap H|$, and hence $b = 0$. It follows trivially from the definition that $\mathrm{Res}^G_{EN}\chi \in \mathcal{A}_1(EN, EN \cap H, N)$ since $EN \in \mathcal{I}_1(EN, EN \cap H)$. Hence it is sufficient to show this holds for $E \in \mathrm{El}_{p,a}(G,H,N)$. □

From the above, one can see that Brauer's induction theorem gives us a simplification to determine whether a given irreducible belongs to $\mathcal{A}_{p^a}(G,H,N)$, namely that we need only show that for suitable prime powers and subgroups that the restriction of this character to each of these subgroups is induced from subgroups of index divisible by the prime powers. In the case that $H = G$, $N = 1$ and $a = 1$, this reduces our problem to restricting to only one maximal elementary subgroup of index prime to p, as we use regularly in the latter chapters. Now we show how $\mathcal{A}_n(G,H,N)$ behaves under restriction.

Lemma 4.8. *Let $H \trianglelefteq G$, $N \trianglelefteq G$ and $K \leq G$ with $N \leq K$. Let $\chi \in \mathcal{A}_n(G,H,N)$, then $\mathrm{Res}^G_K\chi \in \mathcal{A}_m(K, K \cap H, N)$ where $m = n/h$, where $h = (|H : K \cap H|, n)$.*

Proof. Since $\chi \in \mathcal{A}_n(G,H,N)$, we suppose

$$\chi = \sum_{\substack{L \in \mathcal{I}_n(G,H) \\ N \leq L}} \mathrm{Ind}^G_L \phi_{(L)},$$

where $\phi_{(L)} \in \mathcal{C}(L)$. Therefore upon restriction and applying the Mackey formula, we have

$$\mathrm{Res}^G_K\chi = \sum_{\substack{L \in \mathcal{I}_n(G,H) \\ N \leq L}} \sum_{t \in T(L)} \mathrm{Ind}^K_{{}^tL \cap K} \mathrm{Res}^{{}^tL}_{{}^tL \cap K} {}^t\phi_{(L)}.$$

First since $N \leq L$ and $N \trianglelefteq G$, we have that $N \leq {}^tL$ for every $t \in T(L)$. Moreover by assumption, $N \leq K$, hence $N \leq {}^tL \cap K$ for each L. Secondly, we consider $|H \cap K :$

$(H \cap K) \cap ({}^tL \cap K)|$. For simplicity we define $H' := H \cap K$. Since n divides $|H : L \cap H|$, we have that n divides $|H : {}^tL \cap H|$ as $H \trianglelefteq G$. Hence, n divides

$$|H : {}^tL \cap H||{}^tL \cap H : {}^tL \cap H'| = |H : {}^tL \cap H'|.$$

Therefore n divides, $|H : H'||H' : {}^tL \cap H'|$. From the definition of h in the statement of the lemma we have,

$$m = \frac{n}{h} \left| \frac{|H : H'|}{h} \left| H' : {}^tL \cap H' \right| \right.$$

Finally since m and $|H : H'|/h$ are necessarily coprime, we have that m divides $|H' : {}^tL \cap H'|$, i.e. $\mathrm{Res}^G_K \chi \in \mathcal{A}_m(K, H', N)$. □

The previous lemma shows us how $\mathcal{A}_n(G, H, N)$ behaves under restriction, so it is natural to consider now what happens under induction. However, this is easy to see from the definition alone. We give a formal proof below.

Lemma 4.9. *Let G be a group. Let $Z \leq H \leq G$ for $Z \trianglelefteq G$. Suppose that $\chi \in \mathcal{A}_{p^b}(H, Z)$, then $\mathrm{Ind}^G_H \chi \in \mathcal{A}_{p^a}(G, Z)$ where $a := b + v_p(|G : H|)$.*

Proof. Since $\chi \in \mathcal{A}_{p^b}(H, Z)$, we have,

$$\chi = \sum_{\substack{L \in \mathcal{I}_{p^b}(H) \\ Z \leq L}} \mathrm{Ind}^H_L \phi_{(L)},$$

for $\phi_{(L)} \in \mathcal{C}(L)$. Then,

$$\mathrm{Ind}^G_H \chi = \mathrm{Ind}^G_H \left(\sum_{\substack{L \in \mathcal{I}_{p^b}(H) \\ Z \leq L}} \mathrm{Ind}^H_L \phi_{(L)} \right) = \sum_{\substack{L \in \mathcal{I}_{p^b}(H) \\ Z \leq L}} \mathrm{Ind}^G_H \mathrm{Ind}^H_L \phi_{(L)}.$$

Recall that $a = b + v_p(|G : H|)$. Since $\mathcal{I}_{p^b}(H) \subseteq \mathcal{I}_{p^a}(G)$,

$$= \sum_{\substack{L \in \mathcal{I}_{p^a}(G) \\ Z \leq L}} \mathrm{Ind}^G_L \phi_{(L)} \in \mathcal{A}_{p^a}(G, Z),$$

where we take $\phi_{(L)}$ to be 0, if for some $L \in \mathcal{I}_{p^a}(G)$, no corresponding $\phi_{(L)}$ exists. $\square$

The following lemma shows the behaviour of $\mathcal{A}_n(G, H, N)$ with regard to direct products.

Lemma 4.10. *Let G and H be groups with $N \trianglelefteq G$ and $M \trianglelefteq H$. Moreover, let $K \trianglelefteq G$ and $L \trianglelefteq H$ and suppose $\zeta_1 \in \mathcal{A}_{p^{a_1}}(G, K, N)$ and $\zeta_2 \in \mathcal{A}_{p^{a_2}}(H, L, M)$, then $\zeta_1 \times \zeta_2 \in \mathcal{A}_{p^{a_1+a_2}}(G \times H, K \times L, M \times N)$.*

Proof. Since $\zeta_1 \in \mathcal{A}_{p^{a_1}}(G, K, N)$,

$$\zeta_1 = \sum_{\substack{Q \in \mathcal{I}_{p^{a_1}}(G,K) \\ N \leq Q}} \mathrm{Ind}_Q^G \phi_{(Q)},$$

where $\phi_{(Q)} \in \mathcal{C}(Q)$. Similarly,

$$\zeta_2 = \sum_{\substack{R \in \mathcal{I}_{p^{a_2}}(H,L) \\ M \leq R}} \mathrm{Ind}_R^H \psi_{(R)},$$

where $\psi_{(R)} \in \mathcal{C}(R)$. Hence we have that

$$\zeta_1 \times \zeta_2 = \sum_{\substack{Q \in \mathcal{I}_{p^{a_1}}(G,K) \\ N \leq Q}} \sum_{\substack{R \in \mathcal{I}_{p^{a_2}}(H,L) \\ M \leq R}} \mathrm{Ind}_{Q \times R}^{G \times H} \left(\phi_{(Q)} \times \psi_{(R)}\right).$$

Therefore by Proposition 3.33, $\zeta_1 \times \zeta_2 \in \mathcal{A}_{p^{a_1+a_2}}(G \times H, K \times L, M \times N)$. $\square$

The following uses Lemma 3.31. One key step in the reduction theorems and Theorem 4.12 is to reduce to the case that G/N is nilpotent, for some normal subgroup $N \trianglelefteq G$. We do this by showing that $G = EN$ for an elementary subgroup $E \in \mathrm{El}(G)$. However the case when $EN < G$ requires us to refine what we can say about the degree of θ, and to solve this problem we use the theory of isomorphic character triples, see [16, Chapter 11], for details. The following allows us to simplify this section of the proof.

Lemma 4.11. *Let G be a group and $A \leq Z := Z(G)$. Let $\chi \in \mathcal{A}_n(G, Z)$ where $n \in \mathbb{N}$, $n \geq 1$, i.e.*

$$\chi = \sum_{\substack{K \in \mathcal{I}_n(G) \\ Z \leq K}} \sum_{\phi \in \mathrm{Irr}(K)} a_\phi \mathrm{Ind}_K^G \phi,$$

for some $a_\phi \in \mathbb{Z}$. Suppose that $\chi \in \mathrm{Irr}(G||\lambda)$ for $\lambda \in \mathrm{Irr}(A)$, then

$$\chi = \sum_{\substack{K \in \mathcal{I}_n(G) \\ Z \leq K}} \sum_{\phi \in \mathrm{Irr}(K||\lambda)} a_\phi \mathrm{Ind}_K^G \phi. \tag{4.1}$$

Proof. Let ζ be the sum in Equation (4.1). Moreover we define,

$$\psi := \sum_{\substack{K \in \mathcal{I}_n(G) \\ Z \leq K}} \sum_{\substack{\phi \in \mathrm{Irr}(K||\lambda') \\ \lambda' \neq \lambda \in \mathrm{Irr}(A)}} a_\phi \mathrm{Ind}_K^G \phi$$

so that $\chi = \zeta + \psi$. By Lemma 3.31, we have that $1 = \langle \chi, \chi \rangle = \langle \chi, \zeta + \psi \rangle = \langle \chi, \zeta \rangle$ and for any $\theta \in \mathrm{Irr}(G||\lambda)$ with $\theta \neq \chi$, $0 = \langle \theta, \chi \rangle = \langle \theta, \zeta + \psi \rangle = \langle \theta, \zeta \rangle$. A further application of Lemma 3.31 gives that for $\mu \in \mathrm{Irr}(G||\lambda')$ for $\lambda' \neq \lambda \in \mathrm{Irr}(A)$, we have $0 = \langle \mu, \zeta \rangle$. Since A is central, this covers all irreducibles of G and $\chi = \zeta$. □

The final step before proving Proposition 4.21 is to show that Conjecture 2.12 holds for p-soluble groups. This is often a starting point for reduction theorems in Representation Theory and its influence on the reduction theorems in Sections 6.1 and 6.2 is notable.

Theorem 4.12. *Let p be a prime. Suppose that G is a p-soluble group and $\chi \in \mathrm{Irr}(G)$. Then $\chi \in \mathcal{A}_{p^a}(G, Z(G))$ where $a := v_p(\chi(1))$.*

Proof. Assume that (G, χ) is a minimal counterexample to Theorem 4.12 with respect to the partial order on subgroups of G given by Definition 3.1. Therefore we can assume the following.

Claim 4.13. *Without loss of generality, χ is primitive.*

Proof. Suppose that χ is not primitive, i.e. there exists a proper subgroup H of G and an irreducible character ϕ of H such that $\chi = \mathrm{Ind}_H^G \phi$. Note that by Proposition 3.32, we have that $Z(G) \leq H$. If p^a divides $|G : H|$ then, $\chi \in \mathcal{A}_{p^a}(G, Z(G))$, which contradicts that G is a minimal counterexample, so we can assume $b := v_p(|G : H|) < a$.

Note that we have p^{a-b} divides $\phi(1)$ and ϕ is irreducible. Since $|H : Z(H)| < |G : Z(G)|$, $H \prec G$. As G is a minimal counterexample and H is p-soluble by Lemma 3.5, $\phi \in \mathcal{A}_{p^{a-b}}(H, Z(H)) \subseteq \mathcal{A}_{p^{a-b}}(H, Z(G))$. Finally we have $\chi \in \mathcal{A}_{p^a}(G, Z(G))$ by Lemma 4.9. □

Claim 4.14. *$G/Z(G)$ is not simple.*

Proof. Suppose therefore that $G/Z(G)$ is simple. Moreover, suppose for a contradiction that $G/Z(G)$ is abelian. Since $G/Z(G)$ is simple and abelian, we have that $G/Z(G)$ is cyclic and hence G is abelian. This contradicts our choice of χ, since in this case all characters would be linear. Therefore $G/Z(G)$ is non-abelian.

Since G is p-soluble, $G/Z(G)$ is a p'-group, by Remark 3.4. Applying [16, Corollary (11.29)], applied to $N = Z(G)$, we have for any irreducible constituent $\lambda \in \mathrm{Irr}(Z(G))$ of $\mathrm{Res}^G_{Z(G)}\chi$, that $\chi(1)/\lambda(1)$ divides $|G : Z(G)|$. Since λ is linear, we have that $\chi(1)$ divides $|G : Z(G)|$, which by assumption is a p'-group, but this contradicts that p^a divides $\chi(1)$. Hence $G/Z(G)$ is not simple. □

By Claim 4.14, we assume that $N/Z(G) \trianglelefteq G/Z(G)$ is a non-trivial minimal normal subgroup. By Claim 4.13 and Lemma 3.28, we have that there exists $\theta \in \mathrm{Irr}(N)$ such that $\chi \in \mathrm{Irr}(G\|\theta)$. We assume that $\mathrm{Res}^G_N\chi = e\theta$ for $e \in \mathbb{N}$, $e \geq 1$. We therefore have that θ satisfies the following.

Claim 4.15. *θ is G-invariant, i.e. $I_G(\theta) = G$.*

Proof. Suppose that θ is not G-invariant, i.e. $T := I_G(\theta) < G$. We know in this case there exists a $\phi \in \mathrm{Irr}(T)$ such that $\chi = \mathrm{Ind}^G_T\phi$. However since T is a proper subgroup of G, this contradicts Claim 4.13. □

We now show that we can assume, without loss, that $e\theta(1)$ satisfies the following divisibility conditions.

Claim 4.16. *Without loss of generality, p^a divides $\theta(1)$.*

Proof. Suppose that $0 \leq v_p(\theta(1)) < a$. Therefore $0 < v_p(e) \leq a$. We consider the following character triple, (G, N, θ) and we introduce an isomorphic character triple, given by Theorem 3.36, say (Γ, A, λ), such that (Γ, π) is a finite central extension of G/N having the projective lifting property, $A \leq Z(\Gamma)$ and λ is a linear character of $\mathrm{Irr}(A)$. Using Definition 3.35, we denote the isomorphism by (τ, σ). We show $|\Gamma : Z(\Gamma)| < |G : Z(G)|$, so that $\Gamma \prec G$ and Γ satisfies Theorem 4.12. Now,

$$|\Gamma : Z(\Gamma)| = \frac{|G : N|}{|Z(\Gamma) : A|} \leq |G : N| < |G : N||N : Z(G)| = |G : Z(G)|$$

using $\Gamma/A \cong G/N$, $A \leq Z(\Gamma)$, so $|Z(\Gamma) : A| \geq 1$ and that $N/Z(G)$ is non-trivial. Hence the induction hypothesis can be applied to Γ . We define $\psi := \sigma_G(\chi)$ and $B := v_p(\psi(1))$, so that $\psi \in \mathcal{A}_{p^B}(\Gamma, Z(\Gamma))$. We know

$$\frac{\chi(1)}{\theta(1)} = \frac{\psi(1)}{\lambda(1)}$$

and therefore

$$v_p(\chi(1)) - v_p(\theta(1)) = v_p(\chi(1)) - v_p((1/e)\chi(1)) = v_p(e) = v_p(\psi(1)),$$

since λ is linear. By our assumption on $\theta(1)$, $0 < B \leq a$. Note also that Γ is p-soluble since G is p-soluble and that $A \leq Z(\Gamma)$. Applying the statement of Theorem 4.12 to ψ we have

$$\psi = \sum_{\substack{K \in \mathcal{I}_{p^B}(\Gamma) \\ K \geq A}} \sum_{\phi \in \mathrm{Irr}(K)} a_{\phi,K} \mathrm{Ind}_K^\Gamma \phi.$$

Note that $A \leq K$ since $A \leq Z(\Gamma) \leq K$. Since $\chi \in \mathrm{Irr}(G||\theta)$, we have that $\mathrm{Res}_A^\Gamma \psi \in \mathrm{Irr}(\Gamma||\lambda)$. Hence by Lemma 4.11, we have that

$$\psi = \sum_{\substack{K \in \mathcal{I}_{p^B}(\Gamma) \\ A \leq K}} \sum_{\phi \in \mathrm{Irr}(K||\lambda)} a_{\phi,K} \mathrm{Ind}_K^\Gamma \phi.$$

We are now able to use the bijection (τ, θ) to obtain a suitable decomposition for χ. We let $H \leq G$ such that $(H/N)^\tau = K/A$, then using the bijection we have that there exists $\mu \in \mathrm{Irr}(H||\theta)$ such that $\sigma_H(\mu) = \phi$, for each $\theta \in \mathrm{Irr}(K||\lambda)$. Using the Correspondence Theorem, we know $|\Gamma : K| = |G : H|$, so H has index divisible by p^b in G. Furthermore since $N \leq H$ and $Z(G) \leq N$, we have $Z(G) \leq H$. Using [16, Definition 11.23]

$$\sigma_G(\chi) = \sum_{\substack{K \in \mathcal{I}_{p^B}(\Gamma) \\ K \geq A}} \sum_{\phi \in \mathrm{Irr}(K)} a_{\phi,K} \mathrm{Ind}_K^\Gamma \phi = \sum_{\substack{H \in \mathcal{I}_{p^B}(G) \\ H \geq Z(G)}} \sum_{\mu \in \mathrm{Irr}(H)} a_{\mu,H} \mathrm{Ind}_K^\Gamma \left(\sigma_H(\mu)\right)$$

$$= \sum_{\substack{H \in \mathcal{I}_{p^B}(G) \\ H \geq Z(G)}} \sum_{\mu \in \mathrm{Irr}(H)} a_{\mu,H} \sigma_G \left(\mathrm{Ind}_H^G \mu\right).$$

Moreover since inner products and sums are conserved under σ_G using the definition of isomorphic character triples, we have

$$\chi = \sum_{\substack{H \in \mathcal{A}_{p^B}(G) \\ Z(G) \leq H}} \sum_{\mu \in \mathrm{Irr}(H)} a_{\mu,H} \mathrm{Ind}_H^G \mu, \tag{4.2}$$

Hence we have an expression for χ. Now recall $0 < B \leq a$. We are therefore finished if $B = a$, so suppose $B < a$. Since each $\mu \in \mathrm{Irr}(H)$ restricts to a multiple of θ and we have $v_p(\theta(1)) = v_p(\chi(1)) - B \geq a - B$, we are able to conclude p^{a-B} divides $\mu(1)$.

Note that $Z(G) = Z(G) \cap H \leq Z(H) \leq H$, i.e. $|H : Z(H)| \leq |G : Z(G)|$. Since $|H| < |G|$, G is a minimal counterexample, $\mu \in \mathrm{Irr}(H)$ and H is p-soluble (as subgroups of p-soluble groups are p-soluble), we can apply Theorem 4.12 to each μ, i.e. $\mu \in \mathcal{A}_{p^{a-B}}(H, Z(G))$. Hence using Lemma 4.9, $\mathrm{Ind}_H^G \mu \in \mathcal{A}_{p^a}(G, Z(G))$. Hence substituting these expressions into our expression for χ given by (4.2), we have $\chi \in \mathcal{A}_{p^a}(G, Z(G))$. □

We have that N has the following structure.

Claim 4.17. *N is nilpotent.*

Proof. Since G is p-soluble, either $N/Z(G) \cong C_l \times \cdots \times C_l$ for some prime l, or $N/Z(G)$ is a p'-group. By the proof of Claim 4.14, we have that the former must hold, with $l = p$. If q is a prime with $q \neq p$, then $|N|_q = |Z(G)|_q$. Hence, for each $q \neq p$, there is a unique Sylow q-subgroup of N and is central. If $P \in \mathrm{Syl}_p(N)$, then $PZ(G) = N$, i.e. $P \trianglelefteq N$, therefore N is a direct product of its Sylow subgroups and is therefore nilpotent. $\square$

By Proposition 4.6, where $H = G$ and $N = Z(G)$, it is sufficient to show that for all $E \in \mathrm{El}(G)$, that $\mathrm{Res}^G_{EZ(G)}\chi \in \mathcal{A}_{p^{a-b}}(EZ(G), Z(G))$, where $b := v_p\left(|G : EZ|\right)$. By Claim 4.17, $N := N_p \times N_{p'}$ where $N_p \in \mathrm{Syl}_p(N)$. Since $\theta \in \mathrm{Irr}(N)$, we have that there exist $\alpha \in \mathrm{Irr}(N_p)$ and $\beta \in \mathrm{Irr}(N_{p'})$ such that $\theta = \alpha \times \beta$. Since $\chi \in \mathrm{Irr}(G||\theta)$ and $\theta \in \mathrm{Irr}(N||\alpha)$, we have $\chi \in \mathrm{Irr}(G||\alpha)$. We note also that $v_p\left(\theta(1)\right) = v_p\left(\alpha(1)\right)$.

Claim 4.18. $v_p\left(\left|\frac{N_p}{P \cap N_p}\right|\right) \leq b$ *where* $P \in \mathrm{Syl}_p(EZ(G))$.

Proof. Note first that $P \cap N_p = P \cap N$ and therefore we need only consider the p-adic valuation of $|N : P \cap N|$. Note that,

$$v_p(|N : P \cap N|) = v_p(|G : EZ(G)|) + v_p(|EZ(G) : P|) + v_p(|P : P \cap N|) - v_p(|G : N|).$$

Note that $v_p(|EZ(G) : P|) = 0$ and by the Second Isomorphism Theorem $v_p(|P : P \cap N|) = v_p(|PN : N|) \leq v_p(|G : N|)$ i.e., $v_p(|P : P \cap N|) - v_p(|G : N|) \leq 0$. Hence $v_p(|N : P \cap N|) \leq b$. $\square$

If γ is an irreducible constituent of $\mathrm{Res}^{N_p}_{N_p \cap P}\alpha$, then α is an irreducible constituent of $\mathrm{Ind}^{N_p}_{N_p \cap P}\gamma$. Hence $\alpha(1) \leq |N_p : N_p \cap P|\gamma(1)$ and

$$v_p(\gamma(1)) \geq v_p(\alpha(1)) - v_p(|N_p : N_p \cap P|) \geq a - b.$$

Suppose that ζ is an irreducible constituent of $\mathrm{Res}^G_{EZ}\chi$. Suppose also that $\phi \in \mathrm{Irr}(N_p \cap P)$ is a constituent of $\mathrm{Res}^{EZ}_{P \cap N_p}\zeta = \mathrm{Res}^{N_p}_{N_p \cap P}\mathrm{Res}^{EZ}_{N_p}\zeta$. Since $\chi \in \mathrm{Irr}(G||\alpha)$ we have that

$\zeta \in \mathrm{Irr}(EZ||\alpha)$ by Lemma 3.30 . Therefore ϕ is a positive integer multiple of an irreducible constituent of $\mathrm{Res}^{N_p}_{N_p\cap P}\alpha$. Hence $v_p(\phi(1)) \geq a-b$, and $v_p(\zeta(1)) \geq a-b$. We will be done therefore if we can assume $EZ(G) < G$. We note first the following.

Claim 4.19. *For all $E \in \mathrm{El}(G)$, $E < G$.*

Proof. Suppose for a contradiction that $E = G$ for some $E \in \mathrm{El}(G)$, namely G is elementary. Suppose that $E = R \times Q$ where R is a p-group and Q is a p'-group, i.e. E is p-decomposable. In this case, $\chi = \eta \times \mu$ for some $\eta \in \mathrm{Irr}(R)$ and $\mu \in \mathrm{Irr}(Q)$. Since Q is a p'-group, p^a divides $\eta(1)$. Hence, since $\eta \in \mathrm{Irr}(R)$ and R is a p-group, η is not primitive. Let $\eta = \mathrm{Ind}^R_S\eta'$ for some $S < R$ and $\eta' \in \mathrm{Irr}(S)$. Therefore, $\chi = \mathrm{Ind}^{R\times Q}_{S\times Q}\eta' \times \mu$, by definition. This contradicts Claim 4.13. □

Claim 4.20. *$EZ(G) < G$.*

Proof. Suppose that $G = EZ(G)$. Define $\varphi : E \times Z(G) \to G$, by $\varphi(e,z) := ez$ and let $A := \ker(\varphi)$. Define $\widetilde{E} := E \times Z(G)$. Hence via the First Isomorphism Theorem, we have an isomorphism $\overline{\varphi} : \widetilde{E}/A \to G$, given by $\overline{\varphi}((e,z)A) := \varphi(e,z) = ez$. Define $\theta \times \lambda := \mathrm{Inf}^{\widetilde{E}}_G\chi \in \mathrm{Irr}(\widetilde{E})$. Note that $A \subseteq \ker(\theta\times\lambda)$ and this is equivalent to saying $\theta \times \lambda \in \mathrm{Irr}(\widetilde{E}||1_A)$.

Furthermore, if $(e,z) \in A$, then $e = z^{-1}$, i.e. $e \in E \cap Z(G) \leq Z(E)$. Hence, $A \leq Z(E)\times Z(G) = Z(\widetilde{E})$. Also, $|E : Z(E)| \leq |E : E\cap Z(G)| = |EZ(G) : Z(G)| = |G : Z(G)|$, and by Claim 4.19, $|E| < |G|$, we have that $E \prec G$.

Since E is p-soluble by Lemma 3.5 and $\theta(1) = \chi(1)$, we have that $\theta \in \mathcal{A}_{p^a}(E, Z(E))$. Hence by Lemma 4.10, $\theta \times \lambda \in \mathcal{A}_{p^a}(\widetilde{E}, Z(E)\times Z(G)) = \mathcal{A}_{p^a}(\widetilde{E}, Z(\widetilde{E}))$, taking $\lambda \in \mathcal{A}_1(Z(G), Z(G))$. Moreover we can use Lemma 4.11 to give,

$$\theta \times \lambda = \sum_{\substack{K\in\mathcal{I}_{p^a}(\widetilde{E})\\ Z(\widetilde{E})\leq K}} \sum_{\phi\in\mathrm{Irr}(K||1_A)} a_\phi \mathrm{Ind}^{\widetilde{E}}_K\phi,$$

i.e. $A \leq \ker(\phi)$, for all $\phi \in \mathrm{Irr}(K||1_A)$. Hence

$$\chi = \mathrm{Def}_G^{\widetilde{E}}(\theta \times \lambda) = \sum_{\substack{K \in \mathcal{I}_{p^a}(\widetilde{E}) \\ Z(\widetilde{E}) \leq K}} \sum_{\phi \in \mathrm{Irr}(K||1_A)} a_\phi \mathrm{Def}_G^{\widetilde{E}} \mathrm{Ind}_K^{\widetilde{E}} \phi$$

$$= \sum_{\substack{K \in \mathcal{I}_{p^a}(\widetilde{E}) \\ Z(\widetilde{E}) \leq K}} \sum_{\phi \in \mathrm{Irr}(K||1_A)} a_\phi \mathrm{Ind}_{K/A}^G \mathrm{Def}_{K/A}^K \phi.$$

Viewing K/A as a subgroup of G, we will clearly have that K/A will be of index divisible by p^a in G. Finally, as $\overline{\varphi}(K/A) \geq \overline{\varphi}(Z(\widetilde{E})/A) = Z(E)Z(G) \geq Z(G)$, we have that $\chi \in \mathcal{A}_{p^a}(G, Z(G))$. □

Since $EZ(G) < G$, we can apply Theorem 4.12 to each ζ, i.e. $\zeta \in \mathcal{A}_{p^{a-b}}(EZ(G), Z(G))$. Therefore $\mathrm{Res}_{EZ}^G \chi \in \mathcal{A}_{p^{a-b}}(EZ(G), Z(G))$ as claimed. □

We are now in a position to prove Proposition 4.21, which we recall links the work of Evseev to this thesis.

Proposition 4.21. *Let G be a group and $P \in \mathrm{Syl}_p(G)$ for a prime p. Suppose $\chi \in \mathrm{Irr}(G)$ such that p divides $\chi(1)$. Moreover, suppose (pRes-Syl) holds for the pair $(G, N_G(P))$. Then $\chi \in \mathcal{A}_p(G)$.*

Proof. Define $H := N_G(P)$ and $\mathcal{S} = \mathcal{S}(G, P, H)$. As (pRes-Syl) holds for $(G, N_G(P))$ we have $\mathrm{Res}_H^G \chi = \psi + \theta$ where $\psi \in \mathcal{C}^p(H)$ and $\theta \in \mathcal{I}(H, P, \mathcal{S})$. Since $P \in \mathrm{Syl}_p(G)$, the proof of Lemma 4.2 shows that subgroups in $\mathcal{S}$ have index divisible by p. Therefore $\theta \in \mathcal{A}_p(H)$.

Since H is p-soluble, we have that for each irreducible constituent μ of ψ, $\mu \in \mathcal{A}_p(H)$ using Theorem 4.12. Therefore $\mathrm{Res}_H^G \chi \in \mathcal{A}_p(H)$. Since $M := P.C_G(P) \leq H$, we may apply Lemma 4.8, applied in the case $G = H$, $N = 1$ and $K = M$, to obtain $\mathrm{Res}_M^G \chi \in \mathcal{A}_p(M)$, noting that $(|H : M|, p) = 1$. Note that for all $E \in \mathrm{El}_{p,1}(G, G, 1)$, P is conjugate (in G) to $Q \in \mathrm{Syl}_p(E)$. Hence, E is a subgroup of a conjugate of M in G by Lemma 3.9 and $(|M : E|, p) = 1$. Hence Lemma 4.8, gives $\mathrm{Res}_E^G \chi \in \mathcal{A}_p(E)$. The result now follows from Corollary 4.7. □

The following lemma shows that in fact (pRes-Syl) is a transitive property, meaning we can verify the condition in stages.

Lemma 4.22. *Let G be a group, H, K be subgroups of G and $P \in \mathrm{Syl}_p(G)$ such that $P \leq K \leq H \leq G$. Suppose that (pRes-Syl) holds for the pairs (G, H) and (H, K). Then (pRes-Syl) holds for the pair (G, K).*

Proof. Let $\chi \in \mathcal{C}^p(G)$. Since (pRes-Syl) holds for (G, H) and (H, K) we have

$$\mathrm{Res}^G_H (\mathcal{C}^p(G)) \subseteq \mathcal{C}^p(H) + \mathcal{I}(H, P, \mathcal{S}(G, P, H)), \tag{4.3}$$

$$\mathrm{Res}^H_K (\mathcal{C}^p(H)) \subseteq \mathcal{C}^p(K) + \mathcal{I}(K, P, \mathcal{S}(H, P, K)). \tag{4.4}$$

We show $\mathrm{Res}^G_K \chi = \chi_1 + \chi_2$ where $\chi_1 \in \mathcal{C}^p(K)$ and $\chi_2 \in \mathcal{I}(K, P, \mathcal{S}(G, P, K))$. By (4.3) we have,

$$\mathrm{Res}^G_K \chi = \mathrm{Res}^H_K \mathrm{Res}^G_H \chi = \mathrm{Res}^H_K \chi_p + \mathrm{Res}^H_K \chi_{\mathcal{I}},$$

where $\chi_p \in \mathcal{C}^p(H)$ and $\chi_{\mathcal{I}} \in \mathcal{I}(H, P, \mathcal{S}(G, P, H))$. Since (4.4) holds,

$$\mathrm{Res}^H_K \chi_p = \theta_p + \theta_{\mathcal{I}},$$

where $\theta_p \in \mathcal{C}^p(K)$ and $\theta_{\mathcal{I}} \in \mathcal{I}(K, P, \mathcal{S}(H, P, K))$. For simplicity, we define $\chi_1 := \theta_p$ and $\chi_2 := \theta_{\mathcal{I}} + \mathrm{Res}^H_K \chi_{\mathcal{I}}$. We are therefore done provided $\chi_2 \in \mathcal{I}(K, P, \mathcal{S}(G, P, K))$. Since $|H : K|$ is prime to p, we have by [10, Lemma 5.10], that $\mathrm{Res}^H_K \chi_{\mathcal{I}} \in \mathcal{I}(K, P, \mathcal{S}(H, P, K))$. Moreover, it follows by definition that $\mathcal{S}(H, P, K) \subseteq \mathcal{S}(G, P, K)$ and hence
$\mathcal{I}(K, P, \mathcal{S}(H, P, K)) \subseteq \mathcal{I}(K, P, \mathcal{S}(G, P, K))$. □

4.2 Other useful lemmas

Lemma 4.23. *Let G be a group, $H \trianglelefteq G$ and $N \trianglelefteq G$. Let $n \in \mathbb{N}$, $n \geq 1$. Suppose that $\chi \in \mathcal{A}_n(G, H, N)$. Then for all $\phi \in \mathcal{C}(G)$, $\chi \cdot \phi \in \mathcal{A}_n(G, H, N)$.*

Proof. Since $\chi \in \mathcal{A}_n(G,H,N)$, we have

$$\chi = \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}_K^G \theta_{(K)},$$

where $\theta_{(K)} \in \mathcal{C}(K)$ for each $K \leq G$. Therefore using Proposition 3.22 we have,

$$\chi \cdot \phi = \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \left(\mathrm{Ind}_K^G \theta_{(K)} \cdot \phi\right) = \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}_K^G \left(\theta_{(K)} \cdot \mathrm{Res}_K^G \phi\right),$$

hence by definition, $\chi \cdot \phi \in \mathcal{A}_n(G,H,N)$. □

Corollary 4.24. *Let G be a group, $H \trianglelefteq G$ and $N \trianglelefteq G$. Let $n \in \mathbb{N}$, $n \geq 1$. Then $\mathcal{A}_n(G,H,N)$ is an ideal of $\mathcal{C}(G)$.*

Proof. It is clear from the definition of $\mathcal{A}_n(G,H,N)$ that it is an additive subgroup of $\mathcal{C}(G)$. The rest follows from Lemma 4.23. □

Lemma 4.25. *Let G be a group, $K \trianglelefteq G$, $H \trianglelefteq G$ and $N \trianglelefteq G$ such that $N \leq K \cap H$. If $\chi \in \mathcal{A}_n(G/N, H/N, K/N)$ then $\mathrm{Inf}_{G/N}^G \chi \in \mathcal{A}_n(G,H,K)$.*

Proof. If $L \leq G$ and $N \leq L$, we write $\overline{L}$ for L/N. Assume that χ has the following decomposition,

$$\chi = \sum_{\substack{\overline{L} \in \mathcal{I}_n(\overline{G},\overline{H}) \\ \overline{K} \leq \overline{L}}} \mathrm{Ind}_{\overline{L}}^{\overline{G}} \phi_{\overline{L}},$$

where $\phi_{\overline{L}} \in \mathcal{C}(\overline{L})$. It follows that

$$\mathrm{Inf}_{\overline{G}}^G \chi = \sum_{\substack{\overline{L} \in \mathcal{I}_n(\overline{G},\overline{H}) \\ \overline{K} \leq \overline{L}}} \mathrm{Inf}_{\overline{G}}^G \mathrm{Ind}_{\overline{L}}^{\overline{G}} \phi_{\overline{L}} = \sum_{\substack{\overline{L} \in \mathcal{I}_n(\overline{G},\overline{H}) \\ \overline{K} \leq \overline{L}}} \mathrm{Ind}_L^G \mathrm{Inf}_{\overline{L}}^L \phi_{\overline{L}}.$$

Now clearly $K \leq L$ and $\mathrm{Inf}_{\overline{L}}^L \phi_{\overline{L}} \in \mathcal{C}(L)$, so we consider whether n divides $|H : L \cap H|$. However this follows, since n divides $|\overline{H} : \overline{L} \cap \overline{H}|$ and $\overline{L} \cap \overline{H} = \overline{L \cap H}$. □

Lemma 4.26. *Let G be a group with normal subgroups $N \trianglelefteq G$ and $K \trianglelefteq G$ such that $K \leq N$ and p does not divide $|G:N|$. Then $\chi \in \mathcal{A}_p(G,K)$ if and only if $\chi \in \mathcal{A}_p(G,N,K)$.*

Proof. We show that $L \in \mathcal{I}_p(G)$ if and only if $L \in \mathcal{I}_p(G,N)$ where in both situations $K \leq L$. Let $L \leq G$ such that p divides $|G:L|$, then p divides $|G:L\cap N|$, and p divides $|N:L\cap N|$ since p does not divide $|G:N|$, i.e. $L \in \mathcal{I}_p(G,N)$. Now suppose that $L \in \mathcal{I}_p(G,N)$, then p divides $|N:L\cap N|$ and hence p divides $|G:L\cap N|$. Since $N \trianglelefteq G$, then Second Isomorphism Theorem yields

$$\frac{L}{L\cap N} \cong \frac{LN}{N},$$

which as p does not divide $|G:N|$, p does not divide $|LN:N|$, hence p divides $|G:L|$. Therefore $L \in \mathcal{I}_p(G)$. The result now follows. □

4.3 Reduction to p-groups

This section is key to simplifying our GAP and MAGMA computations in Chapter 9. We show that to test Conjecture 2.5 for any group G, it is enough to test whether a certain generalised character belongs to $\mathcal{A}_p(P/P')$ where $P \in \mathrm{Syl}_p(G)$. This means computation time for groups where the fusion information between G and P is known is significantly reduced. As suggested, if the fusion information is not known, it is likely that this approach actually reduces computation time, however this is not been explicitly verified.

The results in this section also motivate the work carried out in Chapter 7, where we investigate the structure of $\mathcal{A}_p(P)$ for elementary abelian p-groups of small rank. We reduce to the study of p-groups in this section, but the purpose of Chapter 7 is to show that even in the simplest of cases, the difference between $\mathcal{C}^p(G)$ and $\mathcal{A}_p(G)$ is quite significant, showing that Conjecture 2.5 is not trivial. First we derive the following result, which is fundamental for Chapter 7.

Lemma 4.27. *If G is an abelian p-group where p is a prime and* $\mathrm{rank}(G) \geq 2$, *then* $p\,\mathcal{C}(G)$ *is an ideal of* $\mathcal{A}_p(G)$.

Proof. Note that by Corollary 4.24, it is enough to show that $p1_G \in \mathcal{A}_p(G)$, where 1_G is the principal character of G. First we suppose that $G = C_p \times C_p$. Let $H_1, \ldots, H_{p+1}$ be the set of maximal subgroups of G. Consider $\theta := \left(\sum_{i=1}^{p+1} \mathrm{Ind}_{H_i}^G 1_{H_i}\right) - \rho$, where ρ is the regular character of G. Then $\theta(x) = p$ for all $x \in G$. Hence $p1_G = \theta \in \mathcal{A}_p(G)$, since $\rho = \mathrm{Ind}_{\{1\}}^G 1_{\{1\}}$.

Now let G be any abelian p-group of rank at least 3, then there exists $N \trianglelefteq G$ such that $G/N \cong K := C_p \times C_p$. By the above, $p1_K \in \mathcal{A}_p(K)$, hence $\mathrm{Inf}_K^G p1_K = p1_G \in \mathcal{A}_p(G, N) \subseteq \mathcal{A}_p(G)$ by Lemma 4.25. □

We now begin by letting $\chi \in \mathrm{Irr}(G)$ such that p divides $\chi(1)$. We want to know when $\chi \in \mathcal{A}_p(G)$, where G is an arbitrary group. By Corollary 4.7, it is enough to show that $\mathrm{Res}_E^G \chi \in \mathcal{A}_p(E)$, for all $E \in \mathrm{El}_{p,1}(G, G, 1)$. Since p does not divide $|G : E|$, we have that if $P \in \mathrm{Syl}_p(G)$, then $P \leq E$ without loss of generality. By an argument used to prove Proposition 4.21, it is therefore sufficient to show $\mathrm{Res}_M^G \chi \in \mathcal{A}_p(M)$, where $M = PC_G(P)$, which in turn belongs to $\mathrm{PD}_{\max}(G)$ by Lemma 3.8, i.e. $M \cong P \times F$ where F is a p'-group. Recall the definition of $\widetilde{\pi}_\theta$ from Definition 3.17.

Lemma 4.28. *Let* $\chi \in \mathcal{C}(M)$, *then* $\chi \in \mathcal{A}_p(M)$ *if for all* $\theta \in \mathrm{Irr}(F)$, $\widetilde{\pi}_\theta(\chi) \in \mathcal{A}_p(P)$.

Proof. Let $\chi \in \mathcal{C}(M)$. Since $M \cong P \times F$, we have that

$$\chi = \sum_{\theta \in \mathrm{Irr}(F)} \left(\varphi_\theta \times \theta\right),$$

where $\varphi_\theta \in \mathcal{C}(P)$. Note also that for each $\theta \in \mathrm{Irr}(F)$, $\widetilde{\pi}_\theta(\chi) = \varphi_\theta$ by definition. Since $\widetilde{\pi}_\theta(\chi) \in \mathcal{A}_p(P)$, we have that

$$\varphi_\theta = \widetilde{\pi}_\theta(\chi) = \sum_{H_\theta \in \mathcal{I}_p(P)} \mathrm{Ind}_{H_\theta}^P \zeta_{(H_\theta)},$$

where $\zeta_{(H_\theta)} \in \mathcal{C}(H_\theta)$. Hence,

$$\chi = \sum_{\theta \in \mathrm{Irr}(F)} \left(\sum_{H_\theta \in \mathcal{I}_p(P)} \mathrm{Ind}_{H_\theta}^{P} \zeta_{(H_\theta)} \right) \times \theta = \sum_{\theta \in \mathrm{Irr}(F)} \sum_{H_\theta \in \mathcal{I}_p(P)} \left(\mathrm{Ind}_{H_\theta}^{P} \zeta_{(H_\theta)} \times \theta \right)$$

$$= \sum_{\theta \in \mathrm{Irr}(F)} \sum_{H_\theta \in \mathcal{I}_p(P)} \mathrm{Ind}_{H_\theta \times F}^{P \times F} \left(\zeta_{(H_\theta)} \times \theta \right).$$

The result now follows as $H_\theta \in \mathcal{I}_p(P)$ gives $H_\theta \times F \in \mathcal{I}_p(M)$ for each $\theta \in \mathrm{Irr}(F)$. $\square$

The above lemma therefore reduces our study to $\mathcal{A}_p(P)$ where P is a p-group. We show now that in fact we can go on stage further, and show that we need only study $\mathcal{A}_p(P)$ when P is an abelian p-group. Recall the definition of $\widetilde{\pi}_L$ from Definition 3.20.

Lemma 4.29. *Let $\chi \in \mathcal{C}(P)$. Then $\chi \in \mathcal{A}_p(P)$ if $\mathrm{Def}_{P/P'}^{P}(\widetilde{\pi}_L(\chi)) \in \mathcal{A}_p(P/P')$ where P' is the derived subgroup of P.*

Proof. Let $\chi \in \mathcal{C}(P)$ and suppose that $\chi = \sum_{\theta \in \mathrm{Irr}(P)} a_\theta \theta$. Let $\zeta := \chi - \widetilde{\pi}_L(\chi)$. Since P is a p-group, it is an M-group. Therefore every irreducible θ of P of degree divisible by p satisfies, $\theta = \mathrm{Ind}_{H_\theta}^{P} \lambda_\theta$ where $\lambda_\theta \in \mathrm{Irr}(H_\theta)$, $\lambda_\theta(1) = 1$ and $H_\theta < P$. Note that since $H_\theta < P$, we have that p divides $|P : H_\theta|$. Since every irreducible constituent of ζ has degree divisible by p, we have $\zeta \in \mathcal{A}_p(P)$. Consider

$$\widetilde{\pi}_L(\chi) = \sum_{\substack{\theta \in \mathrm{Irr}(P) \\ \theta(1)=1}} a_\theta \theta.$$

Recall that $P' < P$. Since $P' \subseteq \ker(\theta)$ for every $\theta \in \mathrm{Irr}(P)$ with $\theta(1) = 1$, we can deflate $\widetilde{\pi}_L(\chi)$ to a generalised character of P/P'. However $\mathrm{Def}_{P/P'}^{P}(\widetilde{\pi}_L(\chi)) \in \mathcal{A}_p(P/P')$. Therefore, by Lemma 4.25, $\widetilde{\pi}_L(\chi) = \mathrm{Inf}_{P/P'}^{P} \mathrm{Def}_{P/P'}^{P}(\widetilde{\pi}_L(\chi)) \in \mathcal{A}_p(P, P') \subseteq \mathcal{A}_p(P)$. Hence $\chi \in \mathcal{A}_p(P)$ as required.

$\square$

Therefore we can reduce our study of $\mathcal{A}_p(G)$ to $\mathcal{A}_p(P)$ for P an abelian p-group. In Chapter 7 we do this, focusing on elementary abelian p-groups of small rank.

CHAPTER 5

REPRESENTATION THEORIES AND $\mathcal{A}_N(G, H, N)$

In this chapter we develop some properties of $\mathcal{A}_n(G, H, N)$, in particular its behaviour in wreath products and symmetric groups. For our proof of Theorem 8.1, we use many details from [10]. In this paper, Evseev introduces a ring of characters κ_w that we need to relate to our character ring, $\mathcal{A}_n(G, H, N)$. Using Lemma 4.2, it suffices to relate κ_w to $\mathcal{I}(G, P, \mathcal{S})$ for suitable groups, G, P and set of subgroups $\mathcal{S}$.

The representation theory of wreath products is a key tool in our reduction theorems. Our goal in this chapter is to prove that if $\chi \in \mathcal{A}_{p^a}(G, H, N)$, then an irreducible character of $G \wr S_n$, $\widetilde{\chi^{\times n}}$, belongs to $\mathcal{A}_{p^{an}}(G \wr S_n, H^n, N^n)$. In particular, if we can embed our initial group into a wreath product, then it reduces our problem to understanding the induced character ring for the base group. However this technique cannot be applied quite so easily to our reduction theorems. We require a more general statement related to the representation theory of wreath products over twisted group algebras. We prove the statement first for the usual group algebra as it clearly highlights the statements from [10] that need to be generalised to twisted group algebras.

Our focus therefore for this chapter will be to develop the representation theory of wreath products and the properties that we use in the reduction theorems. In particular we generalise the above result to projective characters, i.e. characters over twisted group algebras, see Theorem 5.37.

Finally, as highlighted in the introduction, we use many techniques from the proofs of many of the Global-Local conjectures in our reduction theorem. It is therefore not surprising that Brauer's Height Zero Conjecture plays a role in our understanding of $\mathcal{A}_n(G, H, N)$. We give an overview to modular representation theory to prove that if the defect group of the block containing our character is abelian, therefore by Brauer's Height Zero Conjecture every character has height zero in its block, then Conjecture 2.12 holds.

5.1 Representation theory of symmetric groups

Lemma 5.1. *Let $G = S_{pw+e}$ where p is a prime, $w, e \in \mathbb{Z}_{\geq 0}$ with $e < p$. Let $P \in \mathrm{Syl}_p(G)$. Then*

$$\kappa_w \subseteq \mathcal{I}\left(S_p \wr S_w, P, \mathcal{S}\left(S_{pw+e}, P, S_p \wr S_w\right)\right).$$

Proof. Using [10, Theorem 4.1], we have $\kappa_w \subseteq \mathcal{I}\left(S_p \wr S_w, P, \mathcal{S}_e\right)$, where $\mathcal{S}_e = \mathcal{S}\left(S_{pw+e}, P, (S_p \wr S_w) \times S_e\right)$. We are therefore done provided $\mathcal{S}_e \subseteq \mathcal{S}(S_{pw+e}, P, S_p \wr S_w)$. Let $Q \in \mathcal{S}_e$, then $Q \leq P \cap {}^tP$ for some $t \notin (S_p \wr S_w) \times S_e$. Then $t \notin S_p \wr S_w$ and we are done. $\square$

5.2 Representation theory of wreath products

We set $\mathrm{Irr}(G) := \{\phi_1, \ldots, \phi_s\}$. We define n_i to be the number of times ϕ_i appears as a factor of a particular irreducible, ϕ of $G^{\times n}$. Note $\sum_i n_i = n$. Let J_i denote the indices j such that the restriction of ϕ to G_j is ϕ_i, so that $|J_i| = n_i$. We denote by $\phi_i^{\times n_i}$, the character whose j-th factor is ϕ_i if $j \in J_i$ and the trivial character of G otherwise. It is clear that this character is $G^{\times n}$-invariant, as each individual factor is G-invariant, but moreover, $\phi_i^{\times n_i}$ is in fact $\mathrm{Sym}(J_i)$-invriant. Therefore one can easily show $\phi_i^{\times n_i}$ is $G \wr S_{n_i}$-invariant. However, one key feature of the character theory of wreath products is that this character extends to $G \wr S_{n_i}$. We work to define the values that this extension takes. Note that the following definitions and notation correspond to that used in [10].

Definition 5.2. [10, Section 2.3] *A marked cycle in S_n is defined to be either a non-identity cycle in S_n or an element of* $[1,n]$.

For a marked cycle, $\sigma \in S_n$, we define $\text{supp}(\sigma)$ *to be the set of points of* $[1,n]$ *that are not fixed by σ if σ is a non-identity cycle, and otherwise if $\sigma = i \in [1,n]$, then* $\text{supp}(\sigma) = \{i\}$. *Moreover, the order of a marked cycle σ is defined to be* $|\text{supp}(\sigma)|$.

Defining marked cycles in this way, we are able to say that every element $\sigma \in S_n$, decomposes as a product of marked cycles. We note that to multiply such cycles, we replace occurrences of elements of $[1,n]$ by the identity permutation. We now describe how to associate to a marked cycle an element of $G \wr S_n$.

Definition 5.3. [10, Section 2.3] *Let σ be a marked cycle, and let j denote the minimal element of* $\text{supp}(\sigma)$. *Then for $g \in G$, we define $y_\sigma(g) := (1,\ldots,1,g,1,\ldots,1;\sigma) \in G \wr S_n$, where g occurs in the j-th position.*

Let A be a $\mathbb{C}$-algebra, and M be an A-module. Let $G \leq S_n$. We define $A \wr G := A^{\otimes n} \otimes \mathbb{C}G$, where the multiplication is defined as follows,

$$(a_1 \otimes \cdots \otimes a_n \otimes \sigma)(b_1 \otimes \cdots \otimes b_n \otimes \tau) := \left(a_1 b_{\sigma^{-1}(1)} \otimes \cdots \otimes a_n b_{\sigma^{-1}(n)} \otimes \sigma\tau\right),$$

where $a_i \in A$, $b_i \in A$ and σ, $\tau \in T$ and extend by linearity. We give $M^{\otimes n}$ the structure of an $A \wr G$-module via the following action,

$$(a_1,\ldots,a_n,\sigma)(m_1 \otimes \cdots \otimes m_n) := a_1 m_{\sigma^{-1}(1)} \otimes \cdots \otimes a_n m_{\sigma^{-1}(n)},$$

where $\sigma \in T$, $a_i \in A$, $m_i \in M$ and again extend by linearity. We denote the resulting module by $M^{\tilde{\otimes} n}$. We denote the character afforded by this module by $\chi^{\tilde{\times} n}$. The following lemma describes the values taken by this character on marked cycles.

Lemma 5.4. [10, Lemma 2.3] *Let $\sigma_1,\ldots,\sigma_r$ be disjoint marked cycles in S_n, with*

$\sum_i o(\sigma_i) = n$. *Let* $g_1, \dots, g_r \in G$. *Then for* $\phi \in \mathrm{CF}(G)$,

$$\phi^{\widetilde{\times} n}\left(y_{\sigma_1}(g_1) \dots y_{\sigma_r}(g_r)\right) = \phi(g_1) \dots \phi(g_r).$$

It can be checked that the above formula defines a class function of $G \wr S_n$, which we will denote by $\phi^{\widetilde{\times} n}$.

We also require the description of $\mathrm{Irr}(G \wr S_n)$. The reader is referred to [21, Section 4.3], for a detailed description of how this theorem is obtained, in particular, the discussion after Lemma 4.3.33. We use the notation found in [10, §2.3].

Theorem 5.5. (See [21, Theoreom 4.3.34]) *Every irreducible character of* $\mathrm{Irr}(G \wr S_n)$, *has the form,*

$$\mathrm{Ind}_{\prod_i (G \wr S_{n_i})}^{G \wr S_n} \prod_{i=1}^{s} \left(\phi_i^{\widetilde{\times} n_i} \cdot \mathrm{Inf}_{S_{n_i}}^{G \wr S_{n_i}} \chi_i\right),$$

where $\phi_i \in \mathrm{Irr}(G)$ *and* $\chi_i \in \mathcal{C}(S_{n_i})$.

We are now in a position to prove our claim.

Theorem 5.6. *Suppose* $\phi \in \mathcal{A}_{p^a}(G, K, Z(G))$, *where* $K \trianglelefteq G$, *then* $\phi^{\widetilde{\times} n} \in \mathcal{A}_{p^{an}}(G \wr S_n, K^n, Z(G)^n)$.

Proof. By [10, Lemma 2.7], it is clearly true when $\phi = \mathrm{Ind}_H^G \theta$ for $\theta \in \mathcal{C}(H)$. We prove the theorem in the case when $\phi = \mathrm{Ind}_{H_1}^G \theta_1 + \mathrm{Ind}_{H_2}^G \theta_2$, where H_1, $H_2 \leq G$, $|K : K \cap H_i|$ is divisible by p^a and $\theta_i \in \mathcal{C}(H_i)$ for $i = 1, 2$. The result will follow by induction on the number of summands of ϕ. Since $\mathrm{Ind}_{H_i}^G \theta_i$ are class functions and $a_1 = a_2 = 1$, using [10, Lemma 2.5] we have

$$\phi^{\widetilde{\times} n} = \sum_{j=0}^{n} \mathrm{Ind}_{(G \wr S_j) \times (G \wr S_{n-j})}^{G \wr S_n} \left(\left(\mathrm{Ind}_{H_1}^G \theta_1\right)^{\widetilde{\times} j} \times \left(\mathrm{Ind}_{H_2}^G \theta_2\right)^{\widetilde{\times}(n-j)}\right).$$

Now using [10, Lemma 2.7],

$$\left(\mathrm{Ind}_{H_i}^G \theta_i\right)^{\widetilde{\times} w} = \mathrm{Ind}_{H_i \wr S_w}^{G \wr S_w} \left(\theta_i^{\widetilde{\times} w}\right),$$

for any $w \in \mathbb{N}$ and $i = 1, 2$. Therefore,

$$\phi^{\tilde{\times} n} = \sum_{j=0}^{n} \mathrm{Ind}_{(H_1 \wr S_j) \times (H_2 \wr S_{n-j})}^{G \wr S_n} \left(\theta_1^{\tilde{\times} j} \times \theta_2^{\tilde{\times} (n-j)} \right).$$

We clearly have $Z(G)^n$ is contained in each of the subgroups. Finally consider the intersection of K^n with $((H_1 \wr S_j) \times (H_2 \wr S_{n-j}))$. Now,

$$(H_1 \wr S_j) \times (H_2 \wr S_{n-j}) \cong \left(H_1^j \times H_2^{n-j} \right) \rtimes (S_j \times S_{n-j}),$$

where an element $(\sigma, \tau) \in S_j \times S_{n-j}$ acts on $H_1^j \times H_2^{n-j}$ via the action of σ on H_1^j and τ on H_2^{n-j}. We view K^n as a subgroup of $G \wr S_n$ via the embedding $(k_1, \ldots, k_n) \to (k_1, \ldots, k_n; 1)$. Therefore viewing K^n in this way, it intersects trivially with S_n and,

$$\left(\left(H_1^j \times H_2^{n-j} \right) \rtimes (S_j \times S_{n-j}) \right) \cap K^n = (H_1 \cap K)^j \times (H \cap K)^{n-j}.$$

Hence, $|K^n : K^n \cap ((H_1 \wr S_j) \times (H_2 \wr S_{n-j}))| = |K|^j |K|^{n-j} / |H_1 \cap K|^j |H_2 \cap K|^{n-j}$. Moreover, since p^a divides the index of $|K : H_i \cap K|$, we have that p^{an} divides the index above. $\square$

5.3 Representation theory of twisted group algebras

As stated in the introduction to this chapter, we will now develop the representation theory of twisted group algebras of wreath products. The main issue that we encounter in the case when G has trivial centre, is that we would like to extend an irreducible character to its inertia subgroup, which is not necessarily possible. However we are able to extend an "ordinary" irreducible character to a projective character, i.e. an irreducible character of some twisted group algebra. Moreover, in the case when G is an arbitrary group, with no restriction on the centre, we use the outline developed in the trivial centre case, applied to $G/Z(G)$. This requires a character of $G/Z(G)$ to work with, which not only relates back to our original character of G, but this relationship must also behave well with respect to induction. This is described by Theorem 5.19 and the subsequent results show that this relationship is indeed well behaved. Moreover, what the reader should notice is that to achieve such a goal, we again work with twisted group algebras.

Our next problem is that we are now working with the theory of twisted group algebras, so we need to develop analogous results to those in ordinary representation theory to use in our proofs. In particular, we need to generalise the notions of induction, Mackey decomposition and the representation theory of wreath products to modules over twisted group algebras. This chapter will explicitly give and prove these constructions. We follow the notation and definitions in [16] to introduce the notion of twisted group algebras.

Definition 5.7. [16, Definition 11.1] *A projective $\mathbb{C}$-representation of G is a map $\mathfrak{X} : G \to \mathrm{GL}\,(n, \mathbb{C})$, where for all $g, h \in G$, there exists $\alpha : G \times G \to \mathbb{C}$, called the associated factor set of $\mathfrak{X}$, such that*

$$\mathfrak{X}\,(g)\,\mathfrak{X}\,(h) = \alpha(g, h)\mathfrak{X}\,(gh)\,.$$

Definition 5.8. [22, Page 8] *Let ρ be a projective representation. Then $\chi : G \to \mathbb{C}$, defined via $\chi(g) := \mathrm{Tr}(\rho(g))$ is the projective character of G afforded by ρ.*

Note that in both the definitions above we have suppressed the use of the factor set. Where pertinent, we may refer to the above as projective α-representations/characters if this is more clear.

Definition 5.9. [16, Definition 11.4] *A $\mathbb{C}^*$-factor set of G is a function $\alpha : G \times G \to \mathbb{C}^*$ such that,*

$$\alpha(gh, k)\alpha(g, h) = \alpha(g, hk)\alpha(h, k).$$

A simple check shows us that the associated factor set to a projective representation $\mathfrak{X}$ is indeed a $\mathbb{C}^*$-factor set. Furthermore, we are able to construct $\mathbb{C}^*$-factor sets as follows.

Proposition 5.10. (See [16, Page 178]) *Suppose $\mu : G \to \mathbb{C}^*$ is an arbitrary function.*

Then $\delta(\mu) : G \times G \to \mathbb{C}^*$ defined by,

$$\delta(\mu)(g,h) := \mu(g)\mu(h)\mu(gh)^{-1},$$

is a $\mathbb{C}^*$-factor set.

It is also worth noting that the set of $\mathbb{C}^*$-factor sets of G form a group under pointwise multiplication. The resulting group is called the group of 2-cocycles, denoted $Z^2(G, \mathbb{C}^*)$, and has applications in cohomology theory. Furthermore, δ in the above proposition can be viewed as a homomorphism from the group of all $\mathbb{C}^*$-valued functions into the group of 2-cocycles. The image of δ is referred to as the group of 2-coboundaries, denoted $B^2(G, \mathbb{C}^*)$. This forms a normal subgroup, and the resulting quotient group, $M(G) := H^2(G, \mathbb{C}^*)$ is called the **Schur multiplier** of G. If two elements of $Z^2(G, \mathbb{C}^*)$ differ by an element of $B^2(G, \mathbb{C}^*)$, i.e. giving rise to the same class in the Schur multiplier, then these elements are called cohomologous. We note that if G is a non-abelian simple group then the Schur multiplier behaves as follows.

Lemma 5.11. (See [4, (8.2) and (8.3)]) *If S is a non-abelian simple group and $n \geq 1$, then*

$$M(S^n) \cong \underbrace{M(S) \times \cdots \times M(S)}_{n}.$$

Let α be a $\mathbb{C}^*$-factor set. We let $\mathbb{C}^\alpha[G]$ be the $\mathbb{C}$-vector space with basis $\{a_g : g \in G\}$. Define a multiplication on $\mathbb{C}^\alpha[G]$ via $a_g \cdot a_h := a_{gh}\alpha(g,h)$ and extend by linearity. Note that due to Definition 5.9, we have that the multiplication defined in this way is indeed associative. We define $\mathbb{C}^\alpha[G]$ to be the **twisted group algebra** associated with α. First we note the following theorem.

Theorem 5.12. *Let α, $\beta \in Z^2(G, \mathbb{C}^*)$ be cohomologous and let $\mathbb{C}^\alpha[G]$ and $\mathbb{C}^\beta[G]$ respectively be their associated twisted group algebras. Then $\mathbb{C}^\alpha[G] \cong \mathbb{C}^\beta[G]$.*

Proof. As α and β are cohomologous, there exists $t : G \to \mathbb{C}^*$ such that $\alpha = (\delta t)\,\beta$ where for g, $h \in G$, $(\delta t)(g,h) := t(g)t(h)t(gh)^{-1}$. Define $\psi : \mathbb{C}^\alpha[G] \to \mathbb{C}^\beta[G]$ by $\psi\,(a_g) := t(g)a_g$

and extend $\mathbb{C}$-linearly. We simply show that ψ respects the multiplication. If a_g, a_h are basis elements of $\mathbb{C}^\alpha[G]$, then

$$\psi\left(a_g a_h\right) = \alpha(g,h)\psi\left(a_{gh}\right) = \alpha(g,h)t(gh)a_{gh} = \alpha(g,h)t(gh)\beta(g,h)^{-1}a_g a_h$$

$$t(g)t(h)t(gh)^{-1}\beta(g,h)t(gh)\beta(g,h)^{-1}a_g a_h = t(g)a_g t(h)a_h = \psi(a_g)\psi(a_h).$$

$\square$

Due to Theorem 5.12, when we are discussing the factor set associated to a twisted group algebra we may therefore consider it as an element of the Schur multiplier, as up to coboundaries the resulting algebras are isomorphic. The above will also be of importance to us to show when a twisted group algebra we introduce to guarantee extensibility, is in fact the original twisted group algebra on our group. We now introduce the notion of a G-graded algebra and show that twisted group algebras are G-graded.

Definition 5.13. [28, §1] *Let G be a group and A be an algebra. Let $A = \bigoplus_{g\in G} A_g$ be a direct sum decomposition of A into subspaces A_g. This decomposition is called a G-grading if $A_g A_h \subseteq A_{gh}$ for g, $h \in G$.*

Note $\mathbb{C}^\alpha[G]$ is a G-graded algebra with $\mathbb{C}^\alpha[G] = \sum_{g\in G} \mathbb{C}^\alpha[G]_g$, where $\mathbb{C}^\alpha[G]_g := \mathbb{C}[a_g]$ is a 1-dimensional subspace of $\mathbb{C}^\alpha[G]$. Using this, we define for a subset $H \subseteq G$, $\mathbb{C}^\alpha[H] := \sum_{h\in H} \mathbb{C}^\alpha[G]_h$, the twisted group algebra of H associated with $\alpha|_{H\times H}$. The next two lemmas describe the key extensibility condition that we require in the proof when G has trivial centre. One key issue we encounter is generalising Lemma 5.14 when the group algebra on N is not necessarily the usual group algebra. The reader is directed to Chapter 6 for further detail, but in brief we require Condition 6.2 to deal with this issue.

Lemma 5.14. *Let G be a group, $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ such that $I_G(\theta) = G$. Then there exists a twisted group algebra $\mathbb{C}^\alpha[G]$ such that $\mathbb{C}^\alpha[N] \cong \mathbb{C}N$ and the representation affording θ can be extended to a projective representation of G.*

Proof. We denote the representation affording θ by $\mathfrak{Y}$, then using [16, Theorem 11.2], we have that there exists a projective representation $\mathfrak{X}$ of G, such that for all $n \in N$ and $g \in G$, $\mathfrak{X}(n) = \mathfrak{Y}(n)$, i.e. $\mathfrak{X}$ is the extension of $\mathfrak{Y}$ to G. In particular if we consider the factor set, α of $\mathfrak{X}$, restricted to $N \times N$, we have for all n_1, $n_2 \in N$,

$$\mathfrak{Y}(n_1n_2) = \mathfrak{Y}(n_1)\mathfrak{Y}(n_2) = \mathfrak{X}(n_1)\mathfrak{X}(n_2) = \alpha\left(n_1, n_2\right)\mathfrak{X}\left(n_1n_2\right) = \alpha\left(n_1, n_2\right)\mathfrak{Y}\left(n_1n_2\right).$$

Therefore $\alpha(n_1, n_2) = 1$ for all n_1, $n_2 \in N$. Moreover, considering the twisted group algebra with this factor set defining its multiplication, we have $\mathbb{C}^{\alpha}[N] \cong \mathbb{C}N$. $\square$

Lemma 5.15. *Suppose all the conditions of Lemma 5.14 hold and let $\mathbb{C}^{\alpha}[G]$ denote the resulting twisted group algebra. Suppose moreover that θ is extendible to H, where $N \leq H \leq G$. Then $\mathbb{C}^{\alpha}[H] \cong \mathbb{C}H$.*

Proof. We denote the representation affording θ by $\mathfrak{Y}$ and the ordinary representation of H extending $\mathfrak{Y}$ by $\widetilde{\mathfrak{Y}}$. Moreover, we denote the projective representation extending $\mathfrak{Y}$ by $\mathfrak{X}$. For $n \in N$ and $h \in H$, the three conditions in [16, Theorem 11.2] hold since

(a) $\widetilde{\mathfrak{Y}}(n) = \mathfrak{Y}(n)$, by definition.

(b) $\widetilde{\mathfrak{Y}}(hn) = \widetilde{\mathfrak{Y}}(h)\widetilde{\mathfrak{Y}}(n)$ since $\widetilde{\mathfrak{Y}}$ is an ordinary representation.

(c) This proof is identical to (b).

Note further that $\mathfrak{X}$ restricted to H is another projective representation satisfying these three conditions. Therefore by [16, Theorem 11.2] we have that there exists $\mu : G \to \mathbb{C}^*$ such that, $\widetilde{\mathfrak{Y}}(g) = \mathfrak{X}(g)\mu(g)$. If h_1, $h_2 \in H$, then

$$\mu\left(h_1\right)\mu\left(h_2\right)\mathfrak{X}\left(h_1\right)\mathfrak{X}\left(h_2\right) = \mu\left(h_1\right)\mu\left(h_2\right)\alpha\left(h_1, h_2\right)\mathfrak{X}\left(h_1h_2\right)$$

and

$$\mu\left(h_1\right)\mu\left(h_2\right)\mathfrak{X}\left(h_1\right)\mathfrak{X}\left(h_2\right) = \widetilde{\mathfrak{Y}}\left(h_1\right)\widetilde{\mathfrak{Y}}\left(h_2\right) = \widetilde{\mathfrak{Y}}\left(h_1h_2\right) = \mathfrak{X}\left(h_1h_2\right)\mu\left(h_1h_2\right).$$

Therefore combining these two equations, we obtain

$$\alpha\left(h_1, h_2\right)=\mu^{-1}\left(h_2\right) \mu^{-1}\left(h_1\right) \mu\left(h_1 h_2\right).$$

Therefore by Proposition 5.10, we have that α^{-1} is a coboundary. Since the 2-coboundaries of G form a group, we have that α is a coboundary and we have that $\mathbb{C}^\alpha[H] \cong \mathbb{C}H$ by Theorem 5.12.

□

The following result is a generalisation of the above result which we use in the proof of Theorem 6.3.

Lemma 5.16. *Let G be a group and $N \trianglelefteq G$. Suppose that $H^1(N, \mathbb{C}^*) = 1$. Let $\alpha \in H^2(N, \mathbb{C}^*)$ be G-invariant. Let $\beta \in H^2(G, \mathbb{C}^*)$ and $\gamma \in H^2(G, \mathbb{C}^*)$ such that $\mathrm{Res}_N^G \beta = \alpha$ and $\mathrm{Res}_N^G \gamma = \alpha$. Let $\theta \in \mathrm{Irr}(\mathbb{C}^\alpha[N])$ and suppose that θ extends to $\chi \in \mathrm{Irr}(\mathbb{C}^\beta[G])$ and $\chi' \in \mathrm{Irr}(\mathbb{C}^\gamma[G])$ then $\mathbb{C}^\beta[G] \cong \mathbb{C}^\gamma[G]$.*

Proof. By [15, Theorem 2], we have that the sequence

$$H^2\left(G/N, \mathbb{C}^*\right) \underset{l}{\rightarrow} H^2\left(G, \mathbb{C}^*\right) \underset{r}{\rightarrow} H^2\left(N, \mathbb{C}^*\right)^G \underset{t}{\rightarrow} H^3\left(G/N, \mathbb{C}^*\right),$$

is exact. Since α extends to both β and γ, we have $r(\beta\gamma^{-1}) := \mathrm{Res}_N^G(\beta\gamma^{-1}) = 1$, i.e. $\beta\gamma^{-1} \in \ker(r)$ so by exactness $\beta\gamma^{-1} \in \mathrm{im}(l)$. Therefore $\beta\gamma^{-1} = l(\mu) = \mathrm{Inf}(\mu)$ for some $\mu \in H^2(G/N, \mathbb{C}^*)$. Hence $\beta = \gamma\mathrm{Inf}(\mu)$. Recall that θ extends to χ' a projective γ-character of $\overline{G}$ and χ a projective $\gamma\mathrm{Inf}(\mu)$-character of $\overline{G}$. Hence by [22, Theorem 3.1], we have that μ is cohomologous to $1 \in B^2(G/N, \mathbb{C}^*)$. Hence, $\mathrm{Inf}(\mu)$ is cohomologous to $\mathrm{Inf}(1) = 1 \in B^2(G, \mathbb{C}^*)$ and hence β is cohomologous to γ. Therefore by Theorem 5.12, the associated twisted group algebras are isomorphic. □

For the following, up to Lemma 5.20, see Appendix D, which are lecture notes due to Evseev. We let A denote a twisted G-algebra. From our understanding of twisted group algebras, we have that A is therefore G-graded. Suppose that $\pi : G \to G/Z$ for a normal subgroup Z (think $Z := Z(G)$) of G, denotes the canonical surjection. We can give A a G/Z-grading, i.e. for each $g \in G/Z$,

$$A_g := \sum_{h \in \pi^{-1}(g)} A_h.$$

Writing $\overline{G} := G/Z$, we call this algebra $A\left[\overline{G}\right]$. We also introduce the following notation. Suppose that $\chi \in \mathrm{Irr}(A)$ and $\lambda \in \mathrm{Irr}(A[Z])$, we say that χ lies over λ if λ is a constituent of $\mathrm{Res}^A_{A[Z]}\chi$, and we denote by $\mathrm{Irr}(A|\lambda)$ the set of all irreducible characters of A which lie over θ. In our case, $A = \mathbb{C}G$ and $\mathrm{Irr}(A|\theta)$ coincides with our earlier definition.

Definition 5.17. [Appendix D] *Let $\lambda \in \mathrm{Irr}(A[Z])$. We define $A(\lambda) := C_A\left(A[Z]\right)e_\lambda$.*

Definition 5.18. [Appendix D] *For $\chi' \in \mathrm{Irr}(A)$,*

$$e_{\chi'} := \sum_{g \in G} \chi'(g^{-1})g.$$

We note that if λ is $\overline{G}$-invariant then, $A(\lambda)$ can be viewed as a twisted $\overline{G}$-algebra, with the grading inherited from that of $A\left[\overline{G}\right]$. We now state a fundamental theorem, which helps us change our setting from group algebras to that of twisted group algebras.

Theorem 5.19. [Appendix D] *Suppose that $\lambda \in \mathrm{Irr}(A[Z])$ is $\overline{G}$-invariant. There is a one-to-one correspondence between $\mathrm{Irr}(A|\lambda)$ and $\mathrm{Irr}(A(\lambda))$, where two characters $\chi \in \mathrm{Irr}(A|\lambda)$ and $\chi' \in \mathrm{Irr}(A(\lambda))$ correspond if and only if $e_\chi = e_{\chi'}$.*

One can also check that the bijection above is given by $\phi : \mathrm{Irr}(A(\lambda)) \to \mathrm{Irr}(A|\lambda)$, where $\phi\left(\chi'\right) := \mathrm{Res}^{A[Z]}_{A[Z]e_\lambda}\lambda \times \chi'$. Throughout we will use the prime notation, to denote the image of $\chi \in \mathrm{Irr}(A|\lambda)$ under this bijection. Note that we can $\mathbb{Z}$-linearly extend the map ϕ to elements of $\mathcal{C}(G)$ and $\mathbb{Z}[\mathrm{Irr}(A)]$. We now prove our first result.

Lemma 5.20. *Let $Z := Z(G)$ and $N \trianglelefteq G$ such that $Z \trianglelefteq N \trianglelefteq G$. Suppose that $\mathrm{Res}^G_N\chi = \theta \in \mathrm{Irr}(N)$. Then $\mathrm{Res}^A_{A[\overline{N}]}\chi' = \theta'$.*

Proof. First note that there exists some $\lambda \in \mathrm{Irr}(Z)$ such that $\chi \in \mathrm{Irr}(G|\lambda)$ where λ is G/Z invariant, hence the above theorem can indeed be applied in this case. Furthermore, from the correspondence theorem above, $e_\chi = e_{\chi'}$ and $e_\theta = e_{\theta'}$. Note that χ' lies above θ' if and only if $e_{\chi'}e_{\theta'} \neq 0$. This is clear since $e_\chi e_\theta \neq 0$. Moreover since λ is linear, the degrees of χ' and θ' are equal (see ϕ above) and we are done. $\square$

To finish we are need to show that a more general statement holds, namely that the diagram

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{Irr}(B\left[\overline{H}\right])] & \xrightarrow{\phi|_{B[\overline{H}]}} & \mathbb{Z}[\mathrm{Irr}(A[H]|\lambda)] \\ {\scriptstyle \mathrm{Ind}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ind}} \\ \mathbb{Z}[\mathrm{Irr}(B)] & \xrightarrow{\phi} & \mathbb{Z}[\mathrm{Irr}(A|\lambda)] \end{array} \tag{5.1}$$

is commutative. Here, $H \leq G$ such that $Z := Z(G) \leq H$, and $\lambda \in \mathrm{Irr}(Z)$. Note that when $A := \mathbb{C}G$, we have that $\mathrm{Irr}(A|\lambda) = \mathrm{Irr}(G|\lambda)$ and $B := A(\lambda) = C_{\mathbb{C}G}(\mathbb{C}Z)e_\lambda = \mathbb{C}Ge_\lambda$ is a twisted $\overline{G}$-algebra. We first define induction of modules over twisted group algebras.

Definition 5.21. *Let $H \leq G$ and let M be a $\mathbb{C}^\alpha[H]$-module. We define,*

$$\mathrm{Ind}_H^G M := \mathbb{C}^\alpha[G] \otimes_{\mathbb{C}^\alpha[H]} M,$$

to be the induced module $\mathbb{C}^\alpha[G]$-module of M.

The following are properties of induction that we will be using but proved for modules over twisted group algebras. The following can be compared with the statement of Frobenius reciprocity and is known as the Nakayama relation.

Theorem 5.22. (See [5, Proposition 2.8.3]) *Let G be a group and $H \leq G$. Let N be a $\mathbb{C}^\alpha[G]$-module and M a $\mathbb{C}^\alpha[H]$-module then,*

$$\mathrm{Hom}_{\mathbb{C}^\alpha[H]}\left(M, \mathrm{Res}_H^G N\right) \cong \mathrm{Hom}_{\mathbb{C}^\alpha[G]}\left(\mathrm{Ind}_H^G M, N\right).$$

Lemma 5.23. *Let A be a semisimple $\mathbb{C}$-algebra and $B \subseteq A$, be a subalgebra. Then for* *an B-module M,*

$$\mathrm{Ind}_B^A M = \sum_{a\in A} a \otimes M.$$

Where coset representatives make sense, this sum can also be made to be the direct sum over coset representatives of B in A. We now prove our key result regarding the commutativity of Diagram (5.1).

Lemma 5.24. *(5.1) is commutative.*

Proof. We prove the result when $\theta' \in \mathrm{Irr}(B\left[\overline{H}\right])$ and the result follows from the linearity of induction and the bijection ϕ. We define $\theta := \phi(\theta')$, i.e. $\theta = \lambda' \times \theta'$, where $\lambda' \in \mathrm{Irr}(Z|\lambda) = \{\lambda\}$. Let $\chi' := \mathrm{Ind}_{B\left[\overline{H}\right]}^{B}\theta' \in \mathbb{Z}[\mathrm{Irr}(B)]$. Finally let $\chi := \phi(\chi') = \lambda' \times \chi'$. We claim that $\chi = \mathrm{Ind}_{A[H]}^{A}\theta$.

Let M be the $\mathbb{C}Ze_\lambda$-module aferding λ' and N be the $B\left[\overline{H}\right]$-module affording θ'. The claim follows once we show,

$$\mathrm{Ind}_{\mathbb{C}Ze_\lambda\otimes_\mathbb{C} B\left[\overline{H}\right]}^{\mathbb{C}Ze_\lambda\otimes_\mathbb{C} B}(M \otimes_\mathbb{C} N) \cong M \otimes_\mathbb{C} \mathrm{Ind}_{B\left[\overline{H}\right]}^{B} N. \tag{5.2}$$

If Equation (5.2) holds, then $\mathrm{Ind}_{A[H]}^{A}\theta = \mathrm{Ind}_{A[H]}^{A}(\lambda' \times \theta') = \lambda' \times \mathrm{Ind}_{B\left[\overline{H}\right]}^{B}\theta' = \lambda' \times \chi' = \chi$. Let T be a left transversal for $\overline{H}$ in $\overline{G}$. Then,

$$\mathrm{Ind}_{B\left[\overline{H}\right]}^{B} N = B \otimes_{B\left[\overline{H}\right]} N = \left(\bigoplus_{t\in T} tB\left[\overline{H}\right]\right) \otimes_{B\left[\overline{H}\right]} N = \bigoplus_{t\in T} t \otimes_{B\left[\overline{H}\right]} N.$$

Moreover,

$$M \otimes_\mathbb{C} \mathrm{Ind}_{B\left[\overline{H}\right]}^{B} N = \bigoplus_{t\in T}\left(M \otimes_\mathbb{C}\left(t \otimes_{B\left[\overline{H}\right]} N\right)\right).$$

For the left hand side of Equation (5.2),

$$\mathrm{Ind}_{\mathbb{C}Ze_\lambda\otimes_\mathbb{C} B\left[\overline{H}\right]}^{\mathbb{C}Ze_\lambda\otimes_\mathbb{C} B}(M \otimes_\mathbb{C} N)$$

$$= (\mathbb{C}Ze_\lambda \otimes_\mathbb{C} B) \otimes_{\mathbb{C}Ze_\lambda\otimes_\mathbb{C} B\left[\overline{H}\right]} (M \otimes_\mathbb{C} N)$$

$$= \left(\mathbb{C}Ze_\lambda \otimes_{\mathbb{C}} \left(\bigoplus_{t \in T} tB\left[\overline{H}\right] \right) \right) \otimes_{\mathbb{C}Ze_\lambda \otimes_{\mathbb{C}} B\left[\overline{H}\right]} (M \otimes_{\mathbb{C}} N)$$
$$= \bigoplus_{t \in T} \Big((\mathbb{C}Ze_\lambda \otimes_{\mathbb{C}} t) \otimes_{\mathbb{C}Ze_\lambda \otimes_{\mathbb{C}} B\left[\overline{H}\right]} (M \otimes_{\mathbb{C}} N) \Big)$$

Define f_t, for each $t \in T$, by

$$f_t\left((a \otimes t) \otimes (m \otimes n)\right) := am \otimes (t \otimes n).$$

This induces a $\mathbb{C}$-vector space isomorphism between each summand of the left and right hand sides of Equation (5.2). One can also check that the induced map is an isomorphism of $\mathbb{C}Ze_\lambda \otimes_{\mathbb{C}} B$-modules. □

We now look to generalising results from the ordinary representation theory of $\mathbb{C}$, to modules over twisted group algebras. We begin by proving an analogue to the Mackey decomposition formula.

Lemma 5.25. *Let H, K be subgroups of G and let T denote a complete set of (K, H)-double coset representatives in G. Let M be a $\mathbb{C}^\alpha[H]$-module. Then,*

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H M \cong \bigoplus_{t \in T} \mathrm{Ind}^K_{{}^tH \cap K} \mathrm{Res}^{{}^tH}_{{}^tH \cap K} M^t.$$

Proof. Note we can write

$$\mathbb{C}^\alpha\left[KgH\right] = \sum_{x \in KgH} \mathbb{C}^\alpha\left[KgH\right]_x,$$

and by definition $G = \sqcup_{t \in T}(KtH)$. First, we prove that $\mathbb{C}^\alpha\left[KgH\right]$ is a $(\mathbb{C}^\alpha\left[K\right], \mathbb{C}^\alpha\left[H\right])$-bimodule. If $a_x \in \mathbb{C}^\alpha\left[KgH\right]$, $a_h \in \mathbb{C}^\alpha\left[H\right]$ and $a_k \in \mathbb{C}^\alpha\left[K\right]$ are basis elements, then

$$(a_k a_x)\, a_h = \alpha(k, x)\, a_{kx} a_h = \alpha(k, x)\, \alpha(kx, h)\, a_{(kx)h} \in \mathbb{C}^\alpha\left[KgH\right].$$

Furthermore, we have that $\mathbb{C}^\alpha\left[G\right] = \bigoplus_{t \in T} \mathbb{C}^\alpha\left[KtH\right]$. We now define a map ϕ_x for each

$x \in G$ and prove this is an isomorphism between $\mathbb{C}^{\alpha}[K] \otimes_{\mathbb{C}^{\alpha}[{}^xH \cap K]} \mathrm{Res}^{{}^xH}_{{}^xH\cap K} {}^xM$ and $\mathbb{C}^{\alpha}[KxH] \otimes_{\mathbb{C}^{\alpha}[H]} M$ as $\mathbb{C}^{\alpha}[K]$ modules. We define the map ϕ_x by $\phi_x(a_k \otimes m) := a_{kx} \otimes m$. Let $a_{kxh} \otimes m \in \mathbb{C}^{\alpha}[KxH] \otimes_{\mathbb{C}^{\alpha}[H]} M$. Then,

$$a_{kxh} \otimes m = (\alpha(kx,h))^{-1} a_{kx} \otimes a_h m = \phi_x\left((\alpha(kx,h)^{-1} a_k \otimes a_h m\right).$$

Now since M is a $\mathbb{C}^{\alpha}[H]$-module, we have $a_h m \in M$, as $a_h \in \mathbb{C}^{\alpha}[H]$ and we have that ϕ_x is surjective. Let $\{k_1, \ldots, k_l\}$ be a set of coset representatives of left $K \cap {}^xH$ cosets in K, so therefore $\{k_1 x, \ldots, k_l x\}$ is a a basis of $\mathbb{C}^{\alpha}[KxH]$ as a right $\mathbb{C}^{\alpha}[H]$-module. Hence,

$$\dim_{\mathbb{C}}\left((\mathbb{C}^{\alpha}[KxH]) \otimes_{\mathbb{C}^{\alpha}[H]} M\right) = l \dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}\left(\mathbb{C}^{\alpha}[K] \otimes_{\mathbb{C}^{\alpha}[K\cap {}^xH]} \mathrm{Res}^{{}^xH}_{{}^xH\cap K} {}^xM\right),$$

therefore ϕ_x is injective. Finally,

$$\mathbb{C}^{\alpha}[G] \otimes_{\mathbb{C}^{\alpha}[H]} M \cong \bigoplus_{t\in T} \mathbb{C}^{\alpha}[KtH] \otimes_{\mathbb{C}^{\alpha}[H]} M \cong \bigoplus_{t\in T} \mathbb{C}^{\alpha}[K] \otimes_{\mathbb{C}^{\alpha}[K\cap {}^tH]} \mathrm{Res}^{{}^tH}_{{}^tH\cap K} {}^tM,$$

as $\mathbb{C}^{\alpha}[K]$-modules. □

Next, we generalise Lemma 4.23 to a similar statement for projective characters. Note that to see how this generalises Lemma 4.23, we take $\alpha = 1$ and $\mu = 1$.

Lemma 5.26. *Let G be a group with normal subgroups, $H \trianglelefteq G$ and $N \trianglelefteq G$. Let $A = \mathbb{C}^{\alpha\mu}[G]$ where α, $\mu \in Z^2(G, \mathbb{C}^*)$, $A' = \mathbb{C}^{\alpha}[G]$ and $B = \mathbb{C}^{\mu^{-1}}[G]$. Suppose that $\chi \in \mathcal{A}_n(A, H, N)$ and $\lambda \in \mathcal{C}(B)$, then $\chi \cdot \lambda \in \mathcal{A}_n(A', H, N)$.*

Proof. Since $\chi \in \mathcal{A}_n(A, H, N)$, we have that

$$\chi = \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}^G_K \phi_K,$$

where $\phi_K \in \mathcal{C}(A[K])$. Then using [22, Chapter 1, Proposition 9.8] we have that,

$$\chi \cdot \lambda = \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}_K^G \phi_K \cdot \lambda = \sum_{\substack{K \in \mathcal{I}_n(G,H) \\ N \leq K}} \mathrm{Ind}_K^G \left(\phi_K \mathrm{Res}_K^G \lambda \right).$$

Now ϕ_K is a projective $\alpha\mu$ character of K and $\mathrm{Res}_K^G \lambda$ is a projective μ^{-1} character of K. Hence their product is a projective α character of K. Therefore $\chi \cdot \lambda \in \mathcal{A}_n(A', H, N)$. □

As stated in the introduction to this chapter, our goal is to generalise the results from Section 5.2 to modules over twisted group algebras and projective characters.

Theorem 5.27. [26, (A4)] *Let $G \leq S_n$. Every simple $A \wr G$-module, L, is, up to isomorphism, of the form $\mathrm{Ind}_T^G (V \otimes_{\mathbb{C}} M)$, where T is the inertia subgroup of M, a simple $A^{\otimes n}$-module, in G and V is a simple $\mathbb{C}T$-module.*

We now change our focus to projective characters and further develop the character theory of twisted group algebras of wreath products. We first address the issue of existence of such class functions. We aim to define an analogue for $\phi^{\widetilde{\times} n}$ (as in 5.4), for $A \wr S_n$ where A is a semisimple $\mathbb{C}$-algebra. We begin with the case $A = M_n(\mathbb{C})$ and then work towards the general setting. Suppose that $\phi \in \mathbb{C}[\mathrm{Irr}(A)]$. We aim to find an element $\phi^{\widetilde{\times} n} \in \mathbb{C}[\mathrm{Irr}(A \wr S_n)]$ such that,

$$\phi^{\widetilde{\times} n} \left(y_\sigma(\underline{a}) \right) = \phi(a_1) \dots \phi(a_r). \tag{5.3}$$

where if $\sigma = \sigma_1 \cdots \sigma_r$ and $\underline{a} := (a_1, \dots, a_r)$ then,

$$y_\sigma(\underline{a}) = y_{\sigma_1}(a_1) \cdots y_{\sigma_r}(a_r).$$

Note that the only irreducible A-module is $V = \mathbb{C}^n$ and using the above definition, we may obtain a $A \wr S_n$ module denoted $V^{\widetilde{\otimes} n}$. Therefore using Theorem 5.27, we have that $\left\{ V^{\widetilde{\otimes} n} \otimes_{\mathbb{C}} Z^\lambda : \lambda \vdash n \right\}$ is a complete and irredundant set of $A \wr S_n$-modules, where Z^λ is the simple $\mathbb{C}S_n$-module labelled by λ. We denote by χ the character afforded by V, so that ϕ is equal to $c\chi$ for some $c \in \mathbb{C}$. Therefore Equation (5.3) says that we are aiming to prove,

$$\phi^{\widetilde{\times} n}(y_\sigma(\underline{a})) = c^r \chi(a_1) \dots \chi(a_r),$$

for any $a_i \in A$, where σ_i are disjoint marked cycles with orders summing to n. From the above, we have that $\mathrm{Irr}(A \wr S_n)$ is given by

$$\mathrm{Irr}(A \wr S_n) = \left\{ \chi^{\widetilde{\times} n} \cdot \eta^\lambda : \lambda \vdash n \right\},$$

where for $(a; \sigma) \in A \wr S_n$,

$$\left(\chi^{\widetilde{\times} n} \cdot \eta^\lambda \right) (a; \sigma) := \chi^{\widetilde{\times} n}(a; \sigma) \eta^\lambda(\sigma).$$

Therefore as,

$$\left(\chi^{\widetilde{\times} n} \cdot \eta^\lambda \right) (y_\sigma(\underline{a})) = \chi(a_1) \dots \chi(a_r) \eta^\lambda(\sigma),$$

and we are required to find $d_\lambda \in \mathbb{C}$ such that,

$$\phi^{\widetilde{\times} n} = \sum_{\lambda \vdash n} d_\lambda \left(\chi^{\widetilde{\times} n} \cdot \eta^\lambda \right),$$

evaluating both sides on $y_\sigma(\underline{a})$, the d_λ must satisfy

$$\sum_{\lambda \vdash n} d_\lambda \chi(a_1) \cdots \chi(a_r) \eta^\lambda(\sigma) = c^r \chi(a_1) \cdots \chi(a_r).$$

Therefore, $\sum_{\lambda \vdash n} d_\lambda \eta^\lambda(\sigma) = c^r = c^{l(\sigma)}$, where $l(\sigma)$ denotes the number of disjoint cycles in σ. Using the notation following [10, Theorem 2.4], we have that $\kappa_{n,c}(\sigma) = c^{l(\sigma)}$, and this is indeed a class function of S_n. Since η^λ runs over $\mathrm{Irr}(S_n)$, such d_λ exist. Therefore, $\phi^{\widetilde{\times} n}$ is indeed a class function on $A \wr S_n$.

In general, $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$. Supposing that $\mathrm{Irr}(A) = \{\chi_1, \ldots, \chi_k\}$, we have that $\phi \in \mathbb{C}[\mathrm{Irr}(A)]$ takes the form, $\phi = \sum_{i=1}^{k} c_i\chi_i$ for $c_i \in \mathbb{C}$. We will define,

$$\phi^{\widetilde{\times} n} := \sum_{\substack{n_i \geq 0 \\ n = n_1 + \cdots + n_k}} \mathrm{Ind}_{A \wr S_{n_1} \times \cdots \times S_{n_k}}^{A \wr S_n} \left[(c_1\chi_1)^{\widetilde{\times} n_1} \times \cdots \times (c_k\chi_k)^{\widetilde{\times} n_k} \right],$$

where $(c\chi)^{\widetilde{\times} n} := \chi^{\widetilde{\times} n} \cdot \kappa_{c,n}$, where for an element, $g_\lambda \in S_n$ of cycle type λ, $\kappa_{c,n}(g_\lambda) = c^{l(\lambda)}$. We also make the following definition.

Definition 5.28. *Let $\alpha \in \mathbb{C}[\mathrm{Irr}(A \wr S_j)]$ and $\beta \in \mathbb{C}[\mathrm{Irr}(A \wr S_k)]$. We define,*

$$\alpha \circ \beta := \mathrm{Ind}_{(A \wr S_j) \times (A \wr S_k)}^{A \wr S_{j+k}} (\alpha \times \beta).$$

Note that this definition will also be used in the setting where we drop the algebra and simply perform induction in the symmetric group. We now prove some preliminary results using the new definitions from above.

Lemma 5.29. *Let $\alpha \in \mathbb{C}[\mathrm{Irr}(S_j)]$, $\beta \in \mathbb{C}[\mathrm{Irr}(S_{n-j})]$ and $\chi \in \mathrm{Irr}(A)$ then*

$$\left(\chi^{\widetilde{\times} j} \cdot \alpha\right) \circ \left(\chi^{\widetilde{\times} (n-j)} \cdot \beta\right) = \chi^{\widetilde{\times} n} \cdot (\alpha \circ \beta).$$

Proof. Suppose first that $\alpha \in \mathrm{Irr}(S_j)$ and $\beta \in \mathrm{Irr}(S_{n-j})$. Let M be the simple A-module affording χ, V_α be the simple $\mathbb{C}S_j$-module affording α and V_β be the simple $\mathbb{C}S_{n-j}$-module affording β. We therefore are required to show that,

$$\mathrm{Ind}_{(A \wr S_j) \times (A \wr S_{n-j})}^{A \wr S_n} \left(\left(M^{\widetilde{\otimes} j} \otimes_{\mathbb{C}} V_\alpha \right) \otimes_{\mathbb{C}} \left(M^{\widetilde{\otimes} (n-j)} \otimes_{\mathbb{C}} V_\beta \right) \right) \cong$$

$$M^{\widetilde{\otimes} n} \otimes_{\mathbb{C}} \left(\mathrm{Ind}_{S_j \times S_{n-j}}^{S_n} (V_\alpha \otimes_{\mathbb{C}} V_\beta) \right). \tag{5.4}$$

We define a map,

$$\phi\left((t_1, \ldots, t_n; \tau) \otimes (m_1 \otimes \cdots \otimes m_j \otimes v_\alpha \otimes m_{j+1} \otimes \cdots \otimes m_n \otimes v_\beta)\right) :=$$

$$\left(t_1 m_{\tau^{-1}(1)} \otimes \cdots \otimes t_n m_{\tau^{-1}(n)}\right) \otimes \left(\tau \otimes v_\alpha \otimes v_\beta\right),$$

for $t_i \in A$, $\tau \in S_n$, $m_i \in M$, $v_\alpha \in V_\alpha$ and $v_\beta \in V_\beta$. Note we view the left hand side of Equation (5.4) as,

$$\sum_{t \in A \wr S_n} t \otimes \left(\left(M^{\widetilde{\otimes} j} \otimes_{\mathbb{C}} V_\alpha\right) \otimes_{\mathbb{C}} \left(M^{\widetilde{\otimes}(n-j)} \otimes_{\mathbb{C}} V_\beta\right)\right),$$

and similarly for the induced module on the right hand side of Equation (5.4). We show that ϕ is indeed a module homomorphism. Let $a := (a_1, \ldots, a_n; \sigma) \in A \wr S_n$, $t := (t_1, \ldots, t_n; \tau)$ and $m := (m_1 \otimes \cdots \otimes m_j \otimes v_\alpha \otimes m_{j+1} \otimes \cdots \otimes m_n \otimes v_\beta)$. Then,

$$a \cdot (t \otimes m) = \left(a_1 t_{\sigma^{-1}(1)}, \ldots, a_n t_{\sigma^{-1}(n)}; \sigma\tau\right) \otimes m.$$

Hence,

$$\phi\left(a \cdot (t \otimes m)\right) = \left(a_1 t_{\sigma^{-1}(1)} m_{(\sigma\tau)^{-1}(1)}, \ldots, a_n t_{\sigma^{-1}(n)} m_{(\sigma\tau)^{-1}(n)}\right) \otimes \left(\sigma\tau \otimes v_\alpha \otimes v_\beta\right).$$

Moreover,

$$a \cdot \phi\left(t \otimes m\right) = \left(a_1 t_{\sigma^{-1}(1)} m_{\tau^{-1}\sigma^{-1}(1)}, \ldots, a_n t_{\sigma^{-1}(n)} m_{\tau^{-1}\sigma^{-1}(n)}\right) \otimes \left(\sigma\tau \otimes v_\alpha \otimes v_\beta\right).$$

Hence ϕ is indeed a module homomorphism. Extending $\mathbb{C}$-linearly, we obtain a vector space homomorphism, so we construct an inverse map. Define,

$$\psi\left((m_1 \otimes \cdots \otimes m_n) \otimes (\tau \otimes v_\alpha \otimes v_\beta)\right) :=$$

$$(1, \ldots, 1; \tau) \otimes \left(m_{\tau(1)} \otimes \ldots \otimes m_{\tau(j)} \otimes v_\alpha \otimes m_{\tau(j+1)} \otimes \cdots \otimes m_{\tau(n)} \otimes v_\beta\right).$$

We claim that ψ is indeed a module homomorphism. Using the definition of a above,

$t := (m_1 \otimes \ldots \otimes m_n) \otimes (\tau \otimes v_\alpha \otimes v_\beta)$,

$$a \cdot t = \left(a_1 m_{\sigma^{-1}(1)}, \ldots, a_n m_{\sigma^{-1}(n)}\right) \otimes (\sigma\tau \otimes v_\alpha \otimes v_\beta),$$

therefore,

$$\psi(a \cdot t) = (1, \ldots, 1; \sigma\tau) \otimes$$

$$\left(a_{\sigma\tau(1)} m_{\tau(1)} \otimes \cdots \otimes a_{\sigma\tau(j)} m_{\tau(j)} \otimes v_\alpha \otimes a_{\sigma\tau(j+1)} m_{\tau(j+1)} \otimes \cdots \otimes a_{\sigma\tau(n)} m_{\tau(n)} \otimes v_\beta\right).$$

Moreover,

$$a \cdot \psi(t) = (a_1, \ldots, a_n; \sigma\tau) \otimes \left(m_{\tau(1)} \otimes \cdots \otimes m_{\tau(j)} \otimes v_\alpha \otimes m_{\tau(j+1)} \otimes \cdots \otimes m_{\tau(n)} \otimes v_\beta\right),$$

and the result follows since $(a_1, \ldots, a_n; \sigma\tau) = (1, \ldots, 1; \sigma\tau)\left(a_{\sigma\tau(1)}, \ldots, a_{\sigma\tau(n)}\right)$. Finally, we show that the maps are indeed inverses. Defining $t := (t_1, \ldots, t_n; \tau)$ and

$$m := m_1 \otimes \cdots \otimes m_j \otimes v_\alpha \otimes m_{j+1} \otimes \cdots \otimes m_n \otimes v_\beta,$$

we have

$$\psi(\phi(t \otimes m)) = \psi\left(\left(t_1 m_{\tau^{-1}(1)} \otimes \cdots \otimes t_n m_{\tau^{-1}(n)}\right) \otimes (\tau \otimes v_\alpha \otimes v_\beta)\right)$$

$$= (1, \ldots, 1; \tau) \otimes \left(t_{\tau(1)} m_1 \otimes \cdots \otimes t_{\tau(j)} m_j \otimes v_\alpha \otimes t_{\tau(j+1)} m_{j+1} \otimes \cdots \otimes t_{\tau(n)} m_n \otimes v_\beta\right)$$

$$= \left((1, \ldots, 1; \tau)(t_{\tau(1)}, \ldots, t_{\tau(n)})\right) \otimes m = t \otimes m.$$

Therefore the result indeed holds in the case of irreducible α and β. Using this we now consider the general case. We denote $\mathrm{Irr}(S_j) := \{\alpha_1, \ldots, \alpha_k\}$ and $\mathrm{Irr}(S_{n-j}) := \{\beta_1, \ldots, \beta_l\}$. We suppose that $\alpha = \sum_{i=1}^k f_i \alpha_i$ and $\beta = \sum_{j=1}^l g_j \beta_j$ for $f_i \in \mathbb{C}$ and $g_j \in \mathbb{C}$. Then,

$$\chi^{\widetilde{\times} j} \cdot \alpha = \sum_{i=1}^k f_i \left(\chi^{\widetilde{\times} j} \cdot \alpha_i\right),$$

and similarly for $\chi^{\widetilde{\times}(n-j)} \cdot \beta$. Moreover,

$$\left(\chi^{\widetilde{\times}j} \cdot \alpha\right) \times \left(\chi^{\widetilde{\times}(n-j)} \cdot \beta\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} f_i g_j \left(\left(\chi^{\widetilde{\times}j} \cdot \alpha_i\right) \times \left(\chi^{\widetilde{\times}(n-j)} \cdot \beta_j\right)\right).$$

Therefore, by linearity of induction, we have,

$$\left(\chi^{\widetilde{\times}j} \cdot \alpha\right) \circ \left(\chi^{\widetilde{\times}(n-j)} \cdot \beta\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} f_i g_j \left(\left(\chi^{\widetilde{\times}j} \cdot \alpha_i\right) \circ \left(\chi^{\widetilde{\times}(n-j)} \cdot \beta_j\right)\right).$$

Now applying the irreducible case,

$$\left(\chi^{\widetilde{\times}j} \cdot \alpha\right) \circ \left(\chi^{\widetilde{\times}(n-j)} \cdot \beta\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} f_i g_j \left(\chi^{\widetilde{\times}n} \cdot (\alpha_i \circ \beta_j)\right)$$

$$= \chi^{\widetilde{\times}n} \cdot \left(\left(\sum_{i=1}^{k} f_i \alpha_i\right) \circ \left(\sum_{j=1}^{l} g_j \beta_j\right)\right) = \chi^{\widetilde{\times}n} \left(\alpha \circ \beta\right).$$

□

Lemma 5.30. $\sum_{i=0}^{n} \kappa_{c,i} \circ \kappa_{d,n-i} = \kappa_{c+d,n}$.

Proof. Let $g_\lambda \in S_n$ be of cycle type $\lambda := (\lambda_1, \ldots, \lambda_r) \vdash n$. Define $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$. For each $j \in [r]$, let σ_j be the set of elements $i \in [n]$ such that i is in the cycle of length λ_j. Clearly for each j, we associate a distinct cycle of g_λ. Also for each $I \subseteq [r]$, define $\sigma_I := \bigcup_{j \in I} \sigma_j$ and $S_I := S_{\sigma_I}$. Moreover, $S_{I^c} := S_{[n] \setminus \sigma_I}$.

First, we note that if $g_\lambda \in S_J \times S_{J^c}$ and $I \subseteq [r]$ with $I \neq J$ such that $|\sigma_I| = |\sigma_J|$, then there exists $x \in S_n$, where x is a transposition and $x \notin S_J \times S_{J^c}$ (the stabiliser in S_n of σ_J), such that $x g_\lambda x^{-1} \in S_I \times S_{I^c}$. Therefore we have the following,

$$\sum_{i=0}^{n} \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n} \left(\kappa_{c,i} \times \kappa_{d,n-i}\right)(g_\lambda) = \sum_{i=0}^{n} \sum_{\substack{x \in S_n \\ x g_\lambda x^{-1} \in S_i \times S_{n-i}}} \left(\kappa_{c,i} \times \kappa_{d,n-i}\right)(x g_\lambda x^{-1})$$

$$= \sum_{i=0}^{n} \sum_{\substack{I \subseteq [r] \\ |\sigma_I| = i}} c^{|I|} d^{r-|I|} = \sum_{I \subseteq [r]} c^{|I|} d^{r-|I|} = \sum_{i=0}^{r} \binom{r}{i} c^i d^{r-i} = (c+d)^r = \kappa_{c+d,n}(g_\lambda).$$

$\square$

Lemma 5.31. *If $\chi \in \operatorname{Irr}(B)$ and $\psi \in \mathbb{C}[\operatorname{Irr}(S_n)]$, then*

$$\operatorname{Ind}_{B \wr S_n}^{A \wr S_n}\left(\chi^{\widetilde{\times} n} \cdot \psi\right) = \left(\operatorname{Ind}_{B \wr S_n}^{A \wr S_n} \chi^{\widetilde{\times} n}\right) \cdot \psi.$$

Proof. First we suppose that $\psi \in \operatorname{Irr}(S_n)$ and use this to prove the general case. In this setting we denote the simple $\mathbb{C}S_n$-module affording ψ by V and the simple B-module affording χ by M. We therefore prove that,

$$\operatorname{Ind}_{B \wr S_n}^{A \wr S_n}\left(M^{\widetilde{\otimes} n} \otimes_{\mathbb{C}} V\right) \cong \left(\operatorname{Ind}_{B \wr S_n}^{A \wr S_n} M^{\widetilde{\otimes} n}\right) \otimes_{\mathbb{C}} V.$$

We define the following maps and state without proof that they are vector space isomorphisms. Define,

$$\varphi_1 : \operatorname{Ind}_{B \wr S_n}^{A \wr S_n}\left(M^{\widetilde{\otimes} n} \otimes_{\mathbb{C}} V\right) \to A^{\otimes n} \otimes_{B^{\otimes n}} \left(M^{\widetilde{\otimes} n} \otimes_{\mathbb{C}} V\right),$$

via

$$\varphi_1\left((b;\tau) \otimes (m \otimes v)\right) := b \otimes (1;\tau)m \otimes \tau v.$$

Moreover,

$$\varphi_2 : A^{\otimes n} \otimes_{B^{\otimes n}} \left(M^{\widetilde{\otimes} n} \otimes_{\mathbb{C}} V\right) \to \left(A^{\otimes n} \otimes_{B^{\otimes n}} M^{\widetilde{\otimes} n}\right) \otimes_{\mathbb{C}} V,$$

since

$$\varphi_2\left(a \otimes (m \otimes v)\right) := (a \otimes m) \otimes v.$$

Finally,

$$\varphi_3 : \left(A^{\otimes n} \otimes_{B^{\otimes n}} M^{\widetilde{\otimes} n}\right) \otimes_{\mathbb{C}} V \to \left(\operatorname{Ind}_{B \wr S_n}^{A \wr S_n} M^{\widetilde{\otimes} n}\right) \otimes_{\mathbb{C}} V,$$

as

$$\varphi_3\left((a \otimes m) \otimes v\right) := ((a;1) \otimes m) \otimes v.$$

We define $\phi := \varphi_3\varphi_2\varphi_1$, a vector space isomorphism, and we show that it is compatible with the module structure. We remark that,

$$\phi\left((b;\tau)\otimes(m\otimes v)\right):=((b;\tau)\otimes m)\otimes\tau v.$$

Let $(a;\sigma)\in A\wr S_n$. Then,

$$(a;\sigma)\cdot((b;\tau)\otimes(m\otimes v))=(a(b^{\sigma});\sigma\tau)\otimes(m\otimes v),$$

$$\phi\left(a(b^{\sigma});\sigma\tau\right)\otimes(m\otimes v))=((a(b^{\sigma});\sigma\tau)\otimes m)\otimes\sigma\tau v.$$

Moreover,

$$(a;\sigma)\cdot(((b;\tau)\otimes m)\otimes\tau v)=((a;\sigma)(b;\tau)\otimes m)\otimes\sigma\tau v,$$

and we are therefore done. Now consider the general case. Suppose that $\mathrm{Irr}(S_n):=\{\zeta_1,\ldots,\zeta_n\}$ and $\psi=\sum_{i=1}^{n}c_i\zeta_i$ where $c_i\in\mathbb{C}$. Then,

$$\mathrm{Ind}_{B\wr S_n}^{A\wr S_n}\left(\chi^{\widetilde{\times}n}\cdot\psi\right)=\sum_{i=1}^{n}c_i\left(\mathrm{Ind}_{B\wr S_n}^{A\wr S_n}\left(\chi^{\widetilde{\times}n}\cdot\zeta_i\right)\right)$$

$$=\sum_{i=1}^{n}c_i\left(\left(\mathrm{Ind}_{B\wr S_n}^{A\wr S_n}\chi^{\widetilde{\times}n}\right)\cdot\zeta_i\right)=\left(\mathrm{Ind}_{B\wr S_n}^{A\wr S_n}\chi^{\widetilde{\times}n}\right)\cdot\psi.$$

□

Recall that $S_{\lambda_j}:=S_j\times S_{n-j}$.

Lemma 5.32. $\mathrm{Res}_{S_{\lambda_j}}^{S_n}\kappa_{c,n}=\kappa_{c,j}\times\kappa_{c,n-j}$.

Proof. Let $\sigma\in S_{\lambda_j}$, i.e. $\sigma=\sigma_1\times\sigma_2$ with $\sigma_1\in S_j$ and $\sigma_2\in S_{n-j}$. Hence

$$\left(\kappa_{c,j}\times\kappa_{c,n-j}\right)(\sigma_1\times\sigma_2)=\kappa_{c,j}(\sigma_1)\kappa_{c,n-j}(\sigma_2)=c^{l(\sigma_1)}c^{l(\sigma_2)}=c^{l(\sigma_1)+l(\sigma_2)}=c^{l(\sigma)}.$$

□

Lemma 5.33. *Let* $\chi \in \mathrm{Irr}(A \wr S_{\lambda_j})$ *and* $\psi \in \mathbb{C}[\mathrm{Irr}(S_n)]$*, then*

$$\left(\mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \chi\right) \cdot \psi = \mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left(\chi \cdot \left(\mathrm{Res}_{S_{\lambda_j}}^{S_n} \psi\right)\right).$$

Proof. Suppose first that ψ is irreducible and suppose that V is a simple $\mathbb{C}S_n$-module affording ψ. Moreover let M denote the $A \wr S_{\lambda_j}$-module affording χ. We will show that,

$$\left(\mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} M\right) \otimes_{\mathbb{C}} V \cong \mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left(M \otimes_{\mathbb{C}} \mathrm{Res}_{S_{\lambda_j}}^{S_n} V\right).$$

Define φ as follows, $\varphi\left(((a;\tau) \otimes m) \otimes v\right) := (a;\tau) \otimes (m \otimes \tau^{-1}v)$. We claim that φ is a module homomorphism and then construct an inverse, φ^*. First note that the left hand side can be expressed as

$$\sum_{\sigma \in T} ((1;\sigma) \otimes M) \otimes V,$$

where T is a left transversal for S_{λ_j} in S_n. If $(a;\sigma) \in A \wr S_n$, then

$$(a;\sigma)\left(((1;\tau) \otimes m) \otimes v\right) = ((a;\sigma\tau) \otimes m) \otimes \sigma v,$$

and therefore,

$$\varphi\left((a;\sigma)\left(((1;\tau) \otimes m) \otimes v\right)\right) = (a;\sigma\tau) \otimes \left(m \otimes \tau^{-1}\sigma^{-1}\sigma v\right) = (a;\sigma\tau) \otimes \left(m \otimes \tau^{-1}v\right).$$

Moreover,

$$(a;\sigma)\varphi\left(((1;\tau) \otimes m) \otimes v\right) = (a;\sigma)\left((1;\tau) \otimes (m \otimes \tau^{-1}v)\right) = (a;\sigma\tau) \otimes \left(m \otimes \tau^{-1}v\right).$$

Define $\varphi^*\left((a;\tau) \otimes (m \otimes v)\right) := ((a;\tau) \otimes m) \otimes \tau v$. One can check that φ^* is also a module homomorphism and the inverse of φ. Hence the isomorphism indeed holds. Suppose that

$\mathrm{Irr}(S_n) = \{\chi_1, \ldots, \chi_k\}$, and $\psi = \sum_{i=1}^{k} c_i\chi_i$ where $c_i \in \mathbb{C}$. Using the previous case,

$$\left(\mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \chi^{\widetilde{\times}n}\right) \cdot \psi = \sum_{i=1}^{k} \left(c_i \left(\mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \chi^{\widetilde{\times}n}\right) \cdot \chi_i\right)$$

$$= \sum_{i=1}^{k} c_i \mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left(\chi^{\widetilde{\times}n} \cdot \mathrm{Res}_{S_{\lambda_j}}^{S_n} \chi_i\right) = \mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left(\chi^{\widetilde{\times}n} \cdot \mathrm{Res}_{S_{\lambda_j}}^{S_n} \psi\right)$$

□

Lemma 5.34. *Let $\chi \in \mathbb{C}[\mathrm{Irr}(A)]$, then,*

$$\chi^{\widetilde{\times}n} \cdot \kappa_{c,n} = (c\chi)^{\widetilde{\times}n}.$$

Proof. If $\chi \in \mathrm{Irr}(A)$ and $c, d \in \mathbb{C}$, then,

$$(d\chi)^{\widetilde{\times}n} \cdot \kappa_c = \chi^{\widetilde{\times}n} \cdot \kappa_c\kappa_d = \chi^{\widetilde{\times}n} \cdot \kappa_{cd} = (cd\chi)^{\widetilde{\times}n}.$$

Suppose that $\chi = d\chi_1 + e\chi_2$ where χ_1, $\chi_2 \in \mathrm{Irr}(A)$ and d, $e \in \mathbb{C}$. Then,

$$(d\chi_1 + e\chi_2)^{\widetilde{\times}n} \cdot \kappa_{c,n} = \sum_{j=0}^{n} \left(\left(\mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left((d\chi_1)^{\widetilde{\times}j} \times (e\chi_2)^{\widetilde{\times}(n-j)}\right)\right) \cdot \kappa_{c,n}\right)$$

$$= \sum_{j=0}^{n} \mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left(\left((d\chi_1)^{\widetilde{\times}j} \times (e\chi_2)^{\widetilde{\times}(n-j)}\right) \cdot \mathrm{Res}_{S_{\lambda_j}}^{S_n} \kappa_{c,n}\right).$$

using the Tensor Product Formula. Now using Lemma 5.32 and the previous case, we have,

$$(d\chi_1 + e\chi_2)^{\widetilde{\times}n} \cdot \kappa_{c,n} = \sum_{j=0}^{n} \mathrm{Ind}_{A\wr S_{\lambda_j}}^{A\wr S_n} \left((d\chi_1)^{\widetilde{\times}j} \cdot \kappa_{c,j} \times (e\chi_2)^{\widetilde{\times}(n-j)} \cdot \kappa_{c,n-j}\right)$$

$$= \sum_{j=0}^{n} (cd\chi_1)^{\widetilde{\times}j} \circ (ce\chi_2)^{\widetilde{\times}(n-j)} = (c(d\chi_1 + e\chi_2))^{\widetilde{\times}n}.$$

Using these facts and induction on the number of irreducible constituents of χ, the general case holds. $\square$

We now work towards generalising the results from [10] to arbitrary class functions for twisted group algebras.

Lemma 5.35. *Let* $\varphi_1, \varphi_2 \in \mathbb{C}[\mathrm{Irr}(A)]$*. Then,*

$$(\varphi_1 + \varphi_2)^{\widetilde{\times} n} = \sum_{j=0}^{n} \varphi_1^{\widetilde{\times} j} \circ \varphi_2^{\widetilde{\times}(n-j)}.$$

Proof. We begin by showing that equality holds when $\varphi_1 = c_1\chi$ and $\varphi_2 = c_2\chi$ for $\chi \in \mathrm{Irr}(A)$. The left hand side becomes,

$$(c_1\chi + c_2\chi)^{\widetilde{\times} n} = ((c_1 + c_2)\chi)^{\widetilde{\times} n} = \chi^{\widetilde{\times} n} \cdot \kappa_{c_1+c_2,n},$$

by definition. For a fixed j on the right hand side,

$$(c_1\chi)^{\widetilde{\times} j} \circ (c_2\chi)^{\widetilde{\times}(n-j)} = \left(\chi^{\widetilde{\times} j} \cdot \kappa_{c_1,j}\right) \circ \left(\chi^{\widetilde{\times}(n-j)} \cdot \kappa_{c_2,n-j}\right).$$

Now using Lemma 5.29 we have

$$\left(\chi^{\widetilde{\times} j} \cdot \kappa_{c_1,j}\right) \circ \left(\chi^{\widetilde{\times}(n-j)} \cdot \kappa_{c_2,n-j}\right) = \chi^{\widetilde{\times} n} \cdot \left(\kappa_{c_1,j} \circ \kappa_{c_2,n-j}\right).$$

Taking the sum over all j,

$$\sum_{j=0}^{n} (c_1\chi)^{\widetilde{\times} j} \circ (c_2\chi)^{\widetilde{\times}(n-j)} = \chi^{\widetilde{\times} n} \cdot \left(\sum_{j=0}^{n} \kappa_{c_1,j} \circ \kappa_{c_2,n-j}\right),$$

which by Lemma 5.30 yields the result. Now considering the general case, where we denote $\mathrm{Irr}(A) = \{\chi_1, \ldots, \chi_m\}$. Suppose $\varphi_1 = \sum_{i=1}^{m} c_i\chi_i$ and $\varphi_2 = \sum_{i=1}^{m} d_i\chi_i$. Then by definition,

$$(\varphi_1 + \varphi_2)^{\widetilde{\times} n} = \sum_{\substack{n_i \geq 0 \\ n = n_1 + \ldots + n_k}} \left(((c_1 + d_1)\chi_1)^{\widetilde{\times} n_1} \circ \ldots \circ ((c_k + d_k)\chi_k)^{\widetilde{\times} n_k}\right)$$

$$= \sum_{\substack{n_i \geq 0, n'_i \geq 0 \\ n = n_1 + n'_1 + \ldots + n_k + n'_k}} \left((c_1\chi_1)^{\widetilde{\times} n_1} \circ (d_1\chi_1)^{\widetilde{\times} n'_1} \circ \ldots \circ (c_k\chi_k)^{\widetilde{\times} n_k} \circ (d_k\chi_k)^{\widetilde{\times} n'_k} \right),$$

using the first case considered in the proof. Since $\circ$ is commutative, we can re-arrange the above formula and obtain

$$(\varphi_1 + \varphi_2)^{\widetilde{\times} n} = \sum_{\substack{n_i \geq 0, n'_i \geq 0 \\ n = n_1 + n'_1 + \ldots + n_k + n'_k}} \left((c_1\chi_1)^{\widetilde{\times} n_1} \circ \ldots \circ (c_k\chi_k)^{\widetilde{\times} n_k} \circ (d_1\chi_1)^{\widetilde{\times} n'_1} \circ \ldots \circ (d_k\chi_k)^{\widetilde{\times} n'_k} \right)$$

$$= \sum_{j=0}^{n} \left(\varphi_1^{\widetilde{\times} j} \circ \varphi_2^{\widetilde{\times} (n-j)} \right).$$

□

Theorem 5.36. *If* $\chi \in \mathbb{C}[\mathrm{Irr}(B)]$ *where* $B \subseteq A$ *then,*

$$\mathrm{Ind}_{B \wr S_n}^{A \wr S_n} \chi^{\widetilde{\times} n} = \left(\mathrm{Ind}_B^A \chi \right)^{\widetilde{\times} n}.$$

Proof. Similar to the previous proof, we prove the result in the case when $\chi = c\varphi$ where $c \in \mathbb{C}$ and $\varphi \in \mathrm{Irr}(B)$. We have,

$$\mathrm{Ind}_{B \wr S_n}^{A \wr S_n} (c\varphi)^{\widetilde{\times} n} = \mathrm{Ind}_{B \wr S_n}^{A \wr S_n} \left(\varphi^{\widetilde{\times} n} \cdot \kappa_{c,n} \right).$$

Using Lemma 5.31, we have,

$$\mathrm{Ind}_{B \wr S_n}^{A \wr S_n} \left(\varphi^{\widetilde{\times} n} \cdot \kappa_{c,n} \right) = \left(\mathrm{Ind}_{B \wr S_n}^{A \wr S_n} \varphi^{\widetilde{\times} n} \right) \cdot \kappa_{c,n}.$$

Using [10, Lemma 2.8],

$$\left(\mathrm{Ind}_{B \wr S_n}^{A \wr S_n} \varphi^{\widetilde{\times} n} \right) \cdot \kappa_{c,n} = \left(\left(\mathrm{Ind}_B^A \varphi \right)^{\widetilde{\times} n} \right) \cdot \kappa_{c,n}.$$

Finally using Lemma 5.34,

$$\left(\left(\mathrm{Ind}_B^A\varphi\right)^{\widetilde{\times} n}\right)\cdot\kappa_{c,n}=\left(c\mathrm{Ind}_B^A\varphi\right)^{\widetilde{\times} n}=\left(\mathrm{Ind}_B^A\chi\right)^{\widetilde{\times} n}.$$

In general, suppose that $\mathrm{Irr}(B)=\{\chi_1,\ldots,\chi_k\}$ and $\chi=\sum_{i=1}^k c_i\chi_i$ with $c_i\in\mathbb{C}$. Then,

$$\mathrm{Ind}_{B\wr S_n}^{A\wr S_n}\left(\left(\sum_i c_i\chi_i\right)^{\widetilde{\times} n}\right)=\mathrm{Ind}_{B\wr S_n}^{A\wr S_n}\sum_{\substack{n_i\geq 0\\ n=n_1+\ldots+n_k}}(c_1\chi_1)^{\widetilde{\times} n_1}\circ\ldots\circ(c_k\chi_k)^{\widetilde{\times} n_k}$$

$$=\sum_{\substack{n_i\geq 0\\ n=n_1+\ldots+n_k}}\mathrm{Ind}_{A\wr\left(S_{n_1}\times\ldots\times S_{n_k}\right)}^{A\wr S_n}\left(\mathrm{Ind}_{B\wr S_{n_1}}^{A\wr S_{n_1}}(c_1\chi_1)^{\widetilde{\times} n_1}\times\ldots\times\mathrm{Ind}_{B\wr S_{n_k}}^{A\wr S_{n_k}}(c_k\chi_k)^{\widetilde{\times} n_k}\right)$$

$$=\sum_{\substack{n_i\geq 0\\ n=n_1+\ldots+n_k}}\left(\left(\mathrm{Ind}_B^A c_1\chi_1\right)^{\widetilde{\times} n_1}\circ\ldots\circ\left(\mathrm{Ind}_B^A c_k\chi_k\right)^{\widetilde{\times} n_k}\right)=\left(\mathrm{Ind}_B^A\chi\right)^{\widetilde{\times} n},$$

using the previously considered case and Lemma 5.35. □

Finally we obtain the desired generalisation of Theorem 5.6 to projective characters.

Theorem 5.37. *Let A be a twisted G-algebra and let $\chi\in\mathcal{A}_{p^a}(A,K,Z(G))$ where $K\leq G$. Then $\chi^{\widetilde{\times} n}\in\mathcal{A}_{p^{an}}(A\wr S_n,K^n,Z(G)^n)$.*

Proof. We suppose that $\chi=\mathrm{Ind}_{A[H_1]}^A\theta_1+\mathrm{Ind}_{A[H_2]}^A\theta_2$ where H_1, $H_2\leq G$ and p^a divides the index of these subgroups in G, $|H_i:K\cap H_i|$ is divisible by p^a and $\theta_i\in\mathcal{C}(A[H_i])$ for $i=1$, 2. The result will follow by induction on the number of summands of χ. By Lemma 5.35 we have that

$$\chi^{\widetilde{\times} n}=\sum_{j=0}^n\left(\mathrm{Ind}_{A[H_1]}^A\theta_1\right)^{\widetilde{\times} j}\circ\left(\mathrm{Ind}_{A[H_2]}^A\theta_2\right)^{\widetilde{\times}(n-j)}.$$

Moreover, by Theorem 5.36, defining $A_i:=A[H_i]$,

$$\chi^{\widetilde{\times} n}=\sum_{j=0}^n\left(\mathrm{Ind}_{A_1\wr S_j}^{A\wr S_j}\theta_1^{\widetilde{\times} j}\right)\circ\left(\mathrm{Ind}_{A_2\wr S_{n-j}}^{A\wr S_{n-j}}\theta_2^{\widetilde{\times}(n-j)}\right).$$

Hence, by definition of $\circ$,

$$\chi^{\widetilde{\times} n} = \sum_{j=0}^{n} \mathrm{Ind}_{(A_1 \times S_j) \otimes (A_2 \wr S_{n-j})}^{A \wr S_n} \left(\theta_1^{\widetilde{\times} j} \times \theta_2^{\widetilde{\times}(n-j)} \right).$$

The proof of Theorem 5.6, gives us that the subgroups satisfy the divisibility condition, hence the result follows. □

5.4 Modular representation theory

In this section we develop some results from Modular Representation Theory, focusing in particular on block theory and vertices of modules. In all previous chapters, we have seen that our goal is to reduce Conjecture 2.6 to Brauer-good subgroups. An understanding of vertices of modules and other aspects of block theory leads to a simplification which means that we do not have to verify Conjecture 2.6 for an ordinary irreducible character of G if the index in G of the defect group of the block that the character lies in is divisible by the highest power of p dividing character degree. We aim to build up the theory to understand the statement of [30, Theorem 3.4], phrased using the terminology of [11, Theorem 2.3], and then use this theorem to deduce our simplification. We will then discuss the verification of Conjecture 2.6 for the case, $G = J_1$, and $p = 2$.

The study of modular representation theory arises when you deviate from the group algebra over $\mathbb{F}$ where $\mathbb{F}$ has characteristic 0 as is most commonly studied. In this situation, we have due to Maschke that modules over the group algebra are semisimple [1, Corollary 12.8] i.e., they have direct sum decompositions into simple modules. This is due to the fact that the characteristic of the field does not divide the group order. However, if we change the characteristic of the field then the semisimplicity of these modules wil no longer hold. Due to this we introduce the notion of indecomposability. To this end, we let k be a field of characteristic p where p divides the group order and we denote by A a finite dimensional k-algebra.

Definition 5.38. [2, §4] *An A-module is called indecomposable if it does not have a non-trivial direct sum decomposition into submodules*[†]*.*

The following, the Krull–Schmidt Theorem, gives us the reason for introducing the notion of indecomposability.

Theorem 5.39. (See [2, Theorem 4.3]) *Every A-module M has a unique direct sum decomposition into indecomposable modules, up to isomorphism.*

Therefore, just as the simple modules were the object of study in the characteristic 0 setting, the indecomposable modules are the equivalent in this setting. It is important to note that for the purposes of this project, there exists a notion of induction for kG-modules in this setting.

Definition 5.40. [2, §8] *Let $H \leq G$. Let M and V be kG-modules and N be a kH-submodule of M. We say that M is relatively H-free if any kH-module homomorphism $\phi : N \to V$, extends uniquely to a kG-module homomorphism, $\widetilde{\phi} : M \to V$.*

Remark 5.41. We note here that if $H = 1$ then we recover a characterisation of free modules, where by a free module we mean one which is isomorphic to the direct sum of a number of copies of its underlying algebra, viewed as a module over itself.

The point of introducing such a notion is that these relatively free modules have a description in terms of tensor products. The usual construction yields a quotient space $kG \otimes_{kH} M$ of $kG \otimes M$, where M is a kH-module, spanned by the elements $(kh \otimes m - k \otimes hm)$ for $k \in kG$, $h \in H$ and $m \in M$. We give the structure of a kG-module to this quotient space, where for $g \in kG$ and $k \otimes m \in kG \otimes M$, $g(k \otimes m) := gk \otimes m$. One can check that this determines a bilinear action.

As is conventional, we denote this module by $\mathrm{Ind}_H^G M$. In particular, we have that $\mathrm{Ind}_H^G M$ is relatively H-free and all the "usual" properties (i.e. Nakayama relations, Mackey formula) hold for this definition of induction (See [2, Lemma 8.4] and [2, Lemmas

[†]Note that every A-module trivially decomposes into the direct sum of itself and 0.

8.5]). The main reason for our study into modular representation theory is that of the vertices of a module and their relationship with defect groups of blocks of kG. We now work towards this goal.

Definition 5.42. [2, Definition 9.1] *Let $H \leq G$ and let M be a kG-module. M is called relatively H-projective if M is a direct summand of a relatively H-free module.*

Remark 5.43. An alternative definition one could use is that M is a direct summand of $\mathrm{Ind}_H^G\left(\mathrm{Res}_H^G M\right)$. Note that this definition implies our definition above, as $\mathrm{Ind}_H^G\left(\mathrm{Res}_H^G M\right)$ is indeed a relatively H-free module.

We now introduce the notion of vertices and sources of a module. Vertices are essentially subgroups of G which in some sense control when a module is relatively H-projective for $H \leq G$. In particular the smaller the vertex (in terms of order), the closer a module is to being projective, i.e. a direct summand of a free module.

Theorem 5.44. (See [2, Theorem 9.4]) *Let M be an indecomposable kG-module and $H \leq G$. Then,*

(a) There exists a p-subgroup $Q \leq G$, called a vertex of M, which is unique up to G-conjugacy such that M is relatively H-projective if and only if there exists a $g \in G$ such that ${}^gQ \leq H$.

(b) Let Q denote a vertex of M. There exists an indecomposable kQ-module S, which is unique up to $N_G(Q)$-conjugacy, such that M is a direct summand of $\mathrm{Ind}_Q^G S$.

We are now in a position to look in detail at the blocks of A and in particular, we specialise to the case when $A = kG$, and in this setting define the defect group of a block of kG. First we have that every algebra A has a unique decomposition into a direct sum of indecomposable subalgebras. Note that by indecomposable algebra, we mean one which cannot be decomposed as a sum of non-trivial subalgebras.

Definition 5.45. *If $A = A_1 + \cdots + A_r$ where A_i are indecomposable subalgebras called blocks, then we say an A-module M lies in A_i if $A_i M = M$ and $A_j M = 0$ for all $j \neq i$.*

From the decomposition in the above definition, we are also able to obtain idempotents $e_i \in A_i$, called block idempotents, such that $1 = e_1 + \cdots + e_r$. If M lies in A_i then e_i acts trivially on M and each e_j, $j \neq i$ annihilates M, namely $e_i M = M$ and $e_j M = 0$ for all $j \neq i$. The importance of the notion of blocks is given by the following Proposition.

Proposition 5.46. (See [2, Proposition 13.2]) *If M is an A-module and $A = A_1 + \cdots + A_r$ with A_i indecomposable, then $M = M_1 \oplus \cdots \oplus M_r$ (uniquely) where each M_i lies in the block A_i.*

Therefore understanding A-modules reduces to understanding A_i-modules. We now specialise to the case when $A = kG$. The first question which naturally arises is what are the blocks of kG?

To see this, we view kG as a $k[G \times G]$-module. For $a \in kG$ and g, $h \in G$, we define the action of $k[G \times G]$ on kG to be, $(g, h)a := gah^{-1}$ and extend by linearity. In particular using the proposition above, we have that kG decomposes into indecomposable $k[G \times G]$-submodules of kG and in fact, the $k[G \times G]$-submodules of kG are ideals of kG and these are the blocks of kG. We have now developed sufficient theory to discuss the defect groups of a block of kG.

Theorem 5.47. (See [2, Theorem 13.4]) *Let $\delta : G \to G \times G$ be a group homomorphism defined by $\delta(g) := (g, g)$. If B is a block of kG, then B has a vertex (viewed as a $k[G \times G]$-module) of the form δD for a p-subgroup $D \leq G$.*

The subgroup D, given by the above theorem is called a **defect group** of B. The use of "a" here is important as the defect groups are in fact a G-conjugacy class of p-subgroups. This is because the vertex of a module is defined up to conjugacy and if for H, $K \leq G$, such that δH is $(G \times G)$-conjugate to δK, then by projection onto the first (or second) component, we can see that H is G-conjugate to K.

The next question which arises is, why do we need the defect groups of a block? To answer this, we require an understanding of the relationship between the modules lying in a block and the defect groups of said block.

Theorem 5.48. (See [5, Proposition 6.1.2]) *Suppose that B is a block of kG with defect group $D(B)$, then every kG-module lying in B is relatively $D(B)$-projective.*

Definition 5.56 (see [27]) shows that provided we know that the blocks that each of the ordinary characters of G lie in, then we can compute the defect of the block and hence know some details about $|G : D(B)|$. Proposition 5.48, as well as Theorem 5.60, will give us our simplification.

The question which first arises however is what do we mean by blocks of the ordinary characters, given that all the results stated were over characteristic $p > 0$ fields? We introduce the notions of a discrete valuation ring and some of its basic features, as well as that of a p-modular system and an $\mathcal{O}G$-lattice, which will explain why we can talk about defect groups and vertices in the characteristic 0 setting.

Definition 5.49. [9, §4C] *The following are definitions regarding valuations.*

(i) A valuation on a field K is a map $\varphi : K \to \mathbb{R}_{\geq 0}$ such that for all $a, b \in K$,

  (a) $\varphi(a) = 0$ if and only if $a = 0$.

  (b) φ is multiplicative.

  (c) $\varphi(a + b) \leq \varphi(a) + \varphi(b)$.

(ii) φ is called non-archimedean if φ satisfies $\varphi(a + b) \leq \max(\varphi(a) + \varphi(b))^{\dagger}$.

(iii) φ is called a discrete valuation if its value group $\{\phi(a) : a \in K\backslash\{0\}\}$ is an infinite cyclic group.

To each valuation we may now associate a metric on K and hence the topology induced by said metric. Two valuations are said to be equivalent if they give rise to the same topology on K and we define a **prime** of K to be an equivalence class of valuations of K under this relation.

†Note that this is satisfied, this implies condition (c) above.

Definition 5.50. [9, §4C] *Let φ be a non-archimedean prime on K. Then we define the valuation ring of φ to be,*

$$\mathcal{O} := \{a \in K : \varphi(a) \leq 1\}.$$

Note that $\mathcal{O}$ has a unique maximal ideal denoted $P = \{a \in K : \phi(a) < 1\}$. The **residue class field** of $\mathcal{O}$ is the quotient by this maximal ideal. If φ is a discrete valuation then $\mathcal{O}$ is called a **discrete valuation ring**.

Definition 5.51. [9, §16A] *A p-modular system $(K, \mathcal{O}, k)$ consists of a d.v.r., $\mathcal{O}$, whose field of fractions is given by K, and whose residue class field is k of characteristic p.*

The reason for the introduction of p-modular systems, is that within every KG-module, we have the existence of a $\mathcal{O}G$-lattice, which affords the same representation (character) as the KG-module.

Definition 5.52. [11, Section 1.2] *A $\mathcal{O}G$-lattice is a left $\mathcal{O}G$-module which is free as a $\mathcal{O}$-module.*

Definition 5.53. [9, Theorem (16.11)] *Let V be a finitely generated left KG-module. A full $\mathcal{O}G$-lattice M in V is an $\mathcal{O}G$-lattice M contained in V such that $V = KM \cong K \otimes_{\mathcal{O}} M$.*

The following two results, explain why every irreducible character over $\mathbb{C}$ can be afforded by an $\mathcal{O}G$-lattice. The first states that for a p-modular system, if we can find a full $\mathcal{O}G$-lattice then each representation over K can be realised over $\mathcal{O}$ and the second result is an existence result, saying that for a p-modular system we can always find such an $\mathcal{O}G$-lattice.

Lemma 5.54. (See [9, Corollary (16.14)]) *Every representation over G of K is K-equivalent to a representation of G over $\mathcal{O}$.*

Proposition 5.55. (See [9, Proposition (16.15)]) *Every finitely generated KG-module contains full $\mathcal{O}G$-lattices.*

Therefore we can now see that discussing blocks in the characteristic 0 setting is indeed sensible and hence the following definition from [27] can be applied.

Definition 5.56. [27, Definition (3.15)] *Define $A := v_p(|G|)$. If B is a p-block of G then we define the defect $d(B)$ of the block, as the integer satisfying*

$$p^{A-d(B)} = \min\left\{p^{v_p(\chi(1))} : \chi \in \mathrm{Irr}(B)\right\},$$

where $\mathrm{Irr}(B) = \mathrm{Irr}(G) \cap B$.

In addition to this definition we may introduce the notion of the height of a character.

Definition 5.57. [27, Definition (3.15)] *Let $A := v_p(|G|)$. Let B is a p-block of G and $\chi \in \mathrm{Irr}(B)$. The height of χ in B is the non-negative integer h, such that*

$$v_p(\chi(1)) = A - d(B) + h.$$

We now briefly deviate from the current discussion about the computational simplification to discuss Brauer's height zero conjecture, as this will also yield a simplification for our conjecture. Brauer's height zero conjecture states the following.

Conjecture 5.58. *Let p be a prime, B be a p-block of G and $\chi \in \mathrm{Irr}(B)$. Then χ has height zero if and only if the defect groups of B are abelian.*

Due to Kessar and Malle, [23], we have that if the defect group of B is abelian, then χ has height zero. We see later how this yields a simplification. We now show that if we can calculate the defect of a block, then we are able to determine the order of the defect group.

Theorem 5.59. [27, Theorem (4.6)] *If $D(B)$ denotes the defect group of a block B then, $|D(B)| = p^{d(B)}$, where $d(B)$ denotes the defect of the block.*

Recall Proposition 5.48 also. These facts are beneficial to us, and lead to a simplification due to the following result of Willems, [30]. We will state the Theorem in the terminology used by Evseev in [11, Theorem 2.3], but as we require a more specific result we will reprove it here.

Theorem 5.60. [30, Theorem 3.4] *Let Q be a p-subgroup of G and define $a := v_p(|G : Q|)$. Suppose that M is a Q-projective $\mathcal{O}G$-lattice. Suppose also that p does not divide $|Z(G)|$. If χ is the character afforded by M then $\chi \in \mathcal{A}_{p^a}(G, Z(G))^{\dagger}$.*

Proof. We first prove the result when G is a p-decomposable, namely $G = G_p \times G_{p'}$. where $Q \leq G_p$. Note that since $Z(G) = Z(G_p) \times Z(G_{p'})$ and p does not divide $|Z(G)|$, we have $Z(G) = 1 \times Z(G_{p'})$. Following the proof of [11, Theorem 2.3], we have that $M = \mathrm{Ind}_{QG_{p'}}^{G} N$ where N is an indecomposable $\mathcal{O}QG_{p'}$-lattice and $\chi \in \mathcal{I}(G, Q, \mathcal{A}(Q))$. Note $Z(G) = Z(G_{p'}) \leq QG_{p'}$ and since $\mathcal{A}(Q)$ consists of all subgroups of Q and p^a divides $|G : Q|$, we have $\chi \in \mathcal{A}_{p^a}(G, Z(G))$. We now let G be arbitrary. By Brauer's Induction Theorem we have,

$$1_G = \sum_{E \in \mathrm{El}(G)} \sum_{\phi \in \mathrm{Irr}(E)} a_\phi \mathrm{Ind}_E^G \phi.$$

Moreover, without loss of generality, we can assume that this sum is taken over maximal p-decomposable subgroups of G (since elementary subgroups are p-decomposable). Recall that every p-decomposable subgroup is a direct product of a p-group and a p'-group for a fixed prime p. Then using the Tensor Product Formula we have,

$$\chi = \sum_{E \in \mathrm{PD}_{\max}(G)} \sum_{\phi \in \mathrm{Irr}(E)} a_\phi \mathrm{Ind}_E^G \left(\left(\mathrm{Res}_E^G \chi \right) \phi \right).$$

Defining $r = r(E, g) := v_p\left(|E : {}^gQ \cap E|\right)$ for $g \in G$, we follow the proof in [11] and the above to show that for each E, $\left(\mathrm{Res}_E^G \chi\right) \phi \in \mathcal{A}_{p^r}\left(E, Z(E)\right)$, i.e.

$$\left(\mathrm{Res}_E^G \chi\right) \phi = \sum_{\substack{L \in \mathcal{I}_{p^r}(E) \\ Z(E) \leq L}} \sum_{\theta \in \mathrm{Irr}(L)} b_\theta \mathrm{Ind}_L^E \theta.$$

Now let $L \leq E$ where E is a maximal p-decomposable subgroup such that $Z(E) \leq L$. We claim that $Z(G) \leq L$. We clearly have that $Z(G) \cap E \leq Z(E) \leq L$, so we show that

[†]Note that if we ignore the centre and consider whether χ belongs to $\mathcal{A}_p(G)$, then the proof of Theorem 5.60 gives an analogous result for arbitrary groups.

$Z(G) \leq E$. Since E is a p-decomposable subgroup of G, $E = P \times F$ for p-group P and p'-group F. In particular, $F \leq C_G(P)$. Since $Z(G) \trianglelefteq C_G(P)$, $FZ \leq C_G(P)$.

Hence $E = P \times F \leq P \times FZ$. Since p does not divide $|Z(G)|$, FZ is a p'-group and $P \times FZ$ is p-decomposable. Since E is a maximal p-decomposable subgroup $FZ = F$, i.e. $Z(G) \leq F \leq E$. Hence $Z(G) = E \cap Z(G) \leq Z(E) \leq L$. Therefore substituting the expressions for $\left(\mathrm{Res}^G_E \chi\right) \phi$ and using transitivity of induction, we have that $\chi \in \mathcal{A}_{p^{r+v_p(|G:E|)}}(G, Z(G)) = \mathcal{A}_{p^a}(G, Z(G))$. $\square$

Moreover, the following lemma shows us that we need only to consider characters of G that lie in blocks of maximal defect.

Lemma 5.61. *Let p be a prime and G be a group and $\chi \in \mathrm{Irr}(G)$. Suppose χ lies in a block B of* $\mathrm{Irr}(G)$. *If the defect group of B, $D = D(B)$, is not maximal, i.e. $D \notin \mathrm{Syl}_p(G)$, then $\chi \in \mathcal{A}_p(G)$.*

Proof. Suppose $P \in \mathrm{Syl}_p(G)$ such that $D < P$. Denote by $\mathcal{X}$, the set of elementary subgroups of G whose p-part is a subgroup of D. Using [30, Corollary 3.5], we have

$$\chi = \sum_{E \in \mathcal{X}} \mathrm{Ind}^G_E \lambda_{(E)},$$

where $\lambda_E \in \mathcal{C}(E)$ whose irreducible constituents are linear. Let $E \in \mathcal{X}$, then $v_p(|G:E|) \geq v_p(|G:D|) > v_p(|G:P|) = 0$. Hence p divides $|G:E|$ and $\chi \in \mathcal{A}_p(G)$, by definition. $\square$

We are now in a position to discuss the simplification. Suppose that G is an almost simple group, i.e. there exists a non-abelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$. Note that $Z(G) = 1$. Let $\chi \in \mathrm{Irr}(B)$ for some p-block B, with defect group $D(B)$. Let M be a $\mathcal{O}G$-lattice affording χ. Note that by Proposition 5.48, we have that M is relatively $D(B)$-projective. Moreover, suppose that $a := v_p(\chi(1)) = v_p(|G : D(B)|)$, i.e. χ is of height zero in $B^\dagger$. Hence by Theorem 5.60, we have that $\chi \in \mathcal{A}_{p^a}(G, 1)$.

†Recall we always have that $v_p(\chi(1)) \geq v_p(|G : D(B)|)$ and the difference between these values is the height of χ in B.

If we look at Definition 2.10 and Conjecture 2.11, then this is indeed what we would like to prove for almost simple groups. Therefore we can eliminate the verification of certain cases, if the degree of the character divides the index of the defect group of the block it lies in. We proceed by giving an example.

Example 5.62. We consider the case when $G = J_1$ and $p = 2$. Note that $v_2(|G|) = 3$. We denote the irreducible characters of G, by

$$\mathrm{Irr}(G) := \{\chi_1, \chi_2, \ldots, \chi_{15}\},$$

with character degree set, $\{1, 56, 56, 76, 76, 77, 77, 77, 120, 120, 120, 133, 133, 133, 209\}$. We use GAP to obtain the character table of G, using the command `CharacterTable(G)`, and we obtain the 2-blocks of G via `PrimeBlocks(CharacterTable(G),2)`. This command yields seven blocks. The following list shows the distribution of these characters into each block, and the order of the defect group of each block.

(1) $B_0 := \{\chi_1, \chi_6, \chi_7, \chi_8, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}\}$. This is the principal block and the defect group of the principal block is always a Sylow p-subgroup, for the given prime p, i.e. $|D(B_0)| = 8$.

(2) $B_1 := \{\chi_2\}$. Since $v_2(\chi_2(1)) = 3$, we have that the defect of B_1 is 0, and hence $|D(B_1)| = 1$.

(3) $B_2 := \{\chi_3\}$. Since $v_2(\chi_3(1)) = 3$, we have that the defect of B_2 is 0, and hence $|D(B_2)| = 1$.

(4) $B_3 := \{\chi_4, \chi_5\}$. Since $v_2(\chi_4(1)) = v_2(\chi_5(1)) = 2$, we have that the defect of B_3 is 1, and hence $|D(B_3)| = 2$.

(5) $B_4 := \{\chi_9\}$. Since $v_2(\chi_9(1)) = 3$, we have that the defect of B_4 is 0, and hence $|D(B_4)| = 1$.

(6) $B_5 := \{\chi_{10}\}$. Since $v_2(\chi_{10}(1)) = 3$, we have that the defect of B_5 is 0, and hence $|D(B_5)| = 1$.

(7) $B_6 := \{\chi_{11}\}$. Since $v_2(\chi_{11}(1)) = 3$, we have that the defect of B_6 is 0, and hence $|D(B_6)| = 1$.

From the list of character degrees, we see that all characters of degree divisible by some power of 2, lie outside of the principal block. For the characters χ_2, χ_3, χ_9, χ_{10} and χ_{11}, as the defect groups are trivial, we trivially have that the character degree divides the index of the defect group of the block in G. For the remaining characters, χ_4 and χ_5, the index $|G : D(B_3)| = 87780$ is divisible by 4 also. Therefore the simplification we have illustrated means that for $p = 2$, we do not have to verify any characters of J_1.

As stated after Conjecture 5.58, we have that if the defect group of the block is abelian, then the irreducibles in that block necessarily have height zero due to [23]. Note that in the above example, the defect groups $D(B_i)$ for $i = 1, \ldots, 6$ are all abelian and therefore each character in the block has height zero, and our simplification is immediate.

CHAPTER 6

THE REDUCTION THEOREMS

In this chapter we prove the two key results of this thesis. We recall from the Introduction and Chapter 2, that we are aiming to prove Conjecture 2.12. From the formulation of Conjecture 2.12, we see that this is a reduction theorem to Brauer-good groups. However we are unable to obtain a full reduction theorem without additional assumptions on the initial group G. In Section 6.1, we prove the following theorem.

Theorem 6.1. *Let G be a group. Suppose Conjecture 2.11 holds and $Z(G) = 1$, then G is not a minimal counterexample to Conjecture 2.12.*

The above theorem essentially tells us that provided Conjecture 2.12 is true for Brauer-good groups, then the minimal counterexample cannot have trivial centre. The proof of this theorem gives us an outline for how the full reduction theorem should proceed. As support of this claim, we are able to generalise the proof of Theorem 6.1, to give us the proof of the following theorem. Before its statement however we must introduce an "extensibility" condition that G must satisfy.

Condition 6.2. *Let G be an almost simple group and S be its socle. If $\alpha \in H^2(S, \mathbb{C}^*)$ then α extends to G, i.e. there exists $\beta \in H^2(G, \mathbb{C}^*)$ such that $\beta(s,t) = \alpha(s,t)$ for all $s, t \in S$.*

Suppose that G is a subgroup of a group of the form $M \wr S_n$. For each $1 \leq i \leq n$, we define

$$L_i := \{l_i \in M : \exists \sigma \in S_n, \sigma(i) = i, l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n \in M \text{ with } (l_1, \ldots, l_n; \sigma) \in G\}. \tag{6.1}$$

We are now in a position to state the theorem we prove in Section 6.2.

Theorem 6.3. *Let G be a group. Suppose that Conjecture 2.11 holds and Condition 6.2 holds for any almost simple group, then G is not a minimal counterexample to Conjecture 2.12.*

We will see that in the proof of Theorem 6.1, that if G has trivial centre, Condition 6.2 is satisfied for L_1, hence Theorem 6.3 is a true generalisation of Theorem 6.1.

6.1 A reduction theorem for groups with trivial centre

We now prove Theorem 6.1. We make an initial note that some results we state here hold for G, when G does not necessarily have trivial centre. These will be used in the proof of Theorem 6.3, hence the results will be stated in their full generality. Let G be a group with irreducible character χ such that p^a divides $\chi(1)$, where p is a prime, such that (G, χ) is a minimal counterexample to Conjecture 2.12 with respect to the partial order $\prec$ given in Definition 3.1. Note that we are implicitly assuming that G is non-abelian, as if G were abelian then no non-linear characters would exist and there would be nothing to prove. In particular, $Z := Z(G) \neq G$. If G/Z were simple then we would invoke Conjecture 2.11 provided G were perfect, so we show the following.

Lemma 6.4. *Let G be a group such that G/Z is simple. Suppose that (G, χ) is a minimal counterexample to Conjecture 2.12 with respect to $\prec$, then G is perfect, i.e. $G' = G$.*

Proof. Suppose that $G' < G$. Then as $G' \trianglelefteq G$ and $Z \trianglelefteq G$, we have that $G'Z \trianglelefteq G$. Suppose that $G'Z \neq G$. Then $G'Z/Z \trianglelefteq G/Z$, however since G/Z is simple and $G'Z \neq G$, $G'Z = Z$, i.e. $G' \leq Z$. In this case, G/Z would be simple and abelian as the derived subgroup is the smallest subgroup such that the quotient is abelian, i.e. cyclic and hence G would be abelian, which is a contradiction. Therefore we may assume that $G'Z = G$.

We claim now that $Z(G') = G' \cap Z$. Clearly $G' \cap Z \leq Z(G')$. So to this end let $x \in Z(G')$ and we need to show that $x \in Z$. Let $g \in G$, then $g = hz$ for $h \in G'$ and $z \in Z$. Then $xg = x(hz) = h(xz) = (hz)x = gx$, since $x \in Z(G')$ and $z \in Z$, hence $Z(G') = G' \cap Z$. Using this we have the following,

$$\frac{G}{Z} = \frac{G'Z}{Z} \cong \frac{G'}{G' \cap Z} = \frac{G'}{Z(G')}.$$

Since $|G'| < |G|$, we have that the induction hypothesis holds for G', in other words $G' \prec G$. Now we define $H := G' \times Z$ and define a map $\varphi : H \to G$ by $\varphi(g,z) := gz$. Defining $A := \ker(\varphi) = \{(g,z) \in H : g = z^{-1}\}$, this induces an isomorphism $\overline{\varphi} : H/A \to G$, using the First Isomorphism Theorem, given by $\overline{\varphi}\left((g,z)A\right) := \varphi(g,z) = gz$.

Since $\chi \in \mathrm{Irr}(G)$, we may form $\mathrm{Inf}_{H/A}^{H}\chi := \theta \times \lambda \in \mathrm{Irr}(H)$ where $\theta \in \mathrm{Irr}(G')$, $\lambda \in \mathrm{Irr}(Z)$ and $A \subseteq \ker(\theta \times \lambda)$. Moreover, $(\theta \times \lambda)(1) = \theta(1)\lambda(1) = \theta(1) = \chi(1)$. Hence since G' satisfies Conjecture 2.12, we have that $\theta \in \mathcal{A}_{p^a}(G', Z(G'))$. Moreover, $\theta \times \lambda \in \mathcal{A}_{p^a}(H \times Z, Z(G') \times Z)$. Note that $Z(H) = Z(G') \times Z$.

We claim now that $A \subseteq Z(H)$. Let $(g,z) \in A$, namely $g = z^{-1}$. Then $g \in G'$ and $g = z^{-1} \in Z$, so $g \in G' \cap Z = Z(G')$. Therefore $(g,z) \in Z(G') \times Z = Z(H)$. We know that $A \subseteq \ker(\theta \times \lambda)$ if and only if $\theta \times \lambda \in \mathrm{Irr}(H||1_A)$. Hence using Lemma 4.11, we are able to assume that

$$\theta \times \lambda = \sum_{\substack{K \in \mathcal{I}_{p^a}(H) \\ Z(H) \leq K}} \sum_{\phi \in \mathrm{Irr}(K||1_A)} a_\phi \mathrm{Ind}_K^G \phi,$$

namely that for each ϕ in the decomposition, $A \subseteq \ker(\phi)$. Hence we are able to deflate each ϕ in the above expression. Deflating the expression, where we set $\overline{K} := K/A$, $\overline{Z} := Z(H)/A$, $\mathrm{Def}_{\overline{K}}^{K}\phi =: \overline{\phi} \in \mathrm{Irr}(K)$, $a_{\overline{\phi}} := a_\phi$, and using that deflation is a $\mathbb{Z}$-linear operation and applying Lemma 3.21, we can show

$$\chi = \mathrm{Def}^H_G(\theta \times \lambda) = \sum_{\substack{\overline{K} \in \mathcal{I}_{p^a}(G) \\ \overline{Z} \leq \overline{K}}} \sum_{\phi \in \mathrm{Irr}\left(\overline{K}\right)} a_{\overline{\phi}} \mathrm{Ind}^G_{\overline{K}} \overline{\phi}.$$

Finally, $\overline{Z} = Z(H)/A = \overline{\varphi}(Z(H)A) = \varphi(Z(H)) = \varphi(Z(G') \times Z) = Z(G')Z \geq Z$, hence $\chi \in \mathcal{A}_{p^a}(G, Z)$. □

Using Lemma 6.4 we have that G is perfect and hence quasi-simple. As this case is assumed to be true by Conjecture 2.11, we may continue under the assumption that G/Z is not simple. Therefore, we introduce a minimal normal subgroup, say N/Z, of G/Z, as in other reduction arguments of this type. We are able to deduce by [14, Proposition 3.2], that $N/Z \cong S \times \cdots \times S$, i.e. isomorphic to the direct product of isomorphic copies of a simple group S. We now look to the proof of Theorem 4.12. Note that Claim 4.13 holds for G (as we do not need the assumption that G is p-solvable). We let $\theta \in \mathrm{Irr}(N)$ be such that $\chi \in \mathrm{Irr}(G||\theta)$, which exists as χ is primitive [16, Corollary 6.12]. We therefore assume that $\mathrm{Res}^G_N \chi = e\theta$ for some $e \in \mathbb{Z}$. Moreover Claim 4.15 holds (again as G being p-solvable doesn't change the proof), namely $I_G(\theta) = G$. In addition to this Claim 4.16 holds (as this property is independent of G being p-solvable), namely p^a divides $\theta(1)$. The following corollary gives us that it is sufficient to restrict to EN.

Claim 6.5. *Let $E \in \mathrm{El}(G)$. Define $b_1 := b(E, N, G, a)$ and $b_2 := b(E, Z, G, a)$. Suppose $\mathrm{Res}^G_{EN} \chi \in \mathcal{A}_{p^{b_1}}(EN, Z)$. Then $\mathrm{Res}^G_{EZ} \chi \in \mathcal{A}_{p^{b_2}}(EZ, Z)$.*

Proof. If $\mathrm{Res}^G_{EN} \chi \in \mathcal{A}_{p^{b_1}}(EN, Z)$ then by Lemma 4.8, applied in the case $H = G$ and $N = Z$, we have $\mathrm{Res}^G_{EZ} \chi = \mathrm{Res}^{EN}_E \mathrm{Res}^G_{EN} \chi \in \mathcal{A}_m(EZ, Z)$, where $m = \left(p^{b_1}\right)/h$, and $h := \left(|EN : EZ|, p^{b_1}\right)$. Therefore we have

$$v_p(|EN : EZ|) = v_p\left(|G : EZ|\right) - v_p\left(|G : EN|\right) = a - b_2 - a + b_1 = b_1 - b_2 \leq b_1,$$

since $b_2 \geq 0$. Hence $h = p^{b_1 - b_2}$ and $m = p^{b_2}$. □

Claim 6.6. *Without loss of generality, $G = EN$, for some $E \in \mathrm{El}(G)$.*

Proof. Suppose that $EN < G$ for all $E \in \mathrm{El}(G)$. For an arbitrary $E \in \mathrm{El}(G)$, define $b_1 := b(E, N, G, a)$ and ζ be an irreducible constituent of $\mathrm{Res}^G_{EN}\chi$. Using Lemma 3.30, we have $\zeta \in \mathrm{Irr}(EN||\theta)$ and therefore $\zeta(1)$ is divisible by p^a. Therefore using that G is a minimal counterexample, noting that $EN \prec G$ since $Z \leq Z(EN)$, Conjecture 2.12 holds for each such ζ, i.e. $\zeta \in \mathcal{A}_{p^a}(EN, Z(EN))$. By using the decomposition of $\mathrm{Res}^G_{EN}\chi$ into irreducibles we have $\mathrm{Res}^G_{EN}\chi \in \mathcal{A}_{p^a}(EN, Z)$. Since $b_1 \leq a$, we have that $\mathrm{Res}^G_{EN}\chi \in \mathcal{A}_{p^{b_1}}(EN, Z)$. By Corollary 6.5, $\mathrm{Res}^G_{EZ}\chi \in \mathcal{A}_{p^{b_2}}(EZ, Z)$ where $b_2 := b(E, Z, G, a)$. Since E is arbitrary, we have that by Proposition 4.6 that $\chi \in \mathcal{A}_{p^a}(G, Z)^\dagger$, which contradicts the fact that G is a minimal counterexample, hence we may assume that $G = EN$ for some $E \in \mathrm{El}(G)$. $\square$

We now assume that $N/Z \cong C_l \times \cdots \times C_l$ for some prime l, i.e. that S is a cyclic group. We note that in this situation we have that N is nilpotent. If N/Z were a p'-group then using [16, Corollary 11.2,] we have, that for each irreducible constituent $\lambda \in \mathrm{Irr}(Z)$ of $\mathrm{Res}^N_Z\theta$, that $\theta(1)/\lambda(1) = \theta(1)$ divides $|N : Z|$, which by Claim 4.16, namely that p^a divides $\theta(1)$, we have a contradiction. Therefore we may assume that $l = p$.

We prove Theorem 6.1 in this case, using Proposition 4.6. We let E be an arbitrary elementary subgroup. Defining $b := v_p(|G : EZ|)$, then by Proposition 4.6, it is enough to show that, $\mathrm{Res}^G_{EZ}\chi \in \mathcal{A}_{p^{a-b}}(EZ, Z)$. Recall that by Claim 6.6, $G = EN$. Let $P \in \mathrm{Syl}_p(EZ)$. Since N is nilpotent, we have $N = N_p \times N_{p'}$ where $N_p \in \mathrm{Syl}_p(N)$. Since $\theta \in \mathrm{Irr}(N)$, we have that there exist $\alpha \in \mathrm{Irr}(N_p)$ and $\beta \in \mathrm{Irr}(N_{p'})$ such that $\theta = \alpha \times \beta$. Since $\chi \in \mathrm{Irr}(G||\theta)$ and $\theta \in \mathrm{Irr}(N||\alpha)$, we have $\chi \in \mathrm{Irr}(G||\alpha)$. We note also that $v_p(\theta(1)) = v_p(\alpha(1))$.

If $\gamma \in \mathrm{Irr}(N_p \cap P)$ is an irreducible constituent of $\mathrm{Res}^{N_p}_{N_p \cap P}\alpha$, then α is an irreducible

†We note that in the statement of Proposition 4.6, $b(E, Z, G) = a - v_p(|G : EZ \cap G|) = a - v_p(|G : EZ|) = b_2$.

constituent of $\mathrm{Ind}_{N_p \cap P}^{N_p}\gamma$. Hence $\alpha(1) \leq |N_p : N_p \cap P|\gamma(1)$ and by Claim 4.18,

$$v_p(\gamma(1)) \geq v_p(\alpha(1)) - v_p(|N_p : N_p \cap P|) \geq a - b. \tag{6.2}$$

Suppose therefore that ζ is an irreducible constituent of $\mathrm{Res}_{EZ}^{G}\chi$. Suppose also that ϕ is an irreducible constituent of $\mathrm{Res}_{P \cap N_p}^{EZ}\zeta$, then ϕ is an irreducible constituent of $\mathrm{Res}_{N_p \cap P}^{N_p}\mathrm{Res}_{N_p}^{EZ}\zeta$. Since $\chi \in \mathrm{Irr}(G||\alpha)$ we have that $\zeta \in \mathrm{Irr}(EZ||\alpha)$ by Lemma 3.30. Therefore ϕ is is a positive integer multiple of an irreducible constituent of $\mathrm{Res}_{N_p \cap P}^{N_p}\alpha$ by Equation (6.2). Hence $v_p(\phi(1)) \geq a - b$, and $v_p(\zeta(1)) \geq a - b$. Since $EZ < G$, by Claims 4.19 and 4.20 (which both do not require that G is p-solvable), we can apply Conjecture 2.12 to each ζ, i.e. $\zeta \in \mathcal{A}_{p^{a-b}}(EZ, Z)$. Therefore $\mathrm{Res}_{EZ}^{G}\chi \in \mathcal{A}_{p^{a-b}}(EZ, Z)$ as claimed. Therefore we now assume that S is a non-abelian simple group. The following corollaries are now clear from Claim 6.6.

Claim 6.7. *G/N is nilpotent.*

Proof. Use Claim 6.6, the Second Isomorphism Theorem and that quotients of nilpotent groups are nilpotent. □

Claim 6.8. *$\mathrm{Res}_{N}^{G}\chi = \theta$.*

Proof. Using the note following Definition 3.26 and Claim 6.7, we have that G is a relative M-group with respect to N. In particular, we have that χ is a relative M-character, with respect to N. From Definition 3.26, this is a contradiction to Claim 4.13, unless the subgroup $H = G$. The result then follows. □

From this point on we will be using the assumption that $Z(G) = 1$. We define $C := C_G(N)$. Firstly, using Theorem 4.12, we are able to show the following.

Claim 6.9. *$C = 1$.*

Proof. Suppose that $C \neq 1$. Since $Z(N) = 1$, $C \cap N = 1$. Therefore,

$$C \cong C/1 = C/(C \cap N) \cong NC/N,$$

via the Second Isomorphism Theorem. Hence C is isomorphic to a subgroup of G/N which is nilpotent by Claim 6.7. Hence C is nilpotent.

If C were a minimal normal subgroup of G, we may replace our choice of N by C and apply our proof of Theorem 4.12 (and the case where S is cyclic), to give us that $\chi \in \mathcal{A}_{p^a}(G,1)$. If C were not minimal, then there exists a minimal normal subgroup, say M, of G, such that $M \leq C$. Since M is also nilpotent the above argument applies also. Hence without loss, we may assume that $C = 1$. $\square$

Using Claim 6.9, we have that G embeds into $\mathrm{Aut}(N) \cong \mathrm{Aut}(S) \wr S_n$. We define $M := \mathrm{Aut}(S)$ and $K := G \cap M^n$. Note that θ can be identified with $\theta_1 \times \cdots \times \theta_n$ where $\theta_i \in \mathrm{Irr}(S)$ for each i. Recall the definition of L_i, Equation (6.1). Using this, we define $\widetilde{G} := G\,(L_1 \times \ldots \times L_n)$. The following lemma shows that $\widetilde{G}$, is in fact a well-defined subgroup. Moreover, Lemma 6.11 gives some useful properties of $\widetilde{G}$.

Lemma 6.10. $G \leq N_{M \wr S_n}\,(L_1 \times \cdots \times L_n)$.

Proof. Let $g = (g_1, g_2, \ldots, g_n; \tau) \in G$, with $\tau(i) = j$, for some i and j, and let $(k_1, \ldots, k_n) \in L_1 \times \ldots \times L_n$. As $g = (g_1, \ldots, g_n)\,(1, \ldots, 1; \tau)$,

$$^{g}\,(k_1, \ldots, k_n) = \left({}^{g_1}k_{\tau^{-1}(1)}, \ldots, {}^{g_j}\,k_{\tau^{-1}(j)}, \ldots, {}^{g_n}\,k_{\tau^{-1}(n)}\right).$$

We show that ${}^{g_j}k_i \in L_j$. Since $k_i \in L_i$, we have that there exists $l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n \in M$ and $\sigma \in S_n$, with $\sigma(i) = i$ such that,

$$h := (l_1, \ldots, l_{i-1}, k_i, l_{i+1}, \ldots, l_n; \sigma) \in G.$$

In particular, we have $ghg^{-1} \in G$. Moreover,

$$ghg^{-1} = \left({}^{g_1}l_{\tau^{-1}(1)}, \ldots, {}^{g_j}\,k_i, \ldots, {}^{g_n}\,l_{\tau^{-1}(n)}; \tau\sigma\tau^{-1}\right),$$

where as $\tau(i) = j$, k_i gets permuted from the i-th position to the j-th. Hence as $\tau\sigma\tau^{-1}(j) = j$, we have that ${}^{g_j}k_i \in L_j$ by definition. $\square$

Lemma 6.11. *The following hold;*

(1) $\widetilde{G} \cap M^n = L_1 \times \cdots \times L_n$.

(2) $L_1 \times \cdots \times L_n \trianglelefteq \widetilde{G}$.

Proof. (1) We have that as $\widetilde{G} \leq M \wr S_n$, the elements take the form $(m_1, \ldots, m_n; \sigma)$ for $m_i \in M$ for all i, and $\sigma \in S_n$. Clearly the elements of M^n satisfy $\sigma = 1$. Hence we have that elements of $\widetilde{G} \cap M^n$, have the above form for $\sigma = 1$. Since, $\widetilde{G} = G\,(L_1 \times \cdots \times L_n)$, we have that,

$$(m_1, \ldots, m_n; 1) = (g_1, \ldots, g_n; 1)\,(l_1, \ldots, l_n; 1)\,,$$

for $g_i \in M$ and $l_i \in L_i$. However by definition of L_i, we have that $(g_1, \ldots, g_n) \in L_1 \times \cdots \times L_n$. Hence $m_i = g_i l_i \in L_i$, and $\widetilde{G} \cap M^n \leq L_1 \times \cdots \times L_n$. The reverse containment is clear from the definitions of $L_i \leq M$ and $\widetilde{G}$.

(2) Note $M^n \trianglelefteq M \wr S_n$, therefore $\widetilde{G} \cap M^n \trianglelefteq \widetilde{G}$ by standard results from group theory. The result follows from part (1).

□

Define an action of G on $\{1, \ldots, n\}$ as follows. We set $g \star i = j$ if ${}^{g}S^{(i)} = S^{(j)}$.

Lemma 6.12. *Without loss of generality, G acts transitively on $\{1, ..., n\}$.*

Proof. Suppose for a contradiction that G does not act transitively. Then there exists some subset Y of $\{1, ..., n\}$, of size k, $1 \leq k < n$, on which G acts transitively. As $G \leq \mathrm{Aut}(S) \wr S_n$, we are able to assume that in fact $G \leq \mathrm{Aut}(S) \wr (S_k \times S_{n-k})$, where the copy of S_k acts on the indices belonging to Y. However if this holds, then $\underbrace{S \times \cdots \times S}_{k}$, when identified as a subgroup of S^n in G, is normal in G. This contradicts the minimality of S^n. □

We now consider the natural projection of $M \wr S_n \twoheadrightarrow S_n$, denoted Π. Denote by H the image of G under Π. Note that $\ker(\Pi) = M^n$. We adopt the notation that for $W \leq M \wr S_n$, $\Pi|_W$ is the restriction of Π to W.

Lemma 6.13. *The following hold;*

(1) $G/K \cong H$.

(2) $\widetilde{G}/(L_1 \times \ldots \times L_n) \cong H$.

Proof. (1) Note that by the First Isomorphism Theorem we have that

$$\frac{G}{\ker(\Pi|_G)} \cong H.$$

We claim that $\ker(\Pi|_G) = K$. We have that $K = G \cap M^n \leq \ker(\Pi|_G)$, by definition of $\Pi|_G$. Now let $g := (g_1, \ldots, g_n; \sigma) \in \ker(\Pi|_G)$, i.e. $(g_1, \ldots, g_n; \sigma) \in G$. Since $\Pi|_G((g_1, \ldots, g_n; \sigma)) = 1$ if and only if $\sigma = 1$, we have $g \in M^n$. Hence $g \in G \cap M^n = K$.

(2) Note first that as $\Pi(G) = H$, we have $G \leq M \wr H$, and as $L_1 \times \ldots \times L_n \leq M^{\times n}$, $\widetilde{G} \leq M \wr H$. Denote by $\Pi^{(H)}$ the natural projection $M \wr H \twoheadrightarrow H$. We determine $\operatorname{im}\left(\Pi^{(H)}|_{\widetilde{G}}\right)$ and $\ker\left(\Pi^{(H)}|_{\widetilde{G}}\right)$.

By definition, $\operatorname{im}\left(\Pi^{(H)}|_{\widetilde{G}}\right) \leq H$. Now let $\sigma \in H$. Since $H = \operatorname{im}(\Pi|_G)$, we have that there exists $g_i \in M$ such that $(g_1, \ldots, g_n; \sigma) \in G \leq \widetilde{G}$, and therefore $\sigma \in \operatorname{im}\left(\Pi^{(H)}|_{\widetilde{G}}\right)$. An identical proof to (1), using Lemma 6.11, gives that $\ker\left(\Pi^{(H)}|_{\widetilde{G}}\right) = L_1 \times \cdots \times L_n$.

□

Now note that as G acts transitively on $\{1, \ldots, n\}$ we have that H acts transitively on $\{1, \ldots, n\}$. We denote by σ_i a fixed element of H that maps 1 to i and $\omega_i := (g_{i1}, \ldots, g_{in}; \sigma_i) \in \widetilde{G}$ with $g_{ij} \in M$. For a group G, we denote by $G^{(i)}$ the group, $1 \times \cdots \times 1 \times G \times 1 \times \cdots \times 1$, where G appears in the i-th component. We denote by $c_{\omega_i} : L_1^{(1)} \to L_i^{(i)}$ conjugation by ω_i. Note that the image of this map does indeed yield an element of $L_i^{(i)}$. For $l \in L_1$, $c_{\omega_i}(l, 1, \ldots, 1) = (1, \ldots, 1, {}^{g_{ii}}l, 1, \ldots, 1)$. As $L_1^{(1)} \leq L_1 \times \cdots \times L_n$, we have by Lemma 6.11 (2), ${}^{\omega_i}(l, 1, \ldots, 1) \in L_1 \times \cdots \times L_n$, i.e. ${}^{g_{ii}}l \in L_i$.

It is easy to see that the c_{ω_i} are in fact isomorphisms. Furthermore, define $\phi_{i,j} : L_i^{(i)} \to L_j^{(j)}$ by $\phi_{i,j} := c_{\omega_j} \circ c_{\omega_i}^{-1}$. Moreover, define $\pi_i : L_i^{(i)} \to L_i$ to be the natural projection map. Finally, define $\psi_{i,j} : L_i \to L_j$ by $\psi_{i,j} := \pi_j \circ \phi_{i,j} \circ \pi_i^{-1}$. Since the $\phi_{i,j}$ and π_i are clearly isomorphisms, we have that $L_i \cong L_j$ for any $1 \leq i, j \leq n$. Noting this, our goal is to find a complement to $L_1 \times \cdots \times L_n$ in $\widetilde{G}$, which we denote by $\widetilde{H}$.

To do so, let $\alpha \in \widetilde{G}$ and let $\Pi^{(H)}(\alpha) := \sigma \in H$. We aim to find $k_1, \ldots, k_n$ with $k_i \in L_i$ such that for $\alpha' := \alpha(k_1, \ldots, k_n; 1)$ we have $c_{\alpha'}|_{L_i^{(i)}} = \phi_{i,\sigma(i)}$, where for $g \in \widetilde{G}$, $c_g : \widetilde{G} \to \widetilde{G}$ denotes conjugation by g. Moreover for $l \in M$, we denote by $l^{(i)} := (1, \ldots, 1, l, 1, \ldots, 1) \in M^{(i)}$.

Lemma 6.14. *If $g \in \widetilde{G}$, and ${}^{g}L_i^{(i)} = L_i^{(i)}$, for some i, then $g = (l_1, \ldots, l_n; \sigma) \in M \wr H$, where $\sigma(i) = i$, $l_i \in L_i$ and $c_g|_{L_i^{(i)}} = c_{l_i^{(i)}}$.*

Proof. We have that $g = (l_1, \ldots, l_n; \sigma)$ with $l_i \in M$ for all i and $\sigma \in H$. Let $(1, \ldots, 1, l, 1, \ldots, 1) \in L_i^{(i)}$. Then ${}^{g}(1, \ldots, 1, l, 1, \ldots, 1)$ can be expressed as

$$(l_1, \ldots, l_n)(1, \ldots, 1; \sigma)(1, \ldots, 1, l, 1, \ldots, 1)\left(1, \ldots, 1; \sigma^{-1}\right)\left(l_1^{-1}, \ldots, l_n^{-1}\right)$$

$$= (l_1, \ldots, l_n)(1, \ldots, 1, l, 1, \ldots, 1)\left(l_1^{-1}, \ldots, l_n^{-1}\right) = \left(1, \ldots, 1, {}^{l_{\sigma(i)}}l, 1, \ldots, 1\right), \tag{6.3}$$

As $(1, \ldots, 1, l, 1, \ldots, 1) \in L_1 \times \cdots \times L_n$ and $g \in \widetilde{G}$, Lemma 6.11 (2) and Equation 6.3 give us that ${}^{l_{\sigma(i)}}(l) \in L_{\sigma(i)}$. Furthermore, we have shown that ${}^{g}L_i^{(i)} = L_{\sigma(i)}^{(\sigma(i))}$; however by assumption ${}^{g}L_i^{(i)} = L_i^{(i)}$, and hence $\sigma(i) = i$. It follows from this and Equation 6.3 that $c_g|_{L_i^{(i)}} = c_{l_i^{(i)}}$. We are left to show that $l_i \in L_i$.

Note that as $g \in \widetilde{G}$, $g = g_1 g_2$ for $g_1 \in G$ and $g_2 \in L_1 \times \cdots \times L_n$. Let $g_1 = (h_1, \ldots, h_n; \sigma)$ for $h_i \in M$ and $g_2 = (k_1, \ldots, k_n) \in L_1 \times \cdots \times L_n$. In particular, $l_i = h_i k_{\sigma^{-1}(i)}$, for every i.

However as $\sigma^{-1}(i) = i$, $k_i \in L_i$ and we only need to show that $h_i \in L_i$. As $\sigma(i) = i$ and $g_1 \in G$, we have that $h_i \in L_i$ by definition. Hence $l_i \in L_i$ as claimed.

□

As stated before Lemma 6.14, we now aim to determine k_i. We find k_i such that

$$c_\alpha^{-1} \circ c_{\alpha'}\big|_{L_i^{(i)}} = c_{\alpha^{-1}}\big|_{L_{\sigma(i)}^{(\sigma(i))}} \circ \phi_{i,\sigma(i)}. \tag{6.4}$$

The left hand side of Equation 6.4, simplifies to

$$c_\alpha^{-1} \circ c_{\alpha'}\big|_{L_i^{(i)}} = c_{\alpha^{-1}\alpha'}\big|_{L_i^{(i)}} = c_{(k_1,\dots,k_n)}\big|_{L_i^{(i)}} = c_{k_i^{(i)}}.$$

The right hand side of Equation 6.4 simplifies to

$$c_{\alpha^{-1}}\big|_{L_{\sigma(i)}^{(\sigma(i))}} \circ \phi_{i,\sigma(i)} = c_{\alpha^{-1}} \circ c_{\omega_{\sigma(i)}} \circ c_{\omega_i^{-1}}\Big|_{L_i^{(i)}} = c_g\big|_{L_i^{(i)}},$$

for some $g \in \widetilde{G}$, explicitly, $g = \alpha^{-1}\omega_{\sigma(i)}\omega_i^{-1}$. We claim that ${}^{g}L_i^{(i)} = L_i^{(i)}$. To prove this we simply need that ${}^{\alpha^{-1}}L_{\sigma(i)}^{(\sigma(i))} = L_i^{(i)}$. Using the argument used in the proof of Lemma 6.14, we obtain this result. Therefore by Lemma 6.14, we know that $g = (l_1, \dots, l_n; \tau)$ where $l_i \in L_i$, $l_j \in M$, $j \neq i$, $\tau(i) = i$ and most importantly, $c_g|_{L_i^{(i)}} = c_{l_i^{(i)}}$. Hence taking $k_i = l_i$, Equation 6.4 holds. Finally from Equation 6.4, we are able to deduce that for this definition of α', $c_{\alpha'} = \phi_{i,\sigma(i)}$. Furthermore we take our complement $\widetilde{H}$ to be

$$\widetilde{H} := \left\{ g \in \widetilde{G} : g = (l_1, \dots, l_n; \sigma) \text{ such that } \forall i,\ c_g\big|_{L_i^{(i)}} = \phi_{i,\sigma(i)} \right\}.$$

Lemma 6.15. *$\widetilde{H}$ is a complement to $L_1 \times \cdots \times L_n$ in $\widetilde{G}$.*

Proof. By the construction of $\widetilde{H}$ above, we have that $\widetilde{H}$ maps surjectively onto H, under the restriction of $\Pi^{(H)}$ to $\widetilde{H}$. We consider the kernel of this map. By the proof of Lemma 6.13 (2) we see that $\ker\left(\Pi^{(H)}|_{\widetilde{H}}\right) = \widetilde{H} \cap (L_1 \times \cdots \times L_n)$.

Suppose that $g := (l_1, \dots, l_n; \sigma) \in \widetilde{H} \cap (L_1 \times \cdots \times L_n)$. Since $L_1 \times \cdots \times L_n \leq M^n$, we have that σ is the identity permutation and that $l_i \in L_i$ for each i. Furthermore, $c_g|_{L_i^{(i)}} = c_{l_i^{(i)}} = \phi_{i,\sigma(i)} = \mathrm{id}$. Since $Z(L_i) = 1$ for each i, as the L_i are almost simple we have that $l_i = 1$ for every i and hence $\widetilde{H} \cap (L_1 \times \cdots \times L_n) = 1$. Therefore $\widetilde{H} \cong H$.

Since $L_1 \times \cdots \times L_n \trianglelefteq \widetilde{G}$, by Lemma 6.11 (2) we have that $\widetilde{H}\,(L_1 \times \cdots \times L_n) \leq \widetilde{G}$ and equality follows from Lemma 6.13 (2). $\square$

Lemma 6.16. *$\widetilde{G} \cong L_1 \wr H$.*

Proof. We define $\alpha : L_1 \times \cdots \times L_1 \to \widetilde{G}$ by

$$\alpha\,(l_1, \dots, l_n) := (\psi_{1,1}(l_1), \dots, \psi_{1,n}(l_n))\,.$$

Moreover, we define $\beta : H \to \widetilde{G}$ by,

$$\beta\,(\sigma) := \left(\Pi^{(H)}\right)^{-1}(\sigma).$$

Note that by Lemma 6.15, we have that $\left(\Pi^{(H)}\right)^{-1}(\sigma) \in \widetilde{H}$. If $\sigma \in H$ and $(l_1, \dots, l_n) \in L_1 \times \cdots \times L_1$, then

$$\beta(\sigma)\alpha\,(l_1, \dots, l_n)\,\beta(\sigma)^{-1} = c_{\beta(\sigma)}\,(\psi_{1,1}(l_1), \dots, \psi_{1,n}(l_n)) \tag{6.5}$$

$$= c_{\beta(\sigma)}\,(\psi_{1,1}(l_1), 1, \dots, 1) \dots c_{\beta(\sigma)}\,(1, \dots, 1, \psi_{1,n}(l_n))$$

$$= c_{\beta(\sigma)}\left(\pi_1^{-1} \circ \psi_{1,1}(l_1)\right) \dots c_{\beta(\sigma)}\left(\pi_n^{-1} \circ \psi_{1,n}(l_n)\right)$$

$$= c_{\beta(\sigma)}\left(\phi_{1,1} \circ \pi_1^{-1}(l_1)\right) \dots c_{\beta(\sigma)}\left(\phi_{1,n} \circ \pi_1^{-1}(l_n)\right).$$

Now by the definition of $\widetilde{H}$, as $\beta(\sigma) \in \widetilde{H}$, we have $c_{\beta(\sigma)}|_{L_i^{(i)}} = \phi_{i,\sigma(i)}$. Hence,

$$\beta(\sigma)\alpha\,(l_1, \dots, l_n)\,\beta(\sigma)^{-1} = \left(\phi_{1,\sigma(1)} \circ \phi_{1,1} \circ \pi_1^{-1}(l_1)\right) \dots \left(\phi_{n,\sigma(n)} \circ \phi_{1,n} \circ \pi_1^{-1}(l_n)\right).$$

Now for $1 \leq i \leq n$, $\phi_{i,\sigma(i)} \circ \phi_{1,i} = c_{\omega_{\sigma(i)}} \circ c_{\omega_i}^{-1} \circ c_{\omega_i} \circ c_{\omega_1}^{-1} = c_{\omega_{\sigma(i)}} \circ c_{\omega_1}^{-1} = \phi_{1,\sigma(i)}$. Hence

$$\beta(\sigma)\alpha\left(l_1,\ldots,l_n\right)\beta(\sigma)^{-1}=\left(\phi_{1,\sigma(1)}\circ\pi_1^{-1}(l_1)\right)\ldots\left(\phi_{1,\sigma(n)}\circ\pi_1^{-1}(l_n)\right)$$

$$=\phi_{1,\sigma(1)}\left(l_1,1,\ldots,1\right)\ldots\phi_{1,\sigma(n)}\left(l_n,1,\ldots,1\right),$$

where for $1\leq i\leq n$, $\phi_{1,\sigma(i)}\left(l_i,1,\ldots,1\right)\in L_{\sigma(i)}^{(\sigma(i))}$. Hence the $\sigma(i)$-th co-ordinate of (6.5) is $\pi_{\sigma(i)}\circ\phi_{1,\sigma(i)}\circ\pi_1^{-1}(l_i)=\psi_{1,\sigma(i)}(l_i)$. Hence,

$$\beta(\sigma)\alpha\left(l_1,\ldots,l_n\right)\beta(\sigma)^{-1}=\left(\psi_{1,1}\left(l_{\sigma^{-1}(1)}\right),\ldots,\psi_{1,n}\left(l_{\sigma^{-1}(n)}\right)\right)$$

$$=\alpha\left(l_{\sigma^{-1}(1)},\ldots,l_{\sigma^{-1}(n)}\right).$$

Therefore by Lemma 3.10, there exists a homomorphism $\varphi:L_1\wr H\to\widetilde{G}$, given by $\varphi\left(l_1,\ldots,l_n;\sigma\right):=\alpha\left(l_1,\ldots,l_n\right)\beta(\sigma)$. We show that φ is an isomorphism. Let $(l_1,\ldots,l_n;\sigma)$ $\in\ker(\varphi)$. We clearly must have that $\sigma=1$ and therefore since conjugation is bijective it follows that $l_i=1$ for $1\leq i\leq n$.

For surjectivity, we show that every element of $\widetilde{G}$, takes the form $(l_1,\ldots,l_n)h$ for some $h\in\widetilde{H}$ and $l_i\in L_i$. Since by Claim 6.15, $\widetilde{H}$ is a complement to $L_1\times\cdots\times L_n$ in $\widetilde{G}$, for $g\in\widetilde{G}$, $g=(l_1,\ldots,l_n)h$ (in which case we are done) or $g=h(l_1,\ldots,l_n)$. Supposing that $\Pi_{(H)}(h)=\sigma$ the result follows since

$$g=h(l_1,\ldots,l_n)h^{-1}h=c_h(l_1,\ldots,l_n)h=\left(c_h\left(l_1,1,\ldots,1\right)\ldots c_h\left(1,\ldots,1,l_n\right)\right)h.$$

Since $h\in\widetilde{H}$, we have $c_h|_{L_i^{(i)}}=\phi_{i,\sigma(i)}$. Hence,

$$g=\left(\phi_{1,\sigma(1)}(l_1,1,\ldots,1)\ldots\phi_{n,\sigma(n)}(1,\ldots,1,l_n)\right)h$$

$$=\left(\phi_{1,\sigma(1)}\circ\pi_1^{-1}(l_1)\ldots\phi_{n,\sigma(n)}\circ\pi_n^{-1}(l_n)\right)h.$$

Similar to above, we have that for $1\leq i\leq n$, the $\sigma(i)$-th co-ordinate of g is $\pi_{\sigma(i)}\circ\phi_{i,\sigma(i)}\circ$

$\pi_i^{-1}(l_i) = \psi_{i,\sigma(i)}(l_i) \in L_{\sigma(i)}$. Hence,

$$g = \left(\psi_{\sigma^{-1}(1),1}\left(l_{\sigma^{-1}(1)}\right), \ldots, \psi_{\sigma^{-1}(n),n}\left(l_{\sigma^{-1}(n)}\right)\right) h,$$

which takes the desired form. Surjectivity of φ now follows from the definition. $\square$

Using this isomorphism, we identify $G \leq \widetilde{G}$ with its image $G' \leq L_1 \wr H \leq L_1 \wr S_n$. Moreover $S^n \leq G$ maps to $S^n \leq L_1^n \leq L_1 \wr H$, where the latter group is simply n copies of the first copy of S in G. We now show that in this setting each of the components of θ coincide.

Lemma 6.17. $\theta_1 = \theta_2 = \cdots = \theta_n = \gamma \in \mathrm{Irr}(S)$.

Proof. First we show that $\theta_1 \times \cdots \times \theta_n$ is $L_1 \wr H$-invariant. We do this by showing that $\theta_1 \times \cdots \times \theta_n$ is $\widetilde{G}$-invariant. Since $\theta_1 \times \cdots \times \theta_n$ is G-invariant by Claim 4.15, it is sufficient to show that $\theta_1 \times \cdots \times \theta_n$ is $L_1 \times \cdots \times L_n$-invariant. Let $(l_1, \ldots, l_n) \in L_1 \times \cdots \times L_n$. By the definition of the L_i, for each i there exists $l_{i1}, \ldots, l_{ii} = l_i, \ldots, l_{in} \in M$ and $\sigma_i \in S_n$ such that $k_i := (l_{i1}, \ldots, l_i, \ldots, l_{in}; \sigma_i) \in G$ and $\sigma_i(i) = i$. Therefore for each i,

$$^{k_i}\left(\theta_1 \times \cdots \times \theta_n\right) = \theta_1 \times \cdots \times \theta_n.$$

Let $t_i := s_i^{(i)} \in S^n$. Evaluating the i-th equation at t_i we have, $^{l_i}\left(\theta_i(s_i)\right) = \theta_i(s_i)$. Therefore for $(s_1, \ldots, s_n) \in S^n$,

$$^{(l_1,\ldots,l_n)}\left(\theta_1 \times \cdots \times \theta_n\right)(s_1, \ldots, s_n) = \prod_{i=1}^{n} {}^{l_i}\theta_i(s_i) = \prod_{i=1}^{n} \theta_i(s_i)$$

$$= \left(\theta_1 \times \cdots \times \theta_n\right)(s_1, \ldots, s_n).$$

Hence $\theta_1 \times \cdots \times \theta_n$ is $L_1 \times \cdots \times L_n$-invariant. Moreover, we have that $\theta_1 \times \cdots \times \theta_n$ is $L_1 \wr H$-invariant. Suppose $i < j$ and let $\tau := (1, \ldots, 1; \sigma) \in L_1 \wr H$, with $\sigma(i) = j$. It follows that

$$\theta_1 \times \cdots \times \theta_j \times \cdots \times \theta_i \times \cdots \times \theta_n = {}^{\tau}\left(\theta_1 \times \cdots \times \theta_i \times \cdots \times \theta_j \times \cdots \times \theta_n\right)$$

$$= \theta_1 \times \cdots \times \theta_i \times \cdots \times \theta_j \times \cdots \times \theta_n \tag{6.6}$$

Evaluating both sides of (6.6) on $s^{(i)} \in S^n$; for every $s \in S$ we have that $\theta_i = \theta_j$. The result follows since H acts transitively on $\{1, \ldots, n\}$ by Lemma 6.12. □

Since $G' \cong G$, χ can be viewed as an irreducible of G'. Moreover, $\mathrm{Res}^{G'}_{S^{\times n}}\chi = \gamma^{\times n}$. By Lemma 5.14, there exists $\alpha \in H^2(L_1, \mathbb{C}^*)$ such that for $A := \mathbb{C}^{\alpha}[L_1]$, $A[S] \cong \mathbb{C}S$. This shows that in this case the "extensibility" condition, Condition 6.2, is satisfied for L_1. Moreover there exists $\zeta \in \mathrm{Irr}(A)$ such that $\mathrm{Res}^{L_1}_{S}\zeta = \gamma$. Moreover since L_1 is almost simple, Conjecture 2.11 applies and gives us that $\zeta \in \mathcal{A}_{p^c}(A, S, 1)$, where $a = nc$.

Using the results in Section 5.3, $\zeta^{\widetilde{\times n}} \in \mathrm{Irr}(A \wr S_n)$ and $\mathrm{Res}^{L_1 \wr S_n}_{S^n}\zeta^{\widetilde{\times n}} = \mathrm{Res}^{L_1^n}_{S^n}\zeta^{\times n} = \gamma^n$. Define $B := A \wr S_n$. Since the restriction to $S^{\times n}$ is irreducible, we have that $\mathrm{Res}^{L_1 \wr S_n}_{G'}\zeta^{\widetilde{\times n}} \in \mathrm{Irr}(B[G'])$, extending $\gamma^{\times n}$. Hence by Lemma 5.15, $B[G'] \cong \mathbb{C}G'$. Moreover, by the proof of Lemma 5.16, we see that the factor set associated to $B[G']$ is $\mathrm{Inf}(\mu)$ where $\mu \in B^2(G'/S^{\times n}, \mathbb{C}^*)$. Hence from [22, Chapter 2], using the remarks following Theorem 1.1 (taking the obstruction cocycle to be μ) and Theorem 3.3, we have that since $\mathrm{Res}^{L_1 \wr S_n}_{G'}\zeta^{\widetilde{\times n}} \in \mathrm{Irr}(B[G'])$ restricts (irreducibly) to $\gamma^{\times n}$, if we define $\zeta' := \mathrm{Res}^{L_1 \wr S_n}_{G'}\zeta^{\widetilde{\times n}}$, then $\chi = \zeta' \mathrm{Inf}^{G'}_{G'/S^{\times n}}\lambda$, where λ is a (linear) projective μ^{-1}-character of $G'/S^{\times n}$. Hence by Lemma 5.26, the result follows provided $\zeta' \in \mathcal{A}_{p^a}(B[G'])$. By Theorem 5.37, since $\zeta \in \mathcal{A}_{p^c}(A, S, 1)$, we have that $\zeta^{\widetilde{\times n}} \in \mathcal{A}_{p^a}(A \wr S_n, S^n, 1^n)$.

Lemma 6.18. $\zeta' \in \mathcal{A}_{p^a}(B[G'])$.

Proof. Since $\zeta^{\widetilde{\times n}} \in \mathcal{A}_{p^a}(A \wr S_n, S^n, 1^n)$ we can assume

$$\zeta^{\widetilde{\times n}} = \sum_{H \in \mathcal{I}_{p^a}(L_1 \wr S_n, S^{\times n})} \mathrm{Ind}^{L_1 \wr S_n}_{H} \phi_{(H)},$$

where $\phi_{(H)} \in \mathcal{C}(B[H])$. By Lemma 5.25, we have

$$\zeta' = \sum_{H \in \mathcal{I}_{p^a}(L_1 \wr S_n, S^{\times n})} \sum_{t \in T(H)} \mathrm{Ind}_{{}^tH \cap G'}^{G'} \mathrm{Res}_{{}^tH \cap G'}^{{}^tH} {}^t\phi_{(H)},$$

where $T(H)$ denotes a set of (H, G')-double coset representatives in $L_1 \wr S_n$. Hence we are done provided p^a divides $|G' : {}^tH \cap G'|$. Since p^a divides $|S^{\times n} : H \cap S^{\times n}|$, p^a divides $|S^{\times n} : {}^tH \cap S^{\times n}|$ as $S^{\times n} \trianglelefteq L_1 \wr S_n$. Moreover,

$$|G' : {}^tH \cap G'| = \frac{|G' : S^{\times n}|}{|{}^tH \cap G' : {}^tH \cap S^{\times n}|} |S^{\times n} : {}^tH \cap S^{\times n}|.$$

Hence p^a divides $|G' : {}^tH \cap G'|$ provided the quotient is an integer. Note that $({}^tH \cap G') \cap S^{\times n} \trianglelefteq {}^tH \cap G'$ and therefore by the Second Isomorphism Theorem,

$$\frac{|{}^tH \cap G'|}{|{}^tH \cap S^{\times n}|} = \frac{|{}^tH \cap G'|}{|({}^tH \cap G') \cap S^{\times n}|} = \frac{|({}^tH \cap G') S^{\times n}|}{|S^{\times n}|}.$$

Due to this, the quotient simplifies to $|G' : ({}^tH \cap G') S^{\times n}|$. Since $S^{\times n} \trianglelefteq G'$, $({}^tH \cap G') S^{\times n} \leq G'$ so the index is indeed in $\mathbb{Z}$. Therefore p^a divides $|G' : {}^tH \cap G'|$ for all $H \in \mathcal{I}_{p^a}(L_1 \wr S_n, S^{\times n})$ and $t \in T(H)$. Therefore $\zeta' \in \mathcal{A}_{p^a}(B[G'])$. □

Formally Theorem 6.1 now follows by applying the isomorphism given in Lemma 6.16.

6.2 Generalisations of the reduction theorem

In this section we prove Theorem 6.3 using the framework obtained in the previous section that proved Theorem 6.1. Theorem 6.3 removes the restriction that $Z(G) = 1$ and to follow the framework developed in Section 6.1, we apply this to $\overline{G} := G/Z$ where $Z := Z(G)$. Also the use of Condition 6.2 will be used to generalise the results following Lemma 6.17, where, as we are working with $\overline{G}$ instead of G, we will be working not with the group algebra on G but a twisted group algebra on $\overline{G}$, hence the extensibility of the irreducibles of the minimal normal subgroup to L_1 is not guaranteed. A discussion of this condition follows this section.

We begin the proof by letting G be a group with an irreducible character χ of degree divisible by p^a where p is a prime. Suppose that (G, χ) is a minimal counterexample to Conjecture 2.12 with respect to the partial order given in Definition 3.1. Suppose also that Conjecture 2.11 holds. Again we are implicitly assuming that G is non-abelian and that by Lemma 6.4, $\overline{G}$ is not simple. We let $\overline{N} := N/Z$ be a minimal normal subgroup of $\overline{G}$. Following the proof of Theorem 6.1, Claim 4.13 holds for G and we let $\theta \in \mathrm{Irr}(N)$ such that $\chi \in \mathrm{Irr}(G||\theta)$. Moreover Claim 4.15, Corollary 6.5, Claim 6.6, Claim 6.7, Claim 6.8 and the details in between all hold. Therefore Theorem 5.19 (where we take A in the theorem to be $\mathbb{C}G$), we obtain a character $\chi' \in \mathrm{Irr}(A)$ (where A now denotes $A(\theta)$, a twisted $\overline{G}$ algebra) such that $e_\chi = e_{\chi'}$. Moreover, by Lemma 5.20, we have that $\mathrm{Res}^A_{A[\overline{N}]}\chi' = \theta'$. By Theorem 4.12 and its proof $\overline{N} \cong S \times \cdots \times S$ where S is a non-abelian simple group and Claim 6.9 can easily be generalised to to show that $C := C_{\overline{G}}\left(\overline{N}\right) = 1$. Hence $\overline{G}$ embeds as a subgroup of $\mathrm{Aut}(S) \wr S_n$. Again set $M := \mathrm{Aut}(S)$. Recall the definition of L_i from Equation (6.1). We generalise this definition to our purposes, replacing any use of the group G, by $\overline{G}$. Due to this one can see the details of Condition 6.2 in practice.

Since $\overline{N} \cong S \times \cdots \times S$, $A\left[\overline{N}\right] \cong A[S \times \cdots \times S]$. Suppose that A has factor set α. Since $A\left[S^{\times n}\right]$ is a twisted $S^{\times n}$-algebra and $M\left(S^{\times n}\right) \cong M(S)^{\times n}$, $\alpha|_{S^{\times n} \times S^{\times n}} = \alpha_1 \times \cdots \times \alpha_n$ and $A[S^{\times n}] \cong B_1 \otimes \cdots \otimes B_n$ where B_i are twisted S-algebras with S-factor sets α_i respectively. Moreover, $\theta' = \theta_1 \times \cdots \times \theta_n$ where $\theta_i \in \mathrm{Irr}(B_i)$. Since θ' is extendible to A, we have that θ' is $\overline{G}$-invariant. Again we define $\widetilde{G} := \overline{G}\left(L_1 \times \cdots \times L_n\right)$ and note that its properties from Lemma 6.10 to the details before Lemma 6.17 all hold, replacing occurrences of G by $\overline{G}$, as they are simply properties of the group that are independent of the fact that G has trivial centre.

Lemma 6.19. *Without loss of generality, $\alpha_1 = \alpha_2 = \cdots = \alpha_n \in H^2(S, \mathbb{C}^*)$.*

Proof. We define $B := B_1 \otimes \cdots \otimes B_n$. We denote by $\left\{u_g : g \in \overline{G}\right\}$ the basis of A which gives rise to the factor set α. We denote by $\{u_s : s \in S\}$ the basis of B_1 induced by B. We choose $u_{(1,\ldots,1)} = 1$ without loss of generality. First we note that for $1 \leq i \leq n$ and

$s \in S$,

$$\phi_{1,i}(s^{(1)}) = c_{\omega_i} \circ c_{\omega_1^{-1}}(s^{(1)}) = \omega_i \omega_1^{-1}(s^{(1)})\omega_1\omega_i^{-1}.$$

We now define a new basis of B. For $s \in S$, set

$$v_{s^{(1)}} := u_{s^{(1)}},$$

$$v_{s^{(i)}} := u_{\omega_i} u_{\omega_1}^{-1} u_{s^{(1)}} u_{\omega_1} u_{\omega_i}^{-1} = a_{s,i} u_{\phi_{1,i}(s^{(1)})},$$

where $a_{s,i} \in \mathbb{C}$. Note that we are using the fact that $u_g^{-1} = \alpha(g, g^{-1})u_{g^{-1}}$ above, to simplify the expression where the additional constants are incorporated into the term $a_{s,i}$. For $(s_1, \ldots, s_n) \in S^{\times n}$, $v_{(s_1,\ldots,s_n)} := v_{s_1^{(1)}} \ldots v_{s_n^{(n)}}$. Based on the definitions, $\left\{v_{(s_1,\ldots,s_n)} : (s_1, \ldots, s_n) \in S^{\times n}\right\}$ is a $\mathbb{C}$-basis of B. We claim that for $j < i$, $v_{s_i^{(i)}} v_{s_j^{(j)}} = v_{s_j^{(j)}} v_{s_i^{(i)}}$.

First note that since $u_{(1,\ldots,1)} = 1$, we have that $u_1 = 1$ in B_1. Since, for $s \in S$, $u_s u_1 = \alpha_1(s, 1)u_s$, we have that $\alpha_1(s, 1) = \alpha_1(1, s) = 1$. Moreover, this argument holds for $\alpha_i(s, 1) = \alpha_i(1, s)$ for $1 \leq i \leq n$. For $1 \leq i \leq n$, we define $m_i := \phi_{1,i}(s_i, 1, \ldots, 1)$, then

$$v_{s_i^{(i)}} v_{s_j^{(j)}} = a_{s_i,i} a_{s_j,j} u_{m_i} u_{m_j}.$$

Since $m_i \in L_i^{(i)}$ and $m_j \in L_j^{(j)}$,

$$v_{s_i^{(i)}} v_{s_j^{(j)}} = a_{s_i} a_{s_j} \alpha_i\left(\pi_i(m_i), 1\right) \alpha_j\left(1, \pi_j(m_j)\right) u_{m_i m_j} = a_{s_i} a_{s_j} u_{m_i m_j}$$

$$= a_{s_i} a_{s_j} u_{m_j} u_{m_i} = v_{s_j^{(j)}} v_{s_i^{(i)}}.$$

By repeated application of this result, for $(s_1, \ldots, s_n)$, $(t_1, \ldots, t_n) \in S^{\times n}$,

$$v_{(s_1,\ldots,s_n)} v_{(t_1,\ldots,t_n)} = v_{s_1^{(1)}} \ldots v_{s_n^{(n)}} v_{t_1^{(1)}} \ldots v_{t_n^{(n)}} = v_{s_1^{(1)}} v_{t_1^{(1)}} \ldots v_{s_n^{(n)}} v_{t_n^{(n)}}.$$

If $1 \leq i \leq n$, then

$$v_{s_i^{(i)}} v_{t_i^{(i)}} = u_{\omega_i} u_{\omega_1}^{-1} u_{s_i^{(1)}} u_{\omega_1} u_{\omega_i}^{-1} u_{\omega_i} u_{\omega_1}^{-1} u_{t_i^{(1)}} u_{\omega_1} u_{\omega_i}^{-1} = \alpha_1(s_i, t_i) u_{\omega_i} u_{\omega_1}^{-1} u_{s_i t_i^{(1)}} u_{\omega_1} u_{\omega_i}^{-1}$$

$$= \alpha_1(s_i, t_i) v_{s_i t_i^{(i)}}.$$

Therefore,

$$v_{(s_1,\ldots,s_n)} v_{(t_1,\ldots,t_n)} = \prod_{i=1}^{n} \alpha_1(s_i, t_i) v_{s_i t_i^{(i)}} = \prod_{i=1}^{n} \alpha_1(s_i, t_i) v_{(s_1 t_1,\ldots,s_n t_n)}.$$

Therefore the new basis has factor set $\alpha_1 \times \cdots \times \alpha_1$. Since we have simply changed a basis the resulting twisted group algebras are isomorphic and we can assume $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. $\square$

Let $\beta := \alpha_1 = \alpha_2 = \cdots = \alpha_n \in H^2(S, \mathbb{C}^*)$, so that $B_1 \cong B_2 \cong \cdots \cong B_n \cong \mathbb{C}^{\beta}[S]$ and $A[S^{\times n}] \cong \mathbb{C}^{\beta^{\times n}}[S^{\times n}]$. The following lemma shows us that $\beta^{\times n}$ appears in the natural long exact sequence in cohomology, the details of which can be found in [15].

Lemma 6.20. *$\beta^{\times n}$ is $\overline{G}$-invariant.*

Proof. Let ρ be the $\beta^{\times n}$-representation affording θ'. Since ρ is extendible to A, we have that ρ is $\overline{G}$-invariant, i.e. ${}^g\rho = \rho$ for all $g \in \overline{G}$. Let x, $y \in S^{\times n}$,

$${}^g\rho(x) {}^g\rho(y) = \rho\left(g^{-1}xg\right) \rho\left(g^{-1}yg\right) = \beta^{\times n}(g^{-1}xg, g^{-1}yg) {}^g\rho(xy) = {}^g\left(\beta^{\times n}\right)(x,y) {}^g\rho(xy).$$

However, due to $\overline{G}$-invariance, we also have

$${}^g\rho(x) {}^g\rho(y) = \rho(x)\rho(y) = \beta^{\times n}(x,y)\rho(xy) = \beta^{\times n}(x,y) {}^g\rho(xy).$$

Since ${}^g\rho(xy)$ is invertible, we have that ${}^g\left(\beta^{\times n}\right)(x,y) = \beta^{\times n}(x,y)$. Finally since x, y are arbitrary, we have that ${}^g\left(\beta^{\times n}\right) = \beta^{\times n}$, i.e. $\beta^{\times n}$ is $\overline{G}$-invariant. $\square$

We finally prove two lemmas which show the use of Condition 6.2 in the proof of

Theorem 6.3. The reader is referred to Lemma 6.17 and the subsequent proof to see how Condition 6.2 generalises the case of trivial centre.

Lemma 6.21. $\theta_1 = \cdots = \theta_n = \eta \in \mathrm{Irr}(\mathbb{C}^\beta[S])$.

Proof. Recall that due to extensibility, $\theta' = \theta_1 \times \cdots \times \theta_n$ is $\overline{G}$-invariant. We aim to show that θ' is $\widetilde{G}$-invariant. Let $(l_1, \ldots, l_n) \in L_1 \times \cdots \times L_n$ and fix $1 \leq i \leq n$. By definition of L_i, there exists $l_{i1}, \ldots, l_{ii} = l_i, \ldots, l_{in} \in M$ and $\sigma_i \in S_n$, with $\sigma_i(i) = i$, such that $k_i := (l_{i1}, \ldots, l_{in}; \sigma_i) \in \overline{G}$. Since θ' is $\overline{G}$-invariant, ${}^{k_i}\theta' = \theta'$. Let $t_i := s_i^{(i)} \in S^{\times n}$, then ${}^{k_i}\theta'(t_i) = \theta'(t_i)$, in particular ${}^{l_i}\theta_i(s_i) = \theta_i(s_i)$. Hence for $(s_1, \ldots, s_n) \in S^{\times n}$,

$$^{(l_1,\ldots,l_n)}\theta'(s_1, \ldots, s_n) = \prod_i {}^{l_i}\theta_i(s_i) = \prod_i \theta_i(s_i) = \theta'(s_1, \ldots, s_n).$$

Therefore θ' is indeed $(L_1 \times \cdots \times L_n)$-invariant and is therefore $\widetilde{G}$-invariant. Applying the isomorphism in Lemma 6.16, we have that θ' is $L_1 \wr H$-invariant. Let $1 \leq i < j \leq n$, and $\tau := (1, \ldots, 1; \sigma) \in L_1 \wr H$ such that $\sigma(i) = j$, then

$$\theta_1 \times \cdots \times \theta_n = {}^\tau\theta' = \theta_1 \times \cdots \times \theta_j \times \cdots \times \theta_i \times \cdots \times \theta_n.$$

By evaluating both sides of the above equation on $t_i \in S^{\times n}$, we obtain $\theta_i(s_i) = \theta_j(s_i)$, and hence $\theta_i = \theta_j$. Moreover, since H acts transitively on $\{1, \ldots, n\}$, we have the desired result. $\square$

Lemma 6.22. *η is extendible to an irreducible character ζ of a twisted group algebra on L_1.*

Proof. By Condition 6.2, we have that there exists $\gamma \in H^2(L_1, \mathbb{C}^*)$ such that $\mathrm{Res}_S^{L_1}\gamma = \beta$. By [22, Theorem 3.1] we have that there exists some $\zeta \in \mathrm{Irr}(\mathbb{C}^\delta[L_1])$, where $\delta = \gamma\mathrm{Inf}(\omega) = \gamma\mathrm{Inf}(\omega_{L_1}(\eta))$, such that $\mathrm{Res}_S^{L_1}\zeta = \eta$. $\square$

We now follow the proof of Theorem 6.1. We denote by G' the image under ϕ in Lemma 6.16 of $\overline{G}$ and view χ' as an irreducible character of G'. We also note that A can be viewed as a twisted G'-algebra under ϕ. Define $R := \mathbb{C}^{\delta}[L_1]$. Since L_1 is almost simple, we can use Conjecture 2.11 to give us that $\zeta \in \mathcal{A}_{p^c}(R, S, 1)$ where $a = nc$, implicitly using Lemma 6.21. The results of Chapter 5.3, give us a character $\widetilde{\zeta^{\times n}} \in \mathrm{Irr}(R \wr S_n)$ such that $\mathrm{Res}^{L_1 \wr S_n}_{L_1^{\times n}} \widetilde{\zeta^{\times n}} = \zeta^{\times n}$. Define $B := R \wr S_n$. Since the restriction of $\widetilde{\zeta^{\times n}}$ to $S^{\times n}$ is irreducible, we have that $\zeta' := \mathrm{Res}^{L_1 \wr S_n}_{G'} \widetilde{\zeta^{\times n}} \in \mathrm{Irr}(B[G'])$ extending $\eta^{\times n}$. Denote by δ' the factor set associated to $B[G']$. We show that under these assumptions we recover our original twisted group algebra.

By Lemma 6.20, $\beta^{\times n}$ is $\overline{G}$-invariant. Moreover we have that $\eta^{\times n}$ extends to both $\chi' \in \mathrm{Irr}(A)$ and $\zeta' \in \mathrm{Irr}(B[G'])$. Finally since $H^1(\overline{N}, \mathbb{C}^*) = 1$, as $H^1(S, \mathbb{C}^*) = 1$, Lemma 5.16 applies and $A \cong B[G']$. Note moreover that by the proof of Lemma 5.16 we see that $\delta' = \alpha \mathrm{Inf}(\mu)$ where $\mu \in B^2(G'/S^{\times n}, \mathbb{C}^*)$

Hence from [22, Chapter 2], using the remarks following Theorem 1.1 (taking the obstruction cocycle to be μ) and Theorem 3.3, we have that since $\mathrm{Res}^{L_1 \wr S_n}_{G'} \widetilde{\zeta^{\times n}} \in \mathrm{Irr}(B[G'])$ restricts (irreducibly) to $\gamma^{\times n}$, then $\chi' = \zeta' \mathrm{Inf}^{G'}_{G'/S^{\times n}} \lambda$, where λ is a (linear) projective μ^{-1}-character of $G'/S^{\times n}$. Hence by Lemma 5.26, the result follows provided $\zeta' \in \mathcal{A}_{p^a}(B[G'])$. However, by Theorem 5.37, Lemma 6.18 and its proof, we obtain that $\zeta' \in \mathcal{A}_{p^a}(B[G'])$. Hence $\chi' \in \mathcal{A}_{p^a}(A)$. Hence by Lemma 5.24, we have $\chi \in \mathcal{A}_{p^a}(G, Z)$.

6.3 A discussion of Condition 6.2

We now discuss the restrictions of Condition 6.2. First we prove a corollary that follows from Lemma 6.20. Note the reader is required to read Section 6.2 for the relevant notation.

Corollary 6.23. *β is L_1-invariant.*

Proof. If $l \in L_1$, then by definition there exists $\sigma \in S_n$ such that $\sigma(1) = 1$ and $l_2, \dots, l_n \in M$ such that $g := (l, l_2, \dots, l_n; \sigma) \in \overline{G}$. Let $(s, 1, \dots, 1)$ and $(t, 1, \dots, 1) \in S^{\times n}$. By Lemma 6.20 and that $\sigma(1) = 1$,

$$\left(\beta^{\times n}\right)^g ((s, 1, \dots, 1), (t, 1, \dots, 1)) = \beta^{\times n}((s, 1, \dots, 1), (t, 1, \dots, 1)).$$

Hence,

$$\beta^{\times n}\left(\left({}^{l}s,1,\ldots,1\right),\left({}^{l}t,1,\ldots,1\right)\right)=\beta({}^{l}s,{}^{l}t)=\beta(s,t).$$

Therefore $\beta^{l}(s,t)=\beta(s,t)$ and we are done since s, t are arbitrary. $\square$

Now, by [15, Theorem 2], we have that the following sequence

$$H^2\left(L_1/S,\mathbb{C}^*\right)\underset{l}{\rightarrow}H^2\left(L_1,\mathbb{C}^*\right)\underset{r}{\rightarrow}H^2\left(S,\mathbb{C}^*\right)^{L_1}\underset{t}{\rightarrow}H^3\left(L_1/S,\mathbb{C}^*\right)\rightarrow\ldots,$$

is exact and by Corollary 6.23, $\beta\in H^2(S,\mathbb{C}^*)^{L_1}$. To complete the proof we require $\gamma\in H^2(L_1,\mathbb{C}^*)$ such that $r(\gamma)=\mathrm{Res}_S^{L_1}\gamma=\beta$. By exactness this is equivalent to showing $t(\beta)=1$, i.e. $\beta\in\ker(t)$. By considering the simplest example, namely the case $L_1/S\cong C_2$, say $L_1=M_{10}$, $S=A_6$, we have that $H^2(L_1/S,\mathbb{C}^*)=1$, $H^2(L_1,\mathbb{C}^*)=C_3$, $H^2(S,\mathbb{C}^*)^{L_1}\leq C_6$ and $H^3(L_1/S,\mathbb{C}^*)=C_2$. Hence the sequence becomes,

$$1\underset{l}{\rightarrow}C_3\underset{r}{\rightarrow}C_6\underset{t}{\rightarrow}C_2\rightarrow\cdots.$$

Note that since $\mathrm{im}(r)$ is isomorphic to C_3, we know that $C_3\leq H^2(S,\mathbb{C}^*)^{L_1}\leq C_6$. Moreover, clearly the factor set of order 2 is fixed by the action of M_{10} (as the conjugate factor set wuld also have order 2), hence $C_2\leq H^2(S,\mathbb{C}^*)^{L_1}$ and $H^2(S,\mathbb{C}^*)^{L_1}=C_6$. Since $\beta\in H^2(S,\mathbb{C}^*)^{L_1}$ we have that it extends if the order of β is 3 (or as in the previous case, if it is the trivial factor set), but does not necessarily extend otherwise (i.e., the factor sets of order 2 or 6). Hence the Condition 6.2 cannot obviously be removed. It is the case that it may extend to L_1 for another reason, however from the above sequence alone it is not guaranteed. Therefore the removal of Condition 6.2 is an ongoing problem to be solved in the future.

CHAPTER 7

RING STRUCTURE AND DIMENSION

From this point we consider the case when a in Conjecture 2.5 is 1 i.e., we will be considering the structure of $\mathcal{A}_p(G)$. Recall that in Section 4.3, we have seen that it is enough to consider the structure of $\mathcal{A}_p(P)$ for an abelian p-group P. In this chapter, we will focus on the structure of $\mathcal{A}_p(P)$ when P is an elementary abelian p-group of small rank, explictly $C_p \times C_p$. Moreover, we focus more in depth on $p = 2$ for generic rank,, as in this case we will prove the following theorem.

Theorem 7.1. *Let G be a group and $P \in \mathrm{Syl}_2(G)$. Define $M := PC_G(P) = P \times F$ for some p'-group F. Let $\chi \in \mathrm{Irr}(G)$ such that 2 divides $\widetilde{\pi}_\theta \left(\mathrm{Res}^G_M \chi\right)(1)$ for all $\theta \in \mathrm{Irr}(F)$. If P/P' is elementary abelian, then $\chi \in \mathcal{A}_2(G)$.*

Remark 7.2. Note that in Theorem 7.1, if $PC_G(P) = P$ then $F = 1$, and the condition on χ becomes $\mathrm{Res}^G_P\chi(1)$ is divisible by 2, which holds for any irreducible character of G of even degree.

Using the above remark, we simplify the verification of many simple groups for the prime $p = 2$, in particular all the sporadic simple groups, see Chapter 9.

7.1 Dimension of the induced character ring for p-groups

We suppose now that P is an elementary abelian p-group and consider the structure of $\mathcal{A}_p(P)$. Suppose first that $P = C_p$. Here the description of $\mathcal{A}_p(P)$ is simple, namely $\mathcal{A}_p(P) = \mathbb{Z}[\rho_P]$, where ρ_P is the regular character of P. Hence $\dim(\mathcal{A}_p(P)) = 1$ as a $\mathbb{Z}$-module.

Hence, we continue under the assumption that $\operatorname{rank}(P) := n \geq 2$. Several of these results hold for generic n, however the main structual result is only known for $n = 2$. Therefore we split into two cases later in this section, one to complete the proof of $n = 2$ and the other to show some computational evidence for $n > 2$. Note that by Lemma 4.27, we have that $p\mathcal{C}(P) \subseteq \mathcal{A}_p(P) \subseteq \mathcal{C}(P)$. We denote by $A := \mathcal{C}(P)/(p.\mathcal{C}(P))$. Setting $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$, we also define $B := \mathbb{Z}_p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.

Proposition 7.3. *There exists an $\mathbb{Z}_p$-algebra isomorphism $\theta : A \to B$.*

Proof. Since P is elementary abelian, we have a group isomorphism between $\operatorname{Irr}(P)$ and P, sending $\lambda^{i_1} \times \cdots \times \lambda^{i_n}$ to $(g^{i_1}, \ldots, g^{i_n})$, where $1_P \neq \lambda \in \operatorname{Irr}(C_p)$ and $C_p = \langle g \rangle$. Extending $\mathbb{Z}$-linearly, we obtain a ring isomorphism between $\mathcal{C}(P)$ and $\mathbb{Z}[P]$. This then induces an $\mathbb{Z}_p$-algebra homomorphism, θ_1, between A and $\mathbb{Z}_p[P]$.

We define $g_i := (1, \ldots, 1, g, 1, \ldots, 1)$ where g appears in the i-th component. We have that B is isomorphic as a $\mathbb{Z}_p$-algebra to $\mathbb{Z}_p[P]$ sending $x_1^{i_1} \cdots x_n^{i_n} + (x_1^p, \ldots, x_n^p)$ to $(g_1 - 1)^{i_1} \cdots (g_n - 1)^{i_n}$ and extending $\mathbb{Z}_p$-linearly. We define this map to be θ_2. Hence defining $\theta := \theta_2^{-1} \circ \theta_1$, we have the desired isomorphism. $\square$

Remark 7.4. Note that for $\mu := \lambda^{i_1} \times \cdots \times \lambda^{i_n} \in A$, $\theta(\mu) = (x_1 + 1)^{i_1} \ldots (x_n + 1)^{i_n}$.

We define $K := (x_1, \ldots, x_n) \subset \mathbb{Z}_p[x_1, \ldots, x_n]$. Note that since $\mathcal{C}(G)$ has an inner product $\langle \cdot, \cdot \rangle : \mathcal{C}(G) \times \mathcal{C}(G) \to \mathbb{Z}$, it induces a symmetric non-degenerate $\mathbb{Z}_p$-valued bilinear form on A namely, $\langle \cdot, \cdot \rangle_p : A \times A \to \mathbb{Z}_p$, given by $\langle a, b \rangle_p := \langle a, b \rangle \bmod p$. Therefore, using Proposition 7.3, we have $(\cdot, \cdot) : B \times B \to \mathbb{Z}_p$ defined by $(a, b) := \langle \theta^{-1}(a), \theta^{-1}(b) \rangle_p$.

Moreover the conjugation map $\bar{\cdot} : A \to A$ sending λ to $\overline{\lambda}$ induces an automorphism $* : B \to B$.

Lemma 7.5. $x_i^* = \displaystyle\sum_{k=0}^{p-2} \binom{p-1}{k+1} x_i^{k+1}$.

Proof. Define $\lambda_i := 1 \times \cdots \times 1 \times \lambda \times 1 \times \cdots \times 1$ where $\lambda \neq 1_P \in \mathrm{Irr}(C_p)$ appears in the i-th component. Then,

$$x_i^* = \theta\left(\overline{\theta^{-1}(x_i)}\right) = \theta\left(\overline{\lambda_i - 1}\right) = \theta\left(\lambda_i^{p-1} - 1\right) = \theta\left(\lambda_i\right)^{p-1} - 1 = (x_i + 1)^{p-1} - 1,$$

and hence the result follows. □

Lemma 7.6. *Let a, b and $c \in B$, then $(ab, c) = (a, b^*c)$.*

Proof. Note that $\overline{\theta^{-1}(b)} = \theta^{-1}(b^*)$. Hence,

$$(ab, c) = \left\langle \theta^{-1}(a)\theta^{-1}(b), \theta^{-1}(c)\right\rangle_p = \left\langle \theta^{-1}(a), \overline{\theta^{-1}(b)}\theta^{-1}(c)\right\rangle_p$$

$$= \left\langle \theta^{-1}(a), \theta^{-1}(b^*c)\right\rangle = (a, b^*c).$$

□

Using the above, we have that for $a, b \in B$, $(a, b) = (1, a^*b)$. Note also that if $b = 0$, $(a, b) = 0$. Now suppose that a and b are homogeneous of degree l and m respectively, then if $l + m > n(p-1)$, $a^*b \equiv 0 \bmod (x_1^p, \ldots, x_n^p)$. This is since if a has degree l, Lemma 7.5 shows that each term of a^* has degree at least l, hence each term of a^*b has degree at least $l + m$ which implies that in each term of a^*b, there exists some $1 \leq i \leq n$ such that x_i appears with degree at least p, i.e. is 0 mod $(x_1^p, \ldots, x_n^p)$.

Hence, $(a, b) = (1, a^*b) = 0$, since any a and $b \in B$ are sums of homogenous polynomials. Define I to be the ideal of B generated by all monomials of degree $p-1$, and J to be the ideal of B generated by all monomials of degree at least $(n-1)p - (n-2)$. We define $\overline{T} := T/(x_1^p, \ldots, x_n^p)$ for an ideal T of $\mathbb{Z}_p[x_1, \ldots, x_n]$.

Lemma 7.7. $\overline{I}^\perp = \overline{J}$.

Proof. Let $a = x_1^{i_1} \dots x_n^{i_n} \in \overline{I}$ and $b = x_1^{j_1} \dots x_n^{j_n} \in \overline{J}$. Since $\sum_{k=1}^n i_k + j_k \geq np - (n-1) > np - n$, we have that $(a, b) = 0$. Now $\dim_{\mathbb{Z}_p}(\overline{I})$ is the number of $x_1^{i_1} \dots x_n^{i_n}$ with $i_1 + \ldots + i_n \geq p-1$. We note that this is the same number as the number of vectors $i \in \mathbb{Z}_p^n$, $i = (i_1, \ldots, i_n)$ with $\sum_{j=1}^n i_j \geq p-1$. Defining $C := \left\{ i \in \mathbb{Z}_p^n : 0 \leq i_1 + \ldots + i_n \leq p-2 \right\}$, then the number of such vectors is $p^n - |C|$.

Define $C_k = \left\{ i \in \mathbb{Z}_p^n : \sum_{j=1}^n i_j = k \right\}$. Then we have $|C_k| = \binom{k+n-1}{n-1}$. Using that C is the disjoint union of C_k with $0 \leq k \leq p-2$ we have

$$|C| = \sum_{k=0}^{p-2} \binom{k+n-1}{n-1} = \binom{((p-2)+n-1)+1}{n} = \binom{n+p-2}{n}.$$

Hence $\dim_{\mathbb{Z}_p}(\overline{I}) = p^n - \binom{n+p-2}{n}$. For the dimension of $\overline{J}$, we note that $i \in \mathbb{Z}_p^n$ satisfies $0 \leq \sum_{k=1}^n i_k \leq p-2$ if and only if for $j \in \mathbb{Z}_p^n$, with $j_k := (p-1) - i_k$, j satisfies $(n-1)p - (n-2) \leq \sum_{k=1}^n j_k \leq n(p-1)$. Using the correspondence above, relating the dimension to the number of such vectors, we have that the number of such i determines the dimension of $\overline{J}$, namely $\dim_{\mathbb{Z}_p}(\overline{J}) = |C|$. Hence $\dim_{\mathbb{Z}_p}\left(\overline{I}\right) + \dim_{\mathbb{Z}_p}\left(\overline{J}\right) = p^n - |C| + |C| = p^n = \dim_{\mathbb{Z}_p}(B)$. $\square$

Recall the definition of K, the maximal ideal of $\mathbb{Z}_p[x_1, \ldots, x_n]$. Clearly we have that B is a local ring, hence B has a unique maximal ideal, $J(B)$, the Jacobson Radical. However since $B/\overline{K} \cong \mathbb{Z}_p$, we have that $\overline{K}$ is maximal and hence $\overline{K} = J(B)$. Note also that $\overline{I} = \overline{K}^{p-1}$ and $\overline{J} = \overline{K}^{(n-1)p-(n-2)}$. Moreover, by Lemma 7.7, we have that $\overline{K}^{p-1}$ and $\overline{K}^{(n-1)p-(n-2)}$ are orthogonal complements. We now consider $L := \theta^{-1}(K)$. Note that $\overline{L}$ is a maximal ideal of A and $\left(\overline{L}^{(n-1)p+(n-2)}\right)^{\perp} = \overline{L}^{p-1}$.

Remark 7.8. Here by $\overline{I}^q$, we are denoting the ideal $I^q/(x_1^p, \ldots, x_n^p)$ or $I^q/p\mathcal{C}(P)$ depending on which ring we are working in.

Lemma 7.9. $\overline{L} = M := \langle 1_P - \mu : \mu \in \mathrm{Irr}(P) \rangle_{\mathbb{Z}_p}$.

Proof. Since $\overline{L} = J(A)$ and A is a local ring, this follows if we show that M is maximal also. We define $\psi : A \to \mathbb{Z}_p$ by

$$\psi\left(a_1\chi_1 + \cdots + a_{p^n}\chi_{p^n}\right) := a_1 + \cdots + a_{p^n}.$$

We claim that $M = \ker(\psi)$. We assume without loss that $\chi_1 = 1_P$. Note that

$$\ker(\psi) = \left\{\chi \in A : a_1 \equiv -(a_2 + \cdots + a_{p^n}) \bmod p\right\}.$$

Hence if $\chi \in \ker(\psi)$ and $\chi = a_1\chi_1 + \cdots + a_{p^n}\chi_{p^n}$, we have that

$$\chi \equiv -a_2(\chi_1 - \chi_2) + \cdots + -a_{p^n}(\chi_1 - \chi_{p^n}) \bmod p\,\mathcal{C}(P) \in M.$$

The reverse inclusion is clear and moreover ψ is clearly surjective. $\square$

Lemma 7.10. $\overline{\mathcal{A}_p(P)} \subseteq \overline{L}^{p-1}$.

Proof. Define $m := (n-1)p - (n-2)$. Let $\alpha \in \overline{L}^m$, then $\alpha = \alpha_1 \ldots \alpha_m$, with $\alpha_i \in \overline{L}$. Let $H < P$ be a maximal subgroup, namely $H \cong C_p^{n-1}$. Then $\mathrm{Res}^P_H\alpha = \mathrm{Res}^P_H\alpha_1 \ldots \mathrm{Res}^P_H\alpha_m$. Let M_H be the unique maximal ideal of $\mathcal{C}(H)/p\mathcal{C}(H)$, isomorphic to $(x_1, \ldots, x_{n-1})/\left(x_1^p, \ldots, x_{n-1}^p\right)$. Moreover, Lemma 7.9 can now be generalised to show that $M_H = \langle 1 - \mu : \mu \in \mathrm{Irr}(H)\rangle$. Hence if $\alpha \in \overline{L} = M$, $\mathrm{Res}^P_H\alpha_i \in M_H$, since $\mathrm{Res}^P_H\left(\mathrm{Irr}(P)\right) \subseteq \mathrm{Irr}(H)$ (as P is abelian) and $\mathrm{Res}^G_H\alpha \in M_H^m$. We claim that $M_H^m = 0$. Note that,

$$M_H^m = \left(x_1^{i_1} \ldots x_{n-1}^{i_{n-1}} : \sum_{j=1}^{n-1} i_j = m\right).$$

We choose an arbitrary generator and suppose for a contradiction that for all $1 \leq j \leq n-1$, $i_j \leq p-1$. Then $\sum_{j=1}^{n-1} i_j \leq (n-1)(p-1)$. Suppose additionally that $(n-1)(p-1) \geq m$, then $(n-1)(p-1) \geq (n-1)(p-1) + (n-1) - (n-2) = (n-1)(p-1) + 1$ which is a contradiction. Hence $(n-1)(p-1) < m$ which contradicts $\sum_{j=1}^{n-1} i_j = m$. Hence there exists $1 \leq j \leq n-1$ such that $i_j \geq p$ and $x_1^{i_1} \ldots x_n^{i_n} = 0$. Moreover, $M_H^m = 0$.

Therefore $\mathrm{Res}^G_H\alpha = 0$. If $\chi := \mathrm{Ind}^G_H\lambda \in \overline{\mathcal{A}_p(P)}$, where $\lambda \in \mathrm{Irr}(H)$ and $H < G$ is a maximal subgroup, then

$$\langle \chi, \alpha \rangle = \left\langle \lambda, \mathrm{Res}_H^G \alpha \right\rangle = \langle \lambda, 0 \rangle = 0.$$

Therefore since $\overline{\mathcal{A}_p(P)}$ is spanned by such χ, by linearity of induction and the inner product, every element of $\overline{L}^m$ is orthogonal to every element of $\overline{\mathcal{A}_p(P)}$, hence $\overline{L}^m \subseteq \overline{\mathcal{A}_p(P)}^{\perp}$, and therefore $\overline{\mathcal{A}_p(P)} \subseteq \left(\overline{L}^m\right)^{\perp} = \overline{L}^{p-1}$. $\square$

Definition 7.11. $S := \overline{\mathcal{A}_p(P)} + \overline{L}^m$.

Lemma 7.12. *The inner product $\langle ., . \rangle_S$ on $S/\overline{L}^m$, defined by $\langle a, b \rangle_S := \langle a, b \rangle_p$ for a, $b \in S$ is well-defined.*

Proof. If $\alpha_1, \alpha_2, \beta \in S$ such that $\alpha_1 + \overline{L}^m = \alpha_2 + \overline{L}^m$, then $\alpha_1 = \alpha_2 + \gamma$ for some $\gamma \in \overline{L}^m$. We show that $\langle \alpha_1, \beta \rangle_S = \langle \alpha_2, \beta \rangle_S$. We have that

$$\langle \alpha_1, \beta \rangle_S = \langle \alpha_2, \beta \rangle_S + \langle \gamma, \beta \rangle_S .$$

We claim that $\langle \gamma, \beta \rangle_S = 0$. Note that

$$\langle \gamma, \beta \rangle_S = \langle \gamma, \beta \rangle_p = (\theta(\gamma), \theta(\beta)) .$$

Now as $\beta \in S \subseteq \overline{L}^{p-1}$, we have that $\theta(\beta) \in \theta\left(\overline{L}^{p-1}\right) = \overline{K}^{p-1}$, hence has degree at least $p-1$. Similarly, $\theta(\gamma) \in \overline{K}^m$, hence has degree at least m. Since $(p-1)+m = np-(n-1) = np-n+1 = n(p-1)+1 > n(p-1)$, we have that $(\theta(\gamma), \theta(\beta)) = 0$. $\square$

Lemma 7.13. $\dim_{\mathbb{Z}_p}\left(\overline{L}^{p-1}/\,\overline{L}^m\right) = p^n - 2\binom{n+p-2}{n}$.

Proof. We can see that a basis for $\overline{L}^{p-1}/\,\overline{L}^m$ are the polynomials $x_1^{i_1}\dots x_n^{i_n}$ where $p-1 \leq \sum_{j=1}^n i_j \leq m-1$. By the proof of Lemma 7.7, we can see that there are $p^n - 2|C|$ such polynomials, hence yielding the result. $\square$

7.1.1 Specific case

We now specify the case $n = 2$. Here we have $m = p$.

Lemma 7.14. $S/\,\overline{L}^p = \overline{L}^{p-1}/\,\overline{L}^p$.

Proof. We have by Lemma 7.10, that the left hand side is contained in the right hand side. By Lemma 7.13, since $n = 2$, we have that $\dim_{\mathbb{Z}_p}\left(\overline{L}^{p-1}/\,\overline{L}^p\right) = p^2 - 2\binom{p}{2} = p$. Using Lemma 7.12, we have a well-defined inner product on $S/\,\overline{L}^p$. Consider the following set of characters,

$$\mathcal{I} := \left\{\mathrm{Ind}_H^P 1_H + p\mathcal{C}(P) \;:\; 1 < H < P \text{ is maximal}\right\} \subseteq \overline{\mathcal{A}_p(P)}.$$

By abuse of notation, we will identify $\mathrm{Ind}_H^P 1_H$ with its corresponding coset. We have

$$\left\langle \mathrm{Ind}_{H_1}^P 1_{H_1} + \overline{L}^m, \mathrm{Ind}_{H_2}^P 1_{H_2} + \overline{L}^m \right\rangle_S = \left\langle \mathrm{Ind}_{H_1}^P 1_{H_1}, \mathrm{Ind}_{H_2}^P 1_{H_2} \right\rangle_p = \begin{cases} 0 & \text{if } H_1 = H_2, \\ 1 & \text{if } H_1 \neq H_2. \end{cases}$$

If $H_1 = H_2 = H$ then $\left\langle \mathrm{Ind}_H^P 1_H, \mathrm{Ind}_H^P 1_H \right\rangle_p$ is the rank of the permutation representation on cosets of H, which in turn is the number of orbits of the stabiliser of a point i.e. equals p. Hence modulo p this is 0. If $H_1 \neq H_2$, then $H_1 \cap H_2 = \{1\}$ and

$$\left\langle \mathrm{Ind}_{H_1}^P 1_{H_1}, \mathrm{Ind}_{H_2}^P 1_{H_2} \right\rangle_p = \frac{1}{|P|}\mathrm{Ind}_{H_1}^P 1_{H_1}(1)\mathrm{Ind}_{H_2}^P 1_{H_2}(1) = \frac{1}{p^2}p^2 \equiv 1 \bmod p.$$

Therefore the Gram matrix of $\mathcal{I}$, $\mathbb{G} \in \mathrm{GL}_{p+1}(\mathbb{Z}_p)$, corresponding to this inner product is given by,

$$\mathbb{G} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

Let the characteristic polynomial corresponding to $\mathbb{G}$ be denoted by $\chi_{\mathbb{G}}(t)$. Moreover let

$\lambda := t+1$ and $\mathbb{1} := (a_{ij})$ with $a_{ij} = 1$ for $1 \leq i,j \leq p+1$. We therefore have

$$\chi_{\mathbb{G}}(t) = \det(\mathbb{G} - tI) = \det(\mathbb{1} - I - tI) = \det(\mathbb{1} - (t+1)I) = \det(\mathbb{1} - \lambda I) =$$

$$= \chi_{\mathbb{1}}(\lambda) = (\lambda - p - 1)\lambda^p = (t-p)(t+1)^p = t(t+1)^p.$$

Hence $\operatorname{rank}(\mathbb{G}) = p$ and $p = \dim_{\mathbb{Z}_p}(\mathcal{I}) \leq \dim_{\mathbb{Z}_p}\left(S/\,\overline{L}^p\right)$. Therefore the result follows since

$$p \leq \dim_{\mathbb{Z}_p}\left(S/\,\overline{L}^p\right) \leq \dim_{\mathbb{Z}_p}\left(\overline{L}^{p-1}/\,\overline{L}^p\right) = p.$$

□

Lemma 7.15. $\overline{\mathcal{A}_p(P)} = \overline{L}^{p-1}$.

Proof. We have by Lemma 7.14 that $S = \overline{L}^{p-1}$. Hence $\overline{\mathcal{A}_p(P)} + \overline{L}^p = \overline{L}^{p-1}$. Now treating $\overline{L}^{p-1}$ as an A-module, $J(\overline{L}^{p-1}) = J(A)\overline{L}^{p-1} = \overline{L}^p$. Hence by Nakayama's Lemma, [24, Lemma 4.2], $\overline{\mathcal{A}_p(P)} = \overline{L}^{p-1}$. □

Theorem 7.16. $\mathcal{A}_p(P) \cong K^{p-1}$.

Proof. By Lemma 7.15, the Correspondence Theorem and the fact that θ is an isomorphism, we obtain the above result. □

Corollary 7.17. $\dim(C(P)/\mathcal{A}_p(P)) = \binom{p}{2} = p(p-1)/2$ *as an* $\mathbb{Z}_p$*-vector space.*

Proof. From the previous theorem, we have that $\dim(\mathcal{C}(P)/\mathcal{A}_p(P)) =$ $\dim(\mathbb{Z}_p[x_1,\ldots,x_n]/K^{p-1})$. Since $\overline{K}^{p-1} = I$, by the Third Isomorphism Theorem we have $\dim(\mathcal{C}(P)/\mathcal{A}_p(P) = \dim(B/\,\overline{I})$. Since $\overline{I}^{\perp} = \overline{J}$, $\dim(B/\,\overline{I}) = \dim(\overline{J})$ and the result follows from the proof of Lemma 7.7. □

7.1.2 Induced character ring for elementary abelian 2-groups

Recall the definition of $\mathcal{C}_2(G)$ from Definition 2.1(vii). In this section we give an alternative proof that $\dim(C(P)/\mathcal{A}_p(P)) = 1$ when P is an elementary abelian 2-group, i.e. $\mathcal{C}_2(P) =$

$\mathcal{A}_2(P)$ for such P. Using this fact, combined with the results from the previous section we prove Theorem 7.1. Let $P := \underbrace{C_2 \times \cdots \times C_2}_{n}$, where $n \geq 2$. We note that the proof strategy we apply in this section is constructive, in the sense that it can be generalised to produce a GAP code to determine the invariant factors of $\mathcal{C}(P)/\mathcal{A}_p(P)$ for arbitrary elementary abelian p-groups.

We denote $\mathrm{Irr}(P) = \{\chi_k : 1 \leq k \leq 2^n\}$. Note that there are $2^n - 1$ maximal subgroups of P, each of which is isomorphic to C_2^{n-1}. We denote these subgroups by H_i with $i \in \{1, \ldots, 2^n - 1\}$. Moreover, for $j \in \{0, \ldots, 2^{n-1} - 1\}$, we denote by θ_{ij}, the j-th irreducible of H_i. For any appropriate i and j, we have

$$\left\langle \mathrm{Ind}_{H_i}^P \theta_{ij}, \mathrm{Ind}_{H_i}^P \theta_{ij} \right\rangle = \left\langle \theta_{ij}, \mathrm{Res}_{H_i}^P \mathrm{Ind}_{H_i}^P \theta_{ij} \right\rangle = \langle \theta_{ij}, 2\theta_{ij} \rangle = 2.$$

Therefore every induced character is a sum of two irreducibles of G. Let $m = 2^{n-1}$. Form the matrix $A = (a_{r,s})$, where $a_{m(i-1)i+j,k} := \left\langle \mathrm{Ind}_{H_i}^P \theta_{ij}, \chi_k \right\rangle$. Note that A has $(2^n - 1)(2^{n-1}) = \binom{2^n}{2}$ rows. Therefore the rows of A are all permutations of $(1, 1, 0, \ldots, 0)$. Denote by $b_{r,s}$, the permutation of $(1, 1, 0, \ldots, 0)$ which has 1 in the r-th and s-th position. Note that $b_{1,m}, \ldots, b_{m-1,m}$ is clearly a set of linearly independent rows of A. Moreover, for any $s \neq m$, $c_{r,s} := -b_{r,s} + b_{r,m} + b_{s,m} = (0, \ldots, 0, 2)$. Therefore we can choose a basis of $\mathcal{C}(G)/\mathcal{A}_2(G)$ to be the induced characters (modulo $\mathcal{A}_2(G)$) corresponding to $b_{1,m}, \ldots, b_{m-1,m}, c_{r,s}$ for a fixed r and s, with $s \neq m$.

Furthermore, the matrix consisting of these rows is in Smith Normal Form, therefore the diagonal entries correspond to the invariant factors from the structure theorem for finitely generated modules over $\mathbb{Z}$, [24, Chapter 3, Section 7], hence we have that $\mathcal{C}(G)/\mathcal{A}_2(G) \cong \mathbb{Z}/2\mathbb{Z}$ (since the matrix has a single 2 on the diagonal). Note that $\mathcal{A}_2(G) \subseteq \mathcal{C}_2(G) \subseteq \mathcal{C}(G)$. Since $\mathcal{C}_2(G) \neq \mathcal{C}(G)$, we therefore have that $\mathcal{A}_2(G) = \mathcal{C}_2(G)$, however this also holds since an $\mathbb{F}_2$-basis of $\mathcal{C}(G)/\mathcal{C}_2(G)$ is $\{\mathcal{C}_2(G), 1_G + \mathcal{C}_2(G)\}$.

7.1.3 Proof of Theorem 7.1

Now the proof of Theorem 7.1 is immediate. By the proof of Proposition 4.21, it is sufficient to show $\mathrm{Res}^G_M \chi \in \mathcal{A}_2(M)$. By Lemmas 4.28 and 4.29, we need to show that for all $\theta \in \mathrm{Irr}(F)$, $\psi_\theta := \mathrm{Def}^P_{P/P'} \left(\widetilde{\pi}_L \left(\widetilde{\pi}_\theta \left(\mathrm{Res}^G_M(\chi) \right) \right) \right) \in \mathcal{A}_2(G)$. Since for all $\theta \in \mathrm{Irr}(F)$, 2 divides $\widetilde{\pi}_\theta \left(\mathrm{Res}^G_M \chi \right)(1)$, 2 divides $\psi_\theta(1)$ for all $\theta \in \mathrm{Irr}(F)$. Since $\psi_\theta \in \mathcal{C}_2(G)$ for all $\theta \in \mathrm{Irr}(F)$, $\psi_\theta \in \mathcal{A}_2(G)$ by the final paragraph of Section 7.1.2 and we are done.

7.1.4 Computational evidence for $n > 2$

Here we simply list some computational evidence that the rank of $\mathcal{C}(P)/\mathcal{A}_p(P)$ is $\binom{n+p-2}{n}$ in the case $n > 2$, $p > 2$. The algorithm we use is detailed in the proof used in the case $p = 2$, previous section. The code for this algorithm can be found in Appendix A.

For $p = 3$, the code for the function `CPModApP` shows us that the rank of $\mathcal{C}(P)/\mathcal{A}_3(P)$ is 4, 5 and 6, for $n = 3$, $n = 4$ and $n = 5$ respectively.

For $p = 5$, the code for the function `CPModApP` shows us that the rank of $\mathcal{C}(P)/\mathcal{A}_5(P)$ is 20 for $n = 3$.

For $p = 7$, the code for the function `CPModApP` shows us that the rank of $\mathcal{C}(P)/\mathcal{A}_7(P)$ is 56 for $n = 3$.

CHAPTER 8

RESULTS FOR SYMMETRIC GROUPS

In this chapter, we continue with the assumption that $a = 1$ and prove that Conjecture 2.5 holds for the symmetric groups in this case. The necessary results for this proof can be found in Section 5.1 and [10].

Theorem 8.1. *Suppose $G = S_n$ with $n \in \mathbb{N}$. If $\chi \in \mathrm{Irr}(G)$ such that p divides $\chi(1)$, where p is prime, then $\chi \in \mathcal{A}_p(G)$.*

Proof. We proceed by induction on n, namely if $\chi \in \mathrm{Irr}(S_n)$ such that p divides $\chi(1)$, then $\chi \in \mathcal{A}_p(S_n)$. Clearly the base case, $n = 1$, is trivial. Assume the inductive hypothesis holds for any $k < n$. Let w, $e \in \mathbb{Z}_{\geq 0}$ with $e < p$ be such that $n = pw + e$.

As $\chi \in \mathrm{Irr}(G)$, we have $\chi = \chi^\lambda$ for some partition $\lambda \vdash pw + e$. We denote by B the block of $\mathrm{Irr}(G)$ containing χ. Recall $B = B_{w',\rho}$ is labelled by a p-weight, w', and a p-core, ρ, with $|\rho| = e'$. This follows from [21, Theorem 6.1.21]. Note ρ here is the p-core of λ.

Using Lemma 5.61, we may assume, without loss of generality, that $D := D(B)$, the defect group of the block B, is a Sylow p-subgroup of G, i.e. B is a block of maximal defect. Moreover, using Lemma 3.14, we may assume $P \leq N_G(P) \leq S_p \wr S_w \times S_e \leq G$. We define $L = S_p \wr S_w$. We now claim λ has p-weight $w' = w$, and $|\rho| = e' = e$.

Suppose for a contradiction that $w' < w$. Then by [21, Theorem 6.2.45], D is isomorphic to the Sylow p-subgroup of $S_{pw'}$. However, $v_p((pw')!) = w' + v_p(w'!) < w + v_p(w!)$, which contradicts B having maximal defect. Now suppose $w' > w$. Since $pw+e = pw'+e'$, we have $e' = p(w - w') + e < 0$ as $e < p$. Hence $w' = w$ and since $pw + e = pw' + e' =$

$pw + e'$, we have $e' = e$. Note now by the proof of Proposition 4.21, we need only show $\mathrm{Res}^G_{P.C_G(P)}\chi \in \mathcal{A}_p(G)$. Furthermore using Lemma 4.8, applied in the case $G = L \times S_e$, $H = G$, $K = PC_G(P)$ and $N = 1$, this follows if we show $\mathrm{Res}^G_{L\times S_e}\chi \in \mathcal{A}_p(L \times S_e)$. Consider the following,

$$\mathrm{Res}^G_{S_{pw}\times S_e}\chi = \sum_{\sigma\vdash e} \zeta_\sigma \times \chi^\sigma, \tag{8.1}$$

where $\zeta_\sigma \in \mathcal{C}(S_{pw})$ and $\chi^\sigma \in \mathrm{Irr}(S_e)$, for each $\sigma \vdash e$. Upon restriction to $L \times S_e$ we obtain, from (8.1),

$$\begin{aligned}\mathrm{Res}^G_{L\times S_e}\chi &= \mathrm{Res}^{S_{pw}\times S_e}_{L\times S_e}\left(\sum_{\sigma\vdash e} \zeta_\sigma \times \chi^\sigma\right)\\ &= \sum_{\substack{\sigma\vdash e\\ \sigma\neq\rho}} \left(\left(\mathrm{Res}^{S_{pw}}_L \zeta_\sigma\right) \times \chi^\sigma\right) + \left(\left(\mathrm{Res}^{S_{pw}}_L \zeta_\rho\right) \times \chi^\rho\right). \end{aligned} \tag{8.2}$$

We begin by considering the term labelled by ρ. Using the notation of $\tilde{\pi}_\rho$, $F_{p,w,\rho}$ and $\mathrm{Irr}_{\mathrm{pri}}(L)$ introduced in [10], we have by (8.1) that $\tilde{\pi}_\rho\left(\mathrm{Res}^G_{S_{pw}\times S_e}\chi\right) = \zeta_\rho$. We denote by $\theta := F_{p,w,\rho}(\chi) \in \mathrm{Irr}_{\mathrm{pri}}(L)$. Using [10, Theorem 3.7] we have,

$$\theta \equiv \mathrm{Res}^{S_{pw}}_L \zeta_\rho \mod \kappa_w.$$

Using Lemma 4.2 and Lemma 5.1, we have $\kappa_w \subseteq \mathcal{I}\left(L, P, \mathcal{S}(G, P, L)\right) \subseteq \mathcal{A}_p(L)$. Hence $\theta = \mathrm{Res}^{S_{pw}}_L \zeta_\rho + \psi$, where $\psi \in \mathcal{A}_p(L)$. Moreover, using [10, Proposition 3.10], the signed bijection $F_{p,w,\rho}$ preserves heights. Therefore since $\mathrm{ht}(\chi) > 0$, we have that $\mathrm{ht}(\theta) > 0$ and therefore $\theta(1) \equiv 0 \bmod p$. Hence

$$\left(\mathrm{Res}^{S_{pw}}_L \zeta_\rho\right) \times \chi^\rho = (\theta \times \chi^\rho) - (\psi \times \chi^\rho), \tag{8.3}$$

where $\psi \times \chi^\rho \in \mathcal{A}_p\left(L \times S_e\right)$ and $\theta \times \chi^\rho \in \mathrm{Irr}^p(L \times S_e)$.

We now consider $\theta \in \mathrm{Irr}^p(L)$, and show that $\theta \in \mathcal{A}_p(L)$. Given a partition $\tau \vdash w$, by $S_p \wr S_\tau$ we denote the subgroup $\prod_i S_p \wr S_{\tau_i}$, where τ_i denotes the i-th part of τ. Using [21, Theorem 4.3.34] we have that θ decomposes as

$$\theta = \mathrm{Ind}_{S_p \wr S_\tau}^{S_p \wr S_w} \left(\prod_i \widetilde{\phi_i^{\times \tau_i}} \cdot \mathrm{Inf}_{S_{\tau_i}}^{S_p \wr S_{\tau_i}} \alpha_i \right),$$

where $\phi_i \in \mathrm{Irr}(S_p)$, $\alpha_i \in \mathrm{Irr}(S_{\tau_i})$ and $\tau \vdash w$. We may assume p does not divide $|S_w : S_\tau|$, as otherwise $\theta \in \mathcal{A}_p(L)$. Since p divides $\theta(1)$, there exists an i such that p divides the degree of $\theta_i := \widetilde{\phi_i^{\times \tau_i}} \cdot \mathrm{Inf}_{S_{\tau_i}}^{S_p \wr S_{\tau_i}} \alpha_i$.

By Lemma 3.16 and transitivity of induction, we have $\theta \in \mathcal{A}_p(L)$ if $\theta_i \in \mathcal{A}_p(S_p \wr S_{\tau_i})$. Hence, without loss of generality, we may assume $\tau = (w)$ and

$$\theta = \widetilde{\phi^{\times w}} \cdot \mathrm{Inf}_{S_w}^{L} \alpha,$$

where $\phi \in \mathrm{Irr}(S_p)$ and $\alpha \in \mathrm{Irr}(S_w)$. Now by [27, Theorem (9.26)], we have that the block B' of L containing θ covers a block b' of $S_p^{\times w}$ with defect group $C_p^{\times w}$, since by Lemma 5.61 we can assume B' has maximal defect. Hence by [27, Theorem (9.2)], $\phi^{\times w}$, the irreducible constituent of $\mathrm{Res}_{S_p^{\times w}}^{L} \theta$ lies in b'. Hence ϕ lies in a block of S_p of maximal defect, i.e. the principal block of S_p and hence has degree prime to p. Hence since p divides $\theta(1)$, p divides $\alpha(1)$. Moreover, it is enough to show $\alpha \in \mathcal{A}_p(S_w)$, by Lemma 4.25 and Corollary 4.24.

However since $w < pw + e = n^{\dagger}$, we may apply the inductive hypothesis to $\alpha \in S_w$, namely $\alpha \in \mathcal{A}_p(S_w)$. Therefore $\theta \in \mathcal{A}_p(L)$ by the above considerations. Moreover, using (8.3), we have $\left(\mathrm{Res}_{L}^{S_{pw}} \zeta_\rho\right) \times \chi^\rho \in \mathcal{A}_p(L \times S_e)$.

Now we consider the terms in (8.2) labelled by $\sigma \vdash e$, $\sigma \neq \rho$. Recall that $N_G(P) \leq L \times S_e$, therefore the Brauer correspondent of B is defined. Let b be the block of $\mathrm{Irr}(L \times S_e)$ such that $b^G = B$ (see [27, Problem (4.3)]). Moreover, we have $b = c \otimes d$ for some block c of L and d of S_e. Note that since b has defect group P (this follows from the bijection given by Brauer's First Main Theorem) and S_e is a p'-group that c is a block of L of maximal defect. Therefore $c = c_0$, the principal block of $\mathrm{Irr}(L)$, by the proof of [10, Theorem 1.3].

†As if $w = 0$, $S_n = S_e$ would have no characters of degree divisible by p.

Furthermore, by Nakayama's Conjecture (see [21, 6.1.21]) we have that blocks of S_e are labelled by their p-cores and p-weights. Since $e < p$, each $\sigma \vdash e$ is a p-core partition of p-weight 0, i.e. $d = d_{\sigma,0}$ for some $\sigma \vdash e$. Since χ has p-core ρ, we have $b = c_0 \otimes d_{\rho,0}$. Moreover, as the Brauer Correspondence is a bijection, for any other block e of $L \times S_e$, with $e \neq b$, we have that $e^G \neq B$. Let e be a block of $L \times S_e$ with $e = c \otimes d_{\sigma,0}$ where $\sigma \vdash e$, $\sigma \neq \rho$. Using the map Proj_e, as defined in [11, Section 2.2], we have from (8.2)

$$\mathrm{Proj}_e(\mathrm{Res}^G_{L\times S_e}\chi) = \left(\mathrm{Res}^{S_{pw}}_L \zeta_\sigma\right) \times \chi^\sigma.$$

Finally by [11, Proposition 2.11 (i)], we have

$$\mathrm{Proj}_e\left(\mathrm{Res}^G_{L\times S_e}\chi\right) \in \mathcal{I}\left(L \times S_e, P, \mathcal{S}(G, P, L \times S_e)\right) \subseteq \mathcal{A}_p(L \times S_e).$$

Combining our results we have $\mathrm{Res}^G_{L\times S_e}\chi \in \mathcal{A}_p(L \times S_e)$ and we are done. $\square$

8.1 Alternating groups

In this section we discuss the progress that we have made towards the verification of Conjecture 2.11 for alternating groups. We have that for odd primes the conjecture follows from Theorem 8.1 provided the partition labelling the character of A_n is not self-conjugate and for $p = 2$, Conjecture 2.5 holds as the abelianisations of the Sylow 2-subgroups of A_n are elementary abelian.

By [21, Theorem 2.5.7], we have that if $\left\{\chi^\lambda : \lambda \vdash n\right\}$ is the labelling of the irreducible characters of S_n, and if $\lambda \neq \lambda'$, i.e. λ is not a self-conjugate partition then $\mathrm{Res}^{S_n}_{A_n}\chi^\lambda \in \mathrm{Irr}(A_n)$.

Corollary 8.2. *Suppose that $\lambda \vdash n$ such that $\lambda \neq \lambda'$ and p divides $\chi^\lambda(1)$ where p is odd, then $\mathrm{Res}^{S_n}_{A_n}\chi^\lambda \in \mathcal{A}_p(A_n)$.*

Proof. By Theorem 8.1, we have that $\chi^\lambda \in \mathcal{A}_p(S_n)$. Define $H := S_n$ and $K := A_n$ then by Lemma 4.8, applied in the case $H = S_n$, $K = A_n$ and $N = 1$, $\mathrm{Res}^{S_n}_{A_n}\chi^\lambda \in \mathcal{A}_{p/h}(A_n)$ where $h = (|S_n : A_n|, p) = 1$. $\square$

We now show that the Sylow 2-subgroups of A_n for $n \geq 4$ have elementary abelian, abelianisations. This will be proved in two steps, first showing this holds for the Sylow 2-subgroups of S_n for $n \geq 2$.

Proposition 8.3. *Let $P \in \mathrm{Syl}_2(S_n)$ with $n \geq 2$, then P/P' is elementary abelian.*

Proof. By [21, 4.1.20] and [21, 4.1.22], P is isomorphic to the direct product of elementary abelian 2-groups with wreath products of cyclic 2-groups. Hence since the abelianisation of a direct product is the direct product of the abelianisations, we need only consider the abelianisations of iterated wreath products. Suppose that $P := \underbrace{C_2 \wr C_2 \wr \cdots \wr C_2}_{i}$ where $i \geq 1$. Clearly this holds when $i = 1$. Hence we consider the wreath product of i copies of C_2. Note that this follows by induction and Lemma 3.13 since,

$$\underbrace{C_2 \wr C_2 \wr \cdots \wr C_2}_{i} \cong \underbrace{(C_2 \wr \cdots \wr C_2)}_{i-1} \rtimes C_2,$$

using that direct products and quotients of elementary abelian groups are elementary abelian. $\square$

Proposition 8.4. *If $Q \in \mathrm{Syl}_2(A_n)$ with $n \geq 4$, then Q/Q' is elementary abelian.*

Proof. Let $m \in \mathbb{Z}_{\geq 0}$. First, by [21, 4.1.20] and [21, 4.1.22], if $P \in \mathrm{Syl}_2(S_{4m+1})$ or $P \in \mathrm{Syl}_2(S_{4m+3})$ then P can be viewed as a Sylow 2-subgroup of S_{4m} or S_{4m+2} respectively. Hence this holds also for Sylow 2-subgroups of A_n. Suppose that $Q \in \mathrm{Syl}_2(A_{4m+2})$. Since S_{4m} can be viewed as a subgroup of A_{4m+2} and

$$|A_{4m+2} : S_{4m}| = (2m+1)(4m+1),$$

is odd, $Q \in \mathrm{Syl}_2(S_{4m})$ and the result follows from Proposition 8.3. Hence we are reduced to considering $Q \in \mathrm{Syl}_2(A_{4m})$. Consider the subgroup

$$S_2 \wr S_{2m} \cong \langle (12), \ldots, (4m-1\ 4m) \rangle \rtimes S_{2m} \leq S_{4m}.$$

Since $v_2\left((4m)!\right) = v_2\left(2^{2m}(2m)!\right)$, we have $|S_{4m} : S_2 \wr S_{2m}|$ is odd. Hence we may take Q to be the index 2 subgroup $S_2^{2m-1} \rtimes R$ where $R \in \mathrm{Syl}_2(S_{2m})$. The result now follows from Lemma 3.13 and Proposition 8.3. □

We now prove a further lemma required to show that Conjecture 2.5 holds for A_n, $n \geq 4$ and $p = 2$.

Lemma 8.5. *Let $G = A_n$ where $n \geq 4$. If $Q \in \mathrm{Syl}_2(G)$, then $QC_G(Q) = Q$.*

Proof. Suppose that $x \in C_G(Q)$ is an element odd prime order, say p. Let a be the number of p-cycles in x and b be such that $n = ap + b$. We show that $C_{S_n}(x)$ is not contained in A_n, as this gives us that $C_{S_n}(x)$ is not a subgroup of Q, which gives that $x \notin C_{S_n}(Q)$. Moreover this gives is that $x \notin C_G(Q)$. We do this by showing that $C_{S_n}(x)$ contains an odd element of S_n. Note first that $C_{S_n}(x) \cong C_p \wr S_a \times S_b$.

If $a \geq 2$, i.e. x has at least two p-cycles, then there is an odd transposition, σ, of S_n swapping these cycles, so that $\sigma \in C_{S_n}(x)$. Hence without loss of generality, $a = 1$. Moreover if $b \geq 2$, then S_b contains an odd permutation, so $0 \leq b \leq 1$. Hence $C_{A_{p+b}}(x) \cong C_p \leq A_{p+b}$. Note that $|A_{p+b} : C_p|$ is even provided $p \geq 5$ for $b = 0$ and $p \geq 3$ for $b = 1$. However this holds since $n \geq 4$.

However, due to our assumption $Q \leq C_G(x)$, hence $|G : C_G(x)| = |A_{p+b} : C_p|$ is odd, which yields our contradiction. Hence $C_G(Q)$ is a 2-group and $QC_G(Q) = Q$. □

Theorem 8.6. *Let $G = A_n$ where $n \geq 5$. If $\chi \in \mathrm{Irr}(G)$ and $\chi(1)$ is even, then $\chi \in \mathcal{A}_2(G)$.*

Proof. Using Theorem 7.1, Remark 7.2, Proposition 8.4 and Lemma 8.5, the result is immediate. □

CHAPTER 9

SPORADIC GROUPS, THEIR AUTOMORPHISM GROUPS AND THEIR CENTRAL EXTENSIONS

From the definition of Brauer-good groups, we must check Definition 2.10 for groups G which are quasisimple or almost simple. In particular, we consider when $G/Z(G) \cong S$ or $S \leq G \leq \mathrm{Aut}(S)$ where S is a non-abelian sporadic simple group. For these 26 cases, we use both GAP and Magma programs to verify Definition 2.10. We divide this section into the verification for the underlying simple groups, then the almost simple groups and finally the central extensions of sporadic simple groups as we have different approaches in each case.

9.1 Simple sporadic groups

We work on simplifying the number of cases we are required to verify. First we use the reduction to p-groups found in Section 4.3 and in particular Theorem 7.1, to show we can exclude the case $p = 2$. First we address Remark 7.2.

Proposition 9.1. *Let S be a sporadic group and let $P \in \mathrm{Syl}_p(S)$, for a prime p, be non-abelian. Then unless $S = \mathrm{J}_4$ and $p = 3$, $PC_S(P) = P$.*

Proof. We aim to show that $C_S(P)$ is a p-group. First, we list the orders of $C_S(P)$ that can be verified using MAGMA for odd primes.

S	p	$\|C_S(P)\|$
M_{12}	3	3
J_2	3	3
HS	5	5
M_{24}	3	3
McL	3	3
McL	5	5
He	3	3
He	7	7
Ru	3	3
Ru	5	5

S	p	$\|C_S(P)\|$
Suz	3	9
ON	7	7
Co_3	3	3
Co_3	5	5
Co_2	3	3
Co_2	5	5
Fi_{22}	3	3
Fi_{23}	3	3
Co_1	3	3
Co_1	5	5

Second we note that for all $x \in P$, $C_S(P) \leq C_S(x)$. Hence it is sufficient to find a conjugacy class of p-elements whose centraliser is a p-group. We verify this using the Atlas and display the results below. We denote by C_S the desired conjugacy class of S, if one exists.

S	p	C_S	$\|C_S(x)\|$ for $x \in C_S$
J_3	3	$9A$	27
HN	3	$9A$	27
HN	5	$25A$	25
Ly	5	$25A$	25
Th	3	$27A$	27
Fi_{24}'	3	$27A$	81
B	3	$27A$	27
B	5	$25A$	25
M	3	$27B$	243

For the following cases, we use the fact that $x \in C_S(P)$ if and only if $P \leq C_S(x)$. Therefore, if $q \neq p$ is prime and x is a q-element such that P is not a subgroup of $C_S(x)$ then $x \notin C_S(P)$. As an example, we consider Ly and $p = 3$. We note that if x is a $9A$-element, then $|C_S(x)| = 54$. We show that no elements in $C_S(P)$ have order 2, namely 2 cannot divide $|C_S(P)|$, hence $|C_S(P)|$ divides 27.

Suppose for a contradiction that there exists $x \in C_S(P)$ where $o(x) = 2$. Therefore by the above, $P \leq C_S(x)$. However, x is a $2A$-element and $v_3\left(|C_S(x)|\right) = 4$, hence P is not a subgroup of $C_S(x)$, which is a contradiction. The below table displays the group S, the prime p, the class of p elements we are interested in (i.e. the 9A class in the Ly example), the centraliser order of the class, the primes, q, we are trying to rule out, the set of possible p-adic valuations of the centraliser orders of the classes of q elements(i.e. the 2A class in Ly example) and the p-adic valuation is must be less than.

Th	3	$9A$	54	{2}	{4}	10
Th	5	$5A$	3000	{2,3}	{0,1}	3
Fi_{24}'	7	$7B$	2058	{2,3}	{0,1}	3
M	5	$25A$	250	{2}	{4,6}	9
M	7	$7B$	84707	{2,3,5}	{1,2,3}	6
M	13	$13B$	52728	{2,3}	{1}	3
ON	3	$3A$	3240	{2,5}	{2}	4
J_4	11	$11B$	242	{2}	{1}	3

Using the above techniques, we now exhibit the results for $p = 2$.

S	C_S	$\lvert C_S(x)\rvert$ for $x \in C_S$
M_{11}	$8A$	8
M_{12}	$8A$	8
M_{22}	$8A$	8
M_{23}	$8A$	8
M_{24}	$8A$	16
HS	$8A$	16
J_2	$8A$	8
Co_1	$16A$	64
Co_2	$16A$	32
Co_3	$8C$	32
McL	$8A$	8

Suz	$8C$	32
He	$8A$	16
HN	$8B$	64
Fi_{22}	$16A$	32
Fi_{23}	$16A$	32
Fi_{24}'	$16A$	32
B	$16G$	256
M	$16C$	8192
ON	$8A$	32
J_3	$8A$	8
Ru	$8C$	32
J_4	$16A$	32

For the remaining cases, one can verify in MAGMA that if $P \in \mathrm{Syl}_2(\mathrm{J}_1)$ then $|C_G(P)| = 8$, for Th one can check that if x is an $8B$-element that $|C_G(x)| = 96$, but P cannot be contained in the centraliser of any 3-element and similarly for Ly. $\square$

The above shows that in particular for $p = 2$, that if $P \in \mathrm{Syl}_2(S)$ for a sporadic simple group S, then Theorem 7.1 holds provided P/P' is elementary abelian. The following result gives us this.

Lemma 9.2. *Let S be a sporadic simple group and $P \in \mathrm{Syl}_2(S)$. Then $P' = \Phi(P)$.*

Proof. The first table in Appendix C, shows that we have verified in MAGMA that the derived subgroup coincides with the Frattini subgroup of P. $\square$

As stated prior to Lemma 9.2, using Theorem 7.1 and Remark 7.2, Proposition 9.1 and the above give us that Definition 2.10 holds for sporadic simple groups and $p = 2$.

Using the above result, we now have that to verify Definition 2.10 for simple sporadic groups and odd primes. For now we exclude the case when $S = \mathrm{J}_4$ and $p = 3$. By the proof of Proposition 4.21 and Proposition 9.1, we need only restrict characters of degree divisible by p to the Sylow p-subgroup. We use the function `ExTestCharacter(P,chi,M)` from Appendix A.

This function takes inputs `P`, which will be a Sylow p-subgroup of S, `chi` which is the restriction of an irreducible character from `P` to S of degree divisible by p, and `M` which is any set of subgroups of `P` of index divisible by p. We take `M` to be `MaximalSubgroups(P)`, but to save computational time we could take a different set here. The code works as follows.

Given the list of subgroups `M`, we take each subgroup and add to the list `L2`, all the induced characters from these subgroups. We also introduce a counter, `j` which counts the total number of induced characters. The variable `c`, in this loop can be altered to improve computation time. The value currently assigned to `c` allows the for loop to run over all irreducible characters of each subgroup in `M`, however this value could be changed.

We then produce the array `MR` $= (m_{ij})$, where m_{ij} is the inner product of the i-th induced character[†], with the j-th irreducible character of `P`. `M1` is then the corresponding matrix described by the array `MR`.

In a similar fashion, we produce the 1-dimensional array `v` $= (v_i)$, where v_i is the inner product of `chi`, with the i-th irreducible of `P`. `v1` is therefore the corresponding vector described by the array `v`. `sol`, then computes whether the system `M1`$x =$ `v1` has a solution x, i.e. whether our vector `v` is in the $\mathbb{Z}$-span of the rowspace of the matrix `M1`. As we have forced the matrices to be computed over `Z` $= \mathbb{Z}$, `sol` should be a vector in $\mathbb{Z}$ also. Since the vector `v1` is non-zero, we also have that if a solution is outputted it will be non-zero.

[†]The order here is with respect to the default ordering of the subgroups in `M` and its irreducible characters by MAGMA, this may obviously differ per person, but this simply leads to a permutation of the rows.

`Falses` is an array which will count the number of entries of `sol` which are not integers, just to check that the computation is as I have claimed above, outputting true if all entries are integral and false otherwise.

We now must simply create a simple for loop which runs over all irreducible characters of the group S. However the highlighted boxes in the table of results, Appendix C, shows that the computation is not always this simple. Problems occur when the character table of the group S cannot be constructed in MAGMA. In this case, we build the restricted character of the Sylow p-subgroup independently, but to do this we must compute how the conjugacy classes of S split on restriction to P.

Note moreover that the results from Chapter 7 tell us that we can reduce the problem to one on the abelianisation of P. Clearly, we are able still able to run `ExTestCharacter` on inputs P/P' and M where M is now the list of maximal subgroups of P/P', however we must express our restricted character χ, as $\chi_1 + \chi_2$ where every irreducible consituent of χ_1 is linear, and then deflate this generalised character to P/P' to input this into `ExTestCharacter`.

To do so, we need only use the `LiftCharacter` function in MAGMA and extend this $\mathbb{Z}$-linearly to determine the deflation of χ_1 to P/P'. This procedure is one that we use to test the function as it makes a significant difference to the computation time, in comparison to the other suggestions made above.

For the remaining case, $G = \mathrm{J}_4$ and $p = 3$, we simply run `ExTestCharacter` over all characters of degree divisible by 3 and the other inputs are taken to be $PC_G(P)$ and M is a list of subgroups, maximal subject to the condition that 3 divides their index in S. One can check that there are 44 such characters. Clearly $|PC_G(P)| = 54$ and one finds that the elements in T are $2A$, $3A$, $6A$ and $6B$ elements. From here it is sufficient to build a character of $PC_G(P)$ directly to reduce computation time. Our results can be summarized by the following theorem. Note that M and $p = 3$ is not necessarily a counterexample. The current code requires further refinement to reduce computation cost (in time and memory) before we can succesfully verify the group.

Theorem 9.3. *Let S be a simple sporadic group and $\chi \in \mathrm{Irr}(S)$ of degree divisible by p, a prime then, unless $S = M$ and $p = 3$, $\chi \in \mathcal{A}_p(G)$.*

Proof. The results of the computation are displayed in Table 1 of Appendix C. □

9.1.1 Computing the splitting of conjugacy classes

We now discuss the procedure to determine the splitting of the conjugacy classes between S and P. We will do this through a series of examples. First we describe the computation for the Baby Monster, B, and $p = 5$. We use the representation of B as a group of 4370×4370 matrices over $GF(2)$ using the online Atlas generators. We begin by reducing computation time by constructing a maximal subgroup of B which contains $\mathtt{P} \in \mathrm{Syl}_5(S)$. We choose $\mathtt{H} := 5^{1+4}.2^{1+4}.A_5.4$. We determine a list, `XH5` of conjugacy class representatives of order 5 and 25.

Then for each representative x in `XH5`, we determine which representatives of `P` are conjugate to x in `H`, i.e., determining the splitting of the conjugacy classes between `H` and `P`. Our task now is to build a representation of `P`, hence we must must use the Atlas and this requires us to determine distinguishing properties between the conjugacy classes of elements of order 5 and 25 in S. Considering the $5A$ and $5B$ elements of S, we note that elements of S of order 70 power into $5A$, hence we take a random element y of S of order 70 and consider the dimension of the fixed space (1-eigenspace) of y^{14}. For $5A$ the dimension of the fixed space is 890. For B, since there are only 2 classes of elements of order 5 and there exists an element of G of order 5 where dimension of the fixed space is 870, i.e. distinct from 890, we are able to say that the dimension of the fixed space of $5B$ elements is 870. Note we are using that the dimension of the fixed spaces of conjugate (similar) matrices is the same. We discuss after this example how we deal with the case when these have no obvious distinguishing properties. Since there is only one class of elements of order 25 it is clear which value the classes of 25 elements take.

Finally, now this has been computed we are able to determine the splitting of the conjugacy classes between S and P so we are able to run over the list of values of representations of S on the identity, $5A$, $5B$ and $25A$ classes and build characters of P with these values using the `ClassFunctionSpace(S)` function in MAGMA. Note that we compute the list of values using the stored tables in GAP (for simplicity). We can then implement the procedure described above for `ExTestCharacter`.

Now we discuss the procedure we use when we aren't able to distinguish the two classes obviously. This in practice rarely happens, so for example take the case of the Monster, M and $p = 7$. We note that another problem to discuss arises here given that we are not able to build M to test the fixed space dimensions, but this will be discussed following this. We take H to be the maximal subgroup, $7^{1+4} : (3 \times 2S_7)$. We note that H has 7 classes of elements of order 7, which we need to determine which are $7A$ and $7B$. We run over all possible assignments of these 7 classes as $7A$ and $7B$ elements and check whether the assignment is correct by building the character of $P \in \mathrm{Syl}_7(S)$ and testing whether it is an actual character, i.e. the inner product with each irreducible is a non-negative integer. In practice this only occurs for one possible combination of the $7A$ and $7B$ elements. Hence we have the splitting of the conjugacy classes.

For the Monster since we are only able to build maximal subgroups, we have to use the assignment procedure described above since we are not necessarily able to find elements of the group of certain orders to determine distinguishing factors between the conjugacy classes. This in practice causes an (obvious) increase in computation time and this is why in the table (Appendix C), the Monster and $p = 3$ is not complete as this code is still being checked. In Appendix B, I give the fusion information for the conjugacy classes of p-elements I was required to compute. In most cases this will be a list of unique distinguishing features of these classes in S.

9.2 Almost simple sporadic groups

Let $S < G \leq \mathrm{Aut}(S)$, where S is a sporadic simple group. Note that from the Atlas, since S is sporadic, $|G : S| = 2$. We therefore begin by considering what happens for odd primes p. To this end, let p be an odd prime and $P \in \mathrm{Syl}_p(G)$. Let $\chi \in \mathrm{Irr}(G)$ such that p divides $\chi(1)$. Using Corollary 4.7, we need to show that for all $E \in \mathrm{El}_{p,1}(G, S, 1)$, that $\mathrm{Res}^G_E \chi \in \mathcal{A}_p(E, E \cap S, 1)$. To save computing all such elementary subgroups, we determine another set of subgroups, which is easier to construct, to which we restrict.

Consider $E \in \mathrm{El}_{p,1}(G, S, 1)$ and recall that p does not divide $|S : E \cap S|$. Let $R \in \mathrm{Syl}_p(E)$. By Lemma 3.9, $E \leq T := R.C_G(R)$. Moreover, by Lemma 4.8, if $\mathrm{Res}^G_T \chi \in \mathcal{A}_p(T, T \cap S, 1)$ then $\mathrm{Res}^G_E \chi \in \mathcal{A}_{p/h}(E, E \cap S, 1)$, where $h := (|T \cap S : E \cap S|, p)$. Since p does not divide $|S : E \cap S|$ we have that $h = 1$. This shows that that in order to determine whether $\mathrm{Res}^G_E \chi \in \mathcal{A}_p(E, E \cap S, 1)$, we need only restrict to $R.C_G(R)$ for some p-group R. We determine which p-group we require. Note that the following lemma also determines which groups we require for $p = 2$.

Lemma 9.4. *Let G be almost simple with underlying sporadic simple group S. Let $E \in \mathrm{El}_{p,1}(G, S, 1)$ and $Q \in \mathrm{Syl}_p(S) \leq P \in \mathrm{Syl}_p(G)$, where p is prime. Let $R \in \mathrm{Syl}_p(E)$. Then,*

(i) If p is even then, $Q^s \leq R \leq P^h$ for some $s \in S$ and $h \in G$.

(ii) If p is odd then, $R = P^s$ for some $s \in S$.

Proof. Let $L \in \mathrm{Syl}_p(S \cap E)$ such that $L \leq R$. Since p does not divide $|S : S \cap E|$, there exists some $s \in S$ such that $L = Q^s$. Since $E \leq G$, there exists $h \in G$ such that $R \leq P^h$. Therefore, $Q^s \leq R \leq P^h$, proving (i). For (ii), since p is odd, p does not divide $|P^h : Q^s|$, hence $Q^s = R = P^h$. Finally since $P = Q$ for odd p, $R = P^s$. $\square$

Using the above, we have that for odd primes we need only verify the conjecture for the restriction of our irreducible characters to $P^s C_G(P^s) \cong P C_G(P)$. We now show that under this assumption, the result for odd primes follows from the results from the previous section on the underlying simple group.

Theorem 9.5. *Let N be a normal subgroup of a group G such that p does not divide $|G : N|$. Let $P \in \mathrm{Syl}_p(N)$. Suppose that Conjecture 2.6 holds for N. If $\chi \in \mathrm{Irr}(G)$ with $\chi(1)$ divisible by p, then $\mathrm{Res}^G_P \chi \in \mathcal{A}_p(P)$.*

Proof. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of $\mathrm{Res}^G_N \chi$. Denote the conjugates of θ in G by $\theta = \theta_1, \ldots, \theta_t$ where $t = |G : I_G(\theta)|$. By Clifford's Theorem we have,

$$\mathrm{Res}^G_N \chi = e \sum_{i=1}^{t} \theta_i.$$

Since e divides $|G : N|$, we have that p does not divide e. Moreover, $\chi(1) = et\theta_i(1)$. Suppose for a contradiction that p divides t. Since $N \leq I_G(\theta)$ we have that p divides $|G : N|$, which is a contradiction. Hence $(t, p) = 1$. Hence we must have that p divides $\theta_i(1)$ for all i. Since Conjecture 2.6 holds for N, $\theta_i \in \mathcal{A}_p(N, Z(N)) \subseteq \mathcal{A}_p(N)$ for all i. Therefore $\mathrm{Res}^G_N \chi \in \mathcal{A}_p(N)$. Finally, we have that $\mathrm{Res}^G_P \chi \in \mathcal{A}_{p/h}(P)$ by Lemma 4.8, taking $G = P$, $H = N$, $N = 1$, $K = P$, where $h = (|N : P|, p) = 1$. Hence $\mathrm{Res}^G_P \chi \in \mathcal{A}_p(P)$. □

Corollary 9.6. *Let $S < G \leq \mathrm{Aut}(S)$ where S is a sporadic simple group and $P \in \mathrm{Syl}_p(S)$ for an odd prime p. Suppose that $T := P.C_G(P) = P$. Then $\mathrm{Res}^G_T \chi \in \mathcal{A}_p(T, T \cap S, 1)$.*

Proof. Recall that $|G : S| = 2$. Hence for an odd prime p, p does not divide $|G : S|$, namely $P \in \mathrm{Syl}_p(S)$. Moreover, using Appendix C, we have that Conjecture 2.6 holds for S. Hence using Theorem 9.5 and that $\mathcal{A}_p(T, T \cap S, 1) = \mathcal{A}_p(P, P \cap S, 1) = \mathcal{A}_p(P, P, 1) = \mathcal{A}_p(P)$, we are done. □

Now we see that the condition $PC_G(P) = P$ is not too restrictive for almost simple sporadic groups, namely we note that there is one exception to the condition that $PC_G(P) = P$, that being He : 2 and the prime $p = 3$, as in this case $P < P.C_G(P)$.

Proposition 9.7. *Let $S < G \leq \mathrm{Aut}(S)$ and p be a prime such that p^3 divides $|G|$. Let $P \in \mathrm{Syl}_p(G)$, then unless $G = \mathrm{He} : 2$ and $p = 3$, $PC_G(P) = P$.*

Proof. This can be checked using MAGMA or the Atlas using the techniques in Proposition 9.1. □

The above two results show that for odd primes and $G \neq \mathrm{He}:2$, we are done. However we are required to test Conjecture 2.6 for He : 2. Recall that we are required to show for $\chi \in \mathrm{Irr}(\mathrm{He}:2)$, such that 3 divides $\chi(1)$, that $\mathrm{Res}^G_T\chi \in \mathcal{A}_p(T, T\cap S, 1)$. Since $S \trianglelefteq G$,

$$\frac{T}{T\cap S} \cong \frac{TS}{S} \leq \frac{G}{S},$$

so p does not divide $|T : T\cap S|$. Therefore using Lemma 4.26, we need only show that $\mathrm{Res}^G_T\chi \in \mathcal{A}_p(T)$. Before we prove our key result on odd primes, namely Theorem 9.9, we note that in Proposition 9.7, we do not consider what happens when $v_p(|G|) \leq 2$, i.e. the Sylow p-subgroup is abelian. We show that this case follows from Theorem 5.60.

Corollary 9.8. *Let $S < G \leq \mathrm{Aut}(S)$ where S is sporadic. Let p be an odd prime, and p divide $\chi(1)$ for $\chi \in \mathrm{Irr}(B)$ for some block B. Suppose that $D(B)$, the defect group of B, is abelian. Then $\chi \in \mathcal{A}_p(G, S, 1)$.*

Proof. Note that as p does not divide $|G : S|$, then by Lemma 4.26, we need to show $\chi \in \mathcal{A}_p(G)$. However, this follows from Theorem 5.60. □

Hence we have our first main result.

Theorem 9.9. *Let $S < G \leq \mathrm{Aut}(S)$ where S is sporadic and let p be an odd prime. Let $\chi \in \mathrm{Irr}(G)$ such that p divides $\chi(1)$, then $\chi \in \mathcal{A}_p(G, S, 1)$.*

Proof. From all the previous discussion, we only need to show for $p = 3$ and $G = \mathrm{He}:2$, for all $\chi \in \mathrm{Irr}(G)$ such that 3 divides $\chi(1)$, that $\mathrm{Res}^G_T\chi \in \mathcal{A}_3(T)$. This can be verified using a refined version of `ExTestCharacter` in MAGMA. One can check that there are 27 such characters. □

Now we consider what happens if $p = 2$. From Lemma 9.4, we must restrict the irreducibles of even degree to both $PC_G(P)$ and $QC_G(Q)$ where $Q \in \mathrm{Syl}_2(S)$ and $P \in \mathrm{Syl}_2(G)$.

Lemma 9.10. *Let $S < G \leq \mathrm{Aut}(S)$ and $Q \in \mathrm{Syl}_2(S)$. Suppose that $QC_G(Q) = Q$. Then $\mathrm{Res}^G_Q\chi \in \mathcal{A}_2(Q, Q\cap S, 1) = \mathcal{A}_2(Q)$.*

Proof. Recall the definition of $\mathcal{C}_2(Q)$ from Definition 2.1. Moreover since $Q \in \mathrm{Syl}_2(S)$ we have by Lemma 9.2 that Q/Q' is elementary abelian, hence by Theorem 7.1 and Remark 7.2, $\mathrm{Res}_Q^G \chi \in \mathcal{C}_2(Q) = \mathcal{A}_2(Q)$. □

Now we see that the condition $QC_G(Q) = Q$ is not restrictive. Our aim in the proof of Lemma 9.12 is to show that if $G = S : 2$, the outer automorphism (of order 2) cannot centralise a subgroup of the Sylow 2-subgroup of S. The proof will simply give a list of subgroups which, using the following facts, are sufficient to prove this claim.

Remark 9.11. Let $G = S : 2$ and K be a complement to S in G. Then,

(i) Suppose there exists $H \leq S$ such that H is an abelian 2-group, then if $K \leq C_G(H)$, HK is abelian. If there exists a subgroup of S containing an abelian 2-subgroup H such that HK is non-abelian then K is not a subgroup of $C_G(H)$.

(ii) In addition to (i), if there exists a quotient of a subgroup $H \leq S$, which is an abelian 2-group such that QK is non-abelian then K is not a subgroup of $C_G(H)$, as if $L \in \mathrm{Syl}_2(H)$ and $K \leq C_G(H)$ then K centralises any quotient of L.

(iii) If there are 2 (or more) conjugacy classes of 2-elements of S that fuse in G, then K cannot centralise these elements. Let $x,\ y \in S$ be representatives of distinct conjugacy classes which fuse in G, namely ${}^g x = y$ for some $g \in G$. Since $G = SK$, we have that $g = sk$ for some $s \in S$ and $k \in K$. If $K \leq C_G(x)$ then $y = {}^{sk}x = {}^s x$, which is a contradiction to the choice of x and y. Hence K cannot centralise x.

Lemma 9.12. *If $S < G \leq \mathrm{Aut}(S)$ where S is a sporadic group and $Q \in \mathrm{Syl}_2(S)$. Then $QC_G(Q) = Q$.*

Proof. The subgroups we state here are from the Atlas, [8].

- Define $G := \mathrm{M}_{12} : 2$. Let K denote the complement to M_{12} in G. Suppose that $K \leq C_G(Q)$. Using [8], there exists a maximal subgroup of M_{12} isomorphic to $A_4 \times S_3$. Note that $K \cap Q = \{1\}$ and that $(A_4 \times S_3)\, K = S_4 \times S_3$ is a maximal subgroup of G.

  In particular for $C_2 \times C_2 \leq A_4$, $(C_2 \times C_2)\, K \leq S_4$. By assumption, $K \leq\leq C_G(Q) \leq C_G(C_2 \times C_2)$ and $(C_2 \times C_2)\, K$ is abelian. Moreover, $(C_2 \times C_2)\, K \in \mathrm{Syl}_2(S_4)$ but $D_8 \in \mathrm{Syl}_2(S_4)$ is non-abelian. Therefore using Remark 9.11 (i), we have a contradiction. The remaining cases will follow the same proof so will be less explicit.

- For $\mathrm{M}_{22} : 2$, we use (i), applied to $C_2 \times C_2 \leq L_2(11)$ in S which becomes $D_8 \leq L_2(11) : 2$ in G.

- For $\mathrm{HS} : 2$, we use (i), applied to M_{22} in S which becomes $\mathrm{M}_{22} : 2$ (above) in G.

- For $\mathrm{J}_2 : 2$, we use (i), applied to $C_2 \times C_2 \leq A_5$ in S which becomes $D_8 \leq S_5$ in G.

- For $\mathrm{McL} : 2$, we use (ii) applied to $2.A_8$ in S which becomes $2.S_8$ in G. Note that one can verify that $QC_G(Q) = Q$ in MAGMA for $Q \in \mathrm{Syl}_2(A_8)$ and $G = S_8$ to deduce that K cannot centralise Q.

- For $\mathrm{Suz} : 2$, we use (ii) and (i), applied to $2^{2+8}.(A_5 \times S_3)$ in S which becomes $2^{2+8}.(S_5 \times S_3)$ in G, noting that A_5 becomes S_5 as a subgroup of the quotient.

- For $\mathrm{He} : 2$, we use (ii) and (i), applied to $5^2 : 4A_4$ in S which becomes $5^2 : 4S_4$ in G.

- For $\mathrm{HN} : 2$, we use (ii) and (i), applied to $3^{1+4} : 4A_5$ in S which becomes $3^{1+4} : 4S_5$ in G.

- For $\mathrm{Fi}_{22} : 2$, we use (i), applied to M_{12} in S which becomes $\mathrm{M}_{12} : 2$ in G using the above case on M_{12}.

- For $\mathrm{Fi}_{24}' : 2$, we use (i) applied to $\big(A_4 \times O_8^+(2) : 3\big) : 2$ in S which becomes $\big(S_4 \times O_8^+(2)\big) : S_3$ in G.

- For ON : 2, we use (iii) and the fact there are 4 classes of 16 elements in S but only 2 in G, giving that these classes must fuse in G.
- For $\mathrm{J}_3 : 2$, we use (ii) and (i), applied to $2^{1+4} : A_5$ in S which becomes $2^{1+4} : S_5$ in G.

$\square$

We are therefore left to show that for $T := PC_G(P)$, that $\mathrm{Res}^G_T\chi \in \mathcal{A}_p(T, T \cap S, 1)$.

Corollary 9.13. *Let $S < G \leq \mathrm{Aut}(S)$ for a sporadic group S and $P \in \mathrm{Syl}_2(G)$, then $PC_G(P) = P$.*

Proof. This follows directly from Lemma 9.12, since for $Q \in \mathrm{Syl}_2(S)$ such that $Q \leq P$, $C_G(P) \leq C_G(Q)$ is a 2-group. $\square$

Therefore using this we must show that $\mathrm{Res}^G_P\chi \in \mathcal{A}_2(P, P \cap S, 1)$. Choosing $Q \in \mathrm{Syl}_2(S)$ such that $Q \leq P$, we have that $Q = P \cap S$. Hence we must show that $\mathrm{Res}^G_P\chi \in \mathcal{A}_2(P, Q, 1)$. We need to determine the subgroups $L \leq P$ which are maximal subject to the condition that 2 divides $|Q : Q \cap L|$.

Lemma 9.14. *Let $G = \mathrm{Aut}(S)$ where S is a sporadic simple group, $Q \in \mathrm{Syl}_2(S)$ and $P \in \mathrm{Syl}_2(G)$ such that $Q < P$. If $L \leq P$ is such that $Q \cap L$ is maximal in Q then either*

- *L is a maximal subgroup of P,*
- *L is a maximal subgroup of Q.*

Proof. Since $Q \cap L$ is maximal in Q, we have that $|Q : Q \cap L| = 2$. So either $L \leq Q$, in which case L is maximal in Q, or L is not a subgroup of Q, so $|L : L \cap Q| = |LQ : Q| = 2$ and therefore $|P : L| = 2$, i.e. L is maximal in P. $\square$

Therefore we obtain our second key result. Note that $\mathrm{Fi}_{24}' : 2$ is not necessarily a counterexample, but the current computation cost (in time and memory) to verify it is too large and requires refinement before running.

Theorem 9.15. *Let $S \leq G \leq \mathrm{Aut}(S)$ where S is sporadic. If $\chi \in \mathrm{Irr}(G)$ is of even degree, then unless $G = \mathrm{Fi}_{24}' : 2$, $\chi \in \mathcal{A}_2(G, S, 1)$.*

Proof. The reader is referred to the second table in Appendix C for the verification using the MAGMA function BrauerAlmostSimple(G). □

9.3 Central extensions of sporadic groups

In this section, we consider the results of testing Conjecture 2.12 for quasisimple groups where the related simple group is sporadic. First we note that by Theorem 5.60, we have that if p does not divide the order of $Z = Z(G)$ then we are done in the case of abelian defect. These results can be seen in the third table in Appendix C denoted by AD. We also note the following generalisation of Theorem 7.1 to rule out certain groups for $p = 2$.

Proposition 9.16. *Let G be a group and $Z := Z(G)$. Let $P \in \mathrm{Syl}_2(G)$. If the following conditions hold then $\chi \in \mathcal{A}_2(G, Z)$.*

(i) $PC_G(P) = P$.

(ii) $Z \leq P'$.

(iii) $\Phi(P) = P'$, *i.e. the abelianisation of P is elementary abelian.*

Proof. Using Corollary 4.7, we need to show that for all $E \in \mathrm{El}_{2,1}(G, G, Z)$ that $\mathrm{Res}^G_{EZ} \in \mathcal{A}_{2^b}(EZ, EZ \cap G, Z)$. First note that $b = b(E, Z, G) = 1 - v_2(|G : EZ|)$. Since $E \in \mathrm{El}_{2,1}(G, G, Z)$ we have that 2 does not divide $|G : EZ|$, hence $b = 1$. Therefore we are required to show that $\mathrm{Res}^G_{EZ}\chi \in \mathcal{A}_2(EZ, Z)$.

Recall that if $E \in \mathrm{El}_{2,1}(G, G, Z)$ then $E = R \times Q$ where R is a 2-group and Q is a $2'$-group. Hence since Z is a 2-group, $EZ = RZ \times Q$ and moreover as $|G : EZ|$ is odd, there exists $g \in G$ such that $RZ = P^g$. Therefore, we can assume $EZ \leq PC_G(P) = P$ without loss. Hence we show that $\mathrm{Res}^G_P\chi \in \mathcal{A}_2(P, Z)$.

Using Lemma 4.25, taking $H = P$ and $K = P'$, we see that if $\mathrm{Def}^{P}_{P/P'}\left(\widetilde{\pi}_L\left(\mathrm{Res}^G_P\chi\right)\right) \in \mathcal{A}_2(P/P')$, then $\widetilde{\pi}_L\left(\mathrm{Res}^G_P\chi\right) \in \mathcal{A}_2(P, P')$. In addition to this, since $Z \leq P'$, $\widetilde{\pi}_L\left(\mathrm{Res}^G_P\chi\right) \in \mathcal{A}_2(P, Z)$. Since P/P' is elementary abelian and 2 divides $\widetilde{\pi}_L\left(\mathrm{Res}^G_P\chi\right)(1)$, we may use the results of Section 7.1.2 to give us that indeed $\widetilde{\pi}_L\left(\mathrm{Res}^G_P\chi\right) \in \mathcal{A}_2(P/P')$.

Finally, using Proposition 3.32 and the proof of Lemma 4.29, setting $\zeta := \mathrm{Res}^G_P\chi - \widetilde{\pi}_L\left(\mathrm{Res}^G_P\chi\right)$, we have $\zeta \in \mathcal{A}_2(P, Z)$ since $Z \leq Z(P) \leq H_\theta$, for each irreducible constituent θ of $\mathrm{Res}^G_P\chi$ of even degree. □

The characters that are verified using this approach are denoted by 9.16 in the table in Appendix C. We also note that this yields a computational simplification for primes other than 2, for groups satisfying these three conditions, for example $3\mathrm{Fi}'_{24}$ and $p = 3$.

For the remaining characters, we use the function BrauerQuasisimple(G,p) found in Appendix A. The results of this function can be found also in the third table in Appendix C, where the number denotes the number of irreducible characters of G of degree divisible by p. We note that the exceptions to the theorem do not occur as they are counterexamples, simply because the code requires further refinement to reduce the memory cost and computation time to verify these groups. We denote by $\mathcal{Z} := \{(6\mathrm{Fi}_{22}, 2), (6\mathrm{Fi}_{22}, 3), (3\mathrm{Fi}'_{24}, 2), (3\mathrm{Fi}'_{24}, 3), (3\mathrm{Fi}'_{24}, 7), (2\mathrm{B}, 2), (2\mathrm{B}, 3), (2\mathrm{B}, 5)\}$ to be the set of such unverified cases. Note that 2B has not been verified since [8] (online) does not provide generators for it.

Theorem 9.17. *Let G be a group such that G is quasisimple and G/Z is a sporadic simple group. Let $\chi \in \mathrm{Irr}(G)$ of degree divisible by p, a prime. If $(G, p) \notin \mathcal{Z}$ then $\chi \in \mathcal{A}_p(G, Z)$.*

Proof. The reader is referred to the third table in Appendix C for the verification using the MAGMA function BrauerQuasisimple(G,p). □

APPENDIX A

CODE

First we exhibit the code for the function ExTestCharacter.

```
ExTestCharacter:=function(P,chi,M)
local XP,b,i,m,j,Sub,Xm,c,L2,theta,psi,v,MR,R,M1,v1,sol,num,M4,Z;
num:=0;
R:=[];
Z:=IntegerRing();
MR:=[];
L2:=[];
XP:=CharacterTable(P);
b:=#XP;
j:=0;
        for m in M do
        Sub:=m`subgroup;
        Xm:=CharacterTable(Sub);
        c:=#Xm;
                for i in [1..c] do
                j:=j+1;
                Append(~L2,Induction(Xm[i],P));
```

```
                end for;
        end for;
        for theta in L2 do
                for psi in XP do
                Append(~R,InnerProduct(theta,psi));
                end for;
        Append(~MR,R);
        R:=[];
        end for;
M1:=Matrix(Z,j,b,MR);
v:=[];
        for psi in XP do
        Append(~v,InnerProduct(chi,psi));
        end for;
v1:=Matrix(Z,1,b,v);
sol:=[];
sol:=Solution(M1,v1);
Falses:=[];
        for i in [1..NumberOfColumns(sol)] do
                if not sol[1][i] in Z then
                Append(~Falses,1);
                end if;
        end for;
        if #Falses eq 0 then
        return true;
        else
        return false;
        end if;
```

```
end function;;
```

Second, we exhibit the code used to verify Theorem 9.15.

```
BrauerAlmostSimple:=function(G)
local S,Q,P,x,M,M2,m1,Chars,XG,DG,i,j,k,ResChars,L,T;
S:=MinimalNormalSubgroups(G)[1];
P:=SylowSubgroup(G,2);
Q:=P meet S;
        if Index(P,Q) eq 2 then
        M:=MaximalSubgroups(P);
        M2:=MaximalSubgroups(Q);
        M1:=[];
        L:=0;
                for i in [1..#M] do
                T:=Q meet M[i]`subgroup;
                        if 2 in Divisors(Index(Q,T)) then
                        Append(~M1,M[i]);
                        else
                        L:=L+1;
                        end if;
                end for;
                if L gt 0 then
                        for i in [1..#M2] do
                        Append(~M1,M2[i]);
                        end for;
                end if;
        Chars:=[];
```

```
        XG:=CharacterTable(G);
        DG:=CharacterDegrees(G);
        i:=0;
                for j in [1..#DG] do
                k:=DG[j][2];
                        for l in [1..k] do
                        i:=i+1;
                                if 2 in Divisors(DG[j][1]) then
                                Append(~Chars,XG[i]);
                                end if;
                        end for;
                end for;
        ResChars:=[];
                for i in [1..#Chars] do
                Append(~ResChars,Restriction(Chars[i],P));
                end for;
        print(#ResChars);
                for chi in ResChars do
                ExTestCharacter(P,chi,M);
                end for;
        else
        print("Index False");
        end if;
return true;
end function;;
```

The following is code used in Section 9.3 to verify the quasisimple cases.

```
BrauerQuasisimple:=function(G,p)
local P,CGP,T,Chars,XG,DG,i,j,k,l,ResChars,M1,Z,M2,M,Subs,M3,Results;
P:=SylowSubgroup(G,p);
CGP:=centraliser(G,P);
T:=ProductSubgroups(G,P,CGP);
Chars:=[];
XG:=CharacterTable(G);
DG:=CharacterDegrees(G);
i:=0;
        for j in [1..#DG] do
        k:=DG[j][2];
                for l in [1..k] do
                i:=i+1;
                        if p in Divisors(DG[j][1]) then
                        Append(~Chars,XG[i]);
                        end if;
                end for;
        end for;
ResChars:=[];
        for i in [1..#Chars] do
        Append(~ResChars,Restriction(Chars[i],T));
        end for;
print(#ResChars);
M1:=MaximalSubgroups(T);
Z:=Center(G);
M2:=[];
        for i in [1..#M1] do
                if Z subset M1[i]`subgroup then
```

```
                Append(~M2,M1[i]);
                end if;
        end for;
M:=[];
Subs:=[];
j:=1;
M3:=[];
        for i in [1..#M2] do
                if p in Divisors(Index(T,M2[i]`subgroup)) then
                Append(~M,M2[i]);
                else
                Append(~M3,M2[i]);
                end if;
        end for;
        while #M3 gt 0 do
                for i in [1..#M3] do
                M1:=MaximalSubgroups(M3[i]`subgroup);
                        for i in [1..#M1] do
                        Append(~Subs,M1[i]);
                        end for;
                end for;
        M3:=[];
                if p in Divisors(Index(T,Subs[i]`subgroup)) then
                Append(~M,Subs[i]);
                else
                Append(~M3,Subs[i]);
                end if;
        end while;
```

```
Results:=[];
        for chi in ResChars do
        Append(~Results,ExTestCharacter(T,chi,M));
        end for;
return Results;
end function;;
```

The next code exhibited relates to Section 7.1.4, for computing the rank of $\mathcal{C}(P)/\mathcal{A}_p(P)$. The code outputs the dimension of the described matrix, which will be in the form p^r, where r is the required rank. We also note that the following function is a GAP code.

```
CPModApP:=function(p,n)
local H,i,G,L,S,Q,theta,T,M,chi,M1,R,N,m,psi,XH;
H:=CyclicGroup(p);
m:=n-1;
        for i in [1..m] do
        G:=DirectProduct(H,CyclicGroup(p));
        H:=G;
        od;
L:=MaximalSubgroups(H);
S:=[];
XH:=CharacterTable(H);
        for Q in L do
                for theta in Irr(CharacterTable(Q)) do
                Add(S,InducedClassFunction(theta,XH));
                od;
         od;
T:=[];
```

```
M:=[];
        for psi in S do
                for chi in Irr(XH) do
                Add(T,ScalarProduct(psi,chi));
                od;
        Add(M,T);
        T:=[];
        od;
M1:=HermiteNormalFormIntegerMat(M);
M:=[];
        for R in M1 do
                if ForAll(R,IsZero) = false then
                Add(M,R);
                else
                continue;
                fi;
        od;
N:=SmithNormalFormIntegerMat(M);
PrintFactorsInt(Determinant(N));
end;;
```

APPENDIX B

CONJUGACY CLASS FUSION IN SYLOW P-SUBGROUPS OF SPORADIC GROUPS

In this appendix, we detail distinguishing features of the conjugacy classes of p-elements of sporadic simple groups G. This is used in Chapter 9 to build the restriction of irreducible characters of G to $P \in \mathrm{Syl}_p(G)$ without having to build the character table of G itself. The information will be presented in tables, giving the name of the group, the representation of it that we use, the desired prime and the classes of p-elements corresponding to said prime and where possible a distinguishing feature of that class. Note that in some cases the classes are very hard to distinguish, so in the computation we would run the assignment procedure that we described for the Monster over the remaining classes.

If we are using a matrix representation for G, then a number in the box related to a conjugacy class refers to the dimension of the 1-eigenspace of an element from the designated conjugacy class.

Group	Prime	Matrix Representation	$3A$	$3B$	$9A$
HN	3	132×132 matrices over GF(4)	48	42	14

Group	Prime	Matrix Representation	$5A$	$5B$	$5C$	$5D$	$5E$	$25A$	$25B$
HN	5	132×132 matrices over GF(4)	20	32	*	*	28	6	6

For the above group, we note that the $5C$ and $5D$ classes both have dimension 26. However running the following code gives the eigenvalues, EG, over GF(16) which distinguishes the 2 classes. It is not clear however which class is which. Hence if we use this information we must assume it is one of the two classes and see which yields a contradiction. Note also that for our purposes, it does not matter which class is $25A$ or $25B$, since all characters of HN of degree divisible by 5 take the same value on these classes.

```
for v in GF(16) do
Append(~t,v);
end for;
v:=t[3];
EG:={<v^3,29>,<v^12,29>,<v^6,24>,<1,26>,<v^9,24>};
```

Group	Prime	Matrix Representation	$3A$	$3B$	$9A$
Ly	3	111×111 matrices over GF(5)	21	39	13

Group	Prime	Matrix Representation	$5A$	$5B$	$25A$
Ly	5	111×111 matrices over GF(5)	25	23	5

Group	Prime	Matrix Representation	$3A$	$3B$	$3C$	$9A$	$9B$	$9C$	$27A$	$27B$	$27C$
Th	5	248×248 matrices over GF(2)	92	86	80	32	26	30	10	8	8

Above, we note again that since the values of the characters of degree divisible by 3 are the same on the $27B$ and $27C$ classes, we need no further information for our purposes.

Group	Prime	Matrix Representation	$11A$	$11B$
J_4	11	112×112 matrices over GF(2)	2	12

For Fi_{24}' (below) and $p = 3$, we use the representation of Fi_{24}' as a subgroup of S_{306936}.

Class	Cycle Structure
$3A$	$[<3, 101232>, <1, 3240>]$
$3B$	$[<3, 101817>, <1, 1485>]$
$3C$	$[<3, 102186>, <1, 378>]$
$3D$	$[<3, 102258>, <1, 162>]$
$3E$	$[<3, 102312>]$
$9A$	$[<9, 33939>, <3, 486>, <1, 27>]$
$9B$	$[<9, 33939>, <3, 480>, <1, 45>]$
$9C$	$[<9, 33939>, <3, 483>, <1, 36>]$
$9D$	$[<9, 33939>, <3, 495>]$
$9E$	$[<9, 33939>, <3, 492>, <1, 9>]$
$9F$	$[<9, 34086>, <3, 52>, <1, 6>]$
$27A$	$[<27, 11313>, <9, 162>, <3, 8>, <1, 3>]$
$27B$	$[<27, 11313>, <9, 165>]$
$27C$	$[<27, 11313>, <9, 165>]$

Group	Prime	Matrix Representation	$3A$	$3B$	$9A$	$9B$	$27A$
B	3	4370×4370 matrices over GF(2)	1508	1454	482	488	*

Clearly, for the above the elements of $27A$ are distinguished by the fact they have order 27.

Group	Prime	Matrix Representation	$5A$	$5B$	$25A$
B	5	4370×4370 matrices over GF(2)	890	870	*

Clearly, for the above the elements of $25A$ are distinguished by the fact they have order 25. For HN : 2, below, taking $p = 2$, we use its representation as 133×133 matrices over GF(5). We take the maximal subgroup $H := 2^{1+8}.(A_5 \times A_5).2.2$. We denote by E_1, the 1-eigenspace of an element from the corresponding conjugacy class.

Class	Feature
$2A$	$\dim(E_1) = 77$
$2B$	$\dim(E_1) = 69$
$2C$	$\dim(E_1) = 70$
$4A$	$\dim(E_1) = 37$
$4B$	$\dim(E_1) = 39$
$4C$	$\dim(E_1) = 33$
$4D$	$\dim(E_1) = 56$
$4E$	$\dim(E_1) = 40$
$4F$	$\dim(E_1) = 34$
$8A$	$\dim(E_1) = 19$ and $\lvert C_H(x)\rvert = 32$
$8B$	$\dim(E_1) = 19$ and $\lvert C_H(x)\rvert \in \{64, 128\}$
$8C$	$\dim(E_1) = 22$
$8D$	$\dim(E_1) = 18$
$8E$	$\dim(E_1) = 20$
$8F$	$\dim(E_1) = 16$

For $G := \mathrm{Fi}_{22} : 2$, below, taking $p = 2$, we use the representation of G as a subgroup of S_{3510}. We define $C_n(\sigma)$ to be the number of n-cycles in the cycle decomposition of σ. We define ρ_{78} to be the representation afforded by the G-module constructed using the generators of $\mathrm{Fi}_{22} : 2$ as 78×78 matrices over $\mathrm{GF}(7)$ and ρ_{429} to be the representation afforded by the G-module constructed using the generators of $\mathrm{Fi}_{22} : 2$ as 429×429 matrices over $\mathrm{GF}(7)$. We denote by χ_{78} and χ_{429} their corresponding characters.

Class	Feature
$2A$	$C_2(x) = 1408$
$2B$	$C_2(x) = 1664$
$2C$	$C_2(x) = 1728$
$2D$	$C_2(x) = 1575$
$2E$	$C_2(x) = 1719$
$2F$	$C_2(x) = 1723$
$4A$	$C_2(x) = 72$
$4B$	$C_2(x) = 76$
$4C$	$C_2(x) = 88$
$4D$	$C_2(x) = 24$
$4E$	$C_2(x) = 84$
$4F$	$C_2(x) = 55$
$4G$	$C_2(x) = 71$
$4H$	$C_2(x) = 83$
$4I$	$C_2(x) = 87$
$4J$	$C_2(x) = 27$
$8A$	Not $8C$, $8D$, $8G$, $8H$ and $\chi_{78}(x) = 2$ and $\chi_{429}(x) = 1$
$8B$	Not $8C$, $8D$, $8G$, $8H$ and $\chi_{78}(x) = 3$ and $\chi_{429}(x) = 1$
$8C$	$C_8(x) = 416$ and $C_2(x) = 2$
$8D$	$C_8(x) = 438$ and $C_2(x) = 2$
$8E$	Not $8C$, $8D$, $8G$, $8H$ and $\chi_{78}(x) = 2$ and $\chi_{429}(x) = 3$
$8F$	Not $8C$, $8D$, $8G$, $8H$ and $\chi_{78}(x) = 3$ and $\chi_{429}(x) = 3$
$8G$	$C_8(x) = 416$ and $C_2(x) = 2$
$8H$	$C_8(x) = 432$ and $C_2(x) = 3$
$16A$	$*$
$16B$	$*$

Again we do not need to distinguish between the 16-classes for our purposes. For $G :=$

$\mathrm{Fi}_{24}' : 2$, below, taking $p = 2$, we use the representation of G as a subgroup of S_{306936}. We take the maximal subgroup $H := 2^{12}.\mathrm{M}_{24}$.

Class	Feature
$2A$	$C_2(x) = 151712$
$2B$	$C_2(x) = 153216$
$2C$	$C_2(x) = 137632$
$2D$	$C_2(x) = 153120$
$4A$	$C_2(x) = 192$
$4B$	$C_2(x) = 1724$
$4C$	$C_2(x) = 248$
$4D$	$C_2(x) = 1576$
$4E$	$C_2(x) = 1720$
$4F$	$C_2(x) = 232$
$4G$	$C_2(x) = 252$
$8A$	$\|C_H(x)\| = 4608$
$8B$	$C_2(x) = 2$*
$8C$	$C_2(x) = 4$
$8D$	$C_2(x) = 50$
$8E$	$\|C_H(x)\| = 9216$
$8F$	$C_2(x) = 2$*
$16A$	*
$16B$	*

Again we do not need to distinguish between the 16-classes for our purposes. For the $8B$ and $8F$ classes, we must use the random assignment procedure (as described for the Monster) to determine these classes. However, as there are only two classes of P not belonging to the other classes, this is relatively simple.

APPENDIX C

SPORADIC GROUPS VERIFICATION

The following are the tables of results relating to the results from Section 9.1, in particular Theorems 9.3, 9.15 and 9.17.

For the first table, any prime $p \geq 17$, the defect groups for these primes are abelian, so therefore the result is true using Theorem 5.60, since $Z(S) = 1$, for S non-abelian simple. The table is therefore restricted in size but this is not important. We denote by $\Phi := \Phi(P)$ where P is the Sylow 2-subgroup of G, to be the Frattini subgroup of P. The first column shows that we have checked whether $\Phi(P) = P'$.

Any cell containing a number refers to the number of height zero characters that we were required to verify, if we used GAP, or the number of characters of degree divisible by p that we verified, if we used MAGMA, as in MAGMA the code does not work out the distribution of the irreducible characters into blocks, however this can easily be implemented.

S \ p	2	3	5	7	11	13	17	19	23	29
M_{11}	$\Phi = P'$	AD	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
M_{12}	$\Phi = P'$	2	AD	N/A	AD	N/A	N/A	N/A	N/A	N/A
J_1	$\Phi = P'$	AD	AD	AD	AD	N/A	N/A	AD	N/A	N/A
M_{22}	$\Phi = P'$	AD	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
J_2	$\Phi = P'$	4	AD	AD	N/A	N/A	N/A	N/A	N/A	N/A
M_{23}	$\Phi = P'$	AD	AD	AD	AD	N/A	N/A	N/A	AD	N/A
HS	$\Phi = P'$	AD	4	AD	AD	N/A	N/A	N/A	N/A	N/A
J_3	$\Phi = P'$	12	AD	N/A	N/A	N/A	AD	AD	N/A	N/A
M_{24}	$\Phi = P'$	4	AD	AD	AD	N/A	N/A	N/A	AD	N/A
McL	$\Phi = P'$	9	6	AD	AD	N/A	N/A	N/A	N/A	N/A
He	$\Phi = P'$	4	AD	3	N/A	N/A	AD	N/A	N/A	N/A
Ru	$\Phi = P'$	27	16	AD	N/A	AD	N/A	N/A	N/A	AD
Suz	$\Phi = P'$	25	AD	AD	AD	AD	N/A	N/A	N/A	N/A
O'N	$\Phi = P'$	AD	AD	10	AD	N/A	N/A	AD	N/A	N/A
Co_3	$\Phi = P'$	9	22	AD	AD	N/A	N/A	N/A	AD	N/A
Co_2	$\Phi = P'$	33	40	AD	AD	N/A	N/A	N/A	AD	N/A
Fi_{22}	$\Phi = P'$	47	AD	AD	AD	AD	N/A	N/A	N/A	N/A
HN	$\Phi = P'$	35	34	AD	AD	N/A	N/A	AD	N/A	N/A
Ly	$\Phi = P'$	26	28	AD	AD	N/A	N/A	N/A	N/A	N/A
Th	$\Phi = P'$	33	26	AD	N/A	AD	N/A	N/A	N/A	N/A
Fi_{23}	$\Phi = P'$	71	AD	AD	AD	AD	AD	N/A	AD	N/A
Co_1	$\Phi = P'$	73	75	AD	AD	AD	N/A	N/A	AD	N/A
J_4	$\Phi = P'$	44	AD	AD	18	N/A	N/A	N/A	AD	AD
Fi'_{24}	$\Phi = P'$	66	AD	68	AD	AD	AD	N/A	AD	AD
B	$\Phi = P'$	157	159	AD	AD	AD	AD	AD	N/A	N/A
M	$\Phi = P'$	$*$	114	145	AD	139	AD	AD	N/A	N/A

The following is the table of results relating to Theorem 9.15. The second column here represents the number of irreducible characters of G we are required to verify the result for.

G	$\#\chi \in \mathrm{Irr}(G)$
$\mathrm{M}_{12} : 2$	13
$\mathrm{M}_{22} : 2$	5
$\mathrm{J}_2 : 2$	19
$\mathrm{HS} : 2$	23
$\mathrm{J}_3 : 2$	22
$\mathrm{McL} : 2$	25
$\mathrm{He} : 2$	37
$\mathrm{Suz} : 2$	52
$\mathrm{ON} : 2$	37
$\mathrm{Fi}_{22} : 2$	80
$\mathrm{HN} : 2$	62
$\mathrm{Fi}_{24}' : 2$	$*$

The following is the table of results relating to Section 9.3, in particular Theorem 9.17. This table follows the same structure as the first table in Appendix C. Here we also note that 9.16, states that for $p = 2$, we use Proposition 9.16 to verify the result.

G \ p	2	3	5	7	11	13	17	19	23	29
$2M_{12}$	18	8	AD	N/A	AD	N/A	N/A	N/A	N/A	N/A
$2M_{22}$	15	AD	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
$3M_{22}$	10	28	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
$4M_{22}$	31	31	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
$6M_{22}$	41	53	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
$12M_{22}$	85	85	AD	AD	AD	N/A	N/A	N/A	N/A	N/A
$2J_2$	30	20	AD	AD	N/A	N/A	N/A	N/A	N/A	N/A
2HS	34	AD	16	AD	AD	N/A	N/A	N/A	N/A	N/A
$3J_3$	31	46	AD	N/A	N/A	N/A	AD	AD	N/A	N/A
3McL	42	54	27	AD	AD	N/A	N/A	N/A	N/A	N/A
2Ru	53	43	21	AD	N/A	AD	N/A	N/A	N/A	AD
2Suz	60	40	AD	AD	AD	AD	N/A	N/A	N/A	N/A
3Suz	71	101	AD	AD	AD	AD	N/A	N/A	N/A	N/A
6Suz	162	174	AD	AD	AD	AD	N/A	N/A	N/A	N/A
3ON	56	62	AD	20	AD	N/A	N/A	AD	N/A	N/A
$2Fi_{22}$	9.16	78	AD	AD	AD	AD	N/A	N/A	N/A	N/A
$3Fi_{22}$	117	147	AD	AD	AD	AD	N/A	N/A	N/A	N/A
$6Fi_{22}$	$*$	$*$	AD	AD	AD	AD	N/A	N/A	N/A	N/A
$2Co_1$	9.16	74	76	AD	AD	AD	N/A	N/A	AD	N/A
$3Fi'_{24}$	$*$	$*$	AD	$*$	AD	AD	AD	AD	AD	N/A
2B	$*$	$*$	$*$	AD	AD	AD	AD	AD	AD	N/A

APPENDIX D

EVSEEV'S NOTES ON TWISTED GROUP ALGEBRAS

The following are lecture notes for a study group written by Evseev on twisted group algebras. We will not write up the entire series of notes, just the relevant chapter on subalgebras and centralisers of twisted group algebras. Let Γ be a group. Let A be a twisted Γ algebra over $\mathbb{C}$ that has characteristic 0 and is "large enough", i.e. such that $A[H]$ is split semisimple for every $H \leq \Gamma$. We will be interested in the case $A = \mathbb{C}\Gamma$. Let Z be a normal subgroup of Γ and define $G := \Gamma/Z$. Let $\eta : \Gamma \to G$ be the canonical surjection. We turn A into a G-graded algebra by setting,

$$A_g = \sum_{\sigma \in \eta^{-1}(g)} A_\sigma,$$

for each $g \in G$. We denote this G-graded algebra by $A[G]$. Clearly, $A[G]$ is a crossed product of G with identity component $A[Z]$. If $\chi \in \mathrm{Irr}(A)$ and $\lambda \in \mathrm{Irr}(A[Z])$, we say that χ lies over λ, if λ is an irreducible constituent of $\mathrm{Res}^A_{A[Z]}\chi$. We write,

$$\mathrm{Irr}(A|\lambda) = \{\chi \in \mathrm{Irr}(A) : \chi \text{ lies over } \lambda\}.$$

If A is the usual group algebra of Γ over $\mathbb{C}$, we have $\mathrm{Irr}(A|\lambda) = \mathrm{Irr}(\Gamma|\lambda)$. The subalgebra $C_A(A[Z])$ of A is a G-graded algebra with identity component $Z(A[Z])$. The algebra $A[Z]$ is a direct sum of matrix algebras, with components corresponding to irreducible characters of $A[Z]$, and for each $\lambda \in \mathrm{Irr}(A[Z])$, we denote by e_λ the identity elements of the component corresponding to λ. Then the idempotents e_λ, for $\lambda \in \mathrm{Irr}(A[Z])$, form a basis of $Z(A[Z])$. Moreover, we have,

$$1 = \sum_{\lambda \in \mathrm{Irr}(A[Z])} e_\lambda,$$

and this leads to the decomposition

$$C_A(A[Z]) = \bigoplus_{\lambda \in \mathrm{Irr}(A[Z])} C_A(A[Z])e_\lambda$$

as G-graded algebras. For $\lambda \in \mathrm{Irr}(A[Z])$, define $A(\lambda) := C_A(A[Z])e_\lambda$. For each $\chi \in \mathrm{Irr}(A)$ let e_χ be the corresponding primitive central idempotent of A. It is not difficult to show that χ lies over $\lambda \in \mathrm{Irr}(A[Z])$ if and only if $e_\chi e_\lambda \neq 0$. One can show that G acts on the idempotents e_λ, and hence on $\mathrm{Irr}(A[Z])$, with the action given by $\lambda^g(a) = \lambda(u_g a u_g^{-1})$ where u_g is a graded unit of $A[G]$ of degree g. One can therefore generalise Clifford theory to this situation, considering the inertia group of each $\lambda \in \mathrm{Irr}(A[Z])$. However, we will assume (for now, at least) that λ is G-invariant. Our aim is to prove the following result.

Theorem D.1. *Suppose $\lambda \in \mathrm{Irr}(A[Z])$ is G-invariant. Then*

(i) $A(\lambda)$ is a twisted G-algebra, with the grading inherited from that of $A[G]$.

(ii) $A(\lambda)[H]$ is split semisimple for every subgroup $H \leq G$.

(iii) There is a natural bijection between $\mathrm{Irr}(A|\lambda)$ and $\mathrm{Irr}(A(\lambda))$.

Evseev then proves a series of Lemma's to prove this result, however we are interested in (iii) of the above Theorem. This result is summarised by the following proposition.

Proposition D.2. *Suppose $\lambda \in \mathrm{Irr}(A[Z])$ is G-invariant. Then $Z(A)e_\lambda = Z(A(\lambda))$. Hence, there is a one-to-one correspondence between $\mathrm{Irr}(A|\lambda)$ and $\mathrm{Irr}(A(\lambda))$: two characters $\chi \in \mathrm{Irr}(A|\lambda)$ and $\chi' \in \mathrm{Irr}(A(\lambda))$ correspond if and only if $e_\chi = e_{\chi'}$.*

Moreover, we have that $Ae_\lambda = A[Z]e_\lambda \otimes_{\mathbb{C}} A(\lambda)$, because $A[Z]e_\lambda$ is isomorphic to a matrix algebra and contains the identity element of Ae_λ, whereas $A(\lambda)$ is the centraliser of $A[Z]e_\lambda$ in Ae_λ. So the bijection above may be described as

$$\chi' \to \lambda' \times \chi',$$

where $\chi' \in \mathrm{Irr}(A(\lambda))$ and λ' is the restriction of λ to $A[Z]e_\lambda$.

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