ZJ-type Theorems for Fusion Systems

by

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Abstract

In this thesis we prove two major ZJ-type group-theoretic results before generalising them to the category of fusion systems. We also study alternative proofs of the fact that when a fusion system $\mathcal{F}$ on a finite p-group P is sparse and p is odd or $\mathcal{F}$ is S_4-free, $\mathcal{F}$ is constrained.

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Contents

Introduction

In the following p is always assumed to be prime. We begin by stating an important theorem:

Theorem 0.0.1 (Glauberman's ZJ theorem) *[11, 2.11] Let G be a finite group and let p be an odd prime. Let $P \in Syl_p(G)$. Suppose that G is p-stable and has characteristic p. Then $Z(J(P)) \trianglelefteq G$.*

The proof runs over several chapters in [11] and for applications, the fact that $Z(J(P))$ *exists* and is characteristic is more important than its definition. The result can be turned on its head: Given a p-group P, there is a characteristic subgroup of P, $W(P)$, in which for certain finite groups, G, where $P \in \mathrm{Syl}_p(G)$, $W(P)$ is normal. A new proof of this fact was found by Stellmacher ([19]) in the 1990s as a byproduct of his result which generalises this theorem to the case $p = 2$:

Theorem 0.0.2 (Theorem A) *[14, 9.4.4] Let p be an odd prime and let P be a p-group. Then there is a characteristic subgroup $W(P)$ of P which satisfies:*

(a) $\Omega(Z(P)) \leqslant W(P) \leqslant \Omega(Z(J(P)))$.

(b) If G is a p-stable group of characteristic p, with $P \in Syl_p(G)$, then $W(P) \trianglelefteq G$.

(c) $W(P^\eta) = W(P)^\eta$, where η is any automorphism of P.

Both of these theorems have famous corollaries which we respectively state below:

Theorem 0.0.3 (Glauberman-Thompson Normal p-complement Theorem) *[11, 3.1] Let G be a group, p an odd prime and $P \in Syl_p(G)$. Then $N_G(Z(J(P)))$ has a normal p-complement iff G has a normal p-complement.*

Theorem 0.0.4 (Theorem B) *[14, 9.4.7] Let G be a group, p an odd prime and $P \in Syl_p(G)$. Then $N_G(W(P))$ has a normal p-complement if and only if G has a normal p-complement.*

Now Frobenius' Theorem (2.1.37) tells us that the notion of a group having a normal p-complement is related to the concept of *fusion* (conjugation) inside that group. So, loosely speaking, we can understand the fusion that goes on inside a group G by looking at the normalisers of some non-identity p-subgroups of that group.

It turns out that to every finite group, G, one can associate a *saturated fusion system* on a Sylow p-subgroup of G. A fusion system, $\mathcal{F}$, on P is a small category whose objects are the subgroups of P, and whose arrows are certain injective group homomorphisms. Saturation of a fusion system is an important technical furnishing which is satisfied by all fusion systems of finite groups. Since the converse to the above statement is not true, that is, there exist saturated fusion systems, $\mathcal{F}$, on P which do not arise from groups G with $P \in \mathrm{Syl}_p(G)$ (we write $\mathcal{F} \neq \mathcal{F}_P(G)$ in this case [1]) the category of saturated fusion systems is larger than the category of finite groups. This makes them an interesting object of study since results about fusion systems are a *generalisation* of results about finite groups.

If such generalisations are to be possible, a basic requirement is the need to define saturated *subsystems* of a fusion system; to replace subgroups of a group. For example $N_{\mathcal{F}}(S)$ instead of $N_G(S)$, etc. In Section 2.1.3 we will define these systems and show that they are saturated. This is summed up in the following result due to Puig [17] where different subgroups K, of $\mathrm{Aut}(Q)$ give rise to different subsystems, $N_{\mathcal{F}}^K(Q)$ of $\mathcal{F}$:

[1] $\mathcal{F}$ is called an *exotic* fusion system.

Theorem 0.0.5 (Theorem C) *Let $Q \leqslant P$ and $K \leqslant Aut(Q)$. If Q is fully K-normalised in $\mathcal{F}$, then $N_{\mathcal{F}}^K(Q)$ is a saturated fusion system on $N_P^K(Q)$.*

The categorical approach to fusion exemplified by fusion systems lends itself well to being studied via homotopic means. Indeed this has been done extensively in [5] by Carlos Broto, Ran Levi and Bob Oliver. A highlight of this approach is the following result of a subsequent paper which states that for certain objects Q in the fusion system, $\mathcal{F}$ on P, one can associate a finite group, G (called a model) so that every map in $\mathcal{F}$ is realisable as G-conjugation:

Theorem 0.0.6 (Theorem D) *[5, 4.3] Let $\mathcal{F}$ be a fusion system on a finite p-group, P. For $Q \in \mathcal{F}^{fc}$, there is, up to isomorphism, a unique finite group $G = G_Q^{\mathcal{F}}$ with $N_P(Q) \in Syl_p(G)$ such that $Q \trianglelefteq G, C_G(Q) \leqslant Q$ and $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(G)$.*

For example, if $\mathcal{F} = N_{\mathcal{F}}(Q)$, for some $Q \in \mathcal{F}^c$, then the above result applies to show that $\mathcal{F}$ cannot be exotic so $\mathcal{F}$ has a model and we can apply results from finite group theory to $\mathcal{F}$. This idea forms the backbone of the proofs of the following two results, where the finite group theoretic results in question are those stated above. A fusion system is called constrained if it possesses a normal, centric subgroup, and it is called H-free if every element of the set $\{G_Q^{\mathcal{F}} \mid Q \in \mathcal{F}^{fc}\}$ is. Also note that a sparse fusion system is a fusion system whose only proper subsystem is $\mathcal{F}_P(P)$. We have the following remarkable fact:

Theorem 0.0.7 (Theorem E) *[10, 2.6] Let $\mathcal{F}$ be a sparse fusion system over a finite p-group P. If p is odd or $\mathcal{F}$ is S_4-free then $\mathcal{F}$ is constrained.*

Using Theorem E, we are able to prove the following result of Kessar and Linckelman:

Theorem 0.0.8 (Theorem F) *[12, Theorem A] Suppose that p is odd and let $\mathcal{F}$ be a fusion system on a finite p-group P. Then $N_{\mathcal{F}}(W(P)) = \mathcal{F}_P(P)$ if and only if $\mathcal{F} = \mathcal{F}_P(P)$ and $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ if and only if $\mathcal{F} = \mathcal{F}_P(P)$.*

Theorem F is a generalisation of the Normal p-complement Theorem and Theorem B. In fact Theorem F follows almost immediately from Theorem E since it turns out that a minimal counterexample to the forward direction of each part of Theorem F is a sparse fusion system.

Our final result, also due to Kessar and Linckelman, is a generalisation of Theorem A to arbitrary fusion systems:

Theorem 0.0.9 (Theorem G) *Let P be a finite p-group and let $\mathcal{F}$ be a $Qd(p)$-free fusion system over P. Then there exists a non-trivial characteristic subgroup, $W(P)$ of P which is normal in $\mathcal{F}$.*

The absence of any condition on the prime, p, is a consequence of the fact that W is a Glauberman Functor for every prime p, a condition which the functor $Z(J(-))$ does not enjoy.

Chapter 1

Finite Group Theory

1.1 Basic results

In our first section we review some elementary but important results in p-local group theory. From now on, all groups are assumed to be finite unless explicitly stated otherwise.

1.1.1 Sylow's Theorems

Sylow's Theorems are fundamental to all results involving fusion in finite groups. Assuming basic notation, we will prove them in this section. p denotes an arbitrary prime number.

The following is useful and well known:

Lemma 1.1.1 *[14, 3.1.7] Let a finite p-group G act on a set Ω. Then $|\Omega| \equiv |Fix_\Omega(G)|$ mod p.*

Proof. Set $\Omega' := \Omega - \mathrm{Fix}_\Omega(G)$. Then no element of Ω' is fixed by every element of G so for every $\alpha \in \Omega', G_\alpha < G$ and p divides $|G : G_\alpha|$. i.e. p is a divisor of the size of every orbit in Ω' and hence divides $|\Omega'|$ itself. Thus:

$$|\Omega'| = |\Omega| - |\mathrm{Fix}_\Omega(G)| \equiv 0 \bmod p$$

as needed. $\square$

We now prove Sylow's famous result. Recall that a p-subgroup, P, is called *Sylow* if P is not a proper subgroup of any other p-subgroup of G. We write $\mathrm{Syl}_p(G)$ to denote the set of Sylow p-subgroups of G.

Theorem 1.1.2 (Sylow) *Suppose that p is a prime divisor of a finite group G. Then Sylow p-subgroups exist, have maximal p-power order and are all conjugate in G. Furthermore,*

$$|Syl_p(G)| = |G : N_G(P)| \textit{ for all } P \in Syl_p(G) \textit{ and } |Syl_p(G)| \equiv 1 \textit{ mod } p.$$

Proof. By Cauchy's Theorem, G contains a subgroup of order p. Let Q be a subgroup of G of order p^i. Letting Q act on the left cosets G/Q by left multiplication, we have that $|N_G(Q)/Q| \equiv |G/Q| \bmod p$ (Lemma 1.1.1). If $p||G : Q|$ then $N_G(Q)/Q$ has a subgroup of order p whose preimage in $N_G(Q)$ has order p^{i+1}, thus Sylow p-subgroups exist.

Let $P_1, P_2 \in \mathrm{Syl}_p(G)$ and let P_2 act on $\Omega := G/P_1$ by left multiplication. By Lemma 1.1.1, $|\mathrm{Fix}_\Omega(P_2)| \neq 0$ so pick $xP_1 \in \mathrm{Fix}_\Omega(P_2)$. Then for all $y \in P_2$, $yxP_1 = xP_1$ so that $P_2^x \leqslant P_1$. Comparing orders we see that $P_2^x = P_1$ as needed.

Finally let $P \in \mathrm{Syl}_p(G)$ act on the set $\mathrm{Syl}_p(G)$ by conjugation. We claim that the number of fixed points for this action is 1 and apply Lemma 1.1.1. Indeed, if there is $T \in \mathrm{Syl}_p(G)$ not equal to P such that $T^x = T$ for all $x \in P$ then $P, T \in \mathrm{Syl}_p(N_G(T))$ and hence P and T are conjugate in $N_G(T)$ by what we have just proved. However since $T \trianglelefteq N_G(T)$, we must have $T = P$. $\square$

Lemma 1.1.3 (Frattini) *Let G be a finite group and let $N \trianglelefteq G$. Suppose N acts transitively on a set Ω. Then for every $\alpha \in \Omega$, $G = G_\alpha N$. In particular when $P \in Syl_p(N)$, we have $G = N_G(P)N$.*

Proof. The transitive action of N on Ω means that for $\alpha \in \Omega, x \in G$, there is a $y \in N$ such that $\alpha^x = \alpha^y$. Then $xy^{-1} \in G_\alpha$ implies that $x \in G_\alpha N$ and thus $G = G_\alpha N$. The second part follows by setting $\Omega = \mathrm{Syl}_p(N)$, since N acts transitively on the set of its Sylow p-subgroups. $\square$

1.1.2 Fusion in Finite Groups

We begin with some definitions and results from finite group theory. Throughout this section, let G be a finite group and p be any prime. First, recall the theorem of Schur-Zassenhaus:

Theorem 1.1.4 (Schur-Zassenhaus) *Let $N \trianglelefteq G$ be such that $(|N|, |G : N|) = 1$. Then there exists $H \leqslant G$ such that $G = HN$ and $H \cap N = 1$. Further, if either of the groups N or G/N are solvable, then all such H are conjugate in G.*

Using Theorem 1.1.3 we get:

Lemma 1.1.5 *[14, 6.2.2 (a)] Let G act on a set Ω and suppose that $K \trianglelefteq G$ is such that $(|K|, |G : K|) = 1$, either K or G/K is soluble and K acts transitively on Ω. Then for every complement H of K in G, $Fix_\Omega(H) \neq \emptyset$.*

Proof. The transitive action of K on Ω implies that Ω is the unique orbit, and so $|\Omega|$ divides $|K|$. By Lemma 1.1.3, for any $\beta \in \Omega$ we have $G = G_\beta K$ and an Isomorphism Theorem implies that $G/K \cong G_\beta/(K \cap G_\beta)$. Now apply Theorem 1.1.4 to G_β and $K \cap G_\beta$ to find a complement H' of $K \cap G_\beta$ in G_β. But then H' must also complement K in G and since $H' \leqslant G_\beta$, $\beta \in \mathrm{Fix}_\Omega(H')$. Theorem 1.1.4 also implies that there is $g \in G$ such that $H'^g = H$ so also $\mathrm{Fix}_\Omega(H) \neq \emptyset$ and we are done. $\square$

Definition 1.1.6 We say that G possesses a *normal p-complement*, N, if there exists $N \trianglelefteq G$ such that $G = NP$ and $P \cap N = 1$ for some $P \in \mathrm{Syl}_p(G)$.

Notice that we must have $O_{p'}(G) = N = O^p(G)$ in this case.

Example 1.1.7 If G is p-closed[1], then Theorem 1.1.4 tells us that G has a normal p-complement.

Here is an easy lemma:

Lemma 1.1.8 (Burnside) *Let $P \in Syl_p(G)$. Then any two normal subsets of P that are conjugate in G are conjugate in $N_G(P)$.*

Proof. Let $X, Y \subseteq P$ and suppose $Y = X^g$, some $g \in G$. Certainly $P \leqslant N_G(X)$, so that $P^g \leqslant N_G(X)^g = N_G(X^g) = N_G(Y)$. But $P \leqslant N_G(Y)$ so by Sylow's Theorem there is $z \in N_G(Y)$ such that $P = P^{gz}$ which implies that $gz \in N_G(P)$ and $X^{gz} = Y^z = Y$ as required. □

We will study the relationship between control of fusion, i.e. when conjugacy in one group implies conjugacy in another, and the possession of a normal p-complement in later sections. Notice that under the hypothesis that P is abelian, every subset is normal, so in particular, Lemma 1.1.8 applies to two G-conjugate elements of P.

The following definitions will be important later so we state them now:

Definition 1.1.9 Let G be a finite group.

(a) We say that G is *p-reduced* if $O_p(G) = 1$.

(b) A proper subgroup H of G is *strongly p-embedded* if H contains a Sylow p-subgroup of G and p does not divide $|H \cap H^g|$ for $g \in G - H$.

Remark 1.1.10 Notice that if there exists some subgroup H of G satisfying (b) then G is p-reduced since $O_p(G)$ is the intersection of all elements of $\mathrm{Syl}_p(G)$.

[1] i.e. there is a unique Sylow p-subgroup which must therefore be normal.

Definition 1.1.11 A group G is said to be p-soluble if every composition factor of G is either a p'-group or a p-group.

Lemma 1.1.12 *Suppose a group is p-soluble and that $O_{p'}(G) = 1$. Then $C_G(O_p(G)) \leqslant O_p(G)$.*

Proof. An equivalent conclusion would be that $Z(O_p(G)) = C_G(O_p(G))$, so suppose that $Z(O_p(G)) < C_G(O_p(G))$. Let $\overline{G} := G/Z(O_p(G))$ and suppose that $\overline{C_G(O_p(G))} \neq 1$. Let $\overline{N}$ be a minimal normal subgroup of $\overline{G}$ contained in $\overline{C_G(O_p(G))}$. Since G is p-soluble, $\overline{N}$ is either a p-group or a p'-group. Let N denote the inverse image of $\overline{N}$ in G and note that $N \leqslant C_G(O_p(G))$. If $\overline{N}$ is a p-group, then N is a p-group so $N \leqslant O_p(G)$. But then $N \leqslant C_G(O_p(G))$ implies that $N \leqslant Z(O_p(G))$, a contradiction. If $\overline{N}$ is a p'-group then $Z(O_p(G)) \in \mathrm{Syl}_p(N)$ and $Z(O_p(G)) \leqslant Z(N)$ so Theorem 1.1.4 implies that $N = Z(O_p(G)) \times O_{p'}(N)$, and $1 \neq O_{p'}(N) \leqslant O_{p'}(G)$, a contradiction. □

Definition 1.1.13 Let G be a finite group. For $S, T \leqslant G$, with $S \trianglelefteq T$, we call the quotient S/T a *section* of G. A group A is said to be *involved* in the group G if it is isomorphic to some section of G. For a finite group H, G is said to be *H-free* if H is not involved in G.

Definition 1.1.14 A subgroup M of G is a *p-local subgroup* of a finite group G if there is a non-trivial p-subgroup P of G such that $N_G(P) = M$.

Example 1.1.15 If $G \cong \mathrm{Sym}(4)$, the Sylow 2-subgroup, $S \cong \mathrm{Dih}(8)$ is itself p-local since $S = N_G(S)$.

Definition 1.1.16 We say that a group G has *characteristic p* or equivalently that G is *p-constrained* if $C_G(O_p(G)) \leqslant O_p(G)$.

Lemma 1.1.17 *[14, 3.2.8] For a finite group, G, let $N \trianglelefteq G$ and set $\overline{G} := G/N$. If Q is a p-subgroup of G with $(|N|, p) = 1$, we have $N_{\overline{G}}(\overline{Q}) = \overline{N_G(Q)}$.*

Proof. It is evident that $N_{\overline{G}}(\overline{Q}) = \overline{N_G(NQ)}$ (by definition). Now $(|N|, p) = 1$ means that $Q \in \mathrm{Syl}_p(NQ)$ so Lemma 1.1.3 applied to $N_G(NQ)$ yields the factorisation: $N_G(NQ) = NQN_{N_G(NQ)}(Q) = NN_{N_G(NQ)}(Q)$. Since $NN_G(Q) \leqslant N_G(NQ)$ we get $NN_{N_G(NQ)}(Q) = NN_G(Q)$ so that $\overline{N_G(NQ)} = \overline{N_G(Q)}$, hence the claim. □

Lemma 1.1.18 *[14, 3.1.10] Let G be a finite group and let P be a p-subgroup of G. If p divides $|G : P|$. Then $P < N_G(P)$.*

Proof. Set $\Omega = G/P$, and let P act on Ω by right multiplication. By hypothesis $|\Omega| \equiv 0 \mod p$ and by Lemma 1.1.1, we get $0 \equiv |\Omega| \equiv |\mathrm{Fix}_\Omega(P)| \mod p$. Now $|\mathrm{Fix}_\Omega(P)| \neq 0$ since the coset P is fixed by every element of P. So $|\mathrm{Fix}_\Omega(P)| \geqslant p \geqslant 2$. In particular, there is some $Pg \in \mathrm{Fix}_\Omega(P)$, with $P \neq Pg$. So $g \notin P$ and we get $(Pg)P = Pg$. So $gPg^{-1} = P$ and $g \in N_G(P) - P$, which proves the result. □

Lemma 1.1.19 *[14, 3.1.11] Let P be a p-group and N be such that $1 \neq N \trianglelefteq P$. Then $Z(P) \cap N \neq 1$.*

Proof. Setting $\Omega := N$, we have that P acts on Ω by conjugation and $\mathrm{Fix}_\Omega(P) = Z(P) \cap N$. Since P is a p-group, Lemma 1.1.1 implies that, $0 \equiv |\Omega| \equiv |\mathrm{Fix}_\Omega(P)| \mod p$. Since $1 \in \mathrm{Fix}_\Omega(P)$, $|\mathrm{Fix}_\Omega(P)| \geqslant p$ as needed. □

1.1.3 Frobenius' Theorem for Finite Groups

In this section we prove a fundamental result concerning fusion. Loosely it demonstrates that the structure of p-local subgroups of a group, G, relates strongly to that of G. Not only is it important in the proof of Theorem B, there is also a formulation in terms of fusion systems which we prove later (Theorem 2.1.37). In fact many of the following definitions and results have analogues in the context of fusion systems which we encounter in the next chapter.

As in the previous section, let p be a prime and G be a finite group.

Definition 1.1.20 Suppose $Z \leqslant P \leqslant G$. Then Z is *weakly closed in* P with respect to G if for $g \in G$, $Z^g \leqslant P$ implies that $Z^g = Z$.

We state the following result concerning weakly closed subgroups without a proof [2]:

Theorem 1.1.21 (Grün) *[[11], 7.5.2] Let $P \in Syl_p(G)$ and $Z \leqslant Z(P)$ be weakly closed in P with respect to G. Let $N := N_G(Z)$. Then*

$$G \neq O^p(G) \textit{ if and only if } N \neq O^p(N).$$

Lemma 1.1.22 *[14, 7.1.9] Let $P \in Syl_p(G)$ and $Z \leqslant P$ be such that $Z \trianglelefteq N_G(P)$. Then Z is weakly closed in P with respect to G if and only if Z is normal in every Sylow p-subgroup which contains it.*

Proof. Suppose $R \in \mathrm{Syl}_p(G)$ is such that $Z \leqslant R = P^{g^{-1}}$, some $g \in G$. Then $Z^g \leqslant P$ implies $Z^g = Z$ and $Z^{g^{-1}} = Z$. Let $r \in R$. Then writing $r = x^{g^{-1}}$ for $x \in P$, gives $Z^r = Z^{gxg^{-1}} = Z^{xg^{-1}} = Z^{g^{-1}} = Z$ and so $Z \trianglelefteq R$. Conversely if $Z^g \leqslant P$ then $Z \leqslant P^{g^{-1}} \in \mathrm{Syl}_p(G)$ so $Z \trianglelefteq P^{g^{-1}}$ and $Z^g \trianglelefteq P$. By Lemma 1.1.8, there is $y \in N_G(P)$ such that $Z^y = Z^g$ showing $Z^g = Z^y = Z$ as needed.

One more simple observation is required before the proof of Frobenius' Theorem:

Lemma 1.1.23 *[14, 7.2.3] Suppose there is a normal p-complement, N say, of G and $P \in Syl_p(G)$. Then every normal subgroup of P is weakly closed in P with respect to G.*

Proof. Let $g \in G$ be written as $g = yx$ for some $y \in N, x \in P$. Suppose $Z \trianglelefteq P$ and that $Z^g \leqslant P$. Since $Z^y \trianglelefteq P$, $P \geqslant Z^{yx} = Z^y$ implying that $[z, y] = z^{-1}y^{-1}zy \in N \cap P = 1$, so $y \in C_G(Z)$. Finally $Z^g = Z^{yx} = Z^x = Z$. $\square$

[2] The proof requires a concept known as *transfer*.

Theorem 1.1.24 *[14, 7.2.4] Let $P \in Syl_p(G)$. Then G has a normal p-complement if and only if every p-local subgroup has a normal p-complement*

Proof. The forward direction is obvious since if $G = K \rtimes P$ for some $K \trianglelefteq G$, then for any subgroup H of G, $H \cap K = O_{p'}(H)$ is a normal p-complement for H. Setting $H = N_G(Q)$ for any $Q \leqslant P$ then gives the result. To prove the converse, we proceed by induction on $|G|$. Notice that we may assume $P \neq 1$, so that $Z := Z(P) \neq 1$. Now $N := N_G(Z)$ has a normal p-complement (by hypothesis), so $O^p(N) \neq N$. Next, we claim that Z is weakly closed in P. If this is the case, then Theorem 1.1.21 implies that $G \neq O^p(G)$ and by induction, a normal p-complement for $O^p(G)$ is also one for G. By Lemma 1.1.22 it suffices to show that Z is normal in every Sylow p-subgroup containing it. Assume $Z \leqslant R \in \mathrm{Syl}_p(G)$ but $Z \ntrianglelefteq R$ and that R was chosen so that $S := N_R(Z)$ is as large as possible. Let $T \in \mathrm{Syl}_p(N_G(Z))$ contain S. Then we have $S < R, T$ and so by Lemma 1.1.18, $S < N_R(S), N_T(S)$. Now pick $T' \in \mathrm{Syl}_p(N_G(S))$ so that $N_T(S) \leqslant T'$. Since $S < T'$, the maximality of S implies that $Z \trianglelefteq T'$. But by our hypothesis on p-local subgroups, Lemma 1.1.23 applies to $N_G(S)$ so Z is weakly closed in T' with respect to $N_G(S)$. So by Lemma 1.1.22, $Z \trianglelefteq N_R(S)$ (a Sylow p-subgroup of $N_G(S)$) contradicting $S = N_R(Z) < N_R(S)$, as needed.

1.2 Group Action

1.2.1 Coprime Action

Recall that if two finite groups A and G are such that A acts by conjugation on G then there exists a homomorphism, $\phi : A \to \mathrm{Aut}(G) \leqslant S_{|G|}$ which gives rise to a semidirect product $A \ltimes G$. The section concerns the case when the orders of A and G are coprime and either group is soluble. Our proofs closely follow those found in Chapter 8 of [14].

We let G be a finite group on which another finite group A acts. We begin by recalling some conventions:

Definition 1.2.1 We define the following groups:

(i) For $a \in A$, $C_G(a) := \{g \in G \mid g^a = g\}$.

(ii) For $B \subseteq A, C_G(B) := \bigcap_{b \in B} C_G(b)$ is the set of *fixed points* of B in G.

(iii) For $B \subseteq A, C_B(G) := \{b \in B \mid g^b = g$ for all $g \in G\}$ is the *kernel of the action of B on G.*

Notation 1.2.2 For $g \in G, a \in A$ and $U \subseteq G, B \subset A$:

(i) $[g, a] := g^{-1}g^a$.

(ii) $[U, B] := \langle [u, b] \mid u \in U, b \in B \rangle$.

Remark 1.2.3 Notice that $U \leqslant C_G(A)$ if and only if $[U, A] = 1$.

Notation 1.2.4 For elements of a group, a, b, c, write $[a, b, c] := [[a, b], c]$ and similarly for groups A, B, C, set $[A, B, C] := [[A, B], C]$.

We have the following:

Lemma 1.2.5 *For $g \in G, a, x \in A$, $[g, ax] = [g, x][g, a]^x$ and for $g, x \in G$ and $a \in A, [gx, a] = [g, a]^x[x, a]$. In particular when G is abelian, $[g, A] = [G, A]$.*

Lemma 1.2.6 (Three Subgroups Lemma) *For subgroups X, Y, Z of A or G:*

$$[X, Y, Z] = [Y, Z, X] = 1 \textit{ implies that } [Z, X, Y] = 1.$$

Consider the following result in the case where A is a p-group:

Lemma 1.2.7 *Suppose that A is a p-group. Then there is an A-invariant Sylow p-subgroup of G.*

Proof. Pick $\overline{P} \in \mathrm{Syl}_p(AG)$ so that $A \leqslant \overline{P}$. Then $P := \overline{P} \cap G$ is A-invariant and is a Sylow p-subgroup of G. $\square$

We now define coprime action:

Definition 1.2.8 (Coprime Action) Let the group A act on the group G. The action is called *coprime* if $(|A|, |G|) = 1$ and either A is soluble or G is soluble.

We immediately see two applications of Theorem 1.1.4:

Lemma 1.2.9 *Suppose the group A acts coprimely on the group G. Then every subgroup of order $|A|$ in AG is conjugate to A.*

Proof. The resulting product, AG, to which this action gives rise is clearly a semi-direct product, $A \ltimes G$. So the result follows directly from Theorem 1.1.4. $\square$

Lemma 1.2.10 *[14, 8.2.1] Let $U \leqslant G$ be A-invariant, and the action of A on G be coprime. If $g \in G$ is such that Ug is A-invariant, then we can find $c \in C_G(A)$ such that $Ug = Uc$.*

Proof. Since both U and Ug are A-invariant, we get $Ug = (Ug)^A = U^A g^A = Ug^A$, so $g^a g^{-1} \in U$ for every $a \in A$. In other words, $a^{g^{-1}} \in aU$ and $A^{g^{-1}} \leqslant AU$. By Lemma 1.2.9 and Theorem 1.1.4, we have that A and $A^{g^{-1}}$ are conjugate in AU, so there is $u \in U$ such that $A^u = A^{g^{-1}}$. Clearly $c := ug \in Ug$ so $Ug = Uc$. Also notice $[A, c] \leqslant A \cap G = 1$, so that $c \in C_G(A)$, as required. $\square$

Lemma 1.2.11 *[14, 8.2.2] Let $N \trianglelefteq G$ be A-invariant, and the action of A on N be coprime. Then $C_{G/N}(A) = C_G(A)N/N$ and $[N, A] = [G/N, A] = 1$ imply $[G, A] = 1$.*

Proof. Setting $U := N$, and using the result for cosets in the previous lemma, gives that $C_{G/N}(A) = C_G(A)N/N$. Now $[G/N, A] = 1$ implies $C_{G/N}(A) = G/N$ so we get $C_G(A)N = G$. Now $[N, A] = 1$ implies $C_G(A) = N$ and so $C_G(A) = G$ as required. □

Recall that $\Phi(G)$ is the *Frattini Subgroup of* G and is defined to be the intersection of all maximal subgroups of G. Also, recall that any proper subgroup of G is contained in a maximal subgroup. A special case of Lemma 1.2.11 is the following:

Proposition 1.2.12 *Suppose the action of A on $\Phi(G)$ is coprime and that $[G/\Phi(G), A] = 1$. Then $[G, A] = 1$.*

Proof. Lemma 1.2.11 implies that $G = \Phi(G)C_G(A)$ and so the definition of $\Phi(G)$ yields $G = C_G(A)$, as needed. □

The last result in this section is an elementary consequence of Lemma 1.1.5:

Lemma 1.2.13 *[14, 8.2.3] Let $p \in \pi(|G|)$ and suppose that A acts coprimely on G. Then there exists an A-invariant Sylow-p subgroup of G.*

Proof. The group AG acts on $\Omega :=\mathrm{Syl}_p(G)$ by conjugation and $G(\trianglelefteq AG)$ acts transitively on Ω by Sylow's Theorem. Since A is a complement for G in AG, Lemma 1.1.5 implies $\mathrm{Fix}_\Omega(A) \neq \emptyset$ and we are done. □

1.2.2 Quadratic Action

It will turn out, in our definition of p-stability in the next section, that we require a certain condition to hold, namely that G acts *quadratically* on a subgroup. If we are to understand a p-stable action, we must at the very least understand when our group is acting quadratically. Thanks to Proposition 1.2.17, we are able to make a precise statement about when this happens. We base our proofs on those found in Chapter 9 of [14]. In this section, V will always denote an arbitrary elementary abelian p-group and G, a group.

Definition 1.2.14 (Quadratic Action) We say that a group A acts *quadratically* on V if:

$$[V, A, A] = 1.$$

Remark 1.2.15 Note that any elementary abelian p-group (of rank n, say), can be viewed as a vector space $\mathbb{F}_p^n$ over the field $\mathbb{F}_p$ in a natural manner. Indeed, no difficulty arises if we view V in this way.

Example 1.2.16 If $|G| = 2 = p$, then $[v, a]^a = v^{-a}v = [v, a]^{-1} = [v, a]$. Thus for any $a \in G, v \in V$ we have $[v, a, a] = [[v, a], a] = [v, a]^{-1}[v, a]^a = 1$. Hence in this case G acts quadratically on V.

For proofs in later sections, we will require conditions that guarantee quadratic action. As a preliminary observation, notice that we certainly have for $A^* := C_A([V, A])$ that $[V, A, A^*] = 1$, so that A^* acts quadratically on V. Notice also that if A acts quadratically on V then $A/C_A(V)$ is an elementary abelian p-group. In fact we get the following:

Proposition 1.2.17 *[14, 9.2.1] Suppose a group A acts on V, so that $A/C_A(V)$ is abelian. Then there is $A^* \leqslant A$ so that either*

(i) $|A||C_V(A)| < |A^*||C_V(A^*)|$; or

(ii) for any $U \leqslant V$, $A^* = C_A([U, A])$, $C_V(A^*) = [U, A]C_V(A)$ and $|A||C_V(A)| = |A^*||C_V(A^*)|$.

Proof. Suppose (i) is not true. Then for all $B \leqslant A$, we have $|A||C_V(A)| \geqslant |B||C_V(B)|$. For any subgroup U of V, set $A^* = C_A([U, A])$ so trivially $[U, A, A^*] = 1$. Also, the fact that $A/C_A(V)$ is abelian implies that $[A, A] \leqslant C_A(V)$ and so $[A, A^*, U] \leqslant [A, A, V] = 1$. By the Three-Subgroups Lemma 1.2.6 we get $[U, A^*, A] = 1$ which means $[U, A^*] \leqslant C_V(A)$. We now set $X := [U, A]$ and $Y := C_V(A)$ and claim that

$$|A : A^*| \leqslant |XY|/|Y|.$$

To prove this we consider two cases: **Case 1:** $|U| = p$. Set $U = \langle u \rangle$. Now the result would follow if the map $\varphi : A/A^* \to XY/Y$ with $aA^* \longmapsto [u,a]Y$ were well-defined and injective. It is well defined since for any $c \in A^*$, Lemma 1.2.5 gives us

$$[u, ca] = [u, a][u, c]^a \in [u, a]Y$$

since $[U, A^*] \leqslant C_V(A)$. It is injective since if $a_1, a_2 \in A$ are such that $[u, a_1]Y = [u, a_2]Y$, then $[u, a_1][u, a_2]^{-1} = u^{a_1}u^{-a_2} \in Y$. This implies

$$[u, a_1a_2^{-1}] = u^{-1}u^{a_1a_2^{-1}} = (u^{-a_2}u^{a_1})^{a_2^{-1}} = u^{a_1}u^{-a_2} \in Y.$$

So $[u, a_1a_2^{-1}, A] = 1 = [a_1a_2^{-1}, A, u]$ and the Three-Subgroups Lemma 1.2.6 gives $a_1a_2^{-1} \in C_A([u, A]) = C_A([U, A]) = A^*$. (Since U is cyclic, Lemma 1.2.5 implies that $[u, A] = [U, A]$). **Case 2:** $|U| > p$. In this case we may write $U = U_1\langle u \rangle$, some $u \in U$. Write

$$X_1 := [U_1, A], X_2 := [\langle u \rangle, A]$$

and let $A_i = C_A(X_i)$ for $i \in \{1, 2\}$. We have

$$X_1X_2C_V(A) = XC_V(A), A^* = A_1 \cap A_2 \text{ and } X_1C_V(A) \cap X_2C_V(A) \leqslant C_V(A_1A_2).$$

By induction we may assume that $|A||C_V(A)| = |A_i||X_iC_V(A)|$ for $i \in \{1, 2\}$. So

$$|A||C_V(A)| \geqslant |A_1A_2||C_V(A_1A_2)| \geqslant |A_1A_2||X_1C_V(A) \cap X_2C_V(A)| = |A|^2|C_V(A)|^2/|A^*||XC_V(A)|,$$

completing the second case. Since

$$|A^*||C_V(A^*)| \leqslant |A||Y| \leqslant |A^*||XY| \leqslant |A^*||C_V(A^*)|,$$

the result follows. $\square$

We get a useful consequence of this result, a famous 'replacement' theorem of Timmesfeld which we state without proof:

Theorem 1.2.18 (Timmesfeld) *[14, 9.2.3] Let $A \leqslant G$ and suppose that for all $A^* \leqslant A$, $|A||C_V(A)| \geqslant |A^*||C_V(A^*)|$ and that $A/C_A(V)$ is an elementary abelian p-group. Then for all subgroups U of V:*

$$[V, A] \neq 1 \textit{ implies } [V, C_A([U, A])] \neq 1.$$

Remark 1.2.19 Suppose A satisfies the hypotheses of Theorem 1.2.18. Then Theorem 1.2.17 yields that:

$$|A||C_V(A)| = |C_V(A^*)||A^*| \text{ and } C_V(A^*) = [V, A]C_V(A).$$

We now introduce two subgroups which will play key roles in the next section:

Definition 1.2.20 For any p-group G,

$$\Omega_i(G) := \langle x \in G \mid x^{p^i} = 1 \rangle.$$

For simplicity, we set $\Omega(G) := \Omega_1(G)$.

Remark 1.2.21 Note that $\Omega_i(G)$ is $\mathrm{Aut}(G)$-invariant and hence a characteristic subgroup of G. Also note that when G is abelian, $\Omega(G)$ is elementary abelian, since for $x, y \in \Omega(G), (xy)^p = x^p y^p = 1$.

Definition 1.2.22 Let $\mathcal{E}(G)$ be the set of all elementary abelian p-subgroups of G. Let

$m :=\max\{|A||A \in \mathcal{E}(G)\}$ and $\mathcal{A}(G) := \{A \in \mathcal{E}(G)||A| = m\}$. Then set:

$$J(G) := \langle A \mid A \in \mathcal{A}(G)\rangle.$$

Then $J(G)$ is called the *Thompson subgroup*[3] of G with respect to p.

We collect:

Lemma 1.2.23 *(a) $J(G)$ char G.*

(b) If $J(G) \leqslant U \leqslant G$, then $J(G) = J(U)$.

(c) If P is a p-group, then $\Omega(Z(P)) \leqslant J(P)$.

Proof. Clearly A char G for every $A \in \mathcal{E}(G)$ so (a) follows. $J(G) \leqslant U \leqslant G$ clearly implies $J(U) \leqslant J(G)$. The reverse inclusion follows from the fact that U contains every maximal abelian subgroup of G, proving (b). To prove (c), notice that every maximal elementary abelian subgroup of P is self-centralising, thus $\Omega(Z(P)) = \bigcap C_P(A) \leqslant J(P)$. □

It turns out that every $A \in \mathcal{A}(G)$ satisfies the hypothesis of Theorem 1.2.18. This is because if $A^* \leqslant A \in \mathcal{A}(G)$ then $A^*C_V(A^*) \in \mathcal{E}(G)$ and so:

$$|A| \geqslant |A^*C_V(A^*)| = |A^*||C_V(A^*)|/|A^* \cap V| \geqslant |A^*||C_V(A^*)|/|C_V(A)|.$$

This means that Timmesfeld's result can be applied directly to obtain:

Lemma 1.2.24 *[14, 9.2.10] Let $A \in \mathcal{A}(G)$ and set $A_0 := [V, A]C_A([V, A])$. If $[V, A] \neq 1$ then $[V, A_0] \neq 1$.*

The following fact concerning A_0 is also very important:

[3]There is an alternative characterisation of the Thompson subgroup exclusively for p-groups, P, generated by all abelian subgroups of P whose orders coincide with that of the largest abelian subgroup.

Theorem 1.2.25 *Let A_0 be defined as in the previous lemma. Then $A_0 \in \mathcal{A}(G)$ and A_0 acts quadratically on V.*

Proof. We prove the second statement first. Certainly A_0 is elementary abelian and by definition, $[V, A_0, A_0] \leqslant [V, A, A_0] = 1$. Next we show that $|A| = |A_0|$. First decompose A_0 as A^*X where $X := [V, A]$ and $A^* = C_A(X)$. Notice that $C_V(A) = V \cap A = V \cap A^*$ and by definition, $X \cap A = X \cap A^*$. Now by these observations and Remark 1.2.19 we get $|A||A \cap V| = |A||C_V(A)| = |XC_V(A)||A^*|$ and so

$$|A| = |A^*||XC_V(A)|/|C_V(A)| = |A^*||X|/|X \cap C_V(A)| = |A^*||X|/|X \cap A^*| = |A^*X| = |A_0|,$$

as required. □

1.2.3 p-stable Groups

The notion of a p-stable group has varied historically. A group is p-stable if it acts p-*stably* on certain of its subgroups, and the definition varies depending on precisely which subgroups these are chosen to be. The definition we use will allow us to prove Theorem A most easily.

Definition 1.2.26 Let V be an elementary abelian p-group. We say that the action of a group G on V is *(weakly) p-stable* if for every $a \in G$,

$$[V, a, a] = 1 \text{ implies } aC_G(V) \in O_p(G/C_G(V)).$$

We can now define what it means for a group to be p-stable:

Definition 1.2.27 We say that a group G is *p-stable*[4] if for $P \in \mathrm{Syl}_p(G)$:

[4]In [19], this definition is a variant of what is termed weak p-stability (of a finite group).

(a) G is p-stable on V whenever V is an elementary abelian p-group with $V \trianglelefteq G$ and also on $O_p(G)/\Phi(O_p(G))$.

(b) $N_G(J(P))$ is p-stable on V whenever $V \trianglelefteq N_G(J(P))$ and $V \leqslant \Omega(Z(J(P)))$.

This definition seems rather technical, but it is precisely what is needed to prove Theorem A. Fortunately there are certain conditions on a group which are easier to describe that imply p-stability:

Proposition 1.2.28 *[14, 9.4.5] Suppose $p \neq 2$. If any of the following conditions holds then G is p-stable:*

(a) $|G|$ is odd.

(b) G has abelian Sylow 2-subgroups.

(c) $SL_2(p)$ is not involved in G.

Where in fact (b) implies (c). [5] Now consider the following subgroup:

Definition 1.2.29 We define $Qd(p) := (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes SL_2(p)$, where we are viewing $\mathbb{Z}_p \times \mathbb{Z}_p$ as a vector space on which $SL_2(p)$ acts naturally.

We show that the group $Qd(p)$ is not p-stable:

Example 1.2.30 [9, 11.4] Set G=$Qd(p)$, $S \in \mathrm{Syl}_p(G)$. Clearly $|S| = p^3$ since $|SL_2(p)| = p(p+1)(p-1)$. If $p = 2$,then S is dihedral. If p is odd, the characteristic subgroups of S are $1, Z(S)$, and S and for $p = 2$, we also have the cyclic group of order 4. Now $H := O_p(G)$ is elementary abelian of order p^2 and so $[H, S, S] = 1$. Also, $G/H \cong SL_2(p)$ and $C_G(H) = H$. This means that $S/H \nsubseteq O_p(G/H)$ so G is not p-stable on H and hence G is not p-stable.

[5] This is a result due to Dickson whose proof can be found in [11], 2.8.4.

In fact we have the following:

Proposition 1.2.31 (Glauberman) *[9, 14.6] For p, an odd prime and G a finite group, the following are equivalent:*

(a) No section of G is isomorphic to $Qd(p)$.

(b) Every section of G is p-stable.

Thus when a group is $Qd(p)$-free, it is also p-stable.

1.3 The ZJ-Theorems for Finite Groups

1.3.1 Glauberman's ZJ-Theorem

We are now ready to prove Theorem A. The proof is due to Stellmacher and will closely follow that found in Chapter 9 of [14]. The idea is to construct a characteristic subgroup, $W(P)$ of a p-group P containing $Z(P)$, which is normal in certain groups G for which it is a Sylow p-subgroup. This approach is justified since we are interested only in the fact this group *exists* under certain conditions; computing $W(P)$ would be no more difficult than computing $Z(J(P))$ for practical applications. We have the following setup for $W(P)$:

Definition 1.3.1 We denote by $\mathcal{C}_J(P)$ the class of all embeddings (if they exist) (τ, H) which satisfy the following:

$\mathcal{C}_1$ H is a group of characteristic p and $\tau : P \to H$ is a monomorphism.

$\mathcal{C}_2$ $P^\tau \in \mathrm{Syl}_p(H)$.

$\mathcal{C}_3$ $J(P^\tau) \trianglelefteq H$.

$\mathcal{C}_4$ H is p-stable on V whenever $V \leqslant \Omega(Z(J(P^\tau)))$.

We define the subgroup $W(P)$ as follows. For any p-group, Q let

$$A(Q) := \Omega(Z(Q)) \text{ and } B(Q) := \Omega(Z(J(Q))).$$

Notice that these groups are elementary abelian by Remark 1.2.21 and that $A(Q) \leqslant B(Q)$. Notice also that for any automorphism, η of Q, we have $B(Q^\eta) = \Omega(Z(J(Q^\eta))) = \Omega(Z(J(Q)^\eta)) = \Omega(Z(J(Q))^\eta) = \Omega(Z(J(Q)))^\eta = B(Q)^\eta$, since the groups $J(X)$, $Z(X)$ and $\Omega(X)$ are all characteristic in G, for $X \leqslant G$. A similar argument shows that $A(Q^\eta) = A(Q)^\eta$.

Now set $W_0 := A(P)$ and assume that we have defined W_k for $0 \leqslant k \leqslant i-1$. If ${W_{i-1}}^\tau \trianglelefteq H$ for every (τ, H) in $\mathcal{C}_J(P)$, then set $W(P) = W_{i-1}$. If this is not the case, then we can pick some (τ_i, H_i) in $\mathcal{C}_J(P)$ for which ${W_{i-1}}^\tau \ntrianglelefteq H$. Now define W_i so that:

$$W_i^{\tau_i} = \langle ({W_{i-1}}^{\tau_i})^{H_i} \rangle.$$

Then we have:

$$A(P^{\tau_i}) \leqslant {W_{i-1}}^{\tau_i} < W_i^{\tau_i} \leqslant B(P^{\tau_i}) \trianglelefteq H_i.$$

The final normality statement follows here since $J(P^{\tau_i}) \trianglelefteq H_i$ and $\Omega(Z(N)) \trianglelefteq G$ for $N \trianglelefteq G$. So applying τ^{-1}, gives us:

$$A(P) \leqslant W_{i-1} < W_i \leqslant B(P).$$

This recursive definition must end by the fact that $B(P)$ is a finite upper bound, so for $m \in \mathbb{N}$ there must exist a chain of subgroups:

$$A(P) = W_0 < W_1 < ... < W_m = W(P) \leqslant B(P)$$

so that $W(P)^{\tau} \trianglelefteq H$ for every (τ, H) in $\mathcal{C}_J(P)$.

Lemma 1.3.2 *The definition of $W(P)$ is independent of the choice of the pairs (τ_i, H_i) used in its construction. In particular $W(P)$ is characteristic in P.*

Proof. Suppose we have defined $W'(P)$ analagously, so that

$$W_0 = W_0' < ... < W_{m'}' =: W'(P)$$

for $(\tau_i', H_i') \in \mathcal{C}_J(P)$. We claim that $W(P) \leqslant W'(P)$. If $W(P) \nleqslant W'(P)$ then there exists a $j \in \mathbb{N}$ such that $W_j \leqslant W'(P)$ and $W_{j+1} \nleqslant W'(P)$. Notice that $W_j^{\tau_{j+1}} \leqslant W'(P)^{\tau_{j+1}} \trianglelefteq H_{j+1}$, so $X := \langle (W_j^{\tau_{j+1}})^{H_{j+1}} \rangle \leqslant W'(P)^{\tau_{j+1}}$. This means that $X^{\tau_{j+1}^{-1}} \leqslant W'(P)$, a contradiction to its definition. A similar argument implies that $W'(P) \leqslant W(P)$ and hence $W(P) = W'(P)$. $\square$

We will need two fairly technical lemmas to prove Theorem A. Here is the first:

Lemma 1.3.3 *For $x \in P$ satisfying $[W(P), x, x] = 1$, we have that $[W(P), x] = 1$.*

Proof. Clearly $[W_0, x] = 1$ for $x \in P$ since $W_0 = A(P) \leqslant Z(P)$. Suppose P is a counterexample. Then for some $i \in \mathbb{N}$ and $y \in P$, $[W_i, y, y] = 1$ implies that $[W_i, y] \neq 1$. Let this i be minimal, so in particular, $[W_{i-1}, y, y] = 1$ will satisfy $[W_{i-1}, y] = 1$. Applying τ_i (as defined above) we get:

$$[W_i^{\tau_i}, y^{\tau_i}, y^{\tau_i}] = 1 \text{ but } [W_i^{\tau_i}, y^{\tau_i}] \neq 1.$$

Set $a := y^{\tau_i}$ and consider $C := C_{H_i}(W_i^{\tau_i})$. Let L be a the normal subgroup of H_i containing C such that $L/C = O_p(H_i/C)$. Then using the p-stability criterion $(\mathcal{C}_4)$ applied to $W_i^{\tau_i}$ we have that $aC \in O_p(H_i/C)$. Recall $P^{\tau_i} \in \mathrm{Syl}_p(H_i)$ so clearly $Q := P^{\tau_i} \cap L \in \mathrm{Syl}_p(L)$. Thus

$L = CQ$ since C is a p'-group. Now, by Lemma 1.1.3 we get $H_i = N_{H_i}(Q)L = N_{H_i}(Q)C$ since $Q \leqslant N_{H_i}(Q)$. So

$$W_i{}^{\tau_i} = \langle ({W_{i-1}}^{\tau_i})^{H_i} \rangle = \langle ({W_{i-1}}^{\tau_i})^{N_{H_i}(Q)} \rangle$$

(The last equality follows from the fact that $C_{H_i}(W_i^{\tau_i}) \leqslant N_{H_i}(W_{i-1}^{\tau_i})$). So there must be some $h \in N_{H_i}(Q)$ with $[({W_{i-1}}^{\tau_i})^h, a] \neq 1$, i.e $[W_{i-1}, x] \neq 1$ for $x := (a^{h^{-1}})^{\tau_i{}^{-1}}$. But

$$[W_{i-1}, x, x] = [({W_{i-1}}^{\tau_i})^h, a, a]^{h^{-1}\tau_i^{-1}} \leqslant [W_i^{\tau_i}, a, a]^{h^{-1}\tau_i^{-1}} = 1.$$

Thus, we have contradicted $[W_{i-1}, x, x] \neq 1$, as required. $\square$

In the proof of Theorem A we will consider separately the cases when $J(P^\tau) \trianglelefteq H$ and $J(P^\tau) \ntrianglelefteq H$. For the second case, we need to prove a result about a similar but slightly modified class of embeddings:

Definition 1.3.4 We denote by $\mathcal{C}'_J(P)$ the class of all embeddings (if they exist) (τ, H) which satisfy the following:

- $\mathcal{C}'_1$ H is a group of characteristic p and τ is monomorphism from P into H.
- $\mathcal{C}'_2$ $P^\tau \in \mathrm{Syl}_p(H)$.
- $\mathcal{C}'_3$ $J(P^\tau) \ntrianglelefteq H$ and $(\tau, N_H(J(P^\tau))) \in \mathcal{C}_J(P)$.
- $\mathcal{C}'_4$ H is p-stable on V whenever V is elementary abelian with $V \trianglelefteq H$ and also on $O_p(H)/\Phi(O_p(H))$.

The technicalities in $\mathcal{C}'_4$ are necessary to make the proof work. They are also the reason we defined p-stability as we did in Definition 1.2.27. This lemma is the crux of the proof:

Lemma 1.3.5 *$W(P)^\tau \trianglelefteq H$ for $(\tau, H) \in \mathcal{C}'_J(P)$*

Proof. For $(\tau, H) \in \mathcal{C}'_J(P)$, write $W := W(P)^\tau$. Since, $O_p(H)^{\tau^{-1}}, W(P) \trianglelefteq P$, we have

$$[O_p(H)^{\tau^{-1}}, W(P)] \leqslant O_p(H)^{\tau^{-1}} \cap W(P)$$

which implies that

$$[O_p(H), W] \leqslant O_p(H) \cap W.$$

In particular, $[O_p(H), W, W] = 1$ so $[V, W, W] = 1$ for $V := O_p(H)/\Phi(O_p(H))$. By p-stability in $\mathcal{C}'_4$, we get that

$$WC_H(V)/C_H(V) \leqslant O_p(H/C_H(V))$$

which by Lemma 1.2.11 implies that $W \leqslant O_p(H)$. But then $W^{\tau^{-1}} \leqslant O_p(H)^{\tau^{-1}}$, and since $W^{\tau^{-1}} = W(P) \trianglelefteq P$, W is normal in every Sylow p-subgroup of H and so $W \trianglelefteq O_p(H)$ char H. So $W \trianglelefteq H$ and for $h \in H$, $W = W^h \trianglelefteq O_p(H)$. Now since $[W, W^h] \leqslant W \cap W^h$, we get that $[W, W^h, W^h] = 1$ and applying τ^{-1}, then gives $[W(P), (W^h)^{\tau^{-1}}, (W^h)^{\tau^{-1}}] = 1$. Applying Lemma 1.3.3 gives $[W(P), (W^h)^{\tau^{-1}}] = 1$ so $[W, W^h] = 1$. In particular $[W^h, W^g] = 1$, for all $h, g \in H$ and the group $W^* := \langle W^h \mid h \in H \rangle$ is elementary abelian. We finish off the proof by considering two cases: **Case 1** $[W^*, J(P^\tau)] = 1$. The H-invariance of W^* implies that $[W^*, J(P^\tau)^h] = 1$ for every $h \in H$. In particular, $[W^*, J(H)] = 1$ and $J(H) = C_H(W^*)$. Now $P^\tau \leqslant H$ implies that $J(P^\tau) \leqslant J(H)$, so there is a $T \in \mathrm{Syl}_p(J(H))$ with $J(P^\tau) = J(T)$. Lemma 1.1.3 then gives the factorisation: $H = J(H)N_H(T) = C_H(W^*)N_H(J(P^\tau))$. So $W^* := \langle W^h \mid h \in H \rangle = \langle W^h \mid h \in N_H(J(P^\tau)) \rangle$. Our condition $\mathcal{C}'_3$ now gives $(\tau, N_H(J(P^\tau))) \in \mathcal{C}_J(P)$, so that $W = W(P)^\tau \trianglelefteq N_H(J(P^\tau))$. But then $W^* := \langle W^h \mid h \in N_H(J(P^\tau) = W$ and $W = W(P)^\tau \trianglelefteq H$ as required.

Case 2 $[W^*, J(P^\tau)] \neq 1$ We show that this leads to a contradiction. We begin similarly as in the proof of Lemma 1.3.3, by setting L to be the group satisfying $C_H(W^*) \leqslant L \trianglelefteq H$

and $L/C_H(W^*) = O_p(H/C_H(W^*)$. Let $Q := P^\tau \cap L$ be in $\mathrm{Syl}_p(L)$ so that $L = QC_H(W^*)$ and apply the Frattini Lemma to get $H = LN_H(Q) = C_H(W^*)N_H(Q)$. So $W^* = \langle W^h \mid h \in N_H(Q)\rangle$, by a familiar argument. By Theorem 1.2.24, there is some $A^* \in \mathcal{P}^\tau$ such that $[W^*, A^*] \neq 1$, $A^* \leqslant Q$. Then $A^* \leqslant J(Q) \leqslant J(P^\tau)$ implying $[W, J(Q)] = 1$. But then

$$1 \neq [W^*, A^*] \leqslant [Q^*, J(Q)] = 1,$$

a contradiction. □

Our main result is now almost obvious from what we have proved:

Theorem 1.3.6 (Theorem A) *Let P be a p-group. Then there is a characteristic subgroup $W(P)$ of P which satisfies:*

(a) $\Omega(Z(P)) \leqslant W(P) \leqslant \Omega(Z(J(P)))$.

(b) If G is a p-stable group of characteristic p, with $P \in Syl_p(G)$, then $W(P) \trianglelefteq G$.

(c) $W(P^\eta) = W(P)^\eta$, where η is any automorphism of P.

Proof. Only (b) remains to be seen. Suppose G is as stated. If $J(P) \trianglelefteq G$, then, then we certainly have $(id, G) \in \mathcal{C}_J(P)$ (see Definition 1.2.27). Thus, by its construction $W(P) \trianglelefteq G$ and we are done. If $J(P) \ntrianglelefteq G$, a quick check shows that $(id, G) \in \mathcal{C}'_J(P)$ (again by Definition 1.2.27 $(id, N_G(J(P))) \in \mathcal{C}_J(P)$) and in this case Lemma 1.3.5 implies $W(P) \trianglelefteq G$. □

In Theorem G, we see that in fact this result holds for all saturated fusion systems. Consider the following definitions:

Definition 1.3.7 A *positive characterstic p functor* is a map, U from a non-trivial p-group, P to $U(P)$ satisfying $U(P\phi) = U(P)\phi$ for all $\phi \in \mathrm{Aut}(P)$. A *Glauberman functor* is a positive characterstic p functor, U, such that whenever $P \in \mathrm{Syl}_p(G)$, where G is a $Qd(p)$-free group of characteristic p, $U(P) \trianglelefteq G$.

Example 1.3.8 It is clear that the functors J, W and $Z(J(-))$ are all positive characterstic p functors. Theorem A and Glauberman's ZJ-Theorem show that when p is odd, W and $Z(J(-))$ are Glauberman functors. There are other examples of such functors, some of which are listed in Section 14 of [9].

When $p = 2$, $G := Qd(2) \cong V_4 \rtimes S_3 \cong S_4$ and it was shown in Example 1.2.30 that G is not 2-stable and hence Glauberman's ZJ-Theorem does not hold. However with the functor $Z(J(-))$ replaced by W, the result does hold, thanks to Stellmacher's main result in [18] [6]:

Theorem 1.3.9 (Stellmacher) *[16, 6.8] Let G be an S_4-free ($Qd(2)$-free) finite group. Let $P \in Syl_2(G)$. Suppose that G has characteristic 2. Then there exists a non-trivial characteristic subgroup, $W(P)$ of P which is normal in G.*

Combining this with Theorem A, we get the following result, independent of the nature of the prime p:

Theorem 1.3.10 *Let G be an $Qd(p)$-free finite group. Let $P \in Syl_p(G)$ and suppose that G has characteristic p. Then there exists a non-trivial characteristic subgroup, $W(P)$ of P which is normal in G.*

1.3.2 Glauberman and Thompson's p-nilpotency Theorem

Having proved Theorem A it is in fact possible to derive Theorem B as a consequence:

Theorem 1.3.11 (Theorem B) *Let G be a group, p an odd prime and $P \in Syl_p(G)$. Then $N_G(W(P))$ has a normal p-complement if and only if G has a normal p-complement.*

[6] For a brief discussion of the technical reasons that allow Stellmacher's result to be reformulated in this way, see Lemmas 6.5 and 6.6 and Remark 6.7 in [16].

Proof. Note first that since the property of having a normal p-complement is inherited by subgroups (and quotients), we only need to prove the forward direction. Let G be a minimal counterexample and let $N := O_{p'}(G)$. We claim $N = 1$. To see this, suppose $N \neq 1$ and set $\overline{G} := G/N$. Then $\overline{P} \in \mathrm{Syl}_p(\overline{G})$ and in fact $\overline{P} \cong P$ since the natural surjection $P \to \overline{P}$ is also injective. So by 1.3.2 above, we have $W(\overline{P}) = \overline{W(P)}$. Furthermore by Lemma 1.1.17, $N_{\overline{G}}(W(\overline{P})) = \overline{N_G(W(P))}$. Thus since $\overline{N_G(W(P))}$ must have a normal p-complement, so must $N_{\overline{G}}(W(\overline{P}))$ and this means $\overline{G}$ also satisfies the hypothesis. But now if $|\overline{G}| < G$, then the minimality of G implies $\overline{G}$ has a normal p-complement. But the preimage of such a complement is also one for G (by the definition of $O_{p'}(G) = N$), a contradiction. This proves the claim.

Now, Since G does not possess a normal p-complement, Frobenius' Theorem 1.1.24 implies that the set of non-trivial p-subgroups, whose normalisers have no normal p-complement is non-empty. Let S be such a subgroup, chosen so that $|N_G(S)|_p$ is as large as possible. We claim that $S \trianglelefteq G$. If not then $G' := N_G(S) < G$. By Sylow's Theorem, we can assume (after conjugating by a suitable element of G), that $T := N_G(S) \cap P \in \mathrm{Syl}_p(G')$. Certainly $T < P$, else $N_G(S)$ has a normal p-complement by the minimality of G, a contradiction. So by Lemma 1.1.18 $T < N_P(T)$. If U char T, we have $T < N_P(T) \leqslant N_P(U) \leqslant N_G(U)$ and so the maximality of $|N_G(S)|_p$ implies that $N_G(W(T))$ has a normal p-complement, as then does $N_{G'}(W(T))$. So G' satisfies the hypothesis of the theorem and since $|G'| < |G|$, G' has a normal p-complement. Finally this implies S has a normal p-complement, a contradiction. In particular, we have proved that $O_p(G) \neq 1$.

Now set $\overline{G} := G/O_p(G)$ and let N be the preimage in G of the subgroup $N_{\overline{G}}(W(\overline{P}))$ of $\overline{G}$. Now $|\overline{G}| < |G|$ and $1 \neq W(\overline{P}) \ntrianglelefteq \overline{G}$, since $O_p(\overline{G}) = 1$ which means $\overline{N} < \overline{G}$ so that $N < G$. The minimality of G implies that $\overline{G}$ has a normal p-complement and so G is

p-soluble. Also, recall $O_{p'}(G) = 1$ so by Lemma 1.1.12 we get:

$$C_G(O_p(G)) \leqslant O_p(G) \text{ and } \overline{G} \text{ has a normal } p\text{-complement, } \overline{K}\ .$$

If G is p-stable then by Theorem A, $G = N_G(W(P))$, a contradiction. So G is not p-stable and by Proposition 1.2.28, $\overline{G}$ and hence $\overline{K}$ have non-abelian Sylow 2-subgroups. Since $\overline{P}$ is acting coprimely on $\overline{K}$, Lemma 1.2.13 implies the existence of a $\overline{S}$-invariant Sylow 2-subgroup, $\overline{T}$ of $\overline{K}$. In particular, $Z(\overline{T})$ is $\overline{S}$-invariant. Now consider the preimage, U of the group $Z(\overline{T})\overline{S}$ in G. Since T is non-abelian, $Z(\overline{T}) < \overline{K}$ and so this preimage cannot be all of G. So $U < G$ means U possesses a non-trivial normal p-complement U', say, satisfying, $[U', O_p(G)] \leqslant U' \cap O_p(G) = 1$. But then $U' \leqslant C_G(O_p(G))$ and $U' \nleqslant O_p(G)$, implying that $C_G(O_p(G)) \nleqslant O_p(G)$ which is a contradiction. Thus G is not a counterexample and we are done.

Chapter 2

Fusion Systems

2.1 The Theory of Fusion Systems

2.1.1 Motivation and Definitions

Fusion systems arose gradually over the course of the last two decades and the idea is attributed to Puig [1]. They are an attempt to axiomatise the notion of fusion in the context of both finite groups and p-blocks [2], by encoding the fusion data in a category. The focus will be the generalisation of group-theoretic results (involving fusion) to arbitrary fusion systems. This idea has been employed, most recently by Kessar and Linckelman (see, [12], [13]) amongst others. It has also been used by Aschbacher in [3] in an effort to simplify results used in the CFSG [3]. This followed some speculative observations that recognising simple fusion systems may be easier than recognising finite simple groups. The goal of this section will be to develop the basic theory of fusion systems, proving some important results for our later discussion.

For any group, G, let $c_g \in \mathrm{Aut}(G)$ denote conjugation by an element of G. For $A, B \leqslant G$, define $\mathrm{Hom}_G(A, B) := \{c_g|_A \mid A^g \leqslant B\}$ to be the set of all injective group

[1] In [17], Puig called them 'Full Frobenius Systems.'

[2] p-blocks arise in modular representation theory. For a group G and field, K, of characteristic p, such that $p \mid |G|$ they are defined as indecomposable two-sided ideals of the group algebra KG.

[3] The Classification of Finite Simple Groups.

homomorphisms given by conjugation by an element of G. Set $\mathrm{Aut}_G(A) := \mathrm{Hom}_G(A, A)$. For $Q \leqslant G$ and $H \leqslant N_G(Q)$, we write H^* for the group of automorphisms induced by elements of H, so for example, $N_G(Q)^* :=\mathrm{Aut}_G(Q)$, etc. [4]

Definition 2.1.1 Let $H \leqslant G$. A subgroup K of G is said to *control G-fusion in H* provided: whenever two subsets A, B of H are conjugate via a map $c_g : A \to B$, for some $g \in G$ then there is a $k \in K$ such that $c_k|_A = c_g$.

Definition 2.1.2 Given any group, P, we define a *fusion system*, $\mathcal{F}$ to be a category whose objects are the subgroups of P and whose morphisms, $\mathrm{Hom}_{\mathcal{F}}(Q, R)$, satisfy:

(a) $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R) \subseteq \mathrm{Inj}(Q, R)$ [5],

(b) for every $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, the map $\phi : Q \to Q\phi$ is in $\mathrm{Hom}_{\mathcal{F}}(Q, Q\phi)$, and

(c) if $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ is an isomorphism, $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$.

We will be particularly interested in the case where P is a finite p-group.

Definition 2.1.3 When $P \in \mathrm{Syl}_p(G)$, we write $\mathcal{F}_P(G)$ for the fusion system on P whose morphisms satisfy $\mathrm{Hom}_{\mathcal{F}_{P(G)}}(Q, R) = \mathrm{Hom}_G(Q, R)$.

Example 2.1.4 Suppose $P \in \mathrm{Syl}_p(G)$ and N is such that $P \leqslant N \leqslant G$. If N controls G-fusion in P then $\mathcal{F}_P(N) = \mathcal{F}_P(G)$. Thus by Theorem 1.1.8, $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$ when P is abelian.

2.1.2 Saturation

As can be seen from its definition, a fusion system (in itself), has fairly limited structure; it requires a certain furnishing termed *saturation*. To define what this means, some definitions are required:

[4]Note that when $A \leqslant H \leqslant G$, by $\mathrm{Aut}_H(A)$, we mean $N_H(A)^*$ and not $N_H(A)/C_H(A)$.

[5]This is the set of all injective group homomorphisms from Q to R.

Write $\mathrm{Iso}_{\mathcal{F}}(Q,R)$ for the set of isomorphisms from Q to R in $\mathcal{F}$. Suppose that $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q,R)$. One can ask: What is the largest subgroup of P to which φ can possibly extend? The following lemma provides the answer:

Lemma 2.1.5 *[6, 4.7] Let $\mathcal{F}$ be a fusion system on a finite p-group P. Suppose that there is $\varphi \in Iso_{\mathcal{F}}(Q,R)$ for subgroups Q,R of P. If $\overline{\varphi} : S \to P$, with $S \leqslant N_P(Q)$ then $S\overline{\varphi} \leqslant N_P(R)$. In particular $S^* \leqslant Aut_P(Q) \cap\ Aut_P(R)^{\varphi^{-1}}$.*

Proof. Let $x \in S$. Then clearly $g^x \in Q$ for all $g \in Q$ and so $(g\overline{\varphi})^{x\overline{\varphi}} = (x\overline{\varphi})^{-1}g\overline{\varphi}x\overline{\varphi} = (g^x)\overline{\varphi} \in R$ which implies that $x\overline{\varphi} \in N_P(R)$. Thus $S\overline{\varphi} \leqslant N_P(R)$. In particular, $(S^*)^{\varphi} \leqslant \mathrm{Aut}_P(R)$, which proves the second part. □

The following subgroup is thus the largest subgroup of P to which φ can possibly extend:

Definition 2.1.6 Let $\mathcal{F}$ be a fusion system over a finite p-group, P and let $Q \leqslant P$. For every $\varphi : Q \to R$ we define the *control of* φ to be the subgroup of $N_P(Q)$ satisfying $N_{\varphi}^* = \mathrm{Aut}_P(Q) \cap \mathrm{Aut}_P(Q\varphi)^{\varphi^{-1}}$, or equivalently:

$$N_{\varphi} = \{y \in N_P(Q) \mid \text{ there is } z \in N_P(R) \text{ such that } (u^y)\varphi = (u\varphi)^z \text{ for every } u \in Q\}.$$

Remark 2.1.7 Clearly $QC_P(Q) \leqslant N_{\varphi} \leqslant N_P(Q)$.

We say that two subgroups, Q and R are $\mathcal{F}$*-conjugate* provided there exists an isomorphism from Q to R. Also define:

$$Q^{\mathcal{F}} := \{Q\phi \mid R \in \mathcal{F} \text{ and } \phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)\}$$

to be the $\mathcal{F}$*-conjugacy class* of Q in $\mathcal{F}$.

Definition 2.1.8 Let $\mathcal{F}$ be a fusion system over a finite p-group, P.

(a) A subgroup, Q, of P is called *fully $\mathcal{F}$-centralised* if $|C_P(R)| \leqslant |C_P(Q)|$ for any $R \in Q^{\mathcal{F}}$.

(b) A subgroup, Q, of P is called *fully $\mathcal{F}$-normalised* if $|N_P(R)| \leqslant |N_P(Q)|$ for any $R \in Q^{\mathcal{F}}$.

Put another way, a subgroup Q is fully $\mathcal{F}$-normalised ($\mathcal{F}$-centralised) if its normaliser (centraliser) is of maximal order among all subgroups in its $\mathcal{F}$-conjugacy class. In particular, every subgroup is $\mathcal{F}$-isomorphic to a fully $\mathcal{F}$-normalised ($\mathcal{F}$-centralised) subgroup.

We define $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$. The following definitions will be very important in what follows.

Definition 2.1.9 Let $\mathcal{F}$ be a fusion system on a finite p-group, P. Then for $Q \leqslant P$ we say that:

(a) Q is *$\mathcal{F}$-centric* if $C_P(R) \leqslant R$ for any $R \in Q^{\mathcal{F}}$.

(b) Q is *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(Q)$ is p-reduced.

(c) Q is *$\mathcal{F}$-essential* if $\mathrm{Out}_{\mathcal{F}}(Q)$ contains a strongly p-embedded subgroup.

Notice by the remark following Definition 1.1.9, every $\mathcal{F}$-essential subgroup is $\mathcal{F}$-radical. As is customary, we will denote by $\mathcal{F}^f, \mathcal{F}^z, \mathcal{F}^c, \mathcal{F}^r$ and $\mathcal{F}^e$, the set of fully $\mathcal{F}$-normalised, fully $\mathcal{F}$-centralised, $\mathcal{F}$-centric, $\mathcal{F}$-radical and $\mathcal{F}$-essential subgroups of $\mathcal{F}$ respectively. Also for any $X \subseteq \{f, z, c, r, e\}$, $\mathcal{F}^X := \bigcap_{x \in X} \mathcal{F}^x$.

We now define what it means for a fusion system to be saturated:

Definition 2.1.10 Let $\mathcal{F}$ be a fusion system over a finite p-group, P. Then we say that $\mathcal{F}$ is *saturated* if the following two conditions are satisfied:

(a) **Sylow Axiom:** $\mathrm{Aut}_P(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(b) **Extension Condition:** Every morphism $\varphi : Q \to P$ for which $Q\varphi \in \mathcal{F}^f$ extends [6] to a morphism $\psi : N_\varphi \to P$.

The standard definition of a saturated fusion system has been a subject of debate. The one presented here is equivalent to that given in [4] where in part (b), the condition that $Q\varphi$ is fully $\mathcal{F}$-normalised is strengthened to the condition that it is fully $\mathcal{F}$-centralised. We prove this fact in Lemma 2.1.15.

We will soon see (in Theorem 2.1.35) that $\mathcal{F}_P(G)$ is saturated, providing us with our first example.

To conclude this section, we demonstrate that saturated fusion systems form a category; that is, there exist *morphisms* between them:

Definition 2.1.11 [7, 5.4] Let P and Q be p-groups and $\mathcal{F}$ and $\mathcal{G}$ be fusion systems on P and Q respectively. A *morphism* $\Phi : \mathcal{F} \to \mathcal{G}$ is a pair $(\phi, \{\phi_{R,S} \mid R, S \leqslant P\})$, where $\phi : P \to Q$ is a group homomorphism and for all $R, S \leqslant P, \phi_{R,S}$ is a function

$$\phi_{R,S} : \mathrm{Hom}_{\mathcal{F}}(R, S) \to \mathrm{Hom}_{\mathcal{G}}(R\phi, S\phi)$$

so that Φ is a functor.

Notice that the underlying group homomorphism, ϕ, determines the morphism completely.

2.1.3 Saturated Subsystems of Fusion Systems

Of fundamental importance will be the facility to form new fusion systems from old. In this section we develop the important notions of normaliser and centraliser fusion subsystems of a fusion system, $\mathcal{F}$, which give rise to normal subgroups and centres of $\mathcal{F}$. We also prove the important result that when $\mathcal{F}$ is saturated, so are certain subsystems.

[6]We mean that $\psi|_Q = \varphi$.

Our development is based on that in [8] and [15]. Throughout this section we let P be a p-group and let $\mathcal{F}$ be a saturated fusion system on P.

The following result and its corollary are both very useful:

Proposition 2.1.12 *[15, 2.5] Let $Q \in \mathcal{F}$. Then $Q \in \mathcal{F}^f$ if and only if $Q \in \mathcal{F}^z$ and $Aut_P(Q) \in Syl_p(Aut_{\mathcal{F}}(Q))$.*

Proof. Suppose $Q \in \mathcal{F}^f$. To see that $Q \in \mathcal{F}^z$, pick $R \in \mathcal{F}^z$ such that there is an isomorphism $\varphi : R \to Q$. By Definition 2.1.10 (ii), there is $\psi : RC_P(R) \to P$ such that $\psi|_R = \varphi$. So $C_P(R)$ is mapped to $C_P(Q)$. Thus since $R \in \mathcal{F}^z$, $|C_P(R)| = |C_P(Q)|$ and then $Q \in \mathcal{F}^z$. For the second part let Q, be a maximal counterexample. Certainly $Q < P$, by Definition 2.1.10 (i). Pick $A \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$ so that $\mathrm{Aut}_P(Q) \trianglelefteq A$. Let $\varphi \in A - \mathrm{Aut}_P(Q)$. Then since $\mathrm{Aut}_P(Q) \cap \mathrm{Aut}_P(Q\varphi)^{\varphi^{-1}} = \mathrm{Aut}_P(Q)$, $N_\varphi = N_P(Q)$. Definition 2.1.10 now implies that there is some morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(N_P(Q))$, such that $\psi|_Q = \varphi$. Furthermore we may assume to have chosen ψ to be p-element. Let $\tau : N_P(Q) \to P$ be picked so that $N_P(Q)\tau \in \mathcal{F}^f$. Then ψ^τ is also a p-element, hence conjugate to an element of $\mathrm{Aut}_P(N_P(Q)\tau)$. So τ may be chosen so that there is some $y \in N_P(N_P(Q)\tau)$ such that $(x)\psi^\tau = x^y$ for each $x \in N_P(Q)\tau$. Notice ψ^τ stabilises $Q\tau$ (since $\psi|_Q = \varphi$), so in fact $y \in N_P(Q\tau)$. Since $Q \in \mathcal{F}^f, N_P(Q\tau) \leqslant (N_P(Q))\tau$, so $u\psi = u^{y\tau^{-1}}$ for every $u \in N_P(Q)$, But $y\tau^{-1} \in N_P(Q)$ implies $\varphi \in \mathrm{Aut}_P(Q)$, a contradiction to its choice. For the converse, note that $|C_P(Q)| \geqslant |C_P(R)|$ for $R \in Q^{\mathcal{F}}$, so:

$$|N_P(Q)| = |\mathrm{Aut}_P(Q)||C_P(Q)| \geqslant |\mathrm{Aut}_P(R)||C_P(R)| = |N_P(R)|$$

as needed. ☐

Corollary 2.1.13 *[15, 2.6] For $Q, R \in \mathcal{F}$, let $\varphi \in Iso_{\mathcal{F}}(Q, R)$ with $R \in \mathcal{F}^f$. Then there exists $\psi \in Iso_{\mathcal{F}}(Q, R)$ such that $N_\psi = N_P(Q)$ and a factorisation $\varphi = \beta^{-1} \circ \psi|_Q$ for some $\beta \in Aut_{\mathcal{F}}(Q)$.*

Proof. Clearly $\mathrm{Aut}_P(Q)^\varphi$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$. By Proposition 2.1.12, $\mathrm{Aut}_P(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R))$ and so there exists some $\beta \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\mathrm{Aut}_P(Q)^{\varphi\beta} \leqslant \mathrm{Aut}_P(R)$. Setting $\psi := \beta\varphi$, this means that for any $y \in N_P(Q)$, there is $z \in N_P(R)$ such that $c_y^\psi = c_z$, i.e. $N_\psi = N_P(Q)$, as needed. □

This means that we may always assume to have picked an $\mathcal{F}$-isomorphism which fully extends when the target lies in $\mathcal{F}^f$. Here is a result of a similar nature which we require later:

Lemma 2.1.14 *Suppose that $Q \in \mathcal{F}^f$ and $K \leqslant Q$ is such that $N_P(K) \geqslant N_P(Q)$. Then there exists $\chi \in Hom_{\mathcal{F}}(Q, P)$ such that $K\chi, Q\chi \in \mathcal{F}^f$.*

Proof. Assume we have picked $\phi \in \mathrm{Hom}_{\mathcal{F}}(K, P)$ so that $K\phi \in \mathcal{F}^f$. By Lemma 2.1.12, $\mathrm{Aut}_P(K\phi) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(K))$. Hence $\mathrm{Aut}_P(K)^\phi$ is conjugate to some subgroup of $\mathrm{Aut}_P(K\phi)$, say $\mathrm{Aut}_P(K)^{\phi\circ\tau} \leqslant \mathrm{Aut}_P(K\phi)$, for some $\tau \in \mathrm{Aut}_{\mathcal{F}}(K\phi)$. Set $\chi := \phi \circ \tau$. Then $N_\chi = N_P(K) \geqslant N_P(Q)$, with $K\chi = K(\phi \circ \tau) = K\phi \in \mathcal{F}^f$ and $Q\chi = Q(\phi \circ \tau) = Q\phi \in \mathcal{F}^f$. □

A consequence of Corollary 2.1.13 is the reassurance that our definition of saturation is equivalent to that in [4]:

Lemma 2.1.15 *[15, 2.7] Let $Q \in \mathcal{F}$. Then every morphism $\varphi : Q \to P$ for which $Q\varphi \in \mathcal{F}^z$ extends to a morphism $\psi : N_\varphi \to P$*

Proof. Let φ be as stated in the hypothesis and let $\rho : Q\varphi \to P$ be such that $(Q\varphi)\rho \in \mathcal{F}^f$, chosen so that $N_\rho = N_P(Q\varphi)$ (Corollary 2.1.13). So ρ extends to a morphism $\overline{\rho} : N_P(Q\varphi) \to P$. This implies that $N_\varphi \subseteq N_{\varphi\rho}$ since if $\overline{\varphi}$ extends φ to some subgroup of $N_P(Q)$, $\overline{\varphi} \circ \overline{\rho}$ extends $\varphi \circ \rho$ to the same subgroup since $N_\rho = N_P(Q\varphi)$. This means that $\varphi\rho$ extends to a morphism $\theta : N_\varphi \to P$. Then observe that $\theta \circ \overline{\rho}^{-1}|_{N_{\varphi\theta}} : N_\varphi \to P$ is the desired map. □

We now define a subsystem of a fusion system, **making no assumptions about saturation**:

Definition 2.1.16 A *fusion subsystem*, $\mathcal{G}$ of $\mathcal{F}$ is a fusion system over a subgroup, $\tilde{P}$ of P such that for all $Q, R \leqslant \tilde{P}$, $\mathrm{Hom}_{\mathcal{G}}(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$.

The following terminology will be useful:

Definition 2.1.17 Let $R, Q \leqslant P$ and let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$. We say that φ *K-stably extends to Q* if $QR \leqslant P$ and there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QR,P)$ such that:

(a) $\overline{\varphi}|_R = \varphi$, and

(b) $\overline{\varphi}|_Q \in K \leqslant \mathrm{Aut}(Q)$.

If $K = \langle id_Q \rangle$ then we say that φ *centrally extends* to Q, and if $K = \mathrm{Aut}_{\mathcal{F}}(Q)$, we say that φ *stably extends to Q*.

The following definition provides us with our first example of a (not necessarily saturated) fusion subsystem:

Definition 2.1.18 For $Q \leqslant P$ and $K \leqslant \mathrm{Aut}(Q)$:

(a) The *K-normaliser of Q, in P*, is the following subgroup of $N_P(Q)$:

$$N_P^K(Q) = \{y \in N_P(Q) \mid c_y|_Q \in K\}.$$

(b) The *K-normaliser of Q in $\mathcal{F}$* is the category $N_{\mathcal{F}}^K(Q)$ whose objects are subgroups of $N_P^K(Q)$ and for two objects R, S, we have:

$$\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(R,S) = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,S) \mid \varphi \text{ } K\text{-stably extends to } Q\}.$$

Notation 2.1.19 We set $\mathrm{Aut}_P^K(Q) := K \cap \mathrm{Aut}_P(Q)$ and $\mathrm{Aut}_{\mathcal{F}}^K(Q) := K\cap \mathrm{Aut}_{\mathcal{F}}(Q)$.

Notice that $\mathrm{Aut}_P^K(Q) = N_P^K(Q)/C_P(Q)$. Observing that the group $K^\varphi := \varphi^{-1}\circ K\circ\varphi \leqslant \mathrm{Aut}(Q\varphi)$, means we can also consider the K^φ-normaliser of $Q\varphi$, in P. This gives rise to the following important definition:

Definition 2.1.20 We say that Q is *fully K-normalised in $\mathcal{F}$* if

$$|N_P^{K^\varphi}(Q\varphi)| \leqslant |N_P^K(Q)| \text{ for any } \varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P).$$

We write $\mathcal{F}_K^f$ for the set of fully K-normalised subgroups of P.

Notice that this generalises Definition 2.1.8. We now generalise Proposition 2.1.12 for all K:

Proposition 2.1.21 *Let $Q \in \mathcal{F}$. Then $Q \in \mathcal{F}_K^f$ if and only if $Q \in \mathcal{F}^z$ and $Aut_P^K(Q) \in Syl_p(Aut_{\mathcal{F}}^K(Q))$.*

Proof. Suppose that $Q \in \mathcal{F}_K^f$. By Corollary 2.1.13, there is some isomorphism $\varphi : Q \to R$ with $R \in \mathcal{F}^f$ so that φ extends to $\psi : N_P(Q) \to P$. Write $L := K^\varphi$ and notice that $N_P^K(Q)\psi \leqslant N_P^L(R)$. But $Q \in \mathcal{F}_K^f$ implies that $N_P^K(Q) \cong N_P^L(R)$, and by restriction, $C_P(Q) \cong C_P(R)$. Now Proposition 2.1.12 implies $R \in \mathcal{F}^z$ so we also have $Q \in \mathcal{F}^z$. For the second part we have that the p-group $(\mathrm{Aut}_P^K(Q))^\varphi \leqslant \mathrm{Aut}_{\mathcal{F}}^L(R)$ and by Proposition 2.1.12 $\mathrm{Aut}_P(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R))$. So Sylow's Theorem implies there is some $\beta \in\mathrm{Aut}_{\mathcal{F}}(R)$ such that $\mathrm{Aut}_P(R)^\beta\cap \mathrm{Aut}_{\mathcal{F}}^L(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^L(R))$. And moreover this β can be chosen in order that this Sylow p-subgroup contains $(\mathrm{Aut}_P^K(Q))^\varphi$. Conjugating by β^{-1} yields $\mathrm{Aut}_P(R)\cap \mathrm{Aut}_{\mathcal{F}}^{L^{\beta^{-1}}}(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^{L^{\beta^{-1}}}(R))$. Lastly $Q \in \mathcal{F}_K^f$ and $\mathrm{Aut}_P^K(Q) = N_P^K(Q)/C_P(Q)$ imply that $|\mathrm{Aut}_P^K(Q)| \geqslant |\mathrm{Aut}_P^{L^{\beta^{-1}}}(R)|$ so $\mathrm{Aut}_P^K(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^K(Q))$.

To prove the converse take φ, R and L as above and notice that the fact that $\mathrm{Aut}_{\mathcal{F}}^K(Q) \cong \mathrm{Aut}_{\mathcal{F}}^L(R)$ means $|\mathrm{Aut}_P^L(R)| \leqslant |\mathrm{Aut}_P^K(Q)|$. This, together with the fact that $Q \in \mathcal{F}^z$, imply that:

$$|N_P^L(R)| = |C_P(R)||\mathrm{Aut}_P^L(R)| \leqslant |C_P(Q)||\mathrm{Aut}_P^K(Q)| \leqslant |N_P^K(Q)|$$

and $Q \in \mathcal{F}_K^f$, as needed. $\square$

The following table neatens up notation for certain subgroups, K, of $\mathrm{Aut}(Q)$:

Notation 2.1.22 Let K be a subgroup of $\mathrm{Aut}(Q)$.

K	$\mathcal{F}_K^f$	$N_P^K(Q)$	$N_\mathcal{F}^K(Q)$
$\langle id_Q \rangle$	$\mathcal{F}^z$	$C_P(Q)$	$C_\mathcal{F}(Q)$
$\mathrm{Aut}_\mathcal{F}(Q)$	$\mathcal{F}^f$	$N_P(Q)$	$N_\mathcal{F}(Q)$
$\mathrm{Aut}_Q(Q)$		$QC_P(Q)$	$QC_\mathcal{F}(Q)$
$\mathrm{Aut}_P(Q)\dagger$		$PC_P(Q)$	$PC_\mathcal{F}(Q)$

$\dagger$ We require that $Q \trianglelefteq P$ in this case.

Lemma 2.1.23 *Let G be a finite group and $P \in Syl_p(G)$. Let $\mathcal{F} = \mathcal{F}_P(G)$. For any $Q \leqslant P$, $Q \in \mathcal{F}_K^f$ if and only if $N_P^K(Q) \in Syl_p(N_G^K(Q))$.*

Proof. Suppose $S \in \mathrm{Syl}_p(N_G^K(Q))$ is such that $N_P^K(Q) \leqslant S$. Since $P \in \mathrm{Syl}_p(G)$, there exists $x \in G$ such that $(QS)^x \leqslant P$ (by Sylow's Theorem). Since $Q^x \leqslant P$, $c_x \in \mathrm{Hom}_\mathcal{F}(Q, P)$, thus c_x induces a group homomorphism $\mathrm{Aut}_P^K(Q) \to \mathrm{Aut}_P^{K^{c_x}}(Q^x)$. This means that $S^x \leqslant N_P^{K^{c_x}}(Q^x)$. If $Q \in \mathcal{F}_K^f$ then

$$|S| = |S^x| \leqslant |N_P^{K^{c_x}}(Q^x)| \leqslant |N_P^K(Q)| \leqslant |S|,$$

so $S = N_P^K(Q)$. Under the same notation, if $N_P^K(Q) \in \mathrm{Syl}_p(N_G^K(Q))$, $|N_P^K(Q)| = |S|$ is the order of a Sylow p-subgroup of $N_G^{K^{c_x}}(Q^x)$ so we have $|N_P^{K^{c_x}}(Q^x)| \leqslant |N_P^K(Q)|$ as needed. $\square$

We use this result to prove that the definition of $N_\mathcal{F}^K(Q)$ makes sense when $\mathcal{F} = \mathcal{F}_P(G)$:

Lemma 2.1.24 *[6, 4.25] Let G be a finite group, $P \in Syl_p(G)$ and $\mathcal{F} = \mathcal{F}_P(G)$. If $Q \in \mathcal{F}_K^f$ then $\mathcal{F}_{N_P^K(Q)}(N_G^K(Q)) = N^K_{\mathcal{F}_P(G)}(Q)$.*

Proof. Write $\mathcal{F} := \mathcal{F}_P(G)$. Note first that by Lemma 2.1.23, $N_P^K(Q) \in \mathrm{Syl}_p(N_G^K(Q))$ so the fusion systems are on the correct subgroups. If $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(A, B)$ then by definition there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QA, QB)$ with $\overline{\varphi}|_Q \in \mathrm{Aut}_{\mathcal{F}}^K(Q) = \mathrm{Aut}_G^K(Q)$, i.e. there is $g \in G$ such that $\overline{\varphi}|_Q = c_g|_Q$. This means that $g \in N_G^K(Q)$ so $c_g : A \to B$ is a morphism in $\mathcal{F}_{N_P^K(Q)}(N_G^K(Q))$. Conversely if $\varphi : A \to B$ lies in $\mathcal{F}_{N_P^K(Q)}(N_G^K(Q))$, there is $g \in N_G^K(Q)$ such that $\varphi = c_g|_A$. Clearly $(QA)^g = QB$ so $c_g : QA \to QB$ extends φ and acts on Q implying $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(A, B)$, as needed. □

Luckily, we have the following important result:

Theorem 2.1.25 (Theorem C) *Let $Q \leqslant P$ and $K \leqslant Aut(Q)$. If Q is fully K-normalised in $\mathcal{F}$, then $N_{\mathcal{F}}^K(Q)$ is a saturated fusion system on $N_P^K(Q)$.*

We will prove this shortly, but first require some further results. In what follows, we assume that $Q \leqslant P$ and that $K \leqslant \mathrm{Aut}(Q)$ as in the hypothesis of the theorem.

Observation 2.1.26 *For an $\mathcal{F}$-isomorphism, $\varphi : Q \to R$, set $L = K^\varphi$. If $R \in \mathcal{F}_L^f$ then there is a morphism:*

$$\tau : QN_P^K(Q) \to P \textit{ such that } \tau|_Q = \kappa \circ \varphi \textit{ for some } \kappa \in K.$$

Proof. Let $X := (QN_P^K(Q))^*$. Our goal is to show that some conjugate of X^φ is contained in $\mathrm{Aut}_P(R)$. We do this by considering the factors of X: First, since $Q^* \trianglelefteq \mathrm{Aut}(Q), Q^{*\varphi} \trianglelefteq \mathrm{Aut}(R)$ and is contained in $\mathrm{Aut}_P(R)$, thus any conjugate of $Q^{*\varphi}$ also has this property.

Second, since $R \in \mathcal{F}_L^f$, $(N_P^K(Q)^*)^\varphi = \mathrm{Aut}_P^K(Q)^\varphi \leqslant \mathrm{Aut}_P^L(R)$. By 2.1.21, $\mathrm{Aut}_P^L(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^L(R))$ and so there is some $\lambda \in \mathrm{Aut}_{\mathcal{F}}^L(R)$ so that $(\mathrm{Aut}_P^K(Q)^\varphi)^\lambda \leqslant \mathrm{Aut}_P^L(R)$ which is a subgroup of $\mathrm{Aut}_P(R)$.

Putting these together we have $(X^{\varphi})^{\lambda} \leqslant \mathrm{Aut}_P(R)$ which means that $QN_P^K(Q) \leqslant N_{\varphi\lambda}$. But now we are done by letting $\tau := \varphi\lambda$ and $\kappa := \lambda^{\varphi^{-1}}$. $\square$

Observation 2.1.27 *Let $\psi : QN_P^K(Q) \to P$ be a morphism in $\mathcal{F}$. Then $Q \in \mathcal{F}_K^f$ implies that $Q\psi \in \mathcal{F}_{K^\psi}^f$.*

Proof. Since ψ maps $N_P^K(Q)$ to $N_P^{K^\psi}(Q\psi)$, injectively, we get $|N_P^K(Q)| \leqslant |N_P^{K^\psi}(Q\psi)|$. Since $Q \in F_K^f$ we have equality and the result holds. $\square$

Observation 2.1.28 *For every $K \leqslant Aut_P(Q), Q \in \mathcal{F}^f$ implies that $Q \in \mathcal{F}_K^f$.*

Proof. For such K, $|N_P^K(Q)| = |C_P(Q)||K|$ (every element of K is realisable by conjugation by an element of $N_P(Q)$), and by Proposition 2.1.12, $Q \in \mathcal{F}^z$ so the result is clear. $\square$

Now for $R \leqslant N_P(Q)$ and $L \leqslant \mathrm{Aut}(R)$, define a set:

$$K \bullet L := \{\alpha \in \mathrm{Aut}(QR) : \alpha|_Q \in K \text{ and } \alpha|_R \in L\}.$$

We have that:

$$N^L_{N_P^K(Q)}(R) = N_P^{K\bullet L}(QR) = N_P^K(Q) \cap N_P^L(R),$$

is just the set of $g \in P$ inducing automorphisms of both Q in K and R in L. We get the following:

Observation 2.1.29 *The restriction map $Aut_{\mathcal{F}}^{K\bullet L}(QR) \to Aut^L_{N_{\mathcal{F}}^K(Q)}(R)$ is surjective.*

Proof. For every $\beta \in \mathrm{Aut}^L_{N_{\mathcal{F}}^K(Q)}(R)$ there is an extension of $\beta, \overline{\beta} \in \mathrm{Aut}_{\mathcal{F}}^L(QR)$ with $\overline{\beta}|_Q \in K$. Since $\beta = \overline{\beta}|_R$, the result follows. $\square$

The following result turns out to be very useful:

Proposition 2.1.30 *Suppose $Q \in \mathcal{F}_K^f$. Then for any $R \leqslant N_P^K(Q)$ and $L \leqslant Aut(R)$, there is a morphism in $\mathcal{F}$:*

$$\varphi : QR \to QN_P^K(Q) \text{ with } \varphi|_Q \in K \text{ and } (QR)\varphi \in \mathcal{F}^f_{(K\bullet L)^\varphi}.$$

Proof. We need to build such a map so start with $\rho : QR \to P$ with $(QR)\rho \in \mathcal{F}^f_{(K\bullet L)^\rho}$ and define $\sigma := \rho|_Q$. Since $Q \in \mathcal{F}_K^f$, Observation 2.1.27 implies that $Q\sigma \in \mathcal{F}^f_{K^\sigma}$ and so we can apply Observation 2.1.26 with $\varphi = \sigma^{-1}$, Q replaced by $Q\sigma$ and $K = K^{\sigma^{-1}}$, implying the existence of a morphism:

$$\tau : (Q\sigma)N_P^{K^{\sigma^{-1}}}(Q\sigma) \to P \text{ such that } \tau|_{Q\sigma} = \kappa \circ \sigma^{-1} \text{ for some } \kappa \in K^{\sigma^{-1}}.$$

Writing $\varphi := \rho \circ \tau$, we have $\varphi|_Q = \rho|_Q \circ \tau = \sigma \circ \tau = \sigma \circ \kappa \circ \sigma^{-1} \in K$ since $\sigma \in \mathrm{Aut}(Q)$.

Since

$$(QR)\rho N_P^{(K\bullet L)^\rho}((QR)\rho) \leqslant (Q)\sigma N_P^{K\sigma-1},$$

we have $(QR)\rho \in \mathcal{F}^f_{(K\bullet L)^\rho}$. The last part follows on applying Observation 2.1.27 to τ and $(QR)\rho$. ☐

As a consequence of this, we have the following:

Corollary 2.1.31 *Let $R \leqslant N_P^K(Q)$ and $L \leqslant Aut(R)$. Then:*

$$R \in N_{\mathcal{F}}^K(Q)_L^f \text{ implies that } QR \in \mathcal{F}^f_{K\bullet L}.$$

Proof. By Proposition 2.1.30 there is $\varphi : QR \to QN_P^K(Q)$ with $\varphi|_Q \in K$ and $(QR)\varphi \in$

$\mathcal{F}^f_{(K\bullet L)^\varphi}$. By an earlier remark, $N_P^{K\bullet L}(QR) = N^L_{N_P^K(Q)}(R)$ and $R \in N^K_{\mathcal{F}}(Q)^f_L$ means that

$$|N^L_{N_P^K(Q)}(R)| \geqslant |N^{L^\varphi}_{N_P^K(Q)}(R\varphi)| = |N_P^{K\bullet L^\varphi}((QR)\varphi)|,$$

where the final equality is a consequence of the same remark. Since $\varphi|_Q \in K$, we have $K \bullet L^\varphi = (K \bullet L)^\varphi$, so

$$|N_P^{K\bullet L}(QR)| \geqslant |N_P^{(K\bullet L)^\varphi}((QR)\varphi)|,$$

as needed. □

Finally we are ready to prove Theorem C:

Proof (of Theorem C). **The Sylow Axiom:** Our goal is to show that for $R := N_P^K(Q)$, $\mathrm{Aut}_R(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{N^K_{\mathcal{F}}(Q)}(R))$. Let $A :=\mathrm{Aut}(R)$ and $L :=\mathrm{Aut}_R(R)$. By Observation 2.1.29, the map $\mathrm{Aut}^{K\bullet A}_{\mathcal{F}}(QR) \to\mathrm{Aut}^A_{N^K_{\mathcal{F}}(Q)}(R)$ surjectively maps $\mathrm{Aut}^{K\bullet A}_{\mathcal{F}}(QR)$ onto L. So it will suffice to demonstrate the existence of a Sylow p-subgroup of $\mathrm{Aut}^{K\bullet A}_{\mathcal{F}}(QR)$ in $\mathrm{Aut}^{K\bullet L}_{\mathcal{F}}(QR)$. By Proposition 2.1.30, with $R = N_P^K(Q)$, we get a morphism $\varphi : N_P^K(Q) \to QN_P^K(Q)$ with $\varphi|_Q \in K$ and $(N_P^K(Q))\varphi \in \mathcal{F}^f_{(K\bullet A)^\varphi} = \mathcal{F}^f_{(K\bullet A)}$. Proposition 2.1.21 then implies that $\mathrm{Aut}^{K\bullet A}_P(QR) \in \mathrm{Syl}_p(\mathrm{Aut}^{K\bullet A}_{\mathcal{F}}(QR))$. Also $\mathrm{Aut}^{K\bullet A}_P(QR) =\mathrm{Aut}^{K\bullet L}_P(QR) \subseteq \mathrm{Aut}^{K\bullet L}_{\mathcal{F}}(QR)$ (since $R = N_P^K(Q)$) and we are done.

The Extension Axiom: Let $R \leqslant N_P^K(Q)$ and pick $\varphi : R \to N_P^K(Q)$, a morphism in $N^K_{\mathcal{F}}(Q)$ so that $R\varphi \in N^K_{\mathcal{F}}(Q)^f$. We are searching for a morphism with source

$$N_\varphi := \{y \in N_{N_P^K(Q)}(R) \mid \text{ there is } z \in N_{N_P^K(Q)}(R\varphi) \text{ such that } (u^y)\varphi = (u\varphi)^z \text{ for every } u \in R\}$$

which extends φ. It is clear that for $L :=\mathrm{Aut}_{N\varphi}(R)$, N_φ can be rewritten as $N^L_{N_P^K(Q)}(R) = N_P^{K\bullet L}(QR)$. Notice also that since $L^\varphi \subseteq\mathrm{Aut}_{N_P^K(Q)}(R\varphi)$, we can apply Observation 2.1.28 and get $R\varphi \in N^K_{\mathcal{F}}(Q)^f_{L^\varphi}$. An application of Corollary 2.1.31, then gives $Q(R)\varphi \in \mathcal{F}^f_{K\bullet L^\varphi}$. Now, by the definition of $N^K_{\mathcal{F}}(Q)$, there is a morphism ψ in $\mathcal{F}$, extending φ, with $\psi|_Q \in K$, so in fact $(QR)\psi = Q(R)\varphi \in \mathcal{F}^f_{K\bullet L^\varphi}$. Since $(K \bullet L^\varphi)^{\psi^{-1}} = \psi \circ K \bullet L\varphi^{-1} = K \bullet L$ ($\varphi \circ \psi^{-1}|_R = \mathrm{Id}_R$ and $\psi|_Q \in K$), Observation 2.1.26 applied to ψ and QR implies the existence of a morphism:

$$\tau : QRN_P^{K\bullet L}(QR) \to P \text{ such that } \tau|_{QR} = \kappa \circ \varphi \text{ for some } \kappa \in K \bullet L.$$

Now recall that $N_P^{K\bullet L}(QR) = N_\varphi$ and notice that since $\kappa|_R \in L, \kappa|_R = c_y$ for some $y \in R$.
Claim: The morphism $c_y^{-1} \circ \tau|_{N_\varphi} : N_\varphi \to P$ lies in $N_{\mathcal{F}}^K(Q)$ and extends φ.

Proof of Claim: We must first show that there is a morphism in $\mathcal{F}$ which K-stably extends $c_y^{-1} \circ \tau|_{N_\varphi}$. We claim that $c_y^{-1} \circ \tau|_{QN_\varphi} : QN_\varphi \to P$ is such a morphism. It clearly extends and its restriction to Q is

$$c_y^{-1} \circ \tau|_Q = c_y^{-1}|_Q \circ \kappa|_Q \circ \psi|_Q \in K.$$

Lastly, the restriction of $c_y^{-1} \circ \tau|_{N_\varphi}$ to R is

$$c_y^{-1}|_R \circ \kappa|_R \circ \psi|_R = \psi|_R = \varphi.$$

Thus we have completed the proof.

Trivially if $N_{\mathcal{F}}^K(Q) = \mathcal{F}$ then $N_P^K(Q) = P$. We have:

Lemma 2.1.32 *Let $Q, R \in \mathcal{F}$ with $K \leqslant Aut(Q)$ and $J \leqslant Aut(R)$. Suppose that $\mathcal{F} = N_{\mathcal{F}}^K(Q) = N_{\mathcal{F}}^J(R)$. Then $\mathcal{F} = N_{\mathcal{F}}^{K\bullet J}(QR)$.*

Proof. Let $\phi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(A, B)$. Then $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, B)$ and there is $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(AQ, BQ)$ such that $\overline{\phi}|_Q \in K$ and $\overline{\phi}|_A \in \phi$. But this implies there is $\tilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(AQR, BQR)$ such that $\tilde{\phi}|_{AQ} = \overline{\phi}$ and $\tilde{\phi}|_R \in J$. This means $\tilde{\phi}|_A = \overline{\phi}_A = \phi$ and $\tilde{\phi}|_Q = \overline{\phi}_Q \in K$. i.e. $\tilde{\phi}$ $(K \bullet J)$-stably extends ϕ as needed. $\square$

Thus we may define the following:

Definition 2.1.33 We denote by $Z(\mathcal{F})$ the largest central subgroup of P such that $C_{\mathcal{F}}(Z(\mathcal{F})) = \mathcal{F}$. We denote by $O_p(\mathcal{F})$ the largest normal subgroup of P such that $N_{\mathcal{F}}(O_p(\mathcal{F})) = \mathcal{F}$.

Notation 2.1.34 We will frequently write $Q \trianglelefteq \mathcal{F}$ in place of $N_{\mathcal{F}}(Q) = \mathcal{F}$.

We now have more than enough machinery to prove the following important fact:

Theorem 2.1.35 *[15, 2.11] Let G be a finite group and $P \in Syl_p(G)$. Then $\mathcal{F}_P(G)$ is a saturated fusion system.*

Proof. We check the axioms from Definition 2.1.10 since clearly $\mathcal{F}_P(P) \subseteq \mathcal{F}_P(G)$. For a fully $\mathcal{F}$-normalised (and hence centralised by Proposition 2.1.12) subgroup Q of P, we get from 2.1.23 that

$$\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q) \cong N_P(Q)C_G(Q)/C_G(Q) \in \mathrm{Syl}_p(N_G(Q)/C_G(Q)) = \mathrm{Syl}_p(\mathrm{Aut}_G(Q)).$$

For the extension condition, let $Q \leqslant P$ and $\varphi : Q \to P$ be a morphism in $\mathcal{F}$, assuming that $R = Q\varphi \in \mathcal{F}^f$. Suppose $\varphi = c_x$, with $x \in G$. Then:

$$N_\varphi = \{y \in N_P(Q) \mid \exists\ z \in N_P(R) \text{ such that } (u^y)\varphi = (u\varphi)^z\ \forall\ u \in Q\}$$

$$= \{y \in N_P(Q) \mid \exists\ z \in N_P(R) \text{ such that } u^{yx} = u^{xz}\ \forall\ u \in Q\}.$$

Now $u^{yx} = u^{xz}$ implies that $yxz^{-1}x^{-1} \in C_G(Q)$ which in turn gives $x^{-1}yxz^{-1} \in C_G(Q^x) = C_G(R)$. i.e. $x^{-1}yx = cz$, for some $c \in C_G(R)$. So $N_\varphi^x \subseteq C_G(R)N_P(R)$. Since $R \in \mathcal{F}^f$, Lemma 2.1.23 implies that $N_P(R) \in \mathrm{Syl}_p(N_G(R))$ so $N_P(R) \in \mathrm{Syl}_p(C_G(R)N_P(R))$. The fact that N_φ is a p-group and Sylow's Theorem imply there is some $d \in C_G(R)$ such that ${N_\varphi}^{xd} \subseteq N_P(R)$. We claim the morphism $\psi : N_\varphi \to P$ which sends y to y^{xd} extends φ. Indeed for $u \in Q$, $(u)\psi = u^{xd} = (u\varphi)^d = (u)\varphi$, since $d \in C_G(Q\varphi)$. $\square$

Consider the following two examples:

Example 2.1.36 Let G be a finite group, p, a prime and $P \in \mathrm{Syl}_p(G)$.

(1) Suppose for some $Q \leqslant P$ that $\mathcal{F} = \mathcal{F}_P(G) = N_\mathcal{F}(Q)$. Is it true that $Q \trianglelefteq G$? In fact the answer is no. As a counterexample, take G =Alt(4) and $P = \langle(123)\rangle$. Then $\mathcal{F}_P(G) = N_\mathcal{F}(P)$ since $\mathrm{Aut}_\mathcal{F}(1)$= $\mathrm{Aut}_\mathcal{F}(P) = 1$. Clearly $P \ntrianglelefteq G$.

(2) Suppose N is such that $P \leqslant N \leqslant G$ and $\mathcal{F}_P(N) = \mathcal{F}_P(G)$. Does $N = G$? Again the answer is no. Take $G = \mathrm{Sym}(p)$ and $N = N_G(P)$ for $p \geqslant 5$. Then since P is cyclic, by Burnside's theorem, $\mathcal{F}_P(G) = \mathcal{F}_P(N)$. But clearly $N_G(P) \neq G$.

2.1.4 The Theorems of Frobenius and Alperin

In this section we prove analogues of the theorems of Frobenius and Alperin for saturated fusion systems. The latter provides us with important information concerning the structure of the fusion system, saying that we need only consider a small class of subgroups to understand it completely. This class forms an example of what we term a *conjugation family*, which we define to be any set of subgroups which determines a fusion system.

From now on, unless explicitly stated, the term 'fusion system' will be used to mean 'saturated fusion system'. We begin with a useful reformulation of Theorem 1.1.24:

Theorem 2.1.37 (Frobenius) *[15, 1.4] Let G be a finite group, p, a prime and $P \in$ $Syl_p(G)$. Then the following three statements are equivalent:*

(a) G has a normal p-complement.

(b) $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.

(c) $Aut_G(Q)$ is a p-group for any $Q \leqslant P$.

Proof.

(a) implies (b): Suppose that G has a normal p-complement, K. Let $x \in G, u \in P$ be such that $u^x \in P$. Then since $G = PK$ we may write $x = zy, z \in P, y \in K$. Thus $u^{-z}y^{-1}u^z y = [u^z, y] \in P \cap K$ since $K \trianglelefteq G$ and $u^{zy} \in P$. Since $P \cap K = 1$, $[u^z, y] = 1$ so $u^x = u^z$, so every morphism in $\mathcal{F}_P(G)$ is in $\mathcal{F}_P(P)$ as required.

(b) implies (c): By (b), $\mathrm{Aut}_G(Q) = \mathrm{Aut}_P(Q)$ is a p-group.

(c) implies (a): We proceed by induction on $|G|$, and assume that every proper subgroup of G has a normal p-complement (notice that for $Q \leqslant P \leqslant H \leqslant G$, $\mathrm{Aut}_H(Q) \leqslant \mathrm{Aut}_G(Q)$, so our hypothesis carries down to subgroups of G). Let $Q \leqslant P$. If $G = N_G(Q)$ then by induction G/Q has a normal p-complement. Set $\overline{G} := G/Q$ and let $Q \leqslant L \leqslant G$ be such that $\overline{L} = O_{p'}(\overline{G})$. Since Q is a normal Sylow p-subgroup of L, we can write $L = K \ltimes Q$ where K is a p'-subgroup of L. We claim that $K \trianglelefteq L$. Now $G/C_G(Q)$ is a p-group so $C_G(Q)$ must contain every p' subgroup of G. Hence $K \leqslant C_G(Q)$ so $[K, Q] = 1 = K \cap Q$ and $K \trianglelefteq L$. So $L = K \times Q$ implies $K = C_G(Q)$ and K is a normal p-complement in G. So we are reduced to considering the case $O_p(G) = 1$. But in this case $N_G(Q)(\neq G)$ has a normal p-complement for all $Q \leqslant P$ and Theorem 1.1.24 delivers the result. $\square$

Alperin's Fusion Theorem was first proved for finite groups in [1] in 1967. To understand it, we need a new definition:

Definition 2.1.38 Let G be a finite group, and $P, Q \in \mathrm{Syl}_p(G)$, for some prime p. The intersection $R := P \cap Q$ is called *tame* if $\{N_P(R), N_Q(R)\} \subseteq \mathrm{Syl}_p(R)$.

Theorem 2.1.39 (Alperin) *[11, 7.2.6] Let G be a finite group, and $P \in Syl_p(G)$, for some prime p. For subsets $A, B \subseteq P$ satisfying $A = B^g$ for some $g \in G$, there exist $Q_i \in Syl_p(G)$, p-elements $x_i \in N_G(P \cap Q_i)$ for $1 \leqslant i \leqslant n$ and $y \in N_G(P)$ with:*

(a) $g = x_1 ... x_n y$;

(b) $P \cap Q_i$ *is tame; and*

(c) $A^{x_1 ... n_i} \subseteq P \cap Q_{i+1}$ *for every* $0 \leqslant i \leqslant n-1$.

Example 2.1.40 Notice that in the case where P is abelian, $A = \{x\}$ and $B = \{y\}$, by Theorem 1.1.8, we get a factorisation of g so that $x_1 ... x_n y \in N_G(P)$.

We can view Theorem 2.1.39 as saying that in a finite group the fusion is always dependent upon certain p-local subgroups, namely the $N_G(P \cap Q_i)$'s. There is a result of a very similar flavour for fusion systems. It says that any $\mathcal{F}$-isomorphism can be factored as restrictions of automorphisms of subgroups in $\mathcal{F}^{fcr}$. In other words we need only consider a small class of objects of $\mathcal{F}$ to understand $\mathcal{F}$ completely.

First, we introduce the concept of a *conjugation family*:

Definition 2.1.41 A *conjugation family* for $\mathcal{F}$ is a set $\mathcal{C}$ of subgroups of P such that for any $Q, Q' \leqslant P$ and $\phi \in \mathrm{Iso}_{\mathcal{F}}(Q, Q')$, there is:

(a) a sequence of $\mathcal{F}$-conjugate subgroups $Q = Q_0, Q_1, ..., Q_n = Q'$,

(b) a sequence $S_1, S_2, ..., S_n$ where $S_i \in \mathcal{C}$ and $Q_{i-1}, Q_i \leqslant S_i$ for all i, and

(c) a sequence $\tilde{\phi}_i \in \mathrm{Aut}_{\mathcal{F}}(S_i)$ such that, for $\phi_i := \tilde{\phi}_i|_{Q_{i-1}}$:

$$Q_{i-1}\phi_i = Q_i \text{ and } (\phi_1 \phi_2 \phi_3 ... \phi_n)|_Q = \phi.$$

We have the following result concerning conjugation families:

Proposition 2.1.42 *[8, 2.10] Let* $\mathcal{C} := \{A_1, ... A_n\}$ *be a conjugation family for* $\mathcal{F}$*. Suppose that* $B_i \in A_i^{\mathcal{F}}$*. Then* $\mathcal{C}' := \{B_1, ... B_n\}$ *is also a conjugation family for* $\mathcal{F}$*.*

Proof. Pick $A_i \in \mathcal{C}$ so that $A_i \neq B_i$. Then $A_i < P$. It suffices to show that

$$\mathrm{Aut}_{\mathcal{F}}(A_i) \leqslant \langle \mathrm{Aut}_{\mathcal{F}}(B_i) \mid 1 \leqslant i \leqslant n \rangle =: X.$$

By induction we may assume that $\mathrm{Aut}_{\mathcal{F}}(T) \leqslant X$ for $P < T \in \mathcal{C}$. Let $\varphi \in \mathrm{Iso}_{\mathcal{F}}(A_i, B_i)$. As $\mathcal{C}$ is a conjugation family, φ can be written as the composition of isomorphisms extending to groups $\mathrm{Aut}_{\mathcal{F}}(Q)$, where $|Q| > |A_i|$ and $Q \in \mathcal{C}$. By induction each such isomorphism is the restictrion of an isomorphism in X. This means that $\varphi \in X$. Taking $\psi \in \mathrm{Aut}_{\mathcal{F}}(A_i)$, we get $\chi := \psi^{\varphi^{-1}} \in \mathrm{Aut}_{\mathcal{F}}(B_i)$, which means $\psi = \chi^{\varphi} \in X$, as needed. $\square$

Theorem 2.1.43 (Alperin's Fusion Theorem) *[5, A.10] Let $\mathcal{F}$ be a fusion system on a finite p-group, P. Then $\mathcal{F}^{fcr}$ is a conjugation family for $\mathcal{F}$.*

Proof. We closely follow the argument presented in, for example [5] Theorem A.10. We proceed by downward induction on the order of Q, with a view to repeatedly applying part (b) in Definition 2.1.10. Note that it is clear that $P \in \mathcal{F}^{fcr}$ ($P \in \mathcal{F}^r$ follows trivially from part (a) of Definition 2.1.10.) Let $Q < P$. Pick any $\tilde{Q} \in Q^{\mathcal{F}}$, such that $\tilde{Q} \in \mathcal{F}^f$ and let $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, \tilde{Q})$. Then the result will follow for $\phi = \psi \circ (\phi^{-1} \circ \psi)^{-1}$ if it follows for both ψ and for $\psi \circ \phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(Q', \tilde{Q})$, where both target groups lie in $\mathcal{F}^f$. So we may assume that $Q' \in \mathcal{F}^f$.

We may now apply Corollary 2.1.13 to obtain a morphism extending ϕ, $\overline{\phi} : N_P(Q) \to P$ and a factorisation $\phi = \beta^{-1} \circ \overline{\phi}|_Q$ for some $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Since $Q < P$, by Lemma 1.1.18, $N_P(Q) > Q$, so the result follows for ϕ if and only if it holds for $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$. So we are reduced to considering the case where $\phi : Q \cong Q' \in \mathcal{F}^f$.

Suppose $Q \notin \mathcal{F}^c$. Since $Q \in \mathcal{F}^f, Q \in \mathcal{F}^z$ by Theorem 2.1.12 so by 2.1.15, ϕ extends to a morphism $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(C_P(Q)Q, P)$. Since such a map must act trivially on both $C_P(Q)$ and Q, we may assume $\overline{\phi} \in \mathrm{Aut}_{\mathcal{F}}(C_P(Q)Q)$. Since $Q \notin \mathcal{F}^c$, $C_P(Q)Q > Q$ so by induction, such a factorisation must exist in this case.

Now suppose $Q \notin \mathcal{F}^r$ so that $K := O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) > \mathrm{Aut}_Q(Q)$. Since $Q \in \mathcal{F}^f$, by Lemma 2.1.23 we have $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ so the p-subgroup K is contained in $\mathrm{Aut}_P(Q)$. Recall that $N_P^K(Q) := \{y \in N_P(Q) \mid c_y \in K\}$. Since $K > \mathrm{Aut}_Q(Q)$, $N_P^K(Q) > Q$. Obviously $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$ so that for $g \in N_P^K(Q)$, $c_g^\phi \in K \leqslant \mathrm{Aut}_P(Q)$. This means that N_ϕ contains $N_P^K(Q)$, i.e. ϕ extends to a morphism from $N_P^K(Q) > Q$ so by induction, we are also done in this case.

The final case to consider is when $Q \in \mathcal{F}^{frc}$, but then the result follows trivially. □

In fact it turns out that $\mathcal{F}^{fe}$ is also a conjugation family and since $\mathcal{F}^e \subseteq \mathcal{F}^{cr}$, this generalises Theorem 2.1.43. See [15] Theorem 5.2 for a proof of this fact.

The following useful result (which we will need later) concerns elements of $\mathcal{F}^{cr}$:

Lemma 2.1.44 *For $Q \leqslant P$, suppose that $\mathcal{F} = N_{\mathcal{F}}(Q)$. Then $Q \leqslant R$ for all $R \in \mathcal{F}^{cr}$.*

Proof. Let $R \in \mathcal{F}^{cr}$. First, we show that $\mathrm{Aut}_{QR}(R) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$. By the definition of $\mathcal{F}$, we can extend φ to an automorphism, $\overline{\varphi}$ in $\mathrm{Aut}_{\mathcal{F}}(QR)$, such that both Q and R are invariant under φ. This implies that $N_{QR}(R)$ is invariant under φ. Now clearly $g \in N_{QR}(R)$ if and only if $c_g \in \mathrm{Aut}_{QR}(R)$. For any $x \in R$,

$$x(\varphi^{-1}c_g\varphi) = (x\varphi^{-1})c_g\varphi = (g^{-1}x\varphi^{-1}g)\varphi = g^{-1}\varphi^{-1}xg\varphi = (g\varphi)^{-1}x(g\varphi),$$

implying $\mathrm{Aut}_{QR}(R) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$. Since $R \in \mathcal{F}^r$, this gives us $\mathrm{Aut}_{QR}(R) = \mathrm{Aut}_R(R)$. Taking preimages we get $N_{QR}(R) \leqslant RC_P(R)$ and $R \in \mathcal{F}^c$ means $N_{QR}(R) = R$, i.e. $QR = R$ or equivalently $Q \leqslant R$. □

We also have the following lemma which helps us to find some elements of $\mathcal{F}^c$:

Lemma 2.1.45 *Let $\mathcal{F}$ be a fusion system on a finite p-group, P and suppose that $Q \in \mathcal{F}^c$. If $|Q| = p$ then $|P| = p$.*

Proof. Since $Q \in \mathcal{F}^c$, $C_P(Q) \leqslant Q$. Now $|Q| = p$ and $Z(P) \leqslant C_P(Q)$ imply that $Q = Z(P)$, and hence $Q = C_P(Q) = P$. $\square$

Example 2.1.46 We claim that there are exactly three $\mathcal{F}$-centric subgroups in $\mathcal{F}_P(G)$, where $G \cong$Alt(6) and $P \in \mathrm{Syl}_2(G)$. Since $|\mathrm{Alt}(6)| = 2^3 3^2 5$, $|P| = 8$ and an easy calculation shows that we can choose $P = \langle (12)(34), (23)(56) \rangle \cong$Dih(8). Of the four non-trivial normal subgroups, only the centre is not centric by the above lemma.

Example 2.1.47 We use Theorem 2.1.43 to show that for $\mathcal{F}_1 := \mathcal{F}_P(\mathrm{Alt}(4))$ and $\mathcal{F}_2 := \mathcal{F}_P(\mathrm{Alt}(5))$, we have $\mathcal{F}_1 \cong \mathcal{F}_2$. (In each case P is a Sylow 2-subgroup isomorphic to V_4). Indeed by the previous Lemma, $\mathcal{F}_i^{frc} = \{P\}$ so it suffices to check that $\mathrm{Aut}_{\mathcal{F}_1}(P) \cong \mathrm{Aut}_{\mathcal{F}_2}(P)$. Writing $G_1 :=$Alt(4) and $G_2 :=$Alt(5), we have $N_{G_i}(P) \cong \mathrm{Alt}(4)$ so $\mathrm{Aut}_{\mathcal{F}_i}(P) \cong N_{G_i}(P)/C_{G_i}(P)$ is isomorphic a cyclic group of order 3 for $i = 1, 2$, as needed.

We remark that it is probably the case that for all $n \in \mathbb{N}$, $\mathcal{F}_P(\mathrm{Alt}(2^n)) \cong \mathcal{F}_P(\mathrm{Alt}(2^n + 1))$ when P is a Sylow 2-subgroup.

We will need one further example of a conjugation family:

Definition 2.1.48 Let $\mathcal{F}$ be a fusion system on a finite p-group and U be a positive characteristic p-functor [7]. For $Q \leqslant P$, set $U_1(Q) = Q$ and $U_{i+1}(Q) = U(N_S(U_i(Q)))$ for $i \geqslant 2$. If $U_i(Q) \in \mathcal{F}^f$ for all i then we say that Q is $(\mathcal{F}, U)$-*well placed.*

Concerning this situation, we have:

Proposition 2.1.49 *[16, 3.3] For $1 \leqslant i \leqslant n$, let $U_i \leqslant P$ be such that:*

(a) U_{i+1} char $N_P(U_i)$ for $1 \leqslant i \leqslant n-1$; and

(b) $U_i \in \mathcal{F}^f$ for $1 \leqslant i \leqslant n-1$.

[7] see Definition 1.3.7.

Then there is a morphism $\varphi : N_P(U_n) \to P$ such that $U_i\varphi|_{U_i} \in \mathcal{F}^f$ for all $1 \leqslant i \leqslant n$. In particular, $N_P(U_i\varphi) = N_P(U_i)\varphi$. Moreover for $X \subseteq \{c, r\}, U_i \in \mathcal{F}^X$ implies that $U_i\varphi \in \mathcal{F}^X$ for any $1 \leqslant i \leqslant n$.

Proof. By Lemma 2.1.13, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_P(U_n), P)$ such that $U_n\varphi \in \mathcal{F}^f$. Since U_{i+1} char $N_P(U_i)$ for $1 \leqslant i \leqslant n-1$, $N_P(U_i) \leqslant N_P(U_{i+1})$. This means $N_P(U_i) \leqslant N_P(U_n)$ so we may restrict φ to $N_P(U_i)$ for every $1 \leqslant i \leqslant n$. Let $1 \leqslant i \leqslant n-1$. Clearly $N_P(U_i)\varphi \leqslant N_P(U_i\varphi)$ and since φ is injective we get $|N_P(U_i)| \leqslant |N_P(U_i)\varphi|$. Since $U_i \in \mathcal{F}^f$, $|N_P(U_i)| \geqslant |N_P(U_i)\varphi|$ so $|N_P(U_i)| = |N_P(U_i)\varphi|$ and $U_i\varphi \in \mathcal{F}^f$ for all $1 \leqslant i \leqslant n-1$. In particular, $N_P(U_i\varphi) = N_P(U_i)\varphi$, for $1 \leqslant i \leqslant n-1$ so $U_n\varphi, U_n \in \mathcal{F}^f$.

Now since for $1 \leqslant i \leqslant n$, $U_i \in \mathcal{F}^f$, we have $U_i \in \mathcal{F}^z$ so $C_P(U_i\varphi) = C_P(U_i)\varphi$. Thus $U_i \in \mathcal{F}^c$ implies that $U_i\varphi \in \mathcal{F}^c$. Since $\mathrm{Aut}_{\mathcal{F}}(U_i)^\varphi = \mathrm{Aut}_{\mathcal{F}}(U_i\varphi)$ and $\mathrm{Aut}_P(U_i)^\varphi = \mathrm{Aut}_P(U_i\varphi)$, $U_i \in \mathcal{F}^r$ implies that $U_i\varphi \in \mathcal{F}^r$, as needed. □

Corollary 2.1.50 *[12, 5.2] Let U be a positive characteristic p-functor and $Q \leqslant P$. Then there is a morphism $\varphi : N_P(Q) \to P$ such that $Q\varphi$ is $(\mathcal{F}, U)$ well-placed.*

Proof. For every $i \in \mathbb{N}$, set $U_i := U_i(Q)$. Assuming that $U_i \in \mathcal{F}^f$ for $1 \leqslant i \leqslant n-1$, Proposition 2.1.49 yields the existence of the required morphism and the result follows by induction. □

Theorem 2.1.51 *[8, 2.12] The $(\mathcal{F}, U)$-well placed subgroups of P form a conjugation family for $\mathcal{F}$.*

Proof. By the previous result, every element of an arbitrary conjugation family, $\mathcal{C}$ is $\mathcal{F}$-conjugate to an $(\mathcal{F}, U)$ well-placed subgroup of P. The claim now follows from Proposition 2.1.42. □

To conclude this section we prove a technical lemma which we require later:

Proposition 2.1.52 *[12, 5.3] Suppose that for* $1 \neq Q \in \mathcal{F}^f$, $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}(Q)}}(U(N_P(Q)))$. *Then* $\mathcal{F} = N_{\mathcal{F}}(U(P))$.

Proof. Suppose the conclusion does not hold. Then by Theorem 2.1.43, there is a proper subgroup of P, $Q \in \mathcal{F}^{fcr}$ such that $\mathrm{Aut}_{N_{\mathcal{F}(U(P))}}(Q) \subsetneq \mathrm{Aut}_{\mathcal{F}}(Q)$. Setting $U_1 := Q$ and $U_i := U(N_P(U_{i-1}))$, we have that U_i char $N_P(U_{i-1})$ so by Proposition 1.1.18, $N_P(U_{i-1}) < N_P(N_P(U_{i-1})) \leqslant N_P(U_i)$. This means that there is a sequence

$$N_P(U_1) < N_P(U_2) < ... < N_P(U_n)$$

which must terminate at P, since if $N_P(U_{i-1}) = N_P(U_i)$ then $N_P(U_i) = N_P(N_P(U_i)) \leqslant P$ so $N_P(U_i) = P$. By Proposition 2.1.49, we may assume that $U_i \in \mathcal{F}^f$ so $N_{\mathcal{F}}(U_i)$ is saturated for all $1 \leqslant i \leqslant n-1$. By assumption, $N_{\mathcal{F}}(U_{i-1}) \subseteq N_{\mathcal{F}}(U_i)$ for all $1 \leqslant i \leqslant n-1$, but this means that $N_{\mathcal{F}}(Q) = N_{\mathcal{F}}(U_1) \subseteq N_{\mathcal{F}}(U_n) = N_{\mathcal{F}}(U(P))$. Now $\mathrm{Aut}_{\mathcal{F}}(Q) \subseteq \mathrm{Aut}_{N_{\mathcal{F}(U(P))}}(Q)$, a contradiction. □

2.1.5 Quotients of Fusion Systems

We now turn our attention to the notion of a quotient fusion system. Our introduction will follow that found in [7] and [15]. We begin with some important definitions:

Definition 2.1.53 Let $\mathcal{F}$ be a fusion system over a finite p-group P and let $Q \leqslant P$.

(a) Q is called *weakly $\mathcal{F}$-closed* if $Q^{\mathcal{F}} = \{Q\}$.

(b) Q is called *strongly $\mathcal{F}$-closed* if for any $R \leqslant Q$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ we have $R\phi \leqslant Q$.

Notice that if $\langle x \rangle \leqslant P$ is weakly $\mathcal{F}$-closed, then $x \in Z(P)$ since $\mathcal{F}_P(P) \subseteq \mathcal{F}$ and so when $\mathcal{F} = \mathcal{F}_P(G)$, for a finite group G with $P \in \mathrm{Syl}_p(G)$, weakly $\mathcal{F}$-closed is the same as weakly closed in the sense of Definition 1.1.20.

The following result will be useful in later sections:

Lemma 2.1.54 *[8, 2.8] Let $\mathcal{F}$ be a fusion system over a finite p-group P. The set of weakly $\mathcal{F}$-closed elements is precisely $Z(\mathcal{F})$.*

Proof. Let $x \in Z(\mathcal{F})$. Since $\mathcal{F} = C_{\mathcal{F}}(\langle x \rangle)$, x must be weakly $\mathcal{F}$-closed (by definition of $C_{\mathcal{F}}(\langle x \rangle)$). Conversely suppose that $x \in P$ is weakly $\mathcal{F}$-closed. Pick any $Q \in \mathcal{F}^{frc}$. Then since $x \in Z(P)$, $x \in C_P(Q) = Z(Q)$, so $x \in Q$. Now for every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, $\varphi|_{\langle x \rangle} = \mathrm{Id}_{\langle x \rangle}$ and so by Alperin's Theorem 2.1.43 any morphism in $\mathcal{F}$ is the identity on $\langle x \rangle$, i.e. $\langle x \rangle \leqslant Z(\mathcal{F})$. □

Definition 2.1.55 Let $\mathcal{F}$ be a fusion system over a finite p-group P and let $Q \trianglelefteq P$. Let $\mathcal{F}/Q$ be a category on P/Q such that for subgroups $R, S \leqslant P$ containing Q, $\mathrm{Hom}_{\mathcal{F}/Q}(R/Q, S/Q)$ is the set of morphisms induced by $\mathrm{Hom}_{\mathcal{F}}(R, S)$ which leave Q invariant.

Lemma 2.1.56 *Let $\mathcal{F}$ be a fusion system on a finite p-group, P. Let $R \leqslant P$ and $Q \leqslant P$ be weakly $\mathcal{F}$-closed. Suppose that $R/Q \in (\mathcal{F}/Q)^f$. Then $R \in \mathcal{F}^f$.*

Proof. Setting $\overline{G} := G/Q$, let R' and ϕ be such that $\phi \in \mathrm{Iso}_{\mathcal{F}/Q}(\overline{R}, \overline{R'})$ so that $\overline{R'} \in (\mathcal{F}/Q)^f$. Since Q is weakly $\mathcal{F}$-closed, $Q \leqslant T$ for every $T \in R^{\mathcal{F}}$. Trivially $Q \leqslant N_P(T)$. Since $\overline{N_P(R)} = N_{\overline{P}}(\overline{R})$,

$$|N_P(T)| = |N_{\overline{P}}(\overline{T})||Q| \leqslant |N_{\overline{P}}(\overline{R'})||Q| = |N_P(R')|$$

so $R' \in \mathcal{F}^f$ and hence $R \in \mathcal{F}^f$, as needed. □

That $\mathcal{F}/Q$ defines a (not necessarily saturated) fusion system on P/Q is immediate from its definition. Saturation is dependent on a condition on Q:

Proposition 2.1.57 *[7, 5.3] Let $\mathcal{F}$ be a saturated fusion system over a finite p-group P and let $Q \trianglelefteq P$ be weakly $\mathcal{F}$-closed. Then $\mathcal{F}/Q$ is a saturated fusion system on P/Q.*

Proof. **Sylow Axiom:** Since elements of $\mathrm{Aut}_{\mathcal{F}/Q}(P/Q)$ are induced by elements of $\mathrm{Aut}_{\mathcal{F}}(P)$, $\mathrm{Aut}_P(P)$ is surjectively mapped to $\mathrm{Aut}_{P/Q}(P/Q)$ in $\mathrm{Aut}_{\mathcal{F}/Q}(P/Q)$ as needed.

Extension Condition: Let $\psi : R/Q \to S/Q$ be an isomorphism in $\mathcal{F}/Q$ so that $S/Q \in (\mathcal{F}/Q)^f$. By Lemma 2.1.56, $S \in \mathcal{F}^f$. Before dealing with the general case, consider $\phi \in \mathrm{Aut}_{\mathcal{F}/Q}(R/Q)$ and let $\psi \in \mathrm{Aut}_{\mathcal{F}}(R)$ be its preimage under the canonical homomorphism. Certainly $N_\psi/Q \leqslant N_\phi$ but equality may not necessarily hold. Let K denote the kernel of the canonical surjection $\mathrm{Aut}_{\mathcal{F}}(R) \to \mathrm{Aut}(R/Q)$. We claim that there exists $\chi \in K$ and $\theta : R \to R$ with $\psi = \chi\theta$ and $N_\theta/Q = N_\phi$. If this follows, then ϕ extends to N_ϕ since θ and ψ are both mapped to ϕ in $\mathcal{F}/Q$. To prove the claim, notice $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$ and $\mathrm{Aut}_P(R) \in \mathrm{Syl}_p(\ \mathrm{Aut}_{\mathcal{F}}(R))$ imply that $\mathrm{Aut}_P^K(R) \in \mathrm{Syl}_p(K)$. Theorem 1.1.3 implies that $\mathrm{Aut}_{\mathcal{F}}(R) = KN_{Aut_{\mathcal{F}}(R)}(\mathrm{Aut}_P^K(R))$. Let $X := N_{Aut_{\mathcal{F}}(R)}(\mathrm{Aut}_P^K(R))$ so that $\mathrm{Aut}_P(R) \in \mathrm{Syl}_p(X)$. By the Second Isomorphism Theorem, $X/(X \cap K) \cong KX/X = \mathrm{Aut}_{\mathcal{F}}(R)/K \cong \mathrm{Aut}_{\mathcal{F}/Q}(R/Q)$. Also since $S := \mathrm{Aut}_P^K(Q)$ is a normal Sylow p-subgroup of $X \cap K$, we have:

$$\mathrm{Aut}_{\mathcal{F}/Q}(R/Q) \cong X/X \cap K \cong (X/S)/(X \cap K/S).$$

Now N_ϕ is the preimage in P/Q of $\mathrm{Aut}_{P/Q}(R/Q) \cap \mathrm{Aut}_{P/Q}(R/Q)^{\phi^{-1}}$, which is the intersection of two Sylow p-subgroups of $\mathrm{Aut}_{\mathcal{F}/Q}(R/Q)$. Since $(X \cap K)/S$ is a p'-group there exist $A/S, B/S \in \mathrm{Syl}_p(X/S)$ and a coset $Sg \in X/S$ such that $A/S \cap (B/S)^{Sg^{-1}}$ maps onto this intersection. We may (in turn) take the pre-image of this group in $X(= N_{Aut_{\mathcal{F}}(R)}(\mathrm{Aut}_P^K(R))$. Putting this together we get an element $\theta \in X$, so that $\mathrm{Aut}_P(R) \cap \mathrm{Aut}_P(R)^{\theta^{-1}}$ maps surjectively to $\mathrm{Aut}_{P/Q}(R/Q) \cap \mathrm{Aut}_{P/Q}(R/Q)^{\phi^{-1}}$. This means the map $N_\theta \to N_\phi$ is surjective and so $N_\theta/Q = N_\phi$. Since $\mathcal{F}$ is saturated, θ extends to N_θ and so ϕ extends to N_ϕ as needed.

It remains to consider the general case. Let $\phi \in \mathrm{Hom}_{\mathcal{F}/Q}(S/Q, R/Q)$ with $(R/Q) \in$

$(\mathcal{F}/Q)^f$. Then there is a map $\psi \in \mathrm{Hom}_{\mathcal{F}}(S, R)$ with $R \in \mathcal{F}^f$, so that by Lemma 2.1.13, there is $\theta \in \mathrm{Hom}_{\mathcal{F}}(S, R)$, with $N_\theta = N_P(S)$. Let χ denote the image of θ in $\mathcal{F}/Q$. Then ϕ extends to N_ϕ if χ extends to N_χ and $\chi^{-1}\phi$ ($\in \mathrm{Aut}_{\mathcal{F}}/Q(R/Q)$) extends to $N_{\chi^{-1}\phi}$. The first case trivially holds since $N_\chi = N_{P/Q}(S/Q) = N_P(S)/Q$ and the second case was dealt with above. This completes the proof. $\square$

Notice that by Definition 2.1.11, the homomorphism $\alpha : P \to P/Q$, gives rise to a morphism of saturated fusion systems $\overline{\alpha} : \mathcal{F} \to \mathcal{F}/Q$. Furthermore, when $R/Q \in (\mathcal{F}/Q)^f$ and $Q \trianglelefteq R$, we get an isomorphism $N_{\mathcal{F}}(R)/Q \to N_{\mathcal{F}/Q}(R/Q)$. Note that both these fusion systems are saturated by Lemma 2.1.56 and Proposition 2.1.57.

Definition 2.1.58 A fusion system $\mathcal{F}$ on a finite p-group P is called *trivial* if $\mathcal{F} = \mathcal{F}_P(P)$.

We will need the following result later:

Proposition 2.1.59 *[12, 3.4] Let $\mathcal{F}, \mathcal{G}$ be fusion systems over a finite p-group P, such that $\mathcal{G} \subseteq \mathcal{F}$. Suppose that Q, $R \trianglelefteq P$ are such that $Q \leqslant R$, Q is weakly $\mathcal{F}$-closed and $\mathcal{F}$=$PC_{\mathcal{F}}(Q)$. Then $N_{\mathcal{F}}(R) = \mathcal{G}$ if and only if $N_{\mathcal{F}/Q}(R/Q)$=$\mathcal{G}/Q$. In particular:*

(a) When $\mathcal{G} = \mathcal{F}_R(R)$, $N_{\mathcal{F}}(R) = \mathcal{F}_R(R)$ if and only if $N_{\mathcal{F}/Q}(R/Q)$=$\mathcal{F}_{R/Q}(R/Q)$

(b) $\mathcal{F}$ is trivial if and only if $\mathcal{F}/Q$ is trivial.

First note that since Q and R are both normal subgroups of P, they are in particular fully $\mathcal{F}$-normalised and hence all fusion systems in question are saturated by Theorem C and Theorem 2.1.57. To prove Proposition 2.1.59 we will proceed by induction on $|Q|$, considering first the group Q/Z where $1 \neq Z \leqslant Z(P)$. We require a series of results (as in [12]) concerning this situation. In each case $\mathcal{F}$ is a fusion system on a finite p-group P.

Proposition 2.1.60 *[12, 3.1] Let $Z \leqslant Z(P)$ be such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ and write $\overline{P} :=$ P/Z, $\overline{\mathcal{F}} := \mathcal{F}/Z$. Let Q be a subgroup of P containing Z. Then there exists a surjective group homomorphism $\rho : Aut_{\mathcal{F}}(Q) \to Aut_{\overline{\mathcal{F}}}(\overline{Q})$ whose kernel is an abelian p-group.*

Proof. It is clear that the canonical map $Q \to \overline{Q}$ induces the homomorphism $\rho : \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{Q})$. Furthermore if $\phi \in \ker\ \rho$ then ϕ fixes both Z and $\overline{Q}$ pointwise, so for each $u \in Q$, $u\phi = u(u)\chi$, where $\chi \in \mathrm{Hom}(Q, Z)$. Since the latter group is an abelian p-group, the result follows. □

Corollary 2.1.61 *[12, 3.1] Assuming the same hypotheses as Proposition 2.1.60, the following implications hold:*

(a) $Q \in \mathcal{F}^r$ implies $\overline{Q} \in \overline{\mathcal{F}}^r$;

(b) $\overline{Q} \in \overline{\mathcal{F}}^c$ implies $Q \in \mathcal{F}^c$; and

(c) $Q \in \mathcal{F}^{cr}$ implies $\overline{Q} \in \overline{\mathcal{F}}^{cr}$.

Proof. By Proposition 2.1.60 there is $\rho : \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{Q})$. Then since $\ker\ \rho \leqslant \mathrm{Aut}_Q(Q)$, $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\ \mathrm{Aut}_Q(Q) \cong \mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{Q})/\ \mathrm{Aut}_{\overline{Q}}(\overline{Q}) = \mathrm{Out}_{\overline{\mathcal{F}}}(\overline{Q})$ so (a) follows. If $\overline{Q} \in \overline{\mathcal{F}}^c$ then $C_{\overline{P}}(\overline{R}) \leqslant \overline{R}$ for each $R \in Q^{\mathcal{F}}$, so $C_P(R) \leqslant R$ which proves (b).

For (c), notice we may assume that $Q \in \mathcal{F}^{crz}$. By Proposition 2.1.60, $K :=\ker\ \rho \leqslant \mathrm{Aut}_Q(Q)$. If C is the preimage in P of $C_{\overline{P}}(\overline{Q})$, we have $\mathrm{Aut}_C(Q) \leqslant K$, since the image in $\overline{P}$ of elements of C centralise $\overline{Q}$. This means $C \leqslant QC_P(Q) = Q$ and hence $\overline{C} \leqslant \overline{Q}$ so $\overline{Q} \in \overline{\mathcal{F}}^{cr}$, as needed. □

Proposition 2.1.62 *[12, 3.2] Let $\mathcal{G} \subseteq \mathcal{F}$ be a fusion system on P and $Z \leqslant Z(P)$ be such that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Then:*

$$\mathcal{F} = \mathcal{G} \text{ if and only if } \mathcal{F}/Z = \mathcal{G}/Z.$$

Proof. The forward implication is trivial. For the converse, suppose that $\mathcal{F}/Z = \mathcal{G}/Z$ and let $Q \in \mathcal{F}^{cr}$. By Proposition 2.1.60 there is $\rho : \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}/Z}(Q/Z)$ with $\ker\ \rho \leqslant \mathrm{Aut}_Q(Q)$. But then $\ker\rho=\ker\ \rho'$, where $\rho' : \mathrm{Aut}_{\mathcal{G}}(Q) \to \mathrm{Aut}_{\mathcal{G}/Z}(Q/Z)$. Since by hypothesis, im $\rho = $ im ρ', $|\mathrm{Aut}_{\mathcal{F}}(Q)| = |\mathrm{Aut}_{\mathcal{G}}(Q)|$ which means $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{G}}(Q)$ since $\mathcal{G} \subseteq \mathcal{F}$. But now by Alperin's Theorem 2.1.43, $\mathcal{F} = \mathcal{G}$. □

Corollary 2.1.63 *[12, 3.3] Let $\mathcal{G} \subseteq \mathcal{F}$ be a fusion system on P and $Z \leqslant Z(P)$ be such that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Suppose Q satisfies $Z \leqslant Q \trianglelefteq P$. Then:*

$$\mathcal{G} = N_{\mathcal{F}}(Q) \text{ if and only if } \mathcal{G}/Z = N_{\mathcal{F}/Z}(Q/Z).$$

Proof. Again, the forward implication is trivial. Write $\overline{\mathcal{G}} := \mathcal{G}/Z$, $\overline{\mathcal{F}} := \mathcal{F}/Z$ and $\overline{Q} := Q/Z$ and suppose that $\overline{\mathcal{G}} = N_{\overline{\mathcal{F}}}(\overline{Q})$. By Lemma 2.1.44, $\overline{Q} \leqslant \overline{R}$ for all $\overline{R} \in \overline{\mathcal{G}}^{cr}$. Then by Corollary 2.1.61, $Q \leqslant R$ for all $R \in \mathcal{G}^{cr}$ so $\mathcal{G} = N_{\mathcal{G}}(Q) \subseteq N_{\mathcal{F}}(Q)$. Since $N_{\mathcal{F}}(Q)/Z = N_{\overline{\mathcal{F}}}(\overline{Q})$, $\mathcal{G} = N_{\mathcal{F}}(Q)$ by Proposition 2.1.62. □

Proof (of Proposition 2.1.59). The forward implication is again trivial. Suppose that $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$. As remarked earlier, to prove that $\mathcal{G} = N_{\mathcal{F}}(R)$ we proceed by induction on $|Q|$. When $|Q| = 1$ the result trivially holds. By Lemma 1.1.19, $Z := Q \cap Z(P) \neq 1$ and since $\mathcal{F} = PC_{\mathcal{F}}(Q)$ it is clear that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Set $\overline{\mathcal{F}} := \mathcal{F}/Z$, $\overline{\mathcal{G}} := \mathcal{G}/Z$, and $\overline{P} := P/Z$. Notice that:

$$\overline{\mathcal{F}} = \overline{P}C_{\overline{\mathcal{F}}}(\overline{Q}) \text{ and } \overline{G} \subseteq \overline{F}.$$

Also, via the isomorphism $\overline{P}/\overline{Q} \cong P/Q$ we get:

$$\overline{\mathcal{G}}/\overline{Q} \cong \mathcal{G}/Q \text{ and } N_{\overline{\mathcal{F}}/\overline{Q}}(\overline{R}/\overline{Q}) \cong N_{\mathcal{F}/Q}(R/Q).$$

This means that $\overline{\mathcal{G}}/\overline{Q} = N_{\overline{\mathcal{F}}/\overline{Q}}(\overline{R}/\overline{Q})$ and so we may apply induction to get that $\overline{\mathcal{G}} = N_{\overline{\mathcal{F}}}(\overline{R})$. But now Corollary 2.1.63 implies that $\mathcal{G} = N_{\mathcal{F}}(R)$, as needed. □

2.1.6 Models for Fusion Systems

In this section we introduce the notion of a *model*, for a fusion system $\mathcal{F}$ on a finite p-group, P, a term frequently used in the literature referring to the existence of a group, G so that $\mathcal{F} = \mathcal{F}_P(G)$. We present a basic outline of some of the ideas used in the proof of Theorem D, and discuss the result's importance for our purposes. We will also introduce convenient notation of Aschbacher's found in, for example [2].

$\mathcal{F}$ will always denote a saturated fusion system on a finite p-group P.

Definition 2.1.64 We say that $\mathcal{F}$ is *constrained* if there exists $Q \trianglelefteq \mathcal{F}$, such that $Q \in \mathcal{F}^c$

Example 2.1.65 In the case where $\mathcal{F} = \mathcal{F}_P(G)$ and $C_G(O_p(G)) \leqslant O_p(G)$, it is clear that $\mathcal{F}$ is constrained, since $N_{\mathcal{F}}(O_p(G)) = \mathcal{F}$.

The following notation of Aschbachers' will be useful:

Definition 2.1.66 Let $\mathcal{G}(\mathcal{F})$ denote the class of finite groups G such that $P \in \mathrm{Syl}_p(G)$, $C_G(O_p(G)) \leqslant O_p(G)$, and $\mathcal{F} = \mathcal{F}_P(G)$. Call elements of $\mathcal{G}(\mathcal{F})$, *models* of $\mathcal{F}$.

What Example 2.1.65 essentially shows is that if $\mathcal{G}(\mathcal{F}) \neq \emptyset$ then $\mathcal{F}$ is constrained. What is alarming is that the converse is also true:

Theorem 2.1.67 (Theorem D) *For $Q \in \mathcal{F}^{fc}$, there is, up to isomorphism, a unique finite group $G = G_Q^{\mathcal{F}}$ with $N_P(Q) \in Syl_p(G)$ such that $Q \trianglelefteq G, C_G(Q) \leqslant Q$ and $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(G)$.*

Definition 2.1.68 As in the above theorem we define $\{G_Q^{\mathcal{F}} \mid Q \in \mathcal{F}^{fc}\}$ to be the set of *potential models for $\mathcal{F}$.*

It is clear that every model for $\mathcal{F}$ is a potential model. To recapitulate:

Proposition 2.1.69 *$\mathcal{G}(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{F}$ is constrained.*

Proof. To prove the reverse direction, we may choose $Q \in \mathcal{F}^{fc}$, maximal with respect to satisying the conclusion of Theorem D, i.e. $Q := O_p(G)$ and $\mathcal{G}(\mathcal{F}) \neq \emptyset$, as needed. $\square$

We introduce the important notions of a *centric linking system* and a *p-local finite group* associated with $\mathcal{F}$:

Definition 2.1.70 A *centric linking system* associated to $\mathcal{F}$ is a category, $\mathcal{L}$ whose object set is $\mathcal{F}^c$, together with a functor $\pi : \mathcal{L} \to \mathcal{F}^c$ and monomorphisms $\delta_Q : Q \to \mathrm{Aut}_{\mathcal{L}}(Q)$ for $Q \in \mathcal{F}^c$ satisfying:

- π is the identity on objects of $\mathcal{L}$. For $Q, R \in \mathcal{L}$, $\delta_Q(Z(Q)) \leqslant \mathrm{Aut}_{\mathcal{L}}(Q)$ acts on $\mathrm{Mor}_{\mathcal{L}}(Q, R)$ so that $\mathrm{Mor}_{\mathcal{L}}(Q, R)/Z(Q) \cong \mathrm{Hom}_{\mathcal{F}}(Q, R)$.
- For $Q \in \mathcal{F}^c$ and $x \in Q$, $\pi(\delta_Q(x)) = c_x \in \mathrm{Aut}_{\mathcal{F}}(Q)$.
- For $f \in \mathrm{Mor}_{\mathcal{L}}(Q, R)$, $x \in Q$, $f \circ \delta_Q(x) = \delta_Q((x)) \circ f$.

Definition 2.1.71 A *p-local finite group* is a triple $(P, \mathcal{F}, \mathcal{L})$, where $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, a fusion system over P.

It is not at all obvious that a centric linking systems (and hence p-local finite groups) should exist for a given fusion system. Luckily we have:

Proposition 2.1.72 *[4, 4.2] Let $\mathcal{F}$ be constrained. Then there is a centric linking system associated to $\mathcal{F}$ and it is unique up to isomorphism of categories.*

In terms of proving Theorem D, the fact that this centric linking system, $\mathcal{L}$, exists, allows one to recover a group G, satisfying $\mathcal{F} = \mathcal{F}_P(G)$, as the group $\mathrm{Aut}_{\mathcal{L}}(Q)$ of $\mathcal{L}$-automorphisms of Q. This fact can be deduced quite quickly (see [4], 4.3), while the proof of Proposition 2.1.72 uses many of the deep ideas found in [5].

If we are to transfer results about H-free finite groups to fusion systems, we need an analogous notion:

Definition 2.1.73 For any finite group H, $\mathcal{F}$ is called *H-free* if no potential model of $\mathcal{F}$ involves H.

The following two results, due to Kessar and Linckelman are concerned with the inheritance of the property of being H-free. They are important in the proof of Theorem G and their proofs can be found in [12], Section 6:

Proposition 2.1.74 *[12, 6.3] Let H be a finite group. Suppose that $Q \in \mathcal{F}^f$. If $\mathcal{F}$ is H-free then $N_{\mathcal{F}}(Q)$ is H-free.*

Proposition 2.1.75 *[12, 6.4] Let H be a finite group. Suppose that $Q \trianglelefteq P$ $\mathcal{F}$ is such that $\mathcal{F} = N_{\mathcal{F}}(Q)$. If $\mathcal{F}$ is H-free then $\mathcal{F}/Q$ is H-free.*

2.2 The ZJ-Theorems for Fusion Systems

2.2.1 Sparse Fusion Systems and Glauberman and Thompson's p-nilpotency Theorem

In this section, we prove Theorems E and F. In [12], the authors proceed by showing that a minimal counterexample is constrained in order to apply Theorem D and use the corresponding result from group theory to derive a contradiction. In [10], Adam Glesser streamlines the approach by introducing the notion of a 'sparse fusion system', and noticing that such a counterexample is sparse. He proves that under certain conditions, sparse systems are constrained. In this section we provide two different proofs of this result, one of which is independent of the results in [8]. Let $\mathcal{F}$ be a fusion system over a finite p-group, P.

Definition 2.2.1 A non-trivial fusion system $\mathcal{F}$ over a finite p-group P is called *sparse* if the only proper subsystem of $\mathcal{F}$ on P is the trivial fusion system, $\mathcal{F}_P(P)$.

We begin with some important lemmas:

Lemma 2.2.2 *[8, 2.5] Let $Q \leqslant P$ be such that $Q \trianglelefteq \mathcal{F}$. Suppose that $Aut_{\mathcal{F}}(Q)$ is a p-group. Then $\mathcal{F} = PC_{\mathcal{F}}(Q)$.*

Proof. Saturation implies that $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ so clearly $\mathrm{Aut}_P(Q)$=$\mathrm{Aut}_{\mathcal{F}}(Q)$. So since $\mathcal{F} = N_{\mathcal{F}}(Q)$, every morphism in $\mathcal{F}$ $\mathrm{Aut}_P(Q)$-stably extends to Q and so $\mathcal{F} = PC_{\mathcal{F}}(Q)$. □

The following tells us how to 'split up' a fusion system:

Definition 2.2.3 For some indexing set I, given a collection of subsytems of $\mathcal{F}$: $\{\mathcal{F}_i \mid i \in I\}$, we define:

$$\langle \mathcal{F}_i \mid i \in I \rangle := \bigcap_{\mathcal{H} \in X} \mathcal{H}, \text{ where } X := \{\mathcal{H} \mid \mathcal{F}_i \subseteq \mathcal{H} \text{ for all } i \in I\}$$

to be the smallest (not necessarily saturated) fusion system containing all the $\mathcal{F}_i$'s.

We remark that this notation permits a nice restatement of Theorem 2.1.43, telling us that every fusion system is generated by constrained fusion systems:

$$\mathcal{F} = \langle N_{\mathcal{F}}(U) \mid U \in \mathcal{F}^{frc} \rangle.$$

The next result, although easy to prove, is powerful and has led to considerable shortenings of the proofs of the ZJ-theorems:

Lemma 2.2.4 (Stancu) *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group, P. If $Q \trianglelefteq \mathcal{F}$ then*

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle.$$

Proof. Since $\mathcal{F} = N_{\mathcal{F}}(Q)$, Lemma 2.1.44 implies $Q \leqslant R$, for all $R \in \mathcal{F}^{fcr}$. Pick $\phi \in \mathrm{Aut}_{\mathcal{F}}(R)$, so that $\psi := \phi|_Q$, is an automorphism of Q. Clearly then, $R, QC_P(Q) \leqslant N_\psi$, so there is some $\theta \in \mathrm{Hom}_{\mathcal{F}}(RQC_P(Q), P)$ such that $\theta|_Q = \psi$. We factorise ϕ as

$$\theta|_R \circ ((\theta|_R)^{-1} \circ \phi).$$

Now $\theta|_R$ is a morphism in $N_{\mathcal{F}}(QC_P(Q))$ since $\theta(QC_P(Q)) = QC_P(Q)$, and it is clear that $(\theta|_R)^{-1} \circ \phi$ is a morphism in $PC_{\mathcal{F}}(Q)$. So $\phi \in \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q))\rangle$, and since R was chosen arbitrarily, Theorem 2.1.43 delivers the result. □

Notice that if $\mathcal{F}$ is constrained, $\mathcal{F} = N_{\mathcal{F}}(QC_P(Q))$ for some normal subgroup Q of $\mathcal{F}$. Thus Lemma 2.2.4 is only interesting when $\mathcal{F}$ is not constrained.

In Theorems 4.1 and 4.5 of [8], the authors prove analogues for fusion systems of the following two results, due to Glauberman:

Theorem 2.2.5 *[9, 14.10] Let G be a finite group, and $P \in Syl_p(G)$. Suppose that $x \in P \cap Z(N_G(J(P)))$ and that any of the following hold:*

(a) p is odd.

(b) $x \in (Z(P))^p$.

(c) G is S_4-free.

Then x is weakly closed in P with respect to G.

Theorem 2.2.6 *[9, 14.11] Suppose p is odd or G is S_4-free. If $C_G(Z(P))$ and $N_G(J(P))$ both have normal p-complements, then so does G.*

The original proof of [8], Theorem 4.1 is longer than necessary. The proof presented here uses less group theory:

Theorem 2.2.7 *[8, 4.1] Let $\mathcal{F}$ be a fusion system on a finite p-group, P. Suppose $x \in Z(P) \cap Z(N_{\mathcal{F}}(J(P)))$ and that any of the following hold:*

(a) p is odd.

(b) $x \in (Z(P))^p$.

(c) $p = 2$ and $\mathcal{F}$ is S_4-free.

Then x is weakly $\mathcal{F}$-closed. In particular, $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(P)))$.

Proof. Let $\mathcal{F}$ be a minimal counterexample, so $x \in Z(P) \cap Z(N_{\mathcal{F}}(J(P)))$ and x is not weakly $\mathcal{F}$-closed, i.e. by Lemma 2.1.54, $x \notin Z(\mathcal{F})$. We let $\mathcal{C}$ denote the conjugation family of all $(\mathcal{F}, J)$ well-placed subgroups of $\mathcal{F}$ (recall that J defines a positive characteristic p functor). Now $x \notin Z(\mathcal{F})$ means there is $\varphi \in \mathcal{F}$ with $x\varphi \neq x$, so there is $Q \in \mathcal{C}$, $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ with $x \in Q$ and $x\psi \neq x$. From amongst all such $Q \in \mathcal{C}$, pick Q so that $|N_P(Q)|$ is maximal. By definition of $\mathcal{C}$, we have $Q, J(N_P(Q)) \in \mathcal{F}^f$ and so $J(N_P(Q)) \in N_{\mathcal{F}}(Q)^f$, so their respective normaliser fusion systems, $N_{\mathcal{F}}(Q)$, $N_{\mathcal{F}}(J(N_P(Q)))$ and $N_{N_{\mathcal{F}}(Q)}(J(N_P(Q)))$, by Theorem C, are all saturated. Suppose that $\mathcal{F} \neq N_{\mathcal{F}}(Q)$. Since $x\psi \neq x$, $x \notin Z(N_{\mathcal{F}}(Q))$ and $x \in Z(P)$ implies $x \in Z(N_P(Q))$, so by the minimality of $\mathcal{F}$, the hypothesis must fail when applied to $N_{\mathcal{F}}(Q)$, i.e. $x \notin Z(N_{N_{\mathcal{F}}(Q)}(J(N_P(Q))))$. But then $x \notin Z(N_{\mathcal{F}}(J(N_P(Q))))$, in particular, $N_P(Q) \neq P$. But now by Lemma 1.1.18, $|N_P(J(N_P(Q)))| > |N_P(Q)|$, a contradiction to the choice of Q. So $\mathcal{F} = N_{\mathcal{F}}(Q)$. Now Stancu's Lemma 2.2.4 applies to give $\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q))\rangle$. Since $x \in Z(PC_{\mathcal{F}}(Q)), x \notin Z(N_{\mathcal{F}}(QC_P(Q)))$. Since $QC_P(Q) \trianglelefteq P$, we have $QC_P(Q) \in \mathcal{F}^f$ so $N_{\mathcal{F}}(QC_P(Q))$ is saturated. Also $x \in Z(N_{N_{\mathcal{F}}(QC_P(Q))}(J(P))$ and the minimality of $\mathcal{F}$ imply that $\mathcal{F} = N_{\mathcal{F}}(QC_P(Q))$. Then $QC_P(Q) \in \mathcal{F}^c$ so $\mathcal{F}$ is constrained and has a model, G say. But the hypotheses of Theorem 2.2.5 are now satisfied for G. So $x \in Z(\mathcal{F}_P(G)) = Z(\mathcal{F})$, as needed for the final contradiction. $\square$

The fusion system equivalent of Theorem 2.2.6, [8], Theorem 4.5 is an easy corollary. By Theorem 2.1.37 this is the required generalisation.

Corollary 2.2.8 *[8, 4.5] Let $\mathcal{F}$ be a fusion system on a finite p-group, P. Assume p is odd or $p = 2$ and $\mathcal{F}$ is S_4-free. Then:*

$$C_{\mathcal{F}}(Z(P)) = N_{\mathcal{F}}(J(P)) = \mathcal{F}_P(P) \text{ implies that } \mathcal{F} = \mathcal{F}_P(P)$$

Proof. By Theorem 2.2.7, $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(P))) = Z(\mathcal{F}_P(P)) = Z(P)$, the last equality following from the fact that every element in $Z(P)$ is fixed by every inner automorphism of P. The result follows trivially since $\mathcal{F} = C_{\mathcal{F}}(Z(\mathcal{F})) = C_{\mathcal{F}}(Z(P)) = \mathcal{F}_P(P)$. □

Using Corollary 2.2.8, we can now prove Theorem E. The idea is due to Glesser.

Theorem 2.2.9 (Theorem E) *Let $\mathcal{F}$ be a sparse fusion system over a finite p-group P. If p is odd or $\mathcal{F}$ is S_4-free then $\mathcal{F}$ is constrained.*

Proof (1). If $J(P), Z(P) \ntrianglelefteq \mathcal{F}$, then since $\mathcal{F}$ is sparse, $C_{\mathcal{F}}(Z(P)) = N_{\mathcal{F}}(J(P)) = \mathcal{F}_P(P)$ and by Corollary 2.2.8, $\mathcal{F} = \mathcal{F}_P(P)$, a contradiction. Thus $Q := O_p(\mathcal{F}) \neq 1$. If $Q \in \mathcal{F}^c$, we are done, so assume that $Q \notin \mathcal{F}^c$. Then $Q < QC_P(Q) \ntrianglelefteq \mathcal{F}$ so $N_{\mathcal{F}}(QC_P(Q)) = \mathcal{F}_P(P)$ which, by Lemma 2.2.4 means that $\mathcal{F} = PC_{\mathcal{F}}(Q)$.

Next recall that by Lemma 2.1.75, if $\mathcal{F}$ is H-free, so is $\mathcal{F}/Q$. By Proposition 2.1.59, if $\mathcal{F}/Q = \mathcal{F}_{P/Q}(P/Q)$ then $\mathcal{F} = \mathcal{F}_P(P)$, so $\mathcal{F}/Q \neq \mathcal{F}_{P/Q}(P/Q)$. By Corollary 2.2.8 applied to $\mathcal{F}/Q$, $N_{\mathcal{F}/Q}(R/Q) \neq \mathcal{F}_{P/Q}(P/Q)$, for some $R > Q$. A further application of Proposition 2.1.59 yields $N_{\mathcal{F}}(R) \neq \mathcal{F}_P(P)$, but since $\mathcal{F}$ is sparse, $Q < R \trianglelefteq \mathcal{F}$, a contradiction. □

We get the following result as a corollary:

Corollary 2.2.10 *Suppose that p is odd or $\mathcal{F}$ is S_4-free and . If $N_{\mathcal{F}}(Q) = \mathcal{F}$ or $N_{\mathcal{F}}(Q) = \mathcal{F}_P(P)$ for all $Q \in \mathcal{F}^f$, then $\mathcal{F}$ is constrained.*

Proof. Suppose not. Then by Theorem E, $\mathcal{F}$ is not sparse. However we can pick $\mathcal{G} \supseteq \mathcal{F}_P(P)$, sparse. Then $\mathcal{G}$ is constrained so $N_{\mathcal{F}}(O_p(\mathcal{G})) \supseteq \mathcal{G}$ with $C_P(O_p(\mathcal{G})) \subseteq O_p(\mathcal{G})$. But $O_p(\mathcal{G}) \subseteq O_p(\mathcal{F})$, so $C_P(O_p(\mathcal{F})) \subseteq C_P(O_p(\mathcal{G})) \subseteq O_p(\mathcal{G}) \subseteq O_p(\mathcal{F})$ and $\mathcal{F}$ is constrained, contradicting our initial assumption. $\square$

We also get a swift proof of Theorem F, as noticed by Glesser in [10]:

Theorem 2.2.11 (Theorem F) *Suppose that p is odd and let $\mathcal{F}$ be a fusion system on a finite p-group P. Then $N_{\mathcal{F}}(W(P)) = \mathcal{F}_P(P)$ if and only if $\mathcal{F} = \mathcal{F}_P(P)$*

Proof. Let $\mathcal{F}$ be a minimal counterexample with respect to the number of morphisms in $\mathcal{F}$, i.e $N_{\mathcal{F}}(W(P)) = \mathcal{F}_P(P)$ but $\mathcal{F} \neq \mathcal{F}_P(P)$. Now let $\mathcal{G} \subseteq \mathcal{F}$, be a proper subfusion system of $\mathcal{F}$, so that $N_{\mathcal{G}}(W(P)) \subseteq N_{\mathcal{F}}(W(P)) = \mathcal{F}_P(P)$. So $N_{\mathcal{G}}(W(P)) = \mathcal{F}_P(P)$ and by the minimality of $\mathcal{F}$ we get $\mathcal{G} = \mathcal{F}_P(P)$. So $\mathcal{F}$ is sparse. By Theorem E, $\mathcal{F}$ is constrained so $\mathcal{F}$ has a model, G, say. But now $\mathcal{F}_P(N_G(W(P)))) = N_{\mathcal{F}}(W(P)) = \mathcal{F}_P(P)$ implies $N_G(W(P))$ has a normal p-complement and hence so does G by Theorem B. In other words, $\mathcal{F} = \mathcal{F}_P(G) = \mathcal{F}_P(P)$, a contradiction, as needed. $\square$

In some ways the proof of Theorem E, was unsatisfactory as its proof required results which involve fusion systems. Remarkably, there is a very different proof requiring only one ZJ-type theorem from *finite group theory*, namely Glauberman's ZJ-Theorem. We demonstrate this now:

Theorem 2.2.12 (Theorem E) *Let $\mathcal{F}$ be a sparse fusion system over a finite p-group P. If p is odd or $\mathcal{F}$ is S_4-free then $\mathcal{F}$ is constrained.*

Proof (2). Let $\mathcal{F}$ be a minimal counterexample with respect to the number of morphisms and set $Q = O_p(\mathcal{F})$. If $\mathcal{F} = N_{\mathcal{F}}(QC_P(Q))$. Then $QC_P(Q)$ is normal and we must have $Q = QC_P(Q)$ so $Q \in \mathcal{F}^c$ and $\mathcal{F}$ is constrained in this case. Hence we may assume that

$\mathcal{F} \neq N_{\mathcal{F}}(QC_P(Q))$. Since $\mathcal{F}$ is sparse, $N_{\mathcal{F}}(QC_P(Q))$=$\mathcal{F}_P(P)$. Then Stancu's Lemma implies that $\mathcal{F}$=$PC_{\mathcal{F}}(Q)$.

We first show that $Q = 1$. Suppose not. Since Q is weakly $\mathcal{F}$-closed, we may let $\mathcal{G}$ be a fusion system on P/Q satisfying $\mathcal{F}/Q \supseteq \mathcal{G} \supseteq \mathcal{F}_{P/Q}(P/Q)$ so that $\mathcal{G}$ is sparse. Then $R/Q := O_p(\mathcal{G}) \neq Q$ as $\mathcal{G}$ is constrained, by inclusion and the minimality of $\mathcal{F}$. But then R/Q is normal in P/Q and its preimage, R is normal in P. In particular, R is fully normalized and $N_{\mathcal{F}}(R)$ is saturated. Since $\mathcal{F}_{P/Q}(P/Q) \neq \mathcal{G} \subseteq N_{\mathcal{F}/Q}(R/Q)$ by Theorem 2.1.59, $N_{\mathcal{F}}(R) \neq \mathcal{F}_P(P)$ so, as $\mathcal{F}$ is sparse, $R \trianglelefteq \mathcal{F}$ which is a contradiction as $R > Q = O_p(\mathcal{F})$, proving the claim.

Assuming $O_p(\mathcal{F}) = 1$, now choose $T \leqslant P$, fully $\mathcal{F}$-normalised, so that $R := N_P(T)$ is of maximal order subject to the restriction that $N_{\mathcal{F}}(T) \neq \mathcal{F}_R(R)$. Notice such T exist by Alperin's Theorem. Choose $\mathcal{U}$, a fusion system on on R, sparse, so that $\mathcal{F}_R(R) \subseteq \mathcal{U} \subseteq N_{\mathcal{F}}(T)$. Notice that by Lemma 2.1.74, $\mathcal{U}$ is $Qd(p)$-free. By the minimality of $\mathcal{F}$, and the fact that $\mathcal{U}$ is sparse, $\mathcal{U}$ is constrained with $O_p(\mathcal{U}) = T$ and thus there exists a $Qd(p)$-free model, H for $\mathcal{U}$.

We may now apply Glauberman's ZJ-Theorem to get that $Z(J(R)) \trianglelefteq H$. Since $Z(J(R))$ char R, Lemma 2.1.14 implies that $Z(J(R)) \in \mathcal{F}^f$ and hence that $N_{\mathcal{F}}(Z(J(R)))$ is saturated. Now $N_{\mathcal{F}}(Z(J(R))) \supseteq \mathcal{F}_R(H)$ implies that $N_P(Z(J(R)))$ is not a p-group contradicting the maximal choice of T as $R < N_P(R) \leqslant N_P(Z(J(R))$. This final contradiction implies that $\mathcal{F}$ is constrained as needed. □

2.2.2 Glauberman's ZJ-Theorem for Fusion Systems

In this section, $\mathcal{F}$ will always denote a fusion system on a finite p-group, P. Recall that in Section 1.3, we saw that the following was true for the group $W(P)$ defined in the first chapter:

Theorem 2.2.13 *Let G be an $Qd(p)$-free finite group and let $P \in Syl_p(G)$. Suppose that*

G has characteristic p. Then there exists a non-trivial characteristic subgroup, $W(P)$ of P which is normal in G.

In the fusion system case it is possible to bypass treating the odd/even prime case separately by applying this Theorem. The proof idea is attributed to Stancu and Onofrei. First an analogous definition of $W(P)$ in terms of fusion systems is needed. We define this now:

Definition 2.2.14 Let P be a finite p-group. Let $\mathcal{C}_J$ be the class of fusion systems $\mathcal{F}$ on P satisfying:

$\mathcal{C}_1$ $J(P) \trianglelefteq \mathcal{F}$ for every $\mathcal{F} \in \mathcal{C}_J$.

$\mathcal{C}_2$ $\mathcal{F}$ is $Qd(p)$-free.

Set $W_0 := \Omega(Z(P))$ and let $W := \langle W_0\varphi \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(J(P), P) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle$.

Proposition 2.2.15 *[16, 4.3] The following hold:*

(a) $1 \neq W$ *char* P*;*

(b) $W = W(P)$*; and*

(c) for every $\mathcal{F} \in \mathcal{C}_J, \mathcal{F}$ *has a model, in which* W *is normal.*

Proof. Let $\alpha \in \mathrm{Aut}(P)$ and $\mathcal{F}$ be a fusion system on P. Let $\mathcal{F}^\alpha$ denote the fusion system on P with morphisms:

$$\mathrm{Hom}_{\mathcal{F}^\alpha}(Q, R) := \alpha^{-1} \circ \mathrm{Hom}_{\mathcal{F}}(Q\alpha, R\alpha) \circ \alpha.$$

Then $\mathcal{F} \in \mathcal{C}_J$ implies that $\mathcal{F}^\alpha \in \mathcal{C}_J$ [8] and:

$$W\alpha = \langle W_0\varphi\alpha \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(J(P), P) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle$$

[8] $\mathcal{F}$ and $\mathcal{F}^\alpha$ are *isotypically equivalent.*

$$= \langle (W_0\alpha)\alpha^{-1}\varphi\alpha \mid \varphi^\alpha \in \mathrm{Hom}_{\mathcal{F}^{\alpha^{-1}}}(J(P)\alpha, P\alpha) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle.$$

Set $\overline{\varphi} := \varphi^\alpha$. By the fact that $W_0, J(P)$ char P, this becomes

$$\langle W_0\overline{\varphi} \mid \overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}^{\alpha^{-1}}}(J(P), P) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle$$

which is clearly contained in W. The injectivity of α implies that $|W| = |W\alpha|$, so $W = W\alpha$ and part (a) follows.

Part (b) follows from [16], Section 4.2, where an analogous construction for $W(P)$ is given in terms of fusion systems.

For any $\mathcal{F} \in \mathcal{C}_J$, $J(P) \in \mathcal{F}^c$, thus $\mathcal{F}$ is constrained and has a $Qd(p)$-free model, L say. Part (c) now follows from part (b) and Theorem A. □

Having constructed, W, we can now prove Theorem G:

Theorem 2.2.16 (Theorem G) *Let P be a finite p-group and let $\mathcal{F}$ be a $Qd(p)$-free fusion system over P. Then there exists a non-trivial characteristic subgroup $W(P)$ of P which is normal in $\mathcal{F}$.*

Proof. This is trivially true for $\mathcal{F}_P(P)$. Let $\mathcal{F}$ be a minimal counterexample and set $Q := O_p(\mathcal{F})$, $R := QC_P(Q)$. If $Q = 1$ then for every non-trivial $X \in \mathcal{F}^f$, $N_\mathcal{F}(X) < \mathcal{F}$ and by Proposition 2.1.74, since $\mathcal{F}$ is $Qd(p)$-free, $N_\mathcal{F}(X)$ is $Qd(p)$-free. Thus by the minimality of $\mathcal{F}$, $N_\mathcal{F}(X)$ satisfies the conclusion of the theorem, i.e $N_\mathcal{F}(X) = N_{N_\mathcal{F}(X)}(W(N_P(X)))$. But then Proposition 2.1.52, implies that $\mathcal{F} = N_\mathcal{F}(W(P))$, a contradiction.

Claim: $Q \notin \mathcal{F}^c$. *Proof of Claim:* If not, then $\mathcal{F}$ is constrained, which, by Theorem D means that $\mathcal{G}(\mathcal{F}) \neq \emptyset$, i.e. there is a $Qd(p)$-free finite group, G with Sylow p-subgroup P, such that $\mathcal{F}_P(G) = \mathcal{F}$. By Theorem 1.3.10, $G = N_G(W(P))$ so

$$\mathcal{F} = \mathcal{F}_P(G) = \mathcal{F}_P(N_G(W(P))) = N_\mathcal{F}(W(P)),$$

a contradiction.

Claim: $\mathcal{F} \neq PC_{\mathcal{F}}(Q)$. *Proof of Claim:* Suppose not. Since $\mathcal{F}$ is $Qd(p)$-free, so is $\mathcal{F}/Q$ by Proposition 2.1.75 so by the minimality of $\mathcal{F}$, $\mathcal{F}/Q = N_{\mathcal{F}/Q}(W(P/Q))$. If V denotes the inverse image of $W(P/Q)$ in P, then by Proposition 2.1.59, $\mathcal{F} = N_{\mathcal{F}}(V)$ implying that $V \leqslant Q$. So $W(P/Q) = QV/Q$ is trivial. But this is a contradiction since $Q < R \leqslant P$ means that P/Q is non-trivial, as then is $W(P/Q)$.

Claim $\mathcal{F}$ is not a counterexample. *Proof of Claim:* Since $\mathcal{F} = N_{\mathcal{F}}(Q)$, Lemma 2.2.4 imples that $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$, for $\mathcal{F}_1 := PC_{\mathcal{F}}(Q), \mathcal{F}_2 := N_{\mathcal{F}}(R)$. Since $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, by induction, $W(P) \trianglelefteq \mathcal{F}_1, \mathcal{F}_2$. Since $W(P) \in \mathcal{F}^f$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_P(W(P)), P)$, such that $(W(P))\varphi \in \mathcal{F}^f$. Since $W(P)$ char P, $(W(P))\varphi = W(P) \in \mathcal{F}^f$. Thus, since

$$\mathcal{F} = \langle N_{\mathcal{F}_1}(W(P)), N_{\mathcal{F}_2}(W(P)) \rangle \subseteq N_{\mathcal{F}}(W(P)) \subseteq \mathcal{F},$$

we have

$$\mathcal{F} = N_{\mathcal{F}}(W(P))$$

as needed. $\square$

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