USING SEMI-INFINITE OPTIMISATION TO CALCULATE PRICE BOUNDS FOR BASKET OPTIONS

by

Zubair Ahmad

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College of Engineering &
Physical Sciences
School of Mathematics
University of Birmingham
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Abstract

The use of optimisation within financial markets is rapidly increasing. There is a growing demand for a class of new and improved methods to accurately price financial options. Semi-infinite optimisation (SIO) has become a vivid research area in mathematical optimisation during the recent two decades. This is due to the fact that there are many new theoretical advances as well as a broad variety of real-life problems where this mathematical model can be applied. This research thesis considers particular applications of SIO to finding upper and lower bounds on the prices of various types of basket options. In particular, new and original results have been derived for:

- Finding a lower bound on European basket call option prices.
- Calculating a lower bound on European basket call option prices, incorporating bid-ask prices within the model; thus making it more realistic.
- Analysing price bounds on various types of American basket options.
- Deriving an upper bound on the price of a discretely sampled arithmetic average Asian basket call option.
- Extending this model by finding an upper bound on the price of a discretely sampled arithmetic average Asian basket call option, but incorporating bid-ask prices.
- The final result is concerned with calculating an upper bound on the price of an Altiplano Mountain Range option which is closely related to basket options.

The models and results obtained in this thesis could directly be used in financial markets by investors, investment banks and hedge funds amongst others.

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Contents

1 Introduction

The use of options in financial markets is ever present. From speculating to hedging, more and more investors are starting to use options in their financial activity. Investors are likely to purchase options because the potential to make profit from them is much greater in comparison to investing in stocks and shares alone (so called *gearing*). However, this should be done with some care. Where there is the opportunity to make bigger profits, of course there is also the risk of making a bigger loss. Investing in an option is no different. The primary question of interest is, 'how much should one pay for an option?' This has and still is perhaps the most widely studied area of mathematical finance because when pricing an option many factors need to be taken into account. Calculating the correct current price of an option is important because it stops any potential *arbitrage* opportunities, as well considering the following questions. Will the underlying share price increase or decrease? What is the risk to the holder (buyer)? What is the risk to the writer (seller)? The current price of a financial option ensures that the holder and writer are getting a fair price and a price that reflects the risk that is being taken by both parties. So, for example if there is more risk to the writer, that is, it is more likely that the writer will have to pay out a huge payoff to the holder then this should be reflected in a higher price for the option since there is a lesser risk for the holder. Thus, there are many factors that need to be taken into account when pricing a financial option. Of course, this price should reflect the risk being taken by both the holder and the writer and should take other factors into account.

Hence, it is clear that correctly pricing financial options is of the up-most importance but the question that still remains is, how do we calculate such a fair current price? For many years this problem has been studied in mathematical finance and right up until the present day, mathematicians, economists and financial firms amongst many others have been researching and trying to produce valid theory and methods to correctly price financial options.

There are many theoretical ideas and models derived in mathematical finance aimed at finding the price of an option. Of course we are already aware of the famous *Black-Scholes* partial differential equation derived by Fischer Black and Myron Scholes to price options (see sub-section 2.1.4), as well as other methods such as the binomial method (see sub-section 2.1.4) which uses probabilities under a risk-neutral assumption to construct a binomial lattice tree. However these pricing models are not without their problems and the paper [1] highlights the problems that using such mathematical models may have in the real world. It should be remembered that a model is just a model which makes many assumptions about the real world that may not hold. Further, it may simplify reality too much and so it must be used with real care, especially when implementing the results in the real financial market.

It is thus clear that although the pricing models that currently exist are very credible, they do have some potential problems and any new advances within this area of financial mathematics would be greatly welcomed. This leads us to considering another area of mathematics which has been developing rapidly in the last decade, and that is using *optimisation* to price options. We will present and discuss in detail some important results showing how optimisation can be used to calculate price bounds on options throughout this thesis, in particular in sections 4, 5, 7 and 8. As an introductory remark, we note that when using optimisation to find price bounds on options, the basic steps to model and solve the problem are very similar regardless of the option under consideration. Using an optimisation approach we can obtain upper and lower bounds for the current prices of various options and using market data it can then be shown that these bounds are indeed valid and in some instances it can be shown that the obtained bounds are sharp, that is, they are the tightest and thus the best possible bounds for the current price of the option under consideration. Of course as is the case when modelling any real world problem as an optimisation problem, many factors affect the size, complexity, tractability and solvability of the optimisation but that is something we shall discuss later on in the thesis. We do note here that combining optimisation with this widely studied area of mathematical finance has produced impressive results [11, 18, 19, 20, 21, 22].

Having this as motivation, this thesis is dedicated to analysing how mathematical optimisation can be used to find upper and lower bounds on the current price of a particular type of option. However it should be remembered that although we are looking for bounds on the current price of an option, ultimately the price paid is an agreement between the writer and the holder and whether that is at the current price or not is up to the two parties.

The goal of this PhD thesis is to present some of our own, new and original results. As we will see a bit later, the problem of finding upper/lower bounds on the current price of a *basket option* may be modelled as a *semi-infinite optimisation* (SIO) problem. We then look at various reformulation techniques that can be used to solve the formulated semi-infinite problems and thus yield an upper/lower bound on the current price of the basket option we are interested in.
In particular, the topics which we have derived new results for are,

- Lower bounds for European basket options.
- A lower bound incorporating bid-ask prices for European basket options.
- Finding price bounds on American basket options.
- Finding upper price bounds on Asian basket options.
- An upper bound for Asian basket options incorporating bid-ask prices.
- An extension to finding upper price bounds on Altiplano Mountain Range options.

Besides these new and original results, we present a literature overview about existing results on this topic. The thesis is organised as follows. In Section 2 we present some preliminaries from mathematical finance and mathematical optimisation which we will use throughout this thesis. The work presented in this section is important because it is here where we present the most basic and fundamental ideas for which the rest of this thesis is based on. In Section 3 we give a literature overview and present some existing results on how semi-infinite optimisation has been used to find upper and lower bounds on the price of a European basket call option. The remainder of the thesis is dedicated to looking at our own, new and original results and is organised in the following way. In Section 4 we look at new results concerning price bounds on European basket options. In Section 5 we consider finding price bounds on American basket options. In Section 7 we demonstrate how price bounds on Asian basket options may be found. In Section 8 we present an extension to finding price bounds on an Altiplano Mountain Range option. Finally, in Section 9 we conclude this thesis and look ahead to what could be researched further.

2 Preliminaries

In this section we present some preliminary and fundamental ideas which are needed for the rest of this thesis. We note here that these ideas form the base of this thesis and everything we present in this thesis can be linked back to these fundamental ideas in one form or another. Our topic of interest combines together two huge areas of mathematics. Namely, we are considering a combination of mathematical finance and mathematical optimisation, and, in particular, we are considering how mathematical optimisation techniques can be used to solve problems from mathematical finance. Thus, it is absolutely essential that we are familiar with the basic ideas and definitions from these two areas of mathematics.

2.1 Preliminaries from Mathematical Finance

We begin by briefly introducing the main ideas from mathematical finance which will be needed and referred back to throughout this thesis.

2.1.1 Introduction

The first step is to explain some main ideas from *mathematical finance*. Now, put simply, mathematical finance is an area of applied mathematics. Confronting a real-world problem, the task is to model it mathematically. We then use numerical mathematical techniques to solve the problem and obtain a solution. Finally, we must interpret the solution in the context of the original problem and make an appropriate conclusion in the context of the problem under consideration. The area of mathematical finance is no different. Here the 'real world' problem under consideration is the financial markets. Thus, mathematical finance is concerned with modelling problems from the financial markets mathematically, and then, using mathematical techniques, to obtain a solution that can be interpreted within the context of the problem. It is worth noting here that this area of mathematics has been rapidly growing, especially in the last decade or so. There are many reasons for this. Firstly, is the fact that we are looking at how to use mathematics on a 'real world' problem; the solutions we obtain can be directly interpreted and used in real by investors in financial markets. The second reason is the current state of the economy and financial sector. After the financial crisis of 2007, many people including private investors, financial firms and investment banks are turning to more sophisticated models to implement in financial markets. Mathematical finance provides investors with alternative and successful methods which may be implemented in real and as such, it is a hugely welcome area. By using advanced and sophisticated mathematical techniques to model and solve financial problems, mathematical finance provides an alternative to other, standard financial techniques.
In what follows we present some ideas and well-known results from mathematical finance which will be useful for the work that follows later in this thesis. We assume here that the reader is familiar with basic definitions from finance, such as *stocks, shares, dividends, arbitrage* and so on.

2.1.2 Financial Options

The base of this thesis is formed from the idea of financial options which we define below.

Definition 2.1. *A financial option is a financial product, or a contract, between two parties, issued by the seller, known as the **writer** from here on in which gives the buyer, known as the **holder** from here on in, the right but not the obligation to purchase or sell a prescribed asset, known as the **underlying asset**, for a prescribed price, known as the **exercise price** at a prescribed future time, known as the **expiry date**.*

Note: The word *option* comes from the fact that the holder has a choice. The holder may choose to buy/sell the underlying asset if he wishes to do so. Conversely, the other party of the option is the *writer*. They have the **obligation** to sell/buy the underlying asset as agreed in the option if the holder wishes to exercise the option.

Observation: Since the holder has the choice and not the obligation, then it makes intuitive sense that it should cost the holder something to enter this contract. This is the idea behind pricing financial options; something which we will come back to in sub-section 2.1.4.

The *payoff* of the option is the quantity or amount of money that the holder obtains when exercising the option.

The payoff depends on the exercise price and the value of the underlying asset(s) at the moment of exercising but *not* on the price of the option at the moment the holder bought it.

Now, there are many different types of options. As the financial markets continue to grow further, more types of options will become available to best suit investor's needs. However, the simplest type of option is a European vanilla option. These types of options are split into the following two categories. European vanilla call options and European vanilla put options. Put simply, a European vanilla call option gives the holder the right to buy the underlying asset, at a pre-agreed fixed price (exercise price), on a pre-agreed fixed, future date (expiry date). At no other time is the holder allowed to choose to buy the underlying asset (apart from the pre-agreed expiry date, that is). A European vanilla put option is exactly the same as a European vanilla call option except that here the holder has the right to sell the underlying asset.

We now present the payoff functions for European vanilla call and put options. For more information including diagrams and a thorough explanation of these types of options, we refer the interested reader to [2], pp. 35-38.

We first need to introduce some notation. In what follows let $C_{\mathcal{E}}$ denote the current price of a European vanilla call option, and let $P_{\mathcal{E}}$ denote the current price of a European vanilla put option. Let S denote the price of the underlying asset at expiry and let E denote the exercise price of the option. Let time be denoted by t, so that time $t = T$ is the expiry date. Then, since the price of the call/put option depends upon the price of the underlying asset at expiry and time (see [2] for full details), we may write $C_{\mathcal{E}} = C_{\mathcal{E}}(S,t)$ and $P_{\mathcal{E}} = P_{\mathcal{E}}(S,t)$.

This allows us to define the payoff of a European vanilla call option as, (see [2])

$$C_{\mathcal{E}}(S,T) = \max(S - E, 0). \tag{2.2}$$

Similarly, we may define the payoff of a European vanilla put option as, (see [2])

$$P_{\mathcal{E}}(S,T) = \max(E - S, 0). \tag{2.3}$$

Now, as already mentioned, European vanilla options are the simplest type of options that we will encounter. However, extensions of these options do exist. The most natural extension leads us to *American* options. Here the basic idea is the same as European vanilla options but the holder has the choice to exercise the option at any time up to and including the expiry date. Of course, since the holder has more freedom as to when they can exercise the option (if they exercise it all, that is), we should expect an American option to cost at least as much as its European option 'counterpart'. More information on the basic ideas of American options can be found in [2].

Now we return to the point made earlier about how rapidly the options market is growing to keep up with demand to meet different investor's needs. If only simple European and American options existed the use of complicated and impressive mathematical models within this area of finance would not be needed. Where the use of sophisticated mathematics comes into finance is when we consider *exotic options*. Now, put simply 'an exotic option is an option that is not a vanilla put or a call' [2], p.198. Thus the simplest way to spot an exotic option is to look at its corresponding payoff. If the payoff is of the form (2.2) or (2.3), then this option is a simple vanilla call or put option and if not, then this option is an exotic option. Of course, the range of exotic options is huge and we only need to look on the internet, for example, to see the broad range of exotic options that exist and are readily available to trade in. Also, we note here that different exotic options have different properties and investors may choose to trade in various different options to best suit their needs.

Some examples of exotic options are: binaries and digitals, compound options, chooser (or 'as you like it') options, barrier options, Asian options, look-back options, shout options, power options, basket options,

exchange options, extendible options, range options, spread options, forward-start options, swing options and rainbow options. We note here that this is by no means a complete list of exotic options that are available for trading. There are many more which can most commonly be found on the internet. For more information regarding these, and other exotic options we refer the reader to [2] or [3].
Now, it is clear that there exist a whole range of exotic options available for trading. However, in order to carry out some meaningful and in-depth research we need to concentrate on one area. In this thesis we will be working with *basket options*. As we will see in later sections, the reason for choosing this particular type of exotic option is that the area of semi-infinite optimisation can be applied to this type of option in an efficient way.

2.1.3 Basket Options

In this section we introduce the main ideas and properties of basket options. Put simply, a basket option is an exotic option whose payoff depends on multiple assets. Each asset is assigned a weight and then the overall sum of the assets with their corresponding weights is taken into account. The value of this weighted sum is then compared with the fixed exercise price to determine whether or not the basket option should be exercised. Of course, there exist many types of basket options such as European or American basket options, for example, as well as basket call options and basket put options. We will explain the type of basket option we are considering.
Next we introduce the payoff of a basket option. For this, consider a basket option written on n underlying assets. Recall that there then exists a weight attached to each asset and the payoff depends on a combination of the prices of these n assets and their corresponding weights as well as the exercise price of the basket option. Let the exercise price of the basket option under consideration be given by $E \geqslant 0$. Define $S_i \in \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$ to be the price of the i^{th} underlying asset at expiry, for $i = 1, 2, \ldots, n$ and let $\omega_i \in \mathbb{R}$ be the corresponding weight for the i^{th} asset in the basket option. Then for given weights $\omega_1, \omega_2, \ldots, \omega_n$ the payoff of a European basket call option is given by

$$C(S_1, S_2, \ldots, S_n, T) = \max\left(\sum_{i=1}^{n} \omega_i S_i - E, 0\right). \tag{2.4}$$

Also, the payoff of a European basket put option for given weights $\omega_1, \omega_2, \ldots, \omega_n$ is given by

$$P(S_1, S_2, \ldots, S_n, T) = \max\left(E - \sum_{i=1}^{n} \omega_i S_i, 0\right). \tag{2.5}$$

We note here that sometimes it may be more convenient for us to use a vector form of (2.4) and (2.5). We may define this as follows. Let the weights vector be given by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \in \mathbb{R}^n$, and the asset price vector (at expiry) be given by $S = (S_1, S_2, \ldots, S_n)^T \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geqslant 0,\ i = 1, 2, \ldots, n\}$. We may then write (2.4) equivalently as

$$C(S, T) = \max\left(\omega^T S - E, 0\right), \tag{2.6}$$

and (2.5) equivalently as

$$P(S, T) = \max\left(E - \omega^T S, 0\right). \tag{2.7}$$

Basket options may be used by investors for various reasons. Perhaps the most common reason is *diversification*. By investing in a basket option, an investor is taking a hedging approach. This is because, since the basket option depends on a combination of numerous, n, assets, if the price of one of the assets falls but the price of another asset in the basket rises then the investor is in a sense covered by any potential loss he may make, (of course this is also dependent on the weights given to the assets). For this reason basket options are popular amongst investors as it allows them to diversify their investment without directly investing in numerous stocks and shares alone, where transaction costs need to be paid.
Another reason is that basket options allow investors to essentially make a bet on an index. For example,

suppose we had a basket option written on the assets of the top 100 companies in the UK. Then the holder of this basket option could choose to exercise this option and lock in a profit by observing the value of the FTSE 100 at the time of expiry for European options or at any time before expiry for American options. Thus, for these reasons, investing in basket options seems attractive and sensible for many investors.

A final note we make here is the comparison between the payoff of a basket option and the payoff of a simple vanilla option. If we compare (2.2) with (2.6) and (2.3) with (2.7), we can see that the payoffs are similar and almost identical except that in the payoff of the basket option the asset price is not dependent on the price of a single asset (as is the case with simple vanilla options). Instead it is dependent on the weighted sum of n assets, thus showing us the similarities yet vital differences between the two types of options.

In the remainder of this thesis we will concentrate on basket options. Interested readers who wish to consider other types of options as well as basket options are referred to study the work done in [2] or [3].

2.1.4 The Option Pricing Problem

Recall the observation made in sub-section 2.1.2. It was argued there that, on the one hand, since the holder of an option has a choice but not an obligation to exercise an option it should cost them something to hold an option. On the other hand, if the holder chooses to exercise the option, the writer has an obligation to fulfill. They must sell/buy the underlying asset depending on the type of option that has been agreed. Thus, it would make sense that since the holder has the freedom of choice it should cost them something to hold an option. This is one of the basic ideas behind option pricing in mathematical finance.

Now, option pricing is one of the most widely studied areas of mathematical finance [2]. There are many reasons for this. The main reason is linked with the idea above. When pricing a financial option there are many factors that need to be taken into account. The writer of an option may face the risk of paying out a potentially large payoff to the holder. In some sense the writer needs to be compensated for taking this risk. This risk should be reflected in the price of the option. That is, this risk should be reflected in the amount of money the holder pays the writer to enter the option. Thus the primary question of interest to us is, what is the price of an option? How can we work out what the price of an option should be such that the risk to the writer and the freedom of choice to the holder are captured? This is the main question of interest to us and we will consider this question throughout this thesis.

In this section we state some of the most famous mathematical models that exist to find the price of an option. However before we proceed to doing so, we take note of the following. Firstly, a mathematical model is just a model and should be used with care and common sense. Many mathematical models make numerous assumptions and some of these assumptions may be violated in real life in which case, we should be very careful when using the model to solve and price options in the real world. Secondly, we should take note of the results that the mathematical model produces. Financial markets are dependent upon the laws of supply and demand. The options market is no different. Ultimately, the price paid for an option by the holder to the writer is the price agreed by both parties. If both parties agree to a price even though this price is different to the current price, then this is the price that will be paid for the option. However, the mathematical models which we consider here give a starting point. Using these models to price an option produces results which take the risk that the writer is taking into account and captures other factors. In a sense the mathematical models aim to find the 'fair' price of an option. They find what the price 'ought to be' on the market under many different, but realistic assumptions about the financial market.

We now recall some well-known mathematical models that exist to find the price of an option. We will not give in-depth detail here since derivations of the models presented can be found in [2]. Instead we merely state the models.

The first of these is the well-known *Black-Scholes model*. In order to present this model we first introduce some notation. Let S denote the price of the underlying asset at expiry. We will let $V = V(S,t)$ denote the price of the option under consideration, which depends on the asset price at expiry and time, t. Further, we will let σ be the volatility of the underlying asset (which captures the standard deviations of future prices), μ be a measure for the average growth rate of the underlying asset (known as the drift rate), E be the exercise price of the option, T be the expiry date and r be the risk free interest rate.

Then if W is a Brownian motion (see [2] for a formal definition), using the following stochastic differential equation,

$$\frac{dS}{S} = \sigma dW + \mu dt,$$

or

$$dS = S(\sigma dW + \mu dt), \tag{2.8}$$

and under some assumptions stated in [2], we may derive the following partial differential equation (PDE) for V.

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - rV + rS\frac{\partial V}{\partial S} = 0. \tag{2.9}$$

The PDE (2.9) is called the *Black-Scholes partial differential equation*, and its solution $V(S,t)$ of this PDE is the price of the option.
In general the PDE (2.9) must be solved numerically, but there do exist some cases (see [2]) where (2.9) can be solved analytically to give a solution.
Another model which we present here, and is perhaps more linked to what follows later in this thesis that can be used to price certain financial options is known as the *binomial method*. This method is important with regards to the work that follows in the main sections of this thesis because the idea of *risk neutrality*, which is fundamental to the binomial method is utilised in the model set-up which is used in later sections. The basic idea of this method can be summarised as follows.

- Here we assume that the stochastic differential equation (2.8) can be modelled by a discrete random walk such that the following assumptions hold. The asset price changes only at discrete time points, given by δt, $2\delta t$, $3\delta t$, $\ldots$, $M\delta t = T$. If the asset price has value S^m at time point $m\delta t$, then at the time point $(m+1)\delta t$, S^{m+1} can take one of two prices. Either it can take a price of $S^{m+1} = \tilde{u}S^m > S^m$, or it can take a price of $S^{m+1} = \tilde{d}S^m < S^m$, so that $\tilde{u} > 1$ and $0 < \tilde{d} < 1$. Further, we assume that S^m can move to $S^{m+1} = \tilde{u}S^m$ with a probability p, and so S^m can move to $S^{m+1} = \tilde{d}S^m$ with probability $(1-p)$. The parameters $\tilde{u}, \tilde{d}$ and p are chosen in such a way so that the statistical properties of the discrete random walk coincide with (2.8).

- The second basic idea is based upon the idea of a risk neutral world. This means that (and see [2] for full details) an investor's risk preferences become irrelevant when pricing options. Consequently this means that we can replace the drift rate μ in (2.8) by the interest rate r. This gives the risk-neutral random walk,

  $$dS = S(\sigma dW + rdt). \tag{2.10}$$

  The idea of a risk neutral world is a very important concept in mathematical finance and in particular when pricing options. We will see how this will play an important role in the main work of this thesis in later sections.

Now that we have introduced the basic ideas of this method, we are in a position to give a brief overview of how to practically implement it when pricing options. Full details of the binomial method for pricing options can be found in [2]. Essentially, what we do is build up a tree of possible prices that the underlying asset can take. At the end (terminal) nodes which represent the expiry date we evaluate the payoff of the option to be priced. We then use the fact that the price of an option at expiry is equal to its payoff (by arbitrage arguments) and working back down the tree we can evaluate the current price of the option at time $t = 0$. Indeed, if V^m denotes the price of the option at time $m\delta t$, then using the idea of risk neutrality we can show that (see [2])

$$V^m = \mathbb{E}[\mathrm{e}^{-r\delta t}V^{m+1}], \tag{2.11}$$

where $\mathbb{E}$ denotes expectation. For a full derivation of this, we refer the reader to [2]. Then using (2.11) and the method described above, the task is to find the value V^0, which is the current price of the option.
Now, we have stated the famous Black-Scholes model and the binomial method used for pricing options. One of the restrictions that both of these methods possess is that in their current form they can only be used to price options with a single underlying asset. However, we have already mentioned that the main topic

of this thesis is to concentrate on looking at how the prices of a basket option can be found. The natural question to consider now is if and how we can extend the Black-Scholes model to allow for the pricing of options written on many underlying assets, such as a basket option. This is done in [3], where we consider the *multi-asset Black-Scholes equation.*
Here instead of considering an option written on one underlying asset we consider an option written on n underlying assets. This allows us to write down a stochastic differential equation for the i^{th} asset, for $i = 1, 2, \ldots, n$ where we extend (2.8) to

$$dS_i = S_i(\sigma_i dW_i + \mu_i dt). \tag{2.12}$$

Under similar assumptions (see [3]) and following what was done for the single asset case, we can arrive at the following PDE, see [3] for full details. This PDE is known as the *multi-asset Black-Scholes partial differential equation* (MABSE), and is given by

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i \partial S_j} + \frac{\partial V}{\partial t} - rV + r\sum_{i=1}^{n}S_i\frac{\partial V}{\partial S_i} = 0, \tag{2.13}$$

where ρ_{ij} denotes the correlation between asset i and j, for $i, j = 1, 2, \ldots, n$. We note here that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$, for all $i, j = 1, 2, \ldots, n$. Thus, we can use (2.13) to price options written on many underlying assets.
Now, the main concern of this thesis is the pricing of basket options. We have just introduced the multi-asset Black-Scholes equation that can be used to price options written on many underlying assets. Of course we are free to go ahead and try to use (2.13) to price a basket option of interest. Unfortunately, it turns out that (2.13) is very difficult to solve numerically, when it comes to basket options. This is highlighted in [3], p.283, where it is pointed out that when deriving the MABSE formula for basket options we need to delta hedge with knowledge about correlations between the assets. However, it is very difficult to measure and predict the correlations between each asset and so delta hedging basket options becomes an almost impossible task. So although the MABSE formula is valid and can be used to price any multi-asset option, in reality for the pricing of basket options it is of very little use. Thus, we are in need of more mathematical models or techniques that can be practically implemented to help us find the price of a basket option. This highlights one of the reasons as to why specifically we are considering the pricing of basket options in this thesis.

2.1.5 Some other existing methods to price Basket Options

Now, option pricing is a widely studied area so researchers have not just observed the limited use of equation (2.13) when it comes to pricing basket options, but they have tried to produce some more results. We will give a short overview of some of these results in this sub-section, referring the reader to study the full details in [4]. In all of the methods mentioned below and derived in [4] we consider pricing a European basket call option. We mention the following methods.

- *Beisser's conditional expectation technique.* The basic idea here is to estimate the price of the basket call option under consideration by using the known prices of simple European vanilla calls to obtain a lower bound on the required price. For a full in-depth derivation of this method we refer the reader to [4]. In addition, as we will see in later sections, we will adopt a similar technique but use mathematical optimisation as a solution technique to find upper/lower bounds on the price of a basket option.

- *Gentle's approximation by geometric average.* This method is based on the following observation. In the payoff for the basket call option we have the weighted arithmetic average of the prices of the underlying assets at expiry. In this method, in the payoff, we replace this weighted arithmetic average by the weighted geometric average of the underlying assets at expiry.

- *Levy's log-normal moment matching approach.* Here, the basic idea is to approximate the distribution of the basket option of interest by matching the first two moments of the required distribution with the first two moments of the original distribution of the weighted sum of the stock prices. For full details of this pricing method we refer the reader to [4].

- Finding the price of a European basket call option based upon *Taylor expansions*. In particular, here we consider *Ju's Taylor expansion* from [4]. The basic idea is to define a characteristic function and then we Taylor expand the ratio of this characteristic function with an arithmetic average which is defined in [4].
- *Inverse gamma approximation by Milevsky and Posner*. Here the idea is to use the inverse gamma distribution (see [5]) as an approximation for the distribution of the assets in the basket option.
- Another technique derived by Milevsky and Posner. The basic idea here is to find the price of the basket call option by higher order moments using state prices (which are also called *all or nothing options*). This option pays out (a certain amount) if a scenario occurs and nothing otherwise. This allows us to define a state price density which is a density that captures the likelihood of all possible future scenarios. It may be thought of as a probabilistic weight being assigned to each possible scenario. Here, we consider matching state price densities to higher moments of the distribution of the weighted sum of the underlying assets in the payoff of the basket option to be priced.

That concludes this sub-section on presenting preliminaries from mathematical finance and option pricing. In particular we have introduced some basic ideas about basket options which are fundamental to the rest of this thesis.
Obviously, the need to find valid pricing techniques to accurately price basket options is very much in demand. This is the motivation behind the research presented in this thesis.

2.2 Preliminaries from Mathematical Optimisation

Along with the ideas from mathematical finance, this thesis uses mainly ideas from mathematical optimisation. In this sub-section we present some preliminaries and basic definitions from mathematical optimisation which are needed later.

2.2.1 Introduction

Optimisation is a mathematical technique aimed at finding the maximum or minimum of a function subject to a particular domain or region. In the following, we will look at some classes of optimisation problems.

2.2.2 Classes of optimisation problems

Many optimisation problems possess certain characteristics which allow us to define a range of different classes of optimisation problems. In this sub-section we briefly present some of these different types of problems.
Mathematical optimisation problems can be put into one of the following two groups. A problem can either be an *unconstrained optimisation problem*, or it can be a *constrained optimisation problem*. We formalise these concepts as follows. Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, given by $f(x)$, where $x = (x_1, x_2, \ldots, x_n)^T$. We will assume that f is k times continuously differentiable and this will be denoted by $f \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R})$, for $k \geqslant 0,\ k \in \mathbb{N}$.
We now present some important definitions which are fundamental to rest of this thesis.

Definition 2.14. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then $x^* \in \mathbb{R}^n$ is said to be a* ***local minimiser*** *of f, if there exists a neighbourhood $\mathcal{U}(x^*)$ of x^* such that*

$$f(x^*) \leqslant f(x), \text{ for all } x \in \mathcal{U}(x^*).$$

Definition 2.15. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then $x^* \in \mathbb{R}^n$ is said to be a* ***global minimiser*** *of f if*

$$f(x^*) \leqslant f(x), \text{ for all } x \in \mathbb{R}^n.$$

Then, the global unconstrained optimisation problem is to find an $x^* \in \mathbb{R}^n$ such that

$$f(x^*) \leqslant f(x), \text{ for all } x \in \mathbb{R}^n.$$

We write this problem as

$$\underset{x}{\text{minimise}}\, f(x). \tag{$\mathcal{UO}$}$$

This can be done without loss of generality because even if we wanted to find the maximum of f we could use optimisation to find the minimum of $-f$ and then observe that the maximum of f is equal to the negative minimum of $-f$.
Now we present what is meant by a *constrained optimisation problem.*
Let $f \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R})$ which is called the *objective function.* Let $X \subset \mathbb{R}^n$ be the so called *feasible set.* If $x \in X$, then x is said to be a *feasible point.* Then $(\mathcal{CO})$, given by

$$\begin{aligned} &\underset{x}{\text{minimise}} && f(x) \\ &\text{subject to} && x \in X, \end{aligned} \tag{$\mathcal{CO}$}$$

is a constrained optimisation problem.
Analogous to the unconstrained problem we look to minimise the function f, again without loss of generality (see above), but this time we have the condition or *constraints* that $x \in X$.
This allows to present the following definitions for $(\mathcal{CO})$.

Definition 2.16. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then $x^* \in \mathbb{R}^n$ is said to be a* ***local minimiser*** *of $(\mathcal{CO})$, if there exists a neighborhood $\mathcal{U}(x^*)$ of x^* such that*

$$f(x^*) \leqslant f(x), \text{ for all } x \in \mathcal{U}(x^*) \cap X.$$

Definition 2.17. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then $x^* \in \mathbb{R}^n$ is said to be a* ***global minimiser*** *of $(\mathcal{CO})$if*

$$f(x^*) \leqslant f(x), \text{ for all } x \in X.$$

We note here that quite often the *feasible set* X may take the form

$$X = \{x \in \mathbb{R}^n | h_i(x) = 0, i = 1, 2, \ldots, m,\ g_j(x) \leqslant 0, j = 1, 2, \ldots, r\}, \tag{2.18}$$

where $h_i, g_j \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R})$, for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$.

We now distinguish between two important cases. In the case where the function f and the functions h_i, g_j, for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$ are linear, $(\mathcal{CO})$ is called a *linear optimisation problem.* In contrast, if the objective function f or at least one of the constraints h_i, g_j are not linear, then $(\mathcal{CO})$ is called a *non-linear optimisation problem.* It is well known in mathematical optimisation that linear problems are easier to solve than constrained non-linear problems. This is because mathematically the linear problem is the simplest case.
An important class of optimisation problems which we consider here is the class of *convex optimisation problems.* In order to define this we need the following definition of a convex function.

Definition 2.19. *Let $\tilde{c} : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that $\tilde{c}$ is a* ***convex function*** *if*

$$\tilde{c}(\tilde{\alpha}x + (1 - \tilde{\alpha})y) \leqslant \tilde{\alpha}\tilde{c}(x) + (1 - \tilde{\alpha})\tilde{c}(y), \text{ for all } x, y \in \mathbb{R}^n \text{ and } \tilde{\alpha} \in [0, 1].$$

Then a convex optimisation problem is a problem where the objective function f is convex and the feasible set, X is convex. Convexity of a function is highly desirable in optimisation because for convex optimisation problems we have the following well-known theorem. (See [6]).

Theorem 2.20. *Suppose that in $(\mathcal{CO})$ the objective function f is convex and all the constraints h_i, for $i = 1, 2, \ldots, m$ are linear and g_j for $j = 1, 2, \ldots, r$ are convex. Then all local minimisers of $(\mathcal{CO})$ are also global minimisers of $(\mathcal{CO})$.*

2.2.3 Introduction to Semi-Infinite Optimisation

Above we introduced and presented some basic fundamental ideas about mathematical optimisation in general. In this sub-section we look at a particular type of optimisation method that will be used throughout this thesis. Namely, we will look at *semi-infinite optimisation.*

Here we consider an extension of ($\mathcal{CO}$). Observe that in ($\mathcal{CO}$) together with (2.18) we have a **finite** dimensional variable, $x \in \mathbb{R}^n$, where n is finite and we also have a **finite** number of constraints. That is, we have a total of $(m + r)$ constraints where both m and r are finite. An extension of this would be to consider the case where either the dimension of the variable is infinite or the number of inequality constraints is infinite but **not** both. It is this idea which we consider here.

Definition 2.21. *A* ***semi-infinite optimisation problem****, (SIO problem) is an optimisation problem of the form ($\mathcal{CO}$) where either the dimension of the variable or the number of inequality constraints is infinite but* ***not*** *both.*

Observation: The word 'semi-infinite' is of vital importance here. It means that both the dimension of the variables and the number of constraints can not simultaneously be infinite.

Thus, a semi-infinite optimisation problem is a generalisation of a standard, finite optimisation problem, where in the latter we have finitely many constraints and x is finite dimensional.

Now, the constraints in (2.18) are defined by $h_i(x)$ and $g_j(x)$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r$. In particular, the indices i and j, for a fixed value, each represent one particular constraint. In order to extend this notion, and to capture infinitely many inequality constraints we introduce an *index set*, $\mathcal{I}$. Here, $\mathcal{I} \subset \mathbb{R}^{\tilde{p}}$, where $\tilde{p}$ is a (finite) integer. If $\mathcal{I}$ is a finite set then we have a standard optimisation problem. If, however, $\mathcal{I}$ is an infinite set then we encounter infinitely many constraints. This index set $\mathcal{I}$ will play an important role throughout the rest of this thesis.

We are now in a position to introduce the basic, primal semi-infinite optimisation problem. For this we assume that the index set $\mathcal{I}$ is an infinite set, and we introduce the following notation. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $g_\tau : \mathbb{R}^n \to \mathbb{R}$, where $\tau \in \mathcal{I}$ and $\mathcal{I}$ is the index set. Then, the basic (primal) SIO problem (as given in [7]) is given by,

$$\begin{aligned} \max_{x} \quad & f(x) \\ \text{subject to} \quad & g_\tau(x) \leqslant 0, \forall\, \tau \in \mathcal{I}. \end{aligned} \tag{2.22}$$

Now, as was the case with standard optimisation, semi-infinite problems may be classed into one of the following two categories. Linear semi-infinite optimisation problems (LSIO) or non-linear semi-infinite optimisation problems (NLSIO). Where of course, analogously to standard optimisation problems a SIO problem is said to be *linear* if f and g_τ are linear functions in the variable x. If f or g_τ are not linear for at least one $\tau \in \mathcal{I}$ then we have a non-linear semi-infinite optimisation problem. Of course, linear semi-infinite problems are easier to solve than non-linear semi-infinite problems, since a linear SIO problem is the simplest case of this type of optimisation problem.

As we will see later, linear semi-infinite optimisation will play a vital role in what we consider in this thesis. Thus, for that reason we only consider linear SIO problems for the rest of this thesis. For NLSIO problems we refer the interested reader, for example to [8].

2.2.4 Linear Semi-Infinite Optimisation (LSIO)

Here we consider and present some preliminaries from the area of *linear semi-infinite optimisation.* Recall that a linear semi-infinite optimisation problem arises when we have a linear objective function f subject to linear constraints g_τ, for all $\tau \in \mathcal{I}$ in the variable x in the general semi-infinite optimisation problem (2.22). This allows us to re-write the general SIO problem (2.22) as a general linear semi-infinite optimisation problem, LSIO. For this we define the following. Let $a : \mathcal{I} \to \mathbb{R}^n$, and $b : \mathcal{I} \to \mathbb{R}$ be continuous functions on $\mathcal{I}$ (recall here that $\mathcal{I}$ is an index set; it may an interval which is a subset of $\mathbb{R}$ or any subset of $\mathbb{R}^{\tilde{p}}$ as described above. Note that $\mathcal{I}$ can not be a set consisting of a finite number of isolated points because this would then give a finite (and not a semi-infinite) optmisation problem, because if $\mathcal{I}$ did consist of a finite number of isolated points this would give a finite number of constraints. $\mathcal{I}$ can however consist of an infinite number of isolated points) and define the vector $c \in \mathbb{R}^n$. Then the linear semi-infinite optimisation problem is given by, (see [7])

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{subject to} \quad & a(\tau)^T x \leqslant b(\tau), \forall\, \tau \in \mathcal{I}. \end{aligned} \tag{2.23}$$

Throughout this thesis, and unless otherwise stated, we will refer to (2.23) as the linear semi-infinite *primal* problem. This will play an important role when we present some preliminaries from linear semi-infinite duality theory. We refer to [9] for examples of the linear SIO model.

2.2.5 LSIO Duality Theory

We are already aware from standard, finite optimisation how important the concept of *duality* is. Often it may be difficult to solve a given primal optimisation problem in its current form. One way to overcome this difficulty is to formulate the *dual* problem. We can then solve the dual problem if it is easier to solve then the original primal problem. Then we can use the dual solution to either directly get the solution to the primal problem or to find bounds on the optimal objective function value of the primal problem.
Duality theory extends to the semi-infinite setting which we are considering. For this thesis we present some preliminaries from linear semi-infinite optimisation duality theory. For duality results on other types of finite and semi-infinite optimisation problems we refer the reader to [6] and [8], respectively.
We define the dual problem to the primal linear semi-infinite optimisation problem (2.23).
We define the *dual* problem to (2.23) as follows. Suppose that the index set $\mathcal{I}$ is equal to a set consisting of infinitely many isolated points. We define a function $\pi : \mathcal{I} \to \mathbb{R}_+$ given by $\pi(\tau)$ which takes the value 0 everywhere except on some finite subset of $\mathcal{I}$. This finite subset of $\mathcal{I}$, where π takes positive values is called the *supporting set* of π, (see [7] for more on this), and is denoted by

$$\text{supp}(\pi) = \{\tau \in \mathcal{I} | \pi(\tau) \neq 0\}.$$

If we let $\mathbb{R}_+^{|\mathcal{I}|}$ denote the set of all such functions π, then we can define the dual problem to the semi-infinite optimisation problem (2.23) as given in [9] as:

$$\begin{aligned} \min_{\pi(\tau)} \quad & \sum_{\tau \in \mathcal{I}} b(\tau)\pi(\tau) \\ \text{subject to} \quad & \sum_{\tau \in \mathcal{I}} a(\tau)\pi(\tau) = c \\ & \pi \in \mathbb{R}_+^{|\mathcal{I}|}. \end{aligned} \tag{2.24}$$

Before proceeding we make some remarks about the problems (2.23) and (2.24). In (2.23) we are optimising an objective function subject to infinitely many constraints. In contrast, in (2.24) we are optimising an objective function depending on the infinite dimensional variable, $\pi(\tau)$ subject to a finite number of constraints. In particular our aim in (2.24) is to find a function $\pi(\tau)$. This highlights the connection between the number of constraints in the primal problem and the dimension of the variables in the dual problem, something which we are familiar with from linear optimisation duality theory. Throughout this thesis we will refer to problems taking the form of the problem (2.23) as the *primal linear semi-infinite optimisation problem* and problems taking the form (2.24) as the *dual of the linear semi-infinite optimisation problem.* We note here that the dual problem itself is **not** a semi-infinite optimisation problem because it does not have the properties of problem (2.23). That is, it is not a semi-infinite optimisation problem because we are not considering a linear optimisation problem on a finite dimensional variable with infinitely many constraints. Therefore, it makes sense to refer to (2.23) as a linear SIO problem and (2.24) as the dual to a linear SIO problem.
However, on closer analysis of (2.24) some potential problems could arise. One of these is concerned with

the index set $\mathcal{I}$. In (2.24) we have a sum over all $\tau \in \mathcal{I}$.
Now, this sum only makes sense if the index set $\mathcal{I}$ is described discretely. If we have a continuous description of the index set $\mathcal{I}$, such as $\mathcal{I} = [0, 1]$, for example and we now define $\pi : \mathcal{I} \to \mathbb{R}_+$, given by $\pi(\tau)$ to be a function that can now take non-zero values anywhere on $\mathcal{I}$ then it is impossible to 'sum' up over all $\tau \in \mathcal{I}$. Therefore, to complete the definition of duality we consider the following. In the case when the index set $\mathcal{I}$ is a continuous set, intuitively we could take integrals instead of 'sums' in the problem (2.24). We can now define the dual problem to (2.23) where the index set $\mathcal{I}$ is described continuously.
Referring the reader to [7] and references given there-in, we define the 'alternative' dual problem to the linear SIO problem (2.23) in terms of Lebesgue integrals as

$$\begin{aligned} \min_{\pi(\tau)} \quad & \int_{\mathcal{I}} b(\tau) \mathrm{d}\pi(\tau) \\ \text{subject to} \quad & \int_{\mathcal{I}} a(\tau) \mathrm{d}\pi(\tau) = c \\ & \pi(\tau) \in M^+(\mathcal{I}), \end{aligned} \tag{2.25}$$

where $M^+(\mathcal{I})$ is the set of non-negative Borel measures on $\mathcal{I}$, (see [7] for more on this).
From a practical point of view the appropriate formulation of the dual problem to the linear SIO problem (2.23) will depend on the index set $\mathcal{I}$. If $\mathcal{I}$ is equal to a set consisting of infinitely many isolated points then we use (2.24) as the dual to (2.23). If $\mathcal{I}$ is equal to an interval, for example then we use (2.25) as the dual to (2.23). Throughout this thesis we will see how important the integral form of the dual problem to the linear SIO problem that models our specific options pricing problem is.
Now that we have presented the dual to the linear semi-infinite primal problem we can now present some uses of this dual. The primary reason for considering the dual problem is to help us obtain the optimal objective function value to the primal problem. Frequently in optimisation problems it turns out that the original primal problem is difficult to solve but its corresponding dual problem may be easier to solve. Using appropriate duality results we may use the dual optimal objective function value to arrive at the primal optimal objective function value.
If $f^*((2.23))$ denotes the optimal value of the objective function in (2.23) and $f^*((2.25))$ denotes the optimal value of the objective function in (2.25), then we are interested in the following question, under what conditions can it be guaranteed that $f^*((2.23)) = f^*((2.25))$? (Note that we can replace (2.25) with (2.24) and all of the following analysis would still hold. We consider (2.25) here as the integral form of the dual problem is most relevant to this thesis). Indeed if we have the relation $f^*((2.23)) = f^*((2.25))$, then solving the dual problem (which may be easier than solving the primal problem directly) would give us the solution directly to the primal problem.
In what follows we present some important results from linear semi-infinite duality theory which will be used later in this thesis.
We may observe that the optimal objective function value of the dual problem provides an upper bound to the optimal objective function value of the primal problem. That is, the relation $f^*((2.23)) \leqslant f^*((2.25))$ holds [7]. This is known as *weak duality* and can be summarised in the following theorem.

Theorem 2.26 (Weak duality)**.**

$$f^*((2.23)) \leqslant f^*((2.25)).$$

Proof. See [7]. □

Theorem 2.26 gives us a link between the optimal objective function values of the linear SIO problem (2.23) and its corresponding dual problem (2.25). That is, it gives us an upper bound on the optimal value of the objective function for (2.23).
Now the primary question that still remains is, under what conditions does the relation $f^*((2.23)) = f^*((2.25))$ hold? We now investigate when this is the case.

To present the case when $f^*((2.23)) = f^*((2.25))$ we appeal to the following definition given in [10].

Definition 2.27. *We define the* ***first moment cone*** *by*

$$\mathcal{M}_{n+1} = \left\{ w = \int_{\mathcal{I}} \begin{pmatrix} b(\tau) \\ a(\tau) \end{pmatrix} \mathrm{d}\pi(\tau) | \pi(\tau) \in M^+(\mathcal{I}) \right\}.$$

The following result from [10] gives the conditions for the desired equality $f^*((2.23)) = f^*((2.25))$. This result is known as the *strong duality* theorem.

Theorem 2.28 (Strong duality)**.** *Suppose that*
(i) $f^*((2.23))$ *is finite, and*
(ii) $\mathcal{M}_{n+1}$ *is closed.*
Then $f^*((2.23)) = f^*((2.25))$ *and* (2.25) *has a solution.*

Proof. See [10]. □

Before proceeding we observe that in (*ii*) of Theorem 2.28 we have the condition that $\mathcal{M}_{n+1}$ is closed. In relation to the linear SIO problem this condition is exactly satisfied when the conditions of Theorem 2.30 presented below are satisfied.
We note here that the proof of Theorem 2.30 uses the following lemma taken from [23].

Lemma 2.29. *Let* $\hat{A} \subset \mathbb{R}^{\tilde{p}}$ *be a compact set. Then its convex hull,* $conv(\hat{A})$*, is also compact.*

Proof. See [23]. □

Then we have the following result taken from [23].

Theorem 2.30. *Suppose that the index set* $\mathcal{I}$ *is a compact subset of* $\mathbb{R}^n$*, and the real valued functions* $a_1, a_2, \ldots, a_n, b$ *which are defined on* $\mathcal{I}$ *are continuous. Further, assume that* (2.23) *meets the Slater condition, that is, there exists a vector* $\tilde{x} \in \mathbb{R}^n$ *such that*

$$\sum_{r=1}^{n} a_r(\tau)\tilde{x}_r > b(\tau), \quad \forall\, \tau \in \mathcal{I}.$$

Then, the first moment cone $\mathcal{M}_{n+1}$ *is closed.*

Proof. See [23]. □

Now we may observe the following. We have seen in Theorem 2.28 above, some conditions for $f^*((2.23)) = f^*((2.25))$ holding.
Also we have just stated Theorem 2.30 from [23] which states that if (2.23) meets the Slater condition and $\mathcal{I}$ is compact, with a, b being continuous on $\mathcal{I}$, then $\mathcal{M}_{n+1}$ is closed. Combining these results together gives the following theorem which can be used in the context of our problem later and can be found in [23].

Theorem 2.31. *Consider the primal-dual pair* (2.23) *and* (2.25)*. Assume the following.*

- $\mathcal{I}$ *is a compact subset of* $\mathbb{R}^n$ *and the real-valued functions* $a_1, a_2, \ldots, a_n, b$ *are continuous.*
- (2.23) *meets the Slater condition.*
- $f^*((2.23))$ *is finite.*

Then, (2.25) *is solvable and* $f^*((2.23)) = f^*((2.25))$.

Observation: Theorem 2.31 is just a mixture of Theorem 2.30 and Theorem 2.28. As such, we may view Theorem 2.31 as an alternative to the strong duality theorem.

That allows us to conclude all of the results needed from LSIO duality theory. As we will see in later sections, we have utilised the duality theory for LSIO described above for the basket option pricing problem.

2.2.6 Some solution methods to solve LSIO problems

Above we have introduced some preliminary ideas from linear semi-infinite optimisation and the important concept of duality.
The next area that we consider is concerned with some solution methods for solving the linear semi-infinite primal problem (2.23). In what follows here we present some selective solution techniques that exist to solve (2.23). For a more comprehensive and in-detail approach of the methods presented here, and for more methods that exist to solve (2.23) we refer the reader to [9].
The first solution method we consider is the *local reduction approach*. For a detailed discussion on this method we refer the reader to [9]. The basic idea of this solution method is as follows. Consider the linear semi-infinite primal optimisation problem (2.23). We assume a particular structure of the index set $\mathcal{I}$. The local reduction method aims to solve (2.23) by using Carathéodory's Theorem for cones to construct a system of non-linear equations which produce a necessary condition for the existence of an optimal solution x^* to (2.23). In this method, whilst finding optimal solutions to (2.23), we simultaneously find optimal solutions for the dual of the problem (2.23) too.
The next solution method which we consider is the most important one for the purposes of this thesis, and it is called the *generic discretisation method*. Put simply, this method is one of the simplest types of discretisation methods that exist to solve (2.23). It aims to solve (2.23) by iteratively replacing the infinite index set $\mathcal{I}$ by a finite subset $\mathcal{U}_k \subset \mathcal{I}$. This means that at each step of this solution algorithm we are solving a standard, finite linear problem because $\mathcal{U}_k$ is a *finite* index set. The idea is that in the limit as $k \to \infty$ the optimal solutions of the solved finite linear problems will iteratively converge to the optimal solution of (2.23). For more insights and examples on this type of solution method, we refer the reader to [11].
The above two described methods are part of the simplest solution methods that exist to solve (2.23). Some other solution methods that could be used to solve (2.23) can be found in [9]. Other solution methods include the following.

- A cutting plane discretisation method
- The three phase method
- Simplex like and exchange methods
- Descent methods

That concludes the preliminary ideas which we need from mathematical optimisation for this thesis.

2.3 Using Mathematical Optimisation in Mathematical Finance

In the previous sub-section we have discussed the main ideas from mathematical optimisation which will be used in this thesis. Now we briefly explain some uses of mathematical optimisation, and in particular, uses of optimisation within mathematical finance. We recall here that our aim is to show how mathematics can be used to model and solve real world financial problems.
The main interest of this thesis is seeing how optimisation can be used to solve problems from mathematical finance. Although we only consider how to use optimisation to solve one particular problem from mathematical finance, namely that of option pricing, there are many other problems from various areas of mathematical finance which can be solved using optimisation. Some of these can be found in [12], where various optimisation methods are used to solve problems including, asset/liability cash flow matching, volatility estimation,

portfolio optimisation and constructing an index fund. The use of optimisation to solve real world problems are not just limited to mathematical finance. Optimisation has wide range of uses to solve many other problems, some of which can be found in [13].
As a final remark we should remember that a model is just a model and when modelling real world problems mathematically many assumptions are made. We should take care when interpreting what the mathematical results mean in the context of the problem. We should also remember to use any mathematical model with care, especially if the assumptions it makes are violated in real life.

2.3.1 Using Semi-Infinite Optimisation to find bounds on the prices of financial options

As already mentioned previously, the aim of this thesis is to investigate how semi-infinite optimisation can be used to find bounds on the current price of a basket option.
Before we proceed to doing this, we first need to introduce some fundamental ideas which combine optimisation and finance and are vitally important as they will be used in the remainder of this thesis.
We introduce the idea behind how optimisation can be used to find bounds on the price of a financial option here. In particular, we will introduce some important definitions which will be implemented throughout this thesis when we see how SIO can be used for the bounding of prices of basket options.
We have seen in sub-section 2.1.4 that one possible way to find the price of a basket option is to use the multi-asset Black-Scholes model, where we assume that the price of the basket option satisfies the partial differential equation (2.13). However, as we stated previously, solving this PDE for a basket option in reality is difficult due to the correlations between each of the assets in the basket. Thus the need for more sophisticated and alternative mathematical techniques to price basket options is highly in demand.
Recall that from mathematical finance an *arbitrage opportunity* means that it is possible for an investor to make an instantaneous risk-free profit. Now we give a slightly different but equivalent definition which will be useful to us in this thesis, taken from [12].

Definition 2.32 (Arbitrage)**.** *An* ***arbitrage*** *is a trading strategy that*

- *type A- has a positive initial cash inflow and has no risk of a loss later, or*
- *type B- requires no initial cash input, has no risk of a loss, and has a positive probability of making profits in the future.*

When using optimisation to find bounds on the prices of financial options we are concerned with using an *arbitrage pricing* technique. Put simply this means that the only underlying assumption we make in our model is the assumption of no arbitrage and we use optimisation to produce upper and lower price bounds using only the knowledge of known prices of other, related financial products such as options written on the same underlying assets for example, [12]. Here we present the main ideas of how to use optimisation to find bounds on the price of general financial options and then consider how optimisation can be used to solve the specific basket option pricing problem in the subsequent sections.
In order to derive a general setting so that we can efficiently use optimisation to find bounds on the prices of options, we consider a one period model where we are currently at time $t = 0$ and we wish to model the asset prices at time $t = 1$. We start by presenting the following definitions and results, which can be found in [12].
We consider a set of possible future states which model the possible prices that the underlying assets can take at a future time. Let these states be given by $\hat{\sigma}_j$ for $j = 1, 2, \ldots, m$ and let the set of all states be given by $\hat{\Sigma}$. That is,

$$\hat{\Sigma} = \{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}.$$

To each state $\hat{\sigma}_j$ we assign a corresponding probability p_j of that state occurring, for $j = 1, 2, \ldots, m$.
Now, let S_i, for $i = 1, 2, \ldots, n$ denote the prices of n underlying assets. In particular, we will let S_i^0 denote the current price of the i^{th} underlying asset. We will let $S_i^1(\hat{\sigma}_j)$ denote the price of the i^{th} underlying asset at the future time $t = 1$, under state $\hat{\sigma}_j$ for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. We will also assume the

existence of a 'riskless' security (such as a government bond) which pays out interest $r \geqslant 0$ between time $t = 0$ and $t = 1$. We will denote this riskless security by S_0. Without loss of generality we may set the current price of the riskless security as, $S_0^0 = 1$ (see [12] for full details why) and define $R = 1 + r$, so that the price of the riskless security at the future time, $t = 1$ is given by

$$S_0^1(\hat{\sigma}_j) = R = 1 + r, \text{ for all } j = 1, 2, \ldots, m.$$

This allows us to define the following, (see [12]).

Definition 2.33. *A* ***risk-neutral probability measure*** *on the set*

$$\hat{\Sigma} = \{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}$$

is a vector of positive numbers $(p_1, p_2, \ldots, p_m)^T \in \mathbb{R}_+^m$ *such that*

$$\sum_{j=1}^{m} p_j = 1,$$

and for every security/asset S_i *for* $i = 0, 1, \ldots, n$*, we have*

$$S_i^0 = \frac{1}{R}\left(\sum_{j=1}^{m} p_j S_i^1(\hat{\sigma}_j)\right) = \frac{1}{R}\mathbb{E}[S_i^1],$$

where $\mathbb{E}[S_i^1]$ *denotes the expectation of the random variable* S^1 *under the probability distribution* $(p_1, p_2, \ldots, p_m)^T$*.*

We now present some standard results from linear optimisation taken from [12, 13] which are used in the proof of Theorem 2.39 which we will use later in this thesis.

Theorem 2.34 (Strong duality)**.** *If a primal linear problem has an optimal solution, then so does its dual problem and the respective objective function values are equal.*

Theorem 2.35. *If the dual linear problem has a non-empty feasible set then the corresponding primal problem has a finite objective function value.*

Theorem 2.36 (Goldman and Tucker)**.** *When both the primal and dual linear optimisation problems*

$$\begin{aligned} \min_x \quad & c^T x \\ s.t. \quad & Ax = b \quad x \geqslant 0, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \max_y \quad & b^T y \\ s.\ t. \quad & A^T y \leqslant c, \end{aligned} \tag{2.38}$$

have feasible solutions, they have optimal solutions satisfying the strict complementarity condition. That is, there exist x^* *and* y^* *optimal for the respective problems such that*

$$x^* + (c - A^T y^*) > 0.$$

Referring to [12] for full details, these results give the following theorem which is fundamental when using optimisation models to find bounds on the prices of financial options. It is known as *the first fundamental theorem of asset pricing.*

Theorem 2.39. *A risk-neutral probability measure exists if and only if there are no arbitrage opportunities.*

Proof. See [12]. □

Remark: We will see in the main sections of this thesis, that the implementation of the above theorem is vital when formulating the optimisation model that is used to find price bounds on the option of interest.

In order for the reader to familiarise themselves on how the above results can be implemented to a practical financial problem, we refer them to study the example presented in [12] pp.187-191 where optimisation is used to find bounds on the price of a *forward-start option*.
After studying the example from [12], we note that the methodology employed in the example is very popular. To model the option pricing problem as a dual of a SIO problem and then to consider the SIO problem and then to equivalently re-write this SIO problem as a finite optimisation problem which can then be solved to yield an optimal solution and hence the bounds on the price of the option under consideration, is the *standard* way to use optimisation to find price bounds on options. We shall see this in the remainder of this thesis when we consider how semi-infinite optimisation can be used to find bounds on the price of a basket option.
That concludes all preliminary material which is needed to understand the rest of this thesis.

3 Literature Review

In this section we present some existing results from various references about finding price bounds on European basket options using semi-infinite optimisation.
In all of the results we present in this section, we are interested in using semi-infinite optimisation to find upper and lower bounds on the price of a European basket call option. As we will see in later sections, these existing results provide motivation and a base for which we may derive our own results given in sections 4, 5, 7 and 8. We start by giving an introduction to the problem under consideration.

3.1 Introduction

We begin by recalling the basic definition and properties of a basket option as given in sub-section 2.1.3. We recall that, a basket option is an exotic option whose payoff depends on multiple assets. Each asset is assigned a weight and then the overall sum of the assets with their corresponding weights is taken into account. The value of this weighted sum is then compared with the exercise price to determine whether or not the basket option should be exercised.
Also, recall that associated with any financial option is its current price. This current price reflects the risk being taken by both the holder and the writer of the option. One of the most important and interesting topics of mathematical finance is the pricing of options. We are particularly interested in seeing how to find price bounds on basket options. Recalling from sub-section 2.1.4 that although there are many different methods that can be used to price a basket option, these methods are not without their problems. The multi-asset dependence of the payoff of the basket option makes existing pricing techniques very difficult to execute in practice.
Thus, the need to find a sophisticated and valid mathematical technique to price basket options is very much in demand, not only by mathematicians but by investors and financial firms as well. The way in which we tackle the basket option pricing problem is to use mathematical optimisation. The use of optimisation in pricing financial options has already been highlighted in sub-section 2.3.1 and further uses of optimisation within the financial world can be found in [12].
Hence, the primary question of interest and under consideration in this thesis is, how can we use optimisation to find price bounds on basket options? In this thesis we will concentrate on looking at how semi-infinite optimisation can be used to find upper and lower bounds on the prices of basket options. We present the basic model set-up and specification of the problem later in this section. Then we present some work from a range of sources which have considered this problem in detail, in the succeeding sub-sections.
Before proceeding we remark here that basket options consist of many different types. They may be European or American and they may be call options or put options. When we consider finding price bounds on a basket option we will clearly specify the type of basket option under consideration.
We first present some fundamental ideas and important concepts from measure theory which can be found in [14, 15, 16, 17] and will play an important role in the work that follows. We then move on to presenting existing results on how semi-infinite optimisation can be used to find bounds on basket option prices.

3.2 Concepts from Measure Theory

The concept of *measure* stems from Euclidean geometry where we consider how to measure a solid body in various dimensions. In one dimension we think of this *measure* as length. In two dimensions we think of it as area and in three dimensions we think of it as volume. However, outside of this geometrical setting the concept of *measure* becomes unclear. Measure theory aims to generalise the concepts of length, area and volume into arbitrary dimensions. To start with we need the following definition of a *set function*.

Definition 3.1. *A **set function** is a function whose domain is the power set of the set of real numbers, denoted by $2^{\mathbb{R}}$ and whose co-domain is the set of real numbers $\mathbb{R}$. That is, the mapping $\mathcal{S}_f : 2^{\mathbb{R}} \to \mathbb{R}$ is called a set function.*

We will give a definition of a *measure* below, (see Definition 3.3). Before presenting that however, we may think of a measure, informally as follows. A measure on a set is a set function which assigns non-negative values or ∞ to each possible subset of the original set. The non-negative (or ∞) value which is assigned to the subset is done with respect to the size of the subset in relation to the original set. It is this concept of a measure that allows us to extend the notion of length, area and volume into arbitrary dimensions.
Before formally defining what we mean by a measure, we need the following definition of a $\sigma-$*algebra*.

Definition 3.2. *Let $\mathcal{X}$ be a set, and let $2^{\mathcal{X}}$ denote its* ***power set****. Then the set $\Sigma \subset 2^{\mathcal{X}}$ is said to be a $\sigma-$**algebra** on the set $\mathcal{X}$ if the following three conditions hold.*

1. $\varnothing \in \Sigma$
2. *If $G \in \Sigma$ then its complement $G' \in \Sigma$.*
3. *If $G_1, G_2, G_3, \ldots$ are a collection of sets in Σ, then it holds for their union that $\bigcup_{i=1}^{\infty} G_i \in \Sigma$.*

Then we may present the definition of a measure as follows.

Definition 3.3. *The set function $\hat{\mu} : \Sigma \subset 2^{\mathcal{X}} \to \mathbb{R}_+ \cup \{\infty\}$ is a* ***measure*** *on the set $\mathcal{X}$ if it satisfies the following properties.*

1. $\hat{\mu}(\hat{x}) \geqslant 0,$ *for all* $\hat{x} \in \Sigma$
2. $\hat{\mu}(\varnothing) = 0$
3. *For disjoint sets $\{\hat{x}_i\}_{i\in\mathbb{N}} \in \Sigma$, we have*

$$\hat{\mu}\left(\bigcup_{i\in\mathbb{N}} \hat{x}_i\right) = \sum_{i\in\mathbb{N}} \hat{\mu}(\hat{x}_i).$$

Moreover, if the above conditions hold, $(\mathcal{X}, \Sigma, \hat{\mu})$ is said to be a ***measurable space****.*

We will refer to $(\mathcal{X}, \Sigma, \hat{\mu})$ as a *triple*, where $\mathcal{X}$ is a set, Σ is a $\sigma-$algebra on $\mathcal{X}$ and $\hat{\mu}$ is a measure on the set $\mathcal{X}$.
For the purposes of this thesis, we are interested in a particular area of measure theory. Namely we are interested in *probabilistic measure theory.*

Definition 3.4. *Consider the triple $(\Gamma, \mathcal{F}, \pi)$ as described above. Then the set function $\pi : \mathcal{F} \to [0, 1]$ is a* ***probability measure*** *on the set Γ if it satisfies the following properties.*

1. $\pi(\bar{\gamma}) \in [0, 1],$ *for all* $\bar{\gamma} \in \mathcal{F}$
2. $\pi(\varnothing) = 0$
3. $\pi(\Gamma) = 1$
4. *For disjoint sets $\{\bar{\gamma}_i\}_{i\in\mathbb{N}} \in \mathcal{F}$, we have*

$$\pi\left(\bigcup_{i\in\mathbb{N}} \bar{\gamma}_i\right) = \sum_{i\in\mathbb{N}} \pi(\bar{\gamma}_i).$$

Moreover, if the above conditions hold, $(\Gamma, \mathcal{F}, \pi)$ is said to be a ***probability space****.*

For the remainder of this thesis, in order to distinguish between general and probabilistic measures we will let $\hat{\mu}$ denote a general measure and let π denote a probability measure.

Note: The condition that $\pi(\Gamma) = 1$ and $\pi(\varnothing) = 0$ implies that $\pi(\bar{\gamma}) \in [0, 1]$, for all $\bar{\gamma} \in \mathcal{F}$.

Before proceeding to see how we may apply the above definitions to the basket option pricing problem we need the following definitions taken from [16].

Definition 3.5. *Let the set $\tilde{A}$ be contained in some space X. Then the* ***indicator function*** *of $\tilde{A}$, written $\chi_{\tilde{A}}$, is the function defined by: $\chi_{\tilde{A}}(x) = 1$ for $x \in \tilde{A}$, and $\chi_{\tilde{A}}(x) = 0$ for $x \in \tilde{A}'$.*

Definition 3.6. *A non-negative function $\varphi(x)$ taking only a finite number of different values is called a* ***simple function****. If $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n$ are the distinct values taken by φ, and $A_i = \{x | \varphi(x) = \hat{a}_i\}$, then clearly*

$$\varphi(x) = \sum_{i=1}^{n} \hat{a}_i \chi_{A_i}(x).$$

Definition 3.7. *Let f be a simple function defined on the measurable space $(\mathcal{X}, \Sigma, \hat{\mu})$, taking a finite number of non-negative distinct values. If $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n$ are the distinct values of f, then we may express f as, $f = \sum_{i=1}^{n} \hat{\alpha}_i \chi_{\tilde{A}_i}$, where $\tilde{A}_i = \{x | f(x) = \hat{\alpha}_i\}$. Then the* ***Lebesgue integral*** *of f with respect to the measure $\hat{\mu}$ is given by*

$$\int_{\mathcal{X}} f \, \mathrm{d}\hat{\mu} = \sum_{i=1}^{n} \hat{\alpha}_i \hat{\mu}(\tilde{A}_i).$$

Also, the convention $0 \times \infty = 0$ is to be understood in this definition [16].

We will also use the following definition of expectation for a random variable with respect to a probability measure as presented in [17].

Definition 3.8. *Let Z be a simple function defined on the probability space $(\Gamma, \mathcal{F}, \pi)$. Then the* ***expectation*** *of Z, with respect to the probability measure π is defined as*

$$\mathbb{E}_{\pi}[Z] = \int_{\Gamma} Z \, \mathrm{d}\pi,$$

[17].

We are now in a position to utilise what we have just introduced from probabilistic measure theory for the basket option pricing problem. In particular we note here that the probability measure π will play an important role in the set-up and formulation of the optimisation model for the problem we are considering.

3.3 Modelling the basket option pricing problem as an optimisation problem

In what follows, for the remainder of this section, we present results from literature which aim to find price bounds on a European basket call option.

We start by formally setting some notation. Suppose that we are interested in finding price bounds on a European basket call option written on n underlying assets, with exercise price E, where $E \geqslant 0$. Suppose further that for each asset, we know its respective current price. We denote the current price of the i^{th} asset by S_i^0 for $i = 1, 2, \ldots, n$. Then we may define the vector $S^0 \in \mathbb{R}_+^n$ by $S^0 = \left(S_1^0, S_2^0, \ldots, S_n^0\right)^T$. Further, suppose that the basket option has an expiry date given by T. Then we denote the random asset prices at expiry T, by S_i for $i = 1, 2, \ldots, n$. Then, the vector $S \in \mathbb{R}^n_+$ given by $S = (S_1, S_2, \ldots, S_n)^T$ is an n-dimensional non-negative real valued random variable. $\mathcal{F}$ is a $\sigma-$algebra on the set $\mathbb{R}^n_+$, and π is a probability measure which assigns a probability to a vector S. We have the property that $\pi(\mathbb{R}^n_+) = 1$, indicating that the probability of the vector S having non-negative values, that is, the probability that at expiry all assets have a non-negative price is 1. This property is captured in the constraints of our optimisation problem as we will see below, via an appropriate Lebesgue integral. Also, we assume that for each asset i, for $i = 1, 2, \ldots, n$, there exists one European vanilla call option with expiry date T and exercise price $E_i \in \mathbb{R}_+$, for $i = 1, 2, \ldots, n$. Then the payoff of each of these i European vanilla call options is given by $\max(S_i - E_i, 0)$ for each $i = 1, 2, \ldots, n$. For our work we assume that we know the current prices of each of these i European vanilla call options and we denote this known price by $C_i^0 \in \mathbb{R}_+$ for $i = 1, 2, \ldots, n$. Again, by convention we set $C^0 \in \mathbb{R}^n_+$, where $C^0 = (C_1^0, C_2^0, \ldots, C_n^0)^T$.

Now recall the binomial method for pricing options from sub-section 2.1.4. There, we saw that the pricing of an option depended on the parameters $\tilde{u}$, $\tilde{d}$ and p. In particular, we explained how the probability p was to be used within the method to obtain the price of an option. One of the main assumptions of this method was the idea of risk-neutrality. Under this assumption we introduced the formula (2.11), which in principle states that the current price of an option is equal to the value of the discounted expected payoff of that option under a risk neutral probability measure. This was then used to accurately price the option. Using a similar idea we will see how we can derive an optimisation model aimed at finding bounds on the price of a European basket call option. The assumption of risk-neutrality meant that the probability p from the binomial method was risk-neutral. We now extend this idea to the basket option setting.

To start with, consider an option which gives a future random payoff of Y. If this option has an associated risk-neutral probability measure π for the returns of its underlying assets, then the current price of this option would be given by

$$\mathbb{E}_\pi[\mathrm{e}^{-rT} Y]. \tag{3.9}$$

That is, the current price of the option would be the discounted expected payoff of the option under the risk-neutral probability measure π. This is confirmed by formula (2.11), which essentially states the same thing under the risk-neutrality assumption, with probability p and in a more general setting.

Thus, if we assume risk-neutrality, then the probability measure π associated with the basket option written on n underlying assets, with future values $S \in \mathbb{R}^n_+$, is a risk-neutral measure. Also, it is known that (see [11]) if we assume the market to be arbitrage free, then at least one such risk-neutral measure π exists. This was also presented in Theorem 2.39 in sub-section 2.3.1.

Thus, under the risk-neutrality assumption we can find arbitrage-free upper and lower bounds on the current price of a basket option. Indeed, for an option with payoff Y, if we take the infimum/supremum over all probability measures π of $\mathbb{E}_\pi[\mathrm{e}^{-rT} Y]$, respectively we would get no-arbitrage lower/upper bounds on the price of the option, respectively. In terms of the basket option pricing problem, we consider the random future payoff, $\max(\omega^T S - E, 0)$, so that we are interested in optimising the following expected value with respect to the risk-neutral probability measure π,

$$\mathbb{E}_\pi[\mathrm{e}^{-rT} \max(\omega^T S - E, 0)].$$

We also need some restrictions on the probability measure π to be found. For the basic set-up of the problem we include the conditions that we know the current prices of one European vanilla call option per asset, with exercise price E_i, for $i = 1, 2, \ldots, n$ and these known prices are given by C_i^0 for $i = 1, 2, \ldots, n$. Under the assumption of π being a risk-neutral probability measure, we have the condition that $\mathbb{E}_\pi[\mathrm{e}^{-rT} \max(S_i - E_i, 0)] = C_i^0$ for $i = 1, 2, \ldots, n$. Also, we may obviously assume that we know the current prices of each of the n underlying assets. This gives the restriction that $\mathbb{E}_\pi[\mathrm{e}^{-rT} S_i] = S_i^0$ for $i = 1, 2, \ldots, n$. Alongside all of these we also require the condition that $\mathbb{E}_\pi[1] = 1$, so that this restriction guarantees that the measure π to be found is a probability measure.

This then allows us to model the basic problem of finding arbitrage-free upper bounds on the price of a

European basket call option as

$$
\begin{aligned}
&\sup_{\pi} && \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(\omega^T S - E, 0)] \\
&\text{subject to} && \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(S_i - E_i, 0)] = C_i^0, \quad \text{for } i = 1, 2, \ldots, n \\
& && \mathbb{E}_{\pi}[\mathrm{e}^{-rT} S_i] = S_i^0, \quad \text{for } i = 1, 2, \ldots, n \\
& && \mathbb{E}_{\pi}[1] = 1.
\end{aligned} \tag{3.10}
$$

Similarly, we may model the task of finding arbitrage-free lower bounds on the price of a European basket call option as

$$
\begin{aligned}
&\inf_{\pi} && \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(\omega^T S - E, 0)] \\
&\text{subject to} && \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(S_i - E_i, 0)] = C_i^0, \quad \text{for } i = 1, 2, \ldots, n \\
& && \mathbb{E}_{\pi}[\mathrm{e}^{-rT} S_i] = S_i^0, \quad \text{for } i = 1, 2, \ldots, n \\
& && \mathbb{E}_{\pi}[1] = 1.
\end{aligned} \tag{3.11}
$$

Now, (3.10) and (3.11) are one of the basic optimisation problems that can be used to model the task of finding upper/lower bounds on the price of a European basket call option. In the next sub-sections we will consider results which have been derived from (3.10) and (3.11) as well as modifications of these problems. In particular, we will see how we can obtain bounds on the price of a basket option when we modify the constraints of the above by assuming that we know the prices of numerous European vanilla call options per asset with different exercise prices. We will see how to modify the constraints in the above problems so that instead of assuming that we know the prices of European vanilla call options, we assume that we know the prices of various other basket call options written on the same n underlying assets, and obtain upper/lower bounds in this way. We will also consider how to incorporate *bid-ask prices* into the model and we will also consider the case when the interest rate r is zero, that is when $r = 0$. In that case we observe that

$$\mathrm{e}^{-rT} = \mathrm{e}^0 = 1.$$

Before proceeding we observe that (3.10) and (3.11) may be written in integral form by utilising the definitions introduced above.
Observing that the variable S is associated with the probability space $\left(\mathbb{R}_+^n, \mathcal{F}, \pi\right)$, using Definition 3.8 the objective function of (3.10) and (3.11) to be optimised may be written as,

$$\mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(\omega^T S - E, 0)] = \int_{\mathbb{R}_+^n} \mathrm{e}^{-rT}\max(\omega^T S - E, 0)\ \mathrm{d}\pi.$$

Similarly we may re-write the expectation in the constraints as,

$$\mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(S_i - E_i, 0)] = \int_{\mathbb{R}_+^n} \mathrm{e}^{-rT}\max(S_i - E_i, 0)\ \mathrm{d}\pi$$

and

$$\mathbb{E}_{\pi}[\mathrm{e}^{-rT} S_i] = \int_{\mathbb{R}_+^n} \mathrm{e}^{-rT} S_i\ \mathrm{d}\pi,$$

respectively, for $i = 1, 2, \ldots, n$.
For the final constraint $\mathbb{E}_{\pi}[1] = 1$, we argued above that this constraint ensured that the measure π to be found is a probability measure. To see why this is so we observe the following.
From Definition 3.8 we may write,

$$\mathbb{E}_{\pi}[1] = \int_{\mathbb{R}_+^n} 1\ \mathrm{d}\pi.$$

Now, we may define the constant function $\bar{c} : \mathbb{R}^n_+ \to \mathbb{R}$ given by $\bar{c}(S) = 1$. Then we may observe that $\bar{c}(S)$ is a simple function because we may write,

$$\bar{c}(S) = 1 \times \chi_{A_1}(S),$$

where χ_{A_1} is the indicator function defined by

$$\chi_{A_1}(S) = \begin{cases} 1 & \text{if } S \in A_1 \\ 0 & \text{otherwise,} \end{cases}$$

and A_1 is the set $A_1 = \{S | \bar{c}(S) = 1\}$.
Now, we observe that $A_1 \equiv \mathbb{R}^n_+$ because $\bar{c}(S) = 1$, for all $S \in \mathbb{R}^n_+$ and so $\chi_{A_1} = 1$, since $S \in \mathbb{R}^n_+ \equiv A_1$.
Therefore, using Definition 3.7 we have,

$$\int_{\mathbb{R}^n_+} 1 \; d\pi = 1 \times \pi(A_1) = 1 \times \pi(\mathbb{R}^n_+) = \pi(\mathbb{R}^n_+).$$

Therefore, we have

$$\mathbb{E}_\pi[1] = \pi(\mathbb{R}^n_+) \; (= 1),$$

and so the final constraint does indeed ensure that the measure π to be found is a probability measure.
Finally, we note here that (3.10) and (3.11) and all possible modifications of these problems which we consider in this thesis are duals of semi-infinite optimisation problems. They are duals because as we will see later, the dual problem to (3.10) and (3.11) is a linear semi-infinite optimisation problem with a finite dimensional variable and infinitely many constraints. That is, the dual problem to (3.10) and (3.11) is a problem of the form (2.23) and so is a linear semi-infinite optimisation problem.

3.3.1 Solving the basic problems (3.10) and (3.11)

Now, (3.10) and (3.11) are the most basic optimisation problems aimed at finding upper and lower bounds on the current price of a European basket call option, respectively. Existing literature has observed that (3.10) and (3.11) are duals to semi-infinite optimisation problems and hence they have utilised results from duality theory which we introduced in section 2.2.5 to solve these problems. In particular, in [18] the upper bound problem (3.10) is solved for the general n asset case and the lower bound problem (3.11) is solved for the specific $n = 2$ asset case. In [19], problem (3.11) is solved for the general n asset case by equivalently re-writing the SIO problem for which (3.11) is its dual problem, as a finite LO problem. Finally, in [20] problems (3.10) and (3.11) are solved for the general n asset case and with the risk-free interest rate $r = 0$. In this case the derived upper and lower bounds are the same and so the derived result is actually the price of the European basket call option.
It is worth noting here that this is **not** the only model where the obtained bounds are equal to the exact current price of the basket option. In the following cases the obtained bounds are the exact current basket option price.

1. When computing an upper bound on the price of the basket option given that we know one forward (expected) price and one vanilla call option price constraint per asset.
2. When computing an upper bound on the price of the basket option, given two different exercise prices of a vanilla call option per asset and thus two prices per option, per asset as the constraints. There are *no* forward (expected) prices in this model.
3. When computing a lower bound on the price of the basket option, given one forward (expected) price and one vanilla call option price constraint, per asset.
4. When computing a lower bound on the price of the basket option, given one vanilla call price constraint per asset, but *no* forward (expected) prices.

The constraints of problems (3.10) / (3.11) can be modified to give new optimisation models which aim to solve the same problem. By considering these modifications we can introduce different solution techniques and results and compare these to see what the tightest upper/lower bound is.

3.4 Modification 1: Assuming that multiple vanilla call option prices are known, per asset

We first consider perhaps the most natural extension of the constraints from (3.10) / (3.11). Recall that when we first formulated the basket option pricing problem as an optimisation problem, we assumed that we knew the current prices of **one** European vanilla call option per asset, and so we knew a total of n prices. Now we consider an extension of this model. We now assume that we know the current prices of **multiple** European vanilla call options per asset. Then, using this information and the assumption of no-arbitrage we look to find bounds on the current price of a European basket call option of interest. In what follows, we present the work done in [11].
We consider working with the basic upper bound problem (3.10). Recall this optimisation problem as

$$\begin{array}{ll} \sup\limits_{\pi} & \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(\omega^T S - E, 0)] \\ \text{subject to} & \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(S_i - E_i, 0)] = C_i^0, \quad \text{for } i = 1, 2, \dots, n \\ & \mathbb{E}_{\pi}[\mathrm{e}^{-rT} S_i] = S_i^0, \quad \text{for } i = 1, 2, \dots, n \\ & \mathbb{E}_{\pi}[1] = 1. \end{array} \tag{3.10}$$

We present a simple extension of (3.10), as done in [11]. Instead of assuming that we know the current prices of **one** European vanilla call option per asset, given by C_i^0, for $i = 1, 2, \dots, n$; we now assume that we know the current prices of q European vanilla call options per asset, given by C_i^l, with exercise prices E_i^l, for $i = 1, 2, \dots, n =$ and $l = 1, 2, \dots, q$. This means that we know a total of $(n \times q)$ current European vanilla call option prices.
This gives the optimisation problem (3.12) below, which is a dual to a linear SIO problem, taken from [11], and which may be viewed as an extension to (3.10).

$$\begin{array}{ll} \sup\limits_{\pi} & \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(\omega^T S - E, 0)] \\ \text{subject to} & \mathbb{E}_{\pi}[\mathrm{e}^{-rT} S_i] = S_i^0, \quad \text{for } i = 1, 2, \dots, n \\ & \mathbb{E}_{\pi}[\mathrm{e}^{-rT}\max(S_i - E_i^l, 0)] = C_i^l, \quad \text{for } i = 1, 2, \dots, n \text{ and } l = 1, 2, \dots, q \\ & \mathbb{E}_{\pi}[1] = 1. \end{array} \tag{3.12}$$

We now observe that the linear SIO problem for which (3.12) is its dual is given by

$$\begin{array}{ll} \inf\limits_{u^l, z, v} & \sum\limits_{l=1}^{q} (u^l)^T C^l + z + v^T S^0 \\ \text{subject to} & \sum\limits_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \ \ \forall\, S \in \mathbb{R}_+^n. \end{array} \tag{3.13}$$

We note here that the index set $\mathcal{I} = \mathbb{R}_+^n$ is not compact. However, as the next proposition taken from [11] shows, we may restrict $\mathcal{I}$ in (3.13), to a compact set without changing the feasible set of the problem.
For the purposes of this thesis, we find it necessary to present the proof of the next proposition. This is because the proof of our new and original results, presented in later sections use similar ideas to those presented in the following proof.

Proposition 3.14. *Suppose that the exercise prices E_i^l are ordered such that $0 \leqslant E_i^1 \leqslant E_i^2 \leqslant \cdots \leqslant E_i^q$, for all $i = 1, 2, \ldots, n$. Define the index set $\mathcal{I}_{(3.15)} = \bigtimes_{i=1}^{n} [0, E_i^q]$. Then the following optimisation problem* (3.15)*, is equivalent to* (3.13) *in the sense that both problems have the same feasible set, and, hence the same optimal solution and optimal objective function value.*

$$\begin{aligned} \underset{u^l, z, v \in \mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n}{\text{min}} \quad & \sum_{l=1}^{q} (u^l)^T C^l + z + v^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q} u^l + v \geqslant \max\{\omega, 0\} \\ & \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \;\; \forall\, S \in \mathcal{I}_{(3.15)}. \end{aligned} \tag{3.15}$$

Proof. We start by observing that the objective functions of (3.13) and (3.15) are the same. Thus in order to show these two problems are equivalent we must show that their respective feasible regions are the same. Let $\mathcal{F}_{(3.13)}$ and $\mathcal{F}_{(3.15)}$ denote the feasible regions of (3.13) and (3.15), respectively. We then show that $\mathcal{F}_{(3.13)} = \mathcal{F}_{(3.15)}$.
The proof comes in two parts.
(i) $\underline{\mathcal{F}_{(3.13)} \subset \mathcal{F}_{(3.15)}}$: Take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(3.13)}$. We then show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(3.15)}$. Now, since $\mathcal{I}_{(3.15)} = \bigtimes_{i=1}^{n} [0, E_i^q]$, then $\mathcal{I}_{(3.15)}$ forms an n-dimensional 'rectangle'. That is, it forms a 'rectangle' in n-dimensional non-negative space and so $\mathcal{I}_{(3.15)} \subset \mathbb{R}_+^n$. From the constraint in (3.13) we have

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \;\; \forall\, S \in \mathbb{R}_+^n.$$

It then follows that since $\mathcal{I}_{(3.15)} \subset \mathbb{R}_+^n$, the constraint in (3.13) obviously still holds for all $S \in \mathcal{I}_{(3.15)}$. That is,

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \;\; \forall\, S \in \mathcal{I}_{(3.15)},$$

and so the second constraint from (3.15) holds.
To show that the first constraint holds we have the following. Recall that $(u^1, u^2, \ldots, u^q, z, v)$ satisfies the constraint

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0),$$

from (3.13), for all $S \in \mathbb{R}_+^n$. This means that for the asset price vector S, with $S_i = \eta$, for some $\eta > 0$ and all other components equal to 0, for $i = 1, 2, \ldots, n$, the constraint still holds. That is, the constraint holds for the vector $S = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \eta \\ 0 \end{pmatrix}$, where η is in the i^{th} position, for $i = 1, 2, \ldots, n$.

We have

$$\begin{aligned} & \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \;\; \forall\, S \in \mathbb{R}_+^n, \\ \Longleftrightarrow & \sum_{l=1}^{q} \sum_{k=1}^{n} (u_k^l) \max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k \geqslant \max\left(\sum_{k=1}^{n} \omega_k S_k - E, 0\right), \;\; \forall\, S \in \mathbb{R}_+^n. \end{aligned}$$

Now, for each $i = 1, 2, \ldots, n$, $S_i = \eta$ and all other components are equal to 0.
For a particular i, $\implies$

$$\sum_{l=1}^{q}(u_i^l)\max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + v_i S_i \geqslant \max(\omega_i S_i - E, 0)$$

$$\iff \sum_{l=1}^{q}(u_i^l)\max(\eta - E_i^l, 0) + z\mathrm{e}^{rT} + v_i\eta \geqslant \max(\omega_i\eta - E, 0)$$

$$\iff \sum_{l=1}^{q}(u_i^l)\eta\max\left(1 - \frac{E_i^l}{\eta}, 0\right) + z\mathrm{e}^{rT} + v_i\eta \geqslant \eta\max\left(\omega_i - \frac{E}{\eta}, 0\right),$$

and so, if we divide both sides by η we get (since $\eta > 0$),

$$\implies \sum_{l=1}^{q}(u_i^l)\max\left(1 - \frac{E_i^l}{\eta}, 0\right) + \frac{z\mathrm{e}^{rT}}{\eta} + v_i \geqslant \max\left(\omega_i - \frac{E}{\eta}, 0\right),$$

and if $\eta \to \infty$, then $\frac{E_i^l}{\eta} \to 0$, $\frac{z\mathrm{e}^{rT}}{\eta} \to 0$ and $\frac{E}{\eta} \to 0$. This gives, in the limit as $\eta \to \infty$,

$$\sum_{l=1}^{q}(u_i^l)\max(1, 0) + 0 + v_i \geqslant \max(\omega_i, 0)$$

$$\iff \sum_{l=1}^{q}(u_i^l) + v_i \geqslant \max(\omega_i, 0),$$

which holds for all $i = 1, 2, \ldots, n$.
This in vector form is just

$$\sum_{l=1}^{q} u^l + v \geqslant \max\{\omega, 0\}, \tag{3.16}$$

and so the first constraint from (3.15) holds.
(ii) $\underline{\mathcal{F}_{(3.15)} \subset \mathcal{F}_{(3.13)}}$: Now we prove the converse. So, take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(3.15)}$. Then, in order to show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(3.13)}$, we must show that

$$\sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \ \ \forall\, S \in \mathbb{R}_+^n.$$

Thus, it suffices to show that

$$\max_{S\in\mathbb{R}_+^n}\left\{\max(\omega^T S - E, 0) - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S\right\} =$$
$$\max_{S\in\mathcal{I}_{(3.15)}}\left\{\max(\omega^T S - E, 0) - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S\right\},$$

since $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(3.15)}$ it holds that

$$\sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \ \ \forall\, S \in \mathcal{I}_{(3.15)}.$$

Thus,

$$\max(\omega^T S - E, 0) - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \leqslant 0 \ \ \forall\, S \in \mathcal{I}_{(3.15)},$$

and so,

$$\max_{S\in\mathcal{I}_{(3.15)}} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\} \leqslant 0,$$

and so if

$$\max_{S\in\mathbb{R}_+^n} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\} =$$
$$\max_{S\in\mathcal{I}_{(3.15)}} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\},$$

it means that

$$\max_{S\in\mathbb{R}_+^n} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\} \leqslant 0$$

and so

$$\max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \leqslant 0, \;\; \forall\, S \in \mathbb{R}_+^n,$$
$$\iff \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \;\; \forall\, S \in \mathbb{R}_+^n,$$

in which case the proposition is proved.
We now show that

$$\max_{S\in\mathbb{R}_+^n} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\} =$$
$$\max_{S\in\mathcal{I}_{(3.15)}} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\}.$$

Define the function $\psi : \mathbb{R}_+^n \to \mathbb{R}$, given by

$$\psi(S) = \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S.$$

Consider $\nabla\psi(S)$, for all $S \notin \mathcal{I}_{(3.15)}$. Then we have the following.
(a) If $\omega^T S - E < 0$,

$$\nabla\psi(S) = \frac{\mathrm{d}\psi}{\mathrm{d}S} = -\sum_{l=1}^{q} u^l - v.$$

(b) If $\omega^T S - E > 0$,

$$\nabla\psi(S) = \frac{\mathrm{d}\psi}{\mathrm{d}S} = \omega - \sum_{l=1}^{q} u^l - v.$$

Now, from the first constraint in (3.15) however, we observe that

$$-\sum_{l=1}^{q} u^l - v \leqslant -\max\{\omega, 0\} \leqslant 0, \text{ and } \sum_{l=1}^{q} u^l + v \geqslant \max\{\omega, 0\} \geqslant \omega.$$

This means $\sum_{l=1}^{q} u^l + v \geqslant \omega$ which gives $-\sum_{l=1}^{q} u^l - v \leqslant -\omega$ and so $\omega - \sum_{l=1}^{q} u^l - v \leqslant 0$.

$\Longrightarrow$ In case (a) and (b), for all $S \notin \mathcal{I}_{(3.15)}$,

$$\nabla\psi(S) \leqslant 0 \implies \psi(S) \text{ is non-increasing for all } S \notin \mathcal{I}_{(3.15)}.$$

This means that $\psi(S)$ must attain its maximum value for a value of $S \in \mathcal{I}_{(3.15)}$ and so it holds that

$$\max_{S\in\mathbb{R}^n_+}\left\{\max(\omega^T S - E, 0) - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S\right\} =$$
$$\max_{S\in\mathcal{I}_{(3.15)}}\left\{\max(\omega^T S - E, 0) - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S\right\},$$

and the proposition is proved. □

We now show that the semi-infinite optimisation problem (3.15) can be re-formulated as a finite linear problem. Again we consider the work carried out in [11].
We start be recalling (3.15) as

$$\begin{aligned} \min_{u^l,z,v} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q} u^l + v \geqslant \max\{\omega, 0\} \\ & \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \max(\omega^T S - E, 0), \;\; \forall\, S \in \mathcal{I}_{(3.15)}. \end{aligned} \tag{3.15}$$

Now the second constraint is equivalent to the following two semi-infinite constraints, (3.17) and (3.18), by observing that

$$\max(\omega^T S - E, 0) \geqslant 0, \text{ and } \max(\omega^T S - E, 0) \geqslant \omega^T S - E.$$

$\Longrightarrow$

$$\sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \omega^T S - E, \;\; \forall\, S \in \mathcal{I}_{(3.15)} \tag{3.17}$$

and

$$\sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant 0, \;\; \forall\, S \in \mathcal{I}_{(3.15)}. \tag{3.18}$$

Which is equivalent to

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i \geqslant \sum_{i=1}^{n} \omega_i S_i - E, \;\; \forall\, S \in \mathcal{I}_{(3.15)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i \geqslant 0 \;\; \forall\, S \in \mathcal{I}_{(3.15)},$$

respectively.
We may re-write these as

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i - \sum_{i=1}^{n} \omega_i S_i + z\mathrm{e}^{rT} + E \geqslant 0, \;\; \forall\, S \in \mathcal{I}_{(3.15)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i + z\mathrm{e}^{rT} \geqslant 0 \ \ \forall\ S \in \mathcal{I}_{(3.15)},$$

respectively.
Switching the order of summation gives

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i\right)}_{(\star)} + z\mathrm{e}^{rT} + E \geqslant 0, \ \ \forall\ S \in \mathcal{I}_{(3.15)},$$

and

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i\right)}_{(\star\star)} + z\mathrm{e}^{rT} \geqslant 0 \ \ \forall\ S \in \mathcal{I}_{(3.15)},$$

respectively.
Now we choose $\alpha, \beta \in \mathbb{R}^n$ to be such that α_i provides a lower bound to $(\star)$ and β_i provides a lower bound to $(\star\star)$, for all $i = 1, 2, \ldots, n$. That is, we choose $\alpha_i, \beta_i,$ for all i, such that

$$\alpha_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i,$$

and

$$\beta_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i,$$

for all $i = 1, 2, \ldots, n$.
$\Longrightarrow$ The semi-infinite constraints of (3.15) become

$$\begin{cases} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i \geqslant \alpha_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n \\ \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} \alpha_i + z\mathrm{e}^{rT} + E \geqslant 0 \\ \sum_{i=1}^{n} \beta_i + z\mathrm{e}^{rT} \geqslant 0. \end{cases} \tag{3.19}$$

This means that a vector $(u^1, u^2, \ldots, u^q, z, v)$ is feasible for (3.15) if and only if $(u^1, u^2, \ldots, u^q, \alpha, \beta, z, v) \in \mathbb{R}^{n\times q} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is feasible for system (3.19). We observe here that by writing the semi-infinite constraints of (3.15) as system (3.19), we have that the last two constraints are standard (finite) linear constraints.
Now consider the semi-infinite constraints from system (3.19). Then, we have

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i \geqslant \alpha_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n,$$

and

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n.$$

Then we observe that both of these constraints are piece-wise linear constraints. Thus, the minimum value of the left hand side of both inequalities over all values of $S_i \in [0, E_i^q]$, for all $i = 1, 2, \ldots, n$ occurs at one of the break points. That is, it occurs exactly when $S_i = 0$ or $S_i = E_i^1$ or $S_i = E_i^2$ or $\ldots$ or $S_i = E_i^q$, for all $i = 1, 2, \ldots, n$. Therefore, we may consider these semi-infinite constraints for the $(q + 1)$ values $S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}$, for $i = 1, 2, \ldots, n$.
Thus, it holds that

$$\min_{S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i$$
$$= \min_{S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i,$$

and

$$\min_{S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i$$
$$= \min_{S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i.$$

Therefore, each of the semi-infinite constraints of system (3.19) can now be replaced by $(q+1)$ finite piece-wise linear constraints. That is, we may replace the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i \geqslant \alpha_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n$$

by

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i \geqslant \alpha_i, \ \text{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n,$$

and the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n$$

by

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \ \text{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n.$$

This leads us to the following theorem, which can be found in [11].

Theorem 3.20. *The semi-infinite optimisation problem* (3.15) *is equivalent to the following finite linear* *optimisation problem*

$$\begin{array}{ll}
\underset{(u^1,u^2,\ldots,u^q,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n,\alpha,\beta,\in\mathbb{R}^n}{min} & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\
\textit{subject to} & \sum_{l=1}^{q} u^l + v \geqslant \max\{\omega, 0\} \\
& \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \omega_i S_i \geqslant \alpha_i, \\
& \textit{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\},\ \forall\ i = 1, 2, \ldots, n \\
& \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \\
& \textit{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\},\ \forall\ i = 1, 2, \ldots, n \\
& \sum_{i=1}^{n} \alpha_i + z\mathrm{e}^{rT} + E \geqslant 0 \\
& \sum_{i=1}^{n} \beta_i + z\mathrm{e}^{rT} \geqslant 0,
\end{array} \tag{3.21}$$

in the sense that both optimisation problems have the same feasible region and hence the same optimal solution and the same optimal objective function value.

Observation: In comparison to the semi-infinite problem (3.15); we observe here that (3.21) has $n+n = 2n$ additional variables and a total of

$$n + n(q+1) + n(q+1) + 2 = n + 2n(q+1) + 2 = n + 2nq + 2n + 2 = 3n + 2nq + 2 = n(2q+3) + 2$$

linear constraints. The advantage of solving (3.21) in comparison to (3.15) is that we are solving a standard, finite linear problem in comparison to a semi-infinite one; something which can be easily implemented on an appropriate LO software solver, even for large values of n.
This concludes presenting results from [11] which look at finding an upper bound on the current price of a European basket call option under this particular model setting.
The proof of Proposition 3.14 and the technique used to derive the finite LO problem in Theorem 3.20 is vital for the remainder of this thesis. In particular, a similar technique used in the proof of Proposition 3.14 is used in section 4.1 to re-write a SIO problem with a non-compact index set as a SIO problem with a compact index set when considering finding lower bounds on the price of a European basket call option. Furthermore, the technique used in the proof of Proposition 3.14 and the technique used to obtain the finite LO problem in Theorem 3.20 is extended in sections 6, 7.2 and 8.2 of this thesis. In particular, using similar methods to those outlined in Proposition 3.14 and Theorem 3.20, we obtain upper bounds on the price of a Bermuda basket put option in section 6. We obtain upper bounds on the price of an Asian basket call option in section 7.2. We obtain upper bounds on the price of an Altiplano Mountain Range option in section 8.2. Finally, we note that a lower bound for the general n asset case is derived in [19] under this model setting. The exact problem considered here is the same as problem (3.12) but with an inf objective function, and the risk-free interest rate r set equal to 0.
We conclude by noting here that the methodology employed in this section and in the proof of Proposition 3.14 is vital to the remainder of this thesis. In this section we modelled the problem of finding an upper bound on the current price of a European basket call option as a dual of a SIO problem. We then considered the SIO problem for which the original problem was its dual; and observed that the index set of the SIO problem was not compact. Through Proposition 3.14 we equivalently re-wrote the SIO problem as a SIO problem with a compact index set. Finally, we managed to re-write the SIO problem with a compact index

set as a finite, solvable LO problem. We will employ a similar methodology to the one outlined above when we present our own results on this topic in sections 4.1,6,7.2 and 8.2.
This concludes presenting work from literature which looks at finding basket option price bounds given that we know the current prices of numerous vanilla call options per asset.

3.5 Modification 2: Assuming that other basket option prices are known

In this sub-section we consider another modification of the basic optimisation problems (3.10) and (3.11). Here, the basic model set-up is the same as in section 3.3, but the only difference is that instead of assuming the current prices of n European vanilla call options, one per asset are known; we assume that we know the current prices of a certain number of European basket call options, all written on the same n underlying assets. These underlying assets are the same as those written on the basket option whose price we are bounding.
We consider the result obtained in [19], which looks at obtaining a lower bound on the price of a European basket call option.
In what follows, we will use the notation defined below.

- $B^0 \in \mathbb{R}^r_+$ represents the current, known prices of the various r basket options, written on n underlying assets.
- Let $E \in \mathbb{R}_+$ be the exercise price of the basket call option whose price we are bounding.
- Let $\omega \in int(\mathbb{R}^n_+)$ represent the positive weights vector for the basket call option whose price we are bounding.
- Let $E_i \in \mathbb{R}_+$ be the exercise price of the i^{th} basket call option whose current price we know, for $i = 1, 2, \dots, r$.
- Let $\omega^i \in int(\mathbb{R}^n_+)$ represent the weights of the i^{th} basket option whose current price we know, for $i = 1, 2, \dots, r$.

Then, taking the risk-free interest rate, $r = 0$, our problem may be modelled as the following optimisation problem which is a dual to a SIO problem and is given as

$$\begin{array}{ll} \inf\limits_{\pi} & \mathbb{E}_\pi[\max(\omega^T S - E, 0)] \\ \text{subject to} & \mathbb{E}_\pi[1] = 1 \\ & \mathbb{E}_\pi[\max({\omega^i}^T S - E_i, 0)] = B^0_i, \quad \text{for } i = 1, 2, \dots, r \\ & \pi \text{ is a distribution in } \mathbb{R}^n_+. \end{array} \tag{3.22}$$

Defining the variables $\begin{pmatrix} z \\ y_i \end{pmatrix} \in \mathbb{R}^{r+1}$, for $i = 1, 2, \dots, r$, the linear SIO problem for which (3.22) is its dual is given by

$$\begin{array}{ll} \sup\limits_{z,y} & z + \sum\limits_{i=1}^{r} B^0_i y_i \\ \text{subject to} & z + \sum\limits_{i=1}^{r} y_i \max({\omega^i}^T S - E_i, 0) \leqslant \max(\omega^T S - E, 0) \quad \forall\, S \in \mathbb{R}^n_+ \\ & y \in \mathbb{R}^r,\ z \in \mathbb{R} \text{ are `free'}. \end{array} \tag{3.23}$$

We are interested in using (3.23) to obtain a solution and in particular we wish to see when the optimal objective function values of the problem (3.23) and the problem (3.22) coincide. The conditions for exactly this to occur are given in the next proposition.

Proposition 3.24. *The optimal objective function values of* (3.22) *and* (3.23) *coincide if at least one of the following two conditions hold.*
(i) Strict primal feasibility,

$$\begin{pmatrix} 1 \\ B^0 \end{pmatrix} \in int \left(\left(\begin{matrix} \mathbb{E}_\pi[1] \\ \mathbb{E}_\pi[\max({\omega^i}^T S - E_i, 0)] \end{matrix} \right) : \pi \text{ is a distribution in } \mathbb{R}^n_+ \right).$$

In particular, strong duality holds provided the prices B^0 are arbitrage-free and remain arbitrage-free after slight perturbations.
(ii) Strict dual feasibility.
There exists $(\hat{z}, \hat{y}) \in \mathbb{R}^{r+1}$ such that

$$(\hat{z}, \hat{y}) \in int \left((z, y) \in \mathbb{R}^{r+1} : z + \sum_{i=1}^{r} y_i \max({\omega^i}^T S - E_i, 0) \leqslant \max(\omega^T S - E, 0), \forall\ S \in \mathbb{R}^n_+ \right).$$

In particular, strong duality holds provided that, for each asset, at least the current price of one vanilla option is known [19].

We now present one way to solve the SIO problem (3.23) as outlined in [19].
For this we fix the following notational convention. Denote Ω to be the $(r \times n)$ matrix whose j^{th} row is the vector $(\omega^j)^T$, for $j = 1, 2, \ldots, r$ and also denote $\tilde{E} \in \mathbb{R}^r_+$ to be the vector $(E_1, E_2, \ldots, E_r)^T$. Furthermore, let I be a finite index set and suppose that we are given a vector $\nu \in \mathbb{R}^{|I|}$. Then, for $J \subseteq I$, let $\nu_J \in \mathbb{R}^{|J|}$ denote the vector formed by the entries ν_j of ν with $j \in J$. Similarly, suppose that a matrix Λ, is given such that the rows of Λ are indexed by the set I. For $J \subseteq I$, let Λ_J denote the matrix formed by the rows of Λ indexed by J. Also, let J' denote the set $I \backslash J$. The set I will be equal to $\{1, 2, \ldots, r\}$, for some $r > 0$ and $r \in \mathbb{Z}$.
Using this notation and given $J \subseteq \{1, 2, \ldots, r\}$ we define

$$\mathcal{P}_J = \mathcal{P}_J(\Omega, \tilde{E}) = \left\{ S : \Omega_J S \geqslant \tilde{E}_J,\ \Omega_{J'} S \leqslant \tilde{E}_{J'},\ S \geqslant 0 \right\}$$

and the set $\mathcal{J}$ as

$$\mathcal{J} = \{ J \subseteq \{1, 2, \ldots, r\} : \mathcal{P}_J \neq \varnothing \}.$$

Then, in order to solve (3.23), we may equivalently re-write (3.23) as a finite linear problem. In order to re-write (3.23) as a finite LO problem, we use the following lemma.

Lemma 3.25. *Let $\tilde{P} = \{x | Qx \leqslant \bar{d}\} \subseteq \mathbb{R}^n$ be a non-empty polyhedron.*
(a) Assume that $\bar{a} \in \mathbb{R}^n$ and $\bar{\alpha} \in \mathbb{R}$ are given. Then the following two conditions are equivalent.
(i) For all $x \in \tilde{P}, \bar{a}^T x \leqslant \bar{\alpha}$
(ii) There exists $y \in \mathbb{R}^r_+$, such that

$$\bar{a} = Q^T y \text{ and } \bar{\alpha} \geqslant \bar{d}^T y.$$

(b) Assume $\bar{a}, \bar{b} \in \mathbb{R}^n$ and $\bar{\alpha}, \bar{\beta} \in \mathbb{R}$ are given. Then the following two conditions are equivalent.
(i) For all $x \in \tilde{P}$, either $\bar{a}^T x \leqslant \bar{\alpha}$ or $\bar{b}^T x \leqslant \bar{\beta}$
(ii) There exists $\zeta \in [0, 1]$ such that for all $x \in \tilde{P}$,

$$(\zeta \bar{a} + (1 - \zeta) \bar{b})^T x \leqslant (\zeta \bar{\alpha} + (1 - \zeta) \bar{\beta}).$$

Proof. See [19]. □

Then, using this lemma, the SIO problem (3.23) can be re-written as a finite linear problem.

Proposition 3.26. *The SIO problem* (3.23) *can be re-written equivalently as the following finite linear problem,*

$$\begin{aligned}
\max_{z,y} \quad & z + {B^0}^T y \\
\text{subject to} \quad & -\zeta\omega + \Omega_J^T y_J = -\Omega_J^T \gamma^J + \Omega_{J'}^T \beta^{J'}, \quad J \in \mathcal{J} \\
& -z - \zeta E + y_J^T \tilde{E}_J \geqslant -\tilde{E}_J^T \gamma^J + \tilde{E}_{J'}^T \beta^{J'}, \quad J \in \mathcal{J} \\
& \zeta \leqslant 1 \quad J \in \mathcal{J} \\
& y \in \mathbb{R}^r, z \in \mathbb{R}, \zeta \in \mathbb{R}_+, \gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}, \quad J \in \mathcal{J}.
\end{aligned} \tag{3.27}$$

Proof. We have the following SIO problem for which its dual is (3.22) and is given by

$$\begin{aligned}
\sup_{z,y} \quad & z + \sum_{i=1}^r B_i^0 y_i \\
\text{subject to} \quad & z + \sum_{i=1}^r y_i \max({\omega^i}^T S - E_i, 0) \leqslant \max(\omega^T S - E, 0) \quad \forall\, S \in \mathbb{R}_+^n.
\end{aligned} \tag{3.23}$$

Now consider the constraint of the above SIO problem (3.23). We do not know today whether the basket option whose price we are bounding expires in or out of the money. We consider both cases and use part (*b*) of Lemma 3.25.

Then we have that either

$$z + y^T \max(\Omega S - \tilde{E}, 0) \leqslant \omega^T s - E, \ \forall\, S \in \mathbb{R}_+^n, \tag{3.28}$$

or

$$z + y^T \max(\Omega S - \tilde{E}, 0) \leqslant 0, \ \forall\, S \in \mathbb{R}_+^n. \tag{3.29}$$

Re-write (3.28) as

$$\begin{aligned}
& -(\omega^T S - E) + y^T \max(\Omega S - \tilde{E}, 0) \leqslant -z, \ \forall\, S \in \mathbb{R}_+^n \\
\iff & -\omega^T S + E + y_J^T(\Omega_J S - \tilde{E}_J) \leqslant -z, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J} \\
\iff & -\omega^T S + E + y_J^T \Omega_J S - y_J^T \tilde{E}_J \leqslant -z, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J}.
\end{aligned}$$

This is the same as

$$(-\omega + \Omega_J^T y_J)^T S \leqslant -z - E + y_J^T \tilde{E}_J, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J}. \tag{3.30}$$

We can also re-write (3.29) as

$$\begin{aligned}
& z + y_J^T(\Omega_J S - \tilde{E}_J) \leqslant 0, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J} \\
\iff & y_J^T \Omega_J S - y_J^T \tilde{E}_J \leqslant -z, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J}.
\end{aligned}$$

This is the same as

$$(\Omega_J^T y_J)^T S \leqslant -z + y_J^T \tilde{E}_J, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J}. \tag{3.31}$$

We use part (*i*) of (*b*) with

$$\begin{aligned}
\bar{a} = (-\omega + \Omega_J^T y_J), \quad x = S, \quad \bar{\alpha} &= -z - E + y_J^T \tilde{E}_J, \\
\bar{b} = \Omega_J^T y_J, \quad \bar{\beta} &= -z + y_J^T \tilde{E}_J,
\end{aligned}$$

along with

$$Q = \begin{pmatrix} -\Omega_J \\ \Omega_{J'} \end{pmatrix} \text{ and } \bar{d} = \begin{pmatrix} -\tilde{E}_J \\ \tilde{E}_{J'} \end{pmatrix}.$$

Then from (i) in part (b), (3.30) and (3.31) holding is equivalent to there existing a $\zeta \in [0,1]$ such that $\forall\, S \in \mathcal{P}_J$

$$\left(\zeta(-\omega + \Omega_J^T y_J) + (1-\zeta)\Omega_J^T y_J\right)^T S \leqslant \left(\zeta(-z - E + y_J^T \tilde{E}_J) + (1-\zeta)(-z + y_J^T \tilde{E}_J)\right)$$
$$\iff \left((-\zeta\omega + \zeta\Omega_J^T y_J) + \Omega_J^T y_J - \zeta\Omega_J^T y_J\right)^T S \leqslant -\zeta z - \zeta E + y_J^T \tilde{E}_J \zeta - z + y_J^T \tilde{E}_J + \zeta z - y_J^T \tilde{E}_J \zeta.$$

This is equivalent to

$$\left(-\zeta\omega + \Omega_J^T y_J\right)^T S \leqslant -z + y_J^T \tilde{E}_J - \zeta E, \ \forall\, S \in \mathcal{P}_J, \ J \in \mathcal{J}. \tag{3.32}$$

Now apply part (a) with

$$\bar{a} = (-\zeta\omega + \Omega_J^T y_J) \ \text{ and } \ \bar{\alpha} = -z + y_J^T \tilde{E}_J - E\zeta,$$

and Q and $\bar{d}$ as before.

Then (3.32) holding is equivalent to there existing a vector $\begin{pmatrix} \gamma^J \\ \beta^{J'} \end{pmatrix} \in \mathbb{R}_+^n$ (where $\gamma^J \in \mathbb{R}_+^{|J|}$ and $\beta^{J'} \in \mathbb{R}_+^{|J'|}$) such that

$$-\zeta\omega + \Omega_J^T y_J = \begin{pmatrix} -\Omega_J^T & \Omega_{J'} \end{pmatrix} \begin{pmatrix} \gamma^J \\ \beta^{J'} \end{pmatrix}$$

and

$$-z + y_J^T \tilde{E}_J - E\zeta \geqslant \begin{pmatrix} -\tilde{E}_J^T & \tilde{E}_{J'} \end{pmatrix} \begin{pmatrix} \gamma^J \\ \beta^{J'} \end{pmatrix}$$

for $J \in \mathcal{J}$.

This is equivalent to

$$-\zeta\omega + \Omega_J^T y_J = -\Omega_J^T \gamma^J + \Omega_{J'}^T \beta^{J'}, \ J \in \mathcal{J}$$

and

$$-z + y_J^T \tilde{E}_J - \zeta E \geqslant -\tilde{E}_J^T \gamma^J + \tilde{E}_{J'}^T \beta^{J'}, \ J \in \mathcal{J}.$$

Recalling that $\zeta \leqslant 1$, the proof is complete. □

Then, (3.27) can be solved to yield an optimal solution and hence a lower bound on the current price of the basket option of interest.

We note here that the technique used in the proof of Proposition 3.26 above to re-write the SIO problem (3.23) as a finite LO problem (3.27) is vital for the remainder of this thesis. In particular, when deriving our own results in sections 4.2 and 7.3 , we have used similar techniques as used in the proof of Proposition 3.26. In section 4.2 we used a similar technique to the one outlined above to obtain lower bounds on the price of a European basket call option, incorporating bid-ask prices. In section 7.3 we again used similar techniques as those in the proof of Proposition 3.26 above to obtain results regarding upper bounds on the price of an Asian basket call option, again incorporating bid-ask prices. We note here that our obtained results extend the techniques used in the proof of Proposition 3.26 as we incorporate bid-ask prices in our results and we change the technique so we can solve an upper bound problem in section 7.3.

For the final part of this sub-section, we consider a solution method outlined in [21] which aims to solve the optimisation problem (3.22) and the corresponding upper bound problem which is the same as problem (3.22) but with *sup* replacing *inf* in the objective function.

The way to find bounds on the current price of a European basket call option using the above described model what we consider is via a suitable Dantzig-Wolfe decomposition.

Recall that we have seen, and will also see again, that finding a lower bound on the current price of a basket call option presents a significantly harder and more difficult problem in comparison to the corresponding upper bound problem. In particular, when we look to find a lower bound on the current price of a basket option we are presented with various challenges which were not encountered when solving the upper bound problem.

As highlighted in [21], one advantage of using a Dantzig-Wolfe decomposition solution approach to solve

the basket option pricing problem is that the upper and lower bounds can be obtained in a similar way. In particular, if an algorithm is designed for solving the lower bound problem, we only have to slightly modify the algorithm so that it can be used to solve the upper bound problem. In addition to this, by using this specific algorithm, which we present below, we can incorporate the use of *bid-ask prices* into the model, making it more realistic.
In what follows, we present a Dantzig-Wolfe decomposition solution approach (taken from [21]) to solve the lower bound problem for the current price of a basket option. The upper bound problem can be solved similarly by making slight changes as given in Remark 3.38.
For convenience, in the remainder of this sub-section we will use the following notation. Let $\omega^0 \in \mathbb{R}^n_+$ denote the weights vector of the basket option whose current price we are bounding, (instead of ω) and let $E_0 \in \mathbb{R}_+$ replace E as the exercise price of the basket option whose current price we are bounding. All other notation will stay the same. To start with, we recall the basic lower bound optimisation problem under the model we are considering as

$$\begin{aligned} \inf_{\pi} \quad & \mathbb{E}_\pi[\max({\omega^0}^T S - E_0, 0)] \\ \text{subject to} \quad & \mathbb{E}_\pi[1] = 1 \\ & \mathbb{E}_\pi[\max({\omega^i}^T S - E_i, 0)] = B_i^0, \quad \text{for } i = 1, 2, \ldots, r \\ & \pi \text{ is a distribution in } \mathbb{R}^n_+. \end{aligned} \tag{3.33}$$

To proceed we use the following notational convention, as done in [21]. Let $\bar{\mathcal{J}} = \{J \subseteq \{0, 1, 2, \ldots, r\}\}$ be the set of all subsets J, which is an index set. Define $\bar{\Omega} \in \mathbb{R}^{(r+1)\times n}$, to be an $(r+1) \times n$ matrix whose j^{th} row is the vector $(\omega^j)^T$, for all $j = 0, 1, \ldots, r$, so that the matrix $\bar{\Omega}$ captures the weights of the basket option whose current price we are bounding, as well as the weights of the basket options whose current price we know. Further, we also let $\hat{E} \in \mathbb{R}^{(r+1)}_+$ be given by $(E_0, E_1, \ldots, E_r)^T$, so that the vector $\hat{E}$ captures all the exercise prices that we know.
Now, we assume that we know an upper bound for the price of each asset S_i, for $i = 1, 2, \ldots, n$ in the basket option at expiry. In particular, we will let $U_i > 0$, for all $i = 1, 2, \ldots, n$ be an upper bound for the asset price S_i at expiry. This means that for the i^{th} asset, we have that $0 \leqslant S_i \leqslant U_i$, for all $i = 1, 2, \ldots, n$. Then, we may use this to define the vector $U \in \mathbb{R}^n_+$ as $(U_1, U_2, \ldots, U_n)^T$.
Now we further set the following notational convention, similar to what was done above. For a vector $\bar{\nu} \in \mathbb{R}^{(r+1)}$, and $J \in \bar{\mathcal{J}}$, let $\bar{\nu}_J \in \mathbb{R}^{|J|}$ denote the vector formed by the entries $\bar{\nu}_j$, for $j \in J$. Similarly, for a matrix $\bar{\Lambda} \in \mathbb{R}^{(r+1)\times n}$, let the rows of $\bar{\Lambda}$ be indexed by the set $\{0, 1, \ldots, r\}$. Then, for $J \in \bar{\mathcal{J}}$, the sub-matrix $\bar{\Lambda}_J$ is the matrix $\bar{\Lambda}_J \in \mathbb{R}^{|J|\times n}$, where $\bar{\Lambda}_J$ has its rows constructed by the rows from $\bar{\Lambda}$ indexed by the set J. Also, let $J' = \{0, 1, \ldots, r\}\backslash J$.
Then, in order to use a suitable Dantzig-Wolfe decomposition solution approach for this problem, we consider the probability measure π to be from $\tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}}$ are the box constraints given by

$$\tilde{\mathcal{B}} = [0, U_1] \times [0, U_2] \times \cdots \times [0, U_n].$$

We use the following proposition.

Proposition 3.34. *Let I be an index set, and R_i, for $i \in I$ be a partition of $\tilde{\mathcal{B}}$. For any piece-wise linear function, $f : \tilde{\mathcal{B}} \to \mathbb{R}$, such that when f is restricted to R_i, it is linear for each $i \in I$; we have that*

$$\mathbb{E}_\pi[f(S)] = \sum_{i\in I} \mathbb{E}_\pi[f(S)|S \in R_i]\pi(S \in R_i) = \sum_{i\in I} f[\mathbb{E}_\pi(S|S \in R_i)]\pi(S \in R_i).$$

If R_i is convex and bounded for each i, we further have that $\mathbb{E}_\pi[S|S \in R_i] \in R_i$.

Then, using Proposition 3.34, we may conclude that the probability measure π in a certain region of $\tilde{\mathcal{B}}$ can be concentrated to a single point in that region [21]. That is, in a single 'box' of $\tilde{\mathcal{B}}$, there exists only one point where π has a non-zero, and hence positive probability value.

In particular, using the piece-wise linearity of the payoff of the basket options, we may partition $\tilde{\mathcal{B}}$ by $\mathcal{P}_J$, where

$$\mathcal{P}_J = \{S \in \mathbb{R}^n_+ : \bar{\Omega}_J S \geqslant \hat{E}_J, \bar{\Omega}_{J'} S \leqslant \hat{E}_{J'}, 0 \leqslant S \leqslant U\},\ \forall\ J \in \bar{\mathcal{J}}.$$

Using this partition of $\tilde{\mathcal{B}}$ and the result in Proposition 3.34, we may re-write the lower bound problem (3.33) which is a dual to a linear SIO problem, as a finite non-linear optimisation problem.
If $S^J \in \mathbb{R}^n_+$ represents the expected price of the asset prices at expiry. That is, we set $\mathbb{E}_\pi[S] = S^J$ and let the probability of this expected outcome occurring be given by θ^J, where $0 \leqslant \theta^J \leqslant 1$, and $\theta^J \in \mathbb{R}$, then we have that the problem (3.33) is equivalent to the following finite non-linear optimisation (NLO) problem,

$$\begin{aligned}
\min_{\theta^J} \quad & \sum_{J\in\bar{\mathcal{J}}:0\in J} \max({\omega^0}^T S^J - E_0, 0)\theta^J \\
\text{subject to} \quad & \sum_{J\in\bar{\mathcal{J}}} \theta^J = 1 \\
& \sum_{J\in\bar{\mathcal{J}}:j\in J} \max({\omega^j}^T S^J - E_j, 0)\theta^J = B^0_j, \quad \text{for } j = 1, 2, \ldots, r, \\
& S^J \in \mathcal{P}_J, \quad \text{for } J \in \bar{\mathcal{J}} \\
& \theta^J \geqslant 0, \quad \text{for } J \in \bar{\mathcal{J}}.
\end{aligned} \tag{3.35}$$

To solve this non-linear optimisation problem we appeal to the Dantzig-Wolfe, (D-W for short) decomposition algorithm. For the basic ideas and definitions regarding the (D-W) decomposition algorithm, we refer the reader to [21].
To apply the (D-W) decomposition algorithm to the basket option lower bound pricing problem (3.35) we first tackle the non-linearity of the problem. To do this we define a new variable $u^J = S^J\theta^J$, for all $J \in \bar{\mathcal{J}}$. This would then allow us to apply the (D-W) decomposition algorithm to obtain an optimal solution to (3.35).
For the *sub-problems* we consider the block constraints

$$S^J \in \mathcal{P}_J,\ \forall\ J \in \bar{\mathcal{J}}$$

and

$$\theta^J \geqslant 0,\ \forall\ J \in \bar{\mathcal{J}},$$

from (3.35). The remainder of (3.35) will form the *master problem.* (See [21] for more on this).
Now, observing that $\mathcal{P}_J$ is a polyhedron, since it is just a combination of a finite number of linear constraints, then for each $J \in \bar{\mathcal{J}}$, if $S^{\tilde{J},k}$, for $J \in \bar{\mathcal{J}}$, and $k = 1, 2, \ldots, N_J$, where N_J is the number of extreme points of the polyhedron $\mathcal{P}_J$, for all $J \in \bar{\mathcal{J}}$, is the set of extreme points for $\mathcal{P}_J$, then, and referring the reader to [21] for full details, we may re-write (3.35) as the following equivalent problem

$$\begin{aligned}
\min_{\lambda_{J,k},\theta^J} \quad & \sum_{J\in\bar{\mathcal{J}}:0\in J} \left({\omega^0}^T \left(\sum_{k=1}^{N_J} \lambda_{J,k} S^{\tilde{J},k}\right) - E_0\right)\theta^J \\
\text{subject to} \quad & \sum_{J\in\bar{\mathcal{J}}} \theta^J = 1 \\
& \sum_{J\in\bar{\mathcal{J}}:j\in J} \left({\omega^j}^T \left(\sum_{k=1}^{N_J} \lambda_{J,k} S^{\tilde{J},k}\right) - E_j\right)\theta^J = B^0_j, \quad \text{for } j = 1, 2, \ldots, r \\
& \sum_{k=1}^{N_J} \lambda_{J,k} = 1,\ \forall\ J \in \bar{\mathcal{J}} \\
& \lambda_{J,k} \geqslant 0, \quad \text{for } J \in \bar{\mathcal{J}} \\
& \theta^J \geqslant 0, \quad \text{for } J \in \bar{\mathcal{J}}.
\end{aligned} \tag{3.36}$$

Further, if we define $\theta^{J,k} = \theta^J \lambda_{J,k}$, for all $J \in \bar{\mathcal{J}}$ and for all $k = 1, 2, \ldots, N_J$, then (3.36) becomes

$$\begin{aligned}
\min_{\theta^{J,k}} \quad & \sum_{J \in \bar{\mathcal{J}} : 0 \in J} \sum_{k=1}^{N_J} ({\omega^0}^T S^{\tilde{J},k} - E_0) \theta^{J,k} \\
\text{subject to} \quad & \sum_{J \in \bar{\mathcal{J}}} \sum_{k=1}^{N_J} \theta^{J,k} = 1 \\
& \sum_{J \in \bar{\mathcal{J}} : j \in J} \sum_{k=1}^{N_J} ({\omega^j}^T S^{\tilde{J},k} - E_j) \theta^{J,k} = B_j^0, \quad \text{for } j = 1, 2, \ldots, r \\
& \theta^{J,k} \geqslant 0, \ \forall \ J \in \bar{\mathcal{J}}, \ \forall \ k = 1, 2, \ldots, N_J.
\end{aligned} \tag{3.37}$$

We observe that (3.37) is now a large-scale linear optimisation problem in the variable $\theta^{J,k}$, and it is equivalent to (3.36) which is used to find a lower bound on the current price of a basket option which we are considering here.
We may now apply the (D-W) decomposition algorithm to the large scale LO problem (3.37) as done in [21]. Consider a subset of the extreme points of the set $\mathcal{P}_J$, for all $J \in \bar{\mathcal{J}}$, given by

$$\hat{\mathcal{X}} = \bigcup_{J \in \bar{\mathcal{J}}} \{S^{\tilde{J},k} : k = 1, 2, \ldots, M_J\} \subseteq \bigcup_{J \in \bar{\mathcal{J}}} \{S^{\tilde{J},k} : k = 1, 2, \ldots, N_J\}.$$

Then we define the *master problem* as

$$\begin{aligned}
\min_{\theta^{J,k}} \quad & \sum_{J \in \bar{\mathcal{J}} : 0 \in J} \sum_{k=1}^{M_J} ({\omega^0}^T S^{\tilde{J},k} - E_0) \theta^{J,k} \\
\text{subject to} \quad & \sum_{J \in \bar{\mathcal{J}}} \sum_{k=1}^{M_J} \theta^{J,k} = 1 \\
& \sum_{J \in \bar{\mathcal{J}} : j \in J} \sum_{k=1}^{M_J} ({\omega^j}^T S^{\tilde{J},k} - E_j) \theta^{J,k} = B_j^0, \quad \text{for } j = 1, 2, \ldots, r \\
& \theta^{J,k} \geqslant 0, \ \forall \ J \in \bar{\mathcal{J}}, \ \forall \ k = 1, 2, \ldots, M_J.
\end{aligned} \tag{$3.37_{\hat{\mathcal{X}}}$}$$

Further, defining an indicator variable $\mathbb{I}_{0 \in J}$, where

$$\mathbb{I}_{0 \in J} = \begin{cases} 1 & \text{if } 0 \in J \\ 0 & \text{otherwise,} \end{cases}$$

and recalling that the *sub-problems* captured the constraints

$$S^J \in \mathcal{P}_J, \ \ \forall \ J \in \bar{\mathcal{J}}, \text{ and } \ \theta^J \geqslant 0, \ \forall \ J \in \bar{\mathcal{J}},$$

from (3.35), we can now define the *sub-problems* as follows.
Given $\tilde{\tau}$ and $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_{\tilde{\tau}})$, and $J \in \bar{\mathcal{J}}$, as in [21], the *sub-problems* are given by

$$\begin{aligned}
\min_{\tilde{\tau}, \tilde{\rho}} \quad & ({\omega^0}^T S - E_0) \mathbb{I}_{0 \in J} - \tilde{\tau} - \sum_{j \in J, j > 0} \tilde{\rho}_j ({\omega^j}^T S - E_j) \\
\text{subject to} \quad & S \in \mathcal{P}_J.
\end{aligned} \tag{$3.37_{\mathcal{S}}$}$$

We may now solve $(3.37_{\hat{\mathcal{X}}})$ to obtain an optimal solution and hence a lower bound on the current price of the basket option under consideration. For full detailed explanation on the solution algorithm used to solve $(3.37_{\hat{\mathcal{X}}})$ we refer the reader to [21].

Remark 3.38. *All of the analysis presented above on using the (D-W) decomposition algorithm to obtain a solution for the bounds on the current price of a basket option has been concerned with a* ***lower bound*** *problem. As already mentioned, one advantage of using this approach is that we may solve the upper bound problem in identical fashion by making the following modifications.*

- *In all optimisation problems, replace min by max.*
- *Make suitable changes to the (D-W) decomposition solution algorithm as highlighted in [21].*

We note that the lower bound problem (3.22) and its corresponding upper bound problem obtained by replacing *inf* with *sup* in the objective function is also solved in [20] using a different solution approach than the one considered here.
Again, we conclude by noting here that the methodology employed in the proof of Proposition 3.26 is vital to the remainder of this thesis. In this section we modelled the problem of finding a lower bound on the current price of a European basket call option as a dual of a SIO problem. We then considered the SIO problem for which the original problem was its dual. We then re-wrote the semi-infinite constraint of the SIO problem (3.23) as finitely many linear constraints so that the SIO problem (3.23) could equivalently be re-written as a finite and solvable LO problem. We will employ a similar methodology to the one outlined in the proof of Proposition 3.26 when we present our own results on this topic in sections 4.2 and 7.2.

3.6 Other modifications of (3.10) / (3.11)

The modifications of the constraints of problems (3.10) / (3.11) discussed in sections 3.4 and 3.5 are arguably the most natural modifications that we can make to the basic optimisation model. In this sub-section we briefly consider some other modifications of (3.10) / (3.11) that have been looked at in existing literature.
In [11], upper bounds on the price of a European basket call option are derived by assuming that we know the current prices of one European vanilla call option, per asset and the current prices of other European exotic call options written on single assets only, where these single assets are S_i. That is, those assets that form the basket option; then it is possible to repeat the analysis from sub-section 3.4 and obtain an upper bound on the current price of a basket call option under this assumption. Some examples of European exotic call options written on single assets that could be used are *k-power options, binaries and digitals* and so on. For a full list we refer the reader to [3].
For a simple example on how to implement what is described above, we refer the reader to [11], where the above ideas have been implemented for a 2-*power option*.
Another modification of (3.10) / (3.11) is considered in [21], where a lower bound on the price of a European basket call option is derived given that we know the current *bid-ask prices* of m other European basket call options written on the same n underlying assets. If B_i^{ask} represents the current, known *ask* price for the i^{th} basket option and B_i^{bid} represents the current, known *bid* price for the i^{th} basket option, for $i = 1, 2, \dots, m$, then the following optimisation problem, which is a dual to a linear SIO problem, is solved in [21].

$$\begin{array}{ll} \inf\limits_{\pi} & \mathbb{E}_\pi[\max(\omega^T S - E, 0)] \\ \text{subject to} & \mathbb{E}_\pi[1] = 1 \\ & \mathbb{E}_\pi[\max({\omega^i}^T S - E_i, 0)] \leqslant B_i^{ask}, \quad \text{for } i = 1, 2, \dots, m \\ & \mathbb{E}_\pi[\max({\omega^i}^T S - E_i, 0)] \geqslant B_i^{bid}, \quad \text{for } i = 1, 2, \dots, m \\ & \pi \text{ is a distribution in } \mathbb{R}^n_+. \end{array} \tag{3.39}$$

3.6.1 Assuming that the bid-ask prices of vanilla options is known

The final optimisation model which we consider is taken from [22]. Recall that in sub-section 3.3, when setting up the basic optimisation problems (3.10) / (3.11) we assumed that we knew the current prices of European vanilla call options. Here, we implicitly assumed that we knew the current *mid-market prices* of these options. However, in reality this assumption does not hold. Instead when we look for current prices of options, say on the internet or in the Financial Times, for example, we are given two prices; a *bid price* and an *ask price*. The *bid price* corresponds to how much we may sell the option for on the market and the *ask price* is the price that we may buy the option for from the market, where *ask price* is greater than or equal to the *bid price*, of course.

We start by setting up the optimisation model as done in [22]. Suppose that the basket call option whose current price we are bounding is written on n assets, with $\omega_i \in \mathbb{R}$ representing the weight of the i^{th} asset, for $i = 1, 2, \ldots, n$. E is the exercise price of this basket option, with $E \in \mathbb{R}$ and $E \geqslant 0$. $S_i \in \mathbb{R}_+$ represents the possible prices of the i^{th} asset at expiry, for $i = 1, 2, \ldots, n$. Furthermore, we also suppose that we know the current *ask prices* and current *bid prices* of $(m+1) \times n$ European vanilla options. That is, we assume that the current *bid-ask* prices of $(m+1)$ European vanilla call options, per asset (for a total of n assets) is known. Let p_{ij}^{ask} denote the current *ask price* of the j^{th} European vanilla call option, written on asset i, for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m$. Similarly, let p_{ij}^{bid} denote the current *bid price* of the j^{th} European vanilla call option, written on asset i, for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m$. Of course we also assume that $p_{ij}^{ask} \geqslant p_{ij}^{bid}$, for all $i = 1, 2, \ldots, n$, and for all $j = 0, 1, \ldots, m$. Finally, we also let $E_{ij} \geqslant 0$ denote the exercise price of the j^{th} European vanilla call option written on asset i, for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m$. We also assume that **all** options have the same expiry date, given by T; and the risk-free interest rate, $r = 0$. If π is a probability measure, then the task of finding a no-arbitrage upper bound on the current price of a European basket call option is given by,

$$\begin{aligned}
\sup_{\pi} \quad & \mathbb{E}_\pi \left[\max \left(\sum_{i=1}^{n} \omega_i S_i - E, 0 \right) \right] \\
\text{subject to} \quad & \mathbb{E}_\pi[\max(S_i - E_{ij}, 0)] \leqslant p_{ij}^{ask}, \quad i = 1, 2, \ldots, n, \ j = 0, 1, \ldots, m \\
& \mathbb{E}_\pi[\max(S_i - E_{ij}, 0)] \geqslant p_{ij}^{bid}, \quad i = 1, 2, \ldots, n, \ j = 0, 1, \ldots, m \\
& \mathbb{E}_\pi[1] = 1.
\end{aligned} \tag{3.40}$$

Now, we observe that (3.40) is the dual to a linear SIO problem.

So if we define $y = y^{ask} - y^{bid} \in \mathbb{R}^{n \times (m+1)}$, we may deduce the linear SIO problem for which (3.40) is its dual as

$$\begin{aligned}
\inf_{y^{ask}, y^{bid}, y, z} \quad & z + \sum_{i=1}^{n} \sum_{j=0}^{m} p_{ij}^{ask} y_{ij}^{ask} - \sum_{i=1}^{n} \sum_{j=0}^{m} p_{ij}^{bid} y_{ij}^{bid} \\
\text{subject to} \quad & z + \sum_{i=1}^{n} \sum_{j=0}^{m} y_{ij} \max(S_i - E_{ij}, 0) \geqslant \max \left(\sum_{i=1}^{n} \omega_i S_i - E, 0 \right), \quad \forall \, S \in \mathbb{R}_+^n \\
& y = y^{ask} - y^{bid} \\
& y \in \mathbb{R}^{n \times (m+1)} \\
& y^{ask}, y^{bid} \in \mathbb{R}_+^{n \times (m+1)} \\
& z \in \mathbb{R}.
\end{aligned} \tag{3.41}$$

We start by re-writing (3.41) in vector form. We adopt the following notational convention. For any vector $[a_{ij}]_{i=1,2,\ldots,n}$, we will write a^j, where $a^j \in \mathbb{R}^n$. That is, the vector with components (a_{ij}), for $i = 1, 2, \ldots, n$ will be written as a^j, with $a^j \in \mathbb{R}^n$, for all $j = 0, 1, \ldots, m$. Furthermore, $\cdot$ denotes the usual vector 'dot' product between two vectors, defined as follows.

$$\text{For } b^j, c^j \in \mathbb{R}^n, \ b^j \cdot c^j = \sum_{i=1}^{n} b_{ij} c_{ij}, \text{ for all } j = 0, 1, \ldots, m.$$

Then, in vector form, (3.41) can be written as

$$\begin{aligned}
\inf_{y^{ask},y^{bid},y,z} \quad & z + \sum_{j=0}^{m} p^{ask^j} \cdot y^{ask^j} - \sum_{j=0}^{m} p^{bid^j} \cdot y^{bid^j} \\
\text{subject to} \quad & z + \sum_{j=0}^{m} y^j \cdot \max(S - E^j, 0) \geqslant \max(\omega \cdot S - E, 0), \quad \forall\, S \in \mathbb{R}^n_+ \\
& y = y^{ask} - y^{bid} \\
& y \in \mathbb{R}^{n\times(m+1)} \\
& y^{ask}, y^{bid} \in \mathbb{R}^{n\times(m+1)}_+ \\
& z \in \mathbb{R}.
\end{aligned} \tag{3.42}$$

To solve (3.42) the following definitions and results taken from [22] are utilised.

Definition 3.43. *Assume that* $\bar{E} = [E^0, E^1, \ldots, E^m] \in \mathbb{R}^{n\times(m+1)}$, $\tilde{b} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$ *are given. We define the set of* ***super-replicating strategies***, $SR(\bar{E}, \tilde{b}, c)$ *as*

$$SR(\bar{E}, \tilde{b}, c) = \{(y, z) = (y^0, y^1, \ldots, y^m, z) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R} : z + \sum_{j=0}^{m} y^j \cdot \max(S - E^j, 0) \geqslant$$

$$\tilde{b} \cdot S - c,\ \forall\, S \in \mathbb{R}^n_+\}. \tag{3.44}$$

Then, using Definition 3.43, we may equivalently re-write the SIO problem (3.42) as the following optimisation problem

$$\begin{aligned}
\inf_{y^{ask},y^{bid},y,z} \quad & z + \sum_{j=0}^{m} p^{ask^j} \cdot y^{ask^j} - \sum_{j=0}^{m} p^{bid^j} \cdot y^{bid^j} \\
\text{subject to} \quad & (y, z) \in SR(\bar{E}, \omega, E) \\
& (y, z) \in SR(\bar{E}, 0, 0) \\
& y = y^{ask} - y^{bid} \\
& y \in \mathbb{R}^{n\times(m+1)} \\
& y^{ask}, y^{bid} \in \mathbb{R}^{n\times(m+1)}_+ \\
& z \in \mathbb{R},
\end{aligned} \tag{3.45}$$

so that solving (3.45) would now yield an upper bound on the current price of the basket option, which is what we seek.
We now work with (3.45). The way in which we solve (3.45) is to re-write it as an equivalent finite linear problem. To do this, we proceed as in [22].

Definition 3.46. *Let* $u, v \in \mathbb{R}^n$. *The* ***Hadamard product*** *of* u *and* v *is denoted by* $u \circ v$ *and is defined as*

$$u \circ v = (u_1 v_1, u_2 v_2, \ldots, u_n v_n)^T.$$

Therefore, the *Hadamard product* of two vectors results in a vector whose components are the multiplication of the original two vectors multiplied component-wise.

Observation: We note here that the *Hadamard product* is closely linked with the 'dot' product for vectors in the following way.

Let $u, v \in \mathbb{R}^n$ and $e \in \mathbb{R}^n$ be the vector whose components are all equal to 1.
Then,

$$(u \circ v) \cdot e = \sum_{i=1}^{n} (u \circ v)_i = \sum_{i=1}^{n} u_i v_i = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u \cdot v.$$

To present the next definition we set the following notational convention. Given a matrix $A \in \mathbb{R}^{n\times(m+1)}$, we define $A^i \in \mathbb{R}^n$ to be the vector which is the i^{th} column of the matrix A for $i = 0, 1, \ldots, (m-1), m$.

Definition 3.47. *We define $LSR(\bar{E}, \tilde{b}, c)$ to be the set of points $(y, z, \hat{\gamma}, \hat{\beta}, \xi) \in \mathbb{R}^{n\times(m+1)} \times \mathbb{R} \times \mathbb{R}_+^{n\times(m+1)} \times \mathbb{R}_+^{n\times m} \times \mathbb{R}^n$, which satisfy*

$$\begin{aligned} \sum_{j=0}^{m} y^j - \tilde{b} = \hat{\gamma}^i - \hat{\beta}^i, \quad i = 0, 1, \ldots, (m-1), \\ \sum_{j=0}^{m} y^j - \tilde{b} = \hat{\gamma}^m, \\ \sum_{j=0}^{i} \left(E^j \circ y^j\right) \leqslant \xi + \left(E^i \circ \hat{\gamma}^i\right) - \left(E^{i+1} \circ \hat{\beta}^i\right), \quad i = 0, 1, \ldots, (m-1), \\ \sum_{j=0}^{m} \left(E^j \circ y^j\right) \leqslant \xi + \left(E^m \circ \hat{\gamma}^m\right), \\ -z - c \leqslant -e \cdot \xi. \end{aligned} \tag{3.48}$$

Using these definitions we can state the main result of [22]. We note here that it is this result which allows us to re-write the SIO problem (3.45) as a finite and solvable LO problem. For the proof of this result we refer the reader to [22].

Theorem 3.49. *Assume that $0 = E^0 \leqslant E^1 \leqslant \cdots \leqslant E^m \in \mathbb{R}^n$ component-wise, and $\tilde{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are given. Then $(y, z) \in SR(\bar{E}, \tilde{b}, c)$ if and only if there exist $\hat{\gamma} \in \mathbb{R}_+^{n\times(m+1)}$, $\hat{\beta} \in \mathbb{R}_+^{(n\times m)}$, and $\xi \in \mathbb{R}^n$, such that $(y, z, \hat{\gamma}, \hat{\beta}, \xi) \in LSR(\bar{E}, \tilde{b}, c)$.*

Proof. See [22]. □

Thus, using Theorem 3.49 we can re-write the SIO problem (3.45) as a finite linear problem whose number of variables and constraints is proportional to $((m+1) \times n)$ [22]. This result can be summarised in the following corollary, taken from [22].

Corollary 3.50. *The super-replication problem* (3.45) *can equivalently be re-written as the following finite optimisation problem*

$$\begin{aligned} \min_{z,y,y^{ask},y^{bid},\hat{\gamma},\hat{\beta},\xi,\tilde{\gamma},\tilde{\beta},\tilde{\xi}} \quad & z + \sum_{j=0}^{m} (p^{ask^j} \cdot y^{ask^j} - p^{bid^j} \cdot y^{bid^j}) \\ \textit{subject to} \quad & (y, z, \hat{\gamma}, \hat{\beta}, \xi) \in LSR(\bar{E}, \omega, E) \\ & (y, z, \tilde{\gamma}, \tilde{\beta}, \tilde{\xi}) \in LSR(\bar{E}, 0, 0) \\ & y = y^{ask} - y^{bid} \\ & y^{ask}, y^{bid} \in \mathbb{R}_+^{n\times(m+1)}. \end{aligned} \tag{3.51}$$

Theorem 3.49 gives a stronger result for the special case when $m = 0$. That is, when we consider knowing *one* European vanilla option price per asset (and so one exercise price, per option, per asset) so that we know the current bid-ask prices of a total of n European vanilla call options. In this case, it is shown in [22] that Theorem 3.49 gives the following result.

Theorem 3.52. *Assume that* $0 = E^0 \leqslant E^1 = E' \in \mathbb{R}^n$ *and* $\tilde{b} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$ *are given. Then* $(y^0, y^1, z) \in SR(\bar{E}, \tilde{b}, c)$ *if and only if there exists* $\check{\gamma} \in \mathbb{R}^n_+$*, such that*

$$\check{\gamma} \geqslant y^0 - \tilde{b} \geqslant -y^1, \text{ and} - z - c \leqslant (y^0 - \tilde{b}) \cdot E' - (\check{\gamma} \cdot E').$$

Proof. See [22]. □

We notice here that the upper bound obtained in [22] for this particular case (for $m = 0$) using Theorem 3.52 is an alternative and more general derivation for the upper bound obtained in [20]. This is because in [20] we ignored bid-ask prices and used mid-market prices. The result obtained here for the $m = 0$ case is more general, and it may be shown that (see [22]) the bound obtained in [20] can be obtained here too as a special case when mid-market prices are assumed. That is, when we take the price to be the average of the bid-ask prices, so we use

$$p_{ij} = \frac{p_{ij}^{ask} + p_{ij}^{bid}}{2}, \text{ for all } i = 1, 2, \ldots, n \text{ and } j = 0 \iff p_i = \frac{p_i^{ask} + p_i^{bid}}{2} \text{ for all } i = 1, 2, \ldots, n.$$

Before concluding, we note here that in [22] only the upper bound problem is considered. The corresponding lower bound problem is more difficult and is open for future research.
That concludes presenting all existing models from literature which we consider in this thesis concerning the calculation of upper and lower bounds on the current price of a European basket call option.

3.7 Link with LSIO duality theory

Before we present the main sections of this thesis, we note here that in all of the existing models we have considered, we are interested in finding price bounds on a European basket call option. As we have seen, such an option has a piece-wise linear payoff given by, $\max(\omega^T S - E, 0)$. Furthermore, a standard European vanilla call option also has a piece-wise linear payoff given by, $\max(S_i - E_i, 0)$. Hence in all models, we are faced with solving an optimisation problem which is a dual to a piece-wise linear semi-infinite optimisation problem. We tackled this problem by transforming the linear SIO problem into an equivalent finite LO problem. In sub-section 2.2.4 we presented results from linear semi-infinite duality theory and we also considered when the optimal objective function values of the primal semi-infinite problem and its respective dual problem coincided. Recall that the underlying assumption behind all of these results was the assumption of a compact index set, $\mathcal{I}$. However, in the basket option pricing problem which we are considering, we have a non-compact index set $\mathcal{I} \equiv \mathbb{R}^n_+$, in the SIO problem. However, in sub-section 3.4 and in particular when considering finding an upper bound on the current price of a basket option, given that we know the current prices of q other European vanilla call options, per asset, we introduced Proposition 3.14. There we saw how to equivalently re-write the SIO problem with a non-compact index set $\mathbb{R}^n_+$, as a different SIO problem with one extra constraint and a compact index set $\mathcal{I}_{(3.15)}$. Thus, provided that the other assumptions from Theorem 2.31 are met (which all of the existing literature does), we may apply Theorem 2.31 to the SIO problem with the compact index set and conclude that the associated SIO problem and its dual problem have the same optimal objective function value, and so solving the linear SIO problem would also solve its corresponding dual problem, and so we would find an upper bound on the current price of the basket option in this way.
There are some other results that consider finding price bounds on basket options. In [24, 25, 26, 27, 28, 29, 30] the results are concerned with finding price bounds on options. In particular, in [24] upper bounds on a European basket call option are found using a model-independent approach to formulate a super-replicating

portfolio consisting of European vanilla call options whose exercise prices we may choose. In [25] upper bounds on general exotic options using the theory of integral stochastic orders and the theory of co-monotonic risks are derived. The method used here may of course be applied to basket options which are a type of exotic option. In [26], upper and lower bounds for derivatives written on two underlying assets are derived through finding a joint distribution under which the option price is equal to the hedging portfolio's value. The method used here may be applied to find price bounds on a basket option written on 2 underlying assets. In [27], upper and lower bounds on the current price of European basket call options for a general class of continuous-time financial models are derived. In [28] upper bounds on a European call option written on many underlying assets using a semi-parametric method are derived. The methodology introduced here may be used to find price bounds for European basket options. In [29] upper and lower model-independent bounds on various classes of exotic options using infinite-dimensional linear optimisation methods are derived. The methodology used here is similar to that as presented above in this section but for different types of options. In [30], bounds on the current prices of exotic options, in particular the lookback option using a model-independent approach are derived. The model set-up here is similar to that as presented in this section but for a different type of option. Further, in [31, 32, 33, 34, 35, 36] some results for the actual current prices of options are derived. In particular, in [31] an approximate price of a European basket call option using a jump-diffusion model is derived. In [32], an approximation formula for the current prices of European basket call options using a local-stochastic volatility model with jumps is derived. In [33], prices of European basket call options using simulation methods; including a Quasi- Monte Carlo method are derived. In [34], the price of a European basket call option using the reciprocal gamma distribution is derived. In [35], a new approach to valuing and hedging basket and spread options using log-normal distributions is derived. Finally, in [36], approximate basket option prices using a Bernoulli jump process is derived.
Now we present our own results on this topic. In particular we will see how to obtain bounds on the current price of various classes of basket options. The work that follows are the main sections of this thesis, and they are our own results. Any existing results which have been used to obtain our own, new and original results have been clearly cited and referenced.

4 Finding price bounds on European Basket Options

In this section we present our own new and original results regarding price bounds for European basket call options. Before proceeding to that however, we begin by noting that there are other results given in [37, 38] which aim to find lower bounds on the current price of a European basket call option using a different solution approach. In [37], lower bounds on the current price of European basket call options using a Fréchet copula approach are derived. In [38], a lower bound on the current price of a European basket call option under the Black-Scholes framework using a conditioning method is derived.
Our approach uses a different solution methodology so we present our own results for lower bounds on the current price of European basket call options.

4.1 A lower bound price result for European basket call options

We have seen previously, in Proposition 3.14 and in [11], that it is possible to equivalently re-write the SIO problem (3.13) as a different SIO problem with an extra constraint and a compact index set $\mathcal{I}_{(3.15)}$. However, this result is only valid when considering upper bounds on the current price of a basket option. The natural question that arises now is 'what happens for the lower bound?'. Unfortunately, the lower bound problem is more complex and presents different challenges which were not encountered in the upper bound problem. Thus, it is very difficult to obtain a general result in the case of the lower bound problem. However, under a set of very specific circumstances and assumptions it is possible to derive a similar result to Proposition 3.14 for the lower bound problem. This is done as follows.
Again, we consider the same setting as before. We consider finding a lower bound on the current price of a European basket call option written on n underlying assets. We assume that for each of the n underlying assets, the current prices of q European vanilla call options where the underlying is the asset itself are known. Further, we also assume that we know the current prices of each of the n underlying assets. This allows us to define the following notation for this sub-section.

- Let $\omega_i \in \mathbb{R}$ denote the weight of the i^{th} asset in the basket option, for $i = 1, 2, \ldots, n$.
- Let $S_i \in \mathbb{R}_+$ denote the price of the i^{th} underlying asset at expiry, for $i = 1, 2, \ldots, n$.
- Let $E \in \mathbb{R}_+$ be the exercise price of the basket option whose price we are bounding.
- Let $E_i^l \in \mathbb{R}_+$ denote the l^{th} exercise price of the i^{th} European vanilla call option whose current price we know and is given by $C_i^l \in \mathbb{R}_+$, for $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, n$.
- Let $S_i^0 \in \mathbb{R}_+$ denote the current price of the i^{th} underlying asset, for $i = 1, 2, \ldots, n$.

For convenience we adopt the following notational convention. We let $\omega \in \mathbb{R}^n$, $S \in \mathbb{R}_+^n$, $E^l \in \mathbb{R}_+^n$, $C^l \in \mathbb{R}_+^n$ and $S^0 \in \mathbb{R}_+^n$, for $l = 1, 2, \ldots, q$.
Also, for each $l = 1, 2, \ldots, q$ by writing $\max(S - E^l, 0) \in \mathbb{R}_+^n$, we mean this to be the vector with components $\max(S_i - E_i^l, 0)$ for $i = 1, 2, \ldots, n$.
Then, the problem of finding a lower bound on the current price of the European basket call option is given by the following problem (4.1), which is a dual to a linear SIO problem, where π is a risk-neutral probability measure to be found, $r > 0$ is the interest rate and T is the expiry date of **all** options, which is assumed to be the same.

$$\begin{aligned} \inf_{\pi} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(\omega^T S - E, 0)] \\ \text{subject to} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(S - E^l, 0)] = C^l, \text{for } l = 1, 2, \ldots, q \\ & \mathbb{E}_\pi[1] = 1 \\ & \mathbb{E}_\pi[\mathrm{e}^{-rT} S] = S^0. \end{aligned} \tag{4.1}$$

To solve (4.1) we appeal to the linear SIO problem for which (4.1) is its dual. This SIO problem is given by

$$\begin{aligned} \sup_{(u^1,u^2,\ldots,u^q,z,v)} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0)\mathrm{e}^{-rT} + z + v^T S\mathrm{e}^{-rT} \leqslant \mathrm{e}^{-rT}\max(\omega^T S - E, 0), \\ & \forall\ S \in \mathbb{R}^n_+. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \sup_{(u^1,u^2,\ldots,u^q,z,v)} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S \leqslant \max(\omega^T S - E, 0), \\ & \forall\ S \in \mathbb{R}^n_+. \end{aligned} \tag{4.2}$$

Now, we observe that the SIO problem has a natural financial interpretation. It aims to find the most expensive portfolio consisting of European vanilla call options, cash and the underlying assets themselves such that the overall value of the portfolio is always less than or equal to the payoff of the basket option. Furthermore, we may interpret the variables $(u^1, u^2, \ldots, u^q, z, v)$ as follows. The vectors $u^l \in \mathbb{R}^n$ for $l = 1, 2, \ldots, q$ give the amount of the l^{th} priced European vanilla call option that the investor holds. The variable z is the amount of cash that is in the portfolio and the vector $v \in \mathbb{R}^n$ represents the number or amount of underlying assets that have been invested in. We note here that for each u^l and v we have adopted the notational convention described previously.
The question of importance to us is, when does strong duality hold between (4.1) and (4.2)? We discuss this concept now.
Duality theory from semi-infinite optimisation indicates that strong duality would hold between (4.1) and (4.2) under some mild assumptions, (see Theorem 2.31), and the index set being compact. In our case the index set in (4.2) is $\mathbb{R}^n_+$, which is not compact since it is not a bounded set. Fortunately however, under some additional assumptions we can equivalently re-write (4.2) as a semi-infinite optimisation problem with a compact index set. Then using duality theory from SIO, strong duality would then hold between (4.1) and this 'new' semi-infinite problem with a compact index set. We formalise this approach in the next theorem. Before proceeding to that however, we note that the lower bound result is very specific and is here only proved under the assumptions as given in Theorem 4.3. This is in contrast to the upper bound result where no additional assumptions are needed and the problem can be re-written using a compact index set more generally.
Recalling that if the basket option yields a payoff of $\omega^T S - E$ at expiry, we term the option *in the money* and if the basket option yields a payoff of 0 at expiry we term the option *out of the money*, then for the lower bound case we restrict our attention to the specific case that when the basket option is *in the money* at expiry, the value of each weight ω_i is at least the total amount of the number of vanilla calls written on the underlying asset S_i, and the amount of the underlying S_i that is held, for all $i = 1, 2, \ldots, n$. Here, some u_i^l and/or v_i may be negative due to short selling, for $l = 1, 2, \ldots, q$ and for all $i = 1, 2, \ldots, n$. That is, we assume that,

$$\omega \geqslant \sum_{l=1}^{q} u^l + v.$$

When the basket option is *out of the money* at expiry, we assume that the investor has short sold at least one vanilla call option written on S_i, and/or the underlying asset S_i itself, implying that u_i^l and/or $v_i < 0$, for some $l = 1, 2, \ldots, q$ and for all $i = 1, 2, \ldots, n$; and the magnitude at which the short selling has been

done, is such that the overall holdings in the sub-replicating portfolio are non-positive.
That is, we assume

$$\sum_{l=1}^{q} u^l + v \leqslant 0.$$

We are now ready to present the following theorem.

Theorem 4.3. *Suppose that when the basket option has payoff equal to $\omega^T S - E$ at expiry, the position that the investor takes in the sub-replicating portfolio is such that*

$$\omega - \sum_{l=1}^{q} u^l - v \geqslant 0,$$

and assume that when the basket option has payoff equal to 0 at expiry, the position that the investor takes in the sub-replicating portfolio is such that

$$\sum_{l=1}^{q} u^l + v \leqslant 0.$$

Consider the problem

$$\begin{array}{ll} \underset{(u^1,u^2,\ldots,u^q,z,v)}{max} & \displaystyle\sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\ \textit{subject to} & \displaystyle\sum_{l=1}^{q} u^l + v \leqslant \max\{\omega, 0\} \\ & \displaystyle\sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \leqslant \max(\omega^T S - E, 0), \\ & \forall\ S \in \mathcal{I}_{(4.4)}. \end{array} \tag{4.4}$$

Then the semi-infinite optimisation problems (4.4), *and* (4.2) *are equivalent, in the sense that both problems have the same feasible set (and hence the same optimal solution and same optimal objective function value). Here* $\mathcal{I}_{(4.4)} = \bigtimes_{i=1}^{n} [0, E_i^q]$, *where we have assumed, without loss of generality that the exercise prices are arranged in a non-restrictive way such that* $0 \leqslant E_i^1 \leqslant \cdots \leqslant E_i^q$, *for all* $i = 1, 2, \ldots, n$.

Remarks

1. *We first remark here that the objective functions of* (4.2) *and* (4.4) *are the same. This is because the objective function in* (4.2) *represents the total cost of the sub-replicating portfolio at current time,* $t = 0$. *Since this is a cost which is paid, the maximal such cost is attained since what ever the maximum cost is, the investor pays it. Therefore, the* ***sup*** *in* (4.2) *is attained and can be replaced by* ***max*** *as done in* (4.4).

2. *The extra constraint in* (4.4) *also has a significant financial meaning.*
   The constraint

   $$\sum_{l=1}^{q} u^l + v \leqslant \max\{\omega, 0\},$$

   which is equivalent to

   $$\sum_{l=1}^{q} u_i^l + v_i \leqslant \max\{\omega_i, 0\},\ \forall\ i = 1, 2, \ldots, n,$$

means

(a) If the weight $\omega_i > 0$*, this implies* $\sum_{l=1}^{q} u_i^l + v_i \leqslant \omega_i$*, for all* $i = 1, 2, \ldots, n$*. This means that for a strictly positive weight* ω_i*, the total amount of European vanilla calls written on the* i^{th} *asset plus the amount of the* i^{th} *asset which we buy, is at most* ω_i*. Note that when the basket option expires in the money, we assume that this particular constraint holds, for all* $i = 1, 2, \ldots, n$*.*

(b) If the weight $\omega_i < 0$*, this implies* $\sum_{l=1}^{q} u_i^l + v_i \leqslant 0$*, for all* $i = 1, 2, \ldots, n$*. This means that for a strictly negative weight* ω_i*, the total amount of European vanilla calls written on the* i^{th} *asset plus the amount of the* i^{th} *asset which we hold is non-positive. This means that some or all of the European vanilla call options plus the underlying itself have been short sold. We note here that when the basket option expires out of the money, we assume that this particular constraint holds, for all* $i = 1, 2, \ldots, n$*.*

3. *We note here that the additional assumptions are consistent with the additional constraint*

$$\sum_{l=1}^{q} u^l + v \leqslant \max\{\omega, 0\}.$$

4. *When the assumptions are satisfied, the advantage of solving* (4.4) *is that the index set in the semi-infinite constraint of* (4.4) *is compact, albeit* (4.4) *containing additional constraints. Problem* (4.4) *allows us to solve* (4.2) *by considering the asset price values in the compact set* $\mathcal{I}_{(4.4)}$ *rather than* $\mathbb{R}_+^n$ *along with some restrictions on how the sub-replicating portfolio must be constructed. These restrictions are captured in the assumptions of the theorem and the extra constraints in* (4.4)*.*

5. *This theorem is only valid for the particular case when the stated assumptions are satisfied. This result does* ***not*** *hold for the lower bound in general, but it does hold for the particular case we are considering here. Thus, when the assumptions are satisfied we can use this result to obtain lower bounds on the current price of a European basket call option of interest.*

Proof of Theorem 4.3. Let $\mathcal{F}_{(4.2)}$ and $\mathcal{F}_{(4.4)}$ denote the feasible sets of (4.2) and (4.4), respectively. We show that $\mathcal{F}_{(4.2)} = \mathcal{F}_{(4.4)}$. The proof comes in two parts.

1. $\underline{\mathcal{F}_{(4.2)} \subset \mathcal{F}_{(4.4)}}$: Take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(4.2)}$. Then we show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(4.4)}$. We start by observing that, since $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(4.2)}$, then we have

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \leqslant \max(\omega^T S - E, 0) \tag{4.5}$$

holds for all $S \in \mathbb{R}_+^n$.

Now, $\mathcal{I}_{(4.4)} = \times_{i=1}^{n} [0, E_i^q]$ is just an n-dimensional 'rectangle' in $\mathbb{R}_+^n$ and so $\mathcal{I}_{(4.4)} \subset \mathbb{R}_+^n$. Therefore, since (4.5) holds for all $S \in \mathbb{R}_+^n$, it certainly holds for all $S \in \mathcal{I}_{(4.4)} \subset \mathbb{R}_+^n$ and so the second constraint from (4.4) holds. To show that the first constraint holds in (4.4) consider (4.5) again. Now (4.5) holds for all $S \in \mathbb{R}_+^n$. In particular, it holds for the vector S with $S_i = \eta$, for $\eta > 0$ and all other components equal to 0, for for all $i = 1, 2, \ldots, n$. That is, (4.5) holds for the asset price vector given by

$$S = \begin{pmatrix} 0 \\ 0 \\ \eta \\ \vdots \\ 0 \end{pmatrix}$$

where η is in the i^{th} position, for all $i = 1, 2, \ldots, n$.
Substituting this asset price vector into (4.5) gives

$$\sum_{l=1}^{q} \sum_{k=1}^{n} u_k^l \max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k \leqslant \max\left(\sum_{k=1}^{n} \omega_k S_k - E, 0 \right).$$

Since for any $i = 1, 2, \ldots, n$, $S_i = \eta$ and all other components are 0,
it follows that

$$\sum_{l=1}^{q} u_i^l \max(\eta - E_i^l, 0) + z\mathrm{e}^{rT} + v_i \eta \leqslant \max(\omega_i \eta - E, 0), \ \forall\ i = 1, 2, \ldots, n,$$

which is equivalent to

$$\sum_{l=1}^{q} u_i^l \eta \max\left(1 - \frac{E_i^l}{\eta}, 0\right) + z\mathrm{e}^{rT} + v_i \eta \leqslant \eta \max\left(\omega_i - \frac{E}{\eta}, 0\right), \quad \forall\ i = 1, 2, \ldots, n.$$

Dividing by $\eta > 0$, we obtain

$$\sum_{l=1}^{q} u_i^l \max\left(1 - \frac{E_i^l}{\eta}, 0\right) + \frac{z\mathrm{e}^{rT}}{\eta} + v_i \leqslant \max\left(\omega_i - \frac{E}{\eta}, 0\right), \ \forall\ i = 1, 2, \ldots, n.$$

Taking the limit as $\eta \to \infty$, we have that $\frac{E_i^l}{\eta} \to 0$, $\frac{z\mathrm{e}^{rT}}{\eta} \to 0$ and $\frac{E}{\eta} \to 0$. This gives

$$\sum_{l=1}^{q} u_i^l \max(1, 0) + v_i \leqslant \max(\omega_i, 0), \ \forall\ i = 1, 2, \ldots, n,$$

$$\iff \sum_{l=1}^{q} u^l + v \leqslant \max\{\omega, 0\},$$

and so the first constraint from (4.4) holds too.

$$\therefore \mathcal{F}_{(4.2)} \subset \mathcal{F}_{(4.4)}.$$

2. $\underline{\mathcal{F}_{(4.4)} \subset \mathcal{F}_{(4.2)}}$: To show the converse, take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(4.4)}$. Then we show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(4.2)}$.
Now, if we can show that

$$\min_{S \in \mathbb{R}_+^n} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\} \tag{4.6}$$

$$= \min_{S \in \mathcal{I}_{(4.4)}} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\},$$

then we are done because,

$$\max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \geqslant 0, \ \forall\ S \in \mathcal{I}_{(4.4)},$$

$$\text{implies,} \ \min_{S \in \mathcal{I}_{(4.4)}} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S \right\} \geqslant 0$$

and so if (4.6) holds, then

$$\min_{S \in \mathbb{R}^n_+} \left\{ \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z e^{rT} - v^T S \right\} \geqslant 0,$$

$$\text{which implies, } \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z e^{rT} - v^T S \geqslant 0, \ \forall\, S \in \mathbb{R}^n_+,$$

that is, $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(4.2)}$.
Now we show that (4.6) is indeed true.
Define $\psi(S) = \max(\omega^T S - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z e^{rT} - v^T S$, and consider $S \notin \mathcal{I}_{(4.4)}$.
Then we have the following two cases for $\frac{d\psi}{dS}$.

__Case 1__: The basket option expires in the money so that the payoff is given by $\max(\omega^T S - E, 0) = \omega^T S - E > 0$,

$$\implies \frac{d\psi}{dS} = \omega - \sum_{l=1}^{q} u^l - v \geqslant 0,$$

by supposition.
__Case 2__: The basket option expires out of the money so that the payoff is given by $\max(\omega^T S - E, 0) = 0$,

$$\implies \frac{d\psi}{dS} = -\sum_{l=1}^{q} u^l - v \geqslant 0,$$

by supposition.
Hence in either case, we have that $\frac{d\psi}{dS} \geqslant 0$, for all $S \notin \mathcal{I}_{(4.4)}$. That is, $\psi(S)$ is non-decreasing for all $S \notin \mathcal{I}_{(4.4)}$ and so the minimum of $\psi(S)$ is for sure attained in $\mathcal{I}_{(4.4)}$.
Therefore, (4.6) holds true, and the proof is complete. □

We observe here that in the proof of our Theorem 4.3 we have employed a similar methodology to the proof in Proposition 3.14 in sub-section 3.4. We have considered the same model set up as that described in sub-section 3.4 but for the lower bound problem.
We note here that due to the disjunctive structure of the underlying feasible set in the SIO problem (4.4), it is still very difficult to solve, albeit the problem having a compact index set. Nevertheless, our derived theorem, which allows us to re-write the SIO problem (4.2) with a non-compact index set $\mathbb{R}^n_+$, as the SIO problem (4.4) which has a compact index set is a start and a step in the right direction to solve this more difficult problem. The solution techniques to solve (4.4) remain open for further research, but even though we have not fully solved the problem we have derived a theorem which should help in obtaining the final solution; especially since known, standard techniques can be more readily applied to a SIO problem with a compact index set in comparison to a SIO problem with a non-compact index set.

Now we present a lower bound result for European basket call options using current *bid-ask prices* of other European basket call options.

4.2 A lower bound derived with bid-ask prices

When finding price bounds on a European basket option, we have seen the unrealistic assumption of knowing current mid-market prices. One way to overcome this assumption, is to incorporate *bid-ask prices* within the optimisation model. In this sub-section we consider one way to find a lower bound on the current price of a European basket call option, given that we know the current bid-ask prices of other basket options. We have mentioned that this was done in [21]. Here, we a present a new solution approach also for the lower

bound.
We consider an optimisation model similar to, but **not** identical to the model given in [21]. We then employ a similar solution approach to [19], where an upper bound on the current basket option price was found, but neglecting bid-ask prices.
We consider calculating a lower bound on the current price of a European basket call option written on n underlying assets, given that we know the current bid-ask prices of r other European basket call options, written on the same n underlying assets. Furthermore, we also incorporate a positive interest rate, $r > 0$ into the model.
For convenience, we will adopt the following notation for the remainder of this sub-section.

- Let $\omega^0 \in \mathbb{R}^n_+$ denote the vector of the weights of the basket option whose price we are bounding.
- Let $E_0 \in \mathbb{R}_+$ be the exercise price of the basket option whose price we are bounding.
- Let $S \in \mathbb{R}^n_+$ denote the vector of the asset prices at expiry, T.
- Let $\omega^j \in \mathbb{R}^n_+$ denote the vector of the weights for the j^{th} basket option whose current bid-ask price we know, for $j = 1, 2, \ldots, r$.
- Let $E_j \in \mathbb{R}_+$ be the exercise price of the j^{th} basket option whose current bid-ask price we know, for $j = 1, 2, \ldots, r$.
- Let p_j^{ask}, $p_j^{bid} \in \mathbb{R}_+$ denote the current, known ask, bid prices of the j^{th} basket option, respectively, for $j = 1, 2, \ldots, r$. Here we have that $p_j^{ask} \geqslant p_j^{bid}$, for all $j = 1, 2, \ldots, r$.

Then, if π denotes a risk-neutral probability measure, and the risk-free interest rate is given by $r > 0$, the task of finding a lower bound on the current price of a European basket call option is given by

$$\begin{aligned}
\inf_{\pi} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(\omega^0 \cdot S - E_0, 0)] \\
\text{subject to} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(\omega^j \cdot S - E_j, 0)] \leqslant p_j^{ask}, \quad \text{for } j = 1, 2, \ldots, r \\
& \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(\omega^j \cdot S - E_j, 0)] \geqslant p_j^{bid}, \quad \text{for } j = 1, 2, \ldots, r \\
& \mathbb{E}_\pi[1] = 1 \\
& \pi \text{ is a probability measure in } \mathbb{R}^n_+.
\end{aligned} \tag{4.7}$$

Then, using the definition of duality, we may find the linear SIO problem for which (4.7) is its dual as

$$\begin{aligned}
\sup_{z, y^{ask}, y^{bid}} \quad & z + \sum_{j=1}^{r} \left(p_j^{ask} y_j^{ask} - p_j^{bid} y_j^{bid}\right) \\
\text{subject to} \quad & z + \sum_{j=1}^{r} y_j^{ask}(\mathrm{e}^{-rT} \max(\omega^j \cdot S - E_j, 0)) - \sum_{j=1}^{r} y_j^{bid}(\mathrm{e}^{-rT} \max(\omega^j \cdot S - E_j, 0)) \leqslant \\
& \mathrm{e}^{-rT} \max(\omega^0 \cdot S - E_0, 0) \quad \forall\, S \in \mathbb{R}^n_+ \\
& y^{ask}, y^{bid} \in \mathbb{R}^r_+, \; z \in \mathbb{R}.
\end{aligned}$$

This problem is equivalent to

$$\begin{aligned}
\sup_{z, y^{ask}, y^{bid}} \quad & z + \sum_{j=1}^{r} \left(p_j^{ask} y_j^{ask} - p_j^{bid} y_j^{bid}\right) \\
\text{subject to} \quad & z\mathrm{e}^{rT} + \sum_{j=1}^{r} (y_j^{ask} - y_j^{bid})(\max(\omega^j \cdot S - E_j, 0)) \leqslant \max(\omega^0 \cdot S - E_0, 0) \quad \forall\, S \in \mathbb{R}^n_+ \\
& y^{ask}, y^{bid} \in \mathbb{R}^r_+, \; z \in \mathbb{R}.
\end{aligned}$$

Defining $y \in \mathbb{R}^r$ as $y = y^{ask} - y^{bid}$, we may equivalently re-write the above SIO problem as

$$\begin{aligned}
\sup_{z,y,y^{ask},y^{bid}} \quad & z + \sum_{j=1}^{r} \left(p_j^{ask} y_j^{ask} - p_j^{bid} y_j^{bid}\right) \\
\text{subject to} \quad & z e^{rT} + \sum_{j=1}^{r} y_j(\max(\omega^j \cdot S - E_j, 0)) \leqslant \max(\omega^0 \cdot S - E_0, 0) \quad \forall\, S \in \mathbb{R}_+^n \\
& y = y^{ask} - y^{bid} \\
& y \in \mathbb{R}^r,\ y^{ask}, y^{bid} \in \mathbb{R}_+^r,\ z \in \mathbb{R}.
\end{aligned} \tag{4.8}$$

Therefore the linear SIO problem for which (4.7) is its dual is (4.8).
We assume that strong duality holds between (4.7) and (4.8). That is, we assume that the conditions of the following lemma, taken from [19] but incorporating bid-ask prices, are satisfied.

Lemma 4.9 ([19], Proposition 2.1)**.** *The optimal values of* (4.7) *and* (4.8) *coincide if at least one of the following two conditions holds.*
(i) Strict primal feasibility.

$$\begin{pmatrix} 1 \\ p^{ask} \\ p^{bid} \end{pmatrix} \in int\left(\left\{\begin{pmatrix} \mathbb{E}_\pi[1] \\ \left(\mathbb{E}_\pi[e^{-rT}\max(\omega^j \cdot S - E_j, 0)]\right)_{j=1,2,\ldots,r} \\ \left(\mathbb{E}_\pi[e^{-rT}\max(\omega^j \cdot S - E_j, 0)]\right)_{j=1,2,\ldots,r} \end{pmatrix} : \pi \text{ is a distribution in } \mathbb{R}_+^n\right\}\right).$$

In particular, strong duality holds provided the bid-ask prices, p^{ask} and p^{bid}, are arbitrage free and remain arbitrage free after slight perturbations.
(ii) Strict dual feasibility.
There exists $(\hat{z}, \hat{y})^T \in \mathbb{R}^{r+1}$ such that

$$(\hat{z}, \hat{y}) \in int\left(\left\{(z, y)^T \in \mathbb{R}^{r+1} :\ z e^{rT} + \sum_{j=1}^{r} y_j(\max(\omega^j \cdot S - E_j, 0)) \leqslant \max(\omega^0 \cdot S - E_0, 0) \quad \forall\, S \in \mathbb{R}_+^n\right\}\right).$$

In particular, strong duality holds provided that, for each asset, at least one vanilla option price is known.

The goal of this sub-section is to re-write the semi-infinite optimisation problem (4.8) as a finite linear problem. In order to do this, we employ a similar methodology to what was done in [19] and in sub-section 3.5. We note that our result is different because it deals with the lower bound problem by incorporating bid-ask prices. Further, when re-writing the semi-infinite constraint from problem (4.8) as a set of finite linear constraints, we have that our finite linear constraints consist only of inequality constraints and no equality constraints as was derived in [19] and sub-section 3.5. We note here that the results presented in this sub-section are new in the sense that modelling the problem as (4.7) and solving its specific associated linear SIO problem (4.8) using this technique has not been done before.
We start by setting the following notational convention.

- Let Ω denote the $(r \times n)$ matrix whose j^{th} row is the vector $(\omega^j)^T$, for $j = 1, 2, \ldots, r$.
- Let $\bar{\Omega}$ be the $((r+1) \times n)$ matrix whose j^{th} row is the vector $(\omega^j)^T$, for $j = 0, 1, \ldots, r$.
- Let $\tilde{E} \in \mathbb{R}_+^r$ be the vector $(E_1, E_2, \ldots, E_r)^T$.
- Let $\hat{E} \in \mathbb{R}_+^{(r+1)}$ be the vector $(E_0, E_1, \ldots, E_r)^T$.
- Let I be a finite index set with $J \subseteq I$. Define a vector $\bar{\nu} \in \mathbb{R}^{|I|}$. Then by $\bar{\nu}_J \in \mathbb{R}^{|J|}$ we mean the vector formed by the entries $\bar{\nu}_j$, for $j \in J$.

- Similarly, if the rows of a matrix, $\bar{\Lambda}$ whose indices belong to the set I, then by $\bar{\Lambda}_J$ we mean the matrix formed by the rows of $\bar{\Lambda}$ whose indices $j \in J$.
- Finally, let J' denote the set $I \backslash J$ and the index set I will be equal to $\{0, 1, \ldots, r\}$.

Now we are ready to transform (4.8). To start, we define the following sets.
Let $J \subseteq \{0, 1, \ldots, r\}$. We define $\mathcal{P}_J$ as

$$\mathcal{P}_J = \{S : \bar{\Omega}_J S \geqslant \hat{E}_J,\ \bar{\Omega}_{J'} S \leqslant \hat{E}_{J'},\ S \geqslant 0\},$$

and let

$$\bar{\mathcal{J}} = \{J \subseteq \{0, 1, \ldots, r\} : \mathcal{P}_J \neq \varnothing\}.$$

Then we will show that (4.8) can equivalently be re-written as a finite linear problem, which may be solved by an appropriate software program to yield the optimal objective function value which is a lower bound on the current price of the basket call option of interest.
We first need the following lemma which applies the same techniques as in [19], Lemma 3 for upper bounds.

Lemma 4.10. *Let $\bar{\Omega} \in \mathbb{R}^{((r+1)\times n)}, \hat{E} \in \mathbb{R}_+^{(r+1)}$ and let $J \subseteq \{0, 1, \ldots, r\}$ be arbitrarily chosen and fixed. Denote the set $\mathcal{P}_J = \mathcal{P}_J(\bar{\Omega}, \hat{E})$ as above. If $\mathcal{P}_J \neq \varnothing$ then for $\hat{\psi} \in \mathbb{R}^{(r+1)}, a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ we have*

$$\hat{\psi}^T \max(\bar{\Omega} S - \hat{E}, 0) \leqslant a^T S - b, \quad \textit{for all } S \in \mathcal{P}_J, \tag{4.11}$$

if and only if there exist $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$ such that

$$(\bar{\Omega}_J)^T \hat{\psi}_J - a \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'} \ \textit{and}\ \hat{\psi}_J^T \hat{E}_J - b \geqslant -(\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'}. \tag{4.12}$$

Proof. For all $S \in \mathcal{P}_J$, we have

$$\hat{\psi}^T \max(\bar{\Omega} S - \hat{E}, 0) - a^T S + b = \hat{\psi}_J^T (\bar{\Omega}_J S - \hat{E}_J) - a^T S + b \quad = ((\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S - (\hat{\psi}_J^T \hat{E}_J - b),$$

since

$$\bar{\Omega}_J S - \hat{E}_J \geqslant 0, \text{ and } \bar{\Omega}_{J'} S - \hat{E}_{J'} \leqslant 0.$$

So, if we consider the linear problem

$$\begin{aligned} \max_S \quad & ((\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S \\ \text{subject to} \quad & -\bar{\Omega}_J S \leqslant -\hat{E}_J \\ & \bar{\Omega}_{J'} S \leqslant \hat{E}_{J'} \\ & S \geqslant 0, \end{aligned} \tag{4.13}$$

it follows that (4.11) holds if and only if the optimal value of the linear problem (4.13) is at most $(\hat{\psi}_J^T \hat{E}_J - b)$, because if this is the case, then

$$\begin{aligned} & \max_{S \in \mathcal{P}_J} \{(\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S\} \leqslant \hat{\psi}_J^T \hat{E}_J - b \\ \Longrightarrow \ & ((\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S \leqslant \hat{\psi}_J^T \hat{E}_J - b \quad \forall\ S \in \mathcal{P}_J. \\ \Longleftrightarrow \ & ((\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S - \hat{\psi}_J^T \hat{E}_J + b \leqslant 0, \quad \forall\ S \in \mathcal{P}_J. \end{aligned}$$

This means that $\hat{\psi}^T \max(\bar{\Omega} S - \hat{E}, 0) - a^T S + b \leqslant 0, \quad \forall\ S \in \mathcal{P}_J,$

so that (4.11) holds.
Now, by linear optimisation duality, the dual to (4.13) is obtained as follows. Define the dual variables $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$. Then the dual to (4.13) is given by

$$\begin{aligned} \min_{\gamma^J, \beta^{J'}} \quad & -(\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \\ \text{subject to} \quad & -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'} \geqslant (\bar{\Omega}_J)^T \hat{\psi}_J - a \\ & \gamma^J \in \mathbb{R}_+^{|J|} \\ & \beta^{J'} \in \mathbb{R}_+^{|J'|}. \end{aligned} \tag{4.14}$$

Now, by the weak duality theorem, we have that

$$\max_{S \in \mathcal{P}_J} \{((\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S\} \leqslant \min_{\gamma^J, \beta^{J'}} \{-(\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'}\}, \tag{4.15}$$

but if we impose the condition

$$-(\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \leqslant \hat{\psi}_J^T \hat{E}_J - b,$$

then by (4.15) we have

$$\max_{S \in \mathcal{P}_J} \{((\bar{\Omega}_J)^T \hat{\psi}_J - a)^T S\} \leqslant \hat{\psi}_J^T \hat{E}_J - b,$$

and since the constraint in (4.14) must hold true, we have that the optimal value of the linear problem (4.13) is at most $\hat{\psi}_J^T \hat{E}_J - b$ if and only if there exists $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$ such that (4.12) holds and the proof is complete. □

This leads us to the main result of this sub-section, which is summarised in the next proposition.

Proposition 4.16. *(i) The SIO problem* (4.8) *can equivalently be re-written as the following finite linear problem*

$$\begin{aligned} \max_{z, y, y^{ask}, y^{bid}} \quad & z + (p^{ask})^T y^{ask} - (p^{bid})^T y^{bid} \\ \textit{subject to} \quad & (\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ y \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'}, \quad J \in \bar{\mathcal{J}} \\ & z e^{rT} - \begin{pmatrix} -1 \\ y \end{pmatrix}_J^T \hat{E}_J \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'}, \quad J \in \bar{\mathcal{J}} \\ & y = y^{ask} - y^{bid} \\ & y \in \mathbb{R}^r, \ y^{ask} \in \mathbb{R}_+^r, \\ & y^{bid} \in \mathbb{R}_+^r, \ z \in \mathbb{R}, \\ & \gamma^J \in \mathbb{R}_+^{|J|}, \ \ \beta^{J'} \in \mathbb{R}_+^{|J'|}, \quad J \in \bar{\mathcal{J}}. \end{aligned} \tag{4.17}$$

(ii) In particular, if $0 \in J$*, the finite linear problem* (4.17) *can be written as*

$$
\begin{aligned}
\max_{z, y^{ask}, y^{bid}} \quad & z + \sum_{j=1}^{r} (p_j^{ask} y_j^{ask} - p_j^{bid} y_j^{bid}) \\
\textit{subject to} \quad & \sum_{\substack{j \in J \\ j \neq 0}} (\omega^j y_j^{ask} - \omega^j y_j^{bid}) + \bar{\Omega}_J^T \gamma^J - \bar{\Omega}_{J'}^T \beta^{J'} \leqslant \omega^0, \quad J \in \bar{\mathcal{J}} \\
& z e^{rT} + \sum_{\substack{j \in J \\ j \neq 0}} (-y_j^{ask} E_j + y_j^{bid} E_j) - (\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \leqslant -E_0, \quad J \in \bar{\mathcal{J}} \\
& y^{ask} \in \mathbb{R}_+^r, \ y^{bid} \in \mathbb{R}_+^r \\
& z \in \mathbb{R}, \ \gamma^J \in \mathbb{R}_+^{|J|}, \ \ \beta^{J'} \in \mathbb{R}_+^{|J'|}, \quad J \in \bar{\mathcal{J}}.
\end{aligned}
\tag{4.18}
$$

Proof. The proof comes in two parts.
(i) We start by observing that the objective function in (4.8) is the same as the objective function in (4.17). This is because the objective function in (4.8) represents the total cost of the sub-replicating portfolio consisting of cash and other European basket call options. The maximal total cost is a cost which is paid by the investor. Therefore, the optimal objective function value of (4.8) is attained and the **sup** in (4.8) can be replaced by **max** as done in (4.17).
For the constraints of (4.17) we have the following.
Recall the constraint from (4.8) as

$$z e^{rT} + \sum_{j=1}^{r} y_j (\max(\omega^j \cdot S - E_j, 0)) \leqslant \max(\omega^0 \cdot S - E_0, 0) \quad \forall\, S \in \mathbb{R}_+^n.$$

This is equivalent to

$$-\max((\omega^0)^T S - E_0, 0) + y^T \max(\Omega S - \tilde{E}, 0) \leqslant -z e^{rT}, \quad \forall\, S \ \in \mathcal{P}_J, \ J \in \bar{\mathcal{J}},$$

which is the same as

$$\begin{pmatrix} -1 \\ y \end{pmatrix}^T \max(\bar{\Omega} S - \hat{E}, 0) \leqslant -z e^{rT}, \quad \forall\, S \ \in \mathcal{P}_J, \ J \in \bar{\mathcal{J}}.$$

Now we use Lemma 4.10 with the following

$$\hat{\psi} = \begin{pmatrix} -1 \\ y \end{pmatrix} \ a = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \text{ and } b = z e^{rT}.$$

Then this means

$$\begin{pmatrix} -1 \\ y \end{pmatrix}^T \max(\bar{\Omega} S - \hat{E}, 0) \leqslant -z e^{rT}, \quad \forall\, S \ \in \mathcal{P}_J, \ J \in \bar{\mathcal{J}},$$

if and only if there exists $\gamma^J \in \mathbb{R}_+^{|J|}$ and $\beta^{J'} \in \mathbb{R}_+^{|J'|}$ such that

$$(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ y \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'},$$

and

$$\begin{pmatrix} -1 \\ y \end{pmatrix}_J^T \hat{E}_J - z e^{rT} \geqslant -(\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'},$$

for $J \in \bar{\mathcal{J}}$.
The latter inequality is equivalent to

$$ze^{rT} - \begin{pmatrix} -1 \\ y \end{pmatrix}_J^T \hat{E}_J \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'},$$

for $J \in \bar{\mathcal{J}}$. This gives the constraints

$$(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ y \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'}, \ J \in \bar{\mathcal{J}}$$

and

$$ze^{rT} - \begin{pmatrix} -1 \\ y \end{pmatrix}_J^T \hat{E}_J \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'}, \ J \in \bar{\mathcal{J}},$$

and so (i) is proved.
(ii) We start by observing that the objective functions of (4.17) and (4.18) are the same.
For the constraints we have the following.
From (4.17) we have

$$(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ y \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'}, \ J \in \bar{\mathcal{J}}.$$

Since $0 \in J$, this is equivalent to

$$-\omega^0 + (\Omega_J)^T y_J + \bar{\Omega}_J^T \gamma^J - \bar{\Omega}_{J'}^T \beta^{J'} \leqslant 0, \ J \in \bar{\mathcal{J}},$$

which is the same as

$$\sum_{\substack{j \in J \\ j \neq 0}} (\omega^j y_j) + \bar{\Omega}_J^T \gamma^J - \bar{\Omega}_{J'}^T \beta^{J'} \leqslant \omega^0, \ J \in \bar{\mathcal{J}}.$$

Now substitute in $y_j = y_j^{ask} - y_j^{bid}$, to get

$$\sum_{\substack{j \in J \\ j \neq 0}} \left(\omega^j (y_j^{ask} - y_j^{bid})\right) + \bar{\Omega}_J^T \gamma^J - \bar{\Omega}_{J'}^T \beta^{J'} \leqslant \omega^0, \ J \in \bar{\mathcal{J}}.$$

This is equivalent to

$$\sum_{\substack{j \in J \\ j \neq 0}} (\omega^j y_j^{ask} - \omega^j y_j^{bid}) + \bar{\Omega}_J^T \gamma^J - \bar{\Omega}_{J'}^T \beta^{J'} \leqslant \omega^0, \ \ J \in \bar{\mathcal{J}},$$

and so the first constraint of (4.18) is proved.
For the second constraint we have the following.
Recall from (4.17) the constraint

$$ze^{rT} - \begin{pmatrix} -1 \\ y \end{pmatrix}_J^T \hat{E}_J \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'}, \ \ J \in \bar{\mathcal{J}}.$$

Since $0 \in J$, this is equivalent to

$$ze^{rT} + E_0 - y_J^T \tilde{E}_J - (\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \leqslant 0, \ \ J \in \bar{\mathcal{J}},$$

which is the same as

$$ze^{rT} + \sum_{\substack{j \in J \\ j \neq 0}} (-y_j E_j) - (\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \leqslant -E_0, \ \ J \in \bar{\mathcal{J}}.$$

Now substitute in $y_j = y_j^{ask} - y_j^{bid}$, to get

$$z\mathrm{e}^{rT} + \sum_{\substack{j \in J \\ j \neq 0}} \left(-E_j(y_j^{ask} - y_j^{bid})\right) - (\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \leqslant -E_0, \;\; J \in \bar{\mathcal{J}}.$$

This is equivalent to

$$z\mathrm{e}^{rT} + \sum_{\substack{j \in J \\ j \neq 0}} (-y_j^{ask} E_j + y_j^{bid} E_j) - (\gamma^J)^T \hat{E}_J + (\beta^{J'})^T \hat{E}_{J'} \leqslant -E_0, \;\; J \in \bar{\mathcal{J}},$$

and so the second constraint from (4.18) also holds, and the proof is complete. □

We may now solve the *finite* linear problem (4.17) (instead of the semi-infinite problem (4.8)), to yield the optimal objective function value and hence a lower bound on the current price of a European basket call option.
To conclude this sub-section we make the following observation about the size of the linear problem (4.17). We observe that in (4.17) we have a total of

$$1 + r + r + |J| + |J'| = 1 + 2r + r + 1 = 3r + 2$$

variables.
For the constraints we have the following. Observe that each constraint requires $J \in \bar{\mathcal{J}}$, where $J \subseteq \{0, 1, \ldots, r\}$. However, for some $J \subseteq \{0, 1, \ldots, r\}$, $\mathcal{P}_J = \varnothing$ and $J \notin \bar{\mathcal{J}}$ even though $J \subseteq \{0, 1, \ldots, r\}$. Thus, we can obtain upper bounds on the size, or amounts of the constraints of the linear problem (4.17). In particular, for the set $\{0, 1, 2, \ldots, r\}$ there exists a total of 2^{r+1} subsets. Since $J \subseteq \{0, 1, \ldots, r\}$ we have the following. The first constraint in (4.17) actually represents n constraints. Therefore the first constraint can give at most $n(2^{r+1})$ constraints. Similarly, the second constraint in (4.17) may yield at most 2^{r+1} constraints.
Hence in total the number of constraints in (4.17) has the following upper bound

$$n(2^{r+1}) + 2^{r+1} = 2^{r+1}(n + 1).$$

Thus, the total number of constraints of (4.17) is at most $2^{r+1}(n + 1)$.
Although the number of constraints depends exponentially on r, and for large values of r, (4.17) may become large, it is still finite and we have removed the problem of having infinitely many constraints as in (4.8). Further, from a practical point of view, we may choose r so that (4.17) remains solvable via an appropriate linear optimisation software solver. Also, when solving the optimisation problem (4.17), we do not know today which subsets of J are in $\bar{\mathcal{J}}$. That is, we do not currently know whether the European basket call option whose current price we are bounding and the basket options whose current bid-ask prices we know will expire in/out of the money. In order to solve (4.17) to obtain a lower bound on the current price of the European basket call option of interest; we take **all** subsets of J to be in $\bar{\mathcal{J}}$ and solve (4.17) in this way. We note here that the above inequality regarding the number of constraints of problem (4.17) in this case holds as an equality.
We observe here that the model set-up and approach to finding a lower bound on the price of a European basket call option presented above is similar to the model set-up presented in sub-section 3.5. Further, the methodology employed in the proof of Lemma 4.10, to obtain a finite and solvable LO problem can be viewed as an extension to the methodology to obtain a finite and solvable LO problem employed in the proof of Proposition 3.26 in sub-section 3.5. It is an extension because in the proof of Lemma 4.10 we obtained a lower bound on the price of a European basket call option by incorporating bid-ask prices in the model.
That concludes this sub-section on considering finding lower bounds on the current price of a European basket call option given that we know the current bid-ask prices of other basket options.

5 Finding price bounds on American Basket Options

In all of the analysis which we have considered so far, we have only seen how to find price bounds on *European* basket call options. In this section we consider an extension to the European basket option pricing problem. Namely, we consider how to find current price bounds on *American* basket options.

5.1 Introduction

We start by recalling what is meant by an *American* option. Put simply an American option is the same as a European option but with the additional property that an American option can be exercised at *any* point in time before or on the expiry date. We observe that if an American option is **not** exercised before the expiry date, but is exercised **on** the expiry date then this 'American' option becomes identical to a corresponding European option.

Since American options have this extra property of *early exercise*, then the corresponding current price of the option should take this extra property into account. As such, intuitively it is clear that the current price of an American option, in general, is not the same as its European option counter-part.

In fact, we may observe the following link between the current prices of European and American options. An American option gives the holder the choice of exercising the option early. In turn, this means that the holder has more freedom and choice as to when to exercise the option. Thus, it is possible for the holder to obtain a large pay-out at a time before expiry that suits them. The current price of the American option must take this early exercise property into account. Since the writer has a higher likelihood of a larger payout in comparison to a European option, then the current price of an American option should **never** be less than its corresponding European option counterpart. That is, for a European option and an American option written on the same underlying assets, with the same exercise price and same expiry dates; the current price of the European option provides a lower bound to the current price of the corresponding American option. More formally, if the current price of a European option is given by V and the current price of a corresponding American option is given by $V_{\mathcal{A}}$, then the relation

$$V \leqslant V_{\mathcal{A}}$$

holds. For a more in-depth explanation, including a no-arbitrage argument on why the above inequality must hold for corresponding pairs of European and American options we refer the reader to the relevant chapters given in [2].

The question of interest to us is, 'how can we accurately find price bounds on an American basket option?' From above, we are already aware that a lower bound on the current price of an American basket option is given by its corresponding European basket option current price. Thus, we now investigate if and how we may apply similar ideas and methodologies from Section 3 to obtain an upper bound on the current price of an American basket option of interest.

Before proceeding, we note here that the results presented in this section are our own results and are new and original results. Previous and old results which have been used to obtain our new results have been clearly referenced.

We start by considering finding bounds on the current price of an American basket call option and then turn our attention to American basket put options.

Thus, in all that follows we retain the same notation from sub-section 2.1 and define the following.

- Let $C_{\mathcal{A}}(t)$ denote the price of an American basket call option at time t, for $t \in [0, T]$.
- Set $C_{\mathcal{A}}(0) = C_{\mathcal{A}}$, so that $C_{\mathcal{A}}$ is the current price of an American basket call option.
- Let $P_{\mathcal{A}}(t)$ denote the price of an American basket put option at time t, for $t \in [0, T]$.
- Set $P_{\mathcal{A}}(0) = P_{\mathcal{A}}$, so that $P_{\mathcal{A}}$ denotes the current price of an American basket put option.
- Let $S(t)$ be the vector of the prices of the n underlying assets on which the basket option is written on at time t, for $t \in [0, T]$.

- Set $S = S(T)$, which is the vector of the prices of the n underlying assets at expiry, $t = T$.
- Furthermore, let $C(t)$ denote the price of a corresponding European basket call option, at time t, for $t \in [0, T]$.
- Let $P(t)$ denote the price of a corresponding European basket put option, at time t, for $t \in [0, T]$.
- These European calls and puts are written on the same underlying assets, with the same weights and same exercise price.
- Also set $C(0) = C$ and $P(0) = P$, which are the current prices of the European call and put basket options, respectively .

Now, since we are considering American basket options, we denote the time of early exercise (if it is done, of course) by $\tilde{t}$, where $0 \leqslant \tilde{t} < T$. That is, we assume that $\tilde{t}$ is a point in time such that $0 \leqslant \tilde{t} < T$ and if early exercise is done, it is done at time $t = \tilde{t}$.
Now we are in a position to consider methods to find upper and lower bounds on the current price of American basket options of interest.
Before proceeding we note here that there do exist other results in [39, 40] which find prices and price bounds, respectively of various types of American options. In particular, in [39], the prices of American basket options using a multi-GPU adaptation of a specific Monte-Carlo based method is derived. In [40], upper bounds on the current prices of general American options using linear semi-infinite optimisation are derived. However, here a different solution methodology is employed and this paper does not specifically cover basket options.
Our new and original results, which we present now; use a different solution methodology to that used in [39, 40].

5.2 Pricing bounds on American basket call options

In this sub-section we consider finding price bounds on an American basket call option.
We consider the following two cases. The first case is when **none** of the n underlying assets pay out any dividends. The second case is when some or all of the n underlying assets pay out dividends.

5.2.1 Assuming that all assets do not pay any dividends

Here we consider finding price bounds on an American basket call option under a specific setting. That is, we assume that **none** of the n underlying assets on which the basket option is written on pay out any dividends. Under this setting we will show that the current price of an American basket call option is the same as its corresponding European counter-part. We note here that the result obtained in this section uses similar ideas from the derivation of showing that a European and American vanilla call option, where the underlying pays out no dividends, are equivalent in price. Thus, as such we may view this result which follows for basket options as an extension from the simple vanilla call option case.
We consider holding a portfolio Π, consisting of one American basket call option, short holding the weighted sum of the n underlying assets on which the basket option is written on, and a cash amount.
Then, if $\Pi(t)$ denotes the value of the portfolio at time t, for $t \in [0, T]$, we have

$$\Pi(t) = C_{\mathcal{A}}(t) - \omega^T S(t) + E\mathrm{e}^{-r(T-t)}.$$

Then we have two cases to consider for the value of the portfolio $\Pi(t)$.
Staying consistent with the notation introduced above, we observe that $S(\tilde{t})$ is the vector of the asset prices at time $\tilde{t}$. That is, $S(\tilde{t})$ denotes the vector of the asset prices at the time of early exercise, $\tilde{t}$ if it is done.
(a) Early exercise. We consider the value of the portfolio at time $t = \tilde{t}$.

$$\Pi(\tilde{t}) = C_{\mathcal{A}}(\tilde{t}) - \omega^T S(\tilde{t}) + E\mathrm{e}^{-r(T-\tilde{t})} = \max(\omega^T S(\tilde{t}) - E, 0) - \omega^T S(\tilde{t}) + E\mathrm{e}^{-r(T-\tilde{t})}.$$

We observe that

$$\max\left(\omega^T S(\tilde{t}) - E, 0\right) = \omega^T S(\tilde{t}) - E,$$

since we are exercising early.
This gives

$$\Pi(\tilde{t}) = \omega^T S(\tilde{t}) - E - \omega^T S(\tilde{t}) + E\mathrm{e}^{-r(T-\tilde{t})} = E\mathrm{e}^{-r(T-\tilde{t})} - E = E(\mathrm{e}^{-r(T-\tilde{t})} - 1) < 0.$$
$$\therefore \Pi(\tilde{t}) < 0.$$

(b) Not exercised early. We consider the value of the portfolio $\Pi(t)$ at expiry, $t = T$.
Here there are two sub-cases to consider.
(i) $\omega^T S > E$

$$\begin{aligned}\Pi(T) = C_{\mathcal{A}}(T) - \omega^T S(T) + E\mathrm{e}^{-r(T-T)} &= C_{\mathcal{A}}(T) - \omega^T S + E = \\ \max(\omega^T S - E, 0) - \omega^T S + E &= \omega^T S - E - \omega^T S + E = 0. \\ \therefore \Pi(T) = 0.\end{aligned}$$

(ii) $\omega^T S \leqslant E$

$$\begin{aligned}\Pi(T) &= C_{\mathcal{A}}(T) - \omega^T S(T) + E\mathrm{e}^{-r(T-T)} = C_{\mathcal{A}}(T) - \omega^T S + E = \max(\omega^T S - E, 0) - \omega^T S + E \\ &= 0 - \omega^T S + E = E - \omega^T S \geqslant 0. \\ &\therefore \Pi(T) \geqslant 0.\end{aligned}$$

Comparing (a) with (b) we can see that in the early exercise case, the portfolio makes a loss. If we do not exercise early, the portfolio either gives a zero or positive return. Therefore, it follows that we would **never** exercise the American basket call option early (since if we did, our portfolio would make a loss). Hence if we never exercise the American basket call option early, it becomes identical to its European counter-part.
Therefore, we may conclude that an American basket call option where all n underlying assets pay out **no** dividends has identical properties to a corresponding European basket call option where all n underlying assets pay out **no** dividends.
So, the current price of this particular American basket call option is equal to the current price of the corresponding European basket call option. That is, the current price of an American basket call option where all n underlying assets pay out **no** dividends is equal to the current price of a corresponding European basket call option where all of the n underlying assets pay out **no** dividends. Hence we have that $C_{\mathcal{A}} = C$ in this case.
Therefore, the results derived in Section 3 for the price bounds on a European basket call option are also valid bounds for the current price of a corresponding American basket call option written on the same n underlying assets, which pay out **no** dividends, with the same weights ω and same exercise price E.

5.2.2 Assuming that all or some assets pay dividends

The result above that $C = C_{\mathcal{A}}$ is only valid under the assumption that **none** of the n underlying assets pay out any dividends. For the case when some or all of the underlying assets pay out dividends the above result may not hold. We explain why this is so in this sub-section.
To start we set the following notation and assume that all of the n underlying assets pay out a dividend at a single fixed point in time t_d, where $0 < t_d < T$. Let $D_i \in \mathbb{R}_+$ denote the dividend amount paid out by the i^{th} asset at time t_d, for $i = 1, 2, \ldots, n$. Since we are considering a basket option we assume that for each of the underlying assets i, the investor receives the dividend D_i adjusted by its corresponding weight ω_i, for $i = 1, 2, \ldots, n$. That is, for each asset i, the investor receives a dividend $\omega_i D_i$ at time t_d, so that the total amount of dividends received from the basket option would be $\sum_{i=1}^{n} \omega_i D_i$.
Define the dividends vector $D \in \mathbb{R}^n_+$, so that at time t_d, the assets from the basket option pay out a dividend equal to $\omega^T D$.

Now we explain what happens to the current price of an American basket call option when the underlying assets pay out dividends.
If we were to exercise the American basket call option early, say at time $\tilde{t}$ where $0 \leqslant \tilde{t} < T$ then we would pay E to obtain a basket worth $\omega^T S(\tilde{t})$.
If instead we did not exercise the American basket call option early, then we could put the amount E in the bank until expiry, $t = T$ and we would receive an amount $E\mathrm{e}^{r(T-\tilde{t})}$ in return. Therefore we would make a profit of $E\mathrm{e}^{r(T-\tilde{t})} - E$.
By exercising the American basket call option early we can not make the profit $E\mathrm{e}^{r(T-\tilde{t})} - E$ which is made by investing E in the bank at time $\tilde{t}$ until T.
Thus, it is only worth exercising the American basket call option early if the dividend received is greater than the profit received by investing E in the bank at time $\tilde{t}$ until T. That is, we should exercise the American basket call option early if

$$\omega^T D > E\mathrm{e}^{r(T-\tilde{t})} - E,$$

and not exercise early if

$$\omega^T D \leqslant E\mathrm{e}^{r(T-\tilde{t})} - E,$$

since we can make more money by investing E in the bank at time $\tilde{t}$ until T.
We observe here that in the early exercise case we assume $\tilde{t} \leqslant t_d$, because the assets need to be held to receive the dividend $\omega^T D$ at time t_d.
Thus, since there is a possibility of early exercise we conclude that $C_{\mathcal{A}} = C$ may now **not** always be the case.
Therefore, we may conclude that for an American basket call option where some or all of the underlying assets pay out dividends, the relation

$$C \leqslant C_{\mathcal{A}}$$

holds.
Note: It is possible to extend the above analysis to cover the case when the n underlying assets all pay out dividends on multiple dates.
In what follows next, we focus our attention on finding pricing bounds for an American basket put option.

5.3 Pricing bounds on American basket put options

In the previous sub-section we derived bounds on the current price of an American basket call option. In particular, we have shown that the current price of an American basket call option where all underlying assets pay out **no** dividends is equal to the current price of its corresponding European counter-part. This is because in this case early exercise makes no sense. In the case of dividends we explained how early exercise could make sense.
The question which we answer in this sub-section is 'how can we find bounds on the current price of an American basket put option?' To start we utilise the following *put-call parity* result for European basket options. This result is analogous to the put-call parity for European vanilla options, which can be found in [2]. After, we prove a *put-call parity inequality* for American basket options, which is an extension of the *put-call parity inequality* for American vanilla options, given in [41].

5.3.1 Put-call parity for European basket options

Consider holding a portfolio, Π, consisting of the weighted sum of the n underlying assets, a European basket put option and short a European basket call option, written on the same n underlying assets with the same weights and same exercise price, as well as the same expiry date, $t = T$.
Then, at any time t, the portfolio has value $\Pi(t)$, given by

$$\Pi(t) = \omega^T S(t) + P(t) - C(t).$$

Therefore, the current value of the portfolio, (that is, at time $t = 0$) is

$$\Pi(0) = \omega^T S(0) + P(0) - C(0) = \omega^T S(0) + P - C.$$

Hence to hold the portfolio Π today it would cost $\Pi(0)$.
Now consider the value of $\Pi(t)$ at expiry, $t = T$.
We have the value of $\Pi(T)$ as

$$\begin{aligned}\Pi(T) &= \omega^T S(T) + P(T) - C(T)\\ &= \omega^T S + \max(E - \omega^T S, 0) - \max(\omega^T S - E, 0).\end{aligned}$$

Now, there are the following two cases to consider.

1. If $\omega^T S > E$
$$\implies \Pi(T) = \omega^T S - \omega^T S + E = E.$$
2. If $\omega^T S \leqslant E$
$$\implies \Pi(T) = \omega^T S + E - \omega^T S - 0 = E.$$

Therefore, the value of the portfolio at expiry is always equal to E, and in particular is independent of the prices of the underlying assets at expiry.
That is, the value of the portfolio $\Pi(T)$ is independent of S. Now, the current price of the portfolio is dependent upon the asset prices at expiry and time. However as we have just seen, the price of $\Pi(T)$ is independent of S. Therefore, we may let the price of the portfolio be equal to $V(t)$, which is dependent on time, t but **not** dependent on the prices of the assets at expiry, S.
Substituting this into the multi-asset Black-Scholes equation (MABSE) given in sub-section 2.1.4, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} - rV = 0, \text{ and so } \frac{\mathrm{d}V}{\mathrm{d}t} = rV.$$

$$\text{This gives, } \frac{1}{V}\,\mathrm{d}V = r\,\mathrm{d}t, \text{ and so integrating, } \int \frac{1}{V}\,\mathrm{d}V = \int r\,\mathrm{d}t. \text{ Hence, } \ln(V) = rt + k,$$

where k is an arbitrary constant.

$$\implies V = \mathrm{e}^{rt+k} = \mathrm{e}^{rt}\mathrm{e}^{k}.$$

Let $\mathrm{e}^k = A$, then we have

$$V = A\mathrm{e}^{rt}.$$

Now, at expiry $t = T$ we have $V(t) = V(T) = E$.

$$\implies E = A\mathrm{e}^{rT} \implies A = E\mathrm{e}^{-rT}.$$

$$\text{This gives, } V = E\mathrm{e}^{-rT}\mathrm{e}^{rt}, \text{ which is equivalent to, } V = E\mathrm{e}^{-r(T-t)}.$$

$$\text{Therefore, } V(t) = E\mathrm{e}^{-r(T-t)}.$$

$\implies$ At time t, the portfolio $\Pi(t)$ has value

$$\begin{aligned}V(t) &= E\mathrm{e}^{-r(T-t)}\\ \implies \omega^T S(t) + P(t) - C(t) &= E\mathrm{e}^{-r(T-t)}.\end{aligned}$$

So that the current value of the portfolio, that is at time $t = 0$, is

$$\begin{aligned}V(0) &= E\mathrm{e}^{-rT}\\ \implies \omega^T S(0) + P(0) - C(0) &= E\mathrm{e}^{-rT}\\ \text{which is, } \omega^T S(0) + P - C &= E\mathrm{e}^{-rT}.\end{aligned}$$

This gives us the *put-call parity* for European basket options as

$$\omega^T S(0) + P - C = E\mathrm{e}^{-rT}. \tag{5.1}$$

The put-call parity (5.1) gives us a link between the current prices of a European basket call option and a European basket put option written on the same n underlying assets with the same weights and same exercise price, as well as the same expiry date, $t = T$.
We may now use (5.1) to find price bounds on American basket put options, as we describe below.

5.3.2 A put-call parity inequality

We have derived (5.1) which is a put-call parity for European basket call and put options. This allows us to link the current prices of European basket put options and call options. Unfortunately, there does not exist a corresponding put-call parity equation for American basket call and put options.
However, it is possible to re-write the put-call parity equation (5.1) as an inequality which holds for American basket options. We will call this a *put-call parity inequality* and this put-call parity inequality follows from the fact that an American basket option should cost at least as much as its European counter-part.
We summarise the above ideas in the following theorem. The proof of the following theorem uses similar techniques as in [41], where a result was derived for American vanilla options.

Theorem 5.2. *The* ***put-call parity inequality*** *for American basket options, without dividends is given by*

$$\omega^T S(0) - E \leqslant C_{\mathcal{A}} - P_{\mathcal{A}} \leqslant \omega^T S(0) - E\mathrm{e}^{-rT}.$$

Proof. The proof comes in two parts.
Upper bound
From (5.1) we have

$$\omega^T S(0) + P - C = E\mathrm{e}^{-rT}$$
$$\iff C - P = \omega^T S(0) - E\mathrm{e}^{-rT}.$$

Now, $P = C - \omega^T S(0) + E\mathrm{e}^{-rT}$, but, $P_{\mathcal{A}} \geqslant P$

$$\implies P_{\mathcal{A}} \geqslant C - \omega^T S(0) + E\mathrm{e}^{-rT}.$$

However $C = C_{\mathcal{A}}$ and so we have

$$P_{\mathcal{A}} \geqslant C_{\mathcal{A}} - \omega^T S(0) + E\mathrm{e}^{-rT}.$$
$$\text{Therefore, } C_{\mathcal{A}} - P_{\mathcal{A}} \leqslant \omega^T S(0) - E\mathrm{e}^{-rT},$$

and so the upper bound is proved.
Lower bound
For the lower bound, we have the following.
Consider holding the following two portfolios, Π_A and Π_B. Π_A consists of one European basket call option and an amount of money, specifically E. Π_B consists of an American basket put option and the weighted sum of the n underlying assets on which the basket options are written on. Then at time $t \in [0, T]$, the values of Π_A and Π_B are given by

$$\Pi_A(t) = C(t) + E \text{ and } \Pi_B = P_{\mathcal{A}}(t) + \omega^T S(t).$$

In particular, the current prices of the portfolios Π_A and Π_B are given by

$$\Pi_A(0) = C(0) + E = C + E,$$

and

$$\Pi_B(0) = P_{\mathcal{A}}(0) + \omega^T S(0) = P_{\mathcal{A}} + \omega^T S(0),$$

respectively.
Now we consider the following two cases.

1. No early exercising. We consider the prices of $\Pi_A(t)$ and $\Pi_B(t)$ at expiry $t = T$. Then we have

$$\Pi_A(T) = C(T) + E = \max(\omega^T S - E, 0) + E = \max(\omega^T S, E) - E + E = \max\{\omega^T S, E\}.$$
$$\Pi_B(T) = P_{\mathcal{A}}(T) + \omega^T S(T) = \max(E - \omega^T S, 0) + \omega^T S = \max\{\omega^T S, E\}.$$

The prices or payoffs of the portfolios Π_A and Π_B are equal at expiry. This means that, since Π_A and Π_B both give the same payoff (that is, they have the same value) at expiry, then to avoid arbitrage opportunities they should have the same current value. That is, the equality $\Pi_A(0) = \Pi_B(0)$, must hold.
This gives

$$\Pi_A(0) = \Pi_B(0)$$
$$C + E = P_{\mathcal{A}} + \omega^T S(0).$$

However $C = C_{\mathcal{A}}$ since none of the assets pay out dividends. This gives us

$$C_{\mathcal{A}} + E = P_{\mathcal{A}} + \omega^T S(0)$$
$$\iff \quad C_{\mathcal{A}} - P_{\mathcal{A}} = \omega^T S(0) - E.$$

2. Early exercising. Here we assume that we exercise any American option early, at time $t = \tilde{t} \in [0, T)$. We consider the prices of the portfolios $\Pi_A(t)$ and $\Pi_B(t)$ at time $t = \tilde{t}$.
For portfolio A we have

$$\Pi_A(\tilde{t}) = C(\tilde{t}) + E,$$

and for portfolio B we have

$$\Pi_B(\tilde{t}) = P_{\mathcal{A}}(\tilde{t}) + \omega^T S(\tilde{t}) = \max(E - \omega^T S(\tilde{t}), 0) + \omega^T S(\tilde{t}) = E - \omega^T S(\tilde{t}) + \omega^T S(\tilde{t}) = E.$$

Now, we observe that $\Pi_A(\tilde{t}) = C(\tilde{t}) + E > E = \Pi_B(\tilde{t})$, since $C(\tilde{t}) > 0$.
Therefore, in the early exercise case we have that

$$\Pi_A(\tilde{t}) > \Pi_B(\tilde{t}).$$

However, since $\tilde{t} \in [0, T)$ is arbitrary, it can take any value between 0 and T, including 0 but not T. Thus, we have that for *any* early exercise time $\tilde{t}$, where $0 \leqslant \tilde{t} < T$,

$$\Pi_A(\tilde{t}) > \Pi_B(\tilde{t}).$$

In particular, since this inequality holds for $\tilde{t} = 0$ we have

$$\Pi_A(0) > \Pi_B(0)$$
$$\iff \quad C(0) + E > P_{\mathcal{A}}(0) + \omega^T S(0),$$

but $C(0) = C_{\mathcal{A}} = C$, so we have

$$C_{\mathcal{A}} + E > P_{\mathcal{A}} + \omega^T S(0),$$
$$C_{\mathcal{A}} - P_{\mathcal{A}} > \omega^T S(0) - E.$$

Combining the results for the non-early and early exercising cases, gives

$$\omega^T S(0) - E \leqslant C_{\mathcal{A}} - P_{\mathcal{A}}$$

and so the lower bound also holds true.

Therefore, we have shown that

$$\omega^T S(0) - E \leqslant C_{\mathcal{A}} - P_{\mathcal{A}} \leqslant \omega^T S(0) - E\mathrm{e}^{-rT},$$

and the theorem is proved. □

Observation: We may now use the *put-call parity inequality* from Theorem 5.2 to find upper/lower bounds on the current price of an American basket put option, $P_{\mathcal{A}}$. Since we know that $C_{\mathcal{A}} = C$, that is, we know that the current price of the American basket call option is equal to the current price of the corresponding European basket call option and we have already considered methods to find bounds on the current price of a European basket call option. Then, we may use the bounds on $C_{\mathcal{A}} = C$ to obtain bounds on $P_{\mathcal{A}}$.
More formally, if C_{UB} is an upper bound on C and C_{LB} is a lower bound on C, so that $C_{LB} \leqslant C \leqslant C_{UB}$, as obtained in Section 3, then we may use Theorem 5.2 to find bounds on $P_{\mathcal{A}}$ as follows.
For the upper bound we have

$$\begin{aligned} \omega^T S(0) - E &\leqslant C_{\mathcal{A}} - P_{\mathcal{A}} \\ P_{\mathcal{A}} &\leqslant C_{\mathcal{A}} - \omega^T S(0) + E, \end{aligned}$$

but $C_{\mathcal{A}} = C$ so that

$$\begin{aligned} P_{\mathcal{A}} &\leqslant C - \omega^T S(0) + E \\ &\leqslant C_{UB} - \omega^T S(0) + E. \end{aligned}$$

$$\text{Hence } P_{\mathcal{A}} \leqslant C_{UB} - \omega^T S(0) + E.$$

For the lower bound we have

$$\begin{aligned} C_{\mathcal{A}} - P_{\mathcal{A}} &\leqslant \omega^T S(0) - E\mathrm{e}^{-rT} \\ C_{\mathcal{A}} - \omega^T S(0) + E\mathrm{e}^{-rT} &\leqslant P_{\mathcal{A}}. \end{aligned}$$

However $C_{\mathcal{A}} = C$, so that we have

$$C - \omega^T S(0) + E\mathrm{e}^{-rT} \leqslant P_{\mathcal{A}}.$$

This gives

$$\begin{aligned} C_{LB} - \omega^T S(0) + E\mathrm{e}^{-rT} &\leqslant C - \omega^T S(0) + E\mathrm{e}^{-rT} \\ &\leqslant P_{\mathcal{A}}. \end{aligned}$$

$$\text{Hence, } C_{LB} - \omega^T S(0) + E\mathrm{e}^{-rT} \leqslant P_{\mathcal{A}}.$$

Therefore the upper/lower bounds on $P_{\mathcal{A}}$ are given by

$$C_{LB} - \omega^T S(0) + E\mathrm{e}^{-rT} \leqslant P_{\mathcal{A}} \leqslant C_{UB} - \omega^T S(0) + E.$$

We note here that although this is one possible way to find price bounds on an American basket put option, it is limited to the specific case of all underlying assets paying out no dividends as described above, and it may give a wide interval, since we are essentially bounding $P_{\mathcal{A}}$ by other bounds (on $C_{\mathcal{A}} = C$) obtained in Section 3.
With this in mind, we now consider a different setting and see how optimisation can be used to find more accurate and credible bounds on the current price of a particular type of American basket put option.

6 Using optimisation to find upper bounds on $P_{\mathcal{A}}$

In this section we consider finding upper bounds on the current price of a certain type of American basket put option using optimisation. In particular, for this section we consider finding an upper bound on the current price of a *Bermuda basket put option.* Put simply a Bermuda option is a certain type of American option. In particular a Bermuda option gives the holder the right to exercise the option at fixed, pre-agreed times prior to expiry as well as exercising the option at expiry if it has not been exercised early. In this section, we let $P_{\mathcal{A}}$ denote the current price of a Bermuda basket put option. Then we consider finding upper bounds on $P_{\mathcal{A}}$ using similar techniques to that in sub-section 3.4 for a European basket call option.

Before proceeding we note here that there exist other results regarding pricing and finding price bounds of Bermuda options, which can be found in [42, 43, 44, 45]. In particular, in [42], pricing methods for Bermudan options dependent on a large number of underlying assets are derived. The results obtained in this paper may be used to calculate prices for Bermudan basket options. In [43] upper and lower bounds on Bermuda options using new variance reduction techniques are derived. The results obtained from this paper could also be extended to the Bermuda basket option case. In [44] lower bounds on multi-dimensional Bermudan options using a stochastic grid bundling method are derived. Finally, in [45] upper and lower bounds on the prices of Bermuda basket options using Monte Carlo simulations are found.

Our new and original result uses a solution approach which is different to the approaches used in the references described above. It is different because we model the problem as a dual of a linear semi-infinite optimisation problem, and then re-write this linear SIO problem as a finite and solvable linear problem in a way which has not been done before. The optimal objective function value is then an upper bound on the current price of a Bermuda basket put option. In particular, we follow a similar model set-up to what was presented in sub-section 3.4. So, we consider finding an upper bound on the current price of a Bermuda basket put option, given that we know the prices of q European vanilla call options per asset and the expected (forward) price per asset, under a risk-neutral probability measure.

Further, assuming that the holder of this Bermuda put option may exercise this option either at one fixed, pre-agreed point in time prior to the expiry date or they may exercise it at expiry, and retaining all notation from sub-section 3.4 we consider the following model set-up.

- Let the risk free interest rate be given by $r > 0$.
- Let t^* be a point in time such that $0 \leqslant t^* < T$, be the pre-agreed, fixed time point at which the Bermuda basket put option may be exercised early.
- Set $S(t^*) = S_B$, so that $S_{B_i} \in \mathbb{R}^n_+$ is the price of the i^{th} asset at time t^*, for $i = 1, 2, \ldots, n$.
- Recall that $S(T) = S \in \mathbb{R}^n_+$, is the vector of the prices of the n underlying assets at expiry.

Then we may now present the optimisation model.

The task of finding an upper bound on the current price of a Bermuda basket put option is modelled by the following problem, with the variable π, which is a dual of a linear SIO problem.

$$\begin{aligned}
\sup_{\pi} \quad & \mathbb{E}_\pi[\max\{\mathrm{e}^{-rt^*}\max(E-\omega^T S_B,0), \mathrm{e}^{-rT}\max(E-\omega^T S,0)\}] \\
\text{subject to} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT}\max(S_i - E_i^l, 0)] = C_i^l, \quad \text{for } i = 1,2,\ldots,n \text{ and } l = 1,2,\ldots,q \\
& \mathbb{E}_\pi[\mathrm{e}^{-rT} S_i] = S_i^0, \quad \text{for } i = 1,2,\ldots,n \\
& \mathbb{E}_\pi[\mathrm{e}^{-rt^*} S_{B_i}] = S_i^0, \quad \text{for } i = 1,2,\ldots,n \\
& \mathbb{E}_\pi[1] = 1.
\end{aligned} \tag{6.1}$$

We solve (6.1) using a similar methodology to that what was done for the European basket call option pricing problem.

Before proceeding we highlight the key differences between a Bermuda basket put option and a European basket option and the new challenges we face with trying to find price bounds on this particular instrument.

The main difference is that of early exercise. The Bermuda basket put option may either be exercised early, at time t^* or it can be exercised at expiry, $t = T$. In comparison, the European option can only be exercised at expiry, $t = T$. This vital difference is captured in the objective function of (6.1). The payoff of the Bermuda basket put option is the maximum of the payoff at expiry or at the time of early exercise. We face new challenges when finding price bounds on the Bermuda basket put option, in the sense that we have to consider different cases: when the Bermuda option is exercised early and when the Bermuda option is not exercised early.

In particular, when carrying out pricing analysis for this particular type of option, we have to consider both the values of the asset price vector at early exercise, given by S_B as well as the values of the price vector at expiry, given by S. We note here that this additional challenge was not encountered when we found price bounds on European basket options. The feature of early exercise that the Bermuda option gives makes finding price bounds on this option more complicated than what was seen for European basket options.

We start by deriving the linear SIO problem for which (6.1) is its dual, and this is given by

$$\begin{aligned} \inf_{(u^l,v,y,z)} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 + y^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0)\mathrm{e}^{-rT} + z + v^T S \mathrm{e}^{-rT} + y^T S_B \mathrm{e}^{-rt^*} \geqslant \\ & \max\{\mathrm{e}^{-rt^*}\max(E - \omega^T S_B, 0), \mathrm{e}^{-rT}\max(E - \omega^T S, 0)\}, \ \ \forall\ S_B,\ S \in \mathbb{R}^n_+. \end{aligned}$$

This problem is equivalent to

$$\begin{aligned} \inf_{(u^l,v,y,z)} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 + y^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant \\ & \max\{\mathrm{e}^{r(T-t^*)}\max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \ \ \forall\ S_B,\ S \in \mathbb{R}^n_+. \end{aligned} \tag{6.2}$$

We observe that the SIO problem (6.2) and the variables (u^l, v, y, z) have a natural financial interpretation. We may interpret the variables (u^l, v, y, z) as follows. The components of u^l, given by u^l_i denote the amount of European call options, written on asset i, for $i = 1, 2, \ldots, n$ which we buy today at price C^l_i and contributes $\max(S_i - E^l_i, 0)$ to the super-replicating portfolio, for $l = 1, 2, \ldots, q$. z represents a cash amount. The components of v, given by v_i, for $i = 1, 2, \ldots, n$ represent the amount of the i^{th} asset we buy at a price of S^0_i and contributes S_i to the super-replicating portfolio. The components of y, given by y_i, for $i = 1, 2, \ldots, n$ represent the amount of the i^{th} asset we buy at a price of S^0_i and contributes S_{B_i} to the super-replicating portfolio.

Problem (6.2) aims to find the cheapest cost portfolio consisting of European vanilla calls, cash, an amount of the underlying assets and another amount of the underlying assets such that the value of this portfolio super-replicates the payoff of the basket option whose current price we are bounding, for all possible non-negative values of S and S_B.

Now, we note that the index set $\mathcal{I} = \mathbb{R}^n_+$ in (6.2) is not compact. However, we may derive the following proposition whereby we can restrict $\mathcal{I}$ in (6.2) to a compact set without changing the feasible set of (6.2).

Proposition 6.3. *Suppose without loss of generality that the exercise prices E^l_i are ordered such that $0 \leqslant E^1_i \leqslant E^2_i \leqslant \cdots \leqslant E^q_i$, for all $i = 1, 2, \ldots, n$. Define the index set $\mathcal{I}_{(6.4)} = \bigtimes_{i=1}^{n} [0, E^q_i]$. Then the following optimisation problem* (6.4)*, is equivalent to* (6.2) *in the sense that both problems have the same feasible set,* *and, hence the same optimal solution and optimal objective function value.*

$$\begin{aligned}
\min_{u^l,v,y,z\,\in\mathbb{R}^{n\times q}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 + y^T S^0 \\
\text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant \\
& \max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \quad \forall\, S_B,\ S \in \mathcal{I}_{(6.4)} \\
& y \geqslant \max\{-\omega, 0\} \\
& \sum_{l=1}^{q} u^l + v \geqslant \max\{-\omega, 0\}.
\end{aligned} \tag{6.4}$$

Remarks

1. *We first remark that the objective functions of* (6.2) *and* (6.4) *are the same. This is because the objective function in* (6.2) *represents the total cost of the super-replicating portfolio at the current time,* $t = 0$. *Since this is a cost which is paid, the minimal such cost is attained since what ever the minimum cost is, the investor pays it. Therefore, the* ***inf*** *in* (6.2) *is attained and can be replaced by* ***min*** *as in* (6.4).

2. *The extra constraints in* (6.4) *have a significant financial meaning.*
The constraint
$$y \geqslant \max\{-\omega, 0\},$$
which is equivalent to
$$y_i \geqslant \max\{-\omega_i, 0\},\ \forall\, i = 1, 2, \ldots, n,$$
means:
(a) If the weight $\omega_i > 0$*, this implies,* $y_i \geqslant 0$*, for all* $i = 1, 2, \ldots, n$. *This means that for a strictly positive weight* ω_i*, the corresponding amount of the* i^{th} *asset which is bought at a price of* S_i^0 *today and contributes* S_{B_i} *to the super-replicating portfolio is non-negative. That is, we do* ***not*** *short sell the asset* i *in this case.*
(b) If the weight $\omega_i < 0$*, this implies,* $y_i \geqslant \omega_i$*, for all* $i = 1, 2, \ldots, n$. *This means that for a strictly negative weight* ω_i*, the amount at which we buy the* i^{th} *underlying asset at a price of* S_i^0 *today and contributes* S_{B_i} *in the super-replicating portfolio, is at least* ω_i.
Further, the constraint
$$\sum_{l=1}^{q} u^l + v \geqslant \max\{-\omega, 0\}$$
which is equivalent to
$$\sum_{l=1}^{q} u_i^l + v_i \geqslant \max\{-\omega_i, 0\},\ \forall\, i = 1, 2, \ldots, n$$
means:
(a) If the weight $\omega_i > 0$*, this implies,* $\sum_{l=1}^{q} u_i^l + v_i \geqslant 0$*, for all* $i = 1, 2, \ldots, n$. *This means that for a strictly positive weight* ω_i*, the total amount of European vanilla calls written on the* i^{th} *asset plus the amount of the* i^{th} *asset which contributes* S_i *to the super-replicating portfolio is non-negative.*
(b) If the weight $\omega_i < 0$*, this implies,* $\sum_{l=1}^{q} u_i^l + v_i \geqslant \omega_i$*, for all* $i = 1, 2, \ldots, n$. *This means that for a strictly negative weight* ω_i*, the total amount of European vanilla calls written on the* i^{th} *asset plus the amount of the* i^{th} *asset which contributes* S_i *to the super-replicating portfolio is at least* ω_i.

3. *The extra constraints give conditions or restrictions on how the super-replicating portfolio should be constructed.*

4. *The advantage of solving* (6.4) *instead of* (6.2) *is that the index set in the semi-infinite constraint of* (6.4) *is compact, albeit* (6.4) *containing additional constraints. Problem* (6.4) *allows us to construct the minimal cost super-replicating portfolio, which is the optimal objective function value of the SIO problem* (6.2)*, by considering a compact index set* $\mathcal{I}_{(6.4)}$ *and constraints which impose conditions on how the super-replicating portfolio is to be constructed. Thus,* (6.4) *ensures that the value of the portfolio always super-replicates the payoff of the basket option whose current price we are bounding for all non-negative values of* S *and* S_B *by considering a compact set to which* S *and* S_B *can take values in, as well as some extra restrictions on how to construct the super-replicating portfolio which are to be obeyed.*

Proof of Proposition 6.3. We start by observing that the objective functions of (6.2) and (6.4) are the same. Thus in order to show these two problems are equivalent we must show that their respective feasible regions are the same. Let $\mathcal{F}_{(6.2)}$ and $\mathcal{F}_{(6.4)}$ denote the feasible regions of (6.2) and (6.4), respectively. We then show that $\mathcal{F}_{(6.2)} = \mathcal{F}_{(6.4)}$.
The proof comes in two parts.
(i) $\underline{\mathcal{F}_{(6.2)} \subset \mathcal{F}_{(6.4)}}$: Take any $(u^1, u^2, \ldots, u^q, v, y, z) \in \mathcal{F}_{(6.2)}$. We then show that $(u^1, u^2, \ldots, u^q, v, y, z) \in \mathcal{F}_{(6.4)}$. Now, since $\mathcal{I}_{(6.4)} = \bigtimes_{i=1}^{n} [0, E_i^q]$, then $\mathcal{I}_{(6.4)}$ forms an n-dimensional 'rectangle'. That is, it forms a 'rectangle' in n-dimensional non-negative space and so $\mathcal{I}_{(6.4)} \subset \mathbb{R}^n_+$. Thus, from the constraint in (6.2) we have

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant$$
$$\max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \;\; \forall\, S_B,\; S \in \mathbb{R}^n_+.$$

It then follows that since $\mathcal{I}_{(6.4)} \subset \mathbb{R}^n_+$, the constraint in (6.2) obviously still holds for all $S_B,\; S \in \mathcal{I}_{(6.4)}$. That is,

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant$$
$$\max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \;\; \forall\, S_B,\; S \in \mathcal{I}_{(6.4)},$$

and so the first constraint from (6.4) holds.
To show that the final two constraints hold we have the following. Recall that $(u^1, u^2, \ldots, u^q, v, y, z)$ satisfies the constraint

$$\begin{aligned}&\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant \\ &\max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \;\; \forall\, S_B,\; S \in \mathbb{R}^n_+,\end{aligned} \tag{6.5}$$

from (6.2), for all $S_B,\; S \in \mathbb{R}^n_+$.
Then we consider the following two cases.
<u>Case 1</u>: The American basket put option is exercised early. In this case we are interested in the price S_B. The price of the assets at expiry, given by S is irrelevant in this case and may be treated as an unknown constant.
Now, since (6.5) holds for all $S_B \in \mathbb{R}^n_+$, it certainly holds for the particular asset price price vector S_B, with $S_{B_i} = \mathrm{e}^{-r(T-t^*)}\eta$, for some $\eta > 0$ and all other components equal to 0, for $i = 1, 2, \ldots, n$. That is, the constraint holds for the vector $S_B = \mathrm{e}^{-r(T-t^*)} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \eta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \mathrm{e}^{-r(T-t^*)}\eta \\ 0 \end{pmatrix}$, where η is in the i^{th} position, for

$i = 1, 2, \ldots, n$.
This gives

$$\sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant$$
$$\max\{\mathrm{e}^{r(T-t^*)}\max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \quad \forall\, S_B \in \mathbb{R}_+^n$$
$$\iff \sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k + \mathrm{e}^{r(T-t^*)}\sum_{k=1}^{n} y_k S_{B_k} \geqslant$$
$$\max\left\{\mathrm{e}^{r(T-t^*)}\max\left(E - \sum_{k=1}^{n}\omega_k S_{B_k}, 0\right), \max\left(E - \sum_{k=1}^{n}\omega_k S_k, 0\right)\right\}, \quad \forall\, S_B \in \mathbb{R}_+^n.$$

Now, for each $i = 1, 2, \ldots, n$, $S_{B_i} = \mathrm{e}^{-r(T-t^*)}\eta$ and all other components are equal to 0.
For a particular i, $\implies$

$$\sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \geqslant$$
$$\max\left\{\mathrm{e}^{r(T-t^*)}\max\left(E - \omega_i S_{B_i}, 0\right), \max\left(E - \sum_{k=1}^{n}\omega_k S_k, 0\right)\right\},$$
$$\iff \sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k + \mathrm{e}^{r(T-t^*)} y_i \mathrm{e}^{-r(T-t^*)}\eta \geqslant$$
$$\max\left\{\mathrm{e}^{r(T-t^*)}\max\left(E - \omega_i \mathrm{e}^{-r(T-t^*)}\eta, 0\right), \max\left(E - \sum_{k=1}^{n}\omega_k S_k, 0\right)\right\},$$
$$\iff \sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k + \eta y_i \geqslant$$
$$\max\left\{\eta\max\left(\frac{E\mathrm{e}^{r(T-t^*)}}{\eta} - \omega_i, 0\right), \max\left(E - \sum_{k=1}^{n}\omega_k S_k, 0\right)\right\}$$

and so, if we divide both sides by η we get, (since $\eta > 0$)

$$\implies \frac{\sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(S_k - E_k^l, 0)}{\eta} + \frac{z\mathrm{e}^{rT}}{\eta} + \frac{\sum_{k=1}^{n} v_k S_k}{\eta} + y_i \geqslant$$
$$\max\left\{\max\left(\frac{E\mathrm{e}^{r(T-t^*)}}{\eta} - \omega_i, 0\right), \frac{1}{\eta}\max\left(E - \sum_{k=1}^{n}\omega_k S_k, 0\right)\right\},$$

and if $\eta \to \infty$, then, $\frac{\sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(S_k - E_k^l, 0)}{\eta} \to 0$, $\frac{z\mathrm{e}^{rT}}{\eta} \to 0$, $\frac{\sum_{k=1}^{n} v_k S_k}{\eta} \to 0$, $\frac{E\mathrm{e}^{r(T-t^*)}}{\eta} \to 0$ and $\frac{1}{\eta}\max\left(E - \sum_{k=1}^{n}\omega_k S_k, 0\right) \to 0$. This gives, in the limit as $\eta \to \infty$,

$$y_i \geqslant \max\{\max(-\omega_i, 0), 0\} = \max\{-\omega_i, 0\}$$
$$\therefore\ y_i \geqslant \max\{-\omega_i, 0\},$$

which holds for all $i = 1, 2, \ldots, n$.
This in vector form is just

$$y \geqslant \max\{-\omega, 0\}. \tag{6.6}$$

Case 2: The American basket put option is **not** exercised early. In this case we are interested in the asset price vector at expiry, given by $S \in \mathbb{R}^n_+$. The price of the assets at any time before T, that is, the price of the assets at any time before expiry, and in particular at any potential S_B, is irrelevant in this case and may be treated as an unknown constant.
Now, since (6.5) holds for all $S \in \mathbb{R}^n_+$, it certainly holds for the particular asset price price vector S, with $S_i = \eta$, for some $\eta > 0$ and all other components equal to 0, for $i = 1, 2, \ldots, n$. That is, the constraint holds for the vector $S = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \eta \\ 0 \end{pmatrix}$, where η is in the i^{th} position, for $i = 1, 2, \ldots, n$.

This gives

$$\begin{aligned}
&\sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant \\
&\max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\}, \quad \forall\, S \in \mathbb{R}^n_+ \\
\iff &\sum_{l=1}^{q}\sum_{k=1}^{n}(u^l_k) \max(S_k - E^l_k, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k + \mathrm{e}^{r(T-t^*)} \sum_{k=1}^{n} y_k S_{B_k} \geqslant \\
&\max\left\{\mathrm{e}^{r(T-t^*)} \max\left(E - \sum_{k=1}^{n} \omega_k S_{B_k}, 0\right), \max\left(E - \sum_{k=1}^{n} \omega_k S_k, 0\right)\right\}, \quad \forall\, S \in \mathbb{R}^n_+.
\end{aligned}$$

Now, for each $i = 1, 2, \ldots, n$, $S_i = \eta$ and all other components are equal to 0.
For a particular i, $\implies$

$$\begin{aligned}
&\sum_{l=1}^{q}(u^l_i) \max(S_i - E^l_i, 0) + z\mathrm{e}^{rT} + v_i S_i + \mathrm{e}^{r(T-t^*)} \sum_{k=1}^{n} y_k S_{B_k} \geqslant \\
&\max\left\{\mathrm{e}^{r(T-t^*)} \max\left(E - \sum_{k=1}^{n} \omega_k S_{B_k}, 0\right), \max(E - \omega_i S_i, 0)\right\} \\
\iff &\sum_{l=1}^{q}(u^l_i) \max(\eta - E^l_i, 0) + z\mathrm{e}^{rT} + v_i \eta + \mathrm{e}^{r(T-t^*)} \sum_{k=1}^{n} y_k S_{B_k} \geqslant \\
&\max\left\{\mathrm{e}^{r(T-t^*)} \max\left(E - \sum_{k=1}^{n} \omega_k S_{B_k}, 0\right), \max(E - \omega_i \eta, 0)\right\} \\
\iff &\sum_{l=1}^{q}(u^l_i)\eta \max\left(1 - \frac{E^l_i}{\eta}, 0\right) + z\mathrm{e}^{rT} + v_i \eta + \mathrm{e}^{r(T-t^*)} \sum_{k=1}^{n} y_k S_{B_k} \geqslant \\
&\max\left\{\mathrm{e}^{r(T-t^*)} \max\left(E - \sum_{k=1}^{n} \omega_k S_{B_k}, 0\right), \eta \max\left(\frac{E}{\eta} - \omega_i, 0\right)\right\},
\end{aligned}$$

and so, if we divide both sides by η we get, (since $\eta > 0$)

$$\Longrightarrow \sum_{l=1}^{q}(u_i^l)\max\left(1-\frac{E_i^l}{\eta},0\right)+\frac{z\mathrm{e}^{rT}}{\eta}+v_i+\frac{\mathrm{e}^{r(T-t^*)}\sum_{k=1}^{n}y_kS_{B_k}}{\eta}\geqslant$$
$$\max\left\{\frac{1}{\eta}\mathrm{e}^{r(T-t^*)}\max\left(E-\sum_{k=1}^{n}\omega_kS_{B_k},0\right),\max\left(\frac{E}{\eta}-\omega_i,0\right)\right\}$$
,

and if $\eta\to\infty$, then, $\frac{\mathrm{e}^{r(T-t^*)}\sum_{k=1}^{n}y_kS_{B_k}}{\eta}\to 0$, $\frac{z\mathrm{e}^{rT}}{\eta}\to 0$, $\frac{E_i^l}{\eta}\to 0$, $\frac{E}{\eta}\to 0$ and $\frac{1}{\eta}\mathrm{e}^{r(T-t^*)}\max\left(E-\sum_{k=1}^{n}\omega_kS_{B_k},0\right)\to 0$. This gives, in the limit as $\eta\to\infty$,

$$\sum_{l=1}^{q}(u_i^l)\max(1,0)+v_i\geqslant\max\{0,\max(0-\omega_i,0)\}$$
$$\Longleftrightarrow \sum_{l=1}^{q}u_i^l+v_i\geqslant\max\{-\omega_i,0\},$$

which holds for all $i=1,2,\ldots,n$.
This in vector form is just

$$\sum_{l=1}^{q}u^l+v\geqslant\max\{-\omega,0\}. \tag{6.7}$$

Thus we have shown that (6.6) and (6.7) hold and so the final two inequalities from (6.4) hold, thus proving that $\mathcal{F}_{(6.2)}\subset\mathcal{F}_{(6.4)}$.
(ii) Now we prove the converse, $\underline{\mathcal{F}_{(6.4)}\subset\mathcal{F}_{(6.2)}}$. So, take any $(u^1,u^2,\ldots,u^q,v,y,z)\in\mathcal{F}_{(6.4)}$. Then, in order to show that $(u^1,u^2,\ldots,u^q,v,y,z)\in\mathcal{F}_{(6.2)}$, we must show that

$$\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)+z\mathrm{e}^{rT}+v^TS+y^TS_B\mathrm{e}^{r(T-t^*)}\geqslant$$
$$\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^TS_B,0),\max(E-\omega^TS,0)\},\ \ \forall\ S_B,\ S\in\mathbb{R}_+^n.$$

To this end, it suffices to show that; in the early exercise case

$$\max_{S_B\in\mathbb{R}_+^n}\left\{\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^TS_B,0),\max(E-\omega^TS,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}\right.$$
$$\left.-v^TS-y^TS_B\mathrm{e}^{r(T-t^*)}\right\}=$$
$$\max_{S_B\in\mathcal{I}_{(6.4)}}\left\{\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^TS_B,0),\max(E-\omega^TS,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}\right.$$
$$\left.-v^TS-y^TS_B\mathrm{e}^{r(T-t^*)}\right\},$$

and in the non-early exercise case, that

$$\max_{S\in\mathbb{R}^n_+}\Bigg\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}$$
$$-v^T S-y^T S_B e^{r(T-t^*)}\Bigg\}=$$
$$\max_{S\in\mathcal{I}_{(6.4)}}\Bigg\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}$$
$$-v^T S-y^T S_B e^{r(T-t^*)}\Bigg\}.$$

This is because, since $(u^1,u^2,\ldots,u^q,v,y,z)\in\mathcal{F}_{(6.4)}$, it holds that

$$\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)+ze^{rT}+v^T S+y^T S_B e^{r(T-t^*)}\geqslant$$
$$\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\},\quad\forall\, S_B,\ S\in\mathcal{I}_{(6.4)}.$$

Thus,

$$\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}$$
$$-v^T S-y^T S_B e^{r(T-t^*)}\leqslant 0\ \ \forall\, S_B,\ S\in\mathcal{I}_{(6.4)},$$

and so,

$$\max_{S_B\in\mathcal{I}_{(6.4)}}\Bigg\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)$$
$$-ze^{rT}-v^T S-y^T S_B e^{r(T-t^*)}\Bigg\}\leqslant 0,$$

in the early exercise case, and

$$\max_{S\in\mathcal{I}_{(6.4)}}\Bigg\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)$$
$$-ze^{rT}-v^T S-y^T S_B e^{r(T-t^*)}\Bigg\}\leqslant 0,$$

in the non-early exercise case.
So, if in the early exercise case we have

$$\max_{S_B\in\mathbb{R}^n_+}\Bigg\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}$$
$$-v^T S-y^T S_B e^{r(T-t^*)}\Bigg\}=$$
$$\max_{S_B\in\mathcal{I}_{(6.4)}}\Bigg\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}-$$
$$v^T S-y^T S_B e^{r(T-t^*)}\Bigg\},$$

and in the non-early exercise case we have

$$\max_{S\in\mathbb{R}^n_+}\left\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}\right.$$
$$\left.-v^T S-y^T S_B e^{r(T-t^*)}\right\}=$$
$$\max_{S\in\mathcal{I}_{(6.4)}}\left\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}-\right.$$
$$\left.v^T S-y^T S_B e^{r(T-t^*)}\right\},$$

it means that

$$\max_{S_B\in\mathbb{R}^n_+}\left\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}-\right.$$
$$\left.v^T S-y^T S_B e^{r(T-t^*)}\right\}\leqslant 0,$$

and

$$\max_{S\in\mathbb{R}^n_+}\left\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}-\right.$$
$$\left.v^T S-y^T S_B e^{r(T-t^*)}\right\}\leqslant 0,$$

respectively.
Thus,

$$\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}-$$
$$v^T S-y^T S_B e^{r(T-t^*)}\leqslant 0\ \ \forall\ S_B,\ S\in\mathbb{R}^n_+,$$
$$\iff \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)+ze^{rT}+v^T S+y^T S_B e^{r(T-t^*)}\geqslant$$
$$\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\},\ \ \forall\ S_B,\ S\in\mathbb{R}^n_+,$$

in which case the proposition is proved.
We now show that, in the early exercise case

$$\max_{S_B\in\mathbb{R}^n_+}\left\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}\right.$$
$$\left.-v^T S-y^T S_B e^{r(T-t^*)}\right\}=$$
$$\max_{S_B\in\mathcal{I}_{(6.4)}}\left\{\max\{e^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-ze^{rT}\right.$$
$$\left.-v^T S-y^T S_B e^{r(T-t^*)}\right\},$$

and in the non-early exercise case

$$\max_{S\in\mathbb{R}^n_+}\Big\{\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}$$

$$-v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}\Big\}=$$

$$\max_{S\in\mathcal{I}_{(6.4)}}\Big\{\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}$$

$$-v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}\Big\},$$

is indeed the case.
Again we consider two cases.
Case 1: The American basket put option is exercised early. In this case we are interested in the asset price S_B. The price of the assets at expiry, given by S, is irrelevant in this case and may be treated as an unknown constant.
In the early exercise case, we have that $E-\omega^T S_B>0$ since early exercise of the Bermuda basket put option would only take place if it were *in the money*.
We define the function $\psi:\mathbb{R}^n_+\to\mathbb{R}$, given by

$$\psi(S_B)=\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}-$$
$$v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}.$$

In this case the function $\psi(S_B)$ becomes,

$$\psi(S_B)=\max\{\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B),\max(E-\omega^T S,0)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}-$$
$$v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}.$$

Consider $\nabla\psi(S_B)$, for all $S_B\notin\mathcal{I}_{(6.4)}$. Then we have the following cases to consider.
(a) If $E-\omega^T S>0$, then $\psi(S_B)$ becomes,

$$\psi(S_B)=\max\{\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B),(E-\omega^T S)\}-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}-$$
$$v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}.$$

Then we consider two sub-cases as follows.
(*i*) If $\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B)>(E-\omega^T S)$, then $\psi(S_B)$ becomes,

$$\psi(S_B)=\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B)-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}-v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}.$$

This means

$$\nabla\psi(S_B)=\frac{\mathrm{d}\psi}{\mathrm{d}S_B}=-\mathrm{e}^{r(T-t^*)}\omega-\mathrm{e}^{r(T-t^*)}y=\mathrm{e}^{r(T-t^*)}(-\omega-y).$$

(*ii*) If $(E-\omega^T S)>\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B)$, then $\psi(S_B)$ becomes,

$$\psi(S_B)=(E-\omega^T S)-\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0)-z\mathrm{e}^{rT}-v^T S-y^T S_B\mathrm{e}^{r(T-t^*)}.$$

This means

$$\nabla\psi(S_B) = \frac{\mathrm{d}\psi}{\mathrm{d}S_B} = -\mathrm{e}^{r(T-t^*)}y.$$

(b) If $E - \omega^T S < 0$, then $\psi(S_B)$ becomes,

$$\psi(S_B) = \max\{\mathrm{e}^{r(T-t^*)}(E - \omega^T S_B), 0\} - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

This is the same as

$$\psi(S_B) = \mathrm{e}^{r(T-t^*)}(E - \omega^T S_B) - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)},$$

since we only exercise early if the option is in the money, (see above). This means,

$$\nabla\psi(S_B) = \frac{\mathrm{d}\psi}{\mathrm{d}S_B} = -\mathrm{e}^{r(T-t^*)}\omega - \mathrm{e}^{r(T-t^*)}y = \mathrm{e}^{r(T-t^*)}(-\omega - y).$$

Now, from (6.6) however,we observe that

$$y \geqslant \max\{-\omega, 0\} \geqslant -\omega \implies y \geqslant -\omega \iff -y \leqslant \omega$$

$$\text{so } -\omega - y \leqslant 0. \text{ This means } \mathrm{e}^{r(T-t^*)}(-\omega - y) \leqslant 0.$$

$$\text{Also, } y \geqslant \max\{-\omega, 0\} \geqslant 0 \implies y \geqslant 0 \iff -y \leqslant 0.$$

$$\text{This means } -\mathrm{e}^{r(T-t^*)}y \leqslant 0.$$

$\implies$ In all cases (a) and (b), for all $S_B \notin \mathcal{I}_{(6.4)}$,

$$\nabla\psi(S_B) \leqslant 0 \implies \psi(S_B) \text{ is non-increasing for all } S_B \notin \mathcal{I}_{(6.4)}.$$

For case 2 we have the following.
Case 2: The American basket put option is **not** exercised early. In this case we are interested in the asset price vector at expiry, given by $S \in \mathbb{R}^n_+$. The price of the assets at any time before T, that is, the price of the assets at any time before expiry, and in particular at any potential S_B, is irrelevant in this case and may be treated as an unknown constant.
We define the function $\tilde{\psi} : \mathbb{R}^n_+ \to \mathbb{R}$, given by

$$\tilde{\psi}(S) = \max\{\mathrm{e}^{r(T-t^*)}\max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\} - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

Consider $\nabla\tilde{\psi}(S)$, for all $S \notin \mathcal{I}_{(6.4)}$. Then we have the following cases to consider.
(a) If $E - \omega^T S_B > 0$, then $\tilde{\psi}(S)$ becomes

$$\tilde{\psi}(S) = \max\{\mathrm{e}^{r(T-t^*)}(E - \omega^T S_B), \max(E - \omega^T S, 0)\} - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

Then we consider the following two sub-cases.
(*i*) If $E - \omega^T S > 0$, then $\tilde{\psi}(S)$ becomes,

$$\tilde{\psi}(S) = \max\{\mathrm{e}^{r(T-t^*)}(E - \omega^T S_B), (E - \omega^T S)\} - \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

Then we have the following.
1) If $\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B) > (E-\omega^T S)$, then $\tilde{\psi}(S)$ becomes

$$\tilde{\psi}(S) = \mathrm{e}^{r(T-t^*)}(E-\omega^T S_B) - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

This means

$$\nabla\tilde{\psi}(S) = \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}S} = -\sum_{l=1}^{q} u^l - v.$$

2) If $(E-\omega^T S) > \mathrm{e}^{r(T-t^*)}(E-\omega^T S_B)$, then $\tilde{\psi}(S)$ becomes

$$\tilde{\psi}(S) = (E-\omega^T S) - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

This means,

$$\nabla\tilde{\psi}(S) = \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}S} = -\omega - \sum_{l=1}^{q} u^l - v.$$

(ii) If $E-\omega^T S < 0$, then $\tilde{\psi}(S)$ becomes,

$$\tilde{\psi}(S) = \max\{\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B), 0\} - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

However since $\mathrm{e}^{r(T-t^*)} > 0$, then $\implies$ $\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B) > 0$ for this case since $E-\omega^T S_B > 0$. This means $\tilde{\psi}(S)$ is

$$\tilde{\psi}(S) = \mathrm{e}^{r(T-t^*)}(E-\omega^T S_B) - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

This gives

$$\nabla\tilde{\psi}(S) = \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}S} = -\sum_{l=1}^{q} u^l - v.$$

(b) If $E-\omega^T S_B < 0$, then $\tilde{\psi}(S)$ becomes,

$$\tilde{\psi}(S) = \max\{0, \max(E-\omega^T S, 0)\} - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)}.$$

Then, again we consider the following two sub-cases.
(i) If $E-\omega^T S > 0$, then $\tilde{\psi}(S)$ becomes,

$$\tilde{\psi}(S) = \max\{0, (E-\omega^T S)\} - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)},$$

which in this case is the same as

$$\tilde{\psi}(S) = (E-\omega^T S) - \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)},$$

This gives,

$$\nabla\tilde{\psi}(S) = \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}S} = -\omega - \sum_{l=1}^{q} u^l - v.$$

(ii) If $E - \omega^T S < 0$, then $\tilde{\psi}(S)$ becomes,

$$\tilde{\psi}(S) = \max\{0,0\} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)},$$

which is the same as,

$$\tilde{\psi}(S) = -\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)},$$

This gives

$$\nabla\tilde{\psi}(S) = \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}S} = -\sum_{l=1}^{q} u^l - v.$$

Now, from (6.7) however,we observe that

$$\sum_{l=1}^{q} u^l + v \geqslant \max\{-\omega, 0\} \geqslant 0 \implies \sum_{l=1}^{q} u^l + v \geqslant 0 \iff -\sum_{l=1}^{q} u^l - v \leqslant 0, \text{ and}$$

$$\sum_{l=1}^{q} u^l + v \geqslant \max\{-\omega, 0\} \geqslant -\omega \implies \sum_{l=1}^{q} u^l + v \geqslant -\omega$$

$$\iff -\sum_{l=1}^{q} u^l - v \leqslant \omega \implies -\omega - \sum_{l=1}^{q} u^l - v \leqslant 0.$$

$\implies$ In all cases (a) and (b), for all $S \notin \mathcal{I}_{(6.4)}$,

$$\nabla\tilde{\psi}(S) \leqslant 0 \implies \tilde{\psi}(S) \text{ is non-increasing for all } S \notin \mathcal{I}_{(6.4)}.$$

Thus, in both cases we have established that ψ and $\tilde{\psi}$ are non-increasing for all $S_B,\ S \notin \mathcal{I}_{(6.4)}$, respectively. This means that $\psi(S_B)$ and $\tilde{\psi}(S)$ must attain its maximum value for a value of $S_B,\ S \in \mathcal{I}_{(6.4)}$, respectively. Thus, it holds that; in the early exercise case

$$\max_{S_B \in \mathbb{R}^n_+} \Big\{ \max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)} \Big\} =$$

$$\max_{S_B \in \mathcal{I}_{(6.4)}} \Big\{ \max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)} \Big\},$$

and in the non-early exercise case

$$\max_{S \in \mathbb{R}^n_+} \Big\{ \max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)} \Big\} =$$

$$\max_{S \in \mathcal{I}_{(6.4)}} \Big\{ \max\{\mathrm{e}^{r(T-t^*)} \max(E - \omega^T S_B, 0), \max(E - \omega^T S, 0)\} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S - y^T S_B \mathrm{e}^{r(T-t^*)} \Big\},$$

as was to be shown, and the proposition is proved. □

We now show that the semi-infinite optimisation problem (6.4), can be re-formulated as a finite linear problem.
We recall (6.4) as

$$\begin{aligned}
&\min_{u^l,v,y,z\,\in\mathbb{R}^{n\times q}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} && \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 + y^T S^0 \\
&\text{subject to} && \sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} \geqslant \\
& && \max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0), \max(E-\omega^T S,0)\},\ \ \forall\ S_B,\ S\in\mathcal{I}_{(6.4)} \\
& && y \geqslant \max\{-\omega,0\} \\
& && \sum_{l=1}^{q} u^l + v \geqslant \max\{-\omega,0\}.
\end{aligned} \tag{6.4}$$

Now, the first constraint is equivalent to the following three semi-infinite constraints (6.8), (6.9) and (6.10), by observing that

$$\begin{aligned}
\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0), \max(E-\omega^T S,0)\} &\geqslant \\
\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0) \geqslant \mathrm{e}^{r(T-t^*)}(E-\omega^T S_B)&,
\end{aligned}$$

and

$$\begin{aligned}
\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0), \max(E-\omega^T S,0)\} &\geqslant \\
\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0) \geqslant 0&,
\end{aligned}$$

and

$$\begin{aligned}
\max\{\mathrm{e}^{r(T-t^*)}\max(E-\omega^T S_B,0), \max(E-\omega^T S,0)\} &\geqslant \\
\max(E-\omega^T S,0) \geqslant E-\omega^T S&.
\end{aligned}$$

$\implies$

$$\begin{aligned}
\sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} &\geqslant \\
\mathrm{e}^{r(T-t^*)}(E-\omega^T S_B),\ \ \forall\ S_B,\ S\in\mathcal{I}_{(6.4)}&
\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}
\sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} &\geqslant \\
0,\ \ \forall\ S_B,\ S\in\mathcal{I}_{(6.4)}&
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
\sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S + y^T S_B \mathrm{e}^{r(T-t^*)} &\geqslant \\
E-\omega^T S,\ \ \forall\ S_B,\ S\in\mathcal{I}_{(6.4)}&.
\end{aligned} \tag{6.10}$$

Which is equivalent to

$$\begin{aligned}
\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i-E_i^l,0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i + \mathrm{e}^{r(T-t^*)}\sum_{i=1}^{n} y_i S_{B_i} \geqslant E\mathrm{e}^{r(T-t^*)} - \mathrm{e}^{r(T-t^*)}\sum_{i=1}^{n}\omega_i S_{B_i}&, \\
\forall\ S_B,\ S\in\mathcal{I}_{(6.4)}&,
\end{aligned}$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i-E_i^l,0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i + \mathrm{e}^{r(T-t^*)}\sum_{i=1}^{n} y_i S_{B_i} \geqslant 0,\ \ \forall\ S_B,\ S\in\mathcal{I}_{(6.4)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i + \mathrm{e}^{r(T-t^*)} \sum_{i=1}^{n} y_i S_{B_i} \geqslant E - \sum_{i=1}^{n} \omega_i S_i, \quad \forall\ S_B,\ S \in \mathcal{I}_{(6.4)},$$

respectively.
$\Longleftrightarrow$

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i + \mathrm{e}^{r(T-t^*)} \sum_{i=1}^{n} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \sum_{i=1}^{n} \omega_i S_{B_i} + z\mathrm{e}^{rT} - E\mathrm{e}^{r(T-t^*)} \geqslant 0,$$
$$\forall\ S_B,\ S \in \mathcal{I}_{(6.4)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i + \mathrm{e}^{r(T-t^*)} \sum_{i=1}^{n} y_i S_{B_i} + z\mathrm{e}^{rT} \geqslant 0, \quad \forall\ S_B,\ S \in \mathcal{I}_{(6.4)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i + \mathrm{e}^{r(T-t^*)} \sum_{i=1}^{n} y_i S_{B_i} + \sum_{i=1}^{n} \omega_i S_i + z\mathrm{e}^{rT} - E \geqslant 0, \quad \forall\ S_B,\ S \in \mathcal{I}_{(6.4)},$$

respectively.
Switching the order of summation gives

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \right)}_{(\star)} + z\mathrm{e}^{rT} - E\mathrm{e}^{r(T-t^*)} \geqslant 0,$$

$\forall\ S_B,\ S \in \mathcal{I}_{(6.4)}$,

and

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \right)}_{(\star\star)} + z\mathrm{e}^{rT} \geqslant 0, \quad \forall\ S_B,\ S \in \mathcal{I}_{(6.4)},$$

and

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \right)}_{(\star\star\star)} + z\mathrm{e}^{rT} - E \geqslant 0, \quad \forall\ S_B,\ S \in \mathcal{I}_{(6.4)},$$

respectively.
Before proceeding we observe that the left hand side is identical in (6.8), (6.9) and (6.10). Therefore, it is possible to derive the first constraint in the optimisation problem (6.4), from (6.8), (6.9) and (6.10) by using the following property of the *maximum*.

$$\text{If } a \geqslant b \text{ and } a \geqslant c \implies a \geqslant \max\{b, c\}.$$

Now we choose $\alpha,\ \beta,\ \gamma \in \mathbb{R}^n$ such that α_i provides a lower bound to $(\star)$, β_i provides a lower bound to $(\star\star)$, and γ_i provides a lower bound to $(\star\star\star)$, for all $i = 1, 2, \ldots, n$. That is, we choose $\alpha_i, \beta_i, \gamma_i$, for all i, such that

$$\alpha_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i}$$

and

$$\beta_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i},$$

and

$$\gamma_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i,$$

for all $i = 1, 2, \ldots, n$.
$\Longrightarrow$ The semi-infinite constraints of (6.4) become

$$\begin{cases} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \geqslant \alpha_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n \\ \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \geqslant \beta_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n \\ \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \geqslant \gamma_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} \alpha_i + z\mathrm{e}^{rT} - E\mathrm{e}^{r(T-t^*)} \geqslant 0 \\ \sum_{i=1}^{n} \beta_i + z\mathrm{e}^{rT} \geqslant 0 \\ \sum_{i=1}^{n} \gamma_i + z\mathrm{e}^{rT} - E \geqslant 0. \end{cases} \tag{6.11}$$

This means that a point $(u^1, u^2, \ldots, u^q, v, y, z)$ is feasible for (6.4) if and only if
$(u^1, u^2, \ldots, u^q, v, y, z, \alpha, \beta, \gamma) \in \mathbb{R}^{n\times q} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ is feasible for system (6.11). We further observe that by writing the semi-infinite constraints of (6.4) as system (6.11), we have that the last three constraints are standard (finite) linear constraints and the first $3n$ constraints are semi-infinite but the index set is now the bounded, closed interval $[0, E_i^q]$, for all $i = 1, 2, \ldots, n$.
Now we consider the semi-infinite constraints from system (6.11). Then, we have

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \geqslant \alpha_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n,$$

and

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \geqslant \beta_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n,$$

and

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \geqslant \gamma_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n.$$

The semi-infinite constraints of system (6.11) are piece-wise linear constraints. This means that the minimum value of the left hand side of the inequalities over all values of S_{B_i} and S_i in the interval $[0, E_i^q]$ occurs at one of the breakpoints. That is, it occurs exactly when S_{B_i} and /or $S_i = 0$ or S_{B_i} and/or $S_i = E_i^1$ or S_{B_i} and/or $S_i = E_i^2$ or $\ldots$ or S_{B_i} and/or $S_i = E_i^q$, for all $i = 1, 2, \ldots, n$. Thus, we may consider the semi-infinite constraints above for the values $S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}$, for all $i = 1, 2, \ldots, n$.

Hence, it holds that

$$\begin{aligned}
&\min_{S_{B_i},\, S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \\
= &\min_{S_{B_i},\, S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i},
\end{aligned}$$

and

$$\begin{aligned}
&\min_{S_{B_i},\, S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \\
= &\min_{S_{B_i},\, S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i},
\end{aligned}$$

and

$$\begin{aligned}
&\min_{S_{B_i},\, S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \\
= &\min_{S_{B_i},\, S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i.
\end{aligned}$$

Therefore, each of the semi-infinite constraints can now be replaced by $(q+1)^2$ finite piece-wise linear constraints. That is, we may replace the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \geqslant \alpha_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n,$$

by

$$\begin{aligned}
&\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \geqslant \alpha_i, \text{ for } S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \\
&\forall\ i = 1, 2, \ldots, n
\end{aligned}$$

and the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \geqslant \beta_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n,$$

by

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \geqslant \beta_i, \text{ for } S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\},\ \forall\ i = 1, 2, \ldots, n,$$

and the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \geqslant \gamma_i, \ \forall\ S_{B_i},\ S_i \in [0, E_i^q],\ \forall\ i = 1, 2, \ldots, n,$$

by

$$\begin{aligned}
&\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \geqslant \gamma_i, \text{ for } S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \\
&\forall\ i = 1, 2, \ldots, n.
\end{aligned}$$

We may summarise the above analysis in the following theorem, which we have derived from above.

Theorem 6.12. *The semi-infinite optimisation problem* (6.4) *is equivalent to the following finite linear optimisation problem*

$$\begin{aligned}
\min_{(u^1,u^2,\dots,u^q,v,y,z)\in\mathbb{R}^{n\times q}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R},\alpha,\beta,\gamma\in\mathbb{R}^n} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 + y^T S^0 \\
\textit{subject to} \quad & \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \\
& \mathrm{e}^{r(T-t^*)} \omega_i S_{B_i} \geqslant \alpha_i, \\
& \textit{for } S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \dots, E_i^q\},\ \forall\ i = 1,2,\dots,n \\
& \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} \geqslant \beta_i, \\
& \textit{for } S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \dots, E_i^q\},\ \forall\ i = 1,2,\dots,n \\
& \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i + \mathrm{e}^{r(T-t^*)} y_i S_{B_i} + \omega_i S_i \geqslant \gamma_i, \\
& \textit{for } S_{B_i},\ S_i \in \{0, E_i^1, E_i^2, \dots, E_i^q\},\ \forall\ i = 1,2,\dots,n \\
& \sum_{i=1}^{n} \alpha_i + z\mathrm{e}^{rT} - E\mathrm{e}^{r(T-t^*)} \geqslant 0 \\
& \sum_{i=1}^{n} \beta_i + z\mathrm{e}^{rT} \geqslant 0 \\
& \sum_{i=1}^{n} \gamma_i + z\mathrm{e}^{rT} - E \geqslant 0 \\
& y \geqslant \max\{-\omega, 0\} \\
& \sum_{l=1}^{q} u^l + v \geqslant \max\{-\omega, 0\},
\end{aligned} \tag{6.13}$$

in the sense that both optimisation problems have the same feasible region and hence the same optimal solution and the same optimal objective function value.

Observation: In comparison to the semi-infinite problem (6.4); we observe that (6.13) has $n+n+n=3n$ additional variables and a total of

$$\begin{aligned}
n(q+1)^2 + n(q+1)^2 + n(q+1)^2 + 3 + n + n &= 3n(q+1)^2 + 3 + 2n = 2n + 3n(q+1)^2 + 3 \\
= 2n + 3n(q^2+2q+1) + 3 = 2n + 3nq^2 + 6nq + 3n + 3 &= 5n + 6nq + 3nq^2 + 3 = n(3q^2+6q+5) + 3
\end{aligned}$$

linear constraints. The advantage of solving (6.13) in comparison to (6.4) is that we are solving a standard, finite linear problem in comparison to a semi-infinite one.
Therefore, we have derived a solvable optimisation model which can be used to find upper bounds on the current price of this particular American basket put option.
We observe here that the model set up and approach to finding an upper bound on the price of a Bermuda basket put option presented above is very similar to the model set up presented in sub-section 3.4 which looked at finding an upper bound on the price of a European basket call option. Further, the methodology employed in the proof of Proposition 6.3 can be viewed as an extension to the methodology employed in the proof of Proposition 3.14 in sub-section 3.4. It is an extension because in the proof of Proposition 6.3 we had to consider the early exercise feature of the Bermuda basket put option; something which was not required of course in the proof of Proposition 3.14. Similarly, the ideas used to obtain the finite LO problem (6.13) can also be viewed as an extension to the techniques used in sub-section 3.4 to re-write the SIO problem

(3.13) as a solvable and finite LO problem.
As a concluding remark, we note here that the above model may be extended to the case where there are multiple early exercise dates for the Bermuda basket put option.
This is done by recalling one of the assumptions we made about the Bermuda basket put option at the beginning of this section. We assumed that the Bermuda option had a sole early exercise time given by t^*. In reality however most of the Bermuda options traded have multiple exercise times. Fortunately however, we can derive price bounds on a Bermuda basket put option with multiple early exercise times by proceeding in exactly the same way as we did above.
In this case the objective function and one of the constraints would have to be modified. We assume that the Bermuda basket put option can be exercised at $(k-1)$ early exercise times before expiry or on the expiry date if it has not already been exercised.
In order to derive the optimisation model to capture the feature of multiple early exercise times we introduce the following notation.

- Let t_j denote the fixed time points for which the option can be exercised early, for $j = 1, 2, \dots, k$; where $t_k = T$, which is the expiry date of the option.
- Let $S_i(t_j) \in \mathbb{R}_+$ denote the value of the i^{th} asset at time t_j, for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, k$. Note here that under this notation we have $S_i(t_k) = S_i(T) = S_i$, for $i = 1, 2, \dots, n$.
- Retain all other notation from the beginning of this section.

Then we can model the problem of finding an upper bound on the price of a Bermuda basket put option, with multiple early exercise times as the following optimisation problem which is a dual of a SIO problem.

$$\begin{aligned}
\sup_{\pi} \quad & \mathbb{E}_\pi \left[\max_{j=1,2,\dots,k} \left(\mathrm{e}^{-rt_j} \max(E - \omega^T S(t_j), 0) \right) \right] \\
\text{subject to} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(S_i - E_i^l, 0)] = C_i^l, \quad \text{for } i = 1, 2, \dots, n \text{ and } l = 1, 2, \dots, q \\
& \mathbb{E}_\pi[\mathrm{e}^{-rt_j} S_i(t_j)] = S_i^0, \quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, k \\
& \mathbb{E}_\pi[1] = 1.
\end{aligned} \tag{6.14}$$

We remark here that the constraint

$$\mathbb{E}_\pi[\mathrm{e}^{-rt_j} S_i(t_j)] = S_i^0, \quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, k$$

means that the discounted expected value of the asset price vector at each time point t_j, for $j = 1, 2, \dots, k$ is equal to the current asset price vector, where we recall that $t_k = T$, which is the expiry date of the option.
We can now derive the SIO problem for which (6.14) is its dual and proceed as above to re-write this SIO problem as a SIO problem with a compact index set and then equivalently re-write this SIO problem with a compact index set as a finite and solvable LO problem. This concludes our analysis on using SIO to find upper bounds on the current price of a Bermuda basket put option.

7 Finding price bounds on Asian Basket Options

Recall that in Section 6 we presented one extension to the European basket option pricing problem. Namely, we saw how to calculate price bounds on a *Bermuda* basket put option. In this section we consider another extension to the European basket option pricing problem. That is, we now consider how to find price bounds on *Asian* basket options.

7.1 Introduction

Before proceeding to the analysis on finding price bounds for *Asian* basket options, we first formally define what we mean by an *Asian* option.
Put simply, an *Asian* option is a path-dependent option whose payoff depends upon an average of its underlying asset(s) price. This average is calculated by considering the prices of the underlying asset(s) throughout the life of the option, including the expiry date, T. Thus, *Asian* options are often referred to as, *options on the average.*
The simplest example of an Asian option is an Asian vanilla call option. If $\bar{S}$ is the average price of the underlying asset from time $t = 0$ to $t = T$ which is the expiry date, and $E \geqslant 0$ is the exercise price, then the payoff of an Asian vanilla call option would be given by

$$\max(\bar{S} - E, 0). \tag{7.1}$$

We observe that the way in which the *average* in the payoff of an Asian option is calculated is different from option to option. It is up to the holder and writer of the Asian option under consideration to agree, before the start of the option, how the average will be calculated.
Letting $S(t)$ denote the price of the underlying asset at time t, for all $t \in [0, T]$, then the common ways to calculate the average in an Asian option are given below.

1. *Arithmetic average* in the continuous case. Here we assume that t can take any value between 0 and T. Then under this assumption the value of $\bar{S}$ is given by

$$\bar{S} = \frac{1}{T}\int_0^T S(t)\ \mathrm{d}t. \tag{7.2}$$

2. *Arithmetic average* in the discrete case. Here we assume that the average depends upon the prices of the underlying asset(s) at N discrete time points in the interval $[0, T]$. If these time points are given by $t_1, t_2, \ldots, t_N = T$, then the value of $\bar{S}$ is given by

$$\bar{S} = \frac{1}{N}\sum_{j=1}^{N} S(t_j). \tag{7.3}$$

3. *Geometric average* in the continuous case. In this case we assume that t can take any value between 0 and T. Then the value of $\bar{S}$ is given by

$$\bar{S} = \mathrm{e}^{\frac{1}{T}\int_0^T \ln(S(t))\ \mathrm{d}t}. \tag{7.4}$$

There are other recent results on finding current prices and finding current price bounds on Asian options in [46, 47, 48, 49, 50, 51, 52]. In particular, in [46] current prices for Asian basket options using moment matching procedures are derived. In [47] pricing methods for Asian arithmetic average basket options in a Black-Scholes framework using a Quasi-Monte Carlo method are derived. In [48] prices of Asian basket options using limit distributions of sums of log-normal variables are derived. In [49] an analytical approximation approach is used to find current prices of Asian basket options. In [50] upper and lower bounds on the current price of an Asian basket option are derived by using a conditioning variable approach. In [51] upper and lower bounds on the current price of discrete arithmetic Asian basket options are derived in a Black-Scholes framework.

Finally, in [52] model-independent lower bounds for general arithmetic average Asian options are derived, using a similar approach to what is done here but not specifically for basket options.
Our approach to finding upper bounds on the current price of an Asian basket option is new because we model the problem as a dual to a linear SIO problem and then re-write this linear SIO as a finite and solvable linear problem in a way which has not been done before.

7.2 Upper bound on the price of an Asian basket call option using SIO

We now consider how semi-infinite optimisation can be used to find current price bounds on Asian basket options.
In this sub-section we consider a similar model set-up and solution approach to that given in sub-section 3.4 for European basket call options. We note here that the results obtained in this sub-section are new, in the sense that finding price bounds on an Asian basket option by using semi-infinite optimisation using the solution approach which we consider here has not been done before.
We first describe the model set-up and recall and introduce some notation. In what follows we consider finding an upper bound on the current price of an Asian basket call option, given that we know the current prices of q Asian vanilla call options per asset and the expected (forward) price per asset, under a risk-neutral probability measure, which is to be found.
We consider finding price bounds on an Asian basket call option written on n underlying assets, and we introduce the following notation.

- Let $\omega_i \in \mathbb{R}$ denote the weight of the i^{th} asset, for $i = 1, 2, \ldots, n$.
- Let the price of the i^{th} underlying asset at time t be denoted by $S_i(t) \in \mathbb{R}_+$, for all $i = 1, 2, \ldots, n$.
- Let the exercise price of the Asian basket option whose current price we are bounding be denoted by $E \in \mathbb{R}_+$.
- For the q Asian vanilla options per asset whose current price we know, let the l^{th} exercise price for the i^{th} underlying asset be given by $E_i^l \in \mathbb{R}_+$, for $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, n$.
- Let the l^{th} current price of the Asian vanilla option written on asset i be given by $A_i^l \in \mathbb{R}_+$, for $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, n$.
- Further, the current price of the i^{th} asset will be denoted by $S_i^0 \in \mathbb{R}_+$, for all $i = 1, 2, \ldots, n$.

We will adopt the notational convention for the vector form of these variables as given in Section 3.
Furthermore, we will assume that **all** options in the model have the same expiry date, T. Also, for the purposes of this thesis we consider the case where for **all** options in the model, the average of the n underlying asset prices is calculated by an *arithmetic discrete* average. That is, for all options we consider the asset prices of all n underlying assets at times $t_1, t_2, \ldots, t_N = T$ and calculate the arithmetic mean using these values to obtain $\bar{S}$. We will also assume that the risk-free interest rate is equal to zero.
Then this gives us

$$\bar{S} = \frac{1}{N} \sum_{j=1}^{N} S(t_j),$$

and so the l^{th} payoff for the Asian vanilla call option written on asset i, whose current price we know and is given by A_i^l, is

$$\max\left(\frac{1}{N} \sum_{j=1}^{N} S_i(t_j) - E_i^l, 0\right), \tag{7.5}$$

for all $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, n$.
Similarly, the payoff of the Asian basket call option whose current price we are bounding is given by

$$\max\left(\sum_{i=1}^{n} \omega_i \left(\frac{1}{N} \sum_{j=1}^{N} S_i(t_j)\right) - E, 0\right). \tag{7.6}$$

Using (7.5) and (7.6) and the model setting outlined above, we may model the task of finding an upper bound on the current price of an Asian basket call option as the following optimisation problem which is a dual of a linear SIO problem.

$$
\begin{aligned}
\sup_{\pi} \quad & \mathbb{E}_{\pi}\left[\max\left(\sum_{i=1}^{n}\omega_i\left(\frac{1}{N}\sum_{j=1}^{N}S_i(t_j)\right)-E,0\right)\right] \\
\text{subject to} \quad & \mathbb{E}_{\pi}\left[\max\left(\frac{1}{N}\sum_{j=1}^{N}S_i(t_j)-E_i^l,0\right)\right]=A_i^l, \quad \forall\, i=1,2,\ldots,n \text{ and } l=1,2,\ldots,q \\
& \mathbb{E}_{\pi}[1]=1 \\
& \mathbb{E}_{\pi}[S_i(t_j)]=S_i^0, \quad \forall\, j=1,2,\ldots,N, \quad i=1,2,\ldots,n.
\end{aligned}
\tag{7.7}
$$

Deriving the linear SIO problem for which (7.7) is its dual, we obtain the problem

$$
\begin{aligned}
\inf_{(u^l,z,v(t_j))\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^{n\times N}} \quad & \sum_{l=1}^{q}(u^l)^TA^l+z+\sum_{j=1}^{N}(v(t_j))^TS^0 \\
\text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T\max\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)-E^l,0\right)+z+\sum_{j=1}^{N}(v(t_j))^TS(t_j)\geqslant \\
& \max\left(\sum_{i=1}^{n}\omega_i\left(\frac{1}{N}\sum_{j=1}^{N}S_i(t_j)\right)-E,0\right), \quad \forall\, S(t_j)\in\mathbb{R}_+^n,\ \forall\, j=1,2,\ldots,N.
\end{aligned}
\tag{7.8}
$$

Now, we assume that the investors position in the super-replicating portfolio does not change with respect to time. That is, once the investor has initially decided how many Asian vanilla call options to buy, how much cash to invest and how much of each underlying asset to buy, at time $t=0$, he holds these amounts throughout the duration of the portfolio, that is, until expiry $t=T$. This means that all of the variables u^l, z and $v(t_j)$ in the optimisation model are independent of time t. In particular, we may replace $v(t_j)$ by some vector, independent of t, say v, for all j, so that we have $v(t_j)=v$, for all $j=1,2,\ldots,N$.
Now we recall the definition of the discrete arithmetic average as

$$\bar{S}=\frac{1}{N}\sum_{j=1}^{N}S(t_j),$$

that is,

$$\bar{S}_i=\frac{1}{N}\sum_{j=1}^{N}S_i(t_j),\ \forall\, i=1,2,\ldots,n.$$

Then under these stated assumptions we may equivalently re-write the semi-infinite optimisation problem (7.8) as

$$
\begin{aligned}
\inf_{(u^l,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n} \quad & \sum_{l=1}^{q}(u^l)^TA^l+z+Nv^TS^0 \\
\text{subject to} \quad & \sum_{l=1}^{q}(u^l)^T\max\left(\bar{S}-E^l,0\right)+z+Nv^T\bar{S}\geqslant\max\left(\omega^T\bar{S}-E,0\right), \\
& \forall\, \bar{S}\in\mathbb{R}_+^n, \\
& \bar{S}=\frac{1}{N}\sum_{j=1}^{N}S(t_j).
\end{aligned}
\tag{7.9}
$$

This SIO problem (7.9) and the variables have a natural financial interpretation. We may interpret the variables (u^l,z,v) as follows. u^l which has components u_i^l represents the amount of the l^{th} Asian vanilla

call option written on the i^{th} asset in the super-replicating portfolio, for $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, n$. z represents a cash amount. Nv which has components Nv_i, for $i = 1, 2, \ldots, n$ represents the **total** amount of the i^{th} underlying asset which is currently held, for $i = 1, 2, \ldots, n$. We observe here that v_i represents the amount of the i^{th} asset which is held in the portfolio at **each** respective time point t_j for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N$. However since we buy the same amount of the underlying asset today but for N discrete time points, then Nv_i represents the total amount which we currently hold of the i^{th} asset, for $i = 1, 2, \ldots, n$.
The problem (7.9) itself may be interpreted as follows. We are interested in finding the cheapest cost portfolio consisting of Asian vanilla call options, cash and the underlying assets themselves such that the overall value of the portfolio always super-replicates the payoff of the Asian basket call option whose current price we are finding an upper bound on, for all possible non-negative values of the average asset price.
Our task is to reformulate (7.9) as a solvable linear problem which can be solved to obtain the optimal objective function value which is an upper bound on the current price of the Asian basket call option under consideration.
Now, we note here that the index set $\mathcal{I} = \mathbb{R}^n_+$ in (7.9) is not compact. However, as the next proposition shows, if we impose a restriction on how the super-replicating portfolio is to be constructed, we may restrict $\mathcal{I}$ in (7.9) to a compact set without changing the feasible set of the problem.
In particular, if we impose the constraint

$$\sum_{l=1}^{q} u^l + Nv \geqslant \max\{\omega, 0\},$$

it can be shown, (see Proposition 7.10) that we may replace the non-compact index set $\mathbb{R}^n_+$ by a compact index set.

Proposition 7.10. *Suppose without loss of generality that the exercise prices E_i^l are ordered such that $0 \leqslant E_i^1 \leqslant E_i^2 \leqslant \cdots \leqslant E_i^q$, for all $i = 1, 2, \ldots, n$. Define the index set $\mathcal{I}_{(7.11)} = \bigtimes_{i=1}^{n} [0, E_i^q]$. Then the following optimisation problem* (7.11), *is equivalent to* (7.9) *in the sense that both problems have the same feasible set, and, hence the same optimal solution and optimal objective function value.*

$$\begin{aligned}
\min_{(u^l,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n} \quad & \sum_{l=1}^{q} (u^l)^T A^l + z + Nv^T S^0 \\
\text{subject to} \quad & \sum_{l=1}^{q} u^l + Nv \geqslant \max\{\omega, 0\} \\
& \sum_{l=1}^{q} (u^l)^T \max\left(\bar{S} - E^l, 0\right) + z + Nv^T \bar{S} \geqslant \max\left(\omega^T \bar{S} - E, 0\right), \\
& \forall\ \bar{S} \in \mathcal{I}_{(7.11)}, \\
& \bar{S} = \frac{1}{N} \sum_{j=1}^{N} S(t_j).
\end{aligned} \tag{7.11}$$

Remarks

1. *We first remark that the objective functions of* (7.9) *and* (7.11) *are the same. This is because the objective function in* (7.9) *represents the total cost of the super-replicating portfolio at the current time, $t = 0$. Since this is a cost which is paid, the minimal such cost is attained since what ever the minimum cost is, the investor pays it. Therefore, the **inf** in* (7.9) *is attained and can be replaced by **min** as in* (7.11).

2. *The extra constraint in* (7.11) *has a significant financial meaning.*

The constraint

$$\sum_{l=1}^{q} u^l + Nv \geqslant \max\{\omega, 0\},$$

which is equivalent to

$$\sum_{l=1}^{q} u_i^l + Nv_i \geqslant \max\{\omega_i, 0\},\ \forall\ i = 1, 2, \ldots, n$$

means:

(a) If the weight $\omega_i > 0$, this implies $\sum_{l=1}^{q} u_i^l + Nv_i \geqslant \omega_i$, for all $i = 1, 2, \ldots, n$. This means that for a strictly positive weight, ω_i, the total amount of Asian vanilla calls, written on the average price of the i^{th} underlying asset plus the total amount of the holdings of the i^{th} asset over all time points is at least ω_i, for $i = 1, 2, \ldots, n$.

(b) If the weight $\omega_i < 0$, this implies $\sum_{l=1}^{q} u_i^l + Nv_i \geqslant 0$, for all $i = 1, 2, \ldots, n$. This means that for a strictly negative weight, ω_i, the total amount of Asian vanilla calls, written on the average of the i^{th} underlying asset, plus the total amount of holdings of the i^{th} asset over all time points is non-negative, for $i = 1, 2, \ldots, n$.

3. *This extra constraint gives conditions or restrictions on how the super-replicating portfolio should be constructed.*

4. *The advantage of solving* (7.11) *instead of* (7.9) *is that the index set in the semi-infinite constraint of* (7.11) *is compact, albeit* (7.11) *containing additional constraints. Problem* (7.11) *can be solved to obtain the optimal objective function value of* (7.9) *by considering a compact index set $\mathcal{I}_{(7.11)}$ and additional constraints which impose conditions on how the super-replicating portfolio is to be constructed. Thus,* (7.11) *can be used to solve* (7.9) *by considering the values of $\bar{S}$ in the compact set $\mathcal{I}_{(7.11)} = \bigtimes_{i=1}^{n} [0, E_i^q]$ as well as some restrictions on how to construct the super-replicating portfolio, instead of considering values of $\bar{S}$ in the unbounded set $\mathbb{R}_+^n$.*

Proof of Proposition 7.10. We start by observing that the objective functions of (7.9) and (7.11) are the same. Thus in order to show these two problems are equivalent we must show that their respective feasible regions are the same. Let $\mathcal{F}_{(7.9)}$ and $\mathcal{F}_{(7.11)}$ denote the feasible regions of (7.9) and (7.11), respectively. We then show that $\mathcal{F}_{(7.9)} = \mathcal{F}_{(7.11)}$.

The proof comes in two parts.

(i) $\underline{\mathcal{F}_{(7.9)} \subset \mathcal{F}_{(7.11)}}$: Take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(7.9)}$. We then show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(7.11)}$. Now, since $\mathcal{I}_{(7.11)} = \bigtimes_{i=1}^{n} [0, E_i^q]$, then $\mathcal{I}_{(7.11)}$ forms an n-dimensional 'rectangle'. That is, it forms a 'rectangle' in n-dimensional non-negative space and so $\mathcal{I}_{(7.11)} \subset \mathbb{R}_+^n$. Thus, from the constraint in (7.9) we have

$$\sum_{l=1}^{q} (u^l)^T \max(\bar{S} - E^l, 0) + z + Nv^T \bar{S} \geqslant \max(\omega^T \bar{S} - E, 0),\ \ \forall\ \bar{S} \in \mathbb{R}_+^n.$$

It then follows that since $\mathcal{I}_{(7.11)} \subset \mathbb{R}_+^n$, the constraint in (7.9) obviously still holds for all $\bar{S} \in \mathcal{I}_{(7.11)}$. That is,

$$\sum_{l=1}^{q} (u^l)^T \max(\bar{S} - E^l, 0) + z + Nv^T \bar{S} \geqslant \max(\omega^T \bar{S} - E, 0),\ \ \forall\ \bar{S} \in \mathcal{I}_{(7.11)},$$

and so the second constraint from (7.11) holds.

To show that the first constraint holds we have the following. Recall that $(u^1, u^2, \ldots, u^q, z, v)$ satisfies the constraint

$$\sum_{l=1}^{q}(u^l)^T \max(\bar{S}-E^l,0)+z+Nv^T\bar{S} \geqslant \max(\omega^T\bar{S}-E,0), \quad \forall\, \bar{S}\in\mathbb{R}_+^n$$

$$\iff \sum_{l=1}^{q}\sum_{k=1}^{n}(u_k^l)\max(\bar{S}_k-E_k^l,0)+z+N\sum_{k=1}^{n}v_k\bar{S}_k \geqslant \max\left(\sum_{k=1}^{n}\omega_k\bar{S}_k-E,0\right), \quad \forall\, \bar{S}\in\mathbb{R}_+^n.$$

This means that for the asset price vector $\bar{S}$, with $\bar{S}_i=\eta$, for some $\eta>0$ and all other components equal to 0, for $i=1,2,\ldots,n$, the constraint still holds. That is, the constraint holds for the vector $\bar{S}=\begin{pmatrix}0\\0\\\vdots\\\eta\\0\end{pmatrix}$, where η is in the i^{th} position, for $i=1,2,\ldots,n$.

Now, for each $i=1,2,\ldots,n$, $\bar{S}_i=\eta$ and all other components are equal to 0.

For a particular i, $\implies$

$$\sum_{l=1}^{q}(u_i^l)\max(\bar{S}_i-E_i^l,0)+z+Nv_i\bar{S}_i \geqslant \max(\omega_i\bar{S}_i-E,0)$$

$$\iff \sum_{l=1}^{q}(u_i^l)\max(\eta-E_i^l,0)+z+Nv_i\eta \geqslant \max(\omega_i\eta-E,0)$$

$$\iff \sum_{l=1}^{q}(u_i^l)\eta\max\left(1-\frac{E_i^l}{\eta},0\right)+z+Nv_i\eta \geqslant \eta\max\left(\omega_i-\frac{E}{\eta},0\right),$$

and so, if we divide both sides by η we get, (since $\eta>0$)

$$\implies \sum_{l=1}^{q}(u_i^l)\max\left(1-\frac{E_i^l}{\eta},0\right)+\frac{z}{\eta}+Nv_i \geqslant \max\left(\omega_i-\frac{E}{\eta},0\right),$$

and if $\eta\to\infty$, then, $\frac{E_i^l}{\eta}\to 0$, $\frac{z}{\eta}\to 0$ and $\frac{E}{\eta}\to 0$. This gives, in the limit as $\eta\to\infty$

$$\sum_{l=1}^{q}(u_i^l)\max(1,0)+0+Nv_i \geqslant \max(\omega_i,0)$$

$$\iff \sum_{l=1}^{q}u_i^l+Nv_i \geqslant \max(\omega_i,0),$$

which holds for all $i=1,2,\ldots,n$.

This in vector form is just

$$\sum_{l=1}^{q}u^l+Nv \geqslant \max\{\omega,0\}, \tag{7.12}$$

and so the first constraint from (7.11) holds.

(ii) $\underline{\mathcal{F}_{(7.11)}\subset\mathcal{F}_{(7.9)}}$: Now we prove the converse. So, take any $(u^1,u^2,\ldots,u^q,z,v)\in\mathcal{F}_{(7.11)}$. Then, in order to show that $(u^1,u^2,\ldots,u^q,z,v)\in\mathcal{F}_{(7.9)}$, we must show that

$$\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)+z+Nv^T\bar{S} \geqslant \max(\omega^T\bar{S}-E,0), \quad \forall\, \bar{S}\in\mathbb{R}_+^n.$$

To this end, it suffices to show that

$$\max_{\bar{S}\in\mathbb{R}^n_+}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\}=$$
$$\max_{\bar{S}\in\mathcal{I}_{(7.11)}}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\},$$

since $(u^1,u^2,\ldots,u^q,z,v)\in\mathcal{F}_{(7.11)}$ it holds that

$$\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)+z+Nv^T\bar{S}\geqslant\max(\omega^T\bar{S}-E,0),\ \ \forall\,\bar{S}\in\mathcal{I}_{(7.11)}.$$

Thus

$$\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\leqslant 0,\ \ \forall\,\bar{S}\in\mathcal{I}_{(7.11)},$$

and so

$$\max_{\bar{S}\in\mathcal{I}_{(7.11)}}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\}\leqslant 0,$$

and so if

$$\max_{\bar{S}\in\mathbb{R}^n_+}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\}=$$
$$\max_{\bar{S}\in\mathcal{I}_{(7.11)}}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\},$$

it means that

$$\max_{\bar{S}\in\mathbb{R}^n_+}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\}\leqslant 0$$

and so

$$\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\leqslant 0\ \ \forall\,\bar{S}\in\mathbb{R}^n_+$$
$$\iff\ \sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)+z+Nv^T\bar{S}\geqslant\max(\omega^T\bar{S}-E,0),\ \ \forall\,\bar{S}\in\mathbb{R}^n_+,$$

in which case the proposition is proved.
We now show that

$$\max_{\bar{S}\in\mathbb{R}^n_+}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\}=$$
$$\max_{\bar{S}\in\mathcal{I}_{(7.11)}}\left\{\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}\right\}.$$

Define the function $\psi:\mathbb{R}^n_+\to\mathbb{R}$, given by

$$\psi(\bar{S})=\max(\omega^T\bar{S}-E,0)-\sum_{l=1}^{q}(u^l)^T\max(\bar{S}-E^l,0)-z-Nv^T\bar{S}.$$

Consider $\nabla\psi(\bar{S})$, for all $\bar{S} \notin \mathcal{I}_{(7.11)}$. Then we have
(a) If $\omega^T\bar{S} - E < 0$,

$$\nabla\psi(\bar{S}) = \frac{\mathrm{d}\psi}{\mathrm{d}\bar{S}} = -\sum_{l=1}^{q} u^l - Nv.$$

(b)If $\omega^T\bar{S} - E > 0$,

$$\nabla\psi(\bar{S}) = \frac{\mathrm{d}\psi}{\mathrm{d}\bar{S}} = \omega - \sum_{l=1}^{q} u^l - Nv.$$

Now, from (7.12) however, we observe that

$$\begin{aligned} -\sum_{l=1}^{q} u^l - Nv \leqslant -\max\{\omega, 0\} \leqslant 0, \text{ and} \\ \sum_{l=1}^{q} u^l + Nv \geqslant \max\{\omega, 0\} \geqslant \omega \\ \iff \sum_{l=1}^{q} u^l + Nv \geqslant \omega \iff -\sum_{l=1}^{q} u^l - Nv \leqslant -\omega \\ \iff \omega - \sum_{l=1}^{q} u^l - Nv \leqslant 0. \end{aligned}$$

$\implies$ In case (a) and (b), for all $\bar{S} \notin \mathcal{I}_{(7.11)}$,

$$\nabla\psi(\bar{S}) \leqslant 0 \implies \psi(\bar{S}) \text{ is non-increasing for all } \bar{S} \notin \mathcal{I}_{(7.11)}.$$

This means that $\psi(\bar{S})$ must attain its maximum value for a value of $\bar{S} \in \mathcal{I}_{(7.11)}$ and so it holds that

$$\max_{\bar{S}\in\mathbb{R}^n_+} \left\{ \max(\omega^T\bar{S} - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(\bar{S} - E^l, 0) - z - Nv^T\bar{S} \right\} =$$
$$\max_{\bar{S}\in\mathcal{I}_{(7.11)}} \left\{ \max(\omega^T\bar{S} - E, 0) - \sum_{l=1}^{q} (u^l)^T \max(\bar{S} - E^l, 0) - z - Nv^T\bar{S} \right\},$$

and the proposition is proved. □

Now we show that the semi-infinite optimisation problem (7.11) can be re-formulated as a finite linear problem.
We start be recalling (7.11) as

$$\begin{aligned} \min_{(u^l,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n} \quad & \sum_{l=1}^{q} (u^l)^T A^l + z + Nv^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q} u^l + Nv \geqslant \max\{\omega, 0\} \\ & \sum_{l=1}^{q} (u^l)^T \max\left(\bar{S} - E^l, 0\right) + z + Nv^T\bar{S} \geqslant \max\left(\omega^T\bar{S} - E, 0\right), \\ & \forall\ \bar{S} \in \mathcal{I}_{(7.11)} = \bigtimes_{i=1}^{n} [0, E_i^q], \\ & \bar{S} = \frac{1}{N}\sum_{j=1}^{N} S(t_j). \end{aligned} \tag{7.11}$$

Now, using the following property of the *maximum*,

$$\max\{a,b\} \geqslant a \text{ and } \max\{a,b\} \geqslant b,$$

the second constraint in (7.11) is equivalent to the following two semi-infinite constraints (7.13) and (7.14), by observing that

$$\max(\omega^T \bar{S} - E, 0) \geqslant 0 \text{ and}$$
$$\max(\omega^T \bar{S} - E, 0) \geqslant \omega^T \bar{S} - E.$$

$\Longrightarrow$

$$\sum_{l=1}^{q} (u^l)^T \max(\bar{S} - E^l, 0) + z + N v^T \bar{S} \geqslant \omega^T \bar{S} - E, \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)} \tag{7.13}$$

and

$$\sum_{l=1}^{q} (u^l)^T \max(\bar{S} - E^l, 0) + z + N v^T \bar{S} \geqslant 0, \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)}. \tag{7.14}$$

Which is equivalent to

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(\bar{S}_i - E_i^l, 0) + z + N\sum_{i=1}^{n} v_i \bar{S}_i \geqslant \sum_{i=1}^{n} \omega_i \bar{S}_i - E, \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(\bar{S}_i - E_i^l, 0) + z + N\sum_{i=1}^{n} v_i \bar{S}_i \geqslant 0 \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)},$$

respectively.
$\Longleftrightarrow$

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(\bar{S}_i - E_i^l, 0) + N\sum_{i=1}^{n} v_i \bar{S}_i - \sum_{i=1}^{n} \omega_i \bar{S}_i + z + E \geqslant 0, \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(\bar{S}_i - E_i^l, 0) + N\sum_{i=1}^{n} v_i \bar{S}_i + z \geqslant 0 \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)},$$

respectively.
Switching the order of summation gives

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i\right)}_{(\star)} + z + E \geqslant 0, \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)},$$

and

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i\right)}_{(\star\star)} + z \geqslant 0 \ \ \forall\, \bar{S} \in \mathcal{I}_{(7.11)},$$

respectively.
Now we choose $\alpha, \beta \in \mathbb{R}^n$ such that α_i provides a lower bound to $(\star)$ and β_i provides a lower bound to $(\star\star)$, for all $i = 1, 2, \ldots, n$. That is, we choose α_i, β_i, for all $i = 1, 2, \ldots, n$, such that

$$\alpha_i \leqslant \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i$$

and

$$\beta_i \leqslant \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i,$$

for all $i = 1, 2, \ldots, n$.

$\Longrightarrow$ The semi-infinite constraints of (7.11) become

$$\begin{cases} \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i \geqslant \alpha_i, \ \forall\ \bar{S}_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n \\ \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i \geqslant \beta_i, \ \forall\ \bar{S}_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} \alpha_i + z + E \geqslant 0 \\ \sum_{i=1}^{n} \beta_i + z \geqslant 0. \end{cases} \tag{7.15}$$

This means that a point $(u^1, u^2, \ldots, u^q, z, v)$ is feasible for (7.11) if and only if $(u^1, u^2, \ldots, u^q, \alpha, \beta, z, v) \in \mathbb{R}^{n \times q} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is feasible for system (7.15). We observe that by writing the semi-infinite constraints of (7.11) as system (7.15), we have that the last two constraints are standard (finite) linear constraints.
Now consider the semi-infinite constraints from system (7.15),

$$\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i \geqslant \alpha_i, \ \forall\ \bar{S}_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n,$$

and

$$\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i \geqslant \beta_i, \ \forall\ \bar{S}_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n.$$

Then we observe that these constraints are piece-wise linear constraints. Thus, the minimum value of the left hand side of both inequalities over all values of $\bar{S}_i \in [0, E_i^q]$, for all $i = 1, 2, \ldots, n$ occurs at one of the break points. That is, it occurs exactly when $\bar{S}_i = 0$ or $\bar{S}_i = E_i^1$ or $\bar{S}_i = E_i^2$ or ... or $\bar{S}_i = E_i^q$, for all $i = 1, 2, \ldots, n$. Therefore, we may consider these semi-infinite constraints for the $(q+1)$ values $\bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}$, for $i = 1, 2, \ldots, n$.
Thus, it holds that

$$\begin{aligned} &\min_{\bar{S}_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i \\ = &\min_{\bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i, \end{aligned}$$

and

$$\begin{aligned} &\min_{\bar{S}_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i \\ = &\min_{\bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i. \end{aligned}$$

Therefore, each of the semi-infinite constraints can now be replaced by $(q+1)$ finite piece-wise linear constraints. That is, we may replace the constraint

$$\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i \geqslant \alpha_i, \ \forall\ \bar{S}_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n,$$

by

$$\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i \geqslant \alpha_i, \text{ for } \bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n,$$

and the constraint

$$\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i \geqslant \beta_i, \ \forall\ \bar{S}_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n,$$

by

$$\sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i \geqslant \beta_i, \text{ for } \bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n.$$

We may summarise the above analysis in the following theorem.

Theorem 7.16. *The semi-infinite optimisation problem* (7.11) *is equivalent to the following finite linear optimisation problem*

$$\begin{aligned}
\min_{(u^1, u^2, \ldots, u^q, z, v) \in \mathbb{R}^{n \times q} \times \mathbb{R} \times \mathbb{R}^n, \alpha, \beta, \in \mathbb{R}^n} \quad & \sum_{l=1}^{q} (u^l)^T A^l + z + N v^T S^0 \\
\textit{subject to} \quad & \sum_{l=1}^{q} u^l + N v \geqslant \max\{\omega, 0\} \\
& \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i - \omega_i \bar{S}_i \geqslant \alpha_i, \\
& \textit{for } \bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n \\
& \sum_{l=1}^{q} u_i^l \max(\bar{S}_i - E_i^l, 0) + N v_i \bar{S}_i \geqslant \beta_i, \\
& \textit{for } \bar{S}_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n \\
& \sum_{i=1}^{n} \alpha_i + z + E \geqslant 0 \\
& \sum_{i=1}^{n} \beta_i + z \geqslant 0 \\
& \bar{S}_i = \frac{1}{N} \sum_{j=1}^{N} S_i(t_j), \ \forall\ i = 1, 2, \ldots, n,
\end{aligned} \tag{7.17}$$

in the sense that both optimisation problems have the same feasible region and hence the same optimal solution and the same optimal objective function value.

Observation: In comparison to the semi-infinite problem (7.11); we observe that (7.17) has $n + n = 2n$ additional variables and a total of

$$\begin{aligned}
n + n(q+1) + n(q+1) + 2 + n &= 2n + 2n(q+1) + 2 = 2n + 2nq + 2n + 2 \\
&= 4n + 2nq + 2 = 2(2n + nq + 1) = 2(n(q+2) + 1)
\end{aligned}$$

linear constraints. The advantage of solving (7.17) in comparison to (7.11) is that we are solving a standard, finite linear problem in comparison to a semi-infinite one.
Therefore, solving the linear problem (7.17) would yield an upper bound on the current price of an Asian basket call option.

We observe here that the model set up and approach to finding an upper bound on the price of an Asian basket call option as presented above is very similar to the model set up presented in sub-section 3.4 which looked at finding an upper bound on the price of a European basket call option. Further, the methodology employed in the proof of Proposition 7.10 can be viewed as a modification to the methodology employed in the proof of Proposition 3.14 in sub-section 3.4. It is a modification because in the proof of Proposition 7.10 we considered finding price bounds on an Asian basket call option given that we know the current prices of Asian vanilla call options. This is different to the proof of Proposition 3.14 which considered finding an upper bound on the price of a European basket call option, given that we know the prices of European vanilla call options. Similarly the ideas used to obtain the finite LO problem (7.17) can also be viewed as a modification to the techniques used in sub-section 3.4 to re-write the SIO problem (3.13) as a solvable and finite LO problem.

7.3 An upper bound derived with bid-ask prices

When finding price bounds on an Asian basket option, we have seen in the previous sub-section the unrealistic assumption of knowing current mid-market prices. One way to overcome this, is to incorporate *bid-ask prices* within the optimisation model. In this sub-section we consider one way to find an upper bound on the current price of an Asian basket call option, given that we know the current bid-ask prices of other basket options. We saw how this was done in [22] and sub-section 3.6.1 for European basket options. Here, we present a new and original solution approach by incorporating bid-ask prices within the model and assuming a non-zero interest rate, $r > 0$ within the optimisation model.
We consider an optimisation model similar to, but **not** identical to the model given in sub-section 4.2. We then employ a solution approach similar to what was done in sub-section 4.2, where we found a lower bound on the current European basket option price, incorporating bid-ask prices.
We consider calculating an upper bound on the current price of an Asian basket call option written on n underlying assets, given that we know the current bid-ask prices of r other Asian basket call options, written on the same n underlying assets.
For this sub-section we will adopt the following notation.

- Let $\omega \in \mathbb{R}^n_+$ denote the vector of the weights of the basket option whose current price we are bounding.
- Let $E \in \mathbb{R}_+$ be the exercise price of the basket option whose current price we are bounding.
- Let $\omega^k \in \mathbb{R}^n_+$ denote the vector of the weights for the k^{th} basket option whose current bid-ask price we know, for $k = 1, 2, \ldots, r$.
- Let $E_k \in \mathbb{R}_+$ be the exercise price of the k^{th} basket option whose current bid-ask price we know, for $k = 1, 2, \ldots, r$.
- Let p_k^{ask}, $p_k^{bid} \in \mathbb{R}_+$ denote the current and known ask, bid prices of the k^{th} basket option, respectively, for $k = 1, 2, \ldots, r$. Obviously, here $p_k^{ask} \geqslant p_k^{bid}$, for all $k = 1, 2, \ldots, r$.

We consider the case where the average of **all** Asian options in the model is calculated via a *discrete arithmetic average*. In particular, we assume that the average is calculated by considering the asset price values at N discrete time points given by $t_1, t_2, \ldots, t_N = T$, where T is the expiry date of **all** options which is assumed to be the same. If $S(t_j) \in \mathbb{R}^n_+$ denotes the vector of the asset prices at time t_j, for $j = 1, 2, \ldots, N$; the average asset price, denoted by $\bar{S}$ would be given by

$$\bar{S} = \frac{1}{N} \sum_{j=1}^{N} S(t_j).$$

Then, if π denotes a risk-neutral probability measure which is to be found, and the risk-free interest rate is given by $r > 0$, the task of finding an upper bound on the current price of an Asian basket call option is given by

$$
\begin{aligned}
\sup_{\pi} \quad & \mathbb{E}_\pi\left[\mathrm{e}^{-rT}\max\left(\omega^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E,0\right)\right]\\
\text{subject to} \quad & \mathbb{E}_\pi\left[\mathrm{e}^{-rT}\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E_k,0\right)\right]\leqslant p_k^{ask},\quad \text{for } k=1,2,\ldots,r\\
& \mathbb{E}_\pi\left[\mathrm{e}^{-rT}\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E_k,0\right)\right]\geqslant p_k^{bid},\quad \text{for } k=1,2,\ldots,r\\
& \mathbb{E}_\pi[1]=1\\
& \mathbb{E}_\pi[\mathrm{e}^{-rt_j}S(t_j)]=S^0,\ \ j=1,2,\ldots,N\\
& \pi \text{ is a probability measure in } \mathbb{R}_+^n.
\end{aligned}
\tag{7.18}
$$

We find the linear SIO problem for which (7.18) is its dual as

$$
\begin{aligned}
\inf_{u^{ask},u^{bid},z,v(t_j)} \quad & z+(u^{ask})^Tp^{ask}-(u^{bid})^Tp^{bid}+\sum_{j=1}^{N}(v(t_j))^TS^0\\
\text{subject to} \quad & z+\sum_{k=1}^{r}\mathrm{e}^{-rT}\left(u_k^{ask}\right)\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E_k,0\right)\\
& -\sum_{k=1}^{r}\mathrm{e}^{-rT}\left(u_k^{bid}\right)\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E_k,0\right)\\
& +\sum_{j=1}^{N}\mathrm{e}^{-rt_j}(v(t_j))^TS(t_j)\geqslant\\
& \mathrm{e}^{-rT}\max\left(\omega^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E,0\right),\quad \forall\ S(t_j)\in\mathbb{R}_+^n,\ j=1,2,\ldots,N\\
& u^{ask},u^{bid}\in\mathbb{R}_+^r,\ z\in\mathbb{R},\ v(t_j)\in\mathbb{R}^n,\ j=1,2,\ldots,N.
\end{aligned}
$$

This SIO problem is equivalent to

$$
\begin{aligned}
\inf_{u^{ask},u^{bid},z,v(t_j)} \quad & z+(u^{ask})^Tp^{ask}-(u^{bid})^Tp^{bid}+\sum_{j=1}^{N}(v(t_j))^TS^0\\
\text{subject to} \quad & z\mathrm{e}^{rT}+\sum_{k=1}^{r}\left(u_k^{ask}-u_k^{bid}\right)\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E_k,0\right)\\
& +\sum_{j=1}^{N}\mathrm{e}^{r(T-t_j)}(v(t_j))^TS(t_j)\geqslant\\
& \max\left(\omega^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right)-E,0\right),\quad \forall\ S(t_j)\in\mathbb{R}_+^n,\ j=1,2,\ldots,N\\
& u^{ask},u^{bid}\in\mathbb{R}_+^r,\ z\in\mathbb{R},\ v(t_j)\in\mathbb{R}^n,\ j=1,2,\ldots,N.
\end{aligned}
$$

Defining $u \in \mathbb{R}^r$ as $u = u^{ask} - u^{bid}$, we may equivalently re-write the above SIO problem as

$$
\begin{aligned}
\inf_{u^{ask},u^{bid},u,z,v(t_j)} \quad & z + (u^{ask})^T p^{ask} - (u^{bid})^T p^{bid} + \sum_{j=1}^{N} (v(t_j))^T S^0 \\
\text{subject to} \quad & z\mathrm{e}^{rT} + \sum_{k=1}^{r} u_k \max\left((\omega^k)^T \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) - E_k, 0 \right) \\
& + \sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} (v(t_j))^T S(t_j) \geqslant \\
& \max\left(\omega^T \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) - E, 0 \right), \quad \forall\, S(t_j) \in \mathbb{R}^n_+,\ j = 1, 2, \ldots, N \\
& u = u^{ask} - u^{bid} \\
& u^{ask}, u^{bid} \in \mathbb{R}^r_+,\ z \in \mathbb{R},\ u \in \mathbb{R}^r,\ v(t_j) \in \mathbb{R}^n,\ j = 1, 2, \ldots, N.
\end{aligned}
\tag{7.19}
$$

Now, we assume that the investors position in the super-replicating portfolio does not change with respect to time. That is, once the investor has initially decided how many Asian basket call options to buy, how much cash to invest and how much of each underlying asset to buy, at time $t = 0$, the investor holds these amounts throughout the duration of the portfolio, that is until expiry $t = T$. This means that **all** of the optimisation variables u, u^{ask}, u^{bid}, z and $v(t_j)$ are independent of time t. In particular, we may replace $v(t_j)$ by some vector, independent of t, say v, for all j, so that we have $v(t_j) = v$, for all $j = 1, 2, \ldots, N$. With this in mind, we may re-write the SIO problem (7.19) as the following SIO problem

$$
\begin{aligned}
\inf_{u^{ask},u^{bid},u,z,v} \quad & z + (u^{ask})^T p^{ask} - (u^{bid})^T p^{bid} + N v^T S^0 \\
\text{subject to} \quad & z\mathrm{e}^{rT} + \sum_{k=1}^{r} u_k \max\left((\omega^k)^T \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) - E_k, 0 \right) \\
& + v^T \left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j) \right) \geqslant \\
& \max\left(\omega^T \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) - E, 0 \right), \quad \forall\, S(t_j) \in \mathbb{R}^n_+,\ j = 1, 2, \ldots, N \\
& u = u^{ask} - u^{bid} \\
& u^{ask}, u^{bid} \in \mathbb{R}^r_+,\ z \in \mathbb{R},\ u \in \mathbb{R}^r,\ v \in \mathbb{R}^n.
\end{aligned}
\tag{7.20}
$$

Therefore under the stated assumptions, the problem (7.18) is a dual to the linear SIO problem (7.20). We assume that strong duality holds between (7.18) and (7.20) (this is satisfied under some mild assumptions stated in [19]). That is, we assume that the conditions of the following lemma, taken from [19] but slightly modified to allow for Asian options and bid-ask prices, are satisfied.

Lemma 7.21 ([19], Proposition 2.1)**.** *The optimal values of* (7.18) *and* (7.20) *coincide if at least one of the following two conditions holds.*

(i) Strict primal feasibility,

$$\begin{pmatrix} 1 \\ p^{ask} \\ p^{bid} \\ S^0 \end{pmatrix} \in int\left(\left\{ \begin{pmatrix} \mathbb{E}_\pi[1] \\ \mathbb{E}_\pi\left[\mathrm{e}^{-rT}\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E_k, 0\right)\right]_{k=1,2,\ldots,r} \\ \mathbb{E}_\pi\left[\mathrm{e}^{-rT}\max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E_k, 0\right)\right]_{k=1,2,\ldots,r} \\ \mathbb{E}_\pi[\mathrm{e}^{-rt_j} S(t_j)]_{j=1,2,\ldots,N} \end{pmatrix} : \pi \text{ is a distribution in } \mathbb{R}^n_+ \right\}\right).$$

In particular, strong duality holds provided the current bid-ask prices p^{ask} and p^{bid}, are arbitrage free and remain arbitrage free after slight perturbations.
(ii) Strict dual feasibility.
There exists $(\hat{z}, \hat{u}, \hat{v})^T \in \mathbb{R}^{1+r+n}$ such that

$$(\hat{z}, \hat{u}, \hat{v}) \in int\left(\left\{(z, u, v)^T \in \mathbb{R}^{1+r+n} : z\mathrm{e}^{rT} + \sum_{k=1}^{r} u_k \max\left((\omega^k)^T\left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E_k, 0\right) + v^T\left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j)\right) \geqslant \max\left(\omega^T\left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E, 0\right), \quad \forall\, S(t_j) \in \mathbb{R}^n_+,\ j = 1, 2, \ldots, N\right\}\right).$$

In particular, strong duality holds provided that, for each asset, at least one vanilla option price is known.

The aim of this sub-section is to re-formulate the semi-infinite optimisation problem (7.20) as a finite linear problem. In order to do this, we employ a similar methodology to what was done in sub-section 4.2 for European basket options. We note here that the results presented in this sub-section are new in the sense that modelling the problem as (7.18) and solving its specific, related linear SIO problem (7.20) using this technique has not been done before. Also, obtaining an upper bound on the current price of an Asian basket call option under the presence of bid-ask prices and a non-zero interest rate is a significant result because it improves the previous model which we considered in sub-section 4.1.
We set the following notational convention, similar to what was defined in sub-section 4.2.

- Let Ω denote the $(r \times n)$ matrix whose k^{th} row is the vector $(\omega^k)^T$, for $k = 1, 2, \ldots, r$.
- Let $\bar{\Omega}$ be the $((r+1) \times n)$ matrix which is obtained by adding the vector ω^T on top of the first row of the matrix Ω. That is, $\bar{\Omega}$ is the matrix whose zeroth row is ω^T and whose k^{th} row is the vector $(\omega^k)^T$, for $k = 1, 2, \ldots, r$.
- Let $\hat{E} \in \mathbb{R}^{(r+1)}_+$ be the vector $(E, E_1, E_2, \ldots, E_r)^T$.
- Let I be a finite index set with $J \subseteq I$. Define a vector $\bar{\nu} \in \mathbb{R}^{|I|}$. Then $\bar{\nu}_J \in \mathbb{R}^{|J|}$ is the vector formed by the entries $\bar{\nu}_j$, for $j \in J$.
- Similarly, if the rows of a matrix, $\bar{\Lambda}$ whose indices belong to the set I, then $\bar{\Lambda}_J$ is the matrix formed by the rows of $\bar{\Lambda}$ whose indices $j \in J$.
- Let J' denote the set $I \backslash J$ and the index set I will be equal to $\{0, 1, \ldots, r\}$.
- Here, the first row of $\bar{\Omega}$ will be indexed by 0 and the first entry of $\hat{E}$ by 0.

Now we are ready to transform (7.20). For this, we define the following sets.
Let $J \subseteq \{0, 1, \ldots, r\}$. We define $\mathcal{P}_J$ as

$$\mathcal{P}_J = \left\{ S(t_j) | \bar{\Omega}_J \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) \geqslant \hat{E}_J,\ \bar{\Omega}_{J'} \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) \leqslant \hat{E}_{J'},\ S(t_j) \geqslant 0,\ \forall\ j = 1, 2, \ldots, N \right\},$$

and let

$$\bar{\mathcal{J}} = \{ J \subseteq \{0, 1, \ldots, r\} : \mathcal{P}_J \neq \varnothing \}.$$

Analogously to Lemma 4.10 and Proposition 4.16, we will show that (7.20) can equivalently be re-written as a *finite* linear problem, which may be solved by an appropriate software program to yield the optimal objective function value which is an upper bound on the current price of the Asian basket call option of interest.

Lemma 7.22. *(i) Let $\bar{\Omega} \in \mathbb{R}^{((r+1)\times n)}, \hat{E} \in \mathbb{R}_+^{(r+1)}$ and let $J \subseteq \{0, 1, \ldots, r\}$ be arbitrarily chosen and fixed. Denote the set $\mathcal{P}_J = \mathcal{P}_J(\bar{\Omega}, \hat{E})$ as above. If $\mathcal{P}_J \neq \varnothing$ then for $\begin{pmatrix} -1 \\ u \end{pmatrix} \in \mathbb{R}^{(r+1)}, v \in \mathbb{R}^n$ and $z\mathrm{e}^{rT} \in \mathbb{R}$ we have*

$$\begin{pmatrix} -1 \\ u \end{pmatrix}^T \max\left(\bar{\Omega} \left(\frac{1}{N} \sum_{j=1}^{N} S(t_j) \right) - \hat{E}, 0 \right) + v^T \left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j) \right) \geqslant -z\mathrm{e}^{rT}, \quad \textit{for all } S(t_j) \in \mathcal{P}_J, \tag{7.23}$$

if and only if $v \geqslant 0$ and there exist $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$ such that

$$\begin{aligned} -(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ u \end{pmatrix}_J &\leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'} \\ &\textit{and} \\ \begin{pmatrix} -1 \\ u \end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT} &\leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'}. \end{aligned} \tag{7.24}$$

(ii) The SIO problem (7.20) can equivalently be re-written as the following finite linear problem

$$\begin{array}{ll} \underset{z,u,u^{ask},u^{bid},v}{\textit{min}} & z + (u^{ask})^T p^{ask} - (u^{bid})^T p^{bid} + N v^T S^0 \\ \textit{subject to} & -(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ u \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'},\ \ J \in \bar{\mathcal{J}} \\ & \begin{pmatrix} -1 \\ u \end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT} \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'},\ \ J \in \bar{\mathcal{J}} \\ & u = u^{ask} - u^{bid} \\ & v \geqslant 0 \\ & u \in \mathbb{R}^r,\ u^{ask} \in \mathbb{R}_+^r, \\ & u^{bid} \in \mathbb{R}_+^r,\ z \in \mathbb{R},\ v \in \mathbb{R}^n, \\ & \gamma^J \in \mathbb{R}_+^{|J|},\ \ \beta^{J'} \in \mathbb{R}_+^{|J'|}, \quad J \in \bar{\mathcal{J}}. \end{array} \tag{7.25}$$

Proof. The proof comes in two parts.

(i) For all $S(t_j) \in \mathcal{P}_J$, we have

$$\begin{pmatrix}-1\\u\end{pmatrix}^T \max\left(\bar\Omega\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) - \hat E, 0\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) + z\mathrm{e}^{rT}$$
$$= \begin{pmatrix}-1\\u\end{pmatrix}_J^T \left(\bar\Omega_J\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) - \hat E_J\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) + z\mathrm{e}^{rT}$$
$$= \left(\bar\Omega_J^T \begin{pmatrix}-1\\u\end{pmatrix}_J\right)^T \left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) - \left(\begin{pmatrix}-1\\u\end{pmatrix}_J^T \hat E_J - z\mathrm{e}^{rT}\right),$$

since

$$\bar\Omega_J\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) - \hat E_J \geqslant 0, \text{ and}$$
$$\bar\Omega_{J'}\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) - \hat E_{J'} \leqslant 0.$$

So, if we consider the linear problem

$$\begin{aligned} \min_{S(t_j)} \quad & \left(\bar\Omega_J^T \begin{pmatrix}-1\\u\end{pmatrix}_J\right)^T \left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) \\ \text{subject to} \quad & -\bar\Omega_J\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) \leqslant -\hat E_J \\ & \bar\Omega_{J'}\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) \leqslant \hat E_{J'} \\ & S(t_j) \geqslant 0, \ \forall\, j = 1, 2, \ldots, N, \end{aligned} \tag{7.26}$$

it follows that (7.23) holds if and only if the optimal value of the linear problem (7.26) is at least $\left(\begin{pmatrix}-1\\u\end{pmatrix}_J^T \hat E_J - z\mathrm{e}^{rT}\right)$, because if this is the case, then

$$\min_{S(t_j)\in\mathcal{P}_J}\left\{\left(\bar\Omega_J^T \begin{pmatrix}-1\\u\end{pmatrix}_J\right)^T \left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right)\right\} \geqslant \left(\begin{pmatrix}-1\\u\end{pmatrix}_J^T \hat E_J - z\mathrm{e}^{rT}\right)$$
$$\Longrightarrow \left(\bar\Omega_J^T \begin{pmatrix}-1\\u\end{pmatrix}_J\right)^T \left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) \geqslant \begin{pmatrix}-1\\u\end{pmatrix}_J^T \hat E_J - z\mathrm{e}^{rT}, \ \forall\ S(t_j) \in \mathcal{P}_J$$
$$\Longleftrightarrow \left(\bar\Omega_J^T \begin{pmatrix}-1\\u\end{pmatrix}_J\right)^T \left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) - \begin{pmatrix}-1\\u\end{pmatrix}_J^T \hat E_J + z\mathrm{e}^{rT} \geqslant 0, \ \forall\ S(t_j) \in \mathcal{P}_J.$$

This means that

$$\begin{pmatrix}-1\\u\end{pmatrix}^T \max\left(\bar\Omega\left(\frac{1}{N}\sum_{j=1}^N S(t_j)\right) - \hat E, 0\right) + v^T\left(\sum_{j=1}^N \mathrm{e}^{r(T-t_j)} S(t_j)\right) + z\mathrm{e}^{rT} \geqslant 0, \quad \text{for all } S(t_j) \in \mathcal{P}_J,$$

so that (7.23) holds.
Now, by linear optimisation duality, the dual to (7.26) is obtained as follows. First we make a substitution.

Define $\mathcal{S} \in \mathbb{R}_+^n$ and $\tilde{\mathcal{S}} \in \mathbb{R}_+^n$ as

$$\mathcal{S} = \frac{1}{N}\sum_{j=1}^{N} S(t_j) \text{ and } \tilde{\mathcal{S}} = \sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j),$$

then the LO problem (7.26) becomes

$$\begin{aligned} \min_{\mathcal{S},\tilde{\mathcal{S}}} \quad & \left(\bar{\Omega}_J^T \begin{pmatrix}-1\\ u\end{pmatrix}_J\right)^T \mathcal{S} + v^T\tilde{\mathcal{S}} \\ \text{subject to} \quad & -\bar{\Omega}_J\mathcal{S} \leqslant -\hat{E}_J \\ & \bar{\Omega}_{J'}\mathcal{S} \leqslant \hat{E}_{J'} \\ & \mathcal{S},\ \tilde{\mathcal{S}} \geqslant 0. \end{aligned}$$

Now we define the dual variables $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$. Then the dual to (7.26) is given by

$$\begin{aligned} \max_{\gamma^J,\beta^{J'}} \quad & (\gamma^J)^T\hat{E}_J - (\beta^{J'})^T\hat{E}_{J'} \\ \text{subject to} \quad & -\bar{\Omega}_J^T\gamma^J + \bar{\Omega}_{J'}^T\beta^{J'} \geqslant -(\bar{\Omega}_J)^T\begin{pmatrix}-1\\ u\end{pmatrix}_J \\ & 0 \geqslant -v \iff v \geqslant 0 \\ & \gamma^J \in \mathbb{R}_+^{|J|} \\ & \beta^{J'} \in \mathbb{R}_+^{|J'|}. \end{aligned} \tag{7.27}$$

Now, by the weak duality theorem, we have that

$$\min_{\mathcal{S},\tilde{\mathcal{S}}}\left\{\left(\bar{\Omega}_J^T\begin{pmatrix}-1\\ u\end{pmatrix}_J\right)^T \mathcal{S} + v^T\tilde{\mathcal{S}}\right\} \geqslant \max_{\gamma^J,\beta^{J'}}\left\{(\gamma^J)^T\hat{E}_J - (\beta^{J'})^T\hat{E}_{J'}\right\},$$

and after re-substituting for $\mathcal{S}$ and $\tilde{\mathcal{S}}$ gives,

$$\min_{S(t_j)\in\mathcal{P}_J}\left\{\left(\bar{\Omega}_J^T\begin{pmatrix}-1\\ u\end{pmatrix}_J\right)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right) + v^T\left(\sum_{j=1}^{N}\mathrm{e}^{r(T-t_j)}S(t_j)\right)\right\} \geqslant \max_{\gamma^J,\beta^{J'}}\left\{(\gamma^J)^T\hat{E}_J - (\beta^{J'})^T\hat{E}_{J'}\right\}. \tag{7.28}$$

However, if we impose the condition

$$(\gamma^J)^T\hat{E}_J - (\beta^{J'})^T\hat{E}_{J'} \geqslant \begin{pmatrix}-1\\ u\end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT},$$

then by (7.28) we have

$$\min_{S(t_j)\in\mathcal{P}_J}\left\{\left(\bar{\Omega}_J^T\begin{pmatrix}-1\\ u\end{pmatrix}_J\right)^T\left(\frac{1}{N}\sum_{j=1}^{N}S(t_j)\right) + v^T\left(\sum_{j=1}^{N}\mathrm{e}^{r(T-t_j)}S(t_j)\right)\right\} \geqslant \begin{pmatrix}-1\\ u\end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT},$$

and since the constraint in (7.27) must hold true, we have that the optimal value of the linear problem (7.26) is at least $\begin{pmatrix}-1\\ u\end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT}$ if and only if $v \geqslant 0$ and there exists $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$ such that (7.24) holds and so (i) is proved.

(ii) We start by observing that the objective function in (7.20) is the same as the objective function in (7.25).

This is because the objective function in (7.20) represents the total cost of the super-replicating portfolio consisting of cash, other Asian basket call options and the underlying assets. The minimal total cost is a cost which is paid by the investor. Therefore, the optimal objective function value of (7.20) is attained and the **inf** in (7.20) can be replaced by **min** as done in (7.25).
For the constraints of (7.25) we have the following.
Recall the constraint from (7.20) as

$$z\mathrm{e}^{rT} + \sum_{k=1}^{r} u_k \max\left((\omega^k)^T \left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E_k, 0\right) + v^T \left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j)\right) \geqslant$$
$$\max\left(\omega^T \left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E, 0\right), \quad \forall\ S(t_j) \in \mathbb{R}_+^n,\ j = 1, 2, \ldots, N.$$

This is equivalent to

$$-\max\left(\omega^T \left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E, 0\right) + \sum_{k=1}^{r} u_k \max\left((\omega^k)^T \left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - E_k, 0\right)$$
$$+v^T \left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j)\right) \geqslant -z\mathrm{e}^{rT},\ \forall\ S(t_j)\ \in \mathcal{P}_J,\ J \in \bar{\mathcal{J}},$$

which is the same as

$$\begin{pmatrix} -1 \\ u \end{pmatrix}^T \max\left(\bar{\Omega}\left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - \hat{E}, 0\right) + v^T \left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j)\right) \geqslant -z\mathrm{e}^{rT},$$
$$\text{for all } S(t_j) \in \mathcal{P}_J,\ J \in \bar{\mathcal{J}}.$$

Now we use part (i), so this means

$$\begin{pmatrix} -1 \\ u \end{pmatrix}^T \max\left(\bar{\Omega}\left(\frac{1}{N}\sum_{j=1}^{N} S(t_j)\right) - \hat{E}, 0\right) + v^T \left(\sum_{j=1}^{N} \mathrm{e}^{r(T-t_j)} S(t_j)\right) \geqslant -z\mathrm{e}^{rT},$$
$$\text{for all } S(t_j) \in \mathcal{P}_J,\ J \in \bar{\mathcal{J}}$$

if and only if $v \geqslant 0$ and there exist $\gamma^J \in \mathbb{R}_+^{|J|}, \beta^{J'} \in \mathbb{R}_+^{|J'|}$ such that

$$-(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ u \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'}$$

and

$$\begin{pmatrix} -1 \\ u \end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT} \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'},$$

for $J \in \bar{\mathcal{J}}$.

Therefore, (7.20) is equivalent to the following linear optimisation problem

$$\begin{aligned}
\min_{z,u,u^{ask},u^{bid},v} \quad & z + (u^{ask})^T p^{ask} - (u^{bid})^T p^{bid} + N v^T S^0 \\
\text{subject to} \quad & -(\bar{\Omega}_J)^T \begin{pmatrix} -1 \\ u \end{pmatrix}_J \leqslant -\bar{\Omega}_J^T \gamma^J + \bar{\Omega}_{J'}^T \beta^{J'}, \;\; J \in \bar{\mathcal{J}} \\
& \begin{pmatrix} -1 \\ u \end{pmatrix}_J^T \hat{E}_J - z\mathrm{e}^{rT} \leqslant (\gamma^J)^T \hat{E}_J - (\beta^{J'})^T \hat{E}_{J'}, \;\; J \in \bar{\mathcal{J}} \\
& u = u^{ask} - u^{bid} \\
& v \geqslant 0 \\
& u \in \mathbb{R}^r, \; u^{ask} \in \mathbb{R}^r_+, \\
& u^{bid} \in \mathbb{R}^r_+, \; z \in \mathbb{R}, \; v \in \mathbb{R}^n, \\
& \gamma^J \in \mathbb{R}_+^{|J|}, \;\; \beta^{J'} \in \mathbb{R}_+^{|J'|}, \quad J \in \bar{\mathcal{J}},
\end{aligned}$$

and the proof is complete. □

Remark: In the finite linear problem (7.25) we have the constraint $v \geqslant 0$. This has the following important financial meaning. The vector v which has components v_i, for $i = 1, 2, \ldots, n$ represents the amount of asset i which is put into the portfolio at each time point $t_1, t_2, \ldots, t_N = T$. $v \geqslant 0$ means $v_i \geqslant 0$, for all $i = 1, 2, \ldots, n$ and financially this means that at **each** time point $t_1, t_2, \ldots, t_N = T$, the position taken in the super-replicating portfolio for the i^{th} underlying asset is a **long** position, for all $i = 1, 2, \ldots, n$. Thus, equivalently re-writing the SIO problem (7.20) as a finite linear problem (7.25) gives us a condition, namely that $v \geqslant 0$ in the super-replicating portfolio. If there exists an index i_0 such that $v_{i_0} < 0$, then it can not be guaranteed that the portfolio super-replicates the payoff of the Asian basket call option for all non-negative values of $S(t_j)$, for $j = 1, 2, \ldots, N$. It is advantageous to solve (7.25) as opposed to (7.20) because we are solving a *finite* linear optimisation problem in comparison to a semi-infinite one.
Thus, we may solve the *finite* linear problem (7.25) (instead of the semi-infinite problem (7.20)) to yield the optimal objective function value which is an upper bound on the current price of an Asian basket call option.
To conclude this sub-section we make the following observation about the size of the linear problem (7.25).
We observe that in (7.25) there are a total of

$$1 + r + r + n + |J| + |J'| = 1 + 2r + n + r + 1 = 3r + n + 2$$

variables.
For the constraints we have the following. Observe that each of the first 2 constraints requires $J \in \bar{\mathcal{J}}$, where $J \subseteq \{0, 1, \ldots, r\}$. However, for some $J \subseteq \{0, 1, \ldots, r\}$, $\mathcal{P}_J = \varnothing$ and $J \notin \bar{\mathcal{J}}$ even though $J \subseteq \{0, 1, \ldots, r\}$. Thus, we can obtain upper bounds on the size, or amounts of the constraints of the linear problem (7.25). In particular, for the set $\{0, 1, 2, \ldots, r\}$ there exists a total of 2^{r+1} subsets. Since $J \subseteq \{0, 1, \ldots, r\}$ we have the following. The first constraint in (7.25) actually represents n constraints. Therefore we have that the first constraint can give at most $n(2^{r+1})$ constraints. Similarly, the second constraint in (7.25) may yield at most 2^{r+1} constraints. The constraint $v \geqslant 0$ means $v_i \geqslant 0$, for all $i = 1, 2, \ldots, n$, so this represents n constraints. Hence, in total the number of constraints in (7.25) has the following upper bound

$$n(2^{r+1}) + 2^{r+1} + n = 2^{r+1}n + 2^{r+1} + n = 2^{r+1}(n+1) + n.$$

Therefore, the total number of constraints of (7.25) is at most $2^{r+1}(n+1) + n$.
Although the number of constraints depends exponentially on r, and for large values of r, (7.25) may become large, it is still finite and we have removed the problem of having infinitely many constraints as in (7.20). Further, from a practical point of view, we may choose r so that (7.25) remains solvable using an appropriate linear optimisation software solver. Also, when solving the optimisation problem (7.25), we do not know today which subsets of J are in $\bar{\mathcal{J}}$. That is, we do not currently know which of the basket options expire

in/out of the money. In order to solve problem (7.25) to obtain an upper bound on the current price of the Asian basket option of interest; we take **all** subsets of J to be in $\bar{\mathcal{J}}$ and solve (7.25) in this way. We note here that the above inequality regarding the number of constraints of problem (7.25) in this case holds as an equality.

We observe here that the model set-up and approach to finding an upper bound on the price of an Asian basket call option presented above is similar to the model set-up presented in sub-section 3.5. Further, the methodology employed in the proof of Lemma 7.22, to obtain a finite and solvable LO problem can be viewed as an extension to the methodology to obtain a finite and solvable LO problem employed in the proof of Proposition 3.26 in sub-section 3.5. It is an extension because in the proof of Lemma 7.22 we obtained an upper bound (instead of a lower bound) on the price of an Asian basket call option given that we know the bid-ask prices of other Asian basket call options.

That concludes this sub-section on considering finding upper bounds on the current price of an Asian basket call option given that we know the current bid-ask prices of other basket options.

This concludes our analysis on finding bounds on the current price of an Asian basket option.

8 Extension to price bounds for Mountain Range options

In this section we consider finding an upper bound on the current price of a specific *Mountain Range option*. We first explain what a Mountain Range option is and then we see how to apply similar techniques from previous sections to aid us to find price bounds on these specific types of options.

8.1 Introduction

Introduced in 1998 by the French bank *Société Générale*, *Mountain Range options* are a type of exotic option which are aimed at fulfilling the rising demand for unique and innovative financial products by investors. Put simply, a Mountain Range option combines properties of basket options and range options. In general, Mountain Range options have a payoff which depends upon several assets (just like a basket option) and several time points (like a range option). Since these types of options have basket option like properties they become of interest to us, and in particular the question of how to accurately price a Mountain Range option arises.

In this section we consider how to find an upper bound on the current price of a particular type of Mountain Range option using semi-infinite optimisation. We note here that the results presented in this section are new and original, in the sense that modelling the problem as a SIO problem and re-formulating it the way we do so here for this specific type of option has not been done before.

In particular, we consider finding an upper bound on the current price of an *Altiplano Mountain Range option*. Full details and properties of this type of option are discussed below.

Other types of Mountain Range options which exist but are not discussed in this thesis include *Annapurna, Atlas, Everest* and *Himalayan* Mountain Range options. For a full list of Mountain Range options, and for more information on these we refer the interested reader to [53], for example.

8.2 Upper bound on the price of an Altiplano Mountain Range option

We start this sub-section by explaining some basic properties and the payoff function for an *Altiplano Mountain Range option*, and then we move on to seeing how semi-infinite optimisation may be used to obtain an upper bound on the current price of this particular option.

Put simply, an Altiplano Mountain Range option is an option written on many assets with the following property. An Altiplano Mountain Range option pays out a coupon payment, $\mathcal{C} > 0$, if the ratio of the prices of **all** underlying assets at pre-agreed and pre-specified times with the current respective asset price does not exceed the exercise price, E, of the option. If at least one ratio does exceed E, at any one of the pre-agreed times, then the option pays out what a European basket call option would where the weights are the reciprocal of the current underlying asset price [53].

To formalise this, we define the following notation. Consider an Altiplano Mountain Range option written on n underlying assets, with expiry date T, and with only *one* pre-agreed, pre-specified time point of interest. This time point is the expiry date, $t = T$. Let S_i denote the price of the i^{th} asset at expiry, for $i = 1, 2, \dots, n$ and let S_i^0, denote the current price of the i^{th} asset, for $i = 1, 2, \dots, n$. Then we may define the following quantity, $\omega_i \in \mathbb{R}_+$ as the weight for asset i, for all $i = 1, 2, \dots, n$ as,

$$\omega_i = \frac{1}{S_i^0}, \text{ for } S_i^0 > 0.$$

Thus, we may define the vectors, $S \in \mathbb{R}^n_+$, $S^0 \in int(\mathbb{R}^n_+)$ and $\omega \in int(\mathbb{R}^n_+)$, where $\omega_i = \frac{1}{S_i^0}$, for $i = 1, 2, \dots, n$. If we define a binary variable $\mu_{\mathcal{A}}$ by

$$\mu_{\mathcal{A}} = \begin{cases} 0, & \text{if } \max\limits_{i=1,2,\dots,n} \left\{ \frac{S_i}{S_i^0} \right\} \leqslant E \\ 1, & \text{otherwise} \end{cases}, \tag{8.1}$$

then the payoff of the Altiplano Mountain Range option under consideration is given by, (and see [53] for more details)

$$\mathcal{A} = \mu_{\mathcal{A}} \max\left(\sum_{i=1}^{n} \frac{S_i}{S_i^0} - E, 0\right) + (1 - \mu_{\mathcal{A}})\mathcal{C}.$$

Using the vector notation introduced earlier, this is equivalent to

$$\mathcal{A} = \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C}, \tag{8.2}$$

where $\mu_{\mathcal{A}} \in \{0, 1\}$ is decided by the conditions in (8.1).
Therefore, the payoff of an Altiplano Mountain Range option is (8.2).
Before proceeding it is worth mentioning the similarities, yet vital differences between a 'standard' European basket call option and an Altiplano Mountain Range option. The payoff of an Altiplano Mountain Range option is ultimately decided by the parameter $\mu_{\mathcal{A}}$, which in this case depends upon the prices of the ratios of the underlying assets at expiry, and the current prices. If all underlying assets have ratio below E at expiry, then the Altiplano Mountain Range option pays out a coupon payment $\mathcal{C}$, set by the writer. If however, at least one of the ratio values exceeds E, then the Altiplano Mountain Range option becomes identical to a European basket call option and pays out

$$\max(\omega^T S - E, 0) = \max\left(\sum_{i=1}^{n} \frac{S_i}{S_i^0} - E, 0\right).$$

Thus, depending on the ratio of the asset price values at expiry and the current asset price values; this particular Altiplano Mountain Range option may payout exactly what a 'standard' European basket call option would.
Now that we have considered the basic properties and payoff function for an Altiplano Mountain Range option, we now see how optimisation can be used to find bounds on its current price.
From here-on in we will use the payoff (8.2) for the Altiplano Mountain Range option.
To find an upper bound on the current price we consider a similar model set-up to that in sub-sections 4.1, 6 and 7.2, where we considered current price bounds for European, American and Asian basket options, respectively.
We consider finding upper bounds on the current price of an Altiplano Mountain Range option, given that we know the current prices of q European vanilla call options, **per asset**. That is, we know a total of $(n \times q)$ current European vanilla call option prices. The European vanilla call options are written on the same n underlying assets as in the Altiplano Mountain Range option. In what follows, we let $C_i^l \in \mathbb{R}_+$ denote the l^{th} current and known European vanilla call option price, written on asset i, for $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, q$. Further, $E_i^l \in \mathbb{R}_+$ will denote the exercise price of the l^{th} European vanilla call option, whose current price we know and is written on asset i, for $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, q$.
We will assume that we know the current prices of each of the n underlying assets, and that these current prices are all positive.
Then, if the risk-free interest rate is given by $r > 0$, and π is a risk-neutral probability measure to be found, the task of finding an upper bound on the current price of an Altiplano Mountain Range option can be modelled by

$$\begin{aligned}
\sup_{\pi} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT}\left(\mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C}\right)] \\
\text{subject to} \quad & \mathbb{E}_\pi[\mathrm{e}^{-rT} \max(S_i - E_i^l, 0)] = C_i^l, \quad \text{for } i = 1, 2, \ldots, n \text{ and } l = 1, 2, \ldots, q \\
& \mathbb{E}_\pi[1] = 1 \\
& \mathbb{E}_\pi[\mathrm{e}^{-rT} S_i] = S_i^0, \quad \text{for } i = 1, 2, \ldots, n,
\end{aligned} \tag{8.3}$$

where we recall that $\mu_{\mathcal{A}} \in \{0,1\}$ is chosen in accordance with (8.1).
We may derive the linear SIO problem for which (8.3) is a dual, as the problem

$$\begin{array}{ll} \inf\limits_{u^l,z,v} & \sum\limits_{l=1}^{q} (u^l)^T C^l + z + v^T S^0 \\ \text{subject to} & \sum\limits_{l=1}^{q} \mathrm{e}^{-rT} (u^l)^T \max(S - E^l, 0) + z + \mathrm{e}^{-rT} v^T S \geqslant \mathrm{e}^{-rT} \left(\mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} \right), \\ & \forall\ S \in \mathbb{R}^n_+. \end{array}$$

This SIO problem is equivalent to

$$\begin{array}{ll} \inf\limits_{u^l,z,v} & \sum\limits_{l=1}^{q} (u^l)^T C^l + z + v^T S^0 \\ \text{subject to} & \sum\limits_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\ S \in \mathbb{R}^n_+. \end{array} \tag{8.4}$$

The SIO problem (8.4) has a natural financial interpretation. We may interpret the variables (u^l, z, v) as follows. u^l which has components u^l_i represents the amount of the l^{th} European vanilla call option written on the i^{th} asset in the super-replicating portfolio, for $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, n$. z represents a cash amount. v which has components v_i, for $i = 1, 2, \ldots, n$ represents the amount of the i^{th} underlying asset which is held in the super-replicating portfolio.
The SIO problem (8.4) itself may be interpreted as follows. We are interested in finding the cheapest cost portfolio consisting of European vanilla call options, cash and the underlying assets such that the overall value of the portfolio always super-replicates the payoff of the Altiplano Mountain Range option whose current price we are finding an upper bound on, for all possible non-negative values of the asset prices at expiry.
We now work with (8.4) to obtain the optimal objective function value which is an upper bound on the current price of the Altiplano Mountain Range option under consideration.
We note here that the index set $\mathcal{I} = \mathbb{R}^n_+$ in (8.4) is not compact. However, as the next proposition shows, if we impose a restriction on how the super-replicating portfolio is to be constructed, we may restrict $\mathcal{I}$ in (8.4) to a compact set without changing the feasible set of the problem.
In particular, if we impose the constraint

$$\sum_{l=1}^{q} u^l + v \geqslant \ \mu_{\mathcal{A}} \omega,$$

then it can be shown, (see Proposition 8.5 below) that we may replace the non-compact index set $\mathbb{R}^n_+$ by a compact index set.

Proposition 8.5. *Suppose without loss of generality that the exercise prices E^l_i are ordered such that $0 \leqslant E^1_i \leqslant E^2_i \leqslant \cdots \leqslant E^q_i$, for all $i = 1, 2, \ldots, n$. Define the index set $\mathcal{I}_{(8.6)} = \bigtimes_{i=1}^{n} [0, E^q_i]$. Then the following optimisation problem* (8.6)*, is equivalent to* (8.4) *in the sense that both problems have the same feasible set,* *and, hence the same optimal solution and optimal objective function value.*

$$
\begin{aligned}
\underset{(u^l,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n}{min} \quad & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\
\textit{subject to} \quad & \sum_{l=1}^{q} u^l + v \geqslant \ \mu_{\mathcal{A}}\omega \\
& \sum_{l=1}^{q}(u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C}, \\
& \forall\ S \in \mathcal{I}_{(8.6)}.
\end{aligned}
\tag{8.6}
$$

Remarks

1. *We remark here that the objective functions of* (8.4) *and* (8.6) *are the same. This is because the objective function in* (8.4) *represents the total cost of the super-replicating portfolio at the current time,* $t = 0$*. Since this is a cost which is paid, the minimal such cost is attained since what ever the minimum cost is, the investor pays it. Therefore the* ***inf*** *in* (8.4) *is attained and can be replaced by* ***min*** *as done in* (8.6).

2. *The extra constraint in* (8.6) *has a significant financial meaning.*
   The constraint
   $$\sum_{l=1}^{q} u^l + v \geqslant \mu_{\mathcal{A}}\omega,$$
   which is equivalent to
   $$\sum_{l=1}^{q} u_i^l + v_i \geqslant \mu_{\mathcal{A}}\omega_i,\ \forall\ i = 1,2,\ldots,n$$
   means:
   (a) If $\mu_{\mathcal{A}} = 0$*, this implies,* $\sum_{l=1}^{q} u_i^l + v_i \geqslant 0$*, for all* $i = 1,2,\ldots,n$*. This means that for the* i^{th} *asset, the total amount of European vanilla calls, written on the* i^{th} *underlying asset, plus the amount of the* i^{th} *asset which is held is non-negative, for* $i = 1,2,\ldots,n$.
   (b) If $\mu_{\mathcal{A}} = 1$*, this implies,* $\sum_{l=1}^{q} u_i^l + v_i \geqslant \omega_i = \dfrac{1}{S_i^0}$*, for all* $i = 1,2,\ldots,n$*. This means that for the* i^{th} *asset, the total amount of European vanilla calls, written on the* i^{th} *underlying asset, plus the amount of the* i^{th} *asset which is held is at least* $\frac{1}{S_i^0}$*, for* $i = 1,2,\ldots,n$.

3. *This extra constraint gives conditions or restrictions on how the super-replicating portfolio should be constructed.*

4. *The advantage of solving* (8.6) *instead of* (8.4) *is that the index set in the semi-infinite constraint of* (8.6) *is compact, albeit* (8.6) *containing additional constraints. Problem* (8.6) *allows us to obtain the optimal objective function value of* (8.4) *by considering a compact index set* $\mathcal{I}_{(8.6)}$ *and additional constraints which impose conditions on how the super-replicating portfolio is to be constructed. Thus,* (8.6) *allows us to solve* (8.4) *by considering the values of* S *in the compact set* $\mathcal{I}_{(8.6)} = \bigtimes_{i=1}^{n} [0, E_i^q]$ *as well as some restrictions on how to construct the super-replicating portfolio, instead of considering values of* S *in the unbounded set* $\mathbb{R}_+^n$.

Proof of Proposition 8.5. We start by observing that the objective functions of (8.4) and (8.6) are the same. Thus in order to show these two problems are equivalent we must show that their respective feasible regions are the same. Let $\mathcal{F}_{(8.4)}$ and $\mathcal{F}_{(8.6)}$ denote the feasible regions of (8.4) and (8.6), respectively. We then show

that $\mathcal{F}_{(8.4)} = \mathcal{F}_{(8.6)}$.
The proof comes in two parts.
(i) $\underline{\mathcal{F}_{(8.4)} \subset \mathcal{F}_{(8.6)}}$: Take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(8.4)}$. We then show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(8.6)}$. Now, since $\mathcal{I}_{(8.6)} = \bigtimes_{i=1}^{n} [0, E_i^q]$, then $\mathcal{I}_{(8.6)}$ forms an n-dimensional 'rectangle'. That is, it forms a 'rectangle' in n-dimensional non-negative space and so $\mathcal{I}_{(8.6)} \subset \mathbb{R}_+^n$. Thus, from the constraint in (8.4) we have

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\, S \in \mathbb{R}_+^n.$$

It then follows that since $\mathcal{I}_{(8.6)} \subset \mathbb{R}_+^n$, the constraint in (8.4) obviously still holds for all $S \in \mathcal{I}_{(8.6)}$. That is,

$$\sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\, S \in \mathcal{I}_{(8.6)},$$

and so the second constraint from (8.6) holds.
To show that the first constraint holds we have the following. Recall that $(u^1, u^2, \ldots, u^q, z, v)$ satisfies the constraint

$$\begin{aligned} & \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\, S \in \mathbb{R}_+^n \\ \Longleftrightarrow\ & \sum_{l=1}^{q} \sum_{k=1}^{n} (u_k^l) \max(S_k - E_k^l, 0) + z\mathrm{e}^{rT} + \sum_{k=1}^{n} v_k S_k \geqslant \mu_{\mathcal{A}} \max\left(\sum_{k=1}^{n} \omega_k S_k - E, 0 \right) + (1 - \mu_{\mathcal{A}})\mathcal{C}, \\ & \forall\, S \in \mathbb{R}_+^n. \end{aligned}$$

This means that for the asset price vector S, with $S_i = \eta$, for some $\eta > 0$ and all other components equal to 0, for $i = 1, 2, \ldots, n$, the constraint still holds. That is, the constraint holds for the vector $S = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \eta \\ 0 \end{pmatrix}$, where η is in the i^{th} position, for $i = 1, 2, \ldots, n$.
Now, for each $i = 1, 2, \ldots, n$, $S_i = \eta$ and all other components are equal to 0.
For a particular i, $\Longrightarrow$

$$\begin{aligned} & \sum_{l=1}^{q} (u_i^l) \max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + v_i S_i \geqslant \mu_{\mathcal{A}} \max(\omega_i S_i - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C} \\ \Longleftrightarrow\ & \sum_{l=1}^{q} (u_i^l) \max(\eta - E_i^l, 0) + z\mathrm{e}^{rT} + v_i \eta \geqslant \mu_{\mathcal{A}} \max(\omega_i \eta - E, 0) + (1 - \mu_{\mathcal{A}})\mathcal{C} \\ \Longleftrightarrow\ & \sum_{l=1}^{q} (u_i^l) \eta \max\left(1 - \frac{E_i^l}{\eta}, 0\right) + z\mathrm{e}^{rT} + v_i \eta \geqslant \eta \mu_{\mathcal{A}} \max\left(\omega_i - \frac{E}{\eta}, 0\right) + (1 - \mu_{\mathcal{A}})\mathcal{C}, \end{aligned}$$

and so, if we divide both sides by η we get, (since $\eta > 0$)

$$\Longrightarrow \sum_{l=1}^{q} (u_i^l) \max\left(1 - \frac{E_i^l}{\eta}, 0\right) + \frac{z\mathrm{e}^{rT}}{\eta} + v_i \geqslant \mu_{\mathcal{A}} \max\left(\omega_i - \frac{E}{\eta}, 0\right) + \frac{(1 - \mu_{\mathcal{A}})\mathcal{C}}{\eta},$$

and if $\eta \to \infty$, then, $\frac{E_i^l}{\eta} \to 0$, $\frac{z e^{rT}}{\eta} \to 0$, $\frac{E}{\eta} \to 0$ and $\frac{(1-\mu_{\mathcal{A}})\mathcal{C}}{\eta} \to 0$. This gives, in the limit as $\eta \to \infty$

$$\sum_{l=1}^{q}(u_i^l)\max(1,0) + 0 + v_i \geqslant \mu_{\mathcal{A}}\max(\omega_i, 0) + 0$$
$$\iff \sum_{l=1}^{q} u_i^l + v_i \geqslant \mu_{\mathcal{A}}\max(\omega_i, 0) = \mu_{\mathcal{A}}\omega_i,$$

since $\omega_i = \frac{1}{S_i^0} > 0$, for all $i = 1, 2, \ldots, n$.
Since the above holds for all $i = 1, 2, \ldots, n$, this in vector form is just

$$\sum_{l=1}^{q} u^l + v \geqslant \mu_{\mathcal{A}}\omega,$$

and so the first constraint from (8.6) holds.
(ii) $\underline{\mathcal{F}_{(8.6)} \subset \mathcal{F}_{(8.4)}}$: Now we prove the converse. So, take any $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(8.6)}$. Then, in order to show that $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(8.4)}$, we must show that

$$\sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) + z e^{rT} + v^T S \geqslant \mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C}, \;\; \forall\, S \in \mathbb{R}_+^n.$$

To this end, it suffices to show that

$$\max_{S \in \mathbb{R}_+^n}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z e^{rT} - v^T S\right\} =$$
$$\max_{S \in \mathcal{I}_{(8.6)}}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z e^{rT} - v^T S\right\},$$

since $(u^1, u^2, \ldots, u^q, z, v) \in \mathcal{F}_{(8.6)}$ it holds that

$$\sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) + z e^{rT} + v^T S \geqslant \mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C}, \;\; \forall\, S \in \mathcal{I}_{(8.6)}.$$

Thus,

$$\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z e^{rT} - v^T S \leqslant 0, \;\; \forall\, S \in \mathcal{I}_{(8.6)},$$

and so

$$\max_{S \in \mathcal{I}_{(8.6)}}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z e^{rT} - v^T S\right\} \leqslant 0,$$

and so if

$$\max_{S \in \mathbb{R}_+^n}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z e^{rT} - v^T S\right\} =$$
$$\max_{S \in \mathcal{I}_{(8.6)}}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S - E^l, 0) - z e^{rT} - v^T S\right\},$$

it means that

$$\max_{S\in\mathbb{R}^n_+}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S\right\} \leqslant 0$$

and so

$$\begin{aligned}
&\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S \leqslant 0 \quad \forall\, S\in\mathbb{R}^n_+\\
\iff &\sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C}, \quad \forall\, S\in\mathbb{R}^n_+,
\end{aligned}$$

in which case the proposition is proved.
We now show that

$$\begin{aligned}
&\max_{S\in\mathbb{R}^n_+}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S\right\} =\\
&\max_{S\in\mathcal{I}_{(8.6)}}\left\{\mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S\right\}.
\end{aligned}$$

Define the function $\psi : \mathbb{R}^n_+ \to \mathbb{R}$, given by

$$\psi(S) = \mu_{\mathcal{A}}\max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S.$$

Consider $\nabla\psi(S)$, for all $S \notin \mathcal{I}_{(8.6)}$. Then we have the following.
(a) If $\mu_{\mathcal{A}} = 1$, then $\psi(S)$ becomes

$$\psi(S) = \max(\omega^T S - E, 0) - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S.$$

Now we consider the following two cases.
(i) If $\omega^T S - E < 0$,

$$\nabla\psi(S) = \frac{\mathrm{d}\psi}{\mathrm{d}S} = -\sum_{l=1}^{q}u^l - v.$$

(ii)If $\omega^T S - E > 0$,

$$\nabla\psi(S) = \frac{\mathrm{d}\psi}{\mathrm{d}S} = \omega - \sum_{l=1}^{q}u^l - v.$$

(b) If $\mu_{\mathcal{A}} = 0$, then $\psi(S)$ becomes

$$\psi(S) = \mathcal{C} - \sum_{l=1}^{q}(u^l)^T\max(S-E^l,0) - z\mathrm{e}^{rT} - v^T S.$$

This gives

$$\nabla\psi(S) = \frac{\mathrm{d}\psi}{\mathrm{d}S} = -\sum_{l=1}^{q}u^l - v.$$

Now, from the constraint in (8.6) however,we have that

$$\sum_{l=1}^{q} u^l + v \geqslant \mu_{\mathcal{A}}\omega.$$

Then we have the following.
(a) If $\mu_{\mathcal{A}} = 1$, the constraint from (8.6) reads

$$\sum_{l=1}^{q} u^l + v \geqslant \omega, \text{ which gives, } -\sum_{l=1}^{q} u^l - v \leqslant -\omega, \text{ which means, } \omega - \sum_{l=1}^{q} u^l - v \leqslant 0.$$
$$\text{Also, } \sum_{l=1}^{q} u^l + v \geqslant \omega > 0, \text{ so, } -\sum_{l=1}^{q} u^l - v < 0.$$

(b) If $\mu_{\mathcal{A}} = 0$, the constraint from (8.6) reads

$$\sum_{l=1}^{q} u^l + v \geqslant 0, \text{ which means, } -\sum_{l=1}^{q} u^l - v \leqslant 0.$$

$\Longrightarrow$ In all cases (a) and (b), for all $S \notin \mathcal{I}_{(8.6)}$, we have that

$$\nabla\psi(S) \leqslant 0 \text{ or } \nabla\psi(S) < 0 \implies \psi(S) \text{ is non-increasing or decreasing for all } S \notin \mathcal{I}_{(8.6)}, \text{ respectively.}$$

This means that $\psi(S)$ must attain its maximum value for a value of $S \in \mathcal{I}_{(8.6)}$ and so it holds that

$$\max_{S\in\mathbb{R}^n_+} \left\{\mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S\right\} =$$
$$\max_{S\in\mathcal{I}_{(8.6)}} \left\{\mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} - \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) - z\mathrm{e}^{rT} - v^T S\right\},$$

and the proposition is proved. □

We now show that the semi-infinite optimisation problem (8.6) can be re-formulated as a finite linear problem. We start by recalling (8.6) as

$$\begin{aligned} \min_{(u^l,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n} \quad & \sum_{l=1}^{q} (u^l)^T C^l + z + v^T S^0 \\ \text{subject to} \quad & \sum_{l=1}^{q} u^l + v \geqslant \ \mu_{\mathcal{A}}\omega \\ & \sum_{l=1}^{q} (u^l)^T \max(S - E^l, 0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C}, \\ & \forall\ S \in \mathcal{I}_{(8.6)}. \end{aligned} \tag{8.6}$$

Now, using the following property of the *maximum*,

$$\max\{a, b\} + c \geqslant a + c \text{ and } \max\{a, b\} + c \geqslant b + c,$$

the second constraint is equivalent to the following two semi-infinite constraints (8.7) and (8.8), by observing that

$$\mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant \mu_{\mathcal{A}}(\omega^T S - E) + (1-\mu_{\mathcal{A}})\mathcal{C}, \text{ and}$$
$$\mu_{\mathcal{A}} \max(\omega^T S - E, 0) + (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant (1-\mu_{\mathcal{A}})\mathcal{C}.$$

$\Longrightarrow$

$$\sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S \geqslant \mu_{\mathcal{A}}(\omega^T S - E) + (1-\mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\ S \in \mathcal{I}_{(8.6)} \tag{8.7}$$

and

$$\sum_{l=1}^{q}(u^l)^T \max(S-E^l,0) + z\mathrm{e}^{rT} + v^T S \geqslant (1-\mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\ S \in \mathcal{I}_{(8.6)}. \tag{8.8}$$

Which is equivalent to

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i \geqslant \mu_{\mathcal{A}}\left(\sum_{i=1}^{n}\omega_i S_i - E\right) + (1-\mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\ S \in \mathcal{I}_{(8.6)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + z\mathrm{e}^{rT} + \sum_{i=1}^{n} v_i S_i \geqslant (1-\mu_{\mathcal{A}})\mathcal{C}, \ \ \forall\ S \in \mathcal{I}_{(8.6)},$$

respectively.

$\Longleftrightarrow$

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i - \mu_{\mathcal{A}}\sum_{i=1}^{n}\omega_i S_i + z\mathrm{e}^{rT} + \mu_{\mathcal{A}}E - (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant 0, \ \ \forall\ S \in \mathcal{I}_{(8.6)},$$

and

$$\sum_{l=1}^{q}\sum_{i=1}^{n} u_i^l \max(S_i - E_i^l, 0) + \sum_{i=1}^{n} v_i S_i + z\mathrm{e}^{rT} - (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant 0, \ \ \forall\ S \in \mathcal{I}_{(8.6)},$$

respectively.

Switching the order of summation gives

$$\sum_{i=1}^{n}\underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}}\omega_i S_i\right)}_{(\star)} + z\mathrm{e}^{rT} + \mu_{\mathcal{A}}E - (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant 0, \ \ \forall\ S \in \mathcal{I}_{(8.6)}$$

and

$$\sum_{i=1}^{n}\underbrace{\left(\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i\right)}_{(\star\star)} + z\mathrm{e}^{rT} - (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant 0, \ \ \forall\ S \in \mathcal{I}_{(8.6)},$$

respectively.

Now we choose $\alpha, \beta \in \mathbb{R}^n$ such that α_i provides a lower bound to $(\star)$ and β_i provides a lower bound to $(\star\star)$, for all $i = 1, 2, \ldots, n$. That is, we choose α_i, β_i, for all $i = 1, 2, \ldots, n$, such that

$$\alpha_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}}\omega_i S_i$$

and

$$\beta_i \leqslant \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i,$$

for all $i = 1, 2, \ldots, n$.
$\Longrightarrow$ The semi-infinite constraints of (8.6) become

$$\begin{cases} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}} \omega_i S_i \geqslant \alpha_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n \\ \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} \alpha_i + z e^{rT} + \mu_{\mathcal{A}} E - (1 - \mu_{\mathcal{A}})\mathcal{C} \geqslant 0 \\ \sum_{i=1}^{n} \beta_i + z e^{rT} - (1 - \mu_{\mathcal{A}})\mathcal{C} \geqslant 0. \end{cases} \tag{8.9}$$

This means that a point $(u^1, u^2, \ldots, u^q, z, v)$ is feasible for (8.6) if and only if $(u^1, u^2, \ldots, u^q, \alpha, \beta, z, v) \in \mathbb{R}^{n\times q} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is feasible for system (8.9). We observe here that by writing the semi-infinite constraints of (8.6) as system (8.9), we have that the last two constraints are standard (finite) linear constraints.
Now consider the semi-infinite constraints from system (8.9),

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}} \omega_i S_i \geqslant \alpha_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n,$$

and

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n.$$

Then we observe that these constraints are piece-wise linear constraints. Therefore, the minimum value of the left hand side of both inequalities over all values of $S_i \in [0, E_i^q]$, for all $i = 1, 2, \ldots, n$ occurs at one of the break points. That is, it occurs exactly when $S_i = 0$ or $S_i = E_i^1$ or $S_i = E_i^2$ or $\ldots$ or $S_i = E_i^q$, for all $i = 1, 2, \ldots, n$. Therefore, we may consider these semi-infinite constraints for the $(q + 1)$ values $S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}$, for $i = 1, 2, \ldots, n$.
Thus, it holds that

$$\begin{aligned} &\min_{S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}} \omega_i S_i \\ = &\min_{S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}} \omega_i S_i, \end{aligned}$$

and

$$\begin{aligned} &\min_{S_i \in [0, E_i^q]} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \\ = &\min_{S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}} \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i. \end{aligned}$$

Therefore, each of the semi-infinite constraints can now be replaced by $(q + 1)$ finite piece-wise linear constraints. That is, we may replace the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}} \omega_i S_i \geqslant \alpha_i, \ \forall\ S_i \in [0, E_i^q], \ \forall\ i = 1, 2, \ldots, n,$$

by

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}} \omega_i S_i \geqslant \alpha_i, \ \text{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \ \forall\ i = 1, 2, \ldots, n,$$

and the constraint

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \; \forall \; S_i \in [0, E_i^q], \; \forall \; i = 1, 2, \ldots, n,$$

by

$$\sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \text{ for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \; \forall \; i = 1, 2, \ldots, n.$$

The above analysis can be summarised in the following theorem.

Theorem 8.10. *The semi-infinite optimisation problem* (8.6) *is equivalent to the following finite linear optimisation problem*

$$\begin{array}{ll} \underset{(u^1,u^2,\ldots,u^q,z,v)\in\mathbb{R}^{n\times q}\times\mathbb{R}\times\mathbb{R}^n,\alpha,\beta,\in\mathbb{R}^n}{min} & \sum_{l=1}^{q}(u^l)^T C^l + z + v^T S^0 \\ \text{subject to} & \sum_{l=1}^{q} u^l + v \geqslant \; \mu_{\mathcal{A}}\omega \\ & \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i - \mu_{\mathcal{A}}\omega_i S_i \geqslant \alpha_i, \\ & \textit{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \; \forall \; i = 1, 2, \ldots, n \\ & \sum_{l=1}^{q} u_i^l \max(S_i - E_i^l, 0) + v_i S_i \geqslant \beta_i, \\ & \textit{for } S_i \in \{0, E_i^1, E_i^2, \ldots, E_i^q\}, \; \forall \; i = 1, 2, \ldots, n \\ & \sum_{i=1}^{n} \alpha_i + z\mathrm{e}^{rT} + \mu_{\mathcal{A}} E - (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant 0 \\ & \sum_{i=1}^{n} \beta_i + z\mathrm{e}^{rT} - (1-\mu_{\mathcal{A}})\mathcal{C} \geqslant 0, \end{array} \tag{8.11}$$

in the sense that both optimisation problems have the same feasible region and hence the same optimal solution and the same optimal objective function value.

Observation: In comparison to the semi-infinite problem (8.6); we observe here that (8.11) has $n + n = 2n$ additional variables and a total of

$$\begin{aligned} & n + n(q+1) + n(q+1) + 1 + 1 = n + 2n(q+1) + 2 \\ & = n + 2nq + 2n + 2 = 3n + 2nq + 2 = n(2q+3) + 2 \end{aligned}$$

linear constraints. The advantage of solving (8.11) in comparison to (8.6) is that we are solving a standard, *finite* linear problem in comparison to a semi-infinite one; something which can be implemented on an appropriate LO software solver, even for large values of n.

We may now solve the finite linear problem (8.11) by using an appropriate software program. This will yield the optimal objective function value which is an upper bound on the current price of the Altiplano Mountain Range option under consideration.

In particular, if the optimal objective function value is independent of $\mu_{\mathcal{A}}$, which we treated as a parameter, then this optimal objective function value is a valid upper bound.

In general however this may not be the case and the optimal objective function value may be dependent upon the value of $\mu_{\mathcal{A}}$. In this case, the upper bound is obtained as follows. Since $\mu_{\mathcal{A}} \in \{0, 1\}$, we substitute in $\mu_{\mathcal{A}} = 0$ and $\mu_{\mathcal{A}} = 1$, respectively into the obtained optimal objective function value. The upper bound on the current price of the Altiplano Mountain Range option is the **bigger** of these optimal objective function values.

We observe here that the model set up and approach to finding an upper bound on the price of an Altiplano Mountain Range option presented above is very similar to the model set up presented in sub-section 3.4 which looked at finding an upper bound on the price of a European basket call option. Further, the methodology employed in the proof of Proposition 8.5 can be viewed as a modification to the methodology employed in the proof of Proposition 3.14 in sub-section 3.4. It is a modification because in the proof of Proposition 8.5 we had to consider the consider the value of the parameter $\mu_{\mathcal{A}}$ which determines what payoff the Altiplano Mountain Range option will take; something which was not required of course in the proof of Proposition 3.14. Similarly, the ideas used to obtain the finite LO problem (8.11) can also be viewed as a modification to the techniques used in sub-section 3.4 to re-write the SIO problem (3.13) as a solvable and finite LO problem.

8.3 A numerical example

We conclude this section by presenting a numerical example which shows how the above analysis can be used and implemented within real financial markets.

For this example we consider the $n = 3$ asset case and we assume that we know the current prices of $q = 2$ European vanilla call options **per asset**. In this case the finite linear problem (8.11) has a total of $6+1+3+3+3 = 16$ variables and $n(2q+3)+2 = 3(4+3)+2 = (3\times 7)+2 = 23$ constraints. We start by re-writing (8.11) in a more convenient form and then solve the problem using the software *LiPS- 1.9.4*, which is designed to solve large scale linear and integer optimisation problems. The optimal objective function value is an upper bound on the current price of the Altiplano Mountain Range option under consideration. For more information about the software program LiPS-1.9.4, we refer the interested reader to [54].

We consider the following numerical values for this example.

The current underlying asset prices, given by S_i^0, for $i = 1, 2, 3$ are given in the following table (where p means pence).

Current underlying asset price	S_1^0	S_2^0	S_3^0
Numerical Value	$194p$	$209p$	$218p$

This allows us to define the following weights ω_i, for $i = 1, 2, 3$.

Weight	ω_1	ω_2	ω_3
Numerical Value	$\frac{1}{194}$	$\frac{1}{209}$	$\frac{1}{218}$

For the European vanilla call options whose current prices we know we have the following data, for the exercise prices.

European Vanilla call exercise price	E_1^1	E_2^1	E_3^1	E_1^2	E_2^2	E_3^2
Numerical Value	$150p$	$161p$	$210p$	$174p$	$200p$	$225p$

With the following current prices.

European Vanilla call current option price	C_1^1	C_2^1	C_3^1	C_1^2	C_2^2	C_3^2
Numerical Value	$56p$	$60p$	$13p$	$32p$	$25p$	$5p$

Furthermore, we have that the interest rate $r = 0.01 = 1\%$ and the expiry date $T = 0.5$ years is 6 months from now. The Altiplano Mountain Range option has an exercise price $E = 1.5$ pence and a coupon payment

$\mathcal{C} = 10$ pence.
Using this numerical data we may write (8.11) as the linear optimisation problem (A.1) given in Appendix A.
In order to solve (A.1) using LiPS-1.9.4, we re-write (A.1) in a more convenient and computer friendly form. In this case we define 16 variables x_i, for $i = 1, 2, \ldots, 16$ as follows. Let $u^1 = (x_1, x_2, x_3)^T$, $u^2 = (x_4, x_5, x_6)^T$, $z = x_7$, $v = (x_8, x_9, x_{10})^T$, $\alpha = (x_{11}, x_{12}, x_{13})^T$ and $\beta = (x_{14}, x_{15}, x_{16})^T$. Then (A.1) can be re-written as problem (A.2) which is given in Appendix A.
We solve (A.2) for the values $\mu_{\mathcal{A}} = 0$ and $\mu_{\mathcal{A}} = 1$, respectively. Then an upper bound on the current price of the Altiplano Mountain Range option is given by the **bigger** of the obtained optimal objective function values.
For the $\mu_{\mathcal{A}} = 0$ case, (A.2) becomes (A.3), and for the $\mu_{\mathcal{A}} = 1$ case problem (A.2) becomes problem (A.4); which can all be found in Appendix A.
We solve each of these problems (A.3) and (A.4) in LiPS-1.9.4.
For (A.3) we may input the problem into LiPS-1.9.4 as shown in Figure 1.
Commanding the program to solve the problem gives the output shown in Table I, below.

Table I below shows part of the solution output for problem (A.3) obtained by LiPS-1.9.4.

```
>> Optimal solution FOUND
>> Minimum = 2000/201
```

*** RESULTS ***

Variable	Value	Obj. Cost	Reduced Cost
X1	0	56	-24
X2	0	60	-35
X3	0	13	-8
X4	0	32	0
X5	0	25	0
X6	0	5	0
X7	2000/201	1	0
X8	0	194	-162
X9	0	209	-184
X10	0	218	-213

Table I: Part of the solution output given in LiPS-1.9.4 for problem (A.3).

From Table I we may read off the optimal objective function value of (A.3) as $\frac{2000}{201} = 9.950248\cdots = 9.95$ to 3 significant figures.
We may solve (A.4) by proceeding in similar fashion. The input of (A.4) into LiPS-1.9.4 is shown in Figure 2 in Appendix A.
Again commanding the program to solve the problem gives the output shown in Table II, below.

Table II below shows part of the solution output for problem (A.4) obtained by LiPS-1.9.4.

```
>> Optimal solution FOUND
>> Minimum = 3.00004

                    *** RESULTS ***
```

Variable	Value	Obj. Cost	Reduced Cost
X1	0	56	-848/29
X2	0	60	-19.245
X3	1/70000	13	0
X4	0	32	-32
X5	0	25	-25
X6	0	5	-5
X7	0	1	-1
X8	0.00515517	194	0
X9	0.004785	209	0
X10	0.00458571	218	0
X11	0	0	-97/87
X12	0	0	-209/200
X13	0	0	-41/42
X14	0	0	0

Table II: Part of the solution output given in LiPS-1.9.4 for problem (A.4).

From Table II we may read off the optimal objective function value of (A.4) as $3.00004 = 3.00$ to 3 significant figures.
Referring the reader to study the material presented in Appendix A and in tables I and II, we may use these numerical results to conclude that an upper bound on the current price of the Altiplano Mountain Range option under consideration is given by

$$\max\{9.95, 3.00\} = 9.95\text{pence}.$$

Therefore, the current price of this particular Altiplano Mountain Range option should **not** exceed 9.95pence.
That concludes this section on looking at how to find price bounds on an Altiplano Mountain Range option.

9 Conclusion

In conclusion, we first presented existing models and then derived our own models which aim to find upper and lower bounds on the current price of various classes of basket options. In particular, we have presented some basic and preliminary ideas from mathematical finance and option pricing as well as mathematical optimisation and semi-infinite optimisation. Then we explained how and why optimisation could be used to find current price bounds on financial options. We then presented existing results which consider how semi-infinite optimisation can be used to find upper and lower bounds on the current price of a European basket call option. In particular, we started by introducing the basic model set-up. We aimed to find no-arbitrage and model independent upper/lower bounds on the current price of a European basket call option, written on n underlying assets. Firstly given that we know the current prices of **one** European vanilla call option, per asset and the forward/expected price of each asset, and then by considering an extension of this basic model. We modified the constraints by assuming that we know the current prices of other European basket call options, written on the same n underlying assets. We then extended both of these models further by incorporating *bid-ask prices*, making each model more realistic and more accurate. We then assumed that instead of knowing one European vanilla call option price, per asset; we know the current prices of several European vanilla call option prices, per asset; allowing us to present another model. In the remainder of the thesis we presented our own, new and original results. In particular, we derived important and new results concerning upper and lower bounds for various classes of basket options, including European, American and Asian basket options as well as extending our analysis to finding current price bounds for a certain class of Mountain Range options. The results which we derived are important because on the one hand they extend previous results and on the other hand they can directly be implemented in financial markets by investment banks and other financial firms which are interested in the pricing of options. As a final summary of what has been presented in this thesis we have the following six new results.

1. In section 4.1, we have derived a lower bound for the current price of a European basket call option, under a set of specific assumptions and scenarios In particular, we have derived a semi-infinite optimisation problem with a compact index set which when solved would yield a lower bound on the price of a European basket call option under the scenario outlined in section 4.1. We note here that given the nature of the problem, solution techniques to solve the derived SIO problem (4.4) is still open to further research. Our result has at least transformed the linear SIO problem with a non-compact index set to a linear SIO problem with a compact index set.

2. In section 4.2 we have extended the European basket call option pricing problem by incorporating bid-ask prices within the optimisation model to make the model more realistic and we derived a lower bound from this model. In particular, we have derived a finite linear optimisation problem which can be solved by an appropriate software solver.

3. In section 5.2 and 5.3 we studied in detail, methods to find price bounds on both American basket call and put options, respectively. In particular, we showed in section 5.2 that it does not make sense to exercise an American basket call option early when none of the underlying assets pay out any dividends. Therefore, this type of option is equivalent to its European basket call option counterpart and the price bounds found for this option are also valid for this particular American basket call option. In the same section we further argued that when at least one of the underlying assets pays out dividends then the price of an American basket call option is at least as much as its European basket call option counterpart. In section 5.3 we considered price bounds on American basket put options, where none of the underlying assets pay out any dividends. By first deriving a put-call parity for European basket options, we derived a put-call parity inequality for American basket options. This allowed us to obtain upper and lower bounds on the price of an American basket put option by using the bounds obtained for the price of a European/American basket call option, where none of the underlying assets pay out any dividends.
   In section 6 we then considered a Bermuda basket put option which is a specific type of American option and found an upper bound on this current option price using semi-infinite optimisation. In particular,

we derived a solvable, finite linear optimisation problem which would yield an upper bound on the price of this particular type of basket option. We ended this section by explaining how to extend the given model to find price bounds on a Bermuda basket put option with multiple early exercise dates and explained that following a similar analysis, a finite linear optimisation problem could also be derived in this case.

4. In section 7.2 we found an upper bound on the current price of a certain type of Asian basket call option by using a similar approach to what had been done for the European basket call option. In particular, we derived a solvable, finite linear optimisation problem which when solved (via an appropriate software) would yield an upper bound on the price of this particular Asian basket call option.

5. In section 7.3, we then extended this model for Asian basket call options to derive an upper bound under the presence of bid-ask prices. Here we used a different approach under the more realistic assumption of knowing bid-ask prices for related options to derive a solvable, finite linear optimisation problem which could be used to find upper bounds on the price of an Asian basket call option.

6. Finally, in section 8.2 we extended our ideas to finding an upper bound on a unique type of exotic option. That is, we used semi-infinite optimisation to derive an upper bound on the current price of an Altiplano Mountain Range option; and we also presented a numerical example for this option. Then using an appropriate software solver we obtained a solution. In particular, in section 8.2 we derived a finite, linear optimisation model which can be used to find an upper bound on the price of this type of option. In section 8.3 we explicitly used our derived optimisation model from section 8.2 with market data and a LO software solver to present the reader with a numerical example of how our obtained model can be used in practice.

We note here that the methodology of using optimisation to calculate price bounds on options ahead of alternative methods should be clear. We have seen in sub-section 2.1.4 the problems that the traditional Black-Scholes framework encounters when pricing multi-asset options, such as basket options. The multi-asset Black-Scholes equation struggles to incorporate correlation between the assets which may exist, and as such may produce inaccurate results. Furthermore, it also makes many assumptions, some of which may not hold in reality. In contrast, by using optimisation to price options we assume only the absence of arbitrage and the knowledge of current prices of similar assets, which is reasonable and holds in the real world. Options with complicated payoffs, such as basket options, are often difficult to price. However, mathematical optimisation provides an efficient and accurate way to find bounds and in some cases price an option of interest.

It is worth explaining our choices of models here and why for a given particular type of basket option, we selected the model that we did. In sections 4.1 and 7.3 we chose these particular models (that is, assuming that we know the prices of other basket options) because the solution technique used in these sections is reliant upon the model being specified in the way it has been done so. For the remaining sections the models were down to choice and finding a model where the solution technique in each respective section worked and gave a solution. It is worth noting that no model is perfect and of course all models can be improved. The selection of models presented in sections 4-8 are down to simplicity and staying close to reality. We have models set up so they are are simple to understand and use but are realistic in the sense that the constraints we have picked, we have done so because we know that prices exist for these instruments and are easily accessible or can be easily found.

We have seen many optimisation models throughout this thesis aimed at finding bounds on the current price of a basket option. We have combined theory from mathematical finance and mathematical optimisation to obtain a solution to our formulated models. Thus, this shows how mathematical theory is being used and implemented in the real world to solve real world problems.

We observe that the main results of this thesis are concerned with semi-infinite optimisation and its applications to the basket option pricing problem. In particular, we have presented various semi-infinite optimisation problems with a non-compact index set, $\mathbb{R}^n_+$. This led us to encounter many different and new challenges. Almost all existing results on SIO assume that the index set is compact. In order to utilise this theory, we first had to re-write our basket option pricing problem as a SIO problem with a compact index set. Recall that this was done firstly in Proposition 3.14 for an upper bound on the current price of a European basket call option, and then by ourselves in proceeding sections for various other types of basket options. One possible area for future research would be to consider general results for semi-infinite problems with a non-compact index set.

Further ideas for future research on this topic include extending the upper bound bid-ask prices result in sub-section 3.6.1, to the lower bound problem. That is, how can we find a lower bound on the current price of a European basket call option given that we know the current bid-ask prices of European vanilla call options.

Another idea which could be researched upon is concerned with the computer optimisation part of this thesis. We have introduced many large scale linear optimisation problems which may be solved by appropriate software packages. One interesting area to research would be to compare the speeds and memory usage of various different software packages when solving the LO problems given in this thesis. This is especially important when solving real-world financial problems where minimising time and money are of course desired.

As a final remark, we should be careful and cautious with our results. Although our newly obtained results may be very credible; we are modelling a real-world problem as an optimisation problem under certain assumptions. If we wish to incorporate our results in real financial markets, we should do so with some care. We should ensure that all assumptions of the model are satisfied in reality before using any results obtained here. It should be remembered that a model is just model, and of course, ultimately the laws of supply and demand dictate what the current price of an option should be. In fact, the ultimate price paid for an option is an agreed price between both respective parties. Nevertheless, the results obtained in this thesis may give the holder and writer of a basket option a starting point.

A Appendix

Using the numerical data given in sub-section 8.3, the finite linear optimisation problem (8.11) may be written as follows.

$$
\begin{aligned}
\min_{(u^1,u^2,z,v)\in\mathbb{R}^{3\times 2}\times\mathbb{R}\times\mathbb{R}^3,\alpha,\beta,\in\mathbb{R}^3} \quad & 56u_1^1 + 60u_2^1 + 13u_3^1 + 32u_1^2 + 25u_2^2 + 5u_3^2 + z + 194v_1 + 209v_2 + 218v_3 \\
\text{subject to} \quad & -u_1^1 - u_1^2 - v_1 \leqslant -\frac{1}{194}\mu_{\mathcal{A}} \\
& -u_2^1 - u_2^2 - v_2 \leqslant -\frac{1}{209}\mu_{\mathcal{A}} \\
& -u_3^1 - u_3^2 - v_3 \leqslant -\frac{1}{218}\mu_{\mathcal{A}} \\
& \alpha_1 \leqslant 0 \\
& -150v_1 + \alpha_1 \leqslant -\frac{75}{97}\mu_{\mathcal{A}} \\
& -24u_1^1 - 174v_1 + \alpha_1 \leqslant -\frac{87}{97}\mu_{\mathcal{A}} \\
& \alpha_2 \leqslant 0 \\
& -161v_2 + \alpha_2 \leqslant -\frac{161}{209}\mu_{\mathcal{A}} \\
& -39u_2^1 - 200v_2 + \alpha_2 \leqslant -\frac{200}{209}\mu_{\mathcal{A}} \\
& \alpha_3 \leqslant 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.1)} \\
& -210v_3 + \alpha_3 \leqslant -\frac{105}{109}\mu_{\mathcal{A}} \\
& -15u_3^1 - 225v_3 + \alpha_3 \leqslant -\frac{225}{218}\mu_{\mathcal{A}} \\
& \beta_1 \leqslant 0 \\
& -150v_1 + \beta_1 \leqslant 0 \\
& -24u_1^1 - 174v_1 + \beta_1 \leqslant 0 \\
& \beta_2 \leqslant 0 \\
& -161v_2 + \beta_2 \leqslant 0 \\
& -39u_2^1 - 200v_2 + \beta_2 \leqslant 0 \\
& \beta_3 \leqslant 0 \\
& -210v_3 + \beta_3 \leqslant 0 \\
& -15u_3^1 - 225v_3 + \beta_3 \leqslant 0 \\
& -\alpha_1 - \alpha_2 - \alpha_3 - z\mathrm{e}^{rT} \leqslant \mu_{\mathcal{A}}E - (1-\mu_{\mathcal{A}})\mathcal{C} \\
& -\beta_1 - \beta_2 - \beta_3 - z\mathrm{e}^{rT} \leqslant -(1-\mu_{\mathcal{A}})\mathcal{C}.
\end{aligned}
$$

Using the variables x_i, for $i = 1, 2, \ldots, 16$ as defined in sub-section 8.3, problem (A.1) may equivalently be re-written as,

$$
\begin{aligned}
\min_{x_1, x_2, \ldots, x_{16}} \quad & 56x_1 + 60x_2 + 13x_3 + 32x_4 + 25x_5 + 5x_6 + x_7 + 194x_8 + 209x_9 + 218x_{10} \\
\text{subject to} \quad & -x_1 - x_4 - x_8 \leqslant -\frac{1}{194}\mu_{\mathcal{A}} \\
& -x_2 - x_5 - x_9 \leqslant -\frac{1}{209}\mu_{\mathcal{A}} \\
& -x_3 - x_6 - x_{10} \leqslant -\frac{1}{218}\mu_{\mathcal{A}} \\
& x_{11} \leqslant 0 \\
& -150x_8 + x_{11} \leqslant -\frac{75}{97}\mu_{\mathcal{A}} \\
& -24x_1 - 174x_8 + x_{11} \leqslant -\frac{87}{97}\mu_{\mathcal{A}} \\
& x_{12} \leqslant 0 \\
& -161x_9 + x_{12} \leqslant -\frac{161}{209}\mu_{\mathcal{A}} \\
& -39x_2 - 200x_9 + x_{12} \leqslant -\frac{200}{209}\mu_{\mathcal{A}} \\
& x_{13} \leqslant 0 \\
& -210x_{10} + x_{13} \leqslant -\frac{105}{109}\mu_{\mathcal{A}} \\
& -15x_3 - 225x_{10} + x_{13} \leqslant -\frac{225}{218}\mu_{\mathcal{A}} \\
& x_{14} \leqslant 0 \\
& -150x_8 + x_{14} \leqslant 0 \\
& -24x_1 - 174x_8 + x_{14} \leqslant 0 \\
& x_{15} \leqslant 0 \\
& -161x_9 + x_{15} \leqslant 0 \\
& -39x_2 - 200x_9 + x_{15} \leqslant 0 \\
& x_{16} \leqslant 0 \\
& -210x_{10} + x_{16} \leqslant 0 \\
& -15x_3 - 225x_{10} + x_{16} \leqslant 0 \\
& -x_{11} - x_{12} - x_{13} - 1.005x_7 \leqslant 1.5\mu_{\mathcal{A}} - 10(1 - \mu_{\mathcal{A}}) \\
& -x_{14} - x_{15} - x_{16} - 1.005x_7 \leqslant -10(1 - \mu_{\mathcal{A}}).
\end{aligned}
\tag{A.2}
$$

Problem (A.2) becomes the following problem (A.3) for the $\mu_{\mathcal{A}} = 0$ case.

$$\begin{aligned}
\min_{x_1, x_2, \ldots, x_{16}} \quad & 56x_1 + 60x_2 + 13x_3 + 32x_4 + 25x_5 + 5x_6 + x_7 + 194x_8 + 209x_9 + 218x_{10} \\
\text{subject to} \quad & -x_1 - x_4 - x_8 \leqslant 0 \\
& -x_2 - x_5 - x_9 \leqslant 0 \\
& -x_3 - x_6 - x_{10} \leqslant 0 \\
& x_{11} \leqslant 0 \\
& -150x_8 + x_{11} \leqslant 0 \\
& -24x_1 - 174x_8 + x_{11} \leqslant 0 \\
& x_{12} \leqslant 0 \\
& -161x_9 + x_{12} \leqslant 0 \\
& -39x_2 - 200x_9 + x_{12} \leqslant 0 \\
& x_{13} \leqslant 0 \\
& -210x_{10} + x_{13} \leqslant 0 \\
& -15x_3 - 225x_{10} + x_{13} \leqslant 0 \\
& x_{14} \leqslant 0 \\
& -150x_8 + x_{14} \leqslant 0 \\
& -24x_1 - 174x_8 + x_{14} \leqslant 0 \\
& x_{15} \leqslant 0 \\
& -161x_9 + x_{15} \leqslant 0 \\
& -39x_2 - 200x_9 + x_{15} \leqslant 0 \\
& x_{16} \leqslant 0 \\
& -210x_{10} + x_{16} \leqslant 0 \\
& -15x_3 - 225x_{10} + x_{16} \leqslant 0 \\
& -x_{11} - x_{12} - x_{13} - 1.005x_7 \leqslant -10 \\
& -x_{14} - x_{15} - x_{16} - 1.005x_7 \leqslant -10.
\end{aligned} \tag{A.3}$$

Problem (A.2) becomes the following problem (A.4) for the $\mu_{\mathcal{A}} = 1$ case.

$$\begin{aligned}
\min_{x_1, x_2, \ldots, x_{16}} \quad & 56x_1 + 60x_2 + 13x_3 + 32x_4 + 25x_5 + 5x_6 + x_7 + 194x_8 + 209x_9 + 218x_{10} \\
\text{subject to} \quad & -x_1 - x_4 - x_8 \leqslant -\frac{1}{194} \\
& -x_2 - x_5 - x_9 \leqslant -\frac{1}{209} \\
& -x_3 - x_6 - x_{10} \leqslant -\frac{1}{218} \\
& x_{11} \leqslant 0 \\
& -150x_8 + x_{11} \leqslant -\frac{75}{97} \\
& -24x_1 - 174x_8 + x_{11} \leqslant -\frac{87}{97} \\
& x_{12} \leqslant 0 \\
& -161x_9 + x_{12} \leqslant -\frac{161}{209} \\
& -39x_2 - 200x_9 + x_{12} \leqslant -\frac{200}{209} \\
& x_{13} \leqslant 0 \\
& -210x_{10} + x_{13} \leqslant -\frac{105}{109} \\
& -15x_3 - 225x_{10} + x_{13} \leqslant -\frac{225}{218} \\
& x_{14} \leqslant 0 \\
& -150x_8 + x_{14} \leqslant 0 \\
& -24x_1 - 174x_8 + x_{14} \leqslant 0 \\
& x_{15} \leqslant 0 \\
& -161x_9 + x_{15} \leqslant 0 \\
& -39x_2 - 200x_9 + x_{15} \leqslant 0 \\
& x_{16} \leqslant 0 \\
& -210x_{10} + x_{16} \leqslant 0 \\
& -15x_3 - 225x_{10} + x_{16} \leqslant 0 \\
& -x_{11} - x_{12} - x_{13} - 1.005x_7 \leqslant 1.5 \\
& -x_{14} - x_{15} - x_{16} - 1.005x_7 \leqslant 0.
\end{aligned} \tag{A.4}$$

Figure 1 below shows the input of problem (A.3) into LiPS-1.9.4.

```
min: 56*X1 + 60*X2 + 13*X3 + 32*X4 + 25*X5 + 5*X6 + X7 + 194*X8 + 209*X9 + 218*X10;

Constraint1: -X1 - X4 - X8 <= 0;
Constraint2: -X2 - X5 - X9 <= 0;
Constraint3: -X3 - X6 - X10 <= 0;
Constraint4: X11 <= 0;
Constraint5: -150*X8 + X11 <= 0;
Constraint6: -24*X1 - 174*X8 + X11 <= 0;
Constraint7: X12 <= 0;
Constraint8: -161*X9 + X12 <= 0;
Constraint9: -39*X2 - 200*X9 + X12 <= 0;
Constraint10: X13 <= 0;
Constraint11: -210*X10 + X13 <= 0;
Constraint12: -15*X3 - 225*X10 + X13 <= 0;
Constraint13: X14 <= 0;
Constraint14: -150*X8 + X14 <= 0;
Constraint15: -24*X1 - 174*X8 + X14 <= 0;
Constraint16: X15 <= 0;
Constraint17: -161*X9 + X15 <= 0;
Constraint18: -39*X2 - 200*X9 + X15 <= 0;
Constraint19: X16 <= 0;
Constraint20: -210*X10 + X16 <= 0;
Constraint21: -15*X3 - 225*X10 + X16 <= 0;
Constraint22: -1.005*X7 - X11 - X12 - X13 <= -10;
Constraint23: -1.005*X7 - X14 - X15 - X16 <= -10;
```

Figure 1: Inputting problem (A.3) for the $\mu_{\mathcal{A}} = 0$ case into LiPS- 1.9.4.

Figure 2 below shows the input of problem (A.4) into LiPS-1.9.4.

```
min: 56*X1 + 60*X2 + 13*X3 + 32*X4 + 25*X5 + 5*X6 + X7 + 194*X8 + 209*X9 + 218*X10;

Constraint1: -X1 - X4 - X8 <= -0.00515;
Constraint2: -X2 - X5 - X9 <= -0.00478;
Constraint3: -X3 - X6 - X10 <= -0.00459;
Constraint4: X11 <= 0;
Constraint5: -150*X8 + X11 <= -0.773;
Constraint6: -24*X1 - 174*X8 + X11 <= -0.897;
Constraint7: X12 <= 0;
Constraint8: -161*X9 + X12 <= -0.770;
Constraint9: -39*X2 - 200*X9 + X12 <= -0.957;
Constraint10: X13 <= 0;
Constraint11: -210*X10 + X13 <= -0.963;
Constraint12: -15*X3 - 225*X10 + X13 <= -1.032;
Constraint13: X14 <= 0;
Constraint14: -150*X8 + X14 <= 0;
Constraint15: -24*X1 - 174*X8 + X14 <= 0;
Constraint16: X15 <= 0;
Constraint17: -161*X9 + X15 <= 0;
Constraint18: -39*X2 - 200*X9 + X15 <= 0;
Constraint19: X16 <= 0;
Constraint20: -210*X10 + X16 <= 0;
Constraint21: -15*X3 - 225*X10 + X16 <= 0;
Constraint22: -1.005*X7 - X11 - X12 - X13 <= 1.5;
Constraint23: -1.005*X7 - X14 - X15 - X16 <= 0;
```

Figure 2: Inputting problem (A.4) for the $\mu_{\mathcal{A}} = 1$ case into LiPS- 1.9.4.

References

[1] Stewart, I., 2012. *The mathematical equation that caused the banks to crash*. The Observer. Available at: http://www.theguardian.com/science/2012/feb/12/black-scholes-equation-credit-crunch (Accessed: 5th September 2013).

[2] Wilmott, P, Howison, S and Dewynne, J., 1995. *The Mathematics of Financial Derivatives: A Student Introduction*. Cambridge: Cambridge University Press.

[3] Wilmott, P., 2007. *Paul Wilmott introduces Quantitative Finance: Second Edition*. Chichester: John Wiley.

[4] Krekel, M, de Kock, J, Korn, R and Man, T-K., 2004. An Analysis of Pricing Methods for Baskets Options. *WILMOTT magazine*, Technical Article 5, pp. 82-89.

[5] Cook, J, D., 2008. *Inverse Gamma Distribution*. Available at: http://www.johndcook.com/inverse_gamma.pdf (Accessed: 5th June 2013).

[6] Bertsekas, D, P., 2004. *Nonlinear Programming. Second Edition*. Athena Scientific.

[7] Hettich, R and Kortanek, K. O., 1993. Semi-infinite programming: Theory, methods, and applications. *SIAM Review*, 35 (3), pp. 380-429.

[8] Reemsten, R and Rückmann, J, J., 1998. *Semi-Infinite Programming*. Boston: Kluwer Academic Publishers.

[9] Goberna, M.A. and Lopez, M.A., 1998. *Linear Semi-Infinite Optimization*. Chichester: John Wiley.

[10] Glashoff, K., 1979. Duality theory of Semi-Infinite Programming. *Semi-Infinite Programming, lecture notes in control and information sciences*, 15, pp. 1-16.

[11] Daum, S and Werner, R., 2011. A novel feasible discretization method for linear semi- infinite programming applied to basket option pricing. *Optimization*, 60(10-11), pp. 1379-1398.

[12] Cornuejols, G and Tütüncü, R., 2007. *Optimization Methods in Finance*. Cambridge: Cambridge University Press.

[13] Vanderbei, R, J., 2001. *Linear Programming: Foundations and Extensions. Second Edition*. Boston: Kluwer Academic Publishers.

[14] Tao, T., 2010. *An introduction to measure theory*. Rhode Island: American Mathematical Society.

[15] Epstein, B., 1970. *Linear Functional Analysis: Introduction to Lebesgue integration and infinite dimensional problems*. Philadelphia: W. B. Saunders.

[16] De Barra, G., 1974. *Introduction to Measure Theory*. London: Van Nostrand Reinhold company Ltd.

[17] Capiński, M and Kopp, E., 2004. *Measure, Integral and Probability. Second Edition*. London: Springer-Verlag London Limited.

[18] Laurence, P and Wang, T., 2005. Sharp Upper and Lower Bounds for Basket Options. *Applied Mathematical Finance*, 12 (3), pp. 253-282.

[19] Peña, J, Vera, J and Zuluaga, L., 2010. Static-arbitrage lower bounds on the prices of basket options via linear programming. *Quantitative Finance*, 10, pp. 819-827.

[20] d'Aspremont, A and El Ghaoui, L., 2006. Static Arbitrage Bounds on Basket Option Prices. *Mathematical Programming*, Series A, 106 (3), pp. 467-489.

[21] Peña, J, Saynac, X, Vera, J and Zuluaga, L., 2010. Computing general static-arbitrage bounds for European basket options via Dantzig-Wolfe decomposition. *Algorithmic Operations Research*, 5, pp. 65-74.

[22] Peña, J, Vera, J and Zuluaga, L., 2012. Computing arbitrage bounds on basket options in the presence of bid-ask spreads. *European Journal of Operational Research*, 222, pp. 369-376.

[23] Glashoff, K and Gustafson, S, A., 1983. *Linear Optimization and Approximation*. Singapore: Springer Singapore Pte. Limited.

[24] Hobson, D, Laurence, P and Wang, T., 2005. Static-arbitrage upper bounds for the prices of Basket Options. *Quantitative Finance*, 5(4), pp. 329-342.

[25] Dhaene, J, Chen, X, Deelstra, G and Vanmaele, M., 2008. Static super-replicating strategies for a class of exotic options. *Insurance: Mathematics and Economics*, 42(3), pp. 1067-1085.

[26] Tsuzuki, Y., 2013. On optimal super-hedging and sub-hedging strategies. *International journal of Theoretical and Applied Finance*, 16(6), pp. 1-17.

[27] Caldana, R, Fusai, G, Gnoatto, A and Grasselli, M., 2014. *General closed- form Basket Option pricing bounds*. Available at: http://ssrn.com/abstract=2376134 (Accessed 24th June 2014).

[28] Boyle, P.P and Lin, S.X., 1997. Bounds on contingent claims based on several assets. *Journal of Financial Economics*, 46(3), pp. 383-400.

[29] Beiglböck, M, Henry-Labordère, P and Penkner, F., 2013. Model-independent bounds for option prices-a mass transport approach. *Finance and Stochastics*, 17(3), pp. 477-501.

[30] Hobson, D., 1998. Robust hedging of the lookback option. *Finance and Stochastics*, 2(4), pp. 329-347.

[31] Xu, G and Zheng, H., 2009. Approximate Basket Options valuation for a jump-diffusion model. *Insurance: Mathematics and Economics*, 45(2), pp. 188-194.

[32] Shiraya, K and Takahashi, A., 2014. Pricing Basket Options under local stochastic volatility with jumps. *Center for Advanced Research in Finance, CARF F-Series*, F336.

[33] Dahl, L.O., 2000. Valuation of European call options on multiple underlying assets by using a Quasi-Monte Carlo method. A case with Baskets from Oslo Stock Exchange. *In Proceedings AFIR*, 10, pp. 239-248.

[34] Milevsky, M and Psoner, S., 1998. Closed form approximation for valuing basket options. *Journal of Derivatives*, pp. 54-61.

[35] Borovkova, S, Permana, F.J and Weide, H.v.d., 2007. A closed-form approach to the valuation and hedging of basket and spread options. *Journal of Derivatives*, 14(4), pp. 8-24.

[36] Flamouris, D and Giamouridis, D., 2007. Approximate basket option valuation for a simplified jump process. *Journal of Futures Markets*, 27, pp. 819-837.

[37] Hobson, D, Laurence, P and Wang, T., 2005. Static-arbitrage optimal sub-replicating strategies for Basket Options. *Insurance: Mathematics and Economics*, 37(3), pp. 553-572.

[38] Mudzimbabwe, W, Patidar, K.C and Witbooi, P.J., 2012. European basket option pricing by maximizing over a subset of lower bounds. *Quaestiones Mathematicae*, 35(4), pp. 507-520.

[39] Benguigui, M and Baude, F., 2014. Fast American Basket Option pricing on a multi-GPU cluster. *22nd High Performance Computing Symposium*, pp. 1-8.

[40] Christensen, S., 2014. A method for pricing American Options using semi-infinite linear programming. *Mathematical Finance*, 24(1), pp. 156-172.

[41] Put-Call Parity, 2013. *Put-Call Parity of American Options*. Available at: http://www.putcallparity.net/put-call -parity-of-american-options (Accessed: 14th December 2013).

[42] Hepperger, P., 2013. Pricing high-dimensional Bermudan Options using variance-reduced Monte Carlo methods. *Journal of Computational Finance*, 16(3), pp. 99-126.

[43] Broadie, M and Cao, M., 2008. Improved lower and upper bound algorithms for pricing American Options by simulation. *Quantitative Finance*, 8, pp. 845-861.

[44] Jain, S and Oosterlee, C.W., 2013. *The stochastic grid bundling method: Efficient pricing of Bermudan Options and their Greeks*. Available at: http://ssrn.com/abstract=2293942 (Accessed: 26th June 2014).

[45] Le, Floc'h, F., 2013. *Lower and Upper bounds in the Monte Carlo simulation of Bermudan Basket options*. Available at: http://ssrn.com/abstract=2262259 (Accessed: 25th June 2014).

[46] Brigo, D, Mercurio, F, Rapisarda, F and Scotti, R., 2004. Approximated moment-matching dynamics for Basket options pricing. *Quantitative Finance*, 4, pp. 1-16.

[47] Dahl, L.O and Benth, F.E., 2001. Valuation of Asian Basket options with Quasi-Monte Carlo techniques and singular value decomposition. *Pure Mathematics*, 5, pp. 1-21.

[48] Dufresne, D., 2002. Asian and basket asymptotics. *University of Melbourne Preprint Series*.

[49] Ju, N., 2002. Pricing Asian and Basket Options via Taylor Expansion. *Journal of Computational Finance*, 5(3).

[50] Deelstra, G, Liinev, J and Vanmaele, M., 2004. Pricing of arithmetic basket options by conditioning. *Insurance: Mathematics and Economics*, 34(1), pp. 55-77.

[51] Deelstra, G, Diallo, I and Vanmaele, M., 2008. Bounds for Asian basket options. *Journal of Computational and Applied Mathematics*, 218(2), pp. 215-228.

[52] Albrecher, H, Mayer, P.A and Schoutens, W., 2008. General lower bounds for arithmetic Asian Option prices. *Applied Mathematical Finance*, 15(2), pp. 123-149.

[53] Investopedia, 2013. *Mountain Range Options*. Available at: http://www.investopedia.com/terms/m/mountain_range_option.asp (Accessed: 4th April 2014).

[54] Soft 112, 2013. *Linear Program Solver LiPS-1.9.4*. Available at: http://www.linear-program-solver.soft112.com (Accessed: 26th April 2014).