# THE LARGE-TIME SOLUTION OF NONLINEAR EVOLUTION EQUATIONS

by

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# Abstract

In this thesis we use the method of matched asymptotic coordinate expansions to examine in detail the structure of the large-time solution of a range of initial-value and initial-boundary value problems based on Burgers' equation or the related Burgers-Fisher equation. The normalized nonlinear partial differential equations considered are:

(i) Burgers' equation

$$u_t + uu_x - u_{xx} = 0.$$

(ii) Burgers-Fisher equation

$$u_t + kuu_x = u_{xx} + u(1 - u).$$

Here $x$ and $t$ represent dimensionless distance and time, respectively, while $k$ ($\neq 0$) is a constant. In particular, we are interested in the emergence of coherent structures (for example: expansion waves, stationary states and travelling waves) in the large-time solution of the problems considered.

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# Contents

# Chapter 1

# Introduction

The characteristics of many dynamical systems are determined by the propagation of fronts. In particular, scalar and systems of reaction-diffusion equations or reaction-diffusion-convection equations arise in the study of many branches of science for example, genetics [1, 2, 21, 32, 47, 48], nonlinear differential equations in biology [47, 48], combustion [3, 45], chemistry [12, 43] and physics [18, 16]. In these applications the phenomenon propagating wavefronts is of considerable interest. The study travelling wave solutions has played a vital role in the mathematical analysis of the nonlinear systems of this class of equations, such solutions can often be readily determined and can arise along with other behaviour such as stationary states as the large-time attractors for the solutions to initial-value and boundary value problems for these equations. The study of the evolution of travelling wave solutions in scalar and systems of nonlinear partial differential equations is of fundamental importance in a wide variety of applications. It is the purpose of this thesis to examine this topic via the method of matched asymptotic coordinate expansions.

The well-known scalar nonlinear reaction-diffusion equation regularly used to illustrate the phenomenon of front propagation is:

$$u_t = u_{xx} + M(u), \tag{1.1}$$

where $x,\ t$ represent non-dimensional distance and time and $M(u)$ is a nonlinear reaction term. The specific case when $M(u)$ is concerned with the power case given by

$$M(u) = u - u^k \quad k > 1 \tag{1.2}$$

has been considered by a number of authors (see for example [21, 32]). Here $u = 1$ is the stable state while $u = 0$ is the unstable state. In 1937 equation (1.1) (with (1.2)) when $k = 2$ was proposed by Fisher [21] and Kolmogorov, Petrovsky, and Piscounov [32] as a model of gene dispersion in a population. Equation (1.1) (with (1.2)) when $k = 2$ is often referred to as the Fisher-Kolmogorov equation, while equation (1.1) (with (1.2)) when $k > 2$ is referred to as the generalized Fisher-Kolmogorov equation. When $k = 2$ the existence of travelling wave solution of equation (1.1) (with (1.2)) was established by phase-plane analysis (see for example [21]). Fisher [21] established via phase plane analysis that travelling wave solutions of equation (1.1) (with (1.2)) only exist when the speed of propagation is equal or greater than minimal speed (which is equal to 2). In 1937 Kolmogorov et al. [32] proved that the solution to the initial-value problem (1.1) (with (1.2)) with $k = 2$ and step (or Heaviside) initial data $u(x, 0) = 1$, $x < 0$ and $u(x, 0) = 0$, $x > 0$ approached the minimum speed travelling wave. In 1983 Bramson [6, 7] extended the work of Fisher and Kolmogorov et al. He considered the initial-value problem for (1.1) (with (1.2)) when $k = 2$ and step initial data of the form considered by Kolmogorov. He established that the correction to the wave speed is of $O(1/t)$ as $t \to \infty$. Specifically,

$$v(t) = 2 - \frac{3}{2t} + o\left(\frac{1}{t}\right) \tag{1.3}$$

as $t \to \infty$. In 1989 Murray [48] extended the equation (1.1) by including a convection term to obtain the more general equation,

$$u_t + duu_x = u_{xx} + u(1-u), \tag{1.4}$$

where $d$ is a constant. Further, he proved via phase plane analysis that equation (1.4) admitted travelling wave solutions with speed $c$ for each

$$c \geqslant c_{min}$$

where

$$c_{min} = \begin{cases} 2, & d \leqslant 2, \\ \frac{d}{2} + \frac{2}{d}, & d > 2. \end{cases}$$

We note that this result has also been obtained by a number of other authors including for example Gilding and Kersner [24].

The nonlinear term $M(u)$ can appear in various forms. Other than (1.2) in 1948 Burgers [8] is investigated (1.1) when $M(u) = -\frac{1}{2}(u^2)_x$. The result is known as Burgers' equation is given by

$$u_t + \frac{1}{2}(u^2)_x - u_{xx} = 0, \tag{1.5}$$

which is a fundamental model for the velocity in one-dimensional turbulent flow. In 1974 Whitham [63] considered an initial-value problem for equation (1.5) when the initial data had a discontinuous step. When $M(u) = u(1-u) - duu_x$ equation (1.1) gives the classical Burgers-Fisher equation:

$$u_t + duu_x = u_{xx} + u(1-u), \tag{1.6}$$

where $d \neq 0$ is a constant which has been studied by many authors including Murray [48]. Burgers' equation (1.5) and the related Burgers-Fisher equation (1.6) are well-studied equations in various areas of mathematical physics [21, 9, 10, 11, 17, 39, 62, 34, 33, 41, 52] and biology [12, 48, 45].

Many authors use the terminology 'pushed' and 'pulled' when discussing the propagation of wave fronts (see for example [59, 60, 61]). This terminology arises from the fact that the properties of 'pulled' wave fronts are determined completely by the 'linear' behaviour of the leading edge of the wave front (the wave front is pulled along by the leading edge), while the properties of 'pushed' wave fronts are determined by the 'nonlinear' behaviour within the wave front (the wave front is pushed along from the bulk region of the wavefront). The methodology described in this thesis can deal with both types of wave front and in what follows we do not differentiate between them rather we focus on determining the properties of interest, wave speed the asymptotic correction to the wave speed as $t \to \infty$, and the rate of convergence of the solution to travelling wave solution as $t \to \infty$ to be determined for the problems considered.

In this thesis I will develop using the method of matched asymptotic coordinate expansions the large-time solution of initial-value and initial-boundary value problems based on the scalar nonlinear equations (1.5) and (1.6). Specifically, I consider:

(i) An initial-value problem for equation (1.5) with step initial condition

$$u(x,0) = \begin{cases} u_+, & x \geqslant 0, \\ u_-, & x < 0, \end{cases}$$

where $u_-$ and $u_+$ ($\neq u_-$) are constants. The detailed structure of the large-time solution of this problem is obtained for $u_+ > u_-$ and $u_+ < u_-$.

(ii) An initial-boundary value problem (positive quarter plane problem) for equation (1.5) with initial and boundary conditions given by

$$u(x,0) = u_+, \quad x > 0,$$

$$u(0,t) = u_b, \quad t > 0,$$

where $u_+$ and $u_b$ ($\neq u_+$) are constants. The detailed structure of the large-time solution of this problem is obtained for following subcases:

(a) $-u_b < u_+ < u_b$ with $u_b > 0$,

(b) ($-u_+ > u_b > u_+$ with $u_+ < 0$) or when $0 \geqslant u_+ > u_b$,

(c) $u_+ > u_b$ and $u_b \geqslant 0$,

(d) $u_b < 0$ and $u_+ > 0$.

(iii) An initial-value problem for equation (1.6) is considered with step initial data

$$u(x,0) = \begin{cases} 1 & \text{as} \quad x \leqslant 0, \\ 0 & \text{as} \quad x > 0, \end{cases}$$

$$u(x,t) \to \begin{cases} 1, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \qquad t \geqslant 0.$$

The large-time solution of this problem is a permanent form travelling wave solution. This travelling wave has speed $c^*(k)$, where,

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases} \tag{1.7}$$

The asymptotic structure of the solution of Burgers' equation (1.5) in cases (i) and (ii) depends critically on the values of the problem parameters $(u_+, u_-)$, $(u_+, u_b)$ respectively. The large-time solution of Burgers' equation in case (i) exhibits either a permanent form travelling wave solution or an expansion wave, while in case (ii) the solution can approach a wide range of large-time attractors depending on problem parameters: In subcase (a) the large-time solution of Burgers' equation displays a permanent form travelling solution. In subcase (b) the large-time solution of Burgers' equation displays the formation of a stationary solution. In subcase (c) the large-time solution of Burgers' equation displays an expansion wave solution. In subcase (d) the large-time solution of Burgers' equation displays a combination of an expansive wave and a stationary solution.

The large-time solution of Burgers-Fisher equation in case (iii) exhibits a permanent form travelling wave solution with the minimal available speed (1.7) however the rate of convergence of solution of Burgers-Fisher equation is exponential in $t$ when $k \in (2, \infty)$ and algebraic in $t$ when $k \in (-\infty, 2]$ as $t \to \infty$.

## 1.1 Burgers' Equation

Burgers' equation (named after J.M. Burgers) is given by

$$u_t + uu_x = vu_{xx}, \tag{1.8}$$

where the parameter $v\ (> 0)$ is a measure of viscous diffusion, $u$ is velocity, and $x,\ t$ represent distance and time respectively. Burgers' equation is a canonical equation combining both nonlinear and diffusive effects. Although named after Burgers for his work on turbulence (see for example [8]) equation (1.8) was known to earlier researchers (for example Forsyth [22] and Bateman [4]) and has found applications in fluid dynamics, gas dynamics and acoustics.

It is convenient at this stage to non-dimensionalize equation (1.8). We write

$$x = lx', \quad u = u_0 u', \quad t = \left(\frac{l}{u_0}\right) t', \tag{1.9}$$

where $l$ is a typical length scale and $u_0$ is a typical scale for $u$. On substituting (1.9) into equation (1.8), and dropping the primes for convenience, we obtain the dimensionless equation as

$$u_t + uu_x - Du_{xx} = 0, \tag{1.10}$$

where $D = v/u_0 l$ is a dimensionless parameter. We note that Reynolds number $Re = 1/D$.

We further note that the Cole-Hopf transformation (see [15] and [27]) allows the general solution to (1.10) to be obtained explicity. In particular, following [63], the initial-value problem

$$u_t + uu_x - Du_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{1.11}$$

$$u(x,0) = F(x), \quad -\infty < x < \infty, \tag{1.12}$$

has the solution

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \left(\frac{x-\eta}{t}\right) e^{-\frac{G}{2D}} d\eta}{\int_{-\infty}^{\infty} e^{-\frac{G}{2D}} d\eta}, \tag{1.13}$$

where

$$G(\eta; x, t) = \int_0^{\eta} F(\eta')d\eta' + \frac{(x-\eta)^2}{2t}. \tag{1.14}$$

In the case of discontinuous initial data of the form

$$F(x) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0, \end{cases} \tag{1.15}$$

where $u_- > u_+$ the solution (1.13), (1.14) can be written as

$$u(x,t) = u_+ + \frac{(u_- - u_+)}{1 + h(x,t)\exp\left[\frac{(u_- - u_+)}{2D}\left(x - \frac{(u_- + u_+)}{2}t\right)\right]}, \tag{1.16}$$

where

$$h(x,t) = \frac{\operatorname{erfc}\left(-\frac{x-u_+t}{2\sqrt{Dt}}\right)}{\operatorname{erfc}\left(\frac{x-u_-t}{2\sqrt{Dt}}\right)}. \tag{1.17}$$

For fixed $\frac{x}{t}$ in the range $u_+ < \frac{x}{t} < u_-$, $h \to 1$ as $t \to \infty$, and the solution approaches the Taylor shock (see [56])

$$u = u_+ + \frac{(u_- - u_+)}{1 + \exp\left(\frac{(u_- - u_+)}{2D}[x - \widehat{U}t]\right)}, \quad \text{where} \quad \widehat{U} = \frac{u_+ + u_-}{2}. \tag{1.18}$$

This is a travelling wave solution of the form $u = u(z)$, $z = x - \widehat{U}t$, where $\widehat{U}$ is the wave speed of the travelling wave and $z$ is the travelling wave coordinate. The translational invariance being fixed so that $u(0) = \widehat{U}$. For $x > u_+t$ then $h \to \infty$ and $u = u_+ + o(1)$, while when $x < u_-t$, $h \to 0$ and $u = u_- - o(1)$.

When $D = 0$ equation (1.10) reduces to the inviscid Burgers' equation

$$u_t + uu_x = 0, \tag{1.19}$$

which is hyperbolic and a canonical equation for nonlinear convection. It is instructive to note that the solution to equation (1.19) when the initial data has a discontinuous expansive step, given by

$$u(x,0) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0, \end{cases} \tag{1.20}$$

where $u_+ > u_-$ can readily be obtained (see for example [63]) as

$$u(x,t) = \begin{cases} u_+, & x > u_+ t, \\ \frac{x}{t}, & u_- t < x < u_+ t, \\ u_-, & x < u_- t. \end{cases} \tag{1.21}$$

A solution of this type is known as an expansion or rarefraction wave. We note that $u = x/t$ is a rational solution to equation (1.10).

In what follows it is convenient to rescale equation (1.10). We introduce the new variables

$$x = Dx', \quad t = Dt', \quad u = u', \tag{1.22}$$

in terms of which equation (1.10) becomes (on dropping the primes)

$$u_t + uu_x - u_{xx} = 0. \tag{1.23}$$

Equation (1.23) is the form of Burgers' equation that we will examine in this thesis.

### 1.1.1 Travelling Wave Solutions

In this section we examine the travelling wave solutions of (1.23), and look for a solution of equation (1.23) of the form

$$u = U(z), \quad z = x - ct, \tag{1.24}$$

where $c$ is the wave speed and $z$ is the travelling wave coordinate. We require a solution with

$$U(z) \to \begin{cases} u_+ & \text{as} \quad z \to \infty, \\ u_- & \text{as} \quad z \to -\infty, \end{cases} \tag{1.25}$$

where $u_+ < u_-$ . On writing equation (1.23) in terms of (1.24) we obtain

$$U_{zz} - UU_z + cU_z = 0, \quad -\infty < z < \infty. \tag{1.26}$$

On integrating (1.26) we obtain

$$U_z - \frac{1}{2}U^2 + cU = \mathcal{C}, \tag{1.27}$$

where $\mathcal{C}$ is a constant of integration. On using conditions (1.25) we have that

$$\mathcal{C} = -\frac{1}{2}u_+^2 + cu_+ = -\frac{1}{2}u_-^2 + cu_-, \tag{1.28}$$

giving

$$c = \frac{1}{2}(u_- + u_+), \tag{1.29}$$

and

$$\mathcal{C} = \frac{1}{2}u_+u_-. \tag{1.30}$$

Therefore, equation (1.27) can be written as

$$\frac{dU}{dz} = \frac{1}{2}(U - u_+)(U - u_-), \quad -\infty < z < \infty, \tag{1.31}$$

where $u_- > U(z) > u_+$. We note that $u = u_+$ and $u = u_-$ are constant solutions of (1.31). The travelling wave solution to (1.31) is readily obtained as

$$U(z) = \frac{u_+ + u_- \mathcal{A} e^{-\frac{1}{2}(u_- - u_+)z}}{1 + \mathcal{A} e^{-\frac{1}{2}(u_- - u_+)z}}, \quad -\infty < z < \infty, \tag{1.32}$$

where $\mathcal{A}$ is a constant. The translational invariance is fixed by requiring $U(0) = \frac{u_- + u_+}{2}$ giving that $\mathcal{A} = 1$. Therefore, the travelling wave solution is given by

$$U(z) = \frac{u_+ + u_- e^{-\frac{1}{2}(u_- - u_+)z}}{1 + e^{-\frac{1}{2}(u_- - u_+)z}}, \quad -\infty < z < \infty, \tag{1.33}$$

where $U \in (u_-, u_+)$. Solution (1.33) is the standard Taylor shock (see [56]), and a graph of (1.33) when $u_+ = 0$ and $u_- = 1$ is given for illustration in Figure 1.1. Further, (1.33) can be written in terms of the hyperbolic tangent as

$$U(z) = \frac{1}{2}(u_+ + u_-) + \left(\frac{u_+ - u_-}{2}\right) \tanh\left(\frac{1}{4}(u_- - u_+)z\right), \quad -\infty < z < \infty. \tag{1.34}$$

We note that the solution of (1.31) for $U \in (-\infty, u_+) \bigcup (u_-, \infty)$ is given by

$$U(z) = \frac{u_+ - u_- \mathcal{B} e^{-\frac{1}{2}(u_- - u_+)z}}{1 - \mathcal{B} e^{-\frac{1}{2}(u_- - u_+)z}}, \tag{1.35}$$

where $\mathcal{B}$ is a constant. Clearly, the solution of (1.35) becomes unbounded as $z \to z_c^{\pm}$, where

$$z_c = \frac{2}{u_- - u_+} \ln \mathcal{B}. \tag{1.36}$$

On rewriting (1.35) we obtain

$$U(z) = \frac{u_+ - u_- e^{-\frac{1}{2}(u_- - u_+)[z - z_c]}}{1 - e^{-\frac{1}{2}(u_- - u_+)[z - z_c]}}, \tag{1.37}$$

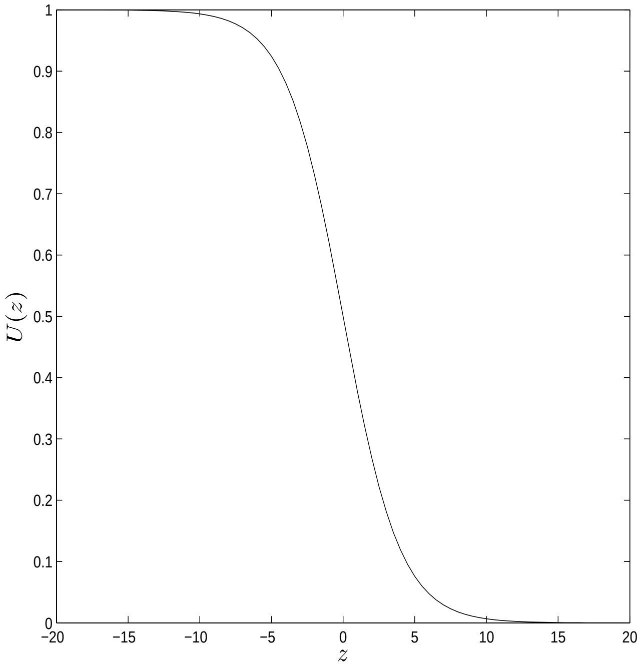


Figure 1.1: A graph of (1.33) when $u_+ = 0$ and $u_- = 1$.

or in terms of the hyperbolic cotangent we have that

$$U(z) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2} \coth\left(\frac{1}{4}(u_- - u_+)[z - z_c]\right). \tag{1.38}$$

A graph of (1.38) when $u_+ = 0$, $u_- = 1$ and $z_c = 0$ is given in Figure 1.2.

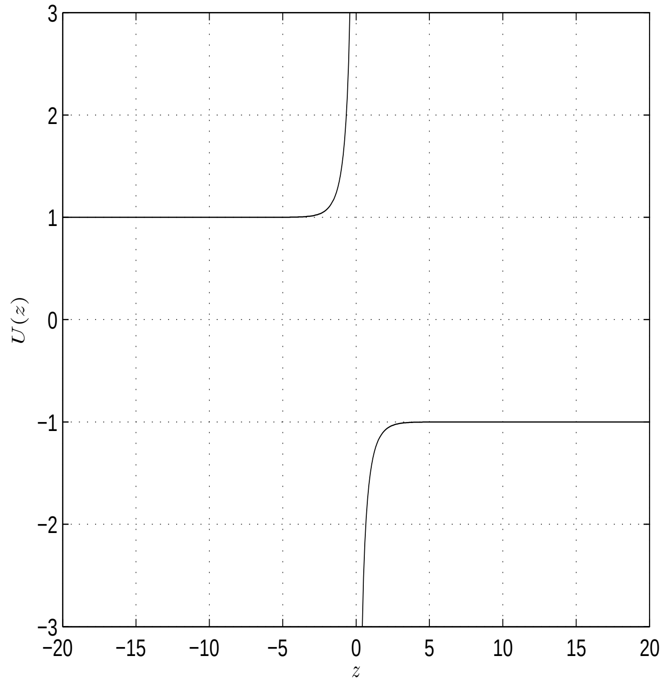


Figure 1.2: Graph of (1.38) when $u_+ = 0$, $u_- = 1$ and $z_c = 0$.

### 1.1.2 Stationary Solutions

In this thesis we will be interested in stationary (time independent) solutions of (1.23) of the form

$$u = U(x), \quad x \geqslant 0. \tag{1.39}$$

In particular, we will require stationary solutions that satisfy the boundary conditions

$$U(0) = u_b \tag{1.40}$$

and

$$U(x) \to u_+(\leqslant 0), \quad \text{as} \quad x \to \infty, \tag{1.41}$$

where $u_b \neq u_+$. On substituting (1.39) into equation (1.23) we obtain, after integrating once and applying boundary condition (1.41), that

$$\begin{aligned} \frac{U^2}{2} + B &= U_x \\ \frac{(u_+)^2}{2} + B &= U_x \end{aligned} \tag{1.42}$$

where $B$ is constant. We note that $U_x \to 0$ as $x \to \infty$ (as $U(x) \to u_+$ as $x \to \infty$) and we obtain

$$B = -\frac{(u_+)^2}{2}.$$

Substitute $B$ into equation $(1.42)_1$ we have that

$$\frac{dU}{dx} = \frac{U^2 - (u_+)^2}{2}, \quad x > 0. \tag{1.43}$$

On integrating (1.43) and applying boundary condition (1.40) we find for $u_+ \leqslant 0$ the following stationary solutions of (1.23) in $x \geqslant 0$:

(i) $u_+ = 0, \quad u_b < 0$

$$U(x) = \frac{2}{\frac{2}{u_b} - x}, \quad x \geqslant 0. \tag{1.44}$$

A graph of (1.44) when $u_b = -4$ is given for illustration in Figure 1.3.

(ii) $u_+ < 0, \quad u_+ < u_b < -u_+$

$$U(x) = u_+ \tanh\left(-\frac{u_+}{2}x + \tanh^{-1}\left(\frac{u_b}{u_+}\right)\right), \quad x \geqslant 0. \tag{1.45}$$

A graph of (1.45) when $u_+ = -2$ and $u_b = 1$ is given for illustration in Figure 1.4.

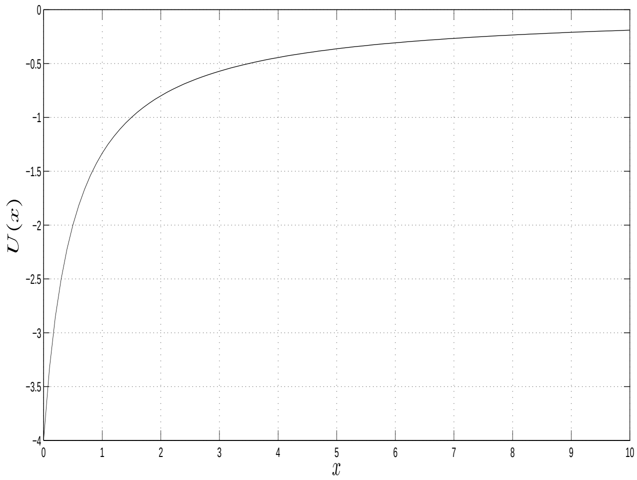


Figure 1.3: Graph of stationary solution of (1.44) when $u_b = -4$.

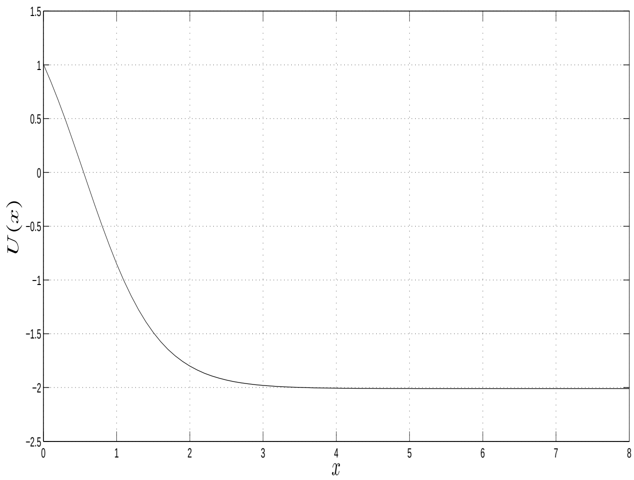


Figure 1.4: Graph of (1.45) when $u_+ = -2$, $u_b = 1$ for $x \geqslant 0$.

(iii) $u_+ < 0, \quad u_b < u_+$

$$U(x) = u_+ \coth\left(-\frac{u_+}{2}x + \coth^{-1}\left(\frac{u_b}{u_+}\right)\right), \quad x \geqslant 0. \tag{1.46}$$

A graph of (1.46) when $u_+ = -1$ and $u_b = -2$ is given for illustration in Figure 1.5.

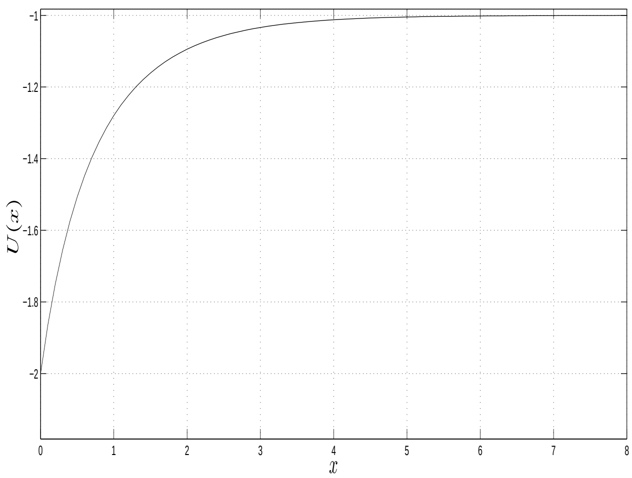


Figure 1.5: Graph of stationary solution of (1.46) when $u_+ = -1$ and $u_b = -2$.

We conclude this section by noting that no stationary solution of (1.23) exists for $x \geqslant 0$ with $U(0) = u_b$ $(> u_+)$ and $U(x) \to u_+$ $(> 0)$ in Chapter 4.

### 1.1.3 Similarity Solutions

The similarity transformation for Burgers equation is given by

$$u = t^{-\frac{1}{2}}U(\xi), \quad \xi = xt^{-\frac{1}{2}}. \tag{1.47}$$

On substitution of (1.47) into equation (1.23) we obtain

$$U_{\xi\xi} - UU_\xi + \frac{\xi}{2}U_\xi + \frac{1}{2}U = 0, \quad -\infty < \xi < \infty. \tag{1.48}$$

We note that $U = \xi$ is an exact solution of (1.48). On integrating (1.48) we obtain

$$U_\xi - \frac{U^2}{2} + \frac{\xi}{2}U = D_1, \tag{1.49}$$

where $D_1$ is a constant. In particular,we are interested in solutions to (1.49) for which

$$U \to 0^-, \quad U_\xi \to 0 \quad \text{as} \quad \xi \to \infty, \tag{1.50}$$

or

$$U \to 0^+, \quad U_\xi \to 0 \quad \text{as} \quad \xi \to -\infty. \tag{1.51}$$

Both conditions (1.50) and (1.51) require $D_1 = 0$ . The solution of (1.49) with $D_1 = 0$ is readily obtained as

$$U(\xi) = \frac{2e^{-\frac{\xi^2}{4}}}{D_2 - \sqrt{\pi}\ \mathrm{erf}\left(\frac{\xi}{2}\right)}, \quad -\infty < \xi < \infty, \tag{1.52}$$

where $D_2$ is a constant.

When $D_2 = \sqrt{\pi}$ (1.52) is the solution of (1.49) subject to condition (1.51), while when $D_2 = -\sqrt{\pi}$ (1.52) is the solution of (1.49) subject to condition (1.50). A graph of (1.52) for $D_2 = \pm\sqrt{\pi}$ is given Figure 1.6.

We note specifically that when $D_2 = \sqrt{\pi}$ that

$$U(\xi) \sim \begin{cases} \xi & \text{as} \quad \xi \to \infty, \\ \frac{1}{\sqrt{\pi}}e^{-\frac{\xi^2}{4}} & \text{as} \quad \xi \to -\infty, \end{cases} \tag{1.53}$$

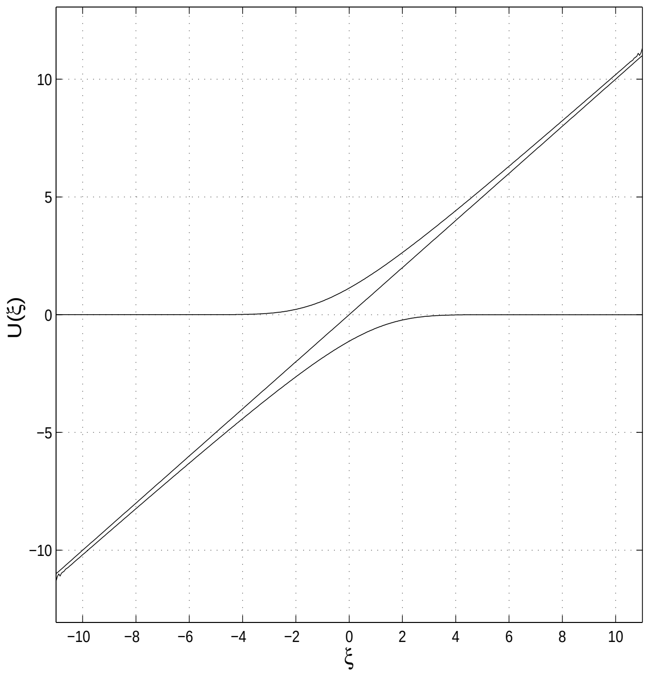


Figure 1.6: Graph of (1.52) when $D_2 = \pm\sqrt{\pi}$.

while when $D_2 = -\sqrt{\pi}$

$$U(\xi) \sim \begin{cases} -\frac{1}{\sqrt{\pi}} e^{-\frac{\xi^2}{4}} \quad \text{as} \quad \xi \to \infty, \\ \xi \quad \text{as} \quad \xi \to -\infty. \end{cases} \tag{1.54}$$

Further, we observe that when $D_2 > \sqrt{\pi}$

$$U(\xi) \to 0^+ \quad \text{as} \quad |\xi| \to \infty. \tag{1.55}$$

In particular, (1.52) is bounded on $-\infty < \xi < \infty$, having a single maximum and with

$$U(\xi) \sim \begin{cases} \frac{2e^{-\frac{\xi^2}{4}}}{D_2 - \sqrt{\pi}} \quad \text{as} \quad \xi \to \infty, \\ \frac{2e^{-\frac{\xi^2}{4}}}{D_2 + \sqrt{\pi}} \quad \text{as} \quad \xi \to -\infty. \end{cases} \tag{1.56}$$

When $D_2 = 0$ the solution (1.21) for $\xi > 0$ is given by

$$U(\xi) = -\frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2/4}}{\operatorname{erf}(\xi/2)}, \quad \xi > 0 \tag{1.57}$$

where

$$U(\xi) \sim \begin{cases} -\frac{2}{\xi} \quad \text{as} \quad \xi \to 0^+, \\ -\frac{2}{\sqrt{\pi}}e^{-\xi^2/4} \quad \text{as} \quad \xi \to \infty. \end{cases} \tag{1.58}$$

Finally, we note that

$$U(\xi) = -\frac{2}{\xi}, \tag{1.59}$$

is an exact solution of (1.48) for $\xi > 0$ (corresponding to $D_1 = -1$ in (1.49)). Therefore, the solution of (1.48) subject to

$$U(\xi) \to \begin{cases} \xi \quad \text{as} \quad \xi \to \infty \\ -\frac{2}{\xi} \quad \text{as} \quad \xi \to 0^+ \end{cases} \tag{1.60}$$

is given directly as

$$U(\xi) = \xi - \frac{2}{\xi}, \quad \xi > 0. \tag{1.61}$$

The graph of $U(\xi)$ against $\xi$ is given in Figure 1.7.

We observe that (1.61) corresponds to $D_1 = 2$ in (1.49). We will return to these solutions later in the analysis.

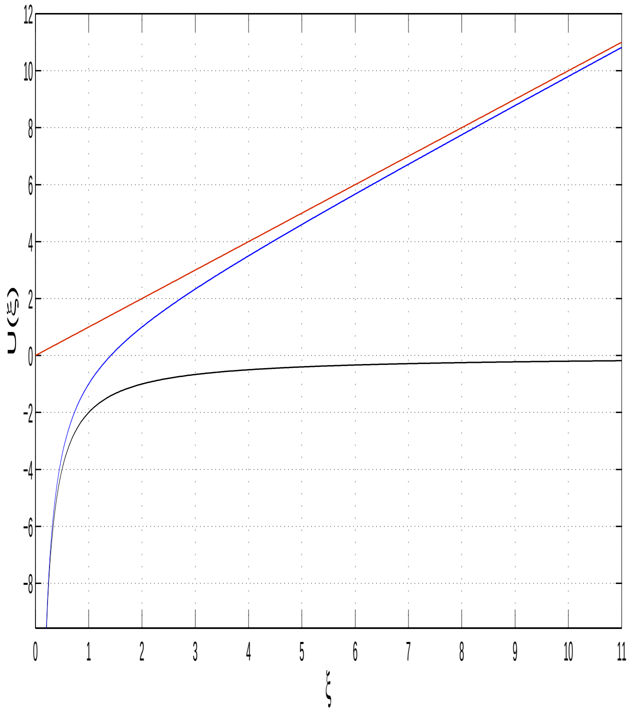


Figure 1.7: A graph of the similarity solution (1.61) (denoted by the blue line). We note that the red line represents the exact solution $U(\xi) = \xi$ of (1.48), while the black line represents the exact solution $U(\xi) = -\frac{2}{\xi}$ of (1.48) for $\xi > 0$.

## 1.2 Burgers-Fisher Equation

The Burgers-Fisher equation is given by

$$u_t + kuu_x = \mu u_{xx} + u(1-u) \tag{1.62}$$

where $\mu$ is the diffusion coefficient and $k$ is a non-zero parameter. The Burgers-Fisher equation is a canonical equation combining reaction, diffusion and convection and as such arises in the modelling of many physical situations including: financial mathematics, gas dynamics, traffic flow, applied mathematics and physics applications (see for example [5, 28, 29, 30, 54, 13, 14, 19, 40, 65, 36, 42, 44]). When $k = 0$, equation (1.62) reduces to the Fisher-Kolmogorov equation [32, 20, 21, 23, 31, 46, 53, 1]

$$u_t = \mu u_{xx} + u(1-u). \tag{1.63}$$

By using spectral analysis method [53] the solutions of equation (1.63) are exponentially small in $t$ were proved. We also note that [57] showed the equation (1.63) displays critical wave form with applying the maximum principle method. By using different methods such as Green's method [46], the spectral method [31], the renormalization group method [23], the $L^1$ weighted energy method together with the Green function method [64] the stability of permanent form travelling waves are defined. Later, Bramson [6, 7] showed the solutions of the equation (1.63) displays the formation of a PTWs. In this thesis I will examine the large-time attractors of problems based on the equation

$$u_t + kuu_x = u_{xx} + u(1-u) \tag{1.64}$$

where $k \neq 0$ is a parameter.

### 1.2.1 Travelling Wave Theory

In this section we examine the travelling wave solutions of the Burgers-Fisher equation, namely,

$$u_t + kuu_x = u_{xx} + u(1-u) \tag{1.65}$$

where $k \neq 0$ is a parameter. We begin by looking for a travelling wave solution of the equation (1.65) we obtain

$$z = x - ct, \quad u = U(z), \tag{1.66}$$

where $c$ is the wave speed. Substituting (1.66) into equation (1.65) gives

$$-cU_z + kUU_z = U_{zz} + U(1-U). \tag{1.67}$$

On writing $U_z = W$, we obtain the dynamical system

$$\begin{aligned} U_z &= W, \\ W_z &= -cW + kUW - U(1-U). \end{aligned} \tag{1.68}$$

Dynamical system (1.68) has been examined by a number of authors including Murray [48]. The arguments presented below follow closely those given in [48]. Therefore, we have that

$$\frac{dW}{dU} = -c + kU - \frac{U(1-U)}{W}. \tag{1.69}$$

Dynamical system (1.68) has two equilibrium points at $M : (0,0)$ and $N : (1,0)$. We require a monotone solution in $0 \leqslant U \leqslant 1$ with $U_z(z) \leqslant 0$. We next classify the equilibrium points by linearization. We first consider the equilibrium point $M : (0,0)$. The associated linear system is given by

$$\left.\begin{aligned} U_z &= W \\ W_z &= -cW - U \end{aligned}\right\} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}. \tag{1.70}$$

Eigenvalues of $A$ and associated eigenvectors are given by

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2 - 4}}{2}, \qquad v_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}. \tag{1.71}$$

Since we require $U \geqslant 0$ these eigenvalues must be real and so

$$c \geqslant 2.$$

Now since $0 > \lambda_+ > \lambda_-$ the point $M : (0, 0)$ is a stable node. Therefore, the linearization Theorem then indicates that the point $M : (0, 0)$ is a stable node for nonlinear system (1.68). Figure 1.8 displays the $(U, W)$ phase plane in the neigbourhood of the equilibrium point $M : (0, 0)$. The stable manifolds $W = \lambda_+ U$ and $W = \lambda_- U$ of the stable node are clearly displayed on the figure. In what follows we label the stable manifold $W = \lambda_- U$ as $W_s^-$.

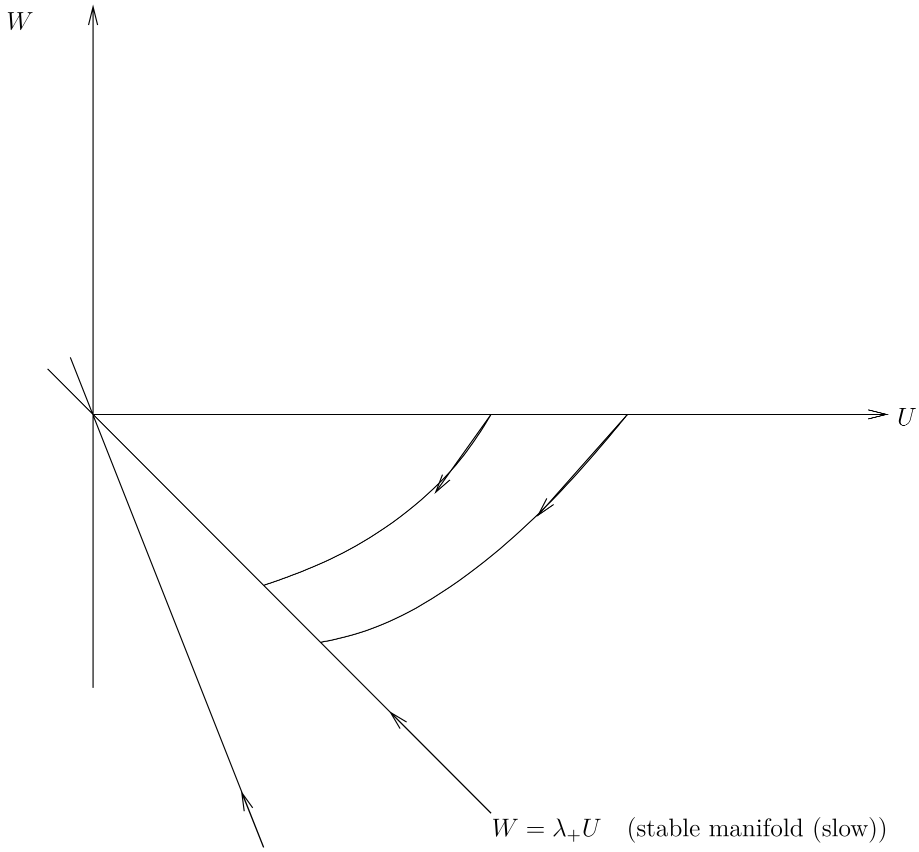


Figure 1.8: $(U, W)$ phase plane in the neigbourhood of the equilibrium point $M : (0, 0)$.

We next consider the equilibrium point $N : (1, 0)$. On writing $\overline{U} = U - 1$ and $\overline{W} = W$ the associated linear system is given by

$$\left.\begin{aligned} \overline{U}_z &= \overline{W} \\ W_z &= \overline{U} + (k-c)\overline{W}, \end{aligned}\right\} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & k-c \end{pmatrix}. \tag{1.72}$$

Eigenvalues of $A$ and associated eigenvectors are given by

$$\widehat{\lambda}_{\pm} = \frac{-(c-k) \pm \sqrt{(c-k)^2 + 4}}{2}, \qquad \widehat{v}_{\pm} = \begin{pmatrix} 1 \\ \widehat{\lambda}_{\pm} \end{pmatrix}. \tag{1.73}$$

Now since $\widehat{\lambda}_+ > 0 > \widehat{\lambda}_-$ the point $N : (1, 0)$ is a saddle point. Therefore, the Linearization

Theorem then indicates that point $N:(1,0)$ is a saddle point for nonlinear system (1.68). Figure 1.9 displays the $(U,W)$ phase plane in the neigbourhood of the equilibrium point $N:(1,0)$. The unstable manifold entering the region where $W<0$ is clearly displayed in the figure. We note that on this unstable manifold

$$U \sim 1 - O(e^{\widehat{\lambda}_+ z}), \quad \text{as} \quad z \to -\infty.$$

In what follows we label this unstable manifold as $W^+$.

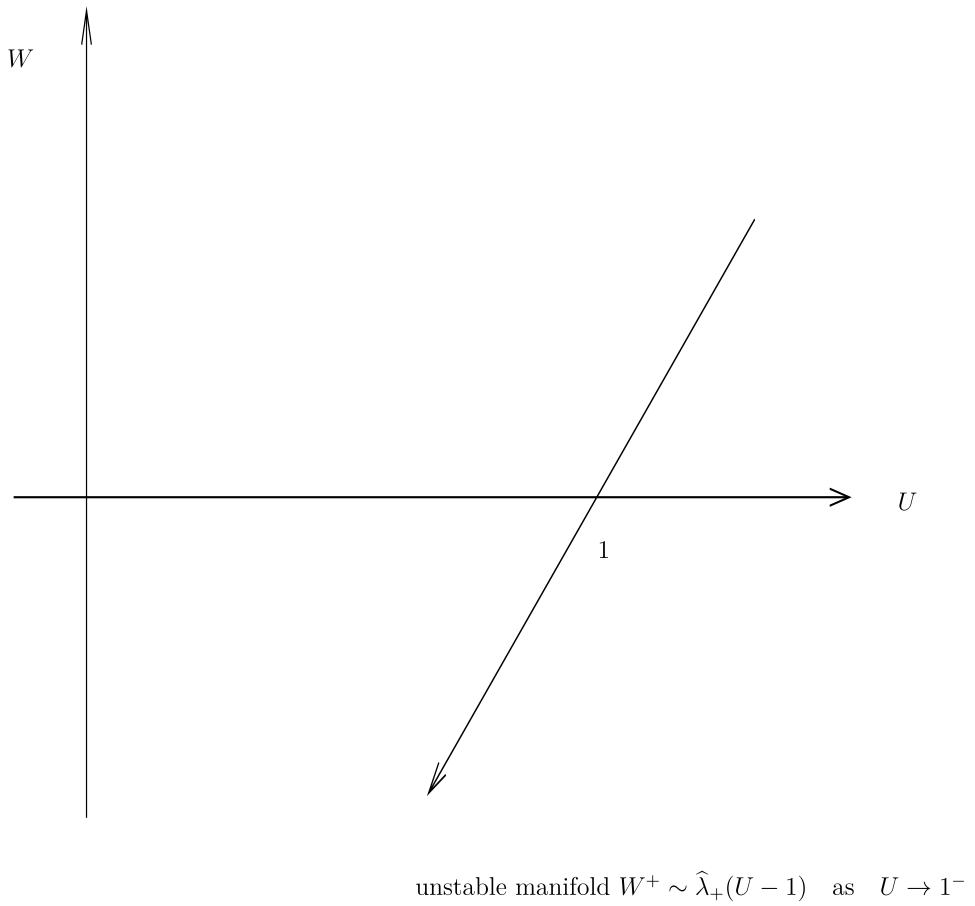


Figure 1.9: $(U,W)$ phase plane in the neigbourhood of the equilibrium point $N:(1,0)$.

We note from (1.69) that since $\frac{d}{dc}\left(\frac{dW}{dU}\right) = -1$ the phase plane rotates clockwise for increasing $c$. The rotation is counterclockwise for $k$ increasing.

We further note that an exact solution of (1.69) exists and is given by

$$W = -\frac{k}{2}U(1-U) \quad \text{when} \quad c = \frac{k}{2} + \frac{2}{k}. \tag{1.74}$$

On recalling that $U_z = W$ we find that the solution of equation (1.74) is given by

$$U(z) = \frac{Ae^{-\frac{k}{2}z}}{1 + Ae^{-\frac{k}{2}z}} \quad \begin{cases} \sim 1 - \frac{1}{A}e^{\frac{k}{2}z} \quad \text{as} \quad z \to -\infty, \\ \sim Ae^{-\frac{k}{2}z} \quad \text{as} \quad z \to \infty, \end{cases} \tag{1.75}$$

where $A$ is a constant. Figure 1.10 displays the $(U, W)$ phase portrait of (1.68) when $k < 2$ and $k \geqslant 2$, respectively. The phase path connecting the equilibrium points $M : (0, 0)$ and $N : (1, 0)$ is given by (1.74). We note that when $k < 2$ this phase path enters $M : (0, 0)$ along the stable manifold $W = \lambda_+ U$, while when $k \geqslant 2$ the phase path enters $M : (0, 0)$ along the stable manifold $W = \lambda_- U$. Consideration of the phase path (1.74) gives that

$$\begin{aligned} \frac{dW}{dU} &= -\frac{k}{2} + kU \\ &\sim -\frac{k}{2} \quad \text{as} \quad U \to 0^+, \end{aligned} \tag{1.76}$$

indicating that when $k \geqslant 2$ the phase path approaches the equilibrium point $M : (0, 0)$ along the stable manifold $W_s^-$.

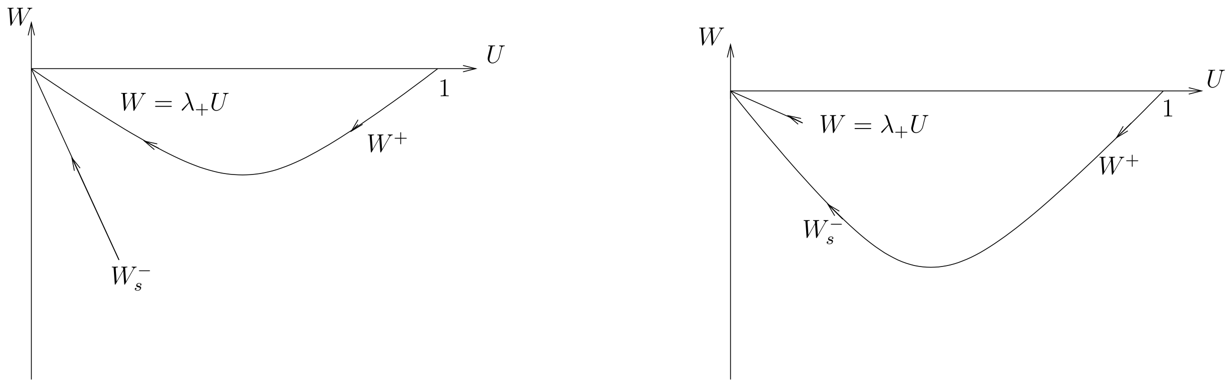


Figure 1.10: The $(U, W)$ phase portrait of dynamical system (1.68) when $k < 2$ and $k \geqslant 2$, respectively.

Before we proceed further with the phase plane analysis it is instructive to examine the asymptotic behaviour of the stable manifold $W_s^-$ for $c \gg 1$ with fixed $k$. It is straightforward as

$$W_s^-(U) = -cU + \left[\frac{(k-1)}{2}U^2 + \frac{U^3}{3}\right] + o(1), \quad U \in (0,1), \tag{1.77}$$

for $c \gg 1$. The stable manifold $W_s^-$ for $c \gg 1$ is sketched in Figure 1.11. As trajectories cannot cross $W_s^-$ the hashed region in Figure 1.11 is a positively invariant region for the dynamical system.

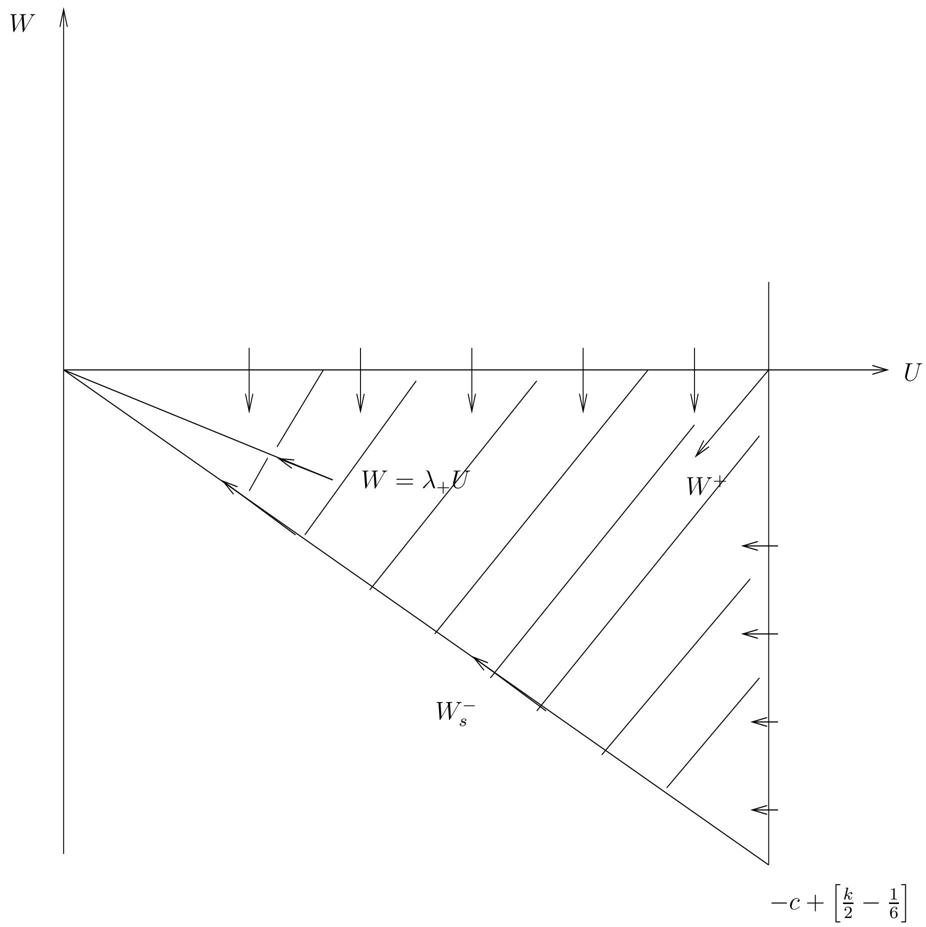


Figure 1.11: The $(U, W)$phase portrait of the dynamical system (1.77) when $c \gg 1$

In what follows we must consider the cases $k \geqslant 2$ and $k < 2$ separetely. We begin with the case when $k \geqslant 2$.

(a) $k \geqslant 2$

In this case the earlier established facts that:

(i) The vector field rotates anticlockwise for decreasing $c$.

(ii) The stable manifold $W_s^-$ crosses the line $U = 1$ at $W = -c + \frac{k}{2} - \frac{1}{6}$ for $c \gg 1$.

(iii) The phase path (1.74) forms a heteroclinic connection between the equilibrium points $M : (0, 0)$ and $N : (1, 0)$ when $c = \frac{k}{2} + \frac{2}{k}$.

allow after consideration of the flow on $(0 < U < 1, W = 0)$ and $(U = 1, W < 0)$

that permanent form travelling wave solutions of (1.65) are only possible when

$$c \geqslant \frac{k}{2} + \frac{2}{k}.$$

(b) $k < 2$

In this case we first show that the phase path $W(U; c = 2, k = 2)$ and the portion of the $U$-axis ($0 < U < 1$, $W = 0$) form a positively invariant region for the unstable manifold emanating from $N : (1, 0)$. Since $\frac{dW}{dU}$ increases with increasing $k$ for sufficiently close to $U = 1$, the phase path $W(U; c, k)$ satisfies

$$W(U; c = 2, k) > W(U; c = 2, k = 2).$$

Now suppose there exists a number $U^*$, where $0 < U^* < 1$ such that

$$W(U^*; c = 2, k = 2) = W(U^*; c = 2, k)$$

where

$$W(U; c = 2, k = 2) < W(U; c = 2, k)$$

for $U^* < U < 1$.

If there phase paths cross (touch) then (using (1.69))

$$\frac{dW}{dU}(U^*; c = 2, k = 2) \leqslant \frac{dW}{dU}(U^*; c = 2, k)\ .$$

However, via (1.69) we obtain

$$-2 + 2U^* - \frac{U^*(1 - U^*)}{W(U^*; c = 2, k = 2)} \leqslant -2 + kU^* - \frac{U^*(1 - U^*)}{W(U^*; c = 2, k)}$$

giving that

$$k \geqslant 2$$

which is a contradiction and no such number $U^*$ exists for $k < 2$. On noting that $W < 0$ on $0 < U < 1$, $W = 0$ we have established that the unstable manifold emanating from $N : (1, 0)$ into $W < 0$ enters a positively invariant region formed by the portion of $U$-axis given by $(0 < U < 1,\ W = 0)$ and the phase path $W(U; c = 2, k = 2)$. Therefore, the unstable manifold emanating from $N : (1, 0)$ must approach $M : (0, 0)$ along $W = \lambda_+ U$ and a permanent form travelling wave solution of (1.65) is possible for $c \geqslant 2$.

### 1.2.2 Summary of Travelling Theory

In this section we review the main results concerning the existence and structure of permanent form travelling waves (PTWs) which may occur in the large-time solution to the following initial-boundary value problem

$$u_t + kuu_x = cu_{xx} + u(1 - u), \quad -\infty < x < \infty, \quad t > 0 \tag{1.78}$$

$$u(x, 0) = \begin{cases} 1, & x \leqslant 0, \\ 0, & x > 0, \end{cases} \tag{1.79}$$

$$u(x, t) \to \begin{cases} 1, & x \to -\infty \\ 0, & x \to \infty \end{cases} \qquad t \geqslant 0. \tag{1.80}$$

On introducing the travelling wave coordinate $z = x - ct$ (with $c > 0$ being constant wave speed) a PTW is a solution to the following nonlinear boundary-value problem

$$u_{zz} - kuu_z + cu_z + u(1-u) = 0, \quad -\infty < z < \infty, \tag{1.81}$$

$$u(z) \geqslant 0, \quad -\infty < z < \infty, \tag{1.82}$$

$$u(z) \to 0 \quad \text{as} \quad z \to \infty, \tag{1.83}$$

$$u(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{1.84}$$

The nonlinear boundary-value problem (1.81)-(1.84) can be regarded as an eigenvalue problem for the travelling wave propagation speed $c$ ($> 0$). Any solution to (1.81) with $c > 0$ provides a permanent form travelling wave solution which could develop as the primary large-time structure in the solution of the initial-value problem (1.78)-(1.80). The nonlinear eigenvalue problem (1.81)-(1.84) has been considered in Section 1.2.1, and it is convenient here to summarize the main results in the following theorem.

**Theorem 1**. *Boundary value* (1.81)-(1.84) *has a unique PTW solution (say $u = u_T(z;c)$ with translational invariance fixed so that $u_T(0;c) = \frac{1}{2}$) for each $c \geqslant c^*(k)$, where*

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

*Moreover,*

*(i) When $k \in (-\infty, 2]$*

$$u_T(z;c) \sim \begin{cases} (A^*z + D^*)e^{-z} & \textit{as} \quad z \to \infty, \quad c = 2, \\ B^* e^{\lambda_+ z} & \textit{as} \quad z \to \infty, \quad c > 2; \end{cases} \tag{1.85}$$

*(ii) When* $k \in (2, \infty)$

$$u_T(z;c) \sim \begin{cases} e^{-\frac{k}{2}z} & as \quad z \to \infty, \quad c = \frac{2}{k} + \frac{k}{2}, \\ B^* e^{\lambda_+ z} & as \quad z \to \infty, \quad c > \frac{2}{k} + \frac{k}{2}; \end{cases} \tag{1.86}$$

*where*

$$\lambda_+ = -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 - 4}.$$

*Further, in each of above cases,*

$$u_T(z;c) = 1 - O(e^{\widehat{\lambda}_+ z}) \quad as \quad z \to -\infty,$$

*where*

$$\widehat{\lambda}_+ = -\frac{c-k}{2} + \frac{1}{2}\sqrt{(c-k)^2 + 4}.$$

**Proof**. See Section 1.2.1 for a sketch proof and [47] and [48] for more detail.

In the above, $A^*$, $B^*$ and $D^*$ are constants which can in principle be determined.

We note that when $k \in (2, \infty)$ the exact solution of the minimum speed travelling wave can be obtained as

$$u_T\left(z; \frac{2}{k} + \frac{k}{2}\right) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}} \begin{cases} 1 - e^{\frac{k}{2}z} & \text{as} \quad z \to -\infty, \\ e^{-\frac{k}{2}z} & \text{as} \quad z \to \infty, \end{cases}$$

where the translational invariance has been fixed so that $u_T\left(0; \frac{2}{k} + \frac{k}{2}\right) = \frac{1}{2}$.

## 1.3 A Short Review of the Method of Matched Asymptotic Coordinate Expansions

Throughout this thesis we use the momenclature of the theory of matched asymptotic expansions given in Van Dyke [58]. We will be primarily concerned with applying the methodology developed by J.A. Leach and D.J. Needham (see for example [37]) in the context of reaction-diffusion equations to Burgers' and related equations. This methodology enables to complete large-time asymptotic structure of the solution to an initial-value or initial-boundary value problem to be obtained by careful consideration of the asymptotic structures as $t \to 0$ and as $|x| \to \infty$ ($t = O(1)$). This approach is applicable to a large class of nonlinear evolution equations when a coherent structure forms the large-time attractor for their solution.

## 1.4 An Overview of the Mathematical Problems Considered

In Chapters 2 and 3 we consider the initial-value problem

$$u_t + uu_x - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{1.87}$$

$$u(x,0) = \begin{cases} u_+, & x \geqslant 0, \\ u_-, & x < 0, \end{cases} \tag{1.88}$$

where $u_+$ and $u_-$ are constants. Specifically, in Chapter 2 we consider the case when $u_+ > u_-$, while in Chapter 3 we consider the case when $u_+ < u_-$. We exclude any discussion of the trivial case $u_+ = u_-$. We will develop, via the method of matched asymptotic coordinate expansions, the complete large-time asymptotic structure of the solution to

(1.87), (1.88). The behaviour of the solution depends critically on the problem parameters $u_-$ and $u_+$. In particular, the large-time attractor of the solution to (1.87), (1.88) is:

(i) An expansion wave when $u_+ > u_-$.

(ii) A travelling wave with positive speed when $u_- > u_+ > -u_-$ with $u_- > 0$.

(iii) A travelling wave with negative speed when $u_+ < u_- < -u_+$ with $u_+ < 0$.

(iv) A stationary solution when $u_+ = -u_-$ with $u_- > 0$.

A sketch of the $(u_-,u_+)$ parameter plane is given in Figure 1.12.

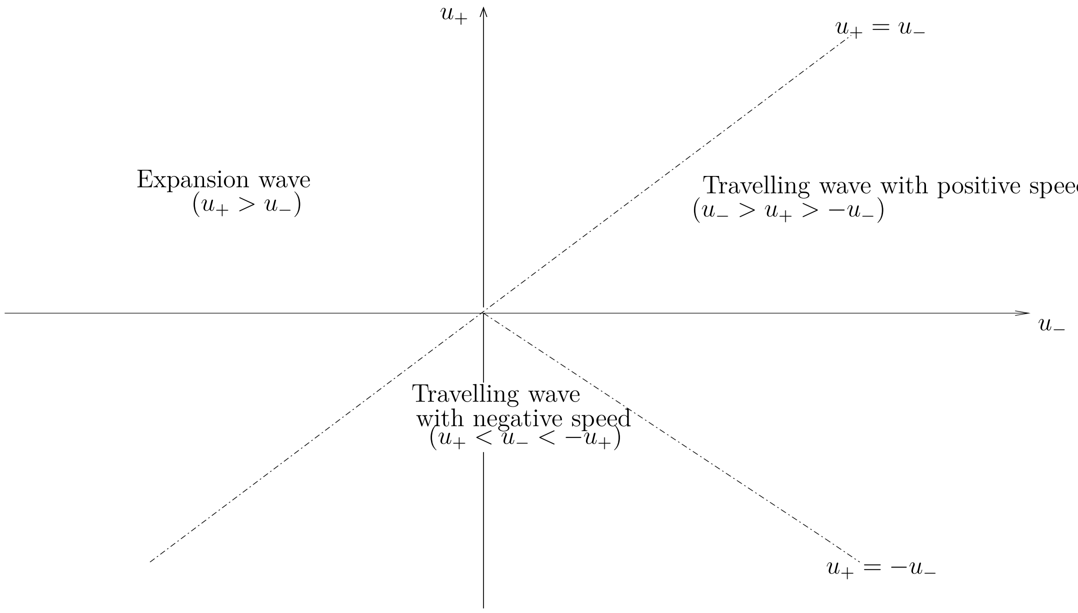


Figure 1.12: The $(u_-,u_+)$ parameter plane

In Chapter 4 we extend the analysis presented in Chapters 2 and 3 by considering a quarter-plane problem for Burgers' equation, namely,

$$u_t + uu_x - u_{xx} = 0, \quad x > 0, \quad t > 0, \tag{1.89}$$

$$u(x,0) = u_+, \quad x > 0, \tag{1.90}$$

$$u(0,t) = u_b, \quad t > 0, \tag{1.91}$$

where $u_+$ and $u_b$ are constants. As in Chapters 2 and 3 we apply the method of matched asymptotic coordinate expansions to obtain the complete large-time asymptotic solution of initial-boundary value problem (1.89)-(1.91). The behaviour of the solution (1.89)-(1.91) is dependent on the problem parameters $u_+$ and $u_b$. Specifically, the large-time attractor of the solution to (1.89)-(1.91) is:

(i) A travelling wave with positive wave speed when $-u_b < u_+ < u_b$ with $u_b > 0$.

(ii) A stationary solution when ($-u_+ > u_b > u_+$ with $u_+ < 0$) or when $0 \geqslant u_+ > u_b$.

(iii) A structure consisting of combination of an expansion wave and a stationary solution when $u_b < 0$ and $u_+ > 0$.

(iv) An expansion wave when $u_+ > u_b$ and $u_b \geqslant 0$.

A sketch of the $(u_b, u_+)$ parameter plane is given in Figure 1.13.

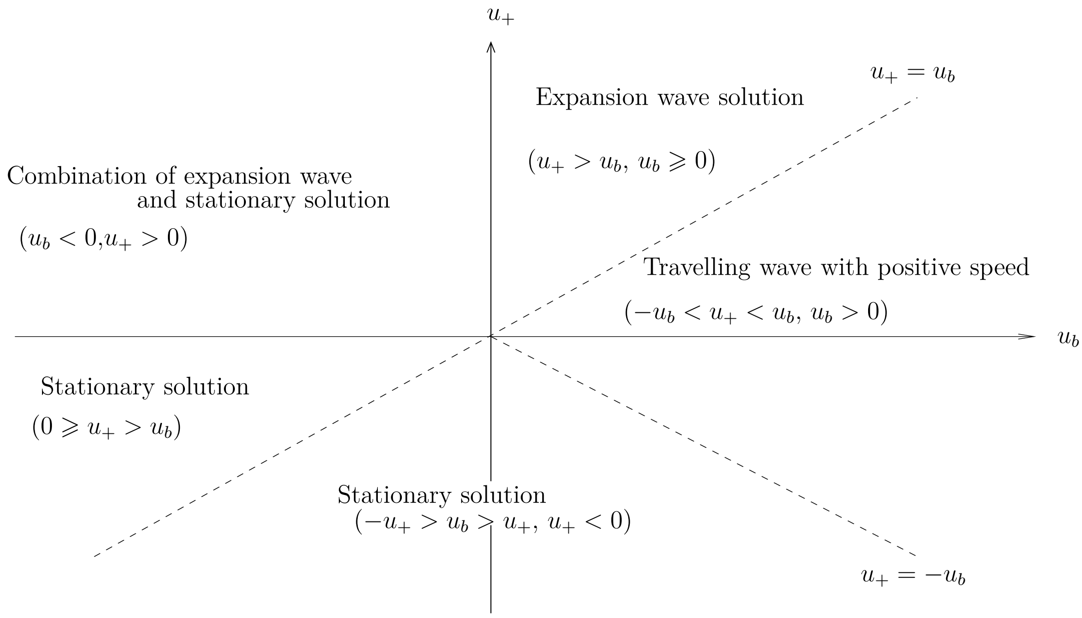


Figure 1.13: The $(u_b,u_+)$ parameter plane

In Chapter 5, we consider an initial value problem for Burgers-Fisher equation, given by

$$u_t + kuu_x = u_{xx} + u(1-u), \quad -\infty < x < \infty, \quad t > 0, \tag{1.92}$$

$$u(x,0) = \begin{cases} 1, & \text{as} \quad x \leqslant 0, \\ 0, & \text{as} \quad x > 0, \end{cases} \tag{1.93}$$

where $k$ $(\neq 0)$ is a parameter. We will establish, via the method of matched asymptotic coordinate expansions, the complete large-time asymptotic structure of the solution to (1.92)-(1.93). In particular, we will establish that the solution of (1.92)-(1.93) exhibits the formation of a permanent form travelling wave solution propagation in the $+\ x$ direction with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

Further, we find that the rate of convergence of the solution of initial-value problem (1.92)-(1.93) the travelling wave is exponential in $t$, as $t \to \infty$,being of

$$O\left(t^{-\frac{3}{2}} \exp\left(-\left(\frac{[c^*(k)^2]}{4} - 1\right) t\right)\right). \tag{1.94}$$

when $k \in (2, \infty)$, while is algebraic in $t$, as $t \to \infty$, being of

$$O\left(t^{-1}\right)$$

when $k \in (-\infty, 2]$. Elements of the work presented in this chapter have been accepted for publication in the Quarterly of Applied Mathematics.

# Chapter 2

# Initial-Value Problem 1 for Burgers' Equation

In this chapter, we consider an initial-value problem for Burgers' equation, namely,

$$u_t + uu_x - u_{xx} = 0, \quad -\infty < x < \infty \tag{2.1}$$

$$u(x,0) = \begin{cases} u_+ & x \geqslant 0, \\ u_- & x < 0, \end{cases} \tag{2.2}$$

where $u_+ > u_-$. The initial distribution is a discontinuous expansive step. For simplicity of exposition in what follows we take $u_+ = 1$ and $u_- = 0$. We label initial-value problem (2.1), (2.2) with $u_+ = 1$ and $u_- = 0$ as IVP1. We develop the structure of the large-time solution of IVP1 using the method of matched asymptotic coordinate expansions. The large-time solution is obtained by careful consideration of the asymptotic structures as $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \geqslant O(1))$. We note that the large-time solution of IVP1 for other values of the problem parameters $u_+$ and $u_-$ $(u_+ > u_-)$ follows straightforwardly, after some minor modification, the analysis presented in this chapter.

We begin by examining the asymptotic structure of the solution to IVP1 as $t \to 0$.

## 2.1 Asymptotic Solution of IVP1 as t → 0

Consideration of initial data (2.2) indicates that the structure of asymptotic solution of IVP1 as $t \to 0$, has three asymptotic regions, namely:

$$\begin{aligned}
&\text{Region I: } x = o(1), && u(x,t) = O(1),\\
&\text{Region II}^+: x = O(1)\ (>0), && u(x,t) = 1 - o(1),\\
&\text{Region II}^-: x = O(1)\ (<0), && u(x,t) = o(1),
\end{aligned}$$

Firstly, we consider region I, in which $x = o(1)$ and $u(x,t) = O(1)$ as $t \to 0$. Therefore, in region I, we introduce the scaled coordinate $\eta = xt^{-\alpha}$ as $t \to 0$, where $\alpha > 0$ and $\eta = O(1)$, and look for an expansion of the form

$$u = \overline{u}(\eta) + o(1) \tag{2.3}$$

as $t \to 0$, with $\eta = O(1)$. We note that matching to region $\text{II}^+$ (as $\eta \to \infty$) and region $\text{II}^-$ (as $\eta \to -\infty$) requires that

$$\overline{u}(\eta) = \begin{cases} 1 - o(1) & \text{as} \quad \eta \to \infty, \\ o(1) & \text{as} \quad \eta \to -\infty. \end{cases} \tag{2.4}$$

On substituting (2.3) into equation (2.1) (when written in terms of $\eta$ and $t$), we obtain, after some calculation [1], that

$$-\frac{\alpha\eta}{t}\overline{u}_\eta + \overline{u}\overline{u}_\eta t^{-\alpha} - \overline{u}_{\eta\eta} t^{-2\alpha} = 0. \tag{2.5}$$

[1] $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$ and $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} - \frac{\alpha\eta}{t}\frac{\partial u}{\partial t}$

We now look for the possible balances in equation (2.5). If $\alpha = 1$, (2.5) becomes

$$-\frac{\eta}{t}\overline{u}_{\eta} + \overline{u}\overline{u}_{\eta}t^{-1} - \overline{u}_{\eta\eta}t^{-2} = 0, \tag{2.6}$$

and at leading order, $O\left(t^{-2}\right)$, we obtain

$$\overline{u}_{\eta\eta} = 0, \tag{2.7}$$

with solution

$$\overline{u}(\eta) = A\eta + B, \tag{2.8}$$

where $A$ and $B$ are constants. Matching with the adjoining regions (regions $\text{II}^{+}$ and $\text{II}^{-}$) is not possible, and we can rule out $\alpha = 1$. Therefore, we left to consider the second possible balance, that is $\alpha = 1/2$, when $\alpha = 1/2$ equation (2.5) becomes

$$\frac{\eta}{2}\overline{u}_{\eta}\frac{1}{t} + \overline{u}\overline{u}_{\eta}\frac{1}{t^{1/2}} - \overline{u}_{\eta\eta}\frac{1}{t} = 0 \tag{2.9}$$

and at leading order $O\left(t^{-1}\right)$ we have that

$$\overline{u}_{\eta\eta} + \frac{\eta}{2}\overline{u}_{\eta} = 0, \quad -\infty < \eta < \infty. \tag{2.10}$$

The solution of (2.10) is readily obtained as

$$\overline{u}(\eta) = A_0\,\mathrm{erfc}\left(\frac{\eta}{2}\right) + B_0, \tag{2.11}$$

where $A_0$ and $B_0$ are constants to be determined, and erfc(.) is the complementary error function [2]. As $\eta \to \infty$ we move out of region I into region II$^+$, and we have from (2.11) that

$$\overline{u}(\eta) \sim B_0 + \frac{2A_0 e^{\frac{-\eta^2}{4}}}{\eta\sqrt{\pi}} + \dots \tag{2.12}$$

as $\eta \to \infty$. Matching with region II$^+$ then, requires that

$$B_0 = 1 \ . \tag{2.13}$$

As $\eta \to -\infty$, we move out of region I into region II$^-$, and we have from (2.11) and (2.13) that

$$\overline{u}(\eta) \sim (2A_0 + 1) + \dots \ . \tag{2.14}$$

Matching with region II$^-$ then, requires that

$$A_0 = -\frac{1}{2} \ . \tag{2.15}$$

Therefore, in region I we have, via (2.3), (2.11), (2.13) and (2.15) that

$$u(\eta, t) = \left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{\eta}{2}\right)\right] + o(1) \tag{2.16}$$

as $t \to 0$ with $\eta = O(1)$.

---

[2] $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2} du$ where

$$\mathrm{erfc}(x) \sim \begin{cases} \frac{e^{-x^2}}{x\sqrt{\pi}}, & x \gg 1, \\ 2 - \frac{e^{-x^2}}{(-x)\sqrt{\pi}}, & (-x) \ll 1 \end{cases}$$

From (2.16) we observe that

$$u(\eta, t) = \begin{cases} 1 - \frac{e^{\frac{-\eta^2}{4}}}{\eta\sqrt{\pi}} + ..., & \text{for} \quad \eta \gg 1, \quad t \to 0, \\ \frac{e^{\frac{-\eta^2}{4}}}{(-\eta)\sqrt{\pi}} + ..., & \text{for} \quad (-\eta) \gg 1, \quad t \to 0. \end{cases} \tag{2.17}$$

As $\eta \to \infty$ we move out region I into region II$^+$, where $x = O(1)(> 0)$ as $t \to 0$. On writing (2.17), in terms of $x$, we obtain

$$u \sim 1 - \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t - \ln x - \ln\sqrt{\pi} + o(1)\right) \quad . \tag{2.18}$$

Equation (2.18) suggests that in region II$^+$, we expand as

$$u(x, t) = 1 - e^{-F(x,t)}, \tag{2.19}$$

where

$$F(x, t) = \frac{f_0(x)}{t} + f_1(x)\ln t + f_2(x) + o(1), \tag{2.20}$$

as $t \to 0$ with $x = O(1)$ $(> 0)$ where $f_0(x)$ $(> 0)$, $f_1(x)$ and $f_2(x)$ are functions to be determined. On substituting (2.19) and (2.20) into equation (2.1), we obtain after some calculation [3] that

$$\begin{aligned} -f_0\frac{1}{t^2} + f_1\frac{1}{t} + \left(f_0'\frac{1}{t} + f_1'\ln t + f_2'\right) - \left(f_0''\frac{1}{t} + f_1''\ln t + f_2''\right) \\ +\left(f_0'^2\frac{1}{t^2} + f_1'^2(\ln t)^2 + f_2'^2 + 2f_0'f_2'\frac{1}{t} + 2f_0'f_1'\frac{\ln t}{t} + 2f_1'f_2'\ln t\right) \sim 0. \end{aligned} \tag{2.21}$$

We now equate at each order in term and solve to obtain the functions $f_i$ $(i = 0, 1, 2)$.

[3] $u_x = F_x e^{-F} \quad u_{xx} = F_{xx}e^{-F} - F_x^2 e^{-F} \quad u_t = F_t e^{-F}$

At $O\left(t^{-2}\right)$, we have that;

$$-f_0(x) + (f_0'(x))^2 = 0, \quad 0 < x < \infty. \tag{2.22}$$

Equation (2.22) has the solution

$$f_0(x) = \frac{(x + c_0)^2}{4},$$

where $c_0$ is constant of integration. Matching with region I as $x \to 0^+$ then requires that $c_0 = 0$, giving

$$f_0(x) = \frac{x^2}{4}, \quad x > 0. \tag{2.23}$$

At $O\left(\ln t/t\right)$ we have that:

$$2f_0'(x)f_1'(x) = 0. \tag{2.24}$$

Substituting (2.23) into equation (2.24) we obtain,

$$\frac{df_1}{dx} = 0.$$

On integrating, we obtain $f_1 = c_1$, where $c_1$ is a constant. Matching with region I, then requires $c_1 = -1/2$ giving that

$$f_1 = -\frac{1}{2} \tag{2.25}$$

Finally, at $O\left(t^{-1}\right)$ we have that:

$$f_1(x) + f_0'(x) - f_0''(x) + 2f_0'(x)f_2'(x) = 0, \quad x > 0. \tag{2.26}$$

Substituting (2.23) and (2.25) into equation (2.26) we obtain,

$$f_2'(x) = -\frac{1}{2} + \frac{1}{x} \quad x > 0, \tag{2.27}$$

On integrating equation (2.27), we obtain

$$f_2(x) = -\frac{x}{2} + \ln x + c_2 \quad x > 0,$$

where $c_2$ is a constant. Matching with region I as $x \to 0^+$ requires that $c_2 = \ln \sqrt{\pi}$ .

Therefore, in region $\text{II}^+$ we have that

$$u(x,t) = 1 - \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t + \frac{x}{2} - \ln x - \ln\sqrt{\pi} + o(1)\right) \tag{2.28}$$

as $t \to 0$ with $x = O(1)$ $(> 0)$.

We now consider region $\text{II}^-$. As $\eta \to -\infty$ we move out of region I into region $\text{II}^-$, where $x = O(1)(< 0)$, as $t \to 0$. (2.17) suggests that in region $\text{II}^-$ we look for an expansion at the form

$$u(x,t) = e^{-\widehat{F}(x,t)}, \tag{2.29}$$

as $t \to 0$ with $x = O(1)$ $(< 0)$ where

$$\widehat{F}(x,t) = \frac{\widehat{f}_0(x)}{t} + \widehat{f}_1(x)\ln t + \widehat{f}_2(x) + o(1), \tag{2.30}$$

where $\widehat{f}_0(x)$ $(> 0)$, $\widehat{f}_1$ and $\widehat{f}_2$ are functions to be determined. Substituting (2.29) and

(2.30) into equation (2.1), we obtain after some calculation [4]:

$$\left(\frac{\widehat{f}_0}{t^2}-\frac{\widehat{f}_1}{t}\right)+\left(\frac{\widehat{f}_0''}{t}+\widehat{f}_1\ln t+\widehat{f}_2''\right)$$
$$-\left(\frac{\widehat{f}_0'^2}{t^2}+2\widehat{f}_0'\widehat{f}_1'\frac{\ln t}{t}+2\frac{\widehat{f}_0'\widehat{f}_2'}{t}+(\widehat{f}_1')^2(\ln t)^2+2\widehat{f}_1'\widehat{f}_2'\ln t+(\widehat{f}_2')^2\right)\sim 0. \tag{2.31}$$

We now equate at each order in turn to find $\widehat{f}_0(x), \widehat{f}_1(x)$ and $\widehat{f}_2(x)$. We first obtain at $O\left(t^{-2}\right)$ that

$$\widehat{f}_0(x)-(\widehat{f}_0'(x))^2=0, \quad x<0. \tag{2.32}$$

Equation (2.32) has the solution

$$\widehat{f}_0(x)=\frac{(x+\widehat{c}_0)^2}{4},$$

where $\widehat{c}_0$ is a constant of integration. Matching with region I as $x\to 0^-$ requires that $\widehat{c}_0=0$, giving

$$\widehat{f}_0(x)=\frac{x^2}{4}, \quad x<0. \tag{2.33}$$

At $O\left(\ln t/t\right)$ we have that

$$2\widehat{f}_0'(x)\widehat{f}_1'(x)=0 \tag{2.34}$$

and substituting (2.33) into equation (2.34), we obtain

$$\frac{d\widehat{f}_1(x)}{dx}=0\ . \tag{2.35}$$

On integrating equation (2.35), we obtain $\widehat{f}_1=\widehat{c}_1$, where $\widehat{c}_1$ is a constant. Matching with

[4] $u_x=-\widehat{F}_x e^{-\widehat{F}} \quad u_{xx}=-\widehat{F}_{xx}e^{-\widehat{F}}+\widehat{F}_x^2 e^{-\widehat{F}} \quad u_t=-\widehat{F}_t e^{-\widehat{F}}$

region I as $x \to 0^-$ requires $\widehat{c}_1 = -1/2$ giving

$$\widehat{f}_1(x) = -1/2. \tag{2.36}$$

Finally, at $O\left(t^{-1}\right)$ we have that

$$-\widehat{f}_1(x) + \widehat{f}_0''(x) - 2\widehat{f}_0'(x)\widehat{f}_2'(x) = 0. \tag{2.37}$$

On substituting (2.33) and (2.36) into equation (2.37) we obtain that

$$\widehat{f}_2'(x) = \frac{1}{x}, \quad x < 0. \tag{2.38}$$

On integrating equation (2.38), we obtain

$$\widehat{f}_2(x) = \ln(-x) + \widehat{c}_2,$$

where $\widehat{c}_2$ is a constant. Matching with region I as $x \to 0^-$ requires that $\widehat{c}_2 = \ln\sqrt{\pi}$ giving

$$\widehat{f}_2(x) = \ln(-x) + \ln\sqrt{\pi}, \quad x < 0.$$

Therefore, we have in region II$^-$ that

$$u(x,t) = \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t - \ln(-x) - \ln\sqrt{\pi} + o(1)\right), \tag{2.39}$$

as $t \to 0$, with $x = O(1)$ $(< 0)$.

The asymptotic structure as $t \to 0$ is now complete with the expansions in regions I, II$^-$ and II$^+$, providing a uniform approximation to solution of IVP1 as $t \to 0$.

## 2.2 Asymptotic Solution of IVP1 as $|x| \to \infty$

Now, we examine the asymptotic structure of the solution to IVP1 as $|x| \to \infty$ with $t = O(1)$. We first consider the structure of solution to IVP1 as $x \to \infty$, with $t = O(1)$. The form expansion (2.28) of region II$^+$ for $x = O(1)(> 0)$ as $t \to 0$ suggests that in this region, which we label as region III$^+$, we write

$$u(x,t) = 1 - e^{G(x,t)} \tag{2.40}$$

where

$$G(x,t) = g_0(t)x^2 + g_1(t)x + g_2(t)\ln x + g_3(t) + o(1), \tag{2.41}$$

as $x \to \infty$ with $t = O(1)$ and where $g_0(t)$ $(< 0)$, $g_1(t)$ and $g_2(t)$ are functions to be determined. On substitution of (2.40) and (2.41) into equation (2.1) we obtain after some calculation, [5] that

$$-\left(\dot{g}_0 x^2 + \dot{g}_1 x + \dot{g}_2 \ln x + \dot{g}_3\right) - \left(2xg_0 + g_1 + g_2\frac{1}{x}\right) + \left(2g_0 - g_2\frac{1}{x^2}\right) + \left(g_1^2 + 4x^2g_0^2 + g_2^2\frac{1}{x^2} + 4g_0g_2 + \frac{2}{x}g_1g_2 + 4xg_0g_1\right) \sim 0, \tag{2.42}$$

We now equate at each order in turn and solve to obtain $g_0(t)$, $g_1(t)$, $g_2(t)$ and $g_3(t)$.

Equating at $O(x^2)$ we have that

$$-\dot{g}_0(t) + 4g_0(t)^2 = 0, \quad t > 0. \tag{2.43}$$

Equation (2.43) has the solution

$$g_0(t) = -\frac{1}{4(t + c_0)},$$

[5] $u_x = -G_x e^G \quad u_{xx} = -G_{xx}e^G - G_x^2 e^G \quad u_t = -G_t e^G$

where $c_0$ is a constant. Matching with region $\mathrm{II}^+$ as $t \to 0^+$ requires that $c_0 = 0$, giving

$$g_0(t) = -\frac{1}{4t}, \quad t > 0. \tag{2.44}$$

At $O(x)$ we have that;

$$-\dot{g}_1(t) - 2g_0(t) + 4g_0(t)g_1(t) = 0. \tag{2.45}$$

On substitution of (2.44) into equation (2.45), we obtain

$$-\dot{g}_1(t) + \frac{1}{2t} - \frac{g_1(t)}{t} = 0. \tag{2.46}$$

Solving equation (2.46), we obtain

$$g_1(t) = \frac{1}{2} + \frac{c_1}{t}, \quad t > 0 \tag{2.47}$$

where $c_1$ is a constant. Matching with region $\mathrm{II}^+$ as $t \to 0^+$ requires $c_1 = 0$, giving

$$g_1(t) = \frac{1}{2}. \tag{2.48}$$

At $O(\ln x)$ we have that

$$\dot{g}_2(t) = 0. \tag{2.49}$$

On integrating equation (2.49), we obtain $g_2(t) = c_2$ where $c_2$ is a constant. Matching with region $\mathrm{II}^+$ as $t \to 0^+$ requires that

$$g_2(t) = -1. \tag{2.50}$$

Finally, at $O(1)$ we have that

$$-\dot{g}_3(t) - g_1(t) + 2g_0(t) + g_1^2(t) + 4g_0(t)g_2(t) = 0. \qquad (2.51)$$

On substitution from (2.44), (2.48) and (2.50) into equation (2.51), we obtain

$$\dot{g}_3(t) = -\frac{1}{4} + \frac{1}{2t}. \qquad (2.52)$$

On integrating equation (2.52), we have

$$g_3(t) = \frac{\ln t}{2} - \frac{t}{4} + c_3,$$

where $c_3$ is a constant. Matching with region II$^+$ as $t \to 0^+$ requires that $c_3 = -\ln\sqrt{\pi}$. Therefore, we have in region III$^+$ that

$$u(x,t) = 1 - \exp\left\{ -\frac{x^2}{4t} + \frac{x}{2} - \ln x + \left( \frac{\ln t}{2} - \frac{t}{4} - \ln\sqrt{\pi} \right) + o(1) \right\} \qquad (2.53)$$

as $x \to \infty$ with $t = O(1)$. Expansion (2.53) remains uniform for $t \gg 1$ provided that $x \gg t$, but becomes non-uniform when $x = O(t)$ as $t \to \infty$.

We now investigate the structure of solution of IVP1 as $x \to -\infty$ with $t = O(1)$, we label this region region III$^-$. The form expansion (2.39) of region II$^-$ suggests that in region III$^-$ we write,

$$u(x,t) = e^{\widehat{G}(x,t)} \qquad (2.54)$$

where

$$\widehat{G}(x,t) = \widehat{g}_0(t)x^2 + \widehat{g}_1(t)\ln(-x) + \widehat{g}_2(t) + o(1), \qquad (2.55)$$

as $x \to -\infty$, with $t = O(1)$ and where $\widehat{g}_0(t)$ $(< 0)$, $\widehat{g}_1(t)$ and $\widehat{g}_2(t)$ are functions to be determined. On substituting (2.54) and (2.55) into equation (2.1), we obtain after some calculation [6] that

$$\left(\dot{\widehat{g}}_0 x^2 + \dot{\widehat{g}}_1 \ln(-x) + \dot{\widehat{g}}_2\right) - \left(2\widehat{g}_0 - \widehat{g}_1 \frac{1}{x^2}\right) - \left(4\widehat{g}_0^2 x^2 + \widehat{g}_1^2 \frac{1}{x^2} + 4\widehat{g}_0\widehat{g}_1\right) \sim 0. \tag{2.56}$$

We now equate at each order in turn to obtain the functions $\widehat{g}_0(t)$, $\widehat{g}_1(t)$ and $\widehat{g}_2(t)$. At $O(x^2)$ we have that

$$\dot{\widehat{g}}_0(t) - 4(\widehat{g}_0(t))^2 = 0. \tag{2.57}$$

Equation (2.57) has the solution

$$\widehat{g}_0(t) = -\frac{1}{4(t + \widehat{c}_0)} \quad t > 0,$$

where $\widehat{c}_0$ is a constant. Matching with region $\text{II}^-$ as $t \to 0^+$ then, requires that $\widehat{c}_0 = 0$, giving

$$\widehat{g}_0(t) = -\frac{1}{4t}, \quad t > 0. \tag{2.58}$$

At $O(\ln(-x))$ we have that

$$\dot{\widehat{g}}_1(t) = 0. \tag{2.59}$$

On integrating equation (2.59), we obtain $\widehat{g}_1(t) = \widehat{c}_1$ where $\widehat{c}_1$ is a constant. Matching with region $\text{II}^-$ as $t \to 0^-$ requires that

$$\widehat{g}_1(t) = -1 \ . \tag{2.60}$$

At $O(1)$ we have that

$$\dot{\widehat{g}}_2(t) - 2\widehat{g}_0(t) - 4\widehat{g}_0(t)\widehat{g}_1(t) = 0. \tag{2.61}$$

[6] $u_x = \widehat{G}_x e^{\widehat{G}} \quad u_{xx} = \widehat{G}_{xx} e^{\widehat{G}} + \widehat{G}_x^2 e^{\widehat{G}} \quad u_t = \widehat{G}_t e^{\widehat{G}}$

On substituting (2.58) and (2.60) into equation (2.61), we obtain

$$\dot{\hat{g}}_2(t) = \frac{1}{2t}, \quad t > 0. \tag{2.62}$$

On integrating equation (2.62), we obtain

$$\hat{g}_2(t) = \frac{\ln t}{2} + \hat{c}_2, \quad t > 0$$

where $\hat{c}_2$ is a constant. Matching with region II$^-$ requires that $\hat{c}_2 = -\ln\sqrt{\pi}$, giving

$$\hat{g}_2(t) = \frac{\ln t}{2} - \ln\sqrt{\pi}, \quad t > 0. \tag{2.63}$$

Therefore, we have in region III$^-$ that

$$u(x,t) = \exp\left\{-\frac{x^2}{4t} - \ln(-x) + \left(\frac{\ln t}{2} - \ln\sqrt{\pi}\right) + o(1)\right\}, \tag{2.64}$$

as $x \to -\infty$ with $t = O(1)$. Expansion (2.64) remains uniform for $t \gg 1$ provided that $(-x) \gg t$, but becomes nonuniform when $(-x) = O(t)$ as $t \to \infty$ .

## 2.3 Asymptotic Solution to IVP1 as $t \to \infty$

As $t \to \infty$, asymptotic expansions (2.53) and (2.64), of regions III$^+$ and III$^-$ respectively continue to remain uniform provided $|x| \gg t$, but become nonuniform when $|x| = O(t)$. We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we define a new region, which we label region IV$^+$, when $x = O(t)$ as $t \to \infty$. To investigate region IV$^+$, we introduce the scaled coordinate,

$$y = \frac{x}{t}, \tag{2.65}$$

where $y = O(1)$ as $t \to \infty$. On substituting (2.65) into (2.53) we find that in region III$^+$ when $x = O(t)$ that

$$u(y,t) = 1 - \exp\left\{\left(-\frac{y^2}{4} + \frac{y}{2} - \frac{1}{4}\right)t - \frac{1}{2}\ln t - \left(\ln y + \ln\sqrt{\pi}\right) + o(1)\right\}. \tag{2.66}$$

The form of expansion (2.66) suggests that in region IV$^+$ we write

$$u(y,t) = 1 - e^{-H(y,t)} \tag{2.67}$$

where

$$H(y,t) = h_0(y)t + h_1(y)\ln t + h_2(y) + o(1), \tag{2.68}$$

as $t \to \infty$ with $y = O(1)$, and where $h_0(y)$ $(> 0)$, $h_1(y)$ and $h_2(y)$ are functions to be determined. On substituting (2.67) and (2.68) into equation (2.1) (when written in terms of $y$ and $t$), we obtain after some calculation [7]

$$
\begin{aligned}
&h_0 + \frac{h_1}{t} + (1-y)\left(h_0' + h_1'\frac{\ln t}{t} + \frac{h_2'}{t}\right) - \left(\frac{h_0''}{t} + h_1''\frac{\ln t}{t^2} + \frac{h_2''}{t^2}\right) \\
&+\left((h_0')^2 + 2h_0'h_1'\frac{\ln t}{t} + 2h_2'h_0'\frac{1}{t} + (h_1')^2\frac{(\ln t)^2}{t^2} + 2h_2'h_1'\frac{\ln t}{t^2} + (h_2')^2\frac{1}{t^2}\right) \sim 0.
\end{aligned}
\tag{2.69}
$$

It is instructive to consider first the leading order problem. At leading order we have

$$(h_0')^2 + (1-y)h_0' + h_0 = 0, \quad y > 0 \tag{2.70}$$

$$h_0(y) > 0, \quad y > 0, \tag{2.71}$$

$$h_0(y) \sim \frac{(y-1)^2}{4}, \quad y \to \infty. \tag{2.72}$$

Condition (2.72) arises from matching expansion (2.67) (with (2.68)) ($y \gg 1$) with expansion (2.53) ($x = O(t)$). Equation (2.70) admits the one-parameter family of linear solutions,

$$h_0(y) = A[y - (A+1)], \quad y \in (-\infty, \infty), \tag{2.73}$$

where $A \in \mathbb{R}$, together with the associated singular envelope solutions

$$h_0(y) = \pm\frac{(y-1)^2}{4}. \tag{2.74}$$

We note that combinations of (2.73) and (2.74) which remain continuous and differentiable also provide solutions to equation (2.70) (envelope-touching solutions). The solution of

[7]

$$u_y = H_y e^H \quad u_{yy} = H_{yy}e^H - H_y^2 e^H \quad u_t = H_t e^H$$

(2.70)-(2.72) is given either by

$$h_0(y) = \frac{(y-1)^2}{4} \quad y > 1, \tag{2.75}$$

or

$$h_0(y) = \begin{cases} \dfrac{(y-1)^2}{4}, & y \geqslant 1 + 2A, \\ A[y - (1+A)], & (1+A) < y < 1 + 2A. \end{cases} \tag{2.76}$$

for each $A > 0$. Each case will have to be considered separately.

(a) We begin by considering the case when $h_0(y)$ is given by (2.76). We note that expansion (2.67) (with (2.68)) then becomes nonuniform as $y \to (1+A)^+$. To examine this nonuniformity we introduce a new region, region TW, where

$$y = (1+A) + O(t^{-1}) \quad \text{as} \quad t \to \infty.$$

Therefore, to examine region TW we introduce the scaled coordinate

$$z = [y - (1+A)]t$$

where $z = O(1)$ as $t \to \infty$ and, via (2.67), (2.68) and (2.76), look for an expansion of the form

$$u(z,t) = U(z) + o(1) \tag{2.77}$$

as $t \to \infty$. We note that the matching condition with region IV$^+$ is given by

$$U(z) \sim 1 - O(e^{-Az}) \quad \text{as} \quad z \to \infty. \tag{2.78}$$

On substituting (2.77) into equation (2.1) (when written in terms of $z$ and $t$) we obtain at leading order that

$$U_{zz} - UU_z + (1+A)U_z = 0, \quad -\infty < z < \infty. \tag{2.79}$$

We note that equation (2.79) has been considered in Section 1.1.1 (where $c = 1 + A$ in this case). On integrating (2.79) we obtain

$$U_z - \frac{1}{2}U^2 + (1+A)U = C, \tag{2.80}$$

where $C$ is a constant. On using condition (2.78) we have that

$$C = 1/2 + A.$$

Therefore, equation (2.80) can be written as

$$\frac{dU}{dz} = \frac{1}{2}(U-1)(U-(1+2A)), \quad -\infty < z < \infty, \tag{2.81}$$

where $1 + 2A > U(z) > 1$. The solution to (2.81) is readily obtained as

$$U(z) = \frac{1 + (1+2A)ke^{-Az}}{1 + ke^{-Az}}, \quad -\infty < z < \infty, \tag{2.82}$$

where $k$ is a constant. We recall from Section 1.1.1 that (2.82) is a travelling wave solution of (2.1) connecting $u = 1$ (as $z \to \infty$) to $u = (1+2A)$ (as $z \to -\infty$). Now as $z \to -\infty$ we move out of region TW into region IV$^-$ where $y = O(1)(\in (-\infty, 1+A))$ as $t \to \infty$, and we have from (2.77) and (2.82) that in region IV$^-$ that we should expand as

$$u(y,t) = (1+2A) - e^{-L(y,t)}$$

where

$$L(y,t) = l_0(y)t + l_1(y) + o(1)$$

as $t \to \infty$ with $y = O(1)(\in (-\infty, 1+A))$, and where $l_0(y)$ $(> 0)$ and $l_1(y)$ are functions to be determined. However, there is no mechanism available to allow an expansion of this form to match to the far field $(-y \gg 1)$ where $u$ is exponentially small in $t$ as $t \to \infty$, and we conclude that this case can be ruled out and that $h_0(y)$ must be given by (2.75).

(b) We now consider the case when $h_0(y)$ is given by (2.75). Continuing expansion (2.67) and (2.68) in region IV$^+$, we obtain at $O(\ln t/t)$ that

$$(1-y)h_1' + 2h_0'h_1' = 0\ . \tag{2.83}$$

On substituting (2.75) into equation (2.83) we have that

$$[2h_0' + (1-y)]h_1' = 0, \tag{2.84}$$

which is identically satisfied since $2h_0' = -(1-y)$. Equating at $O(t^{-1})$ we have that

$$h_1 + h_2'(1-y) - h_0'' + 2h_2'h_0' = 0. \tag{2.85}$$

On substituting (2.75) into equation (2.85), we obtain that

$$h_1(y) = \frac{1}{2}. \tag{2.86}$$

We conclude that

$$h_2(y) = \Psi(y), \quad y > 1$$

where the function $\Psi(y)$ is undetermined.

Therefore, we have in region IV$^+$ that

$$u(y,t) = 1 - \ \exp\left(-\frac{(y-1)^2}{4}t - \frac{1}{2}\ln t - \Psi(y) + o(1)\right) \tag{2.87}$$

as $t \to \infty$ with $y = O(1)[\in (1,\infty)]$. We recall that the function $\Psi(y)$ is undetermined. However matching with region III$^+$ requires that

$$\Psi(y) \sim \ln y + \ln\sqrt{\pi} \quad as \quad y \to \infty, \tag{2.88}$$

and we make the assumption (which we will verify as consistent) that

$$\Psi(y) \sim \ln\frac{1}{\sqrt{\pi}}$$

as $y \to 1^+$.

We note that expansion (2.87) becomes non-uniform as $y \to 1^+$ and to continue the asymptotic structure of the solution to IVP1 as $t \to \infty$ we must introduce a localized region, region A, in which following (2.87), we have that

$$y = 1 + O(t^{-\frac{1}{2}}) \quad \text{as} \quad t \to \infty. \tag{2.89}$$

Thus in region A we write

$$y = 1 + \eta t^{-\frac{1}{2}}, \tag{2.90}$$

where $\eta = O(1)$ as $t \to \infty$. It follows from (2.87) and (2.90) that in region A, we should expand as

$$u = 1 + t^{-\frac{1}{2}}\kappa(\eta) + o(t^{-\frac{1}{2}}) \tag{2.91}$$

as $t \to \infty$ with $\eta = O(1)$, and where $\kappa(\eta) < 0$. On substituting (2.91) into equation (2.1) (when written in terms of $\eta$ and $t$) we obtain at leading order that

$$\kappa_{\eta\eta} - \kappa\kappa_{\eta} + \frac{\eta}{2}\kappa_{\eta} + \frac{\kappa}{2} = 0, \quad -\infty < \eta < \infty. \tag{2.92}$$

Equation (2.92) is to be solved subject to matching with region IV$^{+}$ as $\eta \to \infty$, that is, we require

$$\kappa(\eta) \sim -\frac{1}{\sqrt{\pi}}e^{-\frac{\eta^2}{4}} \quad \text{as} \quad \eta \to \infty. \tag{2.93}$$

We recall from Section 1.1.3 that the solution to (2.92), (2.93) is given by

$$\kappa(\eta) = -\frac{2e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}(1 + \operatorname{erf}\left(\frac{\eta}{2}\right))}, \quad -\infty < \eta < \infty. \tag{2.94}$$

Therefore, in region A, we have

$$u(\eta, t) = 1 - \frac{2e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}\left(1 + \operatorname{erf}(\frac{\eta}{2})\right)}t^{-\frac{1}{2}} + o(t^{-\frac{1}{2}}), \tag{2.95}$$

as $t \to \infty$ with $\eta = O(1)$. Now as $\eta \to -\infty$ we move out of region A into region V. Consideration of expansion (2.95) for $(-\eta) \gg 1$ gives that

$$u \sim 1 + \eta t^{-\frac{1}{2}} \tag{2.96}$$

with $-\eta \gg 1$, which on writing in terms of $y$, gives

$$u \sim y + o(1). \tag{2.97}$$

Therefore, in region V we have that $u = O(1)$ as $t \to \infty$, and we therefore expand as

$$u = F(y) + o(1), \tag{2.98}$$

as $t \to \infty$ with $y = O(1) \in (0, 1)$ where $F(y) = O(1)$. On writing equation (2.1) in terms of $y$ and $t$ we obtain

$$u_t - \frac{y}{t}u_y + \frac{1}{t}uu_y - \frac{1}{t^2}u_{yy} = 0 \ . \tag{2.99}$$

On substituting (2.98) into (2.99) we obtain at leading order that

$$F_y(F - y) = 0, \qquad y < 1, \tag{2.100}$$

$$F(y) \sim y \quad \text{ as } \ y \to 1^- \ , \tag{2.101}$$

with the final condition being the matching condition with region A (see (2.97)). The solution of (2.100) and (2.101) is readily obtained as

$$F(y) = y, \qquad 0 < y < 1. \tag{2.102}$$

We note that (2.98) with (2.102) becomes non-uniform as $y \to 0^+$. Therefore in region V we have that

$$u(y, t) = y + o(1) \tag{2.103}$$

as $t \to \infty$ with $y = O(1)(\in (0, 1))$. As already noted expansion (2.103) becomes non-uniform as $y \to 0^+$ and we must introduce a further localised region, region B, located at $y = 0$. In region B we have, following (2.103), that

$$y = O\left(t^{-\frac{1}{2}}\right) \quad \text{ as } \ t \to \infty. \tag{2.104}$$

To examine region B we introduce the scaled coordinate

$$\eta = yt^{1/2} \tag{2.105}$$

where $\eta = O(1)$ as $t \to \infty$. It follows from (2.103) and (2.105) in region V, that we should expand as

$$u(\eta, t) = t^{-\frac{1}{2}} K(\eta) + o(t^{-\frac{1}{2}}) \quad \text{as} \;\; t \to \infty, \tag{2.106}$$

where $K(\eta) > 0$ and $\eta = O(1)$. On substitution of (2.106) into equation (2.1) (when written in terms of $\eta$ and $t$) we obtain at leading order that

$$K_{\eta\eta} - KK_\eta + \frac{\eta}{2} K_\eta + \frac{K}{2} = 0, \quad -\infty < \eta < \infty. \tag{2.107}$$

which is to be solved subject to matching with region V as $\eta \to \infty$, that is, we require

$$K(\eta) \sim \eta \quad \eta \to \infty. \tag{2.108}$$

The solution to (2.107), (2.108) is readily obtained (see Section 1.1.3) as

$$K(\eta) = \frac{2e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}\left(1 - \operatorname{erf}\left(\frac{\eta}{2}\right)\right)}, \quad -\infty < \eta < \infty. \tag{2.109}$$

Therefore in region B we have

$$u(\eta, t) = \frac{2e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}\left(1 - \operatorname{erf}(\frac{\eta}{2})\right)} t^{-\frac{1}{2}} + o(t^{-\frac{1}{2}}) \tag{2.110}$$

as $t \to \infty$ with $\eta = O(1)$. As $\eta \to -\infty$ we move out of the localised region B into region IV$^-$, where $y = O(1)(\in (-\infty, 0))$ as $t \to \infty$. We now examine the form of expansion

(2.106) for $(-\eta) \gg 1$ (as we move into region IV$^-$). We recall from Section 1.1.3 that

$$K(\eta) \sim \frac{1}{\sqrt{\pi}} e^{-\eta^2/4} \quad \eta \to -\infty.$$

Therefore, we have that

$$u(\eta, t) \sim \frac{1}{\sqrt{\pi}} e^{-\eta^2/4} t^{-1/2} \tag{2.111}$$

with $(-\eta) \gg 1$. When written in terms of $y$, (2.111) becomes

$$u(y, t) \sim \exp\left(-\frac{y^2}{4} t - \frac{1}{2} \ln t - \ln \sqrt{\pi}\right). \tag{2.112}$$

The form of expansion (2.112) suggests that in region IV$^-$ we should look for an expansion of the form

$$u(y, t) = e^{-\widehat{H}(y,t)} \tag{2.113}$$

where

$$\widehat{H}(y, t) = \hat{h}_0(y) t + \hat{h}_1(y) \ln(t) + \hat{h}_2(y) + o(1) \tag{2.114}$$

as $t \to \infty$, with $y = O(1) \in (-\infty, 0)$ where $\hat{h}_0(y) > 0$. On writing (2.1) in terms of $y$ and $t$ we obtain,

$$u_t - \frac{y}{t} u_y + \frac{1}{t} u u_y - \frac{1}{t^2} u_{yy} = 0 \ . \tag{2.115}$$

On substituting (2.113) and (2.114) into equation (2.115), we obtain after some calculation (see footnote [8]),

$$-\widehat{H}_t + \frac{y}{t} \widehat{H}_y + \frac{1}{t^2} \widehat{H}_{yy} - \frac{\widehat{H}_y^2}{t^2} = 0, \tag{2.116}$$

[8] $u_y = -\widehat{H}_y e^{-\widehat{H}} \quad u_{yy} = -\widehat{H}_{yy} e^{-\widehat{H}} + \widehat{H}_y^2 e^{-\widehat{H}} \quad u_t = -\widehat{H}_t e^{-\widehat{H}}$

as $t \to \infty$, with $y = O(1) \in (-\infty, 0)$. On substituting (2.114) in (2.116) we obtain

$$
\begin{aligned}
&-\left(\widehat{h_0} + \widehat{h_1}\frac{1}{t}\right) + \frac{y}{t}\left((\widehat{h_0}')t + \widehat{h_1}'\ln t + \widehat{h_2}'\right) + \left(\widehat{h_0}''\frac{1}{t} + \widehat{h_1}''\frac{\ln t}{t^2} + \widehat{h_2}''\frac{1}{t^2}\right) \\
&-\left((\widehat{h_0}')^2 2\widehat{h_0}'\widehat{h_1}'\frac{\ln t}{t} + (\widehat{h_1}')^2\frac{(\ln t)^2}{t^2} + 2\widehat{h_1}'\widehat{h_2}'\frac{\ln t}{t^2} + 2\widehat{h_0}'\widehat{h_2}'\frac{1}{t} + (\widehat{h_2}')^2\frac{1}{t^2}\right) = 0
\end{aligned}
\tag{2.117}
$$

as $t \to \infty$, with $y = O(1)(\in (-\infty, 0))$ where $\widehat{h_0}(y) > 0$. We now equate at each order in turn to obtain the functions $\widehat{h_0}(y)$, $\widehat{h_1}(y)$ and $\widehat{h_2}(y)$. Equating at $O(1)$, we obtain the leading order problem as,

$$
(\widehat{h_0}')^2(y) - y\widehat{h_0}'(y) + \widehat{h_0}(y) = 0, \quad y < 0, \tag{2.118}
$$

$$
\widehat{h_0}(y) > 0, \quad y < 0, \tag{2.119}
$$

$$
\widehat{h_0}(y) \sim \frac{y^2}{4} \quad \text{as} \quad y \to -\infty, \tag{2.120}
$$

$$
\hat{h}_0(y) \sim \frac{y^2}{4} \quad \text{as} \quad y \to 0^- \ . \tag{2.121}
$$

Condition (2.120) arises from matching expansion (2.113) with (2.114) (for $-y \gg 1$) with expansion (2.64) ($x = O(t)$), while condition (2.121) is the matching condition with region B. The solution to (2.118)-(2.121) is readily obtained as

$$
\hat{h}_0(y) = \frac{y^2}{4}, \quad y < 0.
$$

On continuing expansion (2.113), (2.114) we obtain that

$$
\widehat{h_1}(y) = \frac{1}{2}, \quad y < 0 \tag{2.122}
$$

and

$$
\widehat{h_2} = \hat{\Psi}(y), \quad y < 0 \ , \tag{2.123}
$$

where the function $\widehat{\Psi} : (-\infty, 0) \to \mathbb{R}$ remains undetermined. Therefore, we have in region IV$^-$ that

$$u(y,t) = \exp\left\{-\frac{y^2}{4}t - \frac{1}{2}\ln t - \widehat{\Psi}(y) + o(1)\right\} \tag{2.124}$$

as $t \to \infty$ with $y = O(1)(\in (-\infty, 0))$, and where

$$\widehat{\Psi}(y) \sim \begin{cases} \ln(-y) + \ln\sqrt{\pi} \quad \text{as} \quad y \to -\infty, \\ \ln\sqrt{\pi} \quad \text{as} \quad y \to 0^-, \end{cases} \tag{2.125}$$

to facilitate matching with the adjoining regions. This then completes the asymptotic solution of initial-value problem (2.1), (2.2) as $t \to \infty$, with expansions (2.87) (region IV$^+$), (2.95) (region A), (2.103) (region V), (2.110) (region B) and (2.124) (region IV$^-$) providing a uniform asymptotic solution to (2.1), (2.2) as $t \to \infty$.

## 2.4 Summary

In this chapter we have obtained the complete asymptotic structure of the solution to IVP1 as $t \to \infty$. A uniform approximation to the solution as $t \to \infty$ has been given through regions IV$^+$, A, V, B and IV$^-$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given Figure 2.1. The solution exhibits in $x \geqslant 0$ the formation of an expansion wave. This expansion wave develops in the expansion region, region V, where $y = O(1)(\in (0,1))$ as $t \to \infty$ [that is, $x \in (0,t)$ as as $t \to \infty$]. Regions IV$^-$ and IV$^+$ allow for the transfer of information from the far field $|y| \gg 1$ [that is, $|x| \gg t$] to the near field ($y = O(1)$). We note that at leading order in region IV$^+$ the solution to IVP1, $u$, is $O(1)$ and is given at leading order by the constant value 1 (the value of $u$ ahead of the expansion wave). Regions A and B are localized connection regions connecting regions IV$^+$ and IV$^-$ to the expansion wave, respectively.

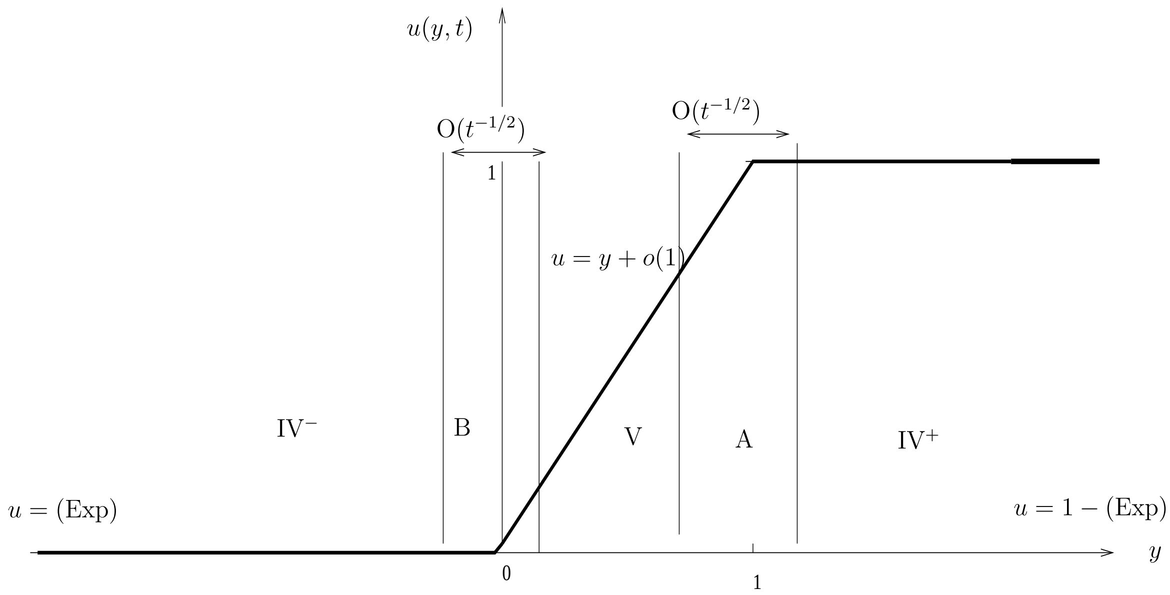


Figure 2.1: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane, as $t \to \infty$. Here (Exp) denotes terms exponentially small in $t$ as $t \to \infty$. We note that $u = 1 + O(t^{-1/2})$ as $t \to \infty$ in region A, while $u = O(t^{-1/2})$ as $t \to \infty$ in region B.

# Chapter 3

# Initial-Value Problem 2 for Burgers' Equation

In this chapter, we consider a second initial-value problem for Burgers' equation, namely,

$$u_t + uu_x - u_{xx} = 0, \ -\infty < x < \infty, \quad t > 0 \tag{3.1}$$

$$u(x,0) = \begin{cases} u_+ & x \geqslant 0, \\ u_- & x < 0, \end{cases} \tag{3.2}$$

where $u_- > u_+$. In what follows we label initial-value problem (3.1), (3.2) as IVP2. In this chapter we develop the large-time structure of the solution of IVP2 using the method of matched asymptotic coordinate expansions. We begin by examining the asymptotic structure of the solution to IVP2 as $t \to 0$.

## 3.1 Asymptotic Solution of IVP2 as $\mathbf{t \to 0}$

Consideration of initial data (3.2) indicates that the structure of the asymptotic solution of IVP2 as $t \to 0$ has three asymptotic regions, namely:

$$
\begin{aligned}
&\text{Region I}: x = o(1), && u(x,t) = O(1)\\
&\text{Region II}^+: x = O(1)\ (>0), && u(x,t) = u_+ + o(1)\\
&\text{Region II}^-: x = O(1)\ (<0), && u(x,t) = u_- - o(1)
\end{aligned}
$$

We first consider region I, in which, $x = o(1)$ and $u(x,t) = O(1)$ as $t \to 0$. To examine region I we introduce the scaled coordinate $\eta = xt^{-\alpha}$ as $t \to 0$, where $\alpha > 0$ and $\eta = O(1)$, and look for an expansion of the form

$$u = \breve{u}(\eta) + o(1) \tag{3.3}$$

as $t \to 0$, with $\eta = O(1)$. We note that matching to region II$^-$ (as $\eta \to -\infty$) and region II$^+$ (as $\eta \to \infty$) requires that

$$\breve{u}(\eta) = \begin{cases} u_+ + o(1), & \eta \to \infty, \\ u_- - o(1), & \eta \to -\infty. \end{cases} \tag{3.4}$$

On substituting (3.3) into equation (3.1) (when written in terms of $\eta$ and $t$), we obtain, after some calculation [1], that

$$-\frac{\alpha\eta}{t}\breve{u}_\eta + \breve{u}\breve{u}_\eta t^{-\alpha} - \breve{u}_{\eta\eta}t^{-2\alpha} = 0, \tag{3.5}$$

[1] $\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)$ and $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t}\right) + \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial t}\right)$ where $\left(\frac{\partial \eta}{\partial t} = \frac{-\alpha\eta}{t}\right)$

as $t \to 0$. We find (see Section 2.1) that a non-trivial leading order balance, requires $\alpha = 1/2$. Therefore, at leading order we have,

$$\breve{u}_{\eta\eta} + \frac{\eta}{2}\breve{u}_{\eta} = 0, \quad -\infty < \eta < \infty. \tag{3.6}$$

The solution of (3.6) is readily obtained as

$$\breve{u}(\eta) = A_1 + B_1 \operatorname{erfc}\left(\frac{\eta}{2}\right), \quad -\infty < \eta < \infty \tag{3.7}$$

where $A_1$ and $B_1$ are constants to be determined on matching, and erfc[.] is the complementary function. Matching conditions (3.4) then require that

$$\breve{u} \sim \begin{cases} u_-, & \eta \to -\infty, \\ u_+, & \eta \to \infty. \end{cases} \tag{3.8}$$

Now as $\eta \gg 1$ we move out of region I into region II$^+$ and we have from (3.7) that

$$\breve{u} \sim A_1 + B_1 \left( \frac{2e^{-\frac{\eta^2}{4}}}{\eta\sqrt{\pi}} \right). \tag{3.9}$$

Matching with region II$^+$ then requires that

$$A_1 = u_+. \tag{3.10}$$

As $(-\eta) \gg 1$ me move out of region I into region II$^-$ and we have from (3.7) that

$$\breve{u} \sim B_1 \left( 2 - \frac{2e^{-\frac{\eta^2}{4}}}{(-\eta)\sqrt{\pi}} \right) + u_+. \tag{3.11}$$

Matching with region II$^-$ then requires $\breve{u} \sim u_-$ as $\eta \to -\infty$ giving that

$$B_1 = \frac{(u_- - u_+)}{2}. \tag{3.12}$$

Therefore via (3.3), (3.7), (3.10) and (3.12), we have in region I that

$$\breve{u}(\eta, t) = \left[\frac{(u_- - u_+)}{2} \operatorname{erfc}\left(\frac{\eta}{2}\right) + u_+\right] + o(1), \quad -\infty < \eta < \infty, \tag{3.13}$$

as $t \to 0$. We can easily see when $u_- > u_+$ expansion in region I being given by (3.13) while when $u_+ > u_-$ and $u_+ = 1$, $u_- = 0$ expansion in region I being given by (2.16). From (3.13), we observe that

$$u(\eta, t) = \begin{cases} u_- - \frac{(u_- - u_+)}{\sqrt{\pi}} \frac{e^{\frac{-\eta^2}{4}}}{(-\eta)} + ..., & \text{for } -\eta \gg 1, \quad t \to 0, \\ u_+ + \frac{(u_- - u_+)}{\sqrt{\pi}} \frac{e^{\frac{-\eta^2}{4}}}{\eta} + ..., & \text{for } \eta \gg 1, \quad t \to 0. \end{cases} \tag{3.14}$$

As $\eta \to \infty$ we move into region II$^+$, where $x = O(1)(> 0)$ as $t \to 0$. The form of expansion (3.13) for $\eta \gg 1$ given by $(3.14)_2$ suggests that in region II$^+$ we write

$$u(x, t) = u_+ + e^{-\widehat{F}(x,t)}, \tag{3.15}$$

as $t \to 0$, where $x = O(1)(> 0)$ and

$$\widehat{F}(x, t) = \widehat{f}_0(x)t^{-1} + \widehat{f}_1(x) \ln t + \widehat{f}_2(x) + o(1), \tag{3.16}$$

where $\widehat{f}_0(x) > 0$ for $x > 0$ and $\widehat{f}_1(x)$, $\widehat{f}_2(x)$ are functions to be determined. On substituting (3.15) and (3.16) into equation (3.1), we obtain after some calculation, [2]

$$\left(\frac{\widehat{f}_0}{t^2} - \frac{\widehat{f}_1}{t}\right) - u_+\left(\frac{\widehat{f}_0'}{t} + \widehat{f}_1' \ln t + \widehat{f}_2'\right) + \left(\frac{\widehat{f}_0''}{t} + \widehat{f}_1'' \ln t + \widehat{f}_2''\right)$$
$$-\left(\frac{(\widehat{f}_0')^2}{t^2} + 2\widehat{f}_0'\widehat{f}_1'\frac{\ln t}{t} + 2\frac{\widehat{f}_0'\widehat{f}_2'}{t}\right) - \left((\widehat{f}_1')^2(\ln t)^2 + 2\widehat{f}_1'\widehat{f}_2' \ln t + (\widehat{f}_2')^2\right) \sim 0,$$

and solving at each order in turn, we find (after matching with (3.13) as $x \to 0^+$) that

$$\begin{aligned} \widehat{f}_0(x) &= \tfrac{x^2}{4}, \\ \widehat{f}_1(x) &= -\tfrac{1}{2}, \\ \widehat{f}_2(x) &= \ln(x) - \ln \tfrac{(u_- - u_+)}{\sqrt{\pi}} - \tfrac{u_+}{2}x. \end{aligned} \qquad (3.17)$$

Therefore, the solution in region II$^+$ is given by

$$u(x,t) = u_+ + \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t + \frac{u_+ x}{2} - \ln(x) + \ln\frac{(u_- - u_+)}{\sqrt{\pi}} + o(1)\right) \qquad (3.18)$$

as $t \to 0$, with $x = O(1)$ $(> 0)$.

As $\eta \to -\infty$ we move into region II$^-$, where $x = O(1)(< 0)$ as $t \to 0$. The form of expansion (3.13) for $(-\eta) \gg 1$ given by $(3.14)_1$ suggests that in region II$^-$ we write

$$u(x,t) = u_- - e^{-\widehat{G}(x,t)}, \qquad (3.19)$$

as $t \to 0$, where $x = O(1)(< 0)$ and

$$\widehat{G}(x,t) = \widehat{g}_0(x)t^{-1} + \widehat{g}_1(x)\ln t + \widehat{g}_2 + o(1), \qquad (3.20)$$

[2] $u_x = \widehat{F}_x e^{\widehat{F}}$ $\quad u_{xx} = \widehat{F}_{xx}e^{\widehat{F}} + \widehat{F}_x^2 e^{\widehat{F}}$ $\quad u_t = \widehat{F}_t e^{\widehat{F}}$

where $\widehat{g}_0(x) > 0$ for $x < 0$ and $\widehat{g}_1(x)$, $\widehat{g}_2(x)$ are functions to be determined. On substituting (3.19) and (3.20) into equation (3.1), gives ( on solving at each order in turn and matching to region I as $x \to 0^-$) that [3],

$$\begin{aligned} \widehat{g}_0(x) &= \tfrac{x^2}{4}, \\ \widehat{g}_1(x) &= -\tfrac{1}{2}, \\ \widehat{g}_2(x) &= -\tfrac{u_- x}{2} + \ln(-x) - \ln\tfrac{(u_- - u_+)}{\sqrt{\pi}}. \end{aligned} \tag{3.21}$$

Therefore the solution in region II$^-$ is given by

$$u(x,t) = u_- - \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t + \frac{u_- x}{2} - \ln(-x) + \ln\frac{(u_- - u_+)}{\sqrt{\pi}} + o(1)\right), \tag{3.22}$$

as $t \to 0$, with $x = O(1)$ $(< 0)$.

The asymptotic structure as $t \to 0$ is now complete with the expansions in regions I, II$^-$ and II$^+$, providing a uniform approximation to the solution of IVP2 as $t \to 0$.

## 3.2 Asymptotic Solution of IVP2 as $|x| \to \infty$

Now, we examine the asymptotic structure of the solution to IVP2 as $|x| \to \infty$ with $t = O(1)$. We first consider the structure of solution to IVP2 as $x \to \infty$, with $t = O(1)$. The form expansion (3.18) of region II$^+$ for $x \gg 1$ as $t \to 0$ suggests that in this region, which we label as region III$^+$, we write,

$$u(x,t) = u_+ + e^{-L(x,t)}, \tag{3.23}$$

[3] $u_x = \widehat{G}_x e^{\widehat{G}}$ $\quad u_{xx} = \widehat{G}_{xx} e^{\widehat{G}} + \widehat{G}_x^2 e^{\widehat{G}}$ $\quad u_t = \widehat{G}_t e^{\widehat{G}}$

as $x \to \infty$ with $t = O(1)$, where

$$L(x,t) = l_0(t)x^2 + l_1(t)x + l_2(t)\ln x + l_3(t) + o(1), \tag{3.24}$$

where $l_0(t) > 0$. $l_1(t)$, $l_2(t)$ and $l_3(t)$ are functions to be determined. On substituting (3.23) and (3.24) into equation (3.1), we obtain after some calculation that [4],

$$\left(\dot{l_0}x^2 + \dot{l_1}x + \dot{l_2}\ln x + \dot{l_3}\right) + u_+\left(2l_0x + l_1 + l_2\frac{1}{x}\right) + \left(-2l_0 + \frac{l_2}{x^2} + 4l_0^2x^2 + 4l_0l_1x + 4l_0l_2 + l_1^2 + \frac{2l_1l_2}{x} + \frac{l_2^2}{x^2}\right) \sim 0$$

and solving at each order in turn, we find (after matching with region II$^+$ as $t \to 0$) that

$$\begin{aligned} l_0(t) &= \tfrac{1}{4t}, \\ l_1(t) &= \tfrac{u_+}{2}, \\ l_2(t) &= 1 \\ l_3(t) &= \tfrac{u_+^2}{4}t - \tfrac{\ln t}{2} - \ln\tfrac{(u_- - u_+)}{\sqrt{\pi}}. \end{aligned} \tag{3.25}$$

Therefore the solution in region III$^+$ is given by

$$u(x,t) = u_+ + \exp\left(-\frac{x^2}{4t} + \frac{u_+x}{2} - \ln(x) + \left(-\frac{u_+^2}{4}t + \frac{1}{2}\ln t + \ln\frac{(u_- - u_+)}{\sqrt{\pi}}\right) + o(1)\right) \tag{3.26}$$

as $x \to \infty$, with $t = O(1)$. Expansion (3.26) remains uniform for $t \gg 1$ provided that $x \gg t$, but become non-uniform when $x = O(t)$ as $t \to \infty$.

We now investigate the structure of solution of IVP2 as $x \to -\infty$ with $t = O(1)$, which we label as region III$^-$. The form expansion (3.22) of region II$^-$ suggests that in

[4] $u_x = L_xe^L \quad u_{xx} = L_{xx}e^L + L_x^2e^L \quad u_t = L_te^L$

region III$^-$ we write,

$$u(x,t) = u_- - e^{-\widehat{L}(x,t)}, \tag{3.27}$$

as $x \to -\infty$ with $t = O(1)$, where

$$\widehat{L}(x,t) = \widehat{l}_0(t)x^2 + \widehat{l}_1(t)x + \widehat{l}_2(t)\ln(-x) + \widehat{l}_3(t) + o(1) \tag{3.28}$$

$\widehat{l}_0(t) > 0$, $\widehat{l}_1(t)$, $\widehat{l}_2(t)$ and $\widehat{l}_3(t)$ are functions to be determined. On substituting (3.27) and (3.28) into equation (3.1), we obtain after some calculation [5], that

$$\begin{aligned}&\left(\dot{\widehat{l}}_0 x^2 + \dot{\widehat{l}}_1 x + \dot{\widehat{l}}_2 \ln(-x) + \dot{\widehat{l}}_3\right) + u_-\left(2\widehat{l}_0 x + \widehat{l}_1 - \widehat{l}_2\frac{1}{x}\right)\\ &+\left(-2\widehat{l}_0 - \widehat{l}_2\frac{1}{x^2} + 4\widehat{l}_0^2x^2 + 4\widehat{l}_0\widehat{l}_1x - 4\widehat{l}_0\widehat{l}_2 + \widehat{l}_1^2 - 2\widehat{l}_1\widehat{l}_2\frac{1}{x} + \widehat{l}_2^2\frac{1}{x^2}\right) \sim 0\end{aligned} \tag{3.29}$$

and solving at each order in turn,we find (after matching with region II$^+$ as $t \to 0$) that

$$\begin{aligned}\widehat{l}_0(t) &= \tfrac{1}{4t},\\ \widehat{l}_1(t) &= -\tfrac{u_-}{2},\\ \widehat{l}_2(t) &= 1,\\ \widehat{l}_3(t) &= \tfrac{u_-^2}{4}t - \tfrac{\ln t}{2} - \ln\tfrac{(u_- - u_+)}{\sqrt{\pi}}.\end{aligned} \tag{3.30}$$

Therefore the solution in region III$^-$ is given by

$$u(x,t) = u_- - \exp\left(-\frac{x^2}{4t} + \frac{u_- x}{2} - \ln(-x) + \left(\frac{1}{2}\ln t - \frac{u_-^2 t}{4} + \ln\frac{(u_- - u_+)}{\sqrt{\pi}}\right) + o(1)\right) \tag{3.31}$$

as $x \to -\infty$, with $t = O(1)$. Expansion (3.31) remains uniform for $t \gg 1$ provided that

[5] $u_x = \widehat{L}_x e^{\widehat{L}}$ $u_{xx} = \widehat{L}_{xx}e^{\widehat{L}} + \widehat{L}_x^2 e^{\widehat{L}}$ $u_t = \widehat{L}_t e^{\widehat{L}}$

$(-x) \gg t$, but becomes non-uniform when $(-x) = O(t)$ as $t \to \infty$.

## 3.3 Asymptotic Solution to IVP2 as $t \to \infty$

The asymptotic expansions (3.26) and (3.31), which are defined in region III$^+$ ($x \to \infty, t = O(1)$) and region III$^-$ ($x \to -\infty$, $t = O(1)$) remain uniform provided $|x| \gg t$ but become nonuniform when $|x| = O(t)$. We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we define a new region, which we label region IV$^+$. To investigate region IV$^+$, we introduce the scaled coordinate:

$$y = \frac{x}{t} \tag{3.32}$$

where $y = O(1)$ as $t \to \infty$. On substituting expansion (3.32) into expansion (3.26), we find as we move into region IV$^+$ when $x = O(t)$ that

$$u(y,t) = u_+ + \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - \ln y + \ln\frac{(u_- - u_+)}{\sqrt{\pi}} + o(1)\right\} . \tag{3.33}$$

The form of (3.33) suggests that in region IV$^+$ we write

$$u(y,t) = u_+ + e^{-C(y,t)} \tag{3.34}$$

where

$$C(y,t) = c_0(y)t + c_1(y)\ln t + c_2(y) + o(1), \tag{3.35}$$

as $t \to \infty$ with $y = O(1)$, and where $c_0(y) > 0$, $c_1(y)$ and $c_2(y)$ are functions to be determined. On substituting (3.34) and (3.35) into equation (3.1) (when written in terms of $y$ and $t$), we obtain the leading order problem as

$$(c_0')^2 + (u_+ - y)c_0' + c_0 = 0, \quad y > 0 \tag{3.36}$$

$$c_0(y) > 0, \quad y > 0, \tag{3.37}$$

$$c_0(y) \sim \frac{(y - u_+)^2}{4} \quad \text{as} \quad y \to \infty. \tag{3.38}$$

Equation (3.36) has a one-parameter family of linear solutions

$$c_0(y) = A[y - (u_+ + A)] \quad y > 0, \tag{3.39}$$

where $A > 0$ together with the associated envelope solution,

$$c_0(y) = \frac{(y - u_+)^2}{4}, \quad y > u_+. \tag{3.40}$$

Combinations of (3.39) and (3.40) which remain continuous and differentiable also provide solutions to (3.36) (envelope touching solutions). Applying condition (3.38) requires us to select the solution

$$c_0(y) = \frac{(y - u_+)^2}{4}, \quad y > u_+ \tag{3.41}$$

or

$$c_0(y) = \begin{cases} \dfrac{(y - u_+)^2}{4}, & y > u_+ + 2A \\ A[y - (u_+ + A)], & u_+ + A < y \leqslant u_+ + 2A \end{cases} \tag{3.42}$$

for each $A > 0$. Each case will have to be considered separately.

(a) We begin by considering the case when $c_0(y)$ is given by (3.41), where

$$c_0(y) = \frac{(y-u_+)^2}{4}, \quad y > u_+. \tag{3.43}$$

Continuing expansion (3.34) and (3.35) in region IV$^+$ gives on solving at each order in turn, and matching with region III$^+$ as $y \to \infty$ that

$$\begin{aligned} c_1(y) &= \tfrac{1}{2}, \\ c_2(y) &= H(y), \end{aligned} \tag{3.44}$$

where the function $H(y)$ remains undetermined. However, matching with the far field requires that

$$H(y) \sim \ln y - \ln \frac{(u_- - u_+)}{\sqrt{\pi}} \quad \text{as} \quad y \to \infty,$$

and we make the assumption (which we will verify as consistent) that,

$$H(y) \sim \beta_0 \quad \text{as} \quad y \to (u_+)^+, \tag{3.45}$$

where $\beta_0$ is constant. Therefore, we obtain in region IV$^+$ that;

$$u(y,t) = u_+ + \exp\left\{ -\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H(y) + o(1) \right\}, \tag{3.46}$$

as $t \to \infty$ with $y = O(1)(\in (u_+, \infty))$. Expansion (3.46) becomes nonuniform when $y = u_+ + O(t^{-1/2})$ as $t \to \infty$ [that is, when $x = u_+ t + O(t^{1/2})$] and we must introduce a further region, region A. To examine region A we introduce the scaled coordinate

$$\eta = (y - u_+)t^{1/2} = O(1)$$

and look for an expansion of the form

$$u(\eta, t) = u_+ + \Psi(t)U(\eta) + o(\Psi(t)) \tag{3.47}$$

as $t \to \infty$ with $\eta = O(1)$, and where $\Psi(t) = o(1)$ as $t \to \infty$ is a gauge function to be determined. Substitution of (3.47) into equation (3.1) (when written in terms of $\eta$ and $t$) indicates that the most structured balance requires that $\Psi(t) = t^{-1/2}$ as $t \to \infty$, giving at leading order that

$$U_{\eta\eta} - UU_\eta + \frac{\eta}{2}U_\eta + \frac{U}{2} = 0, \quad -\infty < \eta < \infty. \tag{3.48}$$

Equation (3.48) is to be solved subject to matching to region IV$^+$ as $\eta \to \infty$, that is,

$$U(\eta) \sim e^{-\beta_0}e^{-\eta^2/4} \quad \text{as} \quad \eta \to \infty. \tag{3.49}$$

We recall from Section 1.1.3 that the solution of (3.49) with $U, \ \ U_\eta \to 0$ as $\eta \to \infty$ is given by (3.8). Further, we require that $D_2 > \sqrt{\pi}$ in this case (as $U \to 0^+$ as $\eta \to \infty$). Therefore, in region A we have that

$$u(\eta, t) = u_+ + \frac{2e^{-\eta^2/4}}{D_2 - \sqrt{\pi}\mathrm{erf}\left(\frac{\eta}{2}\right)}t^{-1/2} + o(t^{-1/2}) \tag{3.50}$$

as $t \to \infty$ with $\eta = O(1)$. Now as $\eta \to -\infty$ we move out of region A into region V, where $y = O(1)(\in (-\infty, u_+))$. From (3.49) we have

$$u(y, t) \sim u_+ + \frac{2}{D_2 - \sqrt{\pi}}e^{-\eta^2/4} \tag{3.51}$$

with $(-\eta) \gg 1$. When written in terms of $y$, (3.51) becomes

$$u(y,t) \sim u_+ + \frac{2}{D_2 - \sqrt{\pi}} e^{-\frac{(y-u_+)^2}{4}t} t^{-1/2}. \tag{3.52}$$

Thus, in region V we look for an expansion of the form

$$u(y,t) = u_+ + e^{-N(y,t)}, \tag{3.53}$$

with

$$N(y,t) = n_0(y)t + n_1(y)\ln t + n_2(y) + o(1) \tag{3.54}$$

where $y = O(1)(\in (-\infty, u_+))$ as $t \to \infty$, and $n_0(y) > 0$, $n_1(y)$ and $n_2(y)$ are functions to be determined. However, there is no mechanism available to allow an expansion of the form $u(y,t) \sim u_+ + Exp$ (in region V) to match to the far field ($-y \gg 1$) where

$$u = u_- - o(1), \quad \text{as} \quad t \to \infty,$$

and we conclude that this case can be ruled out and that $c_0(y)$ is given by (3.42).

(b) We next consider the case when $c_0(y)$ is given by (3.42). In this case region IV$^+$ is replaced by three regions: region IV$^+$(a) $(u_+ + 2A + o(1) < y < \infty)$, region TR$^+$ (transition region) and region IV$^+$(b) $(u_+ + A + o(1) < y < u_+ + 2A - o(1))$. We note that the transition region located at $y = u_+ + 2A$ is required to smooth out the discontinuity of curvature at the point at which the linear solution $(3.42)_2$ meets the envelope solution $(3.42)_1$. We will examine each of these regions in turn.

We begin in region IV$^+$(b) where $c_0(y)$ is given by $(3.42)_2$. On continuing expansion (3.34), (3.35) we obtain in region IV$^+$(b) that

$$u(y,t) = u_+ + \exp\left(-A[y-(u_+ + A)]t - D_R \ln t - D_R \ln|y-(u_+ + 2A)| - E_R + o(1)\right), \tag{3.55}$$

as $t \to \infty$ with $y = O(1)(\in (u_+ + A, u_+ + 2A))$, and where $D_R$ and $E_R$ are constants. We next consider region IV$^+$(a) where $c_0(y)$ is given by $(3.42)_1$. On continuing expansions (3.34), (3.35) we obtain in region IV$^+$(a) that

$$u(y,t) = u_+ + \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H_R(y) + o(1)\right\} \tag{3.56}$$

as $t \to \infty$ with $y = O(1)(\in (u_+ + 2A, \infty))$. The function $H_R(y)$ remains undetermined but matching to the far field $(y \gg 1)$ requires that

$$H_R(y) \sim \ln y - \ln\frac{(u_- - u_+)}{\sqrt{\pi}} \quad \text{as} \quad y \to \infty,$$

and we make the assumption (which we will verify as consistent) that

$$H_R(y) \sim \alpha_1 \ln(y - (u_+ + 2A)) + \beta_1$$

as $y \to (u_+ + 2A)^+$ and $\alpha_1,\ \beta_1$ are constants to be determined. We next consider region $\mathrm{TR}^+$. An examination of expansion (3.56) as $y \to (u_+ + 2A)^+$, and of expansion (3.55) as $y \to (u_+ + 2A)^-$, reveals that in this region $y = (u_+ + 2A) + O(t^{-1/2})$ as $t \to \infty$. Thus, in region $\mathrm{TR}^+$ we introduce the scaled coordinate $\eta = (y - u_+ - 2A)t^{1/2}$, where $\eta = O(1)$ as $t \to \infty$, and expand as

$$u(\eta, t) = u_+ + [F(\eta) + o(1)]t^{\gamma} e^{-A^2 t - A\eta t^{1/2}} \tag{3.57}$$

as $t \to \infty$ with $\eta = O(1)$, where

$$\gamma = -\frac{D_R}{2} = -\frac{1}{2} + \frac{\alpha_1}{2}.$$

On substituting of expansion (3.57) into equation (3.1) (when written in terms of $\eta$ and $t$) gives at leading order that

$$F_{\eta\eta} - \frac{\eta}{2}F_{\eta} - \gamma F = 0, \quad -\infty < \eta < \infty. \tag{3.58}$$

Matching expansion (3.57) (as $\eta \to -\infty$) to expansion (3.55) (as $y \to (u_+ + A)^-$), to leading order, requires that

$$F(\eta) \sim (-\eta)^{-D_R} e^{-E_R} \quad \text{as} \quad \eta \to -\infty. \tag{3.59}$$

Matching expansion (3.57) (as $\eta \to \infty$) to expansion (3.56) (as $y \to (u_+ + 2A)^+$) requires

$$F(\eta) = O(e^{-\eta^2/4}) \quad \text{as} \quad \eta \to \infty. \tag{3.60}$$

However, equation (3.58) has no solutions which satisfy both (3.59) and (3.60) when $\gamma < 0$. We conclude that $\gamma \geqslant 0$. With $\gamma \geqslant 0$, equation (3.58) with conditions (3.59) and

(3.60) has a solution, with

$$F(\eta) \sim \widehat{A}\eta^{-2\gamma-1}e^{-\eta^2/4} \quad \text{as} \quad \eta \to \infty, \tag{3.61}$$

with $\widehat{A}$ undetermined. On matching expansion (3.57) (as $\eta \to \infty$), using (3.61), to expansion (3.56) (as $y \to (u_+ + 2A)^+$) we require, to obtain the least singular behavior in region IV$^+$(a), that

$$\gamma = 0.$$

Therefore, equation (3.58) reduces to

$$F_{\eta\eta} + \frac{\eta}{2}F_\eta = 0, \quad -\infty < \eta < \infty. \tag{3.62}$$

The solution of (3.62) is readily obtained as

$$F(\eta) = A_1 \operatorname{erfc}\left(\frac{\eta}{2}\right) + B_1, \quad -\infty < \eta < \infty. \tag{3.63}$$

Matching expansion (3.58) (with (3.63)) as $\eta \to \infty$ to expansion (3.56) (as $y \to (u_+ + 2A)^+$) requires that

$$B_1 = 0, \quad A_1 = \frac{\sqrt{\pi}}{2}e^{-\beta_1}.$$

Matching expansion (3.58) (with (3.63)) as $\eta \to -\infty$ to expansion (3.55) (as $y \to (u_+ + 2A)^-$) requires

$$D_R = 0, \quad A_1 = \frac{e^{-E_R}}{2}.$$

Therefore, we have that

$$\beta_1 = E_R + \ln\sqrt{\pi},$$

where $E_R$ is still to be determined.

Therefore, in region TR$^+$ we have that

$$u(\eta,t) = u_+ + \left(\frac{1}{2}e^{-E_R}\operatorname{erfc}\left(\frac{\eta}{2}\right) + o(1)\right)e^{-A^2t - A\eta t^{1/2}} \tag{3.64}$$

as $t \to \infty$ with $\eta = O(1)$.

As $\eta \to -\infty$ we move out of region TR$^+$ into region IV$^+$(b). We have that

$$u \sim u_+ + \left(\frac{1}{2}e^{-E_R}\left(2 - \frac{2}{(-\eta)\sqrt{\pi}}e^{-\eta^2/4} + \dots\right)\right)e^{-A^2t - A\eta t^{1/2}} \tag{3.65}$$

when $(-\eta) \gg 1$. On writing (3.65) in terms of $y$ we obtain

$$u \sim u_+ + e^{-A[y-(u_++A)]t - E_R} - \frac{e^{-E_R}t^{-1/2}}{\sqrt{\pi}((u_+ + 2A) - y)}e^{\frac{-(y-u_+)^2t}{4}}.$$

Therefore, the correction to expansion (3.55) is of $O(t^{-1/2}\exp\left\{\frac{-(y-u_+)^2}{4}t\right\})$ as $t \to \infty$, and we find, after some calculation, that in region IV$^+(b)$ we have

$$\begin{aligned} u(y,t) = u_+ + \exp\left\{-A[y-(u_++A)]t - E_R\right\} + t^{-1/2}\overline{K}_R(y)\exp\left\{-\frac{(y-u_+)^2}{4}t\right\} \\ + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_+)^2}{4}t\right\}\right) \end{aligned} \tag{3.66}$$

as $t \to \infty$ with $y = O(1)(\in (u_+ + A, u_+ + 2A))$, where $\overline{K}_R(y)$ remains undetermined, but matching to region TR$^+$ requires that

$$\overline{K}_R(y) \sim \frac{e^{-E_R}}{\sqrt{\pi}((u_+ + 2A) - y)} \quad \text{as} \quad y \to (u_+ + 2A)^- \ .$$

Now as $y \to (u_+ + A)^+$ we move out of region IV$^+(b)$ into region TW an examination of expansion (3.66) as $y \to (u_+ + A)^+$ indicates that in region TW $y = (u_+ + A) + O(t^{-1})$ and $u = O(1)$ as $t \to \infty$. To examine region TW we introduce the scaled variable $z$, via $z = (y - (u_+ + A))t$ as $t \to \infty$ and expand as

$$u(z, t) = U(z) + o(1) \tag{3.67}$$

as $t \to \infty$ with $z = O(1)$. On substituting (3.67) into equation (3.1) (when written in terms of $z$ and $t$) we obtain at leading order that

$$U_{zz} - UU_z + (A + u_+)U_z = 0, \quad -\infty < z < \infty.$$

Equation (3.67) has been considered in Section 1.1.1. We recall that the solution of (3.67) subject to

$$U(z) \to u_+, \quad \text{as} \quad z \to \infty$$

and

$$U(z) \quad \text{remaining bounded} \quad \text{as} \quad z \to -\infty$$

is given by

$$U(z) = \frac{u_+ + (2A + u_+)e^{-Az}}{1 + e^{-Az}}, \quad -\infty < z < \infty, \tag{3.68}$$

where the translational invariance has been fixed by requiring $U(0) = 1/2$, (we note that there is no loss of generality in this as we shall see later). Therefore, in region TW we have at leading order in expansion (3.67) the travelling wave solution of speed $c = u_+ + A$, where the constant $A$ ($A = \frac{1}{2}e^{-E_R} = A_1$) remains undetermined at this stage of analysis. We will return, and complete, region TW later in analysis. We note, via (3.67) and (3.68), that

$$U(z) \sim (2A + u_+) - 2Ae^{Az} \quad \text{as} \quad z \to -\infty. \tag{3.69}$$

We next consider the asymptotic structure as $t \to \infty$ for $x < 0$. To proceed we define a new region, which we label as region IV$^-$, where $y = O(1)(< 0)$ as $t \to \infty$. The details of region IV$^-$ follow, after some minor modification, those for region IV$^+$ and are not repeated here. Therefore, region IV$^-$ is replaced by three regions: region IV$^-$(a) ($-\infty < y < u_- + 2\overline{A} - o(1)$), region TR$^-$ (transition region) and region IV$^-$(b) ($u_- + 2\overline{A} + o(1) < y < u_+ + \overline{A} - o(1)$). The constant $\overline{A}$ remains to be determined. The details of regions IV$^-$(a) and IV$^-$(b) are summarized here for brevity.

Region IV$^-$(b)

$$u(y,t) = u_- - \exp\left\{-\overline{A}[y - (u_- + \overline{A})]t - D_L \ln t - D_L \ln|y - (u_- + 2\overline{A})| - E_L + o(1)\right\} \tag{3.70}$$

as $t \to \infty$ with $y = O(1)(\in (u_- + 2\overline{A}, u_- + \overline{A}))$ and where $D_L$ and $E_L$ are constants.

Region IV$^-$($a$)

$$u(y,t) = u_- - \exp\left\{-\frac{(y - u_-)^2}{4}t - \frac{1}{2}\ln t - H_L(y) + o(1)\right\} \tag{3.71}$$

as $t \to \infty$ with $y = O(1)(\in (-\infty, u_- + 2\overline{A}))$ and where the function $H_L(y)$ remains undetermined. However, matching to far field ($-y \gg 1$) requires

$$H_L(y) \sim \ln(-y) - \ln\frac{(u_- - u_+)}{\sqrt{\pi}} \quad \text{as} \quad y \to -\infty.$$

Finally, we examine region TR$^-$, the details of which follow, after minor modification, those given for region $TR^+$ and summarized here for brevity. Therefore, in region $TR^-$ we have that

$$u(\eta, t) = u_- - e^{-E_L}\left(1 - 1/2\,\mathrm{erfc}\left(\frac{\eta}{2}\right)\right) e^{-\overline{A}^2 t - \overline{A}\eta t^{1/2}} \tag{3.72}$$

as $t \to \infty$ where $\eta = [y - (u_- + 2\overline{A})]t^{1/2} = O(1)$. Matching to regions $IV^-$ (b)(as $\eta \to \infty$) and $IV^-$(a) (as $\eta \to -\infty$) requires

$$D_L = 0, \quad H_L(y) \sim \ln|y - (u_- - 2\overline{A})| + \beta_2 \quad \text{as} \quad y \to (u_- + 2\overline{A})^-$$

with $\beta_2 = E_L + \ln\sqrt{\pi}$, where $E_L$ and $\overline{A}$ remain undetermined. As $\eta \to \infty$ we move out of region $TR^-$ into region $IV^-$(b). The form of (3.72) ( for $\eta \gg 1$) indicates that the correction to expansion (3.70) is of $O\left(t^{-1/2}\exp\left\{-\frac{(y-u_-)^2}{4}t\right\}\right)$ as $t \to \infty$, and we find, after some calculation, that in region $IV^-$(b) we have

$$u(y,t) = u_- - \exp\left\{-\overline{A}[y - (u_- + \overline{A})]t - E_L\right\} + t^{-1/2}\overline{K}_L(y)\exp\left\{-\frac{(y-u_-)^2}{4}t\right\} + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_-)^2}{4}t\right\}\right) \tag{3.73}$$

as $t \to \infty$ with $y = O(1)(\in (u_- + 2\overline{A}, u_- + \overline{A}))$ and where $\overline{K}_L(y)$ remains undetermined, but matching with region $TR^-$ requires that

$$\overline{K}_L(y) \sim \frac{e^{-E_L}}{\sqrt{\pi}(y - (u_- + 2\overline{A}))} \quad \text{as} \quad y \to (u_- + 2\overline{A})^+ \;.$$

We note that as $y \to (u_- + \overline{A})^-$ we move out of region $IV^-$(b) into region TW. Matching expansion (3.67) ( with (3.69)) as $z \to -\infty$ with expansion (3.73) ( as $y \to (u_- + \overline{A})^-$) at leading order requires that

$$A = \frac{u_- - u_+}{2} \;. \tag{3.74}$$

We recall that the wave speed of the travelling wave in region TW is given by

$$c = u_+ + A = \frac{u_+ + u_-}{2}\ . \tag{3.75}$$

We note that

$$c \begin{cases} > 0 \quad \text{when} \quad u_- > u_+ > -u_- \quad \text{with} \quad u_- > 0 \\ \quad \Longrightarrow \text{Travelling wave moves in } +x \text{ direction} \\ < 0 \quad \text{when} \quad u_+ < u_- < -u_+ \quad \text{with} \quad u_+ < 0 \\ \quad \Longrightarrow \text{Travelling wave moves in } -x \text{ direction} \\ = 0 \quad \text{when} \quad u_+ = -u_- \quad \text{with} \quad u_- > 0 \\ \quad \Longrightarrow \text{Stationary wave profile.} \end{cases}$$

We further note on comparing the structure of expansion (3.73) ( as $y \to (u_- + \overline{A})^-$ ) with expansion (3.67) (as $z \to -\infty$) that

$$A = -\overline{A}. \tag{3.76}$$

We conclude the analysis in this case by completing region TW. However, before completing region TW we summarize regions IV$^{\pm}$(b).

Region IV$^-$(b)

$$u(y,t) = u_- - \exp\left\{A[y-c]t - E_L\right\} + t^{-1/2}\overline{K}_L(y)\exp\left\{-\frac{(y-u_-)^2}{4}t\right\} + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_-)^2}{4}t\right\}\right) \tag{3.77}$$

as $t \to \infty$ with $y = O(1)(\in (u_+, c))$ where $c = \frac{u_+ + u_-}{2}$ and $A = \frac{u_- - u_+}{2}$.

Region IV$^+(b)$

$$u(y,t) = u_+ + \exp\{A[y-c]t - E_R\} + t^{-1/2}\overline{K}_R(y)\exp\left\{-\frac{(y-u_+)^2}{4}t\right\} \\ + o\left(t^{-1/2}\exp\left\{-\frac{(y-u_+)^2}{4}t\right\}\right) \quad (3.78)$$

as $t \to \infty$ with $y = O(1)(\in (c, u_-))$.

In region TW, $x \sim s(t)$ and $u$ ( when written in terms of the travelling wave coordinate, $z$ ) has the form ( via (3.67), (3.68) and (3.69))

$$u(z,t) = \frac{u_+ + u_- e^{-Az}}{1 + e^{-Az}} + o(1), \quad (3.79)$$

as $t \to \infty$ with $z = O(1)$ and where $A = \frac{u_- - u_+}{2}$, $z = x - s(t)$ and $s(t) = ct + \widehat{\phi}(t) + \phi_0 + \widehat{\psi}(t)$ as $t \to \infty$. Here $1 \ll \widehat{\phi}(t) \ll t$, $\phi_0$ is a constant and $\widehat{\psi}(t) = o(1)$ as $t \to \infty$ are as yet undetermined gauge functions. We note that (3.79) represents at leading order a permanent from travelling wave solution with speed $c = \frac{u_+ + u_-}{2}$. We further note from (3.79) that

$$u(z,t) = \begin{cases} u_+ + (u_- - u_+)e^{-Az} + O(e^{-2Az}) & \text{as} \quad z \to \infty, \\ u_- - (u_- - u_+)e^{Az} + O(e^{2Az}) & \text{as} \quad z \to -\infty. \end{cases} \quad (3.80)$$

On matching expansion (3.79) (with $(3.80)_1$) with expansion (3.78) (as $y \to c^+$) to $(3.80)_1$ leading order fixes

$$\widehat{\phi}(t) \equiv 0$$

and requires

$$E_R + A\phi_0 = -\ln(u_- - u_+).$$

Since $A = \frac{1}{2}e^{-E_R} = \frac{(u_- - u_+)}{2}$ $(\neq 0)$ we have that $\phi_0 = 0$. Matching expansion (3.79) (with $(3.80)_2$) with expansion (3.77) (as $y \to c^-$) requires

$$E_L - A\phi_0 = -\ln(u_- - u_+).$$

We now make the assumption (which will verify as consistent) that

$$\overline{K}_R(y) \sim \overline{K}_c(y-c)^{\widehat{\gamma}} \quad \text{as} \quad y \to c^+, \tag{3.81}$$

where $\overline{K}_c$ and $\widehat{\gamma}$ are constants. On writing (3.78) in terms of $z$

$$u \sim u_+ + (u_- - u_+)e^{-Az}\left(1 - A\widehat{\psi}(t) + \dots\right) + t^{-\widehat{\gamma}-1/2}\widehat{K}_c z^{\widehat{\gamma}} e^{-A/2[z+\phi_0]} e^{-\frac{(u_- - u_+)^2}{16}t} + \dots \tag{3.82}$$

as $t \to \infty$ with $z \gg 1$. We conclude that in region TW we must have

$$u(z,t) = \frac{u_+ + u_- e^{-Az}}{1 + e^{-Az}} + O(\widehat{\psi}(t)), \tag{3.83}$$

as $t \to \infty$ with $z = O(1)$. On substituting (3.83) into equation (3.1) ( when written in terms of $z$ and $t$ ) we require to obtain a nontrivial balance at $O(\widehat{\psi}'(t))$ that $\widehat{\psi}'(t) = O(\widehat{\psi}(t))$ as $t \to \infty$ and we conclude that $\widehat{\psi}(t)$ must be exponential in $t$ as $t \to \infty$ and we write

$$\widehat{\psi}(t) = \mu t^{\delta} e^{-\lambda t}$$

as $t \to \infty$, where $\mu,\ \delta,\ \lambda\ (>0)$ are to be determined. We now continue the expansion in region TW as

$$u(z,t) = U(z) + U_1(z)\widehat{\psi}(t) + o(\widehat{\psi}(t)) \tag{3.84}$$

as $t \to \infty$ with $z = O(1)$ where $U(z)$ is given by (3.68). On solving at $O(\widehat{\psi}'(t))$ we obtain

$$U_1'' + U_1'(c - U) + U_1(\lambda - U') = \lambda U'. \tag{3.85}$$

Therefore, via (3.85) and (3.68), we have that

$$U_1(z) \sim \begin{cases} (l_0 + l_1 z)e^{-\frac{A}{2}z} - A(u_- - u_+)e^{-Az} & \text{if} \quad \lambda = A^2/4 \\ l_0 e^{m_+ z} + l_1 e^{m_- z} - A(u_- - u_+)e^{-Az} & \text{if} \quad 0 < \lambda < A^2/4 \end{cases}$$

as $z \to \infty$ where $l_0$ and $l_1$ are constants and $m_\pm = -\frac{A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4\lambda}$. We note that $\lambda > A^2/4$ would lead to $U_1(z)$ being oscillatory and is excluded to allow matching with (3.82). Matching (3.84) ( as $z \to \infty$ ) with (3.82) requires that

$$\lambda = \frac{A^2}{4} = \frac{(u_- - u_+)^2}{16}, \quad \widehat{\gamma} = 1, \quad \delta = -\frac{3}{2}, \quad l_0 = 0$$

and

$$\mu = \frac{\overline{K}_c}{l_1} e^{-\frac{A\phi_0}{2}}.$$

In summary, we have in region TW, that

$$u(z,t) = \frac{u_+ + u_- e^{-Az}}{1 + e^{-Az}} + O(t^{-3/2} e^{-\frac{A^2}{4}t}) \tag{3.86}$$

as $t \to \infty$ with $z = o(1)$ where $A = \frac{u_- - u_+}{2}$, $z = x - s(t)$ and

$$s(t) = ct + O\left(t^{-3/2} e^{-\frac{A^2}{4}t}\right) \tag{3.87}$$

as $t \to \infty$. We recall that

$$c = \frac{u_+ + u_-}{2} \begin{cases} > 0 \quad \text{when} \quad u_- > u_+ > -u_- \quad \text{with} \quad u_- > 0 \\ < 0 \quad \text{when} \quad u_+ < u_- < -u_+ \quad \text{with} \quad u_+ < 0 \\ = 0 \quad u_+ = -u_- \quad \text{with} \quad u_- > 0 \end{cases}$$

Therefore, we have established that when:

(i) ($u_- > u_+ > -u_-$ with $u_- > 0$) or ($u_+ < u_- < -u_+$ with $u_+ < 0$). A permanent form travelling wave develops as $t \to \infty$ in the solution of IVP2. The speed of the travelling is $c = \frac{u_+ + u_-}{2}$ which is either negative (when $u_+ < u_- < -u_+$ with $u_+ < 0$) or positive (when $u_- > u_+ > -u_-$ with $u_- > 0$). Further, we have determined the asymptotic correction to the wave speed

$$\dot{s}(t) = c + O\left(t^{-3/2} e^{-\frac{A^2}{4} t}\right)$$

as $t \to \infty$ where $A = \frac{u_- - u_+}{2}$, together with the rate of convergence of the solution to IVP2 onto the travelling wave as $t \to \infty$ which is of $O\left(t^{-3/2} e^{-\frac{A^2}{4} t}\right)$ as $t \to \infty$.

(ii) $u_+ = -u_-$ with $u_- > 0$. A stationary profile located at $x = 0$ develops in the solution of IVP2 as $t \to \infty$. The rate of convergence of the solution to IVP2 onto the stationary profile is of $O\left(t^{-3/2} e^{-\frac{A^2}{4} t}\right)$ as $t \to \infty$ where $A = \frac{u_- - u_+}{2}$.

This then completes the asymptotic structure in this case with regions IV$^+$(a), TR$^+$, IV$^+$(b), TW, IV$^-$(b) TR$^-$ and IV$^-$(a) providing a uniform approximation to the solution of IVP2 as $t \to \infty$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 3.1.

When $u_- > u_+$ a permanent form travelling wave develops as $t \to \infty$ and a schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given

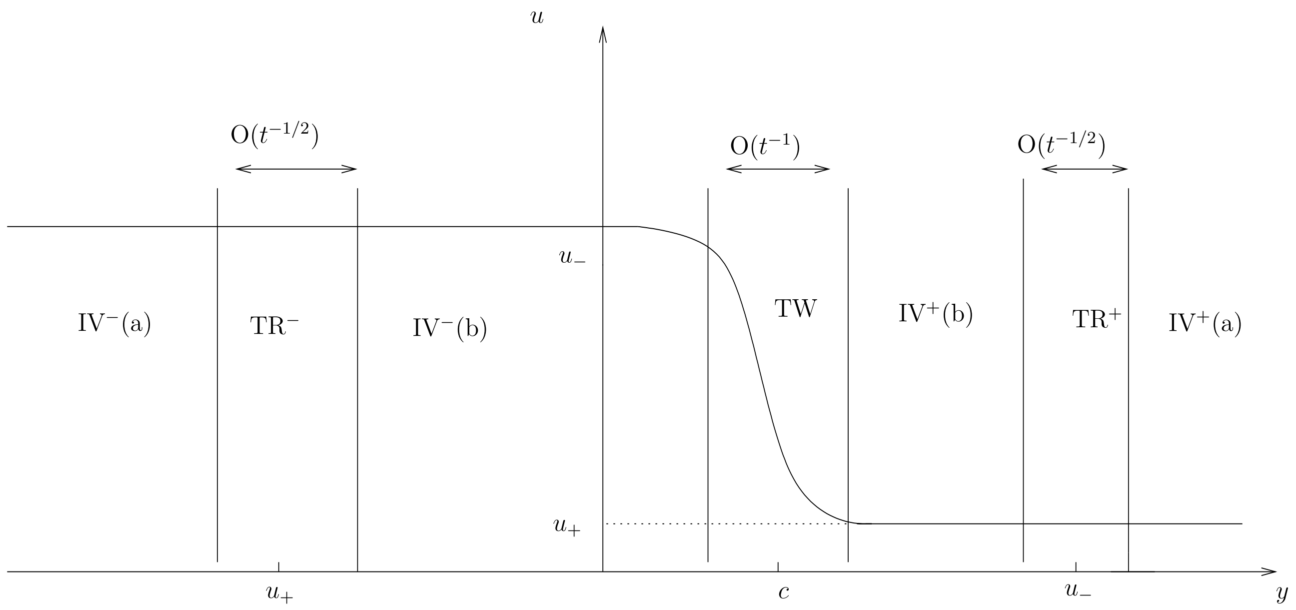


Figure 3.1: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane, as $t \to \infty$. We recall that $c = \frac{(u_+ + u_-)}{2}$.

in Figure 3.1 while when $u_- < u_+$ and $u_+$, $u_-$ are chosen 1, 0 respectively an expansion wave develops in the expansion region, region V, where $y = O(1)(\in (0,1))$ as $t \to \infty$ and a schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 2.1.

## 3.4 Numerical Solutions

In this section representative numerical solutions of IVP2 are presented which confirm the analysis presented in this chapter. We solved IVP2 using the numerical method outlined in [35]. In order to obtain numerical solutions of IVP2 we use a parabolic method with $N = 100$ where $N$ is the number of grid points time step $\triangle t = 0.001$ and the length $\triangle x = 0.005$. Before presenting the numerical solutions we briefly summarize the method given in [35] we begin by writing equation (3.1) in the form

$$u_t + [g(u)]_x = u_{xx} \quad \text{with} \quad g(u) = \frac{u^2}{2}. \tag{3.88}$$

On integrating equation (3.88) from $x_{j-1/2}$ to $x_{j+1/2}$ we obtain that

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u_t \; dx - [u_x]_{x_{j-1/2}}^{x_{j+1/2}} = -[g(u)]_{x_{j-1/2}}^{x_{j+1/2}} \tag{3.89}$$

and we approximate the terms in (3.89) as follows

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u_t \; dx \approx \frac{du}{dt}(x_j, t) \triangle x, \tag{3.90}$$

$$-[u_x]_{x_{j-1/2}}^{x_{j+1/2}} = \Big[u_x(x_{j-1/2}, t) - u_x(x_{j+1/2}, t)\Big] \approx \left[\frac{u(x_j, t) - u(x_{j-1}, t)}{\triangle x} - \frac{u(x_{j+1}, t) - u(x_j, t)}{\triangle x}\right]$$
$$= -\frac{u(x_{j+1,t}) - 2u(x_j, t) + u(x_{j-1}, t)}{\triangle x},$$

$$\text{and} \tag{3.91}$$

$$-[g(u)]_{x_{j-1/2}}^{x_{j+1/2}} = g(u(x_{j-1/2}, t) - u(x_{j+1/2}, t)).$$

On substituting (3.90) into (3.89) we obtain as (dividing by $\triangle x$ )

$$\frac{dU_j}{dt} - \frac{U_{j+1} - 2U_j + U_{j-1}}{\triangle x^2} = \frac{g(U_{j-1/2}) - g(U_{j+1/2})}{\triangle x} \tag{3.92}$$

where $U_j(t)$ represents $u(x_j, t)$ and $g(U_{j\pm1/2})$ is average of $g(U_j)$ and $g(U_{j\pm1})$ . Following [35] we then discretize the time derivative in (3.91) by forward difference to obtain

$$U_j{}^{n+1} = U_j{}^n + \triangle t \left( \frac{U_{j+1}^n - 2U_j{}^n + U_{j-1}^n}{\triangle x^2} \right) + \frac{g(U_{j-1/2}^n) - g(U_{j+1/2}^n)}{\triangle x}$$

In this Section we consider two sets of problem parameters which illustrate the situation when the wave speed $c$ is positive and when it is negative. The two cases, we will consider are:

(i) $u_- = 1, \quad u_+ = 0$

(ii) $u_+ = -1, \quad u_- = 0$

We now consider these two cases in turn.

### 3.4.1 $\mathbf{u_- = 1, \quad u_+ = 0}$

We have established in Section 3.3 that a permanent form travelling wave (PTW) develops in the solution of IVP2 as $t \to \infty$. Further, we have established that the wave speed of this PTW, $c$, is given by

$$c = \frac{u_- + u_+}{2} = \frac{1}{2},$$

and that the rate of convergence the solution of IVP2 to the PTW as $t \to \infty$ is given by

$$u(z + s(t), t) = U_T\left(z, \frac{1}{2}\right) + O(\chi(t)) \tag{3.93}$$

as $t \to \infty$ with $z = O(1)$, where $U_T(z, \frac{1}{2})$ is the PTW (see (3.79)), and $z = x - s(t)$

$$s(t) = \frac{1}{2}t + \phi_0 + O(\chi(t)) \tag{3.94}$$

where

$$\chi(t) = t^{-\frac{3}{2}} e^{-\frac{1}{16}t} \tag{3.95}$$

as $t \to \infty$. We note that the rate of convergence is exponential in $t$ as $t \to \infty$. The asymptotic wave speed, $\dot{s}(t)$, is given by

$$\dot{s}(t) = \frac{1}{2} + O(\chi(t)) \tag{3.96}$$

as $t \to \infty$. We now present numerical evidence to support the above. In Figure 3.2 we plot the numerical solution of IVP2 against $x$ at times $t = 10$ $t = 15$ $t = 20$ and $t = 25$ clearly, the solution converges to the PTW rapidly as $t \to \infty$. This is in line with (3.92) where we expect the rate of convergence to be exponential in $t$ as $t \to \infty$. In Figure 3.3 we plot $\dot{s}(t)$ versus $t$. Clearly, the numerically calculated wave speed rapidly approaches the expected value of $\frac{1}{2}$ as $t \to \infty$. Again the rate of convergence looks to be in agreement with (3.95).

In Figure 3.4 we plot $s(t) - \frac{1}{2}t$ versus $t$. We observe that the numerically calculated curve rapidly approaches $\phi_0 = 0$ as $t \to \infty$. Finally, in Figure 3.5 we plot $\ln(t^{3/2}|\dot{s}(t) - \frac{1}{2}|)$ versus $t$. We observe that the numerically calculated curve rapidly approaches the line of gradient $-\frac{1}{16}$ in line with (3.94) as $t \to \infty$. However, numerical error grows rapidly for $t > 20$. This is an artifact of the numerical method used.

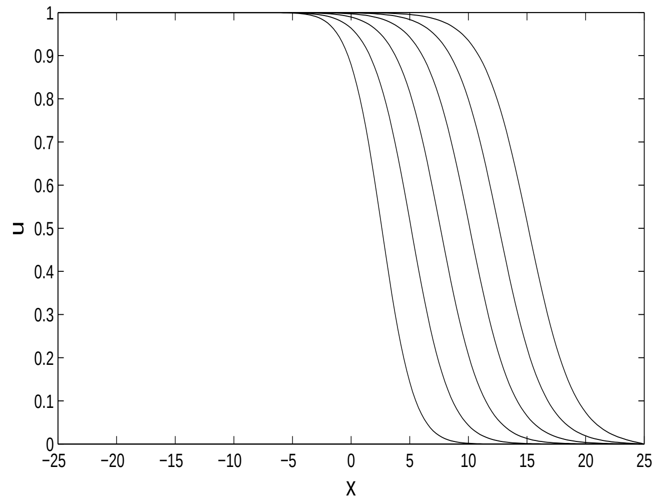


Figure 3.2: Numerical solution of IVP2 at times $t = 10$ $t = 15$ $t = 20$ and $t = 25$.

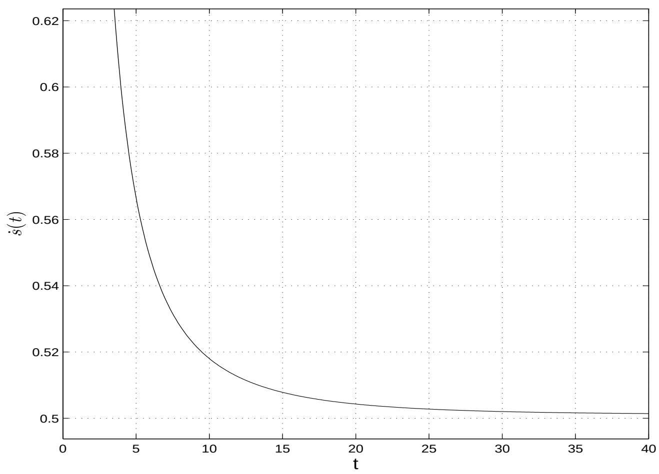


Figure 3.3: Numerical solution of $\dot{s}(t)$ versus $t$.

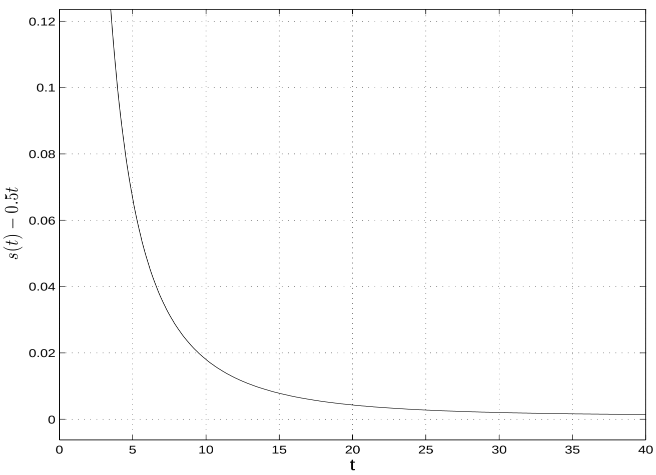


Figure 3.4: Numerical solution of $s(t) - 0.5t$ versus $t$.

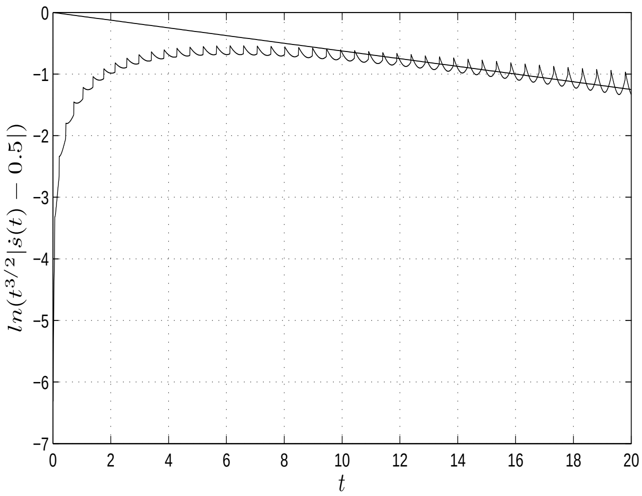


Figure 3.5: Numerical solution of $\ln(t^{3/2}|\dot{s}(t) - \frac{1}{2}|)$ versus $t$

### 3.4.2 $\mathbf{u_- = 0, \quad u_+ = -1}$

We have established in Section 3.3 that a permanent form travelling wave (PTW) develops in the solution of IVP2 as $t \to \infty$. As $t \to \infty$ we have established the wave speed of PTW, $c$, is given by

$$c = \frac{u_- + u_+}{2} = -\frac{1}{2},$$

and that the rate of convergence the solution of IVP2 to the PTW as $t \to \infty$ is given by

$$u(z + s(t), t) = U_T\left(z, -\frac{1}{2}\right) + O(\chi(t)) \tag{3.97}$$

as $t \to \infty$ with $z = O(1)$, where $U_T(z, -\frac{1}{2}) + O(\chi(t))$ $z = x - s(t)$ and

$$s(t) = -\frac{1}{2}t + \phi_0 + O(\chi(t)) \tag{3.98}$$

where

$$\chi(t) = t^{-\frac{3}{2}} e^{-\frac{1}{16}t} \tag{3.99}$$

as $t \to \infty$. We note that the rate of convergence is exponential in $t$ as $t \to \infty$. The asymptotic wave speed, $\dot{s}(t)$, is given by

$$\dot{s}(t) = -\frac{1}{2} + O(\chi(t)) \tag{3.100}$$

as $t \to \infty$. We present numerical evidence to support the above. In Figure 3.6 we plot the numerical solution of IVP2 against $x$ at times $t = 10$ $t = 15$ $t = 20$ and $t = 25$ clearly, the solution converges to the PTW rapidly as $t \to \infty$. This is in line with (3.96) where we expect the rate of convergence to be exponential in $t$ as $t \to \infty$. In Figure 3.7 we plot $\dot{s}(t)$ versus $t$. Clearly, the numerical calculate wave speed rapidly approaches the expected value of $-\frac{1}{2}$ as $t \to \infty$. Again the rate of convergence looks to be in agreement with (3.99). In Figure 3.8 we plot $s(t) + \frac{1}{2}t$ versus $t$. We observe that the numerically calculated curve rapidly approaches $\phi_0 = 0$ as $t \to \infty$.

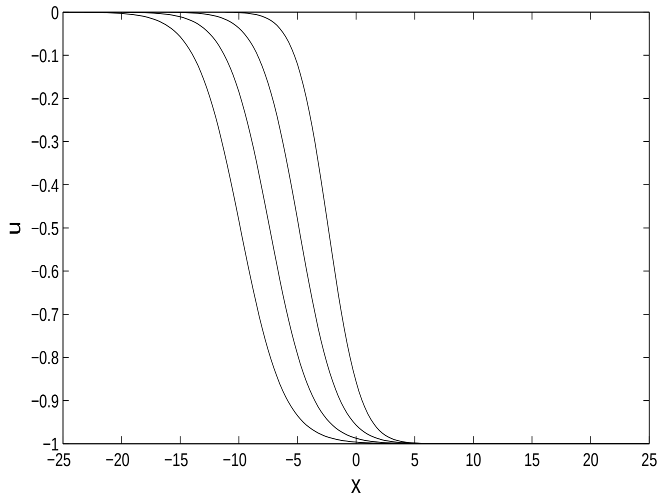


Figure 3.6: Numerical solution of IVP2 at times $t = 10$ $t = 15$  $t = 20$  and  $t = 25$.

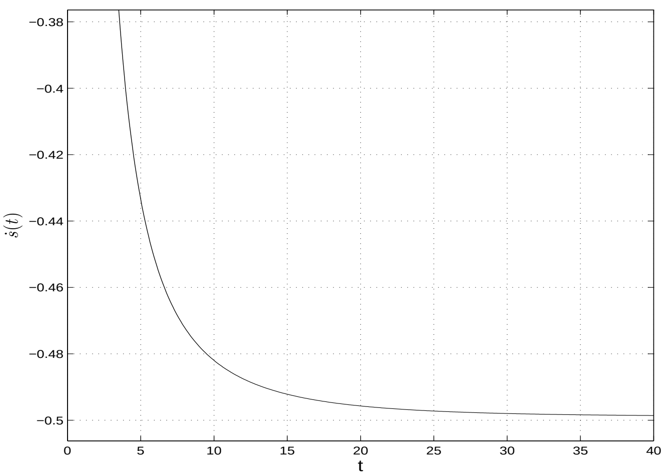


Figure 3.7: Numerical solution of $\dot{s}(t)$ versus $t$.

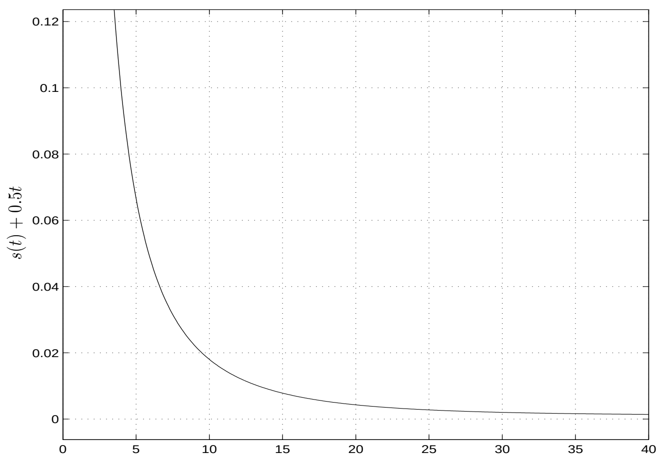


Figure 3.8: Numerical solution of $s(t) + 0.5t$ versus $t$.

## 3.5 Summary

In this chapter we have obtained the asymptotic structure of the solution to IVP2 as $t \to \infty$ when $u_- > u_+$. A uniform approximation has been given through regions IV$^+$(a), TR$^+$, IV$^+$(b), TW, IV$^-$(b), TR$^-$ and IV$^-$(a). A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 3.1. We have demonstrated that the solution to IVP2, $u(x,t)$, has

$$u(z + s(t), t) = \frac{u_+ + u_- e^{-Az}}{1 + e^{-Az}} + O(t^{-3/2} e^{-\frac{A^2}{4} t}) \tag{3.101}$$

as $t \to \infty$ with $z = O(1)$ where $A = \frac{(u_- - u_+)}{2}$ and

$$s(t) = ct + O(t^{-3/2} e^{-\frac{A^2}{4} t})$$

as $t \to \infty$ where $c = \frac{(u_+ + u_-)}{2}$.

In particular, when

(i) ($u_- > u_+ > -u_-$ with $u_- > 0$) or ($u_+ < u_- < -u_+$ with $u_+ < 0$):

A permanent form travelling wave develops as $t \to \infty$ in the solution of IVP2. The speed of the travelling wave, $c$ is either negative (when $u_+ < u_- < -u_+$ with $u_+ < 0$) or positive (when $u_- > u_+ > -u_-$ with $u_- > 0$). The rate of convergence of the solution to IVP2 onto the travelling wave is exponential in $t$, being of $O(t^{-3/2}e^{-\frac{A^2}{4}t})$ as $t \to \infty$.

(ii) $u_+ = -u_-$ with $u_- > 0$. A stationary profile, located at $x = 0$, develops as $t \to \infty$ in the solution of IVP2. The rate of convergence of the solution to IVP2 onto the stationary profile is exponential in $t$ being $O(t^{-3/2}e^{-\frac{{u_-}^2}{4}t})$ as $t \to \infty$.

We note that the envelope solutions in regions $\mathrm{IV}^+$ an $\mathrm{IV}^-$ have the same value when $y = c$ (predicting the location of the travelling wave region and giving the argument of the exponential function in the correction term of $O(t^{-3/2}e^{-\frac{A^2}{4}t})$ while the point at which we switch from the envelope solutions to the linear solutions, region $\mathrm{TR}^\pm$, occur when $y = u_+$ (in $\mathrm{IV}^-$) and $y = u_-$ (in $\mathrm{IV}^+$).

We further note that when $u_- < u_+$ the envelope-touching solutions provide no structure in $u_- < y < u_+$. This gap 'lack of structure' being filled as in Chapter 2 by the expansion wave solution, $u = y$ for $u_- < y < u_+$, which connects the constant status $y \sim u_+$ for $y > u_+$ and $y \sim u_-$ for $y < u_-$.

Finally, it is worth noting that the structure of solution of IVP2 as $t \to \infty$ depends critically on interaction between the selected envelope-touching solution of equation (3.36) in region $\mathrm{IV}^+$ and the selected envelope solution of equivalent equation in region $\mathrm{IV}^-$. These results are in argument with the numerical simulations of Section 3.4.

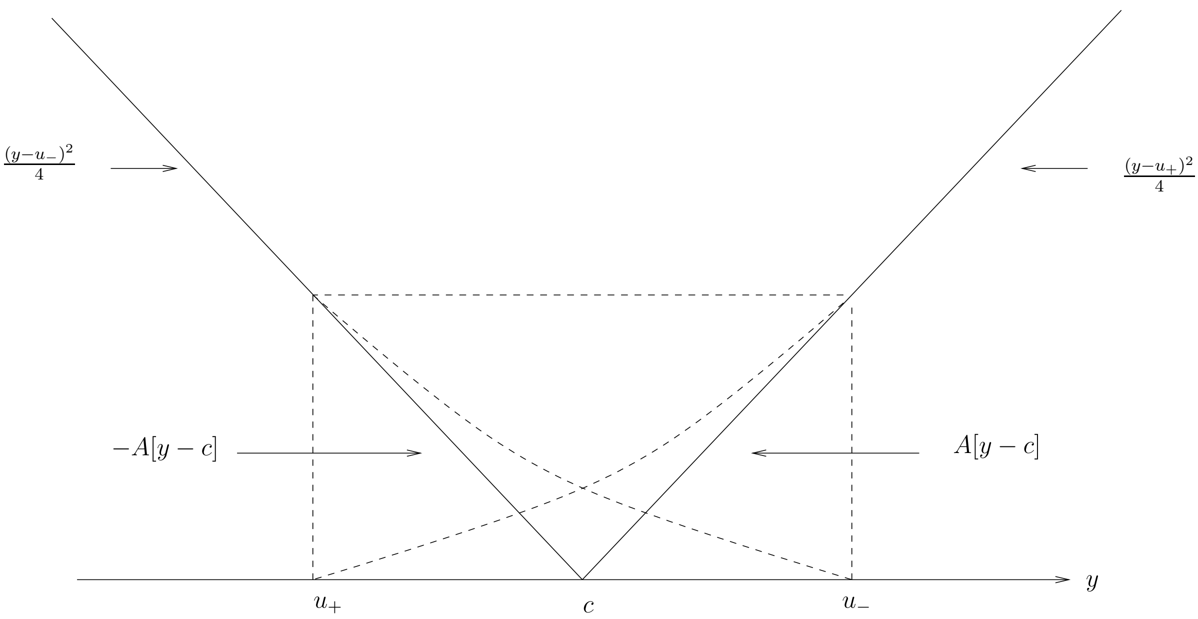


Figure 3.9: The Graph of the envelope solutions in region $IV^{\pm}$ switch to the linear solutions, region $TR^{\pm}$, occur when $y = u_+$ (in $IV^-$) and $y = u_-$ (in $IV^+$).

# Chapter 4

# A Quarter-Plane Problem for Burgers' Equation

In this chapter we examine an initial-boundary value problem for Burgers' equation on the positive quarter-plane. The specific problem under investigation is given by

$$u_t + uu_x - u_{xx} = 0, \quad x > 0, \quad t > 0 \tag{4.1}$$

$$u(x,0) = u_+, \quad x > 0, \tag{4.2}$$

$$u(0,t) = u_b, \quad t > 0, \tag{4.3}$$

where $u_+$ is initial condition and $u_b$ boundary condition ($u_b \neq u_+$) are constants.

In what follows we label initial-boundary value problem (4.1)-(4.3) as QP. In this chapter, we develop the large-time structure of the solution to QP using the method of matched asymptotic coordinate expansions. The behaviour of the large-time solution of QP depends critically on the problem parameters $u_+$ and $u_b$. Specifically, the large-time attractor of the solution to QP is:

(i) A travelling wave with positive wave speed when $-u_b < u_+ < u_b$ with $u_b > 0$.

(ii) A stationary solution when ($-u_+ > u_b > u_+$ with $u_+ < 0$) or when $0 \geqslant u_+ > u_b$.

(iii) A structure consisting of a combination of an expansion wave and a stationary solution when $u_b < 0$ and $u_+ > 0$.

(iv) An expansion wave when $u_+ > u_b \geqslant 0$.

We begin by considering the case when $-u_b < u_+ < u_b$ with $u_b > 0$.

## 4.1 $\mathbf{-u_b < u_+ < u_b, \;\; u_b > 0}$

### 4.1.1 Asymptotic Solution as $\mathbf{t \to 0}$

We begin in region I, where $x = o(1)$ and $u(x,t) = O(1)$ as $t \to 0$. To examine region I we introduce the scaled coordinate $\eta = xt^{-1/2} = O(1)$ as $t \to 0$ and look for an expansion of the form

$$u(\eta, t) = U(\eta) + o(1), \tag{4.4}$$

as $t \to 0$ with $\eta = O(1)$ ($\geqslant 0$). On substitution of (4.4) into equation (4.1) (when written in terms of $\eta$ and $t$) we obtain at leading order that

$$U_{\eta\eta} + \frac{\eta}{2}U_\eta = 0, \quad 0 \leqslant \eta < \infty. \tag{4.5}$$

The solution to (4.5) is readily obtained as

$$U(\eta) = A_0 \,\mathrm{erfc}\left(\frac{\eta}{2}\right) + B_0 \quad 0 \leqslant \eta < \infty, \tag{4.6}$$

where $A_0$, $B_0$ are constants and erfc[.] is complementary error function. As $\eta \to 0^+$

$$U(\eta) \sim (A_0 + B_0) - \frac{\eta}{\sqrt{\pi}} + \dots. \tag{4.7}$$

In order to satisfy boundary condition (4.3) we require $A_0 + B_0 = u_b$. As $\eta \to \infty$,

$$U(\eta) \sim A_0 \left( \frac{2}{\eta\sqrt{\pi}} e^{-\frac{\eta^2}{4}} + \dots \right) + B_0, \tag{4.8}$$

and in order to satisfy the condition that $U(\eta) \to u_+$ as $\eta \to \infty$ we require that $B_0 = u_+$ and hence that $A_0 = (u_b - u_+)$ $(> 0)$. Therefore in region I we have that

$$u(\eta, t) = \left( (u_b - u_+) \operatorname{erfc}\left(\frac{\eta}{2}\right) + u_+ \right) + o(1), \quad 0 \leqslant \eta < \infty, \tag{4.9}$$

as $t \to 0$ with $\eta = O(1)$.

As $\eta \to \infty$ we move out of region I into region II, where $x = O(1)(> 0)$ as $t \to 0$. The form of expansion (4.9) for $\eta \gg 1$ suggests that in region II we look for an expansion of the form

$$u(x, t) = u_+ + e^{-K(x,t)} \quad \text{as} \quad t \to 0, \tag{4.10}$$

where

$$K(x, t) = \frac{k_0(x)}{t} + k_1(x) \ln t + k_2(x) + o(1) \tag{4.11}$$

as $t \to \infty$ with $x = O(1)(> 0)$ where $k_0(x) > 0$ and $k_1(x)$, $k_2(x)$ are functions to be determined. On substitution of (4.10) and (4.11) into equation (4.1) and solving at each order in turn, we find (after matching with (4.9) as $x \to 0^+$) that

$$u(x, t) = u_+ + \exp\left( -\frac{x^2}{4t} + \frac{1}{2} \ln t + \frac{u_+}{2} x - \ln x + \ln \mathcal{A} + o(1) \right) \tag{4.12}$$

as $t \to 0$ with $x = O(1)(> 0)$, and where $\mathcal{A}=\frac{2}{\sqrt{\pi}}(u_b - u_+)(> 0)$. The asymptotic solution of QP as $t \to 0$ is now complete with expansions (4.9) and (4.12) providing a uniform approximation to the solution of QP as $t \to 0$.

### 4.1.2 Asymptotic Solution as $x \to \infty$

Now, we examine the asymptotic structure of the solution to QP as $x \to \infty$ with $t = O(1)$. The form expansion (4.12) of region II for $x \gg 1$ as $t \to 0$ suggests that in this region, which we label as region III, we write

$$u(x,t) = u_+ + e^{-M(x,t)} \quad as\ x \to \infty \tag{4.13}$$

with

$$M(x,t) = m_0(t)x^2 + m_1(t)x + m_2(t)\ln x + m_3(t) + o(1) \tag{4.14}$$

where $t = O(1)$ as $x \to \infty$ and $m_0(t) > 0$, $m_1(t)$, $m_2(t)$ and $m_3(t)$ are functions to be determined. On substituting from (4.13) and (4.14) into equation (4.1) and solving at each order in turn, we find (after matching with (4.12) as $t \to 0^+$) that

$$u(x,t) = u_+ + \exp\left(-\frac{x^2}{4t} + \frac{u_+}{2}x - \ln x + \left(\frac{1}{2}\ln t - \frac{u_+^2}{4}t + \ln \mathcal{A}\right) + o(1)\right) \tag{4.15}$$

as $x \to \infty$ with $t = O(1)(> 0)$. We observe that expansion (4.15) of region III remains uniform for $t \gg 1$ provided that $x \gg t$, but becomes non-uniform when $x = O(t)$ as $t \to \infty$.

### 4.1.3 Asymptotic Solution as $t \to \infty$

As $t \to \infty$, the asymptotic expansion (4.15) of region III continues to remain uniform for $x \gg t$. However, when $x = O(t)$ it becomes non-uniform. To proceed, we introduce a new region, which we label as region IV. To examine region IV we introduce the scaled coordinate $y = \frac{x}{t}$ where $y = O(1)$ as $t \to \infty$ and look for an expansion of the form (as suggested by (4.15))

$$u(y,t) = u_+ + e^{-N(y,t)} \quad \text{as} \quad t \to \infty, \tag{4.16}$$

with

$$N(y,t) = n_0(y)t + n_1(y)\ln t + n_2(y) + o(1), \tag{4.17}$$

where $y = O(1)(> 0)$ as $t \to \infty$ and $n_0(y) > 0$, $n_1(y)$ and $n_2(y)$ are functions to be determined. On substituting (4.16) and (4.17) into equation (4.1) (when written in terms of $y$ and $t$) we obtain the leading order problem as

$$(n_0')^2 - (y - u_+)n_0' + n_0 = 0 \quad y > 0, \tag{4.18}$$

$$n_0(y) > 0 \quad y > 0, \tag{4.19}$$

$$n_0(y) \sim \frac{(y-u_+)^2}{4} \quad \text{as } y \to \infty. \tag{4.20}$$

The final condition (4.20), arises from matching expansion (4.16) ($y \gg 1$) with expansion (4.15) ($x = O(t)$). Equation (4.18) has one-parameter family of linear solutions,

$$n_0(y) = A[y - (A + u_+)], \quad y > 0 \tag{4.21}$$

for any $A \in \mathbb{R}$, together with the associated envelope solution

$$n_0(y) = \frac{(y-u_+)^2}{4}, \quad y > u_+. \tag{4.22}$$

Combinations of (4.21) and (4.22) which remain continuous and differentiable also provide solutions to (4.18) (envelope touching solutions). The solution (4.18)-(4.20) is given either by the envelope solution

$$n_0(y) = \frac{(y-u_+)^2}{4}, \quad y > u_+ \tag{4.23}$$

or by the family of envelope touching solutions

$$n_0(y) = \begin{cases} \frac{(y-u_+)^2}{4}, & y \geqslant u_+ + 2A, \\ \\ A[y-(A+u_+)], & (A+u_+) < y \leqslant u_+ + 2A, \end{cases} \tag{4.24}$$

for each $A > 0$. Each case will have to be considered separately.

(a) We begin by considering the case when $n_0(y)$ is given by (4.23) that is

$$n_0(y) = \frac{(y-u_+)^2}{4}, \quad y > u_+. \tag{4.25}$$

In this case continuing expansion (4.13) gives in region IV that

$$u(y,t) = u_+ + \exp\left(-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H_1(y) + o(1)\right) \tag{4.26}$$

as $t \to \infty$ with $y = O(1)(> 0)$ and where the function $H_1(y)$ is undetermined. However, matching with region III requires that

$$H_1(y) \sim \ln y - \ln \mathcal{A} \quad \text{as} \quad y \to \infty. \tag{4.27}$$

We recall that in this case $-u_b < u_+ < u_b$ with $u_b > 0$ and in what follows we must consider the subcases:

(i) $0 < u_+ < u_b$

(ii) $u_+ = 0$

(iii) $-u_b < u_+ < 0$

separately.

(i) $0 < u_+ < u_b$. We note that expansion (4.26) becomes non-uniform as $y \to$

$u_+{}^+$ $(\in (0, u_b))$ and to continue the large-t asymptotic structure of the solution to QP we introduce a new region, region A1, in which following (4.26), we have that

$$y = u_+ + O(t^{-1/2}) \quad \text{as} \quad t \to \infty. \tag{4.28}$$

Thus in region A1 we can write

$$y = u_+ + \eta t^{-1/2}, \tag{4.29}$$

where $\eta = O(1)$ as $t \to \infty$. To examine region A1 we look (via (4.26) and (4.29)) for an expansion of the form

$$u = u_+ + t^{-1/2} w(\eta) + o(t^{-1/2}), \tag{4.30}$$

as $t \to \infty$ with $\eta = O(1)$ and where $w(\eta) > 0$. On substitution of (4.30) into equation (4.1) (when written in terms of $\eta$ and $t$), we obtain at leading order

$$w_{\eta\eta} - w w_\eta + \frac{\eta}{2} w_\eta + \frac{w}{2} = 0, \quad -\infty < \eta < \infty. \tag{4.31}$$

We require

$$H_1(y) \sim -\ln B \quad \text{as} \quad y \to 0^+$$

where $B > 0$ is a constant, and matching with region IV as $\eta \to \infty$ requires

$$w(\eta) \sim B e^{-\eta^2/4} \quad \text{as} \quad \eta \to \infty. \tag{4.32}$$

Initial-value problem (4.31), (4.32) has been examined in Section 1.1.3, and we recall that the solution of (4.31), (4.32) can be written as

$$w(\eta) = \frac{2e^{-\eta^2/4}}{D_2 - \sqrt{\pi}\text{erf}\,(\eta/2)}, \quad -\infty < \eta < \infty, \tag{4.33}$$

where $D_2$ ($> \sqrt{\pi}$, since we require $w(\eta) > 0$ in $\eta > 0$) is a constant. Matching with region IV requires that $D_2 = \frac{2}{B} + \sqrt{\pi}$. This solution is bounded an $\eta \in (-\infty, \infty)$, having a single maximum, and with

$$w(\eta) = \frac{2e^{-\eta^2/4}}{D_2 + \sqrt{\pi}\text{erf}\,(\eta/2)}, \quad \text{as} \quad \eta \to -\infty, \tag{4.34}$$

Therefore, expansion (4.30) remains uniform for $\eta \in (-\infty, \infty)$. As $\eta \to -\infty$ we move out of region A1 into region V where $y = O(1)$ ($\in (0, u_+)$). The expansion in region V is readily obtained as

$$u(y,t) = u_+ + \exp\left(-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H_2(y) + o(1)\right) \tag{4.35}$$

as $t \to \infty$ with $y = O(1)$ ($\in (0, u_+)$), and where the function $H_2(y)$ remains undetermined, but matching with region A1 requires that

$$H_2(y) \sim \ln\frac{2}{D_2 + \sqrt{\pi}} \quad \text{as} \quad y \to u_+^-.$$

In order to be able to satisfy boundary condition (4.2) expansion (4.35) must become non-uniform as $y \to 0^+$. To examine this possibility we introduce a new region, region SS, where $u = O(1)$ and $y = o(1)$ as $t \to \infty$. To examine region SS we look for an expansion of the form

$$u = U(\xi) + o(1) \tag{4.36}$$

as $t \to \infty$ with $\xi = y\phi^{-1}(t) = O(1)$ ($\geqslant 0$) where $\phi(t) = o(1)$ as $t \to \infty$. On substitution of (4.36) into equation (4.1) (when written in terms of $\xi$ and $t$) we require for the most structured balance that $\phi(t) = t^{-1}$ [giving that $\xi = x = yt = O(1)$ ($\geqslant 0$)] giving the leading problem as

$$U_{xx} - UU_x = 0, \quad x > 0 \tag{4.37}$$

$$U(0) = u_b \ (> u_+) \tag{4.38}$$

$$U(x) \to u_+ \ (> 0) \quad \text{as} \quad x \to \infty. \tag{4.39}$$

However, we have already noted in Section 1.1.2 that no solution to (4.37)-(4.39) exists and we conclude that $n_0(y)$ is given by (4.24).

(ii) $u_+ = 0$. In this case expansion (4.26) becomes non-uniform as $y \to 0^+$ and to continue the large-t asymptotic structure of the solution to QP we introduce, region A2, in which following (4.26) (with $u_+ = 0$), we have that $y = O(t^{-1/2})$ as $t \to \infty$. Thus in region A2 we can write

$$y = \eta t^{-1/2}, \tag{4.40}$$

where $\eta > 0$ as $t \to \infty$. To examine region A2 we look for an expansion of the form

$$u = t^{-1/2} w(\eta) + o(t^{-1/2}), \tag{4.41}$$

as $t \to \infty$ with $\eta = O(1)$ ($> 0$) and where $w(\eta) > 0$. On substitution of (4.41) into equation (4.1) (when written in terms of $\eta$ and $t$), we obtain at leading order

$$w_{\eta\eta} - ww_\eta + \frac{\eta}{2} w_\eta + \frac{w}{2} = 0, \quad 0 < \eta < \infty. \tag{4.42}$$

We require that

$$H_1(y) \sim -\ln B \quad \text{as} \quad y \to 0^+,$$

where $B > 0$ is a constant, and matching with region IV as $\eta \to \infty$ requires

$$w(\eta) \sim Be^{-\eta^2/4} \quad \text{as} \quad \eta \to \infty. \tag{4.43}$$

Initial-value problem (4.42), (4.43) has been examined in Section 1.1.3, and we recall that the solution of (4.42), (4.43) can be written as

$$w(\eta) = \frac{2e^{-\eta^2/4}}{D_2 - \sqrt{\pi}\,\mathrm{erf}\,(\eta/2)}, \quad 0 \leqslant \eta < \infty, \tag{4.44}$$

where $D_2$ ($> \sqrt{\pi}$, since we require $w(\eta) > 0$ in $\eta > 0$) is a constant. Matching with region IV requires that $D_2 = \frac{2}{B} + \sqrt{\pi}$. Therefore, we conclude that for each $B > 0$ the solution of (4.42), (4.43) is bounded on $0 \leqslant \eta < \infty$ and expansion (4.41) remains uniform as $\eta \to 0^+$ and boundary condition (4.2) can not be satisfied. We conclude that when $u_+ = 0$ that $n_0(y)$ is given by (4.24).

(iii) $-u_b < u_+ < 0$. In this case in order to satisfy boundary condition (4.2) expansion (4.26) must become non-uniform as $y \to 0^+$. Following case (i) we introduce a final region, region SS, and look for an expansion of the form

$$u = U(x) + o(1) \tag{4.45}$$

as $t \to \infty$ with $x = O(1)$ ($\geqslant 0$). On substitution (4.45) into (4.1) we obtain leading order problem

$$U_{xx} - UU_x = 0, \quad x > 0 \tag{4.46}$$

$$U(0) = u_b \ (> 0) \tag{4.47}$$

$$U(x) \to u_+ \ (< 0) \quad \text{as} \quad x \to \infty \tag{4.48}$$

However, following Section 1.1.2 no solution to boundary-value (4.46)-(4.48) exists and we conclude that $n_0(y)$ is given by (4.24).

(b) We now consider the case when $n_0(y)$ is given by (4.24). Following the analysis of Section 3.3 we replace region IV by three regions: regions IV(a) ($u_+ + 2A + o(1) < y < \infty$), region TR (transition region) and region IV(b) ($A + u_+ + o(1) < y < u_+ + 2A - o(1)$). The details of regions IV(a), TR, IV(b) follow those given for regions $\text{IV}^+$ (a), $\text{TR}^+$, $\text{IV}^+$(b) of Section 3.3 respectively and are not repeated here in full. Further, we note that to enable boundary condition (4.3) to be satisfied later in the asymptotic analysis we require

$$A = \frac{u_b - u_+}{2} \ (> 0),$$

and for ease of exposition we use this value of A in what follows.

Before continuing the asymptotic development of the solution of (4.1)-(4.3) in this case we summarize regions IV(a), TR and IV(b).

Region IV(a)

$$u(y,t) = u_+ + \exp\left\{ -\frac{(y-u_+)^2}{4} t - \frac{1}{2}\ln t - H_R(y) + o(1) \right\}$$

as $t \to \infty$ with $y = O(1)$ ($\in (u_b, \infty)$). The function $H_R(y)$ remains undetermined but matching to the far field (as $y \to \infty$) and to region TR (as $y \to u_b^+$) requires

$$H_R(y) \sim \begin{cases} \ln y - \ln \mathcal{A} \quad \text{as} \quad y \to \infty, \\ \ln \frac{2(y-u_b)}{\mathcal{A}} \quad \text{as} \quad y \to u_b^+, \end{cases}$$

where $\mathcal{A} = \frac{2}{\sqrt{\pi}}(u_b - u_+)$ $(> 0)$.

Region TR

$$u(\eta, t) = u_+ + \left( \frac{(u_b - u_+)}{2} \operatorname{erfc}\left(\frac{\eta}{2}\right) + o(1) \right) e^{-A^2 t - A\eta t^{1/2}}$$

as $t \to \infty$ with $\eta = (y - u_b)t^{1/2} = O(1)$ and where $A = \frac{u_b - u_+}{2}$.

Region IV(b)

$$u(y, t) = u_+ + \exp\left\{-A[y - c]t + \ln(u_b - u_+)\right\} + t^{-1/2} H_L(y) \exp\left\{ -\frac{(y - u_+)^2}{4} t \right\} + o\left( t^{-1/2} \exp\left\{ -\frac{(y - u_+)^2}{4} t \right\} \right) \quad (4.49)$$

as $t \to \infty$ with $y = O(1)$ ($\in (c, u_b)$) where $c = \frac{u_b + u_+}{2}$ and $A = \frac{u_b - u_+}{2}$. The function $H_L(y)$ remains undetermined but matching to region TR as $y \to u_b^-$ requires

$$H_L(y) \sim \frac{\mathcal{A}}{2(u_b - y)} \quad \text{as} \quad y \to u_b^-.$$

Now as $y \to c^+$ $(= \frac{u_b + u_+}{2})$ expansion (4.49) becomes non-uniform and we must introduce a new region, region TW. The details of region TW, the travelling wave region, follow those given in Section 3.3 and are summarized here:

Region TW

$$u(z, t) = \frac{u_+ + u_b e^{-Az}}{1 + e^{-Az}} + O(t^{-3/2} e^{-\frac{A^2}{4} t}) \quad (4.50)$$

as $t \to \infty$ with $z = O(1)$ where $A = \frac{u_b - u_+}{2}$, $z = x - s(t)$ and

$$s(t) = \frac{u_b + u_+}{2} t + O(t^{-3/2} e^{-\frac{A^2}{4} t}) \quad (4.51)$$

as $t \to \infty$. Therefore, in region TW we have at leading order a travelling wave solution of speed $c = \frac{u_b+u_+}{2}$. As $z \to -\infty$ we wave out of region TW into region V where $y = O(1) = \; [\in (0, c)]$. There are three separate cases to consider

(i) $-u_b < u_+ < 0$. In this case it is straightforward to establish that in region V we have

$$u(y,t) = u_b - \exp\{A[y-c]t + \ln(u_b - u_+) + o(1)\} \tag{4.52}$$

as $t \to \infty$ with $y = O(1) = \; (\in (0, c))$. We note that expansion (4.52) becomes non-uniform when $y = O(t^{-1})$ [that is, when $x = O(1)$] and we must introduce a final region, region B. To examine region B we look for (as suggested by (4.52) when written in terms of $x$) an expansion of the form

$$u(x,t) = u_b - [R(x) + o(1)]e^{-Act} \tag{4.53}$$

as $t \to \infty$ for $x = O(1)$ ($\geqslant 0$). On substituting (4.53) into equation(4.1) we obtain at leading order

$$R_{xx} - u_b R_x + \frac{(u_b - u_+)(u_b + u_+)}{4} = 0, \quad x \geqslant 0. \tag{4.54}$$

The solution of (4.54) is readily obtained as

$$R(x) = E_0 e^{cx} + F_0 e^{Ax}, \quad x \geqslant 0.$$

We require $R(x)$ to satisfy the conditions

$$\begin{aligned} &R(0) = 0 \\ &R(x) \sim (u_b - u_+)e^{Ax} \quad \text{as} \quad x \to \infty, \end{aligned} \tag{4.55}$$

the boundary and matching conditions respectively. Therefore, we have that $E_0 = -F_0 = u_+ - u_b$ giving

$$R(x) = (u_b - u_+)[e^{Ax} - e^{cx}], \quad x \geqslant 0. \tag{4.56}$$

To conclude in this case we note that in region B we have

$$u(x,t) = u_b - [(u_b - u_+)(e^{Ax} - e^{cx}) + o(1)]e^{-Act} \tag{4.57}$$

as $t \to \infty$ for $x = O(1)$ ($\geqslant 0$).

(ii) $u_+ = 0$. In this case we first note that $A = c = \frac{u_b}{2}$ giving in region V that

$$u(y,t) = u_b - \exp\left\{\frac{u_b}{2}\left(y - \frac{u_b}{2}\right)t + \ln(u_b - u_+) + o(1)\right\} \tag{4.58}$$

as $t \to \infty$ with $y = O(1)$ ($\in (0, \frac{u_b}{2})$). An examination of the problem in this case indicates that expansion (4.58) becomes non-uniform when $y = O(t^{-1/2})$ as $t \to \infty$ [that is, when $x = O(t^{1/2})$] and that we should as in case (i) introduce a final region, region B. To examine region B we introduce the scaled coordinate $\xi = yt^{-1/2} = O(1)$ ($\geqslant 0$) and look for an expansion of the form

$$u(\xi,t) = u_b - [F(\xi) + o(1)]e^{-\frac{u_b^2}{4}t + \frac{u_b}{2}\xi t^{1/2}} \tag{4.59}$$

as $t \to \infty$ for $\xi = O(1)$ ($\geqslant 0$). On substituting (4.59) into equation(4.1) (when written in terms of $\xi$ and $t$) we obtain at leading order

$$F_{\xi\xi} + \frac{\xi}{2}F_\xi = 0, \quad \xi > 0. \tag{4.60}$$

Equation (4.60) is to be solved subject to the conditions

$$\begin{aligned} &F(0) = 0, \\ &F(\xi) \sim u_b \quad \text{as} \quad \xi \to \infty, \end{aligned} \tag{4.61}$$

The solution of (4.60)-(4.61) is readily obtained as

$$F(\xi) = u_b \text{erf}\left(\frac{\xi}{2}\right), \quad \xi \geqslant 0.$$

Therefore, we have in region B that

$$u(\xi, t) = u_b - \left(u_b \text{erf}\left(\frac{\xi}{2}\right) + o(1)\right) e^{-\frac{u_b^2}{4}t + \frac{u_b}{2}\xi t^{1/2}} \tag{4.62}$$

as $t \to \infty$ with $\xi = O(1)$ ($\geqslant 0$).

(iii) $0 < u_+ < u_b$. In this case region V must be replaced by three regions: region V(a) ($0 < y < u_+$), region TR (transition region) and region V(b) ($u_+ < y < c$). The details of regions V(b) and TR follow those given for similar regions earlier and are not repeated here. The expansion in region V(a) is given by

$$u(y, t) = u_b - \exp\left\{-\frac{(y - u_b)^2}{4}t - \frac{1}{2}\ln t - H_p(y) + o(1)\right\} \tag{4.63}$$

as $t \to \infty$ with $y = O(1)$ ($\in (0, u_+)$). The function $H_p(y)$ remains undetermined, but matching as $y \to 0$ requires that

$$H_p(y) \sim -\ln y + \beta$$

as $y \to 0^+$. Now as $y \to 0^+$ expansion (4.63) becomes non-uniform $y = O(t^{-1})$ [that is, when $x = O(1)$]. Therefore, to complete the asymptotic structure we must introduce a final region, region B. To examine region B we look for an expansion of the form

$$u(x,t) = u_b - [F(x) + o(1)]t^{-3/2}e^{-\frac{u_b^2}{4}t} \tag{4.64}$$

as $t \to \infty$ for $x = O(1)$ ($\geqslant 0$). On substituting (4.64) into equation(4.1) we obtain at leading order

$$F_{xx} - u_b F_x + \frac{u_b^2}{4}F = 0, \quad x > 0. \tag{4.65}$$

Equation (4.65) is to be solved subject to the conditions

$$\begin{aligned} &F(0) = 0, \\ &F(x) \sim e^{-\beta} x e^{\frac{u_b}{2}x} \quad \text{as} \quad x \to \infty. \end{aligned} \tag{4.66}$$

The solution of (4.65)-(4.66) is readily obtained as

$$u(x,t) = u_b - \left(e^{-\beta} x e^{\frac{u_b}{2}x} + o(1)\right) t^{-3/2} e^{-\frac{u_b^2}{4}t} \tag{4.67}$$

as $t \to \infty$ for $x = O(1)$ ($\geqslant 0$). This then completes the asymptotic structure in this case.

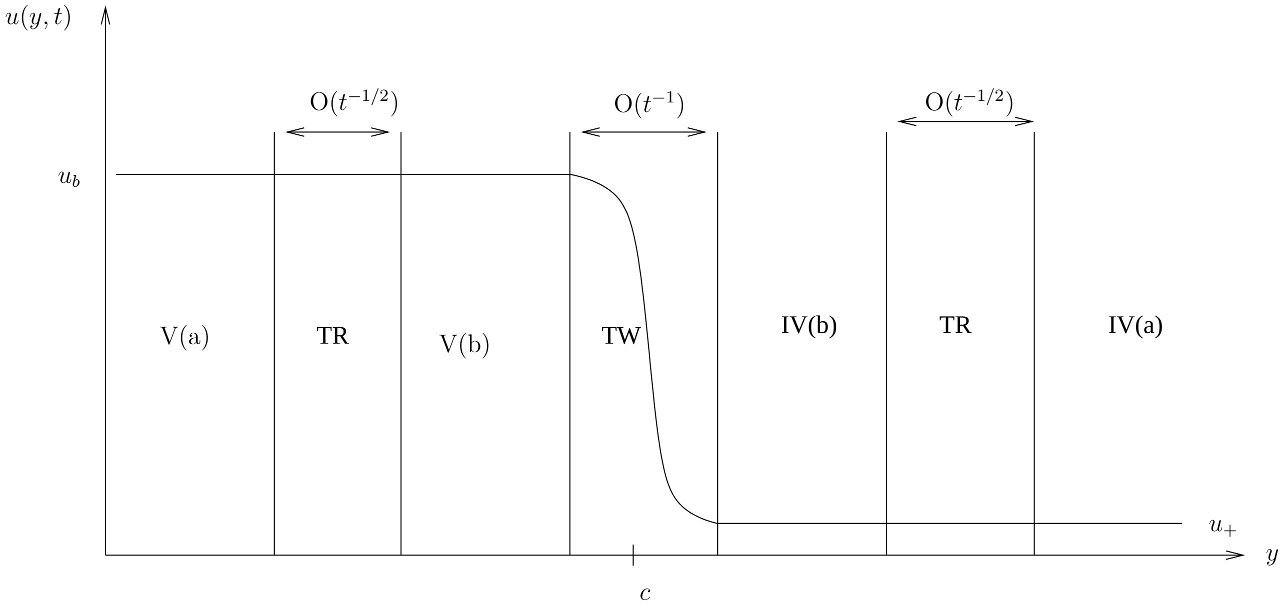


Figure 4.1: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane, as $t \to \infty$.

## 4.2 $-\mathbf{u}_+ > \mathbf{u_b} > \mathbf{u}_+, \quad \mathbf{u}_+ < 0$

The asymptotic solution of QP as $t \to 0$ and as $x \to \infty$ $(t = O(1)(> 0))$ in this case follows directly that given in 4.1.1 and 4.1.2 respectively and is not repeated here.

### 4.2.1 Asymptotic Solution as $t \to \infty$

As in Section 4.1.3 it is straightforward to rule out the envelope solution (4.23) and select the envelope-touching solution (4.24). A consistent asymptotic structure in this case requires that

$$A = -u_+,$$

giving in $y \in (0, \infty)$ the following asymptotic structure:

Region IV(a)

$$u(y,t) = u_+ + \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H_R(y) + o(1)\right\} \tag{4.68}$$

as $t \to \infty$ with $y = O(1)$ $(\in (-u_+, \infty))$ and where the function $H_R(y)$ remains undetermined but matching requires that

$$H_R(y) \sim \begin{cases} \ln y - \ln \mathcal{A} \quad \text{as} \quad y \to \infty \ , \\ \ln(y + u_+) + \beta_1 \quad \text{as} \quad y \to (-u_+)^+ \end{cases}$$

where $\beta_1 = E_R + \ln\sqrt{\pi}$ and $E_R$ is a constant to be determined.

Region TR

$$u(\eta, t) = u_+ + \left(\frac{1}{2}e^{-E_R}\,\mathrm{erfc}\left(\frac{\eta}{2}\right) + o(1)\right)e^{-u_+^2 t + u_+\eta t^{1/2}} \tag{4.69}$$

as $t \to \infty$ with $\eta = [y + u_+]t^{1/2} = O(1)$.

Region IV(b)

$$u(y,t) = u_+ + \exp\{u_+ y t - E_R + o(1)\} \tag{4.70}$$

as $t \to \infty$ with $y = O(1)(\in (0, -u_+))$.

Expansion (4.70) becomes nonuniform when $y = O(t^{-1})$ as $t \to \infty$ [that is, when $x = O(1)$], and we introduce a final region, region V. To examine region V we look for an expansion of the form

$$u(x,t) = F(x) + o(1), \tag{4.71}$$

as $t \to \infty$ with $x = O(1)$ ($\geqslant 0$). On substitution of (4.71) into equation (4.1) we obtain

$$F_{xx} - F F_x = 0, \quad x > 0. \tag{4.72}$$

Equation (4.72) is to be solved subject to

$$F(0) = u_b, \tag{4.73}$$

$$F(x) \to (u_+)^+ \quad \text{as} \quad x \to \infty. \tag{4.74}$$

The solution of (4.73)-(4.74) is readily obtained as

$$F(x) = u_+ \tanh\left(-\frac{u_+}{2}x + \tanh^{-1}\left(\frac{u_b}{u_+}\right)\right), \quad x \geqslant 0. \tag{4.75}$$

We recall from Section 1.1.2 that (4.75) is a stationary solution of (4.1). Therefore, in region V we have that

$$u(x,t) = u_+ \tanh\left(-\frac{u_+}{2}x + \tanh^{-1}\left(\frac{u_b}{u_+}\right)\right) + o(1) \tag{4.76}$$

as $t \to \infty$ with $x \geqslant 0$. We note that it is straightforward to establish that the correction to (4.76) is exponential in $t$ as $t \to \infty$ but we do not pursue this here.

Finally, we note that matching expansion (4.70) as $y \to 0^+$ with expansion (4.76) as $x \to \infty$ fixes

$$E_R = -\ln\left(2(-u_+)\frac{(u_+ - u_b)}{(u_+ + u_b)}\right).$$

The asymptotic structure in this case is now complete with expansion (4.68), (4.69), (4.70) and (4.76) of regions IV(a), TR, IV(b) and V respectively providing a uniform approximation to the solution of QP as $t \to \infty$.

A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4.2.

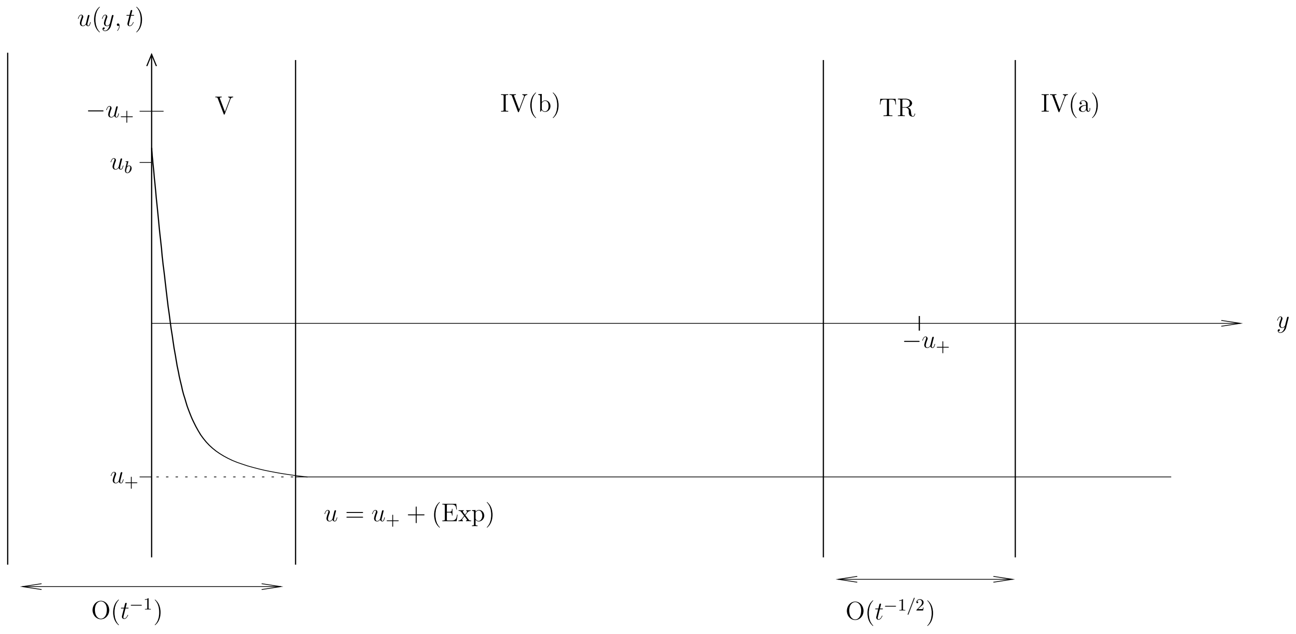


Figure 4.2: A schematic representation of the asymptotic structure of $u(y, t)$ in the $(y, u)$ plane, as $t \to \infty$. Here (Exp) denotes terms exponentially small in $t$ as $t \to \infty$.

## 4.3 $\mathbf{u_b < u_+ < 0, \quad u_b < 0}$

The large time asymptotic structure in this case follows closely that outlined in Section 4.2. We note however that in regions IV(a), TR and IV(b) the expansions (4.68), (4.69) and (4.70) are now slightly modified from those given in Section 4.2 and are given as follows:

Region IV(a)

$$u(y,t) = u_+ - \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H_R(y) + o(1)\right\} \tag{4.77}$$

as $t \to \infty$ with $y = O(1)$ $(\in (-u_+, \infty))$ and where the function $H_R(y)$ remains undetermined but matching requires that

$$H_R(y) \sim \begin{cases} \ln y - \ln \mathcal{A} \quad \text{as} \quad y \to \infty \ , \\ \ln (y + u_+) + \beta_1 \quad \text{as} \quad y \to (-u_+)^+ \end{cases}$$

where $\beta_1 = E_R + \ln \sqrt{\pi}$, $\mathcal{A} = \frac{2}{\sqrt{\pi}}(u_+ - u_b)$ $(> 0)$ and $E_R$ is a constant.

Region TR

$$u(\eta,t) = u_+ - \left(\frac{1}{2}e^{-E_R}\operatorname{erfc}\left(\frac{\eta}{2}\right) + o(1)\right)e^{-u_+^2 t + u_+\eta t^{1/2}} \tag{4.78}$$

as $t \to \infty$ with $\eta = [y + u_+]t^{1/2} = O(1)$.

Region IV(b)

$$u(y,t) = u_+ - \exp\{u_+ yt - E_R + o(1)\} \tag{4.79}$$

as $t \to \infty$ with $y = O(1)(\in (0, -u_+))$.

Expansion (4.79) becomes nonuniform when $y = O(t^{-1})$ as $t \to \infty$ [that is, when $x =$

$O(1)$], and we introduce a final region V. As in Section 4.2 we look for an expansion of the form (4.71) and at leading order obtain the boundary-value problem

$$F_{xx} - FF_x = 0, \quad x > 0, \tag{4.80}$$

$$F(0) = u_b, \tag{4.81}$$

$$F(x) \to (u_+)^- \quad \text{as} \quad x \to \infty. \tag{4.82}$$

The solution of (4.80)-(4.82) is readily obtained as

$$F(x) = u_+ \coth\left(-\frac{u_+}{2}x + \coth^{-1}\left(\frac{u_b}{u_+}\right)\right), \quad x \geqslant 0. \tag{4.83}$$

We recall from Section 1.1.2 that (4.83) is a stationary solution of (4.1). Therefore in region V we have

$$u(x,t) = u_+ \coth\left(-\frac{u_+}{2}x + \coth^{-1}\left(\frac{u_b}{u_+}\right)\right) + o(1), \tag{4.84}$$

as $t \to \infty$ with $x \geqslant 0$. As in Section 4.2 it is straightforward to establish that the correction to (4.84) is exponential in $t$ as $t \to \infty$. Finally, we note that matching expansion (4.79) as $y \to 0^+$ with expansion (4.84) as $x \to \infty$ fixes

$$E_R = -\ln\left(2(-u_+)\frac{(u_b - u_+)}{(u_+ + u_b)}\right).$$

The asymptotic structure in this case is now complete with expansions (4.77), (4.78), (4.79) and (4.84) of regions IV(a), TR, IV(b) and V respectively providing a uniform approximation to the solution of QP as $t \to \infty$.

A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given Figure 4.3.

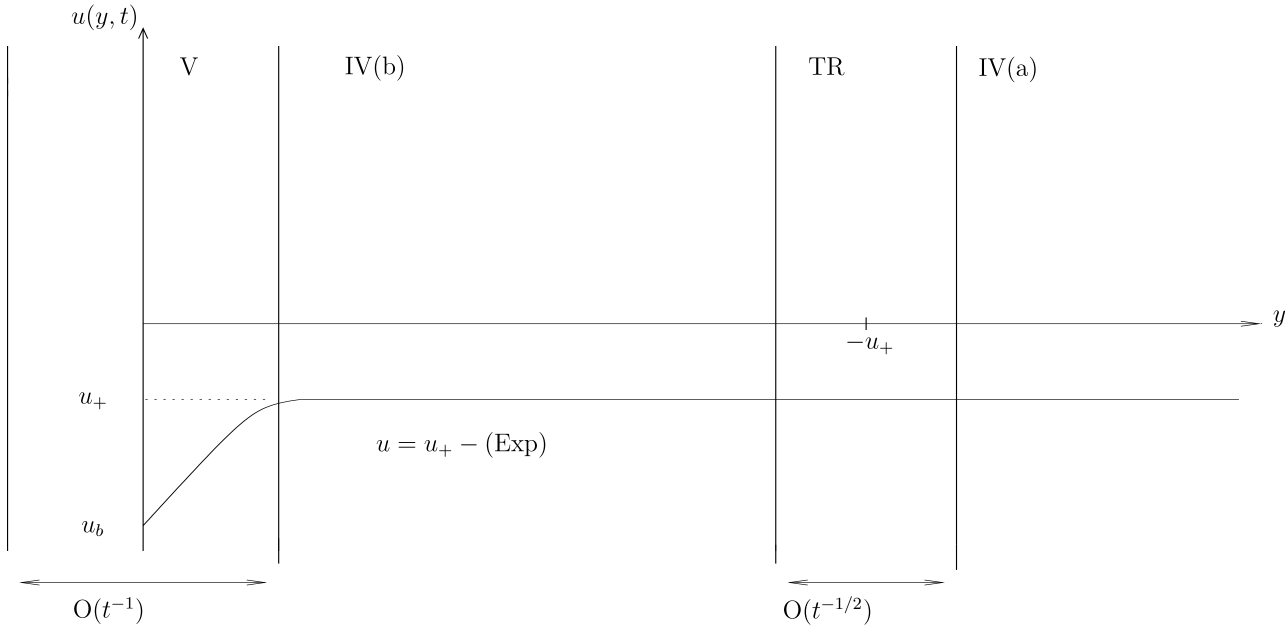


Figure 4.3: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane, as $t \to \infty$. Here (Exp) denotes terms exponentially small in $t$ as $t \to \infty$

.

As $u_+ \to 0^-$ we note that region TR approaches region V, coalescing when $u_+ = 0$, indicating a change of structure should be expected in the asymptotic solution of QP as $t \to \infty$ when $u_+ = 0$ and $u_b < 0$. We consider this case in Section 4.4.

## 4.4 $\mathbf{u_b < 0, \quad u_+ = 0}$

The asymptotic solution of QP as $t \to \infty$ changes from that encountered in Sections 4.2 and 4.3. This is because the transition region, region TR (located at $y = -u_+$) approaches the boundary region as $u_+ \to 0^+$, coalescing at $u_+ = 0$. We begin in region IV

Region IV

$$u(y,t) = -\exp\left\{-\frac{y^2}{4}t - \frac{1}{2}\ln t - H_R(y) + o(1)\right\} \tag{4.85}$$

as $t \to \infty$ with $y = O(1)$ $(\in (0,\infty))$ and where the function $H_R(y)$ is undetermined but matching requires that

$$H_R(y) \sim \begin{cases} \ln y - \ln \mathcal{A} \quad \text{as} \quad y \to \infty \ , \\ -\ln \mathcal{C} \quad \text{as} \quad y \to 0^+, \end{cases}$$

where $\mathcal{A} = \frac{2(-u_b)}{\sqrt{\pi}}$ $(> 0)$ in this case.

As $y \to 0$ expansion (4.85) becomes nonuniform and we introduce a new region, region V. To examine region V we introduce the scaled coordinate $\xi = yt^{1/2} = O(1)$ $(> 0)$ as $t \to \infty$ and look for an expansion of the form

$$u = U(\xi)t^{-1/2} + o(t^{-1/2}) \tag{4.86}$$

as $t \to \infty$ with $\xi = O(1)$ $(> 0)$. On substitution of (4.86) into equation (4.1) (when written in terms of $\xi$ and $t$) we obtain at leading order that

$$U_{\xi\xi} - UU_\xi + \frac{\xi}{2}U_\xi + \frac{1}{2}U = 0, \quad 0 < \xi < \infty. \tag{4.87}$$

Equation (4.87) is to be solved subject to the matching condition with region IV that is

$$U(\xi) \sim -\mathcal{C}e^{-\xi^2/4} \quad \text{as} \quad \xi \to \infty. \tag{4.88}$$

Following the discussion on similarity solutions of Burgers' equation in Section 1.1.3 we see that the solution of (4.87) is given by (1.51). Further, matching condition (4.88) requires that we take $D_2 = 0$ in (1.51) to obtain

$$U(\xi) = -\frac{2}{\sqrt{\pi}} \frac{e^{-\xi^2/4}}{\operatorname{erf}\left(\frac{\xi}{2}\right)}, \quad \xi > 0. \tag{4.89}$$

We note that

$$U(\xi) \sim \begin{cases} -\frac{2}{\sqrt{\pi}} e^{-\xi^2/4} \quad \text{as} \quad \xi \to \infty, \\ -\frac{2}{\xi} \quad \text{as} \quad \xi \to 0^+. \end{cases}$$

Matching expansion (4.86) (as $\xi \to \infty$) with expansion (4.85) (as $y \to 0^+$) then requires that

$$\mathcal{C} = \frac{2}{\sqrt{\pi}}.$$

Therefore, in region V we have that

$$U(\xi, t) = -\frac{2}{\sqrt{\pi}} \frac{e^{-\xi^2/4}}{\operatorname{erf}\left(\frac{\xi}{2}\right)} t^{-1/2} + o(t^{-1/2}) \tag{4.90}$$

as $t \to \infty$ with $\xi = O(1)$ ($> 0$).

We observe that expansion (4.90) becomes nonuniform as $\xi \to 0^+$ [specifically when $\xi = O(t^{-1/2})$ as $t \to \infty$] and we introduce the final asymptotic region, region VI. To examine region VI we look for an expansion of the form

$$u(x, t) = F(x) + o(1) \tag{4.91}$$

as $t \to \infty$ with $x = O(1)$ $(\geqslant 0)$. On substitution of (4.91) into equation (4.1) we obtain at leading order obtain the boundary-value problem

$$F_{xx} - FF_x = 0, \quad x > 0, \tag{4.92}$$

$$F(0) = u_b, \tag{4.93}$$

$$F(x) \to 0^- \quad \text{as} \quad x \to \infty. \tag{4.94}$$

The solution of (4.92)-(4.94) is readily determined (see Section 1.1.2) as

$$F(x) = \frac{2}{\frac{2}{u_b} - x}, \quad x \geqslant 0. \tag{4.95}$$

Therefore, in region VI we have that

$$u(x,t) = \frac{2}{\frac{2}{u_b} - x} + o(1) \tag{4.96}$$

as $t \to \infty$ with $x = O(1)$ $(\geqslant 0)$.

The asymptotic structure of the solution to QP as $t \to \infty$ is now complete with expansions (4.85), (4.90) and (4.96) of regions IV, V and VI respectively providing an uniform asymptotic solution to QP as $t \to \infty$.

A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4.4.

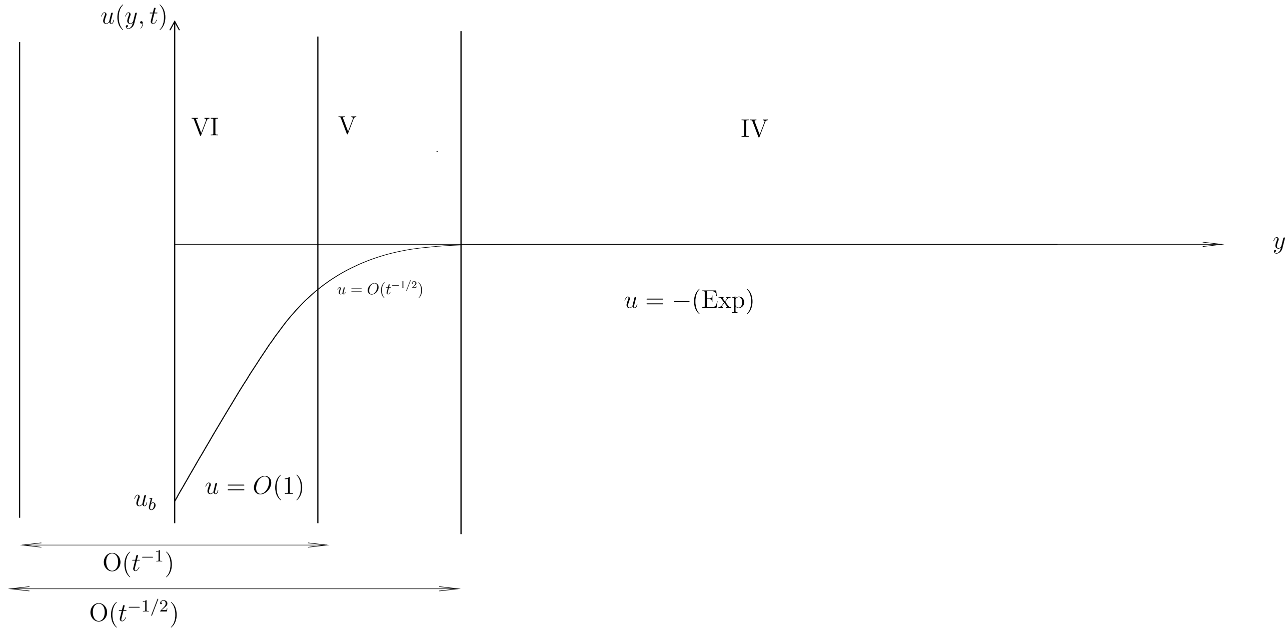


Figure 4.4: A schematic representation of the asymptotic structure of $u(y,t)$ plane, as $t \to \infty$. Here (Exp) denotes terms exponentially small in $t$ as $t \to \infty$.

## 4.5 $\mathbf{u_+ > u_b \geqslant 0}$

The large-time asymptotic structure of the solution to QP in this case follows closely that given in Chapter 2. We begin in region IV corresponding to region IV$^+$ of Chapter 2.

Region IV

$$u(y,t) = u_+ - \exp\left\{-\frac{(y-u_+)^2}{4}t - \frac{1}{2}\ln t - H_R(y) + o(1)\right\} \tag{4.97}$$

as $t \to \infty$ with $y = O(1)$ $(\in (0,\infty))$ and where the function $H_R(y)$ is undetermined but matching requires that

$$H_R(y) \sim \begin{cases} \ln y - \ln \mathcal{A} \quad \text{as} \quad y \to \infty \ , \\ -\ln \frac{1}{\sqrt{\pi}} \quad \text{as} \quad y \to u_+^+, \end{cases}$$

where $\mathcal{A} = \frac{2(u_+ - u_b)}{\sqrt{\pi}}$ $(> 0)$ in this case. We recall that in this case $u_+ > u_b \geqslant 0$ in what follows we must consider the subcases:

(i) $u_+ > u_b > 0$,

and

(ii) $u_b = 0$,

separately.

(i) $u_+ > u_b > 0$. We note that expansion (4.97) becomes non-uniform as $y \to {u_+}^+$ and to continue the large-t asymptotic structure of the solution to QP we introduce a new region, region A, in which following (4.97), we have that

$$y = u_+ + O(t^{-1/2}) \quad \text{as} \quad t \to \infty. \tag{4.98}$$

Thus in region A we can write

$$y = u_+ + \eta t^{-1/2}, \tag{4.99}$$

where $\eta = O(1)$ as $t \to \infty$. To examine region A we look (via (4.97) and (4.99)) for an expansion of the form

$$u = u_+ + t^{-1/2} w(\eta) + o(t^{-1/2}), \tag{4.100}$$

as $t \to \infty$ with $\eta = O(1)$ and where $w(\eta) < 0$. On substitution of (4.100) into equation (4.1) (when written in terms of $\eta$ and $t$), we obtain at leading order

$$w_{\eta\eta} - w w_\eta + \frac{\eta}{2} w_\eta + \frac{w}{2} = 0, \quad -\infty < \eta < \infty. \tag{4.101}$$

Matching with region IV as $\eta \to \infty$ requires

$$w(\eta) \sim -\frac{1}{\sqrt{\pi}} e^{-\eta^2/4} \quad \text{as} \quad \eta \to \infty. \tag{4.102}$$

Initial-value problem (4.101), (4.102) has been examined in Section 1.1.3, and we recall that the solution of (4.101), (4.102) can be written as

$$w(\eta) = \frac{2e^{-\eta^2/4}}{\sqrt{\pi}\,(1 - \operatorname{erf}(\eta/2))}, \quad -\infty < \eta < \infty, \tag{4.103}$$

Therefore, in region A we have that

$$u(\eta, t) = u_+ - \frac{2e^{-\eta^2/4}}{\sqrt{\pi}\operatorname{erfc}(\eta/2)} t^{-1/2} + o(t^{-1/2}) \tag{4.104}$$

as $t \to \infty$ with $\eta = O(1)$. As $\eta \to -\infty$ we move out from region A into region V. Consideration of expansion (4.104) $(-\eta) \gg 1$ gives that

$$u \sim u_+ + \eta t^{-1/2}. \tag{4.105}$$

On rewriting (4.105) in terms of $y$ we obtain

$$u \sim y$$

indicating that in region V we should expand as

$$u(y, t) = \widehat{U}(y) + o(1). \tag{4.106}$$

On substituting (4.106) into equation (4.1) (when written in terms of $y$ and $t$) we obtain that

$$\widehat{U}(y) = y$$

giving in region V that

$$u(y,t) = y + o(1) \tag{4.107}$$

as $t \to \infty$ with $y = O(1)$ ($\in (u_b, u_+)$). We note that expansion (4.107) becomes non-uniform as $y \to u_b^+$ and we must introduce a further region, region B, located at $y = u_b$. To examine region B we introduce scaled coordinate $\eta = (y - u_b)t^{1/2}$ when $\eta = O(1)$ as $t \to \infty$ and we look for an expansion of the form

$$u = u_+ + t^{-1/2}\mathcal{K}(\eta) + o(t^{-1/2}), \tag{4.108}$$

as $t \to \infty$ with $\eta = O(1)$ and where $\mathcal{K}(\eta) > 0$. On substitution of (4.108) into equation (4.1) (when written in terms of $\eta$ and $t$), we obtain at leading order

$$\mathcal{K}_{\eta\eta} - \mathcal{K}\mathcal{K}_\eta + \frac{\eta}{2}\mathcal{K}_\eta + \frac{\mathcal{K}}{2} = 0, \quad -\infty < \eta < \infty. \tag{4.109}$$

which is to be solved subject to matching with region V as $\eta \to \infty$, that is, we require

$$\mathcal{K}(\eta) \sim \eta \quad \eta \to \infty. \tag{4.110}$$

The solution to (4.109), (4.110) is readily obtained (see Section 1.1.3) as

$$\mathcal{K}(\eta) = \frac{2e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}\left(1 - \operatorname{erf}\left(\frac{\eta}{2}\right)\right)}, \quad -\infty < \eta < \infty. \tag{4.111}$$

Therefore in region B we have

$$u(\eta,t) = u_b + \frac{2e^{-\frac{\eta^2}{4}}}{\sqrt{\pi}\left(1-\text{erf}(\frac{\eta}{2})\right)}t^{-\frac{1}{2}} + o(t^{-\frac{1}{2}}) \tag{4.112}$$

as $t \to \infty$ with $\eta = O(1)$. As $\eta \to -\infty$ we move out of the localised region B into region VI, where $y = O(1)(\in (0,u_b))$ as $t \to \infty$, and

$$u(\eta,t) \sim u_b + \frac{1}{\sqrt{\pi}}e^{-\eta^2/4}t^{-1/2}. \tag{4.113}$$

When written in terms of $y$, (4.113) becomes

$$u(y,t) \sim u_b + \exp\left(-\frac{(y-u_b)^2}{4}t - \frac{1}{2}\ln t - \ln\frac{1}{\sqrt{\pi}}\right). \tag{4.114}$$

It is straightforward to then establish in region VI that

$$u(y,t) = u_b + \exp\left(-\frac{(y-u_b)^2}{4}t - \frac{1}{2}\ln t - L(y) + o(1)\right) \tag{4.115}$$

as $t \to \infty$ with $y = O(1)(\in (0,u_b))$ where $L(y)$ is undetermined but matching requires that

$$L(y) \sim \frac{1}{\sqrt{\pi}} \quad \text{as} \quad y \to u_b^-. \tag{4.116}$$

We immediately notice that expansion (4.115) does not satisfy boundary condition (4.3) and we conclude that expansion (4.115) must become non-uniform as $y \to 0^+$. An examination of expansion (4.115) indicates that this nonuniformity occurs when $y = O(t^{-1})$ as $t \to \infty$ [that is, when $x = O(1)$] . To complete the asymptotic analysis with introduce a final asymptotic region, region C. To investigate region C we first assume that

$$L(y) \sim -\ln y + \beta \quad \text{as} \quad y \to 0^+,$$

where $\beta$ is a constant (we will this prove is consistent later). An examination of (4.115) as $y \to 0^+$, when written in terms of $x$ indicates that

$$u \sim u_b + xe^{-\beta}t^{-3/2}e^{-\frac{u_b^2}{4}t}$$

as we move into region C. Therefore, we look for an expansion of the form

$$u(x,t) = u_b + F(x)t^{-3/2}e^{-\frac{u_b^2}{4}t} \tag{4.117}$$

as $t \to \infty$ with $x = O(1)$. Matching with region VI as $x \to \infty$ requires that

$$F(x) \sim xe^{-\beta}e^{\frac{u_b}{2}x} \quad \text{as} \quad x \to \infty. \tag{4.118}$$

On substituting (4.117) into equation (4.1) we obtain at leading order term that

$$F'' - u_bF' + \frac{{u_b}^2}{4}F = 0. \tag{4.119}$$

The solution of (4.119) subject to the matching condition (4.118) and boundary condition (4.3) requires

$$F(x) = e^{-\beta}xe^{\frac{u_b}{2}x}, \quad x \geqslant 0.$$

Therefore, we have in region C that

$$u(x,t) = u_b + xe^{-\beta}e^{\frac{u_b}{2}x}t^{-3/2}e^{-\frac{u_b^2}{4}t} + o(t^{-3/2}e^{-\frac{u_b^2}{4}t}) \tag{4.120}$$

as $t \to \infty$ with $x = O(1)$. This completes the asymptotic structure in this case. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4.5.

(ii) $u_b = 0$. In this subcase the asymptotic structure of the solution of QP given in regions IV,A and V follows on setting $\mathcal{A} = \frac{2u_+}{\sqrt{\pi}}$ and $u_b = 0$ that given above. We note that expansion (4.107) becomes non-uniform as $y \to 0^+$. In fact, this nonuniformity occurs when $y = O(t^{-1})$ as $t \to \infty$ [that is, when $x = O(1)$ as $t \to \infty$] and it is straightforward to establish that the expansion in the final region C is given by

$$u(x,t) = xt^{-1} + o(t^{-1})$$

as $t \to \infty$ with $x = O(1)$. The asymptotic structure of the solution to QP as $t \to \infty$ is now complete with expansions (4.104), (4.107), (4.112), (4.115) and (4.120) of regions A, V, B, VI and C respectively providing an uniform asymptotic solution to QP as $t \to \infty$.

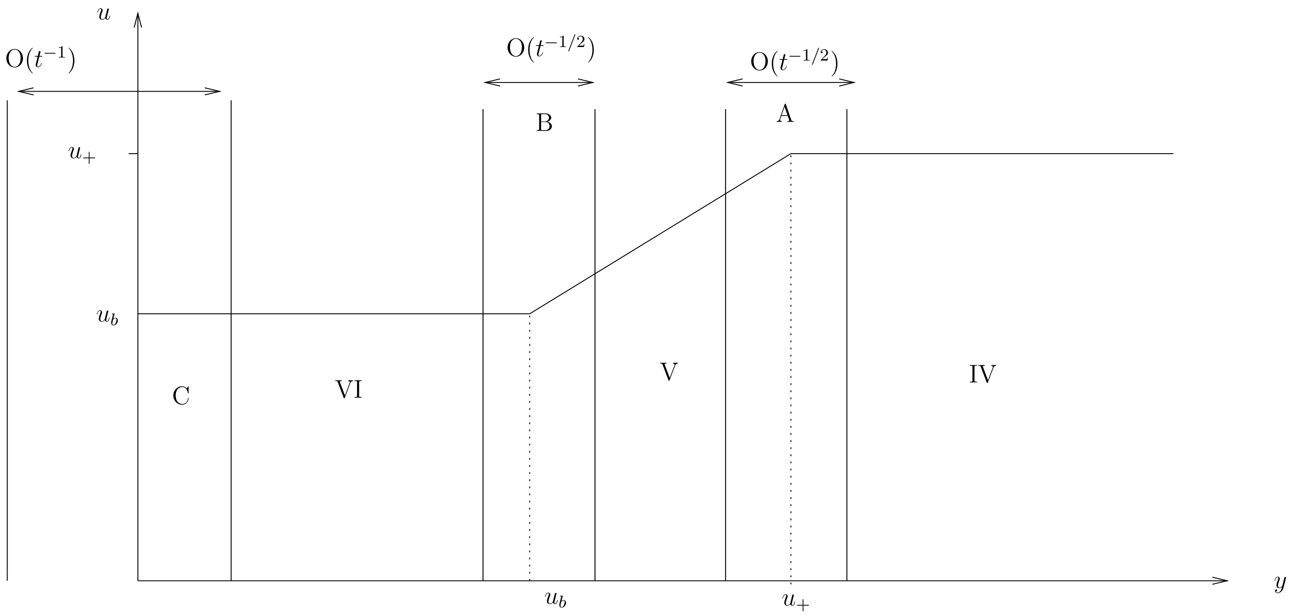


Figure 4.5: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane, as $t \to \infty$. We note that $u = u_+ + O(t^{-1/2})$ as $t \to \infty$ in region A, while $u = u_b + O(t^{-1/2})$ as $t \to \infty$ in region B and $u = u_b + O(1/t)$ as $t \to \infty$ in region C.

## 4.6 $\mathbf{u_b < 0 \quad u_+ > 0}$

In this case regions IV, A and V described in Section 4.5 are still present in the asymptotic solution of QP as $t \to \infty$ and their details are not repeated here. However, the asymptotic structure of the solution of QP as $t \to \infty$ differs from that given in Section 4.5 due to the fact that the parameter $u_b$ is now negative, and we must select, in region B, the following solution (see Section 1.1.3)

$$u(\eta, t) = \left(\eta - \frac{2}{\eta}\right) t^{-1/2} + o(t^{1/2}), \tag{4.121}$$

as $t \to \infty$ with $\eta = O(1)(> 0)$. Expansion (4.121) becomes non-uniform as $\eta \to 0$ and we introduce the final region, region SS. An examination of (4.121) indicates that this nonuniformity occurs when $\eta = O(t^{-1/2})$ as $t \to \infty$ [that is, when $x = O(1)$]. To examine region SS we look for an expansion of the form

$$u(x,t) = F(x) + O(1/t) \tag{4.122}$$

as $t \to \infty$ with $x = O(1)$ ($\geqslant 0$). On substituting (4.122) into equation (4.1) we obtain the leading order problem as

$$F_{xx} - FF_x = 0, \quad x > 0, \tag{4.123}$$

$$F(0) = u_b \ (< 0), \tag{4.124}$$

$$F(x) \to 0^- \quad \text{as} \quad x \to \infty. \tag{4.125}$$

The solution of (4.123)-(4.125) is readily obtained (see Section 1.1.2) as

$$F(x) = \frac{2}{\frac{2}{u_b} - x}, \quad x \geqslant 0. \tag{4.126}$$

Therefore, in region SS we have that

$$u(x,t) = \frac{2}{\frac{2}{u_b} - x} + o(1) \tag{4.127}$$

as $t \to \infty$ with $x = O(1)$ ($\geqslant 0$). The asymptotic structure of the solution to QP as $t \to \infty$ is now complete with expansions (4.121) and (4.127) of regions B and SS respectively providing an uniform asymptotic solution to QP as $t \to \infty$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4.6.

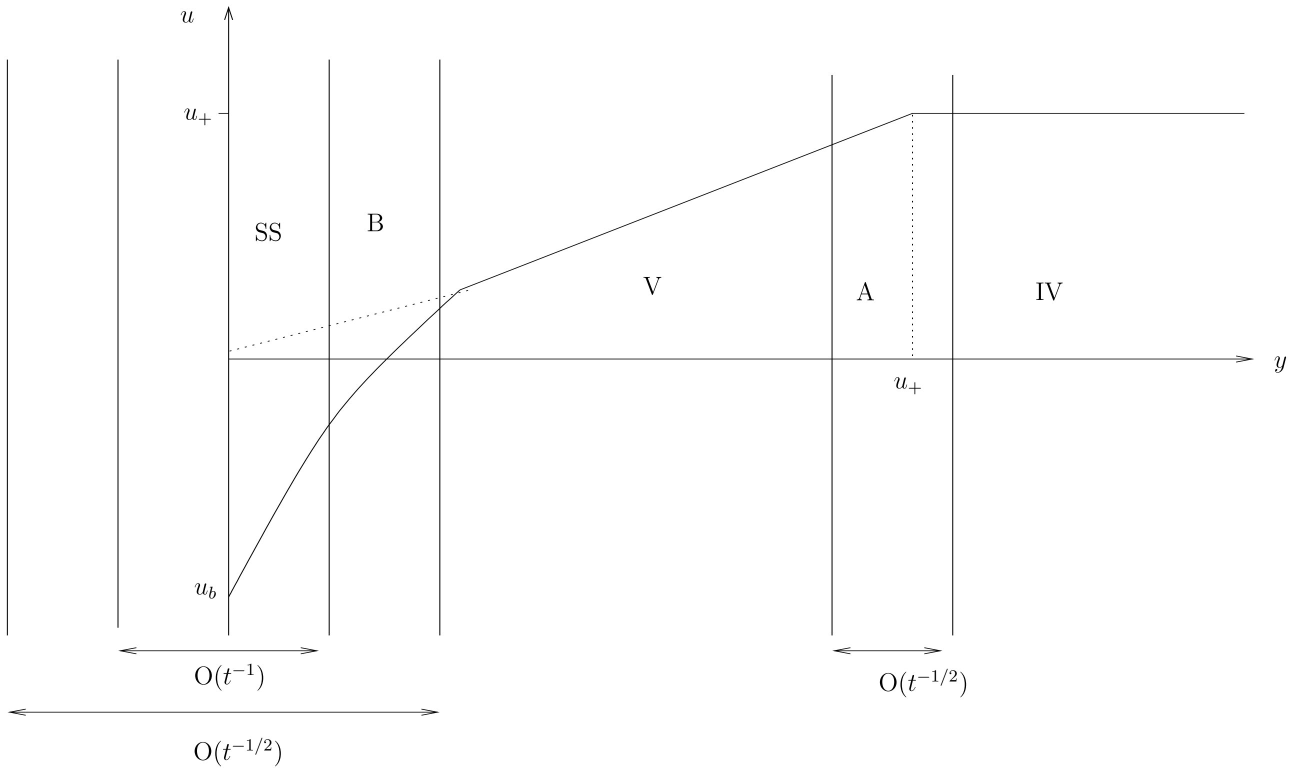


Figure 4.6: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane, as $t \to \infty$. We note that $u = u_+ + O(t^{-1/2})$ as $t \to \infty$ in region A, while $u = O(t^{-1/2})$ as $t \to \infty$ in region B and $u = O(1/t)$ as $t \to \infty$ in region SS.

## 4.7 Numerical Solutions

In this section, we present numerical solutions of initial-boundary value problem QP which illustrate the detailed asymptotic analysis given in this chapter. We again use the numerical method outlined in [35] to solve initial-boundary value problem (see Section 3.4). We recall that there are six cases to consider (see Figure 1.14), in each of these cases (see Sections 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6) we compare numerical simulation of the solution with of QP to the theoretically predicted solution. We consider each case in turn:

(i) $-u_b < u_+ < u_b$ with $u_b > 0$.

In this case we have established in Section 4.1 that a travelling wave develops in the solution of QP as $t \to \infty$. In Figure 4.7 we plot the numerical solution of QP when $u_b = 1$ and $u_+ = 0$ against $x$ at times $t = 5, 10, 15, 20$ and $t = 25$. Clearly, the numerical solution converges rapidly to the expected travelling wave as $t \to \infty$. This is in line with the theory where we expect the convergence to be exponential in $t$ as $t \to \infty$.

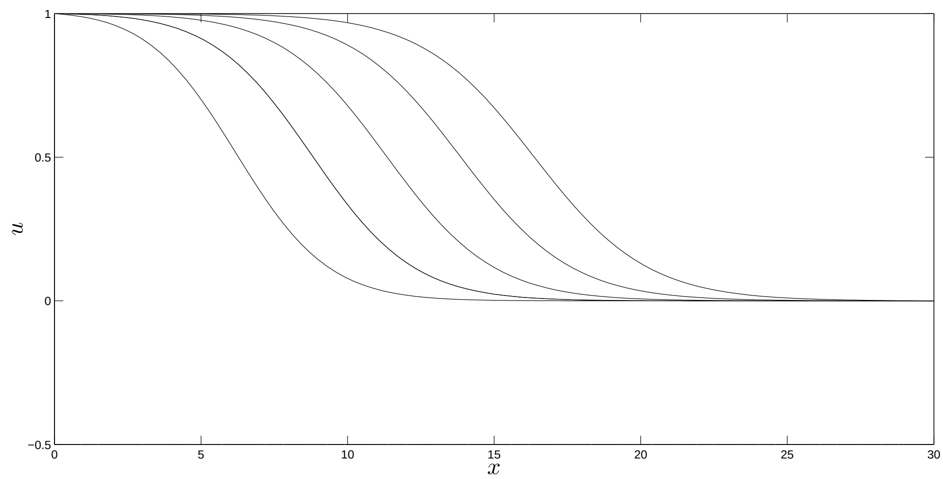


Figure 4.7: Numerical solution of QP when $u_b = 1$ and $u_+ = 0$ at times $t = 5, 10, 15, 20$ and $t = 25$.

Further, we have established that the wave speed of this travelling wave, $c$, is given by

$$c = \frac{u_b + u_+}{2} = \frac{1}{2},$$

and that the rate of convergence the solution of QP to the travelling wave as $t \to \infty$ is given by

$$u(z + s(t), t) = U_T\left(z, \frac{1}{2}\right) + O(\chi(t)) \tag{4.128}$$

as $t \to \infty$ with $z = O(1)$, where $U_T(z, \frac{1}{2})$ is the permanent form travelling wave (see (4.50)), and $z = x - s(t)$

$$s(t) = \frac{1}{2}t + O(\chi(t)) \tag{4.129}$$

where

$$\chi(t) = t^{-\frac{1}{2}} e^{-\frac{1}{16}t} \tag{4.130}$$

as $t \to \infty$. We note that the rate of convergence is exponential in $t$ as $t \to \infty$. The asymptotic wave speed, $\dot{s}(t)$, is given by

$$\dot{s}(t) = \frac{1}{2} + O(\chi(t)) \tag{4.131}$$

as $t \to \infty$. In Figure 4.8 we plot $\dot{s}(t)$ versus $t$. This figure illustrates the rapid convergence of the numerically calculated wave speed to the theoretically predicted wave speed again in line with the theory

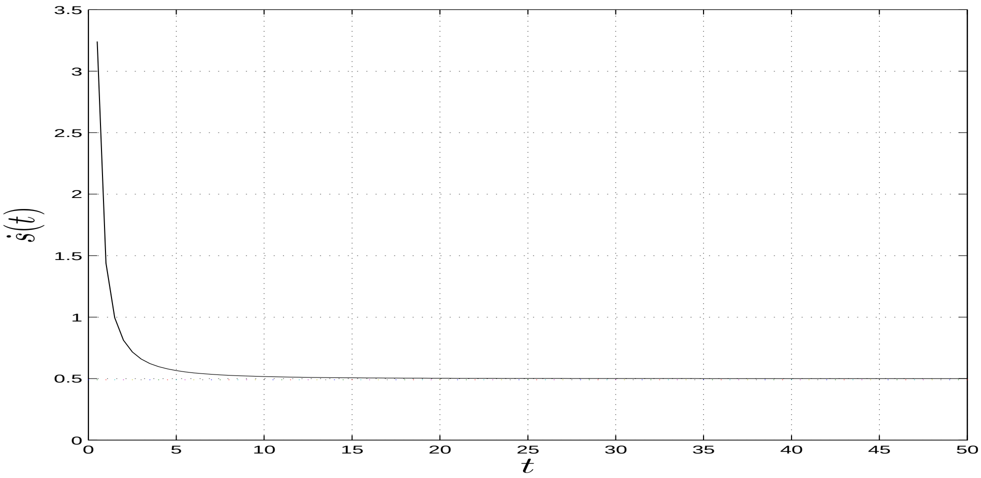


Figure 4.8: Numerical solution of $\dot{s}(t)$ versus $t$.

(ii) $-u_+ > u_b > u_+$ with $u_+ < 0$.

In this case we have established in Section 4.2 that the stationary state profile (4.76) develops in the solution of QP as $t \to \infty$. In Figure 4.9 we plot the numerical solution of QP when $u_b = 1$ and $u_+ = -2$ against $x$ at times $t = 5, 10, 15, 20$ and $t = 25$. The red line represents the theoretically predicted stationary state solution (4.76) (when $u_b = 1$ and $u_+ = -2$) that is

$$u(x) = -2\tanh\left(x + \tanh^{-1}\left(-\frac{1}{2}\right)\right).$$

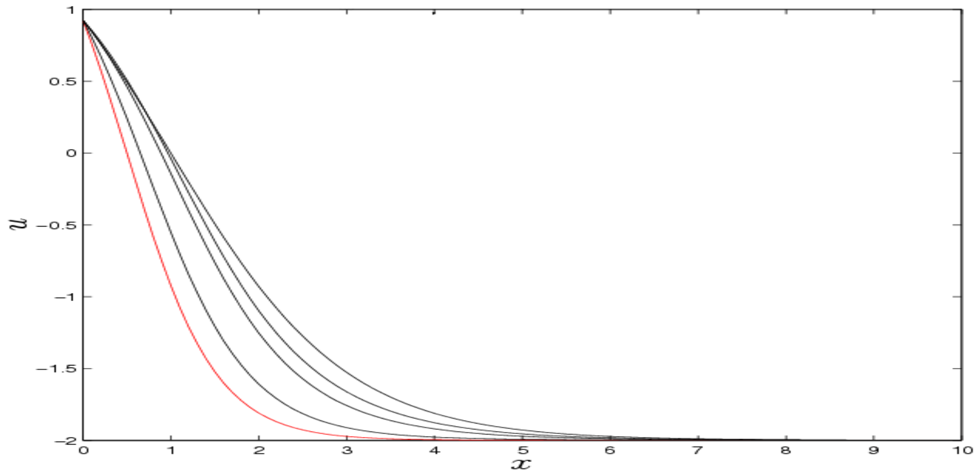


Figure 4.9: Graph of the numerical solution of QP in the $(x,\, u)$ plane when $u_b = 1$ and $u_+ = -2$ at times $t = 5, 10, 15, 20$ and 25. We note that the exact stationary state solution is shown by the red line at $t = 25$.

We can see that the numerical solution approaches the stationary state as $t \to \infty$. In fact, by $t = 25$ there is already good agreement as can be seen from Figure 4.9.

(iii) $u_b < u_+ < 0, \;\; u_b < 0$.

In this case we have established in Section 4.3 that the stationary state profile (4.84) develops in the solution of QP as $t \to \infty$. In Figure 4.10 we plot the numerical solution of QP when $u_b = -2$ and $u_+ = -1$ against $x$ at times $t = 5, 10, 15$ and 20. The red line represents the theoretically predicted stationary state solution (4.84) (when $u_b = -2$ and $u_+ = -1$) that is

$$u(x) = -2 \coth \left( x + \coth^{-1} \left( \frac{1}{2} \right) \right).$$

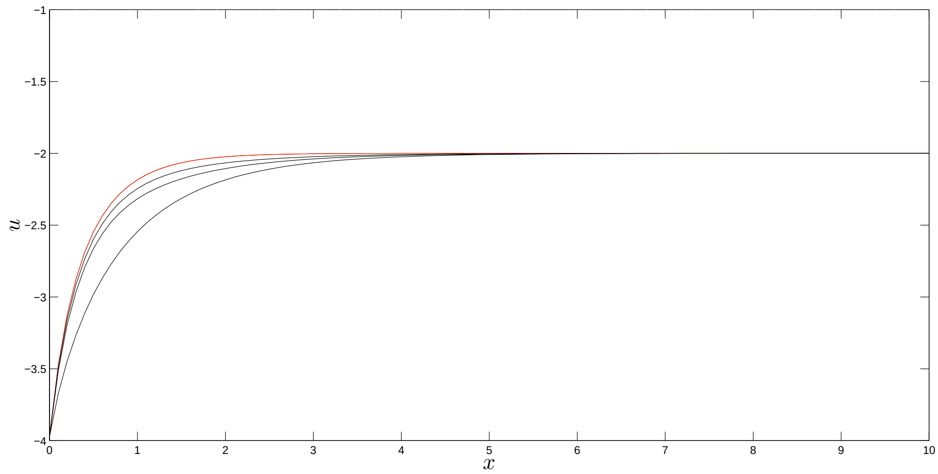


Figure 4.10: Graph of the numerical solution of QP in the $(x,\, u)$ plane when $u_b = -2$ and $u_+ = -1$ at times $t = 5, 10, 15$ and 20. We note that the exact stationary state solution is shown by the red line at $t = 20$.

We can see that the numerical solution approaches the stationary state as $t \to \infty$. In fact, by $t = 20$ there is already good agreement.

(iv) $u_b < 0$ and $u_+ = 0$.

In this case we have established in Section 4.4 that the stationary state profile (4.96) develops in the solution of QP as $t \to \infty$. In Figure 4.11 we plot the numerical solution of QP when $u_b = -1$ and $u_+ = 0$ against $x$ at times $t = 5, 10, 15, 20, 25$ and 30. The red line represents the theoretically predicted stationary state solution(4.96) (when $u_b = -1$ and $u_+ = 0$ ) that is

$$u(x) = -\frac{2}{2 + x}$$

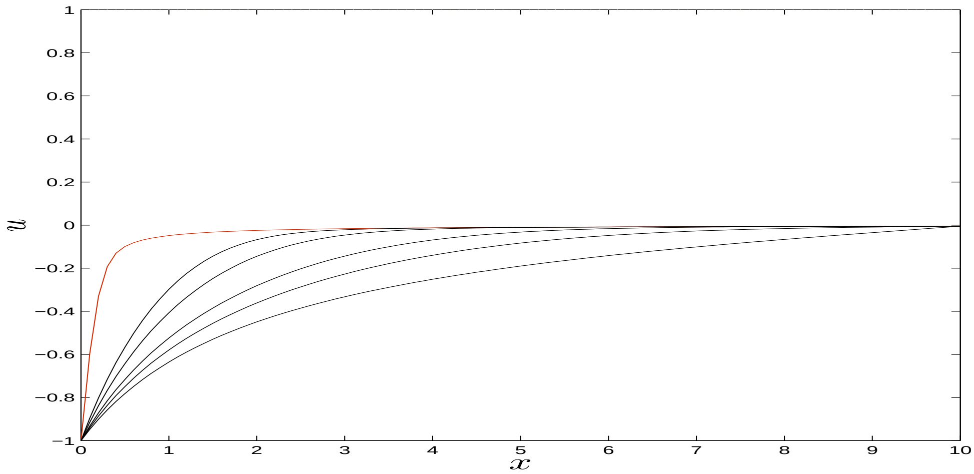


Figure 4.11: Graph of the numerical solution of QP in the $(x,\ u)$ plane when $u_b = -1$ and $u_+ = 0$ at times $t = 5, 10, 15, 20, 25$ and 30. We note that the exact stationary state solution is shown by the red line at $t > 30$.

We can see that the numerical solution approaches the stationary state as $t \to \infty$. In fact, by $t = 30$ there is already good agreement.

(v) $u_+ > u_b \geqslant 0$.

In this case we have established in Section 4.5 that a expansive wave develops in the solution of QP as $t \to \infty$. In Figure 4.12 we plot the numerical solution of QP when $u_b = 1$ and $u_+ = 4$ against $x$ at times $t = 5, 10, 15, 20, 25, 30, 35$ and 40.

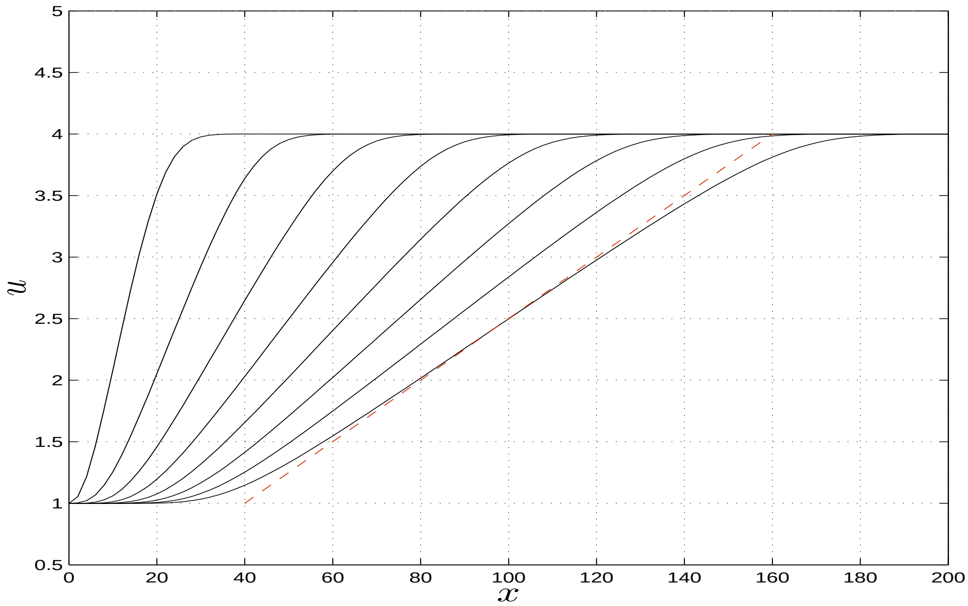


Figure 4.12: Graph of the numerical solution of QP in the $(x,\, u)$ plane when $u_b = 1$ and $u_+ = 4$ at times $t = 5, 10, 15, 20, 25, 30, 35$ and 40. The graph illustrates the development of the expansive wave with the solid lines showing the numerically computed solutions and the red dash line representing the predicted gradient.

We observe that the numerically computed solution of QP when $u_b = 1$ and $u_+ = 4$ in this approaches the predicted large-time attractor, the expansion wave. The gradient of the expansion wave when $t = 40$ is also sketched on Figure 4.11 the red dashed line. We can clearly see the correspondence of the gradient of the numerical solution at $t = 40$ and that of the dashed red line for $x \in (40, 160)$.

(vi) $u_b < 0$ and $u_+ > 0$.

Finally, in this case we have established in Section 4.6 that a expansive wave and a stationary state profile develop in the solution of QP as $t \to \infty$. In Figure 4.13 we plot the numerical solution of QP when $u_b = -1$ and $u_+ = 2$ against $x$ at times $t = 5, 10, 15, 20, 25, 30, 35,$ and 40.

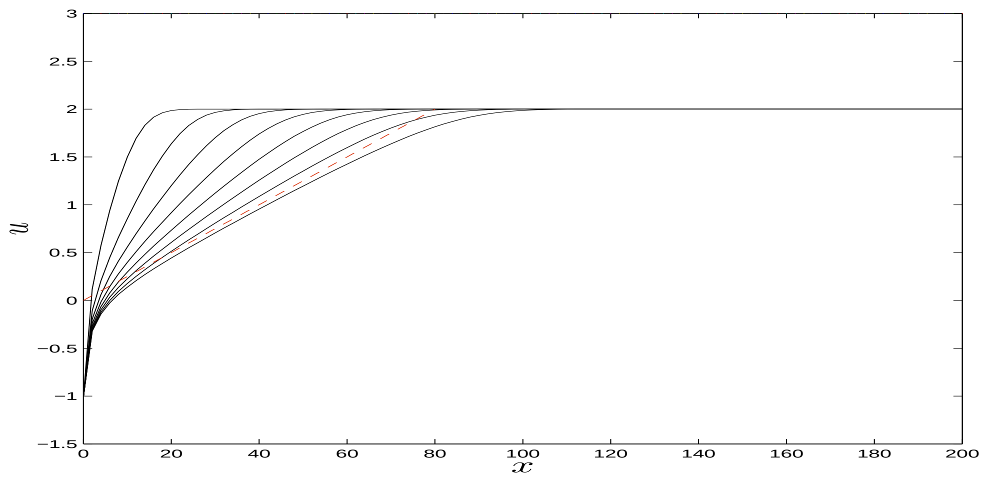


Figure 4.13: Graph of the numerical solution of QP in the $(x, u)$ plane when $u_b = -1$ and $u_+ = 2$ at times $t = 5, 10, 15, 20, 25, 30, 35,$ and 40. The graph illustrates the development of expansive wave and stationary profile with the solid lines show numerically computed solutions and the dash line representing the predicted gradient at $t = 40$.

We observe that the numerically computed solution approaches the predicted large-time attractor which is composed primaly of the stationary state and the expansive wave (see Figure 4.6). The gradient of the stationary state profile when $t = 40$ is also sketched on Figure 4.12 by red dashed line. We can clearly see that when $x \in [at, u_+ t]$ for any fixed number $a \in (0, u_+)$ that the numerical solution approaches the expansive wave. Specifically, at $t = 40$ we see that the gradient of numerical solution is in good agreement with the expected value given by red dashed line.

Further, in Figure 4.14 we present a close up of the numerical solution for $x \in [0, 80]$ when $t = 40$. Again we can see good agreement between the numerical solution and the stationary state when $u_b = -1$ given by

$$u(x) = -\frac{2}{x-2},$$

over the range $x \in [0, bt^{1/2}]$ where $b$ is positive fixed number.

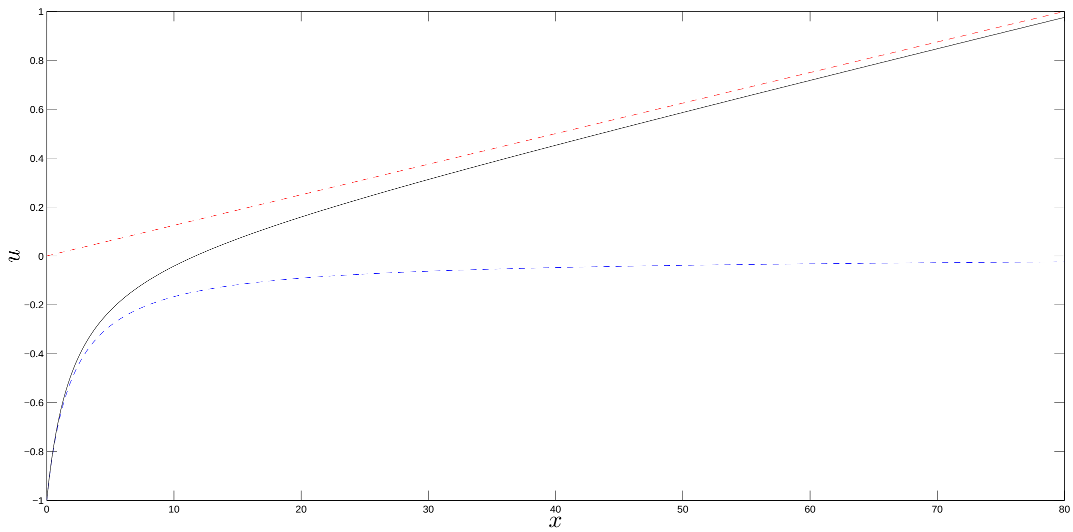


Figure 4.14: Graph of the numerical solution of QP in the $(x,\ u)$ plane for $x \in [0, 80]$ when $u_b = -1$ and when $t = 40$.

## 4.8 Summary

In this chapter we have obtained the complete asymptotic structure of the solution to QP as $t \to \infty$ over all parameter values. The type of large-time attractor which develops as $t \to \infty$ is governed by the parameters $u_+$ and $u_b$. Figure 1.13 gives the $(u_+, u_b)$ parameter plane indicating the parameter ranges over which of the various large-time attractors exist.

In Section 4.1 we develop via the method of matched asymptotic expansion the complete large-time solution of QP when $-u_b < u_+ < u_b$ with $u_b > 0$. A uniform approximation has been given through regions IV(a), TR, IV(b), TW, $\text{IV}^-$(b), TR and $\text{IV}^-$(a). A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 4.1. We have demonstrated that the solution to QP, $u(x,t)$, has

$$u(z + s(t), t) = \frac{u_+ + u_b e^{-Az}}{1 + e^{-Az}} + O(t^{-3/2} e^{-\frac{A^2}{4} t}) \tag{4.132}$$

as $t \to \infty$ with $z = O(1)$ where $A = \frac{(u_b - u_+)}{2}$ and

$$s(t) = ct + O(t^{-3/2} e^{-\frac{A^2}{4} t})$$

as $t \to \infty$ where $c = \frac{(u_b + u_+)}{2}$.

A travelling wave develops as $t \to \infty$ in the solution QP in this case. The rate of convergence of the solution to QP onto the travelling wave is exponential in $t$, being of $O(t^{-3/2} e^{-\frac{A^2}{4} t})$ as $t \to \infty$.

In Section 4.2 we develop via the method of matched asymptotic expansion the complete large-time solution of QP when $-u_+ > u_b > u_+$ and $u_+ < 0$. A uniform approximation has been given through regions IV(a), TR, IV(b), V. A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 4.2. A stationary profile develops in region V, we have demonstrated that

$$u(x,t) = u_+ \tanh\left(-\frac{u_+}{2}x + \tanh^{-1}\left(\frac{u_b}{u_+}\right)\right) + o(1) \tag{4.133}$$

as $t \to \infty$ with $x \geqslant 0$. The rate of convergence of the solution to QP is exponential in $t$ as $t \to \infty$.

In Section 4.3 we develop via the method of matched asymptotic expansion the complete large-time solution of QP when $u_b < u_+ < 0$ and $u_b < 0$. A uniform approximation has been given through regions IV(a), TR, IV(b), V. A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 4.3. We have demonstrated that the solution to QP, $u(x,t)$, has

$$u(x,t) = u_+ \coth\left(-\frac{u_+}{2}x + \coth^{-1}\left(\frac{u_b}{u_+}\right)\right) + o(1) \tag{4.134}$$

as $t \to \infty$ with $x \geqslant 0$. A stationary state solution develops as $t \to \infty$ in the solution QP in this case. The rate of convergence of the solution to QP is exponential in $t$ as $t \to \infty$.

In Section 4.4 we develop via the method of matched asymptotic expansion the complete large-time solution of QP when $u_b < 0$ and $u_+ = 0$. A uniform approximation has been given through regions IV, V and VI. A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 4.4. We have demonstrated that the solution to QP, $u(x,t)$, has

$$u(x,t) = \frac{2}{\frac{2}{u_b} - x} + o(1) \tag{4.135}$$

as $t \to \infty$ with $x = O(1)$ $(\geqslant 0)$. A stationary state solution develops as $t \to \infty$ in the solution QP in this case.

In Section 4.5 we develop via the method of matched asymptotic expansion the complete large-time solution of QP when $u_+ > u_b \geqslant 0$. A uniform approximation has been given through regions IV, A, V, B, VI, C. A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 4.5. The solution exhibits in $x \geqslant 0$ the formation of an expansion wave when $u_+ > u_b \geqslant 0$. This expansion wave develops in expansion region, region V, where $y = O(1)(\in (u_b, u_+))$ as $t \to \infty$. Regions IV and VI allow for the transfer of information from the for field $|y| \gg 1$ [that is, $|x| \gg t$] to the near field ($y = O(1)$). We note that at leading order in region IV the solution to QP, $u$, is $O(1)$ and is given at leading order by the constant value $u_+$ (the value of $u$ ahead of the expansion wave). Regions A and B are localized connection regions connecting regions IV and VI to the expansion wave, respectively.

In Section 4.6 we develop via the method of matched asymptotic expansion the complete large-time solution of QP when $u_b < 0$ and $u_+ > 0$. A uniform approximation has been given through regions IV, A, V, B, SS. A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 4.6. The asymptotic structure of the solution of QP as $t \to \infty$ differs from that given in Section 4.5, there is also a stationary profile in region SS we have demonstrated that

$$u(x,t) = \frac{2}{\frac{2}{u_b} - x} + o(1) \tag{4.136}$$

as $t \to \infty$ with $x = O(1)$ ($\geqslant 0$).

In Section 4.7, we present numerical solutions of QP which confirm and support the asymptotic analysis presented in the sections mentioned above. In all case the numerical simulations are in good agreement with the theory as $t \to \infty$.

# Chapter 5

# An Initial-Value Problem for the Burgers-Fisher Equation

In this chapter, we consider the following initial-value problem for the Burgers-Fisher equation with step initial data, given by

$$u_t + kuu_x = u_{xx} + u(1-u), \quad -\infty < x < \infty, \quad t > 0 \tag{5.1}$$

$$u(x,0) = \begin{cases} 1 & \text{as} \quad x \leqslant 0, \\ 0 & \text{as} \quad x > 0, \end{cases} \tag{5.2}$$

$$u(x,t) \to \begin{cases} 1, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \qquad t \geqslant 0, \tag{5.3}$$

where $k \neq 0$ is a parameter and initial distribution (5.2) is a discontinuous compressive step.

In what follows we label the initial value problem (5.1)-(5.3) as IVP3. Equation (5.1) is a canonical equation combining reaction, diffusion and convection and as such arises in the modelling of many physical phenomena involving reaction-diffusion convection processes. When $k = 0$ equation (5.1) reduces to Fisher-Kolmogorov equation which has been studied extensively (see for example [6], [21], [32], [36], [42] and [44]). Specifically, Bramson [6, 7] considered (5.1) (with $k = 0$) when the initial data has a step function profile (5.2) and determined that the solution displays the formation of a permanent form travelling wave solution with propagation speed is given by $\nu(t) \sim 2 - \frac{3}{2}\frac{1}{t}$ as $t \to \infty$. In this chapter, we develop the large-time structure of the solution to IVP3 using the method of matched asymptotic coordinate expansions. As in previous chapters we employ the methodology developed by the J.A Leach and D.J. Needham in the context of reaction-diffusion equations (see for example the monograph [37]) the structure of the solution to IVP3 obtained by careful consideration of the asymptotic structures as $t \to 0$ ($-\infty < x < \infty$) and as $|x| \to \infty$ ($t \geqslant O(1)$). In particular, we establish that the solution of IVP3 exhibits the formation of a permanent form travelling wave (PTW) propagations in the $+x$ direction with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

Specifically, we establish that:

(i) When $k \in (2, \infty)$ the solution of IVP3 satisfies

$$u(z + s(t), t) = u_T(z; c^*(k)) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$, uniformly in $z$, where, $u_T(z; c^*(k))$ is the permanent for travelling wave solution with propagation speed $c^*(k) = \frac{2}{k} + \frac{k}{2}$ is given by

$$u_T(z; c^*(k)) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}},$$

$z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time $t$) and

$$s(t) = \left(\frac{2}{k} + \frac{k}{2}\right) t + \phi_c + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$, where $\phi_c$ is a constant. We note that the correction to the propagation speed, $\dot{s}(t)$, is exponential in $t$, as $t \to \infty$, being of $O\left(t^{-\frac{3}{2}} \exp\left(-\left(\frac{[c^*(k)]^2}{4} - 1\right) t\right)\right)$. Further, the rate of convergence of the solution of IVP3 to the permanent form travelling wave is exponential in $t$, as $t \to \infty$, being of

$$O\left(t^{-\frac{3}{2}} \exp\left(-\left(\frac{[c^*(k)]^2}{4} - 1\right) t\right)\right).$$

(ii) When $k \in (-\infty, 2]$ the solution of $u(x,t)$ of IVP3 satisfies that

$$u(z + s(t), t) = u_T(z; 2) + O(t^{-1})$$

as $t \to \infty$, uniformly in $z$, where, $u_T(z; 2)$ is the permanent for travelling wave solution with propagation speed 2, $z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time $t$) and

$$s(t) = 2t - \frac{3}{2} \ln t + O(1)$$

as $t \to \infty$. We note that the correction to the propagation speed, $\dot{s}(t)$, is algebraic in $t$, as $t \to \infty$, being of $O(t^{-1})$. Further, the rate of convergence of the solution of IVP3 to the permanent form travelling wave is algebraic in $t$, as $t \to \infty$, being of $O(t^{-1})$.

## 5.1 Asymptotic Solution of IVP3 as t → 0

Consideration of the initial data (5.2) indicates that the structure of the asymptotic solution to IVP3 as $t \to \infty$ has three asymptotic regions for $x \in (-\infty, \infty)$, namely,

$$\left.\begin{array}{lll} \text{Region I: } x = o(1), & u(x,t) = O(1), \\ \text{Region II}^{+}: x = O(1)\ (>0), & u(x,t) = o(1), \\ \text{Region II}^{-}: x = O(1)\ (<0), & u(x,t) = 1 - o(1), \end{array}\right\} \quad \text{as} \quad t \to 0.$$

For brevity we summarize the structure of the solution of IVP3 in each of the above regions (The details following, after some modification, those given in the earlier chapters of this thesis for a similar problem [38]):

**Region I** $\eta = O(1)$ as $t \to 0$,

$$u(\eta, t) = \frac{1}{2}\,\text{erfc}\left(\frac{\eta}{2}\right) + o(1) \tag{5.4}$$

where $\eta = xt^{-1/2} = O(1)$ and where erfc(.) is the standard complementary error function (see Chapter 2).

**Region II$^{-}$** $x = O(1)\ (<0)$ as $t \to 0$,

$$u(x,t) = 1 - \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t + \frac{k}{2}x - \ln(-x) - \ln\sqrt{\pi} + o(1)\right). \tag{5.5}$$

**Region II$^{+}$** $x = O(1)\ (>0)$, as $t \to 0$,

$$u(x,t) = \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t - \ln x - \ln\sqrt{\pi} + o(1)\right). \tag{5.6}$$

The asymptotic solution of IVP3 as $t \to 0$ is now complete with expansions (5.4), (5.5) and (5.6) providing a uniform approximation to the solution of IVP3 as $t \to 0$.

Next, we determine the structure of the asymptotic solution of IVP3 as $|x| \to \infty$ with $t = O(1)$.

## 5.2 Asymptotic Solution of IVP3 as $|x| \to \infty$

For brevity, we summarize the asymptotic structure of the solution to IVP3 as $x \to -\infty$ and $x \to \infty$. (The details following, after some modification, those given in earlier chapters and for a similar problem [38]):

**Region III$^-$** as $x \to -\infty$ with $t = O(1)$,

$$u(x,t) = 1 - \exp\left(-\frac{x^2}{4t} + \frac{k}{2}x - \ln(-x) + \left(\frac{1}{2}\ln t - \left(\frac{k^2}{4} + 1\right)t - \ln\sqrt{\pi}\right) + o(1)\right). \quad (5.7)$$

**Region III$^+$** as $x \to \infty$ with $t = O(1)$,

$$u(x,t) = \exp\left(-\frac{x^2}{4t} - \ln x + \left(\frac{1}{2}\ln t + t - \ln\sqrt{\pi}\right) + o(1)\right). \quad (5.8)$$

We observe that expansion (5.7) (in region III$^-$) and (5.8) (in region III$^+$) remain uniform for $t \gg 1$ provided that $|x| \gg t$ but become non-uniform when $|x| = O(t)$ as $t \to \infty$.

## 5.3 Asymptotic Solution of IVP3 as $t \to \infty$

As $t \to \infty$, the asymptotic expansions (5.8) and (5.7), which are defined in region III$^+$ ($x \to \infty$, $t = O(1)$) and region III$^-$ ($x \to -\infty$, $t = O(1)$) remain uniform provided $|x| \gg t$ whereas non-uniformity occurs when $|x| = O(t)$.

Firstly, we consider the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we define a new region, which we label as region IV$^+$, where $y = O(1)(> 0)$ as $t \to \infty$. We introduce a new scale coordinate as $y = \frac{x}{t}$ where $y = O(1)$ as $t \to \infty$ and look for an expansion of the form (as suggested by (5.8))

$$u(y,t) = e^{-R(y,t)} \quad \text{as} \quad t \to \infty, \tag{5.9}$$

with

$$R(y,t) = r_0(y)t + r_1(y)\ln t + r_2(y) + o(1), \tag{5.10}$$

as $t \to \infty$ with $y = O(1)$ and where $r_0(y) > 0$. On substituting (5.9) and (5.10) into equation (5.1) (when written in terms of $y$ and $t$) we obtain the leading order problem as

$$(r_0')^2 - yr_0' + r_0 + 1 = 0 \quad y > 0, \tag{5.11}$$

$$r_0(y) > 0 \quad y > 0, \tag{5.12}$$

$$r_0(y) \sim \frac{y^2}{4} - 1 \quad \text{as } y \to \infty, \tag{5.13}$$

The final condition (5.13), arises from matching expansion (5.9) ($y \gg 1$) with expansion (5.8) ($x = O(t)$). Equation (5.11) admits the constant solution $r_0(y) = -1$, the one-parameter family of linear solutions,

$$r_0(y) = A\left[y - \left(A + \frac{1}{A}\right)\right], \quad -\infty < y < \infty \tag{5.14}$$

for any $A \in \mathbb{R}$, together with the associated envelope (singular) solution,

$$r_0(y) = \frac{y^2}{4} - 1, \quad -\infty < y < \infty. \tag{5.15}$$

Combinations of (5.14) and (5.15) which remain continuous and differentiable also provide solutions to (5.11) (envelope touching solutions). Therefore, the solution of (5.11)-(5.13) is given either by the envelope solution

$$r_0(y) = \frac{y^2}{4} - 1, \quad 2 < y < \infty. \tag{5.16}$$

or by the family of envelope touching solutions

$$r_0(y) = \begin{cases} \frac{y^2}{4} - 1, & 2A < y < \infty, \\ A\left[y - \left(A + \frac{1}{A}\right)\right], & A + \frac{1}{A} < y \leqslant 2A, \end{cases} \tag{5.17}$$

for each $A > 1$. We conclude a non-uniformity occurs in expansion (5.9), (5.10) as $y \to {y_c}^+ (\geqslant 2)$ where

$$y_c = \begin{cases} 2, & A = 1, \\ A + \frac{1}{A} \ (> 2), & A > 1, \end{cases}$$

for some $A \geqslant 1$ (when $A = 1$, $r_0(y)$ is given by (5.16), whilst when $A > 1$, $r_0(y)$ is given by (5.17)). A consideration of further terms in (5.10) demonstrates that this nonuniformity occurs when

$$y = y_c + O(t^{-1}) \quad \text{with} \quad u = O(1)$$

as $t \to \infty$. We must introduce a further region which we denote as region TW. In this region we write

$$y = y_c + \frac{z}{t}$$

as $t \to \infty$ and expand as

$$u(z,t) = u_c(z) + o(1) \tag{5.18}$$

as $t \to \infty$ with $z = O(1)$. By substituting (5.18) into equation (5.1) (when written in terms of $z$ and $t$) we obtain at leading order that

$$\begin{aligned} &u_{zz} - kuu_z + y_c u_z + u(1-u) = 0, \quad -\infty < z < \infty, \\ &u(z) \geqslant 0, \quad -\infty < z < \infty, \\ &u(z) \to 0 \quad \text{as} \quad z \to \infty, \\ &u(z) \quad \text{bounded as} \quad z \to -\infty. \end{aligned} \tag{5.19}$$

Condition $(5.19)_3$ arises from matching expansion (5.18) (as $z \to \infty$) with expansions (5.9) and (5.10) (as $y \to {y_c}^+$). Furthermore, a phase plane analysis of equation $(5.19)_1$ with conditions $(5.19)_2$ and $(5.19)_3$, allows boundary condition $(5.19)_4$ to be replaced by

$$u_c(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{5.20}$$

We now recognize (5.19) (with (5.20)) as being precisely (1.81)-(1.84), its solutions representing permanent form travelling wave structures. We can now appeal to Theorem 1: boundary value problem (5.19) (with (5.20)) has a unique solution $u_c(z) = u_T(z; y_c)$ for each

$$y_c \in \begin{cases} [2, \infty), & -\infty < k \leqslant 2, \\ \left[\frac{2}{k} + \frac{k}{2}, \infty\right), & k > 2 \end{cases} \tag{5.21}$$

where $y_c$ is the wave speed.

We next match expansion (5.9), (5.10) of region IV$^+$ (as $y \to {y_c}^+$) to expansion (5.18) (as $z \to \infty$) to leading order, in each of the distinct cases $k \in (2, \infty)$ and $k \in (-\infty, 2]$. When $k \in (2, \infty)$ we have, via (5.18) and (1.86), that

$$u \sim \begin{cases} e^{-\frac{k}{2}z} \quad \text{as} \quad z \to \infty, \quad c = \frac{k}{2} + \frac{2}{k}, \\ \\ B^* e^{\lambda_+ z} \quad \text{as} \quad z \to \infty, \quad c > \frac{k}{2} + \frac{2}{k}; \end{cases}$$

On expanding expansion (5.9), (5.10) to $O(1)$ in region TW we obtain that

$$u = O(e^{-Az}) \quad \text{for} \quad z \gg 1, \quad A \geqslant 1.$$

Therefore, matching following the matching principle of Van Dyke [58] requires that

$$A = \frac{k}{2} \quad (> 1),$$

giving that $y_c = \frac{k}{2} + \frac{2}{k}$ and the travelling wave solution of minimum propagation speed $\frac{k}{2} + \frac{2}{k}$ is selected in region TW. When $k \in (-\infty, 2]$ we have, via (5.18) and (1.84), that

$$u \sim \begin{cases} (A^* z + D^*) e^{-z} \quad \text{as} \quad z \to \infty, \quad c = 2, \\ \\ B^* e^{\lambda_+ z} \quad \text{as} \quad z \to \infty, \quad c > 2, \end{cases}$$

In this case matching requires that

$$A = 1$$

giving that $y_c = 2$ and the travelling wave solution of minimum propagation speed 2 is selected in region TW.

Consequently, we have established that the travelling wave solution of minimum speed, $c = c^*(k)$, is selected in region TW where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases} \tag{5.22}$$

Further, when $k \in (-\infty, 2]$, $r_0(y)$ is given by (5.16); while when $k \in (2, \infty)$, $r_0(y)$ is given by (5.17).

In what follows we will consider the cases $k \in (-\infty, 2]$ and $k \in (2, \infty)$ separately.

### 5.3.1 $\mathbf{k} \in (\mathbf{2}, \infty)$

In this case $r_0(y)$ is given by (5.17) with $A = \frac{k}{2}$, that is

$$r_0(y) = \begin{cases} \frac{y^2}{4} - 1, & k < y < \infty, \\ \frac{k}{2}\left(y - \left[\frac{2}{k} + \frac{k}{2}\right]\right), & c^*(k) < y \leqslant k \end{cases} \tag{5.23}$$

and $c^*(k) = \frac{2}{k} + \frac{k}{2}$. Figure 5.1 gives a sketch of the solution (5.23). We observe that although $r_0(y)$ and $r_0'(y)$ are continuous at $y = k$, the second derivative $r_0''(y)$ is discontinuous at the point $y = k$. This indicates that in the point $y = k$ there is a thin transition region, region TR$^+$, is required to smooth out the discontinuity of curvature at the point at which the linear solution $(5.23)_2$ meets the envelope solution $(5.23)_1$. Hence, IV$^+$ is replaced by three regions: region IV$^+$(a) ($k < y < \infty$), region TR$^+$ (transition region) and region IV$^+$(b) ($c^*(k) < y < k$). We will examine each of these regions in turn.

We start in region IV$^+$(a) where $r_0(y)$ is given by $(5.23)_1$. On continuing expansion

(5.9), (5.10) we obtain in region IV$^+$(a) that

$$u(y,t) = \exp\left(-\left(\frac{y^2}{4} - 1\right)t - \frac{1}{2}\ln t - \psi(y) + o(1)\right) \tag{5.24}$$

as $t \to \infty$ with $y = O(1)(\in (k,\infty))$ and where $\psi(y)$ remains undetermined, but having

$$\psi(y) \sim \ln y + \ln\sqrt{\pi} \quad \text{as} \quad y \to \infty.$$

Further, we make the assumption (which we will verify as consistent) that

$$\psi(y) \sim \ln(y-k) + \widehat{\beta}_1 \quad \text{as} \quad y \to k^+$$

where $\widehat{\beta}_1$ is a constant to be determined. We next consider region IV$^+$(b) where $r_0(y)$ is given by $(5.23)_2$. On continuing expansions (5.9), (5.10) we obtain in region IV$^+$(b) that

$$u(y,t) = \exp\left(-\frac{k}{2}\left(y - \left[\frac{2}{k} + \frac{k}{2}\right]\right) - b\ln t - b\ln(k-y) - c_1 + o(1)\right), \tag{5.25}$$

as $t \to \infty$ with $y = O(1)(\in (c^*(k), k))$, and where $b$ and $c_1$ are constants to be determined.

Before considering region TR$^+$ it is instructive to examine the asymptotic structure for $y < c^*(k)$. To proceed we introduce a new region, region IV$^-$, write (as suggested by (5.7))

$$u(y,t) = 1 - e^{-\widehat{R}(y,t)} \tag{5.26}$$

where

$$\widehat{R}(y,t) = \widehat{r}_0(y)t + \widehat{r}_1(y)\ln t + \widehat{r}_2(y) + o(1), \tag{5.27}$$

where $y = O(1)$ $(\in (-\infty, c^*(k)))$ as $t \to \infty$ and $\widehat{r}_0(y) > 0$. On substituting (5.26) and (5.27) into equation (5.1) (when written in terms of $y$ and $t$) we obtain the leading order problem as

$$(\widehat{r}_0')^2 + (k-y)\widehat{r}_0' + \widehat{r}_0 - 1 = 0 \quad y < c^*(k), \tag{5.28}$$

$$\widehat{r}_0(y) > 0 \quad y < c^*(k), \tag{5.29}$$

$$\widehat{r}_0(y) \sim \frac{(y-k)^2}{4} + 1 \quad \text{as} \quad y \to -\infty, \tag{5.30}$$

$$\widehat{r}_0(y) = -\frac{k}{2}(y - c^*(k)) \quad \text{as} \quad y \to c^*(k)^-. \tag{5.31}$$

Condition (5.30) arises from matching expansion (5.26) ($-y \gg 1$) with expansion (5.7)(as $-x = O(t)$), while the final condition (5.31) is the matching condition to allow matching with expansion (5.18) (as $z \to -\infty$) of region TW. Equation (5.28) admits the constant solution $\widehat{r}_0(y) = 1$, the one-parameter family of linear solution,

$$\widehat{r}_0(y) = \widehat{A}\left[y - \left(\widehat{A} - \frac{1}{\widehat{A}} + k\right)\right], \quad -\infty < y < \infty, \tag{5.32}$$

for any $\widehat{A} \in \mathbb{R}$, together with the associated envelope solution

$$\widehat{r}_0(y) = \frac{(y-k)^2}{4} + 1 \quad -\infty < y < \infty. \tag{5.33}$$

Combinations of (5.32) and (5.33) which remain continuous and differentiable also provide solutions to (5.11) (envelope touching solutions). It is straightforward to establish that the required solution of (5.28)-(5.31) is given by the envelope-touching solution

$$\widehat{r}_0(y) = \begin{cases} \frac{(y-k)^2}{4} + 1, & -\infty < y < 0, \\ -\frac{k}{2}(y - c^*(k)) & 0 \leqslant y < c^*(k) \end{cases} \tag{5.34}$$

where $\widehat{A} = -\frac{k}{2}$. Figure 5.1 gives a sketch of the solution (5.34). Therefore, region IV$^-$ is replaced by three regions, region IV$^-$(a) ($-\infty < y < 0$), region TR$^-$ (transition region) and region IV$^-$(b) ($0 < y < c^*(k)$). We note that the transition region located at $y = 0$ is required to smooth out the discontinuity of curvature at the point at which the linear solution $(5.34)_2$ meets the envelope solution $(5.34)_1$. On continuing expansion (5.26), (5.27) in regions IV$^-$(a) and IV$^-$(b) where $\widehat{r}_0(y)$ are given by $(5.34)_1$ and $(5.34)_2$ respectively, we have that:

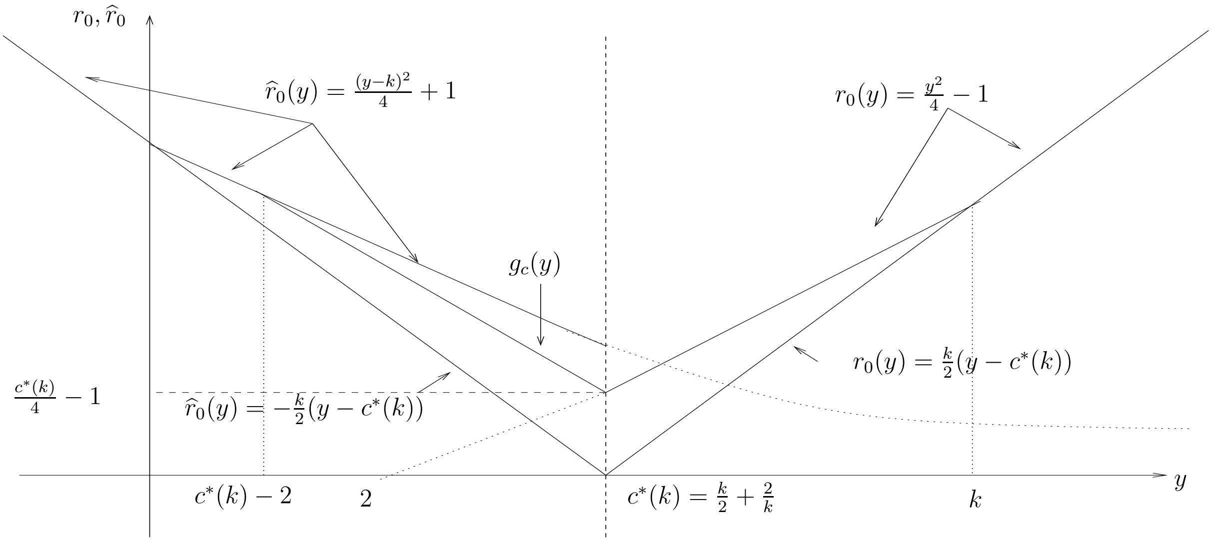


Figure 5.1: A sketch of the functions $r_0(y)$ and $\widehat{r}_0(y)$ when $k \in (2, \infty)$. We note that $g_c(y) = \widehat{A}_c \left( y - \left( \widehat{A}_c + k - \frac{1}{\widehat{A}_c} \right) \right)$ for $y \in (c^*(k) - 2, c^*(k))$ and where $\widehat{A}_c = \frac{1}{2} \left( \frac{2}{k} - \frac{k}{2} \right) - 1$.

**Region IV$^-$(a)**

$$u(y,t) = 1 - \exp\left( -\left( \frac{(y-k)^2}{4} + 1 \right) t - \frac{1}{2} \ln t - \widehat{\psi}(y) + o(1) \right) \tag{5.35}$$

as $t \to \infty$ with $y = O(1)$ $(\in (-\infty, 0))$ where the function $\widehat{\psi}(y)$ remains undetermined, but having

$$\widehat{\psi}(y) \sim \ln(-y) + \ln\sqrt{\pi} \quad \text{as} \quad y \to -\infty.$$

Further, we make the assumption (which we will verify as consistent) that

$$\widehat{\psi}(y) \sim \ln(-y) + \widehat{\gamma}_1 \quad \text{as} \quad y \to k^+,$$

where $\widehat{\gamma}_1$ is a constant to be determined.

**Region IV$^-$(b)**

$$u(y,t) = 1 - \exp\left(-\frac{k}{2}\left[y - c^*(k)\right]t - \widehat{b}\ln t - \widehat{b}\ln y - \widehat{c}_1 + o(1)\right), \tag{5.36}$$

as $t \to \infty$ with $y = O(1)$ $(\in (0, c^*(k)))$, and where $\widehat{b}$ and $\widehat{c}_1$ are constants to be determined.

We will return and complete the transition region, region TR$^-$ once the constant $\widehat{b}$ has been determined.

We now return to region TW and recall, via (5.18), that

$$u(z,t) = u_T(z; c^*(k)) + o(1) \tag{5.37}$$

as $t \to \infty$ with $z = O(1)$ and where $z = x - s(t)$ and $s(t) = c^*(k)t + \theta(t) + \phi_c + \chi(t)$ as $t \to \infty$. Here $1 \ll \theta(t) \ll t$, $\phi_c$ is a constant and $\chi(t) = o(1)$ as $t \to \infty$ and are as yet undetermined gauge functions (to be fixed on matching with regions IV$^\pm$(b)), whilst $u_T(z; c^*(k))$ represents the minimum speed permanent form travelling wave solution. We recall from Section 1.2.2 that

$$u_T(z; c^*(k)) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}}, \tag{5.38}$$

with the following asymptotic properties:

$$u_T(z;c^*(k)) \sim \begin{cases} e^{-\frac{k}{2}z}, & \text{as} \quad z \to \infty, \\ 1 - e^{\frac{k}{2}z}, & \text{as} \quad z \to -\infty. \end{cases} \tag{5.39}$$

Now on matching expansion (5.25) of region IV$^+$(b) (as $y \to c^*(k)^+$) to expansion (5.37) (with $(5.39)_1$) of region TW we obtain that

$$\theta(t) = -\frac{2b}{k}\ln t, \quad \frac{k}{2}\phi_c + c_1 + b\ln\left(k - c^*(k)\right) = 0,$$

$$\chi(t) = \begin{cases} -\frac{4b^2}{k^2}\frac{1}{(k-c^*(k))}\frac{\ln t}{t}, & b \neq 0, \\ O(EXP), & b = 0, \end{cases} \tag{5.40}$$

where $O(EXP)$ is exponentially small in $t$ as $t \to \infty$. On the other hand, matching expansion (5.36) of region IV$^-$(b) (as $y \to c^*(k)^-$) to expansion (5.37) (with $(5.39)_2$) of region TW we obtain that

$$\theta(t) = -\frac{2\widehat{b}}{k}\ln t, \quad \frac{k}{2}\phi_c - \widehat{c}_1 - \widehat{b}\ln c^*(k) = 0,$$

$$\chi(t) = \begin{cases} -\frac{4\widehat{b}^2}{k^2}\frac{1}{c^*(k)}\frac{\ln t}{t}, & \widehat{b} \neq 0, \\ O(EXP), & \widehat{b} = 0. \end{cases} \tag{5.41}$$

Consideration of (5.40) and (5.41) requires for consistency that

$$\widehat{b} = b = 0 \quad \widehat{c}_1 = -c_1 = \frac{k}{2}\phi_c, \tag{5.42}$$

and

$$\chi(t) = O(EXP) \quad \text{as} \quad t \to \infty, \quad \theta(t) = 0. \tag{5.43}$$

Now that we have determined that $b = 0$ and $c_1 = -\frac{k}{2}\phi_c$ we can return to region TR$^+$. An examination of expansion (5.24) (as $y \to k^+$) and (5.25) (as $y \to k^-$) shows that in this region TR$^+$ $y = k + O(t^{-\frac{1}{2}})$ as $t \to \infty$. Therefore, to examine region TR$^+$ we introduce a new scaled coordinate $\eta = (y-k)t^{\frac{1}{2}}$, where $\eta = O(1)$ as $t \to \infty$, and look for an expansion of the form

$$u(\eta, t) = [F(\eta) + o(1)] \exp\left\{-\left(\frac{k^2}{4} - 1\right)t - \frac{k}{2}\eta t^{\frac{1}{2}}\right\} \tag{5.44}$$

as $t \to \infty$ with $\eta = O(1)$ and $F(\eta) < 0$. On substitution (5.44) into equation (5.1) (when written in terms of $\eta$ and $t$) we obtain at leading order

$$F_{\eta\eta} + \frac{\eta}{2}F_\eta = 0, \quad -\infty < \eta < \infty. \tag{5.45}$$

Equation (5.45) is to be solved subject to the matching conditions

$$F(\eta) = \begin{cases} e^{\frac{k}{2}\phi_c}, & \text{as} \quad \eta \to -\infty \\ \frac{e^{-\widehat{\beta}_1}}{\eta} e^{-\frac{\eta^2}{4}} & \text{as} \quad \eta \to \infty. \end{cases} \tag{5.46}$$

The solution to (5.45) subject to conditions (5.46) is readily obtained as

$$u(\eta, t) = \left(e^{\frac{k}{2}\phi_c} \operatorname{erfc}\left(\frac{\eta}{2}\right) + o(1)\right) \exp\left\{-\left(\frac{k^2}{4} - 1\right)t - \frac{k}{2}\eta t^{\frac{1}{2}}\right\} \tag{5.47}$$

as $t \to \infty$ with $\eta = O(1)$ and where the constant $\widehat{\beta}_1 = -\frac{k}{2}\phi_c + \ln\sqrt{\pi}$. The details of the transition region, region TR$^+$, in the case $k \in (2, \infty)$, are now complete.

Now we return to region TR$^-$. The details of region TR$^-$ follow, after some minor modification, those given above for region TR$^+$ are briefly summarized here.

Region TR$^-$

$$u(\zeta,t) = \left(e^{-\frac{k}{2}\phi_c}\left[1-\frac{1}{2}\,\mathrm{erfc}\left(\frac{\zeta}{2}\right)\right] + o(1)\right)\exp\left\{-\left(\frac{k^2}{4}+1\right)t + \frac{k}{2}\zeta t^{\frac{1}{2}}\right\} \tag{5.48}$$

as $t \to \infty$ with $\zeta = yt^{\frac{1}{2}} = O(1)$ and where the constant $\widehat{\gamma}_1 = \frac{k}{2}\phi_c + \ln\sqrt{\pi}$.

Now that we have been able to complete the transition regions, regions TR$^\pm$, we can obtain the correction terms to expansions (5.25) of region IV$^+$(b) and expansion (5.36) of region IV$^-$(b). In particular, the correction term to expansion (5.25) is required in order to be able to determine the rate of convergence of the solution of IVP3 to the PTW. We start by developing expansion (5.25).

The structure of (5.47) for $(-\eta) \gg 1$ (as we move into region IV$^+$(b)) indicates that the correction term to expansion (5.25) is $O\left(t^{-\frac{1}{2}}\exp\left(-\left(\frac{y^2}{4}-1\right)t\right)\right)$ as$t \to \infty$. After some calculation the expansion (5.25) is given as

$$u(y,t) = \exp\left(-\frac{k}{2}\left(y-c^*(k)\right)t + \frac{k}{2}\phi_c\right) - \exp\left(-\left(\frac{y^2}{4}-1\right)t - \frac{1}{2}\ln t - \psi_1(y)\right) + o\left(t^{-\frac{1}{2}}\exp\left(-\left(\frac{y^2}{4}-1\right)t\right)\right) \tag{5.49}$$

as $t \to \infty$ with $y = O(1) (\in (c^*(k), k))$ and where $c^*(k) = \frac{k}{2} + \frac{2}{k}$. The function $\psi_1(y)$ remains undetermined at this order but matching to (5.47) as $y \to k^-$ requires that

$$\psi_1(y) \sim \ln(k-y) - \frac{k}{2}\phi_c + \ln\sqrt{\pi} \quad \text{as} \quad y \to k^-.$$

Further, we make the assumption (which we will verify as consistent) that

$$\psi_1(y) \sim \alpha_0 \ln\left(y - c^*(k)\right) + \alpha_1 \quad \text{as} \quad y \to c^*(k)^+,$$

where $\alpha_0$ and $\alpha_1$ are constants to be determined. As $y \to c^*(k)^+$ we move from region

IV$^+$(b) into region TW. On writing expansion (5.49) in terms of the travelling wave variable $z$ we obtain that

$$u(z,t) \sim e^{-\frac{k}{2}}\left(1 - \frac{k}{2}\chi(t) + ...\right) - \frac{t^{\alpha_0 - \frac{1}{2}}}{z^{\alpha_0} e^{\alpha_1}} e^{-\frac{1}{2}c^*(k)[z+\phi_c]} e^{-\left(\frac{[c^*(k)^2]}{4} - 1\right)} + ... \tag{5.50}$$

as $t \to \infty$ with $z \gg 1$. We conclude from (5.50) that in region TW we must have

$$u(z,t) = u_T(z; c^*(k)) + O(\chi(t)) \tag{5.51}$$

as $t \to \infty$ with $z = O(1)$, and where $u_T(z; c^*(k))$ is given by (5.38). On substituting (5.51) into equation (5.1) (when written in terms of $z$ and $t$) we require

$$\dot{\chi}(t) = O(\chi(t)) \quad \text{as} \quad t \to \infty. \tag{5.52}$$

We conclude, via (5.52), that $\chi(t)$ must be exponential small in $t$ a s $t \to \infty$. Therefore, we set

$$\chi(t) = \overline{A} t^{\varepsilon} e^{-\sigma t}[1 + o(1)] \tag{5.53}$$

as $t \to \infty$ with the constants $\overline{A}$, $\varepsilon$ and $\sigma(> 0)$ to be determined. We now continue the expansion in region TW as

$$u(z,t) = u_T(z; c^*(k)) + u_1(z)\chi(t) + o(\chi(t)) \tag{5.54}$$

as $t \to \infty$ with $z = O(1)$. On substituting (5.54) into equation (5.1) (when written in terms of $z$ and $t$) we obtain at $O(\chi(t))$ that

$$u_1'' + \beta_0(z) u_1' + \beta_1(z) u_1 = \sigma u_T'(z; c^*(k)) \tag{5.55}$$

where $\beta_0(z) = c^*(k) - ku_T(z; c^*(k))$ and $\beta_1(z) = \sigma - ku'_T(z; c^*(k)) - 2u_T(z; c^*(k)) + 1$. We now determine the asymptotic properties of $u_1(z)$ as $|z| \to \infty$. For $z \gg 1$, $\beta_0(z) \sim c^*(k)$ and $\beta_1(z) \sim (\sigma + 1)$ giving that

$$u_1(z) \sim \begin{cases} A_1 e^{s_+ z} + B_1 e^{s_- z} - \frac{k}{2} e^{-\frac{k}{2}z} & \text{if} \quad \sigma < \frac{[c^*(k)]^2}{4} - 1, \\ (A_1 z + B_1) e^{-\frac{c^*(k)}{2} z} - \frac{k}{2} e^{-\frac{k}{2}z} & \text{if} \quad \sigma = \frac{[c^*(k)]^2}{4} - 1, \end{cases} \tag{5.56}$$

as $z \to \infty$ where $A_1$ and $B_1$ are constants and

$$s_\pm = -\frac{c^*(k)}{2} \pm \frac{1}{2}\sqrt{[c^*(k)]^2 - 4\sigma - 4}.$$

We note that the case when $\sigma > \frac{[c^*(k)]^2}{4} - 1$ can be excluded as this would lead to oscillatory solutions and matching with (5.50) would not be possible. For $(-z) \gg 1$, $\beta_0(z) \sim c^*(k) - k$ and $\beta_1(z) \sim (\sigma - 1)$ giving that

$$u_1(z) = O(e^{m_+ z}) \tag{5.57}$$

as $z \to -\infty$, where

$$m_+ = -\frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) + \frac{1}{2}\sqrt{[c^*(k)]^2 - 4\sigma}. \tag{5.58}$$

We also note that $m_+ > 0$ for $k \in (2, \infty)$.

On matching expansion (5.54) ( with (5.53) and (5.56)) as $z \to \infty$ with expansion (5.50) up to exponentially small terms in of $O(t^\varepsilon e^{-\sigma t})$ requires immediately that

$$\sigma = \frac{[c^*(k)]^2}{4} - 1 \tag{5.59}$$

and

$$\varepsilon = -\frac{3}{2}, \quad \alpha_0 = -1, \quad B_1 = 0, \quad \overline{A} = e^{-\frac{c^*(k)}{2}\phi_c - \alpha_1}.$$

Moreover, for this selected value of $\sigma$ we have that

$$m_+ = -\frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) + 1.$$

Therefore, we have established that when $k \in (2, \infty)$ the large-time structure in region TW is dominated by the evolution of PTW with speed $c = c^*(k) = \frac{k}{2} + \frac{2}{k}$ (this being the minimum speed available). In summary, we have in region TW that

$$u(z,t) = u_T(z; c^*(k)) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right) \tag{5.60}$$

as $t \to \infty$ with $z = O(1)$ and where $u_T(z; c^*(k)) = \frac{e^{-\frac{k}{2}z}}{1+e^{-\frac{k}{2}z}}$. We observe from (5.60) that the rate of convergence of the solution of to IVP3 to the PTW is exponential in $t$ as $t \to \infty$.

Although we will not pursue the full details of the correction to expansion (5.36) in region IV$^-$(b) it is instructive to consider the leading order structure of the correction term. On recalling $\widehat{b} = 0$ and $\widehat{c}_1 = \frac{k}{2}\phi_c$, we have in region IV$^-$(b) that

$$u(y,t) = 1 - \exp\left(\frac{k}{2}\left(y - c^*(k)\right)t - \frac{k}{2}\phi_c\right) + e^{-G(y,t)t} \tag{5.61}$$

where

$$G(y,t) = g_c(y) + o(1) \tag{5.62}$$

as $t \to \infty$ with $y = O(1) \in (0, c^*(k))$. It is straightforward to establish that $g_c(y)$ is given by the envelope-touching solution

$$g_c(y) = \begin{cases} \frac{(y-k)^2}{4} + 1, & 0 < y < c^*(k) - 2, \\ \widehat{A}_c \left( y - \left( \widehat{A}_c + k - \frac{1}{\widehat{A}_c} \right) \right), & c^*(k) - 2 \leqslant y < c^*(k), \end{cases} \tag{5.63}$$

where $\widehat{A}_c = \frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) - 1$. The envelope-touching solution (5.63) is sketched in Figure 5.1. This change of structure in $g_c(y)$ is required to facilitate matching with region TW (as $y \to c^*(k)^-$) and region $\mathrm{TR}^-$ as $(y \to 0^+)$. Therefore, to accommodate this change in structure of $g_c(y)$ region $\mathrm{IV}^-$(b) would need to be divided into three further regions, but we do not pursue this here.

This then completes the asymptotic structure of the solution of IVP3 as $t \to \infty$ when $k \in (2, \infty)$. A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ is given in Figure 5.2.

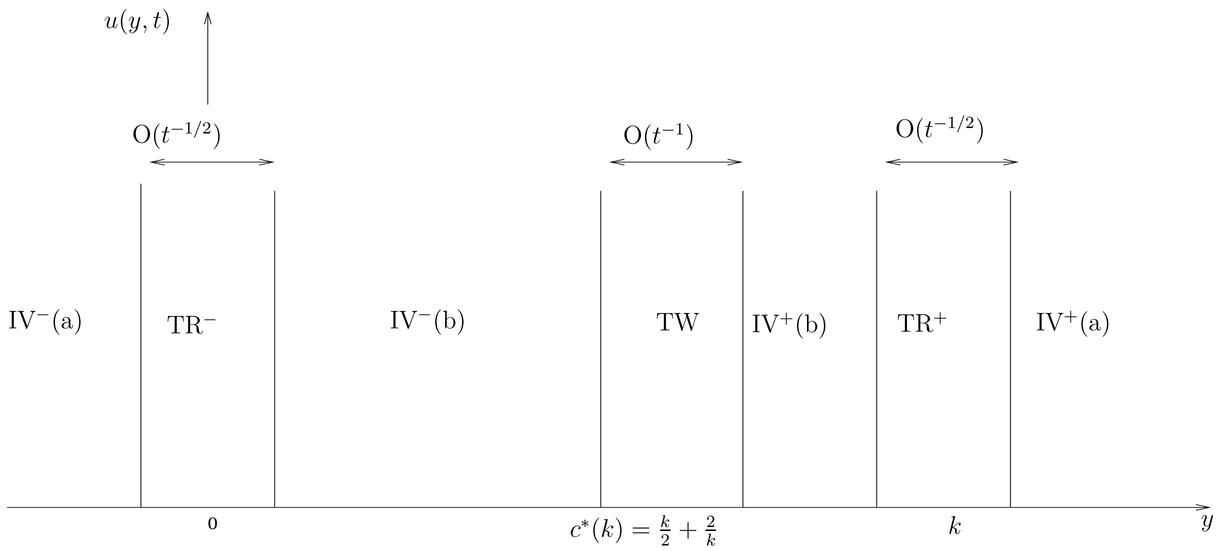


Figure 5.2: A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ when $k \in (2, \infty)$.

### 5.3.2 $\mathbf{k} \in (-\infty, \mathbf{2}]$

In this case $r_0(y)$ is given by (5.16) with $A = 1$, that is

$$r_0(y) = \frac{y^2}{4} - 1, \quad y \in (2, \infty) \tag{5.64}$$

and $c^*(k) = 2$. Figure 5.3 gives a sketch of the solution (5.64) for $y \geqslant 2$.

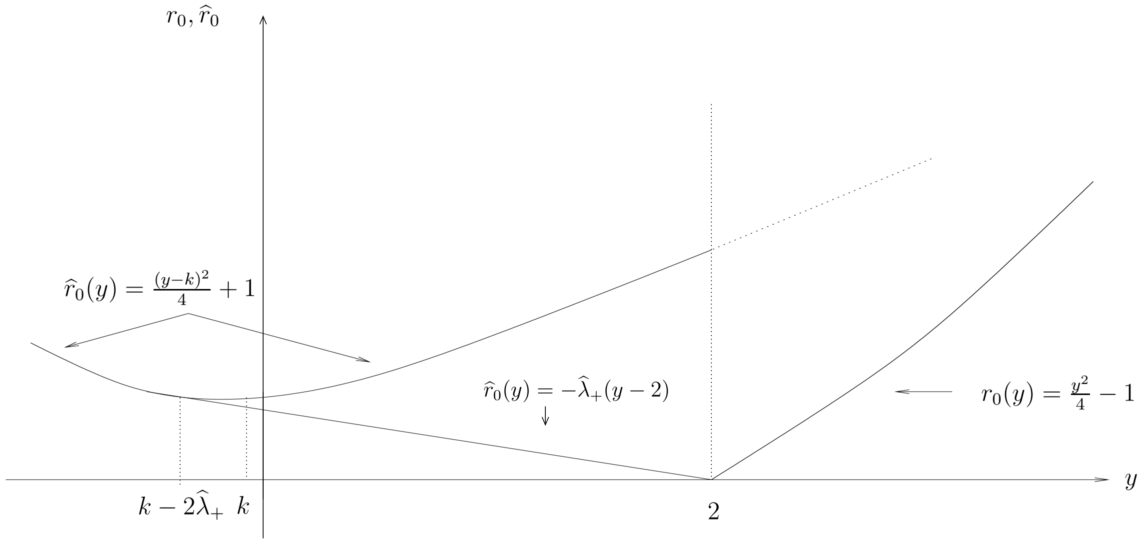


Figure 5.3: A sketch of the functions $r_0(y)$ and $\hat{r}_0(y)$ when $-\infty < k \leqslant 2$.

On continuing expansion (5.9) (with (5.10)) we have in region IV$^+$ that

$$u(y,t) = \exp\left(-\left(\frac{y^2}{4} - 1\right)t - \frac{1}{2}\ln t - \psi(y) + o(1)\right) \tag{5.65}$$

as $t \to \infty$ with $y = O(1)(\in (2, \infty))$, and where the function $\psi(y)$ remains undetermined, but having

$$\psi(y) \sim \ln y + \ln \sqrt{\pi} \quad \text{as} \quad y \to \infty.$$

As $y \to 2^+$ we move out from region IV$^+$ into region TW (the PTW region), where $y = 2 \pm O(t^{-1})$ as $t \to \infty$. In region TW we obtain that

$$u(z,t) = u_T(z,2) + O(\dot{s}(t) - 2) \tag{5.66}$$

as $t \to \infty$ with $z = O(1)$, where $u_T(z,2)$ is the permanent for travelling wave solution with propagation speed 2, $z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time $t$) and

$$s(t) = 2t + \theta(t) + \phi_c$$

as $t \to \infty$, where $\phi_c$ is a constant and $1 \ll \theta(t) \ll t$ as $t \to \infty$. We further recall from Section 1.2.2 that

$$u_T(z;2) \sim (A^* z + D^*)e^{-z} \quad \text{as} \quad z \to \infty. \tag{5.67}$$

On matching expansion (5.66) with (5.67) as $z \to \infty$ to expansion (5.65)(as $y \to 2^+$) requires that

$$\psi(y) \sim \ln(y-2) \quad \text{as} \quad y \to 2^+, \quad \theta(t) = -\frac{3}{2}\ln t, \quad \phi_c = -\ln A^*.$$

Therefore, we have established that when $k \in (-\infty, 2]$ the large-time structure in region TW is dominated by the evolution of PTW with speed $c = c^*(k)(= 2)$ (this being the minimum speed available). In summary, we have in region TW that

$$u(z,t) = u_T(z;2) + O(t^{-1}) \tag{5.68}$$

as $t \to \infty$ with $z = O(1)$ and where $u_T(z; c^*(k))$ is the permanent for travelling wave solution with propagation speed 2, and

$$s(t) = 2t - \frac{3}{2}\ln t - \ln A^*. \tag{5.69}$$

We observe from (5.69) that the rate of convergence of the solution IVP3 to the PTW is algebraic in $t$ as $t \to \infty$.

We conclude this section by summarizing the asymptotic structure of the solution of IVP3 as $t \to \infty$ in $y < 2$. The asymptotic structure of the solution of IVP3 in $y < 0$ in this case follows, after some minor modifications, that given Section 5.3.1. In particular, we note that

$$\widehat{r}_0(y) = \begin{cases} \frac{(y-k)^2}{4} + 1, & -\infty < y < k - 2\widehat{\lambda}_+, \\ -\widehat{\lambda}_+(y-2), & k - 2\widehat{\lambda}_+ < y < 2, \end{cases} \tag{5.70}$$

where $\widehat{A} = -\widehat{\lambda}_+$. Function $\widehat{r}_0(y)$ is sketched in Figure 5.3 for $y \in (-\infty, 2]$. Hence, region IV$^-$ is as in Section 5.3.1 subdivided into three regions IV$^-$(a), TR$^-$ and IV$^-$(b) to accommodate the change in structure of $\widehat{r}_0(y)$. The details of these regions as summarized below:

Region IV$^-$(a)

$$u(y,t) = 1 - \exp\left(-\left(\frac{(y-k)^2}{4} + 1\right)t - \frac{1}{2}\ln t - \widehat{\psi}(y) + o(1)\right) \tag{5.71}$$

as $t \to \infty$ with $y = O(1)(\in (-\infty, k - 2\widehat{\lambda}_+))$, where the function $\widehat{\psi}(y)$ remains undetermined, but having

$$\widehat{\psi}(y) \sim \begin{cases} \ln(-y) + \ln\sqrt{\pi} & \text{as} \quad y \to -\infty, \\ \ln((k - 2\widehat{\lambda}_+) - y) + \frac{k}{2}\phi_c + \ln\sqrt{\pi} & \text{as} \quad y \to (k - 2\widehat{\lambda}_+)^-. \end{cases} \tag{5.72}$$

Region TR$^-$

$$u(\zeta,t) = \left(e^{-\frac{k}{2}\phi_c}\left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{\zeta}{2}\right)\right] + o(1)\right)\exp\left\{-(\widehat{\lambda}_+^2 + 1)t + \widehat{\lambda}_+\zeta t^{\frac{1}{2}}\right\} \tag{5.73}$$

as $t \to \infty$ with $\zeta = (y - (k - \widehat{\lambda}_+))t^{\frac{1}{2}} = O(1)$.

Region IV$^-$(b)

$$u(y,t) = 1 - \exp\left(\widehat{\lambda}_+(y-2)t - \frac{k}{2}\phi_c + o(1)\right) \tag{5.74}$$

as $t \to \infty$ with $y = O(1)(\in (k - 2\widehat{\lambda}_+, 2))$.

This then completes the asymptotic structure of the solution IVP3 as $t \to \infty$ in the case when $k \in (-\infty, 2]$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 5.4.

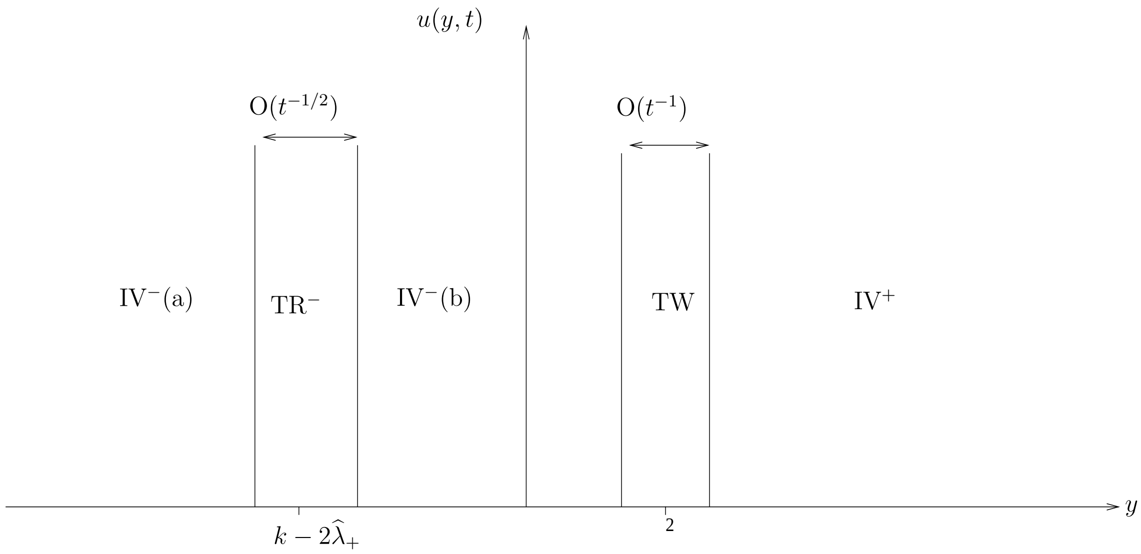


Figure 5.4: A schematic representation of the location and thickness of asymptotic regions as $t \to \infty$ when $k \in (-\infty, 2]$.

## 5.4 Numerical Solutions

In this section we present numerical solutions of IVP3 which both support and illustrate the detailed asymptotic analysis given in the above sections. The numerical simulations were performed using a numerical algorithm based on the method of finite differences,of solving over the spatial interval $(-L, L)$ using $M$ equal width intervals, and over the time interval $(0, T)$ using $N$ equal time steps. We define $\delta x = 2L/M$ and $\delta t = T/N$, typically $\delta x = 0.1$ and $\delta t = 0.005$ was found to give sufficient accuracy, although the $x$ axis in the plots has been shorted in order to better display the features of interest. The interested reader is referred to the following texts[25, 26] for details of the numerical method employed here. We present numerical solutions of IVP3 in two different cases when $k = 5$ ($\in (2, \infty)$) and when $k = -5$ ($\in (-\infty, 2]$). We consider each case in turn:

(i) When $k = 5$ the numerical results are given in Figure 5.5-5.7. In Figure 5.5 the PTW of wave speed $c = c^*(5) = 2.9$ (the minimum wave speed available in this case) is seen to develop rapidly, and the correction to $\dot{s}(t)$ as $t \to \infty$ appears to be exponentially small in $t$ as $t \to \infty$, in line with the theory of Section 5.3.1. Figure 5.6 shows the numerically obtained curve of $\dot{s}(t)$ ($s(t)$ is the $x$ location where $u = 0.5$) against $t$ for $t \in (0, 10]$. As predicted by the theory $\dot{s}(t)$ rapidly approaches the minimum wave speed $c^*(5) = 2.9$ as $t \to \infty$. Figure 5.7 shows the numerically obtained curve of $s(t) - c^*(5)t$ against to $t$ for $t \in (0, 10]$. As predicted by the theory $s(t) - c^*(5)t$ rapidly approaches the constant $\phi_c \sim 0.1406$ as $t \to \infty$.

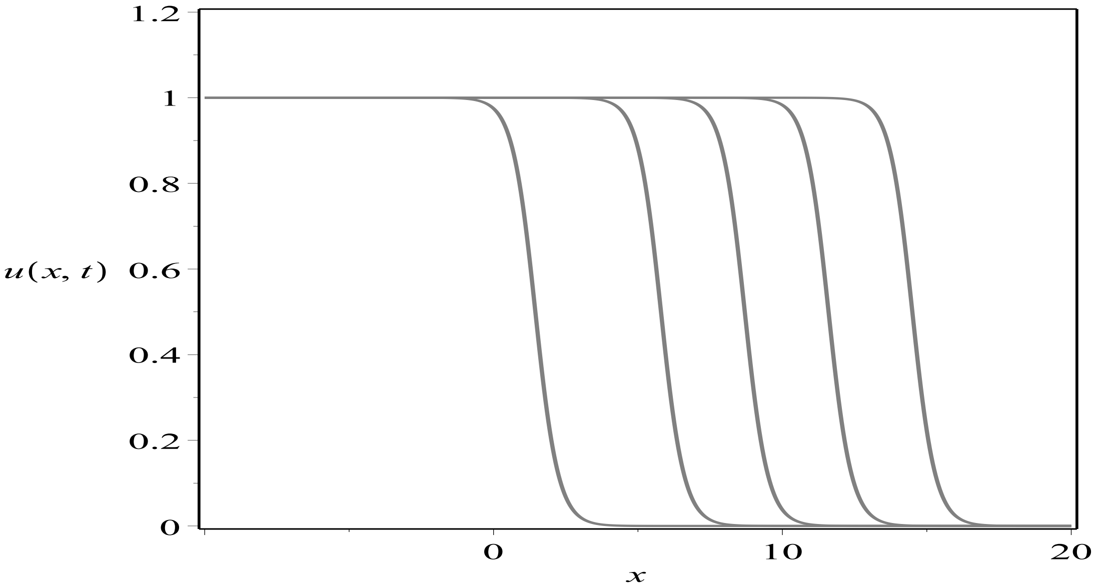


Figure 5.5: Graphs of the solution of IVP3 at times $t = 0.5, 1, 2, 3, 4$.

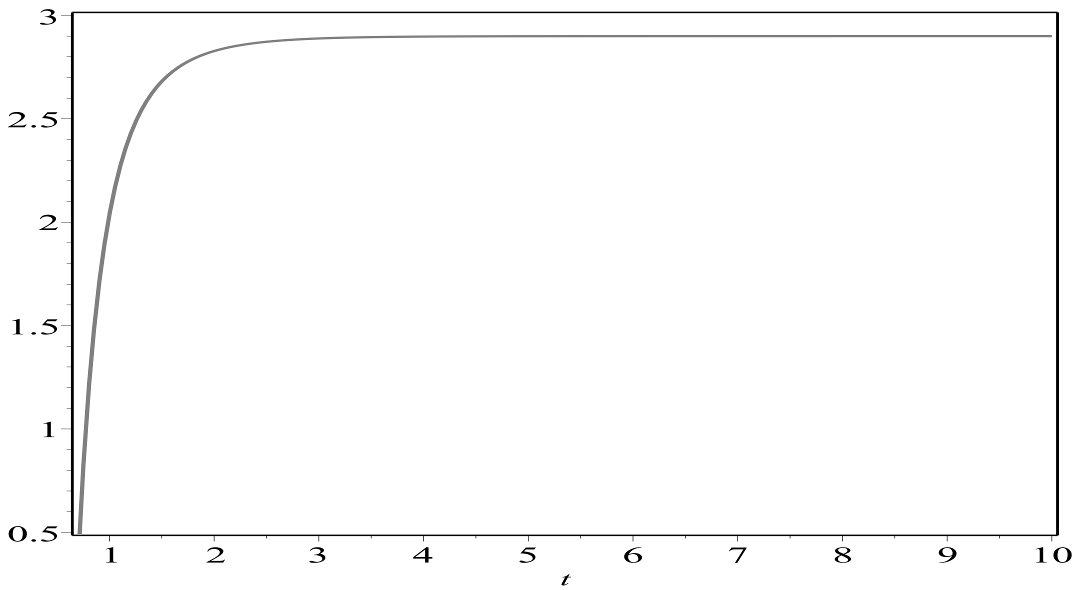


Figure 5.6: The graph of $\dot{s}(t)$ versus $t$ when $k = 5$.

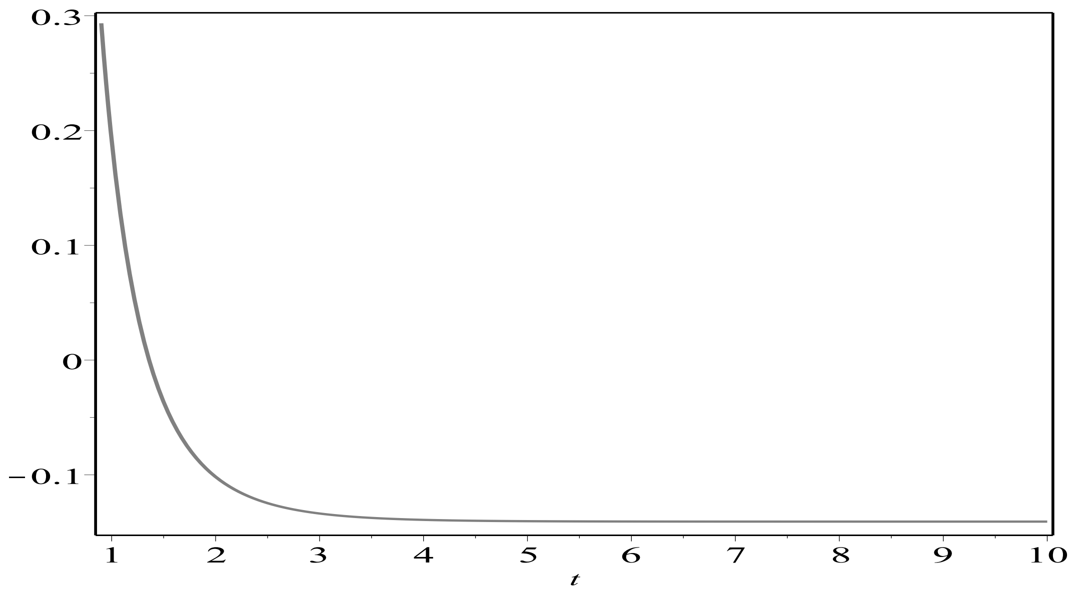


Figure 5.7: The graph of of $s(t) - c^*(5)t$ versus $t$ when $k = 5$ and $\phi_c \sim 0.1406$.

(ii) When $k = -5$ the numerical results are given in Figure 5.8-5.10. In Figure 5.8 the PTW of wave speed $c = c^*(-5) = 2$(the minimum wave speed available in this case) is seen to develop rapidly and the correction to $\dot{s}(t)$ as $t \to \infty$ appears to be algebraically small in $t$ as $t \to \infty$,in line with the theory of Section 5.3.2. This is in contrast to the case when $k = 5$. Figure 5.9 shows the numerically obtained curve of $\dot{s}(t)$ ($s(t)$ is the $x$ location where $u = 0.5$) against $t$ for $t \in (0, 80]$.As predicted by the theory $\dot{s}(t)$ approaches the minimum wave speed 2 as $t \to \infty$. However, the rate of convergence is considerably slower than the case $k = 5$ above, and appears to be algebraically small in $t$ as $t \to \infty$. Finally, Figure 5.10 shows the numerically obtained curve of $s(t) - 2t + \frac{3}{2}\ln t$ against $t$ for $t \in (0, 50]$. As predicted by the theory $s(t) - 2t + \frac{3}{2}\ln t$ approaches a constant as $t \to \infty$. This then confirms (5.69), and supports the theory that the rate of convergence of IVP3 as $t \to \infty$ in this case is algebraically small in $t$ as $t \to \infty$ being of $O(t^{-1})$.

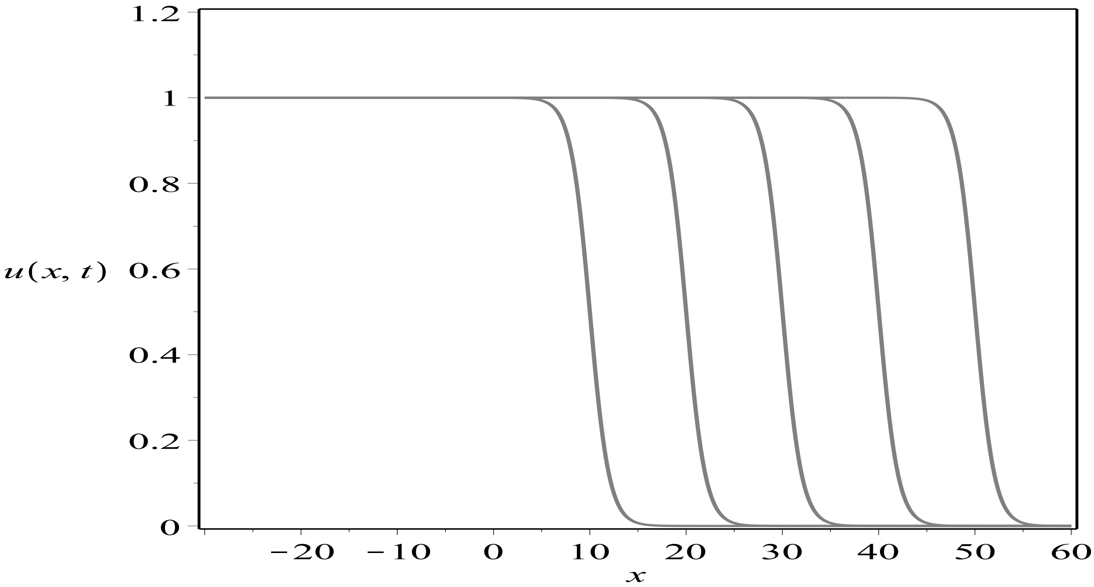


Figure 5.8: Graphs of the solution of IVP3 at times $t = 5, 10, 15, 20, 25$.

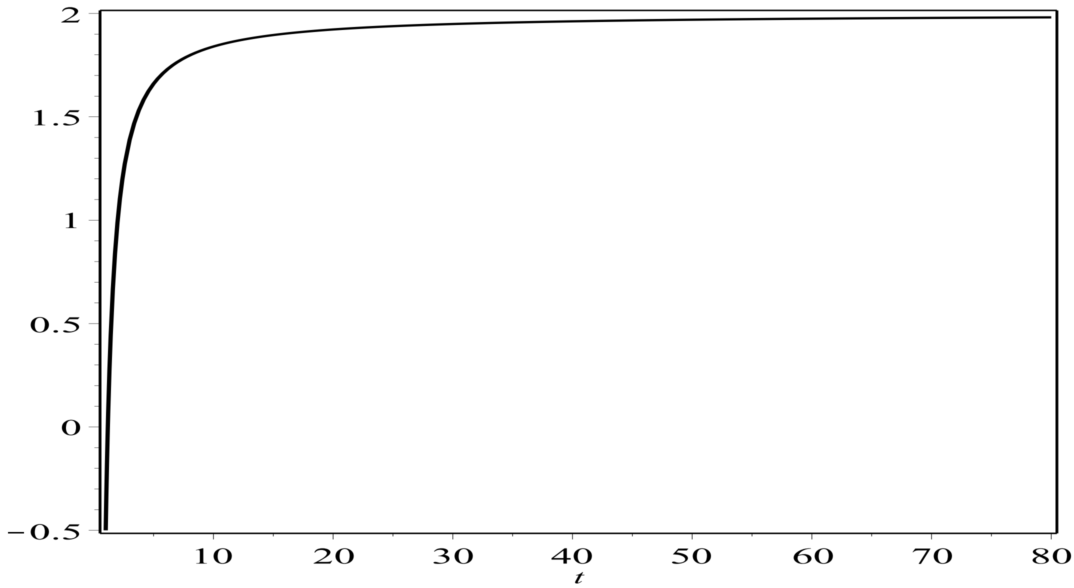


Figure 5.9: The graph of $\dot{s}(t)$ versus $t$ when $k = -5$.

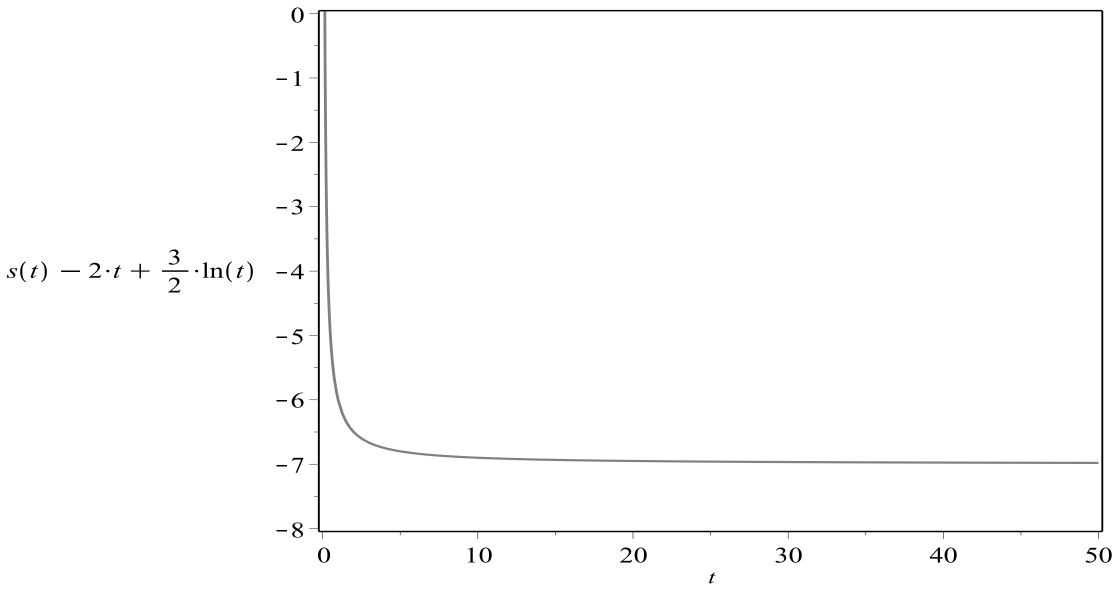


Figure 5.10: The graph of of $s(t) - 2t + \frac{3}{2}\ln t$ versus $t$ when $k = -5$.

## 5.5 Summary

In this chapter we have used the method of matched asymptotic coordinate expansions to develop the complete large-time solution of IVP3 for all values of the parameter $k$ (excluding the case $k = 0$ when equation (5.1) reduces to the Fisher-Kolmogorov equation- The initial-value problem in this case having been considered by a number of authors, see for example [6]). In particular, we have established that the solution of IVP3 exhibits the formation of a permanent form travelling wave propagating in the $+x$ direction with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

In particular, we have established that:

(i) When $k \in (2, \infty)$ the solution to IVP3, $u(x,t)$ satisfies,

$$u(z+s(t),t) = u_T(z;c^*(k)) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)^2]}{4}-1\right)t}\right)$$

as $t \to \infty$, uniformly in $z$, where $u_T(z;c^*(k))$ is the permanent for travelling wave solution with propagation speed $c^*(k) = \frac{k}{2} + \frac{2}{k}$ given by

$$u_T(z;c^*(k)) = \frac{e^{-\frac{k}{2}z}}{1+e^{-\frac{k}{2}z}},$$

$z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time $t$) and

$$s(t) = \left(\frac{k}{2} + \frac{2}{k}\right)t + \phi_c + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)^2]}{4}-1\right)t}\right)$$

as $t \to \infty$ and where $\phi_c$ is a constant. We note that the correction to the propagation speed, $\dot{s}(t)$, is exponential in $t$ as $t \to \infty$, being of $O\left(t^{-\frac{3}{2}} \exp\left(-\left(\frac{[c^*(k)^2]}{4} - 1\right)t\right)\right)$. We further note that, the rate of convergence of the solution of IVP3 to the permanent form travelling wave is exponential in $t$, as $t \to \infty$, being of

$$O\left(t^{-\frac{3}{2}} \exp\left(-\left(\frac{[c^*(k)^2]}{4} - 1\right)t\right)\right). \tag{5.75}$$

(ii) When $k \in (-\infty, 2]$ the solution to IVP3, $u(x,t)$ satisfies,

$$u(z+s(t),t) = u_T(z;2) + O\left(t^{-1}\right)$$

as $t \to \infty$, uniformly in $z$, where $u_T(z;2)$ is the permanent for travelling wave solution with propagation speed 2 and $z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time $t$) and

$$\dot{s}(t) = 2 - \frac{3}{2}\frac{1}{t} + o\left(\frac{1}{t}\right) \tag{5.76}$$

as $t \to \infty$. We note that the correction to the propagation speed, $\dot{s}(t)$, is algebraic in $t$ as $t \to \infty$, being of $O\left(t^{-1}\right)$. We further note that the rate of convergence of the solution of IVP3 to the permanent form travelling wave is algebraic in $t$, as $t \to \infty$, being of $O\left(t^{-1}\right)$. It is worthwhile to point out that the above result (5.76) is in agreement with the result obtained by Bramson [6] who considered (5.1) (with $k = 0$) when the initial data has a step function profile (5.2). These results are supported by the numerical simulations of Section 5.4.

We conclude by noting that in Section 5.3 the large-time solution of IVP3 was obtained by the careful consideration of the functions $r_0(y)$ and $\hat{r}_0(y)$ the solutions of the Clairaut equations (5.11) and (5.28) respectively. Clairaut equations admit constant solutions, one-parameter families and linear solutions and associated envelope (singular) solutions. Further, as already mentioned in Section 5.3 combinations of these solutions which remain continuous and differentiable also provide solutions to the Clairaut equation in question. Figures 5.1 and 5.3 give sketches of the required forms for $r_0(y)$ and $\hat{r}_0(y)$ in the cases $k \in (2, \infty)$ and $k \in (-\infty, 2]$, respectively. It is instructive to note that in the case $k \in (2, \infty)$ (Figure 5.1) the point $\left(c^*(k), \frac{[c^*(k)^2]}{4} - 1\right)$ lying on the envelope solution $r_0(y) = \frac{y^2}{4} - 1$ approaches the point $(2, 0)$ as $k \to 2^+$. The $y$ ordinate of this point being associate with the argument of the exponential correction term to the PTW in this case. $k = 2$ can then be considered a bifurcation point marking a change in structure of the large-time solution of IVP3. When $k \in (2, \infty)$ the correction to the PTW is exponential in $t$ as $t \to \infty$, being given by (5.75); while when $k \in (-\infty, 2]$ is algebraic in $t$, being of $O(t^{-1})$, as $t \to \infty$. Finally, we note that equations of Clairaut type appear to play a central role in the analysis of many problems arising in the areas of mathematical chemistry, biology and physics, and warrant further investigation.

# Chapter 6

# Conclusion and Future Work

## 6.1 Conclusion

In this thesis we have used the method of matched asymptotic coordinate expansions to obtain the complete large-time solution of an initial-value problem and initial-boundary value problem for Burgers' equation and an initial-value problem for the Burgers-Fisher equation. The mathematical problems considered in this thesis are introduced in Section 1.4. The approach developed in this thesis allows the complete large-time solution to be obtained by careful consideration of the small-time solution ($t \to 0$, $x = o(1)$) and then the large $x$ solution ($|x| \to \infty$, $t = O(1)$). We have demonstrated for the examples considered that the structure of the large-time solution is governed by the interplay between the envelope solution and associated one parameter family of linear solutions of the relevant clairaut equation. Although the solution to Burgers' equation can in certain cases be obtained by the Cole-Hopf transformation (see Section 1.1) it provided a simple model for me to start to develop my understanding of the large-time solution of nonlinear PDEs and provided strong foundations for my work on the Burgers-Fisher equation which is covered in Chapter 5. Material from this Chapter has now been accepted for publication in Quarterly of Applied Mathematics and will be published in 2015/2016.

In Chapters 2 and 3 we considered an initial-value problem based on Burgers' equation, namely,

$$u_t + uu_x - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{6.1}$$

$$u(x,0) = \begin{cases} u_+, & x \geqslant 0, \\ u_-, & x < 0, \end{cases} \tag{6.2}$$

where $u_+$ and $u_-$ ($u_+ \neq u_-$) are constants. Specifically, in Chapter 2 we considered the case when $u_+ > u_-$ and focused attention on the case $u_+ = 1$ and $u_- = 0$. Here we found that the large-time solution (6.1), (6.2) approaches the expansion wave solution, that is,

$$u = \begin{cases} 0, & x \leqslant 0 \\ x/t, & x \in (0,t) \\ 1, & x \geqslant t \end{cases} \tag{6.3}$$

as $t \to \infty$. In Chapter 3 we considered the case when $u_+ < u_-$. In this case the large-time attractor of the solution to (6.1), (6.2) is either a permanent form travelling wave solution with speed $c = \frac{u_+ + u_-}{2}$ (when $u_- > u_+ > -u_-$ or $u_+ < u_- < -u_+$) or a stationary solution (when $u_+ = -u_-$ with $u_- > 0$). In both cases the rate of convergence is exponential in $t$ as $t \to \infty$.

Finally, in Section 3.4 numerically calculated solutions of (6.1)-(6.2) are presented for each of the above cases. These numerical simulations support the theoretically predicted results.

In Chapter 4 we considered an initial-boundary value problem based on Burgers' equation, namely,

$$u_t + uu_x - u_{xx} = 0, \quad x > 0, \quad t > 0 \tag{6.4}$$

$$u(x,0) = u_+, \quad x > 0, \tag{6.5}$$

$$u(0,t) = u_b, \quad t > 0, \tag{6.6}$$

where $u_+$ and $u_b$ ($u_b \neq u_+$) are constants. We examine initial-boundary value problem (6.4)-(6.6) depending on parameters $u_+$ and $u_b$, in following cases:

(i) $-u_b < u_+ < u_b$ with $u_b > 0$,

(ii) ($-u_+ > u_b > u_+$ with $u_+ < 0$) or when $0 \geqslant u_+ > u_b$,

(iii) $u_+ > u_b$ and $u_b \geqslant 0$,

(iv) $u_b < 0$ and $u_+ > 0$.

In case (i), the large-time attractor of the solution to (6.4)-(6.6) is a permanent form travelling wave solution with speed $c = \frac{u_b + u_+}{2}$. The rate of convergence of the solution to (6.4)-(6.6) to the travelling wave profile is exponential in $t$ as $t \to \infty$.

In case (ii), the large-time solution of (6.4)-(6.6) exhibits the formation of a stationary solution. The rate of convergence of the solution to (6.4)-(6.6) to this stationary profile is exponential in $t$ as $t \to \infty$.

In case (iii), the large-time solution of (6.4)-(6.6) approaches the expansion wave solution, given by,

$$u = \begin{cases} u_b, & x \leqslant u_b t \\ x/t, & x \in (u_b t, u_+ t) \\ u_+, & x \geqslant u_+ t \ . \end{cases} \tag{6.7}$$

In case (iv), the large-time solution of (6.4)-(6.6) consists of a combination of an expansive wave and a stationary solution.

Finally, in Section 4.7 numerically calculated solutions of (6.4)-(6.6) are presented for each of the above cases. These numerical simulations support the theoretically predicted results.

In Chapter 5 an initial-value problem based on Burgers-Fisher equation is considered, namely,

$$u_t + kuu_x = u_{xx} + u(1-u), \quad -\infty < x < \infty, \quad t > 0 \tag{6.8}$$

$$u(x,0) = \begin{cases} 1 & \text{as} \quad x \leqslant 0, \\ 0 & \text{as} \quad x > 0, \end{cases} \tag{6.9}$$

$$u(x,t) \to \begin{cases} 1, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \qquad t \geqslant 0, \tag{6.10}$$

where $k \neq 0$ is a parameter and the initial distribution (6.9) is a discontinuous compressive step. The large-time solution of the initial-value problem (6.8)-(6.10) approaches a permanent form travelling wave solution with minimum available wave speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

In particular, in Subsection 5.3.1 we considered the case when $k \in (2, \infty)$. In this case the large-time attractor of the solution to initial-value problem (6.8)-(6.10) is a permanent form travelling wave solution with propagation speed $c = c^*(k) = \frac{k}{2} + \frac{2}{k}$, given by

$$u_T(z; c^*(k)) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}},$$

$z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time $t$) and the rate of convergence of the solution of (6.8)-(6.10) to the travelling wave solution $u_T(z; c^*(k))$, specifically

$$u(x,t) = u_T(z; c^*(k)) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$ with $x = O(1)$ is exponential in $t$ as $t \to \infty$.

In Subsection 5.3.2 we considered the case when $k \in (-\infty, 2]$. Here the large-time attractor of the solution to the initial-value problem (6.8)-(6.10) is a permanent form travelling wave solution with propagation speed $c = c^*(k) = 2$, given by

$$u_T(z; 2) \sim (A^* z + D^*) e^{-z}$$

where $A^*$ and $D^*$ are constants and the rate of convergence of the solution of (6.8)-(6.10) to the travelling wave solution $u_T(z; 2)$, specifically,

$$u(x,t) = u_T(z; 2) + O(t^{-1})$$

as $t \to \infty$ with $x = O(1)$ is algebraic in $t$ as $t \to \infty$.

Finally, in Section 5.4 numerically calculated solutions of (6.8)-(6.10) are presented for each of the above cases. These numerical simulations support the theoretically predicted results.

An overview of all the result contained in this thesis can be found in Section 1.4, while a detailed summary of the large-time solution in each case can be found in Sections 2.4, 3.5, 4.8 and 5.5.

## 6.2 Future Work

An important extension to this thesis would be to investigate the large-time solution of the initial-value problem for the Burgers-Fisher equation when the initial data has unbounded support with algebraic or exponential decay in the far field as $|x| \to \infty$. It would be interesting to examine how the correction term to the permanent form travelling wave solution is affected.

However, I intend to use the approach developed in this thesis to investigate the large-time solution of initial-value and initial-boundary value problems arising in mathematical physics and biology on my return to Turkey. I am particularly keen to extend my knowledge to systems of such equations which I consider to be a challenging and important research area.

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