Spin Covers of Maximal Compact Subgroups of Kac-Moody Groups and of Weyl Groups

by

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Dedicated to my father Prof. M. A. Ghatei with love and admiration.

Abstract

The maximal compact subgroup $\mathrm{SO}(n,\mathbb{R})$ of $\mathrm{SL}(n,\mathbb{R})$ admits two double covers $\mathrm{Spin}(n,\mathbb{R})$ and $\mathrm{O}(n,\mathbb{R})$. In this thesis we show that in fact given any simply-laced diagram Δ, with associated split real Kac-Moody group $G(\Delta)$, the 'maximal compact' subgroup $K(\Delta)$ also admits a double cover $\mathrm{Spin}(\Delta)$. We then describe a subgroup of $\mathrm{Spin}(\Delta)$ which is a cover of the Weyl group $W(\Delta)$ associated to $\mathfrak{g}(\Delta)$. This answers a conjecture of Damour and Hillmann. Moreover, the group D which extends W is used in an attempt to classify the images of the so-called generalised spin representations.

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Chapter 1

Introduction

In recent times the hyperbolic Kac-Moody algebra $\mathfrak{e}(10,\mathbb{R})$ has become of interest to physicists due to applications to M-theory ([14], [29], [23]). The algebra $\mathfrak{e}(10,\mathbb{R})$ contains a Lie subalgebra isomorphic to $\mathfrak{so}(10,\mathbb{R})$ which has a representation in the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{10})$. Recently this spin representation was extended in [15] and [17] to give a finite dimensional representation for the infinite dimensional maximal compact subalgebra, that is the fixed subalgebra of the Cartan-Chevalley automorphism ω, $\mathfrak{k} \leqslant \mathfrak{e}(10,\mathbb{R})$, which implies that it has non-trivial finite dimensional quotient.

The recent work set out in [26] generalises the construction for the $\mathfrak{e}(10,\mathbb{R})$ algebra to all maximal compact subalgebras $\mathfrak{k}$ of symmetrisable Kac-Moody algebras $\mathfrak{g}$ over fields of characteristic 0. It is shown in *loc. cit.* that the images of these generalised spin representations are semisimple.

The group $\mathrm{SL}(n,\mathbb{R})$ contains as its maximal compact subgroup $\mathrm{SO}(n,\mathbb{R})$ which in turn admits a double cover called the spin group, $\mathrm{Spin}(n,\mathbb{R})$. This double cover has a flag transitive action on the simply connected geometry $\mathbb{P}(\mathbb{R}^{n-1})$ and this implies that $\mathrm{Spin}(n,\mathbb{R})$ is the universal completion of its rank 1 and rank 2 subgroups. For an arbitrary simply-laced diagram Δ, the split real Kac-Moody group $G := G(\Delta)$ is a universal completion of an amalgam of type SL(2) with respect to Δ. The "maximal compact subgroup" $K := K(\Delta)$

inside G is known to be the completion of an amalgam of type SO(2) with respect to the same diagram [18]. This motivates our main definition.

Definition 1.0.1

For a simply-laced diagram Δ we define the generalised spin group, $\mathrm{Spin}(\Delta)$, *to be the universal completion of the amalgam* $\mathcal{A}(\Delta, \mathrm{Spin}(2))$.

We employ an integrated version of the generalised spin representation to find a finite dimensional representation of the group $\mathrm{Spin}(\Delta)$. What is important here is that the element -1 in $\mathrm{Spin}(\Delta)$ has a non-trivial image in the representation and if we quotient out the subgroup it generates we get the following theorem.

Theorem 1.0.2

The group $Spin(\Delta)$ is a double cover of $K(\Delta)$.

We therefore have a generalised version of the spin groups for arbitrary simply laced diagrams Δ. This answers the conjecture in [16] that was originally for a spin group for the diagram of type E_{10}.

In [16] a group $\mathcal{W}^{spin}$ is conjectured to exist as a subgroup of the group $\mathrm{Spin}(E_{10})$. It is a group on ten generators R_i, $1 \leqslant i \leqslant 10$, thought to have the following properties

- For $1 \leqslant i \leqslant 10, (R_i)^4 = -1$;
- for adjacent i, j, $(R_i R_j)^3 = -1$;
- for non-adjacent nodes the generators commute;
- the elements $(R_i)^2$ generate a non-abelian, normal subgroup D of order 2^{10+1} and
- We have

$$\mathcal{W}^{spin}/D \cong W(E_{10})$$

Since our group Spin(Δ) is an amalgam of copies of Spin(2), we can take the element $\frac{1}{2}(1 - v_1 v_2)$ in each copy. These elements correspond to the R_i in the above definition. Hence we are able to define the subgroups $\mathcal{W}^{spin}$ for all groups Spin(Δ) for a simply laced diagram Δ.

After this introduction we begin in Chapter 2 by introducing the real Clifford algebras and the spin groups, which live 'naturally' inside them. For a fixed Euclidean space we investigate some of the properties of these groups and determine some of the subgroups of Spin(4) that will be necessary later in our work.

Chapter 3 is a study of the maximal compact subalgebras of simply-laced Kac-Moody algebras. The chapter begins with a quick overview of the theory and some of the basic properties of these algebras. In the second section we give a set of generators and relations for the bespoke subalgebras.

A generalistion of the well-known spin representation first discovered by Èlie Cartan is the topic of Chapter 4. We begin by recalling the representation of the Lie algebra $\mathfrak{so}(n, \mathbb{R})$ and then present the generalised version. The chapter continues to provide a finite dimensional algebra into which the representation embeds. Lastly, we classify the larger algebra and show that it is isomorphic to mutually isomorphic copies of simple Lie algebras of type $\mathfrak{so}(n, \mathbb{R})$ or $\mathfrak{sp}(n, \mathbb{R})$, for some $n \in \mathbb{N}$.

In Chapter 5 we give a construction of the Kac-Moody group associated to a simply-laced Kac-Moody algebra by way of completions of amalgams. A similar thing is done for the maximal compact subgroup and its double cover, the spin group. We then use this information to define a generalisation of the spin groups. An integrated form of the spin representation introduced in the previous chapter is used to show that the new spin groups are indeed double covers of the maximal 'compact' subgroups.

The last chapter studies extensions to the Weyl groups associated to simply-laced Kac-Moody algebras. It is these groups that are of interest in Fermionic Kac-Moody Billiards

[16]. We deduce that the groups do exist as subgroups of the generalised spin groups of Chapter 5. Also we show that the extension is actually by the monomial group given by the basis of the algebras used in Chapter 4.

It is the author's hope that the groups constructed herein will certainly find an application as was intended in the work [16]. Moreover, it would be welcome if the generalised versions of the spin groups and the extended Weyl groups prove interesting to mathematicians for there intrinsic properties.

Lastly, we note that in this work square brackets $[\cdot, \cdot]$ are used to denote both the Lie product and the commutator of two elements in a group. We appeal to the reader to use his judgement and hope that no confusion should arise.

Chapter 2

Spin Groups and Clifford Algebras

The associative algebras called Clifford algebras were first introduced by William Kingdon Clifford as a generalisation of the real and complex numbers and Hamilton's quaternions. They have found applications in the fields of differential geometry, theoretical physics and computer imaging.

This first chapter gives the construction of the groups $\mathrm{Spin}(n, \mathbb{R})$. These groups are the double covers of the special orthogonal groups $\mathrm{SO}(n, \mathbb{R})$. The natural setting for these groups is the Clifford algebra $\mathrm{Cl}(V, Q)$ of an n-dimensional vector space V with a non-degenerate quadratic form Q. Therefore we begin the chapter with a construction of the Clifford algebra and then move on to define the pin and spin groups which are embedded inside. The second section deals with those properties of the spin groups that will be essential to us in the sequel.

2.1 Clifford Algebras

The main references of this section are [2] and [42]. We will restrict our attention to the real numbers. For details over other fields we refer to [49], Section 3.9.2.

Definition 2.1.1 ([22], Section 4.1)

Let V be an n-dimensional real vector space over $\mathbb{R}$. A quadratic form on V is a map

$Q : V \to \mathbb{R}$ such that for all $u, v \in V, \lambda \in \mathbb{R}$, we have

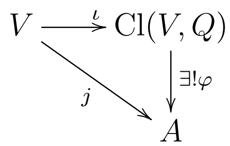
- *$Q(\lambda v) = \lambda^2 Q(v)$ and*
- *$Q(v) = B(v, v)$, where $B : V \times V \to \mathbb{R}$ with $B(u, v) = \frac{1}{2}(Q(u + v) - Q(u) - Q(v))$ is a symmetric bilinear form.*

Under these conditions the pair (V, Q) is called a quadratic space.

Based on his observations in Grassman algebras and Hamilton's quaternions, Clifford in [11] gave a construction of the what he called the geometric algebra. These came to be known as the real Clifford algebras. We give the modern definition below.

Definition 2.1.2 ([21], Definition 1.3)

Let (V, Q) be a quadratic space. Then the Clifford algebra $\mathrm{Cl}(V, Q)$ is the real associative algebra with a 1 and a linear map $\iota : V \to \mathrm{Cl}(V, Q)$, satisfying $\iota(v)^2 = Q(v).1$, for all $v \in V$. Moreover, $\mathrm{Cl}(V, Q)$ is universal in the sense that for any real associative unital algebra A with a linear map $j : V \to A$, satisfying the above condition, there exists a unique algebra homomorphism $\varphi : \mathrm{Cl}(V, Q) \to A$, such that the following diagram is commutative:



that is $j = \varphi \circ \iota$.

We continue by providing an explicit construction of $\mathrm{Cl}(V, Q)$ for a quadratic space (V, Q) see [10], Definition 3.2. The k-th tensor power of V is $T^k(V) = V \otimes \ldots \otimes V$, with the tensor product taken k times, where by convention $T^0(V) = \mathbb{R}$. The tensor algebra of V is the set

$$T(V) = \bigoplus_{i=0}^{\infty} T^i(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots$$

with multiplication given by concatenation and extended linearly.

Now, let I be the two-sided ideal generated by $I = \langle x \otimes y + y \otimes x - B(x,y).\mathbf{1} \mid x, y \in V \rangle$. Then

$$\mathrm{Cl}(V,Q) = \mathrm{Cl}(V) = {}^{T(V)}/_{I}$$

is the Clifford algebra of V with respect to Q. We denote the multiplication in $\mathrm{Cl}(V)$ by juxtaposition.

The map $\iota : V \to \mathrm{Cl}(V)$ is the composition of $V \to T(V) \to \mathrm{Cl}(V)$ and embeds V into $\mathrm{Cl}(V)$. The universal map φ from Definition 2.1.2 is given by defining $\varphi(\iota(v)) = j(v)$ and then extending linearly. We will often consider this embedding and simply write V for $\iota(V)$. Moreover, the space given by $T^0(V) \cong \mathbb{R}$ is a one dimensional subspace of $\mathrm{Cl}(V)$ and we should write $\lambda.\mathbf{1}$ for the image of $\lambda \in \mathbb{R}$ in $\mathrm{Cl}(V)$. However, for brevity, we will just write this as λ. Another notational convention shall be that instead of writing $x_1 \otimes x_2 \otimes \ldots \otimes x_k$, we shall simply write $x_1 x_2 \ldots x_k$. The proof of the following proposition is not hard to verify from the defining relations.

Proposition 2.1.3 ([20], Lemma 20.3)
*For a Euclidean space (V,Q) with $\dim(V) = n$, we have $\dim(\mathrm{Cl}(V)) = 2^n$. In fact if $\{v_1, \ldots, v_n\}$ is a basis for V, then the elements $v_{i_1} v_{i_2} \ldots v_{i_k}$, for $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, along with the element **1** form a basis of $\mathrm{Cl}(V)$.*

Let $\mathrm{Cl}^0(V)$ be the image of $\sum_{i=0}^{\infty} T^{2i}(V)$ under the map $T(V) \to \mathrm{Cl}(V)$ and $\mathrm{Cl}^1(V)$ be the image of $\sum_{i=0}^{\infty} T^{2i+1}(V)$. If $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V, then $\mathrm{Cl}^0(V)$ contains those elements which are sums of basis vectors containing an even number of the v_i and $\mathrm{Cl}^1(V)$ those that are odd. We obtain the decomposition

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$$

which is $\mathbb{Z}_2$-graded in the sense that if $x \in \mathrm{Cl}^i(V)$ and $\mathrm{Cl}^j(V)$, then $xy \in \mathrm{Cl}^m(V)$, where $m \equiv i + j \bmod 2$.

To identify the spin groups inside the Clifford algebras we need to define two certain involutions of the $\mathrm{Cl}(V)$. These will be used to define an action of the invertible elements of the algebra on a certain subspace isomorphic to our original space V.

Definition 2.1.4 ([2], Definition 1.7)
For all $v, u \in V$, define a mapping $\alpha : V \to V$, via $\alpha(v) = -v$. Since $\alpha(vu + uv) = vu + uv = B(v, u)$, this map is well-defined and we can extend it to an automorphism of $\mathrm{Cl}(V)$ which sends the element $x = v_1 v_2 \dots v_k$ to

$$\alpha(x) = (-1)^k v_1 v_2 \dots v_k.$$

It is clear that α is an involution of $\mathrm{Cl}(V)$. Note that the $\mathbb{Z}_2$-grading of $\mathrm{Cl}(V)$ is the eigenspace decomposition with respect to α.

Definition 2.1.5 ([2], pg. 6)
The algebra $\mathrm{Cl}(V)$ admits an antiautomorphism inherited from $T(V)$ namely

$$t : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

$$x = v_1 v_2 \dots v_k \mapsto x^t = v_k \dots v_2 v_1$$

The map $x \mapsto x^t$ is called transposition and x^t is called the transpose of x.

Since $(x \otimes y + y \otimes x - B(x, y))^t = y \otimes x + x \otimes y - B(x, y)$ for each pair $x, y \in V$, this antiautomorphism is well-defined. The next lemma is immediate from the definitions.

Lemma 2.1.6
We have that $\alpha \circ t = t \circ \alpha$.

Definition 2.1.7 ([2], Definition 1.8)

Define an antiautomorphism of Cl(V) *by*

$$x \mapsto \overline{x} = \alpha(x^t) = (\alpha(x))^t.$$

We now have the necessary machinery to define the spin groups. These are subgroups embedded into Clifford algebras which contain the invertible elements of length 1. The aforementioned involutions will be essential in their definition and their action on a subspace V of $\mathrm{Cl}(V, Q)$ which illustrates that they are double covers of $\mathrm{SO}(n, \mathbb{R})$.

2.2 The Spin Groups

In this section we aim to define and investigate some properties of the spin groups. They embed into Clifford algebras and have an action on a subspace that shows that they are double covers of the special orthogonal groups. The definition will be given here and the possible subgroups given by two copies of Spin(2) shall be determined.

We now fix V to be $\mathbb{R}^n$, for $n \in \mathbb{N}$, and let Q be the positive-definite quadratic form on V defined by

$$Q(v) = x_1^2 + x_2^2 + \ldots + x_n^2$$

where $v = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Let $\mathrm{Cl}^{\times}(V)$ be the group of invertible elements of $\mathrm{Cl}(V)$. Recall from the previous section the automorphism α, which for all $v = v_1 \ldots v_k \in \mathrm{Cl}(V)$ gives

$$\alpha(v) = (-1)^k v.$$

We can identify the space V with its image in $\mathrm{Cl}(V)$ and define a subgroup of $\mathrm{Cl}^{\times}(V)$ that acts on it.

Definition 2.2.1 ([2], Definition 3.1)

The set $\mathrm{Cl}^{\times}(V)$ *of invertible elements form a multiplicative group of* $\mathrm{Cl}(V)$ *and we define the subgroup*

$$\mathit{\Gamma} = \{x \in \mathrm{Cl}^{\times}(V) \,|\, \alpha(x)vx^{-1} \in V, \forall v \in V\}$$

to be the Clifford group of $\mathrm{Cl}(V)$.

Note that $\mathit{\Gamma}$ is a subgroup of $\mathrm{Cl}^{\times}(V)$. Moreover, since $\alpha(xy) = \alpha(x)\alpha(y)$ and $(xy)^{-1} = y^{-1}x^{-1}$, for all $x, y \in \mathrm{Cl}(V)$, we get a representation $\rho : \mathit{\Gamma} \to \mathrm{GL}(V)$ where $x \mapsto \rho_x$ is given by

$$\rho_x(v) = \alpha(x)vx^{-1},$$

for all $v \in V$. This map is called the twisted adjoint representation of $\mathit{\Gamma}$. In the following proposition we write $\mathbb{R}^*$ to denote the set of invertible elements of $\mathbb{R}$.

Proposition 2.2.2 ([21], Lemma 1.7)
The kernel of the map $\rho : \mathit{\Gamma} \to \mathrm{GL}(V)$ *is the set* $\mathbb{R}^*$.

Proof. Given $x \in Ker(\rho)$, we have that, for all $v \in V$,

$$\rho_x(v) = \alpha(x)vx^{-1} = v,$$

and so

$$\alpha(x)v = vx.$$

Now we can write $x = x_0 + x_1$ with $x_i \in \mathrm{Cl}^i(V)$. Then

$$x_0v = vx_0$$

since there are an even number of changes and

$$-x_1v = vx_1,$$

similarly. With the standard basis $\{v_1, \ldots, v_n\}$ of V, we can write $x_0 = a_0 + v_1 b_0$, where $a_0 \in \mathrm{Cl}^0(V)$ and $b_0 \in \mathrm{Cl}^1(V)$ and neither term contains the element v_1. By taking $v = v_1$ in the above equations, we see that

$$a_0 + v_1 b_0 = v_1 a_0 v_1^{-1} + v_1^2 b_0 v_1^{-1} = a_0 - v_1 b_0.$$

So $b_0 = 0$. Hence the element x_0 does not contain the element v_1. We can repeat the argument replacing v_1 with any basis vector to see that x_0 does not contain any basis vectors and so is a scalar multiple of 1. A similar argument shows that x_1 cannot contain any basis vectors and so $x = x_0 \in \mathbb{R}$. But x is invertible and so lies in $\mathbb{R}^*$. □

Recall the definition of the bar operation $\overline{x} = \alpha(x^t)$ and define a new map

$$N : \mathrm{Cl}(V) \to \mathrm{Cl}(V),$$

called the norm, where for all $x \in \mathrm{Cl}(V)$,

$$N(x) = x\overline{x}.$$

Proposition 2.2.3 ([21], Proposition 1.8)
If $x \in \Gamma$, then $N(x) \in \mathbb{R}^*$.

Proof. If $x \in \Gamma$, then for all $v \in V$, we have

$$\alpha(x) v x^{-1} = u$$

for some $u \in V$. Since it holds that $u^t = u$, taking transposes yields

$$(x^t)^{-1} v \alpha(x)^t = \alpha(x) v x^{-1}$$

and so

$$v\alpha(x^t)x = x^t\alpha(x)v.$$

This can be rewritten as

$$\begin{aligned} v &= x^t\alpha(x)vx^{-1}\alpha(x^t)^{-1} \\ &= \alpha(\alpha(x^t)x)v(\alpha(x^t)x)^{-1} \end{aligned}$$

as $\alpha^2 = \mathrm{id}$. So $\alpha(x^t)x$ lies in the kernel of ρ and hence is a non-zero scalar by the previous proposition. Taking the transpose shows that $x^t\alpha(x) \in \mathbb{R}^*$, which means that $N(x^t) \in \mathbb{R}^*$. But $x \mapsto x^t$ is an antiautomorphism of Γ and so for all $x \in \Gamma$, $N(x) \in \mathbb{R}^*$. □

Proposition 2.2.4 ([2], Proposition 3.9)
The map $N : \Gamma \to \mathbb{R}^*$ is a homomorphism and $N(\alpha(x)) = N(x)$.

Proof. For $x \in \Gamma$,

$$N(xy) = xy\overline{xy} = xy\overline{y}\,\overline{x} = xN(y)\overline{x} = N(x)N(y).$$

Further,

$$N(\alpha(x)) = \alpha(x)x^t = \alpha(x\alpha(x^t)) = \alpha(N(x)) = N(x).$$ □

Lemma 2.2.5 ([21], Proposition 1.10)
The image $\rho(\Gamma)$ lies in the subgroup $\mathrm{O}(V)$ of $\mathrm{GL}(V)$.

Proof. For all $x \in \Gamma$ and $v \in V$,

$$N(\rho_x(v)) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = N(v).$$

It remains to show that $N(v)$ is equal to the square length of $v \in V$.

$$N(v) = v\alpha(v^{-1})^t = v(v) = v^2 = Q(v)$$

and we are done. $\square$

Definition 2.2.6 ([2], Theorem 3.11)
Let $\mathrm{Pin}(V)$ *be the kernel of the map* $N : \Gamma \to \mathbb{R}^*$. *The group* $\mathrm{Pin}(V)$ *is called the pin group of* $\mathrm{Cl}(V)$.

Proposition 2.2.7 ([2], Theorem 3.11)
We have the following exact sequence

$$\{1\} \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\rho} \mathrm{O}(V) \longrightarrow \{1\}.$$

Proof. To see that ρ is onto, first note that for any basis vector v_i of V, we have $\alpha(v_i)v_i^{-1} = -1$ and for two distinct basis vectors v_i, v_j, we have

$$v_i v_j = -v_j v_i$$

since our basis is orthonormal. Then for the vector v_1, $\alpha(v_1)v_i v_1^{-1}$ is $-v_i$ if $i = 1$ or v_i otherwise. Hence $\rho(v_1)$ is the reflection in the hyperplane orthogonal to v_1. Applying the same argument to all the basis vectors with $Q(x) = 1$ we get the unit sphere, given by $\{x \in V \mid N(x) = 1\}$, is in $\mathrm{Pin}(v)$. This shows that all orthogonal reflections are in $\rho(\mathrm{Pin}(V))$ and as these generate $\mathrm{O}(V)$, ρ is onto. The fact that the kernel of the map is $\{\pm 1\}$ follows immediately from the fact that the kernel of ρ is the set of non-zero scalars of norm 1. $\square$

Definition 2.2.8 ([2], Definition 3.12)

If ρ is the map from $\mathrm{Pin}(V)$ *to* $\mathrm{O}(V)$ *described above, then the subgroup*

$$\mathrm{Spin}(V) = \rho^{-1}(\mathrm{SO}(V))$$

is called the spin group of $\mathrm{Cl}(V)$.

Proposition 2.2.9 ([2], Proposition 3.13)

$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^0(V)$.

Proof. If $x \in \mathrm{Pin}(V)$, then $\rho(x)$ is the product of some reflections, $\rho(x) = s_1 s_2 \ldots s_k$, say. By the previous proposition there are elements $x_i \in \mathrm{Pin}(V)$ with $\rho(x_i) = s_i$, for $1 \leqslant i \leqslant k$ and $x = \pm x_1 x_2 \ldots x_k$ so x is either in $\mathrm{Cl}^0(V)$ or $\mathrm{Cl}^1(V)$. By the Cartan-Dieudonné theorem ([42] 1.4.5.), $x \in \mathrm{Spin}(V)$ if, and only if, $x \in \mathrm{Cl}^0(V)$. □

Lemma 2.2.10

The following diagram commutes

$$\begin{array}{ccc} \mathrm{Spin}(V) & \xrightarrow{\rho} & \mathrm{SO}(V) \\ \downarrow & & \downarrow \\ \mathrm{Pin}(V) & \xrightarrow{\rho} & \mathrm{O}(V) \end{array}$$

Proof. This follows by the definition of $\mathrm{Spin}(V)$ and Proposition 2.2.9. □

Proposition 2.2.11

$\mathrm{Spin}(V)$ *has index* 2 *inside* $\mathrm{Pin}(V)$.

Proof. We can extend the diagram in Lemma 2.2.10 by the determinant map to get

$$\begin{array}{ccccc} \mathrm{Spin}(V) & \xrightarrow{\rho} & \mathrm{SO}(V) & & \\ \downarrow & & \downarrow & & \\ \mathrm{Pin}(V) & \xrightarrow{\rho} & \mathrm{O}(V) & \xrightarrow{\det} & \{\pm 1\} \end{array}$$

The map $\rho \circ \det$ is a homomorphism from $\mathrm{Pin}(n)$ to $\{\pm 1\}$. By definition the kernel of this map is $\mathrm{Spin}(n)$. $\square$

The above proposition gives us a nice description of the group $\mathrm{Pin}(V)$. Every element acts as either a rotation or a reflection on the subspace V. The subgroup $\mathrm{Spin}(V)$ is the set of elements acting by rotations and we can choose a single element in $\mathrm{Pin}(V)$ to be the representative of the second coset consisting of reflections.

Recall that V is an n-dimensional vector space over $\mathbb{R}$ with standard basis $\{v_1, \ldots, v_n\}$ equipped with a quadratic form Q defined as before. Let $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ be respectively the pin and spin groups of $\mathrm{Cl}(V)$.

The following results can be found in Section 1.4 of [21].

Proposition 2.2.12

The group $\mathrm{Spin}(2)$ *is isomorphic to* $\mathrm{U}(1)$, *the group of complex numbers of length one.*

Proof. Elements of $\mathrm{Spin}(2)$ are of the form

$$\lambda + \mu v_1 v_2,$$

with $\lambda, \mu \in \mathbb{R}$ and $\lambda^2 + \mu^2 = 1$. We have

$$v_1 v_2 v_1 v_2 = -v_1^2 v_2^2 = -1.$$

So we can identify the element $\lambda + \mu v_1 v_2$ with the complex number $\lambda + \mu i$ in $\mathrm{U}(1)$. $\square$

Proposition 2.2.13

$\mathrm{Spin}(3)$ *is isomorphic to* $\mathrm{SU}(2)$, *the group of unit quaternions.*

Proof. The group $\mathrm{Spin}(3)$ has elements of the form

$$\lambda_1 + \lambda_2 v_1 v_2 + \lambda_3 v_1 v_3 + \lambda_4 v_2 v_3,$$

where the λ_i are in $\mathbb{R}$ and satisfy $\sum_{i=1}^{4} \lambda_i = 1$. Each $v_l v_k$ squares to -1 by the same argument found in the previous proof. Elements of SU(2) are of the form

$$\lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k,$$

under the same conditions as those of Spin(3). So the identification of v_1v_2 with i, v_1v_3 with j and v_2v_3 with k gives the desired isomorphism. □

We note, however, that Spin(4) is not isomorphic to $\mathbb{O}_1$, the set of unit octonions, which can be seen by the fact that $\mathbb{O}_1$ is non-associative and hence the units in it do not form a group. In fact it is known ([21], Section 1.4) that we have $\text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3)$ but this is not essential for the remainder of this thesis.

Later in this work, we will consider amalgams of groups and therefore we will need to know what the rank one and rank two subgroups will be. It will turn out that the rank one groups are in fact isomorphic to Spin(2) and the rank two parts will be the possible subgroups these can generate. In the final part of this chapter the possible subgroups generated by two copies of Spin(2) will be derived.

For the remainder of this chapter let $n \geqslant 3$, $\{v_i \mid 1 \leqslant i \leqslant n\}$ be a basis of $V = \mathbb{R}^n$, with $\text{Cl}(V)$ the appropriate Clifford algebra and set

$$\text{Spin}_{ij}(2) := \{\lambda + \mu v_i v_j \mid \lambda^2 + \mu^2 = 1\} \leqslant \text{Spin}(n).$$

Suppose we choose two copies of Spin(2) whose monomial part have disjoint indices, *i.e.* , for v_iv_j in one copy and v_lv_k in the other, we have $i, j \notin \{l, k\}$. Then for an arbitrary

$x \in \mathrm{Spin}_{12}(2)$ and $y \in \mathrm{Spin}_{34}(2)$, we have

$$\begin{aligned}
xy &= (\lambda + \mu v_1 v_2)(\alpha + \beta v_3 v_4) \\
&= \lambda\alpha + \mu\alpha v_1 v_2 + \lambda\beta v_3 v_4 + \mu\beta v_1 v_2 v_3 v_4 \\
&= \lambda\alpha + \mu\alpha v_1 v_2 + \lambda\beta v_3 v_4 + \mu\beta v_3 v_4 v_1 v_2 \\
&= \alpha(\lambda + \mu v_1 v_2) + \beta v_3 v_4(\lambda + \mu v_3 v_4) \\
&= (\alpha + \beta v_3 v_4)(\lambda + \mu v_1 v_2) \\
&= yx.
\end{aligned}$$

So these two subgroups of $\mathrm{Spin}(n)$ commute. We denote the subgroup of $\mathrm{Spin}(n)$, $n \geqslant 4$, generated by these two groups as $\mathrm{Spin}(2) \circ \mathrm{Spin}(2)$. Let

$$\theta : \mathrm{Spin}_{12}(2) \times \mathrm{Spin}_{34}(2) \to \mathrm{Spin}(2) \circ \mathrm{Spin}(2),$$

$$(x, y) \mapsto xy.$$

It has been shown that the two subgroups commute and so it is clear from the definition of the map that it is a homomorphism. We have that $Ker\theta = \{(1,1), (-1,-1)\}$. Suppose further that $x \in \mathrm{Spin}(2) \circ \mathrm{Spin}(2)$, since the groups commute, x can be written in the form

$$x = x_{12} x_{34}$$

with $x_{12} \in \mathrm{Spin}_{12}(2)$ and $x_{34} \in \mathrm{Spin}_{34}(2)$. Then the elements $(\pm x_{12}, \pm x_{34}) \in \mathrm{Spin}_{12}(2) \times \mathrm{Spin}_{34}(2)$ maps onto x. We sum this up in the following proposition.

Proposition 2.2.14

$$(\mathrm{Spin}_{12}(2) \times \mathrm{Spin}_{34}(2)) \Big/ \{(\pm 1, \pm 1)\} \cong \mathrm{Spin}(2) \circ \mathrm{Spin}(2).$$

Now if the two indices of the monomials are not equal but not disjoint for example v_1v_2 and v_2v_3, then for $x \in \mathrm{Spin}_{12}(2)$ and $y \in \mathrm{Spin}_{23}(2)$, it follows that

$$\begin{aligned} xy &= (\lambda + \mu v_1 v_2)(\alpha + \beta v_2 v_3) \\ &= \lambda\alpha + \mu\alpha v_1 v_2 + \lambda\beta v_2 v_3 + \mu\beta v_1 v_3, \end{aligned}$$

which is an element lying in a Spin(3) subgroup of Spin(n).

Proposition 2.2.15

Two such copies of Spin(2) *as above generate a copy of* Spin(3).

Proof. Let $\mathrm{SO}_{12}(2)$ be the image of $\mathrm{Spin}_{12}(2)$ under ρ and similarly define $\mathrm{SO}_{23}(2)$. Then there is a 3-dimensional subalgebra $W = \langle v_1, v_2, v_3 \rangle \subset \mathbb{R}^n$ on which these groups act. If we can show that these two groups give a copy of SO(3), then we can use the fact that $\rho^{-1}(\mathrm{SO}(3)) \cong \mathrm{Spin}(3)$.

Let $\sigma \in \mathrm{SO}(3)$, acting on W. Choose $v \in W$ and consider $u = \sigma(v)$. We need to show that we can move from v to u using only the actions of $\mathrm{SO}_{12}(2)$ and $\mathrm{SO}_{23}(2)$.

Let $orb_{ij}(w)$ be the orbit of $w \in W$ under the group $\mathrm{SO}_{ij}(2)$. If we have $orb_{12}(v) = orb_{12}(u)$, we are done and similarly for $orb_{23}(v)$. So suppose they are disjoint.

Note that since $\langle v_1, v_2 \rangle$ and $\langle v_2, v_3 \rangle$ are perpendicular, we have that $orb_{23}(v) \cap \langle v_1, v_2 \rangle$ is non-empty and similary for $orb_{23}(u)$.

Let $\tau_1, \tau_2 \in \mathrm{SO}_{23}(2)$ be such that $\tau_1(v), \tau_2(u) \cap \langle v_1, v_2 \rangle$ are non-empty. And let $\sigma_1 \in \mathrm{SO}_{12}(2)$ the rotation from $\tau_1(v)$ to $\tau_2(u)$. Then the map $\tau_2^{-1}\sigma_1\tau_1$ takes v to u and is therefore equal to σ. □

In this chapter we have met the Clifford algebras and spin groups. We have also derived the properties of each that will be essential in what follows. The even part of the Clifford algebra can be made into a Lie algebra in the usual way for an associative algebra and into this Lie algebra, we can embed the special orthogonal Lie algebra. This property will be generalised in the sequel. Similarly, we will aim to generalise the construction of the Spin groups using the theory of group amalgams so that they can be described in a more general context.

Chapter 3

Kac-Moody Algebras and Their Maximal Compact Subalgebras

Kac-Moody algebras were discovered independently by Victor Kac [35] and Robert Moody [43] in the late 1960s. They are, in general, infinite dimensional Lie algebras that arise from generalised Cartan matrices thus extending the theory of complex semisimple Lie algebras. In recent years they have become popular in mathematical physics. The hyperbolic Kac-Moody algebra $\mathfrak{e}_{10}(\mathbb{R})$ has received special attention due to its connection with M-Theory, an eleven dimensional model of the universe. In fact it is this connection that led to a certain representation being formulated for the maximal compact subalgebra $\mathfrak{k}(\mathfrak{e}_{10})$, which has been the motivation behind our work.

In this chapter we begin with a brief overview of Kac-Moody algebras and some of their well-known properties. We then define what will be for us an important subalgebra that is the set of fixed points under the Cartan-Chevalley involution. This subalgebra plays a key role in what is to follow and a set of generators and relations is given in the simply-laced case.

3.1 Kac-Moody Algebras

A complex semisimple Lie algebra can always be recovered from its Cartan matrix, which were first investigated by Wilhelm Killing. These are positive-definite square matrices. The work of Moody was inspired by what happens when the matrix is no longer positive-definite and this in turn led to his discovery. This motivates our first definition.

Definition 3.1.1 ([38], 1.1; [9], pg. 319)
An $n \times n$ matrix $A = (a_{ij})$ over $\mathbb{Z}$ of rank l is called a Generalised Cartan Matrix (or GCM) if the following conditions are satisfied:

1. $a_{ii} = 2$, *for all* $1 \leqslant i \leqslant n$;
2. $a_{ij} \leqslant 0$, *for all* $i \neq j$;
3. $a_{ij} = 0 \iff a_{ji} = 0$.

For any GCM A we wish to associate a Lie algebra. Before this can be done a further definition is needed.

Definition 3.1.2 ([38], 1.1; [9], Propostion 14.1)
A triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ *is called a realisation of a GCM* $A = (a_{ij})$ *if*

1. $\mathfrak{h}$ *is a* $(2n - l)$ *dimensional vector space over* $\mathbb{C}$;
2. $\Pi^\vee = \{h_1, \ldots, h_n\} \subseteq \mathfrak{h}$ *is a set of* n *linearly independent vectors;*
3. $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{h}^*$ *is a linearly independent set of covectors such that*
4. $\alpha_j(h_i) = a_{ij}$, *for all* $h_i \in \Pi^\vee$ *and* $\alpha_j \in \Pi$.

For each GCM A there is a unique realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ up to isomorphism for details see [38] Proposition 1.1 or [9] Propositions 14.2 and 14.3.

Definition 3.1.3 ([40], Definition 1.1.2)
To the $n \times n$ generalised Cartan matrix $A = (a_{ij})$ with realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ we define the Kac-Moody algebra $\mathfrak{g}$ to be the quotient of the free Lie algebra on the generators $\{e_i, f_i, \mathfrak{h} \,|\, 1 \leqslant i \leqslant n\}$ over the field $\mathbb{C}$ defined by the relations:

1. $[\mathfrak{h}, \mathfrak{h}] = 0$, $\mathfrak{h}$ *is abelian;*
2. $[e_i, f_j] = \delta_{ij} h_i$ *for all* $1 \leqslant i, j \leqslant n$;
3. $[h_i, e_j] = \alpha_j(h_i) e_j$;
4. $[h_i, f_j] = -\alpha_j(h_i) f_j$;
5. $(ad\ e_i)^{1-a_{ij}} e_j = 0 = (ad\ f_i)^{1-a_{ij}} f_j$, *for* $i \neq j$.

The last relation is known as the Serre relations after Jean-Pierre Serre. The standard definition of a Kac-Moody algebra does not involve this last relation and is obtained by taking the remaining relations and factoring out the largest ideal which intersects $\mathfrak{h}$ trivially. However, when the GCM A is symmetrisable, *i.e.* there exists a diagonal matrix D and a symmetric matrix B with $A = DB$, then by the Gabber-Kac Theorem ([38] 9.11), the Serre relations define the subalgebras $\mathfrak{n}_\pm$. Throughout this thesis we will only be interested in certain Kac-Moody algebras obtained from symmetrisable GCMs. To this end we make the following definition.

Definition 3.1.4 ([6], pg. 194)
If we have that $a_{ij} \in \{0, -1\}$ for all $i \neq j$, we call $\mathfrak{g}$ simply-laced.

When $\mathfrak{g}$ is simply-laced, by Definition 3.1.1.3, the matrix A is symmetrisable (in fact it is symmetric) and so our definition of a Kac-Moody algebra will suffice. To a simply-laced GCM A a diagram Δ can be associated which has n vertices n_i, $1 \leqslant i \leqslant n$, and two vertices n_i, n_j are connected by an edge if, and only if, $a_{ij} = a_{ji} = -1$.

We now collect some basic facts about Kac-Moody algebras, details and proofs can be found in [38], Chapter 1, and [9], Chapter 14. The subalgebra $\mathfrak{h}$ is called the Cartan subalgebra of $\mathfrak{g}$. Set $\Phi_{\mathbb{Z}}$ to be the integer lattice in $\mathfrak{h}^*$ spanned by the elements $\alpha_1, \dots, \alpha_n$. For $\alpha \in \Phi_{\mathbb{Z}}$ we define the space

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}.$$

If α is non-zero and $\mathfrak{g}_\alpha$ non-trivial, then they are called a root and root space of $\mathfrak{g}$, respectively. We let $\Gamma = \{\alpha \in Q_{\mathbb{Z}} \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$ be the set of roots of $\mathfrak{g}$. Let

$$\Gamma^+ = \{\alpha = \sum_{i=1}^{n} z_i \alpha_i \in \Gamma \mid z_i \geqslant 0, \forall i\}$$

be the set of positive roots and let $\Gamma^- = -\Gamma^+$ be the set of negative roots. Then we have $\Gamma = \Gamma^+ \cup \Gamma^-$. We can now define $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Gamma^\pm} \mathfrak{g}_\alpha$. Then $\mathfrak{n}_+$ is the subalgebra generated by the e_i, for $1 \leqslant i \leqslant n$ and similarly, $\mathfrak{n}_-$ is generated by the f_i. Note that $\mathfrak{g}_0 = \mathfrak{h}$ and so we have the triangular decompostion

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

which can also be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha.$$

3.2 The Maximal Compact Subalgebra

In this section we introduce the maximal compact subalgebra of a Kac-Moody algebra, which is the subalgebra of fixed points of a certain involution of the Kac-Moody algebra $\mathfrak{g}$. Hence we obtain an eigenspace decomposition of $\mathfrak{g}$. Following the work of [3] and [26] we give a set of generators and relations for them. We introduce the necessary involution which is in some sense related to skew-symmetry.

Definition 3.2.1 ([38], pg. 7; [9], Proposition 14.17)
Let $\mathfrak{g}$ be a Kac-Moody algebra with GCM $A = (a_{ij})$. The Cartan-Chevalley involution $\omega \in Aut(\mathfrak{g})$ is defined by:

1. $\omega(e_i) = -f_i$;

2. $\omega(f_i) = -e_i$;

3. $\omega(h) = -h$, *for all* $h \in \mathfrak{h}$.

So ω interchanges the positive root space $\mathfrak{n}_+$ with $\mathfrak{n}_-$. Note that the involution defined in [3] differs from the one we are using here: We do not include a graph automorphism, *i.e.* a permutation π of the indices $\{1, \dots, n\}$ of the GCM such that $a_{ij} = a_{\pi(i)\pi(j)}$. The paper also has $\omega(e_i) = f_i$ and $\omega(f_i) = e_i$.

Definition 3.2.2 ([26], Section 2.2)
The subalgebra $\mathfrak{k} := \mathfrak{k}(\mathfrak{g}) = \{\mathbf{x} \in \mathfrak{g} \,|\, \omega(\mathbf{x}) = \mathbf{x}\}$ is called the maximal compact subalgebra of $\mathfrak{g}$.

This terminology follows from the theory of Lie groups where in the semisimple case the analytic group K corresponding to $\mathfrak{k}$ is the maximal compact subgroup of the Lie group G corresponding to $\mathfrak{g}$. In the paper [3] these are referred to as involutory subalgebras.

Using the work of [3] and [26] we wish to give a set of generators and relations for $\mathfrak{k}$ when $\mathfrak{g}$ is simply-laced. Recall that a Kac-Moody algebra $\mathfrak{g}$ satisfies the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Proposition 3.2.3

For a simply-laced Kac-Moody algebra $\mathfrak{g}$ *with maximal compact subalgebra* $\mathfrak{k}$, *we have*

$$\mathfrak{k} = \{x + \omega(x) \mid x \in \mathfrak{n}_+\}.$$

Proof. Set $\mathfrak{k}' = \{x + \omega(x) \mid x \in \mathfrak{n}_+\}$. If $y \in \mathfrak{k}'$, then there exists an $y_0 \in \mathfrak{n}_+$ such that $y = y_0 + \omega(y_0)$. Then

$$\omega(y) = \omega(y_0 + \omega(y_0)) = \omega(y_0) + \omega(\omega(y_0)) = \omega(y_0) + y_0 = y.$$

Hence $\mathfrak{k}' \subseteq \mathfrak{k}$.

For the other inclusion choose $x \in \mathfrak{k}$, then by the triangular decomposition we can write $x = x_- + x_h + x_+$ with $x_\pm \in \mathfrak{n}_\pm$ and $x_h \in \mathfrak{h}$. Since $\omega(h) = -h$, for all $h \in \mathfrak{h}$, we get that $x_h = 0$ and x is of the form $x = x_- + x_+$. Now $\omega(x) = x$ and so

$$x = \omega(x) = \omega(x_-) + \omega(x_+).$$

The involution ω interchanges the two subspaces $\mathfrak{n}_+$ and $\mathfrak{n}_-$, so $\omega(x_+) \in \mathfrak{n}_-$ and similarly for $\omega(x_-)$. Then by comparing coefficients, we must have

$$x_\pm = \omega(x_\mp).$$

Hence $x = x_+ + \omega(x_+)$, with $x_+ \in \mathfrak{n}_+$ and so $x \in \mathfrak{k}'$. □

Proposition 3.2.4 ([3], Proposition 1.12)

The maximal compact subalgebra $\mathfrak{k}$ *is generated by* $\{X_i = e_i - f_i \mid 1 \leqslant i \leqslant n.\}$

In the following proof we use the notation $[x_1, x_2, \ldots, x_k]$ for $[\ldots[[x_1, x_2], x_3], \ldots, x_k]$.

Proof. Let $\mathfrak{k}$ be the maximal compact subalgebra of $\mathfrak{g}$ and suppose K is the subalgebra generated by $\{X_i \,|\, 1 \leqslant i \leqslant n.\}$. Since

$$\omega(X_i) = \omega(e_i - f_i) = \omega(e_i) - \omega(f_i) = -f_i + e_i = X_i,$$

we get $K \subseteq \mathfrak{k}$ so we only need show that $\mathfrak{k} \subseteq K$. By Proposition 3.2.3, it suffices to show that for indices $i_1, \ldots, i_r$, $r \geqslant 2$, the element $[e_{i_1}, \ldots, e_{i_r}] + \omega[e_{i_1}, \ldots, e_{i_r}]$ lies in K. This is done by induction. In the base case, for any two distinct indices $r \neq s$, we have

$$[e_r, e_s] + [f_r, f_s] = [e_r - f_r, e_s - f_s] + ([e_r, f_s] + [e_s, f_r])$$

$$= [X_r, X_s] \in K$$

since $[e_r, f_s] = [e_s, f_r] = 0$. Now let $r \geqslant 3$, Set $i = i_r$, $x = [e_{i_1}, \ldots, e_{i_{r-1}}]$ and $y = \omega(x)$. So

$$[e_{i_1}, \ldots, e_{i_r}] + \omega[e_{i_1}, \ldots, e_{i_r}] = [x, e_i] + [y, -f_i]$$

$$= [x + y, e_i - f_i] - ([x, -f_i] + [y, e_i]).$$

By induction $x + y$ is in K and so is $X_i = e_i - f_i$. It is left to show that $[x, -f_i] + [y, e_i]$ also lies in K. We see that

$$[x, -f_i] = -\sum_{t=2}^{r-1} \delta_{i,i_t}(\alpha_{i_1} + \ldots + \alpha_{i_{t-1}})(h_i)[e_{i_1}, \ldots, \widehat{e}_{i_t}, \ldots, e_{i_{r-1}}]$$

$$+\delta_{i,i_1}\alpha_{i_2}(h_i)[e_{i_2}, \ldots, e_{i_{r-1}}]$$

So, by induction, it lies in K. □

Definition 3.2.5

Let $\mathfrak{g}$ be a Kac-Moody algebra. Then the elements $X_i = e_i - f_i$, for $1 \leqslant i \leqslant n$, are called *the Berman generators of the maximal compact subalgebra* $\mathfrak{k}$.

Recall that to any simply-laced Kac-Moody algebra $\mathfrak{g}(A)$ we can associate a diagram Δ. For each node of Δ, we have two generators, e_i and f_i. So for each node n_i of Δ, we get a unique Berman generator X_i of $\mathfrak{k}$. The following definition therefore makes sense.

Definition 3.2.6

Two Berman generators shall be called (non-)adjacent if their corresponding nodes are (non-)adjacent on the graph Δ.

In [3] a set of structure constants are defined and a multiplication is defined for the Berman generators. However, we are only concerned with the simply-laced case and hence can reduce these results to computations. There are only two cases to consider, when the generators are adjacent and when they are not.

Lemma 3.2.7

For adjacent Berman generators X_i *and* X_j*, we have*

$$[X_i, [X_i, X_j]] = -X_j.$$

Proof. Let $X_i = e_i - f_i$ and $X_j = e_j - f_j$. Recall the relations defined in Definition 3.1.3. We have $[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] = 0$ and since $\mathfrak{g}$ is simply-laced $[h_i, e_i] = -e_i$ and

$[h_i, f_i] = f_i$, for all i.

$$\begin{aligned}
[X_i, [X_i, X_j]] &= [e_i - f_i, [e_i - f_i, e_j - f_j]] \\
&= [e_i - f_i, [e_i, e_j] + [f_i, f_j]] \\
&= [e_i, [e_i, e_j]] + [e_i, [f_i, f_j]] - [f_i, [e_i, e_j]] - [f_i, [f_i, f_j]] \\
&= [[e_i, f_i], f_j] + [f_i, [e_i, f_j]] - [[f_i, e_i], e_j] - [e_i, [f_i, e_j]] \\
&= [h_i, f_j] + [h_i, e_j] \\
&= -\alpha_j(h_i) f_j + \alpha_j(h_i) e_j \\
&= f_j - e_j = -X_j. \qquad \square
\end{aligned}$$

Lemma 3.2.8

For non-adjacent Berman generators X_i and X_j,

$$[X_i, X_j] = 0.$$

Proof. For non-adjacent generators $X_i = e_i - f_i$ and $X_j = e_j - f_j$ we have the relations $[e_i, e_j] = [f_i, f_j] = 0$. Hence

$$[X_i, X_j] = [e_i - f_i, e_j - f_j] = [e_i, e_j] + [f_i, f_j] = 0. \qquad \square$$

We are now in a position to state the final result of this chapter. This essentially states that the given generators and relations we have studied do indeed define the maximal compact subalgebra. That is that no other relations are satisfied.

Theorem 3.2.9 ([3], Theorem 1.31)

For a simply-laced Kac-Moody algebra $\mathfrak{g}$ the maximal compact subalgebra $\mathfrak{k}$ is isomorphic to the free Lie algebra generated by Y_i, $1 \leqslant i \leqslant n$, subject to the following relations:

- $[Y_i, [Y_i, Y_j]] = -Y_j$, *if* $|a_{ij}| = 1$ *and*

- $[Y_i, Y_j] = 0$, *otherwise.*

Proof. By Proposition 3.2.4, $\mathfrak{k}$ is generated by the n Berman generators. Let Y be the Lie algebra generated by the Y_i. Since the Berman generators of $\mathfrak{k}$ satisfy the given relations we get a surjection from Y onto $\mathfrak{k}$. Injectivity is much more involved and follows from defining a filtration of both Y and $\mathfrak{k}$. The reader is referred to [3] Theorem 1.31 for details. □

We end this chapter with an example which will play an important part throughout this text. Details can be found in [30].

Example 3.2.10

Let $\mathfrak{sl}_n(\mathbb{R})$ *be the* (n^2-1)*-dimensional special linear Lie algebra over the field* $\mathbb{R}$*. Then we have a generating set* $\{e_i, f_i, h_i \mid 1 \leqslant i \leqslant n-1\}$*. The maximal compact subalgebra* $\mathfrak{k}(\mathfrak{sl}_n(\mathbb{R}))$ *is generated by the elements* $\{e_i - f_i \mid 1 \leqslant i \leqslant n-1\}$*. It is well known that the generators* e_i *of* $\mathfrak{sl}_n(\mathbb{R})$ *can be identified with the matrices* $E_{i,i+1}$ *which have a* 1 *in the* $(i, i+1)^{th}$ *position and zeroes elsewhere. Similarly, the* f_i *are identified with the matrices* $E_{i+1,i}$*. So* $\mathfrak{k}$ *is generated by the matrices* $E_{i,i+1} - E_{i+1,i}$ *and these are precisely the generators of* $\mathfrak{so}_n(\mathbb{R})$*. So we have*

$$\mathfrak{k}(\mathfrak{sl}_n(\mathbb{R})) = \mathfrak{so}_n(\mathbb{R}).$$

Indeed the group $SO(n, \mathbb{R})$ *is known to be the maximal compact subgroup of the Lie group* $SL(n, \mathbb{R})$*.*

So we see that the special linear Lie algebra over $\mathbb{R}$ has the special orthogonal Lie algebra as its maximal compact subalgebra. It will be seen below that the Berman generators of $\mathfrak{so}_n(\mathbb{R})$ in fact play a role in the spin representation and hence in generating the group $\mathrm{Spin}(n, \mathbb{R})$. It is from this observation that we aim to generalise the construction of the spin groups for the A_n Dynkin diagram to any simply-laced Kac-Moody algebra.

Chapter 4

The Spin Representations

It was Élie Cartan in 1913 [8] who was the first to classify the irreducible representations of the complex simple Lie algebras. These so called fundamental representations were constructed on the nodes of the associated Dynkin diagram. However, Cartan noticed that there are some representations of the special orthogonal Lie algebras that cannot be built in that manner. These are the spin representations.

This chapter begins with the construction of the spin representation of the semisimple Lie algebra $\mathfrak{so}(n, \mathbb{R})$. Since Clifford algebras are associative, they can be made into Lie algebras in the natural way. Then we can embed $\mathfrak{so}(n, \mathbb{R})$ into this algebra. In the second section of this chapter we describe the generalised spin representation first given in [26] which generalises the work of [15] and [17]. The next section attempts to set the framework to classify the images of these representations using some geometry and group representation theory.

4.1 The Spin Representations

In this first section we introduce both the classical spin representation of Cartan and then generalised spin representation, which was introduced in [26] building on the finite dimensional representation of the maximal compact subalgebra of the Kac-Moody algebra

$\mathfrak{e}_{10}(\mathbb{R})$ in [15] and [17].

Let (V, Q) be a quadratic space where Q is the standard positive-definite quadratic form on V. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V and $\mathrm{Cl}(V)$ the corresponding Clifford algebra. Since $\mathrm{Cl}(V)$ is an associative algebra, we can define the Lie bracket on it via

$$[x, y] = xy - yx,$$

for all $x, y \in \mathrm{Cl}(V)$. With this multiplication $\mathrm{Cl}(V)$ becomes a Lie algebra which we denote by $\mathcal{L}(V)$. Further, we denote by $\mathcal{L}_2(V)$ the space spanned by the $\binom{n}{2}$ elements of the form $\frac{1}{2}v_iv_j \in \mathcal{L}(V)$, with $i \neq j$.

Proposition 4.1.1

The space $\mathcal{L}_2(V)$ is a Lie subalgebra of $\mathcal{L}(V)$.

Proof. For $1 \leqslant i, j, k, l \leqslant n$, we have

$$\begin{aligned}[\tfrac{1}{2}v_iv_j, \tfrac{1}{2}v_kv_l] &= \tfrac{1}{4}(-\delta_{il}v_jv_k + \delta_{ik}v_jv_l + \delta_{jl}v_iv_k - \delta_{jk}v_iv_l \\ &\quad -\delta_{ki}v_lv_j + \delta_{kj}v_lv_i + \delta_{li}v_kv_j - \delta_{lj}v_kv_i) \\ &= \tfrac{1}{2}(\delta_{il}v_kv_j - \delta_{ik}v_lv_j + \delta_{jl}v_iv_k - \delta_{jk}v_iv_l).\end{aligned}$$

□

The next theorem shows that the n-dimensional special orthogonal Lie algebra $\mathfrak{so}(n, \mathbb{R})$ embeds into $\mathcal{L}(V)$. Recall from Chapter 3 that $\mathfrak{so}(n, \mathbb{R})$ is the maximal compact subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ and is spanned by the elements $\overline{E}_{i,j} = E_{i,j} - E_{j,i}$, for $i < j$, where $E_{i,j}$ is the matrix with entry 1 in the (i, j)th and zeroes elsewhere.

Theorem 4.1.2

The Lie algebra $\mathfrak{so}(n, \mathbb{R})$ is isomorphic to the Lie subalgebra $\mathcal{L}_2(V)$.

Proof. First note that we have $\binom{n}{2}$ elements of the form $\overline{E}_{i,j}$. Define a map $\rho : \mathfrak{so}(n, \mathbb{R}) \to \mathcal{L}_2(V)$ via $\rho(\overline{E}_{i,j}) = \frac{1}{2}v_iv_j$. To see this is a homomorphism note that the elements $\overline{E}_{i,j}$

satisfy the relations

$$[\overline{E}_{i,j}, \overline{E}_{k,l}] = \delta_{il}\overline{E}_{k,j} - \delta_{ik}\overline{E}_{l,j} + \delta_{jl}\overline{E}_{i,k} - \delta_{jk}\overline{E}_{i,l},$$

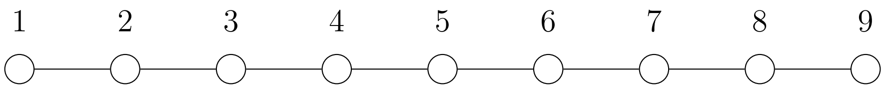
which are identical to those found in the proof of Proposition 4.1.1. □

Definition 4.1.3

The representation ρ defined above is called the spin representation of the special orthogonal Lie algebra, $\mathfrak{so}(n, \mathbb{R})$.

Example 4.1.4

We give an example now to show how the spin representation works. We begin with the Dynkin diagram of type A_9.



Then the associated Lie algebra is $\mathfrak{sl}(10, \mathbb{R})$ and its maximal compact subalgebra $\mathfrak{so}(10, \mathbb{R})$ is given by the ten generators X_i, $1 \leqslant i \leqslant 10$. The spin representation sends

$$X_i \mapsto \frac{1}{2} v_i v_{i+1}.$$

Remark 4.1.5

We note that $\mathfrak{so}(n, \mathbb{R})$ is given by the n Berman generators $X_i = E_{i,i+1} - E_{i+1,i}$ and that under the spin representation we have

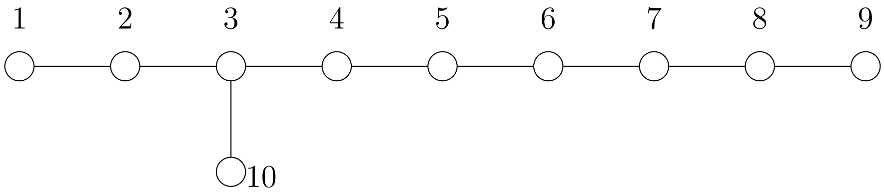
$$\rho(X_i) = \frac{1}{2} v_i v_{i+1}.$$

These elements satisfy the property that

$$\rho(X_i)^2 = \left(\frac{1}{2}v_i v_j\right)^2 = -\frac{1}{4}.\boldsymbol{1}.$$

Example 4.1.6

For the E_{10} *diagram a generalisation of the spin representation was given in* [15]. *We give the construction here. The diagram is of the form:*



Then for the ten berman generators X_i *of* $\mathfrak{k} = \mathfrak{k}(\mathfrak{e}_{10}(\mathbb{R}))$, *numbered as in the graph, we get a generalised spin representation* $\rho : \mathfrak{k} \to \mathrm{Cl}(\mathbb{R}^{10})$, *by defining:*

$$\begin{array}{ccc} X_1 \mapsto \frac{1}{2}v_1v_2, & X_2 \mapsto \frac{1}{2}v_2v_3, & X_3 \mapsto \frac{1}{2}v_3v_4, \\ X_4 \mapsto \frac{1}{2}v_4v_5, & X_5 \mapsto \frac{1}{2}v_5v_6, & X_6 \mapsto \frac{1}{2}v_6v_7, \\ X_7 \mapsto \frac{1}{2}v_7v_8 & X_8 \mapsto \frac{1}{2}v_8v_9 & X_9 \mapsto \frac{1}{2}v_9v_{10}, \\ & X_{10} \mapsto \frac{1}{2}v_1v_2v_3. & \end{array}$$

It can be checked that for all i *and* $A_i = \rho(X_i)$, *we have* $A_i^2 = -\frac{1}{4}.\boldsymbol{1}$. *Moreover, adjacent generators anticommute and nonadjacent ones commute.*

This example is open to generalistion to maximal compact subalgebras of simply-laced Kac-Moody algebras. To start we fix some notation. Let $\Delta = (\mathcal{V}, \mathcal{E})$ be a simply-laced diagram with Kac-Moody algebra $\mathfrak{g}$ and maximal compact subalgebra $\mathfrak{k}$ on the Berman generators X_i, $1 \leqslant i \leqslant n$. Set k to be a field of characteristic 0. If I is a square root of -1, set $\mathbb{F} = k(I)$ and denote by id_s the $s \times s$ identity matrix in $M_s(\mathbb{F})$. Then by the Remark 4.1.5 we can make the following definition.

Definition 4.1.7 ([26], Definition 4.4)

A representation $\rho : \mathfrak{k} \to \mathrm{End}(L^s)$ *is called a generalised spin representation if for every Berman generator* X_i*, we have*

$$\rho(X_i)^2 = -\frac{1}{4}\mathrm{id}_s.$$

Remark 4.1.8 ([26], Remark 4.5)

We collect some properties of the spin representation

1. *Let* ρ *be a generalised spin representation. Then if* $(i,j) \notin \mathcal{E}$*, we must have that* $\rho(X_i)$ *and* $\rho(X_j)$ *commute. Whilst if* $(i,j) \in \mathcal{E}$*, set* $A = \rho(X_i)$ *and* $B = \rho(X_j)$*, then*

$$-B = [A,[A,B]] = A^2B - 2ABA + BA^2 = -\frac{1}{2}B - 2ABA.$$

   By our defining property, we have $A^{-1} = -4A$ *so*

$$4AB = 2AB - 2BA$$

   which implies that A *and* B *anticommute, as required.*

2. *A reversal of the above argument mean that if we have matrices* $A_i \in M_s(\mathbb{F})$ *for* $1 \leqslant i \leqslant n$*, then the mapping* $X_i \mapsto A_i$ *gives a generalised spin representation if we have*

   - $A_i^2 = -\frac{1}{4}\mathrm{id}_s$;
   - $A_iA_j = -A_jA_i$ *if* $(i,j) \in \mathcal{E}$ *and*
   - $A_iA_j = A_jA_i$*, otherwise.*

The final result of this section shows that generalised spin representations indeed exist and gives an *ad hoc* method for their construction.

Definition 4.1.9

Let $\mathfrak{g}$ *and* $\mathfrak{k} = \langle X_1, \ldots, X_n \rangle$ *be as above. Then, for* $1 \leqslant r \leqslant n$*, set*

$$\mathfrak{k}_r = \langle X_1, \ldots, X_r \rangle.$$

Theorem 4.1.10 ([26], Theorem 4.7)

Let $\rho : \mathfrak{k}_r \to End(L^m)$ *be a generalised spin representation, then*

- *if* X_{r+1} *commutes with* $\mathfrak{k}_r$*, then* ρ *can be extended to another generalised spin reprenstation* ρ' *by setting* $\rho'(X_{r+1}) = \frac{1}{2} I.id_{L^m}$ *and*

- *if* X_{r+1} *does not centralise* $\mathfrak{k}_r$*, then* ρ *can again be extended to a representation* ρ' *which is given by*

$$\rho'(\mathfrak{k}_{r+1}) \to End(L^m \oplus L^m).$$

  This is given by

$$\rho'|_{\mathfrak{k}_r} = \rho \oplus (\rho \cdot s_0),$$

  where s_0 *is a sign automorphism, and*

$$\rho'(X_{r+1}) = \frac{1}{2} I \cdot id_{L^m} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right).$$

Proof. The first claim is clear. For the second, we begin by defining the sign automorpism s_0. For $X_i \in \mathfrak{k}_r$ set

$$s_0(X_i) = \begin{cases} -X_i & \text{if } X_i \text{ and } X_{i+1} \text{ are adjacent in } \mathit{\Gamma}, \\ X_i & \text{otherwise.} \end{cases}$$

Then with ρ' given in the statement of the theorem, it is clear that it is an extension of ρ, the images of the Berman geenrators satisfy the requirements and so it is a generalised spin representation. □

Example 4.1.11

Returning to the E_{10} example we get the same representation as for the A_9 diagram as in Example 4.1.4. This embeds $\mathfrak{so}(10,\mathbb{R})$ into $\mathrm{Cl}(\mathbb{R}^{10})$. We then use the theorem above to generalise it. First suppose that $\{v_1,\ldots,v_{10}\}$ is a basis for V and let U be another 10-dimensional real space with basis $\{u_1,\ldots,u_{10}\}$. Then we have the Clifford algebra $\mathrm{Cl}(V)\oplus\mathrm{Cl}(U)$. Then the image of the generator X_{10} associated to the extra node should anticommute with $\rho'(X_3)$ and commute with the rest. So we negate the image of X_3 in the Clifford algebra $\mathrm{Cl}(U)$. The new map ρ' given in Theorem 4.1.10 is given by

$$\begin{aligned}
&X_1\mapsto \tfrac{1}{2}(v_1v_2\oplus u_1u_2),\quad X_2\mapsto \tfrac{1}{2}(v_2v_3\oplus u_2u_3),\quad X_3\mapsto \tfrac{1}{2}(v_3v_4\oplus -u_3u_4),\\
&X_4\mapsto \tfrac{1}{2}(v_4v_5\oplus u_4u_5),\quad X_5\mapsto \tfrac{1}{2}(v_5v_6\oplus u_5u_6),\quad X_6\mapsto \tfrac{1}{2}(v_6v_7\oplus u_6u_7),\\
&X_7\mapsto \tfrac{1}{2}(v_7v_8\oplus u_7u_8),\quad X_8\mapsto \tfrac{1}{2}(v_8v_9\oplus u_8u_9),\quad X_9\mapsto \tfrac{1}{2}(v_9v_{10}\oplus u_9u_{10}),
\end{aligned}$$

and

$$X_{10}\mapsto \frac{1}{2}I\cdot id_s\otimes\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right).$$

4.2 Clifford Monomial Groups

It is the aim of this section to show our preliminary work towards classifying the images of the generalised spin representations of a maximal compact subalgebra $\mathfrak{k}$. We construct a group which will act as a basis for an algebra into which we can embed the image of the representation and then use the analysis of the geometry of the group to classify some images. The results are not complete but it is hoped that they may lead to a full classification. This work is joint with Kieran Roberts and we thank Sergey Shpectorov for bringing our attention to [5].

Definition 4.2.1

Given a simply-laced diagram Δ with associated GCM $A=(a_{ij})$, let $C(\Delta)$ be the group *given by the following presentation*

$$C(\Delta) := \langle z, x_i, 1 \leqslant i \leqslant n \,|\, z^2 = 1; x_i^2 = z; [x_i, x_j] = z^{a_{ij}} \rangle.$$

Then $C(\Delta)$ will be called the Clifford monomial group of Δ.

This definition generalises the monomial basis of a Clifford algebra. We will construct a finite-dimensional algebra from $C(\Delta)$ in which we can embed a generalised spin representation. From now on we fix a diagram Δ and set $C := C(\Delta)$ to be the resulting Clifford monomial group.

Lemma 4.2.2

The element z is in the centre of C.

Proof. First note that every generator x_i has order four and so $x_i^{-1} = x_i^3$. Then

$$[z, x_i] = zx_izx_i^{-1} = zx_izx_i^2x_i = zx_i^2 = z^2 = 1.$$ □

Lemma 4.2.3

Let $Z = \langle z \rangle$, then C/Z is elementary abelian of rank n.

Proof. This follows immediately from the fact that for any two generators x_i, x_j, we have

$$[x_i, x_j] = z^{a_{ij}}.$$ □

Corollary 4.2.4

The Frattini subgroup $\Phi(C)$ is equal to the subgroup Z.

Proof. By Corollary 11.10 of [45] we have that $\Phi(C)$ is the unique smallest normal subgroup of C such that the factor group is elementary abelian. □

Lemma 4.2.5

Every element $c \in C$ can be written in the form $z^{\varepsilon_0} x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ with $\varepsilon_i \in \{0, 1\}$.

Proof. Let c be in C. Then c is of the form, $z^{\varepsilon_0} x_{i_1}^{\varepsilon_1} \ldots x_{i_n}^{\varepsilon_n}$. Let i_k be the smallest index. Since we have $[x_i, x_j] = z^{a_{ij}}$, we can move x_{i_k} until it is in the first position adding only a power of z. We can then continue this process until we have all the x_i in ascending order reading from left to right. Now for each generator we have the following cyclic subgroup $\{1, x_i, z, zx_i\}$ and as z is central, we can move every occurence to the front. □

Lemma 4.2.6

For any $x, y \in C$, we have $[x, y] = z^t$ where $t \in \{0, 1\}$.

Proof. By the previous lemma, we can write $x = z^{\varepsilon_0} x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ and $y = z^{\delta_0} x_1^{\delta_1} \ldots x_n^{\delta_n}$. Since $x_i^{-1} = zx_i$, when we consider $[x, y]$ there will be an even number of occurences of each generator. We can use the rewriting rules to move these elements together and hence we will be left with $[x, y] = z^t$. □

Note in particular that the derived subgroup $C' = [C, C]$ is Z. Hence we have that $Z = \Phi(C) = C'$ and it is contained in the centre $Z(C)$. However, this inclusion is sometimes proper. If, on other the hand, we have equality, then C is an extraspecial 2-group.

Proposition 4.2.7

Let $x \notin Z(C)$ be of order four, then the conjugacy class $x^C = \{x, x^{-1}\}$.

Proof. Since x has order four, we have that $x^{-1} = zx$. Furthermore x is not in the centre, so by Lemma 4.2.6, there exists an element $y \in C$ such that $[y, x] = z$. Then

$$[y, x] = yxy^{-1}x^{-1} = z$$

and so

$$yxy^{-1} = zx = x^{-1}.$$

□

The following definition is taken from Section 4 of [28] and generalises the class of extraspecial groups.

Definition 4.2.8

A group E is called an E_2-group if it is a 2-group and it contains a normal subgroup Z such that E/Z is elementary abelian. The subgroup Z is called the scalar subgroup.

The proof of the following result is immediate.

Proposition 4.2.9

A Clifford monomial group C is an E_2-group with scalar subgroup Z.

Let $\mathbb{C}C$ be the group algebra of a Clifford monomial group C. Let $I := \langle z+1\rangle$ be the two-sided ideal in $\mathbb{C}C$ generated by the element $z+1$ and set $A := \mathbb{C}C/I$. Since A is an associative algebra, as before we can define a Lie bracket on it via $[x,y] = xy - yx$, for all $x, y \in A$. Let $L(C)$ be the Lie algebra obtained in such a manner.

Let A be an $n \times n$ GCM and C be the corresponding Clifford monomial group and $\mathfrak{k}$ the maximal compact subalgebra of the Kac-Moody algebra $\mathfrak{g}'$. Then for all $1 \leqslant i \leqslant n$, C has generators z, x_i and $\mathfrak{k}$ has Berman generators X_i. Define a map $\rho : \mathfrak{k} \to L(C)$ as

$$\rho(X_i) = \frac{1}{2}x_i,$$

for all $1 \leqslant i \leqslant n$.

Theorem 4.2.10

The map $\rho : \mathfrak{k} \to L(C)$ is a generalised spin representation.

Proof. By (2) of Remark 4.1.8 it suffices to show that the images satisfy the following conditions:

- $\rho(X_i)^2 = -\frac{1}{4}.\mathbf{1}_{L(C)}$;

- $\rho(X_i)\rho(X_j) = \rho(X_j)\rho(X_i)$ if $a_{ij} = 0$ and
- $\rho(X_i)\rho(X_j) = -\rho(X_j)\rho(X_i)$, otherwise.

But these are all immediate. □

Now the Lie algebra $L(G)$ is too big in general and we are only interested in the image of ρ.

Proposition 4.2.11

If $x \in im(\rho)$, then x has order four in C.

Proof. Note first that $im(\rho)$ is generated by the elements $\rho(X_i) = \frac{1}{2}x_i$. So up to rescaling we can consider the Lie subalgebra of $L(G)$ on the generators x_i, for $1 \leqslant i \leqslant n$. Consider the product of two generators x, y. If the generators commute in G, they do not contribute a new element to $im(\rho)$. However, if they anticommute, then $[x, y] = xy - yx = 2xy$. Now consider

$$(xy)^2 = xyxy = -x^2y^2 = -1.$$

Hence $[x, y]$ has order four. For elements of longer length the argument is the same. Either they commute and contribute no new element to the algebra or they anticommute. Again, if they anticommute by an identical argument their product will have order four. □

Proposition 4.2.12

If $x \in im(\rho)$, then $x \notin Z(G)$.

Proof. Suppose there are elements $x, y \in L(G)$ such that $[x, y] = xy \in Z(G)$. Then we have $x(xy) = x^2y = -y$. However, $(xy)x = -x^2y = y$, a contradiction. □

4.3 Towards a Classification of the Generalised Spin Representations

In this section we begin work towards a possible classification of the images of the generalised spin representations. The results here are incomplete except under the condition that the Clifford Monomial group is extra-special.

Definition 4.3.1 ([5], Section 1)
A partial linear space $\Pi = (P, L)$ consists of points P and lines L whose point-line incidence graph does not contain any quadrangles, that is a cycle of length four. Π is called connected if its point-line incidence graph is connnected.

The condition that the incidence graph does not contain quadrangles ensures that any two points in P either lie on a unique line or no line. Kaplansky first studied the Lie algebras $\mathcal{L}(\Pi)$ associated to partial linear spaces by defining

$$[x, y] = \begin{cases} z & \text{if } l = \{x, y, z\} \in L \text{ or} \\ 0 & \text{otherwise} \end{cases}$$

which yielded Lie algebras over the field of size two.

To work over arbitrary fields one needs first to give each line $l \in L$, a cyclic orientation $\sigma(l)$. That is for $l = \{x, y, z\} \in L$, we have that $\sigma(l)$ is one of the cycles (x, y, z) or (x, z, y). Such a partial linear space is said to be oriented.

Definition 4.3.2 ([5], Section 1)
For an oriented partial linear space $\Pi = (P, L, \sigma)$ with lines of size three and a field $\mathbb{F}$ of

characteristic not 2 we construct an algebra $\mathcal{L}_{\mathbb{F}}(\Pi, \sigma)$ *with multiplication*

$$[x, y] = \begin{cases} z & \text{if } l = \{x, y, z\} \in L \text{ and } \sigma(l) = (x, y, z), \\ -z & \text{if } l = \{x, y, z\} \in L \text{ and } \sigma(l) = (x, z, y) \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{L}_{\mathbb{F}}(\Pi, \sigma)$ *is called the Kaplansky algebra of the partial linear space* Π*. If a Kaplansky algebra* $\mathcal{L}_{\mathbb{F}}(\Pi, \sigma)$ *is a Lie algebra, then we call* (Π, σ) *a Lie oriented partial linear space.*

Definition 4.3.3 ([5], 1.2)

For a point $p \in P$*, define an automorphism called the flip* σ_p *via,*

$$\sigma_p(l) = \begin{cases} \sigma(l), & \text{if } p \notin l \\ (\sigma(l))^{-1} & \text{if } p \in l. \end{cases}$$

Kaplansky Lie algebras of flipped spaces are isomorphic.

Theorem 4.3.4 ([5], Theorem 1.1)

If $\Pi = (P, L, \sigma)$ *is a connected Lie oriented partial linear space, then* Π *is isomorphic to one of the following*

- $\mathcal{T}(\Omega_1, \Omega_2)$*, containing two disjoint sets* Ω_1 *and* Ω_2 *with points those subsets* $A \subset \Omega_1 \cup \Omega_2$ *with* $|A \cap \Omega_1| = 2$ *and lines given by triples* $\{A, B, C\}$ *such that* $A + B + C = 0$ *in the binary space* $2^{\Omega_1 \cup \Omega_2}$.

- $\mathcal{SP}(V, f)$*, the partial linear space from a vector space* V *over* $\mathbb{F}_2$ *equipped with a symplectic form* f*. The points are the vectors outside the radical of* f *and the lines are the hyperbolic lines.*

- $\mathcal{O}(V, Q)$*, the partial linear space on a vector space* V *over* $\mathbb{F}_2$ *with a a quadratic* *form Q, where again the points are the vectors outside the radical of Q and the lines are the elliptic lines.*

- *$\mathbb{P}V \setminus \mathbb{P}W$, where V is a vector space over $\mathbb{F}_2$ and W is a subspce of codimension 3. The points are those in $\mathbb{P}V$ but not in $\mathbb{P}W$ and the lines are those that are disjoint from $\mathbb{P}W$. We allow the possibility that W could be empty.*

Conversely, the above spaces admit an orientation that is unique, up to flipping, that makes the the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(\Pi, \sigma)$ into a Lie algebra if the characteristic of $\mathbb{F}$ is not 2.

Definition 4.3.5 ([5], Section 6)
For a point $p \in P$ let $p^{\perp}$ be the set of points not collinear to p union with the set $\{p\}$. Define an equivalence relation on P by stating $p \equiv q$ if and only if $p^{\perp} = q^{\perp}$. The partial linear space Π is called reduced if all the equivalence classes contain only one point.

From the quotient space $P/\equiv$ we form the quotient geometry $\Pi/\equiv$ where the points are the $\equiv$-equivalence classes and the lines are those triples of points obtained from the lines of Π.

Theorem 4.3.6 ([5], Proposition 6.1)
For a field $\mathbb{F}$ and connected Lie oriented partial linear space (Π, σ), the Kaplansky Lie algebra $\mathcal{L}_F(\Pi)$ is simple if and only if it is reduced.

Proposition 4.3.7 ([5], Proposition 6.2)
Let Π be a connected Lie oriented partial linear space and $\mathcal{L}_{\mathbb{F}}(\Pi)$ be the Kaplansky Lie algebra of Π over a field $\mathbb{F}$. Then if the characteristic of $\mathbb{F}$ is not equal to 2 and if necessary, extending $\mathbb{F}$ by $\sqrt{2}$, then the algebra $\mathcal{L}_{\mathbb{F}}(\Pi)$ is the direct sum of pairwise commuting copies of $\mathcal{L}_{\mathbb{F}}(\Pi/\equiv)$.

Theorem 4.3.8 ([5], Theorem 6.3)

Let (Π, σ) be a reduced Lie oriented partial linear space and $\mathcal{L}_{\mathbb{F}}(\Pi)$ be the associated Kaplansky Lie algebra over a field $\mathbb{F}$ whose characteristic is not equal to 2. Then $\mathcal{L}_{\mathbb{F}}(\Pi)$ is isomorphic to

- $\mathfrak{so}(n, \mathbb{F})$ *if* Π *is isomorphic to* $\mathcal{T}(\{1, \ldots, n\}, \emptyset)$.
- $\mathfrak{sl}(2^n, \mathbb{F})$ *if* Π *is isomorphic to* $\mathcal{SP}(V, f)$ *for a non-degenrate binary linear space* (V, f) *of dimension* 2^n *and* $\sqrt{-1} \in \mathbb{F}$.
- $\mathfrak{so}(2^n, \mathbb{F})$ *if* Π *is isomorphic to* $\mathcal{O}(V, Q)$ *for a non-degenrate binary orthogonal space* (V, Q) *of dimension* 2^n *with maximal Witt index.*
- $\mathfrak{sp}(2^{n-1}, \mathbb{F})$ *for* Π *isomorphic to* $\mathcal{O}(V, Q)$ *as above except it is of Witt index* $n - 1$ *and also* $\sqrt{-1} \in \mathbb{F}$.

We now describe a way to associate a partial linear space of type $\mathcal{O}(V, Q)$ to an E_2-group. As set out in Section 2.2 of [5] It is shown in [28] that for a binary quadratic space (V, Q) and bilinear form $f : V \times V \to \mathbb{F}_2$ such that $f(v, v) = Q(v)$, for all $v \in V$, then the set $E = V \times \mathbb{F}_2$ with multiplication

$$(v_1, \eta_1)(v_2, \eta_2) = (v_1 + v_2, \eta_1 + \eta_2 + f(v_1, v_2))$$

is an E_2-group and indeed every E_2-group arises in such a manner. Recall that every Clifford monomial group is an E_2-group.

Definition 4.3.9 ([5], Section 2.2)

For an E_2-group, E, we define the geometry of E by taking the points to be the cyclic subgroups of order 4 and the lines to be the quaternion subgroups, Q_8.

4.4 Plesken Lie Algebras

In this section we introduce another Lie algbra based on a finite group. For E_2-groups these algebras are isomorphic to the Kaplansky algebras and some representation theory has been used to classify them.

Definition 4.4.1 ([12], Section 2)
Let G be a finite group with group algebra $\mathbb{C}G$. Let $\widehat{g} = g - g^{-1} \in \mathbb{C}G$. The Plesken Lie algebra $\mathcal{L}(G)$ is the Lie algebra generated by the elements $\widehat{g}$.

The Lie product on $\mathcal{L}(G)$ is well-defined since an elementary calculation shows that

$$[\widehat{g}, \widehat{h}] = \widehat{gh} - \widehat{gh^{-1}} - \widehat{g^{-1}h} + \widehat{g^{-1}h^{-1}}.$$

Definition 4.4.2 ([32], pg. 58)
For an irreducible character χ of a finite group and for the Schur-Frobenius indicator ν we call

- *χ real if $\nu(\chi) = 1$,*
- *χ is symplectic if $\nu(\chi) = -1$ and*
- *χ is complex if $\nu(\chi) = 0$.*

Theorem 4.4.3 ([12], Theorem 5.1)
For a finite group G, the Plesken Lie algebra $\mathcal{L}G$ decomposes as

$$\mathcal{L}(G) \cong \bigoplus_{\chi \in \mathfrak{R}} \mathfrak{so}(\chi(1)) \oplus \bigoplus_{\chi \in \mathfrak{Sp}} \mathfrak{sp}(\chi(1)) \oplus \bigoplus_{\{\chi, \overline{\chi}\} \in \mathfrak{C}} \mathfrak{gl}(\chi(1))$$

with $\mathfrak{R}, \mathfrak{Sp}$ and $\mathfrak{C}$ are the sets of real, symplectic and complex characters, respectively. Where the sums in the general linear case are taken once only for each pair of $\{\chi, \overline{\chi}\} \in \mathfrak{C}$.

Proposition 4.4.4 ([5],Proposition 2.3)

For an E_2-group E and a field $\mathbb{F}$ of characteristic different from 2, the Plesken Lie algebra $\mathcal{L}(E)$ is isomorphic to the Kaplansky Lie algebra of the geometry of E.

We are now in a position to prove our main result of this chapter.

Theorem 4.4.5

Let C be a Clifford monomial group and let $Z(C)$ be its centre. Then the Kaplansky Lie algebra of the geometry of C is isomorphic to at most $|Z(C)|/2$ copies of either $\mathfrak{so}(n, \mathbb{R})$ or $\mathfrak{sp}(n, \mathbb{R})$, for some $n \in \mathbb{N}$.

Proof. By Proposition 4.2.9 C is an E_2-group and so by Proposition 4.4.4 its Plesken algebra is isomorphic to the Kaplansky algebra of its geometry. So the Plesken algebra $\mathcal{L}(C)$ is of the form seen in Theorem 4.4.3 However, by Proposition 4.3.7 the Kaplansky Lie algebra of the geometry of C is the direct sum of mutually commuting copies of its reduced geometry which is either of the form $\mathfrak{so}(n, \mathbb{R})$ or $\mathfrak{sp}(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

Now the number of characters of C is equal to the number of conjugacy classes and since by, Proposition 4.2.7, all non-central elements of C are only conjugate to their inverses, we get $|Z(C)| + |C \setminus Z(C)|/2$ conjugacy classes. By Theorem 17.11 of [34] the number of linear characters is $|C/C'| = |C/Z| = |C|/2$. So the number of non-linear characters is

$$|Z(C)| + \frac{|C \setminus Z(C)|}{2} - \frac{|C|}{2} = \frac{|Z(C)|}{2}.$$

□

Proposition 4.4.6

The Lie algebra $im(\rho)$ embeds into the Kaplansky algebra $\mathcal{L}(C)$.

Proof. By Definition 4.3.9 the Kaplansky algebra of the geometry of an E_2-group is isomorphic is given by the cyclic subgroups of order 4. But by Proposition 4.2.12 the elements in $im(\rho)$ have order 4 and so can be identified with the cyclic subgroups. Two such elements commute or they generate a group isomorphic to the quaternion group Q_8. □

Chapter 5

The Generalised Spin Groups

In this chapter we give a construction of the generalised spin groups and show that they are indeed double covers of the maximal compact subgroups associated to a spilt real Kac-Moody group $G(\Delta)$. We begin by showing that the standard spin groups are the universal completions of an amalgam consisting of the groups Spin(2) and the groups they generate.

5.1 Amalgams of Type H

We start by defining a special type of amalgam. These are of use since our amalgams will be over only one type of group, namely Spin(2), SO(2) or SL(2). In each case we are working over the real numbers $\mathbb{R}$ and so omit the field from the symbol.

Definition 5.1.1

For a finite index set I and a given group H, set $\mathcal{A} := \{G_{ij}, \phi^i_{ij}, \phi^j_{ij} \,|\, i \neq j\}$ where each G_{ij} is a group and $\phi^i_{ij}, \phi^j_{ij} : H \to G_{ij}$ are injective homomorphisms for all $i \neq j$. Then we call $\mathcal{A}$ an amalgam with respect to H. If the groups H and G_{ij}, for all $i, j \in I$, are topological and groups and the maps ϕ^i_{ij} and ϕ^j_{ij} are continuous, then the amalgam is said to be continuous.

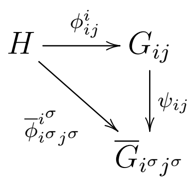
The group H is called the rank-1 subgroup and the groups G_{ij} are the rank-2 groups. Given an amalgam it is possible to construct groups from them. We shall be interested in when the amalgams yield unique groups up to isomorphism.

Definition 5.1.2

Let $\mathcal{A} = \{G_{ij}, \phi^i_{ij}, \phi^j_{ij} \,|\, i \neq j\}$ *and* $\overline{\mathcal{A}} = \{\overline{G}_{ij}, \overline{\phi}^i_{ij}, \overline{\phi}^j_{ij} \,|\, i \neq j\}$ *be amalgams over a group* H. *An isomorphism* $\psi : \mathcal{A} \to \overline{\mathcal{A}}$ *is a system* $\psi = (\sigma, \psi_{ij})_{i \neq j}$ *of isomorphisms where* $\sigma \in \mathfrak{S}(n)$, *the symmetric group on* n *letters, and* $\psi_{ij} : G_{ij} \to \overline{G}_{i^\sigma j^\sigma}$, *satisfying*

$$\psi_{ij} \circ \phi^i_{ij} = \overline{\phi}^{i^\sigma}_{i^\sigma j^\sigma} \text{ and } \psi_{ij} \circ \phi^j_{ij} = \overline{\phi}^{i^\sigma}_{i^\sigma j^\sigma}.$$

For clarity, we represent this as a commutative diagram.



The definition then states that the isomorphims ψ_{ij} respect the embeddings of H into the rank-2 groups. Later we shall associate an amalgam to a simply laced diagram Δ. In a way that will become clear, the permutation σ will relate to a graph automorphism of Δ.

Definition 5.1.3

For an amalgam $\mathcal{A} = \{G_{ij}, \phi^i_{ij}, \phi^j_{ij} \,|\, i \neq j\}$ *over a group* H, *a completion of* $\mathcal{A}$ *is again a system* $(G, \{\phi_{ij} \,|\, i \neq j\})$ *consisting of a group* G *and a set of homomorphisms* $\phi_{ij} : G_{ij} \to G$ *such that*

$$G = \langle \phi_{ij}(G_{ij}) \,|\, i \neq j \rangle$$

and

$$\phi_{ij} \circ \phi^j_{ij} = \phi_{kj} \circ \phi^j_{kj},$$

for all $i \neq j \neq k$.

That is, the diagram

$$\begin{array}{ccc} H & \xrightarrow{\phi^i_{ij}} & G_{ij} \\ {\scriptstyle \phi^j_{kj}}\downarrow & & \downarrow{\scriptstyle \phi_{ij}} \\ G_{kj} & \xrightarrow[\phi_{kj}]{} & G \end{array}$$

commutes. This ensures that group H has a well-defined image as the intersection in all the rank-2 subgroups G_{ij} and G_{jk} in the completion G.

Definition 5.1.4

The universal completion $(G(\mathcal{A}), \{\phi_{ij} \mid i \neq j\})$ *of an amalgam* $\mathcal{A}$ *of type* H *is defined to be*

$$G(\mathcal{A}) := \langle G_{ij} \mid \mathcal{R}(G_{ij}),\ \phi^i_{ij}(x)\phi^j_{ij}(x)^{-1} \rangle$$

where $\mathcal{R}(G_{ij})$ *is the set of relations for the* G_{ij} *and* $\phi_{ij} : G_{ij} \to G(\mathcal{A})$ *is the canonical homomorphism.*

We see that the universal completion of an amalgam is a group $G(\mathcal{A})$ whose elements are words in the different embeddings of H into $G(\mathcal{A})$. Furthermore, two adjacent letters in the word will lie in a group G_{ij} and so can be subjected to a rewriting and this process can be indefinitely repeated using associativity.

The following definition and result allow us to identify when a group is in fact the universal completion of an amalgam based on its action on a geometry.

Definition 5.1.5 ([33], Definition 1.3.1)

An amalgam $\mathcal{A}$ *of finite type and rank* $n \geqslant 2$ *is a set such that for all* $1 \leqslant i \leqslant n$, *there is a subset* A_i *of* $\mathcal{A}$ *and a binary operation* $\odot_i$ *on* A_i *such that*

1. *the pair* $(A_i, \odot_i)$ *is a group for all* $1 \leqslant i \leqslant n$,

2. $\mathcal{A} = \bigcup_{i=1}^{n} A_i$,

3. $|A_i \cap A_j|$ *is finite if* $i \neq j$ *and* $\bigcap_{i=1}^{n} A_i \neq \emptyset$,

4. $(A_i \cap A_j, \odot_i)$ *is a subgroup of* A_i *for all* $1 \leqslant i, j \leqslant n$ *and*

5. *if* $x, y \in A_i \cap A_j$, *then* $x \odot_i y = x \odot_j y$.

The equivalence of this definition with the Definition 5.1.5 is given in [24], Remark 3.3, and is a routine calculation.

Lemma 5.1.6 (Tits' Lemma, [33], Corollary 1.4.6)
Let $\mathcal{G}$ *be a simply connected geometry of rank* $n \geqslant 3$ *and* G *a group acting flag transitively on it. Further, let* $\mathcal{F} = \{x_1, \ldots, x_n\}$ *be a maximal flag and* G_i *the stabiliser of* x_i *in* G. *Let* $\mathcal{A} = \{G_i, 1 \leqslant i \leqslant n\}$ *be the amalgam in the sense of Definition 1.3.1 of* [33]. *Then* G *is isomorphic to the universal completion of* $\mathcal{A}$.

5.2 Amalgams of Type SO(2) and Spin(2)

In this section we give a treatment of the amalgams of type H for H isomorphic to SO(2) and Spin(2). We want to show that the groups $\mathrm{SO}(n, \mathbb{R})$ and $\mathrm{Spin}(n, \mathbb{R})$, for $n \geqslant 4$, are in fact universal completions of amalgams for the A_{n-1} diagrams. It is this result in the Spin(2) case that allows us to define the generalised spin groups.

Definition 5.2.1
Let $\Delta = (\mathcal{V}, \mathcal{E})$ *be a simply-laced diagram with vertex set and edge set* $\mathcal{V}$ *and* $\mathcal{E}$, *respectively. Suppose* $|\mathcal{V}| = n$, *and set* $I = \{1, \ldots, n\}$ *so that* $\sigma : I \to \mathcal{V}$ *is a labelling. An amalgam,* $\mathcal{A}(\Delta, \sigma, \mathrm{SO}(2))$, *of type* SO(2) *with respect to* Δ *and* σ *is an amalgam* $\mathcal{A} := \{G_{ij}\, \phi_{ij}^i, \phi_{ij}^j \,|\, i \neq j \in I\}$ *such that for all* $i \neq j$, *we have*

$$G_{ij} = \begin{cases} \mathrm{SO}(3), & (i^\sigma, j^\sigma) \in \mathcal{E} \\ \mathrm{SO}(2) \times \mathrm{SO}(2), & (i^\sigma, j^\sigma) \notin \mathcal{E}. \end{cases}$$

Furthermore, for all $i < j$,

$$\phi_{ij}^{i}(\mathrm{SO}(2)) = \varepsilon_{12}(\mathrm{SO}(2)), \text{ whilst } \phi_{ij}^{j}(\mathrm{SO}(2)) = \begin{cases} \varepsilon_{23}(\mathrm{SO}(2)), & (i^\sigma, j^\sigma) \in \mathcal{E} \\ \varepsilon_{34}(\mathrm{SO}(2)), & (i^\sigma, j^\sigma) \notin \mathcal{E}. \end{cases}$$

Here ε_{ij} is the canonical homomorphism of SO(2) in a matrix representation.

It is shown in [24] that the amalgam of type $\mathcal{A}(\Delta, \sigma, \mathrm{SO}(2))$ is unique up to isomorphism and independent of the choice of σ. Hence we can omit it and simply write $\mathcal{A}(\Delta, \mathrm{SO}(2))$.

Theorem 5.2.2

For $n \geqslant 4$, we have that the group $\mathrm{SO}(n)$ is the universal completion of the amalgam $\mathcal{A}(A_{n-1}, \mathrm{SO}(2)))$.

Proof. Set $I := \{1, \ldots, n-1\}$. We have that $\mathrm{SO}(n)$ acts flag-transitively, by the action of the Iwasawa decomposition of $\mathrm{SL}(n)$, on the simply-connected ([47], Theorem 13.32) geometry $\mathbb{P}_{n-1}(\mathbb{R})$. A maximal flag is of the form

$$\langle v_1 \rangle \leqslant \ldots \leqslant \langle v_1 \ldots v_{n-1} \rangle.$$

The torus of $\mathrm{SO}(n)$ consists of the diagonal matrices and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ nad has a presentation that has all generators and relations in the rank 2 block diagonal subgroups. An iteration of Tits's Lemma (Lemma 5.1.6) corresponding to the maximal flag above says that $\mathrm{SO}(n)$ is the universal completion of the amalgam of type SO(2) given by $\mathcal{A}(\mathbb{P}(\mathbb{R}^{n-1}), \mathrm{SO}(n)) := \{G_{ij}, \phi_{ij}^{i}, \phi_{ij}^{j} \mid i \neq j \in I\}$, where ϕ_{ij}^{i} are the inclusion maps

$\mathrm{SO}(2) \to G_{ij}$. For $i < j$, we have

$$G_{ij} = \begin{cases} \varepsilon_{i,i+1,i+2}(\mathrm{SO}(3)), & \text{if } j = i+1 \\ \varepsilon_{i,i+1}(\mathrm{SO}(2)) \times \varepsilon_{j,j+1}(\mathrm{SO}(2)), & \text{otherwise.} \end{cases}$$

We see that $\alpha = \{\mathrm{id}_I, \alpha_{ij}, \alpha_i \,|\, i \neq j \in I\}$ with

$$\alpha_{ij} = \begin{cases} \varepsilon_{i,i+1,i+2}, & \text{if } j = i+1, \\ \varepsilon_{i,i+1,j,j+1}, & \text{otherwise} \end{cases}$$

and $\forall i \in I$, $\alpha_i : \mathrm{SO}(2) \to \mathrm{SO}(2)$, is an isomorphism of amalgams cf. Definition 5.1.2. □

We now do the same thing for the groups $\mathrm{Spin}(n)$ and define amalgams of type $\mathrm{Spin}(2)$.

Definition 5.2.3

Let $\Delta = (\mathcal{V}, \mathcal{E})$ be a simply-laced diagram with vertex set and edge set $\mathcal{V}$ and $\mathcal{E}$, respectively. Suppose $|\mathcal{V}| = n$, and set $I = \{1, \ldots, n\}$ so that $\sigma : I \to \mathcal{V}$ is a labelling. An amalgam, $\mathcal{A}(\Delta, \sigma, \mathrm{Spin}(2))$, of type $\mathrm{Spin}(2)$ with respect to Δ and σ is an amalgam $\mathcal{A} := \{G_{ij}\, \phi_{ij}^i, \phi_{ij}^j \,|\, i \neq j \in I\}$ such that for all $i \neq j$, we have

$$G_{ij} = \begin{cases} \mathrm{Spin}(3), & (i^\sigma, j^\sigma) \in \mathcal{E} \\ \mathrm{Spin}(2) \circ \mathrm{Spin}(2), & (i^\sigma, j^\sigma) \notin \mathcal{E}. \end{cases}$$

Furthermore, for all $i < j$,

$$\phi_{ij}^i(\mathrm{Spin}(2)) = \varepsilon_{12}(\mathrm{Spin}(2)), \ \textit{whilst} \ \phi_{ij}^j(\mathrm{Spin}(2)) = \begin{cases} \varepsilon_{23}(\mathrm{Spin}(2)), & (i^\sigma, j^\sigma) \in \mathcal{E} \\ \varepsilon_{34}(\mathrm{Spin}(2)), & (i^\sigma, j^\sigma) \notin \mathcal{E}. \end{cases}$$

Here ε_{ij} is again the canonical homomorphism of $\mathrm{Spin}(2)$ in a matrix representation.

Theorem 5.2.4

For $n \geqslant 4$, the group $\mathrm{Spin}(n)$ is the universal completion of the amalgam $\mathcal{A}(A_{n-1}, \mathrm{Spin}(2))$.

Proof. The proof here is identical to the proof of Theorem 5.2.2 using the action of $\mathrm{Spin}(n)$ on the subspace V of $\mathrm{Cl}(\mathbb{R}^n)$. □

5.3 Spin Covers of Maximal Compact Subgroups of Kac-Moody Groups

In this section we introduce those Kac-Moody groups that are necessary to our work. We also meet the maximal compact subgroups of these groups and quote a result from [18] that will provide some motivation to the methods used in the main result of this chapter.

Definition 5.3.1

Let $\Delta = (\mathcal{V}, \mathcal{E})$ be a simply-laced diagram with vertex set and edge set $\mathcal{V}$ and $\mathcal{E}$, respectively. Suppose $|\mathcal{V}| = n$, and set $I = \{1, \dots, n\}$ so that $\tau : I \to \mathcal{V}$ is a labelling. An amalgam of type $\mathrm{SL}(2)$ with respect to Δ and τ is an amalgam $\mathcal{A} := \{G_{ij}\, \phi^i_{ij}, \phi^j_{ij} \,|\, i \neq j \in I\}$ such that for all $i \neq j$, we have

$$G_{ij} = \begin{cases} \mathrm{SL}(3), & (i^\tau, j^\tau) \in \mathcal{E} \\ \mathrm{SL}(2) \times \mathrm{SL}(2), & (i^\tau, j^\tau) \notin \mathcal{E} \end{cases}$$

furthermore, for all $i < j$,

$$\phi^i_{ij}(\mathrm{SL}(2)) = \varepsilon_{12}(\mathrm{SL}(2)), \ \textit{whilst} \ \ \phi^j_{ij}(\mathrm{SL}(2)) = \begin{cases} \varepsilon_{23}(\mathrm{SL}(2)), & (i^\tau, j^\tau) \in \mathcal{E} \\ \varepsilon_{34}(\mathrm{SL}(2)), & (i^\tau, j^\tau) \notin \mathcal{E}. \end{cases}$$

Definition 5.3.2 ([7], Theorem A and Application)

Let $\mathcal{A} := \mathcal{A}(\Delta, \tau, \mathrm{SL}(2))$ be a standard amalgam of type $\mathrm{SL}(2)$. Then the universal completion $G(\Delta)$ is called the Kac-Moody group of type Δ.

We should be stricter on the type of Kac-Moody group we obtain. However, since in this thesis we are only considering simply-laced diagrams and working exclusively over $\mathbb{R}$, it matters not. See [48] for a comprehensive definition over arbitrary rings. The next Theorem states a result about the maximal compact subgroup $K(\Delta)$ of the Kac-Moody group $G(\Delta)$. This group is the set of fixed points under the Cartain-Chevalley involution. They are generalisations of the subgroup $\mathrm{SO}(n)$ inside $\mathrm{SL}(n)$.

Theorem 5.3.3 ([18], Theorem 1)

Let Δ be a simply-laced diagram, $G(\Delta)$ the corresponding Kac-Moody group and $K(\Delta)$ its maximal compact subgroup. Then $K(\Delta)$ is the universal completion of the amalgam $\mathcal{A}(\Delta, \mathrm{SO}(2))$.

We make a remark about this theorem that will be needed in the proof of the group $Spin(\Delta)$ being a double-cover of the group $K(\Delta)$.

Remark 5.3.4

We see from the above definition that the group $K(\Delta)$ can be considered as a word in the rank 1 subgroups isomorphic to $\mathrm{SO}(2)$ under the rewriting rules coming from the groups $\mathrm{SO}(2) \times \mathrm{SO}(2)$ and $\mathrm{SO}(3)$.

Definition 5.3.3 and Theorem 5.2.4 suggest the following definition.

Definition 5.3.5

For a simply-laced diagram Δ, we define the generalised spin group, $\mathrm{Spin}(\Delta)$, to be the universal completion of the amalgam $\mathcal{A}(\Delta, \mathrm{Spin}(2))$.

It remains to show that the group $\mathrm{Spin}(\Delta)$ exists and is indeed a double cover of $K(\Delta)$. The proof of this result relies on the spin representation and the fact that $K(\Delta)$ is a completion of the amalgam $\mathcal{A}(\Delta, \mathrm{Spin}(2))$.

Proposition 5.3.6

The group $K(\Delta)$ is a completion of the amalgam of type

$$\mathcal{A}(\Delta, \mathrm{Spin}(2)) = \{\widehat{K}_{ij}, \widehat{\phi}^i_{ij}, \widehat{\phi}^j_{ij} \,|\, i \neq j \in I.\}.$$

Proof. Let $\Delta = (\mathcal{V}, \mathcal{E})$, with labelling $\sigma : I \to \mathcal{V}$. By Theorem 5.3.3, $K(\Delta)$ is the universal completion of the amalgam of type $\mathcal{A}(\Delta, \mathrm{SO}(2)) = \{K_{ij}, \phi^i_{ij}, \phi^j_{ij} \,|\, i \neq j \in I\}$, with appropriate morphisms $\psi_{ij} : K_{ij} \to K(\Delta)$. For each K_{ij} in the amalgam we have an associated double cover which is isomorphic to either Spin(3) or Spin(2) $\circ$ Spin(2). Denote by $\widehat{K}_{ij}$ the cover of K_{ij}, then we have morphisms

$$\phi_{ij} := \psi_{ij} \circ \rho : \widehat{K}_{ij} \to K_{ij}$$

for all $i \neq j$. And these morphisms show that $K(\Delta)$ is a completion of the amalgam $\mathcal{A}(\Delta, \mathrm{Spin}(2))$. This can be seen since $\rho \circ \widehat{\phi}^i_{ij} = \rho \circ \widehat{\varepsilon}_{12} = \varepsilon_{12} \circ \rho = \phi^i_{ij}$ and

$$\rho \circ \widehat{\phi}^j_{ij} = \left\{ \begin{array}{ccc} \rho \circ \widehat{\varepsilon}_{23} & = & \varepsilon_{23} \circ \rho \\ \rho \circ \widehat{\varepsilon}_{34} & = & \varepsilon_{34} \circ \rho \end{array} \right\} = \phi^j_{ij} \circ \rho,$$

and so

$$\phi_{ij} \circ \widehat{\phi}^j_{ij} = \psi_{ij} \circ \rho \circ \widehat{\phi}^j_{ij} = \psi_{ij} \circ \phi ij^j \circ \rho = \psi_{kj} \circ \phi^j_{kj} \circ \rho = \psi_{kj} \circ \rho \circ \widehat{\phi}^j_{kj} = \phi_{kj} \circ \widehat{\phi}^j_{kj},$$

for all $i \neq j \neq k \in I$. □

This result ensures we have a surjective homomorphism from $G(\Delta)$ onto $K(\Delta)$. The remainder of this chapter is dedicated to understanding this map and what it tells us about the group $G(\Delta)$.

With these results in place we now integrate the spin representation but focus on the local behaviour of the images of the Berman generators. Given a simply laced diagram Δ with GCM A let $\mathfrak{k}$ be the corresponding maximal compact subalgebra of the Kac-Moody algebra $\mathfrak{g}(A)$. Let ρ be a spin representation of $\mathfrak{k}$ and let X_i be the n Berman generators of $\mathfrak{k}$. Writing $X = \rho(X_i)$, for $X_1, \ldots, X_n$, satisfies the relation $X^2 = -\frac{1}{4}id$ and for $t \in \mathbb{R}$ we have

$$\exp(tX) = \sum_{m=0}^{\infty} \frac{(tX)^m}{m!}.$$

These together yield,

$$\begin{aligned}
&\exp(tX) = 1 + tX - \frac{t^2}{4 \cdot 2!} - \frac{t^3 X}{4 \cdot 3!} + \frac{t^4}{16 \cdot 4!} + \frac{t^5 X}{16 \cdot 5!} - \frac{t^6}{64 \cdot 6!} - \frac{t^7 X}{64 \cdot 7!} + \cdots \\
&= \left(1 - \frac{t^2}{4 \cdot 2!} + \frac{t^4}{16 \cdot 4!} - \frac{t^6}{64 \cdot 6!} - \cdots\right) + \left(t - \frac{t^3}{4 \cdot 3!} + \frac{t^5}{16 \cdot 5!} - \frac{t^7}{32 \cdot 7!} + \cdots\right) X \\
&= \left(1 - \frac{t^2}{4 \cdot 2!} + \frac{t^4}{16 \cdot 4!} - \frac{t^6}{64 \cdot 6!} - \cdots\right) + \left(\frac{t}{2} - \frac{t^3}{8 \cdot 3!} + \frac{t^5}{32 \cdot 5!} - \frac{t^7}{128 \cdot 7!} + \cdots\right) 2X \\
&= \cos(t/2) + 2X \sin(t/2).
\end{aligned}$$

Note that $(2X)^2 = -1$ and so $\langle \exp(tX) \,|\, t \in \mathbb{R} \rangle \cong \mathrm{Spin}(2)$. So we have shown that the image of every Berman generator of $\mathfrak{k}$ yields a copy of Spin(2). It is left to show what these Spin(2) subgroups generate and to find the relations between them.

We know that if two Berman generators, X_i and X_j say, are connected by an edge in the diagram Δ, then $\rho(X_i)$ and $\rho(X_j)$ anti-commute and they commute otherwise. With this is mind (and now writing X_i for $\rho(X_i)$ for convenience) we get that if X_i and X_j are adjacent, then

$$\exp(tX_i)\exp(sX_j) = (\cos(t/2) + 2\sin(t/2)X_i)(\cos(s/2) + 2\sin(s/2)X_j).$$

Set $\alpha = \cos(t/2), \beta = \sin(t/2), \gamma = \cos(s/2)$ and $\delta = \sin(s/2)$. This gives

$$\exp(tX_i)\exp(sX_j) = \alpha\gamma + 2\beta\gamma X_i + 2\alpha\delta X_j + 4\beta\delta X_i X_j.$$

Now

$$(4X_iX_j)^2 = 16X_iX_jX_iX_j = -16X_i^2X_j^2 = -\frac{16}{16} = -1.$$

So the product of $2X_i$ and $2X_j$ is another square root of -1 which can be seen to anticommute with both $2X_i$ and $2X_j$. This tells us that they behave like the Clifford monomials e_1e_2, e_1e_3 and e_2e_3 under the identification $2X_i \mapsto e_1e_2$, $2X_j \mapsto e_1e_3$ and $4X_iX_j \mapsto e_2e_3$. It is not hard to see that the sum of the squares of the coefficients is 1 and now it is clear by our work in Chapter 3 that for adjacent Berman generators X_i and X_j we have

$$\langle \exp(tX_i), \exp(sX_j) \mid s, t \in \mathbb{R} \rangle \cong \mathrm{Spin}(3).$$

If the elements are not adjacent, then we have that $X_iX_j = X_jX_i$ and so for $s, t \in \mathbb{R}$,

$$\exp(tX_i)\exp(sX_j) = (\cos(t/2) + 2X_i\sin(t/2))(\cos(s/2) + 2X_j\sin(s/2)).$$

Playing the same trick as before we get

$$\exp(tX_i)\exp(sX_j) = \alpha\gamma + 2\beta\gamma X_i + 2\alpha\delta X_j + 4\beta\delta X_i X_j.$$

However, this time we have $(4X_iX_j)^2 = 16X_i^2X_j^2 = 1$. So again we see by Chapter 3 that if we have two non-adjacent Berman generators, then

$$\langle \exp(tX_i), \exp(sX_j) \mid s, t \in \mathbb{R} \rangle \cong \mathrm{Spin}(2) \circ \mathrm{Spin}(2),$$

where, as before, $\mathrm{Spin}(2) \circ \mathrm{Spin}(2) \cong (\mathrm{Spin}(2) \times \mathrm{Spin}(2))/\{(\pm 1, \pm 1)\}$.

Theorem 5.3.7

Given a generalised spin representation (V, ϕ) of a maximal compact subalgebra $\mathfrak{k}$ of a simply-laced Kac-Moody algebra $\mathfrak{g}$, we get a representation (V, Φ) of the corresponding group Spin(Δ).

Proof. For each row n_i of the GCM A we have a Berman generator X_i of $\mathfrak{k}$ and a subgroup Spin(2) of Spin(Δ). As we have seen the element $\phi(X_i)$ integrates to a group $R_i = \langle \exp(\theta\phi(X_i)) \,|\, \forall \theta \in \mathbb{R} \rangle$ which is isomorphic to Spin(2). We label the isomorphisms Φ_i. Let $R = \langle R_i \,|\, \forall i \in I \rangle \leqslant GL(V, \phi)$.

Now, by the universality of Spin(Δ) we get an epimorphism $\Phi : \mathrm{Spin}(\Delta) \to R$, which is the desired representation of the generalised spin group. □

Definition 5.3.8

The representation (V, Φ) of Spin(Δ) *given above will also be called a generalised spin representation.*

For any two copies of the group Spin(2) in our amalgam, $\mathrm{Spin}_1(2)$ and $\mathrm{Spin}_2(2)$ say, the intersection $\mathrm{Spin}_1(2) \cap \mathrm{Spin}(2)$ is always $\{\pm 1\}$. The subgroup C of Spin(Δ) generated by the element -1 is of order 2 and we shall use it to show that Spin(Δ) is a double cover of the group $K(\Delta)$. Before attempting this, we need to show that our element -1 survives in the amalgam and we use the result of the previous theorem to do this.

Proposition 5.3.9

For a non-degenerate, simply-laced GCM A and Spin(Δ) *as above, the element* -1 *survives in the completion of the amalgam.*

Proof. Our aim here is to find a representation of Spin(Δ) such that the image of -1 is non-trivial. Let (V, Φ) be a generalised spin representation of the group Spin(Δ). We begin by considering the local isomorphisms Φ_i which comprise Φ. These maps $\Phi_i : \mathrm{Spin}(2) \to R_i$ are defined by

$$\Phi_i : \cos\theta + e_1 e_2 \sin\theta \mapsto \cos\theta + 2\phi(X_i)\sin\theta.$$

Care should be taken here since the element of R_i is more precisely realised as

$$\cos\theta \cdot Id_V + \sin\theta \cdot 2\phi(X_i),$$

where Id_V is the identity map on V. Now the element $-1 \in \mathrm{Spin}(2)$ occurs when $\theta = \pi$. And so for all i, we have

$$\Phi_i(-1) = -Id_V.$$

Hence for the element of -1 of $\mathrm{Spin}(\Delta)$, the image of $\Phi(-1)$ can be taken as any of the Φ_i since they all agree. This element is non-trivial and so -1 is an element distinct from the identity in the group $\mathrm{Spin}(\Delta)$. □

Theorem 5.3.10

The group $\mathrm{Spin}(\Delta)$ *is a double cover of* $K(\Delta)$.

Proof. Since the -1 survives in the amalgam, let $C = \langle -1 \rangle$. The element -1 is central and so C is a normal subgroup of order two. The element also exists locally and is the kernel of the map from each copy of $\mathrm{Spin}(2)$ onto $\mathrm{SO}(2)$ and similarly for $\mathrm{Spin}(3)$ and $\mathrm{Spin}(2) \circ \mathrm{Spin}(2)$.

We consider the factor group $\mathrm{Spin}(\Delta)/C$. Each element $x \in \mathrm{Spin}(\Delta)$ is of the form

$$x = x_1 * x_2 * \ldots * x_n,$$

with each x_i lying in a copy of $\mathrm{Spin}(2)$. So the image of x under the canonical map onto

$\mathrm{Spin}(\Delta)/C$ looks like

$$x_1C * x_2C * \ldots * x_nC.$$

As we have noted the factor group $\mathrm{Spin}(2)/C$ is isomorphic to SO(2) and similarly for Spin(3) and Spin(2)∘Spin(2). Hence the element $x_1C*x_2C*\ldots*x_nC$ corresponds uniquely to an element

$$y_1 * y_2 * \ldots * y_n$$

with the rewriting rules coming from SO(3) and SO(2) × SO(2). But, by Remark 5.3.4, these are precisely the relations for $K(\Delta)$ and so $\mathrm{Spin}(\Delta)/C$ is isomorphic to $K(\Delta)$ and since $|C| = 2$, it is a double cover. □

Chapter 6

Spin Covers of Weyl Groups

In this chapter is about the spin covers of Weyl groups associated to simply-laced diagrams. These extensions are subgroups of the spin groups associated to the diagrams. For the E_{10} case, the extension of the Weyl group was conjectured in Proposition 1 of [16]. We begin by recalling the definition of a Weyl group associated to a simply-laced diagram. Then a more general result than the conjecture is proved. We define a discrete subgroup $\mathcal{W}^{spin}$ of Spin(Δ), for any simply-laced diagram Δ, which is the conjectured extension of the Weyl group $W(\Delta)$. In fact, this extension will be by the Clifford monomial group associated to the diagram.

6.1 The Weyl Group of a Kac-Moody Algebra

Coxeter groups were first introduced by H. S. M. Coxeter in 1934 for which he gave a complete classification of the finite ones in [13]. Weyl groups are subgroups of the isometry groups of the the root system of a semisimple Lie algebra. In this section we introduce the necessary information for the sequel. More extensive treatments can be found in [4], [25] and [31].

As with Kac-Moody algebras, Coxeter groups are given by diagrams and matrices. However, in this case the matrices are symmetric and have non-negative integer entries.

Although we note that the diagram of a Kac-Moody algebra and the diagram of its Weyl group are essentially the same.

Definition 6.1.1

For a finite index set I, a Coxeter matrix $M = (m_{ij})_{i,j\in I}$ is a symmetric $n \times n$-matrix over $\mathbb{N} \cup \{\infty\}$ that satisfies

- $m_{ii} = 1$, *for all $i \in I$ and*
- $m_{ij} = m_{ji} \geqslant 2$, *for $i \neq j$ in I.*

The Coxeter group is then constructed in the following manner.

Definition 6.1.2

Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix and let S to be a set of cardinality $|I|$ consisting of elements s_i for $i \in I$. A Coxeter group is any group W admitting the presentation

$$W = \langle s_i \in S \,|\, (s_i s_j)^{m_{ij}} = 1,\ \forall i, j \in I\rangle.$$

Furthermore, the pair (W, S) is called a Coxeter system.

The question now is how do we associate a Coxeter matrix to a simply-laced GCM.

Definition 6.1.3

Let A be a simply-laced GCM, we define a Coxeter matrix $M_A = (m_{ij})_{i,j\in I}$ by setting

- $m_{ii} = 1$,
- $m_{ij} = 2$ *if* $a_{ij}a_{ji} = 0$ *or*
- $m_{ij} = 3$ *if* $a_{ij}a_{ji} = 1$.

Definition 6.1.4

The Weyl group of a Kac-Moody algebra associated to a GCM A is the Coxeter group obtained from the above procedure via the matrix M_A.

6.2 Spin Covers of Weyl Groups

For any simply-laced diagram Δ we find a subgroup $\mathcal{W}^{spin}$ of $\mathrm{Spin}(\Delta)$. This extends the subgroup predicted in [16] for the group $\mathrm{Spin}(E_{10})$, which is used to describe the behaviour of fermions at the billiard limit. The group turns out to be an extension of the Weyl group $W(\Delta)$ by the Clifford monomial group $C(\Delta)$.

In Proposition 1 of [16] a subgroup $\mathcal{W}^{spin}$ of $\mathrm{Spin}(E_{10})$ is conjectured on ten generators R_i satisfying the following properties:

- For $1 \leqslant i \leqslant 10, (R_i)^4 = -1$;
- for adjacent i, j, $(R_i R_j)^3 = -1$;
- for non-adjacent nodes the generators commute;
- the elements $(R_i)^2$ generate a non-abelian, normal subgroup D of order 2^{10+1} and
- We have

$$\mathcal{W}^{spin}/D \cong W(E_{10})$$

where $W(E_{10})$ is the Weyl group corresponding to the diagram of type E_{10}.

In fact we wish to show that such a subgroup $\mathcal{W}^{spin}$ exists for all simply-laced diagrams Δ, which satisfies generalised versions of the above properties.

For the remainder of this chapter let Δ be a simply-laced diagram of rank n with corresponding Kac-Moody algebra $\mathfrak{g}$. Let $\mathfrak{k}$ be the maximal compact subalgebra with Berman generators $X_1, \ldots, X_n$. Let G be the Kac-Moody group on Δ with maximal compact subgroup K and C the Clifford monomial group. Finally, let $\mathrm{Spin}(\Delta)$ be the generalised spin group that covers K.

Definition 6.2.1

Let $\mathcal{W}_{\Delta}^{spin}$ *be the subgroup of* $\mathrm{Spin}(\Delta)$ *generated by the* n *elements* R_i *corresponding to* $\frac{1}{\sqrt{2}}(1 - v_1v_2)$ *in each copy of the rank-1 subgroup of type* $\mathrm{Spin}(2)$ *in the amalgam.*

When it is clear from context or not necessary to note, we will omit the subscript and so write $\mathcal{W}^{spin}$ in place of $\mathcal{W}_{\Delta}^{spin}$. For the rest of this chapter, we prove the above results in the general case for the group $\mathcal{W}^{spin}$.

Proposition 6.2.2

The elements R_i *are of order eight.*

Proof. We have that

$$\left(\frac{1}{\sqrt{2}}(1 - v_1v_2)\right)^2 = \frac{1}{2}(1 - v_1v_2 - v_1v_2 - 1) = -v_1v_2.$$

The element v_1v_2 squares to -1. The result follows. □

Proposition 6.2.3

Suppose we have two adjacent generators R_i *and* R_j*, then* $(R_iR_j)^3 = -1$.

Proof. The copies of Spin(2) in which R_i and R_j lie are adjacent on the diagram and so lie in a copy of Spin(3). We may then suppose without loss of generality that these correspond to the elements $\frac{1}{\sqrt{2}}(1 - v_1v_2)$ and $\frac{1}{\sqrt{2}}(1 - v_2v_3)$, respectively. Then

$$R_iR_j = \frac{1}{2}(1 - v_1v_2 - v_2v_3 + v_1v_3).$$

Moreover

$$(R_iR_j)^2 = -\frac{1}{2}(1 + v_1v_2 + v_2v_3 - v_1v_3).$$

Then note that $(R_iR_j)^2 = -\overline{R_iR_j}$, that is the negative of its conjugate. Since for all $x \in \mathrm{Spin}(3)$, we have $N(x) = x\overline{x} = 1$, it follows that $(R_iR_j)^3 = -1$. □

Proposition 6.2.4

For two non-adjacent nodes the generators R_i and R_j commute.

Proof. By a similar argument to the proceeding proof but with $\mathrm{Spin}(2) \circ \mathrm{Spin}(2)$ in place of Spin(3), we may consider the elements $\frac{1}{\sqrt{2}}(1 - v_1 v_2)$ and $\frac{1}{\sqrt{2}}(1 - v_3 v_4)$ for R_i and R_j, respectively. Since the 1 is central and $v_1 v_2 v_3 v_4 = v_3 v_4 v_1 v_2$, these elements commute. □

Definition 6.2.5

For the group $\mathcal{W}^{spin}_{\Delta}$ with generators R_i for $1 \leqslant i \leqslant n$, define

$$D_{\Delta} = \langle R_i^2 \,|\, 1 \leqslant i \leqslant n \rangle.$$

We again will write D in place of D_{Δ} when it is not necessary to differentiate.

Proposition 6.2.6

D is a normal subgroup of $\mathcal{W}^{spin}$.

Proof. Consider a generator R_i^2 of D, a generator R_j of $\mathcal{W}^{spin}$, and the conjugate

$$R_j^{-1} R_i^2 R_j.$$

If R_i and R_j lie in non-adjacent copies of Spin(2), by Proposition 6.2.2, they commute so it is enough to consider the case when they are adjacent. In this case they lie in a copy of Spin(3) and as in Proposition 6.2.2 we can identify R_i^2 wth $-v_1 v_2$ and R_j with

$\frac{1}{\sqrt{2}}\left(1 - v_2v_3\right)$. Then we have

$$\begin{aligned}
R_j^{-1}R_i^2R_j &= -\frac{1}{2}(1 + v_2v_3)v_1v_2(1 - v_2v_3) \\
&= -\frac{1}{2}(v_1v_2 - v_1v_3)(1 - v_2v_3) \\
&= -\frac{1}{2}(v_1v_2 - v_1v_3 - v_1v_3 - v_1v_2) \\
&= v_1v_3.
\end{aligned}$$

Similarly, we have $R_j^2 = -v_2v_3$ and so we get $R_i^2R_j^2 = v_1v_2v_2v_3 = v_1v_3$. Hence we see that

$$R_j^{-1}R_i^2R_j = R_i^2R_j^2 \in D.$$

□

We now wish to compute the order of D. The proof of this will rely on a few steps which we outline here. First we find an upper bound for the order of D using the known relations and then by employing the generalised spin representation, we show that this upper bound is in fact attained. Recall by Proposition 6.2.4 two non-adjacent generators commute.

Lemma 6.2.7

For two adjacent generators R_i^2 and R_j^2 of D, we have $R_i^2R_j^2 = -R_j^2R_i^2$.

Proof. First note that by Proposition 6.2.2, we have that $(R_i^2)^{-1} = R_i^6 = -R_i^2$ and by the proof of the previous proposition, we get $R_j^{-1}R_i^2R_j = R_i^2R_j^2$. Now we see

$$\begin{aligned}[R_i^2, R_j^2] &= R_i^2R_j^2R_i^6R_j^6 \\ &= R_i^2R_j^2R_i^2R_j^2 \\ &= R_j^{-1}R_i^2R_jR_j^{-1}R_i^2R_j \\ &= R_j^{-1}R_i^4R_j \\ &= -1. \end{aligned}$$ □

Lemma 6.2.8

$|D| \leqslant 2^{n+1}$.

Proof. For brevity, let each generator of D be written as $r_i = R_i^2$. Then for every word x in the r_i, we have $x = \pm r_{i_1}r_{i_2}\dots r_{i_k}$, for $i_t \in \{1, \dots, n\}$. By Proposition 6.2.4 and Lemma 6.2.7, two generators r_i and r_j either commute or anticommute. Moreover, we have that $r_i^2 = -1$ and $r_i^3 = r_i^{-1} = -r_i$. We can rewrite the x in the form

$$x = (-1)^{\varepsilon_0}r_1^{\varepsilon_1}r_2^{\varepsilon_2}\dots r_n^{\varepsilon_n},$$

with $\varepsilon_i \in \{0, 1\}$ for all $0 \leqslant i \leqslant n$. This gives at most 2^{n+1} elements as required. □

Proposition 6.2.9

$|D| = 2^{n+1}$.

Proof. This follows from Lemma 2.2(a) and Corollary 2.3 of [37]. □

We now quote Von Dyck's Theorem which will be needed in the proofs of the following two results.

Theorem 6.2.10 (von Dyck's Theorem, [46], Theorem 4.84)
If G and H are groups with presentations

$$G = \langle S \mid R \rangle \text{ and } H = \langle S \mid R \cup R' \rangle,$$

with $R' \nsubseteq R$, then H is a quotient of G.

Corollary 6.2.11
The subgroup D is isomorphic to the Clifford Monomial Group C of the diagram Δ.

Proof. By von Dyck's Theorem, we get a surjection from C onto D. However, both groups have order 2^{n+1} and are therefore isomorphic. □

We now prove the last result; that the Weyl group associated to Δ is in fact a quotient of the group $\mathcal{W}^{spin}$. Thanks are due to Max Horn for developing the second half of the proof.

Proposition 6.2.12
Let W be the Weyl group associated to Δ then we have

$$\mathcal{W}^{spin}_G / D_G \cong W.$$

Proof. Since every generator of s_i of W satisfies the relations R_i of $\mathcal{W}^{spin}$, by Von Dyck's Theorem, we get a surjective homomorphism

$$f : \mathcal{W}^{spin} \to W$$

$$R_i \mapsto s_i.$$

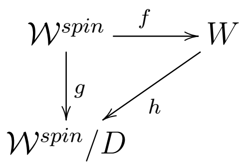
Then note that $f(R_i^2) = s_i^2 = 1_W$ and since D is generated be the squares of the generators, we have $D \subseteq \ker f$. To see the reverse inclusion we need to define another homomorphism

$$h : W \to \mathcal{W}^{spin}/D$$

given by $h(s_i) = R_i D$. Since $h(s_i)h(s_j) = R_i D R_j D$. Then since D is normal, we have $R_i D R_j D = R_i R_j R_j^{-1} D R_j D = R_i R_j D = h(s_i s_j)$ and this map is indeed a homomorphism. Moreover for all i, we have

$$(h \circ f)(R_i) = h(s_i) = R_i D = g(R_i)$$

and so the following diagram commutes



and we have

$$D = \ker g = \ker(h \circ f) \supseteq \ker f.$$

Hence, $D = \ker f$ and the claim follows. □

The result of Proposition 6.2.12 suggests the following definition.

Definition 6.2.13

The group $\mathcal{W}^{spin}$ is called the **spin cover** *of the Weyl group W.*

Hence we have shown that the group conjectured in [16] does indeed exist and has all the relevant properties. And we have gone further as suggested in Section 3.5 of [16] where one can also find a spinor representation of $\mathcal{W}^{spin}_{E_{10}}$.

It has been suggested by Dr. Hoffman and Dr. Levy that it may be of interest to study the action of the group W on the space $D/Z(D)$, since this may provide some insight into the representation theory of the so-called Braid groups. Since this is beyond the scope of the knowledge of the author, it is left as a hope that they will in some way prove useful.

Chapter 7

Further Work

In this final chapter we discuss possible future work that could be carried out.

The structure constants for the generalised spin representations have been calculated in the non simply-laced case in [26]. Similarly, the group $K(\Delta)$ is known to exist in general for certain fields, see [18]. It would therefore be interesting if one could generalise the construction in Definition 5.3.5 to cover the non simply-laced case. The proofs of the existence and double cover contained herein rely on integrating the generalised spin representation of a maximal compact subalgebra $\mathfrak{k}$, however, I believe that the theory of buildings [1] would be more suited to the non simply-laced case.

The spin groups $\mathrm{Spin}(n)$ are known to be simply connected for $n \geqslant 3$, see Corollary 3.15 in [2]. Split real Kac-Moody groups enjoy a topology known as the Kac-Peterson topology, which was first introduced in [36] and studied further in [27]. This topology is also on the subgroup $K(\Delta)$ and when restricted to the rank 1 and rank 2 subgroups gives the natural Lie topology. It could prove of interest to see if the groups $\mathrm{Spin}(\Delta)$ are in fact simply connected under the Kac-Peterson topology.

The even part of a Weyl group is the subgroup generated by the rotations in W. The two papers [39] and [19] give a reprsentation of the even part of certain Weyl groups over the normed division algebras. Specifically, there is a representation of $W^+(E_{10})$, the even part of the E_{10} Weyl group over the 'integers' in the octonions $\mathbb{O}$. It could be asked to what are the full group $W(E_{10})$ and its extension $\mathcal{W}^{spin}$ isomorphic to.

Lastly, the classification in Chapter 4 is incomplete and a full analysis of what the image of a generalised spin representation is would be of interest if one wanted to integrate the representation. It is the author's belief that the algebras given in Theorem 4.3.8 are indeed the images of the generalised spin representations and some GAP calculations have confirmed this for some small cases.

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