INTEGRALITY IN MAX-LINEAR SYSTEMS

by

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Abstract

This thesis deals with the existence and description of integer solutions to max-linear systems. It begins with the one-sided systems and the subeigenproblem. The description of all integer solutions to each of these systems can be achieved in strongly polynomial time.

The main max-linear systems that we consider include the eigenproblem, and the problem of determining whether a matrix has an integer vector in its column space. Also two-sided systems, as well as max-linear programming problems. For each of these problems we construct algorithms which either find an integer solution, or determine that none exist. If the input is finite, then the algorithms are proven to run in pseudopolynomial time. Additionally, we introduce special classes of input matrices for each of these problems for which we can determine existence of an integer solution in strongly polynomial time, as well as a complete description of all integer solutions.

Moreover we perform a detailed investigation into the complexity of the problem of finding an integer vector in the column space. We describe a number of equivalent problems, each of which has a polynomially solvable subcase. Further we prove NP-hardness of related problems obtained by introducing extra conditions on the solution set.

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List of symbols

$\mathbb{R}$	The set of reals	p.1
$\overline{\mathbb{R}}$	Extended reals: $\mathbb{R} \cup \{-\infty\}$	p.1
$\mathbb{Z}$	The set of integers	p.1
$\overline{\mathbb{Z}}$	Extended integers: $\overline{\mathbb{Z}} \cup \{-\infty\}$	p.71
$\overline{\overline{\mathbb{R}}}$	$\overline{\mathbb{R}} \cup \{+\infty\}$	p.13
ε	$-\infty$, also any vector/matrix with all entries $-\infty$	p.1
$\oplus$	Max-algebraic addition: $a \oplus b = \max(a, b)$	p.1
$\otimes$	Max-algebraic multiplication: $a \otimes b = a + b$	p.1
$\oplus'$	Dual addition: $a \oplus' b = \min(a, b)$	p.13
$\otimes'$	Dual multiplication: $a \otimes b = a + b$	p.13
$+, -, \times$	Conventional operations in linear algebra	p.11
$\frac{a}{b}$	Conventional division/a fraction	p.11
M	{1,...,m}	p.11
N	{1,...,n}	p.11
K	{1,...,k}	p.11
e_j	The jth unit vector	p.11
$\mathbf{0}$	The zero vector of appropriate size	p.11
α^{-1}	$-\alpha$	p.11
$fr(a)$	The fractional part: $a - \lfloor a \rfloor$	p.18
$\lfloor a \rfloor$	The lower integer part	p.18
$\lceil a \rceil$	The upper integer part	p.18
$\lfloor A \rfloor$	Matrix with entries $\lfloor a_{ij} \rfloor$	p.18
$\lceil A \rceil$	Matrix with entries $\lceil a_{ij} \rceil$	p.18
$\lambda(A)$	The maximum cycle mean	p.11
A_λ	$\lambda(A)^{-1}A$	p.12

A^+	$A \oplus A^2 \oplus ... \oplus A^n$	p.12
A^*	$I \oplus A^+$	p.12
$\tilde{A}$	Matrix of columns common to both A^+ and A^*	p.12
$A^{(-1)}$	$-A$	p.13
$A^{\#}$	$-A^T$	p.13
A_j	jth column of A	p.14
$(A\|c)$	Matrix A with added column c	p.14
$A[S,T]$	Submatrix of A with rows from S and columns from T	p.12
$A[T]$	$A[T,T]$	p.12
A^{int}	Integer counterpart	p.75
A^{ct}	Column typical counterpart	p.79
D_A	Digraph obtained from A	p.12
$N_C(A)$	Set of critical nodes	p.13
C_A	Critical digraph	p.13
$diag(d_1, ..., d_n)$	Diagonal matrix with diagonal entries $d_1, .., d_n$	p.12
$diag(d)$	Diagonal matrix with diagonal entries $d_1, .., d_n$ where $d = (d_j)$	p.12
I	Identity matrix	p.12
P_n	Set of permutations on N	p.14
$w(\pi, A)$	Weight of a permutation: $\bigotimes_{i \in N} a_{i\pi(i)}$	p.14
$maper(A)$	Max-algebraic permanent: $\bigoplus_{\pi \in P_n} w(\pi, A)$	p.14
$ap(A)$	Set of permutations achieving maximum weight	p.14
$M_j(A,b)$	$\{t \in M : a_{tj} b_t^{-1} = \bigoplus_i a_{ij} b_i^{-1}\}$	p.15
$\bar{x}$	$A^{\#} \otimes' b$	p.15
$\hat{x}$	$\lfloor A^{\#} \otimes' b \rfloor$	p.22
$V^*(A, \lambda)$	The set of finite subeigenvectors with respect to subeigenvalue λ	p.16
$V(A, \lambda)$	The set of finite eigenvectors with respect to eigenvalue λ	p.16
$IV^*(A, \lambda)$	The set of integer subeigenvectors with respect to subeigenvalue λ	p.23
$IV(A, \lambda)$	The set of integer eigenvectors with respect to eigenvalue λ	p.25

$IV(A)$	The set of all integer eigenvectors: $IV(A) = IV(A, \lambda(A))$	p.25
$Im(A)$	The image/column space of A	p.39
$IIm(A)$	The integer image of A	p.39
$IIm^*(A)$	The integer image with integer coefficients	p.71
$X(A)$	$\{x : Ax \in IIm(A)\}$	p.71
$K(A)$	$\lceil \max\{\|a_{ij}\|\}$	p.103
$K(A\|B)$	$K(Y)$ where Y is the matrix $(A\|B)$	p.103
$\leq_p$	Reduction in polynomial time (for decision problems)	p.73
$=_p$	$P1 \leq_p P2$ and $P2 \leq_p P1$	p.73
S	Set of feasible solutions to MLP	p.121
IS	Set of feasible solutions to IMLP	p.121
$IS^{\min}$	Set of feasible solutions to $\text{IMLP}^{\min}$	p.121
$IS^{\max}$	Set of feasible solutions to $\text{IMLP}^{\max}$	p.121
$f(x)$	$f^T \otimes x$	p.120
$f^{\min}$	Optimal objective function value for $\text{IMLP}^{\min}$	p.122
$f^{\max}$	Optimal objective function value for $\text{IMLP}^{\max}$	p.122

Glossary of terms

Finite	Vector/matrix with entries from $\mathbb{R}$	p.11
Active position/entry	Position where the maximum is attained for some max-algebraic equation	p.17
Row $\mathbb{R}$-astic	No ε rows	p.11
Column $\mathbb{R}$-astic	No ε columns	p.11
Doubly $\mathbb{R}$-astic	Both row and column $\mathbb{R}$-astic	p.11
Definite	Matrix with $\lambda(A) = 0$	p.12
Strongly definite	Definite matrix with diagonal entries equal to zero	p.12
Generalised permutation matrix	Matrix obtained by permuting rows/columns of a diagonal matrix	p.12
Irreducible	Matrix for which D_A is strongly connected	p.12
Increasing	Matrix with $a_{ii} \geq 0$	p.16
Column typical	Matrix property defined by each column having entries which do not share a fractional part	p.46
NNI	Matrix property on strongly definite matrices defined by no off diagonal integer entry	p.48
Upper triangular	Matrix with $a_{ij} = \varepsilon$ for $i > j$	p.51
Lower triangular	Matrix with $a_{ij} = \varepsilon$ for $i < j$	p.51
Subspace	A set S containing all max-combinations of its elements (vectors)	p.14
Max-convex	A set S containing the max-algebraic line segment between any two points in it	p.61
Property OneIR	Matrix property defined by existence of exactly one integer entry per row	p.28
Weak Property OneIR	Matrix property defined by existence of at most one integer entry per row	p.29

Property OneFP	Property on a pair of matrices defined by existence of exactly one pair of entries in each row with the same fractional part	p.106
Property ZeroFP	Property on a pair of matrices defined by non-existence of a pair of entries in any row with the same fractional part	p.106
Property One^{+}FP	Property on a pair of matrices defined by existence of at least one pair of entries in each row with the same fractional part	p.106
IIM	Decision problem: Does there exist x such that $Ax \in IIm(A)$?	p.71
IIM-CT	Decision problem: If A is column typical does there exist x such that $Ax \in IIm(A)$?	p.72
IIM-CT-P1	Decision problem: If A is column typical does there exist x such that $Ax \in IIm(A)$ with exactly one active entry per row?	p.72
IIM-P1	Decision problem: Does there exist x such that $Ax \in IIm(A)$ with exactly one active entry per row?	p.72
IIM*	Decision problem: Does there exist integer vector x such that $Ax \in IIm(A)$?	p.72
IIM*-CT	Decision problem: If A is column typical does there exist integer vector x such that $Ax \in IIm(A)$?	p.97
TSS	Two-sided system	p.4
OMLP	One-sided max-linear program	p.117
OMLP$^{\min}$	OMLP minimising objective function	p.117
OMLP$^{\max}$	OMLP maximising objective function	p.117
OIMLP	One-sided integer max-linear program	p.117
OIMLP$^{\min}$	OIMLP minimising objective function	p.117
OIMLP$^{\max}$	OIMLP maximising objective function	p.117
MLP	Max-linear program	p.5
IMLP	Integer max-linear program (two-sided constraints)	p.6
IMLP$^{\min}$	IMLP minimising objective function	p.120
IMLP$^{\max}$	IMLP maximising objective function	p.120

1. Introduction

1.1 The max-algebra

For $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear algebra, that is (assuming compatibility of sizes)

$$
\begin{aligned}
(A \oplus B)_{ij} &= a_{ij} \oplus b_{ij}, \\
(A \otimes B)_{ij} &= \bigoplus_k a_{ik} \otimes b_{kj} \text{ and} \\
(\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij}.
\end{aligned}
$$

We will use ε to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$.

A key advantage of working in max-algebra is that it allows us to write problems which are non linear in conventional linear algebra, as linear problems in the max-algebraic semiring. Background on the history of max-algebra can be found in Section 1.3.

1.2 Thesis overview

This thesis deal with the task of finding integer solutions to max-linear systems, these are systems of equations in the max algebra.

The first set of max-linear systems we consider are one-sided systems, these include the one-sided inequality, $A \otimes x \leq b$, and the one-sided equality, $A \otimes x = b$. Both are defined for $A \in \overline{\mathbb{R}}^{m \times n}$ and $b \in \overline{\mathbb{R}}^{m}$. We also study the max-algebraic eigenproblem and subeigenproblem, these are $A \otimes x = \lambda \otimes x$ and $A \otimes x \leq \lambda \otimes x$ respectively where $A \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \mathbb{R}$. As usual, a vector $x \neq \varepsilon$ satisfying $A \otimes x = \lambda \otimes x$ $[A \otimes x \leq \lambda \otimes x]$ will be called an *eigenvector [subeigenvector]* of A with respect to *eigenvalue [subeigenvalue]* λ.

The problems of finding solutions to each of these four systems are well known [7, 18, 37, 51] and can be solved in low-order strongly polynomial time. However, the question of finding integer solutions to these problems has, to our knowledge, not been studied yet.

In applications, solutions to one-sided systems typically represent starting times of processes that have to meet specified delivery times. Eigenvectors guarantee a stable run of certain systems, for instance a multiprocessor interactive system [37]. Since the time restrictions are usually expressed in discrete terms (for instance minutes, hours or days), it may be necessary to find integer rather than real solutions to these systems. It should be noted that, when all input coefficients are integer, the existing methods for finding general solutions to these systems will find integer solutions. In this thesis we consider finding integer solutions to systems with input coefficients from $\mathbb{R}$ or $\overline{\mathbb{R}}$.

In Section 1.4 we summarise the existing theory necessary for the presentation of our results. In Chapter 2 we show that the description of all integer solutions to one-sided systems follows almost immediately from existing theory, and therefore integer solutions to these systems can be found efficiently. Further, for the question of existence of integer subeigenvectors, we perform a simple transformation of the input matrix that allows us to use known results about general subeigenvectors to give an efficient description of all solutions to the integer subeigenproblem. This is then used to determine a class of matrices for which the integer eigenproblem can be solved efficiently. In general, however, it appears that the integer eigenproblem is not easily solvable by using known results about real eigenvectors. We present a number of additional cases when the integer eigenproblem can be solved in strongly polynomial time, including a generic case we call Property OneIR.

In fact we can give a full description of all solutions in these special cases; Theorem 2.17 outlines the result for column typical matrices. We also show that the set of integer eigenvectors of an $n \times n$ matrix A is equivalent to the set of integer points in the column space of some $n \times k$ ($k \leq n$) matrix easily obtained from A. Further we prove that we only need to consider this equivalent problem for a matrix of dimension, $(n-k) \times k$.

As an extension of the one-sided systems, we define the integer image problem to be the problem of determining whether there exists an integer point in the column span of a matrix. One application of the integer image problem is as follows [31]. Suppose machines $M_1, ..., M_n$ produce components for products $P_1, ..., P_m$. Let x_j denote the starting time of M_j and a_{ij} be the time taken for M_j to complete its component for P_i. Then all components for product P_i are ready at completion time

$$c_i = \max(a_{i1} + x_1, ..., a_{in} + x_n) \;\; i = 1, ..., m.$$

Equivalently this can be written as $Ax = c$. In this context, the integer image problem asks whether there exists a set of starting times for which the completion times are integer (this can easily be extended to ask for any discrete set of values).

In Chapter 3 we propose a solution method for finding integer points in the column space of a matrix, Algorithm 3.1 (INT-IMAGE), which we prove to have pseudopolynomial run time for finite input in Theorem 3.11. It will follow that, for any matrix, integer solutions to $Ax = \lambda x$ can be found in a finite number of steps. Moreover, if A is irreducible, the integer eigenproblem can be solved in pseudopolynomial time. We also present special types of matrices, which includes a new class of matrix we call column typical, for which the integer image set can be fully described in strongly polynomial time. For column typical matrices the description is given in Theorem 3.17. We finish the chapter by looking for equivalent problems to the integer image problem, which include finding integer points in convex sets. A number of the results in Chapters 2 and 3 have been accepted for publication and can be found in [24].

Since none of the methods in Chapter 3 suggest an obvious way of solving the integer image problem in polynomial time, we examine the complexity of the problem in more detail in Chapter 4. Theorem 4.11 proves that we can assume without loss of generality that the matrix is column typical by performing a transformation to a matrix we call the column typical counterpart. This allows us to consider the existence of an integer image with at most one active entry per column, a problem that we prove to be NP-hard for general matrices in Theorem 4.19, but this does not resolve the complexity of the original problem.

We then move on to considering *two-sided max-linear system* (TSS). A TSS is of the form,

$$A \otimes x \oplus c = B \otimes x \oplus d$$

where $A, B \in \overline{\mathbb{R}}^{m \times n}$ and $c, d \in \overline{\mathbb{R}}^{m}$. If $c = d = \varepsilon$, then we say the system is *homogeneous*, otherwise it is called *nonhomogeneous*. Nonhomogeneous systems can be transformed to homogeneous systems [18]. If $B \in \overline{\mathbb{R}}^{m \times k}$, a system of the form

$$A \otimes x = B \otimes y$$

is called a *system with separated variables*.

The problems of finding solutions to $A \otimes x = B \otimes y$ and $A \otimes x = B \otimes x$ have been previously studied; one solution approach is the Alternating Method [18, 38]. If A and B are integer matrices, then the solution found by the Alternating Method is integer, however this cannot be guaranteed if A and B are real.

In Section 5.1 we show that we can adapt the Alternating Method in order to obtain algorithms which determine whether integer solutions to these problems exist for real matrices A and B, and find one if it exists. Note that various other methods for solving TSS are known [6, 22, 63], but none of them has been proved polynomial and there is no obvious way of adapting them to integrality constraints. In Section 5.2 we show that, for a certain class of matrices, which represents a generic case, the problem of finding an integer solution to both systems can be solved in strongly polynomial time. These are matrices satisfying a property we call Property OneFP and Theorem 5.14 gives a complete description of all integer solutions.

If $f \in \overline{\mathbb{R}}^n$ then the function $f(x) = f^T \otimes x$ is called *max-linear*. *Max-linear programming problems* seek to minimise or maximise a max-linear function subject to constraints given by one or two sided systems. Note that, unlike in linear programming, there is no obvious way of converting maximisation of max-linear functions to minimisation of the same type of functions and vice versa. We investigate integer solutions to max-linear programs in Chapter 6.

In Section 6.1 we briefly show that integer solutions to one-sided max-linear programs (these are max linear programs for which the constraint is a one-sided equality) can easily be found by adapting known methods which find real solutions. This shows that integer one-sided max-linear programs are strongly polynomially solvable.

We are more interested in max-linear programs which have constraints in the form of a TSS. For $A, B \in \mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m, f \in \mathbb{R}^n$ the *max-linear program* (MLP) is given by

$$
\begin{aligned}
&f^T \otimes x \to \text{ min or max} \\
&\text{s.t. } A \otimes x \oplus c = B \otimes x \oplus d \\
&x \in \overline{\mathbb{R}}^n.
\end{aligned}
$$

The max-linear programming problem has been used to describe the task of optimising multiprocessor interactive systems [19]. Here the variables x_j correspond to starting times of these systems. If the starting times are restricted to discrete values then the MLP is transformed to an *integer max-linear program* (IMLP).

Solution methods to solve the MLP are known, for example in [18, 19] a bisection method is applied to obtain an algorithm that finds an approximate solution to the MLP when the input matrices are real. Again, an integer solution is found for any instances of the MLP with integer entries, but the problem with integrality constraints is very different if the entries are real. In Section 6.2 we develop two algorithms, 6.19 (INT-MAXLINMIN) and 6.22 (INT-MAXLINMAX), based on the bisection method which will find an optimal solution to the IMLP, or determine that none exist. The algorithms are proven to run in pseudopolynomial time for finite input matrices in Corollaries 6.21 and 6.24. In Section 6.3 we develop a new method for input matrices satisfying Property OneFP. Theorems 6.33 and 6.34 and their corollaries show that the optimal objective value, and a number of optimal integer solutions to the integer max-linear program can be found in strongly polynomial time in this case. The material in Sections 6.2 and 6.3 has been published in [25] and [26].

1.3 Literature review

Papers regarding max-algebra (or tropical algebra) first appeared as early as the 1950's. For 20 years authors in many different areas were independently discovering that max algebra (and other idempotent algebras) were useful in areas such as operations research, scheduling (see for example [35]) and graph theory (see for example [32]) to name a few. The publication [37] is considered by many as the first major work on max-algebra, and it was the first work to develop a "unified account" of max-algebra. Since then, a huge number of mathematicians have contributed to the field, in many different areas and on many different problems, we note here the books [37, 66, 7, 51, 18] which represent only a small sample of the existing literature.

Part of the interest in max-algebra is due to the ability to model real world examples as linear systems. One influence was the study of *discrete event systems* [33] which, in conventional algebra, were nonlinear. A number of examples of modelling problems using max-algebra are given in [7] and include; production/manufacturing; queuing systems; parallel computation; traffic and others. Job-shop scheduling [37] and cellular protein production [12] have also been modelled using max-algebra. More recent applications include a description of how to model the entire Dutch railway system using max-algebra [51]. Further, tropical geometry is used in Klemperer's 2008 Product-Mix Auction [53], used by the Bank of England in the financial crisis.

One of the first problems considered was the question of existence of a solution to the system $A \otimes x = b$ in [35]. A combinatorial approach to describing the set of all solutions to $A \otimes x = b$ can be found in [16]. It is proved in [15] that for every matrix there exist b for which the equation $A \otimes x = b$ has no solutions, and b for which there are infinite solutions. The only other possibility is that the equation has a unique solution; matrices having a unique solution to $A \otimes x = b$ for some b are called strongly regular. These matrices were studied in [14], where the author gives necessary and sufficient conditions for strong regularity. Given a strongly regular matrix [15] describes the set of all b for which a unique solution exists.

Another key question is that of finding max-algebraic eigenvalues and eigenvectors. A full description of the eigenvectors of an irreducible matrix appears in [37], see also [50, 62]. The results for reducible matrices can be found, for example, in [8]. It was proved in [36] that the maximum cycle mean is the only possible eigenvalue for finite matrices. It was later proved that, for a general matrix, the maximum cycle mean is the largest possible eigenvalue (if one exists) [37].

An $\mathcal{O}(n^3)$ (actually $\mathcal{O}(n|E|)$) algorithm, called Karp's algorithm, for computing the maximum cycle mean was designed and proved in [52]. In [41], the authors address Karp's algorithm, noting that it considers many unnecessary nodes and edges when used. Two algorithms based on Karp's original method are proposed, which address this shortfall. Other methods for finding the maximum cycle mean include the power method for irreducible matrices, [43]. Faster results exist for certain special classes of matrix, for example there are $\mathcal{O}(n^2)$ methods for Monge matrices and bivalent matrices [49, 20] and a $\mathcal{O}(n)$ method for circulant matrices [56].

The eigenproblem has also been studied for infinite matrices, see for example, [3].

As an extension of one-sided systems and the eigenproblem, systems of the form $A \otimes x \oplus b = \lambda \otimes x$ were studied in [29] and, independently, in [54]. The authors describe the set of all solutions. Note that in [7] and others the existence conditions were established: that is the description of the least solution (if one exists).

A celebrated result regarding matrix powers in max-algebra is the Cyclicity theorem [33], which proves that every irreducible matrix is ultimately periodic, and that the period is equal to the cyclicity. For finite matrices this result appears in [37]. Other results regarding matrix powers include the study of robust matrices, these are matrices A for which $A^k x$ is an eigenvector of A for all x and some k. The authors of [21] fully characterise irreducible robust matrices as matrices with period equal to 1, which can be checked in $\mathcal{O}(n^3)$ time by results in [48].

The study of two-sided systems began at least as early as the 1980's, and remains an active area of research today. An elimination method for solving these systems (and finding all solutions) appears in [22]. Later, the Alternating method was developed [38]. The Alternating method either finds a solution to a two-sided system or determines that none exist. The algorithm runs in pseudopolynomial time for integer input matrices. The method was generalised in [58] to find a solution to the system $A^{(1)} \otimes x^1 = A^{(2)} \otimes x^2 = ... = A^{(k)} \otimes x^k$ and the pseudopolynomial bound for integer input matrices was extended to cover this more general case. The author also proved that the Alternating method has finite runtime for general input matrices.

A method for finding a solution to a two-sided system based on finding upper bound constraints to subproblems is given in [63] and a pseudopolynomial method based on calculating the Chebyshev distance between Ax and Bx can be found in [47]. Other methods for solving TSS are also known, see for example [6, 22, 63]. In [9] the max-atom problem is studied. This problem is polynomially equivalent to the problem of solving a two-sided system in max-algebra. It is proved that, for integer input, the max-atom problem is pseudopolynomial, but no consideration is made to integer solutions for real input.

Given a tropical polyhedron, represented externally by a system of inequalities in max algebra (a two-sided system), the authors in [6] presented a method to compute a description in terms of extreme points. This was first studied in [22]. In [1], it is proved that determining whether a tropical polyhedron is non empty (that is finding a solution to a system of the form $A \otimes x \oplus c \leq B \otimes x \oplus d$) is equivalent to solving a mean payoff game. This is a well known problem in NP $\cap$ co-NP, so a polynomial algorithm is expected to exist but, to date, none has been found.

Convexity in max-algebra is also an active area of research. Many results from classical convexity have been established; for example analogues of the classical separation theorems appear in [34] and in fact earlier in [65]; the existence of a tropical convex hull of a tropical polytope in [42]. A combinatorial view was introduced in [42], which links also to the area of tropical geometry.

The generalised eigenproblem looks for solutions to $A \otimes x = \lambda \otimes B \otimes x$. Since, for fixed λ, this reduces to a TSS, the area of interest is to describe all generalised eigenvalues. The problem was first studied in [11], where it is proved that, for symmetric matrices, there is at most one eigenvalue. Special solvable cases are presented in [40] as well as upper and lower bounds on the eigenvalues. In [47] the authors present a pseudopolynomial method to describe the set of all eigenvalues.

Linked to the study of two-sided systems is the study of max-linear programming problems. As mentioned previously a bisection method was developed in [18, 19]. Also, a Newton type algorithm has been designed [46] to solve a more general, max-linear fractional programming problem by a reduction to a sequence of mean payoff games. In [4] the authors study tropical linear programming, with constraints of the form $A \otimes x \geq B \otimes x$, and adapt the simplex algorithm to work in the tropical setting.

There are numerous problems for which max/tropical algebra has been used to shed light on the solutions in conventional linear algebra. This link between linear algebra and max-algebra was first observed in [44]. The authors in [29] use max-algebra to describe the set of all solutions in nonnegative linear algebra to $Ax + b = x$. In [30] it is proved that the sequence of the eigencones of successive powers of A is periodic in both max algebra and conventional, nonnegative algebra. In [28] it is demonstrated that max-algebra (specifically max-times algebra, which is isomorphic to the max-plus algebra) can be used to provide a complete description of all solutions to $X^{-1}AX \leq \mu E$ in conventional algebra; previously a full description did not exist.

In [2], eigenvalues are defined as the tropical roots of the characteristic polynomial of A. The authors prove that the absolute value of the normal eigenvalues of a complex matrix can be bounded by the tropical eigenvalues of A. This was motivated by a tropical interpretation of work by Hadamard and Ostrowski which bounds the absolute value of the product of the complex roots of a complex polynomial by functions of the tropical roots of an associated tropical polynomial. Also, [55] contains proofs that the tropical roots of a tropical polynomial can provide a good approximation to the conventional eigenvalues of a matrix polynomial. The advantage of using tropical algebra here is that the max-algebraic roots can be calculated in linear time, and can then be used as starting points for algorithms which search for conventional roots/eigenvalues.

1.4 Preliminary results

As in conventional algebra, it is common to omit the $\otimes$ symbol from calculations. We note here that, except for complexity arguments, all multiplications where the symbol has been omitted are in max-algebra. In some cases we convert back to using symbols max and + instead of $\oplus$ and $\otimes$ for ease of understanding the calculations, but it should be understood that we are still working in the max-algebra. When the symbol $\times$ is used it is understood to be conventional multiplication in linear algebra and, similarly, when we write $\frac{a}{b}$ we are referring to conventional division or a usual fraction.

We will use the following standard notation. For positive integers m, n, k we denote $M = \{1, ..., m\}$, $N = \{1, ..., n\}$ and $K = \{1, ..., k\}$. A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite*. A vector whose j^{th} component is zero and every other component is ε will be called a *unit vector* and denoted e_j. The zero vector, of appropriate size, is denoted $\mathbf{0}$. If a matrix has no ε rows (columns) then it is called *row* (*column*) $\mathbb{R}$-*astic* and it is called *doubly* $\mathbb{R}$-*astic* if it is both row and column $\mathbb{R}$-astic. Note that the vector Ax is sometimes called a *max combination* of the columns of A. For $\alpha \in \mathbb{R}$, α^{-1} is simply $-\alpha$ in conventional notation.

If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$, then $\lambda(A)$ denotes the *maximum cycle mean*, that is,

$$\lambda(A) = \max \left\{ \frac{a_{i_1 i_2} + ... + a_{i_k i_1}}{k} : (i_1, ..., i_k) \text{ is a cycle}, k = 1, ..., n \right\}$$

where $\max(\emptyset) = \varepsilon$ by definition. Note that this definition is independent of whether we allow cycles to contain repeated nodes [18]. The maximum cycle mean can be calculated in $\mathcal{O}(n^3)$ time [18, 52].

The following is well known.

Lemma 1.1. *Let $A, B \in \overline{\mathbb{R}}^{n\times n}$. Then*

$$A \leq B \Rightarrow \lambda(A) \leq \lambda(B).$$

It is easily seen that $\lambda(\alpha A) = \alpha\lambda(A)$ and, in particular, $\lambda(\lambda(A)^{-1}A) = 0$ if $\lambda(A) > \varepsilon$. The matrix $(\lambda(A)^{-1}A)$ will be denoted A_λ. If $\lambda(A) = 0$, then we say that A is *definite*. If moreover $a_{ii} = 0$ for all $i \in N$ then A is called *strongly definite*.

An $n \times n$ matrix is called *diagonal*, written $diag(d_1, ..., d_n) = diag(d)$, if its diagonal entries are $d_1, ..., d_n \in \mathbb{R}$ and off diagonal entries are ε. We use I to denote the identity matrix, $I = diag(0, ..., 0)$, of appropriate size. A matrix Q is called a *generalised permutation matrix* if it can be obtained from a diagonal matrix by permuting the rows and/or columns. Generalised permutation matrices are the only invertible matrices in max-algebra [18, 37].

Given $A \in \overline{\mathbb{R}}^{m\times n}$ and sets $S \subseteq M, T \subseteq N$, we write $A[S, T]$ to mean the submatrix of A with rows from S and columns from T. We denote $A[T, T]$ by $A[T]$.

For matrices with $\lambda(A) \leq 0$ we define

$$A^+ = A \oplus A^2 \oplus ... \oplus A^n \text{ and}$$
$$A^* = I \oplus A \oplus ... \oplus A^{n-1}.$$

Further, if A is definite at least one column in A^+ is the same as the corresponding column in A^* and we define $\tilde{A}$ to be the matrix consisting of columns identical in A^+ and A^*. The matrix A^* is often called the *Kleene star*. The matrices $\tilde{B}$, B^+ where $B = A_\lambda$ will be denoted $\tilde{A}_\lambda$ and A_λ^+ respectively. Using the Floyd-Warshall algorithm; see, e.g., [18], A^* can be calculated in $\mathcal{O}(n^3)$ time.

By D_A we mean the weighted digraph (N, E, w) where $E = \{(i, j) : a_{ij} > \varepsilon\}$ and the weight of the edge (i, j) is a_{ij}. A is called *irreducible* if D_A is strongly connected (that is, if there is an $i-j$ path in D_A for any i and j). If σ is a cycle in D_A then we denote its weight by $w(\sigma, A)$, and its length by $l(\sigma)$. A cycle is called *critical* if

$$\frac{w(\sigma, A)}{l(\sigma)} = \lambda(A).$$

We denote by $N_C(A)$ the set of *critical nodes*, that is any node $i \in N$ which is on a critical cycle. The digraph with node set N and edge set equal to the edges on all critical cycles is called the *critical digraph* and denoted C_A.

If $A \in \overline{\mathbb{R}}^{n\times n}$ is interpreted as a matrix of direct-distances in D_A, then A^t (where t is a positive integer) is the matrix of the weights of heaviest paths with t arcs. Following this observation it is not difficult to deduce the following two lemmas.

Lemma 1.2. *[18] Let $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda(A) > \varepsilon$.*

(a) $\tilde{A}_\lambda$ is column $\mathbb{R}$-astic.

(b) If A is irreducible then A_λ^+, and hence also $\tilde{A}_\lambda$, are finite.

Lemma 1.3. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. If $\lambda(A) = 0$ then $\lambda(A^+) = 0$.*

If $a, b \in \overline{\overline{\mathbb{R}}} := \overline{\mathbb{R}} \cup \{+\infty\}$, then we define $a \oplus' b := \min(a, b)$. Moreover $a \otimes' b := a + b$ exactly when at least one of a, b is finite, otherwise

$$(-\infty) \otimes' (+\infty) := +\infty \text{ and } (+\infty) \otimes' (-\infty) := +\infty.$$

This differs from max-multiplication where

$$(-\infty) \otimes (+\infty) := -\infty \text{ and } (+\infty) \otimes (-\infty) := -\infty.$$

The pair of operations $(\oplus', \otimes')$ is extended to matrices and vectors similarly as $(\oplus, \otimes)$.

For a vector γ we use $\gamma^{(-1)}$ to mean the vector with entries γ_i^{-1}. Similarly, for $A \in \overline{\overline{\mathbb{R}}}^{m\times n}$, $A^{(-1)} = (a_{ij}^{-1})$. For $A \in \overline{\overline{\mathbb{R}}}^{m\times n}$ we define $A^{\#} = -A^T \in \overline{\overline{\mathbb{R}}}^{n\times m}$. It can be shown

[18, 37] that $(A \otimes B)^{\#} = B^{\#} \otimes' A^{\#}$. If $A = (a_{ij}) \in \overline{\mathbb{R}}^{m\times n}$ then A_j stands for the j^{th} column of A. Given a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ and a vector $c \in \overline{\mathbb{R}}^{m}$, we use $(A|c)$ to denote the $m \times (n+1)$ matrix obtained from A by adding c as an extra, final, column. The following observation is easily seen.

Lemma 1.4. *Let* $A \in \overline{\mathbb{R}}^{m\times n}$, $x \in \mathbb{R}^n$.

(i) If A *is row* $\mathbb{R}$*-astic then* $A \otimes x$ *is finite.*

(ii) If A *is column* $\mathbb{R}$*-astic then* $A^{\#} \otimes' x$ *is finite.*

A set $S \subseteq \overline{\mathbb{R}}^{n}$, is called a *max-algebraic subspace* if, for any $u, v \in S$ and $\alpha, \beta \in \overline{\mathbb{R}}$, $\alpha u \oplus \beta v \in S$.

We use P_n to denote the set of permutations on N. For $A \in \overline{\mathbb{R}}^{n\times n}$ the *max-algebraic permanent* is given by

$$maper(A) = \bigoplus_{\pi\in P_n} \bigotimes_{i\in N} a_{i,\pi(i)}.$$

For a given $\pi \in P_n$ its *weight* with respect to A is

$$w(\pi, A) = \bigotimes_{i\in N} a_{i,\pi(i)}$$

and the set of permutations whose weight is maximum is

$$ap(A) = \{\pi \in P_n : w(\pi, A) = maper(A)\}.$$

We note here that the set $ap(A)$ is the set of optimal solutions to the assignment problem.

Propositions 1.5-1.12 below are standard results.

Proposition 1.5. *[18, 37] If* $A \in \overline{\mathbb{R}}^{m\times n}$ *and* $x, y \in \overline{\mathbb{R}}^{n}$, *then*

$$x \leq y \Rightarrow A \otimes x \leq A \otimes y \text{ and } A \otimes' x \leq A \otimes' y.$$

Corollary 1.6. *[18, 37] If* $A, B \in \overline{\mathbb{R}}^{m\times n}$ *and* $x \leq y$*, then*

$$B^{\#} \otimes' (A \otimes x) \leq B^{\#} \otimes' (A \otimes y).$$

Corollary 1.7. *[18] If* $f \in \overline{\mathbb{R}}^{n}$ *and* $x, y \in \overline{\mathbb{R}}^{n}$*, then*

$$x \leq y \Rightarrow f^T x \leq f^T y.$$

Note that, if $Ax \leq b$, $x \in \mathbb{Z}^n$ and $b_i = \varepsilon$, then the i^{th} row of A is ε. In such a case the i^{th} inequality is redundant and can be removed. We may therefore assume without loss of generality that b is finite when dealing with integer solutions to one-sided systems.

Definition 1.8. *If* $A \in \overline{\mathbb{R}}^{m\times n}$ *and* $b \in \mathbb{R}^m$*, then, for all* $j \in N$*, define*

$$M_j(A, b) = \{t \in M : a_{tj} \otimes b_t^{-1} = \max_i a_{ij} \otimes b_i^{-1}\}.$$

Proposition 1.9. *[18, 35, 37] Let* $A \in \overline{\mathbb{R}}^{m\times n}$*,* $b \in \mathbb{R}^m$ *and* $\bar{x} = A^{\#} \otimes' b$*.*

(i) $Ax \leq b \Leftrightarrow x \leq \bar{x}$

(ii) $Ax = b \Leftrightarrow x \leq \bar{x}$ *and*

$$\bigcup_{j : x_j = \bar{x}_j} M_j(A, b) = M.$$

By Propositions 1.5 and 1.9 we have the following.

Corollary 1.10. *Let* $A \in \overline{\mathbb{R}}^{m\times n}$*,* $b \in \mathbb{R}^m$ *and* $\bar{x} = A^{\#} \otimes' b$*.*

(i) $\bar{x}$ *is always a solution to* $Ax \leq b$

(ii) $Ax = b$ *has a solution* $\Leftrightarrow$ $\bar{x}$ *is a solution* $\Leftrightarrow A \otimes (A^{\#} \otimes' b) = b$*.*

Since this thesis deals with integer solutions we only summarise here the existing theory of finite eigenvectors and subeigenvectors. A full description of all solutions to $Ax \leq b$, $Ax = b$, $Ax = \lambda x$ and $Ax \leq \lambda x$ can be found e.g. in [18].

It is known [18] that, if $\lambda(A) = \varepsilon$, then A has no finite eigenvectors unless $A = \varepsilon$. We may therefore assume without loss of generality that $\lambda(A) > \varepsilon$ when discussing integer eigenvectors.

For $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \mathbb{R}$ we denote

$$V(A, \lambda) = \{x \in \mathbb{R}^n : Ax = \lambda x\} \text{ and}$$
$$V^*(A, \lambda) = \{x \in \mathbb{R}^n : Ax \leq \lambda x\}$$

Proposition 1.11. *[37] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$. Then $V(A, \lambda) \neq \emptyset$ if and only if $\lambda = \lambda(A)$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic (and hence doubly $\mathbb{R}$-astic).*

If $V(A, \lambda(A)) \neq \emptyset$, then

$$V(A, \lambda(A)) = \{\tilde{A}_\lambda u : u \in \mathbb{R}^k\}$$

where $\tilde{A}_\lambda$ is $n \times k$ for some $k \leq n$.

Proposition 1.12. *[18] Let $A \in \overline{\mathbb{R}}^{n\times n}$, $A \neq \varepsilon$. Then $V^*(A, \lambda) \neq \emptyset$ if and only if $\lambda \geq \lambda(A)$, $\lambda > \varepsilon$.*

If $V^*(A, \lambda) \neq \emptyset$, then

$$V^*(A, \lambda) = \{(\lambda^{-1}A)^* u : u \in \mathbb{R}^n\}.$$

If $\lambda \in \mathbb{Z}$ and A is integer, then $(\lambda^{-1}A)^*$ is integer and hence we deduce the following.

Corollary 1.13. *If $A \in \overline{\mathbb{Z}}^{n\times n}$, then $Ax \leq x$ has a finite solution if and only if it has an integer solution.*

$A \in \overline{\mathbb{R}}^{n\times n}$ is called *increasing* if $a_{ii} \geq 0$ for all $i \in N$. Since $(Ax)_i \geq a_{ii}x_i$ we immediately see that A is increasing if and only if $Ax \geq x$ for all $x \in \overline{\mathbb{R}}^n$. It follows from the definition of a definite matrix that $a_{ii} \leq 0$ for all $i \in N$. Therefore a matrix is strongly definite if and only if it is definite and increasing. It is easily seen [18] that all diagonal entries of all powers of a strongly definite matrix are zero and thus in this case

$$A^+ = A^* = \tilde{A}_\lambda.$$

Hence we have

Proposition 1.14. *If A is strongly definite then $V(A, 0) = V^*(A, 0)$.*

Other basic properties that we will need are given below. Note that Lemma 1.16 is the cancellation law in max-algebra.

Lemma 1.15. *[18] Let $A, B \in \overline{\mathbb{R}}^{m\times n}$, $c, d \in \overline{\mathbb{R}}^m$. Then there exists $x \in \mathbb{R}^n$ satisfying $Ax \oplus c = Bx \oplus d$ if and only if there exists $z \in \mathbb{R}^{n+1}$ satisfying $(A|c)z = (B|d)z$.*

Lemma 1.16. *[18] Let $v, w, a, b \in \mathbb{R}$, $a > b$. Then for any real number x we have*

$$v \oplus a \otimes x = w \oplus b \otimes x \Leftrightarrow v \oplus a \otimes x = w.$$

Finally, for matrices of compatible sizes [18, 37],

$$X \otimes (X^{\#} \otimes' Y) \leq Y \text{ and} \tag{1.1}$$

$$X \otimes (X^{\#} \otimes' (X \otimes Z)) = X \otimes Z. \tag{1.2}$$

In searching for integer solutions to the integer image problem one helpful tool is being able to identify potential active positions.

Given a solution x to $Ax = b$, we say that a position (i, j) is *active* with respect to x if and only if $a_{ij} + x_j = b_i$, it is called *inactive* otherwise. Further, an element/entry a_{ij} of A is *active* if and only if the position (i, j) is active. Related to this definition, we call a column A_j *active* if it contains an active entry. We also say that a component x_j of x is *active* in the equation $Ax = Bx$ if and only if there exists i such that either $a_{ij}x_j = (Bx)_i$ or $(Ax)_i = b_{ij}x_j$. Lastly x_j is *active* in $f^T x$ if and only if $f_j x_j = f^T x$.

For $a \in \mathbb{R}$ the *fractional part* of a is $fr(a) := a - \lfloor a \rfloor$. For a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ we use $\lfloor A \rfloor$ ($\lceil A \rceil$) to denote the matrix with (i,j) entry equal to $\lfloor a_{ij} \rfloor$ ($\lceil a_{ij} \rceil$) for all i, j, and similarly for vectors. We define $\lfloor \varepsilon \rfloor = \varepsilon = \lceil \varepsilon \rceil = fr(\varepsilon)$. We outline a number of simple properties of $fr(\cdot)$ below.

Lemma 1.17. *Let $a, b, c \in \mathbb{R}$, $\delta \in (0,1)$ and $x \in \mathbb{Z}$. Then*

(i) $fr(a) \geq 0$ so $a \geq 0 \Leftrightarrow fr(a) \leq a$.

(ii) $fr(-a) = \begin{cases} 1 - fr(a), & \text{if } a \notin \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$

(iii) $fr(a+b) = fr(fr(a) + fr(b))$.

(iv) $fr(a-b) = fr(fr(a) - fr(b))$.

In fact,

$$fr(a) > fr(b) \Rightarrow fr(a-b) = fr(a) - fr(b), \text{ and}$$
$$fr(a) < fr(b) \Rightarrow fr(a-b) = 1 - fr(b) + fr(a).$$

(v) $\lfloor a+b \rfloor > \lceil a \rceil \Rightarrow b > fr(-a)$.

(vi) $\lfloor a \rfloor \geq \lceil a - \delta \rceil \Rightarrow \delta \geq fr(a)$.

(vii) $fr(x+a) = fr(a)$.

(viii)

$$\lfloor -a \rfloor = \begin{cases} -a & \text{if } a \in \mathbb{Z} \\ -1 - \lfloor a \rfloor & \text{otherwise.} \end{cases}$$

(ix)

$$\lceil -a \rceil = \begin{cases} -a & \text{if } a \in \mathbb{Z} \\ 1 - \lceil a \rceil & \text{otherwise.} \end{cases}$$

(x) If $b + c \in \mathbb{Z}$*, then* $fr(b) = fr(-c)$*.*

(xi) If $b + c \in \mathbb{Z}$*, then*

$$fr(a + c) = fr(a - b).$$

Proof. Note that (i), (ii), (iii), (vi), (viii) and (ix) follow easily from the definitions. Further (vii) follows immediately from (vi). We give proofs for the rest.

(iv) First note that, from (ii),

$$fr(fr(a) + fr(-b)) = \begin{cases} fr(fr(a) - fr(b)), & \text{if } b \in \mathbb{Z}; \\ fr(fr(a) + 1 - fr(b)) = fr(fr(a) - fr(b)), & \text{otherwise.} \end{cases}$$

Using this and (iii) we get,

$$fr(a - b) = fr(a + (-b)) = fr(fr(a) + fr(-b)) = fr(fr(a) - fr(b)).$$

Now assume $fr(a) > fr(b)$. Then $fr(a) - fr(b) \in (0, 1)$. Therefore

$$fr(a) - fr(b) = fr(fr(a) - fr(b)) = fr(a - b).$$

Finally suppose $fr(a) < fr(b)$. Then, similarly as above,

$$\begin{aligned} fr(b) - fr(a) = fr(fr(b) - fr(a)) = fr(b - a) = fr(-(a - b)) \\ = \begin{cases} 1 - fr(a - b), & \text{if } a - b \notin \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

But, if $a-b \in \mathbb{Z}$, then $fr(a) = fr(b)$ which does not fit our assumption. Hence $fr(b) - fr(a) = 1 - fr(a-b)$.

(v) First $a+b \geq \lfloor a+b \rfloor > \lceil a \rceil$. If $a \in \mathbb{Z}$ then, trivially, $b > \lceil a \rceil - a = 0 = fr(-a)$.

Otherwise, $a \notin \mathbb{Z}$ and we obtain $\lfloor a \rfloor + fr(a) + b > \lceil a \rceil$ which implies, $b > 1 - fr(a) = fr(-a)$.

(x) Assume $b + c \in \mathbb{Z}$, so $fr(b+c) = 0$. Clearly $b \in \mathbb{Z} \Leftrightarrow c \in \mathbb{Z}$ and the result holds in these cases. Assume that $b, c \notin \mathbb{Z}$.

Case 1: $fr(b) > fr(c) > 0$.

Here, $0 = fr(b+c) = fr(b-(-c)) = fr(b) - fr(-c)$ by (iv).

Case 2: $fr(c) > fr(b) > 0$.

Now, $0 = fr(b+c) = fr(b-(-c)) = 1 - fr(-c) + fr(b) = fr(c) + fr(b)$ by (iv) and (ii).

Case 3: $fr(b) = fr(c) > 0$.

In this case we conclude $fr(b) = 0.5 = fr(c)$ and therefore also $fr(-b) = 0.5 = fr(-c)$.

(vi) $\lfloor a \rfloor \geq \lceil a - \delta \rceil \geq a - \delta = \lfloor a \rfloor + fr(a) - \delta \therefore 0 \geq fr(a) - \delta$.

(xi) Since $b + c \in \mathbb{Z}$ we have $fr(c) = fr(-b)$ from (x). Then

$$fr(a+c) = fr(fr(a) + fr(c)) = fr(fr(a) + fr(-b)) = fr(a-b)$$

using (iii). □

2. One-sided systems and the integer eigenproblem

We show that integer solutions to $Ax \leq b$, $Ax = b$ and $Ax \leq \lambda x$ can be found in (strongly) polynomial time in Proposition 2.1 and Theorem 2.5. Recall that by an integer solution we mean a finite integer solution. We also give a full description of all integer eigenvectors of a matrix and present some strongly polynomially solvable cases of the integer eigenproblem. This includes a generic case we call Property OneIR, and Theorem 2.17 allows us to fully describe all integer solutions in this case. Much of the material in Sections 2.1, 2.2, the initial part of Section 2.3, Subsections 2.4.1 and 2.4.2 has been published in [24].

2.1 One-sided systems

Proposition 1.9(i) provides an immediate answer to the task of finding integer solutions to $Ax \leq b$, namely all integer vectors not exceeding $A^{\#} \otimes' b$. Integer solutions to $Ax = b$ can also be straightforwardly deduced from Proposition 1.9(ii) and we summarise this in the next result.

Proposition 2.1. *Let* $A \in \overline{\mathbb{R}}^{m \times n}$, $b \in \mathbb{R}^m$ *and* $\bar{x} = A^{\#} \otimes' b$.

(i) An integer solution to $Ax \leq b$ *exists if and only if* $\bar{x}$ *is finite. If an integer solution exists, then all integer solutions can be described as the integer vectors* x *satisfying* $x \leq \bar{x}$.

(ii) An integer solution to $Ax = b$ exists if and only if

$$\bigcup_{j:\bar{x}_j \in \mathbb{Z}} M_j(A, b) = M$$

where $M_j(A, b)$ is defined in Definition 1.8. If an integer solution exists, then all integer solutions can be described as the integer vectors x satisfying $x \leq \bar{x}$ with

$$\bigcup_{j:x_j = \bar{x}_j} M_j(A, b) = M.$$

Corollary 2.2. *Let $A \in \overline{\mathbb{R}}^{m\times n}$, $b \in \mathbb{R}^m$. An integer solution to $Ax \leq b$ always exists.*

We define $\hat{x} = \lfloor A^{\#} \otimes' b \rfloor$. Then, from Proposition 2.1 and (1.2), we conclude:

Corollary 2.3. *Let $A \in \overline{\mathbb{R}}^{m\times n}, b \in \overline{\mathbb{R}}^{m}, c \in \mathbb{Z}^n$. Then the following hold:*

(i) $\hat{x}$ is the greatest integer solution to $Ax \leq b$ (provided $\hat{x}$ is finite).

(ii) $Ax = b$ has an integer solution if and only if $\hat{x}$ is an integer solution.

(iii) $A \otimes \lfloor A^{\#} \otimes' (A \otimes c) \rfloor = A \otimes c$.

Proof. Similar to the proof of Corollary 3.2.3 in [18]. □

Consider the matrix inequality $AX \leq B$ where $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$, $X \in \overline{\mathbb{R}}^{n\times k}$ and let $\hat{X} = \lfloor A^{\#} \otimes' B \rfloor$. This system can be written as a set of inequalities of the form $Ax \leq b$ in the following way using the notation X_r, B_r to denote the r^{th} column of X and B respectively:

$$AX_r \leq B_r, \ r = 1, ..., k.$$

This allows us to state the following result.

Corollary 2.4. *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}, C \in \mathbb{Z}^{n\times k}$. Then the following hold:*

(i) $\hat{X}$ is the greatest integer solution to $AX \leq B$ (provided $\hat{X}$ is finite), that is $A \otimes \lfloor A^{\#} \otimes' B \rfloor \leq B$.

(ii) $AX = B$ *has an integer solution if and only if* $\hat{X}$ *is an integer solution.*

(iii) $A \otimes \lfloor A^{\#} \otimes' (A \otimes C) \rfloor = A \otimes C$.

2.2 Integer subeigenvectors

For $A \in \overline{\mathbb{R}}^{n\times n}$ we define

$$IV^*(A, \lambda) = V^*(A, \lambda) \cap \mathbb{Z}^n.$$

Proposition 1.12 enables us to deduce an answer to integer solubility of the subeigenproblem.

Theorem 2.5. *Let* $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda \in \mathbb{R}$.

(i) $IV^*(A, \lambda) \neq \emptyset$ *if and only if*

$$\lambda(\lceil \lambda^{-1} A \rceil) \leq 0.$$

(ii) If $IV^*(A, \lambda) \neq \emptyset$, *then*

$$IV^*(A, \lambda) = \{\lceil \lambda^{-1} A \rceil^* z : z \in \mathbb{Z}^n\}.$$

Note here that λ and $\lambda(\cdot)$ mean two different things, the first being a scalar, the second a function defining the maximum cycle mean.

Proof. For both (i) and (ii) we will need the following. Assume that $x \in IV^*(A, \lambda)$.

Using the fact that $x_i \in \mathbb{Z}$ for every i we get the equivalences below.

$$\begin{aligned}
&Ax \leq \lambda x \\
\Leftrightarrow &(\lambda^{-1}A)x \leq x \\
\Leftrightarrow &(\forall i, j \in N)\ x_i \otimes x_j^{-1} \geq \lambda^{-1} \otimes a_{ij} \\
\Leftrightarrow &(\forall i, j \in N)\ x_i \otimes x_j^{-1} \geq \lceil \lambda^{-1} \otimes a_{ij} \rceil \\
\Leftrightarrow &\lceil \lambda^{-1}A \rceil x \leq x.
\end{aligned}$$

Thus integer subeigenvectors of A with respect to λ are exactly the integer subeigenvectors of $\lceil \lambda^{-1}A \rceil \in \overline{\mathbb{Z}}^{n \times n}$ with respect to 0.

(i) Now, from Proposition 1.12, we see that a finite subeigenvector of $\lceil \lambda^{-1}A \rceil$ with respect to $\lambda = 0$ exists if and only if $\lambda(\lceil \lambda^{-1}A \rceil) \leq 0$.

Further $\lceil \lambda^{-1}A \rceil$ is integer so, by Corollary 1.13, we have that a finite subeigenvector exists if and only if an integer subeigenvector exists.

(ii) If a finite subeigenvector exists then, again from Proposition 1.12, we know that

$$V^*(\lceil \lambda^{-1}A \rceil, 0) = \{\lceil \lambda^{-1}A \rceil^* u : u \in \mathbb{R}\}.$$

But $\lceil \lambda^{-1}A \rceil$ and therefore $\lceil \lambda^{-1}A \rceil^*$ are integer matrices, meaning that we can describe all integer subeigenvectors by taking max combinations of the columns of $\lceil \lambda^{-1}A \rceil^*$ with integer coefficients.

Observe that it is possible to obtain an integer vector from a max combination of the integer columns of the matrix with real coefficients, but only if the real coefficients correspond to inactive columns. However any integer vectors obtained in this way can also be obtained by using integer coefficients, for example by taking the lower integer part of the coefficients, and thus it is sufficient to only take integer coefficients. $\square$

Corollary 2.6. *For $A \in \overline{\mathbb{R}}^{n\times n}$ it is possible to decide whether $IV^*(A,\lambda) \neq \emptyset$ in $\mathcal{O}(n^3)$ time.*

2.3 Description of all integer eigenvectors

For $A \in \overline{\mathbb{R}}^{n\times n}$ we define

$$IV(A,\lambda) = V(A,\lambda) \cap \mathbb{Z}^n.$$

It appears that the integer eigenproblem cannot be solved as easily as other one sided systems. We can however describe the set of all integer eigenvectors by using Proposition 1.11.

Proposition 2.7. *Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$. If $IV(A,\lambda) \neq \emptyset$, then $\lambda = \lambda(A)$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic (and hence doubly $\mathbb{R}$-astic).*

Further

$$IV(A,\lambda(A)) = \{z \in \mathbb{Z}^n : z = \tilde{A}_\lambda u, u \in \mathbb{R}^k\}.$$

Note that we denote $IV(A,\lambda(A))$ by $IV(A)$ since all integer eigenvectors correspond to $\lambda(A)$.

Proposition 2.7 shows that the problem of finding one integer eigenvector could be solved by finding a criterion for existence of, and a method for obtaining, an integer point in a finitely generated subspace (namely the column space of the doubly $\mathbb{R}$-astic matrix $\tilde{A}_\lambda$). In Section 3.1 we present an algorithm for finding such a point. The algorithm is pseudopolynomial for finite matrices which, in light of Lemma 1.2, solves the question of integer eigenvectors for any irreducible matrix.

2.4 Some strongly polynomially solvable cases of the integer eigenproblem

2.4.1 Integer matrices

We observe that the problem of integer eigenvectors can easily be solved for matrices over $\overline{\mathbb{Z}}$.

Proposition 2.8. *Let $A \in \overline{\mathbb{Z}}^{n\times n}$. Then A has an integer eigenvector if and only if $\lambda(A) \in \mathbb{Z}$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic.*

Proof. First assume that $x \in IV(A)$. From Proposition 1.11, we know the only eigenvalue corresponding to x is $\lambda(A)$. Then $Ax = \lambda(A)x$ where the product on the left hand side is integer. To ensure that the right hand side is also integer we clearly need $\lambda(A) \in \mathbb{Z}$. Further, any integer eigenvector is finite and so $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic by Proposition 1.11.

Now assume that $\lambda(A) \in \mathbb{Z}$ and $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic. Then $A_\lambda \in \overline{\mathbb{Z}}^{n\times n}$, thus all entries of A_λ^+, A_λ^* and $\tilde{A}_\lambda$ belong to $\overline{\mathbb{Z}}$. Again, from Proposition 1.11, we know that all finite eigenvectors are described by max combinations of the columns of $\tilde{A}_\lambda$. Thus we can pick integer coefficients to obtain an integer eigenvector of A by Lemma 1.4. □

Corollary 2.9. *Let $A \in \overline{\mathbb{Z}}^{n\times n}$ be irreducible. A has an integer eigenvector if and only if $\lambda(A) \in \mathbb{Z}$.*

We cannot assume that this result holds for a general matrix $A \in \overline{\mathbb{R}}^{n\times n}$ as the following examples show.

Example 2.10. *$A \in \mathbb{R}^{n\times n}$ has an integer eigenvector $\not\Rightarrow \lambda(A) \in \mathbb{Z}$.*

$$A = \begin{pmatrix} 1.1 & 1.1 \\ 1.1 & 1.1 \end{pmatrix}.$$

Let $x = (1,1)^T \in \mathbb{Z}^n$. *Then* $x \in IV(A, 1.1)$ *but* $\lambda(A) = 1.1 \notin \mathbb{Z}$.

Example 2.11. $A \in \overline{\mathbb{R}}^{n\times n}$ *with* $\lambda(A) \in \mathbb{Z} \nRightarrow A$ *has an integer eigenvector.*

$$A = \begin{pmatrix} 2.9 & 3.5 \\ 2.5 & 2.7 \end{pmatrix}.$$

Then $\lambda(A) = 3 \in \mathbb{Z}$ *but* Ax *is clearly not integer for any integer vector* x.

Further, a matrix does not have to be integer to have an integer eigenvalue or eigenvector, and integer matrices need not have integer eigenvectors.

Example 2.12. $A \in \mathbb{Z}^{n\times n} \nRightarrow A$ *has an integer eigenvector and an integer eigenvalue.*

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}.$$

Then $\lambda(A) = \frac{5}{2} \notin \mathbb{Z}$. *By Corollary 2.9* A *cannot have an integer eigenvector.*

Example 2.13. A *has an integer eigenvector and an integer eigenvalue* $\nRightarrow A \in \mathbb{Z}^{n\times n}$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0.2 \end{pmatrix} \notin \mathbb{Z}^{n\times n}.$$

Then $A(1,1)^T = 1(1,1)^T$ *and thus* A *has an integer eigenvector and an integer eigenvalue.*

In the above counterexample the matrix A has a large number of integer entries, so the question arises whether a real matrix with no integer entries can have both integer eigenvectors and eigenvalues.

Proposition 2.14. *Let* $A \in \overline{\mathbb{R}}^{n\times n}$ *be a matrix such that it has an integer eigenvector corresponding to an integer eigenvalue, then* A *has an integer entry in every row.*

Proof. The only eigenvalue corresponding to integer eigenvectors is $\lambda(A)$, hence, by assumption, $\lambda(A) \in \mathbb{Z}$. Now let $x \in IV(A)$. Then $Ax = \lambda(A)x$ where the right hand side is integer. Therefore $(\forall i \in N)\ \max(a_{ij} + x_j) \in \mathbb{Z}$ which implies that for every $i \in N$ there exists an index j for which $a_{ij} \in \mathbb{Z}$. □

2.4.2 Strongly definite matrices

Theorem 2.5 and Proposition 1.14 allow us to present a solution to the problem of integer eigenvectors for strongly definite matrices. Since $\lambda(\cdot)$ is monotone on $\overline{\mathbb{R}}^{n\times n}$ we have that, for strongly definite matrices A, the inequality $\lambda(\lceil A \rceil) \leq 0$ is equivalent to $\lambda(\lceil A \rceil) = 0$. This gives the following result.

Corollary 2.15. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be strongly definite.*

(i) $IV(A) \neq \emptyset$ if and only if

$$\lambda(\lceil A \rceil) = 0.$$

(ii) If $IV(A) \neq \emptyset$, then

$$IV(A) = \{\lceil A \rceil^* z : z \in \mathbb{Z}^n\}.$$

2.4.3 A generic case: Property OneIR

Since $IV(A, \lambda(A)) = IV(A_\lambda, 0)$ we can assume without loss of generality that A is definite. Note from Proposition 2.14 that, if $Ax = x$, then the active entry in each row is integer. Thus a necessary condition for a definite matrix A to have an integer eigenvector is that it has at least one integer entry in each row. We will focus on the case when there is exactly one.

Definition 2.16. *Let $A \in \overline{\mathbb{R}}^{n\times n}$. If A has exactly one integer entry per row we say that A satisfies* Property OneIR. *For each $i \in N$ we write $c(i)$ to denote the column index of* *the integer entry in row i.*

We say that A weakly satisfies Property OneIR if it has at most one integer entry per row.

Matrices with at most one integer entry in each row represent a generic case since, if we generate a random matrix (with real entries), the probability of there being more than one integer entry in each row is zero.

For an integer eigenvector x, we have that $(\forall j \in N)\ a_{ij} + x_j \leq x_i$ with equality only when $j = c(i)$. This is equivalent to the following set of inequalities;

$$\begin{aligned} &(\forall i, j \in N)\ x_i - x_j \geq \lceil a_{ij} \rceil, \\ &(\forall i \in N)\ x_{c(i)} - x_i \geq -a_{i,c(i)}. \end{aligned} \tag{2.1}$$

Define a matrix $W = (w_{ij})$ by

$$w_{ij} = \begin{cases} \max(\lceil a_{ij} \rceil, -a_{ji}), & \text{if } i = c(j); \\ \lceil a_{ij} \rceil, & \text{otherwise.} \end{cases}.$$

Then the set of inequalities (2.1) is equivalent to saying that $W \otimes x \leq x$. Thus we get the following result.

Theorem 2.17. *Let $A \in \overline{\mathbb{R}}^{n \times n}$ be definite, weakly satisfy Property OneIR and let W be as defined above.*
(i) If $(\exists i \in N)(\forall j \in N) a_{ij} \notin \mathbb{Z}$ then $IV(A) = \emptyset$, else
(ii) A satisfies Property OneIR and $IV(A) \neq \emptyset \Leftrightarrow \lambda(W) \leq 0$.

Further, if an integer eigenvector exists, then $IV(A) = IV^*(W, 0)$.

Remark Since finding integer subeigenvectors can be done in strongly polynomial time, finding integer eigenvectors for matrices weakly satisfying Property OneIR can also be done in strongly polynomial time.

2.5 A strongly polynomial method if n is fixed

Let $A \in \overline{\mathbb{R}}^{n\times n}$ and assume without loss of generality that A is definite. Suppose there exists a single row, t say, with 2 integer entries and that all other rows have a single integer entry. Then, since one entry per row is active, there are two possible choices for the set of active entries with respect to an integer eigenvector.

Let a_{tj} and a_{tl} be the only integer entries in row t. Define A^{δ_j} to be the matrix A but with (t, j) entry equal to $a_{tj} - \delta$ where $0 < \delta < 1$. Similarly define A^{δ_l} to be A but with (t, l) entry equal to $a_{tl} - \delta$. In this way both matrices A^{δ_j} and A^{δ_l} have exactly one integer entry per row and $IV(A) = IV(A^{\delta_j}) \cup IV(A^{\delta_l})$. We can find the integer eigenvectors of both these matrices in strongly polynomial time.

Extending this idea, assume that there are at most 2 integer entries per row, and that the number of rows with 2 integer entries is d. Then the number of sets of possible active elements is at most $2^d 1^{(n-d)} = 2^d$ since we must choose one integer from every row. Let $S_1, ..., S_t$, $t \in \mathbb{N}$ be the (at most 2^d) different sets which each contain the positions of a possible set of active entries with respect to some eigenvector, that is, each S_i contains a different set of positions, one for each row, which correspond to integer entries.

Define A^{S_r} to be the matrix obtained from A by subtracting $0 < \delta < 1$ from every integer entry except those with indices in S_r. So A^{S_r} is a matrix with exactly one integer entry per row. Then we can calculate $IV(A)$ by calculating each of $IV(A^{S_r})$, $1 \leq r \leq t$, which can be done in strongly polynomial time provided that d is a fixed constant.

Example 2.18. *Let*

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -0.5 \end{pmatrix}.$$

We calculate $S_1 = \{(1,1),(2,1)\}$ *and* $S_2 = \{(1,2),(2,1)\}$ *so*

$$A^{S_1} = \begin{pmatrix} 0 & 1-\delta \\ -2 & -0.5 \end{pmatrix} \text{ and } A^{S_2} = \begin{pmatrix} 0-\delta & 1 \\ -2 & -0.5 \end{pmatrix}.$$

Now, using Theorem 2.17, we calculate the matrices W^{S_1} *and* W^{S^2} *which satisfy*

$$IV(A^{S_r}) = IV^*(W^{S_r}), r = 1, 2.$$

This gives

$$W^{S_1} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \text{ and } W^{S_2} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

Note that $\lambda(W^{S_2}) > 0$ *so* $IV^*(W^{S_2}, 0) = \emptyset$. *We conclude that* $IV(A) = IV^*(W^{S_1}) \neq \emptyset$ *since* $\lambda(W^{S_1}) = 0$. *Indeed, using Theorem 2.5,* $\{(2,0)^T, (0,-2)^T\} \subseteq IV(A)$.

Proposition 2.19. *Let* $d \in \mathbb{N}$.

If $A \in \overline{\mathbb{R}}^{n\times n}$ *is a definite matrix such that the number of rows with more than one integer entry is at most d, then the number of sets* S_r*, each describing one possible set of active positions with respect to some integer eigenvector, is t where* $t \leq n^d$. *Further*

$$IV(A) = \bigcup_{r=1}^{t} IV(A^{S_r})$$

and each of the sets on the right hand side can be calculated in strongly polynomial time using Theorem 2.17.

Corollary 2.20. *Assume n is fixed. Then all integer eigenvectors of* $A \in \overline{\mathbb{R}}^{n\times n}$ *can be described in strongly polynomial time.*

2.6 Describing the multipliers of $\tilde{A}$ and finding other special cases

Here we show that the integer eigenproblem can be solved in strongly polynomial time if every node in D_A is critical, or if there are at most two non trivial components of the critical digraph C_A. Note that by a non trivial component we mean a strongly connected component that contains at least one edge. Recall that $N_C(A)$ is the set of critical nodes of D_A. We develop some results about the integer eigenspace of A when $N_C(A) = C$ for some set C with $|C| = c \leq n$.

Recall that, from Proposition 2.7, we have

$$IV(A, 0) = IIm(\tilde{A}). \tag{2.2}$$

First note that, as a consequence of Lemma 1.3, we can assume without loss of generality that $\lambda(A^+) = 0$ for the rest of this section.

A consequence of Proposition 4.1.1 in [18] is the following.

Corollary 2.21. *Suppose $A \in \overline{\mathbb{R}}^{n\times n}$ is definite. Then $IV(A, 0) \subseteq IV(A^+, 0)$.*

We begin with the case when all nodes are critical.

Lemma 2.22. *Suppose A is definite. If $N_C = N$ then $V(A, 0) = V^*(A, 0)$.*

Proof. First note that $A^+ = A^* = \tilde{A}$ under our assumptions. Then, from Propositions 1.11 and 1.12,

$$V(A, 0) = \{\tilde{A}u : u \in \mathbb{R}^n\} = \{A^*u : u \in \mathbb{R}^n\} = V^*(A, 0).$$

□

From this we immediately get that the integer eigenproblem is solvable in strongly polynomial time if $N_C = N$ since the existence and description of integer subeigenvectors can be achieved in strongly polynomial time by Corollary 2.6.

Corollary 2.23. *Suppose* $A \in \overline{\mathbb{R}}^{n\times n}$ *is definite,* $N_C(A) = N$ *and* $\lambda(A^+) = 0$*. Then*

(i) $IV^*(A,0) = IV(A,0) = IV(A^+,0) = IV^*(A^+,0)$.

(ii) $IV(A,0) \neq \emptyset \Leftrightarrow \lambda(\lceil A \rceil) = 0$.

(iii) $IV(A,0) = \{\lceil A \rceil^* u : u \in \mathbb{Z}^n\}$.

Proof. (i) We only show that $IV(A,0) = IV(A^+,0)$ as the other equalities follow from Lemma 2.22 and the fact that A^+ is strongly definite. Firstly, by Corollary 2.21 and using that A^+ is strongly definite,

$$IV(A,0) \subseteq IV(A^+,0) = IV^*(A^+,0). \tag{2.3}$$

Secondly, from (2.2) and using that $A^+ = \tilde{A}$,

$$IV(A,0) = IIm(A^+). \tag{2.4}$$

Finally, using the fact that $\lambda(A) \leq 0$ implies $(\forall k \in \mathbb{N}) A^k \leq A \oplus ... \oplus A^n$, we have

$$\begin{aligned}(A^+)^+ &= A^+ \oplus (A^+)^2 \oplus ... \oplus (A^+)^n \\ &= A \oplus A^2 \oplus ... \oplus A^n \oplus A^2 \oplus ... \oplus A^{n^2} \oplus ... \oplus A^{n^n} \\ &= A \oplus ... \oplus A^n = A^+\end{aligned}$$

and hence

$$IV(A^+,0) = IIm((A^+)^+) = IIm(A^+). \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) gives the result.

(ii) & (iii) From (i), $IV(A,0) = IV^*(A,0)$. The results now follow from Theorem 2.5. □

In the case when there are at most two non trivial strongly connected components of C_A, $IV(A,0) = IIm(\tilde{A})$ where $\tilde{A}$ has at most two distinct columns. It will be shown later that the integer image problem is solvable in strongly polynomial time for $n \times 2$ matrices (see Theorem 3.27). Therefore we can solve this case of the integer eigenproblem in strongly polynomial time also.

We now consider when $2 < |C| < n$.

Recall that $A \in \overline{\mathbb{R}}^{n\times n}$ is called a *Kleene star* if there exists $B \in \overline{\mathbb{R}}^{n\times n}$ such that $A = B^*$. In what follows we use the following two results.

Lemma 2.24. *[18] If $A \in \overline{\mathbb{R}}^{n\times n}$ is increasing, then*

$$A \leq A^2 \leq A^3 \leq ...$$

Lemma 2.25. *[18] $A \in \overline{\mathbb{R}}^{n\times n}$ is a Kleene star if and only if $A^2 = A$ and $a_{ii} = 0$ for all $i \in N$.*

Proposition 2.26. *If A is a Kleene star, then*

$$IIm(A) = IV(A) = IV^*(A,0).$$

Proof. This is clear since $IV(A) = IIm(\tilde{A})$ and $\tilde{A} = A$. □

We now assume that $N_C(A) = C$ and $|C| = c$. The next result describes the multipliers y for which it could happen that $\tilde{A}y = x \in IV(A,0)$.

Theorem 2.27. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be definite, $\lambda(A^+) = 0$ and $N_C(A) = C$. The following are equivalent.*

(1) $IV(A,0) \neq \emptyset$.

(2) $(\exists x \in \mathbb{Z}^n) x = \tilde{A} x[C]$.

(3) $(\exists u \in \mathbb{Z}^c) A^{\bullet} \otimes \lceil A^+[C] \rceil^* \otimes u \in \mathbb{Z}^{n-c}$ *where* $A^{\bullet} \in \overline{\mathbb{R}}^{(n-c)\times c}$ *is the matrix formed of the rows of* $\tilde{A}$ *not in* $A^+[C]$*, that is,* $A^{\bullet} = A^+[N-C, C]$.

Proof. By applying simultaneous permutation of rows and columns to A if necessary we can assume without loss of generality that A^+ and $\tilde{A}$ have the form

$$\begin{pmatrix} A^+[C] & B \\ A^{\bullet} & A^+[N-C] \end{pmatrix} \text{ and } \begin{pmatrix} A^+[C] \\ A^{\bullet} \end{pmatrix} \tag{2.6}$$

respectively. Note that $A^+[C]$ is strongly definite and $A^+[N-C]$ has no zero diagonal entry.

We first show that $A^+[C]$ is a Kleene star. Indeed by Lemma 2.25 we know that $A^+ \oplus I = (A^+ \oplus I)^2$ and hence, substituting in the form from (2.6) for A^+,

$$\begin{pmatrix} A^+[C] & B \\ A^{\bullet} & A^+[N-C] \oplus I \end{pmatrix} = \begin{pmatrix} (A^+[C])^2 \oplus BA^{\bullet} & A(1) \\ A(2) & A(3) \end{pmatrix}$$

for some matrices $A(1), A(2), A(3)$. Therefore $A^+[C] \geq (A^+[C])^2$ and further, by Lemma 2.24, $A^+[C] \leq (A^+[C])^2$. So we have equality. Finally $A^+[C]$ is a Kleene star by Lemma 2.25.

(1)⇒(2)⇒(3):

Suppose that $Ax = x$ for some $x \in \mathbb{Z}^n$. Then $A^+x = x$ by Corollary 2.21, and hence $A^{\bullet}x[C] \oplus A^+[N-C]x[N-C] = x[N-C]$ implying,

$$A^{\bullet}x[C] \leq x[N-C]. \tag{2.7}$$

Further $(\exists y \in \mathbb{R}^c)\tilde{A}y = x$ which means

$$x[N - C] = A^\bullet y. \tag{2.8}$$

Also, since $A^+[C]$ is increasing,

$$x[C] = A^+[C]y \geq y \tag{2.9}$$

and, by Proposition 2.26,

$$x[C] \in IIm(A^+[C]) = IV(A^+[C], 0) = IV^*(A^+[C], 0). \tag{2.10}$$

Combining (2.7), (2.8) and (2.9) gives

$$x[N - C] = A^\bullet y \leq A^\bullet x[C] \leq x[N - C].$$

This, together with (2.10), gives

$$\tilde{A}x[C] = \begin{pmatrix} A^+[C] \\ A^\bullet \end{pmatrix} x[C] = \begin{pmatrix} x[C] \\ x[N - C] \end{pmatrix} = x$$

as required. To show this also implies (3) note that

$$x[C] \in IV^*(A^+[C], 0) = \{\lceil A^+[C] \rceil^* u : u \in \mathbb{Z}^c\}.$$

(3)⇒(1):

Assume $(\exists u \in \mathbb{Z}^c) A^\bullet \otimes \lceil A^+[C] \rceil^* \otimes u \in \mathbb{Z}^{n-c}$ and note that $\lceil A^+[C] \rceil^* u \in IV^*(A^+[C], 0) = IV(A^+[C], 0)$. Therefore $\tilde{A}\lceil A^+[C] \rceil^* \otimes u \in IIm(\tilde{A}) = IV(A, 0)$. □

This tells us that, to determine whether an integer eigenvector of A exists, we need only find such a vector $x[C] \in IV^*(A^+[C], 0)$ satisfying $A^\bullet x[C] \in \mathbb{Z}^{(n-c)}$.

Corollary 2.28. *We can determine whether an integer eigenvector of $A \in \overline{\mathbb{R}}^{n\times n}$ exists in strongly polynomial time when $|C| \in \{1, 2, n-2, n-1, n\}$.*

Proof. $|C| = n$ is given by Proposition 2.23.

In all cases, to determine whether an integer eigenvector exists it is sufficient to determine whether the matrix $A^\bullet \otimes \lceil A^+[C] \rceil^*$ has an integer image by Theorem 2.27(3).

Let $D = A^\bullet \otimes \lceil A^+[C] \rceil^*$.

When $|C| = n-1$ or $|C| = 1$ it is trivial to decide whether D has an integer image since $D \in \overline{\mathbb{R}}^{1\times(n-1)}$ or $D \in \overline{\mathbb{R}}^{(n-1)\times 1}$.

When $|C| = n-2$, then $D \in \overline{\mathbb{R}}^{2\times(n-2)}$ and we find an integer image of a $2 \times n$ matrix in strongly polynomial time by Theorem 3.27.

Finally, when $|C| = 2$, $D \in \overline{\mathbb{R}}^{(n-2)\times 2}$ and we can determine whether an $m \times 2$ matrix has an integer image in strongly polynomial time using Theorem 3.30. □

2.7 Conclusion

In this chapter we showed that, for the one-sided inequality, one-sided equality and the subeigenproblem we can determine whether an integer solution exists in strongly polynomial time, and further that all integer solutions can be described in strongly polynomial time (see Proposition 2.1 and Theorem 2.5).

For the integer eigenproblem it remains open whether it is polynomially solvable. We gave a number of equivalent problems (integer image of $\tilde{A}$ and integer image of a submatrix of $\tilde{A}$) to the integer eigenproblem.

In special cases (strongly definite matrices, matrices satisfying Property OneIR, matrices with $|C| \leq 2$ or $|C| \geq n-2$) we gave methods to determine existence of an integer eigenvector in strongly polynomial time. For Property OneIR and strongly definite matrices a full description could be found in strongly polynomial time. Key results in this chapter include the definition of the special case, Property OneIR, and the complete description of integer eigenvectors under this assumption, Theorem 2.17.

3. Integer points in the column space

Being motivated by the description of integer eigenvectors as integer points in the column space of a matrix, we study in this chapter the *integer image problem*. We are concerned with the question of whether, for a given matrix $A \in \overline{\mathbb{R}}^{m \times n}$, there exists an integer vector z in the column space of A, which we will call the *image* of A. We denote

$$Im(A) = \{y \in \overline{\mathbb{R}}^m : (\exists x \in \overline{\mathbb{R}}^n) Ax = y\} \text{ and } IIm(A) = \{z \in \mathbb{Z}^m : (\exists x \in \overline{\mathbb{R}}^n) Ax = z\}.$$

Observe that, if $A \in \overline{\mathbb{R}}^{m \times n}$ has an ε row, then $IIm(A) = \emptyset$, and if A has an ε column then $IIm(A)$=$IIm(A')$ where A' is obtained from A by removing the ε column. Hence it is sufficient to only consider doubly $\mathbb{R}$-astic matrices for the rest of this chapter.

Key results in this chapter include the algorithm INT-IMAGE, and the proof of its complexity for finite input matrices, Theorem 3.11. We give a number of special cases where the existence of an integer image can be determined in strongly polynomial time, including when $m = 2$ or $n = 2$. Additionally we define the class of column typical matrices, and give a full description of the set of integer images for this class in Theorem 3.17.

Finally we briefly consider the equivalent problem of finding an integer point in a max-convex hull, giving some sufficient conditions.

The material in Section 3.1 and Subsections 3.2.1 and 3.2.3 has been published in [24].

3.1 Algorithm to determine if the column space contains an integer vector

We propose the following algorithm, motivated by the Alternating method from [18, 38]:

Algorithm 3.1. INT-IMAGE

Input: $A \in \overline{\mathbb{R}}^{m\times n}$ *doubly* $\mathbb{R}$*-astic, any starting vector* $x^{(0)} \in \mathbb{Z}^m$.

Output: A vector $x \in IIm(A)$ *or indication that no such vector exists.*

(1) $r := 1$.

(2) $z := A^{\#} \otimes' x^{(r-1)}, y := A \otimes z$.

(3) If $y \in \mathbb{Z}^m$ *STOP:* $y \in IIm(A)$.

(4) $x_i^{(r)} := \lfloor y_i \rfloor$ *for all* $i \in M$.

(5) If $x_i^{(r)} < x_i^{(0)}$ *for all* $i \in M$ *STOP: No integer image.*

(6) $r := r + 1$. *Go to (2).*

Observe that all vectors produced by Algorithm INT-IMAGE are finite due to Lemma 1.4 and the fact that $A^{\#} \otimes' u$ is finite if u is finite since $A^{\#}$ is doubly $\mathbb{R}$-astic.

Theorem 3.2. *The doubly* $\mathbb{R}$*-astic input matrix* $A \in \overline{\mathbb{R}}^{m\times n}$ *has an integer image if and only if the sequence* $\{x^{(r)}\}_{r=0,1,\ldots}$ *produced by Algorithm INT-IMAGE finitely converges.*

To prove this theorem on the correctness of the algorithm we first prove a number of claims and we will also need the following two results. The first follows from Corollary 1.10(ii) and the second from Proposition 1.5.

Lemma 3.3. *[37] Assume that* $u \in \mathbb{R}^m$ *is in the image of* $A \in \overline{\mathbb{R}}^{m\times n}$*. Then*

$$A \otimes (A^{\#} \otimes' u) = u.$$

Lemma 3.4. *[37] Let $A \in \overline{\mathbb{R}}^{m\times n}$, $x, y \in \overline{\mathbb{R}}^{m}$. If $x \geq y$, then*

$$A \otimes (A^{\#} \otimes' x) \geq A \otimes (A^{\#} \otimes' y).$$

Claim 3.5. *The sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is nonincreasing.*

Proof. Note that for each $x^{(r)}$ the algorithm attempts to solve $Av = x^{(r)}$ by finding $z = \bar{v} = A^{\#} \otimes' x^{(r)}$ which, by Corollary 1.10, satisfies $Az \leq x^{(r)}$. If we have equality, then the algorithm halts, otherwise the algorithm calculates $x^{(r+1)} = \lfloor Az \rfloor \leq Az \leq x^{(r)}$. □

Claim 3.6. *If A has an integer image, then the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is bounded below by a vector in $IIm(A)$.*

Proof. Assume $u \in IIm(A)$. Then also $\gamma \otimes u \in IIm(A)$ for all $\gamma \in \mathbb{Z}$. Pick γ small enough so that $\gamma \otimes u \leq x^{(0)}$.

Now assume that $x^{(r)} \geq v$ for some $v \in IIm(A)$. Then, using Lemmas 3.3 and 3.4, we have

$$x^{(r+1)} = \lfloor A \otimes (A^{\#} \otimes' x^{(r)}) \rfloor \geq \lfloor A \otimes (A^{\#} \otimes' v) \rfloor = \lfloor v \rfloor = v$$

and thus our claim holds by induction. □

Claim 3.7. *If $x_i^{(r)} < x_i^{(0)}$ for some r and all i, then A has no integer image.*

Proof. If $u \in IIm(A)$, then, by Claims 3.5 and 3.6, the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is nonincreasing and bounded below. But further, from the proof of Claim 3.6, we can see that we can choose $\gamma \in \mathbb{Z}$ such that:

(*i*) $\gamma \otimes u \in IIm(A)$,

(*ii*) $\gamma \otimes u \leq x^{(r)}$ for all r, and

(*iii*) there exists i such that $(\gamma \otimes u)_i = x_i^{(0)}$.

So we have that $x_i^{(0)} = (\gamma \otimes u)_i \leq x_i^{(r)} \leq x_i^{(0)}$. This implies that the i^{th} component of every $x^{(r)}$ is the same, and so there is never an iteration where all components of $x^{(r)}$ properly decrease. □

Proof of Theorem 3.2. If the matrix has an integer image, then the above results imply that $\{x^{(r)}\}_{r=0,1,...}$ is nonincreasing and bounded below by some integer image z of A. Clearly this implies that the sequence $\{x^{(r)}\}_{r=0,1,...}$ will converge. Further, since it is a sequence of integer vectors, at each step at least one component must decrease in value by at least one until, at the latest, it reaches the corresponding value of z, and thus the convergence must be finite.

If instead the sequence finitely converges, then there exists an s such that for all $r \geq s$, $x^{(r)} = x^{(r+1)}$. It follows that $y = A \otimes (A^{\#} \otimes' x^{(s)}) \in \mathbb{Z}^m$. To see this assume not, then there exists a component i of y which is not an integer, and thus $y_i < x_i^{(s)}$. But then $x_i^{(s+1)} = \lfloor y_i \rfloor < x_i^{(s)}$ which is a contradiction.

Thus $y \in IIm(A)$. □

It should be observed that Algorithm INT-IMAGE will always terminate in a finite number of steps. But for finite matrices we can give an explicit bound. In order to analyse the performance of Algorithm INT-IMAGE for finite matrices we will use a pseudonorm on $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$ we define

$$\Delta(x) = \max_{j \in N} x_j - \min_{j \in N} x_j.$$

Lemma 3.8. *[39] For vectors $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ the following hold:*

(i) $\Delta(x \oplus y) \leq \Delta(x) \oplus \Delta(y)$ and

(ii) $\Delta(\alpha \otimes x) = \Delta(x)$.

Proposition 3.9. *Let $y \in \mathbb{R}^m$ be a vector in the image of $A \in \mathbb{R}^{m\times n}$. Then*

$$\Delta(y) \leq \bigoplus_{j=1}^{n} \Delta(A_j).$$

Proof. Since y is in the image of A there exists a vector $x \in \mathbb{R}^n$ such that $y = Ax$. Then, using Lemma 3.8, we have that

$$y = \bigoplus_{j\in N} x_j A_j \Rightarrow \Delta(y) = \Delta\left(\bigoplus_{j\in N} x_j A_j\right) \leq \bigoplus_{j\in N} \Delta(x_j A_j) = \bigoplus_{j=1}^{n} \Delta(A_j).$$

□

Proposition 3.10. *Let $x^{(r)}$, with $r \geq 1$, be a vector calculated in the run of Algorithm INT-IMAGE. Then $\Delta(x^{(r)}) < \bigoplus_{j=1}^{n} \Delta(A_j) + 1$.*

Proof. We know that $x^{(r)} = \lfloor y \rfloor$ where $y \in Im(A)$. So, by Proposition 3.9, we have

$$\Delta(y) \leq \bigoplus_{j=1}^{n} \Delta(A_j).$$

To complete the proof it remains to show that $\Delta(x^{(r)}) < \Delta(y) + 1$. This is true since

$$\Delta(x^{(r)}) - \Delta(y) = \max_{j=1,\ldots,n} \lfloor y_j \rfloor - \min_{j=1,\ldots,n} \lfloor y_j \rfloor - \max_{j=1,\ldots,n} y_j + \min_{j=1,\ldots,n} y_j < 1.$$

□

We can now prove a bound on the runtime of Algorithm INT-IMAGE for finite input matrices.

Theorem 3.11. *For $A \in \mathbb{R}^{m\times n}$ and starting vector $x^{(0)} \in \mathbb{Z}^m$ Algorithm INT-IMAGE* *will terminate after at most*

$$D = (m-1)\left(2\bigoplus_{j=1}^{n}\Delta(A_j)+1\right)+1$$

iterations.

Proof. First suppose that A has an integer image. It follows from Claim 3.7 that there exists an index, k say, such that the algorithm will find an integer image y of A satisfying $y_k = x_k^{(r)}$ for all r.

Let $C = \bigoplus_{j=1}^{n}\Delta(A_j)$. By Proposition 3.9, $\Delta(y) \leq C$. Thus, for all i, $|y_i - y_k| \leq C$. Similarly, using Proposition 3.10, for all i, $|x_i^{(1)} - x_k^{(1)}| < C+1$. But then, since $y_k = x_k^{(1)}$,

$$x_i^{(1)} - y_i < 2C + 1.$$

Now in every iteration where an integer image is not found, we have that there exists at least one index $i \neq k$ such that $x_i^{(r)} - x_i^{(r+1)} \geq 1$. This is since if no change occurred then we would have found an integer image.

There are at most $m-1$ components of $x^{(1)}$ that will decrease in the run of the algorithm and none will decrease by more than $2C+1$. Further in every iteration at least one of these components decreases by at least 1. Thus the maximum number of iteration needed for the algorithm to get from $x^{(1)}$ to y is

$$(m-1)(2C+1),$$

and we need to add one iteration to get from $x^{(0)}$ to $x^{(1)}$.

Now, if the input matrix has no integer image, and after D iterations the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ has not stabilised, then there would have been an iteration where the k^{th} component decreased, and so the algorithm would have halted and concluded that A has no integer image. $\square$

Remark 3.12. *Each iteration requires $\mathcal{O}(mn)$ operations and so by Theorem 3.11 INT-IMAGE is a pseudopolynomial algorithm requiring $\mathcal{O}(Cm^2n)$ operations if applied to finite matrices, where $C = \bigoplus_{j=1}^{n} \Delta(A_j)$.*

Remark 3.13. *Since $|(\tilde{A}_\lambda)_{ij}| \leq n \max |a_{ij}|$, Algorithm INT-IMAGE can be used to determine whether $IV(A) \neq \emptyset$ for irreducible matrices in pseudopolynomial time.*

Example 3.14. *The algorithm INT-IMAGE is not a polynomial algorithm in general. This can be seen by considering the matrix*

$$A = \begin{pmatrix} 12.5 & 7.3 - k & 16.9 \\ 1.8 & 7.3 & -7.2 \\ -2.6 & 0.1 & 0.9 \end{pmatrix}$$

and starting vector $x^{(0)} = (-k, 0, 0)^T$. For any $k \geq 0$ the algorithm first computes $x^{(1)} = (-k, 0, -8)^T$ and then, in each subsequent iteration, either the second entry of $x^{(r)}$ decreases by 1 or the third entry of $x^{(r)}$ decreases by 1 until the algorithm reaches the vector $(-k,\ -k-9,\ -k-16)^T \in IIm(A)$. So the number of iterations is equal to $1 + |-k-9| + |-k-8| + 1 = 2k + 19$.

In the case that $m = 2$ however, it can be shown that the algorithm INT-IMAGE will terminate after at most 2 iterations. In fact a simple necessary and sufficient condition in this case is given by Theorem 3.27 in the next section.

3.2 Strongly polynomially solvable special cases

Here we describe a number of special classes of matrices for which we can describe the integer image set in strongly polynomial time. Throughout this section we assume without loss of generality that A is doubly $\mathbb{R}$-astic

3.2.1 Column typical matrices

It follows from the definitions that $IV(A,0) \subseteq IIm(A)$ for any $A \in \overline{\mathbb{R}}^{n\times n}$. Here we first present some types of matrices for which equality holds, and further show that, in these cases, we can describe the subspaces efficiently.

Let A be a square matrix. Consider a generalised permutation matrix Q, that is, a matrix which is obtained from a diagonal matrix by permuting the rows and/or columns. It is easily seen that $IIm(A) = IIm(A \otimes Q)$. Further, from [18] we know that for every matrix A with $maper(A) > \varepsilon$ there exists a generalised permutation matrix Q such that $A \otimes Q$ is strongly definite and Q can be found in $\mathcal{O}(n^3)$ time. Therefore when considering the integer image of a matrix with $maper(A) > \varepsilon$, we can assume without loss of generality that the matrix is strongly definite.

Definition 3.15. *A matrix $A \in \overline{\mathbb{R}}^{m\times n}$ is called* column typical *if, for each $j \in N$, we have $fr(a_{ij}) \neq fr(a_{tj})$ for any $i,t \in M$ with $i \neq t$ and $a_{ij}, a_{tj} > \varepsilon$.*

Remark 3.16. *From Corollary 2.15, $\lambda(\lceil A\otimes Q\rceil) = 0$ is a sufficient condition for a matrix A with $maper(A) > \varepsilon$ to have an integer image.*

Theorem 3.17. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a column typical matrix.*

(i) If $maper(A) = \varepsilon$, then $IIm(A) = \emptyset$.

(ii) If $maper(A) > \varepsilon$ and $|ap(A)| > 1$, then $IIm(A) = \emptyset$.

(iii) If $maper(A) > \varepsilon$ and $|ap(A)| = 1$ let Q be the unique generalised permutation matrix such that $A \otimes Q$ is strongly definite. Then

$$IIm(A) = IIm(A \otimes Q) = IV(A \otimes Q) = IV^*(A \otimes Q, 0).$$

Proof. First observe that, if A is column typical and $Ax \in IIm(A)$, then no two active elements of A with respect to x can lie in the same column. This is since the vector x_jA_j can have at most one integer entry. Further, it is obvious that there will be one active element per row. We deduce that there exists a permutation $\pi \in P_n$ such that the active elements of A with respect to x are $a_{i,\pi(i)}$ and no others.

(i) Assume $maper(A) = \varepsilon$. Suppose $Ax \in IIm(A)$. Then $a_{i,\pi(i)} + x_{\pi(i)} \in \mathbb{Z}$ for all $i \in N$ which implies that $a_{i,\pi(i)} \neq \varepsilon$ for all i which is a contradiction.

(ii) Assume $maper(A) > \varepsilon$. Suppose $Ax = z \in IIm(A)$.

Let $\sigma \in P_n$ be different from π. Then

$$\sum_{i=1}^{n} a_{i,\pi(i)} + x_{\pi(i)} > \sum_{i=1}^{n} a_{i,\sigma(i)} + x_{\sigma(i)}. \tag{3.1}$$

To see this note that not all $a_{i,\sigma(i)}$ can be active since there exist $i, k \in N$ with $i \neq k$ such that $\pi(i) = \sigma(k)$. Therefore, if $a_{k,\sigma(k)}$ was active, then $fr(a_{k,\sigma(k)}) = fr(a_{i,\pi(i)})$, which does not happen. Hence we have that

$$a_{i,\sigma(i)} + x_{\sigma(i)} \leq \max_j a_{ij} + x_j = a_{i,\pi(i)} + x_{\pi(i)}$$

for all $i \in N$ and there is at least one i for which equality does not hold.

Finally, from (3.1),

$$\sum_{i=1}^{n} a_{i,\pi(i)} > \sum_{i=1}^{n} a_{i,\sigma(i)}$$

and so $ap(A) = \{\pi\}$.

(iii) Assume $maper(A) > \varepsilon$ and $|ap(A)| = 1$. Let $B = A \otimes Q$. Since B is strongly definite,

$$IV^*(B, 0) = IV(B) \subseteq IIm(B),$$

so it is sufficient to prove that $IIm(B) \subseteq IV(B)$.

Suppose $z \in IIm(B)$. Then there exists $x \in \overline{\mathbb{R}}^n$ such that $Bx = z$ and the only active elements of B with respect to x are $b_{i,\pi(i)}$. Further from the proof of (ii) we see that π is a permutation of maximum weight with respect to B meaning $\pi = id$.

We conclude that $z_i = \max_j(b_{ij} + x_j) = b_{ii} + x_i = x_i$ for all $i \in N$ and therefore $z \in IV(B)$. $\square$

Using Corollary 2.15 we deduce the following.

Corollary 3.18. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is column typical, then the question of whether or not A has an integer image can be solved in strongly polynomial time.*

Above we saw that, if the entries in each column of a strongly definite matrix had different fractional parts, then only the integer (diagonal) entries were active. So we now consider strongly definite matrices for which the only integer entries are on the diagonal to see if the results can be generalised to this class of matrices.

Definition 3.19. *A strongly definite matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is* nearly non-integer *(NNI) if the only integer entries appear on the diagonal.*

Lemma 3.20. *Let $A \in \overline{\mathbb{R}}^{n\times n}$, $n \geq 3$, be strongly definite and NNI. Then there is no x satisfying $Ax = z \in \mathbb{Z}^n$ such that a_{ij} with $i \neq j$ is active.*

Proof. Let A be a strongly definite, NNI matrix. Suppose that there exists a vector x, satisfying $Ax \in IIm(A)$, such that there exists a row $k_1 \in N$ with an off diagonal entry active.

So $\exists k_2 \in N,\ k_2 \neq k_1$ such that a_{k_1,k_2} is active. Then

$$a_{k_1,k_2} + x_{k_2} \geq a_{k_1,k_1} + x_{k_1} = x_{k_1}. \tag{3.2}$$

There is an active element in every row so consider row k_2. Then a_{k_2,k_2} is inactive because $fr(x_{k_2}) = fr(-a_{k_1,k_2}) > 0$ due to Lemma 1.17(x) and the fact that $a_{k_1k_2} \notin \mathbb{Z}$ but $x_{k2} + a_{k_1k_2} \in \mathbb{Z}$. This means $a_{k_2,k_2} + x_{k_2} \notin \mathbb{Z}$. Further a_{k_2,k_1} is inactive since, otherwise, $a_{k_2,k_1} + x_{k_1} > a_{k_2,k_2} + x_{k_2} = x_{k_2}$ which, together with (3.2), would imply that the cycle (k_1, k_2) has strictly positive weight. This contradicts the definiteness of A.

Thus $\exists k_3 \in N,\ k_3 \neq k_1, k_2$, such that a_{k_2,k_3} is active and, similarly as before,

$$a_{k_2,k_3} + x_{k_3} > a_{k_2,k_2} + x_{k_2} = x_{k_2}. \tag{3.3}$$

Consider row k_3. Again it can be seen that both a_{k_3,k_3} and a_{k_3,k_2} are inactive. Further we show that a_{k_3,k_1} is inactive. If it was active then we would have $a_{k_3,k_1} + x_{k_3} > x_{k_1}$ which, together with (3.2) and (3.3), would imply that cycle (k_1, k_2, k_3) has strictly positive weight, a contradiction.

Thus $\exists k_4 \in N,\ k_4 \neq k_1, k_2, k_3$ such that a_{k_3,k_4} is active.

Continuing in this way we see that,

$$(\forall i \in N)(\forall j \in \{1, 2, ..., i\})\ a_{k_i,k_j} \text{ is inactive.}$$

But this means that no element in row k_n can be active, a contradiction. □

Theorem 3.21. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a strongly definite, NNI matrix. Then*

$$IIm(A) = IV(A) = IV^*(A, 0).$$

Proof. If $n = 2$ then A is column typical and the statement follows from Theorem 3.17. Hence we assume $n \geq 3$.

$IV(A) \subseteq IIm(A)$ holds trivially. To prove the converse let $A \in \overline{\mathbb{R}}^{n\times n}$, $n \geq 3$, be strongly definite and NNI. Then, by Lemma 3.20, there is no x satisfying $Ax = z \in \mathbb{Z}^n$ such that a_{ij} with $i \neq j$ is active. Thus only the diagonal elements can be active. Hence for any $z \in IIm(A)$ we have $Ax = z$ for some x with $a_{ii} = 0$ active for all $i \in N$. Therefore $x = z$ and so $z \in IV(A)$. □

Remark 3.22. *Obviously, if the matrix A is strongly definite, then A NNI would imply that A is column typical, and therefore Theorem 3.17(iii) follows from Theorem 3.21.*

However, to obtain the full classification of the integer image space of any *column typical matrix, we do not assume initially that the matrix is strongly definite.*

Extensions to matrix powers

We briefly consider the integer image of powers of square, column typical and NNI matrices.

Observe that, since $A^2x = z \Rightarrow A(Ax) = z$, the following result holds.

Lemma 3.23. *If A is strongly definite then*

$$IIm(A) \supseteq IIm(A^2) \supseteq ... \supseteq IIm(A^{n-1}) = IIm(A^n) = IIm(A^+) = IIm(A^*).$$

As a consequence of Theorem 3.17 we have that:

Proposition 3.24. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be strongly definite and column typical.*

(i) If $maper(A) = \varepsilon$, then $IIm(A^t) = \emptyset$ for all $t \in \mathbb{N}$.

(ii) If $maper(A) > \varepsilon$ and $|ap(A)| > 1$, then $IIm(A^t) = \emptyset$ for all $t \in \mathbb{N}$.

(iii) If $maper(A) > \varepsilon$ and $|ap(A)| = 1$ then, for all $t \in \mathbb{N}$,

$$IV(A) = IIm(A) = IIm(A^t).$$

Proof.

(i) and (ii): $IIm(A) = \emptyset \Rightarrow IIm(A^k) = \emptyset$.

(iii) Using Lemma 3.23, $\forall t \in N$,

$$IV^*(A,0) = IV(A) = IIm(A) \supseteq IIm(A^t) \supseteq IIm(A^*) = IV^*(A,0)$$

□

Further, as a result of Theorem 3.21:

Proposition 3.25. *Let* $A \in \overline{\mathbb{R}}^{n\times n}$ *be strongly definite and NNI. Then, for all* $t \in \mathbb{N}$*,*

$$IV(A) = IIm(A) = IIm(A^t).$$

3.2.2 Upper and lower triangular matrices

We say that A is *upper triangular* if $a_{ij} = \varepsilon$ whenever $i > j$, and *lower triangular* if $a_{ij} = \varepsilon$ whenever $i < j$. We show that, for matrices of this type with finite diagonal, an integer image always exists, and describe a method to find a single integer image. The description of all integer images remains open.

We will discuss upper triangular matrices only, the results for lower triangular matrices follow similar ideas.

Proposition 3.26. *Let* $A \in \overline{\mathbb{R}}^{n\times n}$ *be upper triangular with finite diagonal. Then* $IIm(A) \neq \emptyset$*.*

Proof. By induction on n.

If $n = 1$, then $Aa_{11}^{-1} = 0 \in \mathbb{Z}$.

So assume that $n > 1$ and the result holds for smaller matrices.

Let $A' = A[N - \{1\}]$ and note that A' is upper triangular with finite diagonal. Thus, by induction hypothesis, there exists $x' \in \overline{\mathbb{R}}^{n-1}$ such that $A'x' \in \mathbb{Z}^{n-1}$. Now let α be the smallest integer such that

$$\alpha \geq \begin{pmatrix} a_{12} & ... & a_{1n} \end{pmatrix} x'.$$

Then setting $x = (\alpha a_{11}^{-1}, x'^T)^T$ gives $Ax \in \mathbb{Z}^n$ as required. □

3.2.3 When $m = 2$ or $n = 2$

We now show that, if either m or n is equal to 2, we can straightforwardly decide whether $IIm(A) = \emptyset$.

Theorem 3.27. *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{2\times n}$ be doubly $\mathbb{R}$-astic, and $d_j := a_{1j} - a_{2j}$ for all $j \in N$.*

(i) If any d_j is an integer, then A has an integer image.

(ii) If no d_j is integer, then A has an integer image if and only if

$$(\exists i, j \in N)\lfloor d_i \rfloor \neq \lfloor d_j \rfloor.$$

Proof. (i) Without loss of generality assume $d_1 \in \mathbb{Z}$. Then

$$A \otimes (-a_{11}, \varepsilon, ..., \varepsilon)^T = (0, -d_1)^T \in \mathbb{Z}^2.$$

(ii) Assume without loss of generality that $\lfloor d_1 \rfloor \neq \lfloor d_2 \rfloor$, $d_1 < d_2$ and that $d_1, d_2 \notin \mathbb{Z}$.

Case 1 $d_1, d_2 \in \mathbb{R}$.

Let $d = \lceil d_1 \rceil$ so that $a_{21} + d > a_{11}$ and $a_{22} + d < a_{12}$. Then

$$A \otimes (-a_{21} - d,\ -a_{12},\ \varepsilon, ..., \varepsilon)^T = (0, -d)^T \in \mathbb{Z}^2.$$

Case 2 $d_1 \in \mathbb{R}$, $d_2 = +\infty$.

Then $a_{22} = \varepsilon$ and, for $t \in \mathbb{Z}$ big enough,

$$A \otimes (-fr(a_{21}),\ -fr(a_{12}) + t,\ \varepsilon, ..., \varepsilon) = (\lfloor a_{12} \rfloor + t, \lfloor a_{21} \rfloor)^T \in \mathbb{Z}^2.$$

Case 3 $d_1 = -\infty$, $d_2 \in \mathbb{R}$.

Then $a_{11} = \varepsilon$ and, for $t \in \mathbb{Z}$ big enough,

$$A \otimes (-fr(a_{21}) + t,\ -fr(a_{12}),\ \varepsilon, ..., \varepsilon) = (\lfloor a_{12} \rfloor, \lfloor a_{21} \rfloor + t)^T \in \mathbb{Z}^2.$$

Case 4 $d_1 = -\infty$, $d_2 = +\infty$.

Here $a_{11} = a_{22} = \varepsilon$ and

$$A \otimes (-fr(a_{21}),\ -fr(a_{12}),\ \varepsilon, ..., \varepsilon)^T = (\lfloor a_{12} \rfloor, \lfloor a_{21} \rfloor)^T \in \mathbb{Z}^2.$$

For the other direction, assume that $d = \lfloor d_j \rfloor < d_j$ for all $j \in N$ and suppose, for a contradiction, that there exists $x \in \overline{\mathbb{R}}^n$ such that $Ax = b \in \mathbb{Z}^2$. Without loss of generality we may assume $b = (0, b')^T$ for some $b' \in \mathbb{Z}$.

If $-b' \leq d$, then

$$\begin{aligned}
&(\forall j \in N)\ -b' < d_j = a_{1j} - a_{2j} \\
\therefore &(\forall j \in N)\ a_{1j} > a_{2j} - b' \\
\therefore &(\forall j \in N)\ M_j(A, b) = \{1\} \\
\therefore &\bigcup_{j \in N} M_j(A, b) = \{1\}.
\end{aligned}$$

Thus, by Proposition 2.1, no such x exists.

If instead $-b' > d$, then, since $b' \in \mathbb{Z}$, we have $b' \geq \lfloor d_i \rfloor + 1 > d_i$. Then, similarly as above, $M_j(A, b) = \{2\}$ for all j and we conclude that no such x exists. $\square$

Note that, if $d_i < d_j$, the condition $\lfloor d_i \rfloor \neq \lfloor d_j \rfloor$ means that

$$(\exists z \in \mathbb{Z})\ z \in [d_i, d_j].$$

So an equivalent condition for a finite matrix A to have an integer image is as follows.

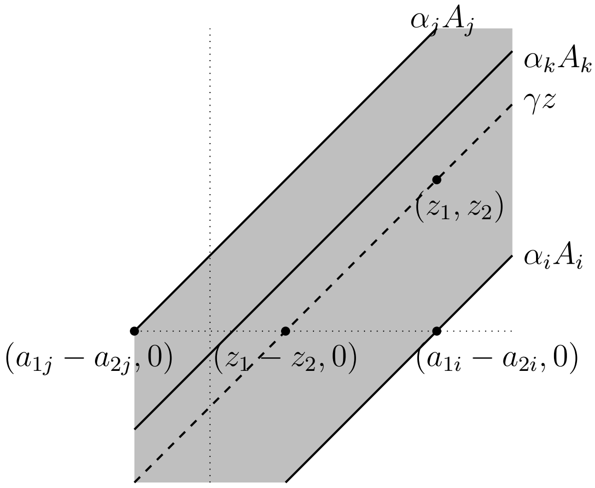


Figure 3.1: Graphical representation for a finite $2 \times n$ matrix to have an integer image.

Proposition 3.28. *$A \in \mathbb{R}^{2\times n}$ has an integer image if and only if that there exists an integer between $\min_j a_{1j} - a_{2j}$ and $\max_j a_{1j} - a_{2j}$.*

We represent this condition graphically in Figure 3.1. In Figure 3.1 the solid lines represent points in $Im(A)$ that are multiples of a single column and the shaded area represents all the points in $Im(A)$. If there exists $z \in IIm(A)$, then also $(z_1 - z_2, 0)^T \in IIm(A)$ and the x-coordinate satisfies, for some i and j,

$$z_1 - z_2 \in [a_{1j} - a_{2j},\ a_{1i} - a_{2i}].$$

Now we deal with matrices for which $n = 2$. It should be noted that these results were also independently discovered in [61]. We start with a lemma whose proof is straightforward.

Lemma 3.29. *Suppose $A \in \overline{\mathbb{R}}^{m\times 2}$.*

(i) If $\exists j \in \{1, 2\}$ such that $(\forall i, t \in M) fr(a_{ij}) = fr(a_{tj})$, then $IIm(A) \neq \emptyset$.

(ii) If $\exists\gamma \in \mathbb{R}$ *such that* $A_1 = \gamma A_2$*, then* $IIm(A) \neq \emptyset$ *if and only if*

$$(\exists j \in \{1,2\})(\forall i,t \in M) fr(a_{ij}) = fr(a_{tj}).$$

Theorem 3.30. *Suppose* $A \in \overline{\mathbb{R}}^{m\times 2}$ *is a doubly* $\mathbb{R}$*-astic matrix not satisfying the conditions in Lemma 3.29. Let* l, r *be the indices such that*

$$a_{l2} - a_{l1} = \min_{i\in M} a_{i2} - a_{i1} \text{ and } a_{r2} - a_{r1} = \max_{i\in M} a_{i2} - a_{i1}.$$

Let

$$\bar{L} = \{i \in M : fr(a_{i1}) = fr(a_{l1})\},$$

$$\bar{R} = \{i \in M : fr(a_{i2}) = fr(a_{r2})\},$$

$L = \bar{L} - \bar{R}$ *and* $R = \bar{R} - \bar{L}$*. Denote* $fr(-a_{l1}) - fr(-a_{r2})$ *by* f*. Then*

(i) If $\bar{L} \cup \bar{R} \neq M$ *then* $IIm(A) = \emptyset$*.*

(ii) Otherwise $IIm(A) \neq \emptyset$ *if and only if*

$$\left\lfloor \min_{i\in L}(a_{i1} - a_{i2}) + f \right\rfloor - \left\lceil \max_{i\in R}(a_{i1} - a_{i2}) + f \right\rceil \geq 0.$$

Proof. We first prove that $fr(x_1) = fr(-a_{l1})$ and $fr(x_2) = fr(-a_{r2})$ for any x satisfying $Ax \in IIm(A)$. We do this by showing that both a_{l1} and a_{r2} are active for any such x and applying Lemma 1.17(x) to the fact that $x_1 + a_{l1}, x_2 + a_{r2} \in \mathbb{Z}$.

Assume for a contradiction that $Ax \in IIm(A)$ but a_{l1} is not active. Then we have that $a_{l1} + x_1 < a_{l2} + x_2 \in \mathbb{Z}$ and therefore

$$x_1 - x_2 < a_{l2} - a_{l1} = \min_{i\in M} a_{i2} - a_{i1}.$$

Moreover there must be an active entry in the first column of A, so $\exists t \in M$ such that $a_{t1} + x_1 \geq a_{t2} + x_2$, equivalently $x_1 - x_2 \geq a_{t2} - a_{t1}$, a contradiction. A similar argument works for a_{r2}.

(i) This is now easily seen to be true since for any x with $fr(x_1) = fr(-a_{l1})$ and $fr(x_2) = fr(-a_{r2})$ there will be at least one index $i \in M$ such that $(Ax)_i \notin \mathbb{Z}$.

(ii) We have already proved that, for any x such that $Ax \in IIm(A)$, it is guaranteed that $fr(x_1) = fr(-a_{l1})$ and $fr(x_2) = fr(-a_{r2})$. Therefore, for any candidate x with these fractional parts, the set $\bar{L} \cap \bar{R}$ contains all the row indices for which we can guarantee that $(Ax)_i \in \mathbb{Z}$, since both $a_{i1}x_1$ and $a_{i2}x_2$ will be integer under our assumptions. We construct a matrix A' from A by removing all rows with indices in $\bar{L} \cap \bar{R}$. We also define sets L' and R' to be the sets of row indices in A' that correspond to the sets L and R respectively. Observe that

$$IIm(A) \neq \emptyset \text{ if and only if } IIm(A') \neq \emptyset.$$

Further

$$\{x \in \mathbb{R}^2 : A \otimes x \in IIm(A)\} = \{x \in \mathbb{R}^2 : A' \otimes x \in IIm(A')\} := X.$$

Since any $x \in X$ has the form

$$\begin{pmatrix} \gamma_1 + fr(-a_{l1}) \\ \gamma_2 + fr(-a_{r2}) \end{pmatrix}$$

for some $\gamma_1, \gamma_2 \in \mathbb{Z}$ we can decide whether $IIm(A') \neq \emptyset$ by determining whether there exists $\alpha \in \mathbb{Z}$ such that

$$x = \begin{pmatrix} fr(-a_{l1}) \\ \alpha + fr(-a_{r2}) \end{pmatrix} \in X.$$

The set L' (R') is exactly the set of row indices i for which a'_{i1} (a'_{i2}) is active for any $x \in X$. So such an α exists if and only if the following sets of inequalities can be satisfied.

$$\begin{cases} (\forall i \in L')a_{i1} + x_1 > a_{i2} + x_2 \\ (\forall i \in R')a_{i2} + x_2 > a_{i1} + x_1 \end{cases}$$
$$\Leftrightarrow \begin{cases} (\forall i \in L')a_{i1} + fr(-a_{l1}) > a_{i2} + fr(-a_{r2}) + \alpha \\ (\forall i \in R')a_{i2} + fr(-a_{r2}) + \alpha > a_{i1} + fr(-a_{l1}) \end{cases}$$
$$\Leftrightarrow \max_{i \in R'} \left(a_{i1} - a_{i2} + f \right) < \alpha < \min_{i \in L'} \left(a_{i1} - a_{i2} + f \right).$$

Therefore $IIm(A') \neq \emptyset$ if and only if there exists an integer

$$\alpha \in \left[\left\lceil \max_{i \in R'}(a_{i1} - a_{i2}) + f \right\rceil, \left\lfloor \min_{i \in L'}(a_{i1} - a_{i2}) + f \right\rfloor \right].$$

□

Remark 3.31. *Note that the proof tells us how to describe all integer images of the matrix* $A \in \overline{\mathbb{R}}^{m \times 2}$, *since we can easily describe all* α *such that*

$$\begin{pmatrix} -fr(a_{l1}) \\ \alpha - fr(a_{r2}) \end{pmatrix} \in X.$$

3.3 Integer image and max-convex hulls: a graphical interpretation

In Proposition 3.28 and Figure 3.1 we viewed the integer image problem for a finite $2 \times n$ matrix as the problem of finding an integer point in an interval. As a consequence of

Proposition 3.28, we can describe the entire set of integer images for $2 \times n$ matrices as shown below.

Corollary 3.32. *Let $A \in \mathbb{R}^{2\times n}$ and suppose $IIm(A) \neq \emptyset$. Let z_U and z_L be the largest and smallest integers respectively contained in the interval*

$$\left[\min_j(a_{1j}^{-1} \otimes a_{2j}), \max_j(a_{1j}^{-1} \otimes a_{2j})\right].$$

Then the set of integer images of A is equal to

$$\left\{\alpha \begin{pmatrix} 0 \\ z_L \end{pmatrix} \oplus \beta \begin{pmatrix} 0 \\ z_U \end{pmatrix} : \alpha, \beta \in \mathbb{Z}\right\}.$$

In this section we study these ideas for general matrices, and show that the integer image problem links to the problem of finding integer vectors in a max-algebraic convex hull.

3.3.1 Necessary conditions using intervals

Let $A \in \mathbb{R}^{m\times n}$. Recall $Im(A) = \left\{y \in \mathbb{R}^m : (\exists x \in \overline{\mathbb{R}}^n)Ax = y\right\}$. Now $Im(A)$ is a subspace and hence the following result is trivial.

Lemma 3.33. *Let $A \in \mathbb{R}^{m\times n}$ where $m \geq 2$. Then $x \in Im(A) \Leftrightarrow x_1^{-1}x \in Im(A)$.*

The above lemma allows us to assume without loss of generality that $z \in IIm(A)$ has $z_1 = 0$.

Theorem 3.34. *Let $m \geq 2$. If an $m \times n$ matrix A has an integer image then, for every $z \in IIm(A)$ with $z_1 = 0$,*

$$(\forall t \in M - \{1\}) z_t \in \left[\min_{j=1,\dots,n}(a_{1j}^{-1} \otimes a_{tj}), \max_{j=1,\dots,n}(a_{1j}^{-1} \otimes a_{tj})\right].$$

Proof. Assume A has an integer image. Thus there exists a vector $x = (x_1, x_2, ..., x_m)^T \in Im(A)$ where $x_i \in \mathbb{Z}, i = 1, ..., m$.

By Lemma 3.33, $(0, z_2, ...z_m)^T \in Im(A)$ where $(\forall t \in M - \{1\})\ z_t = x_t - x_1 \in \mathbb{Z}$.

We will show that

$$(\forall t \in M - \{1\})\ \min_j(a_{1j}^{-1} \otimes a_{tj}) \leq z_t \leq \max_j(a_{1j}^{-1} \otimes a_{tj}). \tag{3.4}$$

For the upper bound note that $0 = \max_j(\alpha_j \otimes a_{1j})$. This means that $\alpha_j \leq a_{1j}^{-1}$ for all j. Also,

$$z_t = \max_j(\alpha_j \otimes a_{tj}) \leq \max_j(a_{1j}^{-1} \otimes a_{tj})$$

by our bounds on α_j.

For the lower bound in (3.4) assume that $z_t < \min_j(a_{1j}^{-1} \otimes a_{tj})$. Since $(0, z_2, ..., z_m)^T \in Im(A)$ we have $z_t = \max_j(\alpha_j \otimes a_{tj})$. Thus

$$\max_j(\alpha_j \otimes a_{tj}) < \min_j(a_{1j}^{-1} \otimes a_{tj}).$$

Which implies,

$$\alpha_j \otimes a_{tj} < a_{1j}^{-1} \otimes a_{tj},$$

$$\therefore \alpha_j < a_{1j}^{-1}.$$

But then $\max(\alpha_j \otimes a_{1j}) < 0$ which contradicts the fact that $(0, z_2, ..., z_m)^T \in Im(A)$. $\square$

Corollary 3.35. *Let $A \in \mathbb{R}^{m \times n}$. If there exists $t \in M - \{1\}$ such that*

$$\left[\min_{j=1,...,n}(a_{1j}^{-1} \otimes a_{tj}), \max_{j=1,...,n}(a_{1j}^{-1} \otimes a_{tj})\right] \cap \mathbb{Z} = \emptyset,$$

then A has no integer image.

We can't say if and only if in Theorem 3.34, as shown by the following example.

Example 3.36. *Let* $A = \begin{pmatrix} 1.4 & 0.8 \\ 2.7 & 5.6 \\ 3.1 & 3.3 \end{pmatrix}.$

Then $2, 3, 4 \in [2.7-1.4,\ 5.6-0.8] = [1.3,\ 4.8]$ *and* $2 \in [3.1-1.4,\ 3.3-0.8] = [1.7,\ 2.5]$ *but A has no integer image because there is no choice of* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\alpha \otimes (1.4, 2.7, 3.1)^T \oplus \beta \otimes (0.8, 5.6, 3.3)^T \in \mathbb{Z}.$$

This is since, with only two multipliers and a column typical matrix, we can set at most two entries to be integers.

Remark 3.37. *Note that if* $m = 1$ *then the matrix always has an integer image, in fact every integer is an image of it. So we must assume that* $m \geq 2$ *in the theorem.*

This gives an obvious (but inefficient) idea for an algorithm to determine whether a given matrix A of size $m \times n$ has an integer image. The algorithm would simply test every set of integer points which are contained within the given intervals to see if they are in the image space of A.

This would involve checking each integer vector of the form, $(z_2, ...z_m)$ with

$$z_t \in \left[\min_j (a_{1j}^{-1} \otimes a_{tj}), \max_j (a_{1j}^{-1} \otimes a_{tj}) \right].$$

There are at most

$$\prod_{t=2}^{m} \left(\lfloor \max_j (a_{1j}^{-1} \otimes a_{tj}) \rfloor - \lfloor \min_j (a_{1j}^{-1} \otimes a_{tj}) \rfloor + 1 \right)$$

such vectors, where the plus 1 is due to the possibility that $\min_j (a_{1j}^{-1} \otimes a_{tj})$ is an integer.

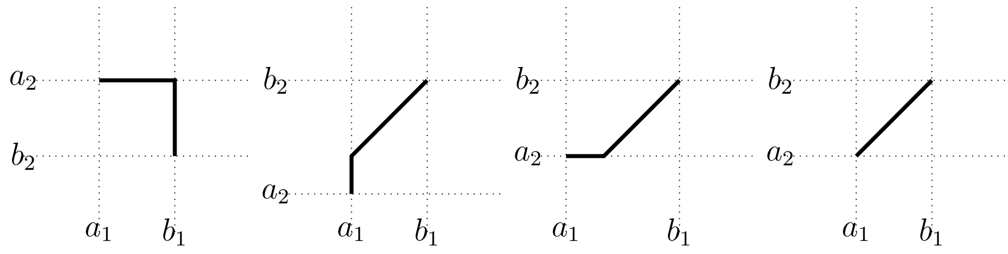


Figure 3.2: Max-algebraic line segments between $(a_1, a_2)^T$ and $(b_1, b_2)^T$

3.3.2 The column space is equal to a max-convex hull

We will move away from considering intervals and consider what can be achieved from considering graphs as in Figure 3.1.

If we use the analogue of the conventional definitions of convexity then we get the following definitions.

A set S is max-convex if, for any two points in S, the max-algebraic line segment between the points is also in S. The max-convex hull of the vectors $x_1, ..., x_n \in \overline{\mathbb{R}}^n$ is the set

$$\left\{ \bigoplus_{i=1}^{n} \alpha_i x_i : \bigoplus_{i=1}^{n} \alpha_i = 0 \right\}.$$

Note that the types of max-algebraic line segments are given in Figure 3.2.

The following result is known, and trivial since $Im(A)$ is a subspace.

Lemma 3.38. *Given a matrix $A \in \mathbb{R}^{m\times n}$ let*

$$B = (a_{11}^{-1} \otimes A_1, a_{12}^{-1} \otimes A_2, ..., a_{1n}^{-1} \otimes A_n) = (b_{ij}) \in \mathbb{R}^{m\times n}.$$

Then $Im(A) = Im(B)$.

Proposition 3.39. *$A \in \mathbb{R}^{m\times n}$ has an integer image if and only if there exist $\alpha_1, ..., \alpha_n \in \overline{\mathbb{R}}$*

with $\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n = 0$ *such that*

$$\bigoplus_{j=1}^{n} \alpha_j \otimes a_{1j}^{-1} \otimes \begin{pmatrix} a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{Z}^{m-1}. \tag{3.5}$$

Remark: Note that this area is the max-algebraic convex hull of the vectors

$$(a_{2j} \otimes a_{1j}^{-1}, a_{3j} \otimes a_{1j}^{-1}, ..., a_{mj} \otimes a_{1j}^{-1})^T, \quad j = 1, ..., n.$$

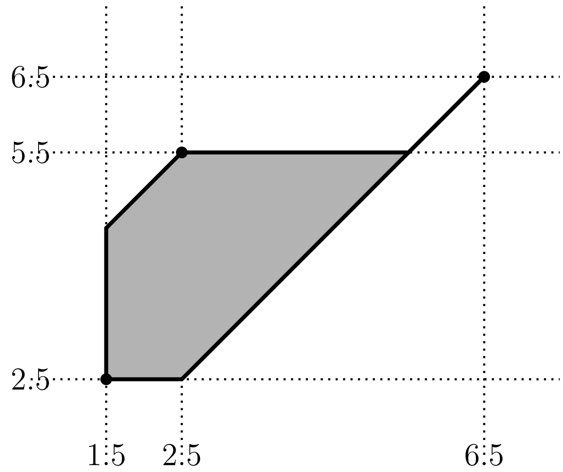
Proof. First assume that there exists $x \in \mathbb{Z}^{m-1}$ of the form in (3.5). We claim $(0, x^T)^T \in IIm(A)$. Indeed

$$\bigoplus_{j=1^n} \alpha_j \otimes a_{1j}^{-1} \otimes A_j = \begin{pmatrix} 0 \\ x \end{pmatrix} \in \mathbb{Z}^m.$$

For the other direction we now assume that A has an integer image. Then, by Lemma 3.38, $B = (a_{11}^{-1} \otimes A_1, a_{12}^{-1} \otimes A_2, ..., a_{1n}^{-1} \otimes A_n) = (b_{ij}) \in \mathbb{R}^{m \times n}$ has an integer image. Suppose $x = (x_j) \in IIm(B)$. By Lemma 3.33, $x_1 \otimes x \in IIm(B)$. Thus

$$\begin{pmatrix} 0 \\ x_2 - x_1 \\ \vdots \\ x_m - x_1 \end{pmatrix} = \alpha_1 \otimes \begin{pmatrix} 0 \\ b_{21} \\ \vdots \\ b_{m1} \end{pmatrix} \oplus \alpha_2 \otimes \begin{pmatrix} 0 \\ b_{22} \\ \vdots \\ b_{m2} \end{pmatrix} \oplus, ..., \oplus \alpha_n \otimes \begin{pmatrix} 0 \\ b_{2n} \\ \vdots \\ b_{mn} \end{pmatrix}$$

for some choice of $\alpha \in \bar{\mathbb{R}}$. Note that this immediately implies that $\alpha_1 \oplus \alpha_2 \oplus, ..., \oplus \alpha_n = 0$. Further, rows 2 to n are exactly (3.5). □



The black points are those obtained by setting two of the coefficients to ε (or to small enough real numbers that they don't influence the solution), the solid lines are those points obtained by setting one of the coefficients to ε. The shaded region is the points obtained when all the coefficients play a role.

Figure 3.3: The convex hull from Example 3.40.

Example 3.40. *Let*

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1.5 & 2.5 & 6.5 \\ 2.5 & 5.5 & 6.5 \end{pmatrix}.$$

By Proposition 3.39, B has an integer image if and only if there exists an integer vector in the max-algebraic convex hull of the points $(1.5, 2.5)^T$, $(2.5, 5.5)^T$, $(6.5, 6.5)^T$. Graphically this is shown in Figure 3.3.

The integer points in the convex hull are: $(2,3)^T, (2,4)^T, (2,5)^T, (3,3)^T, (3,4)^T, (3,5)^T, (4,4)^T, (4,5)^T, (5,5)^T, (6,6)^T$. Thus any point $(0, y, z)^T$ where $(y, z)^T$ is some pair above is an integer point in the image of B, as are all integer multiples of these points.

Corollary 3.41. *Consider a matrix $A \in \mathbb{R}^{m \times n}$. Let C be the max-algebraic convex hull of the points $a_{1j}^{-1}(a_{2j}, ..., a_{mj}), \forall j \in N$. Then,*

$$IIm(A) = \left\{ \gamma \otimes \begin{pmatrix} 0 \\ y \end{pmatrix} : \gamma \in \mathbb{Z} \text{ and } y \in C \cap \mathbb{Z}^{m-1} \right\}.$$

3.3.3 When does the max-convex hull contain an integer point?

We first consider the 3×3 case and assume without loss of generality that

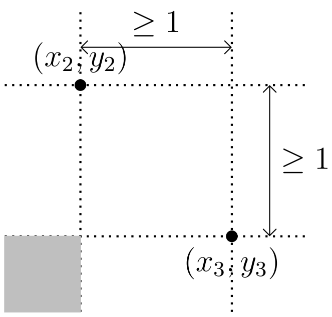
$$A = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \tag{3.6}$$

and $x_1 \leq x_2 \leq x_3$. So, from Lemma 3.39 and Corollary 3.41, $(x, y, z)^T \in IIm(A)$ exactly when $(0, y - x, z - x)^T$ are in the max-convex hull of

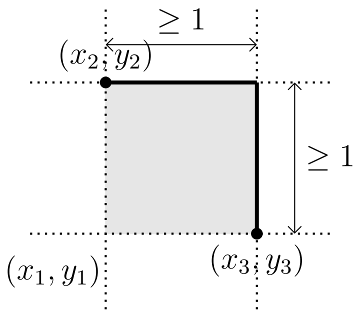
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \tag{3.7}$$

Consider the case when $y_1 \leq y_3 \leq y_2$.

We claim that if $x_3 - x_2 \geq 1$ and $y_2 - y_3 \geq 1$ then there exists an integer point in the max-convex hull.



Here we have placed the points $(x_2, y_2), (x_3, y_3)$ a distance of at least 1 apart as required. The shaded region here is the region in which the point (x_1, y_1) can be located.



Then we know that the bold lines seen here will form part of the boundary of the max-convex hull of the three points, and further that the max line between (x_2, y_2) and (x_1, x_1) will not lie inside the shaded square in this picture, similarly for the max line between (x_3, y_3) and (x_1, x_1)

Thus in this case the max-convex hull of the three points will contain a square of dimension at least 1×1 and this square must contain an integer point.

For other orderings of y_1, y_2, y_3 (when y_3 is not the maximum) we can also find, in a similar way, sufficient conditions for when the max-convex hull will contain a square and thus an integer vector.

We develop sufficient conditions for a matrix to have an integer image based on when the max-convex hull of the vectors in (3.7) contains an integer point.

Proposition 3.42. *Let $A \in \mathbb{R}^{3\times 3}$ have the form in* (3.6) *with $x_1 \leq x_2 \leq x_3$. Then each of the following is a sufficient condition for the max-convex hull to contain an integer point.*

(i) $y_1 \leq y_3 \leq y_2$, $x_3 - x_2 \geq 1$ and $y_2 - y_3 \geq 1$.

(ii) $y_2 \leq y_3 \leq y_1$, $x_3 - x_2 \geq 1$ and $y_1 - y_3 \geq 1$.

(iii) $y_3 \leq y_1 \leq y_2$, $x_3 - x_2 \geq 1$ and $y_2 - y_1 \geq 1$.

(iv) $y_3 \leq y_2 \leq y_1$, $x_3 - x_2 \geq 1$ and $y_1 - y_2 \geq 1$.

Proof. Assume that (i) holds. Let $\alpha_1 = 0, \alpha_2 = -fr(y_2), \alpha_3 = -fr(x_3)$. Then $\alpha_1 \oplus \alpha_2 \oplus$

$\alpha_3 = 0$ and

$$\alpha_1 \otimes \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \alpha_2 \otimes \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \alpha_3 \otimes \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 - fr(y_2) \\ \lfloor y_2 \rfloor \end{pmatrix} \oplus \begin{pmatrix} \lfloor x_3 \rfloor \\ y_3 - fr(x_3) \end{pmatrix}$$
$$= \begin{pmatrix} \lfloor x_3 \rfloor \\ \lfloor y_2 \rfloor \end{pmatrix} \in \mathbb{Z}^2.$$

Thus the max-convex hull contains an integer point.

Similar arguments hold for the other cases.

Finally observe that each of the conditions (i)-(iv) guarantee that the max-convex hull contains a square of dimension 1×1, and hence an integer point. □

We can now generalize this idea to matrices of any size.

Proposition 3.43. *Let $A \in \mathbb{R}^{m\times n}$ be of the form*

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

where $a_{21} \leq a_{22} \leq \dots \leq a_{2n}$. For any $i \in M\backslash\{1\}$ let j_i be the index such that $a_{ij_i} = \max_j a_{ij}$. If

(i) $j_p \neq j_q$ for any $p \neq q \in M\backslash\{1\}$, and

(ii) For all $i \in M\backslash\{1\}$, $a_{ij_i} - fr(a_{ij_i}) \geq a_{it},\ t \neq j_i$.

Then there exists an integer point in the max-convex hull.

Proof. Let $J = \{j_i : i \in M\backslash\{1\}\}$.

For $i \in M\backslash\{1\}$ let $\alpha_{j_i} = -fr(a_{ij_i})$ and set all other $\alpha_j = 0$. Note that this can be done without conflict since by assumption each j_i is unique.

Then $\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n = 0$ and

$$\bigoplus_{j=1}^{n} \alpha_j \otimes \begin{pmatrix} a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix} = \bigoplus_{j_i \in J} \begin{pmatrix} a_{2j_i} - fr(a_{ij_i}) \\ a_{3j_i} - fr(a_{ij_i}) \\ \vdots \\ a_{mj_i} - fr(a_{ij_i}) \end{pmatrix} \oplus \bigoplus_{j \notin J} \begin{pmatrix} a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$= \begin{pmatrix} \lfloor a_{2j_2} \rfloor \\ \lfloor a_{3j_3} \rfloor \\ \vdots \\ \lfloor a_{mj_m} \rfloor \end{pmatrix} \in \mathbb{Z}^{m-1}.$$

This is since we do not increase any value a_{it} when we multiply by α_t and so for each row i $(i \neq 1)$ we know that $\lfloor a_{ij_i} \rfloor \geq a_{it} \geq \alpha_t \otimes a_{it}$ for each $t \neq j_i$.

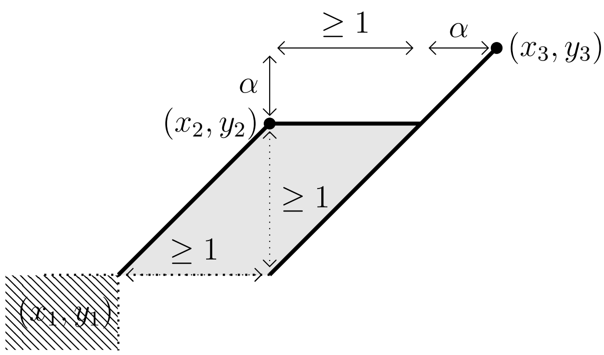
Thus the max-convex hull contains an integer point. $\square$

The above results do not cover the cases when there is more than one row maximum in the same column, so we will return to the 3×3 case and assume $x_1 \leq x_2 \leq x_3$.

Case 1: $y_1 \leq y_2 \leq y_3$.

Case 1a: $x_2 - x_1 \geq 1$, $y_2 - y_1 \geq 1$ and $x_3 - x_2 \geq 1 + y_3 - y_2$.

(x_1, y_1) is located in the marked region and we can guarantee that the shaded area will be within the max-convex hull of the three points, and this area will clearly contain an integer point.



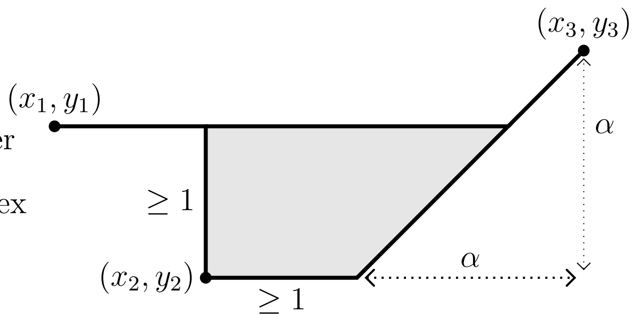
Case 1b: $x_2 - x_1 \geq 1$, $y_2 - y_1 \geq 1$ and $y_3 - y_2 \geq 1 + x_3 - x_2$

Similar

Case 2: $y_2 \leq y_1 \leq y_3$ t.

Case 2a: $y_2 - y_1 \geq 1$ and $x_3 - x_2 \geq 1 + y_3 - y_2$.

There will be an integer point in this max-convex hull.



Case 2b: $x_2 - x_1 \geq 1$ and $y_3 - y_1 \geq 1 + x_3 - x_1$.

Similar.

We summarise these results in the following proposition.

Proposition 3.44. *Let $A \in \mathbb{R}^{3\times 3}$ be of the form*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Assume that $x_1 \leq x_2 \leq x_3$. Then each of the following is a sufficient condition for the max-convex hull to contain an integer point;

(i) $y_1 \leq y_2 \leq y_3$, $x_2 - x_1 \geq 1$, $y_2 - y_1 \geq 1$ and $x_3 - x_2 \geq 1 + y_3 - y_2$.

(ii) $y_1 \leq y_2 \leq y_3$, $x_2 - x_1 \geq 1$, $y_2 - y_1 \geq 1$ and $y_3 - y_2 \geq 1 + x_3 - x_2$.

(iii) $y_2 \leq y_1 \leq y_3$, $y_1 - y_2 \geq 1$ and $x_3 - x_2 \geq 1 + y_3 - y_2$.

(iv) $y_2 \leq y_1 \leq y_3$, $x_2 - x_1 \geq 1$ and $y_3 - y_1 \geq 1 + x_3 - x_1$.

Corollary 3.45. *Let $A \in \mathbb{R}^{3\times n}$, $n \geq 3$, be of the form*

$$A = \begin{pmatrix} 0 & 0 & ... & 0 \\ x_1 & x_2 & ... & x_n \\ y_1 & y_2 & ... & y_n \end{pmatrix}.$$

Assume that $x_1 \leq x_2 \leq ... \leq x_n$ and that $y_n = \max_j y_j$. Then each of the following is a sufficient condition for the max-convex hull to contain an integer point.

(i) If $\exists j_1, j_2 \in N - \{n\}$ with $x_{j_2} - x_{j_1} \geq 1$, $y_{j_2} - y_{j_1} \geq 1$ and $x_n - x_{j_2} \geq 1 + y_n - y_{j_2}$.

(ii) If $\exists j_1, j_2 \in N - \{n\}$ with $x_{j_2} - x_{j_1} \geq 1$, $y_{j_2} - y_{j_1} \geq 1$ and $y_n - y_{j_2} \geq 1 + x_n - x_{j_2}$.

(iii) If $\exists j_1, j_2 \in N - \{n\}$ with $y_{j_1} - y_{j_2} \geq 1$ and $x_n - x_{j_2} \geq 1 + y_n - y_{j_2}$.

(iv) If $\exists j_1, j_2 \in N - \{n\}$ with $x_{j_2} - x_{j_1} \geq 1$ and $y_n - y_{j_1} \geq 1 + x_n - x_{j_1}$.

Proof. Follows from Proposition 3.44 since the max-convex hull of > 3 points from a set S will contain the max-convex hull of any 3 points from S. □

3.4 Conclusion

We began this section by describing Algorithm 3.1 (INT-IMAGE), which, when given a finite input matrix, will determine whether an integer image exists in pseudopolynomial time, see Theorem 3.11. We then moved on to looking for classes of matrices for which an integer image could be found in strongly polynomial time.

We showed that, for upper and lower triangular matrices, the question of existence of an integer point could be solved in strongly polynomial time. For $2 \times n$ and $m \times 2$ matrices we gave necessary and sufficient conditions for the existence of an integer image which could be checked in strongly polynomial time. A key result was the introduction of the class of column typical matrices, for which a full description of the set of integer images could be described in strongly polynomial time, as shown in Theorem 3.17. We extended this result to the class of nearly non-integer matrices and demonstrated that, for these matrices the integer image set is equal to the set of integer subeigenvectors, which can be fully described in strongly polynomial time.

We then described equivalent problems to the integer image problem, in particular the problem of finding an integer point in a max-convex set, and used this to find sufficient conditions for a matrix to have an integer image. There is potentially a large amount of work still to do in investigating integer points in max-convex sets.

A full description of the integer image set remains an open problem. It can be shown that the integer image space is equal to the intersection of the integer subeigenspaces of (up to n^m) matrices, this can be found in the departmental paper [27]. Although each of the integer subeigenspaces can be described efficiently, the maximum number of subeigenspaces is not polynomial.

4. Investigating the complexity of the integer image problem

We study the problem of determining whether there is an integer vector in the image of A,

$$IIm(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{\mathbb{R}}^n)Ax = z\}.$$

We define $X(A)$ to be the set of vectors x for which Ax belongs to the set of integer images, that is

$$X(A) := \{x \in \overline{\mathbb{R}}^n : Ax \in \mathbb{Z}^m\}.$$

A related question is whether $X(A) \cap \overline{\mathbb{Z}}^n$ is nonempty, where $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\varepsilon\}$. We define the integer image with integer coefficients to be

$$IIm^*(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{\mathbb{Z}}^n)Ax = z\}.$$

Note that, since we are looking for integer (finite) vectors, we could assume without loss of generality that the vector x satisfying $Ax \in \mathbb{Z}^m$ is finite.

Let IIM be the problem of determining whether there exists an integer vector in the image space of A, that is,
(IIM) For $A \in \overline{\mathbb{R}}^{m \times n}$, is $IIm(A) \neq \emptyset$?

Here we will consider a number of integer image problems, each with an additional requirement on the set of integer images. These are detailed in the definition below. The main two variants are the column typical (CT) variant, and the Property One (P1) variant. Figure 4.1 outlines the relations between these problems. Recall that, in a column typical matrix all entries in any columns have different fractional parts (see Definition 3.15). Note also that the Property One variant, although it refers to the existence of one position in each row, is not related to the definitions of Property OneIR and Property OneFP found in this thesis. Property OneFP/OneIR impose a condition on the input matrices having one entry per row with some property, whereas Property One here refers to a solution with some property.

Definition 4.1. *Given $A \in \overline{\mathbb{R}}^{m\times n}$ we consider the following problems related to the Integer Image Problem.*

(IIM-CT) If A is column typical does there exist $x \in \overline{\mathbb{R}}^{n}$ such that $Ax \in \mathbb{Z}^m$?

(IIM-CT-P1) If A is column typical does there exist $x \in \overline{\mathbb{R}}^{n}$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?

(IIM-P1) Does there exist $x \in \overline{\mathbb{R}}^{n}$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?

(IIM) Does there exist $x \in \overline{\mathbb{Z}}^{n}$ such that $Ax \in \mathbb{Z}^m$?*

In Section 4.2 we show that determining whether $IIm(A) \neq \emptyset$ reduces to checking whether A is a yes instance of IIM-CT or IIM*. A key result here is the transformation to the column typical counterpart, and the proof that this preserves the set of integer images: Theorem 4.11. From Theorem 3.17 there exist special classes of matrices for which IIM-CT is strongly polynomially solvable. We will further show that there also exist special cases for IIM*. Theorem 4.17 shows that IIM-P1 is NP-hard. What this chapter aims to demonstrate is that, on the one hand, the integer image problem for general matrices is closely related to the integer image problem for column typical matrices $A \in \overline{\mathbb{R}}^{m\times n}$, which is strongly polynomially solvable if either $m \geq n$ or we fix the value of m. On the other

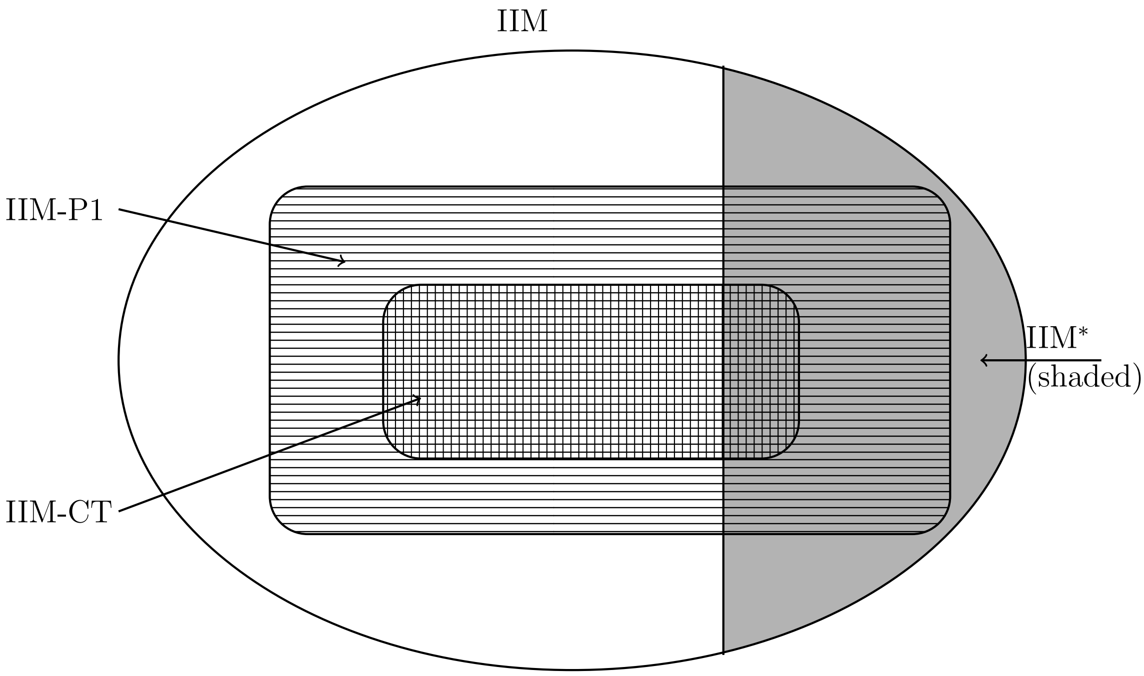


Figure 4.1: Simple relations between the different versions of the integer image problem. Observe that IIM-CT is identical to IIM-CT-P1 (see Theorem 4.13 for details).

hand IIM-CT and IIM-CT-P1 are polynomially equivalent by Theorem 4.13 and, if we remove the assumption that the matrix is column typical, IIM-P1 is NP-hard. So we are in essence approaching the integer image problem from two sides, one a set of problems in P and the other a set of problems that are NP-hard.

4.1 Preliminaries and simple cases

We denote by P the class of all problems which are solvable in polynomial time. The class NP and the definition of an NP-hard problem can be found, for example, in [45]. For general problems P1 and P2 we write $P1 \leq_p P2$ to mean that P1 can be reduced to P2 in polynomial time. Additionally, $P1 =_p P2$ will mean that $P1 \leq_p P2$ and $P2 \leq_p P1$. It is known that if $P1 \leq_p P2$ and $P1$ is NP-hard then $P2$ is NP-hard, if instead $P2 \in P$ then $P1 \in P$.

Recall from Chapter 3 that it is sufficient to only consider doubly $\mathbb{R}$-astic matrices.

Recall, from Theorem 3.17 that if A is a square, column typical matrix then the set of integer images of A can be described in strongly polynomial time. Observe that if $A \in \overline{\mathbb{R}}^{m \times n}$ is column typical with $m \leq n$ then

$$(\exists x)Ax = z \Leftrightarrow (\exists j_1, ..., j_m \in N)(\exists x')A'x' = z$$

where $A' \in \overline{\mathbb{R}}^{m \times m}$ is the matrix formed of columns $A_{j_1}, ..., A_{j_m}$. Therefore if A is column typical with $m \leq n$ then we could simply check each of the $\binom{n}{m}$ square submatrices of A to see if they have an integer image. Checking each submatrix can be achieved in $\mathcal{O}(m^3)$ time by Theorems 2.5 and 3.17.

Corollary 4.2. *For fixed m the integer image problem is solvable in strongly polynomial time.*

4.2 Transformations which preserve the set of integer images

We present two transformations which allow us to assume some structure on the matrix for which we are seeking an integer image. In both cases the transformation can be achieved in strongly polynomial time and we expect that the added structure will help in finding integer images. Indeed for each type of structure described we find a small class of matrices for which we can solve the integer image problem efficiently.

4.2.1 Transformation to matrices with one integer per column

First we describe a (strongly polynomial) transformation $A \rightarrow B$ such that $IIm(A) = IIm^*(B)$. Further, we show in Theorem 4.5, that if a general matrix $A \in \overline{\mathbb{R}}^{m \times n}$ has at most one integer entry in each column then we can decide in $\mathcal{O}(m^3 + n)$ time whether $A \in \text{IIM}^*$.

Given a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ let A^{int} be constructed from A by replacing each column $A_j, j \in N$ with m columns,

$$fr(a_{1j})^{-1}A_j, fr(a_{2j})^{-1}A_j, ..., fr(a_{mj})^{-1}A_j.$$

Example 4.3.

$$A = \begin{pmatrix} 0 & 1.1 \\ 0.5 & -2.3 \\ -0.6 & -0.9 \end{pmatrix}, A^{int} = \begin{pmatrix} 0 & -0.5 & -0.4 & 1 & 0.4 & 1 \\ 0.5 & 0 & 0.1 & -2.4 & -3 & -2.4 \\ -0.6 & -1.1 & -1 & -1 & -1.6 & -1 \end{pmatrix}$$

Note that each column takes at least one entry of the matrix and makes it integer.

Observe that for any $z \in IIm^*(A^{int})$, if the position (i,j) is active then it is necessary that $a_{ij} \in \mathbb{Z}$ since $a_{ij} + x_j = z_i$ where by definition x_j and z_i are integer. Therefore the following result tells us that when considering the integer image problem we can assume without loss of generality that only integer entries can be active.

Theorem 4.4. *$IIm(A) = IIm(A^{int}) = IIm^*(A^{int})$.*

Proof. Let A^{int} have columns $A_{j(i)}$ where $A_{j(i)} = fr(a_{ij})^{-1}A_j$. We first show that $IIm(A) = IIm(A^{int})$.

Suppose $\exists x \in \overline{\mathbb{R}}^n$ such that $Ax = z \in IIm(A)$. Then

$$\begin{aligned} z = \bigoplus_{j\in N} A_j x_j &= \bigoplus_{j\in N}\bigoplus_{i\in M} A_j fr(a_{ij}) fr(a_{ij})^{-1} x_j \\ &= \bigoplus_{j\in N}\bigoplus_{i\in M} A^{int}_{j(i)} (fr(a_{ij})x_j) \in IIm(A^{int}). \end{aligned}$$

For the other inclusion assume that

$$y = (y_{1(1)}, ..., y_{1(m)}, y_{2(1)}, ..., y_{2(m)}, ..., y_{n(1)}, ..., y_{n(m)}) \in \overline{\mathbb{R}}^{mn}$$

satisfies $Ay = z \in IIm(A)$. Then

$$z = \bigoplus_{j \in N} \bigoplus_{i \in M} A_{j(i)} y_{j(i)} = \bigoplus_{j \in N} \bigoplus_{i \in M} A_j fr(a_{ij})^{-1} y_{j(i)}$$
$$= \bigoplus_{j \in N} A_j \left(\bigoplus_{i \in M} fr(a_{ij})^{-1} y_{j(i)} \right) \in IIm(A).$$

Further it is clear that $IIm^*(A^{int}) \subseteq IIm(A^{int})$. This together with $IIm(A) = IIm(A^{int})$ implies $IIm^*(A^{int}) \subseteq IIm(A)$. It remains to show $IIm(A) \subseteq IIm^*(A^{int})$.

Clearly

$$(\exists x \in \overline{\mathbb{R}}^n) Ax = z \in \overline{\mathbb{Z}}^m \Rightarrow (\exists y \in \overline{\mathbb{Z}}^{mn}) A^{int} y = z \in \overline{\mathbb{Z}}^m$$

since if A_j is active with respect to x then $(\exists i \in M) fr(a_{ij}) = fr(-x_j)$ and a_{ij} is active, therefore, using Lemma 1.17(ii),

$$a_{ij} + x_j = \lfloor a_{ij} \rfloor + fr(a_{ij}) + \lfloor x_j \rfloor + fr(x_j) = \lfloor a_{ij} \rfloor + fr(-x_j) + \lfloor x_j \rfloor + fr(x_j) = a_{it}^{int} + \lceil x_j \rceil$$

for some $t \in \{1, .., mn\}$. This means that $x_j A_j = \lceil x_j \rceil (fr(a_{ij})^{-1} A_j) = y_t A_t^{int}$ where $y_t = \lceil x_j \rceil$. Hence the result. $\square$

This transformation is expected to be helpful in solving the integer image problem since it allows us to look for integer images of the matrix for which active positions are (i, j) where $a_{ij} \in \mathbb{Z}$. While it remains unknown whether IIM* is in P or not we can describe one class of matrices for which it is solvable in $\mathcal{O}(m^3 + n)$ time.

Indeed suppose $A \in \overline{\mathbb{R}}^{m \times n}$ has at most one integer entry in each row. Then either the matrix does not satisfy the necessary condition that every row has an active entry, in which case $IIm^*(A) = \emptyset$, or A has exactly one integer in each row. In this case let I be the identity matrix with dimension m. Then

$$IIm^*(A) \neq \emptyset \Leftrightarrow (\exists x \in \mathbb{Z}^n, y \in \mathbb{Z}^m) Ax = Iy.$$

We say that a pair of rectangular matrices (A, B) satisfies *Property OneFP* if, for all $i \in M$ there exists exactly one pair (j, t), $j, t \in N$ such that $fr(a_{it}) = fr(a_{ij}) \neq \varepsilon$ (see Definition 5.8 for full detail). Note that the pair (A, I) satisfies Property OneFP.

We will prove in Section 5.2 that we can find an integer solution to a two-sided system in strongly polynomial time if the system satisfies Property OneFP. The following result is an immediate consequence of Corollary 5.15.

Theorem 4.5. *Let $A \in \overline{\mathbb{R}}^{m \times n}$ have exactly one integer entry in each row. Then we can determine whether $IIm^*(A) \neq \emptyset$ in $\mathcal{O}(m^3 + n)$ time.*

Generally however A^{int} will not satisfy this condition, since it contains at least n integer entries in each row. We finish this subsection by detailing a few observations about $IIm^*(A)$ for an arbitrary matrix A.

Proposition 4.6. *Suppose $A \in \overline{\mathbb{R}}^{m \times n}$.*

(i) $IIm^*(A) \subseteq IIm(\lceil A \rceil)$

(ii) $IIm^*(A) \subseteq IIm(\lfloor A \rfloor)$

Proof. If $Ax = z \in \mathbb{Z}^m$ where $x \in \overline{\mathbb{Z}}^n$ then

$$\max_j (a_{ij} + x_j) = z_i \Rightarrow \max_j (\lceil a_{ij} \rceil + x_j) = z_i.$$

The other result is also trivial to prove. □

For each $i \in M$ let

$$d_i(A) = \max_{j \in N} \lceil a_{ij} \rceil - \max_{j : a_{ij} \in \mathbb{Z}} a_{ij}.$$

Clearly $d_i(A) \geq 0$ for all $i \in M$. Using this we obtain a simple sufficient condition for $IIm^*(A) \neq \emptyset$.

Proposition 4.7. *Let $A \in \overline{\mathbb{R}}^{m\times n}$ have at least one integer in each row. If $(\forall i \in M) d_i(A) = 0$ then $A \otimes \mathbf{0} \in IIm^*(A)$.*

Proof.

$$(\forall i) d_i(A) = 0 \Rightarrow (\forall i \in M)(\exists j(i) \in N) a_{ij(i)} \in \mathbb{Z} \text{ and } a_{ij(i)} = \bigoplus_{t\in N} a_{it}.$$

$$\therefore \begin{pmatrix} a_{1j(1)} \\ a_{2j(2)} \\ \vdots \\ a_{mj(m)} \end{pmatrix} = A \otimes \mathbf{0}.$$

This belongs to $IIm(A)$ since the left hand side is an integer vector. □

4.2.2 Transforming to column typical matrices

Here we show that, for the problem of determining if $IIm(A) \neq \emptyset$, we may assume without loss of generality that A is column typical with $m \leq n$. It follows that in order to solve the problem of whether or not a matrix has an integer vector in its column span it is sufficient to find a method for column typical matrices only.

First observe that if $A \in \overline{\mathbb{R}}^{m\times n}$ is column typical and $Ax \in \mathbb{Z}^m$ then each column A_j contains at most one active entry with respect to x. Since every row contains an active entry it is necessary that at least m columns are active in this equation. We conclude:

Observation 4.8. *Suppose $A \in \overline{\mathbb{R}}^{m\times n}$ is column typical with $m > n$. Then $IIm(A) = \emptyset$.*

Suppose without loss of generality in this subsection that $A \in \overline{\mathbb{R}}^{m\times n}$ is doubly $\mathbb{R}$-astic and no two columns in A are the same. Let

$$J^{ct}(A) = \{j \in N : A_j \text{ is column typical}\}.$$

If $j \in N - J^{ct}(A)$ then define

$$I_j^{ct} = \{i \in M : \exists t \in M, t \neq i \text{ such that } fr(a_{ij}) = fr(a_{tj})\}$$

otherwise set $I_j^{ct} = \{\emptyset\}$.

Definition 4.9. *The* column typical counterpart*, A^{ct}, of A is the*

$$m \times (\sum_{j \in N} |I_j^{ct}|)$$

matrix obtained from A by replacing each column A_j with $|I_j^{ct}|$ columns as follows:

If $I_j^{ct} = \{\emptyset\}$ then add one copy of A_j. Otherwise for each $i \in I_j^{ct}$ add a column with entries

$$\begin{cases} a_{tj} - \delta_t & \textit{if } t \in I_j^{ct} - \{i\} \\ a_{tj} & \textit{otherwise.} \end{cases} \tag{4.1}$$

The values $\delta_i, i \in M$ will satisfy $0 < \delta_i < 1$ and be chosen to ensure that each new column has entries with different fractional parts.

Example 4.10. *The columns*

$$\begin{pmatrix} 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 0.2 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}$$

would be replaced by

$$\begin{pmatrix} 0 & 0-\delta_1 & 0-\delta_1 & 0-\delta_1 & 0-\delta_1 \\ 0-\delta_2 & 0 & 0-\delta_2 & 0-\delta_2 & 0-\delta_2 \\ 0.5-\delta_3 & 0.5-\delta_3 & 0.5 & 0.5-\delta_3 & 0.5-\delta_3 \\ 0-\delta_4 & 0-\delta_4 & 0-\delta_4 & 0 & 0-\delta_4 \\ 0.5-\delta_5 & 0.5-\delta_5 & 0.5-\delta_5 & 0.5-\delta_5 & 0.5 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}$$

where $0 < \delta_s < 1$ *are such that the new matrix is column typical.*

The columns in A^{ct} which replace A_j are called the *counterparts* of A_j. For now we suppose that $\delta_i \in (0,1), i \in M$ satisfy the following assumptions:

(A1) δ_i are distinct;

(A2i) $(\forall j \in N)(\forall i, t \in M) fr(a_{ij}) \neq fr(a_{tj})$ & $a_{ij}, a_{tj} > \varepsilon \Rightarrow fr(a_{ij} - a_{tj}) > \delta_i, \delta_t$;

(A2ii) $(\forall i \in M)(\forall j, p \in N) fr(a_{ij}) \neq fr(a_{ip})$ & $a_{ij}, a_{ip} > \varepsilon \Rightarrow fr(a_{ij} - a_{ip}) > \delta_i$;

(A3) $(\forall i \in M)(\forall j \in N) fr(a_{ij}) \neq 0$ & $a_{ij} > \varepsilon \Rightarrow \delta_i < \min(fr(a_{ij}), fr(-a_{ij}))$.

Theorem 4.11. *Let* $A \in \overline{\mathbb{R}}^{m\times n}$ *be doubly* $\mathbb{R}$*-astic and* A^{ct} *be the column typical counterpart of* A *where* $\delta_i, i \in M$ *satisfy A1-A3. Then* A^{ct} *is column typical and*

$$IIm(A) = IIm(A^{ct}).$$

This will be proved in Section 4.5, where we also prove that $\delta_i, i \in M$ satisfying A1-A3 can be found efficiently. It will follow that A^{ct} can be constructed in $\mathcal{O}((mn)^2)$ time.

Remark 4.12. *Given a matrix* $A \in \overline{\mathbb{R}}^{m\times n}$ *we could first construct* $B = A^{ct} \in \overline{\mathbb{R}}^{m\times mn}$ *and then* $C = B^{int} \in \overline{\mathbb{R}}^{m\times m^2 n}$*. Then* $IIm(A) = IIm^*(C)$ *and further the candidates for active position of* C *are* (i,j) *such that* $c_{ij} \in \mathbb{Z}$ *of which there is exactly one per column since* C

is column typical (it inherits the property from B). Finally note that this transformation can be constructed in strongly polynomial time.

4.3 Problems that are polynomially equivalent to IIM

We show that IIM, IIM-CT and IIM-CT-P1 are polynomially equivalent, and therefore belong to the same complexity class.

Theorem 4.13. *IIM-CT-P1$\in P \Leftrightarrow$ IIM-CT$\in P \Leftrightarrow$ IIM$\in P$, i.e.*

(i) IIM-CT-P1 $=_p$ IIM-CT.

(ii) IIM-CT $=_p$ IIM.

Proof.

(i) We show that if A is column typical then A has an integer image if and only if A has an integer image in which there is exactly one active entry per row.

The sufficient direction is clear. So assume that A has an integer image z. Then $\exists x \in \overline{\mathbb{R}}^n$ such that $Ax = z$. If there exist $t, j \in N$ such that a_{ij} and a_{it} are both active for some $i \in M$ then the vector x' obtained from x by replacing x_t by ε also satisfies $Ax' = z$. This is because A is column typical meaning there is at most one active entry in every column and so removing A_t from the system will not affect active entries in any other row. In this way we can construct a vector x'' such that $Ax'' = z$ and A has exactly one active entry per row.

(ii) IIM-CT $\leq_p$ IIM is trivial. We show IIM $\leq_p$ IIM-CT. Let $A \in \overline{\mathbb{R}}^{m\times n}$. Let $A^{ct} \in \overline{\mathbb{R}}^{m\times k}, k \leq mn$ be the column typical counterpart of A (see Definition 4.9).

We have $IIm(A) \neq \emptyset \Leftrightarrow IIm(A^{ct}) \neq \emptyset$.

Therefore A is an instance of IIM-CT if and only if A^{ct} is an instance of IIM-CT-P1 and A^{ct} can be constructed in $\mathcal{O}((mn)^2)$ time. □

Corollary 4.14. *To show that IIM is NP-hard or $IIM \in P$ it is enough to consider either IIM-CT-P1 or IIM-CT.*

We know from Theorem 3.17 that checking whether a square matrix is in IIM-CT is achievable in strongly polynomial time. But this does not imply that IIM for square matrices is polynomially solvable since in transforming a matrix to its column typical counterpart we increase the number of columns.

4.4 Related NP-hard problems

In this section we consider modifications of the integer image problem which we can prove to be NP-hard. The hardness of these related problems does not imply hardness of IIM, that question remains open. We begin with IIM-P1, which asks for an integer image with exactly one active position per row (see Definition 4.1). Recall that $X(A)$ is the set of vectors x for which $Ax \in IIm(A)$.

We will use the following key result.

Proposition 4.15. *Fix $\alpha \in (0, 1)$. Let $A \in \{0, \alpha^{-1}\}^{m\times n}$ be a matrix in which each column has at least one zero entry. If $z \in IIm(A)$ then*

(i) for any $x \in X(A)$ such that $Ax = z$ all active entries of A are integer (zero),

(ii) z is a constant vector, and

(iii) if $A_j, j \in N$ contains an active position then (i, j) is active for all $i \in M$ such that $a_{ij} = 0$.

Proof. Assume $(\exists z \in \mathbb{Z}^m)(\exists x \in \overline{\mathbb{R}}^n)Ax = z$.

(i) Suppose $a_{ij} = \alpha^{-1}$ is active, so $x_j = z_i\alpha \notin \mathbb{Z}$. By assumption there exists a zero entry in every column so let t be an index such that $a_{tj} = 0$. Then $a_{tj}x_j \notin \mathbb{Z}$, so a_{tj} is inactive and there exists l such that a_{tl} is active. Hence we have

$$\begin{aligned}\alpha^{-1}x_j =& z_i,\\ 0x_j <& z_t,\\ a_{il}x_l \leq& z_i \text{ and}\\ a_{tl}x_l =& z_t.\end{aligned}$$

From the first two equations we obtain $z_i = \alpha^{-1}x_j < x_j < z_t$ and therefore $z_i 1 \leq z_t$. Using this and the last two equations we get

$$a_{tl}x_l = z_t \geq z_i 1 \geq a_{il}x_l 1.$$

This implies that $a_{tl} \geq a_{il}1$, a contradiction with $|a_{il}a_{tl}^{-1}| \leq \alpha < 1$.

(ii) Suppose there exists $x \in \overline{\mathbb{R}}^n$ such that $Ax = z \in \mathbb{Z}^m$ where $(\exists i, t \in M) z_i \neq z_t$. Without loss of generality assume that $z_i > z_t$, in fact $z_i \geq z_t 1$. Let a_{ij}, a_{tl} be active entries in rows i and t respectively. Note that by (i), $a_{ij} = 0 = a_{tl}$, meaning $x_j = z_i$ and $x_l = z_t$. But then

$$A_j x_j = A_j z_i \geq A_j(z_t 1) = (1A_j)z_t > A_l z_t = A_l x_l$$

which implies that A_l is inactive. This is a contradiction since a_{tl} is active.

(iii) Denote $S = \{j :$ There exists an active entry in $A_j\}$. Fix $j \in S$ and suppose $a_{ij}x_j = z_i$. Then by (i), $x_j = z_i$ and hence

$$A_j x_j = A_j z_i \leq \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = \begin{pmatrix} z_i \\ \vdots \\ z_i \end{pmatrix}$$

where the final equality is obtained using (ii). Finally for all $t \in M$ such that $a_{tj} = 0$ we have $a_{tj}x_j = z_i$, therefore every integer (zero) entry in A_j is active. $\square$

It is important to observe that any matrix $A \in \{0, \alpha^{-1}\}^{m \times n}$ with at least one zero entry in each column has an integer image if and only if there is a zero in every row, which occurs if and only if $\mathbf{0} \in IIm(A)$. In fact

$$IIm(A) \neq \emptyset \Leftrightarrow IIm(A) = \{\gamma\mathbf{0} : \gamma \in \mathbb{Z}\}.$$

Thus when we consider whether the matrix has an integer image that also satisfies some specified property (number of active entries per row or column for example) this property will be determined by the vector x such that $Ax = \mathbf{0}$. Note that it can be assumed that $x \in \{0, \varepsilon\}^n$ where the finite components correspond to active columns of A.

We will use a reduction from the following NP-hard problem.

(Monotone 1-in-3 SAT): 1-in-3 SAT is a modification of the SAT problem in which each clause has 3 literals and we ask whether there exists a satisfying assignment such that exactly one literal in each clause is TRUE. The monotone version of the problem satisfies the additional condition that each clause contains only unnegated variables. Note that without loss of generality each literal appears in at least one clause.

Remark 4.16. *1-in-3 SAT is problem L04 in [45], where it is noted that it remains NP complete even if no clause contains a negated literal. The result follows from the classification of NP-hard satisfiability problems in [60].*

Theorem 4.17. *Monotone 1-in-3 SAT*$\leq_p$ *IIM-P1.*

Proof. Let $F = C_1 \wedge ... \wedge C_m$ where every clause contains 3 unnegated literals from $\{y_1, ..., y_n\}$.

Construct an $m \times n$ matrix $A = (a_{ij})$ as follows: For some $\alpha \in (0, 1)$,

$$a_{ij} = \begin{cases} 0 & \text{if } y_j \in C_i \\ \alpha^{-1} & \text{otherwise.} \end{cases}$$

Note that A can be constructed in strongly polynomial time.

Example 4.18. *For* $F = (y_1 \vee y_2 \vee y_3) \wedge (y_1 \vee y_3 \vee y_4)$ *we obtain*

$$A = \begin{pmatrix} 0 & 0 & 0 & \alpha^{-1} \\ 0 & \alpha^{-1} & 0 & 0 \end{pmatrix}.$$

Now assume there exists $z \in \mathbb{Z}^m$ such that $(\exists x \in \overline{\mathbb{R}}^n)\ Ax = z$.

Since A satisfies the conditions of Proposition 4.15 we know that there exists $\gamma \in \mathbb{Z}$ such that $z = \gamma\mathbf{0}$ and active entries are integer, thus $x \in \mathbb{Z}^n$. Further for all $j \in S$,

$$a_{ij} = 0 \Rightarrow a_{ij} \text{ is active}$$

where $S = \{j : \text{ There exists an active entry in } A_j\}$.

If $Ax = z$ with exactly one active entry per row then $y = (y_1, ..., y_n)^T$ is a satisfying assignment of F with exactly one TRUE literal per clause where

$$y_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand if F has a satisfying assignment y in which exactly one literal in each clause is satisfied then for all $j \in N$ let

$$x_j = \begin{cases} 0 & \text{if } y_j = 1 \\ \varepsilon & \text{else.} \end{cases}$$

The vector $x = (x_1, ..., x_n) \in \overline{\mathbb{R}}^n$ is such that $Ax = \mathbf{0}$ and there is exactly one active entry per row.

Therefore F has a satisfying assignment with exactly one TRUE literal per clause if and only if A has an integer image with exactly one active entry per row. $\square$

Corollary 4.19. *IIM-P1 is NP-hard.*

Remark 4.20. *Consider the following problems related to IIM.*
(IIM-P2) Does there exist $x \in \overline{\mathbb{R}}^n$ *such that* $Ax \in \mathbb{Z}^m$ *with exactly two active entries per row with respect to* x*?*
(IIM-P3) Does there exist $x \in \overline{\mathbb{R}}^n$ *such that* $Ax \in \mathbb{Z}^m$ *with at most two active entries per row with respect to* x*?*
(IIM-P4) Given $t \in \mathbb{N}$ *does there exist* $x \in \overline{\mathbb{R}}^n$ *such that* $Ax \in \mathbb{Z}^m$ *with at most* t *active columns of* A *with respect to* x*?*
(IIM-P1) Does there exist* $x \in \overline{\mathbb{Z}}^n$ *such that* $Ax \in \mathbb{Z}^m$ *with exactly one active entry per row with respect to* x*?*

These problems can all be shown to be NP-hard using similar methods and reducing from either Monotone 1-in-3 SAT, the NP-hard problem Monotone NAE-3-SAT (every clause contains 3 unnegated literals and we ask whether there exists a satisfying assignment for which no clause contains only TRUE literals) or the Minimum Cardinality Cover problem (given a universe U*, a family* S *of finite subsets of* U *and a positive integer* t*, does there exist a subfamily* $C \subseteq S$*,* $|C| \leq t$ *such that* C *is a cover of* U*).*

The proofs for IIM-P2 and IIM-P1 use Monotone 1-in-3 SAT, the proof for IIM-P3 uses Monotone NAE-3-SAT, and the minimum cardinality cover problem is reduced to IIM-P4.*

4.5 Proving the validity of column typical counterparts

We prove the results from Subsection 4.2.2 which we repeat below.

Theorem 4.11 states: Let $A \in \overline{\mathbb{R}}^{m\times n}$ be doubly $\mathbb{R}$-astic and A^{ct} be the column typical counterpart of A where $\delta_i, i \in M$ satisfy A1-A3. Then A^{ct} is column typical and

$$IIm(A) = IIm(A^{ct}).$$

Further $\delta_i, i \in M$ satisfying A1-A3 can be found efficiently and A^{ct} constructed in $\mathcal{O}((mn)^2)$ time where assumptions A1-A3 are as follows.

(A1) δ_i are distinct;

(A2i) $(\forall j \in N)(\forall i, t \in M) fr(a_{ij}) \neq fr(a_{tj})$ & $a_{ij}, a_{tj} > \varepsilon \Rightarrow fr(a_{ij} - a_{tj}) > \delta_i, \delta_t$;

(A2ii) $(\forall i \in M)(\forall j, p \in N) fr(a_{ij}) \neq fr(a_{ip})$ & $a_{ij}, a_{ip} > \varepsilon \Rightarrow fr(a_{ij} - a_{ip}) > \delta_i$;

(A3) $(\forall i \in M)(\forall j \in N) fr(a_{ij}) \neq 0$ & $a_{ij} > \varepsilon \Rightarrow \delta_i < \min(fr(a_{ij}), fr(-a_{ij}))$.

The purpose of the values $\delta_i, i \in M$ is to guarantee that the column typical counterpart of A is indeed column typical, and also that it retains the same integer image as A. The idea is to alter the entries of A by small enough values so the the image set is unchanged, so we will eventually choose $\delta_i \leq g$ for some upper bound g dependent on A. Further, we will prove that g will be chosen to not only to alter any entries in a column with the same fractional part, but also in such a way that no new entries share a fractional part.

The assumption A2ii is used to prove that the column typical counterpart has no two columns the same, which is then used in the proofs that A^{ct} is column typical, and shares its integer image with A. Assumptions A1, A2i and A3 are key to the proof that the fractional parts of entries in any column of A^{ct} are indeed different.

4.5.1 Proof of Theorem 4.11

Recall (from Subsection 4.2.2) that we assume $A \in \overline{\mathbb{R}}^{m\times n}$ has no two identical columns. Further, that the columns in A^{ct} which replace A_j are called the *counterparts* of A_j.

Proposition 4.21. *If A has no two identical columns then all columns of A^{ct} are different when $\delta_i, i \in M$ are chosen according to assumptions A1-A3 .*

Proof. If $A^{ct}_{j_1}$ and $A^{ct}_{j_2}$ are both counterparts to A_j and $j_1 \neq j_2$ then by definition $A^{ct}_{j_1} \neq A^{ct}_{j_2}$.

Assume then that $A^{ct}_{c(j)}$ and $A^{ct}_{c(p)}$ are counterparts of A_j and A_p respectively, $j \neq p$. Since $A_j \neq A_p$ there exists i such that $a_{ij} \neq a_{ip}$. We prove that $a^{ct}_{ic(j)} \neq a^{ct}_{ic(p)}$by showing

$$a_{ij} - \delta_i \neq a_{ip} - \delta_i,$$
$$a_{ij} - \delta_i \neq a_{ip} \text{ and}$$
$$a_{ij} \neq a_{ip} - \delta_i.$$

The first is immediate, the second and third are proved in the same way. To see that the third statement holds assume, for a contradiction, that $a_{ij} = a_{ip} - \delta_i$.

Case 1: $fr(a_{ip}) = 0$

Here, $fr(a_{ij}) = fr(-\delta_i) = 1 - \delta_i$ since $fr(\delta_i) = \delta_i$. But then either $\delta_i = 1$ (if $a_{ij} \in \mathbb{Z}$), which is a contradiction, or $\delta_i = fr(-a_{ij})$ by Lemma 1.17(ii), which conflicts with A3. So this case does not occur.

Case 2: $fr(a_{ip}) > 0$

By A3, $fr(a_{ip}) > \delta_i$. Further, using A2 and Lemma 1.17(iv),

$$a_{ip} > a_{ij} = a_{ip} - \delta_i > \lfloor a_{ip} \rfloor \Rightarrow fr(a_{ip}) > fr(a_{ij}) \text{ and}$$
$$fr(a_{ij}) = fr(a_{ip} - \delta_i) = fr(a_{ip}) - \delta_i.$$

But then, again by Lemma 1.17(iv),

$$\delta_i = fr(a_{ip}) - fr(a_{ij}) = fr(a_{ip} - a_{ij}),$$

a contradiction with A2ii. □

We first show that our assumptions imply that A^{ct} is column typical.

Claim 4.22. *If $0 < \delta_i < 1, i \in M$ satisfy A1-A3, then A^{ct} is column typical.*

Proof. Assume that A^{ct} is not column typical, thus $\exists j \in N - J^{ct}(A)$. Then there exist $i, t \in M, i \neq t$ such that

$$fr(a_{ij} - \delta_i) = fr(a_{tj} - \delta_t). \tag{4.2}$$

Since $\delta_i \neq \delta_t$ we conclude, that $fr(a_{ij}) \neq fr(a_{tj})$. Assume without loss of generality that $fr(a_{ij}) > fr(a_{tj})$, then $fr(a_{ij}) \geq \delta_i = fr(\delta_i)$ by A3. Therefore, using Lemma 1.17(iv),

$$fr(a_{ij} - \delta_i) = fr(a_{ij}) - \delta_i \text{ and } fr(a_{tj} - \delta_t) = \begin{cases} fr(a_{tj}) - \delta_t; & \text{if } fr(a_{tj}) > 0; \\ fr(-\delta_i) = -\delta_t; & \text{otherwise.} \end{cases}$$

Case 1: $fr(a_{tj}) > 0$

Substituting the above into (4.2) we obtain, using $fr(a_{ij}) > fr(a_{tj})$ and Lemma 1.17(iv),

$$fr(a_{ij} - a_{tj}) = fr(a_{ij}) - fr(a_{tj}) = \delta_i - \delta_t \Rightarrow \delta_i > fr(a_{ij} - a_{tj})$$

which contradicts A2.

Case 2: $fr(a_{tj}) = 0$

From (4.2) we get $fr(a_{ij}) - \delta_i = 1 - \delta_t$ (since $fr(a_{ij}) > fr(a_{tj}) = 0$), which implies $1 - \delta_t < fr(a_{ij})$. But then $\delta_t > 1 - fr(a_{ij}) = fr(-a_{ij})$, a contradiction with A3. $\square$

We now show that the image set is unaffected by the transformation.

We set

$$r = \sum_{j \in N} |I_j^{ct}|.$$

First assume that $z \in IIm(A) \neq \emptyset$. So there exists $x \in X(A)$ such that $Ax = z$. Observe that the vector

$$x' = (x_1, ..., x_1, x_2, ..., x_2, ..., x_n, ..., x_n)^T \in \overline{\mathbb{R}}^r$$

(where each $x_j, j \in N$ is repeated $|I_j^{ct}|$ times) satisfies $A^{ct}x' = z$.

For the other direction assume that $z \in IIm(A^{ct}), x \in X(A^{ct})$ and let

$$c(1), ..., c(m) \in \{1, ..., r\}$$

be indices of m active columns of A^{ct} such that

$$\bigoplus_{t=1}^{m} A_{c(t)}^{ct} x_{c(t)} = z. \tag{4.3}$$

So $c(1), ..., c(m)$ represent a list of m active columns $A_{c(1)}^{ct}, ..., A_{c(m)}^{ct}$ in A^{ct} with respect to z. Note that there is exactly one active entry in each of the columns in the list as they are column typical.

We prove that there exists some $x' \in \overline{\mathbb{R}}^n$ such that $Ax' = z$ by considering two cases: when the list of these m active columns in A^{ct} contains at most one counterpart of each column $A_j, j \in N$ and when it contains more than one counterpart to some column A_j.

To do this we first need the following claim on the active entries of columns in the list.

Claim 4.23. *Let $c(1), ..., c(m)$ represent a list of m active columns of A^{ct} satisfying (4.3). For each $t \in \{1, ..., m\}$ there exists an index $p(t)$ such that the new list of columns $A^{ct}_{p(1)}, ..., A^{ct}_{p(m)}$ satisfies*

$$\bigoplus_{t=1}^{m} A^{ct}_{p(t)} y_{p(t)} = z$$

for some $y \in X(A^{ct})$ where the active entry of each $A^{ct}_{p(t)}$ has not been altered when moving from A to A^{ct}.

Proof.

Fix $t \in \{1, ..., m\}$ and suppose $A^{ct}_{c(t)}$ is a counterpart to A_j for some $j \in N$. Further suppose that the active entry in $A^{ct}_{c(t)}$ is in row i. If $a^{ct}_{ic(t)} = a_{ij}$ then let $p(t) = c(t)$ and $y_{p(t)} = x_{c(t)}$.

If instead $a^{ct}_{ic(t)} = a_{ij} - \delta_i$ then we know $|I^{ct}_j| \geq 2$ and

$$A^{ct}_{c(t)} x_{c(t)} \leq z$$

with equality only in row i. Defining $\mu = x_{c(t)} - \delta_i$, we obtain

$$\begin{aligned}
&(a_{ij} - \delta_i) + (\mu + \delta_i) = z_i, \\
&a_{sj} + (\mu + \delta_i) < z_s \ \forall s \notin I^{ct}_j \text{ and} \\
&(a_{sj} - \delta_s) + (\mu + \delta_i) < z_s \ \forall s \in I^{ct}_j - \{i\}.
\end{aligned}$$

Therefore

$$a_{ij} + \mu = z_i,$$

$$a_{sj} + \mu < z_s \ \forall s \notin I_j^{ct} \text{ and}$$

$$(a_{sj} - \delta_s) + \mu < z_s \ \forall s \in I_j^{ct} - \{i\}.$$

But this means that there exists a counterpart of A_j in A^{ct}, say A_p^{ct}, such that

$$A_p^{ct}\mu \leq z$$

with equality only for z_i and active entry $a_{ip}^{ct} = a_{ij}$. So set $p(t) = p$ in our choice of columns, and $y_{c(t)} = x_{c(t)} - \delta_i$.

Repeat this for each column in the list. For any unassigned entry of y set $y_l = x_l$. This results in a new list of m distinct columns $A_{p(t)}^{ct}, t \in \{1, ..., m\}$ such that

$$\bigoplus_{t=1}^{m} A_{p(t)}^{ct} y_{p(t)} = z$$

and having active entries which are unaltered from A. It immediately follows that $A^{ct}y = z$ and hence $y \in X(A^{ct})$. □

Hence we can assume that $Ax = z$, and further that there is a list of m active columns $A_{c(t)}^{ct}, t \in \{1, ..., m\}$ satisfying (4.3) with active entries unaltered by some δ_i, i.e. entries such that $a_{ij}^{ct} = a_{ij}$. We use this to describe $x' \in \overline{\mathbb{R}}^n$ such that $Ax' = z$.

Case 1: $(\forall j, l \in \{1, ..., m\}, j \neq l)$ $A_{c(j)}^{ct}$ and $A_{c(l)}^{ct}$ are counterparts to different columns in A.

By rearranging columns in A if necessary, we can assume without loss of generality that $A_{c(t)}^{ct}, t \in \{1, ..., m\}$ is a counterpart to A_t.

Define $x' \in \overline{\mathbb{R}}^n$ by $x'_t = x_{c(t)}$, $t \in \{1, ..., m\}$ and ε otherwise. Observe that $Ax' \geq z$ since if $a^{ct}_{ic(j)}$ is active in A^{ct} with respect to x then, $a^{ct}_{ic(j)} + x_{c(j)} = a^{ct}_{ic(j)} + x'_j = a_{ij} + x'_j = z_i$. It remains to show $Ax' \leq z$.

Assume there exists $i \in M$, $t \in \{1, ..., m\}$ such that $a_{it} + x'_t > z_i$. Then, by definition of A^{ct}, $a_{it} - \delta_i = a^{ct}_{ic(t)}$ and $a^{ct}_{ic(t)}$ is inactive in $A^{ct}x = z$. Note that $A^{ct}_{c(t)}$ is active in (4.3) so there exists $i' \in M, i' \neq i$ such that $a^{ct}_{i'c(t)} + x_{c(t)} = z_{i'}$ and additionally $a^{ct}_{i'c(t)} = a_{i't}$ by our assumptions on the active entries.

Case 1a: $fr(a_{it}) \neq fr(a_{i't})$

We have

$$a_{it} + x'_t > z_i > a^{ct}_{ic(t)} + x_{c(t)} = a_{it} - \delta_i + x'_t. \tag{4.4}$$

Since $z_i \in \mathbb{Z}$ we deduce

$$\lfloor a_{it} + x'_t \rfloor \geq \lceil a_{it} - \delta_i + x'_t \rceil.$$

Lemma 1.17(vi) gives $\delta_i \geq fr(a_{it} + x'_t)$. Further $a_{i't} + x'_t = z_{i'}$ implies $fr(x'_t) = fr(-a_{i't})$. Therefore, using Lemma 1.17(iiv),

$$\delta_i \geq fr(a_{it} + x'_t) = fr(fr(a_{it}) + fr(-a_{i't})) = fr(a_{it} - a_{i't})$$

but this is a contradiction with assumption A2i on δ_i.

Case 1b: $fr(a_{it}) = fr(a_{i't})$

Using $a_{it} + x'_t > z_i$ and $a_{i't} + x'_t = z_{i'}$ we get that $a_{it} + x'_t \in \mathbb{Z}$ and therefore $a_{it} + x'_t \geq z_i + 1$. But then $a^{ct}_{ic(t)} + \delta_i + x_{c(t)} \geq z_i + 1$ which is a contradiction with $a^{ct}_{ic(t)} + x_{c(t)} \leq z_i$ as it suggests $\delta_i \geq 1$.

In both subcases we reach a contradiction and therefore $A_t x'_t \leq z$. Since this argument holds for all i we conclude $Ax' \leq z$ and then that $Ax' = z$ as required.

Case 2: $(\exists j \in N)(\exists s, t \in \{1, ..., m\})$ $A^{ct}_{c(s)}$ and $A^{ct}_{c(t)}$ are counterparts of A_j.

We would like to argue that the same idea as in Case 1 holds here, however to do this we must show that when we go from $A^{ct}_{c(s)}$ and $A^{ct}_{c(t)}$ back to A_j the components $x_{c(s)}$ and $x_{c(t)}$ do not cause a problem.

Both columns have a single active entry in different rows, i_1 and i_2 say.

So, using our assumptions on the active entries,

$$a_{i_1 j} + x_{c(s)} = a^{ct}_{i_1 c(s)} + x_{c(s)} = z_{i_1} \text{ and } a_{i_2 j} + x_{c(t)} = a^{ct}_{i_2 c(t)} + x_{c(t)} = z_{i_2},$$
$$\therefore a^{ct}_{i_1 c(s)} + x_{c(s)} \geq a^{ct}_{i_1 c(t)} + x_{c(t)} \text{ and } a^{ct}_{i_2 c(s)} + x_{c(s)} \leq a^{ct}_{i_2 c(t)} + x_{c(t)}.$$

Note that if $p, q \in \{1, ..., m\}$ and $A_{c(p)}, A_{c(q)}$ are counterparts to the same column in A then

$$(\forall i \in M)|a^{ct}_{ic(p)} - a^{ct}_{ic(q)}| \in \{0, \delta_i\}.$$

Using this we obtain

$$x_{c(t)} - x_{c(s)} \leq a^{ct}_{i_1 c(s)} - a^{ct}_{i_1 c(t)} \leq \delta_{i_1} \text{ and } x_{c(s)} - x_{c(t)} \leq a^{ct}_{i_2 c(t)} - a^{ct}_{i_2 c(s)} \leq \delta_{i_2}$$
$$\therefore -\delta_{i_1} \leq x_{c(s)} - x_{c(t)} \leq \delta_{i_2}.$$

Substituting $x_{c(s)} = z_{i_1} - a_{i_1 j}$ and $x_{c(t)} = z_{i_2} - a_{i_2 j}$ gives

$$-\delta_{i_1} \leq z_{i_1} - a_{i_1 j} - z_{i_2} + a_{i_2 j} \leq \delta_{i_2}.$$

Case 2a: $0 = z_{i_1} - a_{i_1 j} - z_{i_2} + a_{i_2 j}$.

Then $fr(a_{i_1 j}) = fr(a_{i_2 j})$ and, more importantly, $0 = z_{i_1} - a_{i_1 j} - z_{i_2} + a_{i_2 j} = x_{c(s)} - x_{c(t)}$ so $x_{c(s)} = x_{c(t)}$ and there will be no conflict in choosing x'_j. We detail this later.

Case 2b: $0 < z_{i_1} - a_{i_1 j} - z_{i_2} + a_{i_2 j} \leq \delta_{i_2}$.

Then since $\delta_{i_2} < 1$ and $fr(a_{i_2 j}) \neq fr(a_{i_1 j})$ we have (using Lemma 1.17(vii),

$$\delta_{i_2} \geq z_{i_1} - a_{i_1 j} - z_{i_2} + a_{i_2 j} = fr(z_{i_1} - a_{i_1 j} - z_{i_2} + a_{i_2 j}) = fr(a_{i_2 j} - a_{i_1 j}).$$

But this is a contradiction with assumption A2i on δ. So this case does not occur.

Case 2c: $0 < z_{i_2} - a_{i_2 j} - z_{i_1} + a_{i_1 j} \leq \delta_{i_1}$.

Similarly as in Case 2b we can reach a contradiction on the size of δ_{i_1}.

Since only Case 2a can occur we conclude that the active entries of $A^{ct}_{c(s)}$ and $A^{ct}_{c(t)}$ correspond to entries of A_j with the same fractional part, and $x_{c(s)} = x_{c(t)}$. This proves that there is no conflict moving from multipliers $x_{c(s)}$ and $x_{c(t)}$ to a single multiplier x_j.

In general, given a list $A^{ct}_{c(1)}, ..., A^{ct}_{c(m)}$ satisfying (4.3) with active entries unaltered by any δ_i, we construct $x' \in \overline{\mathbb{R}}^n$ as follows:

For each $j \in N$

(1) If no column corresponding to A_j in A^{ct} is in the list then let $x'_j = \varepsilon$.

(2) If exactly one column, $A^{ct}_{c(j)}$ say, corresponding to A_j is in the list set $x'_j = x_{c(j)}$.

(3) If more than one column corresponding to A_j in A^{ct} is in the list then choose any of them, $A^{ct}_{c(j')}$ say, and set $x_j = x_{c(j')}$.

Finally $Ax' = z$ can be shown using similar arguments as in Case 1; $Ax' \geq z$ because $Ax' \geq A^{ct}x = z$ and $Ax' \leq z$ because otherwise there would exist i, t such that

$$a_{it} + x'_t > z_i \geq a^{ct}_{ic(t)} + x_{c(t)} \geq a_{it} - \delta_i + x'_t,$$

which is exactly (4.4) and so we can follow the same argument to reach a contradiction with assumption A2i on δ_i.

This ends the proof of Theorem 4.11.

4.5.2 The choice of δ_i

Given $A \in \overline{\mathbb{R}}^{m\times n}$ we show how to choose $0 < \delta_i < 1, i \in M$ satisfying A1-A3 in $\mathcal{O}((mn)^2)$ time.

We achieve this by showing there exists $g \in (0, 1)$ such that any choice of $\delta_i, i \in M$ satisfying $(\forall i)\delta_i < g$ will satisfy A2 and A3. It follows from A2 and A3 that satisfying

A1 is trivial.

We consider how to choose δ_i such that A2 and A3 hold.

Let

$$F = \{fr(a_{ij}) : i \in M, j \in N\} - \{0, \varepsilon\},$$
$$F' = \{fr(-a_{ij}) : i \in M, j \in N\} - \{0, \varepsilon\} \text{ and}$$
$$G = \{fr(f + f') : f \in F, f' \in F'\} - \{0, \varepsilon\}.$$

So $|F|, |F'| \leq mn$ and $|G| \leq (mn)^2$.

Consider satisfying A2:

To satisfy A2i for each column j of A we need to exclude any $fr(a_{ij} - a_{tj}) \neq 0$ from our choice of δ_i. By Lemma 1.17

$$fr(a_{ij} - a_{tj}) = fr(fr(a_{ij}) - fr(a_{tj}))$$

and hence these excluded values are contained in G. The same argument holds for rows so A2ii is also satisfied by excluding values from G.

To satisfy A3 we additionally exclude the values from $F \cup F'$ from our choice of δ_i.

Now let g be the minimum of the at most $(mn)^2 + 2mn$ values from

$$F \cup F' \cup G.$$

Then any choice of distinct δ_i satisfying $0 < \delta_i < g$ will satisfy our assumptions.

4.6 The integer image problem and the assignment problem

It follows from the results in this chapter that, to solve the integer image problem in polynomial time, it would be enough to find a polynomial time method for finding an integer image of a column typical matrix with integer active entries, that is, find $z \in IIm^*(A)$ for a column typical, rectangular matrix A (see Remark 4.12). We define the problem IIM*-CT as follows.

(IIM*-CT) If A is column typical does there exist $x \in \overline{\mathbb{Z}}^n$ such that $Ax \in \mathbb{Z}^m$?

Here we look at links between IIM*-CT and the assignment problem for rectangular matrices.

Recall from Theorem 3.17 that, when the matrix is square and column typical, if an integer image exists then the active entries form a permutation of maximum weight. An immediate consequence of that theorem is Lemma 4.24 below.

Lemma 4.24. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be column typical. If $IIm^*(A) \neq \emptyset$ then $maper(A) \in \mathbb{Z}$.*

Now, given $A \in \overline{\mathbb{R}}^{m\times n}$ and $f \in (0,1)$ define $A^{(0,f)} := (b_{ij})$ where

$$b_{ij} = \begin{cases} a_{ij}; & \text{if } a_{ij} \in \overline{\mathbb{Z}}, \\ \lfloor a_{ij} \rfloor + f; & \text{otherwise.} \end{cases}$$

Recall that $X(A) := \{x \in \overline{\mathbb{R}}^n : Ax \in Z^m\}$. The following lemma is trivial to prove.

Lemma 4.25. *Let $A \in \overline{\mathbb{R}}^{m\times n}$ and $f \in (0,1)$. Then $IIm^*(A) = IIm^*(A^{(0,f)})$ and $X(A) = X(A^{(0,f)})$.*

Further, note that, if $A \in \overline{\mathbb{R}}^{n\times n}$ is column typical then there will be at most one integer entry in each column of $A^{(0,f)}$, and therefore if $z \in IIm^*(A) = IIm^*(A^{(0,f)})$ then the active entries of $A^{(0,f)}$ form a permutation.

Proposition 4.26. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be column typical and $f = \frac{n}{n+1}$. Then $IIm^*(A) \neq \emptyset$ if and only if $maper(A^{(0,f)}) \in \mathbb{Z}$.*

Proof. Suppose $maper(A^{(0,f)}) \in \mathbb{Z}$ and let $\pi \in ap(A^{(0,f)})$. We observe first that $\forall i, a_{i\pi(i)} \in \mathbb{Z}$. To see this note that

$$\sum_{i\in N} a_{i\pi(i)} = \sum_{i\in N} \lfloor a_{i\pi(i)} \rfloor + \frac{t \times n}{n+1}$$

where $0 \leq t \leq n$ is the number of i such that $a_{i\pi(i)} \notin \mathbb{Z}$ and $\frac{t\times n}{n+1} \in \mathbb{Z} \Leftrightarrow t = 0$.

It remains to show that there exists $z \in IIm^*(A) = IIm^*(A^{(0,f)})$ for which $a_{i\pi(i)}$ are active. Note that, since A was column typical, $A^{(0,f)}$ has exactly one integer entry in each row and column. Let A' be the matrix obtained by rearranging the rows and columns of $A^{(0,f)}$ so that the integer entries are on the diagonal, and then subtract the diagonal entry from each entry in its respective row. Observe that A' is a matrix with zero diagonal, and no other integer entries. Further $maper(A') = 0$ and, importantly,

$$IIm^*(A^{(0,f)}) \neq \emptyset \Leftrightarrow IIm^*(A') \neq \emptyset.$$

We claim that A' is strongly definite, indeed if $\lambda(A') > 0$ then there would exist a permutation σ such that $w(\sigma, A') > 0$ which is a contradiction with the value of $maper(A')$. Hence A' is a NNI matrix (see Section 3.2.1 for the definition). Then, by Theorem 3.21,

$$IIm(A') = IIm^*(A') = IV^*(A', 0) = IV^*(\lceil A' \rceil, 0).$$

To finish the proof we show that $IV^*(\lceil A' \rceil, 0) \neq \emptyset$ by showing that $\lambda(\lceil A' \rceil) = 0$.

If not, there exists a cycle τ such that $w(\tau, \lceil A' \rceil) > 0$. Since every entry of $\lceil A' \rceil$ is integer this means

$$w(\tau, \lceil A' \rceil) \geq 1. \tag{4.5}$$

On the other hand, because $fr(a'_{ij}) \in \{0, f\}$ we have

$$w(\tau, \lceil A' \rceil) \leq w(\tau, A') + n \times (1 - f) \leq 0 + \frac{n}{n+1} < 1,$$

a contradiction with (4.5).

The other direction follows from Lemma 4.24. □

We can extend this result to the rectangular case as shown below.

Corollary 4.27. *Let $A \in \overline{\mathbb{R}}^{m \times n}$ be column typical with $m \leq n$ and $f = \frac{m}{m+1}$. Then $IIm(A) \neq \emptyset$ if and only if there exists $N' \subseteq N$ with $|N'| = m$ such that $maper(A^{(0,f)}[M, N']) \in \mathbb{Z}$.*

Therefore we could solve the integer image problem by solving the following problem:

IRAP: Given $A \in \overline{\mathbb{R}}^{m \times n}$, $m \leq n$, does there exist $N' \subseteq N$ with $|N'| = m$ such that $maper(A[M, N']) \in \mathbb{Z}$?

Unfortunately IRAP is NP-hard as shown by a reduction from Partition, the proof of which is due to R. Burkard [13]. Note that Partition is NP-hard and an instance of the problem is defined as follows.

Partition: Given $a_1, ..., a_{2k}, b \in \mathbb{N}$ with

$$\sum_{i=1}^{2k} a_i = 2b$$

does there exist $I \subseteq \{1, ..., 2k\}$, $|I| = k$, such that

$$\sum_{i \in I} a_i = b?$$

Proposition 4.28. *Partition $\leq_p$ IRAP.*

Proof. Take any $a_1, ..., a_{2k}, b \in \mathbb{N}$ with $\sum_{i=1}^{2k} a_i = 2b$. Let

$$A = \begin{pmatrix} \frac{a_1}{b} & \frac{a_2}{b} & \cdots & \frac{a_{2k}}{b} \\ \frac{a_1}{b} & \frac{a_2}{b} & \cdots & \frac{a_{2k}}{b} \\ \vdots & \vdots & & \vdots \\ \frac{a_1}{b} & \frac{a_2}{b} & \cdots & \frac{a_{2k}}{b} \end{pmatrix} \in \mathbb{R}^{k \times 2k}.$$

Observe that, for any square submatrix of A, every permutation has the same weight. Therefore there exists a square submatrix of A with integer max-algebraic permanent if and only if

$$\exists I \subseteq \{1, ..., 2k\}, |I| = k, \text{ such that } \sum_{i \in I} \frac{a_i}{b} \in \mathbb{Z}.$$

Since $1 \leq \sum_{i \in I} a_i \leq 2b - 1$ we have

$$\sum_{i \in I} \frac{a_i}{b} \in \mathbb{Z} \Leftrightarrow \sum_{i \in I} a_i = b.$$

Hence A has a submatrix with integer max-algebraic permanent if and only if $a_1, ..., a_{2k}, b$ are an instance of Partition. Clearly A can be constructed in strongly polynomial time.

□

Finally we observe that, although the above discussion is more evidence of a fine line between easily solvable cases of the integer image problem and NP-hard problems it tells us nothing about the complexity of the integer image problem itself.

4.7 Conclusion

We have shown in Theorems 4.11 and 4.13, that the problem of determining whether a matrix has a non empty set of integer images can be reduced to the problem of determining whether a column typical matrix has a non empty set of integer images. If the matrix has $m \geq n$ then the column typical version of the problem can be solved in strongly polynomial time, which, on the one hand, gives hope that maybe the integer image problem is polynomially solvable. On the other hand we show that similar problems are hard. The problem of finding an integer image of a column typical matrix is equivalent to determining whether a column typical matrix has an integer image for which there is exactly one active entry per row. The complexity of this problem for column typical matrices remains unresolved but if we remove the assumption that the matrix is column typical, we find that the problem of determining whether a general matrix has an integer image with exactly one active entry per row is NP-hard, see Corollary 4.19. A graphical representation of the results in this chapter are shown in Figure 4.2.

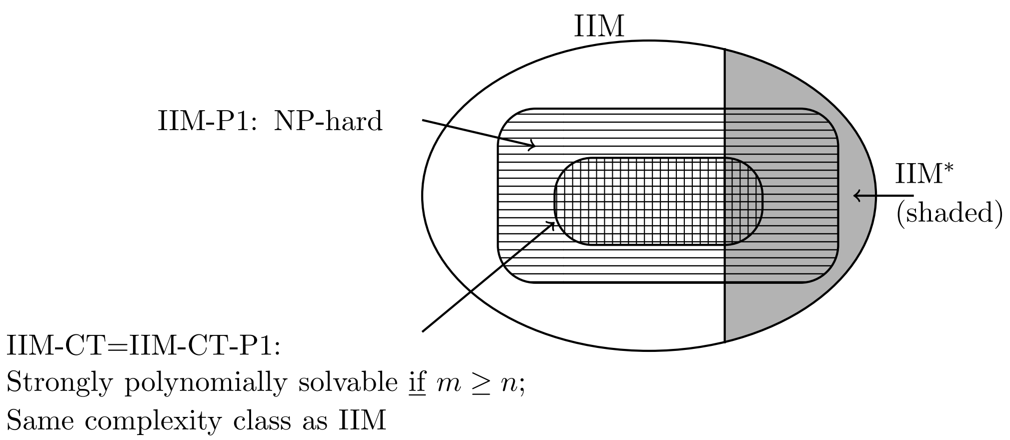


Figure 4.2: The known complexity results about the variants of the integer image problem.

5. Two-sided systems

Recall that a TSS with separable variables has the form $Ax = By$. A general TSS has the form $Ax \oplus c = Bx \oplus d$ which can be written, without loss of generality, in the form $A'y = B'y$ by Lemma 1.15 . In this chapter we investigate integer solutions to $Ax = By$ and $Ax = Bx$. We adapt existing methods for finding real solutions to TSS and develop Algorithms SEP-INT-TSS and GEN-INT-TSS to decide whether an integer solution to these TSSs exist. We then describe a generic case, called Property OneFP, for which we can describe all solutions and determine existence in strongly polynomial time, as proved in Theorem 5.14. The material in Sections 5.1 and 5.2 has been published in [25].

5.1 Algorithm to find integer solutions to two-sided systems

In this section we show that the Alternating Method [18, 38] can be easily adapted to design algorithms that determine whether integer solutions to $Ax = By$ or $Ax = Bx$ exist, and if so find one. We first detail an algorithm to solve systems with separated variables and then a second algorithm to solve general systems. Since the justification behind the construction of the two algorithms in this section is similar to the arguments in [18, 38] we only outline the key results here, full details are available in the departmental paper [23].

If the i^{th} row of either A or B is ε then we have $(Ax)_i = \varepsilon = (By)_i$ which, since x and

y are finite, means that the i^{th} row of the other matrix is also ε. Thus we may remove the redundant i^{th} equation from the equality. If instead either of A or B has an ε column then this column may be removed without affecting the solution. Hence we assume without loss of generality that A, B are doubly $\mathbb{R}$-astic.

For any matrix $Y \in \mathbb{R}^{m\times n}$ let

$$K(Y) := \left\lceil \max\{|y_{ij}| : i \in M, j \in N\} \right\rceil. \tag{5.1}$$

We propose the following algorithm to find integer solutions to the system with separated variables.

Algorithm 5.1. SEP-INT-TSS

Input: $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ *doubly* $\mathbb{R}$*-astic, any starting vector* $x(0) \in \mathbb{Z}^n$.

Output: An integer solution (x, y) *to* $Ax = By$ *or indication that no such solution exists.*

1. $r := 0$.

2. $y(r) := \lfloor B^{\#} \otimes' (A \otimes x(r))\rfloor$.

3. $x(r+1) := \lfloor A^{\#} \otimes' (B \otimes y(r))\rfloor$.

4. If $x_i(r+1) < x_i(0)$ *for all* $i \in N$ *then STOP (no solution).*

5. If $A \otimes x(r+1) = B \otimes y(r)$ *then STOP (solution found).*

6. Go to 2.

Theorem 5.2. *[23] Algorithm SEP-INT-TSS is correct and terminates after*

$$\mathcal{O}(mn(n+k)K(A))$$

operations, if applied to instances where the matrix A *is finite.*

The following statement is obvious.

Proposition 5.3. *Let $A, B \in \overline{\mathbb{R}}^{m\times n}$. The problem of finding $x \in \mathbb{Z}^n$ satisfying $Ax = Bx$ is equivalent to finding $x \in \mathbb{Z}^n, y \in \mathbb{R}^m$ such that*

$$\begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} I \\ I \end{pmatrix} y$$

where $I \in \overline{\mathbb{R}}^{m\times m}$.

We propose the following algorithm to find integer solutions to $Ax = Bx$.

Algorithm 5.4. GEN-INT-TSS

Input: $A', B' \in \overline{\mathbb{R}}^{m\times n}$ doubly $\mathbb{R}$-astic, $I \in \overline{\mathbb{R}}^{m\times m}$, any starting vector $x(0) \in \mathbb{Z}^n$.

Output: A solution $x \in \mathbb{Z}^n$ to $A'x = B'x$ or indication that no such vectors exist.

1. $r := 0$, $A := \begin{pmatrix} A' \\ B' \end{pmatrix}$, $B := \begin{pmatrix} I \\ I \end{pmatrix}$.

2. $y(r) := B^{\#} \otimes' (A \otimes x(r))$.

3. $x(r+1) := \lfloor A^{\#} \otimes' (B \otimes y(r)) \rfloor$.

4. If $x_i(r+1) < x_i(0)$ for all $i \in N$ then STOP (no solution).

5. If $A \otimes x(r+1) = B \otimes y(r)$ then STOP (solution found).

6: Go to (2).

Theorem 5.5. *[23] Algorithm GEN-INT-TSS is correct and terminates after*

$$\mathcal{O}(K(A'|B')mn(m+n))$$

operations, if applied to instances where both of the matrices A', B' are finite.

5.2 Problem is strongly polynomially solvable in a generic case

In this section we give a generic condition on the matrices A, B which, if satisfied, means that we can determine in strongly polynomial time whether an integer solution to any TSS exists, and if so find one. We then show that the method for these matrices can be extended to find integer solutions to any TSS, but at a cost to efficiency.

Recall that we assume without loss of generality that A, B are doubly $\mathbb{R}$-astic.

5.2.1 Property OneFP

The key observation in this section is the following result.

Proposition 5.6. *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$. If an integer solution to $Ax = By$, or (if $n = k$) $Ax = Bx$, exists then*

$$(\forall i \in M)(\exists j \in N, t \in K)\ fr(a_{ij}) = fr(b_{it}) \text{ and } a_{ij}, b_{it} \in \mathbb{R}.$$

Proof. Assume $x \in \mathbb{Z}^n$, $y \in \mathbb{Z}^k$ satisfy $Ax = By$. Then

$$(\forall i \in M)\ \max_j(a_{ij} + x_j) = \max_j(b_{ij} + y_j) \in \mathbb{R}.$$

Note that these values are finite since the matrices are doubly $\mathbb{R}$-astic. Therefore, for each i, there exist $r(i), r'(i) \in N$ such that

$$fr(a_{i,r(i)} + x_{r(i)}) = fr(b_{i,r'(i)} + y_{r'(i)})$$

and $a_{i,r(i)}, b_{i,r'(i)} \in \overline{\mathbb{R}}$. But $fr(a_{i,r(i)} + x_{r(i)}) = fr(a_{i,r(i)})$ and $fr(b_{i,r'(i)} + y_{r'(i)}) = fr(b_{i,r'(i)})$. Hence

$$(\forall i)(\exists j \in N, t \in K)\ fr(a_{ij}) = fr(b_{it}).$$

□

Definition 5.7. *Let $A \in \overline{\mathbb{R}}^{m\times n}$ and $B \in \overline{\mathbb{R}}^{m\times k}$. We say that the pair (A, B) satisfies*

(a) Property *$ZeroFP$ if there exists $i \in M$ such that there is no pair of finite entries (a_{ij}, b_{it}) with the same fractional part.*

(b) Property *$One^{+}FP$ if for each $i \in M$ there is at least one pair of finite entries (a_{ij}, b_{it}) with the same fractional part.*

By Proposition 5.6 a necessary condition for a TSS to have an integer solution is that the input matrices satisfy Property $One^{+}FP$. We will restrict our attention to matrices A and B that have exactly one pair of entries with the same fractional part in each row. Under this assumption note that, without loss of generality, we may assume entries sharing the same fractional part are integer valued (this is since we may subtract a constant from each row of the system without affecting the answer to the question), and that no other entries in the equation for either matrix are integer.

Definition 5.8. *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$. We say (A, B) satisfies* Property OneFP *if for each $i \in M$ there is exactly one pair $(r(i), r'(i))$ such that*

$$a_{ir(i)}, b_{ir'(i)} \in \mathbb{Z}, \text{ and}$$

for all $i \in M$, if $j \neq r(i)$ and $t \neq r'(i)$, then

$$a_{ij}, b_{it} > \varepsilon \Rightarrow fr(a_{ij}) \neq fr(b_{it}).$$

Remark 5.9. *Note that this definition allows for multiple ε entries in each row, for example the pair (I, I) satisfies Property OneFP with $r(i) = i = r'(i)$ for all i.*

Given a TSS we say that the system satisfies Property OneFP if the pair of input matrices satisfy Property OneFP

The aim of this section is to show that when the pair (A, B) satisfies Property OneFP the problem of finding integer solutions can be solved in strongly polynomial time.

We argue that matrices (A, B) satisfying either Property ZeroFP or Property OneFP represent a generic case. This is since the probability of two randomly generated real matrices $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ having multiple entries sharing the same fractional part is zero. Of course, for integer matrices, the existing methods [18] for finding real solutions to the systems discussed will find integer solutions, and hence the interesting case to consider is indeed when the input matrices are not integer.

From the proof of Proposition 5.6, in each row, active entries with respect to an integer solution have the same fractional part. Since, under Property OneFP, there are exactly one pair of entries per row with the same fractional part, the following is immediate.

Corollary 5.10. *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP. Then the entries $a_{i,r(i)}$ [$b_{i,r'(i)}$] are the only possible candidates for active entries in the matrix A [B] with respect to any integer vector x [y] satisfying $Ax = By$.*

5.2.2 Systems with separated variables

Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$. First we consider the question of whether there exist $x \in \mathbb{Z}^n, y \in \mathbb{Z}^k$ such that $Ax = By$ when (A, B) satisfies Property OneFP.

Observe that

$$Ax = z^{(-1)} = By \Leftrightarrow diag(z)Ax = 0 = diag(z)By.$$

Proposition 5.11. *Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP. Then (x, y) is an* *integer solution to* $Ax = By$ *if and only if there exists* $z \in \mathbb{Z}^m$ *such that* (x, y) *satisfy*

$$diag(z) \otimes A \otimes x = 0 \text{ and} \tag{5.2}$$

$$diag(z) \otimes B \otimes y = 0. \tag{5.3}$$

Proposition 5.12. *Let* $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ *satisfy Property OneFP. Suppose* $z \in \mathbb{Z}^m$ *satisfies* (5.2) *and* (5.3) *for some integer vectors* x, y. *If there exists a column of* $diag(z)A$ *containing more than one integer entry then these entries are equal. Similarly for* $diag(z)B$.

Proof. Assume z satisfies (5.2). Then for each i the entry $a_{i,r(i)}$ is the only active entry of A in the i^{th} row (equivalently $z_i + a_{i,r(i)}$ is the only active entry in the i^{th} row of $diag(z)A$). This implies that if there exist $i, t \in M$ such that $r(i) = r(t)$ then

$$z_i + a_{i,r(i)} + x_{r(i)} = 0 = z_t + a_{t,r(t)} + x_{r(t)} \Rightarrow z_i + a_{i,r(i)} = z_t + a_{t,r(t)}.$$

□

Proposition 5.13. *Simultaneously solving* (5.2) *and* (5.3) *is equivalent to the problem of finding* $z \in \mathbb{Z}^m$ *satisfying*

$$(\forall i, t \in M)\ z_i - z_t \geq \lceil a_{t,r(i)} \rceil - a_{i,r(i)} \text{ and}$$
$$(\forall i, t \in M)\ z_i - z_t \geq \lceil b_{t,r'(i)} \rceil - b_{i,r'(i)}.$$

Proof. Consider (5.2). We have that, for each i, the integer entry $a_{i,r(i)} + z_i$ is the only possible active entry of $diag(z)A$ with respect to an integer vector x. From Proposition 2.1, an integer solution to (5.2) exists exactly when the integer column maxima of $diag(z)A$

cover all rows. A similar argument holds for (5.3). Hence we require that

$$a_{i,r(i)} + z_i > a_{t,r(i)} + z_t \text{ for } t \neq i \text{ and } a_{t,r(i)} \notin \mathbb{Z}, \tag{5.4}$$

$$a_{i,r(i)} + z_i = a_{t,r(i)} + z_t \text{ for } a_{t,r(i)} \in \mathbb{Z}, \tag{5.5}$$

$$b_{i,r'(i)} + z_i > b_{t,r'(i)} + z_t \text{ for } t \neq i \text{ and } b_{t,r'(i)} \notin \mathbb{Z} \text{ and}$$

$$b_{i,r'(i)} + z_i = b_{t,r'(i)} + z_t \text{ for } b_{t,r'(i)} \in \mathbb{Z}.$$

For any other column (those not containing integer entries) we do not get any additional constraints since we may set the corresponding entry of x or y to be small enough so that the column has no effect on the product Ax or By.

This set of inequalities is equivalent to

$$(\forall i, t \in M)\ a_{i,r(i)} + z_i \geq \lceil a_{t,r(i)} \rceil + z_t \text{ and} \tag{5.6}$$

$$(\forall i, t \in M)\ b_{i,r'(i)} + z_i \geq \lceil b_{t,r'(i)} \rceil + z_t. \tag{5.7}$$

To see this note that (5.4) and (5.5) imply (5.6). For the other direction assume that (5.6) holds. If $a_{t,r(i)} \notin \mathbb{Z}$ then we have $a_{i,r(i)} + z_i > a_{t,r(i)} + z_i$ as required. If instead $a_{t,r(i)} \in \mathbb{Z}$ then $r(t) = r(i)$ and from (5.6) we have

$$a_{i,r(i)} + z_i \geq \lceil a_{t,r(i)} \rceil + z_t \text{ and } a_{t,r(t)} + z_t \geq \lceil a_{i,r(t)} \rceil + z_i$$

which together imply equality. Similar arguments hold for the inequalities with entries from B.

The result is obtained by rearranging inequalities (5.6) and (5.7). □

Let $W = (w_{ij}) \in \overline{\mathbb{Z}}^{m\times m}$ where

$$w_{ij} = \max\left(\lceil a_{j,r(i)}\rceil - a_{i,r(i)},\ \lceil b_{j,r'(i)}\rceil - b_{i,r'(i)}\right).$$

Then, by Proposition 5.13, to decide if $Ax = By$ has an integer solution, we need to determine whether there exists $z \in \mathbb{Z}^m$ satisfying

$$\begin{aligned}(\forall i, j \in M) z_i - z_j &\geq w_{ij}\\ \Leftrightarrow (\forall i) \max_j (w_{ij} + z_j) &\leq z_i\\ \Leftrightarrow W \otimes z &\leq z.\end{aligned}$$

This is exactly the condition for $z \in IV^*(W, 0)$ which can be checked using Theorem 2.5. We have therefore proved the following result.

Theorem 5.14. *Let $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP. For all $i, j \in M$ let*

$$w_{ij} = \max(\lceil a_{j,r(i)}\rceil - a_{i,r(i)},\ \lceil b_{j,r'(i)}\rceil - b_{i,r'(i)}).$$

Then an integer solution to $Ax = By$ exists if and only if $\lambda(W) \leq 0$. If this is the case then $Ax = By = z^{-1}$ where $z \in IV^*(W, 0)$ and x and y can be found using Proposition 2.1.

From Theorem 5.14 and Corollary 2.6 we obtain the following corollary which shows that, for systems satisfying Property OneFP, integer solutions to TSS can be fully described in strongly polynomial time.

Corollary 5.15. *For $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ satisfying Property OneFP it is possible to decide whether an integer solution to $Ax = By$ exists in $\mathcal{O}(m^3 + n + k)$ time.*

5.2.3 General two-sided systems

We now consider finding integer solutions to $Ax = Bx$ under the condition that (A, B) satisfy Property OneFP. The following statement is obvious.

Proposition 5.16. *Let $A, B \in \overline{\mathbb{R}}^{m\times n}$ satisfy Property OneFP. The problem of finding $x \in \mathbb{Z}^n$ such that $Ax = Bx$ is equivalent to finding $x \in \mathbb{Z}^n, y \in \mathbb{Z}^n$ such that*

$$\begin{pmatrix} A \\ I \end{pmatrix} x = \begin{pmatrix} B \\ I \end{pmatrix} y.$$

Observe that, if (A, B) satisfies Property OneFP, then so does $(\hat{A}, \hat{B})$ where

$$\hat{A} = \begin{pmatrix} A \\ I \end{pmatrix}, \hat{B} = \begin{pmatrix} B \\ I \end{pmatrix}.$$

Thus to solve a general two-sided system satisfying Property OneFP we may convert it into a system with separated variables and solve using Theorem 5.14. By Corollary 5.15 we have:

Corollary 5.17. *For $A, B \in \overline{\mathbb{R}}^{m\times n}$ satisfying Property OneFP we can decide whether an integer solution to $Ax = Bx$ exists in $\mathcal{O}((m+n)^3)$ time.*

Remark 5.18. *The transformation described in Proposition 5.3 is not suitable here since it has $y \in \mathbb{R}$ whereas, to use Theorem 5.14, we want to be able to look for integer solutions. Conversely, the transformation described in Proposition 5.16 is not suitable to use when discussing the Alternating Method since we need at least one of the matrices to be finite for our complexity arguments to hold.*

5.2.4 Some special cases

We now give a couple of cases where we can give simpler conditions, both are described for systems with separated variables.

The first case occurs when we have $r(1) = ... = r(m)$ or $r'(1) = ... = r'(m)$. We assume without loss of generality that the former occurs.

Proposition 5.19. *Assume that $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ satisfy Property OneFP. Suppose further that $r(1) = ... = r(m) = p$. Let A' and B' be the matrices obtained from A and B by max-multiplying the i^{th} row by a_{ip}^{-1}. Then an integer solution to $Ax = By$ exists if and only if*

$$B' \otimes \lfloor (B')^{\#} \otimes' 0 \rfloor = 0.$$

Proof. Observe that an integer solution to $Ax = By$ exists if and only if an integer solution to $A'x = B'y$ exists.

Assume first that x, y are integer vectors satisfying $A'x = B'x$. Now, from Corollary 5.10, we know that the active entries in A' with respect to x are the zero entries in column p. Thus $A' \otimes x = (x_p, x_p, ..., x_p)^T$.

Therefore $B'y = (x_p, x_p, ..., x_p)^T$ which implies that $B(x_p^{-1} \otimes y) = 0$ and hence, using Corollary 2.3 we know that $B' \otimes \lfloor (B')^{\#} \otimes' 0 \rfloor = 0$.

For the other direction assume that $B' \otimes \lfloor (B')^{\#} \otimes' 0 \rfloor = 0$. Choosing $x \in \mathbb{Z}^n$ such that $x = (x_1, ...x_{p-1}, 0, x_{p+1}, ..., x_n)^T$ with x_j small for $j \neq p$ gives us that $A' \otimes x = 0$ and thus setting $y = \lfloor (B')^{\#} \otimes' 0 \rfloor$ gives $A' \otimes x = B' \otimes y$ as required. □

Remark 5.20. *If $r(1) = ... = r(m) = p$ and $r'(1) = ... = r'(m) = q$ then the only candidates for active entries are found in columns A_p and B_q. So if $Ax = By$ then $x_pA_p = y_qB_q$ and the other components of x and y are small enough not to affect the outcome. Thus a solution to $Ax = By$ exists if and only if A_p is a max-multiple of B_q.*

The second case occurs when A, B are square, satisfy Property OneFP, and for one matrix the active entries are spread over all columns. Without loss of generality assume that it is matrix A, so $\{r(1), ..., r(m)\} = M$.

Proposition 5.21. *Assume that $A, B \in \overline{\mathbb{R}}^{m\times m}$ satisfy Property OneFP. Suppose further that $r(i) \neq r(t)$ for all $i, t \in M$ with $i \neq t$. Let A' be obtained from A as follows: For each i max-multiply each entry of row i by $a_{i,r(i)}^{-1}$ and permute the columns so that the zero entries appear on the leading diagonal. If*

$$\lambda(\lceil A' \rceil) \neq 0$$

then no integer solution to $A \otimes x = B \otimes y$ exists.

Proof. Let B' be obtained from B by max-multiplying each entry of row i by $a_{i,r(i)}^{-1}$. Assume an integer solution to $Ax = By$ exists. Then an integer solution to $A'x = B'y$ exists and the active entries in A' are the zeros on the diagonal by Corollary 5.10. Thus $(A'x)_i = a_{ii} + x_i = x_i$ and hence $x \in IV^*(A', 0)$. By Theorem 2.5 we have $\lambda(\lceil A' \rceil) = 0$.

□

5.2.5 A strongly polynomial algorithm for general matrices with fixed m

We end this section by giving a brief description of how the solution method for systems satisfying Property OneFP could be adapted to find integer solutions to any TSS, but that in doing so we may lose efficiency. Since we can convert any general two-sided system into a system with separated variables we discuss systems with separated variables only.

Let $A \in \overline{\mathbb{R}}^{m\times n}, B \in \overline{\mathbb{R}}^{m\times k}$ and suppose (A, B) satisfies Property One$^+$FP. In this case for each row i of $Ax = By$ we will have a number of pairs $(a_{ir(i,s)}, b_{ir'(i,s)})$, some integer $s \leq nk$, such that $fr(a_{ir(i,s)}) = fr(b_{ir'(i,s)})$. Observe that for any $x \in \mathbb{Z}^n, y \in \mathbb{Z}^k$ satisfying

$Ax = By$ we can identify a single pair of active entries for each row of the equation, and hence the pair (x, y) is also an integer solution to the system $Ax = B^{-}y$ where (A, B^{-}) satisfies Property OneFP and B^{-} is obtained from B by slightly decreasing each inactive entry in B with respect to y.

In general $Ax = By$ if and only if there exists an m-tuple $(k_1, ..., k_m)$, $k_i \in \{1, ..., k\}$, a real number $0 < \delta \ll 1$ and a matrix $B^{-} = (b^{-}_{ij}) \in \overline{\mathbb{R}}^{m \times k}$ with

$$b^{-}_{ij} = \begin{cases} b_{ij}, & \text{if } j = k_i; \\ b_{ij} \otimes \delta^{-1}, & \text{otherwise} \end{cases}$$

such that $Ax = B^{-}y$.

Hence given a pair (A, B) satisfying Property One^{+}FP we can generate a number of pairs $(A, B^{(t)}), t \in \mathbb{N}$ such that x, y is an integer solution to $Ax = By$ if and only is there exists t such that $Ax = B^{(t)}y$. Note that each $B^{(t)}$ is obtained from B by decreasing the value of all but one element, $b_{ir'(i,s)}$, per row and the pairs $(A, B^{(t)})$ satisfy Property OneFP. We can therefore determine whether an integer solution to $Ax = B^{(t)}y$ exists in strongly polynomial time.

Unfortunately in the worst case there could be as many as nk pairs per row and thus $(nk)^m$ matrices to check, so the complexity of this method is $\mathcal{O}(m^3 n^m k^m)$. However we can say that, for fixed m, a strongly polynomial method for finding integer solutions to TSS exists.

5.3 Conclusion

We began by constructing Algorithms 5.1 (SEP-INT-TSS) and 5.4 (GEN-INT-TSS) which, for finite input matrices, can determine whether an integer solution to $Ax = By$ and $Ax = Bx$ respectively, exist. Further we proved that these algorithms run in pseudopolynomial time for finite input matrices.

We defined a class of TSSs for which the entire set of integer solutions could be described in strongly polynomial time, a key result being Theorem 5.14. This was any TSS for which the pair of input matrices satisfied Property OneFP. We used this class of matrices to show that, for fixed m, it is possible to find integer solutions to TSSs in strongly polynomial time.

Further research could be done to find more strongly polynomially solvable cases, and to give an efficient full description of all integer solutions to a TSS. It is also known, see [27], that the set of integer solutions to a TSS can be written as an intersection of the solution sets of one sided systems. While the set of integer solutions to each of these simpler systems can be described in strongly polynomial time, the number of systems involved in the description is too large to determine whether an integer solution to the original TSS existed efficiently.

At the time of writing, for two-sided systems which do not satisfy the generic property, it is unknown whether an integer solution, or indeed any solution, can be found in polynomial time. If we remove the integrality requirement, then it is known that finding a solution to a max-algebraic two-sided system is equivalent to finding a solution to a mean payoff game [10]. Mean payoff games are a well known class of problems in NP $\cap$ co-NP, it is expected that a polynomial solution method will be found in the future.

6. Integer max-linear programs

In this chapter we investigate integer solutions to max-linear programming problems. We briefly consider the case when the constraints are in the form of a one-sided equality, showing that methods for finding real solutions can be adapted to find integer solutions, and that the optimal objective value, and an optimal solution, can be found in strongly polynomial time. The main focus of this chapter is problems with two-sided constraints. Using Algorithm 5.4 (GEN-INT-TSS) for solving TSSs, we describe a bisection method to find an optimal solution to an integer max-linear program in pseudopolynomial time, this is Algorithms INT-MAXLINMIN and INT-MAXLINMAX. Finally, we consider the IMLP where the input matrices satisfy Property OneFP. Key results are Theorems 6.33 and 6.34, which describe the optimal objective value and find an optimal solution to finite input systems in strongly polynomial time. These results are then used to prove that, for any input matrix, the problem is strongly polynomially solvable if the input matrices satisfy Property OneFP.

Recall that, in max-algebra, we have to consider the maximisation problem and the minimisation problem independently, as there is no easy way to switch between them.

All of the material in Sections 6.2 and 6.3 has been published in [25] and [26].

6.1 One-sided constraints

Let $A \in \overline{\mathbb{R}}^{m\times n}$ $b \in \overline{\mathbb{R}}^{m}, c \in \overline{\mathbb{R}}^{n}$. We will assume throughout this section that A is doubly $\mathbb{R}$-astic. The one-sided max linear program (OMLP) is stated below.

$$\begin{aligned} & c^T \otimes x \to \text{ min or max} \qquad\qquad (OMLP) \\ & \text{s.t } A \otimes x = b \\ & x \in \mathbb{R}^n \end{aligned}$$

When we additionally require that $x \in \mathbb{Z}^n$ we have the one-sided integer max-linear program (OIMLP). We will show that the OIMLP is strongly polynomially solvable.

We use OMLP$^{\max}$ and OMLP$^{\min}$ to denote the problems maximising and minimising the objective function respectively. Similarly for OIMLP.

6.1.1 Preliminaries

It is known how to solve the OMLP, one solution method is found in [18] and we outline the results here.

Recall Proposition 1.9 and that $\bar{x}$ satisfies $x \leq \bar{x}$ for any feasible x. Thus, since the objective function $c^T \otimes x$ is isotone the following is immediate.

Proposition 6.1. *[18] OMLP$^{\max}$ has a solution if and only if $\bar{x}$ is an optimal solution.*

We now consider OMLP$^{\min}$. The algorithm from [18] to solve the problem of minimizing the objective function (given that the feasible set is non empty) given below. It is proved in [18] that this algorithm requires $\mathcal{O}(mn)$ operations.

Algorithm: ONEMAXLINMIN

Input: A, b, c.

Output: Optimal solution x to OMLP$^{\min}$

1. *Calculate* $\bar{x}$ *and* $M_j = \{i \in M : \bar{x}_j = b_i \otimes a_{ij}^{-1}\}, j \in N$.

2. *Without loss of generality let* $c_1 \otimes \bar{x}_1 \leq c_2 \otimes \bar{x}_2 \leq ... \leq c_n \otimes \bar{x}_n$.

3. $J := \{1\}, r := 1$.

4. *If*

$$\bigcup_{j \in J} M_j = M$$

then STOP. Solution x *exists with* $x_j = \bar{x}_j$ *for* $j \in J$ *and* x_j *small enough otherwise.*

5. $r := r + 1, J := J \cup \{r\}$.

6. *Go to 4.*

In step 4, for a component $x_j, j \notin J$ to be small enough it means that it does not contribute to the objective function value, thus any x_j with $x_j \leq c_j^{(-1)} \otimes c_r \otimes x_r$ will ensure that $c_r \otimes \hat{x}_r = c_r \otimes x_r \geq c_j \otimes x_j$.

6.1.2 One-sided integer max-linear programs

The methods for the OMLP are easily adaptable to solve the OIMLP using Proposition 2.1.

Define $\hat{x}_j = \lfloor \bar{x}_j \rfloor$ for all $j \in N$. Then any feasible $x \in \mathbb{Z}^n$ satisfies $x \leq \hat{x}$.

An immediate corollary of Proposition 6.1 and Corollary 2.3 follows.

Corollary 6.2. *OIMLP*$^{\max}$ *has a solution if and only if* $\hat{x}$ *is an feasible solution. If this is the case then* $\hat{x}$ *is an optimal solution.*

Corollary 6.3. *Solving* $OIMLP^{\max}$ *requires at most* $\mathcal{O}(mn)$ *operations.*

Proof. We need to calculate $\hat{x}$ and then check whether $A \otimes \hat{x} = b$. If it is then $\hat{x}$ is a solution, if it is not then no feasible solution exists. Calculating $\bar{x}$ requires $(2m-1)n = \mathcal{O}(mn)$ operations, taking the integer part to get $\hat{x}$ requires n operations. The final check requires $\mathcal{O}(mn)$ operations. □

To solve the minimization case we propose a simple change Algorithm ONEMAXLIN-MIN; calculating $\hat{x}$ instead of $\bar{x}$. We define $OIS^{\min}$ to be the set of optimal solutions to $OIMLP^{\min}$, and $M'_j = \{i \in M : \hat{x}_j = b_i \otimes a_{ij}^{-1}\}$. This gives us the following algorithm to solve $OIMLP^{\min}$.

Algorithm 6.4. OIMLPMin

Input: A, b, c.

Output: Optimal solution $x \in OIS^{\min}$*, or indication that the feasible set is empty.*

1. Calculate $\hat{x}$ *and* $M'_j, j \in N$*. If* $A\hat{x} \neq b$ *STOP; no feasible solutions.*

2. Order $c_j \otimes \hat{x}_j, j \in N$*: Without loss of generality let*

$$c_1 \otimes \hat{x}_1 \leq c_2 \otimes \hat{x}_2 \leq ... \leq c_n \otimes \hat{x}_n.$$

3. $J := \{1\}, r := 1$.

4. If

$$\bigcup_{j \in J} M'_j = M$$

then STOP. Solution x *exists with* $x_j = \hat{x}_j$ *for* $j \in J$ *and* x_j *small enough otherwise.*

5. $r := r + 1, J := J \cup \{r\}$.

6. Go to 4.

Again, for $x_j, j \notin J$ to be small enough in step 4 we could have $x_j \leq c_j^{(-1)} \otimes c_r \otimes x_r$.

Theorem 6.5. *Algorithm OIMLPMin is correct and its complexity is* $\mathcal{O}(mn^2)$*.*

Proof.

Correctness: From Corollary 2.3 a feasible solution exists if and only if $\hat{x}$ is feasible. From Proposition 2.1, any feasible solution $x \in \mathbb{Z}^n$ satisfies $x_j \leq \hat{x}_j$ and has $x_j = \hat{x}_j$ for $j \in J$ such that $\bigcup_{j \in J} M_j = M$. It is key here to note that, if x_j is active in $Ax = b$, then

$x_j = \hat{x}_j$. It follows that, if $c^T \otimes x = c_j \otimes x_j$, then $x_j = \hat{x}_j$ (x_j is active in $Ax = b$) because we can reduce the value of any x_t which is inactive in $Ax = b$ while remaining feasible.

Therefore, the value of $c^T \otimes x$ is determined by the active entries of x with respect to $Ax = b$, whose value is predetermined. Since we are looking for the minimum possible value, we begin with the smallest values of $c_j \otimes x_j$ first, and test whether a feasible solution exists with just these minimum values active. If yes, we are done, if not we add the next smallest value to our candidates for active entries, and repeat.

Complexity: Step 1 requires $\mathcal{O}(mn)$ operations since calculating $\hat{x}$ needs $(2m-1)n + n$ and calculating each of n $M_j's$ require m comparisons. Step 2 requires $\mathcal{O}(n \log n)$ operations to calculate n products of the form $c_j \otimes x_j$ and then order them. The loop 4,5,6 requires $\mathcal{O}(mn)$ and is repeated at most n times (after n times $J = N$ and so there is nothing left to consider). □

Remark: From Corollary 6.3 and Theorem 6.5 we have that $OIMLP$ is strongly polynomially solvable.

6.2 Two-sided constraints

In this section we develop algorithms to solve the integer max linear program for problems with finite entries, so we assume throughout that A, B, c, d, f are finite. Recall that the IMLP has the form

$$\begin{aligned} & f^T \otimes x = f(x) \to \text{ min or max} \\ & \text{s.t. } Ax \oplus c = Bx \oplus d \\ & x \in \mathbb{Z}^n. \end{aligned}$$

We will write $IMLP^{\max}$ to mean the integer max-linear program which maximises $f(x) := f^T x$, and $IMLP^{\min}$ to denote the program minimising $f^T x$, where $f \in \mathbb{R}^n$. For $A, B \in$

$\mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m$ we denote

$$S = \{x \in \mathbb{R}^n : Ax \oplus c = Bx \oplus d\},$$
$$IS = S \cap \mathbb{Z}^n,$$
$$IS^{\min} = \{x \in IS : f(x) \leq f(z)\ \forall z \in IS\},$$
$$IS^{\max} = \{x \in IS : f(x) \geq f(z)\ \forall z \in IS\}.$$

The method described here is based on the bisection method used to find real optimal solutions to max-linear programs described in [18].

In [18] Proposition 6.6 below explains why it is valid to use a bisection method to solve the MLP.

Proposition 6.6. *[18] If $x, y \in S, f(x) = \alpha < \beta = f(y)$ then for every $\gamma \in (\alpha, \beta)$ there is a $z \in S$ satisfying $f(z) = \gamma$.*

Proposition 6.6 is not strong enough to justify using a bisection method for problems with integrality requirements since it does not ensure that $z \in IS$. We can however construct a similar argument which serves this purpose.

Proposition 6.7. *Let $x, y \in IS$ with $f(x) = \alpha < \beta = f(y)$. Then for all $\gamma \in (\alpha, \beta)$ with $fr(\gamma) = fr(\beta)$ there exists $z \in IS$ such that $f(z) = \gamma$.*

Proof. Take $\lambda = 0$ and $\mu = \gamma \otimes \beta^{-1} \in \mathbb{Z}$. Let $z = \lambda \otimes x \oplus \mu \otimes y$. Now S is a max-convex set [18] and so since $\lambda \oplus \mu = 0$ we have $z \in S \cap \mathbb{Z}^n = IS$. Finally

$$f(z) = \lambda \otimes f(x) \oplus \mu \otimes f(y) = \alpha \oplus \gamma = \gamma.$$

□

We also need the following results. The first follows from the cancellation rule, Lemma 1.16.

Lemma 6.8. *[18] Let $\alpha, \alpha' \in \mathbb{R}$, $\alpha' < \alpha$ and $f(x) = f^T \otimes x, f'(x) = f'^T \otimes x$ where $f'_j < f_j$ for every $j \in N$. Then the following holds for every $x \in \mathbb{R}$: $f(x) = \alpha$ if and only if $f(x) \oplus \alpha' = f'(x) \oplus \alpha$.*

Since the result holds for real vectors x it clearly also holds for integer x. Using this and the cancellation law we can check whether $f(x)$ attains some value α as detailed in the following proposition.

Proposition 6.9. *$f(x) = \alpha$ for some $x \in IS$ if and only if the following integer max-linear system has a solution:*

$$A \otimes x \oplus c = B \otimes x \oplus d,$$

$$f(x) \oplus \alpha' = f'(x) \oplus \alpha,$$

$$x \in \mathbb{Z}^n,$$

where $\alpha' < \alpha$ and $f'_j < f_j$ for every $j \in N$.

We know from Section 5.1 that we can decide whether a two-sided system has a solution by applying Algorithm GEN-INT-TSS.

6.2.1 When the objective function is unbounded

We now consider the question of when optimal solutions exist. We denote the minimum and maximum of f by $f^{\min}$ and $f^{\max}$ respectively. Without loss of generality we assume that $c \geq d$. Following the work in [18], we denote $M^{>} = \{i \in M : c_i > d_i\}$. Let

$$L_r = \min_{t \in N}(f_t \otimes c_r \otimes b_{rt}^{-1})$$

and

$$L = \left\lfloor \max_{r \in M^{>}} L_r \right\rfloor. \tag{6.1}$$

Note that $L = -\infty$ if $M^{>} = \emptyset$.

Lemma 6.10. *If $c \geq d$ then $f(x) \geq L$ for every $x \in IS$.*

Proof. From Lemma 10.2.9 in [18] we have that the result holds for every $x \in S$. □

Theorem 6.11. *[18] Consider MLP$^{\min}$. Then $f^{\min} = -\infty$ if and only if $c = d$.*

Proof. If $c = d$ then $\alpha x \in IS$ for all $x \in \mathbb{Z}^n$ and $\alpha \in \mathbb{Z}$ with α small enough. Letting α tend to $-\infty$ gives the first direction. If $c \neq d$ then $L > -\infty$ and so we can apply Lemma 6.10. □

For the upper bound we need the following results.

Lemma 6.12. *Let $c \geq d$. If $x \in IS$ and $(Ax)_i > c_i$ for all $i \in M$ then $x' = \alpha x \in IS$ and $(Ax')_i \leq c_i \otimes 1$ for some $i \in M$ where*

$$\alpha = \left\lceil \max_{i \in M} \left(c_i (Ax)_i^{-1} \right) \right\rceil. \tag{6.2}$$

Proof. Assume $x \in IS$. Then $Ax = Bx$ since $Ax > c \geq d$. From the choice of α we get that

$$(A(\alpha x))_i = \alpha (Ax)_i \geq c_i$$

for all $i \in M$. Further, since $A(\beta x) = B(\beta x)$ for any $\beta \in \mathbb{Z}$, we have $x' \in IS$.

Finally let $t \in M$ be an index at which the value of α is attained. Then

$$(A \otimes x')_t = \lceil c_t \otimes (A \otimes x)_t^{-1} \rceil (A \otimes x)_t \leq c_t \otimes (Ax)_t^{-1} \otimes 1 \otimes (Ax)_t \leq c_t \otimes 1.$$

□

Let

$$U = \left\lceil \max_{r\in M}\max_{j\in N}(f_j \otimes a_{rj}^{-1} \otimes c_r \otimes 1)\right\rceil. \tag{6.3}$$

Lemma 6.13. *If $c \geq d$ then the following hold:*

(i) If $x \in IS$ and $(Ax)_r \leq c_r \otimes 1$ for some $r \in M$ then $f(x) \leq U$.

(ii) If $Ax = Bx$ has no integer solution then $f(x) \leq U$ for every $x \in IS$.

Proof. (i) For all $j \in N$ we have that $a_{rj} \otimes x_j \leq c_r \otimes 1$. Thus

$$f(x) = \max_{j\in N}(f_j \otimes x_j) \leq \max_{j\in N}(f_j \otimes a_{rj}^{-1} \otimes c_r \otimes 1) \leq U.$$

(ii) If $IS = \emptyset$ then there is nothing to prove so assume that $x \in IS$. Since $A\otimes x \neq B\otimes x$ there exists $r \in M$ such that $(A \otimes x) \leq c_r \leq c_r \otimes 1$ and so we can apply (i). □

Theorem 6.14. *Consider IMLP$^{\max}$. Then $f^{\max} = +\infty$ if and only if $Ax = Bx$ has an integer solution.*

Proof. Without loss of generality $c \geq d$. If $Ax = Bx$ does not have an integer solution then we know from Lemma 6.13 that $f^{\max}$ is bounded from above. Conversely, if $Az = Bz, z \in \mathbb{Z}^n$ then for all large enough $\alpha \in \mathbb{Z}$ we have

$$A(\alpha z) = B(\alpha z) \geq c \geq d.$$

Thus $(\alpha z) \in IS$ and $f(\alpha z) \to +\infty$ as $\alpha \to +\infty$. □

So far we have shown that we can determine immediately when $f^{\min}$ is unbounded and can check whether $f^{\max}$ is unbounded, for example by applying the Algorithm GEN-INT-TSS. We now need to argue that when the objective function value is bounded there exist integer vectors for which $f^{\max}$ and $f^{\min}$ are attained.

6.2.2 Attainment of optimal values

For all $j \in N$ let

$$h_j = \left\lfloor \min\left(\min_{r\in M}(a_{rj}^{-1} \otimes c_j), \min_{r\in M}(b_{rj}^{-1} \otimes d_j), f_j^{-1} \otimes L \right) \right\rfloor,$$
$$h'_j = \left\lfloor \min\left(\min_{r\in M}(a_{rj}^{-1} \otimes c_j), \min_{r\in M}(b_{rj}^{-1} \otimes d_j) \right) \right\rfloor, \tag{6.4}$$

$h = (h_1, ..., h_n)^T$ and $h' = (h'_1, ..., h'_n)^T$. Observe that h' is finite and h is finite if and only if $f^{\min} > -\infty$.

A direct consequence of Propositions 10.2.14 and 10.2.16 in [18] is the following result.

Proposition 6.15. *Let h and h' be as defined above.*

(i) For any $x \in IS$ the vector $x' = x \oplus h \in IS$ satisfies $x' \geq h$ and $f(x) = f(x')$.

(ii)For any $x \in IS$ the vector $x' = x \oplus h' \in IS$ satisfies $x' \geq h'$ and $f(x) \leq f(x')$.

Corollary 6.16. *Let h and h' be as defined above.*

(i)If $f^{\min} > -\infty$ and $IS \neq \emptyset$ then

$$f^{\min} = \min_{x\in \overline{IS}} f(x)$$

where $\overline{IS} = IS \cap \{x \in \mathbb{Z}^n : h_j \leq x_j \leq f_j^{-1} \otimes f(\bar{x}),\ j \in N\}$.

(ii) If $f^{\max} < +\infty$ then

$$f^{\max} = \max_{x\in \overline{IS}'} f(x)$$

where $\overline{IS}' = IS \cap \{x \in \mathbb{Z}^n : h'_j \leq x_j \leq f_j^{-1} \otimes U,\ j \in N\}$.

Proof. Similar to the proof of Corollary 10.2.12 and Corollary 10.2.17 in [18] □

We can therefore conclude that, provided the objective function value is bounded, a feasible solution implies the existence of an optimal solution. We summarise this below.

Corollary 6.17. *If $IS \neq \emptyset$ and $f^{\min} > -\infty$ $[f^{\max} < +\infty]$ then $IS^{\min} \neq \emptyset$ $[IS^{\max} \neq \emptyset]$.*

Finally we need a finite lower bound on $f^{\max}$ since L will not work in the case when $c = d$.

Corollary 6.18. *Let $L' = \lfloor f(h') \rfloor$. If $x \in IS$ then $x' = x \oplus h'$ is such that $f(x') \geq L'$ and hence $f^{\max} \geq L'$.*

6.2.3 The algorithms

Algorithm when minimising the objective function

For minimisation we know that an optimal solution exists provided $c \neq d$. We first check whether $f^{\min} = L$ using Proposition 6.9, if so we are done, if not then we find any feasible x^0 using the Algorithm 5.4 (GEN-INT-TSS) and, if necessary scale it using (6.2) so that $f(x^0) \leq U$. Then we know that $f^{\min} \in (L, f(x^0)]$

Once we know that, for any $x \in IS$, $f(x)$ satisfies

$$L \leq f(x) \leq f(x^0) = U$$

we can set

$$\Theta = \{\theta : \theta \in (L, U] \text{ and } fr(\theta) = fr(U)\}$$

and apply a bisection method on the set Θ as follows.

1. Order $\theta_i \in \Theta$ from smallest to largest and test whether the middle value, θ, is attained for any $x \in IS$.

2. If it is then we have a new upper bound; $f(x) \leq \theta$.

3. If it is not then we have a new lower bound; $f(x) > \theta$.

In 3 we use the fact that if θ is unattainable then no value in $(L, \theta]$ is attainable. To see this note that, if there exists $\alpha \in (L, \theta]$ attainable then by Proposition 6.7 all values in (α, U) with fractional part equal to that of U work, but θ satisfied this condition and was not attainable. In each case we have halved the number of values that we need to test.

Using this idea, we obtain the following algorithm for solving $IMLP^{\min}$.

Algorithm 6.19. INT-MAXLINMIN

Input: $A, B \in \mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m, c \geq d, c \neq d, f \in \mathbb{R}^n$.

Output: $x \in IS$ *such that* $f(x) = f^{\min}$.

1. Calculate L *from* (6.1). *If* $L = f(x)$ *for some* $x \in IS$ *then STOP,* $f^{\min} = L$.

2. Find an $x^0 \in IS$. *If* $(A \otimes x^0)_i > c_i$ *for all* $i \in M$ *then scale* x^0 *by* α *defined in* (6.2).

3. $L(0) := L, U(0) := f(x^0), r := 0$.

4. $\Theta := \{\theta : \theta \in (L(r), U(r)]$ *and* $fr(\theta) = fr(U(r))\}$, $\eta := |\Theta|$. *If* $\eta = 1$ *go to 9.*

5. Take $\theta \in \Theta \cap \left[\frac{L(r)+U(r)}{2} - \frac{1}{2}, \frac{L(r)+U(r)}{2} + \frac{1}{2}\right]$.

6. Check whether $f(x) = \theta$ *for some* $x \in IS$ *and if so find one.*

If yes then $U(r+1) = \theta, L(r+1) = L(r)$.

If no then $U(r+1) := U(r), L(r+1) = \theta$.

8. $r := r + 1$, *go to 4.*

9. Find the smallest $\gamma \in (L(r), U(r)]$ *such that* $(\exists j) fr(f_j) = fr(\gamma)$ *and* $f(x) = \gamma$ *for some* $x \in IS$ *(by checking each of the, at most* n *values). Output* $f^{\min} = \gamma$ *and* x.

Note that, in the run of Algorithm INT-MAXLINMIN, whenever the algorithm checks whether $f(x) = \theta$ for some given θ and $x \in IS$, it solves the TSS described in Proposition 6.9. Solving the TSS can be done by any known method.

Theorem 6.20. *Algorithm INT-MAXLINMIN is correct and terminates after at most* $\mathcal{O}(\log(\lceil U - L \rceil))$ *iterations.*

Let

$$\bar{K} = \left\lceil \max\{|a_{ij}|, |b_{ij}|, |c_i|, |d_i|, |f_j| : i \in M, j \in N\} \right\rceil.$$

Observe that $L, L', U \in [-3\bar{K}, 3\bar{K}]$.

Corollary 6.21. *If the Algorithm GEN-INT-TSS is used to perform the checks in steps 1, 6 and 8 then Algorithm INT-MAXLINMIN has complexity $\mathcal{O}(mn(m+n)\bar{K}\log\bar{K})$.*

Proof. The number of iterations is $\mathcal{O}(\log(\lceil U - L\rceil)) \leq \mathcal{O}(\log 6\bar{K}) = \mathcal{O}(\log\bar{K})$. Each iteration uses Algorithm GEN-INT-TSS which, from Corollary 5.5, requires

$$\mathcal{O}(K(X|Y)m'n'(m'+n'))$$

operations where $K(X|Y)$ is defined in (5.1), $m' = m+1$, $n' = n+1$ and

$$X = \begin{pmatrix} A & c \\ f^T & \alpha' \end{pmatrix}, \; Y = \begin{pmatrix} B & d \\ f'^T & \alpha \end{pmatrix}.$$

We can choose α' and f' so that $\alpha - 1 \leq \alpha' \leq \alpha$, $f_j - 1 \leq f'_j \leq f_j$ and hence $K(X|Y) \leq \bar{K}+1$. Therefore the number of operations in a single iteration is $\mathcal{O}(\bar{K}mn(m+n))$. $\square$

Algorithm when maximising the objective function

For maximisation we cannot assume that $c \neq d$ since this is not the criterion for $f^{\max}$ to be unbounded. We must first check that $f^{\max} < +\infty$ by verifying that $Ax = Bx$ has no integer solution, which can be done using Algorithm GEN-INT-TSS. We then check whether $f^{\max} = U$ (where U is defined in (6.3)) using Proposition 6.9. If not then we find any feasible solution x^0 and, set $x^0 := x^0 \oplus h'$ so that $f(x^0) \geq L'$.

Further when maximising it is no longer enough to only check values in the interval with a single fractional part. This is because the upper bound is not attained, and so we can no longer guarantee that the optimal value shares its fractional part with U. However we do know that there are only a finite number of possible fractional parts that could be attained, these are $fr(f_i)$ for all i because $f^T \otimes x$ for $x \in \mathbb{Z}^n$ can only take its fractional part from the elements of f. Once we know, for all $x \in IS$, $L = f(x^0) \leq f(x) \leq U$ we proceed as follows.

1. Let $[J, J+1)$ be an interval contained halfway between L and U.

2. Test each of the (at most n) values in this interval that share the same fractional part as a component of f to see whether they are attained by some $x \in IS$,

3. If one exists then the largest becomes a new lower bound.

4. If none in the interval are attained then Proposition 6.7 guarantees that no value higher than J can be attained and thus we have a new upper bound.

Continue in this way, each time approximately halving the length of the interval until $U - L \leq 2$. In this case the interval $[J, J+1)$ may not be contained entirely in (L, U) and so testing points in this smaller interval is no longer efficient since we will check unnecessary points, or find L again. So instead check the remaining $\leq 2n$ possible points and choose the one with smallest value.

We obtain the following algorithm for $IMLP^{\max}$.

Algorithm 6.22. INT-MAXLINMAX

Input: $A, B \in \mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m, c \geq d, f \in \mathbb{R}^n$.

Output: $x \in IS$ *such that* $f(x) = f^{\max}$.

1. Calculate U *from* (6.3). *If* $U = f(x)$ *for some* $x \in IS$ *then STOP,* $f^{\max} = U$.

2. Check whether $Ax = Bx$ *has an integer solution. If yes STOP,* $f^{\max} = +\infty$.

3. Find an $x^0 \in IS$. *Set* $x^0 := x^0 \oplus h'$ *as defined in* (6.4).

4. $L(0) := f(x^0), U(0) := U, r := 0$.

5. If $U - L \leq 2$ *go to 8. Else let* $J := \frac{1}{2}(U(r) + L(r))$.

6. Using a bivalent search find the biggest $\sigma \in [J, J+1)$ *such that* $(\exists j) fr(f_j) = fr(\sigma)$

and $f(x) = \sigma$ *for some* $x \in IS$.

If none exist then $U(r+1) := J, L(r+1) = L(r)$.

Otherwise $U(r+1) = U(r), L(r+1) = \sigma$.

7. $r := r+1$, *go to 5.*

8. Using a bivalent search find the biggest $\gamma \in (L(r), U(r))$ *such that* $(\exists j) fr(f_j) = fr(\gamma)$ *and* $f(x) = \gamma$ *for some* $x \in IS$.

If none exist then STOP, $f^{\max} = L(r)$.

Otherwise STOP, $f^{\max} = \gamma$.

Note that, in the run of Algorithm INT-MAXLINMAX, whenever the algorithm checks whether $f(x) = \theta$ for some given θ and $x \in IS$, it solves the TSS described in Proposition 6.9. Solving the TSS can be done by any known method.

Theorem 6.23. *Algorithm INT-MAXLINMAX is correct and terminates after at most* $\mathcal{O}(\log(U - L'))$ *iterations where* $L' = \lfloor f(h') \rfloor$.

Corollary 6.24. *If the Algorithm GEN-INT-TSS is used to perform the checks in steps 1, 6 and 8 then Algorithm INT-MAXLINMAX has complexity* $\mathcal{O}(mn(m+n)\log(n)\bar{K}\log(\bar{K}))$.

Proof. The same as the proof of Corollary 6.21 with L replaced by L' but here we have that each iteration uses the Algorithm GEN-INT-TSS at most $\log(2n)$ times. □

Remark 6.25. *We note here that, for systems satisfying Property OneFP (see Definition 5.8), the use of Algorithm GEN-INT-TSS in both Algorithm INT-MAXLINMIN and INT-MAXLINMAX can be replaced checking the conditions in Theorem 5.14. In this case these algorithms for the IMLP become polynomial, details are contained in [25].*

6.3 Strongly polynomial algorithm for IMLP under Property OneFP

The aim of this section is to develop strongly polynomial methods for solving $IMLP^{\min}$ and $IMLP^{\max}$ under the assumption that Property OneFP holds. Recall that the IMLP has the form

$$
\begin{aligned}
&f^T \otimes x \rightarrow \text{ min or max} \\
&\text{s.t. } Ax \oplus c = Bx \oplus d, \\
&x \in \mathbb{Z}^n,
\end{aligned}
\tag{6.5}
$$

where $A, B \in \mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m, f \in \mathbb{R}^n$. We can write the constraints of the IMLP as

$$
\begin{pmatrix} A|c \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} B|d \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{6.6}
$$

$$x \in \mathbb{Z}^n.$$

Recall $A^{(-1)} := -A \in \overline{\overline{\mathbb{R}}}^{m\times n}$. From Theorems 6.11 and 6.14 we have, $f^{\min} = -\infty$ if and only if $c = d$ and $f^{\max} = +\infty$ if and only if there exists an integer solution to $Ax = Bx$.

We will need the following immediate corollary of Theorem 2.5.

Corollary 6.26. *If A is integer and $\lambda(A) \leq 0$, then*

$$IV^*(A, 0) = \{A^* z : z \in \mathbb{Z}^n\}.$$

Recall from Proposition 5.6 that a necessary condition for the existence of an integer solution to either $Ax = By$ or $Ax = Bx$ is that

$$(\forall i \in M)(\exists j \in N, t \in K)\ fr(a_{ij}) = fr(b_{it}) \text{ and } a_{ij}, b_{it} \in \mathbb{R}.$$

Further, under the assumption that Property OneFP holds, we denote the positions of the pairs of entries with the same fractional parts in row i by $(r(i), r'(i))$. We also assume without loss of generality that the entries $(a_{i,r(i)}, b_{i,r'(i)})$ are integer and no other entries in the equation for either matrix are integer.

From Corollary 5.10, the entries $a_{i,r(i)}$ $[b_{i,r'(i)}]$ are the only possible active entries in the matrix A $[B]$ with respect to any integer vector x $[y]$ satisfying $Ax = By$. Additionally general systems can be converted into systems with separated variables by Corollary 5.16 and that this conversion will preserve Property OneFP. So Corollary 5.10 holds accordingly for general systems. Hence we restrict our attention to the case of separated variables.

6.3.1 Consequences of Property OneFP

Let $z = (x^T, 0)^T \in \mathbb{Z}^{n+1}$. By Proposition 5.16, the constraint (6.6) is equivalent to the condition that there exists $y \in \mathbb{Z}^{n+1}$ such that (z, y) is an integer solution to $A'z = B'y$ where

$$A' := \begin{pmatrix} A|c \\ I \end{pmatrix} \in \overline{\mathbb{R}}^{(m+n+1)\times(n+1)}, B' := \begin{pmatrix} B|d \\ I \end{pmatrix} \in \overline{\mathbb{R}}^{(m+n+1)\times(n+1)}.$$

This is since if (z, y) is an integer solution to $A'z = B'y$ then so is $(z_{n+1}^{-1}z, z_{n+1}^{-1}y)$ where $z_{n+1}^{-1}z = (x^T, 0)^T$ and $z_{n+1}^{-1}y = y_{n+1}^{-1}y = (x^T, 0)^T$.

Proposition 6.27. *Let $A, B \in \mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m$. If there exists a row in which the matrices $(A|c)$ and $(B|d)$ do not have entries with the same fractional part, then the feasible set of $IMLP^{\min}$ is empty.*

Proof. It follows from Proposition 5.6. □

For the rest of the section we will assume that the pair $((A|c),(B|d))$ satisfies Property OneFP, and hence so does (A', B'). Note that an example is provided at the end of this section to clarify many of the concepts that will be introduced in what follows.

Corollary 6.28. *Let A', B' be as defined above. Let*

$$W := (w_{ij}) \in \overline{\mathbb{Z}}^{(m+n+1)\times(m+n+1)}$$

where for all $i, j \in \{1, ..., m+n+1\}$,

$$w_{ij} := (a'_{i,r(i)})^{-1} \lceil a'_{j,r(i)} \rceil \oplus (b'_{i,r'(i)})^{-1} \lceil b'_{j,r'(i)} \rceil.$$

Then a feasible solution to IMLP exists if and only if $\lambda(W) \le 0$. If this is the case, then

$$A'z = B'z$$

where $z_j = \gamma_{m+j}^{-1}$ for any $\gamma \in IV^*(W, 0)$ and $j \in \{1, ..., n+1\}$.

Proof. Existence follows from Theorem 5.14.

Assume that $\lambda(W) \le 0$, hence for all $\gamma \in IV^*(W, 0)$

$$\begin{pmatrix} A|c \\ I \end{pmatrix} z = \gamma^{(-1)} = \begin{pmatrix} B|d \\ I \end{pmatrix} y.$$

Let $\mu \in \mathbb{Z}^{n+1}$ be defined by $\mu_j = \gamma_{m+j}, j = 1, ..., n+1$, and note that since γ is finite so is μ. Then

$$Iz = \mu^{(-1)} = Iy.$$

□

Remark 6.29. *(i) For A', B' as defined above, W can be calculated in $\mathcal{O}((m+n)^2)$ time, $\lambda(W)$ in $\mathcal{O}((m+n)^3)$ time and W^* in $\mathcal{O}((m+n)^3)$ time.*

(ii) Clearly $w_{ii} = 0$ for all $i \in \{1, ..., m+n+1\}$, and so $\lambda(W) \geq 0$. Hence an integer solution to the TSS exists if and only if $\lambda(W) = 0$.

This matrix W, constructed from A' and B', will play a key role in the solution of the IMLP. To construct the i^{th} row of W we only consider columns $A'_{r(i)}$ and $B'_{r'(i)}$. Define $A'' := (A|c)$ and $B'' := (B|d)$. Observe that the i^{th} row is equal to $H(i)^T$ for

$$H(i) = (a'_{i,r(i)})^{-1} \begin{pmatrix} \lceil A''_{r(i)} \rceil \\ I_{r(i)} \end{pmatrix} \oplus (b'_{i,r'(i)})^{-1} \begin{pmatrix} \lceil B''_{r'(i)} \rceil \\ I_{r'(i)} \end{pmatrix}, \tag{6.7}$$

Also,

$$H(i)_t > \varepsilon \text{ for all } i \in \{1, ..., m+n+1\}, t \in \{1, ..., m\}$$

since A and B are finite. Further when $i \in \{m+1, ..., m+n+1\}$, $i = m+j$ say, then $r(i) = j = r'(i)$ and $I_{i,r(i)} = 0 = I_{i,r'(i)}$. Hence

$$H(i) = \begin{pmatrix} \lceil A''_j \rceil \\ I_j \end{pmatrix} \oplus \begin{pmatrix} \lceil B''_j \rceil \\ I_j \end{pmatrix} = \begin{pmatrix} \lceil A''_j \rceil \oplus \lceil B''_j \rceil \\ I_j \end{pmatrix}.$$

Therefore the matrix $W \in \overline{\mathbb{Z}}^{m+n+1}$ has the form

$$\begin{pmatrix} P & Q \\ R & I \end{pmatrix}$$

where $P \in \mathbb{Z}^{m \times m}$, $Q \in \overline{\mathbb{Z}}^{m \times (n+1)}$, $R \in \mathbb{Z}^{(n+1) \times m}$, $I \in \overline{\mathbb{Z}}^{(n+1) \times (n+1)}$.

Moreover each row of Q has either one or two finite entries: for a fixed $i \in \{1, ..., m\}$, the entries $w_{ij}, j \in \{m+1, ..., m+n+1\}$ are obtained by calculating

$$\max(\lceil a'_{j,r(i)} \rceil - a'_{i,r(i)},\ \lceil b'_{j,r'(i)} \rceil - b'_{i,r'(i)})$$

where

$$a'_{j,r(i)}, j \in \{m+1, ..., m+n+1\}$$

form a unit vector, as do

$$b'_{j,r'(i)}, j \in \{m+1, ..., m+n+1\},$$

so at least one will be finite and, if $r(i) \neq r'(i)$, there will be exactly two.

From Corollary 6.28 we have

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = z = \mu^{(-1)}$$

where μ is the vector of the last $n+1$ entries of some $\gamma \in IV^*(W, 0)$. By Corollary 6.26, $\gamma = W^* \omega$ for some integer vector ω. Let $V = (v_{ij})$ be the matrix formed of the last $n+1$ rows of W^*, so that $\mu = V \otimes \omega$ for $\omega \in \mathbb{Z}^{m+n+1}$, equivalently

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = z = V^{(-1)} \otimes' \omega^{(-1)}. \tag{6.8}$$

Now (6.8) can be split into two equations, one for the vector x and one for the scalar 0. Further we would like the second equation to be of the form $\min_t w_t = 0$ for ease of calculations later. This leads to the following definition.

Definition 6.30. *Let $V^{(0)}$ be the matrix formed from $V^{(-1)}$ by max-multiplying each finite column j by $v_{m+n+1,j}$, and then removing the final row (at least one finite column exists* *by Property OneFP). Let* $U \in \overline{\mathbb{R}}^{1\times(m+n+1)}$ *be the row that was removed.*

Note that U contains only 0 or $+\infty$ entries.

Proposition 6.31. *Let* $A, B, c, d, V^{(0)}$ *and* U *be as defined in* (6.5) *and Definition 6.30. Then* $x \in \mathbb{Z}^n$ *is a feasible solution to IMLP if and only if it satisfies*

$$x = V^{(0)} \otimes' \nu$$
$$\textit{where } 0 = U \otimes' \nu$$
$$\textit{for some } \nu \in \mathbb{Z}^{m+n+1}.$$

Proof. By Corollary 6.28 x is feasible if and only if $(x^T, 0)^T = \mu^{(-1)}$ where μ is the vector containing the last $n+1$ components of some $\gamma \in IV^*(W, 0)$. By the above discussion this means that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = V^{(-1)} \otimes' w^{(-1)} = \begin{pmatrix} V^{(0)} \\ U \end{pmatrix} \otimes' \nu.$$

□

We will first consider (in Subsection 6.3.2) solutions to $IMLP$ when W^*, and hence also $V^{(0)}$ and U, are finite. In Subsections 6.3.4 and 6.3.4 we deal with the case when W^* is not finite.

Before this we summarise key definitions and assumptions that will be used throughout the remainder of this section, for easy reference later.

Assumption 6.32. *We assume the following are satisfied.*

(i) $A, B \in \mathbb{R}^{m\times n}, c, d \in \mathbb{R}^m$.

(ii) $A'' := (A|c), B'' := (B|d)$ *and*

$$A' := \begin{pmatrix} A|c \\ I \end{pmatrix}, B' := \begin{pmatrix} B|d \\ I \end{pmatrix}.$$

(iii) The pair (A'', B'') satisfies Property OneFP (and therefore also (A', B')).

(iv) W is constructed from A', B' according to Corollary 6.28.

(v) Without loss of generality $\lambda(W) = 0$.

(vi) V is the matrix containing the last $n+1$ rows of W.

6.3.2 Finding the Optimal Solution to IMLP When W^* is Finite

Theorem 6.33. *Let A, B, c, d satisfy Assumption 6.32 and $V^{(0)}$ be as in Definition 6.30. If W^* is finite, then the optimal objective value $f^{\min}$ is attained for*

$$x^{opt} = V^{(0)} \otimes' \mathbf{0}.$$

Proof. By Proposition 6.31, we know that any feasible x satisfies $x = V^{(0)} \otimes' \nu$ where, by the finiteness of W^* (and also $V^{(0)}$), we have $U = \mathbf{0}$ and hence

$$\nu_1 \oplus' \ldots \oplus' \nu_{m+n+1} = 0.$$

Therefore $x \geq V^{(0)} \otimes' \mathbf{0}$ for any feasible x and further $V^{(0)} \otimes' \mathbf{0}$ is feasible. The statement now follows from the isotonicity of $f^T x$, see Corollary 1.7. □

Theorem 6.34. *Let A, B, c, d satisfy Assumption 6.32 and $V^{(0)}$ be as in Definition 6.30. If W^* is finite, then the optimal objective value $f^{\max}$ is equal to*

$$f^T \otimes V^{(0)} \otimes \mathbf{0}.$$

Further let $y := V^{(0)} \otimes \mathbf{0}$ and j be an index such that $f^{\max} = f_j y_j$. If i is such that $y_j = V^{(0)}_{ji}$, then an optimal solution is $x^{opt} = V^{(0)}_i$.

Proof. By Proposition 6.31, we know that any feasible x satisfies $x = V^{(0)} \otimes' \nu$ where, by the finiteness of W^* (and also $V^{(0)}$), we have $U = \mathbf{0}$ and hence

$$\nu_1 \oplus' ... \oplus' \nu_{m+n+1} = 0.$$

If $\nu_j = 0$, then $x \leq V_j^{(0)}$ and therefore all feasible x satisfy $x \leq y = V^{(0)} \otimes \mathbf{0}$. Note that y may not be feasible.

By isotonicity, $f^T y \geq f^T x$ for any feasible x. We claim that there exists a feasible solution x for which they are equal. Suppose that $f^T y = f_j y_j$. Let i be an index such that $v_{ji}^{(0)} = y_j$. By setting $\nu_i = 0$ and all other components to large enough integers we get a feasible solution $\bar{x}$ such that $\bar{x}_j = y_j$. In fact $\bar{x} = V_i^{(0)}$. Hence

$$f_j \bar{x}_j = f_j y_j = f^T y \geq f^T \bar{x} \geq f_j \bar{x}_j,$$

which implies $f^T y = f^T \bar{x}$ as required. □

It follows from Theorems 6.33 and 6.34 that, if $\lambda(W) \leq 0$ and W^* is finite, then an optimal solution to $\text{IMLP}^{\min}$ and $\text{IMLP}^{\max}$ always exists.

6.3.3 Criterion for Finiteness of W^*

Theorems 6.33 and 6.34 provide explicit solutions to IMLP, which can be found in $\mathcal{O}((m+n)^3)$ time by Remark 6.29 in the case when W^* is finite. We now consider criteria for W^* to be non-finite and show how we can adapt the problem in this case so that IMLP can be solved using the above methods in general.

Proposition 6.35. *Let A, B, c, d satisfy Assumption 6.32.*

Let $e_j \in \overline{\mathbb{R}}^{m+n+1}$ be the j^{th} unit vector. The following are equivalent:

(i) W^* contains an ε entry.

(ii) There exists $j \in \{1, ..., n+1\}$ such that $W^*_{m+j} = e_{m+j}$.

(iii) There exists $j \in \{1, ..., n+1\}$ such that $W_{m+j} = e_{m+j}$.

(iv) There exists $j \in \{1, ..., n+1\}$ such that neither A''_j nor B''_j contain an integer entry.

Further the index j satisfies the condition in (ii) if and only if j satisfies the condition in (iii) if and only if j satisfies the condition in (iv).

Proof. Recall that W has the form

$$\begin{pmatrix} P & Q \\ R & I \end{pmatrix}$$

where $P \in \mathbb{Z}^{m\times m}$, $Q \in \overline{\mathbb{Z}}^{m\times(n+1)}$, $R \in \mathbb{Z}^{(n+1)\times m}$, $I \in \overline{\mathbb{Z}}^{(n+1)\times(n+1)}$.

(ii)$\Rightarrow$(i): Obvious.

$\neg$(iii)$\Rightarrow$ $\neg$(i): Assume that, for all j, $W_j \neq e_j$. We know that the first m columns of W are finite and, by assumption, every column of Q contains a finite entry. This means that W^2 will be finite and thus so will W^*.

(ii)$\Leftrightarrow$(iii): We show $W_{m+j} = e_{m+j}$ if and only if $W^2_{m+j} = e_{m+j}$. Fix j such that $W_{m+j} = e_{m+j}$. Then clearly $W^2_{m+j} = e_{m+j}$ and hence (iii)$\Rightarrow$(ii). Although (ii)$\Rightarrow$ (iii) follows from above we need to also prove that the same index j satisfies both statements. To do this we suppose that $W^2_{m+j} = e_{m+j}$. Then for all $i \in \{1, ..., m\}$ with $i \neq j$ we have

$$\begin{pmatrix} w_{i,1} & \dots & w_{i,m} \end{pmatrix} \otimes \begin{pmatrix} w_{1,m+j} \\ \vdots \\ w_{m,m+j} \end{pmatrix} \oplus \begin{pmatrix} w_{i,m+1} & \dots & w_{i,m+n+1} \end{pmatrix} \otimes I_j = \varepsilon$$

where $w_{i,1}, ..., w_{i,m} \in \mathbb{R}$. Thus

$$w_{1,m+j} = ... = w_{m,m+j} = \varepsilon$$

and hence $W_{m+j} = e_{m+j}$.

(iii) $\Leftrightarrow$ (iv): By the structure of W, (iii) holds if and only if Q contains an ε column. Fix $j \in \{1, ..., n+1\}$. Now, for any $i \in M$,

$$\begin{aligned} & q_{ij} = \varepsilon \\ \Leftrightarrow\ & w_{i,m+j} = \varepsilon \\ \Leftrightarrow\ & a'_{m+j,r(i)} = \varepsilon = b'_{m+j,r'(i)} \\ \Leftrightarrow\ & r(i) \neq j \text{ and } r'(i) \neq j \\ \Leftrightarrow\ & a''_{ij}, b''_{ij} \notin \mathbb{Z}. \end{aligned}$$

Therefore Q contains an ε column if and only if neither $A'' = (A|c)$ nor $B'' = (B|d)$ contain an integer entry. $\square$

Observe that, for each $j \in \{1, ..., n+1\}$, either $W^*_{m+j} = e_{m+j}$ or W^*_{m+j} is finite. Further W^*_t is finite for all $t \in M$ since P and R are finite.

Corollary 6.36. *Let A, B, c, d satisfy Assumption 6.32. W^* is finite if and only if for all $j \in \{1, ..., n+1\}$ either $(A|c)_j$ or $(B|d)_j$ contains an integer entry.*

6.3.4 IMLP When W^* is Non-Finite

Theorems 6.33 and 6.34 solve IMLP when W^* is finite. In this case $U = \mathbf{0}$ and we took advantage of the fact that $\nu_i \geq 0$ held for every component of ν. However, if $W^*_{m+j} = e_{m+j}$ for some $j \in N$, then $U_j = +\infty$ and so ν_j will be unbounded. This suggests that feasible solutions $x = V^{(0)} \otimes' \nu$ are not bounded from below and introduces the question of whether $f^{\min} = \varepsilon$ in these cases. We define the set J to be

$$J := \{j \in N : \text{ Neither } A_j \text{ nor } B_j \text{ contain an integer entry}\}.$$

Clearly this definition of J is independent of whether or not c and d contain integer entries, this is necessary because, by the discussion above, only values ν_j with $j \in N$ may be unbounded (note that $U_{m+n+1} = 0$ regardless of whether or not W^* is finite). In the following sections we will use it to identify 'bad' or inactive columns of A and B which can be removed from the system. First, we consider the case $J = \emptyset$, under which all ν_i are bounded even though W^* may not be finite.

Observe that $J = \emptyset$ if and only if $U = \mathbf{0}$. Further it can be verified that the results in Theorems 6.33 and 6.34 hold when the assumption that W^* is finite is replaced by an assumption that $U = \mathbf{0}$, in fact the same proofs apply without any alterations. The case $J = \emptyset$ is therefore solved as follows.

Proposition 6.37. *Let A, B, c, d satisfy Assumption 6.32 and $V^{(0)}$ be as defined in Definition 6.30. Suppose $J = \emptyset$.*

(1) For IMLP$^{\min}$, the optimal objective value $f^{\min}$ is attained for

$$x^{opt} = V^{(0)} \otimes' \mathbf{0}.$$

(2) For IMLP$^{\max}$, the optimal objective value $f^{\max}$ is equal to

$$f^T \otimes V^{(0)} \otimes \mathbf{0}.$$

Further let $y := V^{(0)} \otimes \mathbf{0}$ and j be an index such that $f^{\max} = f_j y_j$. If i is such that $y_j = V^{(0)}_{ji}$ then an optimal solution is $x^{opt} = V^{(0)}_i$.

It remains to show how to find solutions to $IMLP^{\min}$ and $IMLP^{\max}$ in the case when $U \neq \mathbf{0}$, i.e. when W^* is not finite and $J \neq \emptyset$. We do this in the following subsections.

IMLP$^{\min}$ When W^* is Non-Finite

If $J \neq \emptyset$, then we aim to remove the 'bad' columns $A_j, B_j, j \in J$ from our program and use Theorem 6.33 to solve the problem. The next result allows us to do this when $J \subset N$.

It will turn out that in this case, under Assumption 6.32, an optimal solution always exists, this will be shown in the proof of Proposition 6.40 below. The case $J = N$ will be dealt with in Proposition 6.42.

Proposition 6.38. *Let A, B, c, d satisfy Assumption 6.32 and $f \in \mathbb{R}^n$.*

Suppose $\emptyset \neq J \subset N$. If an optimal solution x exists, then $f^{\min} = f_j x_j$ for some $j \in N - J$.

Proof. Suppose x is a feasible solution of $IMLP^{\min}$ such that $f^T x = f^{\min}$ but $f^{\min} \neq f_l x_l$ for any $l \in N - J$. Let

$$\bar{J} := \{t \in J : f^{\min} = f_t x_t\}.$$

Observe that, for all $t \in \bar{J}$, neither A_t nor B_t contain an integer entry and so, by Proposition 5.10, x_t is not active in the equation $Ax \oplus c = Bx \oplus d$. Thus the vector x' with components

$$x'_j = \begin{cases} x_j & \text{if } j \notin \bar{J} \\ x_j \alpha^{-1} & \text{otherwise} \end{cases}$$

for some integer $\alpha > 0$ is also feasible but $f^T x' < f^T x$, a contradiction. □

Hence we can simply remove all columns $j \in J$ from our system and solve this reduced system using previous methods. Formally, let g be obtained from f by removing entries with indices in J. Let A^-, B^- be obtained from A and B by removing columns with indices in J, so $A^-, B^- \in \overline{\mathbb{R}}^{m \times n'}$ where $n' = n - |J|$. By IMLP_1 and IMLP_2 we mean the integer max-linear programs:

$$\begin{aligned}(IMLP_1)\min\ & f^T\otimes x = f(x)\\ s.t.\ & Ax\oplus c = Bx\oplus d \\ & x\in\mathbb{Z}^n\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}(IMLP_2)\min\ & g^T\otimes y = g(y)\\ s.t.\ & A^{-}y\oplus c = B^{-}y\oplus d \\ & y\in\mathbb{Z}^{n'}\end{aligned} \tag{6.10}$$

where by assumption the pair $((A|c),(B|d))$ satisfies Property OneFP, and therefore so does $((A^{-}|c),(B^{-}|d))$.

To differentiate between solutions to IMLP$_1$ and IMLP$_2$ the matrices W, W^*, $V^{(0)}$, U will refer to those obtained from A, B, c, d . When they are calculated using A^{-}, B^{-}, c, d we will call them $\hat{W}$, $\hat{W}^*$, $\hat{V^{(0)}}$, $\hat{U}$.

In order to prove that an optimal solution always exists we recall the following results which tell us that, for any IMLP, the problem is either unbounded, infeasible or has an optimal solution. Recall

$$IS = \{x\in\mathbb{Z}^n : Ax\oplus c = Bx\oplus d\},$$

$$IS^{\min} = \{x\in IS : f(x)\le f(z)\ \forall z\in IS\} \text{ and}$$

$$IS^{\max} = \{x\in IS : f(x)\ge f(z)\ \forall z\in IS\}.$$

From Theorems 6.11 and 6.14 ,

$$f^{\min} = -\infty \Leftrightarrow c = d \text{ and } f^{\max} = +\infty \Leftrightarrow (\exists x \in \mathbb{Z}^n)Ax = Bx.$$

Proposition 6.39. *[25] Let A, B, c, d, f be as defined in* (6.5)*. If $IS \neq \emptyset$, then $f^{\min} > -\infty \Rightarrow IS^{\min} \neq \emptyset$ and $f^{\max} < +\infty \Rightarrow IS^{\max} \neq \emptyset$.*

Proposition 6.40. *Let A, B, c, d satisfy Assumption 6.32 and $f \in \mathbb{R}^n$. Let A^-, B^-, g be as defined in* (6.10)*. Suppose $\emptyset \neq J \subset N$. Then $f^{\min} = g^{\min}$, x^{opt} can be obtained from its subvector y^{opt} by inserting suitable 'small enough' integer components and IMLP$_2$ can be solved by Theorem 6.33.*

Proof. First, observe that an optimal solution to IMLP$_2$ always exists since $\hat{U} = \mathbf{0}$, so all components of ν are bounded below. This implies that feasible solutions to IMLP$_2$, and therefore also IMLP$_1$, exist. So, by Proposition 6.39, IMLP$_1$ either has an optimal solution or $f^{\min} = \varepsilon$. If $f^{\min} = \varepsilon$, then, by Theorem 6.11, $c = d$ which under Property OneFP means that $c, d \in \mathbb{Z}^m$ and there are no integer entries in A or B. This is impossible since $J \neq N$.

Suppose x^{opt} is an optimal solution to IMLP$_1$ and let y' be obtained from x^{opt} by removing elements with indices in J. Using Property OneFP, we know that components $x_j^{opt}, j \in J$ are inactive in $Ax \oplus c = Bx \oplus d$. Further, from Proposition 6.38, we can assume also that $x_j^{opt}, j \in J$ are inactive in $f^{\min}$ (can decrease their value if necessary without changing the solution). Hence

$$f^{\min} = f^T x^{opt} = g^T y'$$

and

$$A^- y' \oplus c = Ax^{opt} \oplus c = Bx^{opt} \oplus d = B^- y' \oplus d.$$

So y' is feasible for IMLP$_2$. If y' is not optimal then $g^{\min} = g^T y'' < f^{\min}$ for some feasible (in IMLP$_2$) y'' . But letting $x' = (x'_j)$ where for $j \in J$, x'_j corresponds to y''_j and $x'_j, j \notin J$ are set to small enough integers, we obtain a feasible solution to IMLP$_1$ satisfying $f^T x' = g^{\min} < f^{\min}$, a contradiction. Therefore $y' = y^{opt}$. A similar argument holds for the other direction.

We now show how to solve IMLP$_2$. By Proposition 6.31, feasible solutions to IMLP$_2$ satisfy

$$y = \hat{V}^{(0)} \otimes' \nu,$$
$$0 = \hat{U} \otimes' \nu \text{ and}$$
$$\nu \in \mathbb{Z}^{m+n'+1}.$$

Case 1: There exists an integer entry in either c or d.

Observe that IMLP$_2$ can be solved immediately by Theorem 6.33 since $\hat{W}^*$ is finite.

Case 2: Neither c nor d contain an integer entry.

Now $\hat{W}^*$ is not finite. However $\hat{U}$ is finite and

$$\hat{V}^{(0)}_{m+n'+1} = \begin{pmatrix} +\infty \\ \vdots \\ +\infty \end{pmatrix}.$$

All other columns of $\hat{V}^{(0)}$ are finite. The single $+\infty$ column contains no finite entries and will never be active in determining the value of a feasible solution. Hence any feasible solution y still satisfies $y \geq \hat{V}^{(0)} \otimes' \mathbf{0}$ and $y^{opt} = \hat{V}^{(0)} \otimes' \mathbf{0}$ as in the proof of Theorem 6.33. □

Corollary 6.41. *Let A, B, c, d satisfy Assumption 6.32 and $f \in \mathbb{R}^n$. Let A^-, B^-, g and*

$\hat{V}^{(0)}$*be as defined in* (6.10)*. If* $\emptyset \neq J \neq N$*, the optimal objective value* $f^{\min}$ *of* $IMLP_1$ *is equal to* $g^T y^{opt}$ *for*

$$y^{opt} = \hat{V}^{(0)} \otimes' \mathbf{0}.$$

The final case for IMLP$^{\min}$ is when $J = N$.

Proposition 6.42. *Let* A, B, c, d *satisfy Assumption 6.32 and* $f \in \mathbb{R}^n$*. Suppose* $J = N$*. If* $c = d$*, then* $f^{\min} = -\infty$*. If instead* $c \neq d$*, then* $IMLP^{\min}$ *is infeasible.*

Proof. Follows from Theorem 6.11 and the fact that entries in columns with indices in J are never active. □

IMLP$^{\max}$ When W^* is Non-Finite

We will now discuss $IMLP^{\max}$ when $J \neq \emptyset$. The case when neither c nor d contains an integer is trivial and will be described in Proposition 6.46. We first assume that either c or d contain an integer entry. Here we cannot make the same assumptions about active entries in the objective function as in the minimisation case:

Example 6.43. *Suppose we want to maximise* $(0, 1)^T x$ *subject to*

$$\begin{pmatrix} 0 & -1.5 \\ -0.5 & -1.5 \end{pmatrix} x \oplus \begin{pmatrix} -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1.6 \\ -0.6 & -1.6 \end{pmatrix} x \oplus \begin{pmatrix} -0.6 \\ 0 \end{pmatrix}.$$

Note that $J = \{2\}$*. It can be seen that the largest integer vector* x *which satisfies this equality is* $(0, 1)$*.*

Therefore $f^{\max} = 2$*, the only active entry with respect to* $f^T x$ *is* x_2 *and* $2 \in J$*.*

Instead, we give an upper bound y on x for which $f^{\max} = f^T y$ and we can find a feasible x' where $f^T x'$ attains this maximum value. For all $j \in J$ we have $U_j = +\infty$ and also $V_j^{(0)}$ non-finite since $L^*_{m+j} = e_{m+j}$. We will therefore adapt the matrix $V^{(0)}$ to reflect this.

Definition 6.44. *Let $\bar{V}$ be obtained from $V^{(0)}$ by removing all columns $j \in J$.*

Proposition 6.45. *Let A, B, c, d satisfy Assumption 6.32 and $f \in \mathbb{R}^n$. Let $\bar{V}$ be as defined in Definition 6.44. Suppose either c or d contains an integer and $\emptyset \neq J \subseteq N$. Then the optimal objective value $f^{\max}$ is equal to $f^T y$ for*

$$y = \bar{V} \otimes \mathbf{0}.$$

Further let j be an index such that $f^{\max} = f_j y_j$ and i satisfy $y_j = \bar{V}_{ji}$. Then an optimal solution is $x^{opt} = \bar{V}_i$.

Proof. From Proposition 6.31 any feasible x satisfies

$$x = V^{(0)} \otimes' \nu$$
$$0 = \min_{i \in T} \nu_i$$
$$\nu \in \mathbb{Z}^{m+n+1}$$

where

$$T = \{1, ..., m+n+1\} - \{m + j : j \in J\}.$$

Note that T is the set of indices t for which $U_t = 0$ and $|T| = m + n + 1 - |J|$.

Consider an arbitrary feasible solution $x' = V^{(0)} \otimes' \nu'$. Let μ' be the subvector of ν' with indices from T. Then

$$x' = V^{(0)} \otimes' \nu' \leq \bar{V} \otimes' \mu' \leq \bar{V} \otimes \mathbf{0} = y$$

since $\min_i \mu'_i = 0$. Therefore $f^T x' \leq f^T y$.

We claim that there exists a feasible x such that $f^T x = f^T y$ and hence it is an optimal solution with $f^{max} = f^T y$. Indeed let $j \in N$ be any index such that $f^T y = f_j y_j$. Let $i \in T$

be an index such $v_{ji}^{(0)} = y_j$. Then by setting $\nu_i = 0$ and $\nu_j, j \neq i$ to large enough integers we obtain a feasible solution $\bar{x} = V_i^{(0)}$ which satisfies $f^T\bar{x} = f^T y$. $\square$

Proposition 6.46. *Let A, B, c, d satisfy Assumption 6.32 and $f \in \mathbb{R}^n$. Suppose neither c nor d contain an integer entry. If there exists $x \in \mathbb{Z}^n$ such that $Ax = Bx$, then $f^{\max} = +\infty$. If no such x exists, then $IMLP^{\max}$ is infeasible.*

Proof. Follows from Theorem 6.14 and the fact that $c \neq d$ since they do not have any entries with the same fractional part. $\square$

We conclude by noting that all methods for solving the IMLP under Property OneFP described in this section are strongly polynomial.

Corollary 6.47. *Given input A, B, c, d satisfying Assumption 6.32 and $f \in \mathbb{R}^n$, both IMLP$^{\min}$ and IMLP$^{\max}$ can be solved in $\mathcal{O}((m+n)^3)$ time.*

Proof. From A, B, c, d we can calculate $V^{(0)}, \bar{V}$ and U in $\mathcal{O}((m+n+1)^3)$ time by Remark 6.29. Then $V^{(0)} \otimes' \mathbf{0}, V^{(0)} \otimes \mathbf{0}$ or $\bar{V} \otimes \mathbf{0}$ can be calculated in $\mathcal{O}(n(m+n+1))$ time. From this we can calculate $f^{\min}$ or $f^{\max}$ in $\mathcal{O}(n)$ time. Finally, for IMLP$^{\max}$ we can find an optimal solution in $\mathcal{O}(m+n+1)$ time.

In the cases described in Proposition 6.46, we can perform the necessary checks in $\mathcal{O}((m+n)^3)$ time. $\square$

6.3.5 An Example

Suppose we want to find $f^{\min}$ and $f^{\max}$ subject to the constraints $x \in \mathbb{Z}^4$ and

$$\begin{pmatrix} 3 & 0.5 & -1.7 & -2.5 \\ -3.7 & -1.9 & -2.1 & -3.7 \end{pmatrix} x \oplus \begin{pmatrix} -0.3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1.4 & 1.1 & 1 & -1.3 \\ 0.8 & 1 & -1.3 & -2.2 \end{pmatrix} x \oplus \begin{pmatrix} -0.2 \\ -2.4 \end{pmatrix}.$$

Note that $J = \{4\}$ and

$$A^- = \begin{pmatrix} 3 & 0.5 & -1.7 \\ -3.7 & -1.9 & -2.1 \end{pmatrix} \text{ and } B^- = \begin{pmatrix} 1.4 & 1.1 & 1 \\ 0.8 & 1 & -1.3 \end{pmatrix}.$$

We first construct A' and B', these are

$$\begin{pmatrix} 3 & 0.5 & -1.7 & -2.5 & -0.3 \\ -3.7 & -1.9 & -2.1 & -3.7 & -1 \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1.4 & 1.1 & 1 & -1.3 & -0.2 \\ 0.8 & 1 & -1.3 & -2.2 & -2.4 \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

Then

$$W = \begin{pmatrix} 0 & -2 & -3 & \varepsilon & -1 & \varepsilon & \varepsilon \\ 1 & 0 & \varepsilon & -1 & \varepsilon & \varepsilon & 1 \\ 3 & 1 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & 1 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ 1 & -1 & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ -1 & -2 & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ 0 & -1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \text{ and } W^* = \begin{pmatrix} 0 & -2 & -3 & -3 & -1 & \varepsilon & -1 \\ 1 & 0 & -2 & -1 & 0 & \varepsilon & 1 \\ 3 & 1 & 0 & 0 & 2 & \varepsilon & 2 \\ 2 & 1 & -1 & 0 & 1 & \varepsilon & 2 \\ 1 & -1 & -2 & -2 & 0 & \varepsilon & 0 \\ -1 & -2 & -4 & -3 & -2 & 0 & -1 \\ 0 & -1 & -3 & -2 & -1 & \varepsilon & 0 \end{pmatrix}.$$

Note that $\lambda(W) = 0$ and hence feasible solutions exist, further $W^*_{2+4} = e_{2+4}$ as expected from Proposition 6.35. Now, using Definitions 6.30 and 6.44,

$$\bar{V} = \begin{pmatrix} -3 & -2 & -3 & -2 & -3 & -2 \\ -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } \hat{V}^{(0)} = \begin{pmatrix} -3 & -2 & -3 & -2 & -3 & -2 \\ -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$$

(recall that $\hat{V}^{(0)}$ is calculated from A^-, B^- as defined in (6.10)).

Suppose $f^T = (0, -1, 1, 0)$. We first look for $f^{\min}$.

By Corollary 6.41 we have that

$$g^{\min} = (0, -1, 1) \otimes (\hat{V}^{(0)} \otimes' 0) = (0, -1, 1) \otimes (-3, -2, -1) = 0.$$

Hence $f^{\min} = 0$ and $x^{opt} = (-3, -2, -1, x_4)^T$ for any small enough x_4.

Now we look for $f^{\max}$.

By Proposition 6.45 we have that

$$f^{\max} = f^T \otimes y = (0, -1, 1, 0) \otimes (-2, -2, 0, 1)^T = 1.$$

Following the proof of this proposition, we see that the optimum is attained either for $i = 3$ or $i = 4$. For $i = 3$ this relates to columns $2, 4$ or 6 of $\bar{V}$ and hence the optimal solution can be obtained by setting either ν_2, ν_4 or ν_6 to 0. This yields $x^{opt} = (-2, -2, 0, x_4)^T$ for any small enough x_4. If we instead choose $i = 4$, then we conclude that any column of $\bar{V}$ admits an optimal solution.

Finally, observe that $\hat{V}^{(0)}$ can be obtained from $\bar{V}$ by removing rows with indices in J. This is since A^- and B^- differ from A and B only in columns with indices from J, meaning that $\hat{W} = W[N - J]$ and $\hat{W}^* = W[N - J]^*$.

6.4 Conclusion

We began, in Section 6.1 by showing that the OIMLP can be solved in strongly polynomial time by a simple adaptation to known solution methods for the OMLP.

We studied in detail the IMLP with two-sided constraints. First we constructed Algorithms 6.19 (INT-MAXLINMIN) and 6.22 (INT-MAXLINMAX) which, for finite input, solved the IMLP in pseudopolynomial time when Algorithm 5.4 (GEN-INT-TSS) was used to find feasible points. The key to the algorithms was Proposition 6.6. Both algorithms apply a bisection method to reduce the range of optimal objective function values, but in each iteration Algorithm INT-MAXLINMIN checks only one value whereas Algorithm INT-MAXLINMAX needs to check up to n values. It is shown in [25] that, if the system satisfies Property OneFP, we could instead use Theorem 5.14 to find feasible solutions. In this special case the Algorithms INT-MAXLINMIN and INT-MAXLINMAX solve the IMLP in polynomial time.

Finally, in Section 6.3, we presented a strongly polynomial method to determine whether an integer optimal solution exists to a max-linear program when the input matrices satisfy Property OneFP. We gave a necessary condition for existence of an integer feasible solution and further showed that, under this condition, an integer optimal solution always exists. We described how to find an optimal solution in strongly polynomial time for finite input in Theorems 6.33 and 6.34. We then used these results to describe the optimal objective function value, and find an optimal solution, to any IMLP satisfying Property OneFP. Our solution methods can be used to describe many possible integer optimal solutions to the system.

7. Conclusions and future work

In this thesis we explored integer solutions to a range of max-algebraic systems of equations and inequalities. We showed in Proposition 2.1 and Theorem 2.5 that finding integer solutions to the one-sided systems $Ax \leq b$, $Ax = b$, and $Ax \leq \lambda x$ is no more difficult than finding real solutions, and thus that existing theory can be used to describe the entire set of integer solutions to these systems in almost linear time.

For the integer eigenproblem we used existing results to show that the integer eigenvectors of A are equivalent to the points in the integer image of $\tilde{A}_\lambda$. We proved that we could in fact consider the integer image of a smaller matrix in Theorem 2.27. However this was not enough to conclude whether such a vector exists. To solve the problem of existence for the integer eigenproblem we developed Algorithm 3.1 (INT-IMAGE) which, in a finite number of steps, finds a vector in the integer image of a matrix or determine that the integer image was empty. If the input matrix was finite we proved that the algorithm ran in pseudopolynomial time, see Theorem 3.11, and therefore concluded that we could solve the integer eigenproblem for irreducible matrices in pseudopolynomial time. We observed that, if an integer eigenvector of a matrix A exists, then there must be an integer entry in every row of A. In light of this we defined the class of matrices satisfying Property OneIR and, for matrices having at most one integer entry per row, presented a strongly polynomial method to describe all integer eigenvectors in Theorem 2.17. It remains an open problem to find a polynomial algorithm to solve the integer eigenproblem when the matrix has rows containing more than one integer entry.

We introduced the definition of a column typical matrix and proved, in Theorem 3.17, that, for matrices of this type, we can determine whether the integer image is non-empty (and describe all integer solutions) in strongly polynomial time. In the case when an integer image exists, we showed that the set of integer images of a column typical matrix was equivalent to the set of integer eigenvectors. We went on to define a slightly more general set of matrices, NNI matrices, for which the integer image set is equivalent to both the set of integer eigenvectors and the set of integer subeigenvectors.

We noted that the integer image problem can be viewed as the problem of finding an integer point in a max-algebraic convex hull. In Section 3.3 we briefly explored some sufficient conditions for when a max-algebraic convex hull contains an integer point. This is a clear area that would benefit from further research.

Although the complexity of the integer image problem remains unknown, we explored the complexity of related problems. Specifically we showed that we could assume without loss of generality that the matrix is column typical by describing a strongly polynomial transformation to a column typical counterpart, see Theorem 4.11. For column typical matrices we can further assume that, if an integer image exists, then one exists with at most one active entry per column. We defined the problem of finding an integer image with exactly one active entry per row to be the P1 variant of IIM. Theorem 4.19 proves that IIM-P1 is NP-hard. One area of future research would be to determine whether IIM-P1 with column typical input remains NP-hard, or whether a strongly polynomial method exists for column typical matrices with $m \leq n$.

We studied integer solutions to the TSSs $Ax = By$ and $Ax = Bx$. We began by considering the Alternating Method, which is a known algorithm to find real solutions to these systems. We presented an adaptation to the Alternating Method that allowed us to create Algorithms 5.1 (SEP-INT-TSS) and 5.4 (GEN-INT-TSS) which determine whether integer solution to TSS exist in a finite number of steps. If the input matrices are finite then, by Corollaries 6.21 and 6.24, the algorithms terminate after $\mathcal{O}(mn(n+k)K(A))$ and $\mathcal{O}(K(A'|B')mn(m+n))$ respectively and are thus pseudopolynomial.

We then considered cases when we could find integer solutions to TSSs in strongly polynomial time. We defined when a system satisfies Property OneFP and argued that it represented a generic case. For systems satisfying Property OneFP , the results in Theorem 5.14 allow us to describe all integer solutions in strongly polynomial time. We argued that this method could be extended to solve general systems but that this would only be strongly polynomial if m was fixed. It remains an open problem to find a polynomial method for any general system, or to determine if the problem is NP-hard.

The last problem that we considered was the IMLP. We adapted the Bisection Method, a known algorithm to find real solutions to the MLP, to produce Algorithms 6.19 (INT-MAXLINMIN) and 6.22 (INT-MAXLINMAX) which find an integer solution to IMLP^{min} and IMLP^{max} respectively in pseudopolynomial time when the input matrices are finite. These algorithms can be proven to have polynomial runtime under certain input conditions, which include Property OneFP [25]. Additionally, we constructed a new method for systems satisfying Property OneFP. This method allowed us to find the optimal objective function value and a number of optimal solutions to both the IMLP^{min} and IMLP^{max} in strongly polynomial time, see Theorems 6.33 and 6.34 and their corollaries.

Other max-linear systems also exist. Currently, nothing is known about integer solutions to the max-algebraic supereigenproblem, $Ax \geq \lambda x$. At the time of writing there is not much literature on supereigenvectors in max-algebra, but the paper [64] considers the problem for irreducible matrices. The generalised eigenproblem, $Ax = \lambda Bx$ is another possible area of future research. For fixed λ the problem reduces to a two-sided system, but the description of all generalised eigenvalues with respect to any integer solution x remains open.

Another direction for future research would be to consider finding extended integer solutions to each of the max-linear systems studied here, that is solutions with entries from $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\varepsilon\}$. It should be noted that the methods for finding integer solutions to the one-sided systems and the subeigenproblem can be readily extended to give results on finding $\overline{\mathbb{Z}}$-solutions with the same complexity [27]. For the eigenproblem, image problem and TSSs it is possible to extend the methods for special cases (Property OneIR, NNI matrices, Property OneFP) found when looking for integer solutions to obtain strongly polynomial methods for finding $\overline{\mathbb{Z}}$-solutions in these generic cases, details appear in [27]. For general matrices the question of determining existence of $\overline{\mathbb{Z}}$-solutions remains open.

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