DIFFERENTIABILITY AND NEGLIGIBLE SETS IN BANACH SPACES

by

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Abstract

A set S in a Banach space X is called a universal differentiability set if S contains a point of differentiability of every Lipschitz function $f : X \to \mathbb{R}$. The present thesis investigates the nature of such sets. We uncover examples of exceptionally small universal differentiability sets and prove that all universal differentiability sets satisfy certain strong structural conditions. Later, we expand our focus to properties of more general absolutely continuous functions.

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Contents

Chapter 1

Introduction

The differentiability of functions on Banach spaces has been topic of interest going back centuries. Differentiability is a local property; a function being differentiable at a point means that the behaviour of the function in a small neighbourhood of that point can be well approximated by a linear map.

For a Banach space X, we let $\|-\|_X$ denote the norm of X and $S(X)$ denote the unit sphere in X. Let us now give the key definition of differentiability, according to [6, p. 83].

Definition. *Let X and Y be Banach spaces and suppose that U is a subset of X with non-empty interior* $\mathrm{Int}(U)$.

(i) A function $f : U \to Y$ is said to Gâteaux differentiable at a point $x \in \mathrm{Int}(U)$ if there exists a bounded linear operator $L : X \to Y$ such that the following condition holds: For every direction $e \in S(X)$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x + te) - f(x) - tL(e)\|_Y \leq \epsilon\,|t| \ \text{ whenever } t \in (0, \delta). \tag{1.1}$$

If such $\delta > 0$ can be chosen independently of $e \in S(X)$ then we say that f is Fréchet differentiable at x.

(ii) A function $f : U \to Y$ is said to be differentiable at a point $x \in \mathrm{Int}(U)$ in a direction $v \in X$ if the limit

$$\lim_{t \to 0+} \frac{f(x + tv) - f(x)}{t} \ \text{ exists.}$$

When $Y \neq \mathbb{R}$ we sometimes call the function $f : U \to Y$ a mapping. If f is Gâteaux (respectively Fréchet) differentiable at x then the linear operator L witnessing this fact is called the Gâteaux (respectively Fréchet) derivative of f at x and denoted by $f'_G(x)$ (respectively $f'(x)$). Clearly, Fréchet differentiability is a stronger property than Gâteaux differentiability; we have $f'(x) = f'_G(x)$ whenever f is Fréchet differentiable at a point $x \in X$.

When X is a finite dimensional Banach space, the compactness of the unit sphere $S(X)$ causes the notions of Gâteaux and Fréchet differentiability to coincide. In this thesis, we will mainly be concerned with differentiability inside finite dimensional Banach spaces. In such settings we will simply speak about differentiability and derivative and omit the terms Gâteaux and Fréchet.

If f is differentiable at x in the direction v then we denote

$$f'(x, v) = \lim_{t \to 0+} \frac{f(x + tv) - f(x)}{t}.$$

and call this vector a directional derivative of f at x. Note that when f is Gâteaux differentiable at x, we have $f'_G(x)(e) = f'(x, e)$ for all $e \in S(X)$. Conversely, to prove that f is Gâteaux differentiable at x, it is enough to show that all directional derivatives of f at x exist and the map $v \mapsto f'(x, v)$ is a bounded linear operator.

Whenever $f : U \to Y$ is Fréchet differentiable at a point $x \in \mathrm{Int}(U)$, we have that f is continuous at x. However, the same is not true if we replace Fréchet differentiability with Gâteaux differentiability, see [6, p. 83] for an example. Differentiability can be thought of as a rather rare property amongst continuous functions. Indeed, if we consider the Banach space $C([0, 1])$ of all continuous functions on the interval $[0, 1]$, then 'typical' functions are nowhere differentiable. We will discuss the meaning of the word 'typical' in the introduction of Chapter 2.

Considering a stronger form of continuity, namely Lipschitz continuity, the situation becomes very different.

Definition. *A function $f : X \to Y$ is called Lipschitz if the number*

$$\mathrm{Lip}(f) := \sup \left\{ \frac{\|f(y) - f(x)\|_Y}{\|y - x\|_X} \ : \ x, y \in X, \ y \neq x \right\} \text{ is finite.}$$

We refer to the number $\mathrm{Lip}(f)$ *as the Lipschitz constant of* f.

Lipschitz functions are particularly appealing for the study of differentiability. If $f : X \to \mathbb{R}$ is a Lipschitz function and $x_0 \in X$ then the Lipschitz property guarantees that both $\limsup_{x \to x_0} \frac{f(x)-f(x_0)}{\|x-x_0\|_X}$ and $\liminf_{x \to x_0} \frac{f(x)-f(x_0)}{\|x-x_0\|_X}$ exist and have absolute value bounded above by the Lipschitz constant $\mathrm{Lip}(f)$. Moreover, in many settings, we have that Lipschitz functions are differentiable everywhere except for a negligible set. A cornerstone of the theory of differentiation is the following result, known as Rademacher's Theorem [27, p. 101]:

Theorem. *(Rademacher's Theorem)*
Let $f : \mathbb{R}^d \to \mathbb{R}^l$ be a Lipschitz map. Then f is differentiable almost everywhere with respect to the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

It is natural to ask whether Rademacher's theorem admits a converse statement. That is, to consider the following question:

Question 1. *Given $d, l \geq 1$ and a Lebesgue null set $N \subseteq \mathbb{R}^d$, is it always possible to find a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^l$ such that f is nowhere differentiable in N?*

The answer in the case $d = 1$ is positive and has been known for some time, see [34]. However, the case $d \geq 2$ provides a stark contrast and has inspired the modern theory of universal differentiability sets, which will be the central theme of this thesis. The term 'universal differentiability set' was introduced by Doré and Maleva in [14].

Definition. *A set S in a Banach space X is said to have the universal differentiability property if for every Lipschitz function $f : X \to \mathbb{R}$, there exists a point $x \in S$ such that f is Fréchet differentiable at x. If $S \subseteq X$ has the universal differentiability property, then we say that S is a universal differentiability set.*

Clearly, any subset of $\mathbb{R}^d$ with positive Lebesgue measure has the universal differentiability property. The first interesting and non-trivial example of a universal differentiability set was given by Preiss [28] in 1990 and provided a counterexample to the proposed converse to Rademacher's Theorem. Preiss [28] proves that any G_δ subset of $\mathbb{R}^2$ of Lebesgue measure zero, containing all line segments between points with rational co-ordinates has the universal differentiability property.

Whilst Preiss' set is null in the sense of the Lebesgue measure, in other respects it is still rather large. Note that the closure of the set is the whole space $\mathbb{R}^2$. The question of whether the universal differentiability property can be found in much smaller sets has been the focus of recent research. Doré and Maleva [12] verify the existence of a compact universal differentiability set of Lebesgue measure zero in any Euclidean space of dimension at least two. The same authors prove, in [14], the existence of a compact universal differentiability set with Hausdorff dimension one; we define the important notions of Hausdorff measure and dimension presently, according to [27].

Definition. *Given a subset E of a separable Banach space X and real numbers $s, \delta > 0$, we define a quantity*

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s \,:\, E \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) \leq \delta\right\},$$

where $\operatorname{diam}(U)$ *denotes the diameter of a set U. The s-dimensional Hausdorff measure of E is then given by*

$$\mathcal{H}^s(E) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(E).$$

Finally, we define the Hausdorff dimension of E as

$$\dim_H(E) = \inf\{s > 0 \,:\, \mathcal{H}^s(E) = 0\} = \sup\{s > 0 \,:\, \mathcal{H}^s(E) = \infty\}.$$

We remark that all non-empty open subsets of $\mathbb{R}^d$ have Hausdorff dimension d. Moreover, any Borel set $E \subseteq \mathbb{R}^d$ with $\dim_H(E) < d$ is necessarily a set of d-dimensional

Lebesgue measure zero.

To finish our account of the existing research surrounding Question 1, we point out that a complete answer is now known: The answer to Question 1 is positive if and only if $l \geq d$. A new paper [29] of Preiss and Speight establishes that the answer is negative when $l < d$, whilst it is understood that recent discoveries of Alberti, Csörnyei, Preiss [2] and Csörnyei, Jones deal with the case $l \geq d$. The situation has also been studied in infinite dimensional Banach spaces: Doré and Maleva [13] prove that any non-zero Banach space X with separable dual contains a closed and bounded set of Hausdorff dimension one which has the universal differentiability property for Lipschitz functions $f : X \to \mathbb{R}$.

The last mentioned result differs from the finite dimensional case [14] because the null universal differentiability set of [13] cannot be compact. All compact sets in infinite dimensional spaces are porous and we will see that porosity is a forbidden property in universal differentiability sets. Let us state what it means for a set to be porous, according to [6, p. 92].

Definition. *Let E be a subset of a Banach space X.*

(i) We say that E is porous if there exists a number $c \in (0,1)$ such that the following condition holds. For every $x \in E$ and $\epsilon > 0$, there exists a point $h \in X$ and a number $r > 0$ such that $\|h - x\|_X < \epsilon$, $B(h,r) \cap E = \emptyset$ and $r \geq c\,\|h - x\|_X$.

(ii) We say that E is σ-porous if E is a countable union of porous sets.

Clearly, porous sets are nowhere dense. Moreover, as a consequence of the Lebesgue Density Theorem, we have that σ-porous subsets of $\mathbb{R}^d$ have Lebesgue measure zero. For a survey on porosity and σ-porosity see [35].

To demonstrate the connection between porous sets and the differentiability of Lipschitz functions, we highlight that any porous set is a non-universal differentiability set. For a given porous set E, in a Banach space X, the distance function $\operatorname{dist}(\cdot, E) : X \to \mathbb{R}$, defined by

$$\operatorname{dist}(x, E) = \inf \{\|x - y\|_X \,:\, y \in E\},$$

is Lipschitz and nowhere Fréchet differentiable on E. Indeed, if $x_0 \in E$, then the porosity of E guarantees that

$$\limsup_{x \to x_0} \frac{\operatorname{dist}(x, E) - \operatorname{dist}(x_0, E)}{\|x - x_0\|_X} = \limsup_{x \to x_0} \frac{\operatorname{dist}(x, E)}{\|x - x_0\|_X} > 0.$$

Since $\operatorname{dist}(\cdot, E)$ attains its minimum value 0 at the point $x_0 \in E$, we deduce that $\operatorname{dist}(\cdot, E)$ is not Fréchet differentiable at x_0.

We even have that any σ-porous subset F of a separable Banach space X admits a Lipschitz function $f : X \to \mathbb{R}$ such that f is nowhere Fréchet differentiable in the set F, see [24, Theorem 3.4.3]. Hence, any universal differentiability set in a separable Banach space must be non-σ-porous. However, it appears that there is a deep and still rather mysterious relationship between the universal differentiability property and non-σ-porosity. Existing constructions of universal differentiability sets are driven by the aim to destroy porosity in the final set.

We should point out that not all non-σ-porous subsets are universal differentiability sets. Zajíček [35, Proposition 5.2] gives a construction of a non-σ-porous subset V of the interval $[0, 1]$ such that V has Lebesgue measure zero. As a Lebesgue null subset of $\mathbb{R}$, V admits a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that f is nowhere differentiable on V. Moreover, given $d \geq 2$ we have that the set $V \times \mathbb{R}^{d-1}$ is a non-σ-porous subset of $\mathbb{R}^d$ which does not have the universal differentiability property. Writing p_1 for the first co-ordinate projection map on $\mathbb{R}^d$, the Lipschitz function $f \circ p_1$ is nowhere differentiable on $V \times \mathbb{R}^{d-1}$.

Chapter 2

Exceptional sets meeting curves.

2.1 Introduction.

For an infinite dimensional Banach space, there is no useful concept of a translation invariant measure to emulate the Lebesgue measure on Euclidean spaces, see [22]. Therefore, to describe the size of sets in infinite dimensional spaces, it is desirable to look at their intersection with finite dimensional objects such as curves and surfaces. In the current chapter we will investigate the significance of this technique in the study of porous sets and differentiability of Lipschitz functions.

Given a Banach space X, let $\Gamma_n(X)$ denote the Banach space of all C^1 mappings $\gamma : [0,1]^n \to X$. The norm on $\Gamma_n(X)$ is given by $\|\gamma\| = \|\gamma\|_\infty + \|\gamma'\|_\infty$, where $\|-\|_\infty$ denotes the supremum norm. Recall that a subset R of a Banach space Y is called residual if R contains a countable intersection of open, dense subsets of Y. The Baire Category Theorem asserts that any residual subset of a Banach space is dense. Thus residual sets can be thought of as large; elements of a residual set R in a Banach space X are often referred to as typical elements of X.

Lindenstrauss, Preiss and Tišer [24] introduce the following class of negligible sets.

Definition. *Let X be a Banach space. We say that a Borel set $E \subseteq X$ is Γ_n-null if the set*

$$\left\{\gamma \in \Gamma_n(X) \,:\, \gamma^{-1}(E) \text{ has } n\text{-dimesional Lebesgue measure zero}\right\}$$

is a residual subset of $\Gamma_n(X)$*. A non-Borel set* $A \subseteq X$ *is called* Γ_n*-null if* A *is contained in a Borel subset of* X *which is* Γ_n*-null.*

In [24, Theorem 10.4.1], it is proved that every σ-porous subset of a separable Banach space having separable dual, is Γ_1-null. The present author's paper [16] gives a new proof of this result in Hilbert spaces. In this new proof, curves avoiding a fixed σ-porous set in a Hilbert space are constructed explicitly. Replacing curves with higher dimensional C^1 surfaces, one observes contrasting behaviour. Speight [32] proves that if $2 < n < \dim(X)$, where X is a Banach space, then there exists a porous set $P \subseteq X$ which is not Γ_n-null.

In section 2 of the present chapter, we investigate a variant of porosity.

Definition. *Let* X *be a Banach space with norm* $\|-\|$ *and let* $p > 1$*. A set* $F \subseteq X$ *is said to be power-p-porous at a point* $x \in F$ *if for every* $\epsilon > 0$ *there exists* $h \in X$ *and* $r > 0$ *such that* $\|h - x\| < \epsilon$*,* $B(h, r) \cap F = \emptyset$ *and* $r > \|h - x\|^p$*. The set* F *is called power-p-porous if* F *is power-p-porous at every point* $x \in F$*.*

Note that any porous set is power-p-porous for all $p > 1$. We will show that power-p-porous sets can be significantly larger than traditional porous sets. Namely, we prove that there exists a Lebesgue null, power-p-porous subset of $\mathbb{R}^2$ which is not Γ_1-null.

The third section of this chapter is devoted to establishing a sufficient geometric condition for a set to have the universal differentiability property. In Chapter 3 we will construct sets possessing this geometric property and apply our result to prove that they are universal differentiability sets. The condition we describe, in Definition 2.3.1, relates to the behaviour of sets on Lipschitz curves; sets which intersect many curves with large one-dimensional Lebesgue measure are shown to be universal differentiability sets.

Let us stress that we only establish a sufficient condition; it is unknown whether this condition is also necessary. However, it is plausible that universal differentiability sets could be characterised in this way. Alberti, Csörnyei and Preiss [1] establish a geometric description of non-universal differentiability sets according to their intersection with curves.

The first form of the geometric condition presented in this chapter was discovered by Doré and Maleva in [12] and later used in [14],[13] and [17]. Doré and Maleva prove that sets having the property that around each point, one can approximate every 'wedge' (a connected union of two line segments) in a special way by wedges **entirely contained** in the set, must also possess the universal differentiability property. We strengthen this result by no longer requiring complete wedges or even line segments to be contained in the set. Instead, we show that it suffices for the set to intersect many Lipschitz curves in large measure.

The relevance of curves to the differentiability of Lipschitz functions can be seen in the theory of uniformly purely unrectifiable sets. A subset E of a Banach space X is called uniformly purely unrectifiable (u.p.u.) if there exists a Lipschitz function $f : X \to \mathbb{R}$ such that for all $x \in E$ and all $v \in S(X)$, the directional derivative $f'(x, v)$ does not exist, see [2]. Thus, u.p.u. sets can be thought of as even more negligible than non-universal differentiability sets. Alberti, Csörnyei and Preiss [1] establish the following characterisation of u.p.u. sets in $\mathbb{R}^d$ in terms of their measure on curves: A set $E \subseteq \mathbb{R}^d$ is uniformly purely unrectifiable if and only if for any $e \in S^{d-1}$, $\alpha \in (0, \pi)$ and $\epsilon > 0$ there exists an open set G containing E such that whenever $\gamma : [0, 1] \to \mathbb{R}^d$ is a C^1 map satisfying $|\langle \gamma'(t), e\rangle| \geq \|\gamma'(t)\| \cos(\alpha/2)$ for all $t \in [0, 1]$, we have $\mathcal{H}^1(\gamma([0, 1]) \cap G) < \epsilon$.

2.2 Power-porosity and Γ_1.

The research communicated in this section is part of the present author's paper [16]. We prove that, unlike σ-porous sets, power-p-porous sets need not be Γ_1-null.

Recall that porous and σ-porous subsets of a Euclidean space have Lebesgue measure zero. Our first lemma signals that power-p-porous sets are of a different nature. We denote by m the one-dimensional Lebesgue measure on $\mathbb{R}$.

Lemma 2.2.1. *If $p > 1$, there exists a power-p-porous subset C of $[0, 1]$ with positive Lebesgue measure.*

Proof. We shall construct a Cantor type set C with the desired properties.

Fix $\lambda \in (0, 1/3)$. Let C be the Cantor set obtained, following the normal construction. On step n, remove 2^{n-1} intervals, each of length λ^n in the usual manner (instead of length 3^{-n} for the ternary Cantor set). We can compute the Lebesgue measure of C explicitly: $m(C) = \frac{1-3\lambda}{1-2\lambda} > 0$.

We now show that C is a power-p-porous set. Fix $x \in C$ and $\epsilon > 0$. Let N be a natural number large enough so that

$$2^{-(N+1)} < \epsilon \text{ and } N \geq \frac{\log 2}{\log (2^p \lambda)} - 1. \tag{2.1}$$

Since $x \in C$, there exists a remaining interval of the Nth step containing x. Call this interval R. By construction, there is a deleted interval D of the $(N+1)$th step of length λ^{N+1} whose midpoint h coincides with the midpoint of R. Let h denote the midpoint of D and R. Then $D = B(h, \lambda^{N+1}/2)$ is a ball which lies inside the complement of C. Moreover,

$$|h - x| \leq \text{length}(R)/2 \leq 2^{-(N+1)} < \epsilon. \tag{2.2}$$

Together, (2.1) and (2.2) imply that $\frac{\lambda^{N+1}}{2} \geq \left(\frac{1}{2^{N+1}}\right)^p \geq |h - x|^p$. □

The next Lemma reveals that subsets of $[0, 1]$ with positive Lebesgue measure give rise to a class of subsets of the plane which have positive Lebesgue measure on all curves belonging to some open subset of $\Gamma_1 = \Gamma_1(\mathbb{R}^2)$.

Lemma 2.2.2. *Suppose F is a subset of $[0, 1]$ with positive Lebesgue measure and let $S = F \times \mathbb{R}$. There exists an open subset U of Γ_1 such that S has positive Lebesgue measure on all curves in U.*

Proof. Let $\gamma(t) = (t, 0)$ be the horizontal interval. It is now easy to check that for sufficiently small $\delta > 0$, we have $m(\rho^{-1}(S)) > 0$ for all curves $\rho \in B(\gamma, \delta)$. □

From the next Lemma, we see that measure zero subsets of $\mathbb{R}^2$ can fail dramatically to be Γ_1-null.

Lemma 2.2.3. *There is a subset T of $\mathbb{R}^2$ which has Lebesgue measure zero and contains the image of all curves belonging to a residual subset of Γ_1.*

Proof. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a countable dense subset of Γ_1. For each $\epsilon > 0$ and $n \in \mathbb{N}$, let $U_n(\epsilon)$ be an open neighbourhood of $\gamma_n([0,1])$ of Lebesgue measure less than $\epsilon/2^n$. Consider an open neighbourhood of γ_n defined by

$$Y_n(\epsilon) = \{\gamma \in \Gamma_1 : \gamma(t) \in U_n(\epsilon) \text{ for all } t \in [0,1]\}$$

Letting $Y(\epsilon) = \bigcup_{n=1}^{\infty} Y_n(\epsilon)$, we get an open, dense subset of Γ_1 containing the sequence $\{\gamma_n\}_{n=1}^{\infty}$. Finally, letting $Y = \bigcap_{m=1}^{\infty} Y(\epsilon_m)$, where $\epsilon_m \to 0$, we get a residual subset of Γ_1. Hence, the set

$$T = \{\gamma(t) : \gamma \in Y, t \in [0,1]\}$$

has the desired properties. □

We are now ready to prove the main result of this section.

Theorem 2.2.4. *There exists a measure zero, power-p-porous subset A of $\mathbb{R}^2$ such that the set of all curves $\gamma \in \Gamma_1$ on which A has Lebesgue measure zero, is not a residual subset of Γ_1.*

Proof. Let C be the positive measure, power-p-porous subset of $[0,1]$ given by the conclusion of Lemma 2.2.1. Define a set $B \subset \mathbb{R}^2$ by $B = C \times \mathbb{R}$.

It is clear that B is a power-p-porous subset of $\mathbb{R}^2$. Further, we can apply Lemma 2.2.2 to deduce that there is an open subset U of Γ_1 such that B has positive Lebesgue measure on every curve in U.

By Lemma 2.2.3, there exists a subset T of $\mathbb{R}^2$ with measure zero containing the image $\gamma([0,1])$ of all curves γ belonging to a residual subset R of Γ_1. We now define $A = B \cap T$.

Since $A \subseteq B$ we have that A is a power-p-porous subset of $\mathbb{R}^2$. Simultaneously, A has Lebesgue measure zero in $\mathbb{R}^2$ because $A \subseteq T$. Therefore, it only remains to show that the curves in Γ_1, avoiding A, do not form a residual subset of Γ_1.

Suppose $\gamma \in U \cap R$. Since T contains the image of γ, we have

$$\gamma^{-1}(A) = \gamma^{-1}(B) \cap \gamma^{-1}(T) = \gamma^{-1}(B) \cap [0,1] = \gamma^{-1}(B).$$

Recall that $\gamma \in U$ and B has positive measure on all curves in U. Hence, $\gamma^{-1}(A)$ has positive Lebesgue measure and we conclude that A has positive Lebesgue measure on all curves in $U \cap R$. Since U is open and R is residual in Γ_1, the complement of $U \cap R$ is not residual. $\square$

2.3 A curve approximation property.

In this section, we establish a sufficient condition for the universal differentiability property in real Hilbert spaces. However, many of the results of the present section hold in more general Banach spaces, namely Banach spaces with the Radon-Nikodým property. Let us give a definition of this concept, according to [24, p. 12].

Definition. *We say that a Banach space X has the Radon-Nikodým property if every Lipschitz map $f : \mathbb{R} \to X$ is differentiable almost everywhere.*

Given a real Banach space X, let $\|-\|$ denote the norm on X and $S(X)$ denote the unit sphere in X. For a subset V of X we define the closed convex hull of V by

$$\overline{\text{conv}}(V) = \text{Clos}\left\{\sum_{i=1}^{n} \alpha_i v_i \,:\, n \geq 1,\, \alpha_i \in [0,1],\, \sum_{i=1}^{n} \alpha_i = 1 \text{ and } v_i \in V\right\}.$$

The closed convex hull of V is equal to the intersection of all closed convex sets containing V.

We continue to write m for the one-dimensional Lebesgue measure on $\mathbb{R}$. Let $I \subseteq \mathbb{R}$ be a non-degenerate interval and $\gamma : I \to X$ be a Lipschitz mapping. We sometimes refer to such mappings as Lipschitz curves. If γ is differentiable at a point $t \in I$, we will identify the linear operator $\gamma'(t) : \mathbb{R} \to X$ with the unique element $x \in X$ such that $\gamma'(t)(s) = xs$ for all $s \in \mathbb{R}$.

We say that a set $F \subseteq X$ is m_γ-measurable if the set $\gamma^{-1}(F) \subseteq I$ is m-measurable. Given a m_γ-measurable set $F \subseteq X$, we define a quantity $m_\gamma(F)$ by

$$m_\gamma(F) = \frac{m(\gamma^{-1}(F))}{m(I)}.$$

Note that all Borel subsets of X are m_γ-measurable.

The geometric property of sets that we will study in this section is presently stated.

Definition 2.3.1. *Let X be a real Banach space. A collection $(U_\lambda)_{\lambda\in(0,1]}$ of subsets of X has the curve approximation property if the following conditions are met:*

(i) $U_\lambda \subseteq U_{\lambda'}$ whenever $0 < \lambda \leq \lambda' \leq 1$.

(ii) For every $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$ and $\eta \in (0,1)$ there exists $\delta_0 = \delta_0(\lambda, \psi, \eta) > 0$ such that whenever $x \in U_\lambda$, $e \in S(X)$ and $\delta \in (0, \delta_0)$ there exists a Lipschitz curve $\gamma : [0, \delta] \to X$ such that $\|\gamma(0) - x\| \leq \eta\delta$, $\|\gamma'(t) - e\| \leq \eta$ whenever $\gamma'(t)$ exists, and $m_\gamma(U_{\lambda+\psi}) \geq 1 - \eta$.

Remark 2.3.2. *A stronger condition than* (ii) *is the following:*

(ii)′ For every $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$ and $\eta \in (0,1)$ there exists $\delta_0 = \delta_0(\lambda, \psi, \eta) > 0$ such that whenever $x \in U_\lambda$, $e \in S(X)$ and $\delta \in (0, \delta_0)$ there exists $x' \in X$ and $e' \in S(X)$ such that $\|x' - x\| \leq \eta\delta$, $\|e' - e\| \leq \eta$ and $[x', x' + \delta e'] \subseteq U_{\lambda+\psi}$.

To see that (ii)′ *is stronger than* (ii)*, note that, if x, e, δ are given by* (ii)*, we can choose x', e' according to (ii)′ and then the desired curve $\gamma : [0, \delta] \to X$ can be defined by the formula $\gamma(t) = x' + te'$.*

We presently state the main result of this chapter:

Theorem 2.3.3. *Let H be a real Hilbert space and let $(U_\lambda)_{\lambda\in(0,1]}$ be a collection of closed subsets of H with the curve approximation property. Then each set U_λ is a universal differentiability set.*

Let us now work towards a proof of Theorem 2.3.3; we begin by quoting Lemma 3.3 from [28, p. 321].

Lemma 2.3.4. *(Lemma 3.3 [28]) Suppose that $a < \xi < b$, $0 < \theta < \frac{1}{4}$, and $L > 0$ are real numbers, h is a Lipschitz function defined on $[a, b]$, $\mathrm{Lip}(h) \leq L$, $h(a) = h(b) = 0$, and $h(\xi) \neq 0$. Then there is a measurable set $A \subset (a, b)$ such that*

(i) $m(A) \geq \frac{\theta|h(\xi)|}{L}$,

(ii) $h'(\tau) \geq \frac{\theta|h(\xi)|}{(b-a)}$ *for every* $\tau \in A$,

(iii) $|h(t) - h(\tau)| \leq 4(1 + 2\theta)\sqrt{h'(\tau)L}\,|t - \tau|$ *for every* $\tau \in A$ *and every* $t \in [a, b]$.

The next Lemma is a small adaptation of Lemma 3.4 in [28]. We give a proof of this Lemma which closely follows [28, p. 323-326], with only minor changes.

Lemma 2.3.5. *Suppose that $|\xi| < s < \rho$, $0 < \upsilon < \frac{1}{32}$, $\sigma > 0$, and $L > 0$ are real numbers and that φ and ψ are Lipschitz functions defined on the real line such that $\mathrm{Lip}(\varphi) + \mathrm{Lip}(\psi) \leq L$, $\varphi(t) = \psi(t)$ for $|t| \geq s$ and $\varphi(\xi) \neq \psi(\xi)$. Suppose, moreover, that $\psi'(0)$ exists and that*

$$|\psi(t) - \psi(0) - t\psi'(0)| \leq \sigma L\,|t|$$

whenever $|t| \leq \rho$,

$$\rho \geq s\sqrt{\frac{sL}{\upsilon\,|\varphi(\xi) - \psi(\xi)|}},$$

and

$$\sigma \leq \upsilon^3 \left(\frac{\varphi(\xi) - \psi(\xi)}{sL}\right)^2.$$

Then there is a measurable set $D \subset (s, -s) \setminus \{\xi\}$ such that

$$m(D) \geq \frac{4\upsilon \left|\varphi(\xi) - \frac{\psi(s)+\psi(-s)}{2} - \frac{\xi(\psi(s)-\psi(-s))}{2s}\right|}{L}, \tag{2.3}$$

and for all $\tau \in D$, $\varphi'(\tau)$ exists,

$$\varphi'(\tau) \geq \psi'(0) + \frac{\upsilon\,|\varphi(\xi) - \psi(\xi)|}{s}, \tag{2.4}$$

and

$$|(\varphi(\tau + t) - \varphi(\tau)) - (\psi(t) - \psi(0))| \leq 4(1 + 20\upsilon)\sqrt{[\varphi'(\tau) - \psi'(0)]L}\,|t| \tag{2.5}$$

for every $t \in \mathbb{R}$.

Proof. (Following [28, p. 323-326]) Let

$$h(t) := \varphi(t) - \frac{\psi(s) + \psi(-s)}{2} - \frac{t(\psi(s) - \psi(-s))}{2s}.$$

In [28, p. 323-324] it is proved that $|h(\xi)| \geq |\varphi(\xi) - \psi(\xi)|/2 > 0$. We also have $h(s) = h(-s) = 0$ and $\mathrm{Lip}(h) \leq \mathrm{Lip}(\varphi) + \mathrm{Lip}(\psi) \leq L$. Thus, the conditions of Lemma 2.3.4 are satisfied with $a = -s$, $b = s$ and $\theta = 8\upsilon$. Applying, Lemma 2.3.4, we find a measurable set $A \subset (-s, s)$ such that the conditions (i), (ii) and (iii) of Lemma 2.3.4 are satisfied with $a = -s$, $b = s$, $\theta = 8\upsilon$. Setting

$$E = \left[-s, -s + \frac{2\upsilon\,|h(\xi)|}{L}\right] \cup \left[s - \frac{2\upsilon\,|h(\xi)|}{L}, s\right] \cup \{\xi\}$$
$$\cup \{t \in (-s, s) : \varphi'(t) \text{ or } \psi'(t) \text{ does not exist}\},$$

we observe that

$$m(E) \leq \frac{4\upsilon\,|h(\xi)|}{L}. \tag{2.6}$$

We define the set $D \subset (-s, s) \setminus \{\xi\}$ by

$$D = A \setminus E.$$

Clearly, D is a measurable set. By Lemma 2.3.4 (with $\theta = 8\upsilon$) and (2.6) we have

$$m(D) \geq m(A) - \frac{4\upsilon\,|h(\xi)|}{L} \geq \frac{4\upsilon\left|\varphi(\xi) - \frac{\psi(s)+\psi(-s)}{2} - \frac{\xi(\psi(s)-\psi(-s))}{2s}\right|}{L}.$$

This verifies (2.3). Now fix $\tau \in D$. The conditions (2.4) and (2.5) are proved by [28, p. 323-326]. □

In what follows, we will frequently use the forthcoming lemma. The lemma can be thought of as a type of 'mean value theorem' for Lipschitz mappings. Of the three conclusions listed below, the statement (i) should be considered the main conclusion of the lemma. Statements (ii) and (iii) are simple consequence of (i) which will be useful later on.

Lemma 2.3.6. *Suppose X is a Banach space with the Radon-Nikodým property. Let $I \subseteq \mathbb{R}$ be a non-degenerate closed interval, N be a subset of I with Lebesgue measure zero, $\gamma : I \to X$ be a Lipschitz mapping and $y \in X$ be such that*

$$\|\gamma'(t) - y\| \leq \rho \text{ whenever } t \in I \setminus N \text{ and } \gamma'(t) \text{ exists.}$$

Then the following statements hold:

(i) For all $t_0 \in I$ and $t \in \mathbb{R}$ such that $t_0 + t \in I$ we have

$$\|\gamma(t_0 + t) - \gamma(t_0) - ty\| \leq \rho\,|t|\,.$$

(ii) $\mathrm{Lip}(\gamma) \leq \|y\| + \rho$.

(iii) If, in addition, $\rho < \|y\|$ then $\gamma : I \to X$ is injective.

Proof. Fix $t_0 \in I$ and $t \in \mathbb{R}$ such that $t_0 + t \in I$. Without loss of generality suppose

$t > 0$. Define sets $R, \widetilde{R}, C \subseteq X$ by

$$R = \left\{ \frac{\gamma(s_0 + s) - \gamma(s_0)}{s} \,:\, s_0, s_0 + s \in I,\, s > 0 \right\},$$
$$\widetilde{R} = \{\gamma'(u) \,:\, u \in I \setminus N \text{ and } \gamma'(u) \text{ exists}\},$$
$$C = \{x \in X \,:\, \|x - y\| \leq \rho\}.$$

By [23, Lemma 2.12] we have that the closed convex hulls $\overline{\text{conv}}(R)$ and $\overline{\text{conv}}(\widetilde{R})$ coincide. Moreover, C is a closed, convex set containing $\widetilde{R}$. Therefore we have that

$$\frac{\gamma(t_0 + t) - \gamma(t_0)}{t} \in R \subseteq \overline{\text{conv}}(R) = \overline{\text{conv}}(\widetilde{R}) \subseteq C.$$

This verifies statement (i). The conclusions (ii) and (iii) are simple consequences of (i). □

The next lemma is a small adjustment of Lemma 4.2 from [12]. We give a proof of this lemma which closely follows [12, p. 656-659], with only minor changes.

Lemma 2.3.7. *Let $(X, \|-\|)$ be a real Banach space with the Radon-Nikodým property. Let $f : X \to \mathbb{R}$ be a Lipschitz function with $\text{Lip}(f) > 0$ and let $\epsilon \in (0, \text{Lip}(f)/100)$. Suppose $x \in X$, $e \in S(X)$ and $s > 0$ are such that the directional derivative $f'(x, e)$ exists, is non-negative and*

$$|f(x + te) - f(x) - f'(x, e)t| \leq \frac{\epsilon^2}{160\text{Lip}(f)} |t| \;\; \textit{for } |t| \leq s\sqrt{\tfrac{2\text{Lip}(f)}{\epsilon}}. \tag{2.7}$$

Let $\xi \in (-s/2, s/2)$ and $z \in X$ be such that

$$|f(x + z) - f(x + \xi e)| \geq 240\epsilon s, \qquad \|z - \xi e\| \leq s\sqrt{\frac{\epsilon}{\text{Lip}(f)}}. \tag{2.8}$$

Suppose $\gamma : [-s, s] \to X$ is a Lipschitz mapping satisfying

$$\max(\|\gamma(-s) - (x - se)\|, \|\gamma(s) - (x + se)\|) \leq \frac{\epsilon^2}{320\mathrm{Lip}(f)^2} s, \tag{2.9}$$

$$\|\gamma(\xi) - (x + z)\| \leq \frac{\epsilon s}{16\mathrm{Lip}(f)}, \tag{2.10}$$

$$\|\gamma'(t)\| \leq 1 + \frac{\epsilon}{2\mathrm{Lip}(f)} \text{ and } \|\gamma'(t) - e\| \leq 3\sqrt{\epsilon/\mathrm{Lip}(f)} \text{ for a.e. } t \in [-s, s]. \tag{2.11}$$

Then we can find a m_γ-measurable set $T \subseteq \gamma([-s, s])$ such that the following conditons hold:

(i)

$$m_\gamma(T) \geq \epsilon/160\mathrm{Lip}(f). \tag{2.12}$$

(ii) For all $x' \in T$, there exists $e' \in S(X)$ such that $f'(x', e')$ exists,

$$f'(x', e') \geq f'(x, e) + \epsilon \text{ and} \tag{2.13}$$

$$\begin{aligned} &|(f(x' + te) - f(x')) - (f(x + te) - f(x))| \\ &\leq 25\sqrt{(f'(x', e') - f'(x, e))\mathrm{Lip}(f)}\,|t| \text{ for all } t \in \mathbb{R}. \end{aligned} \tag{2.14}$$

Proof. (Following [12, p. 656-659]) We let $L = 4\mathrm{Lip}(f)$, $\upsilon = 1/80$, $\sigma = \epsilon^2/20L^2$ and $\rho = s\sqrt{L/2\epsilon}$. We presently define 'paths' h and $\underline{\gamma}$ in X where h follows the line through x, parallel to e and $\underline{\gamma}$ is an extension of γ. Let $h : \mathbb{R} \to X$ be a mapping such that $h(t) = x + te$ for $t \in [-s/2, s/2]$, $h(\pm s) = \gamma(\pm s)$ and h is affine on each of the intervals $(-\infty, -s/2]$ and $[s/2, \infty)$. Let $\underline{\gamma} : \mathbb{R} \to X$ be the mapping with $\underline{\gamma}(t) = \gamma(t)$ for $t \in [-s, s]$ and $\underline{\gamma}(t) = h(t)$ for $|t| \geq s$. Next, define functions $\psi, \varphi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = f(h(t)) \text{ and } \varphi(t) = f(\underline{\gamma}(t)). \tag{2.15}$$

From the definition of h and (2.9) we deduce

$$\|h'(t) - e\| \leq \frac{2\max(\|\gamma(-s) - (x - se)\|, \|\gamma(s) - (x + se)\|)}{s} \leq \frac{\epsilon^2}{160\mathrm{Lip}(f)^2} \tag{2.16}$$

for $t \in \mathbb{R} \setminus \{-s/2, s/2\}$.

Let us verify that the conditions of Lemma 2.3.5 hold for ξ, s, ρ, υ, σ, L, φ, ψ .

Evidently $|\xi| < s < \rho$, $0 < \upsilon < 1/32$, $\sigma > 0$ and $L > 0$, whilst $\mathrm{Lip}(h) \leq 2$ is implied by (2.16). Now observe that $\mathrm{Lip}(\underline{\gamma}) \leq 2$ follows from $\mathrm{Lip}(h) \leq 2$ and (2.11). Combining with (2.15), we get $\mathrm{Lip}(\varphi) + \mathrm{Lip}(\psi) \leq 4\mathrm{Lip}(f) = L$. Moreover, when $|t| \geq s$ we have $\underline{\gamma}(t) = h(t)$ and therefore $\varphi(t) = \psi(t)$.

Using $\xi \in (-s/2, s/2)$, (2.8) and (2.10) we deduce that

$$\begin{aligned} |\varphi(\xi) - \psi(\xi)| &= |f(\gamma(\xi)) - f(x + \xi e)| \\ &\geq |f(x + z) - f(x + \xi e)| - \mathrm{Lip}(f)\,\|\gamma(\xi) - (x + z)\| \\ &\geq 240\epsilon s - \frac{\epsilon s}{16} \geq 160\epsilon s. \end{aligned} \tag{2.17}$$

Hence $\varphi(\xi) \neq \psi(\xi)$.

The remainder of the conditions are verified exactly as in [12, p. 658].

Let the measurable set $D \subset (-s, s) \setminus \{\xi\}$ be given by the conclusion of Lemma 2.3.5. Then for all $\tau \in D$, $\varphi'(\tau)$ exists and from (2.4), (2.17) and $\psi'(0) = f'(x, e) \geq 0$ we have

$$\varphi'(\tau) \geq \psi'(0) + \upsilon\,|\varphi(\xi) - \psi(\xi)|\,/s \geq f'(x, e) + 2\epsilon > 0, \tag{2.18}$$

Moreover, (2.5) gives

$$|(\varphi(\tau + t) - \varphi(\tau)) - (\psi(t) - \psi(0))| \leq 5\sqrt{(\varphi'(\tau) - f'(x, e))L}\,|t| \text{ for every } t \in \mathbb{R}. \tag{2.19}$$

Let N denote the Lebesgue null subset of $[-s, s]$ where the condition (2.11) fails. In particular, the set N contains all points where the map γ is not differentiable. We define

the set $T \subseteq [-s, s]$ by $T = \gamma(D \setminus N)$. Let us now verify (2.12); we will need to establish several inequalities. First, observe that

$$\begin{aligned}
&|f(\gamma(s)) + f(\gamma(-s)) - 2f(x)| \\
&\quad \leq |f(\gamma(s)) - f(x+se)| + |f(x+se) - f(x) - sf'(x,e)| + |f(\gamma(-s)) - f(x-se)| \\
&\qquad\qquad + |f(x-se) - f(x) + sf'(x,e)| \leq \frac{3\epsilon^2}{160\mathrm{Lip}(f)} s, \quad (2.20)
\end{aligned}$$

using (2.9) and (2.7) for the final inequality. Next, we note that

$$\begin{aligned}
&|f(\gamma(s)) - f(\gamma(-s)) - 2sf'(x,e)| \\
&\qquad \leq |f(\gamma(s)) - f(x+se)| + |f(x+se) - f(x) - sf'(x,e)| \\
&\quad + |f(x) - f(x-se) - sf'(x,e)| + |f(x-se) - f(\gamma(-s))| \leq \frac{3\epsilon^2}{160\mathrm{Lip}(f)} s, \quad (2.21)
\end{aligned}$$

using (2.9) and (2.7) again. We now obtain a lower estimate for $m(D)$: By (2.3),

$$\begin{aligned}
m(D) &\geq \frac{4\upsilon}{L} \left| \varphi(\xi) - \frac{\psi(s) + \psi(-s)}{2} - \frac{\xi(\psi(s) - \psi(-s))}{2s} \right| \\
&= \frac{1}{20L} \left| f(\gamma(\xi)) - \frac{f(\gamma(s)) + f(\gamma(-s))}{2} - \frac{\xi(f(\gamma(s)) - f(\gamma(-s)))}{2s} \right| \\
&\geq \frac{1}{20L} |f(x+z) - f(x+\xi e)| - \frac{1}{20L} |f(\gamma(\xi)) - f(x+z)| \\
&- \frac{1}{20L} |f(x+\xi e) - f(x) - \xi f'(x,e)| \\
&- \frac{1}{40L} |2f(x) - f(\gamma(s)) - f(\gamma(-s))| \\
&- \frac{|\xi|}{40Ls} |2sf'(x,e) - f(\gamma(s)) + f(\gamma(-s))| \\
&\geq \frac{240\epsilon s}{20L} - \frac{\epsilon s}{320L} - \frac{\epsilon^2 |\xi|}{3200L\mathrm{Lip}(f)} - \frac{3\epsilon^2 s}{6400L\mathrm{Lip}(f)} - \frac{3\epsilon^2 |\xi|}{6400L\mathrm{Lip}(f)} \\
&\geq \frac{\epsilon s}{20L}.
\end{aligned}$$

For the penultimate inequality, we apply (2.8), (2.10), (2.7), (2.20) and (2.21). To get the final inequality we use $\epsilon \in (0, \mathrm{Lip}(f)/100)$.

The condition (2.11) together with Lemma 2.3.6 leads us to conclude that γ is injective. It follows that $\gamma^{-1}(T) = \gamma^{-1}(\gamma(D \setminus N)) = D \setminus N$. In particular, T is m_γ-measurable and $m(\gamma^{-1}(T)) = m(D)$. Using the lower bound derived above for $m(D)$, we deduce (2.12):

$$m_\gamma(T) = \frac{m(\gamma^{-1}(T))}{2s} = \frac{m(D)}{2s} \geq \frac{\epsilon}{40L} = \frac{\epsilon}{160\mathrm{Lip}(f)}.$$

Fix $x' \in T$. Then there exists $\tau \in D \setminus N$ such that $x' = \gamma(\tau) = \underline{\gamma}(\tau)$. We define $e' = \underline{\gamma}'(\tau)/\left\|\underline{\gamma}'(\tau)\right\|$, noting that $\left\|\underline{\gamma}'(\tau)\right\| > 0$ follows from (2.11).

Since $\tau \in D$, we have that φ is differentiable at τ. Moreover, we may compute the derivative as

$$\varphi'(\tau) = f'(\underline{\gamma}(\tau), \underline{\gamma}'(\tau)) = f'(x', \left\|\underline{\gamma}'(\tau)\right\| e').$$

In particular, the directional derivative $f'(x', e')$ exists and

$$f'(x', e') = \varphi'(\tau)/\left\|\underline{\gamma}'(\tau)\right\|.$$

From (2.18) we have that $0 < f'(x, e) + \varphi'(\tau) \leq 2\varphi'(\tau) - 2\epsilon$. Applying (2.11) and later $\varphi'(\tau) \leq \mathrm{Lip}(\varphi) \leq 2\mathrm{Lip}(f)$, we deduce

$$\begin{aligned}
\left\|\underline{\gamma}'(\tau)\right\| (f'(x, e) + \varphi'(\tau)) &\leq \left(1 + \frac{\epsilon}{2\mathrm{Lip}(f)}\right)(2\varphi'(\tau) - 2\epsilon) \\
&= 2\varphi'(\tau) + \frac{\epsilon\varphi'(\tau)}{\mathrm{Lip}(f)} - 2\epsilon - \frac{\epsilon^2}{\mathrm{Lip}(f)} \\
&\leq 2\varphi'(\tau) = 2\left\|\underline{\gamma}'(\tau)\right\| f'(x', e').
\end{aligned}$$

After some manipulation, we arrive at

$$f'(x', e') - f'(x, e) \geq \frac{\varphi'(\tau) - f'(x, e)}{2} \geq \epsilon, \tag{2.22}$$

where the latter inequality is simply (2.18). This proves (2.13).

It only remains to verify (2.14) for x' and e'. To this end, we use the definitions of

φ, ψ in combination with $x' = \underline{\gamma}(\tau)$, $L = 4\mathrm{Lip}(f)$, (2.19) and (2.22) to deduce

$$\begin{aligned}
&\left|(f(\underline{\gamma}(\tau+t)) - f(x')) - (f(h(t)) - f(x))\right| \\
&\leq 20\sqrt{(f'(x',e') - f'(x,e))\mathrm{Lip}(f)}\,|t|\,. \qquad (2.23)
\end{aligned}$$

From (2.16), (2.11), Lemma 2.3.6 and $\epsilon \leq \mathrm{Lip}(f)$ it is clear that

$$\begin{aligned}
&\left\|\underline{\gamma}(\tau+t) - \underline{\gamma}(\tau) - te\right\| \leq 3\sqrt{\frac{\epsilon}{\mathrm{Lip}(f)}}\,|t|\,, \\
&\|h(t) - h(0) - te\| \leq \frac{\epsilon^2}{160\mathrm{Lip}(f)^2}\,|t| \leq \sqrt{\frac{\epsilon}{\mathrm{Lip}(f)}}\,|t|
\end{aligned}$$

for all t. Moreover, using $\underline{\gamma}(\tau) = x'$ and $h(0) = x$, we get

$$\begin{aligned}
&\left|f(\underline{\gamma}(\tau+t)) - f(x'+te)\right| \leq 3\sqrt{\epsilon\mathrm{Lip}(f)}\,|t|\,, \\
&|f(h(t)) - f(x+te)| \leq \sqrt{\epsilon\mathrm{Lip}(f)}\,|t|
\end{aligned}$$

for all t.

Combining these estimates with (2.23) and $\epsilon \leq f'(x',e') - f'(x,e)$ yields

$$\begin{aligned}
&|(f(x'+te) - f(x')) - (f(x+te) - f(x))| \\
&\leq 20\sqrt{(f'(x',e') - f'(x,e))\mathrm{Lip}(f)}\,|t| + 3\sqrt{\epsilon\mathrm{Lip}(f)}\,|t| + \sqrt{\epsilon\mathrm{Lip}(f)}\,|t| \\
&\leq 25\sqrt{(f'(x',e') - f'(x,e))\mathrm{Lip}(f)}\,|t|\,.
\end{aligned}$$

This is (2.14). □

In the next two lemmas we reveal how the presence of the curve approximation property in a collection of sets is conducive to constructing special Lipschitz curves which meet the sets in large one-dimensional Lebesgue measure.

Lemma 2.3.8. *Let X be a real Banach space with the Radon-Nikodým property and suppose that the collection $(U_\alpha)_{\alpha\in(0,1]}$ of subsets of X has the curve approximation property.*

Let $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$, $0 < \eta < \eta^{1/2} < \omega < \theta < 1/10000$ *be real numbers and suppose that*

$$0 < s < \frac{1}{2}\delta_0\left(\lambda + \frac{i\psi}{2}, \frac{\psi}{2}, \eta\right) \text{ for } i = 0, 1. \tag{2.24}$$

Let $x \in U_\lambda$, $e \in S(X)$ *and suppose that* $z \in X$ *and* $\xi \in (-s/2, s/2)$ *satisfy*

$$\|z - \xi e\| \le \theta s, \qquad \frac{\|z - \sigma s e\|}{s - \sigma\xi} \le 1 + \omega \text{ for } \sigma = \pm 1. \tag{2.25}$$

Then, for each $\sigma \in \{-1, 1\}$ *there exists a Lipschitz map* $\gamma_\sigma : [0, s - \sigma\xi] \to X$ *such that the following conditions are satisfied.*

$$m_{\gamma_\sigma}(U_{\lambda+\psi}) \ge 1 - \eta, \tag{2.26}$$

$$\|\gamma_\sigma'(r)\| \le 1 + 20\omega \text{ and } \|\gamma_\sigma'(r) + \sigma e\| \le 100\theta \text{ whenever } \gamma_\sigma'(r) \text{ exists,} \tag{2.27}$$

$$\|\gamma_\sigma(0) - (x + \sigma s e)\| \le 6\eta s \text{ and } \|\gamma_\sigma(s - \sigma\xi) - (x + z)\| \le 4\eta s. \tag{2.28}$$

Proof. Fix $\sigma \in \{-1, 1\}$ and let us describe how to obtain the Lipschitz map γ_σ.

Applying Definition 2.3.1 to the point $x \in U_\lambda$, the direction $\sigma e \in S(X)$ and the number $s \in (0, \delta_0(\lambda, \psi/2, \eta))$, we can find a Lipschitz map $\gamma_0 : [0, s] \to X$ such that

$$\|\gamma_0(0) - x\| \le \eta s, \tag{2.29}$$

$$\|\gamma_0'(r) - \sigma e\| \le \eta \text{ whenever } \gamma_0'(r) \text{ exists and,} \tag{2.30}$$

$$m_{\gamma_0}(U_{\lambda+\frac{\psi}{2}}) \ge 1 - \eta. \tag{2.31}$$

From (2.31), it is clear that we can choose $t_1 \in [(1-2\eta)s, s]$, such that $x_1 := \gamma_0(t_1) \in U_{\lambda+\frac{\psi}{2}}$. The point x_1 is close to $(x + \sigma s e)$:

$$\begin{aligned}
\|x_1 - (x + \sigma s e)\| &= \|\gamma_0(t_1) - \gamma_0(0) + \gamma_0(0) - x - \sigma s e\| \\
&\le \|\gamma_0(t_1) - \gamma_0(0) - \sigma t_1 e\| + \|\sigma(t_1 - s)e\| + \|\gamma_0(0) - x\| \\
&\le \eta t_1 + 2\eta s + \eta s \le 4\eta s.
\end{aligned} \tag{2.32}$$

The last step is deduced using (2.30), Lemma 2.3.6, $t_1 \in [(1-2\eta)s, s]$ and (2.29).

We will later require various estimates for the quantity $A_\sigma := \|x_1 - (x+z)\|$ to show that it closely approximates $s - \sigma\xi$. For an upper bound, note that

$$\begin{aligned} A_\sigma &\leq \|z - \xi e\| + \|\xi e - \sigma s e\| + \|(x + \sigma s e) - x_1\| \\ &\leq \theta s + (s - \sigma\xi) + 4\eta s < 2s, \end{aligned} \tag{2.33}$$

whilst a lower bound is given by

$$\begin{aligned} A_\sigma &\geq \|\xi e - \sigma s e\| - \|\xi e - z\| - \|x_1 - (x + \sigma s e)\| \\ &\geq (s - \sigma\xi) - \theta s - 4\eta s > s/4. \end{aligned} \tag{2.34}$$

In the above we use (2.25), (2.32) and $|\xi - \sigma s| = s - \sigma\xi$. Together, (2.33) and (2.34) establish

$$\Big| A_\sigma - (s - \sigma\xi) \Big| \leq 5\theta s, \tag{2.35}$$

and this, in tandem with $s - \sigma\xi > s/2$, leads to

$$\left| \frac{A_\sigma}{s - \sigma\xi} - 1 \right| \leq 10\theta. \tag{2.36}$$

We may also combine (2.32), $s - \sigma\xi > s/2$ and (2.25) to deduce that

$$\begin{aligned} \frac{A_\sigma}{s - \sigma\xi} &= \frac{\|x_1 - (x+z)\|}{s - \sigma\xi} \\ &\leq \frac{\|x_1 - (x + \sigma s e)\|}{s - \sigma\xi} + \frac{\|\sigma s e - z\|}{s - \sigma\xi} \\ &\leq 8\eta + (1 + \omega) < 1 + 9\omega. \end{aligned} \tag{2.37}$$

Setting $v = \frac{(x+z)-x_1}{\|(x+z)-x_1\|} = \frac{(x+z)-x_1}{A_\sigma}$, we observe that

$$\begin{aligned}
\|v+\sigma e\| &= \frac{1}{A_\sigma}\Big\|(x+z)-x_1+A_\sigma\sigma e\Big\| \\
&\le \frac{1}{A_\sigma}\Big(\|z-\xi e\| + \|(x+\sigma s e)-x_1\| + \Big|(s-\sigma\xi)-A_\sigma\Big|\Big) \\
&\le \frac{4}{s}(\theta s + 4\eta s + 5\theta s) < 40\theta, \qquad (2.38)
\end{aligned}$$

where the final estimates are obtained using (2.34), (2.25), (2.32) and (2.35).

Note that $x_1 \in U_{\lambda+\frac{\psi}{2}}$, $v \in S(X)$ and, from (2.33), $0 < A_\sigma < 2s < \delta_0(\lambda+\frac{\psi}{2}, \frac{\psi}{2}, \eta)$. Therefore, applying Definition 2.3.1, we can find a Lipschitz map $\widetilde{\gamma}_\sigma : [0, A_\sigma] \to X$ such that

$$\|\widetilde{\gamma}_\sigma(0) - x_1\| \le \eta A_\sigma < 2\eta s, \qquad (2.39)$$

$$\|\widetilde{\gamma}'_\sigma(r) - v\| \le \eta \text{ whenever } \widetilde{\gamma}'_\sigma(r) \text{ exists, and} \qquad (2.40)$$

$$m_{\widetilde{\gamma}_\sigma}(U_{\lambda+\psi}) \ge 1-\eta. \qquad (2.41)$$

We define the Lipschitz map $\gamma_\sigma : [0, s-\sigma\xi] \to X$ by

$$\gamma_\sigma(t) = \widetilde{\gamma}_\sigma\left(\frac{tA_\sigma}{(s-\sigma\xi)}\right) \qquad \forall t \in [0, s-\sigma\xi]. \qquad (2.42)$$

From (2.41) and (2.42) we deduce (2.26). Combining (2.42), (2.37), (2.40), (2.36) and (2.38), we obtain

$$\begin{aligned}
\|\gamma'_\sigma(r)+\sigma e\| &\le \frac{A_\sigma}{s-\sigma\xi}\left\|\widetilde{\gamma}'_\sigma\left(\frac{rA_\sigma}{s-\sigma\xi}\right) - v\right\| + \left|\frac{A_\sigma}{s-\sigma\xi} - 1\right| \|v\| + \|v+\sigma e\| \\
&\le (1+9\omega)\eta + 10\theta + 40\theta < 60\theta.
\end{aligned}$$

whenever $\gamma'_\sigma(r)$ exists. Moreover, using (2.42), (2.37) and (2.40) we get that

$$\|\gamma'_\sigma(r)\| = \frac{A_\sigma}{s-\sigma\xi}\left\|\widetilde{\gamma}'_\sigma\left(\frac{rA_\sigma}{s-\sigma\xi}\right)\right\| \le (1+9\omega)(1+\eta) \le 1+19\omega,$$

whenever $\gamma'_\sigma(r)$ exists. Thus, (2.27) is established.

Finally, we verify (2.28). The first inequality is obtained using (2.42), (2.39) and (2.32) as follows:

$$\begin{aligned}\|\gamma_\sigma(0) - (x + \sigma se)\| &\leq \|\widetilde{\gamma}_\sigma(0) - x_1\| + \|x_1 - (x + \sigma se)\| \\ &\leq 2\eta s + 4\eta s = 6\eta s.\end{aligned}$$

For the second inequality of (2.28) we use (2.42), the definition of v, (2.40), Lemma 2.3.6, (2.39) and (2.33) to deduce

$$\begin{aligned}\|\gamma_\sigma(s - \sigma\xi) - (x + z)\| &\leq \|\widetilde{\gamma}_\sigma(A_\sigma) - \widetilde{\gamma}_\sigma(0) - A_\sigma v\| + \|\widetilde{\gamma}_\sigma(0) - x_1\| \\ &\leq \eta A_\sigma + 2\eta s \leq 4\eta s.\end{aligned}$$

□

Lemma 2.3.9. *Let X, $(U_\alpha)_{\alpha\in(0,1]}$, λ, ψ, η, ω, θ, s, x, e, z and ξ be given by the conditions of Lemma 2.3.8. Then there exists a Lipschitz map $\gamma : [-s, s] \to X$ such that the following conditions hold.*

$$m_\gamma(U_{\lambda+\psi}) \geq 1 - 2\eta^{1/2}, \tag{2.43}$$

$$\|\gamma(\sigma s) - (x + \sigma se)\| \leq 6\eta s \text{ for } \sigma = \pm 1, \qquad \|\gamma(\xi) - (x + z)\| \leq 6\eta^{1/2} s, \tag{2.44}$$

$$\|\gamma'(r)\| \leq 1 + 25\omega \text{ and } \|\gamma'(r) - e\| \leq 200\theta \text{ for almost every } r. \tag{2.45}$$

Proof. Let the Lipschitz maps γ_{-1}, γ_1 be given by the conclusion of Lemma 2.3.8. For future reference, we observe that the condition (2.27) enables us to apply Lemma 2.3.6 to establish that $\mathrm{Lip}(\gamma_\sigma) \leq 1 + 20\omega \leq 2$ for $\sigma = \pm 1$.

We define $\gamma : [-s, s] \to X$ as the map which is affine on the interval $[\xi - \eta^{1/2}s, \xi + \eta^{1/2}s]$

and satisfies

$$\gamma(t) = \begin{cases} \gamma_{-1}(t+s) & \text{if } t \in [-s, \xi - \eta^{1/2}s], \\ \gamma_1(s-t) & \text{if } t \in [\xi + \eta^{1/2}s, s]. \end{cases}$$

To prove (2.43), we use the definition of γ, (2.26) and $\xi \in (-s/2, s/2)$ to deduce that

$$\begin{aligned} m\left(\{t \in [-s,s] : \gamma(t) \in U_{\lambda+\psi}\}\right) &\geq m\left(\{t \in [0, s+\xi] : \gamma_{-1} \in U_{\lambda+\psi}\}\right) - \eta^{1/2}s \\ &\quad + m\left(\{t \in [0, s-\xi] : \gamma_1(t) \in U_{\lambda+\psi}\}\right) - \eta^{1/2}s \\ &\geq (1-\eta)(s+\xi) - \eta^{1/2}s + (1-\eta)(s-\xi) - \eta^{1/2}s \\ &\geq 2s - 4\eta^{1/2}s. \end{aligned}$$

Dividing through by $m([-s,s]) = 2s$, we obtain (2.43).

Let us now verify (2.44). From (2.28) and the definition of γ, it is clear that the first inequality of (2.44) is satisfied. Since γ is affine on the interval $[\xi - \eta^{1/2}s, \xi + \eta^{1/2}s]$ we have

$$\begin{aligned} \|\gamma(\xi) - (x+z)\| &= \left\| \frac{\gamma_{-1}(s+\xi-\eta^{1/2}s) + \gamma_1(s-\xi-\eta^{1/2}s)}{2} - (x+z) \right\| \\ &\leq \frac{1}{2}\Big(\left\|\gamma_{-1}(s+\xi-\eta^{1/2}s) - \gamma_{-1}(s+\xi)\right\| + \|\gamma_{-1}(s+\xi) - (x+z)\| \\ &\quad + \left\|\gamma_1(s-\xi-\eta^{1/2}s) - \gamma_1(s-\xi)\right\| + \|\gamma_1(s-\xi) - (x+z)\| \Big) \\ &\leq \frac{1}{2}\Big(2\eta^{1/2}s + 4\eta s + 2\eta^{1/2}s + 4\eta s\Big) \leq 6\eta^{1/2}s, \end{aligned} \tag{2.46}$$

using $\mathrm{Lip}(\gamma_\sigma) \leq 2$ and (2.28). This establishes (2.44).

It only remains to prove (2.45). Since γ is affine on the interval $[\xi - \eta^{1/2}s, \xi + \eta^{1/2}s]$,

we have that the following holds for every $r \in (\xi - \eta^{1/2}s, \xi + \eta^{1/2}s)$:

$$\begin{aligned}
\|\gamma'(r) - e\| &= \left\| \frac{\gamma_1(s - \xi - \eta^{1/2}s) - \gamma_{-1}(s + \xi - \eta^{1/2}s)}{2\eta^{1/2}s} - e \right\| \\
&\leq \frac{1}{2\eta^{1/2}s} \Big(\big\|\gamma_1(s - \xi - \eta^{1/2}s) - \gamma_1(s - \xi) - \eta^{1/2}se\big\| + \|\gamma_1(s - \xi) - (x + z)\| \\
&+ \|(x + z) - \gamma_{-1}(s + \xi)\| + \big\|\gamma_{-1}(s + \xi) - \gamma_{-1}(s + \xi - \eta^{1/2}s) - \eta^{1/2}se\big\| \Big) \\
&\leq \frac{1}{2\eta^{1/2}s} \Big(100\theta(\eta^{1/2}s) + 4\eta s + 4\eta s + 100\theta(\eta^{1/2}s) \Big) < 104\theta. \qquad (2.47)
\end{aligned}$$

The final line in the above is deduced using (2.27), Lemma 2.3.6 and (2.28). Further, we assert that every $r \in (\xi - \eta^{1/2}s, \xi + \eta^{1/2}s)$ satisfies

$$\begin{aligned}
\|\gamma'(r)\| &= \left\| \frac{\gamma_1(s - \xi - \eta^{1/2}s) - \gamma_{-1}(s + \xi - \eta^{1/2}s)}{2\eta^{1/2}s} \right\| \\
&\leq \frac{1}{2\eta^{1/2}s} \Big(\big\|\gamma_1(s - \xi - \eta^{1/2}s) - \gamma_1(s - \xi)\big\| + \|\gamma_1(s - \xi) - (x + z)\| \\
&+ \|(x + z) - \gamma_{-1}(s + \xi)\| + \big\|\gamma_{-1}(s + \xi) - \gamma_{-1}(s + \xi - \eta^{-1/2})\big\| \Big) \\
&\leq \frac{1}{2\eta^{1/2}s} \Big((1 + 20\omega)\eta^{1/2}s + 4\eta s + 4\eta s + (1 + 20\omega)\eta^{1/2}s \Big) \\
&\leq 1 + 24\omega, \qquad (2.48)
\end{aligned}$$

using $\mathrm{Lip}(\gamma_\sigma) \leq 1 + 20\omega$ and (2.28) for the penultimate inequality. Putting (2.47) and (2.48) together with (2.27) and the definition of γ we obtain (2.45). □

The next Lemma is an adaptation of [12, Lemma 4.3]. We give a proof which closely follows [12, p. 660-661] with only small changes.

Lemma 2.3.10. *Let H be a real Hilbert space, $(U_\alpha)_{\alpha \in (0,1]}$ be a collection of subsets of H with the curve approximation property and $f : H \to \mathbb{R}$ be a Lipschitz function. Let $\kappa_0 \in (0, 1]$, $\lambda \in (0, \kappa_0)$ and let $(x, e) \in U_\lambda \times S(H)$ be such that the directional derivative $f'(x, e)$ exists, is non-negative and is almost locally maximal in the following sense.*

Almost locally maximal condition: *For all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ and $\lambda_\epsilon \in (\lambda, \kappa_0)$ such that whenever $(x', e') \in U_{\lambda_\epsilon} \times S(H)$ is such that*

(i) the directional derivative $f'(x', e')$ exists and $f'(x', e') \geq f'(x, e)$,

(ii) $\|x' - x\| \leq \delta_\epsilon$, and

(iii) for any $t \in \mathbb{R}$

$$|(f(x' + te) - f(x')) - (f(x + te) - f(x))| \leq 25\sqrt{f'(x', e') - f'(x, e)}\,|t|\,,$$

then we have $f'(x', e') < f'(x, e) + \epsilon$.

Then f is Fréchet differentiable at x.

Proof. (Following [12, p.660-661]) We may assume $\mathrm{Lip}(f) = 1$. Suppose $\epsilon \in (0, 1/(150)^2)$, $\psi = \lambda_\epsilon - \lambda \in (0, \kappa_0 - \lambda)$ and $\eta = \epsilon^2/500^2$.

It suffices to prove that there exists $\Delta > 0$ such that

$$|f(x + ru) - f(x) - f'(x, e)\langle u, e\rangle r| < 20000\epsilon^{1/2} r \tag{2.49}$$

for any $u \in S(H)$ and $r \in (0, \Delta)$. This will establish that the Fréchet derivative $f'(x)$ exists and is given by $f'(x)(h) = f'(x, e)\langle h, e\rangle$ for all $h \in H$.

From the existence of the directional derivative $f'(x, e)$, we can find $\Delta > 0$ such that

$$|f(x + te) - f(x) - f'(x, e)t| < \frac{\epsilon^2}{160}\,|t| \text{ for all } |t| < 200\sqrt{2}\Delta/(3\epsilon). \tag{2.50}$$

We also choose Δ small enough so that $2000\Delta\epsilon^{-1/2}/3 < \delta_\epsilon$ and (2.24) is satisfied for all $s \in (0, 200\Delta\epsilon^{-1/2}/3)$.

Suppose that (2.49) is false. Then there exist $r \in (0, \Delta)$ and $u \in S(H)$ such that

$$|f(x + ru) - f(x) - f'(x, e)\langle u, e\rangle r| \geq 20000\epsilon^{1/2} r. \tag{2.51}$$

Set

$$\theta = 3\epsilon^{1/2}/200, \qquad \omega = \epsilon/50, \qquad s = r\theta^{-1}, \qquad \xi = \langle u, e\rangle r, \qquad z = ru.$$

We verify that $X = H$, $(U_\alpha)_{\alpha\in(0,1]}$, λ, ψ, η, ω, θ, s, x, e, z and ξ satisfy the conditions of Lemma 2.3.8 and Lemma 2.3.9.

First, note that the inequalities $0 < \eta < \eta^{1/2} < \omega < \theta < 1/10000$ are satisfied, whilst $s \in (0, 200\Delta\epsilon^{-1/2}/3)$ so that (2.24) holds. Further,

$$\|z - \xi e\| = r\,\|u - \langle u, e\rangle e\| \leq r = \theta s \tag{2.52}$$

verifies the first inequality of (2.25). To prove the second inequality of (2.25), we observe

$$\frac{z - \sigma s e}{s - \sigma\xi} = -\sigma e + \frac{r}{s - \sigma\xi}(u - \langle u, e\rangle e) \text{ for } \sigma = \pm 1.$$

Since the vectors e and $(u - \langle u, e\rangle e)$ are orthogonal and $s - \sigma\xi > s/2$, the above expression leads to

$$\frac{\|z - \sigma s e\|}{s - \sigma\xi} \leq 1 + \frac{1}{2}\frac{r^2}{(s/2)^2} = 1 + 2\theta^2 \leq 1 + \omega.$$

Hence (2.25) is satisfied.

Let the Lipschitz map $\gamma : [-s, s] \to X$ be given by the conclusion of Lemma 2.3.9. From (2.43), (2.44) and (2.45) we obtain (2.9), (2.10) and (2.11). Since $|\xi| \leq r < \Delta$, the inequality (2.50) holds with $t = \xi$. This, together with (2.51) gives

$$|f(x + ru) - f(x + \xi e)| \geq 20000\epsilon^{1/2} r - \frac{\epsilon^2}{160}|\xi| > 16000\epsilon^{1/2} r = 16000\epsilon^{1/2}\theta s = 240\epsilon s.$$

This, in combination with (2.52), verifies (2.8). Finally, note that the condition (2.50) implies (2.7). We have now established that the conditions of Lemma 2.3.7 are satisfied for $X = H$, f, ϵ, x, e, s, ξ, z and γ.

Let the m_γ-measurable set $T \subseteq \gamma([-s, s])$ be given by the conclusion of Lemma 2.3.7. From (2.43), (2.12) and $\eta = \epsilon^2/500^2$, we deduce that $\gamma^{-1}(T) \cap \gamma^{-1}(U_{\lambda_\epsilon}) \neq \emptyset$. Therefore, $T \cap U_{\lambda_\epsilon} \neq \emptyset$. Let $x' \in T \cap U_{\lambda_\epsilon}$ and let $e' \in S(H)$ be given by the conclusion of Lemma 2.3.7 so that the directional derivative $f'(x', e')$ exists and (2.13) and (2.14) are satisfied. Hence, the pair $(x', e') \in U_{\lambda_\epsilon} \times S(H)$ satisfies the conditions (i) and (iii) in the statement of the

present lemma.

Note that $\mathrm{Lip}(\gamma) \leq 2$ follows from (2.45) and Lemma 2.3.6. Using $x' \in T \subseteq \gamma([-s, s])$, $\mathrm{Lip}(\gamma) \leq 2$, (2.44), $s \in (0, 200\Delta\epsilon^{-1/2}/3)$ and the choice of Δ we get

$$\begin{aligned}\|x' - x\| &\leq \|x' - \gamma(-s)\| + \|\gamma(-s) - (x - se)\| + \|(x - se) - x\| \\ &\leq 4s + 6\eta s + s \leq 10s \leq 2000\Delta\epsilon^{-1/2}/3 \leq \delta_\epsilon.\end{aligned}$$

This verifies that (x', e') satisfies (ii). Therefore, by the almost locally maximal condition in the hypothesis of the present lemma, we have that $f'(x', e') < f'(x, e) + \epsilon$. This, in light of (2.13), provides the desired contradiction. □

The following theorem is a restatement of Theorem 3.1 from [12, p.646]:

Theorem 2.3.11. *([12, Theorem 3.1]) Let H be a real Hilbert space. Let $(U_\alpha)_{\alpha\in(0,1]}$ be a collection of closed subsets of H satisfying $U_\alpha \subseteq U_\beta$ whenever $\alpha \leq \beta$. Suppose $f_0 : H \to \mathbb{R}$ is a Lipschitz function and $\lambda_0, \kappa_0 \in (0, 1]$ with $\lambda_0 < \kappa_0$. Let the pair $(x_0, e_0) \in U_{\lambda_0} \times S(H)$ be such that the directional derivative $f_0'(x_0, e_0)$ exists. Then there exists a Lipschitz function $f : H \to \mathbb{R}$, an index $\lambda \in (\lambda_0, \kappa_0)$ and a pair $(x, e) \in U_\lambda \times S(H)$ such that $(f - f_0)$ is linear and the directional derivative $f'(x, e)$ exists, is strictly positive and satisfies the almost locally maximal condition of Lemma 2.3.10.*

We are now ready to give a proof of Theorem 2.3.3. The statement is repeated here for the reader's convenience.

Theorem. *Let H be a real Hilbert space and let $(U_\lambda)_{\lambda\in(0,1]}$ be a collection of closed subsets of H with the curve approximation property. Then each set U_λ is a universal differentiability set.*

Proof of Theorem 2.3.3. Fix $\kappa_0 \in (0, 1]$. We show that the set U_{κ_0} is a universal differentiability set.

Let $f_0 : H \to \mathbb{R}$ be a Lipschitz function. We are tasked with finding a point $x \in U_{\kappa_0}$ such that f_0 is Fréchet differentiable at x. Pick $\lambda_0 \in (0, \kappa_0)$ and choose $(x_0, e_0) \in U_{\lambda_0} \times$

$S(H)$ such that the directional derivative $f'(x_0, e_0)$ exists. Such a pair (x_0, e_0) can be found because the set U_{λ_0} has positive measure on many Lipschitz curves. The conditions of Theorem 2.3.11 are now satisfied for the Hilbert space H, the sets $(U_\alpha)_{\alpha\in(0,1]}$, the Lipschitz function f_0, the real numbers $\lambda_0, \kappa_0 \in (0, 1]$ and the pair $(x_0, e_0) \in U_{\lambda_0} \times S(H)$. Let the Lipschitz function f, the index $\lambda \in (\lambda_0, \kappa_0)$ and the pair $(x, e) \in U_\lambda \times S(H)$ be given by the conclusion of Theorem 2.3.11. Note that the conditions of Lemma 2.3.10 are satisfied for the Hilbert space H, the sets $(U_\alpha)_{\lambda\in(0,1]}$, the function f, the real numbers $\kappa_0, \lambda \in (0, 1]$ and $(x, e) \in U_\lambda \times S(H)$. Therefore, by Lemma 2.3.10, the function f is Fréchet differentiable at x. Since $f - f_0$ is linear, f_0 is also Fréchet differentiable at x. Finally note that $\lambda \leq \kappa_0$ implies $U_\lambda \subseteq U_{\kappa_0}$, so that $x \in U_{\kappa_0}$. $\square$

Chapter 3

Differentiability inside sets with Minkowski dimension one.

3.1 Introduction.

The research presented in this chapter is joint work with Olga Maleva. The present author contributed to all results. The work has been submitted for publication and a preprint of the paper is available at [17].

The Minkowski dimensions of a bounded subset of $\mathbb{R}^d$ are closely related to the Hausdorff dimension. Whilst the Hausdorff dimension of a set is based on coverings by sets of arbitrarily small diameter, the Minkowski dimensions are defined similarly according to coverings by sets of the same small diameter. For this reason, the Minkowski dimension is often referred to as the box-counting dimension [19, p. 41]. The definition below follows [27, p. 76-77].

Definition 3.1.1. *Given a bounded subset A of $\mathbb{R}^d$ and $\varepsilon > 0$, we define $N_\varepsilon(A)$ to be the minimal number of balls of radius ε required to cover A. That is, $N_\varepsilon(A)$ is the smallest integer n for which there exist balls $B_1, \ldots, B_n \subseteq \mathbb{R}^d$, each of radius ε, such that $A \subseteq \cup_i B_i$.*

The lower Minkowski dimension of A is then defined by

$$\underline{\dim}_M(A) = \inf\left\{p > 0 : \liminf_{\varepsilon \to 0} N_\varepsilon(A)\varepsilon^p = 0\right\}, \tag{3.1}$$

and the upper Minkowski dimension of A is given by

$$\overline{\dim}_M(A) = \inf\left\{p > 0 : \limsup_{\varepsilon\to 0} N_\varepsilon(A)\varepsilon^p = 0\right\}. \tag{3.2}$$

If $\underline{\dim}_M(A) = \overline{\dim}_M(A)$, then this common value is called the Minkowski dimension of A and is denoted by $\dim_M(A)$.

Remark. *Note that in Definition 3.1.1, we can fix any norm $\|\cdot\|'$ on $\mathbb{R}^d$ and consider the balls B_i with respect to $\|\cdot\|'$; passing to a ball in a norm equivalent to the Euclidean norm $\|\cdot\|$ (or a rotation of a ball in a norm equivalent to $\|\cdot\|$) does not change the values of the lower/upper Minkowski dimensions. Moreover, it does not matter whether the balls B_i are open or closed. In what follows, it will be convenient for us to consider coverings of a set by rotated ℓ_∞-balls of radius ε which we refer to as ε-cubes (see Definition 3.3.1).*

Writing $\dim_H$ for the Hausdorff dimension, it is readily verified that for all bounded $A \subseteq \mathbb{R}^d$ it holds that $\dim_H(A) \leq \underline{\dim}_M(A) \leq \overline{\dim}_M(A)$. Moreover, the Hausdorff dimension and Minkowski dimensions can be very different: For example, a countable dense subset of a ball in $\mathbb{R}^d$ has Hausdorff dimension 0 whilst having the maximum Minkowski dimension d. A construction of a set having lower Minkowski dimension strictly less than its upper Minkowski dimension is given in [27, p. 77]. It is also worth noting that, in contrast to the Hausdorff dimension, the Minkowski dimensions behave nicely with respect to closures. For a bounded set $A \subseteq \mathbb{R}^d$, we have that $\overline{\dim}_M(A) = \overline{\dim}_M(\mathrm{Clos}(A))$ and $\underline{\dim}_M(A) = \underline{\dim}_M(\mathrm{Clos}(A))$, where $\mathrm{Clos}(A)$ denotes the closure of A.

In the present chapter, we verify the existence of a compact universal differentiability set with Minkowski dimension one in $\mathbb{R}^d$ for all d. Such a set is constructed explicitly (of course, the case $d = 1$ is trivial as one can simply take a bounded closed interval of positive length). This is an improvement on the result of [14], where a compact universal differentiability set of Hausdorff dimension one is given. We should note that one cannot achieve a better Minkowski dimension as any universal differentiability set has Hausdorff dimension at least one [14, Lemma 1.2], hence the lower Minkowski dimension

of a universal differentiability set must be at least one. However we now go further and show, in Theorem 3.2.2 and Corollary 3.2.3, that every universal differentiability subset of $\mathbb{R}^d$ with $d \geq 2$ has infinite 1-dimensional Hausdorff measure. This means that although $\limsup_{\varepsilon \to 0} N_\varepsilon(A)\varepsilon^p = 0$ for every $p > 1$ we must have $\liminf_{\varepsilon \to 0} N_\varepsilon(A)\varepsilon^1 = \infty$, see Corollary 3.2.3.

The construction in [14] involves considering a G_δ set O of Hausdorff dimension one, containing all line segments between points belonging to a countable dense subset R of the unit ball in $\mathbb{R}^d$ (and hence the Minkowski dimension of O is equal to d). The set O can be expressed as $O = \bigcap_{k=1}^{\infty} O_k$, where each O_k is an open subset of $\mathbb{R}^d$ and $O_{k+1} \subseteq O_k$. For each $k \geq 1$, a set R_k is defined consisting of a finite union of line segments between points from R. Since R_k is then a closed subset of O, it is possible to choose $w_k > 0$ such that $\overline{B}_{w_k}(R_k) \subseteq O_k$. The final sets T_λ, $\lambda \in [0,1]$ are defined by

$$T_\lambda = \bigcap_{k=1}^{\infty} \bigcup_{k \leq n \leq (1+\lambda)k} \overline{B}_{\lambda w_k}(R_k),$$

where $\overline{B}_{\lambda w_n}(R_k)$ denotes a closed 'tubular neighbourhood' of R_k. We explain this notation precisely in Section 3.3.

Observe that, for each $k \geq 1$ and $\lambda \in [0,1]$ the closed set $\bigcup_{k \leq n \leq (1+\lambda)k} \overline{B}_{\lambda w_k}(R_k)$ is contained in O_k. Consequently, $\dim_H(T_\lambda) \leq \dim_H(O) = 1$. There is no non-trivial upper bound for the Minkowski dimensions of the sets T_λ constructed in [14]. For constructing a universal differentiability set with upper or lower Minkowski dimension one, the approach of [14] fails because the set O has the maximum upper and lower Minkowksi dimension, d.

To get a set of lower Minkowski dimension one it would be enough to control the number of δ-cubes (this will refer to a rotated cube with side equal 2δ) for a specific sequence $\delta_n \searrow 0$. Assume $p > 1$ is a fixed number and we want to make sure that the set to be constructed has lower Minkowski dimension less than p. Imagine that we have reached the nth step of the construction where we require that C_n is an upper estimate for

a number of δ_n-cubes needed to cover the final set S, and $C_n\delta_n^p < 1$. The idea for the next step is to divide each δ_n-cube by a $K_n \times \cdots \times K_n$ grid into smaller $\delta_{n+1} = \delta_n/K_n$-cubes. If K_n is big enough, then as $\delta_n^p/\delta_{n+1}^p = K_n^p$, we are free to choose inside the given δ_n-cube any number of δ_{n+1}-cubes up to K_n^p. This could, for example, be $K_n(\log K_n)^{M_n} \ll K_n^p$ for any fixed $p > 1$, see inequality (3.30); $K_n = Q^{s_n}$ and $|\mathcal{E}_n| \leq s_n^{2d}$, $M_n \ll \frac{s_n}{\log s_n}$. We then have that the total number of δ_{n+1}-cubes needed to cover S is bounded above by $C_nK_n^p$, whilst $C_nK_n^p\delta_{n+1}^p = C_n\delta_n^p < 1$. Since this is satisfied for all n, we conclude that $\underline{\dim}_M(S) \leq p$. As this is true for every $p > 1$, we obtain a set of lower Minkowski dimension one.

Getting $\overline{\dim}_M(S) \leq 1$ is less clear. As n grows, the sequence K_n tends to infinity or otherwise we would get many points of porosity inside S. In order to prove that $\overline{\dim}_M(S) \leq p$ we should be able to show that there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ the set S can be covered by a controlled number N_δ of δ-cubes. In other words, $N_\delta\delta^p$ should stay bounded for all δ below a certain threshold. Choosing n such that $\delta_{n+1} < \delta \leq \delta_n$ gives $N_\delta\delta^p \leq N_{\delta_{n+1}}\delta_n^p = N_{\delta_{n+1}}\delta_{n+1}^pK_n^p$, and the factor $K_n^p \to \infty$ makes it impossible to have a constant upper estimate for $N_\delta\delta^p$. The idea here is that we need to leave a 'gap' for an unbounded sequence in the upper estimate for $N_{\delta_{n+1}}\delta_{n+1}^p$ and to make sure that K_n^p fits inside that gap. The realisation of that gap is the inequality (3.33).

We begin, in Section 3.2, by proving that any universal differentiability set in $\mathbb{R}^d$, with $d \geq 2$, has infinite one-dimensional Hausdorff measure. From this we show that our main result, the existence of a universal differentiability set with Minkowski dimension one, is optimal in many respects. Section 3.3 is devoted to the construction of a family of nested closed sets of Minkowski dimension one. Finally, in Section 3.4, we show that these closed sets possess the curve approximation property of Chapter 2. By Theorem 2.3.3, we obtain that each member of this family is a compact universal differentiability set of Minkowski dimension one.

3.2 Optimality.

Before turning our attention to verifying the existence of a compact universal differentiability set of Minkowski dimension one in $\mathbb{R}^d$, let us first demonstrate that, in many ways, this result is the best possible.

Firstly, we emphasise that there are no universal differentiability sets which have Minkowski dimension, or even Hausdorff dimension, smaller than one; from [14, Lemma 1.2] we have that any universal differentiability set $S \subseteq \mathbb{R}^d$ satisfies $\underline{\dim}_M(S) \geq \dim_H(S) \geq 1$.

Supposing there exists a universal differentiability set S with Minkowksi dimension equal to one, we have that $\limsup_{\varepsilon\to 0} N_\varepsilon(S)\varepsilon^p = 0$ whenever $p > 1$. It is then natural to ask whether one can do better: Can we find S with $\limsup_{\varepsilon\to 0} N_\varepsilon(S)\varepsilon = 0$ or even $\liminf_{\varepsilon\to 0} N_\varepsilon(S)\varepsilon < \infty$? In the present section, we will prove that when $d \geq 2$, these stronger conditions are impossible to achieve, and that any universal differentiability set in $\mathbb{R}^d$ must have infinite 1-dimensional Hausdorff measure. Clearly in the case $d = 1$, the interval $[0, 1]$ is an example of a universal differentiability set with finite 1-dimensional Hausdorff measure.

The following lemma is a general statement about universal differentiability sets in a Banach space.

Lemma 3.2.1. *If X is a Banach space and $A, B \subseteq X$ are such that A is a non universal differentiability set and there is a non-zero continuous linear mapping $P : X \to \mathbb{R}$ such that the Lebesgue measure of $P(B)$ is zero, then the union $S = A \cup B$ is a non universal differentiability set.*

Proof. As A is a non universal differentiability set there exists a (nonzero) Lipschitz function $f : X \to \mathbb{R}$ which is not Fréchet differentiable at any $x \in A$.

As $C = P(B) \subseteq \mathbb{R}$ has measure zero, there exists a G_δ set $C' \supseteq C$ of measure zero. By [20, Theorem 1][1] there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ which is differentiable

[1]Paper [20] gives a new proof of the characterisation of sets of non-differentiability points of Lipschitz functions on $\mathbb{R}$. This characterisation was first given by Zahorski in [34]. The existence of the above function g follows from the proof of [34, Lemma 8].

everywhere outside C' and for every $t \in C'$,

$$g'_+(t) = \limsup_{s \to t} \frac{g(s) - g(t)}{s - t} = 1 \text{ and } g'_-(t) = \liminf_{s \to t} \frac{g(s) - g(t)}{s - t} = -1.$$

Let $e \in X$ be such that $Pe = 1$. Define a Lipschitz function $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) = \frac{1}{2\|e\|\mathrm{Lip}(f)} f(x) + g(P(x)).$$

Note that if $x \in S$ and $P(x) \in C'$, then $\widetilde{f}'_+(x, e) - \widetilde{f}'_-(x, e) \geq 1$, where $\widetilde{f}'_\pm(x, e)$ denote directional upper/lower derivatives of $\widetilde{f}$. Thus $\widetilde{f}$ is not Fréchet differentiable at x.

If $x \in S$ and $P(x) \notin C'$, then $x \in A$ which implies that f is not Fréchet differentiable at x. However $P(x) \notin C'$ means that $g(P(\cdot))$ is differentiable at x so that $\widetilde{f}$ is not Fréchet differentiable at x.

This implies that the Lipschitz function $\widetilde{f}$ is not Fréchet differentiable at any $x \in S$, hence S is a non universal differentiability set. □

Theorem 3.2.2. *Let $S \subseteq \mathbb{R}^d$, where $d \geq 2$, be a set of finite 1-dimensional Hausdorff measure*

$$\mathcal{H}^1(S) = \liminf_{\varepsilon \to 0+} \Big\{ \sum \mathit{diam}(S_i) \colon S \subseteq \bigcup_{i=1}^{\infty} S_i \text{ and } \mathit{diam}(S_i) \leq \varepsilon \Big\}.$$

Then S is a non universal differentiability set.

Proof. Since $\mathcal{H}^1(S) < \infty$, we may apply Federer's Structure Theorem [27, Theorem 15.6] and the Besicovitch-Federer Projection Theorem [27, Theorem 18.1]. We conclude that S can be decomposed into a union $S = A' \cup B'$, where A' is $\mathcal{H}^1$-rectifiable and B' projects to a set of 1-dimensional Lebesgue measure zero. By the latter, we mean that there exists a line L such that the projection of B' onto L, $\mathrm{proj}_L(B')$, has 1-dimensional Lebesgue measure zero.

The fact that A' is $\mathcal{H}^1$-rectifiable means that there exists a countable collection of 1-dimensional Lipschitz curves $\gamma_i : [0, 1] \to \mathbb{R}^d$ such that $\mathcal{H}^1(A' \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$, where we let $\Gamma_i = \gamma_i([0, 1])$. Note that the union of curves $A = \bigcup_{i=1}^{\infty} \Gamma_i$ is a σ-porous set (in

fact, a countable union of closed porous sets) as each Γ_i is porous (and closed). By [6, Theorem 6.48] (see also [30]) we can conclude that A is a non universal differentiability set.

Define now $B = B' \cup (A' \setminus A)$. As $\mathrm{proj}_L(B')$ has 1-dimensional Lebesgue measure zero and $\mathcal{H}^1(A' \setminus A) = 0$, we conclude that $\mathrm{proj}_L(B)$ has 1-dimensional Lebesgue measure zero too.

It remains to apply Lemma 3.2.1 to $A, B \subseteq \mathbb{R}^d$ and $P = \mathrm{proj}_L$ and to note that $S \subseteq A \cup B$ to get that S is a non universal differentiability set.

□

Corollary 3.2.3. *Let $S \subseteq \mathbb{R}^d$, where $d \geq 2$, be a universal differentiability set. Then $\liminf_{\varepsilon \to 0} N_\varepsilon(S)\varepsilon = \infty$, where $N_\varepsilon(S)$ is defined according to the Definition 3.1.1.*

Proof. We see that $\liminf_{\varepsilon \to 0} N_\varepsilon(S)\varepsilon \geq \frac{1}{2}\mathcal{H}^1(S)$, and the latter must be infinite for a universal differentiability set by Theorem 3.2.2. □

Remark. *The proof of Corollary 3.2.3 works if $N_\varepsilon(S)$ is the minimal number of Euclidean balls of radius ε needed to cover the set S. We will later switch to covering the set by ε-cubes (see Definition 3.3.1) which are rotated ℓ_∞-balls of radius ε and prove that for the compact universal differentiability set we construct, $\lim\limits_{\varepsilon \to 0} N_\varepsilon^{cubes}(S)\varepsilon^p = 0$ for every $p > 1$. Note that as*

$$N_\varepsilon^{Eucl.\ balls} \geq N_\varepsilon^{cubes} \geq N_{\varepsilon\sqrt{d}}^{Eucl.\ balls}$$

we get $\liminf_{\varepsilon \to 0} N_\varepsilon^{cubes}(S)\varepsilon = \infty$ for any universal differentiability set in $\mathbb{R}^d$, $d \geq 2$ and $\lim\limits_{\varepsilon \to 0} N_\varepsilon^{Eucl.\ balls}(S)\varepsilon^p = 0$ for every $p > 1$ for the compact universal differentiability set we construct.

3.3 The set.

We let $d \geq 2$ and construct a universal differentiability set of upper Minkowski dimension one in $\mathbb{R}^d$. There are many equivalent ways of defining the (upper and lower) Minkowski dimension of a bounded subset of $\mathbb{R}^d$. In addition to Definition 3.1.1 in Section 3.1, several

examples can be found in [27, p. 41-45]. The equivalent definition given below will be most convenient for our use. By an *ε-cube*, with centre $x \in \mathbb{R}^d$, parallel to $e \in S^{d-1}$, we mean any subset of $\mathbb{R}^d$ of the form

$$C(x, \varepsilon, e) = \left\{ x + \sum_{i=1}^{d} t_i e_i \,:\, e_1 = e,\, t_i \in [-\varepsilon, \varepsilon] \right\}. \tag{3.3}$$

where $e_2, \ldots, e_d \in S^{d-1}$ and $\langle e_i, e_j \rangle = 0$ whenever $1 \leq i \neq j \leq d$.

Definition 3.3.1. *Given a bounded subset A of $\mathbb{R}^d$ and $\varepsilon > 0$, we denote by $N_\varepsilon(A)$ the minimum number of (closed) ε-cubes required to cover A. That is, $N_\varepsilon(A)$ is the smallest integer n for which there exist ε-cubes $C_1, C_2, \ldots, C_n$ such that*

$$A \subseteq \bigcup_{i=1}^{n} C_i.$$

As in Definition 3.1.1, we define the lower Minkowski dimension of A by (3.1) *and the upper Minkowski dimension of A by* (3.2).

For a point $x \in \mathbb{R}^d$ and $w > 0$, we shall write $\overline{B}_w(x)$ for the closed ball with centre x and radius w with respect to the Euclidean norm. For a bounded subset V of $\mathbb{R}^d$, we let $\overline{B}_w(V) = \bigcup_{x \in V} \overline{B}_w(x)$. The cardinality of a finite set F shall be denoted by $|F|$. Given a real number α, we write $[\alpha]$ for the integer part of α. Finally, S^{d-1} denotes the unit sphere of $\mathbb{R}^d$ with respect to the Euclidean norm.

Fix two sequences of positive integers (s_k) and (M_k) such that the following conditions are satisfied:

$$3 \leq M_k \leq s_k;\;\; M_k, s_k \to \infty;\;\; \frac{M_k \log s_k}{s_k} \to 0 \tag{3.4}$$

and there exists a sequence $\tilde{s}_k \geq s_k$ such that

$$\frac{\tilde{s}_k - \tilde{s}_{k-1}}{s_k} \to 0. \tag{3.5}$$

Remark 3.3.2. *Before we explain how sequences s_k and M_k satisfying* (3.4) *and* (3.5)

can be chosen, we note that in order to prove that the set U_λ as in (3.31) *is a universal differentiability set we only use that*

$$s_k, M_k \to \infty \text{ and } M_k/s_k \to 0.$$

This can be seen from the proof of Lemma 3.4.3. The rest of conditions in (3.4) *and* (3.5) *are needed to prove that the Minkowski dimension of the set U_λ is equal to* 1.

Note that if the sequence $(\tilde{s}_k - \tilde{s}_{k-1})_{k\geq 2}$ is bounded and $s_k \to \infty$, then (3.5) is satisfied. Hence an example of sequences $(s_k), (\tilde{s}_k)$ satisfying (3.4) and (3.5) is $\tilde{s}_k = ak + b$ with $a > 0$ and any integer sequence $s_k \to \infty$ such that $3 \leq s_k \leq \tilde{s}_k$.

We also remark that if $s_k \to \infty$ is such that

$$\frac{s_k}{s_{k+1}} \to 1, \tag{3.6}$$

then (3.5) is satisfied with $\tilde{s}_k = s_k$. Indeed, in such case $\dfrac{\tilde{s}_k - \tilde{s}_{k-1}}{s_k} = 1 - \dfrac{s_{k-1}}{s_k} \to 0$.

An example of an integer sequence $s_k \to \infty$ satisfying the condition (3.6) is given by $s_k = \max\{3, [F(k)]\}$, where $F(x)$ is a linear combination of powers of x such that the highest power of x is positive and has a positive coefficient. Also, whenever $s_k \to \infty$ satisfies the condition (3.6), the sequence $s'_k = [\log s_k]$ also satisfies this condition and tends to infinity.

Once (s_k) is defined there is much freedom to choose (M_k). For example, we may take $M_k = \max\{3, [s_k^\alpha]\}$ with $\alpha \in (0,1)$ or $M_k = \max\{3, [\log s_k]\}$ etc.

Let now $\mathcal{E}_k$ be a maximal $\frac{1}{s_k}$-separated subset of S^{d-1}. We therefore get a collection of finite subsets $\mathcal{E}_k \subseteq S^{d-1}$ such that

$$|\mathcal{E}_k| \leq s_k^{2d} \qquad \text{and} \qquad \forall e \in S^{d-1} \exists\, e' \in \mathcal{E}_k \text{ s.t. } \|e - e'\| \leq \frac{1}{s_k}. \tag{3.7}$$

Definition 3.3.3. *Given a line segment $l = x + [a,b]e \subseteq \mathbb{R}^d$ and $0 < w < \text{length}(l)/2$, define $\mathcal{F}_w(l)$ to be a finite collection of w-cubes of the form $C(x_i, w, e)$, defined by* (3.3),

with $x_i \in l$, such that

$$\overline{B}_w(l) \subseteq \bigcup_{C \in \mathcal{F}_w(l)} C \text{ and } |\mathcal{F}_w(l)| < \text{length}(l)/w. \tag{3.8}$$

Fix an arbitrary number $Q \in (1,2)$. Let l_1 be a line segment in $\mathbb{R}^d$ of length 1, set $w_1 = Q^{-s_1}$, $\mathcal{L}_1 = \{l_1\}$, $\mathcal{C}_1 = \mathcal{F}_{w_1}(l_1)$ and $\mathcal{T}_1 = \left\{\overline{B}_{w_1}(l_1)\right\}$. We refer to the collection $\mathcal{L}_1$ as 'the lines of level 1', the collection $\mathcal{C}_1$ as 'the cubes of level 1' and the collection $\mathcal{T}_1$ as 'the tubes of level 1'. Note that $\mathcal{C}_1$ is a cover of the union of tubes in $\mathcal{T}_1$. Suppose that $k \geq 2$, and that we have defined real numbers $w_r > 0$ and the collections $\mathcal{L}_r$ of lines, $\mathcal{T}_r$ of tubes and $\mathcal{C}_r$ of cubes of level $r = 1, 2, \ldots, k-1$ in such a way that

$$\mathcal{T}_r = \left\{\overline{B}_{w_r}(l) : l \in \mathcal{L}_r\right\};\ \mathcal{C}_r = \bigcup_{l \in \mathcal{L}_r} \mathcal{F}_{w_r}(l) \text{ is a cover of } \bigcup\{T : T \in \mathcal{T}_r\}.$$

We now describe how to construct the lines, cubes and tubes of the kth level. We start with the definition of the new width. Set

$$w_k = Q^{-s_k} w_{k-1}. \tag{3.9}$$

The collections $\mathcal{L}_k$ of lines, $\mathcal{T}_k$ of tubes and $\mathcal{C}_k$ of cubes will be partitioned into exactly $M_k + 1$ classes and each class will be further partitioned into categories according to the length of the lines. We first define the collections of lines, tubes and cubes of level k, class 0 respectively by

$$\mathcal{L}_{(k,0)} = \mathcal{L}_{k-1}, \mathcal{T}_{(k,0)} = \left\{\overline{B}_{w_k}(l) : l \in \mathcal{L}_{(k,0)}\right\} \text{ and } \mathcal{C}_{(k,0)} = \bigcup_{l \in \mathcal{L}_{(k,0)}} \mathcal{F}_{w_k}(l). \tag{3.10}$$

We will say that all lines, tubes and cubes of level k, class 0 have the empty category. From (3.10) and Definition 3.3.3, we have that $\mathcal{C}_{(k,0)}$ is a cover of the union of tubes in

$\mathcal{T}_{(k,0)}$. Using (3.8), we also have that

$$\left|\mathcal{C}_{(k,0)}\right| = \sum_{l\in\mathcal{L}_{(k,0)}} \left|\mathcal{F}_{w_k}(l)\right| \leq \frac{1}{w_k} \sum_{l\in\mathcal{L}_{(k,0)}} \text{length}(l). \tag{3.11}$$

For each line segment $l \in \mathcal{L}_{(k,0)} = \mathcal{L}_{k-1}$ and each cube $C \in \mathcal{F}_{w_{k-1}}(l) \subseteq \mathcal{C}_{k-1}$, the intersection $l \cap C$ is a line segment of length at most $2w_{k-1}$. Moreover, $\mathcal{F}_{w_{k-1}}(l)$ is a cover of the line segment l. It follows that

$$\sum_{l\in\mathcal{L}_{(k,0)}} \text{length}(l) \leq \sum_{l\in\mathcal{L}_{k-1}} 2w_{k-1} \left|\mathcal{F}_{w_{k-1}}(l)\right| = 2\left|\mathcal{C}_{k-1}\right| w_{k-1}. \tag{3.12}$$

Combining (3.11), (3.12) and (3.9) yields

$$\left|\mathcal{C}_{(k,0)}\right| \leq 2\left|\mathcal{C}_{k-1}\right| Q^{s_k}. \tag{3.13}$$

Definition 3.3.4. *Given a bounded line segment $l \subseteq \mathbb{R}^d$, an integer $j \geq 1$ with* $\text{length}(l) \geq Q^j w_k/s_k$ *and a direction $e \in S^{d-1}$, we define a collection of line segments $\mathcal{R}_l(j,e)$ as follows: Let $\Phi \subseteq l$ be a maximal $\frac{Q^j w_k}{s_k}$-separated set and set*

$$\mathcal{R}_l(j,e) = \{\phi_x : x \in \Phi\},$$

where ϕ_x is the line defined by

$$\phi_x = x + [-1,1]Q^j w_k e. \tag{3.14}$$

We note for future reference that

$$\left|\mathcal{R}_l(j,e)\right| \leq \frac{2s_k \text{ length}(l)}{Q^j w_k}. \tag{3.15}$$

For $j \in \{1,2,\ldots,s_k\}$, we define the collection of lines of level k, class 1, category (j)

by

$$\mathcal{L}^{(j)}_{(k,1)} = \bigcup_{l \in \mathcal{L}_{(k,0)}} \bigcup_{e \in \mathcal{E}_k} \mathcal{R}_l(j,e). \tag{3.16}$$

We emphasise that all the lines in $\mathcal{L}^{(j)}_{(k,1)}$ have the same length. Indeed, from Definition 3.3.4, we get

$$\text{length}(l) = 2Q^j w_k \text{ for all lines } l \in \mathcal{L}^{(j)}_{(k,1)}. \tag{3.17}$$

From (3.15) and (3.16), it follows that

$$\left|\mathcal{L}^{(j)}_{(k,1)}\right| \leq 2w_k^{-1} s_k \left|\mathcal{E}_k\right| Q^{-j} \sum_{l \in \mathcal{L}_{(k,0)}} \text{length}(l). \tag{3.18}$$

Together, (3.18), (3.12) and (3.9) imply

$$\left|\mathcal{L}^{(j)}_{(k,1)}\right| \leq \left|\mathcal{C}_{k-1}\right| (4s_k \left|\mathcal{E}_k\right|) Q^{s_k - j}. \tag{3.19}$$

Let

$$\mathcal{C}^{(j)}_{(k,1)} = \bigcup_{l \in \mathcal{L}^{(j)}_{(k,1)}} \mathcal{F}_{w_k}(l). \tag{3.20}$$

Then, using (3.8), (3.17) and (3.19) we obtain

$$\begin{aligned} \left|\mathcal{C}^{(j)}_{(k,1)}\right| &\leq \sum_{l \in \mathcal{L}^{(j)}_{(k,1)}} \left|\mathcal{F}_{w_k}(l)\right| \leq \frac{1}{w_k} \sum_{l \in \mathcal{L}^{(j)}_{(k,1)}} \text{length}(l) \\ &\leq \frac{\left|\mathcal{L}^{(j)}_{(k,1)}\right| \times 2 \times Q^j w_k}{w_k} \leq 8 \left|\mathcal{C}_{k-1}\right| s_k \left|\mathcal{E}_k\right| Q^{s_k}. \end{aligned} \tag{3.21}$$

The collection of tubes of level k, class 1, category (j) is defined by

$$\mathcal{T}^{(j)}_{(k,1)} = \left\{ \overline{B}_{w_k}(l) : l \in \mathcal{L}^{(j)}_{(k,1)} \right\} \tag{3.22}$$

From Definition 3.3.3, (3.20) and (3.22) it is clear that $\mathcal{C}^{(j)}_{(k,1)}$ is a cover of the union of

tubes in $\mathcal{T}^{(j)}_{(k,1)}$. We can also use (3.22) and (3.19) to conclude that

$$\left|\mathcal{T}^{(j)}_{(k,1)}\right| = \left|\mathcal{L}^{(j)}_{(k,1)}\right| \leq |\mathcal{C}_{k-1}|\,(4s_k|\mathcal{E}_k|)Q^{s_k-j}. \tag{3.23}$$

The collections of lines, cubes and tubes of level k, class 1 are now defined by

$$\#_{(k,1)} = \bigcup_{j=1}^{s_k} \#^{(j)}_{(k,1)},$$

where # stands for $\mathcal{L}$, $\mathcal{C}$ or $\mathcal{T}$. Note that $\mathcal{C}_{(k,1)}$ is a cover of the union of tubes in $\mathcal{T}_{(k,1)}$. Moreover, in view of (3.21), we get

$$\left|\mathcal{C}_{(k,1)}\right| \leq \sum_{j=1}^{s_k} \left|\mathcal{C}^{(j)}_{(k,1)}\right| \leq 2\,|\mathcal{C}_{k-1}|\,(4s_k^2|\mathcal{E}_k|)Q^{s_k}. \tag{3.24}$$

Suppose that $1 \leq m < M_k$ and that we have defined the collections

$$\mathcal{L}_{(k,m)},\ \mathcal{C}_{(k,m)} \text{ and } \mathcal{T}_{(k,m)}$$

of lines, cubes and tubes of level k, class m. Assume that these collections are partitioned into categories

$$\mathcal{L}^{(j_1,\ldots,j_m)}_{(k,m)},\ \mathcal{C}^{(j_1,\ldots,j_m)}_{(k,m)} \text{ and } \mathcal{T}^{(j_1,\ldots,j_m)}_{(k,m)}$$

where the j_i are integers satisfying

$$1 \leq j_{i+1} \leq j_i \leq s_k \text{ for all } i. \tag{3.25}$$

Suppose that the following conditions hold.

$$\text{length}(l) = 2Q^{j_m} w_k \text{ for all lines } l \text{ in } \mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}, \tag{I$_m$}$$

$$\left|\mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}\right| \leq |\mathcal{C}_{k-1}| \left(4s_k|\mathcal{E}_k|\right)^m Q^{s_k - j_m}, \tag{II$_m$}$$

$$\mathcal{T}_{(k,m)}^{(j_1,\dots,j_m)} = \left\{\overline{B}_{w_k}(l) : l \in \mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}\right\}, \tag{III$_m$}$$

$$\mathcal{C}_{(k,m)}^{(j_1,\dots,j_m)} = \bigcup_{l \in \mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}} \mathcal{F}_{w_k}(l), \tag{IV$_m$}$$

$$\left|\mathcal{C}_{(k,m)}^{(j_1,\dots,j_m)}\right| \leq 2\,|\mathcal{C}_{k-1}| \left(4s_k|\mathcal{E}_k|\right)^m Q^{s_k}. \tag{V$_m$}$$

For an integer sequence $(j_1, \dots, j_m, j_{m+1})$ satisfying (3.25) we define the collection of lines of level k, class $(m+1)$, category $(j_1, \dots, j_{m+1})$ by

$$\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})} = \bigcup_{l \in \mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}} \left(\bigcup_{e \in \mathcal{E}_k} \mathcal{R}_l(j_{m+1}, e) \right). \tag{3.26}$$

Note that every line in the collection $\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}$ has the same length. In fact, by Definition 3.3.4 we have that (I$_{m+1}$) is satisfied. Combining (3.15), (I$_m$) and (II$_m$) we deduce the following:

$$\begin{aligned} \left|\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}\right| &\leq 2s_k\,|\mathcal{E}_k|\,w_k^{-1} Q^{-j_{m+1}} \sum_{l \in \mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}} \text{length}(l) \\ &\leq |\mathcal{C}_{k-1}| \left(4s_k|\mathcal{E}_k|\right)^{m+1} \times Q^{s_k - j_{m+1}}. \end{aligned} \tag{3.27}$$

Thus, (II$_{m+1}$) is satisfied. We define the collection of tubes and cubes of level k and class $m+1$, category $(j_1, \dots, j_{m+1})$ by (III$_{m+1}$) and (IV$_{m+1}$). Using (3.8), (I$_{m+1}$) and

(3.27) we obtain

$$\left|\mathcal{C}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}\right| \leq \sum_{l\in\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}} |\mathcal{F}_{w_k}(l)| = \frac{1}{w_k} \sum_{l\in\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}} \text{length}(l)$$
$$\leq \frac{1}{w_k} \times \left|\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}\right| \times 2 \times Q^{j_{m+1}} w_k \leq 2\,|\mathcal{C}_{k-1}|\,(4s_k|\mathcal{E}_k|)^{m+1} Q^{s_k},$$

and this verifies (V_{m+1}). The collections of lines, tubes and cubes of level k, class $m+1$ are given by

$$\#_{(k,m+1)} = \bigcup_{s_k\geq j_1\geq\dots\geq j_{m+1}\geq 1} \#_{(k,m+1)}^{(j_1,\dots,j_{m+1})},$$

where # stands for $\mathcal{L}$, $\mathcal{T}$ or $\mathcal{C}$.

Note that $\mathcal{C}_{(k,m+1)}$ is a cover of the union of tubes in $\mathcal{T}_{(k,m+1)}$. Moreover, in view of (V_{m+1}) and (3.25) we have

$$\left|\mathcal{C}_{(k,m+1)}\right| \leq \sum_{(j_1,\dots,j_{m+1})} \left|\mathcal{C}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}\right|$$
$$\leq 2\,|\mathcal{C}_{k-1}|\,(4s_k|\mathcal{E}_k|)^{m+1} s_k^{m+1} Q^{s_k} = 2\,|\mathcal{C}_{k-1}|\,(4s_k^2|\mathcal{E}_k|)^{m+1} Q^{s_k}, \quad (3.28)$$

and this generalises (3.24) for arbitrary $0 \leq m < M_k$. Finally, the collections of lines, tubes and cubes of level k are given by

$$\mathcal{L}_k = \bigcup_{m=0}^{M_k} \mathcal{L}_{(k,m)},\ \mathcal{T}_k = \bigcup_{m=0}^{M_k} \mathcal{T}_{(k,m)} \text{ and } \mathcal{C}_k = \bigcup_{m=0}^{M_k} \mathcal{C}_{(k,m)}. \quad (3.29)$$

Note that $\mathcal{C}_k$ is a cover of the union of tubes in $\mathcal{T}_k$. Moreover, using (3.13) and (3.28) we get

$$|\mathcal{C}_k| \leq 2(M_k+1)\,|\mathcal{C}_{k-1}|\,(4s_k^2|\mathcal{E}_k|)^{M_k} Q^{s_k}. \quad (3.30)$$

The construction of the lines, tubes and cubes of all levels is now complete.

We now define a collection of closed sets $(U_\lambda)_{\lambda\in(0,1]}$. Eventually, we will show that each of these sets is a compact universal differentiability set of Minkowski dimension one.

The sets U_λ are defined similarly to the sets (T_λ) in [14, Definition 2.3].

Definition 3.3.5. *For $\lambda \in (0,1]$ we let*

$$U_\lambda = \bigcap_{k=1}^{\infty} \left(\bigcup_{0 \le m_k \le \lambda M_k} \left(\bigcup_{l \in \mathcal{L}_{(k,m_k)}} \overline{B}_{\lambda w_k}(l) \right) \right). \tag{3.31}$$

We emphasise that the single line segment l_1 of level 1 is contained in the set U_λ for every $\lambda \in (0,1]$. Hence, every U_λ is non-empty. Note also that $U_{\lambda_1} \subseteq U_{\lambda_2}$ whenever $0 < \lambda_1 \le \lambda_2 \le 1$. Finally, since the unions in (3.31) are finite, it is clear that for each $0 < \lambda \le 1$, the set U_λ is closed.

Lemma 3.3.6. *For $\lambda \in (0,1]$, the set U_λ has Minkowski dimension one.*

Proof. For any $\lambda \in (0,1]$ we have that U_λ contains a line segment. Hence, each of the sets U_λ has lower Mikowski dimension at least one. We also have $U_\lambda \subseteq U_1$ for all $\lambda \in (0,1]$. Therefore, to complete the proof, it suffices to show that the set U_1 has upper Minkowski dimension one.

To show $\overline{\dim}_M(U_1) \le 1$ it is enough to argue that $\overline{\dim}_M(U_1) \le p$ for all $p > 1$. Fix an arbitrary $p \in (1,2)$.

From (3.29), it is clear that for all $k \ge 1$ and $0 \le m \le M_k$

$$\bigcup_{l \in \mathcal{L}_{(k,m)}} \overline{B}_{w_k}(l) \subseteq \bigcup_{l \in \mathcal{L}_k} \overline{B}_{w_k}(l) = \bigcup_{T \in \mathcal{T}_k} T.$$

We conclude, using Definition 3.3.5, that for each $k \ge 1$

$$U_1 \subseteq \bigcup_{T \in \mathcal{T}_k} T. \tag{3.32}$$

Recall that $\mathcal{C}_k$ is a finite collection of w_k-cubes which cover the union of tubes T in $\mathcal{T}_k$. Therefore, in view of (3.32), we have that $\mathcal{C}_k$ is also a cover of U_1. By Definition 3.3.1, this means

$$N_{w_k}(U_1) \le |\mathcal{C}_k| \text{ for all } k \ge 1.$$

We claim that the sequence $|\mathcal{C}_k|\, {w_k}^p Q^{ps_k}$ is bounded, i.e. there exists $H > 0$ such that

$$|\mathcal{C}_k| {w_k}^p Q^{ps_k} \leq H \quad \forall k \geq 1. \tag{3.33}$$

Assume that the claim is valid. Fix an arbitrary $w \in (0, w_1)$. There exists an integer $k \geq 1$ such that $w_{k+1} \leq w < w_k$. This implies $N_w(U_1) \leq N_{w_{k+1}}(U_1)$ so that

$$N_w(U_1) w^p \leq N_{w_{k+1}}(U_1) w_k^p \leq |\mathcal{C}_{k+1}|\, w_{k+1}^p Q^{ps_{k+1}} \leq H.$$

Hence, $N_w(U_1)w^p$ is uniformly bounded from above by a fixed constant H. Since this is true for any arbitrarily small $w \in (0, w_1)$, we conclude that $\overline{\dim}_M(U_1) \leq p$.

It only remains to establish the claim (3.33). We prove a more general statement, namely, that the sequence $|\mathcal{C}_k| w_k^p Q^{p\tilde{s}_k}$ tends to zero.

From (3.4), it follows that $s_k \geq M_k + 1 \geq 4$ for sufficiently large k. Using this, together with (3.30), (3.7) and (3.9) we obtain

$$\begin{aligned} \frac{|\mathcal{C}_k| w_k^p Q^{p\tilde{s}_k}}{|\mathcal{C}_{k-1}| w_{k-1}^p Q^{p\tilde{s}_{k-1}}} &\leq 2(M_k+1)(4s_k^2|\mathcal{E}_k|)^{M_k} Q^{-(p-1)s_k} Q^{p(\tilde{s}_k - \tilde{s}_{k-1})} \\ &\leq s_k^2 s_k^{(3+2d)M_k} Q^{-(p-1)s_k} \times Q^{p(\tilde{s}_k - \tilde{s}_{k-1})} \\ &\leq Q^{(p-1)s_k/2} \times Q^{-(p-1)s_k} \times Q^{p(\tilde{s}_k - \tilde{s}_{k-1})} \end{aligned} \tag{3.34}$$

for k sufficiently large. The latter inequality follows from

$$\frac{M_k \log s_k}{s_k} < \frac{(p-1)\log Q}{2(5+2d)},$$

which is true for k sufficiently large, by (3.4). We then see that the product of the three terms in (3.34) tends to zero as $k \to \infty$, since (3.5) implies that

$$p(\tilde{s}_k - \tilde{s}_{k-1}) < (p-1)s_k/4$$

for k sufficiently large. $\square$

3.4 Main result.

The objective of this section is to prove Theorem 3.4.4 which guarantees, in every finite dimensional space, the existence of a compact universal differentiability set of Minkowski dimension one. In Section 3.2 we established that this result is optimal. We note that we will always assume $d \geq 2$ as the case $d = 1$ is trivial, we can simply take $S = [0, 1]$.

We first establish several lemmas. The statements we prove typically concern a line l of level k, class m, category $(j_1, \ldots, j_m)$ where $0 \leq m \leq M_k$. When $m = 0$, we interpret the category $(j_1, \ldots, j_m)$ as the empty category and assume $j \leq j_m$ for all integers j.

Lemma 3.4.1. *Let $k \geq 2$, $0 \leq m < M_k$ and $l \in \mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}$. Let $e \in \mathcal{E}_k$ and $1 \leq j_{m+1} \leq j_m \leq s_k$. If $x \in l$, then there exists $x' \in l$ such that $\|x' - x\| \leq Q^{j_{m+1}} w_k / s_k$ and*

$$l' = x' + [-1, 1] Q^{j_{m+1}} w_k e \in \mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_m,j_{m+1})}$$

Proof. Using Definition 3.3.4 and (3.26), we observe that $\mathcal{R}_l(j_{m+1}, e) \subseteq \mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_{m+1})}$ has an element l' satisfying the conclusions of this lemma. $\square$

Lemma 3.4.2. *Let $k \geq 2$ and suppose $1 \leq m \leq M_k$. Let $x \in l \in \mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}$ and i_m be an integer with $j_m < i_m \leq s_k$. Then there exists an integer sequence $s_k \geq i_1 \geq \ldots \geq i_{m-1} \geq i_m$ and a line $l' \in \mathcal{L}_{(k,m)}^{(i_1,\ldots,i_m)}$, such that l' is parallel to l and there exists a point $x' \in l'$ with $\|x' - x\| \leq \frac{m \times Q^{i_m} w_k}{s_k}$.*

Proof. Suppose that either

(i) $n = 1$, or

(ii) $2 \leq n \leq M_k$ and the statement of Lemma 3.4.2 holds for $m = 1, \ldots, n-1$.

We prove that in both cases, the statement of Lemma 3.4.2 holds for $m = n$. The proof will then be complete, by induction.

Let the line l, integers $j_1, \ldots, j_n, i_n$ and point $x \in l$ be given by the hypothesis of Lemma 3.4.2 when we set $m = n$. By (3.16) in case (i), or (3.26) in case (ii), there exists a line $l^{(n-1)} \in \mathcal{L}_{(k,n-1)}^{(j_1,\ldots,j_{n-1})}$ and a direction $e \in \mathcal{E}_k$ such that the line l belongs to the collection $\mathcal{R}_{l^{(n-1)}}(j_n, e)$.

Let the line segment $l^{(n-1)}$ be parallel to $f^{(n-1)} \in S^1$. By Definition 3.3.4, the line l has the form

$$l = z + [-1, 1]Q^{j_n} w_k e$$

where $z \in l^{(n-1)}$. Therefore, we may write

$$z = x + \beta e, \tag{3.35}$$

where

$$|\beta| \leq Q^{j_n} w_k. \tag{3.36}$$

We now distinguish between two cases. First suppose that $i_n \leq j_{n-1}$. Note that this is certainly the case if $n = 1$. Setting $i_a = j_a$ for $a = 1, \ldots, n-1$, we get that $s_k \geq i_1 \geq \ldots \geq i_{n-2} \geq i_{n-1} \geq i_n$. The line $l^{(n-1)} \in \mathcal{L}_{(k,n-1)}^{(i_1,\ldots,i_{n-1})}$, the direction $e \in \mathcal{E}_k$, the integer i_n and the point $z \in l^{(n-1)}$ now satisfy the conditions of Lemma 3.4.1. Hence, by Lemma 3.4.1, there is a line l' of level k, class n, category $(i_1, \ldots, i_n)$ and a point z' with

$$\|z' - z\| \leq \frac{Q^{i_n} w_k}{s_k}, \tag{3.37}$$

such that the line segment l' is given by

$$l' = z' + [-1, 1]Q^{i_n} w_k e.$$

Finally, set

$$x' = z' - \beta e,$$

so that $x' \in l'$, using (3.36). We deduce, using (3.37) and (3.35) that $\|x' - x\| \leq \frac{Q^{i_n} w_k}{s_k} \leq$

$\frac{n \times Q^{i_n} w_k}{s_k}$. This completes the proof for the case $i_n \leq j_{n-1}$.

Now suppose that $i_n > j_{n-1}$. In this situation, we must be in case (ii). We set $i_{n-1} = i_n > j_{n-1}$. The conditions of Lemma 3.4.2 are now readily verified for $z \in l^{(n-1)} \in \mathcal{L}_{(k,n-1)}^{(j_1,\ldots,j_{n-1})}$, and the integer i_{n-1}. Therefore, by (ii) and Lemma 3.4.2, there exists an integer sequence $s_k \geq i_1 \geq \ldots \geq i_{n-2} \geq i_{n-1}$ and a line $l'' \in \mathcal{L}_{(k,n-1)}^{(i_1,\ldots,i_{n-1})}$ such that l'' is parallel to $l^{(n-1)}$ and there exists a point $y'' \in l''$ such that

$$\|y'' - z\| \leq \frac{(n-1) \times Q^{i_{n-1}} w_k}{s_k}. \tag{3.38}$$

The conditions of Lemma 3.4.1 are now readily verified for the line $l'' \in \mathcal{L}_{(k,n-1)}^{(i_1,\ldots,i_{n-1})}$, the direction $e \in \mathcal{E}_k$, the integer i_n and the point $y'' \in l''$. Thus, by Lemma 3.4.1, there exists a line $l' \in \mathcal{L}_{(k,n)}^{(i_1,\ldots,i_n)}$ and a point $y' \in l'$ such that

$$\|y' - y''\| \leq Q^{i_n} w_k / s_k, \tag{3.39}$$

and the line l' is given by

$$l' = y' + [-1, 1] Q^{i_n} w_k e.$$

We set

$$x' = y' - \beta e.$$

Using (3.36) and $i_n > j_n$ we get that $x' \in l'$. Moreover, using (3.35), (3.38) and (3.39), we obtain $\|x' - x\| \leq \frac{n \times Q^{i_n} w_k}{s_k}$. □

In the next Lemma, we establish that the sets $(U_\lambda)_{\lambda \in (0,1]}$ have the geometric property described in Remark 2.3.2.

Lemma 3.4.3. *For every $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$ and $\eta \in (0,1)$ there exists a number $\delta_0 = \delta_0(\lambda, \psi, \eta) > 0$ such that whenever $x \in U_\lambda$, $e \in S^{d-1}$ and $\delta \in (0, \delta_0)$, there exists a point $x' \in \mathbb{R}^d$ and a direction $e' \in S^{d-1}$ such that $\|x' - x\| \leq \eta\delta$, $\|e' - e\| \leq \eta$ and*

$[x', x' + \delta e'] \subseteq U_{\lambda+\psi}$.

Proof. Fix $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$, $\eta \in (0,1)$. We may assume that $\eta \in (0, Q-1)$. Next, using (3.4) and $w_k \to 0$, we may choose $\delta_0 \in (0, \psi w_1/Q)$ small enough so that

$$\psi M_k \geq 2, \qquad \frac{(M_k+4)Q^3}{\psi s_k} \leq \eta, \tag{3.40}$$

whenever $\psi w_k \leq Q\delta_0$.

Fix $x \in U_\lambda$, $e \in S^{d-1}$ and $\delta \in (0, \delta_0)$. Since $x \in U_\lambda$, we may conclude, using (3.31), that there exists a sequence of integers $(m_k)_{k\geq 1}$ with $0 \leq m_k \leq \lambda M_k$, and a sequence of line segments $(l_k)_{k\geq 1}$ such that $l_k \in \mathcal{L}_{(k,m_k)}$ and

$$x \in \bigcap_{k=1}^{\infty} \overline{B}_{\lambda w_k}(l_k). \tag{3.41}$$

From $\delta \in (0, \delta_0)$, we know that there is a unique integer $n \geq 2$ such that

$$\psi w_n < Q\delta \leq \psi w_{n-1}. \tag{3.42}$$

Further, in view of the relation (3.9), there exists $t \in \{0, \ldots, s_n - 1\}$ such that

$$\psi Q^{t-1} w_n < \delta \leq \psi Q^t w_n. \tag{3.43}$$

The line segment l_n belongs to the collection $\mathcal{L}_{(n,m_n)}^{(j_1,\ldots,j_{m_n})}$ for some sequence of integers $1 \leq j_{m_n} \leq j_{m_n-1} \leq \ldots \leq j_1 \leq s_n$. Since $x \in \overline{B}_{\lambda w_n}(l_n)$, there exists a point $z \in l_n$, a direction $u \in S^{d-1}$ and $r \in [0, \lambda w_n]$ such that $x = z + ru$. Choose $u' \in \mathcal{E}_n$ such that $\|u' - u\| \leq 1/s_n$. Observe that (3.40), (3.42) and $m_n \leq \lambda M_n$ imply that

$$m_n \leq (\lambda+\psi)M_n - \psi M_n \leq (\lambda+\psi)M_n - 2. \tag{3.44}$$

It is now readily verified that the conditions of Lemma 3.4.1 are satisfied for the integers $n \geq 2$, $0 \leq m_n < M_n$, the line $l_n \in \mathcal{L}_{(n,m_n)}^{(j_1,\ldots,j_{m_n})}$, the direction $u' \in \mathcal{E}_n$ and the integers

$j_{1+m_n} = 1 \leq j_{m_n}$. Applying Lemma 3.4.1 to the point $z \in l_n$, we find a point $z' \in l_n$ and a line $l_n^{(1)} \in \mathcal{L}_{(n,1+m_n)}^{(j_1,\ldots,j_{m_n},1)}$ such that $\|z' - z\| \leq Qw_n/s_n$ and $l_n^{(1)} = z' + [-1,1]Qw_n u'$. Set $x_1 = z' + ru' \in l_n^{(1)}$. Then

$$\|x_1 - x\| \leq \|z' - z\| + \|u' - u\| \, w_n \leq 2Qw_n/s_n. \tag{3.45}$$

We now set $i = \max\{t+1, 2\}$, noting that $1 < i \leq s_n$. The conditions of Lemma 3.4.2 are met for $x_1 \in l_n^{(1)} \in \mathcal{L}_{(n,1+m_n)}^{(j_1,\ldots,j_{m_n},1)}$ and the integers $j_{1+m_n} = 1 < i_{1+m_n} := i \leq s_n$. Therefore, by Lemma 3.4.2, there exists an integer sequence $s_n \geq i_1 \geq \ldots \geq i_{1+m_n} = i$ and a line $l_n^{(2)} \in \mathcal{L}_{(n,1+m_n)}^{(i_1,\ldots,i_{m_n},i)}$ such that $l_n^{(2)}$ is parallel to $l_n^{(1)}$ and there exists a point $x_2 \in l_n^{(2)}$ such that

$$\|x_2 - x_1\| \leq (m_n + 1)Q^i w_n/s_n. \tag{3.46}$$

Choose $e' \in \mathcal{E}_n$ such that $\|e' - e\| \leq 1/s_n$. Note that $\|e' - e\| \leq \eta$ follows from (3.40) and (3.42). Set $i_{2+m_n} = t+1$. The conditions of Lemma 3.4.1 are satisfied for the integers $n \geq 2$, $0 \leq 1 + m_n < M_n$, the line $l_n^{(2)} \in \mathcal{L}_{(n,1+m_n)}^{(i_1,\ldots,i_{m_n},i)}$, the direction $e' \in \mathcal{E}_n$, the integers $1 \leq i_{2+m_n} \leq i = i_{1+m_n} \leq s_n$ and the point $x_2 \in l_n^{(2)}$. Hence, by Lemma 3.4.1, there exists $x' \in l_n^{(2)}$ such that

$$\|x' - x_2\| \leq Q^{t+1} w_n/s_n \tag{3.47}$$

and

$$l' = x' + [-1,1]Q^{t+1} w_n e' \in \mathcal{L}_{(n,2+m_n)}^{(i_1,\ldots,i_{2+m_n})}. \tag{3.48}$$

Define a sequence of line segments $\{l'_k\}_{k=1}^{\infty}$ by

$$l'_k = \begin{cases} l_k & \text{if } 1 \leq k \leq n-1, \\ l' & \text{if } k \geq n. \end{cases} \tag{3.49}$$

Recall that, by construction, we have $\mathcal{L}_{(k,0)} = \mathcal{L}_{k-1}$ for all $k \geq 2$. Accordingly, since $l' \in \mathcal{L}_n$, we get that $l' \in \mathcal{L}_{(k,0)}$ for all $k > n$. Now observe that for each $k \geq 1$ we have

$l'_k \in \mathcal{L}_{(k,m'_k)}$, where

$$m'_k = \begin{cases} m_k & \text{if } 1 \leq k \leq n-1, \\ m_n + 2 & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

From $0 \leq m_k \leq \lambda M_k$ and (3.44), it is clear that $0 \leq m'_k \leq (\lambda + \psi) M_k$ for all k. With reference to (3.31), we conclude that $\bigcap_{k=1}^{\infty} \overline{B}_{(\lambda+\psi)w_k}(l'_k) \subseteq U_{\lambda+\psi}$.

Combining (3.47), (3.46), (3.45) and $i = \max\{t+1, 2\}$ we obtain

$$\begin{aligned} \|x' - x\| \leq \frac{Q^{t+1} w_n}{s_n} + \frac{(m_n+1) Q^i w_n}{s_n} + \frac{2Q w_n}{s_n} &\leq \frac{(M_n+4) Q^i w_n}{s_n} \\ &= \frac{(M_n+4) Q^{i-t+1}}{\psi s_n} \times \psi Q^{t-1} w_n \\ &\leq \frac{(M_n+4) Q^3}{\psi s_n} \times \psi Q^{t-1} w_n \leq \eta \delta. \end{aligned} \tag{3.50}$$

The final line in the above relies on $i = \max\{t+1, 2\}$, (3.40), (3.42) and (3.43).

Using (3.50), (3.41), (3.42), (3.49) and $\eta \in (0, Q-1)$ we get

$$[x', x' + \delta e'] \subseteq \overline{B}_{(\lambda+\psi)w_k}(l'_k) \text{ for } 1 \leq k \leq n-1.$$

Simultaneously, we can combine (3.43) and (3.48) to deduce

$$[x', x' + \delta e'] \subseteq l' = l'_k \text{ for all } k \geq n.$$

Therefore,

$$[x', x' + \delta e'] \subseteq \bigcap_{k=1}^{\infty} \overline{B}_{(\lambda+\psi)w_k}(l'_k) \subseteq U_{\lambda+\psi}.$$

□

We are now ready to prove our main result.

Theorem 3.4.4. *For every $d \geq 1$, $\mathbb{R}^d$ contains compact sets of upper Minkowski dimension one with the universal differentiability property.*

Proof. We will show that each set in the collection $(U_\lambda)_{\lambda\in(0,1]}$ is a compact universal differentiability set with upper Minkowski dimension one.

For $\lambda \in (0,1]$ we have that the set U_λ has upper Minkowski dimension one by Lemma 3.3.6 and is compact by (3.31). It therefore only remains to verify that each U_λ is a universal differentiability set. From Theorem 2.3.3, it suffices to prove that the family of sets $(U_\lambda)_{\lambda\in(0,1]}$ has the curve approximation property (see Definition 2.3.1).

It follows immediately from (3.31) that $U_\lambda \subseteq U_{\lambda'}$ whenever $0 < \lambda \leq \lambda' < 1$. Hence the collection $(U_\lambda)_{\lambda\in(0,1]}$ satisfies condition (i) of Definition 2.3.1. Further, by Lemma 3.4.3 we have that the collection of sets $(U_\lambda)_{\lambda\in(0,1]}$ satisfies that condition (ii)$'$ of Remark 2.3.2. Hence, condition (ii) of Definition 2.3.1 is also satisfied (see Remark 2.3.2). This establishes that the collection $(U_\lambda)_{\lambda\in(0,1]}$ has the curve approximation property and completes the proof. $\square$

Chapter 4

On the structure of universal differentiability sets.

4.1 Introduction.

The research presented in this chapter began with the question of whether there exists a universal differentiability set which may be decomposed as a countable union of non-universal differentiability sets. Given the connection between universal differentiability sets and non-sigma porous sets, it seemed plausible that the answer to this question could be negative; clearly any non-σ-porous set cannot be written as a countable union of porous sets. However, after a short time, we found an example to show that the answer is positive, using a result of Alberti, Csörnyei and Preiss in [1] (see Remark 4.3.2). Nevertheless, it became clear that the universal differentiability property imposes rather strong conditions on the nature of sets possessing it. We establish several structural properties of universal differentiability sets in Euclidean spaces, taking inspiration from the work, [36], of Zelený and Pelant on the structure of non-σ-porous sets. In particular, we prove that, like non-σ-porous sets, universal differentiability sets contain a 'kernel', which in some sense captures the core or essence of the set. In the papers [12], [14] and [13] of Doré and Maleva, it was observed that the universal differentiability sets constructed possess the property that the differentiability points of each Lipschitz function form a dense subset. We reveal that this

is, broadly speaking, an intrinsic property of universal differentiability sets. We go on to establish that no universal differentiability set can be decomposed as a countable union of closed, non-universal differentiability sets. Finally, we discuss an application of this decomposition result for differentiability inside sets of positive measure.

4.2 Notation and preliminary results.

We begin with a summary of the notation that we will use: We fix an integer $d \geq 2$ and let $e_1, e_2, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$. For a point $x \in \mathbb{R}^d$ and $\epsilon > 0$, we let $B(x, e)$ (respectively $\overline{B}(x, e)$) denote the open (respectively closed) ball with centre x and radius ϵ. The corresponding norm $\|-\|$ is the standard Euclidean norm on $\mathbb{R}^d$. By a polyhedral, we mean a compact subset of $\mathbb{R}^d$ given by a finite intersection of half spaces. Given a set $S \subseteq \mathbb{R}^d$, we let $\mathrm{Lip}(S)$ denote the space of Lipschitz functions $f : S \to \mathbb{R}$. The Lipschitz constant of a function $f \in \mathrm{Lip}(S)$ is denoted $\mathrm{Lip}(f)$. Further, we let $\mathrm{Int}(S)$ denote the interior, $\mathrm{Clos}(S)$ denote the closure and ∂S denote the boundary of the set S. For non-empty subsets A and B of $\mathbb{R}^d$ we let

$$\mathrm{diam}(A) = \sup \{\|a' - a\| \ : \ a, a' \in A\} \text{ and}$$

$$\mathrm{dist}(A, B) = \inf \{\|b - a\| \ : \ a \in A, b \in B\}.$$

When $A = \{a\}$ is a singleton, we will just write $\mathrm{dist}(a, B)$ rather than $\mathrm{dist}(\{a\}, B)$. We also adopt the convention $\mathrm{dist}(A, \emptyset) = 1$ for all $A \subseteq \mathbb{R}^d$. The support of a real valued function π is denoted by $\mathrm{supp}(\pi)$.

A subset U of $\mathbb{R}^d$ is called a d-box if there exist closed, bounded intervals $I_1, \ldots, I_d \subseteq \mathbb{R}$ such that $U = I_1 \times I_2 \times \ldots \times I_d$. Writing $I_k = [a_k, b_k]$ for each k, we call a set $Y \subseteq \partial U$ a face of U if there exists $m \in \{1, \ldots, d\}$ and $y \in \{a_m, b_m\}$ such that

$$Y = I_1 \times \ldots \times I_{m-1} \times \{y\} \times I_{m+1} \times \ldots \times I_d.$$

Note that each face of U is a subset of a $(d-1)$-hyperplane which is orthogonal to exactly one of the vectors $e_1, \ldots, e_d$.

In what follows we will often need to construct Lipschitz functions on $\mathbb{R}^d$ and it will be convenient to first tile $\mathbb{R}^d$ by d-boxes. We presently establish some basic facts concerning d-boxes.

Lemma 4.2.1. *Let K be a closed subset of $F \subseteq \mathbb{R}^d$. Then there exists a countable collection $\{S_i\}_{i=1}^{\infty}$ of d-boxes with pairwise disjoint interiors such that*

$$F \cap \bigcup_{i=1}^{\infty} S_i = F \setminus K, \qquad \operatorname{dist}(S_i, K) > 0 \ \forall i. \tag{4.1}$$

Proof. Write $K = F \cap K'$ where K' is a closed subset of $\mathbb{R}^d$. Since the set $\mathbb{R}^d \setminus K'$ is open (in $\mathbb{R}^d$), there exists a collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of d-boxes such that $\bigcup_{i=1}^{\infty} U_i = \mathbb{R}^d \setminus K'$. For example, we could take $\mathcal{U}$ to be the collection of all d-boxes, with corners in $\mathbb{Q}^d$, contained in $\mathbb{R}^d \setminus K'$.

Set $p_0 = 0$. For $n \geq 1$ we choose an integer $p_n \geq 1$ and d-boxes $S_{p_{n-1}+1}, \ldots, S_{p_n}$ with pairwise disjoint interiors, such that $\bigcup_{i=p_{n-1}+1}^{p_n} S_i = U_n \setminus \operatorname{Int}\left(\bigcup_{i=1}^{n-1} U_i\right)$. This defines a collection $\{S_i\}_{i=1}^{\infty}$ of d-boxes with pairwise disjoint interiors such that

$$\bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} U_i = \mathbb{R}^d \setminus K'.$$

The condition (4.1) is now readily verified. $\square$

Lemma 4.2.2. *Suppose that K is a closed subset of $F \subseteq \mathbb{R}^d$ and $\{U_i\}_{i=1}^{\infty}$ is a countable collection of d-boxes with pairwise disjoint interiors such that $F \setminus K \subseteq F \cap \bigcup_{i=1}^{\infty} U_i$. Then,*

there exists a collection $\{T_i\}_{i=1}^{\infty}$ *of* d*-boxes with pairwise disjoint interiors such that*

$$F \cap \bigcup_{i=1}^{\infty} T_i = F \setminus K, \tag{4.2}$$

$$\text{For each index } i \text{ there exists an index } j \text{ such that } T_i \subseteq U_j, \tag{4.3}$$

$$\frac{\operatorname{diam}(T_i)}{\operatorname{dist}(T_i, K)} \to 0 \text{ as } i \to \infty. \tag{4.4}$$

Proof. By Lemma 4.2.1, there exists a countable collection $\{S_i\}_{i=1}^{\infty}$ of d-boxes with pairwise disjoint interiors such that (4.1) holds. For each $i \geq 1$, the collection (U_j) determines a partition of S_i into a countable collection $\{S_{i,k}\}_{k=1}^{\infty}$ of d-boxes with pairwise disjoint interiors such that $\bigcup_{k=1}^{\infty} S_{i,k} = S_i$ and $S_{i,k} = S_i \cap U_k$.

Then $\mathcal{W} = \{S_{i,k}\}_{i,k\geq 1}$ is a countable collection of d-boxes with pairwise disjoint interiors. After relabelling, we can write $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$. Note that $\operatorname{dist}(W_i, K) > 0$ for every i. This follows from the definition of the collection $\mathcal{W}$ and (4.1).

Set $p_0 = 0$. For each $i \geq 1$, partition the d-box W_i into a finite number of d-boxes $T_{p_{i-1}+1}, \ldots, T_{p_i}$ with pairwise disjoint interiors such that

$$\frac{\operatorname{diam}(T_j)}{\operatorname{dist}(T_j, K)} \leq 2^{-i} \text{ for } p_{i-1} + 1 \leq j \leq p_i.$$

The statements (4.2), (4.3) and (4.4) are now readily verified. □

4.3 Decompositions of a universal differentiability set.

Let us open this section by quoting the main result that we shall work towards.

Theorem 4.3.1. *Suppose* $E = \bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{R}^d$*, where each* A_i *is a non-universal differentiability set, and is closed as a subset of* E*. Then* E *is a non-universal differentiability set.*

Remark 4.3.2. *In general, a countable union of non-universal differentiability sets may form a universal differentiability set. Indeed, let* S *be a universal differentiability set in* $\mathbb{R}^2$ *with Lebesgue measure zero (such a set is given in [28]). By a result of Alberti,*

Csyornei and Preiss in [1], there exist Lipschitz functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that f and g have no common points of differentiability inside S. Writing D_f for the set of points of differentiability of f, we get that

$$S = (S \setminus D_f) \cup (S \cap D_f)$$

is a decomposition of S as a union of two non-universal differentiabilty sets. Note that $S \cap D_f$ is a non-universal differentiability set since it contains no differentiability points of g.

The proof of Theorem 4.3.1 will involve constructing a sequence of Lipschitz functions which converge to a Lipschitz function nowhere differentiable on the set E. We will need the next Lemma in order to pick, at each step of the construction, a Lipschitz function nowhere differentiable on part of the set, with small supremum norm, small range and small Lipschitz constant. By controlling these quantities, we are able to ensure that the sequence of functions converges.

Lemma 4.3.3. *Let Ω be a bounded subset of $\mathbb{R}^d$. A set $E \subseteq \mathbb{R}^d$ is a non-universal differentiability set if and only if for every $\epsilon_1 > \epsilon_2 > 0$ and for every $\delta > 0$, there exists a Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d)$ such that*

$$\epsilon_2 \leq f(x) \leq \epsilon_1 \text{ for all } x \in \Omega, \qquad \mathrm{Lip}(f) \leq \delta \tag{4.5}$$

and f is nowhere differentiable in E.

Proof. We focus only on the non-trivial direction. Suppose that there exist $\epsilon_1 > \epsilon_2 > 0$ and $\delta > 0$ such that every Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d)$ satisfying (4.5) has a point of differentiability inside E. We may assume that $\delta < 1$ and $\epsilon_1 - \epsilon_2 < 1$.

Fix $g \in \mathrm{Lip}(\mathbb{R}^d)$. We show that g has a point of differentiability inside E. Writing χ_Ω for the characteristic function of Ω, we note that $\|g\chi_\Omega\|_\infty + \mathrm{Lip}(g)$ is finite since Ω is

bounded. Further, we may assume that $\|g\chi_\Omega\|_\infty + \mathrm{Lip}(g) \neq 0$. Let

$$h(x) = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\delta(\epsilon_1 - \epsilon_2)}{4(\|g\chi_\Omega\|_\infty + \mathrm{Lip}(g))} g(x) \text{ for all } x \in \mathbb{R}^d.$$

We have $h \in \mathrm{Lip}(\mathbb{R}^d)$ and $\left\|h(x) - \frac{\epsilon_1+\epsilon_2}{2}\right\| \leq \frac{\epsilon_1-\epsilon_2}{4}$ for all $x \in \Omega$. Hence $\epsilon_2 \leq h(x) \leq \epsilon_1$ for all $x \in \Omega$. Moreover, $\mathrm{Lip}(h) \leq \frac{\delta}{4} < \delta$. This establishes that the condition (4.5) is satisfied with $f = h$. We conclude that there exists a point $x \in E$ such that h is differentiable at x. But then g is also differentiable at x because $g = \alpha h + \beta$ where $\alpha, \beta \in \mathbb{R}$ are fixed constants. Since $g \in \mathrm{Lip}(\mathbb{R}^d)$ was arbitrary, we deduce that E is a universal differentiability set. □

Using the next Lemma, we will later be able to ignore points lying in the boundaries of d-boxes.

Lemma 4.3.4. *Suppose E is a subset of $\mathbb{R}^d$ and $\{U_i\}_{i=1}^\infty$ is a collection of d-boxes in $\mathbb{R}^d$ such that $E \setminus \bigcup_{i=1}^\infty \partial U_i$ is a non-universal differentiability set. Then E is a non-universal differentiability set.*

Proof. For $j = 1, \ldots, d$, let H_j denote the union of all faces of the d-boxes U_i which are orthogonal to e_j. Then $\bigcup_{i=1}^\infty \partial U_i = \bigcup_{j=1}^d H_j$. Moreover, writing p_j for the j-th co-ordinate projection map on $\mathbb{R}^d$, we have that $p_j(H_j)$ is a subset of $\mathbb{R}$ with one-dimensional Lebesgue measure zero, (in fact it is a countable set) for each $j = 1, \ldots, d$. The result now follows from Lemma 3.2.1. □

We presently arrive at a crucial result which drives the construction in the proof of Theorem 4.3.1. Having covered a certain region of $\mathbb{R}^d$ with d-boxes, this lemma enables us to obtain a Lipschitz function, with desired differentiability and norm properties, by prescribing the function on each individual d-box.

Lemma 4.3.5. *Let E be a subset of $\mathbb{R}^d$ and let $C \subseteq \mathbb{R}^d$ be a d-box. Suppose that $E \cap \mathrm{Int}(C)$ is a non-universal differentiability set and let $\epsilon_1, \epsilon_2 > 0$. Then there exists a Lipschitz function $f : C \to \mathbb{R}$ with the following properties.*

1. *f is nowhere differentiable in $E \cap \mathrm{Int}(C)$,*

2. *$\|f\|_\infty \leq \epsilon_1$ and $\mathrm{Lip}(f) \leq \epsilon_2$,*

3. *$f(x) = 0$ for all $x \in \partial C$.*

Proof. We may assume that C is non-empty. Set $D = \max\{\mathrm{diam}(C), 1\}$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. By Lemma 4.3.3, there exists a function $h \in \mathrm{Lip}(\mathbb{R}^d)$ such that h is nowhere differentiable in $E \cap \mathrm{Int}(C)$,

$$\frac{\epsilon}{8D} \leq h(x) \leq \frac{\epsilon}{4D} \text{ for all } x \in C, \text{ and } \mathrm{Lip}(h) \leq \frac{\epsilon}{32D^2}. \tag{4.6}$$

Define $f \in \mathrm{Lip}(C)$ by

$$f(u) = \mathrm{dist}(u, \partial C)h(u) \qquad \forall u \in C. \tag{4.7}$$

Clearly f has property 3. We now establish 2: For $u, v \in C$ with $u \neq v$ we set

$$A_{u,v} = \frac{\mathrm{dist}(u, \partial C)h(u) - \mathrm{dist}(v, \partial C)h(v)}{\mathrm{dist}(u, v)}.$$

Using the triangle inequality, we get

$$\left| A_{u,v} - \mathrm{dist}(v, \partial C)\frac{h(u) - h(v)}{\mathrm{dist}(u, v)} \right| \leq h(u),$$

and combining this with (4.7) and (4.6) yields

$$\mathrm{Lip}(f) = \sup_{u \neq v} |A_{u,v}| \leq \|h\|_\infty + \mathrm{diam}(C)\mathrm{Lip}(h) \leq \epsilon.$$

Further, (4.6) and (4.7) imply $\|f\|_\infty \leq \mathrm{diam}(C)\,\|h\|_\infty < \epsilon/2$, and this verifies property 2.

It only remains to prove property 1. Let $Q_{1,1}, Q_{1,2} \ldots, Q_{d,1}, Q_{d,2}$ denote the $(d-1)$-hyperplanes containing the faces of C, where $Q_{i,1}, Q_{i,2}$ are orthogonal to e_i and parallel

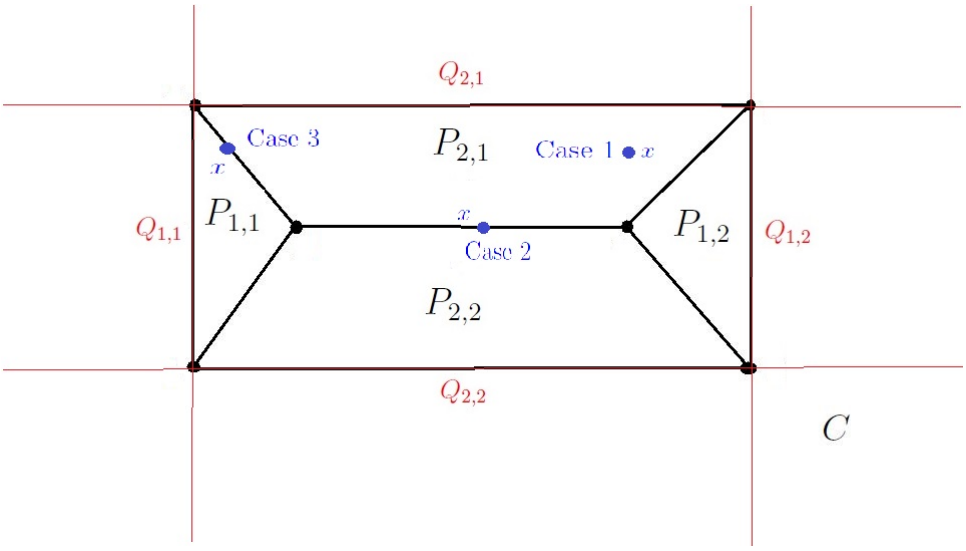

Figure 4.1: The sets C, $P_{i,j}$, $Q_{i,j}$ in dimension $d = 2$.

to e_j for $j \neq i$. Next, partition C into regions $P_{1,1}, P_{1,2}, \ldots, P_{d,1}, P_{d,2}$ such that

$$\operatorname{dist}(u, \partial C) = \operatorname{dist}(u, Q_{i,j}) \text{ for } u \in P_{i,j}, \tag{4.8}$$

(see Figure 4.1). We remark that, writing $C = \prod_{i=1}^{d} [\alpha_{i,1}, \alpha_{i,2}]$ where $\alpha_{i,1} \leq \alpha_{i,2}$ are real numbers, we can take

$$Q_{i,j} = \left\{ x \in \mathbb{R}^d : \langle x, e_i \rangle = \alpha_{i,j} \right\}.$$

Moreover, we have that

$$\operatorname{dist}(x, Q_{i,j}) = \begin{cases} \langle x, e_i \rangle - \alpha_{i,1} & \text{if } j = 1, \\ \alpha_{i,2} - \langle x, e_i \rangle & \text{if } j = 2, \end{cases} \qquad \forall x \in C.$$

Observe that the sets $P_{i,j}$ are closed with pairwise disjoint interiors. It is not difficult to verify that the sets $P_{i,j}$ are polyhedrals: Indeed, we have

$$P_{i,j} = \bigcap_{(k,l) \neq (i,j)} \{ x \in C : \operatorname{dist}(x, Q_{i,j}) \leq \operatorname{dist}(x, Q_{k,l}) \}$$

and each set $\{ x \in C : \operatorname{dist}(x, Q_{i,j}) \leq \operatorname{dist}(x, Q_{k,l}) \}$ is a finite intersection of half spaces.

Let $x \in E \cap \mathrm{Int}(C)$. We need to show that f is not differentiable at x. We distinguish between three cases:

Case 1: First, suppose that $x \in \mathrm{Int}(P_{i,j})$ for some i, j. Choose $\delta > 0$ small enough so that $B(x, \delta) \subseteq P_{i,j}$. Let $e \in S^{d-1}$ and $0 < t < \delta$. Replacing e_i with $-e_i$ if necessary, we have

$$\begin{aligned}\frac{f(x+te)-f(x)}{t} &= \frac{\mathrm{dist}(x+te, Q_{i,j})h(x+te) - \mathrm{dist}(x, Q_{i,j})h(x)}{t} \\ &= \frac{(t\langle e, e_i\rangle + \mathrm{dist}(x, Q_{i,j}))h(x+te) - \mathrm{dist}(x, Q_{i,j})h(x)}{t} \\ &= h(x+te)\langle e, e_i\rangle + \mathrm{dist}(x, Q_{i,j})\frac{h(x+te)-h(x)}{t}. \end{aligned} \tag{4.9}$$

Suppose that there exists $e \in S^{d-1}$ such that the directional derivative $h'(x, e)$ does not exist. Then, from (4.9) we get that the directional derivative $f'(x, e)$ does not exist and consequently f is not differentiable at x. Therefore, we may assume that all directional derivatives of h (and f) exist, and so (4.9) yields

$$f'(x, e) = h(x)\langle e, e_i\rangle + \mathrm{dist}(x, \partial C)h'(x, e) \text{ for all } e \in S^1. \tag{4.10}$$

Since h is not differentiable at x, the map $\mathbb{R}^d \to \mathbb{R}$, $v \mapsto h'(x, v)$ is not linear. This fact, together with (4.10), implies that the map $\mathbb{R}^d \to \mathbb{R}$, $v \mapsto f'(x, v)$ is not linear. Thus, f is not differentiable at x.

Case 2: Now suppose that $x \in P_{i,1} \cap P_{i,2}$ for some i. Consequently, we have that $\mathrm{dist}(x, \partial C) = \mathrm{dist}(x, Q_{i,1}) = \mathrm{dist}(x, Q_{i,2})$. Since the vector e_i is orthogonal to both $Q_{i,1}$ and $Q_{i,2}$ we have that

$$\mathrm{dist}(x + \pi t e_i, \partial C) = \mathrm{dist}(x, \partial C) - t,$$

for $0 < t < \operatorname{dist}(x, \partial C)$ and $\pi = \pm 1$. Therefore, for $0 < t < \operatorname{dist}(x, \partial C)$ we have

$$\frac{f(x + \pi t e_i) - f(x)}{t} = \begin{cases} \operatorname{dist}(x, \partial C) \frac{h(x+te_i)-h(x)}{t} - h(x + te_i) & \text{if } \pi = 1, \\ \operatorname{dist}(x, \partial C) \frac{h(x-te_i)-h(x)}{t} - h(x - te_i) & \text{if } \pi = -1. \end{cases} \tag{4.11}$$

If $h'(x, \pi e_i)$ does not exist for some $\pi \in \{1, -1\}$ then (4.11) gives that $f'(x, \pi e_i)$ does not exist and f is not differentiable at x. Therefore, we may assume that the directional derivatives $h'(x, e_i)$ and $h'(x, -e_i)$ exist. But, in this case, (4.11) and (4.6) yield

$$f'(x, \pm e_i) = \operatorname{dist}(x, \partial C) h'(x, \pm e_i) - h(x) \leq D\operatorname{Lip}(h) - h(x) \leq -3\epsilon/32D.$$

Hence, the map $v \mapsto f'(x, v)$ is not linear and f is not differentiable at x as required.

Case 3: In the remaining case we have that $x \in \partial P_{i,j}$ for some i, j and $x \notin P_{m,1} \cap P_{m,2}$ for any m. Then, replacing e_i with $-e_i$ if necessary, there exists $\eta > 0$ and k, l such that $k \neq i$, $x + te_i \in P_{i,j}$ for $t \in [0, \eta]$ and $x - te_i \in P_{k,l}$ for $t \in [0, \eta]$. In particular, $x \in P_{i,j} \cap P_{k,l}$ so that $\operatorname{dist}(x, \partial C) = \operatorname{dist}(x, Q_{i,j}) = \operatorname{dist}(x, Q_{k,l})$. Moreover, since $k \neq i$, the vector e_i is parallel to $Q_{k,l}$. For $t \in [0, \eta]$, we now observe that

$$\operatorname{dist}(x + te_i, \partial C) = \operatorname{dist}(x, \partial C) - t \text{ and } \operatorname{dist}(x - te_i, \partial C) = \operatorname{dist}(x, \partial C).$$

Hence, for sufficiently small $t > 0$,

$$\frac{f(x + t\pi e_i) - f(x)}{t} = \begin{cases} \operatorname{dist}(x, \partial C) \frac{h(x+te_i)-h(x)}{t} - h(x + te_i) & \text{if } \pi = 1, \\ \operatorname{dist}(x, \partial C) \frac{h(x-te_i)-h(x)}{t} & \text{if } \pi = -1. \end{cases} \tag{4.12}$$

If $h'(x, \pi e_i)$ does not exist for some $\pi \in \{1, -1\}$ then (4.12) gives that $f'(x, \pi e_i)$ does not exist and f is not differentiable at x. Hence, we may assume that $h'(x, \pi e_i)$ exists for

both $\pi = \pm 1$. Then, (4.12) gives that $f'(x, \pi e_i)$ exists for $\pi = \pm 1$ and

$$f'(x, \pi e_i) = \begin{cases} \operatorname{dist}(x, \partial C) h'(x, e_i) - h(x) & \text{if } \pi = 1, \\ \operatorname{dist}(x, \partial C) h'(x, -e_i) & \text{if } \pi = -1. \end{cases} \tag{4.13}$$

For f to be differentiable at x we require $f'(x, e_i) + f'(x, -e_i) = 0$. In view of (4.13) and (4.6), this implies

$$\epsilon/8D \leq h(x) = \operatorname{dist}(x, \partial C)(h'(x, e_i) + h'(x, -e_i)) \leq 2D\operatorname{Lip}(h) \leq \epsilon/16D.$$

Thus, we obtain a contradiction and conclude that f is not differentiable at x. □

Lemma 4.3.6. *Let $E \subseteq \mathbb{R}^d$ and let $\{U_i\}_{i=1}^{\infty}$ be a collection of d-boxes with pairwise disjoint interiors such that and $E \cap \operatorname{Int}(U_i)$ is a non-universal differentiability set for each i. Let $\eta_1, \eta_2 > 0$. Then there exists a function $g \in \operatorname{Lip}(\mathbb{R}^d)$ such that*

$$\|g\|_\infty \leq \eta_1 \text{ and } \operatorname{Lip}(g) \leq \eta_2, \tag{4.14}$$

$$g \text{ is nowhere differentiable in } \bigcup_{i=1}^{\infty} E \cap \operatorname{Int}(U_i), \tag{4.15}$$

$$g(x) = 0 \text{ whenever } x \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \operatorname{Int}(U_i) \right). \tag{4.16}$$

Proof. For each i, the conditions of Lemma 4.3.5 are satisfied for E, $C = U_i$, $\epsilon_1 = \eta_1$ and $\epsilon_2 = 2^{-i}\eta_2$. Let now $f_i \in \operatorname{Lip}(U_i)$ be given by the conclusion of Lemma 4.3.5. We define $g \in \operatorname{Lip}(\mathbb{R}^d)$ by

$$g(x) = \begin{cases} f_i(x) & \text{if } x \in \operatorname{Int}(U_i), i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 4.3.7. *A set $E \subseteq \mathbb{R}^d$ is a non-universal differentiability set if and only if for every $x \in E$, there exists $\epsilon = \epsilon_x > 0$ such that $B(x, \epsilon) \cap E$ is a non-universal differentia-*

bility set.

Proof. The 'only if' direction is trivial; we will prove the 'if' direction. Under the second condition, we may choose a collection $\{U_i\}_{i=1}^{\infty}$ of d-boxes such that $E \subseteq \bigcup_{i=1}^{\infty} U_i$ and $E \cap \mathrm{Int}(U_i)$ is a non-universal differentiability set for all i. We now follow the procedure described in the proof of Lemma 4.2.1 to obtain a collection $\{S_i\}_{i=1}^{\infty}$ of d-boxes with pairwise disjoint interiors such that $E \subseteq \bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} U_i$ and for each $i \geq 1$, there exists $j \geq 1$ such that $S_i \subseteq U_j$. Then for each $i \geq 1$, $E \cap \mathrm{Int}(S_i)$ is a non-universal differentiability set, since it is contained in $E \cap \mathrm{Int}(U_j)$ for some j. The conditions of Lemma 4.3.6 are satisfied for the set E, the collection of d-boxes $\{S_i\}_{i=1}^{\infty}$ and any $\eta_1, \eta_2 > 0$. Therefore, by Lemma 4.3.6, the set $E \cap \bigcup_{i=1}^{\infty} \mathrm{Int}(S_i) = E \setminus \bigcup_{i=1}^{\infty} \partial S_i$ is a non-universal differentiability set. The proof is completed by applying Lemma 4.3.4. □

We now show that the universal differentiability property is preserved by diffeomorphisms.

Lemma 4.3.8. *Suppose O an open subset of $\mathbb{R}^d$ and $f : O \to \mathbb{R}^d$ is a C^1-diffeomorphism. Then, a set $E \subseteq O$ is a universal differentiability set if and only if $f(E)$ is a universal differentiability set.*

Proof. Suppose $f(E)$ is a non-universal differentiability set and pick $g \in \mathrm{Lip}(\mathbb{R}^d)$ such that g is nowhere differentiable on $f(E)$. We show that $g \circ f$ is nowhere differentiable on E.

Let $x \in E$ and note that $f'(x) : \mathbb{R}^d \to \mathbb{R}^d$ is an isomorphism. Given $v \in \mathbb{R}^d$ we have that $f(x + tv) = f(x) + tf'(x)(v) + o(t)$. We can write

$$\frac{g \circ f(x + tv) - g \circ f(x)}{t} = \frac{g[f(x) + tf'(x)(v) + o(t)] - g[f(x) + tf'(x)(v)]}{t} + \frac{g[f(x) + tf'(x)(v)] - g[f(x)]}{t},$$

whilst

$$\left\| \frac{g[f(x) + tf'(x)(v) + o(t)] - g[f(x) + tf'(x)(v)]}{t} \right\| \leq \mathrm{Lip}(g) \left\| \frac{o(t)}{t} \right\| \to 0 \text{ as } t \to 0.$$

Therefore the directional derivative $(g \circ f)'(x, v)$ exists if and only if $g'[f(x), f'(x)(v)]$ exists and $(g \circ f)'(x, v) = g'[f(x), f'(x)(v)]$. Since g is not differentiable at $f(x) \in f(E)$, $v \in \mathbb{R}^d$ is arbitrary and $f'(x) : \mathbb{R}^d \to \mathbb{R}^d$ is an isomorphism, we deduce that $g \circ f$ is not differentiable at x.

The proof is not yet complete since we do not know whether the function $g \circ f$ is Lipschitz. However, fixing $x \in E$ and $\epsilon > 0$ we have that $g \circ f$ restricted to $\overline{B}(x, \epsilon)$ is Lipschitz and nowhere differentiable in $E \cap B(x, \epsilon)$. It follows that $E \cap B(x, \epsilon)$ is a non-universal differentiability set. That E is a non-universal differentiability set now follows from Corollary 4.3.7. □

In the next lemmas, we show that, given a Lipschitz function h and a collection of pairwise disjoint d-boxes, we can slightly modify the function h so that it becomes differentiable everywhere in the interiors of the d-boxes and remains unchanged everywhere else.

Lemma 4.3.9. *Let $C \subseteq \mathbb{R}^d$ be a d-box, $h \in Lip(\mathbb{R}^d)$ and let $\sigma_1, \sigma_2 > 0$. Then there exists a function $f \in Lip(\mathbb{R}^d)$ such that*

$$\|f - h\|_\infty \leq \sigma_1 \text{ and } Lip(f) \leq Lip(h) + \sigma_2, \tag{4.17}$$

$$f \text{ is everywhere differentiable in } \mathrm{Int}(C), \tag{4.18}$$

$$f(x) = h(x) \text{ for all } x \in \mathbb{R}^d \setminus \mathrm{Int}(C). \tag{4.19}$$

Proof. We may and will assume that $\mathrm{Lip}(h) > 0$. Fix a C^∞ function $\pi : \mathbb{R}^d \to \mathbb{R}$ such that $\pi(z) \geq 0$ for all $z \in \mathbb{R}^d$, $\mathrm{supp}(\pi) \subseteq B(0, 1)$ and $\int_{\mathbb{R}^d} \pi(z)dz = 1$. Choose a C^∞ function $\gamma : \mathrm{Int}(C) \to \mathbb{R}$ such that $0 < \gamma(x) \leq \sigma_1/\mathrm{Lip}(h)$ for all $x \in \mathrm{Int}(C)$, $\gamma(x) \to 0$ as $x \to \partial C$

and $\mathrm{Lip}(\gamma) \leq \sigma_2/\mathrm{Lip}(h)$. Define $f \in \mathrm{Lip}(\mathbb{R}^d)$ by

$$f(x) = \begin{cases} \frac{1}{\gamma(x)} \int_{\mathbb{R}^d} h(x-z)\pi(z/\gamma(x))dz & \text{if } x \in \mathrm{Int}(C), \\ h(x) & \text{otherwise.} \end{cases} \tag{4.20}$$

Statement (4.19) is now readily verified.

We presently prove statement (4.17). Fix $x \in \mathrm{Int}(C)$. Then

$$\begin{aligned} \|f(x) - h(x)\| &= \left\| \frac{1}{\gamma(x)} \int_{\mathbb{R}^d} h(x-z)\pi(z/\gamma(x))dz - h(x)\frac{1}{\gamma(x)} \int_{\mathbb{R}^d} \pi(z/\gamma(x))dz \right\| \\ &\leq \frac{1}{\gamma(x)} \int_{\mathbb{R}^d} \|h(x-z) - h(x)\| \, \pi(z/\gamma(x))dz \\ &\leq \mathrm{Lip}(h)\frac{1}{\gamma(x)} \int_{\mathbb{R}^d} \|z\| \, \pi(z/\gamma(x))dz \\ &\leq \mathrm{Lip}(h) \int_{B(0,\gamma(x))} \pi(z/\gamma(x))dz \\ &= \gamma(x)\mathrm{Lip}(h) \leq \sigma_1. \end{aligned} \tag{4.21}$$

This establishes the first part of (4.17). For the second part, fix $x, y \in \mathrm{Int}(C)$. Then

$$\begin{aligned} \|f(y) - f(x)\| &= \left\| \frac{1}{\gamma(y)} \int_{\mathbb{R}^d} h(y-z)\pi(z/\gamma(y))dz - \frac{1}{\gamma(x)} \int_{\mathbb{R}^d} h(x-z)\pi(z/\gamma(x))dz \right\| \\ &= \left\| \int_{\mathbb{R}^d} h(y-\gamma(y)v)\pi(v)dv - \int_{\mathbb{R}^d} h(x-\gamma(x)v)\pi(v)dv \right\| \\ &\leq \int_{B(0,1)} \|h(y-\gamma(y)v) - h(x-\gamma(x)v)\| \, \pi(v)dv \\ &\leq \mathrm{Lip}(h) \int_{B(0,1)} (\|y-x\| + |\gamma(y) - \gamma(x)| \, \|v\|)\pi(v)dv \\ &\leq \mathrm{Lip}(h)(1 + \mathrm{Lip}(\gamma)) \, \|y-x\| \\ &\leq (\mathrm{Lip}(h) + \sigma_2) \, \|y-x\| \, . \end{aligned}$$

Combining the above with (4.20), we may conclude that f restricted to either $\mathrm{Int}(C)$ or $\mathbb{R}^d \setminus \mathrm{Int}(C)$ is Lipschitz with Lipschitz constant bounded above by $\mathrm{Lip}(h) + \sigma_2$. To prove that the global Lipschitz constant $\mathrm{Lip}(f)$ also satisfies this bound, it is enough to show

that f is continuous at all points of ∂C. This follows from (4.20), (4.21) and the condition $\gamma(x) \to 0$ as $x \to \partial C$.

Finally, let us verify (4.18). For each $u \in \mathbb{R}^d$, define a map $\Phi_u : \mathbb{R}^d \to \mathbb{R}$ by

$$\Phi_u(y) = \pi\left(\frac{y-u}{\gamma(y)}\right).$$

Note that each Φ_u is a differentiable function and, for all $x \in \mathrm{Int}(C)$, we can write $f(x) = -\widetilde{f}(x)/\gamma(x)$ where

$$\widetilde{f}(x) = \int_{\mathbb{R}^d} h(u)\Phi_u(x)du.$$

Since γ is C^∞ and strictly positive on $\mathrm{Int}(C)$, it suffices to prove that $\widetilde{f}$ is everywhere differentiable in $\mathrm{Int}(C)$. We show that

$$\widetilde{f}'(x,e) = \int_{\mathbb{R}^d} h(u)\Phi_u'(x,e)du \text{ for all } x \in \mathrm{Int}(C), e \in S^{d-1}.$$

Fix $x \in \mathrm{Int}(C)$ and $e \in S^{d-1}$. Next, choose t_0 small enough so that for all $t \in (0, t_0)$ we have

$$x + te \in \mathrm{Int}(C), \qquad 1/2 \le \gamma(x+te)/\gamma(x) \le 2, \qquad t/\gamma(x+te) \le 1/2. \tag{4.22}$$

Observe that

$$\frac{\widetilde{f}(x+te) - \widetilde{f}(x)}{t} = \int_{\mathbb{R}^d} h(u)\left[\frac{\Phi_u(x+te) - \Phi_u(x)}{t}\right]du,$$

and for each u we have

$$\lim_{t\to 0} h(u)\left[\frac{\Phi_u(x+te) - \Phi_u(x)}{t}\right] = h(u)\Phi_u'(x,e).$$

Therefore, using the Dominated Convergence Theorem [5, p. 44], we only need to show

that there exists $\varphi \in L^1(\mathbb{R}^d)$ such that

$$\left| h(u) \left[\frac{\Phi_u(x+te) - \Phi_u(x)}{t} \right] \right| \leq \varphi(u) \text{ for all } t \in (0, t_0), u \in \mathbb{R}^d.$$

Let $t \in (0, t_0)$. Suppose $\|u - x\| > 4\gamma(x)$. Then, from (4.22),

$$\left\| \frac{x + te - u}{\gamma(x+te)} \right\| \geq \frac{4\gamma(x)}{\gamma(x+te)} - \frac{t}{\gamma(x+te)} \geq 2 - \frac{1}{2} > 1.$$

It follows that $\pi((x + te - u)/\gamma(x + te)) = 0$. Similarly, we have $\|(x - u)/\gamma(x)\| > 4$, so that $\pi((x - u)/\gamma(x)) = 0$. Hence, $(\Phi_u(x + te) - \Phi_u(x))/t = 0$. This establishes that the map $u \mapsto (\Phi_u(x + te) - \Phi_u(x))/t$ has compact support contained in $\overline{B}(x, 4\gamma(x))$.

It now suffices to show that the quantity $\left| h(u) \left[\frac{\Phi_u(x+te) - \Phi_u(x)}{t} \right] \right|$ is uniformly bounded for $u \in \overline{B}(x, 4\gamma(x))$ and $t \in (0, t_0)$.

Fix $u \in \overline{B}(x, 4\gamma(x))$ and $t \in (0, t_0)$. Let $M = \sup_{v \in \overline{B}(x, 4\gamma(x))} |h(v)|$ and note that $M < \infty$. Now, using (4.22) we obtain

$$\begin{aligned}
\left| h(u) \left[\frac{\Phi_u(x+te) - \Phi_u(x)}{t} \right] \right| &\leq M \mathrm{Lip}(\pi) \left\| \frac{x + te - u}{t\gamma(x+te)} - \frac{x - u}{t\gamma(x)} \right\| \\
&\leq M \mathrm{Lip}(\pi) \frac{2}{\gamma(x)^2} \left(\left| \frac{\gamma(x+te) - \gamma(x)}{t} \right| \|x - u\| + \gamma(x) \right) \\
&\leq M \mathrm{Lip}(\pi) \frac{2}{\gamma(x)^2} (\mathrm{Lip}(\gamma) 4\gamma(x) + \gamma(x)).
\end{aligned}$$

□

Lemma 4.3.10. *Let $\{U_i\}_{i=1}^{\infty}$ be a collection of d-boxes in $\mathbb{R}^d$ with pairwise disjoint interiors and let $h \in Lip(\mathbb{R}^d)$. Let σ_1, σ_2 be positive real numbers. Then, there exists a function*

$\widehat{h} \in Lip(\mathbb{R}^d)$ *such that*

$$\left\|\widehat{h} - h\right\|_\infty \leq \sigma_1 \text{ and } \mathrm{Lip}(\widehat{h}) \leq \mathrm{Lip}(h) + \sigma_2, \tag{4.23}$$

$$\widehat{h} \text{ is everywhere differentiable inside } \bigcup_{i=1}^{\infty} \mathrm{Int}(U_i) \text{ and} \tag{4.24}$$

$$\widehat{h}(x) = h(x) \text{ for all } x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \mathrm{Int}(U_i). \tag{4.25}$$

Proof. Using Lemma 4.3.9, we may choose, for each $i \geq 1$, a function $f_i \in \mathrm{Lip}(\mathbb{R}^d)$ such that (4.17), (4.18) and (4.19) hold with $f = f_i$ and $C = U_i$. Define the function $\widehat{h} \in \mathrm{Lip}(\mathbb{R}^d)$ by

$$\widehat{h}(x) = \begin{cases} f_i(x) & \text{if } x \in \mathrm{Int}(U_i), i \geq 1, \\ h(x) & \text{if } x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} \mathrm{Int}(U_i). \end{cases} \tag{4.26}$$

Statements (4.23), (4.24) and (4.25) are now readily verified. □

In our main construction, we are faced with a situation where we would like to slightly modify a Lipschitz function h to obtain a new function f, whilst preserving non-differentiability points. Using the previous lemma, we ensure that f coincides with h on the boundaries of d-boxes. The application of the following important lemma, is to show that if these boundaries become increasingly concentrated around a point x, then the differentiability of f and h at x will coincide.

Lemma 4.3.11. *Let $x \in \mathbb{R}^d$, $e \in S^{d-1}$, $f, h \in \mathrm{Lip}(\mathbb{R}^d)$, $\Delta > 0$ and let $\{(t_{k,1}, t_{k,2})\}_{k=1}^N$, where $0 < t_{k,1} \leq t_{k,2}$ and $N \in \mathbb{N} \cup \{\infty\}$, be a finite or countable collection of open, possibly degenerate, intervals inside $(0, \infty)$ such that the following conditions hold.*

$$\text{If } N = \infty \text{ then } \frac{t_{k,2} - t_{k,1}}{t_{k,1}} \to 0 \text{ as } k \to \infty, \tag{4.27}$$

$$f(x + te) = h(x + te) \qquad \forall t \in \left([0, \Delta) \setminus \left(\bigcup_{k=1}^{N} (t_{k,1}, t_{k,2})\right)\right) \cup \left(\bigcup_{k=1}^{N} \{t_{k,1}, t_{k,2}\}\right). \tag{4.28}$$

Then the directional derivative $f'(x, e)$ exists if and only if $h'(x, e)$ exists and $f'(x, e) =$

$h'(x,e)$.

Proof. Note that the statement is symmetric with respect to f and h. Moreover, we have $f(x) = h(x)$ since $t = 0$ satisfies (4.28). We may assume that $\mathrm{Lip}(h) + \mathrm{Lip}(f) > 0$.

Suppose that the directional derivative $f'(x,e)$ exists. Fix $\epsilon > 0$ and choose $\delta > 0$ so that

$$|f(x+te) - f(x) - tf'(x,e)| \leq \frac{\epsilon}{3}t \text{ whenever } 0 < t < \delta. \tag{4.29}$$

If $N < \infty$ we set $K = N$. Otherwise, using (4.27), we may pick $K \geq 1$ large enough so that

$$\frac{t_{k,2} - t_{k,1}}{t_{k,1}} \leq \frac{\epsilon}{3(\mathrm{Lip}(h) + \mathrm{Lip}(f))} \text{ for every } k \geq K. \tag{4.30}$$

Let $t \in (0, \min\{\delta, \Delta, t_{1,1}, \ldots, t_{K,1}\})$. We may distinguish two cases. First assume that

$$t \in (0,\infty) \setminus \left(\bigcup_{k=1}^{N} (t_{k,1}, t_{k,2}) \right).$$

Then, combining (4.28), (4.29) and $t \in (0, \min\{\delta, \Delta\})$ we get

$$|h(x+te) - h(x) - tf'(x,e)| = |f(x+te) - f(x) - tf'(x,e)| \leq \frac{\epsilon}{3}t.$$

In the remaining case, there exists k such that $t \in (t_{k,1}, t_{k,2})$. Moreover, since $t < t_{l,1}$ for $1 \leq l \leq K$, we must have that $k > K$. Note, in particular, that in this case we must have $N = \infty$. Now, using (4.28), (4.30) and (4.29) we deduce

$$\begin{aligned}
&|h(x+te) - h(x) - tf'(x,e)| \\
&\leq |h(x+te) - h(x+t_{k,1}e)| + |f(x+t_{k,1}e) - f(x+te)| \\
&+ |f(x+te) - f(x) - tf'(x,e)| \\
&\leq (\mathrm{Lip}(h) + \mathrm{Lip}(f))(t - t_{k,1}) + \frac{\epsilon}{3}t \\
&\leq \frac{(\mathrm{Lip}(h) + \mathrm{Lip}(f))(t_{k,2} - t_{k,1})}{t_{k,1}}t + \frac{\epsilon}{3}t \leq \epsilon t.
\end{aligned}$$

We have now established that the directional derivative $h'(x,e)$ exists and equals $f'(x,e)$.

$\square$

Lemma 4.3.12. *Let $F \subseteq \mathbb{R}^d$, let A be a closed subset of F and let O be an open subset of $\mathbb{R}^d$ such that $F \subseteq O$. Let $h \in \mathrm{Lip}(\mathbb{R}^d)$ and let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be a collection of d-boxes with pairwise disjoint interiors such that*

$$F \cap \bigcup_{i=1}^{\infty} U_i = F \setminus A \text{ and } \frac{\mathrm{diam}(U_i)}{\mathrm{dist}(U_i, A)} \to 0 \text{ as } i \to \infty. \tag{4.31}$$

Then the following two statements are satisfied:

1. *Suppose that $F \setminus A$ is a non-universal differentiability set and let $\epsilon_1, \epsilon_2 > 0$. Then there exists a Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d)$ such that the following conditions hold.*

$$f \text{ is nowhere differentiable in } (F \setminus A) \setminus \left(\bigcup_{i=1}^{\infty} \partial U_i \right), \tag{4.32}$$

$$\|f - h\|_{\infty} \leq \epsilon_1 \text{ and } \mathrm{Lip}(f) \leq \mathrm{Lip}(h) + \epsilon_2, \tag{4.33}$$

$$f(y) = h(y) \text{ whenever } y \in O \setminus \bigcup_{i=1}^{\infty} \mathrm{Int}(U_i). \tag{4.34}$$

2. *Suppose $f \in \mathrm{Lip}(\mathbb{R}^d)$ is any function satisfying the condition (4.34) and $x \in A$. Then f is differentiable at x if and only if h is differentiable at x.*

Proof. Let us first verify statement 1. Suppose $F \setminus A$ is a non-universal differentiability set and let $\epsilon_1, \epsilon_2 > 0$. Note that the d-boxes $\{U_i\}_{i=1}^{\infty}$, the function h and $\sigma_j = \epsilon_j/2$, for $j = 1, 2$, satisfy the conditions of Lemma 4.3.10. Let $\widehat{h} \in \mathrm{Lip}(\mathbb{R}^d)$ be the function given by conclusion of Lemma 4.3.10.

The conditions of Lemma 4.3.6 are satisfied for $E = F \setminus A$, the collection $\{U_i\}_{i=1}^{\infty}$ and $\eta_j = \epsilon_j/2$ for $j = 1, 2$. Let $g \in \mathrm{Lip}(\mathbb{R}^d)$ be given by the conclusion of Lemma 4.3.6. We define $f = \widehat{h} + g$.

Statement (4.34) is implied by (4.25) and (4.16). From (4.23) and (4.14) we have that

$$\|f-h\|_\infty \le \left\|\widehat{h}-h\right\|_\infty + \|g\|_\infty \le \epsilon_1, \text{ and}$$
$$\mathrm{Lip}(f) \le \mathrm{Lip}(\widehat{h}) + \mathrm{Lip}(g) \le \mathrm{Lip}(h) + \epsilon_2.$$

This verifies (4.33).

We now prove (4.32). Fix $y \in (F \setminus A) \setminus (\bigcup_{i=1}^\infty \partial U_i)$ and note, using (4.31), that $y \in (F \setminus A) \cap \mathrm{Int}(U_i)$ for some i. Then by (4.24) and (4.15), we have that $f = \widehat{h} + g$ is not differentiable at y.

Finally, we prove statement 2. Suppose $f \in \mathrm{Lip}(\mathbb{R}^d)$ satisfies (4.34) and $x \in A$. Let $e \in S^{d-1}$ be any direction. We show that the directional derivative $f'(x,e)$ exists if and only if $h'(x,e)$ exists and $f'(x,e) = h'(x,e)$. Since $e \in S^{d-1}$ is arbitrary, this suffices.

Let $\{U_{i_k}\}_{k=1}^N$, where $N \in \mathbb{N} \cup \{\infty\}$, be the collection of all d-boxes U_i which intersect the line $x + [0,\infty)e$. Since $x \in A \subseteq F$ and (4.31) holds, we have that $x \notin \bigcup_{i=1}^\infty U_i$. For each $k \ge 1$ we write

$$U_{i_k} \cap \Big(x + (0,\infty)e\Big) = [x + t_{k,1}e, x + t_{k,2}e]$$

where $t_{k,1} \le t_{k,2}$ are strictly positive real numbers. Choose $\Delta > 0$ small enough so that $B(x,\Delta) \subseteq O$. We note that the following conditions hold:

$$x + t_{k,j}e \in \partial U_{i_k} \text{ for } j = 1,2 \text{ and } k \ge 1, \tag{4.35}$$

$$x + te \in O \setminus \left(\bigcup_{i=1}^\infty \mathrm{Int}(U_i)\right) \qquad \forall t \in \left([0,\Delta) \setminus \left(\bigcup_{k=1}^N (t_{k,1}, t_{k,2})\right)\right) \cup \left(\bigcup_{k=1}^N \{t_{k,1}, t_{k,2}\}\right). \tag{4.36}$$

Let us verify that the conditions of Lemma 4.3.11 hold for x, e, f, h, Δ and the intervals $\{(t_{k,1}, t_{k,2})\}_{k=1}^N$.

If $N = \infty$, then using (4.35), we get that

$$0 \leq \frac{t_{k,2} - t_{k,1}}{t_{k,1}} \leq \frac{\operatorname{diam}(U_{i_k})}{\operatorname{dist}(U_{i_k}, A)} \to 0 \text{ as } k \to \infty.$$

This proves (4.27). Now, fix

$$t \in \left([0, \Delta) \setminus \left(\bigcup_{k=1}^{N} (t_{k,1}, t_{k,2}) \right) \right) \cup \left(\bigcup_{k=1}^{N} \{t_{k,1}, t_{k,2}\} \right).$$

Then by (4.36) and (4.34), we have $f(x + te) = h(x + te)$. This verifies (4.28).

Now, by Lemma 4.3.11, we have that the directional derivative $f'(x, e)$ exists if and only if $h'(x, e)$ exists and $h'(x, e) = f'(x, e)$. □

Lemma 4.3.13. *Let $F \subseteq \mathbb{R}^d$ be a universal differentiability set and suppose that A is a closed subset of F. Then either A or $F \setminus A$ is a universal differentiability set.*

Proof. Suppose the contrary for some $A \subseteq F \subseteq \mathbb{R}^d$. We deduce that F is a non-universal differentiability set.

Fix $h \in \operatorname{Lip}(\mathbb{R}^d)$ such that h is nowhere differentiable on A. Applying Lemma 4.2.1 and 4.2.2 with $K = A$, we conclude that there exists a collection $\{U_i\}_{i=1}^{\infty}$ of d-boxes with pairwise disjoint interiors such that (4.31) holds. The conditions of Lemma 4.3.12 are satisfied for F, A, $O = \mathbb{R}^d$, h and $\{U_i\}_{i=1}^{\infty}$. Further, the hypothesis of Lemma 4.3.12, part 1 is satisfied for F, A and arbitrary $\epsilon_1, \epsilon_2 > 0$. Let $f \in \operatorname{Lip}(\mathbb{R}^d)$ be the function given by the conclusion of Lemma 4.3.12, part 1.

Applying Lemma 4.3.12, part 2, we get that the differentiability of f and h coincides at all points of A. Thus f is nowhere differentiable on A. Further, f is nowhere differentiable in $(F \setminus A) \setminus \bigcup_{i=1}^{\infty} \partial U_i$ by (4.32). Hence $F \setminus \bigcup_{i=1}^{\infty} \partial U_i$ is a non-universal differentiability set. By Lemma 4.3.4, we conclude that F is a non-universal differentiability set. □

Before proceeding to the proof of Theorem 4.3.1, we take a short detour to discuss the notions of homogeneity and the kernel of a universal differentiability set. The kernel,

$\ker(S)$, of a universal differentiability set S is defined similarly to the kernel of a non-σ-porous set A in [36, Definition 3.2].

Definition 4.3.14. *(i) We will say that a universal differentiability set S is homogeneous if for every $x \in S$ and every $\epsilon > 0$, the set $B(x, \epsilon) \cap S$ is a universal differentiability set.*

(ii) Given a set $S \subseteq \mathbb{R}^d$, we let

$$\ker(S) = S \setminus \{x \in S : \exists \epsilon > 0 \text{ such that } B(x, \epsilon) \cap S \text{ is a non-UDS}\}.$$

The following theorem shows that the kernel of a universal differentiability set can be thought of as the core of the set. We remark that universal differentiability sets behave similarly to non-σ-porous sets in this respect - see [36, Lemma 3.4].

Theorem 4.3.15. *Suppose $F \subseteq \mathbb{R}^d$ is a universal differentiability set. Then,*

(i) $\ker(F) \subseteq F$ *is a universal differentiability set.*

(ii) $\ker(F)$ *is homogeneous and* $F \setminus \ker(F)$ *is a non-universal differentiability set. Hence F can be decomposed as the union of a homogeneous universal differentiability set and a non-universal differentiability set.*

Proof. Note that $\ker(F)$ is a closed subset of F and $F \setminus \ker(F)$ is a non-universal differentiability set by Corollary 4.3.7. Therefore, we may apply Lemma 4.3.13 with $A = \ker(F)$ to deduce that $\ker(F)$ is a universal differentiability set. This proves (i). For (ii), it only remains to check that $\ker(F)$ is homogeneous. Let $x \in \ker(F)$ and $\epsilon > 0$. Then we observe that

$$B(x, \epsilon) \cap \ker(F) = \ker(B(x, \epsilon) \cap F),$$

and the latter set is a universal differentiability set by part (i), using that $x \in \ker(F)$. $\square$

It is now upon us to prove Theorem 4.3.1. The statement is repeated here for the reader's convenience.

Theorem. *Suppose $E = \bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{R}^d$, where each A_i is a non-universal differentiability set, and is closed as a subset of E. Then E is a non-universal differentiability set.*

Proof. A simple induction argument, using Lemma 4.3.13, proves that $\bigcup_{i=1}^{k} A_i$ is a non-universal differentiability set for each $k \geq 1$. Therefore, we may assume that $A_k \subseteq A_{k+1}$ for each $k \geq 1$. From Corollary 4.3.7 it suffices to show that every bounded subset of E is a non-universal differentiability set. Hence, we may assume E is bounded. Any bounded set can be mapped into $\mathrm{Int}([0,1]^d)$ by a C^1-diffeomorphism. Therefore, in view of Lemma 4.3.8, we may assume that $E \subseteq \mathrm{Int}([0,1]^d)$.

We begin the construction by using Lemma 4.2.1 and Lemma 4.2.2 to find a collection of d-boxes $\left\{U_i^{(1)}\right\}_{i=1}^{\infty}$ with pairwise disjoint interiors such that $E \cap \bigcup_{i=1}^{\infty} U_i^{(1)} = E \setminus A_1$ and $\mathrm{diam}(U_i^{(1)})/\mathrm{dist}(U_i^{(1)}, A_1) \to 0$ as $i \to \infty$. Next, choose a function $f_1 \in \mathrm{Lip}(\mathbb{R}^d)$ such that f_1 is nowhere differentiable on A_1.

Suppose $n \geq 1$, the function $f_n \in \mathrm{Lip}(\mathbb{R}^d)$ and the collections $\left\{U_i^{(l)}\right\}_{i=1}^{\infty}$ of d-boxes with pairwise disjoint interiors are defined for $l = 1, \ldots, n$ such that

$$f_n \text{ is nowhere differentiable on the set } A_n \setminus \left(\bigcup_{l=1}^{n-1} \bigcup_{i=1}^{\infty} \partial U_i^{(l)}\right), \tag{4.37}$$

$$E \cap \bigcup_{i=1}^{\infty} U_i^{(n)} = E \setminus A_n \text{ and } \mathrm{diam}(U_i^{(n)})/\mathrm{dist}(U_i^{(n)}, A_n) \to 0 \text{ as } i \to \infty.$$

Now, let the function $f_{n+1} \in \mathrm{Lip}(\mathbb{R}^d)$ be given by the conclusion of Lemma 4.3.12, part 1, when we take $A = A_n$, $F = A_{n+1}$, $O = \mathbb{R}^d$, $h = f_n$, $U_i = U_i^{(n)}$ for $i \geq 1$ and $\epsilon_1 = \epsilon_2 = 2^{-(n+1)}$. By Lemma 4.3.12, part 1, the function f_{n+1} is nowhere differentiable in $(A_{n+1} \setminus A_n) \setminus \bigcup_{i=1}^{\infty} \partial U_i^{(n)}$. From part 2 of Lemma 4.3.12 we have that the differentiability of f_{n+1} and f_n coincide at all points of A_n. Hence, using (4.37), f_{n+1} is nowhere differentiable in the set $A_{n+1} \setminus \left(\bigcup_{l=1}^{n} \bigcup_{i=1}^{\infty} \partial U_i^{(l)}\right)$.

Let the collection of d-boxes $\left\{U_i^{(n+1)}\right\}_{i=1}^{\infty}$ be given by the conclusion of Lemma 4.2.2 when we take $F = E$, $K = A_{n+1}$ and $U_i = U_i^{(n)}$ for each i. This ensures that we have $E \cap \bigcup_{i=1}^{\infty} U_i^{(n+1)} = E \setminus A_{n+1}$ and $\mathrm{diam}(U_i^{(n+1)})/\mathrm{dist}(U_i^{(n+1)}, A_{n+1}) \to 0$ as $i \to \infty$.

We have defined, for each integer $k \geq 1$, a function $f_k \in \mathrm{Lip}(\mathbb{R}^d)$ and a collection

of d-boxes $\left\{U_i^{(k)}\right\}_{i=1}^{\infty}$ with pairwise disjoint interiors. By construction, the following conditions hold for each $k \geq 2$:

(1_k) f_k is nowhere differentiable on $A_k \setminus \left(\bigcup_{l=1}^{k-1} \bigcup_{i=1}^{\infty} \partial U_i^{(l)}\right)$,

(2_k) $\|f_k - f_{k-1}\|_\infty \leq 2^{-k}$ and $\mathrm{Lip}(f_k) \leq \mathrm{Lip}(f_{k-1}) + 2^{-k}$,

(3_k) $E \cap \bigcup_{i=1}^{\infty} U_i^{(k)} = E \setminus A_k$,

(4_k) $f_k(y) = f_{k-1}(y)$ whenever $y \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \mathrm{Int}(U_i^{(k-1)})\right)$,

(5_k) For each index i, there exists an index j such that $U_i^{(k)} \subseteq U_j^{(k-1)}$,

(6_k) $\dfrac{\mathrm{diam}(U_i^{(k)})}{\mathrm{dist}(U_i^{(k)}, A_k)} \to 0$ as $i \to \infty$.

For the sake of future reference we point out that

$$f_m(y) = f_n(y) \text{ whenever } m \geq n \text{ and } y \in \mathbb{R}^d \setminus \left(\bigcup_{i=1}^{\infty} \mathrm{Int}(U_i^{(n)})\right). \tag{4.38}$$

This follows from (4_k) and (5_k).

For a function $g \in \mathrm{Lip}(\mathbb{R}^d)$, let us write $\widetilde{g}$ for the restriction of g to the unit d-cube $[0,1]^d$. Since (2_k) holds for each $k \geq 2$, we have that the sequence $(\widetilde{f}_k)_{k=1}^{\infty}$ is a Cauchy sequence in the Banach space of all continuous functions on $[0,1]^d$. Hence, there exists a continuous function $\widetilde{f} : [0,1]^d \to \mathbb{R}$ such that $\left\|\widetilde{f}_n - \widetilde{f}\right\|_\infty \to 0$ as $n \to \infty$. We now show that $\widetilde{f}$ is Lipschitz: Fix $x, y \in [0,1]^d$ with $x \neq y$. Using (2_k) we have that

$$\begin{aligned}\left|\widetilde{f}(y) - \widetilde{f}(x)\right| &= \left|\lim_{n\to\infty} f_n(y) - \lim_{n\to\infty} f_n(x)\right| = \lim_{n\to\infty} |f_n(y) - f_n(x)| \\ &\leq \lim_{n\to\infty} \left(\mathrm{Lip}(f_1) + \sum_{k=2}^{n} 2^{-k}\right) \|y - x\| \leq (\mathrm{Lip}(f_1) + 1)\, \|y - x\|.\end{aligned}$$

Hence $\widetilde{f}$ is Lipschitz, with $\mathrm{Lip}(\widetilde{f}) \leq \mathrm{Lip}(f_1) + 1$. Let $f \in \mathrm{Lip}(\mathbb{R}^d)$ be an extension of $\widetilde{f}$.

Using (4.38) and the fact that $f_m \to f$ pointwise on $[0,1]^d$, we deduce that the function

f satisfies

$$f(y) = f_n(y) \text{ whenever } y \in \mathrm{Int}([0,1]^d) \setminus \left(\bigcup_{i=1}^{\infty} \mathrm{Int}({U_i}^{(n)}) \right). \tag{4.39}$$

We are now ready to prove that E is a non-universal differentiability set. In view of Lemma 4.3.4, it is sufficient to show that

$$E' = E \setminus \left(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \partial U_i^{(k)} \right)$$

is a non-universal differentiability set. We will prove that f is nowhere differentiable on E'.

Fix $x \in E'$ and pick n such that $x \in A_n$. From (3_n) and (6_n), we get that the conditions of Lemma 4.3.12 are satisfied for $F = E$, $A = A_n$, $O = \mathrm{Int}([0,1]^d)$, $h = f_n$ and $U_i = {U_i}^{(n)}$ for all i. Moreover, using (4.39), we have that the hypothesis of Lemma 4.3.12, part 2 is satisfied for the function $f \in \mathrm{Lip}(\mathbb{R}^d)$ and the point $x \in A = A_n$. Therefore, by Lemma 4.3.12, part 2, the function f is differentiable at x if and only if f_n is differentiable at x. By (1_n), the proof is complete. □

4.4 Differentiability inside sets of positive measure.

In this section we focus on the following question: Does every subset of $\mathbb{R}^d$ with positive Lebesgue measure contain a universal differentiability set with Lebesgue measure zero? This question was asked by Giles Godefroy after a talk given by Olga Maleva at the 2012 conference 'Geometry of Banach spaces' in CIRM, Luminy. We answer this question positively for sets containing a product set of positive measure.

Theorem 4.4.1. *Suppose $P_1, P_2, \ldots, P_d \subseteq \mathbb{R}$ are sets of positive one-dimensional Lebesgue measure. Then $P_1 \times \ldots \times P_d$ contains a compact universal differentiability set with upper Minkowski dimension one.*

Proof. We may assume that each set P_i is closed. For $k = 0, 1, \ldots, d$, let Π_k be the statement that $P_1 \times P_2 \times \ldots \times P_k \times \mathbb{R}^{d-k}$ contains a compact universal differentiability set

C_k with upper Minkowski dimension one. Note that Π_0 is contained in Theorem 3.4.4. Suppose now that $0 < k \leq d$ and that the statement Π_{k-1} holds. Let us prove the statement Π_k and thus, by induction, Theorem 4.4.1.

Let $\{r_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathbb{R}$ and consider the set

$$F_k = \bigcup_{n=1}^{\infty} (\mathbb{R}^{k-1} \times (P_k + r_n) \times \mathbb{R}^{d-k}).$$

Writing $F_{k,n} = \mathbb{R}^{k-1} \times (P_k + r_n) \times \mathbb{R}^{d-k}$ for each n, we have $F_k = \bigcup_{n=1}^{\infty} F_{k,n}$ and each set $F_{k,n}$ is closed. Further, observe that $p_k(\mathbb{R}^d \setminus F_k)$ is a subset of $\mathbb{R}$ with one-dimensional Lebesgue measure zero, where $p_k : \mathbb{R}^d \to \mathbb{R}$ denotes the kth co-ordinate projection map. We can write

$$C_{k-1} = (C_{k-1} \cap F_k) \cup (C_{k-1} \cap (\mathbb{R}^d \setminus F_k)).$$

Since $\mathbb{R}^d \setminus F_k$ projects to a set of one-dimensional Lebesgue measure zero, we may apply Lemma 3.2.1 to conclude that $C_{k-1} \cap F_k$ is a universal differentiability set. Next, using Theorem 4.3.1, we deduce that there exists n such that $C_{k-1} \cap F_{k,n}$ is a universal differentiability set.

Define $\lambda_n = -r_n e_k = (\underbrace{0, 0, \ldots, 0}_{k-1 \text{ times}}, -r_n, \underbrace{0, \ldots, 0}_{d-k \text{ times}}) \in \mathbb{R}^d$. Setting

$$C_k = (C_{k-1} + \lambda_n) \cap (\mathbb{R}^{k-1} \times P_k \times \mathbb{R}^{d-k}),$$

we observe that

$$C_k = (C_{k-1} \cap F_{k,n}) + \lambda_n.$$

Therefore, by Lemma 4.3.8, the set C_k is a universal differentiability set. Note that $(C_{k-1} + \lambda_n) \subseteq P_1 \times \ldots \times P_{k-1} \times \mathbb{R}^{d-k+1}$. Hence, $C_k \subseteq P_1 \times \ldots \times P_k \times \mathbb{R}^{d-k}$ and the proof of statement Π_k is complete. $\square$

Chapter 5

Properties of absolutely continuous mappings.

5.1 Introduction.

The research presented in this chapter forms part of a joint paper [18]. The investigation was started by Randrianantoanina and Xu, with the present author joining at a later stage. We focus on the part of this research contributed by the present author, namely Theorem 5.2.3 and Theorem 5.2.4. The other results that we mention in Section 5.2 are due to Randrianantoanina and Xu.

Let $I \subseteq \mathbb{R}$ be an interval. Recall, a function $f : I \to \mathbb{R}$ is called absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{k=1}^{K} |f(a_k) - f(b_k)| < \epsilon$ whenever $\{[a_k, b_k]\}_{k=1}^{K}$ is a finite collection of pairwise disjoint intervals inside I with $\sum_{k=1}^{K} \mathcal{L}^1([a_k, b_k]) < \delta$.

It is easy to verify that every Lipschitz function on I is absolutely continuous. A noteworthy property of absolutely continuous functions is that they are differentiable almost everywhere. To prove this, one first shows that a given absolutely continuous function can be expressed as a sum of two monotone functions, [31]. It then suffices to prove that all monotone functions are differentiable almost everywhere. The proof of this fact uses the Vitali Covering Theorem.

Theorem. *(Vitali Covering Theorem [27, Theorem 2.2])*

Let A be Lebesgue measurable subset of $\mathbb{R}^d$ and suppose that $\mathcal{V}$ is a family of closed balls in $\mathbb{R}^d$ such that for every point $x \in A$ and every $\delta > 0$ there exists a ball $V \in \mathcal{V}$ with $x \in V$ and $0 < \operatorname{diam}(V) \leq \delta$. Then there exists a collection $(V_i)_{i=1}^{\infty}$ of pairwise disjoint balls in $\mathcal{V}$ such that

$$\mathcal{L}^d\left(A \setminus \bigcup_{i=1}^{\infty} V_i\right) = 0,$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure.

The balls in the above statement may be taken with respect to any norm on $\mathbb{R}^d$. A family $\mathcal{V}$ satisfying the conditions of the above theorem for a set $A \subseteq \mathbb{R}^d$ is referred to as a Vitali cover of A. We will apply this covering theorem in the proof of a positive differentiability result for functions of a generalised form of absolute continuity (see Theorem 5.2.3).

A second property of absolutely continuous functions on $\mathbb{R}$ that will be of interest to us is the Luzin (N) property. Let us define this concept for more general mappings:

Definition. *A measurable mapping $f : \mathbb{R}^d \to \mathbb{R}^l$, where $l \geq d$, is said to satisfy the Luzin (N) condition if for every measurable set $N \subseteq \mathbb{R}^d$ with $\mathcal{L}^d(N) = 0$, we have that $\mathcal{H}^d(f(N)) = 0$, writing $\mathcal{H}^d$ for the d-dimensional Hausdorff measure on $\mathbb{R}^l$, see [25].*

There have been various different generalisations of the notions of absolute continuity for mappings from $\mathbb{R}^d$ to $\mathbb{R}^l$ with $d \geq 2$, $l \geq 1$. We discuss some examples which have lead to the variant of absolute continuity that we will study.

In [26], Malý introduces the class of d-absolutely continuous mappings:

Definition. *A mapping $f : \mathbb{R}^d \to \mathbb{R}^l$ is called d-absolutely continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\sum_{k=1}^{K} osc(f, B_k)^d < \epsilon$, whenever $\{B_k\}_{k=1}^{K}$ is a finite collection of pairwise disjoint, closed balls with $\sum_{k=1}^{K} \mathcal{L}^d(B_k) < \delta$.*

Here, $\operatorname{osc}(f, B_k)$ denotes the diameter of the set $f(B_k)$. Malý [26] proves that all d-absolutely continuous mappings $f : \mathbb{R}^d \to \mathbb{R}^l$ are differentiable almost everywhere and satisfy the Luzin (N) condition when $l \geq d$.

Instead of using balls in the definition of d-absolutely continuous functions, one can use sets from a general family $\mathcal{F}$. The resulting class of functions is denoted by $\mathcal{F}$-AC. For example, in $\mathbb{R}^2$, $\mathcal{F}$ could be the collection of all squares (i.e. balls with respect to the l_∞ norm). Csörnyei [10] proves that the classes $\mathcal{F}$-AC depend on the family $\mathcal{F}$. Writing $\mathcal{D}$ for the collection of all Euclidean balls in $\mathbb{R}^2$ and $\mathcal{Q}$ for the collection of squares, [10] presents a construction of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which belongs to $\mathcal{D}$-AC but not to $\mathcal{Q}$-AC.

It would therefore be desirable to define a class of mappings of generalised absolute continuity which is independent of the shape of the sets in the family $\mathcal{F}$. Such a concept is given by Hencl in [21]. Hencl's absolute continuity is the following:

Definition. *A mapping $f : \mathbb{R}^d \to \mathbb{R}^l$ is said to belong to the class $AC_H(\mathbb{R}^d, \mathbb{R}^l)$ if there exists $\lambda \in (0,1)$ so that for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{k=1}^{K} osc(f, B(\mathbf{x}_k, \lambda r_k))^d < \epsilon,$$

whenever $\{B(\mathbf{x}_k, r_k)\}_{k=1}^{K}$ is a finite collection of pairwise disjoint, closed balls in $\mathbb{R}^d$.

Here, $\mathrm{osc}(f, B)$ denotes the diameter of the set $f(B)$, whilst $B(\mathbf{x}, r)$ denotes the closed ball with centre $\mathbf{x} \in \mathbb{R}^d$ and radius $r \geq 0$. The balls can be taken with respect to any norm on $\mathbb{R}^d$, [21, Theorem 4.2, Remark 4.3]. Accordingly, Hencl's class is larger than the class of Malý. Nevertheless, Hencl's mappings still retain the traditional properties associated with absolute continuity. Hencl [21] proves that any $f \in AC_H(\mathbb{R}^d, \mathbb{R}^l)$ is differentiable almost everywhere and has the Luzin (N) property if $l \geq d$.

In the present chapter we will study a form of generalised absolute continuity given in [18]. Before stating the definition, we need to establish some notation. The family of sets $\mathcal{F}$ that we will use to define absolute continuity will be a collection of generalised

intervals in $\mathbb{R}^d$. Given points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{b} = \mathbf{a} + t(\underbrace{1, 1, \ldots, 1}_{d \text{ times}})$ for some $t > 0$, we let

$$[\mathbf{a}, \mathbf{b}] := \left\{\mathbf{x} \in \mathbb{R}^d : a_i \leq x_i \leq b_i \text{ for all } 1 \leq i \leq d\right\} = \prod_{i=1}^{d} [a_i, b_i]. \tag{5.1}$$

We shall refer to sets of the form (5.1) as 1-regular intervals. Note that these 1-regular intervals belong to the class of 'cubes' from Chapter 3 and also to the class of 'd-boxes' from Chapter 4. In addition, we highlight that all 1-regular intervals are balls with respect to the l_∞-norm. For a 1-regular interval $[\mathbf{a}, \mathbf{b}] \subseteq \mathbb{R}^d$ and $\lambda \in (0, 1)$, we let ${}^{\lambda}[\mathbf{a}, \mathbf{b}]$ denote the interval with the same centre $(\mathbf{a} + \mathbf{b})/2$ and sides of length $\lambda(b_i - a_i)$. Further, if $f : \mathbb{R}^d \to \mathbb{R}^l$ is a mapping, we let $|f([\mathbf{a}, \mathbf{b}])|$ denote the Euclidean norm $\|f(\mathbf{b}) - f(\mathbf{a})\|$.

Definition 5.1.1. *A mapping $f : \mathbb{R}^d \to \mathbb{R}^l$ is said to belong to the class 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=1}^{K} \mathcal{L}^d([\mathbf{a}_k, \mathbf{b}_k]) < \delta \Rightarrow \sum_{k=1}^{K} |f([\mathbf{a}_k, \mathbf{b}_k])|^d < \epsilon,$$

whenever $\{[\mathbf{a}_k, \mathbf{b}_k]\}_{k=1}^{K}$ is a finite collection of pairwise disjoint, 1-regular intervals in $\mathbb{R}^d$.

We say that f belongs to the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ if for every $\lambda \in (0, 1)$, the following condition holds: For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{K} \mathcal{L}^d([\mathbf{a}_k, \mathbf{b}_k]) < \delta \Rightarrow \sum_{k=1}^{K} \left|f({}^{\lambda}[\mathbf{a}_k, \mathbf{b}_k])\right|^d < \epsilon,$$

whenever $\{[\mathbf{a}_k, \mathbf{b}_k]\}_{k=1}^{K}$ is a finite collection of pairwise disjoint, 1-regular intervals in $\mathbb{R}^d$.

The above definitions are analogues of concepts due to Bongiorno (see [7, Definition 2] and [8, Definition 4]). It is straight-forward to verify that 1-$AC(\mathbb{R}^d, \mathbb{R}^l) \subseteq$ 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$. The sub-index H is attached to the latter class because the role of the parameter λ is similar in Hencl's definition of absolute continuity. Moreover, Randrianantoanina and Xu prove in [18] that an equivalent definition of the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ is obtained if we

instead ask that the ϵ-δ condition above is satisfied for at least one $\lambda \in (0,1)$ rather than for all.

In Theorem 5.2.3, we prove that all mappings in the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ are differentiable almost everywhere in the direction $(1, 1, \ldots, 1) \in \mathbb{R}^d$. Together with a result of Randrianantoanina and Xu, this provides a complete account of the inherent differentiability of such functions. We observe that mappings in the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ do not necessarily have the nice properties of mappings in the previously mentioned classes of absolute continuity. In particular, mappings in 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$ need not be differentiable almost everywhere.

Our proof of Theorem 5.2.3 contains an application of the Stepanov Theorem (see, for example, [4]), which we state below.

Theorem. *(Stepanov Theorem)*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function. Then f is differentiable at almost every point $x \in \mathbb{R}^d$ for which the quantity

$$\mathrm{Lip}(f, x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\|y - x\|}$$

is finite.

Thus, the Stepanov Theorem is a stronger statement than Rademacher's Theorem for Lipschitz functions.

To conclude the present chapter, we restrict our attention to the class 1-$AC_D(\mathbb{R}^d, \mathbb{R}^l)$ of mappings from 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ which are differentiable almost everywhere. However, we show that even this more exclusive class admits wild mappings: We give an example of a mapping in 1-$AC_D(\mathbb{R}^d, \mathbb{R}^l)$ which does not have the Luzin (N) property.

5.2 Mappings of the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$.

From the definition of the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$, it is apparent that their behaviour along lines in the direction $(1, 1, \ldots, 1) \in \mathbb{R}^d$ is somewhat significant. It will therefore be convenient to work with a non-standard basis of $\mathbb{R}^d$: We let $\mathbf{x}_d = (1, 1, \ldots, 1) \in \mathbb{R}^d$ and

extend to an orthogonal basis $\mathbf{x}_1, \ldots, \mathbf{x}_d$ of $\mathbb{R}^d$.

We begin by quoting Theorem 5.4 from [18]. This result is due to Randrianantoanina and Xu and is useful for constructing functions which belong to the class 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$. The theorem is stated for the case $d = 2$ and $l = 1$, however it is clear that the proof given in [18] can be generalised for arbitrary dimensions $d \geq 2$, $l \geq 1$. Since $d = 2$ we may assume, for the following statement, that $\mathbf{x}_1 = (-1, 1)$ and $\mathbf{x}_2 = (1, 1)$.

Theorem 5.2.1. *([18, Theorem 5.1]) Let $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be nonzero measurable functions with support contained in an interval of the form $[-c, c]$ for some $c > 0$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(s\mathbf{x}_1 + t\mathbf{x}_2) = h(s)g(t). \tag{5.2}$$

(Alternatively, in the standard basis of $\mathbb{R}^2$, $f(x, y) = h(\frac{y-x}{2})g(\frac{y+x}{2})$.)

(a) If h is bounded and g is Lipschitz, then f is in 1-$AC(\mathbb{R}^2, \mathbb{R})$.

(b) If $f \in$ 1-$AC_H(\mathbb{R}^2, \mathbb{R})$, then g is Lipschitz.

Since, for part (a), the function h is only subject to a rather weak condition, we can use Theorem 5.2.1 to find many functions in the class 1-$AC_H(\mathbb{R}^2, \mathbb{R})$ which have wild behaviour and violate several properties of the absolutely continuous functions of Malý and Hencl.

Let us summarise some examples given by [18]: We may take h to be bounded but discontinuous everywhere to obtain a function in 1-$AC(\mathbb{R}^2, \mathbb{R})$ which is bounded and discontinuous on its support [18, Corollary 5.3]. Even if h is unbounded, we can still find functions in the class 1-$AC(\mathbb{R}^2, \mathbb{R})$ of the form given by (5.2). If we take a constant $g(t) = 1$ for $t \in [-1, 1,]$ it is straightforward to verify that the function f, given by (5.2), belongs to the class 1-$AC(\mathbb{R}^2, \mathbb{R})$, regardless of the choice of h. Note that such a function f is constant on lines in the direction $(1, 1)$ so that $|f([\mathbf{a}, \mathbf{b}])| = 0$ whenever $[\mathbf{a}, \mathbf{b}]$ is a 1-regular interval. Thus, 1-$AC(\mathbb{R}^2, \mathbb{R})$ contains a function which is supported on a compact set and unbounded [18, Corollary 5.4].

We presently state a result of [18] which nicely complements our positive differentiability theorem (Theorem 5.2.3).

Proposition 5.2.2. *([18, Corollary 5.6]) There exists a continuous function f in the class 1-$AC(\mathbb{R}^2, \mathbb{R})$ such that for every $\mathbf{p}$ in the support of f and every direction $\mathbf{v} \neq (1,1)$ the directional derivative $f'(\mathbf{p}, \mathbf{v})$ does not exist. In particular f is not differentiable anywhere on its support.*

Proposition 5.2.2 is proved by defining a function f of the form (5.2) where h is a variant of the Takagi function, $g(t) = 1 - 2|t|$ and the support of both functions is $[-1/2, 1, 2]$. The Takagi function is a continuous function which is nowhere differentiable, see [33].

The next theorem is the main result of the chapter and the only positive result we found for mappings in the class 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$. From this point forward, we work with arbitrary dimensions $d \geq 2$, $l \geq 1$. We also emphasise that the following theorem holds for the larger class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ and not just for 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$.

Theorem 5.2.3. *Suppose $d \geq 2$ and $l \geq 1$ are integers. Then every mapping f in the class 1-$AC_H(\mathbb{R}^d, \mathbb{R}^l)$ is differentiable a.e. in the direction $\mathbf{x}_d = (1, 1, \ldots, 1) \in \mathbb{R}^d$.*

Proof. In what follows we will identify $\mathbb{R}^{d-1}$ with the $d-1$ dimensional subspace of $\mathbb{R}^d$ spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_{d-1}$, via the correspondence

$$\mathbf{s} \leftrightarrow s_1\mathbf{x}_1 + \ldots + s_{d-1}\mathbf{x}_{d-1}.$$

For each $\mathbf{u} \in \mathbb{R}^d$, we define a mapping $f_{\mathbf{u}} : \mathbb{R} \to \mathbb{R}^l$ by $f_{\mathbf{u}}(t) = f(\mathbf{u} + t\mathbf{x}_n)$. We will show that the set

$$E := \left\{\mathbf{u} \in \mathbb{R}^d : \operatorname{Lip}(f_{\mathbf{u}}, 0) = \infty\right\}$$

has d-dimensional Lebesgue measure zero.

We claim that this suffices: Indeed, if E has measure zero then, using Fubini's Theorem, we get that for almost every $\mathbf{s} \in \mathbb{R}^{d-1}$, $\operatorname{Lip}(f_{\mathbf{s}+t\mathbf{x}_n}, 0) < \infty$ for almost every $t \in \mathbb{R}$.

Observe that $\mathrm{Lip}(f_{\mathbf{s}}, t) = \mathrm{Lip}(f_{\mathbf{s}+t\mathbf{x}_n}, 0)$ for all $\mathbf{s} \in \mathbb{R}^{d-1}$ and $t \in \mathbb{R}$. It follows that $\mathrm{Lip}(f_{\mathbf{s}}, t) < \infty$ for almost every $\mathbf{s} \in \mathbb{R}^{d-1}$ and almost every $t \in \mathbb{R}$. Now, applying the Stepanov Theorem, we conclude that for almost every $\mathbf{s} \in \mathbb{R}^{d-1}$, $f_{\mathbf{s}}$ is differentiable almost everywhere (in $\mathbb{R}$). Clearly f is differentiable at $\mathbf{s} + t\mathbf{x}_n$ in the direction $\mathbf{x}_n$ if and only if $f_{\mathbf{s}}$ is differentiable at t. Hence f is differentiable in the direction $\mathbf{x}_n$ almost everywhere.

We now prove that E has measure zero. Note that E is measurable. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\sum_{k=1}^{K} \mathcal{L}^d([\mathbf{a}_k, \mathbf{b}_k]) < \delta \Rightarrow \sum_{k=1}^{K} \left\| f\left(\frac{\mathbf{a}_k + 3\mathbf{b}_k}{4}\right) - f\left(\frac{3\mathbf{a}_k + \mathbf{b}_k}{4}\right) \right\|^d < \varepsilon, \tag{5.3}$$

whenever $\{[\mathbf{a}_k, \mathbf{b}_k]\}_{k=1}^{K}$ is a finite collection of pairwise disjoint, 1-regular intervals in $\mathbb{R}^d$. For the above, we apply Definition 5.1.1 with $\lambda = 1/2$. Let $\{A_m\}_{m=1}^{\infty}$ be a countable collection of 1-regular intervals with pairwise disjoint, non-empty interiors satisfying the conditions $\mathbb{R}^d = \bigcup_{m=1}^{\infty} A_m$ and $\mathcal{L}^n(A_m) < \delta$ for all m. For each $m \in \mathbb{N}$, we define a collection of 1-regular intervals $\mathcal{V}_m$ by

$$\mathcal{V}_m = \left\{ [\mathbf{a}, \mathbf{b}] \subseteq A_m : \frac{\left\| f\left(\frac{\mathbf{a}+3\mathbf{b}}{4}\right) - f\left(\frac{3\mathbf{a}+\mathbf{b}}{4}\right) \right\|}{\frac{\|\mathbf{b}-\mathbf{a}\|}{2}} \geq m \right\}. \tag{5.4}$$

Note that $\mathcal{V}_m$ is a Vitali cover of $E \cap \mathrm{Int}(A_m)$. Hence, by the Vitali Covering Theorem, we can find a collection $\{I_k^{(m)} = [\mathbf{a}_k^{(m)}, \mathbf{b}_k^{(m)}]\}_{k=1}^{\infty}$ of pairwise disjoint intervals from $\mathcal{V}_m$ such that

$$\mathcal{L}^d\left((E \cap \mathrm{Int}(A_m)) \setminus \bigcup_{k=1}^{\infty} I_k^{(m)} \right) = 0.$$

Choose an integer $N_m \geq 1$ such that $\sum_{k=1}^{N_m} \mathcal{L}^d(I_k^{(m)}) > \mathcal{L}^d(E \cap A_m) - \frac{\varepsilon}{2^m}$. Since the intervals $I_k^{(m)}$ are pairwise disjoint and contained in A_m we have that $\sum_{k=1}^{N_m} \mathcal{L}^d(I_k^{(m)}) < \mathcal{L}^d(A_m) < \delta$.

Therefore, using (5.3) and (5.4), we get

$$\begin{aligned}\varepsilon &> \sum_{k=1}^{N_m} \left\| f\left(\frac{\mathbf{a}_k^{(m)} + 3\mathbf{b}_k^{(m)}}{4}\right) - f\left(\frac{3\mathbf{a}_k^{(m)} + \mathbf{b}_k^{(m)}}{4}\right)\right\|^d \\ &\geq 2^{-d} m^d \sum_{k=1}^{N_m} \left\|\mathbf{b}_k^{(m)} - \mathbf{a}_k^{(m)}\right\|^d \\ &\geq 2^{-d} m^d \sum_{k=1}^{N_m} \mathcal{L}^d(I_k^{(m)}).\end{aligned}$$

Hence,

$$\sum_{k=1}^{N_m} \mathcal{L}^d(I_k^{(m)}) \leq \frac{2^d \varepsilon}{m^d}.$$

We now deduce that

$$\sum_{m=1}^{\infty} \sum_{k=1}^{N_m} \mathcal{L}^d(I_k^{(m)}) \leq 2^d \varepsilon \sum_{m=1}^{\infty} \frac{1}{m^d} < 2^{d+1}\varepsilon,$$

whilst

$$\sum_{m=1}^{\infty} \sum_{k=1}^{N_m} \mathcal{L}^d(I_k^{(m)}) \geq \sum_{m=1}^{\infty} (\mathcal{L}^d(E \cap A_m) - \frac{\varepsilon}{2^m}) \geq \mathcal{L}^d(E) - \varepsilon.$$

Thus $\mathcal{L}^d(E) \leq (2^{d+1}+1)\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have proved that $\mathcal{L}^d(E) = 0$. $\square$

We have seen, in the discussion following Theorem 5.2.1 and in Proposition 5.2.2, that mappings in the class 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$ may fail to possess many of the qualities that one might expect from 'absolutely continuous' mappings. Let us get rid of some of the chaotic mappings and consider only the smaller class 1-$AC_D(\mathbb{R}^d, \mathbb{R}^l)$ of all mappings from 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$ which are differentiable almost everywhere. Even now, we still witness volatile behaviour.

Theorem 5.2.4. *Suppose $d \geq 2$ and $l \geq 1$ are integers. Then there exists a mapping $f \in$ 1-$AC_D(\mathbb{R}^d, \mathbb{R}^l)$ and a set $U \subseteq \mathbb{R}^d$ with $\mathcal{L}^d(U) = 0$ and $\mathcal{L}^l(f(U)) > 0$.*

Remark. *Note that when $l \geq d$, any subset of $\mathbb{R}^l$ with positive l-dimensional Lebesgue measure, has positive d-dimensional Hausdorff measure. In fact, if $l > d$, we have that*

the d-dimensional Hausdorff measure of such a set is necessarily infinite. Therefore, Theorem 5.2.4 shows that functions in the class $1\text{-}AC_D(\mathbb{R}^d, \mathbb{R}^l)$ *may be severely expanding; when* $l \geq d$*, the conclusion of Theorem 5.2.4 is stronger than the assertion that* f *does not have the Luzin (N) property.*

Proof of Theorem 5.2.4. Let $\varphi : \mathbb{R} \to [0,1]$ be an extension of the Cantor function (see [9], [15]) on $[0,1]$ such that φ is constant on $(-\infty, 0)$ and on $(1, \infty)$. Let C denote the standard ternary Cantor set. Recall that $\varphi(C) = [0,1]$ and φ is constant on each connected component of $[0,1] \setminus C$. Hence, φ is differentiable with derivative zero almost everywhere.

Let $p : [0,1] \to \mathbb{R}^{l-1}$ denote a space filling curve with $\mathcal{L}^{l-1}(p([0,1])) > 0$ (see [3]). Note that the function $p \circ \varphi : [0,1] \to \mathbb{R}^{l-1}$ is differentiable with derivative zero almost everywhere.

Define a set $U \cong C \times \mathbb{R}^{d-1} \subseteq \mathbb{R}^d$ by

$$U = \{s_1\mathbf{x}_1 + \ldots + s_d\mathbf{x}_d : s_1 \in C, s_2 \ldots, s_d \in \mathbb{R}\}.$$

We note that U has Lebesgue measure zero. Let $f : \mathbb{R}^d \to \mathbb{R}^l$ be the function defined by

$$f(s_1\mathbf{x}_1 + \ldots + s_d\mathbf{x}_d) = \begin{cases} \varphi(s_1) & \text{if } l = 1, \\ (p \circ \varphi(s_1), s_2) & \text{if } l > 1, \end{cases}$$

so that

$$f(U) = \begin{cases} [0,1] & \text{if } l = 1, \\ p([0,1]) \times \mathbb{R} & \text{if } l > 1. \end{cases}$$

Hence, $\mathcal{L}^l(f(U)) > 0$ (and f does not have the Luzin (N) property). Further, note that f is differentiable almost everywhere. We also emphasise that whenever $[\mathbf{a}, \mathbf{b}]$ is a 1-regular interval and $d > 2$, we have $f(\mathbf{a}) = f(\mathbf{b})$.

It only remains to verify that $f \in 1\text{-}AC(\mathbb{R}^d, \mathbb{R}^l)$. For this, we adapt an argument

which is used for the proof of Theorem 5.2.1, part (a) (see [18, Proof of Theorem 5.1]).

Fix $\epsilon > 0$ and set $\delta = \epsilon$. Let $\{[\mathbf{a}_k, \mathbf{b}_k]\}_{k=1}^K$ be a finite collection of pairwise disjoint 1-regular intervals in $\mathbb{R}^d$. Since the vectors $\mathbf{a}_k, \mathbf{b}_k$ are endpoints of 1-regular intervals, we can find real numbers $s_{k,1}, \ldots, s_{k,d-1}, t_{k,1}, t_{k,2}$ such that

$$\mathbf{a}_k = s_{k,1}\mathbf{x}_1 + \ldots + s_{k,d-1}\mathbf{x}_{d-1} + t_{k,1}\mathbf{x}_d \text{ and } \mathbf{b}_k = s_{k,1}\mathbf{x}_1 + \ldots + s_{k,d-1}\mathbf{x}_{d-1} + t_{k,2}\mathbf{x}_d.$$

The Lebesgue measure of each interval $[\mathbf{a}_k, \mathbf{b}_k]$ is then given by $\mathcal{L}^d([\mathbf{a}_k, \mathbf{b}_k]) = |t_{k,2} - t_{k,1}|^d$. We now observe that

$$\begin{aligned}
\sum_{k=1}^K |f([\mathbf{a}_k, \mathbf{b}_k])|^d &= \sum_{k=1}^K \|f(\mathbf{b}_k) - f(\mathbf{a}_k)\|^d \\
&= \begin{cases} \sum_{k=1}^K |t_{k,2} - t_{k,1}|^2 & \text{if } d = 2 \text{ and } l > 1, \\ 0 & \text{if } d > 2 \text{ or } l = 1, \end{cases} \\
&\leq \sum_{k=1}^K \mathcal{L}^d([\mathbf{a}_k, \mathbf{b}_k]) < \delta = \epsilon.
\end{aligned}$$

Hence, f belongs to the class 1-$AC(\mathbb{R}^d, \mathbb{R}^l)$. □

Chapter 6

Conclusion

With the first example of note given by Preiss in [28] (1990) and the notion only taken further by Doré and Maleva in [12], [14], [13] between 2009 and 2011, universal differentiability sets in Banach spaces are a relatively new object of study. The present thesis contains some advancements in our knowledge of these sets. However, much remains mysterious about them and we envisage extensive future investigation into this area.

The curve approximation property of Chapter 2 is an improvement upon the existing geometric sufficient conditions for the universal differentiability property. With this progression, it is possible to simplify some of the proofs in the papers [14] and [17]. It would be desirable to continue sharpening this geometric guarantor of universal differentiability as it may lead us towards a full geometric characterisation of universal differentiability sets. Alberti, Csörnyei and Preiss [1] announce a geometric description of non-universal differentiability sets in terms of measure on curves and it would be interesting to explore how our curve approximation property relates to this.

In Chapter 3 we explicitly construct universal differentiability sets in $\mathbb{R}^d$ which are, according to Minkowski dimension, smaller than all previously known examples. Indeed, it was not even known that a universal differentiability set could have Minkowski dimension less than that of the whole space. Our universal differentiability sets attain the minimal Minkowski dimension one. By uncovering smaller universal differentiability sets, we hope to gain some intuition about the essential ingredients for the universal differentiability

property. There are multiple directions of future enquiry emerging from this work, which we describe presently.

We plan to consider the effect of replacing, in the definition of Minkowski dimension, the function $x \mapsto x^p$ with a general Hausdorff gauge function [19]. We can then obtain generalised Hausdorff measures and Minkowski dimensions which are bounded below by their traditional counterparts. It would be interesting to discover whether one can characterise the Hausdorff gauge functions which admit universal differentiability sets with generalised Minkowski dimension one. As a consequence of our findings in Chapter 3, Section 3.2, we know that any such gauge function ϕ must satisfy $\phi(x)/x \to 0$ as $x \to 0+$.

Getting a universal differentiability set of generalised Minkowski dimension one with respect to a larger gauge function, might involve changing our construction in Chapter 3, Section 3.3 so that the number of cubes needed to cover the set is significantly reduced. In the current construction, we look at each cube of level $(k-1)$ and side length w and and imagine dividing it into a grid of cubes with side length w/N, where N is some large number (in Chapter 3 this N is Q^{s_k}). We then find that the total number of cubes in this grid covering tubes of the next level is governed by $N \log N$, see (3.30). We would like to get a significantly smaller estimate, for example $N \log(\log N)$. On the other hand, we are also investigating the possibility that the presence of the curve approximation property in our sets makes an estimate of the nature '$N \log N$' or higher unavoidable. However, a proof of this would not show that universal differentiability sets of generalised Minkowski dimension one are not possible. It would only indicate that the current methods are insufficient for constructing such sets.

A second programme of investigation is the connection to the recent work of Preiss and Speight on the converse to Rademacher's theorem. Preiss and Speight prove in [29], that for any $n \geq 2$, there exists a set $E \subseteq \mathbb{R}^n$ of Lebesgue measure zero such that every Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ has a point of differentiability inside E. Furthermore, they establish that such a set can be found with Hausdorff dimension arbitrarily close to $(n-1)$ from above. The question of whether such generalised universal differentiability

sets exist with Hausdorff or Minkowski dimension $(n-1)$ remains open. The set given in [29] is dense in $\mathbb{R}^n$ and one can also ask whether it is possible to make it compact. We intend to explore the possibility that our techniques might extend to constructions of universal differentiability sets for Lipschitz mappings $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$.

The broad objective of Chapter 4 is to describe the general nature of universal differentiability sets. There are not too many existing results in this direction. An observation of [14] is that the image of any universal differentiability set under any projection map has positive Lebesgue measure. In this thesis, we verify that every universal differentiability set in $\mathbb{R}^d$ contains a kernel, which is closed in the subspace topology on E, has the universal differentiability property and is homogeneous (see Definition 4.3.14). Until now, these properties had only been observed in particular examples of universal differentiability sets.

A question that we try to answer is whether every universal differentiability set in $\mathbb{R}^d$ contains a universal differentiability set which is closed with respect to the standard topology on $\mathbb{R}^d$. Zelený and Pelant [36] prove that any Borel non-σ-porous set in a Banach space contains a closed non-σ-porous set. However, their techniques are not applicable in our setting. The main obstruction is the lack of a clear analogue of the notion of a Foran system (see [36, Definition 2.17]) for universal differentiability sets. It is plausible that the geometric description of non-universal differentiability sets given by Alberti, Csörnyei and Preiss [1], is relevant here. This description is reminiscent of the definition of σ-porous sets and thus provides good cause to explore possible connections to the work of Zelený and Pelant.

In Chapter 4, we prove that all universal differentiability sets cannot be decomposed as a countable union of closed, non-universal differentiability sets (see Theorem 4.3.1). Perhaps the strongest statement of this nature that existed beforehand is the following theorem of Kirchheim: All universal differentiability set may not be decomposed as a countable union of porous sets (see [30] or [24]). Questions concerning decompositions of universal differentiability sets are intimately connected to questions of simultaneous differentiability of multiple Lipschitz functions. Lindenstrauss, Tiŝer and Preiss [24] prove that

any two Lipschitz functions on a Hilbert space have a common point of differentiability. However, it is still unknown whether this is true for three Lipschitz functions.

The most natural next step in this work is to investigate whether a universal differentiability set may be decomposed as a union of a G_δ non-universal differentiability set and a non-universal differentiability set. Our feeling is that the answer is negative. A first step towards a possible proof of this could be to show that the G_δ set occurring in such a decomposition must be non-uniformly purely unrectifiable. This would mean that, inside this G_δ set, we could always find, for a given Lipschitz function f, a point x such that the directional derivative $f'(x, e)$ exists for some direction e. It might then be possible to construct a sequence of such points converging to a point of differentiability of f, drawing on the approach in [12]. To our advantage, any G_δ set can be viewed as a complete metric space and this gives us a natural way to ensure that the limit of the sequence stays inside the set.

We also conjecture that no universal differentiability set can be written as a union of a non-universal differentiability set and a σ-porous set. Given that many statements about the differentiability of Lipschitz functions assert that a particular condition holds everywhere except on a σ-porous set (see [30]), a proof of this conjecture could lead to important applications.

Godefroy's question of whether every subset of $\mathbb{R}^d$ with positive Lebesgue measure contains a universal differentiability set with Lebesgue measure zero remains unanswered. We managed to show that the answer is positive for sets containing a product of d sets of positive 1-dimensional Lebesgue measure. Doré and Maleva [13] establish that all sets of the form $P \times \mathbb{R}^{d-1}$ contain null universal differentiability sets, where $P \subseteq \mathbb{R}$ is a set of positive Lebesgue measure. It is unclear whether our techniques can be adapted to get the full result. On the other hand, there are various lines of enquiry which we intend to pursue.

Our first approach to answering Godefroy's question is to try to build a universal differentiability set inside a given set of positive measure, following a similar construction

to that employed in Chapter 3. By carefully positioning line segments or curves so that they intersect the positive measure set, we would aim for a collection of closed sets satisfying the curve approximation property. However, the existing constructions of universal differentiability sets require a certain degree of uniformity which is not present inside a general set of positive measure.

Imposing a type of super-density condition on the given positive measure set makes the situation more promising. We consider Lebesgue measurable sets $E \subseteq \mathbb{R}^2$ satisfying

$$\mathcal{L}^2(B(x,w) \setminus E) = o(w^4) \text{ for a.e. } x \in E, \tag{6.1}$$

where $\mathcal{L}^2$ denotes the 2-dimensional Lebesgue measure and $B(x,w)$ the Euclidean ball in $\mathbb{R}^2$ with centre x and radius w. Note that the left hand side of (6.1) is certainly equal to $o(w^2)$ for a.e. $x \in E$, by the Lebesgue Density Theorem. For a set E satisfying (6.1), we find that deployment of line segments in the manner described above is possible. Sets satisfying the super-density condition (6.1) are currently the focus of research; Delladio [11] proves that any set $E \subseteq \mathbb{R}^2$ with locally finite perimeter satisfies (6.1). We see strong reason to investigate this idea further and to explore the relevance of sets of locally finite perimeter to our problem.

Finally, in Chapter 5 we take a new direction and explore a generalised class of absolutely continuous mappings. Building on research started by Randrianantoanina and Xu, we establish a positive differentiability result for 1-AC mappings and also give an example of a 1-AC mapping which is differentiable almost everywhere and does not have the Luzin (N) property. Our findings suggest that the definition of the class 1-AC does not give rise to a sensible notion of 'absolute continuity', since rather wild mappings are still admitted. An objective of future work is to try to modify this definition so that we get a more controlled class of mappings. One idea is to consider the class 1-AC_{WDN} of all 1-AC mappings which are in the Sobolev space $W^{1,d}_{\mathrm{loc}}$, are differentiable almost everywhere and have the Luzin (N) property. Paper [18] proves a series of containment relations to

show how the class 1-AC_{WDN} compares with previously considered classes of absolute continuity. However, some containment relations, described in [18], are not fully understood and future work is needed to complete the picture.

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