COARSE CORRELATED EQUILIBRIA IN DUOPOLY GAMES

by

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A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

Department of Economics
The University of Birmingham
APRIL 2014

This thesis is dedicated to my Dadun (grandfather), Late Mr. A.B. Sen Gupta - he would have been really happy to see me develop into an academic, as he himself was a great scholar and extremely passionate about learning and teaching.

Acknowledgments

It gives me great pleasure to have finally reached the day when I can officially thank all those people who have been there with me during all these years. The years of my doctoral studies have played an important role in an overall development of my intellect and personality and I am grateful to everyone who has been a part of my journey.

I would first like to start by thanking my (lead) supervisor Indrajit Ray who has been solely responsible for all my professional achievements. He has been a great mentor and has provided me with his guidance throughout. He has always been extremely patient and encouraging and I will always be indebted to him for all the support he has provided me during these years. I have no doubt in saying that he is one of the best supervisors one can get and I am extremely lucky to be his student.

I would also like to thank Martin Jensen for being an excellent (second) supervisor and for providing me with all his guidance and support in all my academic ventures. Many thanks to Matthew Cole, David Maddison, Jayasri Dutta and Siddhartha Bandyopadhyay for all the encouragement and useful advices, and my healt-felt thanks to Emma, Jackie, Wendy, Lisa, Rita and Marleen for all the administrative help they have provided during my Masters and Doctoral studies in Birmingham.

A special thanks to Herve Moulin. I am very fortunate to have such a well respected member of economics community as my co-author, collaborator and mentor. I am highly indebted to him for all the guidance he had provided and I am thankful to him for patiently correcting all my (silly) mistakes. I have learned a lot from him.

I am grateful to Chirantan Ganguly and Herakles Polemarchakis for their valuable suggestions and helpful comments on the papers forming this thesis.

I thank all my colleagues Naoki, Alex, Hyung Chang, Maria, Charlie; for all the interesting conversations during coffee breaks, drinks, and dinner outings. I have had a great time with you all.

I express my gratitude to Herakles Polemarchakis, Colin Rowat, and Siddhartha Bandyopadhyay for kindly accepting to be members of my doctoral committee.

I have been lucky to have a great group of friends and house-mates during my time in Birmingham: Kiran, Mimi, Saptarshi, Arnab, Shashwata, Sandip, and Kushan, who had been a source of joy and laughter all these years. I have thoroughly enjoyed each and every moment spent including watching football and cricket world cups, dinner sessions, trips to London and Malvern hills, swimming lessons, birthday celebrations and many more. I would like to convey my special thanks to Kiran, who has been a great support during my stay in Birmingham. He has always encouraged and motivated me during the tough times. He has played the role of a friend in the best possible way. I also thank Diya di for everything she has done for me and for always being there for me whenever needed. I have always enjoyed spending time with Diya di, Josh and Adi.

Two people who have always been there with me are my two best friends Ankit and Varun, who are like a family to me. I am lucky to have such great friends in my life. My family is the most important source of strength for me. I am always thankful to Pishaba, Pishimum, Mamu, Mamima, Mishti-didi, Dada and Joy for their love and unconditional support. It is due to the blessings of my (late) Dadun, Amma, Dadu, Dimma and Jiju, that I am currently in this position of my life and I hope that this work makes them all proud. I am also thankful to my in-laws Mr. and Mrs. Dokka for all their support.

I finally move on to thanking the three most important people in my life. My dearest husband, Trivikram, who is the person who actually made me believe that I could go on to do a Ph.D. and who has always been my best critic, support, friend and mentor. I hope I can get some of his amazing qualities as a researcher and I thank him with all my heart for always inspiring and motivating and bringing out the best in me.

Lastly, I am deeply thankful to my parents for making me what I am today. They have been responsible for every little and big success I have had in my life. More than anyone in my life, my parents have played many different important roles. They are my best friends when I need to discuss anything, they are my strongest support when I am tired in some way, they become guides who show me the direction when I am lost and so much more for which I have no words to explain. They celebrate my each and every achievement, immaterial of its significance, encouraging and preparing me for the next one. They are the world's best parents and I owe everything to them.

Preface

This thesis comprises of three main chapters (Chapter 2, 3 and 4). Chapter 4 of the thesis is the result of a collaboration with Indrajit Ray. An article corresponding to this chapter has been published in the *International Journal of Game Theory*, see Ray and Sen Gupta (2013). Chapters 2 and 3 are a result of the collaboration with Herve Moulin and Indrajit Ray. The article forming Chapter 2 has been published in the *Journal of Economic Theory*, see Moulin, Ray and Sen Gupta (2014). The article forming Chapter 3 is currently available under the Discussion Papers series (2013) of Department of Economics, University of Birmingham; see Moulin, Ray and Sen Gupta (2013).

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Chapter 1

Introduction

Enforcing cooperative outcomes while respecting the non-cooperative incentives has been the dominant theme of the Game Theory literature for the past three decades. The *dynamic* approach embeds the initial normal form game in a dynamic model (example, by repeating the game) allowing the agents to deploy learning heuristics (i.e. agents can learn or get some information from history of play). The self-enforcing correlation devices known as *correlated equilibrium* - CE - (Aumann 1974, 1987) offer an alternative, *static*, approach where the cooperative benefits are harvested in a one-shot randomized play of the game. A normal form game can be played using a correlation device. The correlation device can be interpreted as a mediator, who is a non-strategic player of the game. The correlation device first sends private messages to each player according to a probability distribution and then the players play the original normal form game. A correlation device is called direct or canonical device if the set of messages is identical to the set of pure strategies of the original game, for each player. A direct CE can best be described as a mediator whose recommendations the players find optimal to follow obediently. Any (pure or mixed) Nash equilibrium (NE) and any convex combination of Nash equilibria can also be viewed as a direct CE. The probability distribution is such that no player should have an incentive to deviate by himself from the strategy the device suggests to him. The concept of CE is interesting because some CE outcomes may be Pareto superior to all NE outcomes of the game; which can be shown using simple examples (even 2×2 normal form games), as below.

Example 1

	A_2	P_2
A_1	$0, 0$	$7, 2$
P_1	$2, 7$	$6, 6$

The Pure Strategy NEs for this game are the strategy profiles (A_1, P_2) *and* (P_1, A_2) *with a joint payoff of* 9 *and the Mixed Strategy NE is the strategy* $(\frac{1}{3}A_1, \frac{2}{3}P_1; \frac{1}{3}A_2, \frac{2}{3}P_2)$ *with an expected individual payoff of* $(4.6, 4.6)$*, and thus a joint payoff of* 9.2*. Let us now consider a correlation device (mediator) which gives recommendations to the players of the game. The original game can be considered as an extended game in the sense that the strategies of the players is to decide whether or not to play an obedient strategy (follow the recommendations of the device).*

	A_2	P_2
A_1	0	$\frac{1}{3}$
P_1	$\frac{1}{3}$	$\frac{1}{3}$

Let us assume that the recommendation for Player 1 (row player) is to play P_1. We want to check whether Player 1 has an incentive to play obedient strategy or not. Given the recommendation of P_1 for Player 1, the recommendations for Player 2 will be (using Bayes' rule), $\Pr(A_2/P_1) = \frac{1}{2} = \Pr(P_2/P_1)$. *Assuming Player 2 (column player) follows the recommendation, the expected values for Player 1 are $E(A_1) = 3.5 < E(P_1) = 4$, thereby implying that Player 1 has a clear incentive to play the obedient strategy. This is also true for a recommendation of A_1 for Player 1. Thus the probability device is a CE for this game, giving an expected payoff of $\frac{1}{3}(7 + 2 + 6) = 5$, to each player. Thus, there is clearly an improvement over the (joint) NE payoff.*

Other simple examples where some CE outcomes improve strictly over the NE outcomes are Battle of Sexes, Stag Hunt games, etc. It was recently discovered that CE cannot help improving upon the NE in many important microeconomic games. Liu (1996) considered an n-firm Cournot oligopoly market with linear demand and asymmetric linear cost function (constant marginal costs). Liu (1996) used the extended definition of CE provided by Hart and Schmeidler (1989) and found that there exists a unique CE for this game, which is same as the NE of the game. Yi (1997) extended the results of Liu to include convex cost functions for the firms and found the same results. This result was later on generalized by Neyman (1997) to conclude that Cournot oligopoly games are games with a smooth and concave potential function, and all such games have a unique CE which coincides with the NE (or a mixture of NE) of the game.

Although CE may not achieve anything more than the NE outcome, as one rightly reckons, a coarsening of the set of correlated equilibria may exist. Indeed, we do have such a coarse concept in the literature, introduced by Moulin and Vial (1978), called the *coarse correlated equilibrium*[1] (CCE, henceforth). The best interpretation of this notion is that of a solicitor who asks the players to either commit to a correlation device or to play any strategy of their own. The agents have to decide whether or not to commit to the solicitor (mediator) and if they decide not to commit, they can choose to play any strategy of their own, even the strategies outside of the ones selected by the correlation device. For an informal description of the concept of CCE (see formal Definitions 1 and 2 in Section 2.2 in Chapter 2), recall first that a CE is a probability distribution over the outcomes of the game, commonly known to the players. Running this lottery yields a profile of *recommended* strategies; the mediator informs each player of his own recommendation, without revealing the recommendation to any other player. The equilibrium property is that it is optimal for any player i to follow this recommendation, if i believes that every other player is doing the same. A CCE also selects the outcome of the game according to a commonly known probability distribution. The difference is that the mediator has now more commitment power. Each player must decide to commit or not to the strategy selected for him by the mediator *before* the mediator runs the lottery. A player who decides not to commit to the mediator may choose to play any strategy of his own; he does not get to know the outcomes chosen by the lottery. The equilibrium property is that it is best to commit to the anticipated outcome of the lottery, if

[1] Moulin and Vial (1978) introduced this equilibrium concept and called it a *correlation scheme*. Young (2004) and Roughgarden (2009) coined the terminology of *coarse correlated equilibrium*, which we adopt, while Forgó (2010) called it a *weak correlated equilibrium*.

one believes that every other player is doing the same. Such a weaker notion of correlation may improve upon the NE of any game, other than *strategically zero-sum games* (Moulin and Vial 1978), even though correlation *a la* Aumann may not. If a NE cannot be improved upon using CCE, then it cannot be improved by CE as well, but the converse is not true. For a game of two players with two strategies each (2×2 normal form game), the CE is the same as the CCE. Let us now consider the following example (Moulin and Vial 1978) where each of the two players have three strategies.

Example 2

	L	C	R
T	$3,3$	$1,1$	$4,1$
M	$1,4$	$5,2$	$0,0$
B	$1,1$	$0,0$	$2,5$

There is a unique NE for this game, $(T, L) = (3, 3)$*, which is also the CE. Thus the expected payoff for each player is* 3*. Let us now consider the following probability device:*

	L	C	R
T	$\frac{1}{3}$	0	0
M	0	$\frac{1}{3}$	0
B	0	0	$\frac{1}{3}$

This device is clearly not a CE because none of the three outcomes (where the device puts a positive probability) are NEs of the game. The dominant strategy equilibrium (by iterative elimination of dominated strategies) and the unique NE *of the game is (T, L), which as mentioned above is also the only CE of the game. We need to now check if the device is a CCE or not. For the device to be a CCE, both the players should have an incentive to 'commit' to the device (mediator). Let us assume that Player* 2 *commits to the device, but Player* 1 *decides not to commit and instead plays the pure strategy* B*. Thus the expected payoff accruing to Player* 1 *by playing this pure strategy is* $E_1(B) = 1(\frac{1}{3}) + 0(\frac{1}{3}) + 2(\frac{1}{3}) = 1$*. Instead, if Player* 1 *commited to the device, the expected payoff achievable would be* $3(\frac{1}{3}) + 5(\frac{1}{3}) + 2(\frac{1}{3}) = \frac{10}{3}$*, thereby implying that Player* 1 *does not have an incentive to deviate from commiting to the device. This is also true for any deviations (*T *or* M*) for Player* 1*. Therefore, the commitment device is a CCE for this game. Thus, we see that there is a clear improvement over the NE payoff via coarse correlation, even though CE coincides with the NE payoff. Moreover, a probability device with probabilities* $\frac{1}{2}$ *on outcomes* (M, C) *and* (B, R) *is also a CCE for this game, giving an even higher payoff of* 3.5 *to each of the two players; implying that for a particular game there can be more than one CCEs.*

The main aim of this thesis, comprising of three chapters, is to consider the concept of CCE in various contexts; quadratic games with two interesting cases of Cournot duopoly and Public good provision (Chapter 2), emission abatement game (Chapter 3) and linear duopoly (Chapter 4). We submit that the concept of coarse correlation has a very natural interpretation in these games. A CCE endows the mediator with more commitment power. As in a CE, the lottery over strategy profiles is known to all players; but now each player must decide to commit or not to the strategy selected for him by the mediator before the mediator runs the lottery. A player who does not commit will then choose his strategy without any information on the outcome of the lottery. The equilibrium property is that it is the best to commit to the anticipated outcome of the lottery, if one believes that every other player is doing the same. The stability requirement in a CCE is strictly weaker than in a CE; hence, CCEs afford more opportunities than CEs to improve upon Nash equilibria. In the following sections we briefly discuss the three chapters, as mentioned above.

1.1 Quadratic Games: Cournot duopoly and Public good provision

The example on the CCE (shown above) we can derive two very important inferences on the importance of CCE: firstly, in games where CE is the same as NE (or mixture of NE), there can exist a CCE which is strictly better than the NE, and secondly, there can exist more than one CCEs for a particular game. We also understand (from our discussion on CE) that there exist many economically meaningful games wherein CE can not do any better than the NE. These results directly bring us to two very interesting (and important) questions: (i) can CCE improve the NE for the games (with concave and smooth potential funtion) where CE does not help?; (ii) can we find an optimal (total utility maximising) CCE for a particular game? The main aim of this chapter is to answer these two questions. For the purpose of doing so, we consider here a class of symmetric two-person games with one-dimensional strategies and quadratic payoff functions.

These include simple versions of the duopoly and public good provision games. These are games with smooth and concave potential payoff functions and therefore, the only CEs in these games are mixtures of pure Nash equilibria. However, they often possess more efficient CCEs. We characterize the optimal (joint utility maximizer) CCE for this class (Theorem 1, Chapter 2): it is a simple symmetric mixture of two pure strategy profiles, as in the following example.

Example 3

Two players contribute non-negative amounts $x_i, i = 1, 2$, to the public good, produced at constant marginal cost, while the benefit from consuming it is concave quadratic (the same for both players). The following is the payoff function of such a game, as an example of the class of games discussed in Section 2.4 *in Chapter* 2.

$$u_1(x_1, x_2) = x_1 + 4x_2 - (x_1 + x_2)^2;\ u_2(x_1, x_2) = u_1(x_2, x_1).$$

Total payoff $(u_1 + u_2)(x_1, x_2)$ is maximal iff $x_1 + x_2 = x^{eff} = 1.25$, with corresponding efficient payoff $u_1 + u_2 = \pi^{eff} = 3.125$. The Nash equilibria are all those profiles such that $x_1 + x_2 = 0.5$, with corresponding payoff $\pi^N = 2$. It is easy to check that the game is a potential game with a concave (however, not strictly concave) potential function $P(x) = x_1 + x_2 - 2x_1x_2 - x_1^2 - x_2^2$. Therefore, any CE of this game is a mixture of Nash equilibrium profiles.

Let us now consider a lottery L, such that, with equal probability, one player contributes $x_i = 1$ while the other contributes nothing. This is clearly not a CE because neither $(1, 0)$ nor $(0, 1)$ is a Nash equilibrium. However it is a CCE: if player 1 *contributes x_1 and assumes player* 2 *is contributing either* 0 *or* 1 *with* *equal probability, his payoff* $\frac{1}{2}[(x_1 - x_1^2) + (x_1 + 4 - (x_1 + 1)^2]$ *is maximized at* $x_1 = 0$ *and gives him* $u_1 = \frac{3}{2}$, *precisely the same as by committing to follow the outcome of* L *(but note that player 2's best response to* $x_1 = 0$ *is not to follow* L*). Our results imply that the total payoff* $\pi^{CC} = 3$ *is indeed the best the players can achieve in any CCE of this game; it yields a* 50% *increase over and above the Nash equilibrium payoff and only incurs a* 4% *efficiency loss.*

Admittedly, the class of games we consider is fairly small. Yet, in addition to the Cournot duopoly and the public good game, it captures quadratic versions of a few other games like Bertrand duopoly, emission game (Barrett 1994)[2], as well as the search game (Diamond 1982). Moreover, even in this small class of games, the proof of our optimality result is not simple; for instance, we have no idea how to generalize it to the three-person case (even the asymmetric two-person case will involve significantly more work). The literature offers no other complete analysis of optimal coarse correlation (or, for that matter, correlation *a la* Aumann) for economically meaningful games where pure strategies are one dimensional.

1.2 Abatement Game

An interesting question in the literature of environmental economics, which still remains (somewhat) unanswered, is how the nations, acting non-cooperatively, can deal with and achieve the goal of global emission abatement, a problem

[2] We analyse this game in a parallel paper (Moulin, Ray and Sen Gupta 2013) - Chapter 3 of the thesis.

which is of great concern since last few decades. An interesting approach towards achievement of this goal could be by way of mediation, a concept which still remains less explored for this problem. Mediation in this context could be useful in achieving better coordination between the nations and direct them towards achieving the goal of emission abatement. CE and CCE are a way of mediation to achieve better outcomes than what could be achieved by the players acting on their own. We, in Chapter 3, analyze the performance of coarse correlation in a well-studied non-cooperative model from the literature in environmental economics. The model, called the *abatement game* (Barrett 1994), is a game played by several players (nations) choosing the level of abatement (pollution). Although several (non-cooperative and cooperative) solutions[3] have already been analyzed for the abatement game, the impact of strategic correlation has not been studied. We submit that the concept of coarse correlation has a very natural interpretation in this game.

In a *CCE*, the mediator requires more commitment from the players: it asks the players, *before running the lottery*, to either commit to the future outcome of the lottery or play any strategy of their own without learning anything about the outcome of the lottery. The equilibrium property is that each player finds it optimal to commit *ex ante* to use the strategy selected by the lottery. In the context of climate change negotiation, and in particular for the abatement game, a correlation device can be interpreted as an independent agency providing

[3] Barrett (2001) and McGinty (2007) studied asymmetric versions of the abatement game. Barrett (1994) also considered the Stackelberg model of abatement which was later analyzed by Rubio and Ulph (2006). Finus (2001) presented generalization of Barrett's results in terms of the number of countries in a stable equilibrium.

a recommendation to all relevant countries towards the ultimate goal of global emission reduction. In a CCE of the abatement game, each country remains free to revert to a non-cooperative emission, but does not benefit from doing so as long as other countries commit to the policy selected by the agency. Correlation, in either the CE or the CCE format, has been mostly ignored by the environmental literature.[4] It might be difficult to imagine a governing body for all countries taken together, but a governing body for smaller groups of countries might be possible (for example European Commission for EU) and such an agency could work as a mediator directing the countries associated towards the achievement of the ultimate goal of emission abatement.

The abatement game is a game with a smooth and concave potential function and hence its only CE is the (unique) Nash equilibrium. However, the game has many more CCEs; in some cases that we identify in the chapter, some of these are strictly more efficient than the Nash equilibrium outcome. We apply the general methodology as in Moulin, Ray and Sen Gupta (2014) and formally characterize the optimal CCE for any 2-player abatement game (Theorem 1, Chapter 3) under the assumption that the benefit parameter (b) is bigger than the cost parameter (c).[5] As in Moulin, Ray and Sen Gupta (2014), we find that the optimal lottery is a 2-dimensional anti-diagonal symmetric lottery[6]. The total payoff at the

[4] Forgó, Fülöp and Prill (2005) and Forgó (2011) recently used (modified versions of) Moulin and Vial's notion of (coarse) correlation in other environmental games. Baliga and Maskin (2003) surveyed some models of mechanisms in this literature.

[5] Gerard-Varet and Moulin (1978) proved that Nash equilibrium can be *locally improvable* by using a concept similar to CCE under a condition, which for this game, perhaps not surprisingly, also turns out to be $b > c$.

[6] The optimal lottery is a symmetric lottery with finite support similar to those studied by Ray and Sen Gupta (2013) who called such a lottery a Simple Symmetric Correlation Device (SSCD) as introduced in Ganguly and Ray (2005) to discuss correlation.

optimal CCE is very close to the efficient payoff for this class of games, with a small improvement above the Nash equilibrium total payoff.

1.3 Linear duopoly game

The third and the final chapter of this thesis is a slight deviation from the first two chapters, wherein we mainly try to achieve the optimal utility maximising CCE in different contexts. In this chapter, as before, we do analyse a specific model and characterise the existence of CCE in that specific model; but the purpose of the analysis is not to see if CCE can improve the NE or not. The notion of CE has been analysed in many strategic situations, primarily to achieve improvement over Nash equilibrium outcomes. Unfortunately, for some games, correlation may not achieve anything more than the Nash outcome. Arguably the most fundamental model of strategic markets, that of oligopoly, indeed provides one such example.

In contrast with this negative result for correlation in strategic markets, the literature does provide, a couple of positive views of correlation in markets. First, as one rightly reckons, a coarser notion of correlation may be able to improve upon the Nash equilibrium. Indeed, the concept of CCE has been used by Gerard-Varet and Moulin (1978) in specific duopoly games to achieve an improvement *(locally)* over the NE payoff.

Second positive result related to correlation in strategic markets is, from a recent literature (Forges and Peck 1995, Dávila 1999), that correlation, like sunspot, matters in strategic market games *a la* Shapley and Shubik (1977). Also, sunspot equilibrium in competitive markets (Azariadis 1981, Cass and Shell 1983) and correlated equilibrium in non-cooperative games (Aumann 1974, 1987), are very similar in nature and indeed closely connected, as noted earlier (Maskin and Tirole 1987; Aumann *et al.* 1988) and formally presented recently by Polemarchakis and Ray (2006).

There are, however, a few gaps worth mentioning in the above strands of literature. First, from the analysis by Gerard-Varet and Moulin (1978), we learn under what conditions Nash equilibrium of the duopoly game can be *locally* improved upon, using a specific notion of improvement with strategies close to the Nash equilibrium; however, we do not know, for improvement or even for existence, whether the support of such a correlation device necessarily has to be close to the Nash equilibrium or not. Second, from the above mentioned literature on correlation and sunspots, we do not find the connection, if there is any, between (coarse) correlated equilibrium and sunspot equilibrium in oligopoly models, in particular, duopoly games.

The purpose of this paper is precisely to bridge these gaps and thus is twofold. We would like to find whether there exists any general (non-local) CCE in strategic markets and if so, whether this equilibrium relates to the sunspot structure. To achieve these two results, we analyze arguably the most fundamental and surely the simplest of models in strategic markets, that of a duopoly game with linear demand and constant marginal cost, called here, the *linear duopoly game*. Establishing whether (coarse) correlation, like sunspots, matters in the strategic market model of the linear duopoly game is, by itself, important because it reveals what might be achieved via pre-play communication in the presence of correlation devices in a duopoly, and such knowledge can also be used to elucidate how players coordinate on an outcome in this game. We achieve the following desired results: we show existence of a CCE which has an obvious sunspot structure, for the linear duopoly game.

The thesis proceeds as follows: Chapter 2 (Improving Nash by Coarse Correlation) precisely concentrates on providing a two-step algorithm to find utility maximising CCEs and uses this algorithm to analyse two interesting economic models of Cournot duopoly and public good provision; Chapter 3 (Coarse correlated equilibria in an abatement game) mainly looks at a very interesting (for environmental economics) game of emission abatement and uses the main algorithm provided in Chapter 2 to analyse this particular game; and finally Chapter 4 (Coarse Correlated Equilibria in Linear Duopoly Games) looks at a very simple linear duopoly game and tries to characterise and show the existence of CCE, which has a special structure - Nash centric device - which always serves as an equilibrium for such a game. All the three chapters start with a very brief intoduction, followed by the model (which includes the definitions, concepts, etc.), results and finally the conclusion of each chapter.

Chapter 2

Improving Nash by Coarse Correlation

2.1 INTRODUCTION

The concept of correlated equilibrium (CE) was introduced by Aumann (1974, 1987) as a way to improve upon the Nash equilibrium in some strategic form games, such as some versions of the Battle of the Sexes, Chicken, Stag Hunt, etc. This is however not possible in many important microeconomic games. Liu (1996) and Yi (1997) proved that the only CEs in a large class of oligopoly games are mixtures of pure Nash equilibria, a result later on generalized by Neyman (1997) and Ui (2008) to all games with a smooth and concave potential function. Recent results show very close connections between the set of CEs and the limit behavior of *regret-based* heuristics[1] (Hart and Mas-Colell 2000, Young 2004, Hart 2005). Thus, the concept of CE has better dynamic behavioral justifications than the ordinary Nash equilibrium (in mixed strategies).

Here, we consider the concept of *coarse correlated equilibrium* – CCE, with strictly weaker stability requirements than the CE. We consider a small but economically relevant class of games which have a smooth and concave potential function and therefore the ordinary CE is not of any use (due to the results just cited). For these games we compute the optimal CCE that achieve substantial cooperative (joint) improvements (of payoffs) for the agents involved. A correlation device is called a coarse correlated equilibrium if it is in no player's interest to choose any alternative strategy of his own, given that other players choose to commit to the device. Just like for CE, the literature discusses the set of CCEs in static one-shot games (Moulin and Vial 1978, Forgó *et al* 2005, Roughgarden 2009, Forgó 2010, Ray and Sen Gupta 2013) and vindicates it as the limit of certain regret-based adaptive dynamics (Hart and Mas-Colell 2003, Young 2004).

The stability requirement in a CCE is strictly weaker than in a CE (see formal Definitions 1 and 2 in Section 2.2.1 below); hence, CCEs afford more opportunities than CEs to improve upon Nash equilibria. The games discussed in this paper are a case in point. They are symmetric two-person games with one-dimensional strategies and quadratic payoff functions that include simple ver-

[1] If each player in the game plays regret-matching, then the joint probability distribution of the play converges to the set of CE of the stage game (Theorem 1, Hart 2005). There is an adaptive procedure which takes place in discrete time specifying that the players adjust strategies probabilistically. Regret matching is defined by the following rule (Hart 2005): switch next period to a different action with a probability that is proportional to the regret for that action, where regret is defined as the increase in payoff had such a change always been made in the past.

sions of the duopoly and games of public good provision. These are games with a smooth and concave potential function and therefore the only CEs in these games are mixtures of pure Nash equilibria (Neyman 1997). However, they often possess more efficient CCEs, as shown in the Example 3 provided in Chapter 1. We characterize the optimal (joint utility maximizer) CCE for this class (Theorem 1): it is a simple symmetric mixture of two pure strategy profiles.

In section 3, we consider symmetric Cournot duopoly games with decreasing linear marginal costs. The profit maximizing outcome has one firm producing the monopoly output, while the other firm stays out. We give examples where these are not NE, so that no randomization over these two outcomes is a CE. Yet, tossing a fair coin to choose which firm acts as a monopolist and which stays out, becomes a CCE. This is because if firm 1 does not commit to the outcome of the lottery, it does not know if firm 2 is active or not, and this incertitude is enough to nullify the benefit of deviating.

Admittedly, the class of games we consider is fairly small, but even in this small class of games, the proof of our optimality result is not simple; for instance, we have no idea how to generalize it to the three-person case (even the asymmetric two-person case will involve significantly more work). The literature offers no other complete analysis of optimal coarse correlation (or, for that matter, correlation *a la* Aumann) for economically meaningful games where pure strategies are one dimensional.

The paper is organized as follows. After a brief literature review, we define CCEs in subsection 2.2.1 and introduce our symmetric quadratic two-person games in subsection 2.2.2, together with our general methodology. Section 2.3 develops full computations for the special case of the Cournot duopoly while section 2.4 does so for the public good provision game and section 2.5 concludes. The Appendix contains the proofs of two key Lemmata.

2.1.1 Related Literature

Two earlier papers are the most relevant to our analysis.

Gerard-Varet and Moulin (1978) consider general smooth[2] two-person games with real-valued strategies x_i and payoff functions $(u_1(x_1,x_2), u_2(x_1,x_2))$[3] and discuss the existence of a CCE improving upon a strict, regular and interior Nash equilibrium $(\overline{x}_1, \overline{x}_2)$.[4] They show that the answer rests (almost) entirely upon the critical parameter

$$\rho = \frac{\frac{\partial^2 u_1}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 u_2}{\partial x_1 \partial x_2}}{\frac{\partial^2 u_1}{\partial^2 x_1} \cdot \frac{\partial^2 u_2}{\partial^2 x_2}}(\overline{x}) \tag{2.1}$$

in the following sense. If $\rho < \frac{1}{4}$, there is no CCE improving upon the Nash equilibrium.[5] If $\rho > \frac{1}{4}$, there is an improving CCE, moreover it can be taken to be a mixture of two pure strategy profiles. This result will be useful in some of the discussion in our paper below.

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[2] u_i is twice continuously differentiable.

[3] Strategy sets X_1, X_2 are real compact intervals.

[4] $\overline{x}_i$ is the only best reply to $\overline{x}_j$; $\frac{\partial^2 u_i}{\partial^2 x_i}(\overline{x}) < 0$; $\frac{\partial u_i}{\partial x_j}(\overline{x}) \neq 0$; $\overline{x}_i$ is interior to the strategy set.

[5] Note that the main Theorem in that paper considers only local improvement. According to the Definition 3 in the paper, a Cournot Nash equilibrium $(\overline{x}_1, \overline{x}_2)$ is said to be locally improvable by correlation if there exists an integer n, n continuous mappings from $[0,1]$ into $X_1 \times X_2$, and n continuous mappings from $(0,1)$ into $\mathbb{R}_+$, such that $L(0)$ is a deterministic lottery on NE outcome, $\delta_{(\overline{x}_1, \overline{x}_2)}$, and $L(t) = \sum_{i=1}^{n} \alpha_i(t) \delta_{(x_1^i(t), x_2^i(t))}$ is a strict CCE for every $t \in]0,1]$ and such that the payoffs $t \to [L(t), u_1]$ and $t \to [L(t), u_2]$ are strictly increasing on $[0,1]$. It is easy to see that if a CCE L improves upon the Nash equilibrium $\overline{x}$, then $\varepsilon L + (1-\varepsilon)\delta_{\overline{x}}$ defines a local improvement; so, our claim follows.

Ray and Sen Gupta (2013) consider a duopoly model with linear demand and constant marginal cost (a member of our class of symmetric quadratic games), and identify a large set of CCEs, dubbed *Nash-centric* correlation devices because their support is symmetric around the Nash equilibrium. These CCEs do not improve upon the Nash equilibrium (as it turns out, no such improvement exists). The Nash-centric devices may choose (put positive probabilities on) quantities which maximise profit, employment, social welfare, etc., and therefore without affecting the profits of the firms (because they still earn Nash equilibrium payoff) other diverse equilibrium opportunities can be achieved that may be preffered over Nash equilibrium. Just like the optimal CCEs we identify below, these are instances of the Simple Symmetric Correlation Devices (SSCD) introduced by Ganguly and Ray (2005) to discuss correlation; that is, these are symmetric and their support is finite.

2.2 MODEL

2.2.1 Correlation and Coarse Correlation

Fix a two-person normal form game, $G = [X_1, X_2; u_1, u_2]$, where the strategy sets are closed real intervals, and the payoff functions $u_i : X_1 \times X_2 \to \mathbb{R}$, $i = 1, 2$, are continuous. We write $\mathbb{C}(X_1 \times X_2)$ for the set of such continuous functions and similarly, $\mathbb{C}(X_i)$ for the set of continuous functions on X_i.

Let $\mathscr{L}(X_1 \times X_2)$ with generic element L and $\mathscr{L}(X_i)$ with generic element ℓ_i be the sets of probability measures on $X_1 \times X_2$ and X_i respectively. That is to say, L is a Radon measure[6] on $X_1 \times X_2$ of mass 1 (similarly, so is ℓ_i, on X_i). We write the mean of $u_i(x_1, x_2)$ w.r.t. L as $u_i(L)$; similarly, $f(\ell_i)$ is the mean of $f(x_i)$ w.r.t. ℓ_i for any $f \in \mathbb{C}(X_i)$. Given $L \in \mathscr{L}(X_1 \times X_2)$, we write L^i for the marginal distribution of L on X_i, defined as follows:

$$\forall f \in \mathbb{C}(X_1),\ f(L^1) = f^*(L), \text{ where } f^*(x_1, x_2) = f(x_1) \text{ for all } x_1, x_2 \in X_1 \times X_2 \tag{2.2}$$

(with a symmetric definition for L^2).

The deterministic distribution at z is denoted by δ_z. We often consider product distributions such as $\delta_{x_1} \otimes \ell_2$ and $\ell_1 \otimes \delta_{x_2}$ and we write $u_i(\delta_{x_1} \otimes \ell_2)$ simply as $u_i(x_1, \ell_2)$.

We now define our equilibrium concept.

Definition 2.1. *A coarse correlated equilibrium (CCE) of the game G is a lottery $L \in \mathscr{L}(X_1 \times X_2)$ such that*

$$u_1(L) \geq u_1(x_1, L^2) \text{ and } u_2(L) \geq u_2(L^1, x_2) \text{ for all } (x_1, x_2) \in X_1 \times X_2. \tag{2.3}$$

Consider the normal form game $\widetilde{G} = [X_1 \cup \{\blacktriangle\}, X_2 \cup \{\blacktriangle\}; \widetilde{u}_1, \widetilde{u}_2]$ where the additional strategy $\blacktriangle$ means "commit to the outcome of L". Payoffs are the same as in G if no player chooses $\blacktriangle$, otherwise are as follows:

$$\widetilde{u}_1(\blacktriangle, \blacktriangle) = u_1(L);\ \widetilde{u}_1(\blacktriangle, x_2) = u_1(L^1, x_2);\ \widetilde{u}_1(x_1, \blacktriangle) = u_1(x_1, L^2)$$

[6] A positive linear functional on $\mathbb{C}(X_1 \times X_2)$.

(with a symmetric definition for L^2). Then, L is a CCE if and only if $(\blacktriangle, \blacktriangle)$ is a Nash equilibrium of $\widetilde{G}$ (as discussed in the Introduction).

The definition of a CCE can be easily rewritten (as shown in Ray and Sen Gupta 2013) for any finite n-person normal form game, $[N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}]$, with set of players, $N = \{1, ..., n\}$, finite pure strategy sets, $X_1, ..., X_n$ with $X = \prod_{i \in N} X_i$, and payoff functions, $u_1, ..., u_n$, $u_i : X \to \Re$, for all i. For such a game, a probability distribution L over X is a CCE if for all i, for all $x'_i \in X_i$, $\sum_{x \in X} L(x) u_i(x) \geq \sum_{x_{-i} \in X_{-i}} L^i(x_{-i}) u_i(x'_i, x_{-i})$, where $L^i(x_{-i}) = \sum_{x_i \in X_i} L(x_i, x_{-i})$ is the *marginal* probability distribution over $x_{-i} \in X_{-i}$, for any deviant $i \in N$ while the others commit to L.

The next concept is correlated equilibrium *a la* Aumann.

Definition 2.2. A correlated equilibrium (CE) of the game G is a lottery $L \in \mathscr{L}(X_1 \times X_2)$ such that for any measurable mapping $\varphi : X_1 \to X_1$, we have

$$u_1(L) \geq u_1^{\varphi}(L), \text{ where } u_1^{\varphi}(x_1, x_2) = u_1(\varphi(x_1), x_2) \text{ for all } (x_1, x_2) \in X_1 \times X_2 \quad (2.4)$$

(and a symmetric property for u_2).[7]

The definition of a CE is more transparent when the strategy sets are finite, so that we can define the conditional distribution $L[x_1] \in \mathscr{L}(X_2)$ of L with respect to strategy x_1 in the support of L^1. Then inequality (2.4) amounts to

$$u_1(x_1, L[x_1]) \geq u_1(x'_1, L[x_1]) \text{ for all } x_1, x'_1 \in X_1$$

[7] Note that u_1^{φ} is bounded and measurable, hence integrable w.r.t. L.

and is interpreted in the usual way: the mediator reveals to player 1 the coordinate x_1 of the outcome of L and then player 1 has no incentive to choose another strategy.

To check that the equilibrium condition for CE is (much) stronger than that for CCE, fix any $\overline{x}_1 \in X_1$ and choose the constant function $\varphi : \varphi(x_1) \equiv \overline{x}_1$. Then (2.2) applied to L^2 implies $u_1^{\varphi}(L) = f(L^2)$ for $f(x_2) = u_1(\overline{x}_1, x_2)$, therefore $u_1^{\varphi}(L) = u_1(\overline{x}_1, L^2)$.

2.2.2 Quadratic Games and CCEs

For most of this paper, we consider symmetric two-person games with strategies $X_1 = X_2 = \mathbb{R}_+$ and quadratic payoffs. Their general form is as follows:

$$u_1(x_1, x_2) = ax_1 + bx_2 + cx_1x_2 + dx_1^2 + ex_2^2 \tag{2.5}$$

where a, b, c, d, e, are constant (and a symmetric u_2).

We always assume $d < 0$ (the sign of e does not matter) to avoid unbounded play, and ensure the existence of at least one Nash equilibrium. Clearly, this game has a potential function $P(x_1, x_2) = a(x_1 + x_2) + cx_1x_2 + d(x_1^2 + x_2^2)$ which is concave (resp. strictly concave) if and only if $|c| \leq 2d$ (resp. $|c| < 2d$).

Our goal is to compute the CCEs maximizing the total payoff $u_1 + u_2$ and to compare this joint payoff with the efficient payoff and Nash equilibrium payoff.

Because our game is symmetric, if L is a CCE then $\widetilde{L}$ obtained by exchanging the role of x_1 and x_2 is also a CCE and so is $\frac{1}{2}L + \frac{1}{2}\widetilde{L}$. Moreover, all three give the same joint payoff. We can thus limit our search to symmetric lotteries L only (because, when we identify an optimal symmetric CCE, we are also capturing an optimal CCE among all CCEs, symmetric or otherwise). We write $\mathscr{L}^{sy}(\mathbb{R}^2_+)$ for the set of symmetric lotteries.

The key to the entire exercise is to write the CCE equilibrium condition (2.3) in terms of three moments of L. If L is the distribution of the symmetric random variable (Z_1, Z_2), these are respectively the expected values of Z_i, Z_i^2, and $Z_1 \cdot Z_2$ as denoted below.

$$\alpha = E_L[Z_i];\ \beta = E_L[Z_i^2];\ \gamma = E_L[Z_1 \cdot Z_2] \tag{2.6}$$

Lemma 2.3. *Any symmetric lottery $L \in \mathscr{L}^{sy}(\mathbb{R}^2_+)$ is a CCE of the game (2.5) if and only if*

$$\max_{z \geq 0}\{(a + c\alpha)z + dz^2\} \leq a\alpha + d\beta + c\gamma$$

and the corresponding utility (for one player) is

$$u_1(L) = (a + b)\alpha + (d + e)\beta + c\gamma.$$

Proof. First note that the expected utility (for one player) from any L for the game (2.5) can be written as

$$u_1(L) = aE_L[Z_1] + bE_L[Z_2] + cE_L[Z_1 \cdot Z_2] + dE_L[Z_1^2] + eE_L[Z_2^2]$$

which by symmetry is

$$\begin{aligned} u_1(L) &= (a+b)E_L[Z_1] + (d+e)E_L[Z_1^2] + cE_L[Z_1 \cdot Z_2] \\ &= (a+b)\alpha + (d+e)\beta + c\gamma. \end{aligned}$$

Now, consider the expected payoff when player 1 plays a pure strategy z while player 2 "commits". This is given by

$$\begin{aligned} & (a + cE_L[Z_2])z + bE_L[Z_2] + dz^2 + eE_L[Z_2^2] \\ = \; & (a + c\alpha)z + dz^2 + b\alpha + e\beta. \end{aligned}$$

Hence, L is a CCE if and only if

$$\max_{z \geq 0}\{(a + c\alpha)z + dz^2\} \leq a\alpha + d\beta + c\gamma$$

rearranging the CCE equilibrium condition (2.3). □

In order to derive the utility maximizing CCEs, we identify the range of the vector (α, β, γ) when $L \in \mathscr{L}^{sy}(\mathbb{R}^2_+)$. We also show that this range is covered by two families of very simple lotteries with at most four strategy profiles in their support.

Let $\mathscr{L}^*$ be the subset of $\mathscr{L}^{sy}(\mathbb{R}^2_+)$ containing the simple lotteries of the form $L = \frac{q}{2}(\delta_{z,z} + \delta_{z',z'}) + \frac{p}{2}(\delta_{z,z'} + \delta_{z',z})$, where z, z', q and p are non-negative and $q + p = 1$.

Let $\mathscr{L}^{**}$ be the subset of $\mathscr{L}^{sy}(\mathbb{R}^2_+)$ of the form $L = q \cdot \delta_{z,z} + q' \cdot \delta_{0,0} + \frac{p}{2}(\delta_{0,z} + \delta_{z,0})$, where z, q, q' and p are non-negative and $q + q' + p = 1$.

Lemma 2.4. *i) For any $L \in \mathscr{L}^{sy}(\mathbb{R}^2_+)$ and the corresponding random variable (Z_1, Z_2), we have*

$$\alpha, \gamma \geq 0;\ \beta \geq \gamma;\ \beta + \gamma \geq 2\alpha^2; \tag{2.7}$$

ii) Equality $\beta = \gamma$ holds if and only if L is diagonal: $Z_1 = Z_2$ (a.e.);

iii) Equality $\beta + \gamma = 2\alpha^2$ holds if and only if L is anti-diagonal: $Z_1 + Z_2$ is constant (a.e.);

iv) For any $(\alpha, \beta, \gamma) \in \mathbb{R}^3_+$ satisfying inequalities (2.7), there exists $L \in \mathscr{L}^* \cup \mathscr{L}^{**}$ with precisely these parameters.

The proof is postponed to the Appendix. Note that (2.7) implies $\beta \geq \alpha^2$, with equality $\beta = \alpha^2$ if and only if L is deterministic, because $\beta = \alpha^2$ implies both $\beta = \gamma$ and $\beta + \gamma = 2\alpha^2$.

Lemmata 2.3 and 2.4 imply the following two-step algorithm to find the utility maximizing CCEs.

Theorem 2.5. *Given the quadratic game (2.5), the following nested programs generate the utility maximizing CCEs:*

Step 1: Fix α non negative, and solve the linear program

$$\max_{\beta,\gamma}\{(d+e)\beta + c\gamma\} \text{ under constraints}$$

$$\beta \geq \gamma \geq 0;\ \beta + \gamma \geq 2\alpha^2;\ d\beta + c\gamma \geq \max_{z \geq 0}\{(a + c\alpha)z + dz^2\} - a\alpha.$$

Step 2: With the solutions $\beta(\alpha), \gamma(\alpha)$ found in Step 1, solve

$$\max_{\alpha}\{(a+b)\alpha + (d+e)\beta(\alpha) + c\gamma(\alpha)\} \text{ under constraints}$$

$$\alpha \geq 0; \ \max_{z\geq 0}\{(a+c\alpha)z + dz^2\} \leq a\alpha + d\beta(\alpha) + c\gamma(\alpha).$$

Moreover, there is a utility maximizing CCE in $\mathscr{L}^* \cup \mathscr{L}^{**}$.

We can finally compare the optimal joint utility in a CCE to that of the Nash equilibrium (or equilibria) of the game.
A complete discussion of these two programs for an arbitrary game given by (2.5) involves too many cases to be of any help. So, in the next two sections, we focus on two simple subclasses with a familiar microeconomic interpretation, where our Theorem 2.5 can be usefully applied.

2.3 COURNOT DUOPOLY

The general form of the symmetric Cournot duopoly with linear demand and quadratic costs is

$$u_i(x_1, x_2) = (A - B(x_1 + x_2))_+ \cdot x_i - f(x_i)$$

(with the notation $(z)_+ = \max\{z, 0\}$), where A and B are positive constants, f is the cost function and x_i is firm i's supply.

The quadratic cost function f may have increasing or decreasing marginal costs. Under increasing marginal costs, we set $f(x_1) = Cx_1 + Dx_1^2$ for all $x_1 \geq 0$, with $C,\ D > 0$. This game has a unique Cournot-Nash equilibrium which is strict, regular and interior (see Subsection 1.1 above). The critical coefficient here is $\rho = \frac{B^2}{[2(B+D)]^2} < \frac{1}{4}$. Therefore, by the result in Gerard-Varet and Moulin (1978) mentioned in Subsection 2.1.1, no CCE can improve upon the Nash equilibrium.

2.3.1 Decreasing Marginal Cost

We turn to the case of decreasing marginal cost in which the cost function[8] f is as follows.

$$f(x_1) = Cx_1 - Dx_1^2 \text{ for } 0 \leq x_1 \leq \frac{C}{2D};\ f(x_1) = \frac{C^2}{4D} \text{ for } x_1 \geq \frac{C}{2D}$$

We assume $A > C$ to ensure a strict incentive to supply some positive amount at the null output. To guarantee that u_i is concave in x_i, we further assume $B > D$.

The optimal monopoly output for this model is $x^m = \frac{A-C}{2(B-D)}$, whenever $\frac{A-C}{2(B-D)}$ is in $(0, \frac{C}{2D})$, i.e., whenever $\frac{A}{C} < \frac{B}{D}$; otherwise, $x^m = \frac{A}{2B}$. To keep our discussion simple, we concentrate on the former standard case, $\frac{A}{C} < \frac{B}{D}$, where the marginal cost is still positive at the monopoly output so that $x^m = \frac{A-C}{2(B-D)}$. Then the monopoly profit is $\pi^m = \frac{(A-C)^2}{4(B-D)}$. Our assumptions on parameters are:

$$1 < \frac{A}{C} < \frac{B}{D} \tag{2.8}$$

[8] Note that this function is, strictly speaking, piecewise-quadratic. However, under the assumptions (2.8), the Nash equilibrium strategies, the efficient strategies, and the support of our optimal CCEs, all belong to the strictly concave part of the domain of f.

It will thus be convenient to express the profit function as follows.

$$u_1(x_1, x_2) = ax_1 - dx_1^2 - cx_1x_2$$

where, $a = A - C$, $d = B - D$ and $c = B$ are all positive; moreover, $D > 0$ implies $\frac{c}{d} > 1$. The critical parameter in our analysis below is indeed $\theta = \frac{c}{d}$, varying in $]1, \infty[$.

The monopoly output is $x^m = \frac{a}{2d}$ with corresponding profit $\pi^m = \frac{a^2}{4d}$. This is the maximal total (of two firms') profit and it is achieved, for instance, by the fair (symmetric) lottery $L^m = \frac{1}{2}\delta_{(x^m,0)} + \frac{1}{2}\delta_{(0,x^m)}$.

The game has a unique *symmetric*[9] Cournot-Nash equilibrium $x^{CN} = \frac{a}{c+2d}$, achieving total profit $\pi^{CN} = (u_1 + u_2)(x^{CN}, x^{CN}) = \frac{2a^2d}{(c+2d)^2}$.

The relative efficiency of this Cournot-Nash equilibrium is $\frac{\pi^{CN}}{\pi^m} = \frac{8}{(\frac{c}{d}+2)^2} = \frac{8}{(\theta+2)^2}$, that decreases rapidly from 88.9%, for $\theta = 1$ (i.e., linear costs with $D = 0$), to 0 as θ increases (i.e., $\frac{B}{D}$ decreases).

The potential function for this game is $P_C(x_1, x_2) = a(x_1 + x_2) - cx_1x_2 - d(x_1^2 + x_2^2)$, which is concave, indeed strictly concave, when $\theta = \frac{c}{d} < 2$. Thus, following Neyman (1997; Theorem 2), when $\theta < 2$, there is a unique CE for this game that coincides with the Cournot-Nash equilibrium x^{CN}.

We now compute the optimal (profit maximizing) CCE with the help of our Theorem 2.5. We distinguish three cases as below, based on the range of $\theta = \frac{c}{d}$.

[9] Asymmetric Cournot-Nash equilibria may arise as well which are discussed below.

Case 1: Strongly decreasing marginal costs: $\theta \geq 2 \Longleftrightarrow 1 < \frac{B}{D} \leq 2$

There are two pure Nash equilibria in each of which firm i supplies the monopoly quantity x^m and crowds firm j out. Indeed, we have $u_2(x^m, x_2) = (a - cx^m)x_2 - dx_2^2 = a(1 - \frac{1}{2}\frac{c}{d})x_2 - dx_2^2$, so that $(x^m, 0)$ is a Nash equilibrium if and only if $\frac{c}{d} > 2$.

Therefore, the "random monopoly" lottery, $L^m = \frac{1}{2}\delta_{(x^m,0)} + \frac{1}{2}\delta_{(0,x^m)}$, which is the average of two Nash equilibria, is a CE (*a la* Aumann). Full efficiency is thus achieved with stronger incentive properties of CE as opposed to CCE.

Case 2: Moderately decreasing marginal costs: $1.171 \leq \theta < 2 \Longleftrightarrow 2 < \frac{B}{D} \leq 6.828$

The monopoly outcome $(x^m, 0)$ is no longer a Nash equilibrium here and thus the random monopoly is not a CE either. However, we show that the latter is a CCE and is fully efficient. On the other hand, the symmetric Cournot-Nash equilibrium (x^{CN}, x^{CN}) is the only Nash equilibrium (the relative efficiency of which, in this interval of θ, decreases from 79.5%, for $\theta = 1.171$, to 50%, for $\theta = 2$).

Proposition 2.6. *For $\theta = \frac{B}{B-D} \in [2(2-\sqrt{2}), 2] \simeq [1.171, 2]$, the random monopoly, $L^m = \frac{1}{2}\delta_{(x^m,0)} + \frac{1}{2}\delta_{(0,x^m)}$, is a fully efficient (hence, optimal) CCE.*

Proof. For the lottery L^m, we have $\alpha = \frac{1}{2}x^m = \frac{a}{4d}$, $\beta = \frac{1}{2}(x^m)^2 = \frac{a^2}{8d^2}$ and $\gamma = 0$. By Lemma 1 (modulo a change of sign for c and d), L^m is a CCE if and only if

$$\max_{z \geq 0}\{(a - \frac{ac}{4d})z - dz^2\} \leq \frac{a^2}{8d}.$$

By assumption $\frac{c}{4d} < 1$, the left-hand side is $\frac{a^2}{4d}(1 - \frac{c}{4d})^2$ and the equilibrium condition therefore is $(1 - \frac{c}{4d})^2 \leq \frac{1}{2}$, which boils down to $\frac{c}{d} \geq 2(2 - \sqrt{2})$. $\square$

Case 3: Lightly decreasing marginal costs: $1 < \theta < 1.171 \iff \frac{B}{D} > 6.828$

Now we assume $\theta \in]1, 2(2-\sqrt{2})[$, which is the most interesting case because the random monopoly is no longer a CCE, and the optimal CCE captures less than the efficient surplus, yet substantially more than the unique Nash equilibrium (x^{CN}, x^{CN}) (the relative efficiency of which, in this interval of θ, decreases from 88.9%, for $\theta = 1$, to 79.5%, for $\theta = 1.171$). The proof of the following result requires the full force of our Theorem 2.5.

Proposition 2.7. *For* $\theta = \frac{B}{B-D} \in]1, 2(2-\sqrt{2})[$, *the optimal CCE is* $L^{CC} = \frac{1}{2}\delta_{(x^{CC},0)} + \frac{1}{2}\delta_{(0,x^{CC})}$, *where* $x^{CC} = \frac{2a}{d(2+\theta+2\sqrt{\theta-1})}$.

Proof. Consider the linear program in Step 1 of Theorem 2.5, where α is fixed, taking into account the change of sign as follows.

$$\min_{\beta,\gamma}\{d\beta + c\gamma\} \text{ under constraints}$$

$$\beta \geq \gamma \geq 0;\ \beta + \gamma \geq 2\alpha^2;\ d\beta + c\gamma \leq a\alpha - \max_{z\geq 0}\{(a - c\alpha)z - dz^2\}.$$

As $c > d$, the optimal choice is clearly $\beta = 2\alpha^2$ and $\gamma = 0$. By Lemma 2.4, this implies that the corresponding lottery L in $\mathscr{L}^{sy}(\mathbb{R}^2_+)$ is anti-diagonal: if (Z_1, Z_2) has probability distribution L, then $Z_1 + Z_2 = 2\alpha$ (a.e.). Moreover, $\gamma = 0$ implies $Z_1 \cdot Z_2 = 0$ (a.e.), so that L takes the simple form $L = \frac{1}{2}\delta_{(2\alpha,0)} + \frac{1}{2}\delta_{(0,2\alpha)}$, i.e., a lottery in $\mathscr{L}^* \cap \mathscr{L}^{**}$.

Note that the equilibrium condition for CCE $d\beta + c\gamma \leq a\alpha - \max_{z \geq 0}\{(a - c\alpha)z - dz^2\}$ places the following constraint on α.

$$\max_{z \geq 0}\{(a - c\alpha)z - dz^2\} \leq a\alpha - 2d\alpha^2 \tag{2.9}$$

Step 2 in Theorem 2.5 now is

$$\max_{\alpha} a\alpha - 2d\alpha^2 \text{ under the constraint (2.9).}$$

We distinguish two cases:

Case 1: $\alpha \geq \frac{a}{c}$. The left-hand-side in (2.9) is 0, so this inequality gives $\alpha \leq \frac{a}{2d}$, which together with $\alpha \geq \frac{a}{c}$ contradicts the assumption $\theta < 2$.

Case 2: $\alpha < \frac{a}{c}$. The left-hand-side in (2.9) is now $\frac{(a-c\alpha)^2}{4d}$; thus, the inequality in (2.9) is

$$(8d^2 + c^2)\alpha^2 - 2a(c + 2d)\alpha + a^2 \leq 0.$$

Upon changing the variable α to $\lambda = \frac{d}{a}\alpha$, the assumption $\alpha < \frac{a}{c}$ becomes $\lambda < \frac{1}{\theta}$, and the program in Step 2 can be written as:

$$\max_{\lambda}\{\lambda - 2\lambda^2\} \text{ under } H(\lambda) = (8 + \theta^2)\lambda^2 - 2(2 + \theta)\lambda + 1 \leq 0.$$

The discriminant of H is $4(\theta - 1) > 0$, with minimum at $\lambda = \frac{2+\theta}{8+\theta^2}$. It is easy to check that $\frac{1}{4} < \frac{2+\theta}{8+\theta^2} < \frac{1}{2}$. Moreover, we have, $H(\frac{1}{2}) = \frac{1}{4}(2 - \theta)^2 > 0$ and $H(\frac{1}{4}) = \frac{1}{16}(\theta^2 - 8\theta + 8)$, also non negative, because the roots of $\theta^2 - 8\theta + 8$ are $2(2 - \sqrt{2})$ and $2(2 + \sqrt{2})$ and by assumption $\theta < 2(2 - \sqrt{2})$. We thus conclude that both roots of H are in $[\frac{1}{4}, \frac{1}{2}]$.

The function $\lambda - 2\lambda^2$ peaks at $\frac{1}{4}$; as we are maximizing it over a subinterval of $[\frac{1}{4}, \frac{1}{2}]$, the optimal λ is the lowest root of H, namely,

$$\widetilde{\lambda} = \frac{2+\theta-2\sqrt{\theta-1}}{8+\theta^2} = \frac{1}{2+\theta+2\sqrt{\theta-1}}.$$

The optimal α is $\widetilde{\alpha} = \frac{a}{d}\widetilde{\lambda}$. Thus, we conclude $x^{CC} = 2\widetilde{\alpha} = \frac{2a}{d(2+\theta+2\sqrt{\theta-1})}$ as desired.

Total profit at the optimal CCE is

$$\pi^{CC} = (u_1+u_2)(L^{CC}) = a\widetilde{\alpha} - 2d\widetilde{\alpha}^2 = \frac{2a^2(\theta+2\sqrt{\theta-1})}{d(2+\theta+2\sqrt{\theta-1})^2}$$

which is easy to check. □

Efficiency performance

We collect the results of the three cases above.

Corollary 2.8. *In the Cournot duopoly with decreasing marginal costs satisfying (2.8), the relative efficiency of the optimal CCE and its relative improvement over the symmetric Cournot-Nash equilibrium only depend upon the parameter* $\theta = \frac{B}{B-D}$, *as follows:*

$$\frac{\pi^{CC}}{\pi^m} = \begin{cases} \frac{8(\theta+2\sqrt{\theta-1})}{(2+\theta+2\sqrt{\theta-1})^2} & \text{for } 1 < \theta \leq 1.171 \\ 1 & \text{for } \theta \geq 1.171 \end{cases}$$

$$\frac{\pi^{CC}}{\pi^{CN}} = \begin{cases} \frac{(\theta+2)^2(\theta+2\sqrt{\theta-1})}{(2+\theta+2\sqrt{\theta-1})^2} & \text{for } 1 < \theta \leq 1.171 \\ \frac{(\theta+2)^2}{8} & \text{for } \theta \geq 1.171 \end{cases}$$

Figure 2.1 illustrates the behavior of the two ratios[10] in Corollary 2.8.

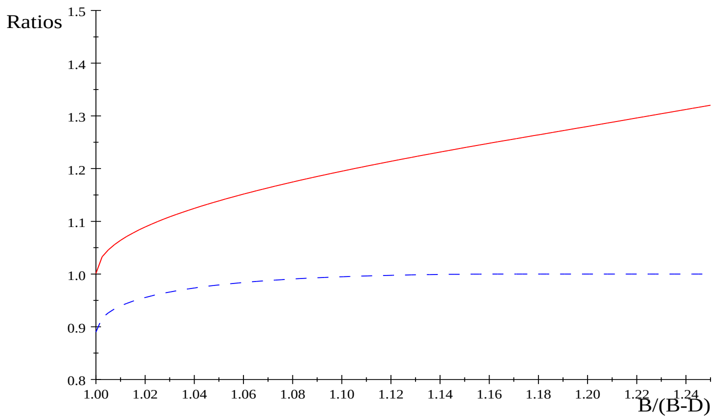

Figure 2.1: $\frac{\pi^{CC}}{\pi^{m}}$ *and* $\frac{\pi^{CC}}{\pi^{CN}}$ *for Cournot Duopoly*

As an example of Case 3, consider $\theta = 1.1$, corresponding, for instance, to the game with payoff function $u_1(x_1, x_2) = 2x_1 - x_1^2 - (1.1)x_1x_2$. The optimal CCE has one firm producing $x^{CC} = 1.07$ while the other firm stays out, alternating roles with equal probabilities. The efficiency ratio is very high: $\frac{\pi^{CC}}{\pi^{M}} = 0.995$, allowing a near 20% improvement over and above the Cournot Nash equilibrium: $\frac{\pi^{CC}}{\pi^{CN}} = 1.192$.

2.4 PUBLIC GOOD PROVISION

In this game of contributions to a public good, we have two identical agents for whom the cost of contributing to the public good is linear and the benefit from

[10] For clarity in presentation, we have trauncated the range of θ in Figure 2.1.

the public good is concave, quadratic in total contributions. Formally,

$$u_1(x_1, x_2) = A(x_1 + x_2) - D(x_1 + x_2)^2 - Cx_1$$

where A, C and D are positive constants. We also assume $A > C$, so that any Nash equilibrium (of voluntary contributions) involves positive contributions.[11] We rewrite the payoff function in a more compact form

$$u_1(x_1, x_2) = ax_1 + bx_2 - d(x_1 + x_2)^2$$

with $a = A - C < b = A$ and $d = D$ (a, b and d are positive).

The efficient level of public good, x^{eff}, maximizes $u_1 + u_2 = (a + b)(x_1 + x_2) - 2d(x_1 + x_2)^2$. Thus $x^{eff} = \frac{a+b}{4d}$ and the efficient total payoff is $\pi^{eff} = \frac{(a+b)^2}{8d}$. A Nash equilibrium is any pair (x_1, x_2) such that $x_1 + x_2 = x^N = \frac{a}{2d}$, with corresponding total (of both players) equilibrium payoff $\pi^N = (a + b)x^N - 2d(x^N)^2 = \frac{ab}{2d}$. Note that the split of the efficient (respectively, Nash equilibrium) contribution between the two players is arbitrary.

The potential function for this game is $P_{PG}(x) = a(x_1 + x_2) - d(x_1 + x_2)^2$, therefore concave but not strictly so. The potential is maximal when $x_1 + x_2 = x^N = \frac{a}{2d}$, implying that any CE of this game is a mixture of Nash equilibrium profiles (Neyman 1997; Theorem 1).

We now look for the optimal CCE for this game. We find that, as in the

[11] If $A < C < 2A$, the Nash equilibrium is $(0, 0)$ but there is still a positive payoff for cooperative players. In this case, computations similar to the ones below (and thus omitted for brevity) show that no CCE yields a positive payoff.

Cournot duopoly example, for a large set of parameters ($\frac{C}{A} \leq \frac{2}{3}$), the optimal symmetric CCE is fully efficient. This is statement $i)$ in our next result.

Proposition 2.9. *$i)$ If $\frac{b}{a} \leq 3 \Leftrightarrow 3C \leq 2A$, the optimal symmetric CCE is $L = \frac{1}{2}\delta_{(x^{eff},0)} + \frac{1}{2}\delta_{(0,x^{eff})}$, dividing equally the efficient total profit $\pi^{CC} = \pi^{eff} = \frac{(a+b)^2}{8d}$. $ii)$ If $\frac{b}{a} \geq 3 \Leftrightarrow 3C \geq 2A$, the optimal symmetric CCE is $L = \frac{1}{2}\delta_{(\frac{a}{d},0)} + \frac{1}{2}\delta_{(0,\frac{a}{d})}$, dividing equally the total profit $\pi^{CC} = \frac{a(b-a)}{d}$.*

Proof. Write the linear program in Step 1 of Theorem 2.5, where α is fixed:

$$\min_{\beta,\gamma}\{2d(\beta+\gamma)\} \text{ under constraints}$$

$$\beta \geq \gamma \geq 0;\ \beta+\gamma \geq 2\alpha^2;\ d(\beta+2\gamma) \leq a\alpha - \max_{z\geq 0}\{(a-2d\alpha)z - dz^2\}.$$

The minimum of both $\beta+\gamma$ and $\beta+2\gamma$ in the region $\{\beta \geq \gamma \geq 0;\ \beta+\gamma \geq 2\alpha^2\}$ is achieved at $\beta = 2\alpha^2$ and $\gamma = 0$.

Therefore, this choice maximizes our objective function and makes the constraint as weak as possible. Exactly as in the proof of Proposition 2.7, the optimal symmetric CCE takes the form $L = \frac{1}{2}\delta_{(2\alpha,0)} + \frac{1}{2}\delta_{(0,2\alpha)}$, $L \in \mathscr{L}^* \cap \mathscr{L}^{**}$, where α solves the program in Step 2 as follows:

$$\max_{\alpha}\{(a+b)\alpha - 4d\alpha^2\} \text{ under constraints}$$

$$\alpha \geq 0;\ \max_{z\geq 0}\{(a-2d\alpha)z - dz^2\} \leq a\alpha - 2d\alpha^2.$$

If $\alpha \geq \frac{a}{2d}$, the constraint reduces to $0 \leq \alpha(a-2d\alpha) \Rightarrow \alpha = \frac{a}{2d}$. So, we can assume $\alpha \leq \frac{a}{2d}$ and rewrite the above constraints as

$$\frac{(a-2d\alpha)^2}{4d} \leq a\alpha - 2d\alpha^2 \Longleftrightarrow (a-2d\alpha)(6d\alpha - a) \geq 0.$$

So, finally the constraints boil down to $\alpha \in [\frac{a}{6d}, \frac{a}{2d}]$.

The unconstrained maximum $\alpha^* = \frac{a+b}{8d}$ of the objective function is in the above interval if and only if $b \leq 3a$ (recall $b > a$); in this case, the optimal symmetric CCE, L, as below, is fully efficient.

$$L = \frac{1}{2}\delta_{(\frac{a+b}{4d},0)} + \frac{1}{2}\delta_{(0,\frac{a+b}{4d})} = \frac{1}{2}\delta_{(x^{eff},0)} + \frac{1}{2}\delta_{(0,x^{eff})}$$

On the other hand if $b > 3a$, we have $\alpha^* > \frac{a}{2d}$ and hence the optimal choice is $\widehat{\alpha} = \frac{a}{2d}$ with the optimal symmetric CCE $L = \frac{1}{2}\delta_{(\frac{a}{d},0)} + \frac{1}{2}\delta_{(0,\frac{a}{d})}$, as in the statement. Following Lemma 2.3, the optimal CCE payoff, π^{CC}, in this case is $\pi^{CC} = 2((a+b)\alpha - 4d\alpha^2) = \frac{a(b-a)}{d}$. □

2.4.1 Efficiency Performance

We present the relative efficiencies of the optimal CCEs for the two cases in Proposition 2.9 above.

Corollary 2.10. *In our game of public good provision, the relative efficiency of the optimal CCE and its relative improvement over the symmetric Nash equilibrium depend only upon the parameter* $\sigma = \frac{C}{A} = \frac{b-a}{b}$, $0 \leq \sigma \leq 1$, *as follows:*

$$\frac{\pi^{CC}}{\pi^{eff}} = \begin{cases} 1 & \text{for } 0 \leq \sigma \leq \frac{2}{3} \\ \frac{8\sigma(1-\sigma)}{(2-\sigma)^2} & \text{for } \frac{2}{3} \leq \sigma \leq 1. \end{cases}$$

$$\frac{\pi^{CC}}{\pi^N} = \begin{cases} \frac{(2-\sigma)^2}{4(1-\sigma)} & for\ 0 \le \sigma \le \frac{2}{3} \\ 2\sigma & for\ \frac{2}{3} \le \sigma \le 1. \end{cases}$$

Figure2.2 below shows the behavior of these two ratios in Corollary 2.10.

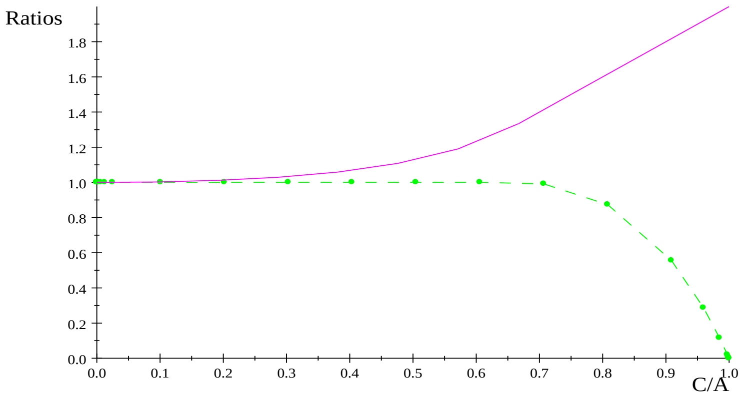

Figure 2.2: $\frac{\pi^{CC}}{\pi^{eff}}$ *and* $\frac{\pi^{CC}}{\pi^N}$ *for Provision of Public Good*

Note that the improvement over the Nash equilibrium is at least 33% whenever $\frac{C}{A} \le \frac{2}{3}$, and reaches 100% as $\frac{C}{A}$ approaches 1.

The example in the Introduction is a game for which $\sigma = \frac{3}{4}$; the results reported there follow directly from the Proposition 2.9 and Corollary 2.10.

2.5 CONCLUSION

We have analyzed coarse correlated equilibria in a class of two-person symmetric games where correlation *a la* Aumann does not offer anything more than the Nash equilibrium. Relying heavily on the quadratic form of the payoff function, we have characterized the utility maximizing CCE and have shown that they have a very simple support with only four deterministic strategy profiles. Such a computation is the first of its kind for coarse correlated equilibria and, as far as we know, there is no similar result for standard correlated equilibria. This is why we regard this exercise as an interesting first step towards more sophisticated computations.

Our methodology can be extended to games where the strategy sets are bounded, such as the search game (Diamond 1982). In the case of the bounded strategy sets, say, $X_1 = X_2 = [0, M]$, Lemma 2.3 is preserved, but we need to adapt the key Lemma 2.4 to describe the range of the three moments (α, β, γ) for all symmetric probability distributions L on $[0, M]^2$. This is done in the Appendix (see Lemma 2.11, more involved than Lemma 2.4).

2.6 APPENDIX

2.6.1 Proof of Lemma 2.4

Statement i): Using the symmetry of L, we have

$$\beta = \frac{1}{2}(E_L[Z_1^2] + E_L[Z_2^2]) = E_L[\frac{1}{2}(Z_1^2 + Z_2^2)] \geq E_L[Z_1 \cdot Z_2] = \gamma.$$

Also, note that $\alpha = \frac{1}{2}(E_L[Z_1] + E_L[Z_2])$ implies $2\alpha^2 = \frac{1}{2}(E_L[Z_1 + Z_2])^2$.

Moreover, $\beta + \gamma = E_L[\frac{1}{2}(Z_1^2 + Z_2^2) + Z_1 \cdot Z_2] = \frac{1}{2}E_L[(Z_1 + Z_2)^2]$. Hence, $\beta + \gamma \geq 2\alpha^2$ follows from $E_L[X^2] \geq (E_L[X])^2$.

Statements ii) and *iii)*: It is straightforward to show that $\beta = \gamma$ holds if and only if L is diagonal; also, $\beta + \gamma = 2\alpha^2$ holds if and only if $Z_1 + Z_2$ is constant.

Statement iv): We compute successively the range of $\mathscr{L}^*$, i.e., the set of triple (α, β, γ) when L is in $\mathscr{L}^*$, and then the range of $\mathscr{L}^{**}$.

Claim 1: The range of $\mathscr{L}^*$ is described by the following system:

$$\gamma \leq \beta \leq 2\alpha^2 \leq \beta + \gamma. \tag{2.10}$$

From the proof of statement *i)*, we have $\beta \geq \gamma$ and $2\alpha^2 \leq \beta + \gamma$. To check $2\alpha^2 \geq \beta$, fix $L \in \mathscr{L}^*$. We have $z + z' = 2\alpha$ and $z^2 + z'^2 = 2\beta$ (by symmetry). Moreover, $z^2 + z'^2 \leq (z + z')^2$, as z and z' are non-negative. Hence, the inequality $\beta \leq 2\alpha^2$ follows.

Conversely, fix (α, β, γ) satisfying (2.10) and check first the existence of non-negative numbers z and z' such that their sum is $S = z + z' = 2\alpha$ and their product is $P = zz' = \frac{1}{2}((z+z')^2 - (z^2 + z'^2)) = 2\alpha^2 - \beta$. Note that both the sum S and the product P are non-negative by assumption, so the existence of z and z' follows from the inequality $S^2 \geq 4P \iff 4\alpha^2 \geq 4(2\alpha^2 - \beta) \iff \beta \geq \alpha^2$. We already noted that system (2.10) implies $\beta \geq \alpha^2$. It remains to choose q and p so that $\gamma = \frac{q}{2}(z^2 + z'^2) + pzz' = q\beta + p(2\alpha^2 - \beta)$, which is clearly possible if and only if $2\alpha^2 - \beta \leq \gamma \leq \beta$.

Claim 2: The range of $\mathscr{L}^{**}$ contains $(0,0,0)$ and the following triples (α, β, γ), where $\alpha > 0$:

$$2\beta - \frac{\beta^2}{\alpha^2} \leq \gamma \leq \beta. \tag{2.11}$$

We first show that these two inequalities imply $2\alpha^2 \leq \beta + \gamma$. If $\beta \geq 2\alpha^2$, there is nothing to prove; so, we assume $\beta \leq 2\alpha^2$. We have $\beta + \gamma \geq 3\beta - \frac{\beta^2}{\alpha^2}$ and $3\beta - \frac{\beta^2}{\alpha^2} \geq 2\alpha^2 \iff (\frac{\beta}{\alpha} - \alpha)(\frac{\beta}{\alpha} - 2\alpha) \leq 0$, which follows from $\alpha^2 \leq \beta \leq 2\alpha^2$. If $\alpha = 0$, then $L = \delta_{0,0}$ which is of course in $\mathscr{L}^{**}$. We thus assume $\alpha > 0$. For any $L \in \mathscr{L}^{**}$, we have:

$$\alpha = (q + \frac{p}{2})z;\ \beta = (q + \frac{p}{2})z^2;\ \gamma = qz^2. \tag{2.12}$$

Therefore, $2\beta - \gamma = (q+p)z^2 \leq z^2 = \frac{\beta^2}{\alpha^2}$, which proves (2.11).

Conversely, we fix (α, β, γ) satisfying (2.11) and show that system (2.12) defines (uniquely) z and a set of convex weights (q, q', p). We have $z = \frac{\beta}{\alpha} > 0$, because $\alpha > 0$; therefore, $\beta > 0$ as well and we compute

$$q + \frac{p}{2} = \frac{\alpha^2}{\beta};\ q = \frac{\alpha^2}{\beta^2}\gamma \Rightarrow p = 2\frac{\alpha^2}{\beta}(1 - \frac{\gamma}{\beta})$$

Clearly, q is non-negative; so is p, because $\gamma \leq \beta$. It only remains to check

$$q + p \leq 1 \Longleftrightarrow 2\frac{\alpha^2}{\beta} - \frac{\alpha^2}{\beta^2}\gamma \leq 1 \Longleftrightarrow 2\beta - \frac{\beta^2}{\alpha^2} \leq \gamma$$

which indeed is (2.11). Hence, Claim 2 is proved.

To conclude the proof of the Lemma, we check finally that the union of the subsets of $\mathbb{R}^3_+$ defined by (2.10) and (2.11) is precisely the set (2.7). We already know the inclusion (2.10) $\cup$ (2.11) $\subseteq$ (2.7). Conversely, if (2.7) holds and $\beta \geq 2\alpha^2$, then we have $\gamma \geq 2\alpha^2 - \beta \geq 2\beta - \frac{\beta^2}{\alpha^2}$, because, the latter inequality is $\frac{\beta^2}{\alpha^2} - 3\beta + 2\alpha^2 \geq 0 \Longleftrightarrow (\frac{\beta}{\alpha} - \alpha)(\frac{\beta}{\alpha} - 2\alpha) \geq 0$ which follows from $\beta \geq 2\alpha^2$. ■

2.6.2 Bounded strategy sets

We assume $X_1 = X_2 = [0, M]$. It turns out that the range of the three moments (α, β, γ) over all symmetric probability distributions L on $[0, M]^2$ contains strictly the range for the subsets of simple lotteries $\mathscr{L}^* \cup \mathscr{L}^{**}$.

Lemma 2.11. *i) For any $L \in \mathscr{L}$, we have $\beta \geq \gamma$; $\beta + \gamma \geq 2\alpha^2$; $\alpha \leq M$; $\beta \leq \alpha M$;*

ii) The range of $\mathscr{L}^*$ is described by the system: $\gamma \leq \beta \leq 2\alpha^2 \leq \beta + \gamma$; $\beta \leq \alpha^2 + (M - \alpha)^2$;

iii) The range of $\mathscr{L}^{**}$ is described by the system: $2\beta - \frac{\beta^2}{\alpha^2} \leq \gamma \leq \beta$; $\beta \leq \alpha M$.

Proof. Statement i) For any $L \in \mathscr{L}$, we have $Z_i^2 \leq Z_i M \Rightarrow \beta = E_L(Z_i^2) \leq E_L(Z_i)M$.

Statement ii) Consider any $L \in \mathscr{L}^*$, where z and z' are both in $[0, M]$. They are the roots of the polynomial $Y^2 - SY + P$ where $S = 2\alpha$ and $P = 2\alpha^2 - \beta$.

As $S \in [0, M]$, this implies $M^2 - SM + P \geq 0 \iff M^2 - 2\alpha M + 2\alpha^2 \geq \beta$ as desired. The converse is also easy to prove. If any (α, β, γ) meets the inequalities in Statement *ii*), one can construct z and z' exactly as in the proof of Claim 1 in Lemma 2.4 and they are both in $[0, M]$. The choice of p and q also proceeds as before.

Statement iii) For any $L \in \mathscr{L}^{**}$, the inequalities in statement *iii*) follow from the system (2.11) and statement *i*). Conversely, fix any (α, β, γ) satisfying the inequalities in the statement and construct z as in Lemma 2.4. Use the system (2.12) to define z, q, q' and p as in Lemma 2.4.Then, $z = \frac{\beta}{\alpha} \leq M$; so, the constructed L has its support in $[0, M]^2$. □

Note that in the statement *i*) in Lemma 2.11, $\beta = \alpha M$ if and only if the support of L is $Z_1 = Z_2 = \{0, M\}$. This is because the equality holds only if $Z_i^2 = Z_i M$ a.e., i.e., $Z_i = 0, M$ a.e.. Also note that the inequalities in statement *ii*) imply $\beta \leq \alpha M$ (which is clear if $M \geq 2\alpha$; and if $M \leq 2\alpha$, observe that $(M - \alpha)(M - 2\alpha) \leq 0 \iff \alpha^2 + (M - \alpha)^2 \leq \alpha M$).

Chapter 3

Coarse Correlated Equilibria in an Abatement Game

3.1 INTRODUCTION

It has been well-developed that the NE of the abatement game (which acts as a prisoner's dilemma) is not a globally optimal solution, and the best response of each nation would be 'not to abate' pollution, as abating pollution involves costs which outweigh the benefits (especially in current period) accruing to each nation. Moreover, since the emissions are a pure public good (bad), a country would still enjoy the benefits of the reduced emissions (if other countries abate). This is referred to as a problem of *free-riding*. This problem of free-riding is because of the lack of coordination among the countries.

A correlation device is a lottery over the outcomes (strategy profiles) of the game. A *correlated equilibrium* (Aumann, 1974, 1987; thereafter CE) is *imple-*

mented[1] by a mediator who selects strategy profiles according to a publicly known probability distribution and sends to each player the private recommendation to play the corresponding realized strategy. The equilibrium property is that each player finds it optimal to follow this recommendation. In a *coarse correlated equilibrium* (Moulin and Vial 1978; thereafter CCE), the mediator requires more commitment from the players: it asks the players, *before running the lottery*, to either commit to the future outcome of the lottery or play any strategy of their own without learning anything about the outcome of the lottery. The equilibrium property is that each player finds it optimal to commit *ex ante* to use the strategy selected by the lottery.

In the context of the climate change negotiations, the mediator should be some sort of governing body, which would provide a direction to all the countries towards the ultimate goal of emission reduction. By correlation, the mediator may not suggest the countries specifically what startegy to pursue, but rather draw their attention towards a small set of strategies worth to be a subject of deeper analysis. Forgó, Fulop and Prill (2005) have tried to analyze the climate change negotiations using the concept of tree-correlated equilibrium[2], and Forgó

[1] However, not *fully*, as shown by Kar, Ray and Serrano (2010).

[2] Tree-correlated equilibrium applies the concept of CCE in extensive form games with perfect information. The players at each decision node have to decide whether to commit to the mediator (in which case the game moves according to the suggestion of the mediator) or not. If the player decides not to commit, then the mediator withdraws from mediation and the game proceeds unattended.

(2011) introduced the idea of soft-correlated equilibrium[3] to analyse such games. Both these equilibrium concepts are (modified) versions of CCE.

First we should note that for the abatement game there is unique CE which coincides with the NE of the game (because the abatement game is a game with a smooth and concave potential function). However, the game has many more CCEs, and we try to derive the optimal CCE using the two-step algorithm introduced in Moulin, Ray and Sen Gupta (2014) to compute the most efficient CCE in a symmetric 2-person game with quadratic payoff functions.

Applying the results of Gerard-Varet and Moulin (1978), where they show that the NE can be *locally* improved by (a concept similar to) CCE under certain parametric condition. In case of the abatement game (equation 3.3) the condition comes out to be that the benefit parameter (b) is bigger than the cost parameter (c). For our formal characterisation of the optimal CCE for any 2-player abatement game (Theorem 3.5) we assume that $b > c$. As in Moulin, Ray and Sen Gupta (2014), we find that the optimal lottery is a 2-dimensional anti-diagonal[4] symmetric lottery.[5] The total payoff at the optimal CCE is very close to the efficient payoff for this class of games, with a small improvement above the Nash

[3] In this, like coarse correlation, the players have to first decide whether or not to commit to the mediator, after which the mediator selects an action profile based upon a commonly known probability distribution. The player who decides not to commit to the mediator can choose any alternate strategy, *except* for the one selected by the mediator; unlike coarse correlation where a deviant can choose to play any strategy (even the ones selected by the mediator).

[4] An anti-diagonal symmetric lottery is the one where only the anti-diagonal elements of the (k x k) probability distribution matrix are strictly positive, i.e., $p_{ij} > 0$, when $i + j = k + 1$ and $p_{ij} = 0$, when $i + j \neq k + 1$.

[5] As in the example, the optimal lottery is a symmetric lottery with finite support similar to those studied by Ray and Sen Gupta (2013) who called such a lottery a Simple Symmetric Correlation Device (SSCD) as introduced in Ganguly and Ray (2005) to discuss correlation.

equilibrium total payoff. [6]

The chapter proceeds in the following manner: we define CCEs in Subsection 3.2.1 and present the two-person abatement game in Subsection 3.2.2. Section 3.3 develops our general methodology to compute the optimal CCE for the 2-player abatement game while Section 3.4 concludes.

3.2 MODEL

3.2.1 Coarse Correlation

This subsection, borrowed from Moulin, Ray and Sen Gupta (2014), is given here for the sake of completeness.

Fix a two-person normal form game, $G = [X_1, X_2; u_1, u_2]$, where the strategy sets are closed real intervals and the payoff functions $u_i : X_1 \times X_2 \to \mathbb{R}$, $i = 1, 2$, are continuous. We write $\mathbb{C}(X_1 \times X_2)$ for the set of such continuous functions and similarly, $\mathbb{C}(X_i)$ for the set of continuous functions on X_i.

Let $\mathscr{L}(X_1 \times X_2)$ with generic element L and $\mathscr{L}(X_i)$ with generic element ℓ_i be the sets of probability measures on $X_1 \times X_2$ and X_i respectively. That is to say, L is a Radon measure[7] on $X_1 \times X_2$ of mass 1 (similarly, so is ℓ_i, on X_i). We write the mean of $u_i(x_1, x_2)$ with respect to L as $u_i(L)$; similarly, $f(\ell_i)$ is the mean of $f(x_i)$ with respect to ℓ_i for any $f \in \mathbb{C}(X_i)$. Given $L \in \mathscr{L}(X_1 \times X_2)$, we

[6] From Corollary 3.6 and footnote 15, the ratio of CCE to efficient payoff is atleast $\frac{3}{4}$, which can also be seen from Figure 3.1. Similarly, as noted in Figure 3.3 and the example in the Conclusion section, the maximum improvement over Nash equilibrium payoff achievable is just above $\frac{1}{2}\%$.

[7] A positive linear functional on $\mathbb{C}(X_1 \times X_2)$.

write L^i for the marginal distribution of L on X_i, defined as follows:

$$\forall f \in \mathbb{C}(X_1),\ f(L^1) = f^*(L), \text{ where } f^*(x_1, x_2) = f(x_1)^8 \text{ for all } x_1, x_2 \in X_1 \times X_2 \tag{3.1}$$

(with a symmetric definition for L^2).

The deterministic distribution at z[9] is denoted by δ_z, and for a product distributions such as $\delta_{x_1} \otimes \ell_2$ we write $u_i(\delta_{x_1} \otimes \ell_2)$ simply as $u_i(x_1, \ell_2)$.

Definition 3.1. *A coarse correlated equilibrium (CCE) of the game G is a lottery $L \in \mathscr{L}(X_1 \times X_2)$ such that*

$$u_1(L) \geq u_1(x_1, L^2) \text{ and } u_2(L) \geq u_2(L^1, x_2) \text{ for all } (x_1, x_2) \in X_1 \times X_2. \tag{3.2}$$

The discussion of the mediator in the Introduction can be formalized by the normal form game $\widetilde{G} = [X_1 \cup \{\blacktriangle\}, X_2 \cup \{\blacktriangle\}; \widetilde{u}_1, \widetilde{u}_2]$ where the additional strategy $\blacktriangle$ means "commit to the outcome of L". Payoffs are the same as in G if no player chooses $\blacktriangle$, otherwise are as follows:

$$\widetilde{u}_1(\blacktriangle, \blacktriangle) = u_1(L);\ \widetilde{u}_1(\blacktriangle, x_2) = u_1(L^1, x_2);\ \widetilde{u}_1(x_1, \blacktriangle) = u_1(x_1, L^2)$$

Then, L is a CCE if and only if $(\blacktriangle, \blacktriangle)$ is a Nash equilibrium of $\widetilde{G}$.

[8] Fixing an x_2 the probability distribution is defined over $f(x_1)$

[9] i.e. with a probability of 1, outcome z is chosen by the lottery.

3.2.2 Abatement Game and CCEs

We consider the model proposed in Barrett (1994) with two countries ($n = 2$). The payoff function of a country is a function of the emission abated by both countries q_1 and q_2. Let us write the total emission abated as $Q = q_1 + q_2$ and therefore we have the benefit function[10] of country i as

$$B_i(Q) = \frac{B}{2}(AQ - \frac{Q^2}{2}).$$

The cost function of each country is a function of its own emission abatement level q_i and is given as $C_i(q_i) = \frac{Cq_i^2}{2}$. The payoff function of country 1 (and similarly for country 2) is thus given by

$$u_1(q_1, q_2) = \frac{AB}{2}(q_1 + q_2) - \frac{B}{4}(q_1 + q_2)^2 - \frac{C}{2}q_1^2, \text{ where } A, B \text{ and } C \text{ are all positive.}$$

We call the above model an *abatement game*. Although the model is similar to the Cournot game, the purpose of this paper is to put forward the relevance of the notion of correlation in environment literature.

We introduce the Barrett's model as it stands and then re-write it in order to compare it with the general quadratic model provided in Moulin, Ray and Sen Gupta (2014). We set $a = \frac{AB}{2}$, $b = \frac{B}{4}$, $c = \frac{C}{2}$ and rewrite the above payoff function in the following form (as in the general model of Moulin, Ray and Sen

[10] Note that the benefit function in the published version of Barrett (1994) has a typo that we have corrected here.

Gupta 2014):

$$u_1(q_1, q_2) = a(q_1 + q_2) - b(q_1 + q_2)^2 - cq_1^2; \; u_2(q_1, q_2) = u_1(q_2, q_1). \tag{3.3}$$

Given q_2, the best response of country 1 (and similarly for country 2) is $BR_1(q_2) = \frac{\partial u_1(q_1,q_2)}{\partial q_1} = a - 2b(q_1 + q_2) - 2cq_1$. Therefore, the Nash equilibrium (q_1^{Neq}, q_2^{Neq}) and corresponding (total) payoff π^{Neq} are

$$q_1^{Neq} = q_2^{Neq} = \frac{a}{2(2b + c)}; \; \pi^{Neq} = \frac{a^2(4b + 3c)}{2(2b + c)^2}.$$

We now compute the efficient profile of emission abatements (q_1^{eff}, q_2^{eff}). To maximize the total payoff $u_1(q_1, q_2) + u_2(q_1, q_2) = 2a(q_1 + q_2) - 2b(q_1 + q_2)^2 - c(q_1^2 + q_2^2)$, we clearly need to choose $q_1 = q_2$ (due to symmetry, the efficient level of abatement should be same for two similar countries); we find

$$q_1^{eff} = q_2^{eff} = \frac{a}{4b + c}; \; \pi^{eff} = \frac{2a^2}{4b + c}.$$

Therefore, the relative efficiency ratio of the Nash outcome is $\frac{\pi^{Neq}}{\pi^{eff}} = \frac{(4+3\lambda)(4+\lambda)}{4(2+\lambda)^2}$, where $\lambda = \frac{c}{b}$, which is plotted below in Figure 3.1.

The abatement game is a game with the potential function $P(q_1, q_2) = a(q_1 + q_2) - b(q_1 + q_2)^2 - c(q_1^2 + q_2^2)$, which is smooth and concave. Therefore, the only CE is the Nash equilibrium q^{Neq} (Neyman 1997).

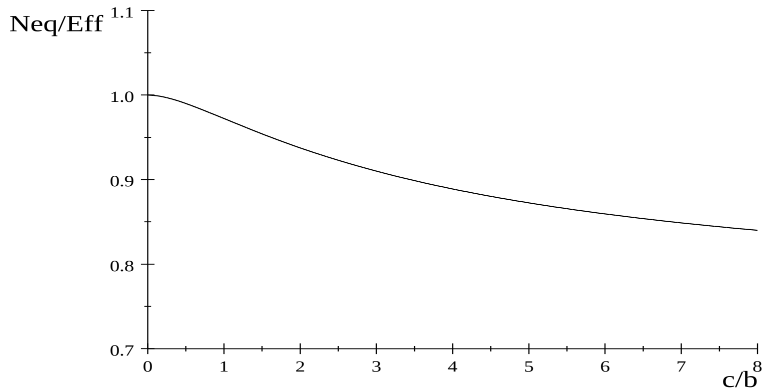

Figure 3.1: $\frac{\pi^{Neq}}{\pi^{eff}}$ *in the abatement game*

3.3 RESULTS

Our goal is to compute the CCEs that maximize the total payoff $u_1 + u_2$. We then compare this joint payoff with the Nash equilibrium payoff and comment on the efficiency of the former. As the abatement game is symmetric, we can limit our search to symmetric lotteries L only (as explained in Moulin, Ray and Sen Gupta 2014, when we identify an optimal symmetric CCE, we are also capturing an optimal CCE among all CCEs, symmetric or otherwise). We denote the set of symmetric lotteries by $\mathscr{L}^{sy}(\mathbb{R}^2_+)$.

We first characterize the equilibrium condition (3.2) presented in Definition 1 in terms of three moments of L. If L is the distribution of the symmetric random variable (Z_1, Z_2), we are interested in the expected values of Z_i, Z_i^2, and $Z_1 \cdot Z_2$ as denoted below.

$$\alpha = E_L[Z_i];\ \beta = E_L[Z_i^2];\ \gamma = E_L[Z_1 \cdot Z_2]$$

Lemma 3.2. *A symmetric lottery $L \in \mathscr{L}^{sy}(\mathbb{R}^2_+)$ is a CCE of the abatement game if and only if*

$$\max_{z \geq 0}\{(a-2b\alpha)z-(b+c)z^2\} \leq a\alpha-(b+c)\beta-2b\gamma \tag{3.4}$$

and the corresponding utility (for one player) is

$$u_1(L) = 2a\alpha - (2b+c)\beta - 2b\gamma.$$

Proof. First note that (by 3.3) the expected utility (for one player) from any lottery $L \in \mathscr{L}^{sy}(\mathbb{R}^2_+)$ can be written as

$$u_1(L) = aE_L[Z_1] + aE_L[Z_2] - bE_L[Z_1^2] - bE_L[Z_2^2] - 2bE_L[Z_1 \cdot Z_2] - cE_L[Z_1^2],$$

which by symmetry is

$$\begin{aligned} u_1(L) &= 2aE_L[Z_1] - (2b+c)E_L[Z_1^2] - 2bE_L[Z_1 \cdot Z_2] \\ &= 2a\alpha - (2b+c)\beta - 2b\gamma. \end{aligned}$$

We write the expected payoff when player 1 plays a pure strategy z and player 2 commits to L, as

$$\begin{aligned} u_1(z, L^2) &= az + aE_L[Z_2] - bz^2 - bE_L[Z_2^2] - 2bzE_L[Z_2] - cz^2 \\ &= (a-2b\alpha)z - (b+c)z^2 + a\alpha - b\beta. \end{aligned}$$

Hence, L is a CCE if and only if

$$\max_{z\geq 0}\{(a-2b\alpha)z-(b+c)z^2\}+a\alpha-b\beta\leq 2a\alpha-(2b+c)\beta-2b\gamma,$$

which, after rearranging, gives us the condition in the statement. □

The next result is due to Moulin, Ray and Sen Gupta (2014). It identifies the range of the vector (α,β,γ) when $L\in\mathscr{L}^{sy}(\mathbb{R}^2_+)$ and also shows that this range is covered by two families of very simple lotteries with at most four strategy profiles in their support.

Let $\mathscr{L}^*$ be the subset of $\mathscr{L}^{sy}(\mathbb{R}^2_+)$ containing the simple lotteries of the form $L=\frac{q}{2}(\delta_{z,z}+\delta_{z',z'})+\frac{p}{2}(\delta_{z,z'}+\delta_{z',z})$, where z, z', q and p are non-negative and $q+p=1$. Let $\mathscr{L}^{**}$ be the subset of $\mathscr{L}^{sy}(\mathbb{R}^2_+)$ of the form $L=q\cdot\delta_{z,z}+q'\cdot\delta_{0,0}+\frac{p}{2}(\delta_{0,z}+\delta_{z,0})$, where z, q, q' and p are non-negative and $q+q'+p=1$. Intuitively, the number of different types of lotteries satisfying 3.5 can only be a discrete/finite set; and it turns out to be just these two types of simple lotteries[11].

Lemma 3.3. *i) For any $L\in\mathscr{L}^{sy}(\mathbb{R}^2_+)$ and the corresponding random variable (Z_1,Z_2), we have*

$$\alpha,\gamma\geq 0;\ \beta\geq\gamma;\ \beta+\gamma\geq 2\alpha^2; \tag{3.5}$$

ii) Equality $\beta=\gamma$ holds if and only if L is diagonal: $Z_1=Z_2$ (a.e.)[12] *;*

iii) Equality $\beta+\gamma=2\alpha^2$ holds if and only if L is anti-diagonal: Z_1+Z_2 is constant (a.e.);

[11] $\mathscr{L}^*$ is not a subset of $\mathscr{L}^{**}$ because $\mathscr{L}^*$ can have any $z'\neq 0$ as well.

[12] $Z_1=Z_2$ implies the lottery L has non-zero probabilities on the same elements of Z_1 and Z_2, i.e. L is a diagonal lottery.

iv) For any $(\alpha, \beta, \gamma) \in \mathbb{R}^3_+$ *satisfying inequalities (3.5), there exists* $L \in \mathscr{L}^* \cup \mathscr{L}^{**}$ *with these parameters.*

Note that (3.5) implies $\beta \geq \alpha^2$, with equality $\beta = \alpha^2$ if and only if L is deterministic, because $\beta = \alpha^2$ implies both $\beta = \gamma$ and $\beta + \gamma = 2\alpha^2$.

Lemma 3.3 is same as the Lemma 2.4 in Chapter 2, and the proof has been provided in the Appendix of Chapter 2. Lemmata 3.2 and 3.3 now imply the following two-step algorithm (Proposition 3.4) to find the utility maximizing CCEs, which is similar to Theorem 1 in Moulin, Ray and Sen Gupta (2014). Theorem 3.5 uses the two-step algorithm provided by Proposition 3.4 to find the optimal CCE.

Proposition 3.4. *Given the abatement game, the following nested programs generate the utility maximizing CCEs*[13]*:*

Step 1: Fix α *non negative, and solve the linear program*

$$\min_{\beta,\gamma}\{(2b+c)\beta + 2b\gamma\} \textit{ under constraints}$$

$$\beta \geq \gamma \geq 0;\ \beta + \gamma \geq 2\alpha^2;\ (b+c)\beta + 2b\gamma \leq a\alpha - \max_{z\geq 0}\{(a-2b\alpha)z - (b+c)z^2\}.$$

Step 2: With the solutions $\beta(\alpha), \gamma(\alpha)$ *found in Step 1, solve*

$$\max_{\alpha}\{2a\alpha - (2b+c)\beta(\alpha) - 2b\gamma(\alpha)\} \textit{ under constraints}$$

$$\alpha \geq 0;\ \max_{z\geq 0}\{(a-2b\alpha)z - (b+c)z^2\} \leq a\alpha - (b+c)\beta(\alpha) - 2b\gamma(\alpha).$$

[13] Theorem 3.5 gives us the specific lottery.

Moreover, there is a utility maximizing CCE in $\mathscr{L}^* \cup \mathscr{L}^{**}$.

3.3.1 Optimal CCE

Proposition 3.4 implies the following characterization of the utility maximizing CCE for our games.

Theorem 3.5. *i) If* $b \le c$, *the Nash equilibrium of the abatement game is its only CCE.*

ii) If $b > c$, *setting* $\lambda = \frac{c}{b}$[14], *the optimal values of the three moments of the utility maximizing* L *are given by* $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$:

$$\widetilde{\alpha} = \frac{a}{b}\frac{2 + 2\lambda - \lambda^2}{2(4 + 5\lambda)},$$

$$\widetilde{\beta} = \frac{a^2}{b^2}\frac{4 + 8\lambda + \lambda^2 - 4\lambda^3}{4(4 + 5\lambda)^2} \text{ and } \widetilde{\gamma} = \frac{a^2}{b^2}\frac{4 + 8\lambda - \lambda^2 - 4\lambda^3 + 2\lambda^4}{4(4 + 5\lambda)^2};$$

while the optimal CCE is $\widetilde{L} = \frac{1}{2}\delta_{(z,z')} + \frac{1}{2}\delta_{(z',z)}$, *with*

$$z, z' = \frac{a}{b}\frac{2 + 2\lambda - \lambda^2 \pm \lambda\sqrt{1 - \lambda^2}}{2(4 + 5\lambda)}.$$

Proof. First, consider the equilibrium condition (3.4) as in Lemma 3.2. We have two cases to consider:

Case (i): $a - 2b\alpha < 0 \iff \alpha > \frac{a}{2b}$. The L.H.S. of the inequality (the maximum over $z \ge 0$) is zero; therefore, (3.4) becomes $a\alpha \ge (b + c)\beta + 2b\gamma$. The R.H.S from the above is $b(\beta + \gamma) + c\beta + b\gamma > b(\beta + \gamma)$. Now, $b(\beta + \gamma) \ge 2b\alpha^2$ (from

[14] We put $\lambda = \frac{c}{b}$ for computational ease later on in the proof.

Lemma 3.3 (i)), which further implies that $\alpha \leq \frac{a}{2b}$, which is a contradiction from where we started.

So, we must have,

Case (ii): $\alpha \leq \frac{a}{2b}$; then the L.H.S. of (3.4) is $\frac{(a-2b\alpha)^2}{4(b+c)}$. The equilibrium condition is now

$$(b+c)\beta + 2b\gamma \leq a\alpha - \frac{(a-2b\alpha)^2}{4(b+c)} = -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c}. \tag{3.6}$$

We now fix α and solve Step 1 in Proposition 1: we must minimize $(2b+c)\beta+2b\gamma$ in the polytope $\Psi = \{(\beta,\gamma) | \beta \geq \gamma, \beta+\gamma \geq 2\alpha^2\}$ under the additional constraint (3.6). Note that Ψ is unbounded from above and bounded from below by the interval $[P, Q]$, where $P = (\alpha^2, \alpha^2)$ and $Q = (2\alpha^2, 0)$. We distinguish two cases here.

Case 1 ($b \leq c$): In this case, the minimum in Ψ of both $(2b+c)\beta + 2b\gamma$ and $(b+c)\beta + 2b\gamma$ is achieved at P. Therefore, if P meets (3.6) it is our optimal pair $(\beta(\alpha), \gamma(\alpha))$; otherwise, there is no CCE for this choice of α. Now, P meets (3.6) if and only if $(3b+c)\alpha^2 \leq -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c}$, which reduces to $[a-(2b+c)\alpha)]^2 \leq 0 \iff \alpha = \frac{a}{2(2b+c)} = q_i^{Neq}$. By Lemma 3.3 the optimal CCE L is diagonal ($\beta = \gamma$) and deterministic ($\beta = \alpha^2$). It is simply the Nash equilibrium $L = \delta_{q^{Neq}}$ of our game.

Case 2 ($b > c$): Here, the minimum of $(b+c)\beta + 2b\gamma$ in Ψ is achieved at Q; so, if Q fails to meet the constraint (3.6) there is no hope to meet it anywhere in

Ψ. Thus, we must choose α such that

$$2(b+c)\alpha^2 \le -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c} \Longleftrightarrow \Lambda(\alpha) = (3b^2+4bc+2c^2)\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4} \le 0 \tag{3.7}$$

The discriminant of the right-hand polynomial $\Lambda(\alpha)$ is $a^2(b^2-c^2)$; therefore, (3.7) restricts α to an interval $[\alpha_-, \alpha_+]$, between the two positive roots of $\Lambda(\alpha)$. For such a choice of α, the constraint (3.6) cuts a subinterval $[R, Q]$ of $[P, Q]$, where R meets (3.6) as an equality. Note that $R = P$ only if $\alpha = q_i^{Neq}$ (from Case 1 and the fact that $\Lambda(q_i^{Neq}) < 0$), otherwise $R \neq P$. Clearly, R is our optimal choice $(\beta(\alpha), \gamma(\alpha))$ and it solves the system

$$\beta + \gamma = 2\alpha^2;\ (b+c)\beta + 2b\gamma = -\frac{b^2\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}}{b+c}.$$

Therefore,

$$\beta(\alpha) = \frac{1}{b^2-c^2}\left[b(5b+4c)\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4}\right] \text{ and}$$

$$\gamma(\alpha) = \frac{1}{b^2-c^2}\left[-(3b^2+4bc+2c^2)\alpha^2 + a(2b+c)\alpha - \frac{a^2}{4}\right].$$

Now in Step 2 of Proposition 3.4, we must maximize $2a\alpha - (2b+c)\beta(\alpha) - 2b\gamma(\alpha)$ under the constraints $\alpha \ge 0$ and $\Lambda(\alpha) \le 0$. Developing this objective function yields the program

$$\frac{1}{b^2-c^2}\max_{\alpha}\{-b^2(4b+5c)\alpha^2 + a(2b^2+2bc-c^2)\alpha - \frac{a^2c}{4}\} \tag{3.8}$$

under the constraints

$$\alpha \geq 0 \text{ and } \Lambda(\alpha) = (3b^2 + 4bc + 2c^2)\alpha^2 - a(2b+c)\alpha + \frac{a^2}{4} \leq 0.$$

The unconstrained maximum of the objective function is achieved at $\widetilde{\alpha} = \frac{a(2b^2+2bc-c^2)}{2b^2(4b+5c)}$.

We now show that $\Lambda(\widetilde{\alpha}) \leq 0$. With the change of variable $\lambda = \frac{c}{b}$, this amounts to

$$\frac{(3+4\lambda+2\lambda^2)(2+2\lambda-\lambda^2)^2}{4(4+5\lambda)^2} - \frac{(2+\lambda)(2+2\lambda-\lambda^2)}{2(4+5\lambda)} + \frac{1}{4} \leq 0$$
$$\iff 4 + 8\lambda - 5\lambda^2 - 12\lambda^3 + 3\lambda^4 + 4\lambda^5 - 2\lambda^6 \geq 0$$

The above polynomial is 0 at $\lambda = 1$; it is also easy to check, numerically, that it is non-negative on $[0, 1]$. The proof is complete if we now express $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma}$ in terms of λ. This is indeed easy for $\widetilde{\alpha} = \frac{a(2b^2+2bc-c^2)}{2b^2(4b+5c)} = \frac{a}{b}\frac{2+2\lambda-\lambda^2}{2(4+5\lambda)}$. One may also verify, using the expression for $\widetilde{\alpha}$ that

$$\begin{aligned}\widetilde{\beta} &= \beta(\widetilde{\alpha}) = \frac{1}{b^2-c^2}\left[b(5b+4c)\widetilde{\alpha}^2 - a(2b+c)\widetilde{\alpha} + \frac{a^2}{4}\right] \\ &= \frac{a^2}{b^2}\frac{4+8\lambda+\lambda^2-4\lambda^3}{4(4+5\lambda)^2} \text{ and}\end{aligned}$$

$$\begin{aligned}\widetilde{\gamma} &= \gamma(\widetilde{\alpha}) = \frac{1}{b^2-c^2}\left[-(3b^2+4bc+2c^2)\widetilde{\alpha}^2 + a(2b+c)\widetilde{\alpha} - \frac{a^2}{4}\right] \\ &= \frac{a^2}{b^2}\frac{4+8\lambda-\lambda^2-4\lambda^3+2\lambda^4}{4(4+5\lambda)^2}.\end{aligned}$$

Finally, we construct the optimal CCE $\widetilde{L}$. From $\widetilde{\beta} + \widetilde{\gamma} = 2\widetilde{\alpha}^2$ and Lemma

3.3(iii), we see that $\widetilde{L}$ is an anti-diagonal lottery of the form $\widetilde{L} = \frac{1}{2}\delta_{(z,z')} + \frac{1}{2}\delta_{(z',z)}$, where z and z' are non-negative numbers such that $z + z' = 2\widetilde{\alpha}$ and $z^2 + z'^2 = 2\widetilde{\beta}$. This implies $2zz' = (2\widetilde{\alpha})^2 - (2\widetilde{\beta}) = 2\widetilde{\gamma}$, hence z, z' solve $Z^2 - 2\widetilde{\alpha}Z + \widetilde{\gamma} = 0$. Thus, $z = z' = \widetilde{\alpha} \pm \sqrt{\widetilde{\alpha}^2 - \widetilde{\gamma}}$. The discriminant is $\widetilde{\alpha}^2 - \widetilde{\gamma} = \widetilde{\beta} - \widetilde{\alpha}^2 = \frac{a^2}{b^2}\frac{\lambda^2(1-\lambda^2)}{4(4+5\lambda)^2}$; thus the expressions for z and z' follow. □

3.3.2 Efficiency Performance

We now compare the optimal CCE (total) profit $\pi^{CC} = 2u_1(\widetilde{L})$, with both efficient and Nash equilibrium (total) profits. From the expression (3.8) of the single player profit $u_1(\widetilde{L})$ and the expression of $\widetilde{\alpha}$ in Theorem 3.5(ii), straightforward computations provide

$$u_1(\widetilde{L}) = \frac{1}{b^2 - c^2}\left[\frac{a^2}{b^2}\frac{(2 + 2\lambda - \lambda^2)^2}{4(4 + 5\lambda)} - \frac{a^2 c}{4b^2}\right] = \frac{a^2}{b}\frac{4 + 4\lambda - \lambda^2}{4(4 + 5\lambda)}.$$

Recalling

$$\pi^{eff} = \frac{2a^2}{4b + c} = \frac{a^2}{b}\frac{2}{4 + \lambda} \text{ and } \pi^{Neq} = \frac{a^2(4b + 3c)}{2(2b + c)^2} = \frac{a^2}{b}\frac{4 + 3\lambda}{2(2 + \lambda)^2},$$

we can now state the following.

Corollary 3.6. *For the abatement game, the relative efficiency of the optimal CCE and its relative improvement over the symmetric Nash equilibrium payoff depend only upon $\lambda = \frac{c}{b}$, as follows:*

$$\frac{\pi^{CC}}{\pi^{eff}} = \frac{(4 + \lambda)(4 + 4\lambda - \lambda^2)}{4(4 + 5\lambda)} \quad \textit{for } 0 \le \lambda \le 1,$$

$$\frac{\pi^{CC}}{\pi^{Neq}} = \begin{cases} \frac{(2+\lambda)^2(4+4\lambda-\lambda^2)}{(4+5\lambda)(4+3\lambda)} & \textit{for } 0 \leq \lambda \leq 1 \\ 1 & \lambda \geq 1 \end{cases}$$

Corollary 3.6 on the behaviour of the efficiency ratios is described in Figures 3.2[15] and 3.3.

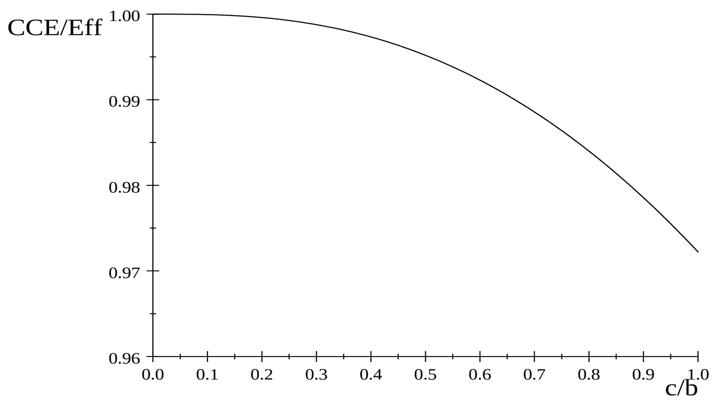

Figure 3.2: $\frac{\pi^{CC}}{\pi^{eff}}$ *in the abatement game*

[15] Note that for $\lambda > 1$, $\frac{\pi^{CC}}{\pi^{eff}} = \frac{\pi^{Neq}}{\pi^{eff}}$ and therefore Figure 3.1 is relevant for $\lambda > 1$.

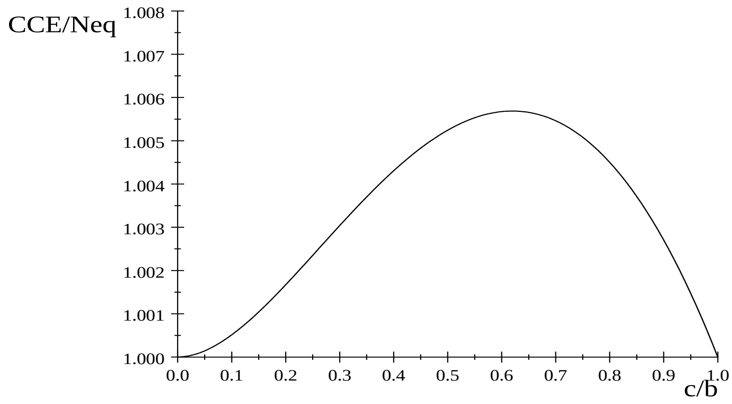

Figure 3.3: $\frac{\pi^{CC}}{\pi^{Neq}}$ *in the abatement game*

3.4 CONCLUSION

We have analyzed coarse correlated equilibria in a class of 2-person symmetric game called the abatement game where correlation *a la* Aumann does not offer anything more than the Nash equilibrium. Incorporating the techniques introduced by Moulin, Ray and Sen Gupta (2014), we have characterized the utility maximizing CCE and have shown that they have a very simple support with only four deterministic strategy profiles. The intuition behind these lotteries comes from 3.5 and the idea that the number of different types of lotteries satisfying this condition is finite and turns out to be the ones described as simple lotteries. Such a computation is the first of its kind for coarse correlated equilibria for the abatement game and, this is why we regard this exercise as an interesting first step towards more sophisticated computations to understand mediation in general for such games.

We conclude by considering an example and illustrating our results more formally. Consider the following values of the parameters, $a = 1$, $b = 2$ and $c = 1$ in the abatement game. Here, $\lambda = \frac{c}{b} = \frac{1}{2} < 1$ and the payoff function[16] is given by $u_1(q_1, q_2) = (q_1 + q_2) - 2(q_1 + q_2)^2 - q_1^2$, with Nash equilibrium quantity, $q^{Neq} = \frac{a}{2(2b+c)} = \frac{1}{10}$.

From Theorem 3.5, the corresponding optimal values of the moments are:

$\widetilde{\alpha} = \frac{11}{104} \approx 0.1057$,

$\widetilde{\beta} = \frac{31}{2704} \approx 0.0114$ and

$\widetilde{\gamma} = \frac{59}{5408} \approx 0.0109$.

The optimal CCE is the lottery $\widetilde{L} = \frac{1}{2}\delta_{(z,z')} + \frac{1}{2}\delta_{(z',z)}$, where $\delta_{(z',z)}$ is the deterministic outcome z, z'. $z, z' = \{\frac{11+\sqrt{3}}{104}, \frac{11-\sqrt{3}}{104}\}$, with $z + z' = \frac{22}{104}$, i.e., with equal probability, one country abates $q_i = \frac{11+\sqrt{3}}{104}$ while the other chooses $\frac{11-\sqrt{3}}{104}$, and vice versa. The above lottery is clearly not a CE because $(\frac{11+\sqrt{3}}{104}, \frac{11-\sqrt{3}}{104})$ is not a Nash equilibrium. But it is a CCE: if player 1 chooses q_1 and assumes player 2 is choosing either $\frac{11+\sqrt{3}}{104}$ or $\frac{11-\sqrt{3}}{104}$ with equal probability, his expected payoff $[\frac{15}{26}q_1 - 3q_1^2 + \frac{11}{104} - (\frac{11+\sqrt{3}}{104})^2 - (\frac{11-\sqrt{3}}{104})^2]$ is maximized at $q_1 = \frac{5}{52}$ and gives him $u_1 = \frac{23}{208}$, precisely the same as by committing to follow the outcome of L, that generates the expected utility of $u_1(L) = (z + z') - 2(z + z')^2 - \frac{1}{2}(z^2 + z'^2) = \frac{23}{208} \approx 0.1105$.

The optimal CCE (total) payoff is $\pi^{CC} = 2u_1(\widetilde{L}) = \frac{23}{104} \approx 0.2211$. One may be surprised to note that this CCE has an improvement ratio $\frac{\pi^{CC}}{\pi^{Neq}}$ of only $\frac{575}{572} \approx 1.0052$, yielding just about $\frac{1}{2}\%$ increase over and above the Nash equilibrium payoff. This is because in this class of games, the Nash equilibrium can be actually

[16] Note that comparing it with the general model (equation 2.5 in Chapter 2), we have the values of the parameters as follows: $a = b = 1$, $c = -4$, $d = -3$ and $e = -2$.

very close to the efficient outcome (which maximizes the joint payoff). For this particular example, the efficient (total) payoff that the players can jointly achieve is $\frac{2}{9} \approx 0.2222$, and the Nash equilibrium (total) payoff 0.22 is 99% efficient. So the optimal CCE only incurs about $\frac{1}{2}\%$ of efficiency loss. Although, not a large amount of improvement over Nash equilibrium payoff is achieved by coarse correlation, one of the positives of using this notion of coarse correlation is that a better coordination amongst the countries could be achieved, which may be helpful in the reduction of *free-riding* incentives and thereby a more effective global emission abatement.

Chapter 4

Coarse Correlated Equilibria in Linear Duopoly Games

4.1 INTRODUCTION

We know from the pioneering works of Azariadis (1981) and Cass and Shell (1983) that extrinsic uncertainty matters in competitive economies. Does it matter in strategic markets as well? The answer we get from the literature is unfortunately partial and thus inconclusive, to some extent.

First of all, we should note that two notions of extrinsic uncertainty, formulated as, sunspot equilibrium in competitive markets (Azariadis 1981, Cass and Shell 1983) and correlated equilibrium in non-cooperative games (Aumann 1974, 1987), are very similar in nature and indeed closely connected, as noted by Maskin and Tirole 1987; Aumann *et al.* 1988, and formally presented by Polemarchakis and Ray (2006), where they proved that correlated equilibrium *a la*

Aumann corresponds to sunspot equilibria in the associated, competitive economy and provided a necessary and sufficient condition for existence of such an effective correlation.

Within the realm of strategic markets, on one hand, we know from a recent literature (Peck 1994, Forges and Peck 1995, Dávila 1999, among others) that indeed sunspot equilibria exist in strategic market games *a la* Shapley and Shubik (1977). On the other hand, in oligopoly models, Liu (1996) and Yi (1997) proved that the only correlated equilibrium for such games is the Cournot-Nash equilibrium of the market. Neyman (1997) generalised Liu's results to conclude that games with a smooth and concave potential function have a unique correlated equilibrium which coincides with the Nash equilibrium of the game. Ui (2008) generalises the results of Neyman (1997) to state a weaker condition which suffices for the uniqueness of a CE. Hence, one may ask whether there exists a link between correlation and sunspots in oligopoly models.

Although correlation *a la* Aumann may not achieve anything more than the Nash equilibrium outcome, as one rightly reckons, a coarsening of the set of correlated equilibria may exist in certain oligopoly models. Indeed, we do have such a coarse concept in the literature, introduced by Moulin and Vial (1978), called the CCE, that has recently evoked interests in several contexts (Young 2004, Forgó *et al.* 2005, Roughgarden 2009, Forgó 2010, 2011, Moulin *et al.* 2013, 2014).This concept has already been used by Gerard-Varet and Moulin (1978) in specific duopoly games to achieve an improvement over the Nash equilibrium payoff.

There are, however, a few gaps worth mentioning in the above strands of literature. First, from the analysis by Gerard-Varet and Moulin (1978), we learn under what conditions Nash equilibrium of the duopoly game can be *locally* improved upon, using a specific notion of improvement with strategies close to the Nash equilibrium; however, we do not know, for improvement or even for existence, whether the support of such a correlation device necessarily has to be close to the Nash equilibrium or not. Second, from the above mentioned literature on correlation and sunspots, we do not find the connection, if there is any, between (coarse) correlated equilibrium and sunspot equilibrium in oligopoly models, in particular, duopoly games.

The purpose of this paper is precisely to bridge these gaps and thus is twofold. We would like to see the existence of a general (non-local) coarse correlated equilibrium in strategic markets and the relation of this equilibrium with sunspots. We consider a specific form of correlation device, that we call a *k-Simple Symmetric Correlation Device* (k-SSCD). The device is named so (by Ganguly and Ray 2005) because the discrete probability distribution is symmetric and the support of it is finite. We apply the notion of coarse correlation *a la* Moulin and Vial (1978) and fully characterise an equilibrium concept that we call *k-Simple Symmetric Coarse Correlated Equilibrium* (k-SSCCE) for the linear duopoly game (Theorem 1). Clearly, the deterministic device that chooses only the Nash equilibrium outcome is a k-SSCCE; however, unlike Aumann's correlated equilibrium, this is not the only equilibrium according to our notion.

We identify a particular sunspot structure, that we call a Nash-centric device, which is a symmetric anti-diagonal probability distribution over equi-distant quantities around the Nash equilibrium point. We prove that any Nash-centric device is a k-SSCCE. This result (Theorem 2) holds for any such Nash-centric devices, regardless of its dimension, probabilities and the distances between the quantity levels, as long as this particular structure is maintained. Moreover, we observe that this is the unique equilibrium among the devices with equi-distant quantities and anti-diagonal probability distributions (Theorem 3). Also, we show that any small unilateral perturbation from this structure is not an equilibrium.

Our results identify a specific random device that the players are willing to commit to, in equilibrium. The Nash-centric device is a public randomisation over strategy profiles, including non-Nash equilibrium points, which is the feature of a sunspot equilibrium. However, we note that the expected payoff from the Nash-centric device is equal to the Nash equilibrium payoff in the linear duopoly game. Thus, unfortunately, this k-SSCCE can not improve upon the Nash equilibrium which may raise questions on the benefits obtained by randomised devices.

We focus on a couple of positive aspects of our results and thus the advantages of analysing coarse correlation in this context. First, it offers a way of generating the Nash equilibrium payoff using a solicitor that the players are willing to commit to, in equilibrium. Second, as we know from the experimental literature, coordinating on Nash equilibrium outcome is not easily achieved in many strategic situations. Our equilibrium, k-SSCCE, offers an explanation of how to achieve the Nash equilibrium payoff, in expected terms, which is also supported by some recent experimental studies (for example, Duffy and Feltovich 2010, Bone *et al.*

2012).

The chapter proceeds as follows: subsection 4.2.1 discusses the definitions of CE and CCE (which in this chapter has been defined in accordance to the purpose of the chapter), 4.2.2, 4.2.3 and 4.2.4 introduce the linear duopoly game, simple devices and the Nash centric devices, respectively; in section 4.3 we collect all our results (which includes the characterisation of the k-SSCCE and uniqueness and robustness results for Nash centric devices); and finally section 4.4 concludes.

4.2 MODEL

4.2.1 Correlation and Coarse Correlation

Fix any finite normal form game, $G = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, with set of players, $N = \{1, ..., n\}$, finite pure strategy sets, $S_1, ..., S_n$ with $S = \prod_{i \in N} S_i$, and payoff functions, $u_1, ..., u_n$, $u_i : S \to \Re$, for all i.

Definition 4.1. A (direct) correlation device μ is a probability distribution over S.

A normal form game, G, can be extended by using a direct correlation device. For correlation *a la* Aumann (1974, 1987), the device first selects a strategy profile $s(= (s_1, ..., s_n))$ according to μ, and then sends the private recommendation s_i to each player i. The extended game G_μ is the game where the correlation device μ selects and sends recommendations to the players, and then the players play the original game G. A (pure)[1] strategy for player i in the game G_μ is a map $\sigma_i : S_i \to S_i$ and the corresponding (ex-ante, expected) payoff is given by, $u_i^*(\sigma_1, ..., \sigma_n) = \sum_{s\in S} \mu(s)u_i(\sigma_1(s_1), ..., \sigma_n(s_n))$. The *obedient* strategy profile is the identity map $\sigma_i^*(s_i) = s_i$, for all i, with payoff to player i given by $u_i^*(\sigma^*) = \sum_{s\in S} \mu(s)u_i(s)$. The device is called a correlated equilibrium (Aumann 1974, 1987) if all the players follow the recommended strategies, i.e., the obedient strategy profile constitutes a Nash equilibrium of the extended game G_μ. Formally, with the notation $s_{-i} \in S_{-i} = \prod_{j\neq i} S_j$,

Definition 4.2. μ is a (direct) correlated equilibrium of the game G if for all i, for all s_i, $t_i \in S_i$, $\sum_{s_{-i}\in S_{-i}} \mu(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \mu(s_i, s_{-i})u_i(t_i, s_{-i})$.

One may use the direct correlation device, μ, in a different way. For a coarser notion of correlation *a la* Moulin and Vial (1978), a game G is extended to a game G'_μ in which the strategies of a player is either to commit to the device or to play any strategy in G. If all the players commit to the device, an outcome is chosen by the device according to the probability distribution. Thus, the expected payoff for any player i, when the device is accepted by all the players, is simply $\sum_{s\in S} \mu(s)u_i(s)$. Note that this is same as the payoff of the obedient strategy profile under the correlated equilibrium *a la* Aumann as above. If one of the players unilaterally deviates, while the others commit to the device, the deviant faces the *marginal* probability distribution μ'_i over $s_{-i} \in S_{-i}$ which is given by $\mu'_i(s_{-i}) = \sum_{s_i\in S_i} \mu(s_i, s_{-i})$. The coarse correlated equilibrium notion suggests that the players will accept the device if the expected payoff from the device is

[1] One can also think of behavioral strategies in any extended game. We, however, in this paper, restrict ourselves to pure strategies only.

higher than that from playing any other strategy, from the entire set of strategies. Formally,

Definition 4.3. μ is a (direct) coarse correlated equilibrium of the game G if for all i, for all $t_i \in S_i$, $\sum_{s \in S} \mu(s) u_i(s) \geq \sum_{s_{-i} \in S_{-i}} \mu'_i(s_{-i}) u_i(t_i, s_{-i})$.

From the system of inequalities[2] in the above definitions, it is clear that the set of coarse correlated equilibria is indeed coarser than the set of correlated equilibria.[3] Also, it is obvious that any Nash equilibrium and any convex combination of several Nash equilibria of any given game G, *corresponds* to a coarse correlated and a correlated equilibrium.[4]

4.2.2 Linear Duopoly

In this paper, we use the simplest form of oligopoly models, that of a duopoly market with linear demand function and constant marginal cost. Consider two quantity-setting firms, each of whose strategy is to choose a quantity level $q \in Q = \{q : q \geq 0\}$ to produce at a constant marginal cost $c \geq 0$ and to sell in a market with an inverse demand function given by $a - b(q^1 + q^2)$, with $a > 0$ and $b > 0$. Thus the profit functions for the firms are given by $\Pi_1(q^1, q^2) = aq^1 -$

[2] Following Aumann (1974) and Moulin and Vial (1978), we have used weak inequalities in our definitions (Definition 2 and Definition 3). We note that strict inequalities may be considered in these definitions; indeed, Gerard-Varet and Moulin (1978) did so in their definition of equilibrium.

[3] It is also easy to prove that the set of correlated and coarse correlated equilibria coincide for the case of 2×2 games. However, as Moulin and Vial (1978) demonstrated, there are games involving 2 players and 3 strategies for each player, for which the set of coarse correlated equilibria is strictly larger than the sets of correlated and Nash equilibria.

[4] Formally, let $NE(G)$, $CONV(G)$, $CE(G)$ and $CCE(G)$ denote, respectively, the sets of all Nash equilibria, convex combination of Nash equilibria, correlated equilibria and coarse correlated equilibria for any game G. Clearly, $NE(G) \subseteq CONV(G) \subseteq CE(G) \subseteq CCE(G)$.

$b(q^1)^2 - bq^1q^2 - cq^1$ and $\Pi_2(q^1, q^2) = aq^2 - b(q^2)^2 - bq^1q^2 - cq^2$, where q^1 and q^2 are quantity choices of firms 1 and 2, respectively. For simplicity and without loss of any generality, for the rest of the paper, we set $c = 0$. Hence, the profit functions are $\Pi_1(q^1, q^2) = aq^1 - b(q^1)^2 - bq^1q^2$ and $\Pi_2(q^1, q^2) = aq^2 - b(q^2)^2 - bq^1q^2$. As it is well-known, the Nash equilibrium outcome of this game is $q^1 = q^2 = q_{NE} = \frac{a}{3b}$ and the Nash equilibrium payoff to each firm is $\frac{a^2}{9b}$. For the rest of the paper this two-person game will be called the *linear duopoly game*.

Liu (1996) analysed an oligopoly model with n firms, each with a constant marginal cost, c^i for firm i $(i = 1, ..., n)$ operating in a market with linear demand, and proved that the only correlated equilibrium of this game is the unique Cournot-Nash equilibrium. Our game, clearly, is a special case of Liu's model with $n = 2$, and $c^1 = c^2 = 0$. Our game is a potential game with a smooth and concave potential function, f, given by, $f(q^1, q^2) = a(q^1 + q^2) - b[(q^1)^2 + q^1q^2 + (q^2)^2]$. Therefore, the linear duopoly game has a unique correlated equilibrium *a la* Aumann that coincides with the Nash equilibrium of the game.

One may also wish to directly apply the result obtained by Gerard-Varet and Moulin (1978), for improvement upon the Nash equilibrium payoff by coarse correlation, in our game. Gerard-Varet and Moulin (1978) introduced a specific notion of improvement using their coarse correlated equilibrium notion (see Footnote 2 above), with correlation devices whose support involve strategies close to the Nash equilibrium outcome. They provided conditions under which such an improvement is attained in duopoly games. Their theorem, unfortunately, does not apply to our linear duopoly game. However, as pointed out in their paper

(Gerard-Varet and Moulin, 1978, page 133), it can be directly proved that Nash equilibrium of this game can not be improved upon by coarse correlation, using their approach.

4.2.3 Simple Devices

We now consider a specific form of correlation device for our game. Although the strategy sets in games, such as the linear duopoly game, are continuous, a device may involve only finitely many strategies, i.e., the support of the probability distribution in the direct correlation device may be finite. The structure of such a *simple* device was used by Ganguly and Ray (2005) in their analysis of simple mediation in cheap-talk games. Here, we consider such simple devices to analyse coarse correlation in the linear duopoly game. Moreover, we impose symmetry in the probability distribution and we restrict the device to use the same quantities for both players. Formally, the specific form of the device we consider in this paper is defined below.

Definition 4.4. A k-Simple Symmetric Correlation Device (k-SSCD), $[P; q_c]$, is a symmetric probability distribution matrix, P, over $q_c \times q_c$, where, $q_c = (q_1, q_2, ..., q_k)$, with $0 < q_1 < q_2 < .... < q_k; q_i \in Q$, and $P = \{(p_{ij})_{i=1,2...,k;j=1,2...,k}\}$ with each $p_{ij} \in [0, 1]$, $p_{ij} = p_{ji}$ and $\sum_{ij} p_{ij} = 1$.

The interpretation of a k-SSCD, $[P; q_c] = [\{(p_{ij})_{i=1,2...,k;j=1,2...,k}\}; (q_i)_{i=1,2...,k}]$, is that the players are given a choice to commit to the device. If both players commit, the device will then pick the strategies q_i and q_j for the two players respectively, with probability p_{ij}. The players do not play the game, however, get the profits $\Pi_1(q_i, q_j)$ and $\Pi_2(q_i, q_j)$, respectively, that correspond to the chosen strategy profile (q_i, q_j). Thus, if both players commit to the device, the expected payoffs for the two players are the same (by symmetry) and is given by $E_P(\Pi) = \sum_{ij} p_{ij}\pi_1(q_i, q_j) = \sum_{ij} p_{ij}\pi_2(q_i, q_j) = \sum_{ij} p_{ij}[aq_i - bq_i^2 - bq_iq_j]$.

A player may decide not to commit to the device, in which case he may play any strategy of his own, while the other commits to the device. Note that, although the device, $[P; q_c]$, involves only finitely many strategies, the deviation for a player is however not restricted; any strategy $q \in Q$ (even outside the domain, q_c, of the device) can be played by a player if he doesn't commit to the device. The deviating player can choose any strategy even outside of the ones selected by the lottery. The deviant faces the *marginal* probability distribution p' over $q_j \in q_c$ which is given by $p'(q_j) = \sum_{q_i \in q_c} p(q_i, q_j)$. Let $E_P(\Pi \mid q)$ denote the expected payoff of any deviating player (by symmetry) from playing q. Clearly, $E_P(\Pi \mid q) = \sum_{q_j \in q_c} p'(q_j)\Pi(q, q_j) = \sum_{q_j \in Q_c} p'(q_j)[aq - bq^2 - bqq_j]$. As mentioned, the equilibrium condition requires that the device be accepted by both players. Formally,

Definition 4.5. A k-SSCD, $[P; q_c]$, is called a k-Simple Symmetric Coarse Correlated Equilibrium (k-SSCCE) if both players commit to the device, i.e., given that the other player is committing to the device, a player does not deviate to play any other strategy $q \in Q$, i.e., $E_P(\Pi) \geq E_P(\Pi \mid q)$, for all $q \in Q$.

4.2.4 Nash-centric Devices

We now define different types of k-SSCD, $[P; q_c]$, which we use later in this paper for proving our results.

A k-SSCD *with equi-distant quantities* is a device for which $q_i = q_1 + (i-1)d$, $1 \leq i \leq k$, and thus can be denoted by $[P; q_1; d]$.

A *public* or *sunspot* k-SSCD is a device for which the probability distribution P is such that whenever $p_{ij} > 0$, the *conditional probability* of q_i is 1, and vice versa (in other words, each row and each column in the probability distribution matrix has one and only one positive element).

An *anti-diagonal* k-SSCD is a device for which the probability distribution P is an *anti-diagonal distribution* in which only the anti-diagonal elements of the probability distribution matrix are strictly positive, i.e., $p_{ij} > 0$, when $i+j = k+1$ and $p_{ij} = 0$, when $i + j \neq k + 1$. Clearly, an anti-diagonal k-SSCD is a special case of a public or sunspot k-SSCD and can be characterised by its positive anti-diagonal elements only.

A *Nash-centric* k-SSCD is a device with equi-distant quantities and anti-diagonal probability distribution for which the quantities are "Nash-centric". We distinguish between the two cases based on the dimension of q_c being odd or even. The Nash equilibrium quantity is included in the middle of the vector q_c, for the "odd" case, however, not in the "even" case.

Formally, for any odd k ($k = 2m + 1$), a *Nash-centric* k-SSCD is given by $q_{m+1} = q_{NE} = \frac{a}{3b}$ and $q_i = q_{NE} - (m + 1 - i)\,\delta$ for $1 \leq i \leq k = 2m + 1$, with $\delta > 0$ and $\delta < \frac{a}{3bm}$ (so that $q_1 > 0$). For any even k ($k = 2m$), a *Nash-centric*

k-SSCD is a device with $q_i = q_{NE} - (2m + 1 - 2i)\delta$ for $1 \leq i \leq k = 2m$, with $\delta > 0$ and $\delta < \frac{a}{3b(2m-1)}$ (so that $q_1 > 0$).

The anti-diagonal probability distribution associated with this device is characterised by its positive anti-diagonal elements only.

Formally, for any odd k ($k = 2m+1$), let p_i, for $1 \leq i \leq m$, be the probability of both strategy profiles $(q_i,\ q_{k+1-i})$ and $(q_{k+1-i},\ q_i)$ and $(1 - 2\sum_{i=1}^{m} p_i) > 0$ be the probability attached to the Nash equilibrium strategy profile, i.e., $p_{i(k+1-i)} = p_{(k+1-i)i} = p_i$ for $1 \leq i \leq m$, $p_{(m+1)(m+1)} = 1 - 2\sum_{i=1}^{m} p_i$ and $p_{ij} = 0$, otherwise. For any even k ($k = 2m$), let p_i, for $1 \leq i \leq m$, be the probability of both strategy profiles $(q_i,\ q_{k+1-i})$ and $(q_{k+1-i},\ q_i)$, with $2\sum_{i=1}^{m} p_i = 1$, i.e., $p_{i(k+1-i)} = p_{(k+1-i)i} = p_i$ for $1 \leq i \leq m$, and $p_{ij} = 0$, otherwise.

A Nash-centric device (for any dimension) thus can be defined by $[k; (p_i)_{1 \leq i \leq m}; \delta]$. We present such a Nash-centric device in a tabular form below

for any odd k ($k = 2m + 1$).

Strategies	q_1	q_2	...	...	q_{m+1}	...	...	....	q_k
$q_1 = q_{NE} - m\delta$	0	0	0	0	0	0	0	0	p_1
$q_2 = q_{NE} - (m-1)\,\delta$	0	0	0	0	0	0	0	p_2	0
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
$q_{m+1} = q_{NE} = \frac{a}{3b}$	0	0	0	0	$1 - 2\sum_{i=1}^{m} p_i$	0	0	0	0
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
$q_{k-1} = q_{NE} + (m-1)\,\delta$	0	p_2	0	0	0	0	0	0	0
$q_k = q_{NE} + m\delta$	p_1	0	0	0	0	0	0	0	0

4.3 RESULTS

4.3.1 Characterisation

We first characterise the condition for a k-SSCD to be a k-SSCCE for the linear duopoly game. Following Definition 4.5, a k-SSCD is in equilibrium when the expected payoff from the device is higher than that from any unilateral deviation by a player. Hence, a necessary and sufficient condition for equilibrium is that the expected payoff from the device is higher than the maximum of the payoffs from any unilateral deviation. A k-SSCCE is thus characterised in the theorem below.

Theorem 4.6. *A k-SSCD, $[P; q_c]$, is a k-SSCCE for the linear duopoly game if and only if $A_P(q_c) \geq 0$, where $A_P(q_c)$ is given by*

$\frac{3a}{2}\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}) - b\sum_{i=1}^{k} q_i^2[\sum_{j=1}^{k} p_{ij} + p_{ii} + \frac{1}{4}(\sum_{j=1}^{k} p_{ij})^2] - b\sum_{i=1}^{k-1}\sum_{j=2}^{k} q_i q_j[2p_{ij} + \frac{1}{2}(\sum_{s=1}^{k} p_{is})(\sum_{s=1}^{k} p_{js})] - \frac{a^2}{4b}$.

Proof. From Definition 4.5, for any k-SSCD, $[P; q_c]$, to be a k-SSCCE in the linear duopoly game, we must have, $E_P(\Pi) \geq E_P(\Pi \mid q)$, for all $q \in Q$, which holds true if and only if $E_P(\Pi) - E_P(\Pi^*) \geq 0$, where, $E_P(\Pi^*) = Max_{q\in Q} E_P(\Pi \mid q)$.

First, we observe

$E_P(\Pi) = a\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}) - b\sum_{i=1}^{k} q_i^2(\sum_{j=1}^{k} p_{ij}) - b\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}q_j)$

and

$E_P(\Pi \mid q) = aq - bq^2 - bq\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})$.

Now, using the first order condition, $\frac{\partial E_P(\Pi|q)}{\partial q} = 0$, $E_P(\Pi \mid q)$ is maximised at $q^* = \frac{1}{2b}[a - b\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})]$, with

$E_P(\Pi^*) = \frac{a^2}{4b} - \frac{a}{2}\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}) + \frac{b}{4}\sum_{i=1}^{k} q_i^2(\sum_{j=1}^{k} p_{ij})^2 + \frac{b}{2}\sum_{i=1}^{k-1}\sum_{j=2}^{k} q_i q_j(\sum_{s=1}^{k} p_{is})(\sum_{s=1}^{k} p_{js})$.

Now define $A_P(q_c) = E_P(\Pi) - E_P(\Pi^*)$, which leads to the characterising inequality in the statement of the theorem. □

Clearly, the Nash equilibrium of the linear duopoly game can be viewed as a device with probability 1 on the Nash equilibrium quantity, $\frac{a}{3b}$, and is trivially a k-SSCCE. This fact is observed in the above characterisation. Note that at the Nash equilibrium point, the value of $A_P(q_c)$ stated in Theorem 4.6 is indeed 0, satisfying the condition weakly. Hence, the unique Nash and correlated equilibrium *a la* Aumann of the linear duopoly game is indeed a k-SSCCE.

One may use the characterisation in Theorem 4.6, of a k-SSCCE, to find the equilibrium condition for different types of k-SSCD, in particular any k-SSCD with equi-distant quantities. The following corollary characterises probability distributions for such a k-SSCCE for the linear duopoly game. The inequality in the following corollary is also used to prove some of our results later in this paper.

Corollary 4.7. *For the linear duopoly game, any probability distribution, P, can be supported as a k-SSCCE with equi-distant quantities, with some q_1 and d, if and only if*

$$2[\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]^2-\sum_{i=1}^{k}(i-1)^2(p_{ii}+\sum_{j=1}^{k}p_{ij})-2\sum_{1\le i<j\le k}(i-1)(j-1)p_{ij}\ge 0.$$

Proof. Given a distribution, P, consider a k-SSCD with equi-distant quantities $[P;q_1;d]$. We will use the equilibrium condition $A_P(q_c)\ge 0$, in Theorem 4.6 for such a k-SSCD. Substituting the values of $q_i=q_1+(i-1)d$, $1\le i\le k$, and simplifying, the expression

$A_P(q_c)$ in Theorem 4.6 becomes

$$\frac{3a}{2}\sum_{i=1}^{k}[q_1+(i-1)d](\sum_{j=1}^{k}p_{ij})-b[q_1+\sum_{i=1}^{k}(i-1)d]^2[\sum_{j=1}^{k}p_{ij}+p_{ii}+\frac{1}{4}(\sum_{j=1}^{k}p_{ij})^2]$$
$$-b\sum_{1\le i<j\le k}[q_1+(i-1)d][q_1+(j-1)d][2p_{ij}+\frac{1}{2}\sum_{s=1}^{k}p_{is}\sum_{s=1}^{k}p_{js}]-\frac{a^2}{4b}$$
$$=\frac{3a}{2}q_1+\frac{3ad}{2}[\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]-bq_1^2[2\sum_{i=1}^{k}\sum_{j=1}^{k}p_{ij}+(\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}p_{ij})^2]$$
$$-bq_1d[4\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}+\frac{1}{2}\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}(\sum_{i=1}^{k}\sum_{j=1}^{k}p_{ij})]$$
$$-bd^2[\sum_{i=1}^{k}(i-1)^2\{p_{ii}+\sum_{j=1}^{k}p_{ij}+\frac{1}{4}(\sum_{j=1}^{k}p_{ij})^2\}+2\sum_{1\le i<j\le k}(i-1)(j-1)p_{ij}]$$
$$-\frac{bd^2}{2}[\sum_{1\le i<j\le k}(i-1)(j-1)\sum_{s=1}^{k}p_{is}\sum_{s=1}^{k}p_{js}]-\frac{a^2}{4b}$$
$$=\frac{3a}{2}q_1+\frac{3ad}{2}[\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]-\frac{9b}{4}q_1^2-\frac{9bd}{2}q_1[\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]$$

$-bd^2[\sum_{i=1}^{k}(i-1)^2\{p_{ii}+\sum_{j=1}^{k}p_{ij}+\frac{1}{4}(\sum_{j=1}^{k}p_{ij})^2\}+2\sum_{1\leq i<j\leq k}(i-1)(j-1)p_{ij}]$
$-\frac{bd^2}{2}[\sum_{1\leq i<j\leq k}(i-1)(j-1)\sum_{s=1}^{k}p_{is}\sum_{s=1}^{k}p_{js}]-\frac{a^2}{4b}$.

To be an equilibrium, the above expression needs to be ≥ 0, for some values of q_1 and d. For any $d > 0$, we can view this expression as a function of q_1. Thus, a necessary and sufficient condition for the existence of such an equilibrium is that the maximum value of this function be ≥ 0.

Using the first order condition, this is maximised at $\widehat{q_1} = \frac{1}{3b}[a - 3bd\{\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}]$.

Using $\widehat{q_1}$, the maximum value of $A_P(q_c)$ becomes

$$bd^2[2\{\sum_{i=1}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}^2-\sum_{i=1}^{k}(i-1)^2(p_{ii}+\sum_{j=1}^{k}p_{ij})-2\sum_{1\leq i<j\leq k}(i-1)(j-1)p_{ij}]$$

The equilibrium condition thus requires the above to be ≥ 0. As both b and d are > 0, this leads to the statement in the corollary. □

The characterisation provided in Corollary 4.7 is important to analyse any k-SSCCE with equi-distant quantities. We will use this characterisation to prove some propositions later in this paper.

4.3.2 Uniqueness

We now show that the unique Nash and correlated equilibrium *a la* Aumann of the linear duopoly game is not the only k-SSCCE for this game. Any Nash-centric device is also a k-SSCCE for the linear duopoly game. Moreover, Nash-centric devices are the only equilibria among the set of k-SSCDs with equi-distant quantities and anti-diagonal probability distribution.

To prove this uniqueness result, we observe a couple of properties of any k-SSCD with equi-distant quantities and anti-diagonal probability distribution.[5] We first note that the expected payoff from any k-SSCD with equi-distant quantities and anti-diagonal probability distribution, $E_d(\Pi)$, can be expressed in terms of the expected quantity of a player, $\overline{q}$, as defined below.

Let us formally define the expected quantity of a player, $\overline{q}$, for a k-SSCD with equi-distant quantities and anti-diagonal probability distribution. For an odd k $(= 2m+1)$, $\overline{q} = \sum_{i=1}^{m} p_i(q_i+q_{2m+2-i})+(1-2\sum_{i=1}^{m} p_i)q_{m+1}$, while for an even k $(= 2m)$, $\overline{q} = \sum_{i=1}^{m} p_i(q_i + q_{2m+1-i})$, with $q_i = q_1 + (i-1)d$, $1 \leq i \leq k$. Clearly, for an odd k $(= 2m + 1)$, $\overline{q} = q_1 + md$, while for an even k $(= 2m)$, $\overline{q} = q_1 + \frac{d(2m-1)}{2}$.

Therefore, note that for a Nash-centric device, for any k, $\overline{q} = q_{NE}$.

Let $\Pi(\overline{q}, \overline{q})$ denote the profit of any player in the linear duopoly game when both players play the quantity $\overline{q}$. The following lemma is in place.

Lemma 4.8. *The expected profit of each player following any k-SSCD with equi-distant quantities and anti-diagonal probability distribution is equal to the profit when both players play their expected quantities in the linear duopoly game, i.e., $E_d(\Pi) = \Pi(\overline{q}, \overline{q})$.*

Proof. The profit of a firm is given by $\Pi(\overline{q}, \overline{q}) = a\overline{q} - 2b\overline{q}^2$.

[5] We thank an anonymous referee for suggesting Lemmata 4.8 and 4.9 to prove Theorems 4.10 and 4.11 directly using them.

For $k = 2m+1$, we have $\overline{q} = \sum_{i=1}^{m} p_i(q_i + q_{2m+2-i}) + (1 - 2\sum_{i=1}^{m} p_i)q_{m+1} = q_1 + md$. Substituting the value of $\overline{q}$, we have, $\Pi(\overline{q}, \overline{q}) = aq_1 - 2bq_1^2 - 4bmdq_1 + amd - 2bm^2d^2$.

The expected payoff from any $k-$SSCD with equi-distant quantities and an anti-diagonal probability distribution is given by,

$E_d(\Pi) = \sum_{i=1}^{m} p_i[a(q_i + q_{2m+2-i}) - b(q_i^2 + q_{2m+2-i}^2) - 2bq_iq_{2m+2-i}] + (1 - 2\sum_{i=1}^{m} p_i)[aq_{m+1} - 2bq_{m+1}^2]$

$= aq_1 - 2bq_1^2 - 4bmdq_1 + amd - 2bm^2d^2 = \Pi(\overline{q}, \overline{q})$.

Similarly, for $k = 2m$, we have $\overline{q} = \sum_{i=1}^{m} p_i(q_i + q_{2m+1-i}) = q_1 + \frac{d(2m-1)}{2}$. In this case,

$\Pi(\overline{q}, \overline{q}) = aq_1 - 2bq_1^2 - 2bd(2m-1)q_1 + \frac{ad(2m-1)}{2} - \frac{bd^2(2m-1)^2}{2}$

and

$E_d(\Pi) = \sum_{i=1}^{m} p_i[a(q_i + q_{2m+1-i}) - b(q_i^2 + q_{2m+1-i}^2) - 2bq_iq_{2m+1-i}]$

$= aq_1 - 2bq_1^2 - 2bd(2m-1)q_1 + \frac{ad(2m-1)}{2} - \frac{bd^2(2m-1)^2}{2} = \Pi(\overline{q}, \overline{q})$. □

From Lemma 4.8, it follows immediately that the expected payoff from a Nash-centric device, $E_{NC}(\Pi)$, is actually equal to that of the Nash equilibrium of the linear duopoly game, that is, $E_{NC}(\Pi) = \frac{a^2}{9b}$.

We now consider the expected payoff of any deviating player from playing any strategy q, $E_d(\Pi \mid q)$, and the maximum payoff from deviating, $Max_{q \in Q} E_d(\Pi \mid q) = E_d(\Pi^*)$ (say). The following lemma confirms that the best response of a deviating player depends only on his (opponent's) expected quantity, i.e.

Lemma 4.9. *$E_d(\Pi^*)$ is a function of $\overline{q}$.*

Proof. For any k, suppose a player deviates from the k-SSCD with equi-distant quantities and an anti-diagonal probability to play an alternate strategy q. We have, $E_d(\Pi \mid q) = aq - bq^2 - bq \sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}) = aq - bq^2 - bq\overline{q}$, which is maximised at $q = \frac{a-b\overline{q}}{2b}$, with

$E_d(\Pi^*) = MaxE_d(\Pi \mid q) = \frac{a^2}{4b} - \frac{a}{2}\overline{q} + \frac{b}{4}\overline{q}^2$, for any k, which proves the lemma. □

Our results (Theorems 4.10 and 4.11) now follow from the above lemmata.

Theorem 4.10. *Any Nash-centric device is a k-SSCCE for the linear duopoly game.*

Proof. From Lemma 4.8, $E_{NC}(\Pi) = \frac{a^2}{9b}$. Let $E_{NC}(\Pi \mid q)$ denote the (expected) payoff of the deviant from playing q. From Lemma 4.9, for any k, $E_{NC}(\Pi \mid q) = aq - bq^2 - bqq_{NE}$.

As in the proof of Theorem 4.6, for the Nash-centric device to be a k-SSCCE, we must have, $E_{NC}(\Pi) \geq E_{NC}(\Pi \mid q)$ for all $q \in Q$, which holds true if and only if $E_{NC}(\Pi) \geq Max_{q \in Q} E_{NC}(\Pi \mid q) = E_{NC}(\Pi^*)$ (say).

Using the proof of Lemma 4.9, $E_{NC}(\Pi \mid q)$ is maximised at $q^* = \frac{a-bq_{NE}}{2b} = \frac{a-b(\frac{a}{3b})}{2b} = \frac{a}{3b} = q_{NE}$, with $E_{NC}(\Pi^*) = aq^* - bq^{*2} - bq^*q_{NE} = E_{NC}(\Pi)$.

Hence, $E_{NC}(\Pi) = E_{NC}(\Pi^*) = Max_{q \in Q} E_{NC}(\Pi \mid q)$, and thus any Nash-centric device is a k-SSCCE. □

As noted in the above proof, $E_{NC}(\Pi) = E_{NC}(\Pi^*)$; hence, any Nash-centric device weakly satisfies the equilibrium condition for a k-SSCCE. Such a device thus does not satisfy the (strict) equilibrium notion in Gerard-Varet and Moulin (1978), as mentioned in Footnote 3.

We observe that Theorem 4.10 holds for any Nash-centric device, i.e., for any dimension k, any appropriate $\delta > 0$ (as long as $q_1 > 0$) and any anti-diagonal probability distribution given by the probabilities p_i, for $1 \leq i \leq m$, maintaining the Nash-centric structure.

We now prove that Nash-centric is the unique equilibrium structure among any k-SSCDs with equi-distant quantities and anti-diagonal probability distributions.

Theorem 4.11. *Nash-centric devices are the only k-SSCCE with equi-distant quantities and an anti-diagonal probability distribution for the linear duopoly game.*

Proof. For any k-SSCD with equi-distant quantities and an anti-diagonal probability distribution to be a k-SSCCE, we must have, $E_d(\Pi) \geq E_d(\Pi^*)$. We consider $E_d(\Pi) - E_d(\Pi^*)$ as a function of q_1 and find the maximum of this function.

For $k = 2m+1$, with $\overline{q} = q_1 + md$, $E_d(\Pi) - E_d(\Pi^*)$ becomes (assuming, $d = \delta$)

$E_d(\Pi) - E_d(\Pi^*) =$

$(aq_1 - 2bq_1^2 - 4bm\delta q_1 + am\delta - 2bm^2\delta^2) - (\frac{a^2}{4b} - \frac{a}{2}q_1 - \frac{am\delta}{2} + \frac{b}{4}q_1^2 + \frac{b\delta^2m^2}{4} + \frac{b\delta m}{2}q_1)$,

which is maximised at $\widehat{q_1} = \frac{a}{3b} - m\delta$, the Nash centric quantity q_1, for odd k, with the distance between quantities, $d = \delta$.

For, $k = 2m$, with $\overline{q} = q_1 + \frac{d(2m-1)}{2}$, $E_d(\Pi) - E_d(\Pi^*)$ becomes (assuming, $d = 2\delta$)

$E_d(\Pi) - E_d(\Pi^*) =$

$[aq_1 - 2bq_1^2 - 4b\delta(2m-1)q_1 + a\delta(2m-1) - 2b\delta^2(2m-1)^2]$

$-[\frac{a^2}{4b} - \frac{a}{2}q_1 - \frac{a\delta(2m-1)}{2} + \frac{b}{4}q_1^2 + \frac{b\delta^2(2m-1)^2}{4} + \frac{b\delta(2m-1)}{2}q_1]$, which is maximised at $\widehat{q_1} = \frac{a}{3b} - \delta(2m-1)$, the Nash centric quantity q_1, for even k, with the distance between quantities, $d = 2\delta$.

Thus, for any k, the Nash-centric quantity q_1 maximises the function $E_d(\Pi) - E_d(\Pi^*)$ and we have, from Theorem 4.10, $E_{NC}(\Pi) = E_{NC}(\Pi^*)$. Hence, the equilibrium condition is weakly satisfied for Nash-centric devices and is not met for any other $k-$SSCD with equi-distant quantities and an anti-diagonal probability distribution. □

As a direct consequence of the above, we can claim that the only $2-$SSCCE for the linear duopoly game is a Nash-centric device with $q_1 = q_{NE} - \delta$ and $q_2 = q_{NE} + \delta$ for any $0 < \delta < \frac{a}{3b}$ (to keep $q_1 > 0$) and $p_{12} = p_{21} = \frac{1}{2}$. To prove this claim one can use the equilibrium characterisation in Corollary 4.7, because a $2-$SSCD can trivially be viewed as a device with equi-distant quantities. The condition in Corollary 4.7, for $k = 2$, becomes $p_{12} - 2MN \geq 0$, where $M = p_{11} + p_{12}$ and $N = 1 - M$. The LHS of this condition can easily be rearranged to $-(p_{22}M + p_{11}N)$, which clearly is always < 0, unless $p_{11} = p_{22} = 0$.

4.3.3 Robustness

In what follows, we deal only with Nash-centric devices. All the results in the rest of the paper relate to the characterisations presented in Theorem 4.6 and Corollary 4.7.

We ask whether Nash-centric devices are robust equilibria for the linear duopoly game or not. To do so, we first check how crucial the linear set-up is.

Let us consider a duopoly market with linear demand function, however with quadratic costs (as analysed by Yi, 1997), where the firms' strategy is to choose a quantity $q \in Q = \{q : q \geq 0\}$. The price and the cost function for firm i are given by $p^i = a - q^i - \gamma q^j$, $j \neq i$, with $0 \leq \gamma \leq 1$, and $C(q^i) = (q^i)^2$. Thus the profit functions are respectively, $\Pi_1(q^1, q^2) = aq^1 - 2(q^1)^2 - \gamma q^1 q^2$ and $\Pi_2(q^1, q^2) = aq^2 - 2(q^2)^2 - \gamma q^1 q^2$, where q^1 and q^2 are the quantity choices of firms 1 and 2. Let us call this game the *quadratic duopoly game.* The Nash equilibrium quantity for the quadratic duopoly game is $q_{NE} = \frac{a}{4+\gamma}$.

We show below that the Nash-centric equilibrium structure is not robust against non-linearity. Indeed, for the quadratic duopoly game, no Nash-centric device is an equilibrium.

Proposition 4.12. *Any Nash-centric device is not a k-SSCCE for the quadratic duopoly game.*

Proof. Consider any Nash centric device for the quadratic duopoly game. The expected payoff from following this device is given by,

$E_{NC}(\Pi) = \frac{2a^2}{[4+\gamma]^2} - 2\delta^2(2-\gamma)[\sum_{i=1}^{m}(m+1-i)^2 p_i]$, for any odd k $(= 2m+1)$,

and,

$E_{NC}(\Pi) = \frac{2a^2}{[4+\gamma]^2} - 2\delta^2(2-\gamma)[\sum_{i=1}^{m}(2m+1-2i)^2 p_i]$, for any even k $(= 2m)$.

Let $E_{NC}(\Pi \mid q)$ denote the expected payoff of the deviant from playing q. Then, for any k, $E_{NC}(\Pi \mid q) = aq - 2q^2 - \gamma q q_{NE}$.

As in the proof of Theorem 4.6, for the Nash centric device to be a $k-$SSCCE, we must have, $E_{NC}(\Pi) \geq E_{NC}(\Pi \mid q)$, for all $q \in Q$, which holds true if and only if $E_{NC}(\Pi) \geq Max_{q \in Q} E_{NC}(\Pi \mid q)$.

$E_{NC}(\Pi \mid q)$ is maximised at $q^* = \frac{a-\gamma q_{NE}}{4}$, and $Max_{q\in Q}E_{NC}(\Pi \mid q) = \frac{2a^2}{(4+\gamma)^2}$.

$E_{NC}(\Pi) - Max_{q\in Q}E_{NC}(\Pi \mid q)$ is clearly < 0, for any k. Hence, the equilibrium condition, $E_{NC}(\Pi) \geq Max_{q\in Q}E_{NC}(\Pi \mid q)$ is never satisfied and thus the Nash centric device is not an equilibrium for this game. □

4.3.4 Perturbations

In this subsection, we prove that the Nash-centric structure is not robust as an equilibrium in the linear duopoly game by showing that any small unilateral perturbation of this device leads to a violation of the equilibrium condition.

We consider small unilateral changes in probabilities and quantities; first in the probability distribution, keeping the quantity levels unchanged, and then in the quantity levels, keeping the probability distribution fixed. We also consider perturbed devices by adding one more quantity level.

Probability

We first consider a small change in the anti-diagonal probability distribution associated with a Nash-centric device, without changing the quantities, which still remain Nash-centric. We divide the perturbation in probabilities into two cases: first, we change one of the zero off-diagonal elements and then we change one of the zero diagonal elements.

First, for simplicity and without loss of generality, we change the first off-diagonal element and make it positive. Let us make $p_{12} = p_{21} > 0$, and as a consequence, (some of) the anti-diagonal probabilities will change. For an odd k, for simplicity, we change only the probability attached to the Nash equilibrium strategy profile, $p_{(m+1)(m+1)}$, to $(1-2\sum_{i=1}^{m} p_i - 2p_{12})$, with the rest of probabilities remaining intact. For an even k, we need not specify the specific changes in the anti-diagonal probabilities, as long as $1-2\sum_{i=1}^{m} p_i = 2p_{12}$. Let us call this new device, for any k, an *off-diagonal-probability-perturbed Nash-centric device* and prove the following desired result.

Proposition 4.13. *Any off-diagonal-probability-perturbed Nash-centric device is not a k-SSCCE for the linear duopoly game.*

Proof. For $k = 2m+1$, from Corollary 4.7, the equilibrium condition for the off-diagonal-probability-perturbed Nash centric device to be a k-SSCCE becomes

$$2[p_{21} + \sum_{i=1}^{m}(i-1)p_i + m(1-2p_{12}-2\sum_{i=1}^{m} p_i) + \sum_{i=1}^{m}(2m+1-i)p_i]^2$$
$$-[p_{21} + \sum_{i=1}^{m}(i-1)^2 p_i + m^2(1-2p_{12}-2\sum_{i=1}^{m} p_i) + \sum_{i=1}^{m}(2m+1-i)^2 p_i]$$
$$-[m^2(1-2p_{12}-2\sum_{i=1}^{m} p_i) + 2\sum_{i=1}^{m}(i-1)(2m+1-i)p_i] \geq 0.$$

After simplification, the LHS of the above turns out to be $(2m-1)^2 p_{21}(2p_{21}-1)$, which is always < 0, unless $p_{21} \geq \frac{1}{2}$, which is not possible.

Similarly, for $k = 2m$, the equilibrium condition turns out to be $4(m-1)^2 p_{21}(2p_{21}-1)$, which is always < 0, unless $p_{21} \geq \frac{1}{2}$, which is not possible.

Hence, the equilibrium condition is violated for any k. □

Now, for simplicity and without loss of generality, let us change the first diagonal element and make it positive, i.e., let us make $p_{11} > 0$, and as a consequence let us change (some of) the anti-diagonal probabilities. As earlier, for an odd

k, for simplicity, we change only the probability attached to the Nash equilibrium strategy profile, $p_{(m+1)(m+1)}$, to $1 - 2\sum_{i=1}^{m} p_i - p_{11}$, keeping the rest of the probabilities intact. For an even k, we need not specify the specific changes in the anti-diagonal probabilities, as long as $1 - 2\sum_{i=1}^{m} p_i = p_{11}$. Let us call this new device, for any k, a *diagonal-probability-perturbed Nash-centric device.*

Proposition 4.14. *Any diagonal-probability-perturbed Nash-centric device is not a k-SSCCE for the linear duopoly game.*

Proof. For $k = 2m + 1$, from Corollary 4.7, the equilibrium condition for the diagonal-probability-perturbed Nash centric device to be a k-SSCCE becomes

$$2[\sum_{i=1}^{m}(i-1)p_i + m(1 - p_{11} - 2\sum_{i=1}^{m} p_i) + \sum_{i=1}^{m}(2m+1-i)p_i]^2$$
$$-[\sum_{i=1}^{m}(i-1)^2 p_i + m^2(1 - p_{11} - 2\sum_{i=1}^{m} p_i) + \sum_{i=1}^{m}(2m+1-i)^2 p_i]$$
$$-[m^2(1 - p_{11} - 2\sum_{i=1}^{m} p_i) + \{2\sum_{i=1}^{m}(i-1)(2m+1-i)p_i\}] \geq 0.$$

Simplifying, the LHS of the above turns out to be $-2m^2 p_{11}(1 - p_{11})$, which is always < 0.

Similarly, for $k = 2m$, the equilibrium condition turns out to be $(2m - 1)^2(\sum_{i=1}^{m} p_i)[2(\sum_{i=1}^{m} p_i) - 1]$, which is ≥ 0, if and only if $\sum_{i=1}^{m} p_i \geq \frac{1}{2}$, which is further possible only when $p_{11} = 0$, and thereby a contradiction.

Hence, the equilibrium condition is violated for any k. □

Quantity

We now consider a small perturbation in the quantity levels, keeping the anti-diagonal probability distribution fixed. For simplicity and without loss of generality, let us change the first quantity. For $k = 2m + 1$, let $q_1 = q_{NE} - m\delta + \varepsilon$, while for $k = 2m$, let $q_1 = q_{NE} - (2m - 1)\delta + \varepsilon$, keeping all other quantities to be Nash-centric and equi-distant from each other. Let us call this new device a *quantity-perturbed device.*

Proposition 4.15. *Any quantity-perturbed Nash-centric device is not a k-SSCCE for the linear duopoly game.*

Note that the above proposition refers to a device that does not involve all equi-distant quantities. Thus, unlike the previous propositions in this subsection, here we can not use the characterisation in Corollary 4.7. The proof, however, directly follows from the characterisation in Theorem 4.6.

Proof. From Theorem 4.6, substituting the values of q_1 and other q_i for $i \neq 1$, (for any k), the expression $A_P(q_c)$ becomes

$\frac{3a}{2}q_{NE} - \frac{9b}{4}q_{NE}^2 + \frac{3ap_1\varepsilon}{2} - b\varepsilon^2(p_1 + \frac{{p_1}^2}{4}) - \frac{9b\varepsilon p_1}{2}q_{NE} - \frac{a^2}{4b}$.

For the quantity-perturbed Nash centric device to be a k-SSCCE, we need the above expression to be ≥ 0. Now substituting $q_{NE} = \frac{a}{3b}$, the expression becomes $-b\varepsilon^2(p_1 + \frac{{p_1}^2}{4})$, which is always < 0. Hence, the equilibrium condition is violated. □

Composition

Finally, we turn to another way of perturbing a Nash-centric device. We consider a new device composed of one additional quantity level (other than the Nash equilibrium quantity) along with the original Nash-centric device. Starting from a Nash-centric k-SSCCE (for any k), we construct a public $(k+1)$-SSCD by adding another quantity $q_\varepsilon > 0$ ($\neq \frac{a}{3b}$) for both players, with probability ε for the strategy profile $(q_\varepsilon, q_\varepsilon)$, coupled with the original Nash-centric device with probability $(1-\varepsilon)$. Formally, given a Nash-centric device, $[k; (p_i)_{1\leq i\leq m}; \delta]$, we construct a $(k+1)$-SSCD, as follows.

For any odd k $(= 2m+1)$, the quantities are $q_1 = q_\varepsilon$, $q_{m+2} = \frac{a}{3b} = q_{NE}$ and $q_i = q_{NE} - (m+2-i)\,\delta$ for $2 \leq i \leq k+1 (= 2m+2)$, while for any even k $(= 2m)$, they are $q_i = q_{NE} - (2m+3-2i)\,\delta$ for $2 \leq i \leq k+1 (= 2m+1)$.

The probabilities are $p_{11} = \varepsilon$, $p_{1j} = p_{j1} = 0$ for $j = 2, ..., k+1$ (for any k). For odd k $(= 2m+1)$, $p_{i(k+3-i)} = p_{(k+3-i)i} = (1-\varepsilon)p_{i-1}$, for $2 \leq i \leq m+1$, $p_{(m+2)(m+2)} = (1-\varepsilon)(1 - 2\sum_{i=1}^{m} p_i)$, and $p_{ij} = 0$, otherwise; for even k $(= 2m)$, $p_{i(k+3-i)} = p_{(k+3-i)i} = (1-\varepsilon)p_{i-1}$ for $2 \leq i \leq m+1$, and $p_{ij} = 0$, otherwise. Note that for any k, $\sum_{i=2}^{k+1}\sum_{j=2}^{k+1} p_{ij} = (1-\varepsilon)$.

We call this device a *composite device.*

Proposition 4.16. *Any composite device is not a* $(k+1)$*-SSCCE for the linear duopoly game.*

Proof. Following Theorem 4.6, for the composite device (for any k) the expression $A_P(q_c)$ becomes

$$\frac{3a\varepsilon}{2}q_\varepsilon + \frac{3a(1-\varepsilon)}{2}q_{NE} - b(2\varepsilon + \frac{\varepsilon^2}{4})q_\varepsilon^2 - \frac{b\varepsilon(1-\varepsilon)}{2}q_\varepsilon q_{NE} - b[2(1-\varepsilon) + (\frac{1-\varepsilon}{2})^2]q_{NE}^2 - \frac{a^2}{4b}.$$

Substituting $q_{NE} = \frac{a}{3b}$, and rearranging, we get, $A_P(q_c) = q_\varepsilon[\frac{a\varepsilon(8+\varepsilon)}{6}] - q_\varepsilon^2[\frac{b\varepsilon(8+\varepsilon)}{4}] - \frac{a^2\varepsilon(8+\varepsilon)}{36b}$, which can be viewed as a (quadratic) function in q_ε. This function is maximised at $q_\varepsilon = \frac{a}{3b}$ (the Nash equilibrium quantity) and the maximised value of the function is 0. Therefore, for any $q_\varepsilon > 0$, other than the Nash

Equilibrium quantity, the value of $A_P(q_c)$ is < 0.

From Theorem 4.6, for the composite device to be a $(k+1)$-SSCCE, we need the above $A_P(q_c)$ to be ≥ 0. However, from above, the value of the above function is < 0, for any $q_\varepsilon > 0$, other than the Nash Equilibrium quantity. Hence, the equilibrium condition is violated. □

4.4 CONCLUSION

In this paper, we have analysed the notion of coarse correlation in the simplest of the oligopoly models, that of a duopoly with linear market demand and constant marginal costs. We have defined and characterised an equilibrium notion, that we call k-SSCCE, for this linear duopoly game. We have identified a particular sunspot structure, that we call a Nash-centric device, which always serves as an equilibrium for such a game; moreover, any small perturbation from this device is not an equilibrium.

Any Nash-centric device is a special type of a public or sunspot k-SSCD, as defined in this paper, and by Theorem 4.10, is also a k-SSCCE. We should point out here that Nash-centric devices assign positive probabilities over non-Nash equilibrium quantities as well. Such a *sunspot structure* is clearly not a correlated equilibrium *a la* Aumann, as it is well-known that a *public* device can only be a correlated equilibrium *a la* Aumann if and only if it is a convexification of pure Nash equilibria.

We however note that the expected payoff from such a device is equal to the Nash equilibrium payoff. We have also pointed out that although any Nash-centric device is a k-SSCCE for the linear duopoly game, such devices may however fail to be so in duopoly models with quadratic costs such as one studied by Yi (1997). For any general quadratic duopoly model, we also note that k-SSCCE, even 2-SSCCE, other than the Nash-centric devices, exists and considerably improves upon the Nash equilibrium. In a parallel paper, Moulin *et al.* (2014) extensively analyse coarse correlated equilibria in quadratic potential games.

One may extend our research to several directions.[6] First, following Young (2004), one may ask under what conditions the regret minimization dynamics converges to the set of Nash-centric devices in our set-up. Second, our main result is similar in spirit to the work by Börgers and Janssen (1995) who identify conditions under which the Nash equilibrium of the oligopolists' game is the only outcome that survives iterated deletion of dominated strategies. It will thus be interesting to formally connect these two strands of literature. Finally, Forgó *et al.* (2005) analysed a related notion, called the *soft correlated equilibrium*, in climate change models. We can similarly analyse our k-SSCCE, in particular, Nash-centric devices in such models, where the objective function to be maximised can be different from the sum of payoffs (such as, reduction in temperature in climate change models). We defer all these issues to future research agenda.

[6] We thank an anonymous referee for suggesting some of these issues.

Bibiliography

[1] Aumann, R. J. (1974), "Subjectivity and correlation in randomized strategies," *Journal of Mathematical Economics*, 1, 67-96.

[2] Aumann, R. J. (1987), "Correlated equilibrium as an expression of Bayesian rationality," *Econometrica*, 55, 1-18.

[3] Aumann, R. J., J. Peck and K. Shell (1988), "Asymmetric information and sunspot equilibria : a family of simple examples," CAE Working Paper #88-34.

[4] Azariadis, C. (1981), "Self-fulfilling prophecies," *Journal of Economic Theory,* 25, 380-396.

[5] Baliga, S. and E. Maskin (2003) "Mechanism design for the environment," in K. Maler and J. Vincent (eds.) *Handbook of Environmental Economics*, Volume 1, Elsevier Science.

[6] Barrett, S. (1994), "Self-enforcing international environmental agreements," *Oxford Economic Papers*, 46, 878-894.

[7] Barrett, S. (2001), "International cooperation for sale," *European Economic Review*, 45, 1835-1850.

[8] Bone, J., M. Drouvelis and I. Ray (2012), "Following recommendations to avoid coordination-failure in 2x2 games," *Discussion Paper* 12-04, Department of Economics, University of Birmingham.

[9] Börgers, T. and M. Janssen (1995), "'On the dominance solvability of large Cournot models," *Games and Economic Behavior*, 8, 297-321.

[10] Cason, T. N. and T. Sharma (2007), "Recommended play and correlated equilibria," *Economic Theory*, 33, 11-27.

[11] Cass, D. and K. Shell (1983), "Do sunspots matter?," *Journal of Political Economy,* 91, 193-227.

[12] Dávila, J. (1999), "On the connection between correlated equilibria and sunspot equilibria," in A. Alkan, C. D. Aliprantis, and N. C. Yannelis (eds.), *Current Trends in Economics: Theory and Applications,* Springer Verlag, 141-156.

[13] Diamond, P. A. (1982), "Aggregate demand management in search equilibrium," *Journal of Political Economy*, 90, 881-894.

[14] Duffy, J. and N. Feltovich (2010), "Correlated equilibria good and bad: an experimental study," *International Economic Review*, 51, 701-772.

[15] Finus, M. (2001), *Game theory and international environmental cooperation*, Edward Elgar.

[16] Forges, F. and J. Peck (1995), "Correlated equilibrium and sunspot equilibrium," *Economic Theory*, 5, 33-50.

[17] Forgó, F., J. Fülöp and M. Prill (2005), "Game theoretic models for climate change negotiations," *European Journal of Operations Research*, 160, 252-267.

[18] Forgó, F. (2010), "A generalization of correlated equilibrium: a new protocol," *Mathematical Social Sciences*, 60, 186-190.

[19] Forgó, F. (2011), "Generalized correlated equilibrium for two-person games in extensive form with perfect information," *Central European Journal of Operations Research*, 19, 201-213.

[20] Ganguly, C. and I. Ray (2005), "Simple mediation in a cheap-talk game," *Discussion Paper* 05-08, Department of Economics, University of Birmingham.

[21] Gerard-Varet, L. A. and H. Moulin (1978), "Correlation and duopoly," *Journal of Economic Theory*, 19, 123-149.

[22] Hart, S. (2005), "Adaptive heuristics," *Econometrica*, 73, 1401-1430.

[23] Hart, S. (2011), "Nash equilibrium and dynamics," *Games and Economic Behavior*, 71, 6-8.

[24] Hart, S. and A. Mas-Colell (2000), "A simple adaptive procedure leading to correlated equilibrium," *Econometrica*, 68, 1127-1150.

[25] Hart, S. and A. Mas-Colell (2003), "Regret-based continuous-time dynamics," *Games and Economic Behavior*, 45, 375-394.

[26] Hart, S. and D. Schmeidler (1989), "Existence of correlated equilibria," *Mathematics of Operations Research*, 14, 18-25.

[27] Kar, A., I. Ray and R. Serrano (2010), "A difficulty in implementing correlated equilibrium distributions," *Games and Economic Behavior*, 69, 189-193.

[28] Liu, L. (1996), "Correlated equilibrium of Cournot oligopoly competition," *Journal of Economic Theory*, 68, 544-548.

[29] Maskin, E. and J. Tirole (1987), "Correlated equilibria and sunspots," *Journal of Economic Theory,* 43, 364-373.

[30] McGinty, M. (2007), "International environmental agreements among asymmetric nations," *Oxford Economic Papers*, 59, 45-62.

[31] Moulin, H. and J. P. Vial (1978), "Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon," *International Journal of Game Theory*, 7, 201-221.

[32] Moulin, H., I. Ray and S. Sen Gupta (2013), "Coarse correlated equilibria in an abatement game," *Discussion Paper* 13-11, Department of Economics, University of Birmingham.

[33] Moulin, H., I. Ray and S. Sen Gupta (2014), "Improving Nash by coarse correlation," *Journal of Economic Theory*, 150, 852-865.

[34] Neyman, A. (1997), "Correlated equilibrium and potential games", *International Journal of Game Theory*, 26, 223-227.

[35] Peck, J. (1994), "Notes on correlated equilibrium and sunspot equilibrium," in J.-F. Mertens and S. Sorin (eds.), *Game Theoretic Methods in General Equilibrium Analysis*, Kluwer Academic Publishers, 249-260.

[36] Polemarchakis, H. and I. Ray (2006), "Sunspots, correlation and competition," *Games and Economic Behavior*, 56, 174-184.

[37] Ray, I. and S. Sen Gupta (2013), "Coarse correlated equilibria in linear duopoly games," *International Journal of Game Theory*, 42, 541-562.

[38] Roughgarden, T. (2009), "Intrinsic robustness of the price of anarchy," *STOC* '09.

[39] Rubio, S. J. and A. Ulph (2006), "Self-enforcing international environmental agreements revisited," *Oxford Economic Papers*, 58, 233-263.

[40] Shapley, L. S. and M. Shubik (1977), "Trade using one commodity as a means of payment," *Journal of Political Economy*, 85, 937-968.

[41] Ui, T. (2008), "Correlated equilibrium and concave games," *International Journal of Game Theory*, 37, 1-13.

[42] Yi, S. S. (1997), "On the existence of a unique correlated equilibrium in Cournot oligopoly," *Economic Letters*, 54, 235-239.

[43] Young, H. P. (2004), *Strategic learning and its limits*, Oxford University Press.