# On a Purely Categorical Framework for Coalgebraic Modal Logic

by

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*Modern algebra also enables one to reinterpret the results of classical algebra, giving them far greater unity and generality.*

G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra*,

Macmillan, New York, 1965

# *Abstract*

A category CoLog of distributive laws is introduced to unify different approaches to modal logic for coalgebras, based merely on the presence of a contravariant functor $P$ that maps a state space to its collection of predicates. We show that categorical constructions, including colimits, limits, and compositions of distributive laws as a tensor product, in CoLog generalise and extend existing constructions given for Set coalgebraic logics and that the framework does not depend on any particular propositional logic or state space.

In the case that $P$ establishes a dual adjunction with its dual functor $S$, we show that a canonically defined coalgebraic logic exists for any type of coalgebras. We further restrict our discussion to finitary algebraic logics and study equational coalgebraic logics. Objects of predicate liftings are used to characterise equational coalgebraic logics.

The expressiveness problem is studied via the mate correspondence, which gives an isomorphism between CoLog and the comma category from the pre-composition to the post-composition with $S$. Then, the modularity of the expressiveness is studied in the comma category via the notion of factorisation system.

*Dedicated to my parents*

# *Acknowledgements*

# Preface

This paragraph is left empty intentionally.

## Structure of Thesis

The main theme of this thesis is explained in Chapter 1.

Chapter 2 consists of an introduction to coalgebras over Set, Hennessy-Milner logic, coalgebraic modal logics over Set, and the motivation for generalising coalgebraic logics. There are barely any new insights in this chapter but mostly existing or slightly improved results or proofs. We will revisit coalgebras in Chapter 4.

Chapter 3 provides the necessary categorical machinery used throughout this thesis. We present factorisation systems and notions related to Kan extension. A section on foundations is included to clarify the difference between set-based functors and finitary functors, and to present to an improper notion of Set-functor.

In Chapter 4, we 're-invent' the theory of coalgebras using factorisation systems and a new operation inspired by the generalised product of Kripke frames. The second section of this chapter contains a categorical transfinite induction for constructing the language of logics and free monads. The third section is mainly a re-working of sifted colimit-preserving functors in elementary category theory for presenting equations in logics.

Chapter 5 is the core of this thesis. We study a category of coalgebraic logics to unify the many different approaches that exist in the literature. See the Introduction for more details. Most of this material is—to the best of my knowledge—completely original.

# Conventions

**Contributions**, or what's new?

If a '*cf.*' appears with a pointer to the literature, then this indicates that a similar notion or a result exists but here it is different, improved, or (maybe trivially) generalised; similarly a 'see' with a pointer indicates an *unoriginal* result, though we may have a different, sometimes simplified, argument. Where there is no annotation, this means that the result is either trivial or well-known, or otherwise, original to this thesis.

For example, the notion of (co)algebras for an endofunctor first appeared in the 1960s, as Richard Bird said in his book [27]:

> ... The notion of $F$-algebras first appeared in the categorical literature during the 1960s, for instance in (Lambek 1968 [78]). Long before the applications to program derivation were realised, numerous authors e.g. (Lehmann and Smyth 1981; Manes and Arbib 1986) pointed out the advantages of $F$-algebras in the area of program semantics. ...

Basic properties of coalgebras for an endofunctor are very well-known in its dual form, for example

> 'Let $T$ be a Set endofunctor. The forgetful functor $\mathsf{Set}_T \to \mathsf{Set}$ creates colimits. In particular, $\mathsf{Set}_T$ is cocomplete.'

However, a complete proof is not easy to find in the literature, or it is only sketched briefly, as in [19]. In such a case, we do not place a pointer in this proposition, but neither do we claim credit. Instead, we give a complete proof, if helpful. As for another well-known result dating back 1968—the *Lambek's Lemma*:

> 'For any endofunctor $T$, every final coalgebra is an isomorphism.'

the proof is widely available and the source is also well-known. In such a case, we only sketch it.

**Assumptions**

⚠ Any assumptions that are meant to remain in force for the reminder of a chapter (a section, or a subsection) will be marked by a dangerous-bend symbol, as in this paragraph.

**Typefaces**

**Bold** typeface is used for new terms in formal definitions; however, such terms may be used before the formal declaration and in such cases are set in *italics*. For example,

> ...transition systems are *coalgebras* for the covariant powerset functor.
> ...Formally a **coalgebra** is a function from $X$ to $TX$ for some endo-functor $T$ ...

**Typefaces for Mathematics**

The formal script typeface, e.g. $\mathscr{C}, \mathscr{D}$, is used for categories, and every specific name for a category is in sans-serif type face, e.g. Set, Top, and Pos. The calligraphic font is mostly used for named functors, e.g. the powerset functor $\mathcal{P}$, the identity functor $\mathcal{I}$, and the Yoneda embedding $\mathcal{Y}$.

Objects of a category $\mathscr{C}$ are denoted by lower case letters $c, d, \ldots$; if a category $\mathscr{C}$ is concrete, then instead objects are in upper case $C, D, \ldots$ and elements are in lower case $c, \ldots \in C$.

Monads are mostly typed in **blackboard boldface**. For example, a monad will be indicated by $\mathbb{M} = (M, \eta, \mu)$ or just $\mathbb{M}$, and the symbol for an arrow $\mathbb{L} \to \mathbb{M}$ between monads, although a natural transformation, is different from that for a general natural transformation $L \overset{\cdot}{\to} M$.

**Notations**

**Hom-set** The hom-set from an object $c$ to an object $d$ in a category $\mathscr{C}$ is indicated by

$$\mathrm{Hom}(c, d) \quad \text{or} \quad \mathscr{C}(c, d).$$

The collection of natural transformations between functors $F, G\colon \mathscr{C} \to \mathscr{D}$ is indicated by $\mathrm{Nat}(F, G)$ or $[\mathscr{C}, \mathscr{D}](F, G)$ where $[\mathscr{C}, \mathscr{D}]$ is the notation for the functor category from $\mathscr{C}$ to $\mathscr{D}$.

**Natural numbers are sets.** Every natural number $n$ is a set and consists of exactly $n$-many elements $\{0, \ldots, n-1\}$. E.g. $0 = \emptyset$, $1 = \{0\} = \{\emptyset\}$.

**One element set**: $1 = \{0\} = \{*\} = \{\checkmark\}$. As stated above, $1$ is $\{0\}$, but to avoid confusion we also use $*$ or $\checkmark$ to indicate the unique element in $1$.

**Two element set** For similar reasons, $2 = \{0, 1\} = \{\bot, \top\}$.

# Contents

# Chapter 1

# Introduction

The aim of this thesis is to provide a unified and novel framework for coalgebraic modal logics in a point-free style. We introduce a category CoLog consisting of one-step interpretations of coalgebraic modalities as objects and pairs of a translation of modalities and a transformation of coalgebras as morphisms. It is based on the observation by Kupke *et al.* in [65] that modalities can be formulated as natural transformations of type $L\mathcal{Q} \dot{\longrightarrow} \mathcal{Q}T$ for the contravariant powerset functor $\mathcal{Q}$ and an endofunctors $T$ for the type of coalgebras and an endofunctor $L$ on the category of Boolean algebras for encoding the syntax.

Our framework is parametric in the contravariant functor $P\colon \mathscr{X} \to \mathscr{A}$ mapping from the 'state' category $\mathscr{X}$ to the 'logical' category $\mathscr{A}$ of predicates. The full strength of our results, however, relies on $P$ establishing a dual adjunction. The category CoLog accommodates different proposed coalgebraic logics, such as Moss' cover modality and Pattinson's predicate liftings, and it allows us to combine these approaches in a uniform way.

Our development does not depend on any assumptions such as concreteness of coalgebras, the use of propositional calculus, or the functor $T$ being finitary functors. For equational coalgebraic logics, the strongest assumption we do use is that the models of propositional logic form a single-sorted variety, although a many-sorted variety may be used as demonstrated in [75].

Despite its generality, the category CoLog faithfully represents various notions and constructions in the literature. For example, the fusion of coalgebraic modal logics is a coproduct, and the composition is a tensor product. It also provides the setting for other novel constructions such as limits of coalgebraic modal logics. A seemingly

useless logic for coalgebras of the identity functor plays an essential role in composition and it makes CoLog a monoidal category. This fact only became apparent within the categorical formulation.

The category itself gives birth to a family of categories indexed by types of coalgebras, and every such a category consists of *families of modalities* for a specific type functor $T$, denoted by $\mathsf{CoLog}_T$ in Chapter 5. By standard categorical techniques, we obtain the *canonically defined* and *most expressive* coalgebraic modal (resp. equational) logic, and we characterise its equational version in terms of generalised predicate liftings. A category of multi-step interpretations is also born from CoLog, which is tentatively proposed to be a categorical formulation of coalgebraic modal logics involving multi-step behaviours. A multi-step interpretation enables us to describe coalgebras for comonads instead of functors, and a free construction is given to construct a multi-step interpretation from a one-step interpretation without using transfinite induction, as one might have expected. Due to its complexity, we leave this line of research for future work. We suggest that CoLog is the right level of abstraction for coalgebraic logic.

Due to the generality of our framework, however, we have to give a detailed background of category theory beyond Mac Lane's textbook and 're-invent' the theory of coalgebras without using *points*. The most important technique throughout this thesis is probably the Kan extension. Other notions in category theory such as *density presentation* will be re-worked in ordinary category theory. As for coalgebras, the usual formulation of behavioural equivalence (also known as bisimilarity under a certain condition) relies on the concreteness of the underlying structure of coalgebras or a regular category where categorical relations are well-behaved. To achieve generality, we characterise behavioural equivalence using factorisation systems and show that it still has the coinduction principle (under mild conditions). Of course, it also covers the classical theory of coalgebras over Set.

To set the scene, we begin with a chapter on the recapitulation of the field of coalgebras and coalgebraic logics. This will serve as the main source of examples and motivations for subsequent chapters. We hope that this chapter will be a pleasing tour for both experts and beginners alike.

# Chapter 2

# Recapitulation

In this chapter, we present a tour on coalgebras and coalgebraic logics from a traditional point of view covering transition systems, homomorphisms, bisimulations, colimit constructions, etc.

We walk through various examples and notions using the language of elementary category theory, for example, *categories*, *functors*, *natural transformations*, etc. Everything we use may be found in any textbook such as [6, 31, 81].

The coalgebraic approach in theoretical computer science has proved to be effective and inspirational. As we will exhibit in Section 2.1, many state-based transition systems can be modelled as coalgebras parametric in endofunctors of Set, and generic constructions exist for every type of coalgebra (under reasonable assumptions). In Section 2.2, we introduce modal logic briefly as a natural logic for describing Kripke frames viewed as coalgebras. In Section 2.3, we generalise classical modal logic to coalgebraic modal logics (parametric in types of Set coalgebra) via Moss' cover modality [82, 83] and Pattinson's predicate liftings [88, 89].

⚜  In this chapter, $T \colon \mathsf{Set} \to \mathsf{Set}$ always denotes an endofunctor of Set.

## 2.1   Coalgebras in Set

**Definition 2.1.1.** A Set **coalgebra** of $T$ is a function $\xi \colon X \to TX$ for some **carrier set** $X$, denoted $\langle X, \xi \rangle$. The functor $T$ is called the **type** of $\langle X, \xi \rangle$. A **coalgebra homomorphism** $f \colon \langle X, \xi \rangle \to \langle Y, \gamma \rangle$ between coalgebras is a function $f \colon X \to Y$

such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \xi\ } & TX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\
Y & \xrightarrow[\ \gamma\ ]{} & TY
\end{array}
$$

commutes.

It is easy to check the following:

**Proposition 2.1.2.** *The collection of $T$-coalgebras with coalgebra homomorphisms and usual function composition forms a category, denoted* $\mathsf{Set}_T$.

Each $\mathsf{Set}_T$ has a forgetful functor $\mathsf{Set}_T \to \mathsf{Set}$ mapping a coalgebra $\langle X, \xi \rangle$ to its carrier $X$ and a coalgebra homomorphism $f \colon \langle X, \xi \rangle \to \langle Y, \gamma \rangle$ to the underlying function $f$; we leave the functor unnamed.

Despite the simplicity of the concept of coalgebras, there are plenty of interesting examples. We will illustrate constructions and notions using these examples throughout this chapter.

**Example 2.1.3.** Let $\mathcal{I}$ denote the identity functor. The following are examples of $T$-coalgebras:

1. Let $A$ be any set and consider $T \coloneqq A \times \mathcal{I}$. Then a $T$-coalgebra consists of a function $\xi$ from $X$ to $A \times X$. Each element of $X$ gives rise to a countably infinite sequence $(a_i)_{i \in \mathbb{N}}$ where $a_i \in A$. Conversely, given a set of such sequences, then we can construct a set $X$ and a function $\xi \colon X \to A \times X$ such that the associated set of streams consists of all the given sequences together with their tails. Similarly, a $T$-coalgebra gives rise to a set of *streams* or *lists* over $A$.

2. Again, assume that $A$ is a set and $T \coloneqq (1 + A \times \mathcal{I} \times \mathcal{I})$. Then a $T$-coalgebra consists of a set $X$ and a function $\xi \colon X \to 1 + A \times X \times X$. Each element of $X$ gives rise to an *ordered binary tree with labels* over $A$. It is called *ordered* because the node $\langle a, t_1, t_2 \rangle$ is distinct from $\langle a, t_2, t_1 \rangle$. To get *unordered* trees, let $TX$ be the quotient of $X \times X$ under the equivalence relation generated by $\langle x_1, x_2 \rangle \sim \langle x_2, x_1 \rangle$ for every $x_1, x_2 \in X$. Then, every $(1 + A \times T)$-coalgebra generates a set of *unordered binary trees with labels*.

3. Let $\mathcal{D}\mathcal{A} \coloneqq 2 \times (-)^A$ for some set $A$, where $(-)^A$ is the $A$-fold product. $\mathcal{D}\mathcal{A}$-coalgebras are *deterministic automata* over alphabets $A$. Each $\mathcal{D}\mathcal{A}$-coalgebra $\langle X, \xi \rangle$ may be viewed as a pair of functions $a \colon X \to 2 = \{0, 1\}$ and $\delta \colon X \to X^A$ where

*a* is the characteristic map of the set of accepting states $\{x \in X \mid a(x) = 1\}$ and $\delta$ is the transition function of the automaton. In particular, a *deterministic finite state automaton* is a $\mathcal{DA}$-coalgebra with an element in the carrier as its initial state and a finite carrier. A more detailed treatment of this example can be found in [94].

4. *Kripke frames, unlabelled transition systems,* or *relations,* are coalgebras of the covariant powerset functor, denoted $\mathcal{P}$, that is, because every relation $\rightarrow \subseteq X \times X$ defines a function from $X$ to the powerset of $X$:

$$f_R \colon x \mapsto \{x' \in X \mid x \rightarrow x'\} \tag{2.1}$$

and vice versa. For a fixed set $\Phi$ of atomic formulae, recall that *Kripke models* over $\Phi$ are triples consisting of a Kripke frame $\langle X, \xi \rangle$ and a valuation $X \rightarrow \mathcal{P}\Phi$. They are coalgebras of the functor $\mathcal{P} \times \mathcal{P}\Phi$.

5. *Labelled transition systems* (*LTS* for short) are coalgebras of the functor $\mathcal{P}(A \times -)$. Denoted $\langle X, A, \rightarrow \rangle$, they are traditionally defined as a relation $\rightarrow \subseteq X \times A \times X$ and $\langle x, a, x' \rangle$ in $\rightarrow$ is written as $x \xrightarrow{a} x'$. A $\mathcal{P}(A \times -)$-coalgebra $\xi$ defines a relation $\rightarrow$ by $x \xrightarrow{a} x'$ if and only if $\langle a, x' \rangle \in \xi(x)$.

6. Transitions may be weighted according to a probability distribution (with a finite support), and we obtain the following. Define the discrete distribution functor $\mathcal{D}$ on a set $X$ by

$$\mathcal{D}X := \{\mu \colon X \rightarrow [0,1] \mid \sum_{x \in X} \mu(x) = 1, \text{ and } \mu \text{ has } \textit{finite support}\}$$

where the support of $\mu$ is the nonzero valued set $\{x \in X \mid \mu(x) \neq 0\}$; and maps a function $f \colon X \rightarrow Y$ to a function $\mathcal{D}f \colon \mathcal{D}X \rightarrow \mathcal{D}Y$ sending any distribution $\mu_X \colon X \rightarrow [0,1]$ to a distribution on $Y$ by

$$\mathcal{D}f(\mu_X) \colon y \mapsto \sum_{fx=y} \mu_X(x).$$

The distribution $\mathcal{D}f(\mu_X)$ has finite support, because $\mu_X$ does. $\mathcal{D}$-coalgebras are *finitely-branching discrete-time Markov chains*. An extensive treatment of this example can be found in [101].

More examples can be found in, e.g. [52, 95] for general Set coalgebras, [47] for neighbourhood frames, and [100, 106] for coalgebras of probability distribution endofunctors.

### 2.1.1 Generic Constructions

One of the benefits of the coalgebra abstraction is the series of constructions uniformly available for all kinds of structures. To start with, colimits in $\mathsf{Set}_T$ are inherited from $\mathsf{Set}$:

**Proposition 2.1.4.** *The forgetful functor* $\mathsf{Set}_T \to \mathsf{Set}$ *creates colimits. In particular,* $\mathsf{Set}_T$ *is cocomplete.*

*Proof.* Let $F\colon \mathscr{D} \to \mathsf{Set}_T$ be a diagram in $\mathsf{Set}_T$. We want to show that $\operatorname{Colim} F$ exists in $\mathsf{Set}_T$. Denote the forgetful functor $\mathsf{Set}_T \to \mathsf{Set}$ by $|-|$. Since $\mathsf{Set}$ is cocomplete, there exists a limiting cocone of $|F|$, denoted by $\langle C, \mu\colon F \dashrightarrow C\rangle$. Define a cocone $v$ from $|F|$ to the set $TC$ by

$$v_i\colon |F_i| \xrightarrow{\ F_i\ } T|F_i| \xrightarrow{\ T\mu_i \circ F_i\ } TC,$$

for each $i \in \mathscr{D}$ so that there exists a unique function from $C$ to $TC$ satisfying

$$
\begin{array}{ccc}
|F_i| & \xrightarrow{\ \mu_i\ } & C \\
{\scriptstyle Fi}\big\downarrow & & \big\downarrow{\scriptstyle k} \\
T|F_i| & \xrightarrow[\ T\mu_i\ ]{} & TC,
\end{array}
\qquad (2.2)
$$

i.e. a $T$-coalgebra on $C$ such that the limiting cocone $\mu$ consists of coalgebra homomorphisms by (2.2).

It is not hard to see that $\langle C, k\rangle$ with $\mu$ is a limiting cocone: Let $v\colon F \dashrightarrow \langle X, \xi\rangle$ be a cocone from $F$ to some coalgebra $\langle X, \xi\rangle$. Consider the diagram

$$
\begin{array}{ccccc}
|F_i| & \xrightarrow{\ \ v_i\ \ } & & & X \\
{\scriptstyle F_i}\big\downarrow & \searrow^{\mu_i} & \overset{\rho}{\nearrow} & & \big\downarrow{\scriptstyle \xi} \\
T|F_i| & \xrightarrow[\ Tv_i\ ]{} & C & \longrightarrow & TX \\
& \searrow_{T\mu_i} & \big\downarrow{\scriptstyle k} & \nearrow_{T\rho} & \\
& & TC & &
\end{array}
$$

where $\rho$ is the mediating function from $C$ to $X$ of the cocone $v$. It remains to show that $T\rho \circ k = \xi \circ \rho$, but it follows from the fact that $\xi \circ \rho$ is the unique function of the cocone $\xi \circ v$ and for each $i$, we have $T\rho \circ k \circ \mu_i = \xi \circ v_i$ by chasing the above diagram. $\qquad\square$

For example, the *coproduct* of two deterministic automata $\langle X_1, \xi_1 \rangle$ and $\langle X_2, \xi_2 \rangle$ is simply an automaton with the disjoint union $X_1 + X_2 := \{(i, x) \mid x \in X_i\}$ as the state space and a transition function $\xi$ defined by

$$\xi(i, x) = \xi_i(x).$$

In the following discussion, we will see many applications of colimit constructions in different occasions.

### 2.1.2 Behavioural Equivalence and Bisimilarity

**Definition 2.1.5.** Let $x$ and $y$ be elements in $T$-coalgebras $\langle X, \xi \rangle$ and $\langle Y, \gamma \rangle$ respectively. We say $x$ and $y$ are **behaviourally equivalent** if there is a pair of homomorphisms $\langle X, \xi \rangle \xrightarrow{f} \langle Z, \zeta \rangle \xleftarrow{g} \langle Y, \gamma \rangle$ such that $fx = gy$.

This is one coalgebraic generalisation of the classical notion of bisimilarity and was first coined by Kurz [73] in the form of an epi cospan in the category of coalgebras.

Coalgebra homomorphisms, by definition, preserve the coalgebraic operations of every coalgebra, similar to the situation in universal algebras:

**Proposition 2.1.6.** *For every coalgebra homomorphism $f : \langle X, \xi \rangle \to \langle Y, \gamma \rangle$ and every $x \in X$, $x$ and $fx$ are behaviourally equivalent.*

By Proposition 2.1.4, two different coalgebras may be joined, so we may consider behavioural equivalence on a single coalgebra. Then, we can show that behavioural equivalence is an equivalence relation on the carrier set:

**Proposition 2.1.7.** *Let $x$ and $y$ be elements of a coalgebra $\langle X, \xi \rangle$. Then $x$ and $y$ are behaviourally equivalent if and only if there is a coalgebra homomorphism $f$ from $\langle X, \xi \rangle$ such that its kernel $B_f := \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ contains $(x, y)$.*

*Proof.* By definition, $x$ and $y$ are behaviourally equivalent if $x \, B_f \, y$.

Conversely, suppose that $x$ and $y$ are behaviourally equivalent, witnessed by $f, g : \langle X, \xi \rangle \rightrightarrows \langle Y, \gamma \rangle$ with $f(x_1) = g(x_2)$. Then, by Proposition 2.1.4, we have the coequaliser $h$ of $f$ and $g$, and $(x_1, x_2)$ are contained in the kernel of $hf = hg$. $\square$

*Remark* 2.1.8. For a direct formulation of $B_f$ in the previous proposition, we may use the following characterisation: An equivalence relation $B \subseteq X \times X$ is a kernel of some coalgebra homomorphism if and only if for any $x$ and $y$

$$x \, B \, y \implies T\pi \circ \xi(x) = T\pi \circ \xi(y) \tag{2.3}$$

where $\pi \colon X \to X / B$ is the projection mapping $x$ to its equivalence class.

Another formulation of bisimulation used widely in the earlier literature, e.g. [4, 83], is defined as follows. A **($T$-)bisimulation** (in a coalgebraic sense) is a relation $R \subseteq X \times Y$ such that there exists a $T$-coalgebra with carrier $R$ such that projections $\pi_X \colon R \to X$ and $\pi_Y \colon R \to Y$ are coalgebra homomorphisms:

$$\tag{2.4}$$

Two elements are said to be **bisimilar** if there exist a bisimulation relating them. Behavioural equivalence and bisimilarity in the coalgebraic sense are defined to unify existing notions of bisimilarity for transition systems and other dynamical systems over Set and their difference is quite minor:

**Proposition 2.1.9.** *The following statements hold:*

1. *Any two bisimilar elements are behaviourally equivalent.*

2. *Conversely, any two behaviourally equivalent elements are bisimilar, provided that the type functor preserves weak pullbacks.*

A direct computation of (2.4) gives the following formulation, which will allow us to link the coalgebraic concept to the traditional counterpart.

**Proposition 2.1.10.** *A relation $R \subseteq X \times Y$ is a bisimulation if and only if*

$$x \, R \, y \implies \exists z \in TR \, . \, \Big[ \, \xi(x) = T\pi_X(z) \quad and \quad \gamma(y) = T\pi_Y(z) \, \Big] \tag{2.5}$$

*where $\pi_X, \pi_Y$ are canonical projections from $X \times Y$ to $X$ and $Y$ respectively.*

**Example 2.1.11.** In the following examples, listed in the same order as Example 2.1.3, $\langle X, \xi \rangle$ and $\langle X, \gamma \rangle$ always denote coalgebras of the specific type and $R \subseteq X \times Y$ is a relation between the carrier sets.

1. For stream coalgebras, bisimulations between $\xi \colon X \to A \times X$ and $\gamma \colon Y \to A \times Y$ are relations $R$ such that for every $x$ and $y$, $x \, R \, y$ implies that $\xi(x) = \langle a, x' \rangle$ and $\gamma(y) = \langle a, y' \rangle$, with *the same label* $a \in A$, and $\langle x', y' \rangle \in R$. It follows that the elements $x$ and $y$ generate the same stream. The stream structure is defined on the bisimulation $R$ by setting $\langle x, y \rangle \mapsto \langle a, \langle x', y' \rangle \rangle$.

2. For tree coalgebras, a bisimulation is a relation satisfying the property that $t \, R \, u$ implies either

   (a) $\xi(t) \in 1$ and $\gamma(u) \in 1$; or

   (b) $\xi(t) = \langle a, t_1, t_2 \rangle$ and $\gamma(u) = \langle a, u_1, u_2 \rangle$ with $t_1 \, R \, u_1$ and $t_2 \, R \, u_2$.

   As in the previous example, it follows that related elements generate the same labelled tree.

3. For deterministic automata, a bisimulation is a relation $R$ such that $x \, R \, y$ implies

   (a) $x$ is an accepting state if and only if $y$ is, i.e. $\xi^{1\text{st}}(x) = \gamma^{1\text{st}}(y)$;

   (b) for every letter $a \in A$, the $a$-transition of $x$ is related to the $a$-transition of $y$, i.e.
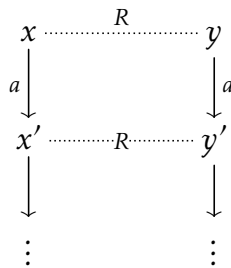
   

   It follows that bisimilar states accept the same language.

4. As for Kripke frames, the coalgebraic definition boils down to the usual bisimulation: A bisimulation is a relation $R$ such that $x \, R \, y$ implies

   (a) for every $x' \leftarrow x$, there is $y' \leftarrow y$ with $x' \, R \, y'$; and conversely

   (b) for every $y' \leftarrow y$, there is $x' \leftarrow x$ with $x' \, R \, y'$.

5. As for $A$-labelled transition systems, the situation is similar to the above: A bisimulation is a relation $R$ such that $x \, R \, y$ implies that for all $a \in A$

(a) for every $x' \overset{a}{\leftarrow} x$, there is $y' \overset{a}{\leftarrow} y$ with $x' R y'$;

(b) for every $y' \overset{a}{\leftarrow} y$, there is $x \overset{a}{\leftarrow} x'$ with $x' R y'$.

6. As for finitely-branching discrete-time Markov chains, a bisimulation is a relation $R$ such that $x R y$ implies there is a distribution $v \colon X \times Y \to [0,1]$ such that

$$\mu_x(x') = \sum_{y \in Y} \{ v(x',y) : x' R y \} \qquad \text{and} \qquad \mu_y(y') = \sum_{x \in X} \{ v(x,y') : x R y' \}$$

where $\mu_x = \xi(x)$ and $\mu_y = \gamma(y)$.

Staton [102] compares four formulations of coalgebraic bisimulation, including $T$-bisimulation (AM-bisimulation *op. cit.*) and behavioural equivalence (kernel bisimulation *op. cit.*), and the latter is the most liberal we have known so far.

## 2.1.3 Final Coalgebras

A **final coalgebra** $\langle Z, \zeta \rangle$ is a final object in $\mathsf{Set}_T$, i.e. for every $T$-coalgebra $\langle X, \xi \rangle$ there exists a unique coalgebra homomorphism $(-)^\dagger \colon \langle X, \xi \rangle \to \langle Z, \zeta \rangle$. It follows that every element in a $T$-coalgebra $\langle X, \xi \rangle$ is mapped to a unique element in $Z$:

**Proposition 2.1.12.** *Suppose that the final coalgebra exists. Let $x$ and $y$ be two elements of a coalgebra $\langle X, \xi \rangle$. The following are equivalent:*

*1. $x$ and $y$ are behaviourally equivalent.*

*2. $x^\dagger = y^\dagger$.*

The above proposition is the **coinduction principle**. For example, in order to show that states in an automaton accept the same language, it suffices to show that they are bisimilar [94]. In non-well-founded set theory, to show that two sets are equal, it suffices to show that the roots of corresponding tree representations are bisimilar [3, 21].

**Example 2.1.13.** The following examples of final coalgebras are listed in the same order as in Example 2.1.3:

1. The collection of countably infinite sequences over $A$, denoted $A^\omega$, can be given a stream structure $s$ by shifting: for any $x = (x_i)_{i \in \mathbb{N}}$

$$(sx)_i = x_{i+1}$$

   where we write $sx$ for $s(x)$ for simplicity. For every stream coalgebra $\langle X, \xi \rangle$, the $A$-sequence associated with each element in $X$ provides the unique coalgebra homomorphism from $\langle X, \xi \rangle$ to $A^\omega$. The situation for $(1 + A \times \mathcal{I})$-coalgebras is similar: First a final coalgebra is given by $A^\infty := A^* \cup A^\omega$ where $A^*$ is the collection of finite sequences. The coalgebraic structure maps non-empty lists and streams to their tails as before, and the empty list is mapped to the unique element in $1 = \{0\}$.

2. The collection of all ordered binary trees with labels from $A$,[1] denoted $OT_A$, also forms a $(1 + A \times \mathcal{I} \times \mathcal{I})$-coalgebra in a natural way. Define a structure map $\zeta$ as follows:

$$\zeta(t) = \begin{cases} 0 & \text{if } t \text{ is a leaf,} \\ \langle a, t_1, t_2 \rangle & \text{if } t \text{ has label } a \text{ with } t_1 \text{ on the left and } t_2 \text{ on the right.} \end{cases}$$

   Clearly, each $(1 + A \times \mathcal{I} \times \mathcal{I})$-coalgebra $\langle X, \xi \rangle$, has a homomorphism to $\langle OT_A, \zeta \rangle$ as we know that each element in $X$ represents a tree. From the definition of homomorphism, any homomorphism $f$ from $\langle X, \xi \rangle$ to $\langle OT_A, \zeta \rangle$ must be the same.

3. Let $\mathcal{L}$ denote the set of all formal languages over $A$, i.e. collections of finite sequences over $A$. The derivative operation, i.e. $L_a = \{ w : aw \in L \}$ for $L \in \mathcal{L}$ and $a \in A$, defines a deterministic automaton $\zeta : \mathcal{L} \to 2 \times \mathcal{L}^A$ by

$$\zeta^{\text{1st}}(L) = \begin{cases} 1 & \epsilon \in L, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \zeta^{\text{2nd}}(L)(a) = L_a.$$

   Given a deterministic automaton $\langle X, \xi \rangle$ over $A$, each state $x$ is in association with a language $L \in \mathcal{L}$ accepted by $x$, and the association is, in fact, the unique homomorphism from $\langle X, \xi \rangle$ to $\langle \mathcal{L}, \zeta \rangle$.

4. For Kripke frames and labelled transition systems, final coalgebras *cannot* exist. See below.

---

[1] We assume that there is a canonical presentation for ordered binary trees. It can be obtained by a final sequence introduced in Chapter 4.

5. For discrete time Markov chains, see [84, 101].

We notice that types of streams, ordered trees, and automata are built upon on constants (e.g. $A$), identities $\mathcal{I}$, products $\times$, coproducts $+$, and exponents $(-)^A$ with a fixed set $A$. A functor defined in this way is called **polynomial** and in fact this class of functors has a nice property, which will be a corollary of Proposition 2.1.33:

**Proposition 2.1.14.** *Every polynomial functor has a final coalgebra.*

However, not every category of coalgebras has a final coalgebra. First we note that every final coalgebra $\zeta\colon Z \to TZ$ is a bijection, an isomorphism in Set:

**Lemma 2.1.15** (Lambek's Lemma [78]). *For any endofunctor $T\colon \mathscr{C} \to \mathscr{C}$, every final coalgebra $\zeta\colon Z \to TZ$ (if any) is an isomorphism in $\mathscr{C}$.*

*Proof.* Consider the following diagram



showing that $\zeta$ is an isomorphism by uniqueness and the right commutative square.

□

Then, by Cantor's diagonal argument, we know that $\mathcal{P}X \ncong X$ and it follows that:

**Corollary 2.1.16.** *There is no final coalgebra for the covariant powerset functor.*

One way to remedy this situation is to restrict the powerset functor to its *finitary* part: Every Set functor $T$ has a finitary coreflection, i.e. there exists an endofunctor $T_\omega$ determined by finite sets only and a natural transformation $T_\omega \overset{\cdot}{\longrightarrow} T$ (satisfying the universal property of coreflection).[2] A particular definition for Set functor will be given in the following and a general definition will be given in Chapter 3.

---

[2]Indeed, every functor on locally finitely presentable categories has a finitary version. We will discuss this in Chapter 3.

We say that a functor $T$ is **finitary** if for every set $X$, the set $TX$ is equal to the following *directed* union:

$$TX = \bigcup \{ T\iota[TS] \subseteq TX \mid \iota \colon S \subseteq_\omega X \}. \tag{2.6}$$

**Example 2.1.17.** Consider the covariant finitary powerset functor $\mathcal{P}_\omega$, i.e. $\mathcal{P}_\omega X$ consists of the finite subsets of $X$ and for every function $f \colon X \to Y$, $\mathcal{P}_\omega f$ is equal to $\mathcal{P}f$. Since a finite subset is obviously a subset, we have an inclusion $\mathcal{P}_\omega X \hookrightarrow \mathcal{P}X$ natural in $X$.

Here is the classic result in the coalgebra theory:

**Theorem 2.1.18** (see [19, 109]). *A $T$-final coalgebra of $T$ exists, if $T$ is finitary.*

*Remark* 2.1.19. Another way to have a final coalgebra for the covariant powerset functor is to consider the category SET of *classes* instead: Aczel and Mendler [4] show that every SET endofunctor determined on *sets* (called **set-based** *op. cit.*) has a final coalgebra.

These two approaches (set-based *vs.* finitary) are essentially the same [17]. In ZF set theory, a class is an informal notion and we may interpret it in a Grothendieck universe, i.e. a set closed under set-theoretical operations, whose existence is equal to the existence of a (strongly) inaccessible cardinal.[3] Finitary Set endofunctors are interpretations of set-based SET endofunctors in the first infinite inaccessible cardinal $\aleph_0$ (or $\omega$). We will discuss this foundation issue in Section 3.5.

## 2.1.4 Homomorphism Factorisation

Every function (in Set) is a composite of an injection and a surjection. Similarly, the factorisation also holds for coalgebras. In [95], it is shown under the assumption that the type functor preserves weak pullbacks, but in fact it holds for every type:

**Proposition 2.1.20.** *The following statements are true for any* Set *functor $T$:*

   *1. $T$ preserves surjections;*

   *2. $T$ preserves injections with a non-empty domain.*

Every (Surjections,Injections)-factorisation of functions can be lifted to the category of $T$-coalgebras: Recall that every function $f \colon X \to Y$ can be written as a composite

---

[3] See Definition 3.5.3.

of a surjection $e\colon X \twoheadrightarrow fX$ and an inclusion $m\colon fX \hookrightarrow Y$, where we write $fX$ for the image of $X$ under $f$. Further, the factorisation is *orthogonal*:

**Proposition 2.1.21.** *Let $m\colon X \to Y$ be an injection and $e\colon A \to B$ a surjection. For every function $u, v$ with $ve = mu$, there exists a unique function $w$ satisfying the following diagram:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ e\ \ } & B \\
\Big\downarrow{\scriptstyle u} & \nearrow{\scriptstyle w} & \Big\downarrow{\scriptstyle v} \\
X & \xrightarrow[\ \ m\ \ ]{} & Y
\end{array}
\tag{2.7}
$$

*Proof.* By commutativity and the surjection $e$, $v[B] = ve[A] = mu[A]$. It follows that $v(b) \in m[X]$, so define a function $w$ from $B$ to $X$ by composing $v$ with the left inverse $m^{-1}\colon m[X] \to X$ of $m$. It is easy to see the uniqueness. $\qquad\square$

The factorisation of Set can be lifted to $\mathsf{Set}_T$:

**Corollary 2.1.22.** *For any coalgebra homomorphism $f\colon \langle X, \xi \rangle \to \langle Y, \gamma \rangle$, the following diagram*

$$
\begin{array}{ccccc}
 & & f & & \\
X & \xrightarrow{\ e\ } & fX & \xrightarrow{\ m\ } & Y \\
\Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \xi'} & & \Big\downarrow{\scriptstyle \gamma} \\
TX & \xrightarrow{\ Te\ } & T(fX) & \xrightarrow{\ Tm\ } & TY \\
 & & Tf & & 
\end{array}
\tag{2.8}
$$

*commutes, i.e. $m$ and $e$ are coalgebra homomorphisms. Also, the factorisation given by classes of injective homomorphisms and surjective homomorphisms has the diagonal fill-in property as in Proposition 2.1.21.*

*Proof.* In the case of coalgebra homomorphisms with a non-empty domain, the injection $m$ always has a non-empty domain, so it follows by Proposition 2.1.20 and Proposition 2.1.21. As for the homomorphism with an empty domain, the factorisation must be trivial, so $\xi'$ is equal to $\xi$. $\qquad\square$

By applying the above factorisation to the unique homomorphism from a given coalgebra $\langle X, \xi \rangle$ to the final coalgebra (if any) we will obtain a coalgebra in which any two distinct elements are not behaviourally equivalent.

## 2.1.5 Preservation of Injections

Although Set endofunctors do not preserve injections in general, the restriction to the class of injection-preserving functors does not affect the coalgebra theory at all due to the following facts:

**Proposition 2.1.23** (see [13, p. 134]). *For every functor $T \colon \mathsf{Set} \to \mathsf{Set}$, there exists an injection-preserving functor $T' \colon \mathsf{Set} \to \mathsf{Set}$ and they are equal on the subcategory $\mathsf{Set}_{\neq \emptyset}$ consisting of non-empty sets.*

**Corollary 2.1.24.** *For every functor $T \colon \mathsf{Set} \to \mathsf{Set}$, there exists an injective-preserving functor $T'$ such that the category of $T$-coalgebras is isomorphic to the category of $T'$-coalgebras.*

*Proof.* There is only one $T$-coalgebra over the empty set, i.e. the empty function $\emptyset \to T\emptyset$. Then, using Proposition 2.1.23, it follows easily. $\qquad \square$

Moreover, every injection-preserving functor $T$ is naturally isomorphic to an *inclusion-preserving* functor $T'$:

**Theorem 2.1.25** ([13, Theorem III.4]). *Every injection-preserving $\mathsf{Set}$ functor is naturally isomorphic to an inclusion-preserving functor.*

The *preservation of inclusions* is sometimes called *standard*, but it is actually weaker then the original definition of standard functor in [13, Definition III.4].

## 2.1.6 Minimality

A coalgebra $\langle X, \xi \rangle$ is called **minimal** [46] if every kernel of $\langle X, \xi \rangle$ is the diagonal $\Delta_X$; or in the contrapositive form, any two distinct elements are not behaviourally equivalent.

**Example 2.1.26.** Let $\langle Z, \zeta \rangle$ be a final coalgebra.

1. $\langle Z, \zeta \rangle$ is a minimal coalgebra.

2. For any coalgebra $\langle X, \xi \rangle$, the image of the unique homomorphism from $\langle X, \zeta \rangle$ to $\langle Z, \zeta \rangle$ is a minimal coalgebra.

Looking at Proposition 2.1.7, we immediately have the following proposition:

**Proposition 2.1.27** (*cf.* [46, 95]). *Let $\langle X, \xi \rangle$ be a coalgebra. $\langle X, \xi \rangle$ is minimal if and only if $\langle X, \xi \rangle$ has no proper quotient, i.e. every surjective homomorphism from $\langle X, \xi \rangle$ is bijective.*

We shall consider the collection of minimal coalgebras as a full subcategory of $\mathsf{Set}_T$, denoted $M(\mathsf{Set}_T)$, and show that every coalgebra can be minimised functorially. The collection of quotients of a coalgebra $\langle X, \xi \rangle$ (i.e. isomorphism classes of surjective homomorphisms from $\langle X, \xi \rangle$) forms a *complete lattice* where pushouts are the lattice sups:

- First, define an order $f \leq g$ on surjective homomorphisms if there exists a homomorphism $h$ such that the diagram

$$
\langle X, \xi \rangle \overset{f}{\twoheadrightarrow} \langle Y, \gamma \rangle
$$
$$
\underset{g}{\searrow} \quad \downarrow h
$$
$$
\langle Z, \zeta \rangle
$$

  commutes, where $h$ must also be surjective and unique. It follows that if $f \leq g$ and $g \leq f$ then the codomains of $f$ and $g$ are isomorphic. Therefore, $\leq$ gives a partial order on quotients of $\langle X, \xi \rangle$.

- Second, since the forgetful functor $\mathsf{Set}_T \to \mathsf{Set}$ creates colimits and $\mathsf{Set}$ is cocomplete, the pushout of a set $S$ of surjective homomorphisms from $\langle X, \xi \rangle$ exists and it is the sup of $S$. There is always a top element, i.e. the pushout of all of them, denoted $(-)^{\ddagger} : \langle X, \xi \rangle \twoheadrightarrow \nabla \langle X, \xi \rangle$: for any surjective homomorphism $\langle X, \xi \rangle \twoheadrightarrow \langle Y, \gamma \rangle$ there is a homomorphism $f$ such that the diagram

$$
\langle X, \xi \rangle \twoheadrightarrow \langle Y, \gamma \rangle
$$
$$
\underset{(-)^{\ddagger}}{\searrow} \quad \downarrow f \tag{2.9}
$$
$$
\nabla \langle X, \xi \rangle
$$

  commutes.

From the complete lattice structure, we obtain a minimisation functor:

**Theorem 2.1.28** (see [46, Theorem 2.3]). *The inclusion from the full subcategory of minimal coalgebras to the category of coalgebras has a left adjoint, i.e. $M(\mathsf{Set}_T)$ is a reflective subcategory of $\mathsf{Set}_T$.*

The left adjoint functor is called *minimisation*, denoted $\nabla$, and the unit of the reflection is the greatest quotient, denoted $(-)^{\ddagger}$. By reflection, we also have the **coinduction principle** :

**Corollary 2.1.29.** *Let $x, y$ be elements in a coalgebra $\langle X, \xi \rangle$. The following are equivalent:*

1. *$x$ and $y$ are behaviourally equivalent.*

2. *$x^{\ddagger} = y^{\ddagger}$.*

As Gumm showed [46] that finite products of minimal coalgebras are *intersections* of coalgebras, we obtain a concrete construction of products of minimal coalgebras in the category $\mathsf{Set}_T$ of coalgebras since the inclusion is a right adjoint.

The class of minimal coalgebras plays an essential role in the construction of a final coalgebra: Every homomorphism to a minimal coalgebra is unique, as otherwise there would be distinct but behaviourally equivalent elements. The existence of a final coalgebra boils down to the existence of a weakly final coalgebra, i.e. a coalgebra to which every coalgebra has at least one homomorphism.

**Corollary 2.1.30.** *Let $T$ be a* $\mathsf{Set}$ *endofunctor. A final coalgebra exists if and only if a weakly final coalgebra exists.*

By the same reasoning, minimisation is stable under repetitions, i.e. for a non-empty set $I$

$$\nabla \coprod_{I} \langle X, \xi \rangle \qquad \text{is isomorphic to} \qquad \nabla \langle X, \xi \rangle. \tag{2.10}$$

The last two facts are the fundamental technique used in the following discussion.

## 2.1.7   Construction of Final Coalgebras

We finish our tour around coalgebras by a final coalgebra construction using the minimisation functor $\nabla$ and a *generating set* of coalgebras:

**Definition 2.1.31.** A **generating set** of coalgebras is a set of coalgebras $\{\langle G_i, \gamma_i \rangle\}_{i \in I}$ such that for each coalgebra $\langle X, \xi \rangle$ the canonical morphism

$$\coprod_{f : \langle G_i, \gamma_i \rangle \to \langle X, \xi \rangle} \langle G_i, \gamma_i \rangle \twoheadrightarrow \langle X, \xi \rangle$$

is surjective.

By definition, for each coalgebra $\langle X, \xi \rangle$ and $x \in X$ we can always find a coalgebra $\langle G_i, \gamma_i \rangle$ in the generating set and an element $g \in G_i$ such that $x$ is behaviourally equivalent to $g$. Hence, since the category $\mathsf{Set}_T$ of coalgebras is cocomplete, we may simply join all of them to derive a final coalgebra as a quotient of $\coprod_{i \in I} \langle G_i, \gamma_i \rangle$ using (2.9) and Corollary 2.1.30 as follows.[4]

**Theorem 2.1.32.** *If there is a generating set $\mathcal{G} = \{ \gamma_i \colon G_i \to TG_i \mid i \in I \}$, then the minimised coalgebra of the coproduct*

$$\langle G, \gamma \rangle \coloneqq \coprod_{i \in I} \langle G_i, \gamma_i \rangle$$

*is a final coalgebra.*

*Proof.* Consider a $T$-coalgebra $\langle X, \xi \rangle$. Let $\mathcal{G}_\xi$ be the coproduct of $\langle G_i, \gamma_i \rangle$ *indexed by* $\langle G_i, \gamma_i \rangle \to \langle X, \xi \rangle$. By definition, the canonical homomorphism from $\mathcal{G}_\xi$ to $\langle X, \xi \rangle$ is surjective, so there is a unique morphism to the minimised coalgebra of $\mathcal{G}_\xi$:

$$
\begin{array}{ccc}
\mathcal{G}_\xi & \twoheadrightarrow & \langle X, \xi \rangle \\
 & \searrow & \downarrow \\
 & & \nabla \mathcal{G}_\xi
\end{array}
$$

by (2.9).

Let $J$ be the subset of $I$ such that for some $i$ the coalgebra $\langle G_i, \gamma_i \rangle$ actually appears in a homomorphism to $\xi$, i.e. $J = \{ j \in I : \exists(\gamma_j \to \xi) \}$. Then, each $\nabla \mathcal{G}_\xi$ always has a unique homomorphism to the minimised coalgebra of the coproduct of $\mathcal{G}$:

$$\nabla \mathcal{G}_\xi \xrightarrow{\ \cong\ } \nabla \coprod_{j \in J} \langle G_i, \gamma_i \rangle \xrightarrow{\ \iota\ } \nabla \coprod_{i \in I} \langle G_i, \gamma_i \rangle$$

where the first isomorphism comes from (2.10), the second homomorphism comes from injections

$$\iota_i \colon \langle G_j, \gamma_j \rangle \to \coprod_{i \in I} \langle G_i, \gamma_i \rangle$$

for $j \in J$, and the uniqueness follows from the minimality. $\qquad\qquad \square$

In particular, we may find a generating set by considering *subcoalgebras generated* by some state $x$, used in the following proposition:

---

[4]The following result is proved without assuming that the type functor preserves weak pullbacks, in contrast to [95].

**Proposition 2.1.33** (See [19, 95]). *Each of the following conditions implies the existence of a generating set of $T$-coalgebras:*

1. *$T$ is a polynomial functor.*

2. *$T$ is accessible, i.e. there exists a regular infinite cardinal[5] $\kappa$ such that*

$$TX = \bigcup \{ Ti[TS] \subseteq TX \mid i \colon S \subseteq_\kappa X \}$$

   *where $S \subseteq_\kappa X$ indicates $S \subseteq X$ with $|S| < \kappa$. In particular, $T$ is finitary if $T$ is $\aleph_0$-accessible.*

The above construction is not algorithmic but rather descriptive. In Chapter 3 we will provide another construction via the so-called *final sequence* which is more instructive and the latter technique has been used to find bisimilarity and minimisation [15].

*Remark* 2.1.34. A final coalgebra can also be constructed by logical methods, see [45] for Set coalgebras and [84] for coalgebras over measurable spaces.

## 2.2 Hennessy-Milner Logics

In the previous section, we described coinduction principles. For stream, ordered trees, and deterministic automata, the coinduction principles are rather obvious: two states in a stream (resp. ordered tree and automaton) are behaviourally equivalent if and only if they produce the same sequence (resp. correspond to the same labelled tree and accept the same language). However, the characterisation in general is not that simple. We provide another characterisation of the coinduction principle for labelled transition systems as an example to be generalised.

In order to show that elements are *not* behaviourally equivalent, by definition, we have to show that there is no such a bisimulation relating the targeted elements, provided that the type functor preserves weak pullbacks. Consider the labelled transition systems in Figure 2.1.

*Proof.* Every $a$-transition (in fact, the only one) of $x$ moves to a state with two possible transitions via labels $b$ and $c$; while every $a$-transition of $y$ moves to a state with only one possible transition via either $b$ or $c$. Therefore, there does not exist any

---

[5] An infinite cardinal $\lambda$ is **regular** if and only if $\sum_{i \in \alpha} \lambda_i < \lambda$ for any $\alpha < \lambda$ and $\lambda_i < \lambda$ for $i \in \alpha$. In the following content, a regular cardinal always means regular infinite.

Figure 2.1: An example of non-bisimilarity

bisimulation relating $x$ and $y$, because every pair of $x$'s and $y$'s successors are not bisimilar. ☐

Like any informal and *ad hoc* mathematical proof, the above argument is error-prone and it is hard to spot a mistake. To avoid any ambiguity, we formalise the informal description using *Hennessy-Milner logic*, a multi-modal logic introduced by Hennessy and Milner [50]:

**Definition 2.2.1.** The language of **Hennessy-Milner logic** over a set $A$ of labels is generated by the following syntax:

$$\varphi := \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi$$

where $a \in A$. We denote the language by $\omega\mathcal{HM}_A$.

The first three $\top$, $\neg\varphi$, and $\varphi \wedge \varphi$, are logical constants and connectives. The last one $\langle a \rangle \varphi$ is a *modal operator*, meaning that the next state after performing some $a$-transition has the property $\varphi$. Write $x \Vdash \varphi$ if the property $\varphi$ holds at state $x$, and $[a]\varphi$ for $\neg\langle a \rangle \neg\varphi$. Modal operators are now interpreted as follows

$$x \Vdash \langle a \rangle \varphi \qquad \text{if and only if} \qquad \exists y \xleftarrow{a} x . y \Vdash \varphi \tag{2.11}$$

$$x \Vdash [a]\varphi \qquad \text{if and only if} \qquad \forall y \xleftarrow{a} x . y \Vdash \varphi \tag{2.12}$$

where $\varphi$ is any Hennessy-Milner formula.

Hennessy-Milner logic is invariant under bisimilarity which is a pleasing property, called **adequacy**. This property can be used in order to show non-bisimilarity:

**Proposition 2.2.2.** *Let $\langle X, A, \rightarrow \rangle$ be a labelled transition system. Suppose that elements $x$ and $y$ in $X$ are bisimilar. Then, for every Hennessy-Milner formula $\varphi$,*

$$x \Vdash \varphi \iff y \Vdash \varphi. \tag{2.13}$$

Figure 2.2: Bisimilarity is contained in logical equivalence

Following this proposition, we say that $x$ and $y$ are **logically equivalent** with respect to Hennessy-Milner logic if (2.13) holds for all $\varphi$.

Given this property, in the contrapositive form, to show that elements are not bisimilar it suffices to find a witness formula distinguishing them. For example, in Figure 2.1 it is easy to verify that the formula

$$\varphi := [a](\langle b \rangle \top \wedge \langle c \rangle \top)$$

distinguishes $x$ and $y$ in Figure 2.1, i.e. $x \Vdash \varphi$ but $y \nVdash \varphi$, so they are not bisimilar by Proposition 2.2.2.

## 2.2.1  Hennessy-Milner Property

On the other hand, we have no guarantee on non-bisimilar states can be distinguished by some Hennessy-Milner formula. Indeed, it requires the converse of Proposition 2.2.2, called **expressiveness** (also known as **expressivity** and **Hennessy-Milner property**) : any two non-bisimilar elements have at least one formula distinguishing them. The class of labelled transition systems does not have expressiveness in general, see Figure 2.2.

However, expressiveness holds for a smaller class of labelled transition systems [49]: We say that a labelled transition system is **image-finite** if for each element $x \in X$ and $a \in A$

the set of successors

$$(x \xrightarrow{a} -) = \{ x' \in X \mid x \xrightarrow{a} x' \}$$

is finite. Then, Hennessy-Milner logic is expressive over the class of image-finite labelled transition systems:

**Proposition 2.2.3.** *For every two elements $x$ and $y$ in an image-finite $A$-labelled transition system, $x$ and $y$ are bisimilar if $x$ and $y$ are logically equivalent.*

To find out where the *image-finiteness* is actually used, we sketch the proof here.

*Proof.* Suppose that $x$ and $y$ are logically equivalent. We would like to show by contradiction that the relation

$$E := \{\langle x, y \rangle \in X \times X \mid \forall \varphi . (x \Vdash \varphi \iff y \Vdash \varphi)\}$$

is a bisimulation, i.e. the following assumption leads to a contradiction: there exists $a \in A$ and $x' \overset{a}{\leftarrow} x$ such that every $y' \overset{a}{\leftarrow} y$ has some $\varphi_{y'}$ with $x' \Vdash \varphi_{y'}$ but $y' \nVdash \varphi_{y'}$.

Note that the set

$$(y \overset{a}{\to} -) = \{y' \in X \mid y \overset{a}{\to} y'\}$$

is *finite* by *image-finiteness*, so we can pick finitely many formulae $\varphi_{y'}$ distinguishing $x'$ and $y'$ for each $y' \overset{a}{\leftarrow} y$. Now define

$$\varphi := \langle a \rangle \bigwedge_{y \overset{a}{\to} y'} \varphi_{y'}.$$

By construction $x \Vdash \varphi$ but $y \nVdash \varphi$, contradicting the assumption that $x$ and $y$ are logically equivalent. It follows that $E$ must be a bisimulation, so logically equivalent elements are bisimilar. $\square$

From the above proof, we have an immediate generalisation by replacing Hennessy-Milner logic with finitary conjunctions (denoted $\omega \mathcal{HM}_A$) by Hennessy-Milner logic with conjunctions of up to $\kappa$-many formulae (denoted $\kappa \mathcal{HM}_A$), or arbitrary conjunctions (denoted $\mathcal{HM}_A$):

**Proposition 2.2.4.** *Let $x$ and $y$ be elements of a labelled transition system $\langle X, A, \to \rangle$. Then,*

1. *$x$ and $y$ are bisimilar if and only if $x$ and $y$ are logically equivalent with respect to $\mathcal{HM}_A$, i.e. Hennessy-Milner logic with arbitrary conjunctions.*

2. *Assume that there exists a cardinality $\kappa$ such that, for all $a \in A$, the set of $a$-successors of every element is bounded by $\kappa$. $x$ and $y$ are bisimilar if and only if $x$ and $y$ are logically equivalent with respect to $\kappa \mathcal{HM}_A$, i.e. Hennessy-Milner logic with conjunctions up to $\kappa$-many formulae.*

**Corollary 2.2.5.** *For each labelled transition system $\langle X, A, \to \rangle$, there exists some cardinality $\kappa$ such that $\kappa \mathcal{HM}_A$ characterises bisimilarity of $\langle X, A, \to \rangle$.*

*Proof.* Let $\kappa$ be the cardinality of $X$. □

## 2.2.2 Algebraic Logics

Boolean algebras provide algebraic semantics for propositional calculus; the same applies to Boolean algebras with operators for Hennessy-Milner logic (normal multimodal logic **K**, see [28]).

**Definition 2.2.6.** Let $I$ be a set of labels. A **Boolean algebra with operators indexed by** $I$ (or, $I$-**BAO** for short) is a Boolean algebra $\mathfrak{A} = \langle A, \bot, \top, \neg, \vee, \wedge \rangle$ with an $I$-indexed family of unary operation $\Diamond_i$ satisfying

**Normality** $\Diamond_i \bot = \bot$ and $\Diamond_i(\varphi \vee \psi) = \Diamond_i(\varphi) \vee \Diamond_i(\varphi)$ for any $\varphi, \psi \in A$.

A homomorphism $f$ between BAOs is a Boolean algebra homomorphism such that $f$ also preserves the additional modal operations, i.e. $f(\Diamond_i \varphi) = \Diamond_i(f \varphi)$ for each $a$.

**Example 2.2.7** (Complex algebra). Given a labelled transition system $\langle X, I, \rightarrow \rangle$, define a Boolean algebra with operators indexed by $I$, called the **complex algebra** of $\langle X, I, \rightarrow \rangle$, as follows. The powerset algebra $2^X$ of $X$ forms a Boolean algebra, and for each $i \in I$, define a modal operation $\Diamond_i^{\rightarrow} \colon 2^X \rightarrow 2^X$ by

$$S \xmapsto{\Diamond_i^{\rightarrow}} \{ x \in X \mid \exists x' \xleftarrow{i} x . x' \in S \} \tag{2.14}$$

for every subset $S \subseteq X$. Clearly, the empty set $\emptyset$ models false $\bot$, the universe $X$ truth $\top$, the intersection conjunction, and so on. Let $[\![ - ]\!] \colon \omega \mathcal{HM} \rightarrow 2^X$ denote the interpretation defined by

$$x \in [\![ \varphi ]\!] \iff x \Vdash \varphi,$$

which can easily be shown to satisfy *i*) $x \in [\![ \top ]\!]$ for all $x \in X$; *ii*) $x \in [\![ \neg \varphi ]\!]$ if and only if $x \notin [\![ \varphi ]\!]$; *iii*) $x \in [\![ \varphi \wedge \psi ]\!]$ if and only if $x \in [\![ \varphi ]\!] \cap [\![ \psi ]\!]$. The inverse image function $f^{-1}[-] \colon 2^Y \rightarrow 2^X$ of any homomorphism $f \colon \langle X, I, \rightarrow \rangle \rightarrow \langle Y, I, \rightarrow \rangle$ between $I$-labelled transition systems is a homomorphism between BAOs.

## 2.2.3 Modal Algebras as Algebras of Endofunctor

For every endofunctor $L \colon \mathscr{A} \rightarrow \mathscr{A}$, the dual notion of coalgebra is also very helpful: an $L$-**algebra** with a carrier $A \in \mathscr{A}$ is a $\mathscr{A}$-morphism $\alpha$ from $LA$ to $A$, denoted

by $\langle A, \alpha \rangle$; an $L$-algebra homomorphism $f \colon \langle A, \alpha \rangle \to \langle B, \beta \rangle$ is a morphism $f$ in $\mathscr{A}$ satisfying $f \circ \alpha = \beta \circ Lf$. The category of $L$-algebras is precisely the opposite category of $L^{\mathrm{op}}$-coalgebras.

In this subsection, we will represent Boolean algebra with operators in this way, not in the category Set, but in the category of Boolean algebras. We characterise the functor part of algebra as follows.

**Definition 2.2.8.** Given a Boolean algebra $A$, define $\mathbb{M}^I A$ by the following presentation[6]

$$\mathsf{BA}\langle \{\blacklozenge_i a\}_{i \in I, a \in A} \mid \blacklozenge_i \bot = \bot; \blacklozenge_i (a \vee b) = \blacklozenge_i a \vee \blacklozenge_i b \rangle$$

and given a Boolean algebra homomorphism $f \colon A \to B$ define a Boolean algebra homomorphism $\mathbb{M}^I f \colon \mathbb{M}^I A \to \mathbb{M}^I B$ on the set of generators

$$\blacklozenge_i a \mapsto \blacklozenge_i f(a).$$

Whenever confusion is unlikely, we suppress the subscript $I$.

**Proposition 2.2.9.** *The mapping $\mathbb{M}$ in Definition 2.2.8 is an endofunctor of* BA.

The modalities of a Boolean algebra with operators $\mathfrak{A} = \langle A, \bot, \top, \neg, \vee, \wedge, (\Diamond_i)_{i \in I} \rangle$ may be viewed as an $\alpha \colon \mathbb{M}^I$-algebra $\mathbb{M}^I A \to A$ on the Boolean algebra reduct $A$ by evaluation:

$$\alpha \colon \blacklozenge_i a \mapsto \Diamond_i(a) \tag{2.15}$$

where $\blacklozenge_i a$ is a generator of $\mathbb{M}_I A$ and $\Diamond_i(a)$ is an actual element in $A$. This is well-defined because the relations that define $\mathbb{M}_I A$ translate to actual equalities in $A$, so the assignment lifts to a Boolean algebra homomorphism from $\mathbb{M}_I A$ to $A$.

**Theorem 2.2.10** (see [1, 68]). *The category of Boolean algebras with operations indexed by $I$ is isomorphic to the category of algebras for $\mathbb{M}_I$, denoted by* $\mathsf{BA}^{\mathbb{M}_I}$.

*Proof.* For brevity assume that the index is singleton. For every Boolean algebra with operator $\mathfrak{A}$, define a Boolean homomorphism $\alpha$ from $\mathbb{M}A$ to $A$ by (2.15); for every homomorphism $f \colon \mathfrak{A} \to \mathfrak{B}$ between Boolean algebras with operator we verify the following diagram

$$\begin{array}{ccc} \mathbb{M}A & \xrightarrow{\;\alpha\;} & A \\ {\scriptstyle \mathbb{M}f} \downarrow & & \downarrow {\scriptstyle f} \\ \mathbb{M}B & \xrightarrow[\;\beta\;]{} & B \end{array}$$

---

[6] A Boolean algebra presentation $\mathsf{BA}\langle \mathcal{G} \mid \mathcal{R} \rangle$, consisting of a set $\mathcal{G}$ and a set of relations $\mathcal{R}$ on terms generated by $\mathcal{G}$, defines the *freest* Boolean algebra generated by $\mathcal{G}$ subject to relations in $\mathcal{R}$

commutes. It suffices to check generators, i.e.

$$(f \circ \alpha)(\blacklozenge a) = f(\lozenge^A a) = \lozenge^B (f a) = \beta(\blacklozenge f a) = (\beta \circ \mathbb{M} f)(\blacklozenge a),$$

so we have defined a functor from the category of Boolean algebras with operator to the category of $\mathbb{M}$-algebras.

Conversely, for every Boolean algebra homomorphism $\alpha \colon \mathbb{M}A \to A$ define a modal operation $\lozenge$ by

$$\lozenge \colon a \mapsto \alpha(\blacklozenge a)$$

and it satisfies normality by the construction of $\mathbb{M}A$. E.g. $\lozenge(a \vee b) = \alpha(\blacklozenge(a \vee b)) = \alpha(\blacklozenge a \vee \blacklozenge b) = \alpha(\blacklozenge a) \vee \alpha(\blacklozenge b) = \lozenge a \vee \lozenge b$ where the second equality holds by the construction of $\mathbb{M}A$ and the last equation holds by the preservation of operations of the Boolean algebra homomorphism $\alpha$. Given an $\mathbb{M}$-algebra homomorphism $f \colon \langle A, \alpha \colon \mathbb{M}A \to A \rangle \to \langle B, \beta \colon \mathbb{M}B \to B \rangle$, it is also straightforward to check the preservation of operators:

$$(f \circ \lozenge)(a) = (f \circ \alpha)(\blacklozenge a) = (\beta \circ \mathbb{M} f)(\blacklozenge a) = \beta(\blacklozenge f a) = \lozenge(f a)$$

for every $a \in A$.

Now it is evident the correspondence is bijective. $\qquad \square$

### 2.2.4  Functorial Construction of Complex Algebras

The naive construction of complex algebras has a deeper connection with the endofunctor $\mathbb{M}$ defined previously. We notice the following fact:

**Proposition 2.2.11.** *For every set $X$ and a Kripke frame, represented as a $\mathcal{P}$-coalgebra $\xi \colon X \to \mathcal{P}X$, we have $\lozenge^{\rightarrow} = \xi^{-1} \circ \lozenge_X$ where $\lozenge_X \colon 2^X \to 2^{\mathcal{P}X}$ is defined by*

$$(U \subseteq X) \mapsto \{ S \subseteq X \mid U \cap S \neq \emptyset \}. \tag{2.16}$$

Equation (2.16) defines a natural transformation not just mere functions:

**Proposition 2.2.12.** *The mapping $\lozenge$ is a natural transformation from the contravariant powerset functor $2^-$ to the composite $2^{\mathcal{P}-}$.*

*Proof.* For every function $f : X \to Y$, the following diagram

$$
\begin{array}{ccc}
2^Y & \xrightarrow{\;\Diamond_Y\;} & 2^{\mathcal{P}Y} \\
{\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f^{-1}} \\
2^X & \xrightarrow[\;\Diamond_X\;]{} & 2^{\mathcal{P}X}
\end{array}
$$

commutes by a simple computation: for every subset $U \subseteq Y$,

$$
\begin{aligned}
&(\mathcal{P}f^{-1} \circ \Diamond_Y)(U) \\
={}& \mathcal{P}f^{-1}\,\{S \subseteq Y \mid U \cap S \neq \emptyset\} \\
={}& \{V \subseteq X \mid U \cap f[V] \neq \emptyset\} \\
={}& \{V \subseteq X \mid f^{-1}(U) \cap V \neq \emptyset\} \qquad\qquad \{\text{see below}\} \\
={}& \Diamond_X(f^{-1}U)
\end{aligned}
$$

where $U \cap f[V] \neq \emptyset$ simply says that there exists some element $v \in V$ such that $fv \in U$, i.e. $v \in f^{-1}(U) \cap V$. $\qquad\square$

Then, the symbol $\blacklozenge$ presenting the modal operation is linked to the aforementioned natural transformation $\Diamond$ in the following way:

**Proposition 2.2.13**. *Let $\mathcal{Q}$ be the contravariant powerset functor to the category of Boolean algebras. There exists a natural transformation $\delta$ from $\mathbb{M}\mathcal{Q}$ to $\mathcal{Q}\mathcal{P}$ defined by*

$$
\blacklozenge S \mapsto \Diamond_X(S)
$$

*on generators of $\mathbb{M}\mathcal{Q}X$ for each component $X$.*

*Proof.* First we notice that $\Diamond_X$ is subject to normality, i.e. $\Diamond_X(\emptyset) = \emptyset$ and $\Diamond_X(U \cup V) = \Diamond_X(U) \cup \Diamond_X(V)$. Therefore, a function $\delta$ from the set of generators of $\mathbb{M}\mathcal{Q}X$ extends to a Boolean algebra homomorphism from $\mathbb{M}\mathcal{Q}X$ to $\mathcal{Q}\mathcal{P}X$ consistent with $\delta$ on generators, using the same symbol $\delta$.

Secondly, we verify that $\delta$ is a natural transformation from $\mathbb{M}\mathcal{Q}$ to $\mathcal{Q}\mathcal{P}$. For every function $f : X \to Y$, consider the following diagram

$$
\begin{array}{ccc}
\mathbb{M}\mathcal{Q}Y & \xrightarrow{\;\delta_Y\;} & \mathcal{Q}\mathcal{P}Y \\
{\scriptstyle \mathbb{M}(f^{-1})}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f^{-1}} \\
\mathbb{M}\mathcal{Q}X & \xrightarrow[\;\delta_X\;]{} & \mathcal{Q}\mathcal{P}X
\end{array}
$$

which commutes since the following

$$(\mathcal{P}f^{-1} \circ \delta_Y)(\blacklozenge U) = \mathcal{P}f^{-1}(\lozenge_Y S) = \lozenge_X(f^{-1}U) = \left(\delta_X \circ \mathbb{M}(f^{-1})\right)(\blacklozenge U)$$

holds for every subset $U \subseteq Y$ by Proposition 2.2.12. $\qquad\qquad\square$

We are now able to show that the construction of complex algebras is *functorial*:

**Theorem 2.2.14.** *In Proposition 2.2.13, the natural transformation $\delta$ defines a contravariant functor $\mathcal{Q}^\delta$ from the category of $\mathcal{P}$-coalgebras to the category of Boolean algebras with operators and $\mathcal{Q}^\delta$ is a lifting of $\mathcal{Q}$, i.e.*

$$
\begin{array}{ccc}
\mathsf{Set}_{\mathcal{P}} & \xrightarrow{\mathcal{Q}^\delta} & \mathsf{BA}^{\mathbb{M}} \\
U \downarrow & & \downarrow U \\
\mathsf{Set} & \xrightarrow{\mathcal{Q}} & \mathsf{BA}
\end{array}
$$

*commutes where the $U$'s are forgetful functors mapping each coalgebra $(X, \xi)$ (resp. $\mathbb{M}$-algebra $(A, \alpha)$) to its carrier $X$ (resp. $A$).*

*Proof.* For each $\mathcal{P}$-coalgebra $\langle X, \xi \rangle$, define an $\mathbb{M}$-algebra by

$$\mathcal{Q}^\delta\langle X, \xi \rangle = \mathbb{M}\mathcal{Q}X \xrightarrow{\delta_X} \mathcal{Q}\mathcal{P}X \xrightarrow{\xi^{-1}} \mathcal{P}X;$$

and for each coalgebra homomorphism $f : \langle X, \xi \rangle \to \langle Y, \gamma \rangle$ define $\mathcal{Q}^\delta f = \mathcal{Q}f = f^{-1}$. $\mathcal{Q}^\delta f$ is an $\mathbb{M}$-algebra homomorphism by the following commutative diagram

$$
\begin{array}{ccccc}
\mathbb{M}\mathcal{Q}Y & \xrightarrow{\delta_Y} & \mathcal{Q}\mathcal{P}Y & \xrightarrow{\gamma^{-1}} & \mathcal{Q}Y \\
\mathbb{M}f^{-1} \downarrow & & \downarrow \mathcal{P}f^{-1} & & \downarrow f^{-1} \\
\mathbb{M}\mathcal{Q}X & \xrightarrow{\delta_X} & \mathcal{Q}\mathcal{P}X & \xrightarrow{\xi^{-1}} & \mathcal{Q}X
\end{array}
$$

using the naturality of $\delta$ and the application of $\mathcal{Q}$ to the coalgebra homomorphism $f$. $\qquad\qquad\square$

*Remark* 2.2.15. In this subsection, we notice the following facts

1. the modal operation $\lozenge^{\to}$ is a composite of the inverse image of $\mathcal{P}$-coalgebra and a natural transformation $\lozenge$;

2. the natural transformation $\lozenge$ defines a natural transformation $\delta$ from $\mathbb{M}\mathcal{Q}$ to $\mathcal{Q}\mathcal{P}$ where $\mathcal{P}$ is the type of coalgebras;

3. to sum up, we derive a lifting $\mathcal{Q}^\delta$ of the contravariant functor $\mathcal{Q}$ along forgetful functors from categories of coalgebras and algebras.

## 2.3 Coalgebraic Logics

Modal logics parametric in endofunctors on Set (or SET), called *coalgebraic logic*, was first explored by Jon Barwise and Lawrence Moss in their book on non-well-founded set theory *Vicious Circles* [21] and Moss' groundbreaking paper *Coalgebraic Logic* [83]. For a more comprehensive coverage including proof systems and various extensions, we refer to [69] and [74].

Two approaches to coalgebraic logic *a*) Moss' cover modality [83], and *b*) Pattinson's predicate lifting [88, 89] are successful in producing a wide range of modal logics *parametric* in endofunctors on Set. To briefly explain the main difference, we provide the definitions with examples first, and then address their properties in subsequent subsections. Logics of predicate liftings and the cover modality can be formulated in an abstract functorial framework which plays the central role when comparing different approaches to coalgebraic logics, and we will introduce it subsequently.

We will focus on coalgebraic modalities without propositional connectives explicitly in the following. Indeed, it is possible to *disregard* every propositional connective, but adequacy and expressiveness still remain [82].

*Remark* 2.3.1. In the study of coalgebraic logic, the Set functor under consideration is usually assumed to be injection-preserving or even inclusion-preserving. This does not affect the category of coalgebras by Corollary 2.1.24 and Theorem 2.1.25.

**Moss' coalgebraic logic**

Moss' idea is to apply the type functor $T$ to the language $\mathcal{M}_T$ and every element $\alpha \in T\mathcal{M}_T$ is taken as an argument of the unique modality of $T$, the so-called **cover modality** $\nabla = \nabla_T,$[7] i.e. a logical connective

$$\nabla\colon T\mathcal{M} \to \mathcal{M}$$

is introduced in the language $\mathcal{M}_T$, whose arity is not a natural number but $T$ itself.

---

[7] The notation $\nabla$ is preferred in recent literature, e.g. [26, 67, 69, 74, 87, 105] in place of the original notation $\Delta$ [21, 83].

To be precise, the language is generated by propositional connectives and the cover modality inductively. As for the satisfaction relation $\Vdash$, propositional connectives are interpreted in the usual way, and the modality $\nabla\alpha$ is interpreted in a Kripke-like semantic: Define the *satisfaction relation* for a $T$-coalgebra $\langle X, \xi \rangle$

$$\Vdash \subseteq X \times \mathcal{M}_T$$

between the carrier $X$ and the language $\mathcal{M}_T$ of Moss' coalgebraic logic such that propositional connectives are interpreted in the usual way; and for every element $x \in X$ and $\alpha \in T\mathcal{M}_T$ define by applying the type functor $T$ to the satisfaction relation $\Vdash$ (that is, $T(\Vdash)$ in the following):

$$\langle X, \xi \rangle; x \Vdash \nabla\alpha \iff \exists w \in T(\Vdash). \begin{cases} (T\pi_X)w &= \xi(x), \\ (T\pi_{\mathcal{M}})w &= \alpha \end{cases} \tag{2.17}$$

where $\pi_X$ and $\pi_{\mathcal{M}}$ are the projections from $X \times \mathcal{M}_T$ to $X$ and $\mathcal{M}_T$ respectively.

*Remark* 2.3.2. The cover modality $\nabla$ and its interpretation depend on nothing but $T$ alone, so the expressive power of Moss' coalgebraic logic is fully determined by the type functor $T$ if propositional connectives are fixed *a priori*.

**Example 2.3.3.** In the following examples of $T$-coalgebras, defined in Example 2.1.3, the symbol $\langle X, \xi \rangle$ always denotes a $T$-coalgebra:

1. Let $T := A \times (-)$ for some fixed set $A$. The corresponding cover modality then is in the following form:
$$\nabla \colon A \times \mathcal{M}_T \to \mathcal{M}_T$$

   which takes an element of $a \in A$ and a formula $\varphi \in \mathcal{M}_T$ such that $x \Vdash \nabla\langle a, \varphi \rangle$ for some $x$ in $\langle X, \xi \rangle$ if and only if

$$\xi(x) = \langle a, y \rangle \quad \text{and} \quad y \Vdash \varphi, \tag{2.18}$$

   i.e. $x$ satisfies $\nabla\langle a, \varphi \rangle$ if the output symbol is exactly $a$ and the next state satisfies $\varphi$. Similarly, when $T := 1 + A \times (-)$, then, $\nabla \colon 1 + A \times \mathcal{M}_T \to \mathcal{M}_T$ either takes a unique element $\checkmark$ from $1$ such that

$$x \Vdash \nabla\checkmark \quad \text{if and only if} \quad \xi(x) = \checkmark$$

   or a pair $\langle a, \varphi \rangle$ as above.

2. Let $T := \mathcal{P}(A \times -)$, i.e. the type of labelled transition systems over $A$. The cover modality of $T$ is an operation

$$\nabla \colon \mathcal{P}(\mathcal{M}_T)^A \to \mathcal{M}_T$$

taking an $A$-indexed collection $(S_a)_{a \in A}$ of sets of formulae such that for any $a \in A$ and $S_a \subseteq \mathcal{M}_T$, a state $x$ satisfies $\nabla \langle S_a \rangle$ if and only if for every $a \in A$:

(a) for every $y \overset{a}{\leftarrow} x$, there is some $\varphi \in S_a$ such that $y \Vdash \varphi$;

(b) for every $\varphi \in S_a$, there exists $y \overset{a}{\leftarrow} X$ such that $y \Vdash \varphi$,

and $\nabla \langle S_a \rangle$ is equal to the following Hennessy-Milner formula

$$\bigwedge_{a \in A} \left( [a]\left(\bigvee S_a\right) \wedge \bigwedge \langle a \rangle S_a \right) \tag{2.19}$$

where $\langle a \rangle S_a$ indicates the set $\{\langle a \rangle \varphi \mid \varphi \in S_a\}$. For unlabelled transition systems, i.e. Kripke frames, (2.19) becomes

$$\Box\left(\bigvee S\right) \wedge \bigwedge \Diamond S,$$

which is exactly the normal form for modal logic without atomic propositions derived by Fine [42].

3. Recall that $\mathcal{D}$ is the discrete probability distribution functor and $\mathcal{D}$-coalgebras are finitely-branching discrete-time Markov chains. The cover modality of $\mathcal{D}$

$$\nabla \colon \mathcal{D}\mathcal{M}_\mathcal{D} \to \mathcal{M}_\mathcal{D},$$

takes a discrete distribution $\mu$ on the *language* as its parameter, and an element $x \in X$ satisfies $\nabla \mu$ if and only if there is $\rho \colon X \times \mathcal{M}_\mathcal{D} \to [0,1]$ with its support in the relation $\Vdash$ such that

$$\nu(y) = \sum_{y \Vdash \varphi} \rho(y, \varphi) \quad \text{and} \quad \mu(\varphi) = \sum_{y \Vdash \varphi} \rho(y, \varphi).$$

where $\nu = \xi(x) \colon X \to [0,1]$.

For example, let $X = \{x_1, x_2\}$ and a $\mathcal{D}$-coalgebra $\xi \colon X \to \mathcal{D}X$ represented by a transition matrix

$$\begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix}$$

where the probability of moving from $x_i$ to $x_j$ is given in the $i$-th row and $j$-th column element. E.g. $(\xi\ x_1)\ x_2 = 0.7$ and $(\xi\ x_2)\ x_2 = 0.6$. Let $\mu$ be a probability distribution on $\mathcal{M}_{\mathcal{D}}$ assigning 0 to everything but 1 to the truth constant $\top$ and is presented by a partial function $\{\langle \top, 1\rangle\}$. Then, we can see that $x_1 \Vdash \nabla\{\langle \top, 1\rangle\}$ because there exists a witness

$$\rho = \{(\langle x_1, \top\rangle, 0.3), (\langle x_2, \top\rangle, 0.7)\}$$

with

$$\xi(x) = \sum_{-\Vdash\top} \rho(-, \top) \quad \text{and} \quad \mu(\top) = 1 = \sum_{y\in X} \rho(y, \top).$$

The statement $\nabla\{\langle \top, 1\rangle\}$ can be seen as saying that the probability of moving to a state satisfying $\top$ is 1.

### Pattinson's predicate liftings

Pattinson's view is that a modal operation for a functor $T$ is a map from predicates over a carrier $X$ to predicates over $TX$, and $T$-coalgebras take predicates over $TX$ back to $X$: A **predicate lifting** for $T$ is a natural transformation from the contravariant powerset functor $2^-$ to the composite $2^T$ of $2^-$ with $T$, i.e. a family of functions $\lambda_X$ for every set $X$ such that

$$\begin{array}{ccc}
2^X & \xrightarrow{\lambda_X} & 2^{TX} \\
{\scriptstyle f^{-1}}\big\uparrow & & \big\uparrow{\scriptstyle (Tf)^{-1}} \\
2^Y & \xrightarrow{\lambda_Y} & 2^{TY}
\end{array}$$

commutes for every function $f : X \to Y$. The composite with the inverse image of any $T$-coalgebra maps any predicate over $X$ to a predicate over $X$ again.

The technical notion of predicate liftings was first devised for generalising induction principles in a fibrational setting by Hermida and Jacobs [51] and found an application in coalgebraic modal logic by Pattinson. Later, predicate liftings were generalised to *polyadic predicate liftings* by Schröder [97] to obtain a general (strong) expressiveness property for any accessible (e.g. finitary) functor.

**Definition 2.3.4.** A **polyadic predicate lifting** for $T$ is a natural transformation

$$\lambda \colon (2^-)^\kappa \longrightarrow 2^T \tag{2.20}$$

for some cardinality $\kappa$ which is the *arity* of $\lambda$. A **finitary predicate** is a polyadic predicate lifting whose arity is a natural number $n \in \aleph_0$.

Given a set $\Lambda$ of predicate liftings, every $\kappa$-ary predicate lifting $\lambda$ in $\Lambda$ introduces an $\kappa$-ary logical connective

$$[\lambda]\colon \mathcal{L}(\Lambda)^\kappa \to \mathcal{L}(\Lambda)$$

in the language $\mathcal{L}(\Lambda)$ generated by propositional connectives along with the set $\Lambda$ of predicate liftings. Besides propositional connectives, the modal formula $[\lambda]\varphi$ is interpreted in a Kripke-like semantics as follows. For any $T$-coalgebra $\langle X, \xi \rangle$, the *satisfaction relation* $\Vdash$ for the modal formula $[\lambda]\varphi$ of some $\kappa$-ary predicate lifting $\lambda \in \Lambda$ is defined by

$$\langle X, \xi \rangle; x \Vdash [\lambda]\varphi \iff \xi(x) \in \lambda_X[\![\varphi]\!] \tag{2.21}$$

for any $x \in X$ and an $\kappa$-indexed set of formulae $\varphi \colon \kappa \to \mathcal{L}(\Lambda)$ where

$$[\![\varphi]\!] := (\{ y \in X \mid y \Vdash \varphi_i \})_{i \in \kappa}$$

is the interpretation of $\varphi$ on the carrier $X$.

*Remark* 2.3.5. In contrast with the cover modality $\nabla$, a coalgebraic logic given by a predicate lifting or a set of predicate liftings depends on not only the type $T$ but also the choice of predicate liftings, *cf.* Remark 2.3.2. However, a canonical choice of predicate liftings can be made: *the collection of all finitary predicate liftings*, if $T$ is finitary.[8] Such a choice is possible because there is only a *proper set* of predicate liftings, see Lemma 2.3.9 below.

**Example 2.3.6.** In the following examples, coalgebraic logics based on the cover modality and predicate liftings, respectively, coincide in their expressive power, if we choose the sets of predicate liftings *properly*: *a*) for sets of streams over $A$, the cover modality $\nabla$ can be expressed as a family of predicate liftings indexed by $A$; *b*) for labelled transition systems, the possibility $\Diamond$ and necessity $\Box$ modal operations in Hennessy-Milner logic are exactly predicate liftings.

1. Consider the type for sets of streams. Define a family $\Lambda_A$ of unary predicate liftings $\bar{a}$ for each $a \in A$ by

$$\bar{a}_X \colon S \mapsto \{a\} \times S = \{ \langle a, s \rangle \in A \times X \mid s \in S \}$$

for any subset $S$ of some set $X$.

---

[8] In general, if $T$ is $\kappa$-accessible for some regular cardinality $\kappa$, then we can take all $\alpha$-ary predicate liftings for any $\alpha < \kappa$. For convenience, we omit this generalisation.

Although it is routine to check the naturality of $\bar{a}$, we still verify it: Let $f: X \to Y$ be a function from $X$ to $Y$. For any subset $U \subseteq X$, verify the following:

$$\begin{aligned}
(\bar{a}_X \circ f^{-1}) U &= \{\langle a, x \rangle \in A \times X \mid fx \in U\} \\
&= (id_A \times f)^{-1} \{\langle a, y \rangle \in A \times Y \mid y \in U\} \\
&= (id_A \times f)^{-1} (\bar{a}_Y U) \\
&= ((id_A \times f)^{-1} \circ \bar{a}_Y) U,
\end{aligned}$$

so $\bar{a}_X$ is natural in $X$.

Given a $(A \times -)$-coalgebra $\langle X, \xi \rangle$, an element $x$ in $X$ satisfies $[\bar{a}]\varphi$ for some formula $\varphi$ if and only if $\xi(x) \in \bar{a}_X [\![\varphi]\!]$, that is

$$\xi(x) = \langle a, y \rangle \quad \text{and} \quad y \Vdash \varphi \tag{2.22}$$

which coincides with the interpretation of the cover modality given in (2.18).

2. For unlabelled transition systems, we can define a predicate lifting $\Diamond$ introduced in (2.16), i.e.
$$\Diamond_X : U \mapsto \{S \subseteq X \mid S \cap U \neq \emptyset\}$$

for any subset $U$ of $X$. The naturality of $\Diamond$ was checked previously in Proposition 2.2.12.

Moreover, the dual operator $\Box\varphi = \neg\Diamond\neg\varphi$ is also a predicate lifting: First note that $[\![\neg\varphi]\!]$ is equal to the complement $[\![\varphi]\!]^{\complement}$. Then for any function $f: X \to Y$ and a subset $U \subseteq Y$, we verify the following by using the naturality of $\langle a \rangle$ and the commutativity between $f^{-1}$ and $(-)^{\complement}$, i.e. $f^{-1}(-)^{\complement} = f^{-1}(-^{\complement})$:

$$\begin{aligned}
(\Box_X \circ f^{-1}) U &= \left(\Diamond_X (f^{-1}U)^{\complement}\right)^{\complement} \\
&= \left(\left(\Diamond_X \circ f^{-1}\right) U^{\complement}\right)^{\complement} \\
&= \left(\left(\mathcal{P}f^{-1} \circ \Diamond_Y\right) U^{\complement}\right)^{\complement} \\
&= \mathcal{P}f^{-1}\left(\Diamond_Y U^{\complement}\right)^{\complement} = (\mathcal{P}f^{-1} \circ \Box_Y) U,
\end{aligned} \tag{2.23}$$

so the naturality of $\Box$ follows.

The interpretation of $\Diamond$ and $\Box$ match (2.11) and (2.12) exactly.

3. Last but not least, consider finitely-branching discrete-time Markov chains. For each real number $p$ in the unit interval $[0, 1]$, define a predicate liftings $\bar{p}$ by

$$\bar{p}\colon S \mapsto \{\mu \in \mathcal{D}X \mid \sum \mu S \geq p\}$$

for each subset $S$ of some set $X$.

The naturality of $\bar{p}$ follows similarly to the previous examples in few easy steps: Let $f\colon X \to Y$ be a function from $X$ to $Y$. Verify the naturality as follows: for each $p \in [0, 1]$ and any subset $U \subseteq X$,

$$\begin{aligned}
(\bar{p}_X \circ f^{-1})\, U &= \{\mu \in \mathcal{D}X \mid \sum \mu(f^{-1}U) \geq p\} \\
&= \{\mu \in \mathcal{D}X \mid \sum (\mathcal{D}f\,\mu)\, U \geq p\} &&(*) \\
&= \mathcal{D}f^{-1}\, \{v \in \mathcal{D}Y \mid \sum vU \geq p\} \\
&= (\mathcal{D}f^{-1} \circ \bar{p}_Y)\, U
\end{aligned}$$

where $(*)$ follows from the identities

$$\sum (\mathcal{D}f\,\mu)\, U = \sum_{y \in U} \sum_{fx=y} \mu(x) = \sum_{fx \in U} \mu(x) = \sum \mu(f^{-1}U).$$

Given a $\mathcal{D}$-coalgebra $\langle X, \xi \rangle$, an element $x$ in $X$ satisfies a formula $[\bar{p}]\varphi$ for some $\varphi$, i.e.

$$x \Vdash [\bar{p}]\varphi \quad \text{if and only if} \quad \sum \mu[\![\varphi]\!] \geq p$$

where $\mu = \xi(x)$. The statement says precisely that the formula $\varphi$ 'will' hold with a probability greater than or equal to $p$ after one transition from $x$.

### 2.3.1 Logics of Predicate Liftings

Now we fix a propositional logic. Assume that the propositional logic we choose is propositional calculus consisting of two truth values, negation, and finitary conjunction and disjunction. Following the terminology in modal logic, a set of symbols with finite arities is called a **similarity type** [28, Definition 1.11] (also known as a *signature* in universal algebra), so an indexed set of finitary predicate liftings for a Set endofunctor is a similarity type *with interpretations.*

Given a similarity type $\Lambda$ for $T$, the language $\mathcal{L}(\Lambda)$ induced by the similarity type $\Lambda$ is generated by the following syntax:

$$\varphi := \bot \mid \top \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid [\lambda](\varphi_i)_{i \in n} \quad (\lambda \in \Lambda, n \text{ is the arity of } \lambda). \quad (2.24)$$

Given a $T$-coalgebra $\langle X, \xi \rangle$, the interpretation $[\![\varphi]\!] = [\![\varphi]\!]_{\langle X, \xi \rangle}$ of a formula $\varphi$ in $\mathcal{L}(\Lambda)$ is a predicate over $X$, i.e. a subset, defined inductively as follows:

$$[\![\bot]\!] = \emptyset \qquad\qquad [\![\top]\!] = X$$
$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \qquad [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![\neg\varphi]\!] = [\![\varphi]\!]^{\complement} \qquad\qquad [\_{i \in n}]\!] = \xi^{-1} \circ \lambda([\![\varphi_0]\!], \ldots, [\![\varphi_{n-1}]\!]).$$

The *satisfaction relation* $\Vdash$ is then simply an *alias* for the membership relation on the interpretation of a given $T$-coalgebra $\langle X, \xi \rangle$: $\langle X, \xi \rangle; x \Vdash \varphi$ if and only if $x \in [\![\varphi]\!]_{\langle X, \xi \rangle}$.

Note that the satisfaction relation matches (2.21) exactly.

Every logic induced by a set of predicate liftings is *adequate*. We begin with a simple observation:

**Lemma 2.3.7.** *Let $\langle X, \xi \rangle$ and $\langle Y, \gamma \rangle$ be $T$-coalgebras, and $f$ a coalgebra homomorphism from $\langle X, \xi \rangle$ to $\langle Y, \gamma \rangle$. Then, every element $x \in X$ is logically equivalent to its image $f x \in Y$, i.e.*

$$\langle X, \xi \rangle; x \Vdash \varphi \iff \langle Y, \gamma \rangle; f x \Vdash \varphi$$

*for any formula $\varphi \in \mathcal{L}(\Lambda)$.*

*Proof.* This will follow from Theorem 2.3.28 in a generalised framework. $\square$

Since two elements are behaviourally equivalent if they can be identified by two coalgebra homomorphisms, the adequacy is an immediate consequence of the above lemma:

**Theorem 2.3.8** (Adequacy). *Let $\langle X, \xi \rangle$ and $\langle Y, \gamma \rangle$ be $T$-coalgebras. Elements $x \in X$ and $y \in Y$ are logically equivalent provided that $x$ and $y$ are behaviourally equivalent.*

From the definition, a predicate lifting can be wild, but there are only set-many of them, [97] (this also appears in [66, Proposition 9]):

**Lemma 2.3.9** (see [97, Proposition 20]). *There is a one-to-one correspondence between the collection of $\kappa$-ary predicate liftings and the powerset of $T 2^{\kappa}$.*

*Proof.* Observe that $2^-$ is naturally isomorphic to $\mathrm{Hom}(-, 2)$. Then, the statement follows from the Yoneda Lemma. $\qquad\square$

The actual bijection between natural transformations and morphisms from $T2^\kappa$ to $2$ is given by 'tracing the identity': Viewing a predicate lifting $\lambda$ as a natural transformation from $\mathrm{Hom}(-, 2^\kappa)$ to $\mathrm{Hom}(T-, 2)$, the correspondence is given by $\lambda \mapsto \lambda_{2^\kappa}(id_{2^\kappa})$ and each $\lambda_X(f)$, for $f : X \to 2^\kappa$, is equal to $\lambda_{2^\kappa}(id_{2^\kappa}) \circ T\chi$. A $\kappa$-indexed collection of subsets of $X$ can be described uniquely by its characteristic function $\chi : X \to 2^\kappa$ and vice versa. It follows that the identity on $2^\kappa$ is the $\kappa$-indexed collection of subsets

$$\pi_i^{-1}\{\top\}$$

where $\pi_i$ is the $i$-th projection function from $2^\kappa$ to $2$. Thus, a predicate lifting $\lambda$, viewed as a natural transformation from $2^-$ to $2^{T-}$, is uniquely determined by the subset

$$\lambda_{2^\kappa}(\pi_i^{-1}\{\top\})_{i\in\kappa} \subseteq T2^\kappa.$$

Conversely, since $\lambda_X(\chi_S) = \lambda_{2^\kappa}(id_{2^\kappa}) \circ T\chi_S$ for any subset $S \subseteq X$ by the Yoneda Lemma, we can translate the equality using subsets:

$$t \in \lambda_X(S) \iff T\chi_S(t) \in \lambda_{2^\kappa}\langle \pi_i^{-1}\{\top\}\rangle_{i\in\kappa}, \tag{2.25}$$

and it follows that any subset of $T2^\kappa$ defines a predicate lifting $\lambda_X(S) := T\chi_S^{-1}[C]$.

The lemma provides another characterisation of predicate liftings in Example 2.3.6:

**Example 2.3.10.** Let $A$ and $X$ be sets. The following examples are listed in the same order as in Example 2.3.6:

1. The predicate lifting $\bar{a}_X$, for some $a \in A$, sending a subset $S$ of $X$ to the subset $\{a\} \times S$ of $A \times X$ is determined uniquely by the subset $\bar{a}_2\{\top\} = \{\langle a, \top\rangle\}$ of $A \times 2$.

2. The predicate liftings $\Diamond_X$ sending a subset $S \subseteq X$ to $\{S \subseteq X \mid S \cap R \neq \emptyset\}$ is determined uniquely by the collection of subsets

$$\Diamond_2\{\top\} = \{S \subseteq 2 \mid S \cap \{\top\} \neq \emptyset\} = \{\{\bot, \top\}, \{\top\}\} \subseteq \mathcal{P}2.$$

Similarly, the predicate lifting $\Box$ is uniquely determined by

$$\Box_2\{\top\} = \{\emptyset, \{\top\}\} \subseteq \mathcal{P}2.$$

3. For each $p \in [0,1]$, the predicate lifting $\bar{p}$ for discrete-time Markov chains, sending a subset $S \subseteq X$ to $\{\mu \in \mathcal{D}X \mid \sum \mu S \geq p\}$, is determined uniquely by the subset of $\mathcal{D}2$:

$$\bar{p}_2\{\top\} = \{\mu \in \mathcal{D}2 \mid \sum \mu\{\top\} \geq p\} = \{\mu \in \mathcal{D}2 \mid \mu(\top) \geq p\}.$$

It is worth mentioning that the characterisation of a $\kappa$-ary predicate lifting for $T$ by a subset of $T2^\kappa$ only involves a subset while the original definition involves a family of functions with naturality.

To obtain expressiveness, the chosen set of predicate liftings needs a certain property:

**Definition 2.3.11** (see [88, 97]). A set $\Lambda$ of $n$-ary predicate liftings $\lambda$ for $T$ has the **separation property** on a set $X$ if the $\Lambda$-indexed family of maps

$$\lambda_X^\flat \colon TX \to 2^{(2^X)^n}$$
$$t \mapsto \{(S \subseteq X)_{i \in n} \mid t \in \lambda_X(S)_i\} \quad (n \text{ is the arity of } \lambda)$$

is jointly injective. We say that $\Lambda$ is **separating** if $\Lambda$ has the separation property for any set.

That is, a set of predicate liftings has the separation property on a set $X$ if any two distinct elements $t, u \in TX$ can be *distinguished* by a predicate lifting with an $n$-ary predicate on $X$.

**Theorem 2.3.12** (see [97, Theorem 41]). *Let $T$ be finitary and $\Lambda$ a separating set of finitary predicate liftings. The logic $\mathcal{L}(\Lambda)$ of predicate liftings is expressive.*

**Example 2.3.13.** Using the above theorem, we proceed to verify the expressiveness of the following logics of predicate liftings given previously:

1. Consider predicate liftings $\bar{a}\colon 2^- \to 2^{A \times -}$ for sets of streams over $A$. Each pair of elements $\langle a, x \rangle, \langle a', x' \rangle \in A \times X$ on a set $X$ with $a \neq a'$ can be distinguished by $\bar{a}$ and any subset $S \subseteq X$. Then, $\langle a, x \rangle \in \bar{a}_X(S)$ but $\langle a', x' \rangle \notin \bar{a}_X(S)$. Otherwise, a pair $\langle a, x \rangle$ and $\langle a, x' \rangle$ with $x \neq x'$ can be distinguished by $\bar{a}(S)$ for a subset $S \subseteq X$ with $x \in S$ but $x' \notin S$.

2. Consider the predicate lifting $\Diamond$ for unlabelled transition systems. Let $U, V \in \mathcal{P}X$ be two distinct subsets of $X$ and $U$ is non-empty, and $x \in U$ but $x \notin V$. Hence the singleton $\{x\}$ has a non-empty intersection with $U$ but an empty intersection with $V$. That is, $U \in \Diamond_X\{x\}$ but $V \notin \Diamond_X\{x\}$.

3. Consider predicate liftings $\bar{p}$, $p \in [0,1]$, for discrete-time Markov chains. Let $\mu$ and $\mu'$ be distributions with finite support on a set $X$ with $p := \mu(x) > \mu'(x)$ for some $x \in X$. It follows that $\mu(x) = \mu \in \bar{p}\{x\}$ but $\mu' \notin \bar{p}\{x\}$ by definition.

The separation property for subsets of $T2$ or $T2^\kappa$ can be formulated by using the translation in Lemma 2.3.9:

**Lemma 2.3.14** (see [97, Corollary 44]). *A* Set *endofunctor $T$ has a set of polyadic predicate liftings bounded by $\kappa$ which has the separation property on $X$ if and only if the family*

$$\{ T\chi \colon TX \to T2^\gamma \mid \gamma < \kappa, \chi \colon X \to 2^\gamma \} \tag{2.26}$$

*is jointly injective on $X$.*

*Proof.* For convenience, we only prove it for unary predicate liftings and the general case follows similarly.

The powerset $\mathcal{P}T2$ is the collection of *all* predicate liftings up to bijection by Lemma 2.3.9. By (2.25), for each subset $S \subseteq X$ and a subset $C \subseteq T2$, the subset $T\chi_S^{-1}[C]$ is the value of $S$ mapped by the predicate lifting induced from $C$. Hence, the set of all predicate liftings $\mathcal{P}T2$ is separating if and only if it is jointly injective, since $\mathcal{P}T2$ distinguishes each pair of distinct elements in $T2$. $\qquad\square$

**Lemma 2.3.15.** *A set $\Lambda$ of (unary) predicate liftings for a finitary and inclusion-preserving* Set *endofunctor is separating if and only if $\Lambda$ has the separation property on all finite sets.*

*Proof.* As for the *if* part: Note that for an inclusion-preserving and finitary functor $T$, the set $TX$ is the directed union of $\{TS \subseteq TX \mid S \subseteq_\omega X\}$, so for any $t_1, t_2 \in TX$, there exists a finite subset $S \subseteq X$ such that $t_1, t_2 \in TS$. Thus there exists a predicate lifting $\lambda \in \Lambda$ and a subset $S' \subseteq S$ with $\chi_{\lambda S'}(t_1) \neq \chi_{\lambda S'}(t_2)$. Moreover, by naturality

$$\begin{array}{ccc}
2^X & \xrightarrow{\lambda_X} & 2^{TX} \\
{\scriptstyle i^{-1}}\downarrow & & \downarrow{\scriptstyle Ti^{-1}} \\
2^S & \xrightarrow{\lambda_S} & 2^{TS}
\end{array}$$

we have $\lambda_X(U)$ for any subset $U$ of $S$ where $i \colon S \subseteq X$ is the inclusion.

The *only-if* part follows by definition. $\qquad\square$

Using the above two facts, a finitary Set endofunctor admits a separating set of finitary predicate liftings if and only if (2.26) holds for all finite sets $X$. The latter is easily established for the set of all finitary predicate liftings:

**Theorem 2.3.16** (see [97, Corollary 45]). *A finitary and inclusion-preserving endofunctor of* Set *admits a separating set of finitary predicate liftings.*

*Proof.* Let $X$ be a finite set. Clearly, there is an injective function $\chi\colon X \to 2^X$ such that

$$\chi(x)(y) = \begin{cases} \top & \text{if } x = y, \\ \bot & \text{otherwise.} \end{cases}$$

By assumption, $T$ preserves injections, so $T\chi$ is also injective. It follows that (2.26) is jointly injective, so the set of all finitary predicate liftings is separating. $\qquad\square$

**Corollary 2.3.17.** *The logic of all finitary predicate liftings for an inclusion-preserving and finitary* Set *endofunctor is expressive.*

### 2.3.2   Logics of Cover Modality

As we have seen, Moss' cover modality provides a modal operator $\nabla = \nabla_T$ only dependent on the type functor $T$. The language $\mathcal{M}_T$ of the corresponding logic for a finitary Set endofunctor $T$ is generated by the following syntax:

$$\varphi \coloneqq \bot \mid \top \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \nabla\alpha \tag{2.27}$$

where $\alpha \in Ti[TS]$ for some finite subset $i\colon S \subseteq \mathcal{M}_T$. Note that, since $T$ is finitary, $T\mathcal{M}_T = \bigcup\{Ti[TS] \mid i\colon S \subseteq_\omega \mathcal{M}_T\}$. For convenience, we may assume further that $T$ preserves inclusions so that the cover modality $\nabla\alpha$ can be equally constructed by

$$\alpha \in T\{\varphi_0, \ldots, \varphi_{n-1}\}$$

for finitely many formulae $\varphi_i \in \mathcal{M}_T$ for $i \in n$.

In (2.17), we have seen the interpretation of the cover modality in terms of the satisfaction relation. This formulation can be captured by *relation lifting*: Given a Set endofunctor $T$, the **relation lifting** $\overline{T}R$ of a relation $R \subseteq X \times Y$ for $T$ is defined to be

the image of the following diagram

$$
\begin{array}{ccc}
& TR & \\
T\pi_X \swarrow & \Big\downarrow & \searrow T\pi_Y \\
TX \xleftarrow{\pi_{TX}} & TX \times TY & \xrightarrow{\pi_{TY}} TY,
\end{array}
\tag{2.28}
$$

i.e. $\overline{T}R := \{\langle u, v \rangle \in TX \times TY \mid \exists w \in TR . T\pi_X(w) = u, T\pi_Y(w) = v\}$. We can rewrite (2.17) using relation liftings as

$$
\langle X, \xi \rangle; x \Vdash \nabla\alpha \iff \xi(x) \, \overline{T}(\Vdash) \, \alpha.
$$

The interpretation $[\![-]\!] = [\![-]\!]_{\langle X, \xi \rangle}$ of Moss' logic $\mathcal{M}_T$, for a $T$-coalgebra $\langle X, \xi \rangle$, as subsets of $X$ is defined as follows:

$$
\begin{aligned}
[\![\bot]\!] &= \emptyset & [\![\top]\!] &= X \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] & [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\neg\varphi]\!] &= [\![\varphi]\!]^{\complement} & [\![\nabla\alpha]\!] &= (\xi^{-1} \circ \nabla_X)((T[\![-]\!])\alpha)
\end{aligned}
$$

where $\nabla_X : T2^X \to 2^{TX}$ is defined by

$$
\nabla_X : \alpha \mapsto \{t \in TX \mid (t, \alpha) \in \overline{T}(\in_X)\}
\tag{2.29}
$$

and $\in_X \subseteq X \times \mathcal{P}X$ is the membership relation on $X$.

**Weak-pullback preservation**

The relation lifting $\overline{T}$ of $T : \mathsf{Set} \to \mathsf{Set}$ is a Rel endofunctor if and only if it preserves *weak pullbacks*, where Rel consists of sets as objects and relations as morphisms with compositions defined by

$$
R \circ S = \{\langle x, z \rangle \mid \exists y . x \, R \, y \quad \text{and} \quad y \, S \, z\}.
$$

**Lemma 2.3.18** (see [18]). *$T$ preserves weak pullbacks if and only if $\overline{T}(R \circ S) = \overline{T}R \circ \overline{T}S$ for any relation $R \subseteq X \times Y$ and $S \subseteq Y \times Z$.*

**Proposition 2.3.19.** *Let $T$ be weak-pullback preserving. The following statements are true:*

   *1. The map $\nabla$ defined in (2.29) is a natural transformation from $T2^{-}$ to $2^{T}$.*

*2. In particular, the logic of the cover modality for $T$ is adequate.*

*Proof.* The naturality of $\nabla\colon T2^- \to 2^T$ amounts to the equation

$$\{t \in TX \mid \langle Tf(t), \beta \rangle \in \overline{T}(\in_Y)\} = \{t \in TX \mid \langle t, T2^f(\beta)\rangle \in \overline{T}(\in_X)\}$$

for any $f\colon X \to Y$ and $\beta \in T2^Y$. However, it suffices to show that in Rel the following diagram on the left commutes; and by Lemma 2.3.18, it follows from the commutative diagram on the right:

$$
\begin{array}{ccc}
TX \xrightarrow{\overline{T}(\in_X)} T2^X & & X \xrightarrow{\in_X} 2^X \\
\text{graph}(Tf)\downarrow \quad \downarrow\text{graph}(T2^f)^{\mathrm{op}} & \text{and} & \text{graph}(f)\downarrow \quad \downarrow\text{graph}(2^f)^{\mathrm{op}} \\
TY \xrightarrow[\overline{T}(\in_Y)]{} T2^Y & & Y \xrightarrow[\in_Y]{} 2^Y
\end{array}
$$

where $\text{graph}(Tf) = \overline{T}(\text{graph}\,f)$ and $\text{graph}(T2^f)^{\mathrm{op}} = \overline{T}(\text{graph}(2^f)^{\mathrm{op}})$ by a direct computation.[9] The above diagram on the right is simply the diagrammatic form of the statement

$$x \in_X f^{-1}(S) \iff f(x) \in_Y S$$

for $x \in X$ and $S \subseteq Y$. Hence, the first statement follows.

As for the second statement, it will follow from the naturality using Theorem 2.3.28. $\qquad\square$

**Theorem 2.3.20** ([83]). *Let $T$ be finitary and preserve weak pullbacks. Then, the logic of the cover modality for $T$ is expressive.*

In particular, if $T$ is a *polynomial functor* defined by the following syntax:

$$T := \mathcal{I} \mid \mathcal{K}_X \mid T + T \mid T \times T \mid \coprod T, \tag{2.30}$$

i.e. built from the identity functor $I$, constant functors $\mathcal{K}_X$ with value $X$, binary products $T \times T$, and coproducts $T + T$ and $\coprod T$, then $\nabla\colon T2^- \to 2^T$ is a family of predicate liftings.

---

[9] The slashed arrow $X \longrightarrow Y$ indicates a relation.

**Moss liftings**

Moss' coalgebraic logic is different from logics of predicate liftings in the sense that there is always a modality by default and the syntax is different. However it is pointed out by Leal [74] that the cover modality for finitary and weak-pullback preserving Set endofunctor essentially introduces a family of polyadic predicate liftings.

We begin with a presentation property of finitary functors:

**Theorem 2.3.21.** *Every finitary functor* $T\colon \mathsf{Set} \to \mathsf{Set}$ *is a coequaliser of the following diagram*

$$\coprod_{f\colon m\to n} Tm \times X^n \xrightarrow[\rho_2]{\rho_1} \coprod_{n\in\omega} Tn \times X^n \xrightarrow{E_X} TX$$

*natural in* $X$ *where for any function* $f\colon m \to n$ *and* $(\sigma, x) \in Tm \times X^n$ *the above two functions are defined by*

$$\rho_1\colon (\sigma, x) \mapsto (Tf(\sigma), x) \quad \textit{and} \quad \rho_2\colon (\sigma, x) \mapsto (\sigma, x \circ f)$$

*respectively.*

*Proof.* See Corollary 3.4.11 □

An **equational presentation** of $T$ is a pair $\langle \Sigma, E \rangle$ consisting of a functor $\Sigma$ from the discrete small category $\aleph_0$ to $\mathsf{Set}$ and a surjective natural transformation $E\colon \coprod \Sigma n \times X^n \dashrightarrow T$ in the functor category. The set $\Sigma n$ is called the **collection of $n$-ary operations** and $E$ the **set of equations**. We call $\langle T, E \rangle$ defined above the **canonical (equational) presentation**.

The canonical presentation is usually not the most *efficient* one:

**Example 2.3.22.** The finitary powerset functor $\mathcal{P}_\omega$ can be presented by operations and equations as follows. For every $n$-tuple $x = \langle x_i \rangle_{i\in n}$ and $m$-tuple $y = \langle y_j \rangle_{j\in m}$ in $X$, we say $\sigma(x) = \sigma(y)$ if and only if for any $i \in n$, there is some $j \in m$ such that $x_i = y_j$ and vice versa, i.e.

$$\{x_0, \dots, x_{n-1}\} = \{y_0, \dots, y_{m-1}\}.$$

It shows that the finitary powerset functor on a set $X$ is a coequaliser

$$R_X \xrightarrow[\pi_2]{\pi_1} \coprod_{n\in\omega} X^n \xrightarrow{\phantom{xxxx}} \mathcal{P}_\omega X$$

where $R_X = \left\{ (x, y) \in (\coprod_{n \in \omega} X^n)^2 \,\middle|\, \{x_0, \dots, x_{n-1}\} = \{y_0, \dots, y_{m-1}\} \right\}$ and $\pi_i$ is the $i$-th projection, for $i = 1$ and $2$. (To see that $R_X$ is a functor, define $R_X \to R_Y$ for $f : X \to Y$ by $(x, y) \mapsto (fx, fy)$. Now, it is not hard to see that $R$ is a functor and $\pi_i$ is natural in $X$.)

On the other hand, the canonical presentation of the finitary powerset functor $\mathcal{P}_\omega$ is given by

$$\coprod_{f : m \to n} \mathcal{P}_\omega(m) \times X^n \xrightarrow[\rho_2]{\rho_1} \coprod_{n \in \omega} \mathcal{P}_\omega(n) \times X^n \xrightarrow{E_X} \mathcal{P}_\omega X$$

with equations

$$(\{ fi \in n \mid i \in S \}, \langle x_0, \dots, x_{n-1} \rangle) = (S, \langle x_{f0}, \dots, x_{f(m-1)} \rangle)$$

for $f : m \to n$, a subset $S$ of $m$, and an $n$-tuple $x$. Informally, a finite subset $X'$ of $X$ is presented by an $n$-tuple $x$ and a subset $S$ of $n$ where $x_i$ is 'picked' in $X'$ if $i \in S$.

Back to the cover modality, given an equational presentation $\langle \Sigma, E \rangle$ of $T$, every operation $\sigma$ in $\Sigma n$ gives rise to an $n$-ary predicate lifting $\nabla_\sigma$ via the following diagram

$$
\begin{array}{ccccc}
(2^X)^n & \xrightarrow{\;i_\sigma\;} & \coprod \Sigma n \times (2^X)^n & \xrightarrow{\;E_X\;} & T2^X \\
{\scriptstyle (\nabla_\sigma)_X} \downarrow & & & & \downarrow {\scriptstyle \nabla_X} \\
2^{TX} & & \xrightarrow[\qquad\qquad id \qquad\qquad]{} & & 2^{TX}
\end{array}
\qquad (2.31)
$$

where $i_\sigma : S \mapsto (\sigma, S)$ is the injection. Conversely, since every element of $T2^X$ is an equivalence class with representative $\langle \sigma, S \rangle$ for some $\sigma \in \Sigma n$ and $(S_i \subseteq X)_{i \in n}$, for $\alpha \in T2^X$, the set $\nabla_X \alpha \subseteq TX$ is equal to $\nabla_\sigma (S_i)_{i \in n}$.

**Definition 2.3.23** (see [74]). Given an equational presentation $\langle \Sigma, E \rangle$ of a finitary functor $T$, a $\langle \Sigma, E \rangle$-**Moss lifting** (or **Moss lifting** if the presentation is clear) is a polyadic predicate lifting equal to some $\nabla_\sigma$ defined in (2.31) for $\sigma \in \Sigma n$.

By Theorem 2.3.21, every finitary and weak-pullback preserving Set endofunctor has a canonical set of Moss liftings

$$\{ \nabla_t : (2^-)^n \to 2^T \mid n \in \omega, t \in Tn \} \cong \{ \nabla_t \langle \pi_i^{-1}\{\top\} \rangle_{i \in n} \mid n \in \omega, t \in Tn \}$$

by Lemma 2.3.9 and the set of Moss liftings can be identified as a subset of $\coprod_{n \in \omega} T(n)$.

### 2.3.3 Abstract Functorial Framework

The two approaches introduced so far can be unified into a single categorical framework. It was first observed in [65] starting from the algebraic semantics of logics of predicate liftings, and discussed as an abstract tool for coalgebra logics in [30, 53, 63, 72, 76, 92, 107] to name but a few. The cover modality was then subsumed into this picture in [74] used to investigate the equational aspects of Moss' coalgebraic logic.

**Definition 2.3.24.** Let BA denote the category of Boolean algebras. An **abstract logic over** BA for a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ is a pair consisting of

- a *syntax*, in the form of an endofunctor $L\colon \mathsf{BA} \to \mathsf{BA}$, and

- an *interpretation*, in the form of a natural transformation $\delta\colon L\mathcal{Q} \dot{\longrightarrow} \mathcal{Q}T$

where $\mathcal{Q}\colon \mathsf{Set} \to \mathsf{BA}$ is the contravariant powerset functor mapping any set $X$ to its powerset (Boolean) algebra; and any function $f\colon X \to Y$ to the inverse image function $f^{-1}$.

**Example 2.3.25.** In the following discussion, we shall note that the contravariant powerset functor $2^-$ is the composite of $P$ with the forgetful functor $U\colon \mathsf{BA} \to \mathsf{Set}$ and denote the left adjoint to $U$ by $F$ for the free Boolean algebra construction.

1. An $n$-ary predicate lifting $\lambda\colon (2^-)^n \dot{\longrightarrow} 2^T$ is understood as a natural transformation from $UP^n$ to $UPT$, and then a set $\Lambda$ of predicate liftings can be identified as a natural transformation:

$$\delta^\Lambda\colon F\left(\bigsqcup_{n\in\omega} \Lambda_n \times U(-)^n\right) \circ P \dot{\longrightarrow} PT$$

where $\Lambda_n$ denotes the set of $n$-ary predicate liftings, i.e. a logic consisting of $F(\bigsqcup \Lambda_n \times U(-)^n)$ as the syntax and $\delta^\Lambda$ as the interpretation.

2. Assume that the coalgebra type functor $T$ is finitary and preserves weak pull-backs. The map $\nabla\colon T2^X \to 2^{TX}$ defined in (2.29) is a natural transformation by Proposition 2.3.19 where $T2^-$ and $2^T$ are equal to $TUP$ and $UPT$ respectively. By the adjointness, $\nabla$ corresponds uniquely to a natural transformation

$$\overline{\nabla}\colon FTUPX \longrightarrow PTX,$$

i.e. a logic consisting of $FTU$ as the syntax and $\overline{\nabla}$ as the interpretation.

The *language* of a given abstract logic is defined as an initial $L$-algebra, if it exists. The initial algebra can be constructed from the initial sequence of $L$:

$$2 \to L2 \to \cdots \to L^i 2 \to \ldots$$

which is equivalent to the construction of an inductively defined language.

**Example 2.3.26.**    1. The language $\mathcal{L}(\Lambda)$ for a set $\Lambda$ of predicate liftings for a functor $T$ subject to the Boolean laws is clearly a Boolean algebra; we denote it by $\overline{\mathcal{L}(\Lambda)}$. Predicate liftings $\lambda \in \Lambda$ define a (trivial) Boolean algebra homomorphism

$$i \colon F\left(\bigsqcup_{n \in \omega} \Lambda_n \times U\overline{\mathcal{L}(\Lambda)^{\mathrm{ar}(\lambda)}}\right) \longrightarrow \overline{\mathcal{L}(\Lambda)}$$

by sending $(\lambda, \varphi)$ to the formula $[\lambda](\varphi)$ for any $n$-ary predicate lifting $\lambda$ and $\varphi = (\varphi_i)_{i \in n}$. The existence of the unique map from $(\overline{\mathcal{L}(\Lambda)}, i)$ to any $L$-algebra $(A, \alpha)$ is shown inductively with the basic step: $\bot$ and $\top$ must be mapped to the bottom element and top element in $A$ respectively. Notably, the semantic map $[\![-]\!] \colon \mathcal{L}(\Lambda) \to 2^X$ is indeed the unique homomorphism.

2. For a finitary functor $T$, the existence of an initial algebra for $FTU \colon \mathsf{BA} \to \mathsf{BA}$, i.e. the syntax for the logic of the cover modality, is shown similarly.

Similar to complex algebras of Kripke frames, any $T$-coalgebra can be turned into an $L$-algebra: Every abstract logic $(L, \delta \colon LP \dashrightarrow PT)$ defines a contravariant functor $\tilde{P}$ from the category of $T$-coalgebras to the category of $L$-algebras via

$$(X \xrightarrow{\xi} TX) \mapsto (LPX \xrightarrow{\delta_X} PTX \xrightarrow{\xi^{-1}} PX) \quad \text{and} \quad f \mapsto f^{-1}$$

for any $T$-coalgebra $(X, \xi)$ and coalgebra homomorphism $f$.

**Definition 2.3.27.** Assume that a language $(\Phi, i)$ for a given abstract logic $(L, \delta)$ for $T$ exists. For any $T$-coalgebra $(X, \xi)$, we say that an element $x \in X$ **satisfies** a formula $\varphi \in \Phi$, if $x \in [\![\varphi]\!]_{(X, \xi)}$ where $[\![-]\!]_{(X, \xi)} \colon \Phi \to PX$ is the unique $L$-algebra homomorphism from the language $(\Phi, i)$ to the complex algebra $\tilde{P}(X, \xi)$. As a formula,

$$(X, \xi); \; x \models \varphi \quad \text{if and only if} \quad x \in [\![\varphi]\!]_{(X, \xi)}. \tag{2.32}$$

For any abstract logic induced by a set $\Lambda$ of predicate liftings and $\lambda \in \Lambda$, the semantics of a formula $\lambda(\varphi)$ for some $\varphi \in \Phi$ is calculated via the following diagram

$$
\begin{array}{ccc}
L\Phi & \xrightarrow{\quad i \quad} & \Phi \\
{\scriptstyle L[\![-]\!]}\Big\downarrow & & \Big\downarrow{\scriptstyle [\![-]\!]} \\
PX & \xrightarrow[\delta_X]{} PTX \xrightarrow[\xi^{-1}]{} & PX
\end{array}
$$

with $i^{-1}$ curving over the top from $\Phi$ back to $L\Phi$.

such that we obtain the following interpretation of $\lambda(\varphi)$:

$$
\begin{aligned}
[\![\lambda(\varphi)]\!] &= (\xi^{-1} \circ \delta_X \circ L[\![-]\!] \circ i^{-1})(\lambda(\varphi)) \\
&= (\xi^{-1} \circ \delta_X \circ L[\![-]\!])(\lambda, \varphi) \\
&= (\xi^{-1} \circ \delta_X)(\lambda, [\![\varphi]\!]) \\
&= (\xi^{-1} \circ \lambda_X)([\![\varphi]\!]) = \xi^{-1}(\lambda_X[\![\varphi]\!])
\end{aligned}
$$

which coincides with the definition given in page 35.

The following theorem generalises Theorem 2.3.8 and Proposition 2.3.19:

**Theorem 2.3.28.** *Let* $(L, \delta)$ *be an abstract logic for* $T$ *such that an* $L$*-initial algebra exists. Then,* $(L, \delta)$ *is adequate.*

*Proof.* Without loss of generality, it suffices to show that for any coalgebra homomorphism $f : (X, \xi) \to (Y, \gamma)$, every $x$ is logically equivalent to its image $fx$.

For brevity, let $[\![-]\!] = [\![-]\!]_{(X,\xi)}$ and $[\![-]\!]' = [\![-]\!]_{(Y,\gamma)}$ respectively. The following diagram

$$
\begin{array}{ccc}
(\Phi, i) & \xrightarrow{\ [\![-]\!]\ } & \tilde{P}(X, \xi) \\
& {\scriptstyle [\![-]\!]'}\searrow & \Big\uparrow{\scriptstyle f^{-1}} \\
& & \tilde{P}(Y, \xi)
\end{array}
$$

commutes by the initiality of $(\Phi, i)$, so $x \in [\![\varphi]\!]$ if and only if $fx \in [\![\varphi]\!]'$ for every $\varphi \in \Phi$. $\qquad\square$

**Expressiveness**

To study expressiveness, it is more convenient to rephrase the statement in terms of morphisms in Set instead of Boolean algebra homomorphisms in Theorem 2.3.28, as

discussed in [63] and formulated using a duality with a factorisation system in [53]. It is possible by using the well-known *Stone duality*: every Boolean algebra $A$ corresponds to the set $\mathsf{Uf}A$ of ultrafilters over $A$, i.e. Boolean algebra homomorphisms from $A$ to the two-element algebra $\{\bot, \top\}$, and each Boolean algebra homomorphism $f$ gives rise to a map between the sets of ultrafilters, defined by precomposition, i.e.

$$(x\colon B \to 2) \mapsto (x \circ f\colon A \to B \to 2)$$

for any $x \in \mathsf{Uf}A$. The resulting contravariant functor is denoted by $S$. Then the contravariant powerset algebra functor $\mathcal{Q}$ with $S$ form a dual adjunction (on the right) $\mathsf{Set}(X, SA) \cong \mathsf{BA}(A, \mathcal{Q}X)$ natural in $X$ and $A$ defined via currying

$$X \to (A \to 2) \cong X \times A \to 2 \cong A \to (X \to 2).$$

Given an abstract logic $(L, \delta)$, the interpretation $\delta\colon L\mathcal{Q} \dashrightarrow \mathcal{Q}T$ can be translated into a natural transformation

$$\delta^*\colon TS \xLongrightarrow{\delta^\flat S} SL\mathcal{Q}S \xLongrightarrow{SL\eta} SL \tag{2.33}$$

called the **mate** of $\delta$ where $\delta^\flat$ is the transpose of $\delta$ by Stone duality and $\eta\colon \mathcal{I} \to PS$ is the unit defined by $a \mapsto \{f\colon A \to 2 \mid f(a) = \top\} \subseteq SA$.

**Theorem 2.3.29** (see [53, 63]). *Let $(L, \delta)$ be an abstract logic for $T\colon \mathsf{Set} \to \mathsf{Set}$ for which the initial $L$-algebra exists. If the mate $\delta^*_\Phi$ on the language $\Phi$ is injective, then the logic $(L, \delta)$ is expressive.*

*Proof.* We will show this in Chapter 5 in a more general setting. □

The correspondence (2.33) is called *mate correspondence*, see Section 5.1.4 for details.

## 2.4 Coalgebras beyond Set

A number of coalgebras over the category Pos of posets with order-preserving functions or related categories have been studied. For example, a real number representation as a final coalgebra of some endofunctor on Pos appears in [91]; Kripke frames for positive modal logic, a negation-free modal logic, are presented as coalgebras of the convex powerset functor on Pos in [24, 86]; a coalgebraic view on the $\pi$-calculus is

given in [29], inspired from [2], which is also an instance of coalgebras over a category of domains.

Further, coalgebras over measurable spaces or metric spaces with distribution or valuation functors are also considered in the literature, e.g. [33, 41, 84, 85, 101, 106].

Coalgebras over topological spaces are also of interest due to the long-standing connection between logic and topology, i.e. Stone duality [103] and Jónsson-Tarski duality [57]. A coalgebraic view on the dual representation of modal algebras is discussed in [68] as coalgebras over Stone spaces, the corresponding ultrafilter extension in [66] and the corresponding notion of bisimulation in [23]. Descriptive Kripke frames, coalgebras for the Vietoris functor on Stone spaces, are generalised and discussed in [22].

For example, compactly branching transition systems can be formulated as coalgebras on Top:

**Definition 2.4.1.** Let $X$ be a topological space. The Vietoris space $\mathbb{V}X$ consists of the collection $\mathcal{K}X$ of compact subsets of $X$ with a topology generated by the following subsets as subbasis:

$$\Box(U) := \{ K \in \mathcal{K}X \mid K \subseteq U \} \quad \text{and} \quad \Diamond(U) := \{ K \in \mathcal{K}X \mid K \cap U \neq \emptyset \}$$

for every open subset $U \subseteq X$. Define the Vietoris (space) functor $\mathbb{V}$ by mapping a space $X$ to $\mathbb{V}X$ and a continuous function $f$ to a continuous function $\mathbb{V}f$ defined by mapping $K$ to $\{ fx \mid x \in K \}$. The functoriality follows from the compactness-preservation.

It is apparent from the definition that every set $\xi(x)$ of successors in a Vietoris coalgebra $\langle x, \xi \rangle$ is compact; Vietoris coalgebras with discrete carrier are exactly coalgebras of the finitary powerset functor.

## 2.5  **Coalgebraic Logic beyond** Set

Compared to the rich collection of coalgebras beyond Set, there are relatively few systematic attempts to define coalgebraic logic in a generic way, i.e. coalgebraic logic for (almost) arbitrary functors.

A notable study of coalgebraic logic over locally finitely presentable categories, where every object is a filtered colimit of 'finite' objects, is given by Klin [63]. In *op. cit.*, the

type of coalgebras is assumed to be finitary and monomorphism-preserving. Following Klin's work, Jacobs studies the expressiveness property for a number of instances in [53].

Similar assumptions are also proposed in the study of coalgebraic logic for Pos coalgebras via order-enriched predicate liftings in [58] where the preservation of injections becomes the preservation of embeddings.[10]

A comprehensive study of coalgebraic logic for coalgebras over measurable spaces via predicate liftings is studied by Doberkat [38–40] with a survey paper comparing them with Set predicate liftings [41].

Some generic approaches to coalgebraic logic for coalgebras over spaces are discussed under dualities between certain spaces and complete lattices with the infinite distributive law, e.g. [30, 71], and coalgebraic logic for Vietoris polynomial functors on Stone spaces is discussed in [68].

---

[10]An **embedding** function between posets is an order-reflecting function, i.e. $f(x) \leq f(y)$ if and only if $x \leq y$.

# Chapter 3

# Categorical Preliminaries

Category theory and its terminologies covered by the textbook by Mac Lane [81] is assumed in the following chapters. To lay down the necessary foundations for the main framework, this chapter serves a gentle walk-through of useful techniques in [6, 10, 31, 61]. In addition, we also aim to clarify assumptions and conventions common in the study of coalgebras and coalgebraic logics, e.g. set-based, the preservation of inclusions, and the definition of finitary Set functor.

The foundation of category theory does not affect the following discussion. The distinction between *set* and *class* is left informal, except Section 3.5. Thus, we only need to know that a category is **small** if the collection of all morphisms is a set; a category is **locally small** if the collection of all morphisms *between each pair of objects* is a set. On the other hand, the collection of all objects might be a *proper class*, e.g. Set.

## 3.1   Factorisation Systems

**Definition 3.1.1.** Given a category $\mathscr{C}$ and morphisms $f, g$, we say that $f$ **is orthogonal to** $g$ and it is written as $f \perp g$ if for any commutative diagram



there is a unique morphism filling the diagonal.

Given a collection $\mathcal{M}$ of morphisms, $^\perp\mathcal{M}$ denotes the collection of morphisms defined by

$$\{e \mid \forall m \in \mathcal{M} \, . \, e \perp m\},$$

and similarly $\mathcal{E}^\perp$, for some collection $\mathcal{E}$ of morphisms, is the collection of morphisms $m$ satisfying $e \perp m$ for every $e \in \mathcal{E}$. Further, write $\mathcal{E} \perp \mathcal{M}$ if $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

**Definition 3.1.2.** Given two collections of morphisms $\mathcal{E}$ and $\mathcal{M}$ of some category $\mathscr{C}$, we say that $(\mathcal{E}, \mathcal{M})$ is a **factorisation system** if *a)* every morphism $f \in \mathscr{C}$ has an $(\mathcal{E}, \mathcal{M})$-factorisation; *b)* $\mathcal{E}$ and $\mathcal{M}$ contain isomorphisms and are closed under composition; *c)* $\mathcal{M} \perp \mathcal{E}$.

**Example 3.1.3.** Let $\mathscr{C}$ be any category in the following examples:

1. The collection $\mathrm{Mor}\mathscr{C}$ of morphisms in $\mathscr{C}$ and the collection $\mathrm{Iso}\mathscr{C}$ of isomorphisms give two factorisation systems: $(\mathrm{Mor}\mathscr{C}, \mathrm{Iso}\mathscr{C})$ and $(\mathrm{Iso}\mathscr{C}, \mathrm{Mor}\mathscr{C})$.

2. The category of sets has the $(\mathsf{Surjection}, \mathsf{Injection})$-factorisation system as shown in Proposition 2.1.21.

3. Every regular category (see Definition 4.3.38), such as a variety of algebras, has a $(\mathsf{RegEpi}, \mathsf{Mono})$-factorisation system, see Theorem 4.3.39.

**Proposition 3.1.4** (see [6, Section 14]). *Given a factorisation system $(\mathcal{E}, \mathcal{M})$ on a category $\mathscr{C}$, the following statements hold:*

1. *$\mathcal{E}$ and $\mathcal{M}$ determine each other: $\mathcal{E} = {}^\perp\mathcal{M}$ and $\mathcal{M} = \mathcal{E}^\perp$.*

2. *Every $\mathcal{E}$-morphism (resp. $\mathcal{M}$-morphisms) is preserved by pushouts (resp. pullbacks).*

3. *The $\mathcal{M}$-class (resp. $\mathcal{E}$-class) is closed under limits (resp. colimits) in the arrow category $\mathcal{C}^\rightarrow$.*

4. *The left cancellation law holds, i.e.*

$$f \circ g \in \mathcal{M} \quad and \quad f \in \mathcal{M} \quad implies \quad g \in \mathcal{M}.$$

5. *Dually, the right cancellation law holds, i.e.*

$$f \circ g \in \mathcal{E} \quad and \quad g \in \mathcal{E} \quad implies \quad f \in \mathcal{E}.$$

Given an $(\mathcal{E}, \mathcal{M})$-factorisation system, we always use $\rightarrowtail$ and $\twoheadrightarrow$ to denote an $\mathcal{M}$-morphism and an $\mathcal{E}$-morphism, respectively.

### 3.1.1 Quotients and Right Factorisation Systems

A factorisation system is called **right** if every $\mathcal{E}$-morphism is epic. This kind of factorisation system generalises the notion of quotient in a natural way:

**Definition 3.1.5.** Given an $(\mathcal{E}, \mathcal{M})$-factorisation system on a category $\mathscr{C}$ where every $\mathcal{E}$-morphism is epic, an $\mathcal{E}$-**quotient** of an object $c$ is an isomorphism class of $\mathcal{E}$-morphisms

$$c \twoheadrightarrow d.$$

For every object $c \in \mathscr{C}$ we define a preorder on $\mathcal{E}$-morphisms out of $c$

$$(c \xrightarrow{e_1} d_1) \le (c \xrightarrow{e_2} d_2)$$

if there exists a morphism $f$ from $d_1$ to $d_2$ such that $f e_1 = e_2$.

Note that there is at most one such morphism from $d_1$ to $d_2$ because $e_1$ is epic, so the preorder is lifted to a partial order on the collection of $\mathcal{E}$-quotients. We may call an $\mathcal{E}$-morphism from $c$ as an $\mathcal{E}$-quotient of $c$ if it only matters up to isomorphism.

Every right $(\mathcal{E}, \mathcal{M})$-factorisation system reveals about the class $\mathcal{M}$ as well: Recall that an **extremal monomorphism** is a monomorphism $m$ satisfying that for every factorisation $m = g \circ e$, the morphism $e$ is epic if and only if $e$ is invertible.

**Proposition 3.1.6** (see [10, Proposition 14.10]). *For every right $(\mathcal{E}, \mathcal{M})$-factorisation system on a category $\mathscr{C}$, every extremal monomorphism in $\mathscr{C}$ is an $\mathcal{M}$-morphism.*

Also, a **strong monomorphism** is a monomorphism contained in $\mathsf{Epi}\mathscr{C}^{\perp}$. Every strong monomorphism is extremal.[1] Conversely, every extremal monomorphism is strong if the category has pushouts.[2]

The extremal condition (*resp.* and the strong condition) provide a general factorisation system for every category with enough limits (*resp.* and pushouts):

**Theorem 3.1.7** (see [6, Theorem 14.19]). *Every category with equalisers and intersections, i.e. the pullback of an arbitrary collection of subobjects, has the* $(\mathsf{Epi}, \mathsf{ExtrMono})$-*factorisation system.*

---

[1] For any factorisation $m = ge$ where $e$ is an epimorphism, apply the diagonalisation property to obtain a morphism $s$ such that $g = mh$ and $id = he$. Since $e$ is epic, we have $id \circ e = ehe$ and thus $id = eh$.

[2] For any extremal monomorphism $m$ and a commutative square $v \circ e = m \circ h$ for some epimorphism $e$, the pushout $e'$ of $e$ along $h$ is epic, so by the universal property $f$ factors through $e'$. By the extremal condition, $e'$ is invertible, so some morphism fills the commutative square. The uniqueness is simple to check.

### 3.1.2 The Reflective Subcategory determined by a Factorisation System

It is well-known that [35] every factorisation system on a category $\mathscr{C}$ with a terminal object determines a reflective subcategory $\mathscr{A}$ consisting of '$\mathcal{M}$-subobjects' of the terminal object:

**Proposition 3.1.8** (see [31, Proposition 5.5.5]). *Every $(\mathcal{E}, \mathcal{M})$-factorisation system on a category $\mathcal{C}$ with a terminal object induces an $\mathcal{E}$-reflective subcategory $\nabla \dashv i : \mathscr{A} \hookrightarrow \mathscr{C}$ such that*

1. *for every object $c \in \mathscr{C}$, the reflection of $c$ is the $\mathcal{E}$-part*

$$!_c : c \twoheadrightarrow \nabla c \rightarrowtail \mathbf{1};$$

   *of the unique morphism to the terminal object;*

2. *every morphism $f \in \mathcal{E}$ is inverted by $\nabla$, i.e. $\nabla f$ is an isomorphism.*

However, considering applications in the category of coalgebras, the existence of a terminal object is a very strong assumption as it states the existence of final coalgebra against the leading example—coalgebras of the powerset functor. First we observe that:

**Lemma 3.1.9.** *Given a category $\mathscr{C}$ with a terminal object $1 \in \mathscr{C}$ and a right $(\mathcal{E}, \mathcal{M})$-factorisation system of $\mathscr{C}$, for every object $c \in \mathscr{C}$ the factorisation of the unique morphism*

$$!_c : c \twoheadrightarrow !c \rightarrowtail \mathbf{1}$$

*consists of a greatest $\mathcal{E}$-quotient of $c$ and an $\mathcal{M}$-subobject of $\mathbf{1}$.*

*Proof.* We only need to show that $e$ is the greatest $\mathcal{E}$-quotient: for any $\mathcal{E}$-morphism from $c$ to some object $d$ the following diagram always commutes

$$
\begin{array}{ccc}
c & \twoheadrightarrow & d \\
{\scriptstyle e}\downarrow & & \downarrow \\
!c & \rightarrowtail & \mathbf{1}
\end{array}
$$

since $\mathbf{1}$ is the terminal object. It follows that there is a unique morphism from $d$ to $!c$ by the diagonalisation property. $\qquad\square$

**Proposition 3.1.10** (*cf.* [31, Proposition 5.5.5]). *Every right $(\mathcal{E}, \mathcal{M})$-factorisation system on a category $\mathcal{C}$ with pushouts and a greatest $\mathcal{E}$-quotient object for each object in $\mathcal{C}$ induces an $\mathcal{E}$-reflective subcategory $\nabla \dashv i \colon \mathcal{A} \hookrightarrow \mathcal{C}$ such that*

1. *the reflective subcategory consists of objects without proper $\mathcal{E}$-quotient;*

2. *for every object $c \in \mathcal{C}$, the reflection is an $\mathcal{E}$-greatest quotient*

$$e_c \colon c \twoheadrightarrow \nabla c;$$

3. *every morphism $f \in \mathcal{E}$ is inverted by $\nabla$, i.e. $\nabla f$ is an isomorphism.*

*Proof.* Let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ consisting of objects without proper $\mathcal{E}$-quotients, i.e. every morphism from an object in $\mathcal{A}$ is an $\mathcal{M}$-morphism.

Every object $c \in \mathcal{C}$ has the greatest $\mathcal{E}$-quotient $\nabla c$. By the Axiom of Choice, there is always a representative $c \twoheadrightarrow \nabla c$ for each isomorphism class and it is easy to see that $\nabla c$ is in $\mathcal{A}$.

To see that the greatest $\mathcal{E}$-quotient is a reflection, consider the following diagram for every morphism $f \colon c \to a$ to some $a \in \mathcal{A}$:

$$
\begin{array}{ccc}
c & \xrightarrow{\;\;e_c\;\;} & \nabla c \\
{\scriptstyle f}\downarrow & {\scriptstyle h}\;\;\nearrow & \downarrow{\scriptstyle \iota_c} \\
a & \xrightarrowtail{\;\;\iota_a\;\;} & a +_c \nabla c
\end{array}
\tag{3.1}
$$

where $(\iota_c, \iota_a)$ is a pushout of $(e_c, f)$. By assumption, $\iota_a$ is an $\mathcal{M}$-morphism, so there exists a unique morphism $h$ filling the diagonal.[3] To see that $h$ is indeed the unique morphism satisfying $h \circ e_c = f$, we show that the commutativity of the upper triangle also implies the lower triangle. Given a morphism $g \colon \nabla c \to a$ satisfying $g \circ e_c = f$, we have $\iota_a \circ g \circ e_c = \iota_c \circ e_c$ by diagram chasing, so, by $e$ being epic, it implies that $\iota_a \circ g = \iota_c$. That is, the lower triangle commutes.

---

[3] The morphism $h$ in (3.1) can be deduced directly as follows. Since $\mathcal{E}$-morphisms are preserved by pushouts, $\iota_a$ is an $\mathcal{E}$-morphism. By assumption $\iota_a$ is an $\mathcal{M}$-morphism, so $\iota_a$ is an isomorphism. It follows that $h$ is equal to the $\iota_a^{-1} \circ \iota_c$ by commutativity.

To show the last statement, let $f\colon c \to d$ be an $\mathcal{E}$-morphism and $\nabla f$ the unique morphism satisfying $f = \nabla f \circ e_c$ in the diagram:

$$
\begin{array}{ccccc}
c & \xrightarrow{\ e_c\ } & \nabla c & & \\
{\scriptstyle f}\Big\downarrow & {\scriptstyle e_d \circ f}\ \searrow & \Big\downarrow {\scriptstyle \nabla f} & \searrow {\scriptstyle \iota_{\nabla c}} & \\
d & \xrightarrow[\ e_d\ ]{} & \nabla d & \xrightarrow[\ \iota_{\nabla d}\ ]{} & \nabla c +_c \nabla d.
\end{array}
$$

Applying the right (*resp.* left) cancellation law for $\mathcal{E}$-morphisms (*resp.* $\mathcal{M}$-morphisms) to $e_d \circ f$ (*resp.* $\iota_{\nabla c}$), the morphism $\nabla f$ is an $\mathcal{E}$-morphism (*resp.* $\mathcal{M}$-morphism). Since $\nabla f$ is at the same time an $\mathcal{E}$-morphism and an $\mathcal{M}$-morphism, it is an isomorphism. $\qquad\square$

### 3.1.3 The Reflective Thin Subcategory by a Proper Factorisation System

Given a right $(\mathcal{E}, \mathcal{M})$-factorisation system, the $\mathcal{M}$-morphisms are often monomorphisms, e.g. in the standard factorisation system for Set; and we call this kind of factorisation system **proper**.

**Lemma 3.1.11.** *Let $(\mathcal{E}, \mathcal{M})$ be a proper factorisation system on a category $\mathscr{C}$ with coequalisers.*

1. *For any greatest $\mathcal{E}$-quotient $c$ and an object $d$, there is at most one morphism from $d$ to $c$.*

2. *In particular, the full subcategory of greatest $\mathcal{E}$-quotients is thin, i.e. for every pair of objects $c$ and $d$ there exists at most one morphism from $c$ to $d$.*

*Proof.* For any two morphisms $f, g$ to a greatest $\mathcal{E}$-quotient $c$ from some object $d$, consider the coequaliser of $f$ and $g$:

$$
d \overset{f}{\underset{g}{\rightrightarrows}} c \overset{h}{\rightarrowtail} e
$$

where $h$ must be an $\mathcal{M}$-morphism, so monic. Thus, $hf = hg$ implies $f = g$. $\qquad\square$

**Proposition 3.1.12.** *Let $\mathscr{C}$ be a cocomplete category with a proper $(\mathcal{E}, \mathcal{M})$-factorisation system such that $\mathscr{C}$ is $\mathcal{M}$-wellpowered. Then, a terminal object exists in $\mathscr{C}$ if and only if there is a small thin reflective subcategory $\nabla \dashv i\colon \mathscr{A} \hookrightarrow \mathscr{C}$ satisfying that*

1. *every object in $\mathscr{A}$ is a greatest $\mathcal{E}$-quotient; and*

2. *for every object $c \in \mathscr{C}$, the reflection is a greatest quotient.*

*Proof.* Proposition 3.1.8 shows that there is a reflective subcategory $\mathscr{A}$ of $\mathscr{C}$ satisfying the above conditions. Every greatest $\mathcal{E}$-quotient is an $\mathcal{M}$-subobject of the terminal object by finality and the fact that every greatest $\mathcal{E}$-quotient has no proper quotient. By the Axiom of Choice, the skeleton of $\mathscr{A}$ exists and by well-poweredness it is small.

Conversely, suppose that there is a small reflective subcategory $\mathscr{A}$ consisting of greatest $\mathcal{E}$-quotients. By assumption, the coproduct of every object in $\mathscr{A}$ exists in $\mathscr{C}$, and each object $c \in \mathscr{C}$ has at least a morphism to the coproduct by the reflection of $c$. Also, the coproduct has a greatest $\mathcal{E}$-quotient by reflection and thus every object $c \in \mathscr{C}$ has a morphism to the greatest $\mathcal{E}$-quotient. By Lemma 3.1.11, every morphism to a $\mathcal{E}$-greatest quotient is unique:

$$c \twoheadrightarrow \nabla c \xrightarrow{\iota_{\nabla c}} \coprod_{a \in \mathscr{A}} a \twoheadrightarrow \nabla \coprod_{a \in \mathscr{A}} a,$$

so it follows. $\qquad\square$

Note that $\mathcal{A}$ is cocomplete, small and thin, i.e. a *complete lattice.*

## 3.2 Canonical Limits

### 3.2.1 End

Many (co-)limits appear in a canonical way, and they are more conveniently computed by *(co-)ends* using a generalised natural transformation:

**Definition 3.2.1.** Given categories $\mathscr{C}$ and $\mathscr{D}$, functors $S$ and $T$ from the product category $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to $\mathscr{D}$, a **diagonally natural transformation** (**dinatural transformation** for short) is a family of morphisms $S(c, c) \to T(c, c)$ for each $c \in \mathscr{C}$ such

that for any morphism $f \colon c \to c'$ the diagram

$$
\begin{array}{ccc}
 & S(c,c) \xrightarrow{\ \tau_c\ } T(c,c) & \\
S(f,c) \nearrow & & \searrow T(c,f) \\
S(c',c) & & T(c,c') \\
S(c',f) \searrow & & \nearrow T(f,c') \\
 & S(c',c') \xrightarrow[\tau_{c'}]{} T(c',c') &
\end{array}
$$

commutes. If either functor $S$ or $T$ is a constant, say $d \in \mathscr{D}$, we call a dinatural transformation from $S$ to $T$ a **wedge**, i.e. a family of morphisms either $\{\tau_c \colon d \to T(c,c)\}_{c \in \mathscr{C}}$ or $\{\tau_c \colon T(c,c) \to d\}$ with commutative diagrams

$$
\begin{array}{ccc}
d & \xrightarrow{\ \tau_c\ } & T(c,c) \\
\tau_{c'} \downarrow & & \downarrow T(c,f) \\
T(c',c') & \xrightarrow[T(f,c')]{} & T(c,c')
\end{array}
\qquad \text{or} \qquad
\begin{array}{ccc}
S(c',c) & \xrightarrow{\ S(f,c)\ } & S(c,c) \\
S(c',f) \downarrow & & \downarrow \tau_c \\
S(c',c') & \xrightarrow[\tau_{c'}]{} & d
\end{array}
$$

respectively for each $f \colon c \to c'$.

**Example 3.2.2** (Every natural transformation is a dinatural transformation). Let $Q \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ be the second projection functor, i.e. $Q(c',c) = c$ and $Q(f,g) = g$. Every natural transformation $\sigma$ from $S \colon \mathscr{C} \to \mathscr{D}$ to $T \colon \mathscr{C} \to \mathscr{D}$ is a dinatural transformation from $SQ$ to $TQ$: for any $f \colon c \to c'$

$$
\begin{array}{ccc}
 & Sc \xrightarrow{\ \sigma_c\ } Tc & \\
id \nearrow & & \searrow Tf \\
Sc & & Tc' \\
Sf \searrow & & \nearrow id \\
 & Sc' \xrightarrow[\tau_{c'}]{} Tc' &
\end{array}
$$

which is equal to the naturality diagram of $\sigma$.

**Definition 3.2.3** (see [81, Definition IX.5]). Given categories $\mathscr{C}$ and $\mathscr{D}$ and a functor $S \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$, an **end** consists of an object $e \in \mathscr{D}$ and a *universal* wedge $\tau$ from $e$ to the functor $S$, i.e. a wedge such that for any wedge $\sigma$ to $S$, there exists a unique

morphism $h$ from $d$ to $e$ with the commutative diagram



for any $f\colon c \to c'$. A **coend** is an end in the opposite category.

By abuse of notation, an end of $S\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$ is written as an object

$$\int_c S(c,c) \quad \text{or} \quad \int_{\mathscr{C}} S$$

in $\mathscr{D}$ without writing down the universal wedge; similarly a coend of $S\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$ is written as $\int^c S(c,c)$ or $\int^{\mathscr{C}} S$.

**Example 3.2.4** (The collection of natural transformations is an end). Let $U, V\colon \mathscr{C} \to \mathscr{D}$ be functors. A natural transformation from $U$ to $V$ consists of morphisms from $Uc$ to $Vc$ for each object $c$, and the association itself is a map from the collection of natural transformations $\mathsf{Nat}(U,V)$ to $\mathsf{Hom}(Uc, Vc)$, denoted $(-)_c$, for each $c$. Using the association $(-)_c$, the naturality can be written as, for $f\colon c \to c'$,

$$
\begin{array}{ccc}
\mathsf{Hom}(Uc', Vc') & \xrightarrow{\;-\circ Uf\;} & \mathsf{Hom}(Uc, Vc') \\
{\scriptstyle (-)_{c'}}\big\uparrow & & \big\uparrow{\scriptstyle Vf\circ -} \\
\mathsf{Nat}(U, V) & \xrightarrow[\;(-)_c\;]{} & \mathsf{Hom}(Uc, Vc)
\end{array}
$$

which exhibits a wedge from $\mathsf{Nat}(U,V)$ to the functor $\mathsf{Hom}(U-_1, V-_2)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$. Every wedge from some set $X$ to $\mathsf{Hom}(U-_1, V-_2)$ defines a natural transformation for every element in $X$, so every element is indeed a natural transformation and a unique function, the inclusion, exists. To sum up, the collection of natural transformations from $U$ to $V$ is an end of the hom-functor $\mathsf{Hom}(U-, V-)$:

$$\mathsf{Nat}(U, V) = \int_c \mathsf{Hom}(Uc, Vc).$$

**Example 3.2.5.** Let $Q\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ be the second projection functor and $S$ a functor from $\mathscr{C}$ to $\mathscr{D}$. A wedge $\langle \sigma_c\colon e \to Sc \rangle$ is a limit for $S$ if and only if $\langle \sigma_c\colon e \to SQ(c,c) = Sc \rangle$ is an end for $SQ$. See [81, Proposition IX.5.3] for the details.

The computation of an end can be reduced to products and equalisers:

**Theorem 3.2.6.** *For categories $\mathscr{C}, \mathscr{D}$ and a functor $S \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$, an end of $S$ exists if the following products*

$$\prod_{c \in \mathscr{C}} S(c,c) \overrightarrow{\phantom{\longrightarrow}} \prod_{f \colon c \to c'} S(c,c') \tag{3.2}$$

*and an equaliser of the parallel morphisms, induced by $S(f,c')$ and $S(c,f)$ for $c,c' \in \mathscr{C}$ and $f \colon c \to c'$, exist.*

In detail, one of the parallel morphisms is given as follows. For any $c \in \mathscr{C}$ and any morphism $f$ from $c$ to some $c'$, there exists a morphism $S(c,f) \colon S(c,c) \to S(c,c')$. By the universal property of product, there is a mediating morphism

$$(S(c,f))_{f \colon c \to c'} \colon S(c,c) \to \prod_{f \colon c \to c'} S(c,c')$$

where the index of the product is the collection of morphisms from $c$. It induces a morphism

$$\prod_{c \in \mathscr{C}} S(c,c) \to \prod_{c \in \mathscr{C}} \prod_{f \colon c \to c'} S(c,c')$$

where the double product is isomorphic to $\prod_{f \colon c \to c'} S(c,c')$.

*Proof.* Suppose the above products exist. Any equaliser $(e, h \colon e \to \prod_c S(c,c))$ of the morphisms induced by $S(c,f)$ and $S(f,c')$, for $c,c' \in \mathscr{C}$ and $f \colon c \to c'$, defines a family of morphisms from $e$ to $S$ by composing with the projections.

The dinaturality follows because $e$ equalises the parallel morphisms: The product $\prod_c \prod_f S(c,c')$ is $\prod_f S(c,c')$. For each $f \colon c \to c'$ the diagram (3.2) gives the following commutative diagram

where $i$ and $j$ denote morphisms induced by $S(c, f)$ and $S(c', f)$ respectively. The commutativity of the above diagram is exactly the dinaturality. The universal property of the equaliser gives the universal property of the end, so it follows. $\qquad\square$

As corollaries, every end of a small diagram in a complete category $\mathscr{C}$ exists; every limit is constructed by an equaliser of products by Example 3.2.5. As an application, we demonstrate another use of the coend formula to simplify complicated arguments:

**Proposition 3.2.7** (see [81, Section III.7]). *Every* $\mathsf{Set}$-*valued functor* $F\colon \mathscr{C} \to \mathsf{Set}$ *from a small category* $\mathscr{C}$ *is a colimit of representable functors.*

*Proof.* The following isomorphisms are natural in $G$

$$[\mathscr{C}, \mathsf{Set}](F, G)$$
$$\cong \int_c \mathscr{C}(Fc, Gc) \qquad\qquad \{\text{by Example 3.2.4}\}$$
$$\cong \int_c \mathscr{C}(Fc, [\mathscr{C}, \mathsf{Set}](\mathscr{C}(c, -), G)) \quad \{\text{by the Yoneda Lemma}\}$$
$$\cong \int_c [\mathscr{C}, \mathsf{Set}](Fc \cdot \mathscr{C}(c, -), G) \qquad \{\text{by the universal property of copower}\}$$
$$\cong [\mathscr{C}, \mathsf{Set}](\int^c Fc \cdot \mathscr{C}(c, -), G), \quad \{\text{every representable functor maps coends to ends}\}$$

so by the Yoneda Lemma, $F$ is naturally isomorphic to the coend of $Fc \cdot \mathscr{C}(c, -)$. Then, it follows from Theorem 3.2.6. $\qquad\square$

### 3.2.2 Kan Extension

All concepts in category theory are subsumed by the notion of *Kan extension* as indicated by Mac Lane [81, Section X.7]. This notion also characterises finitary functors elegantly and naturally as well as other functors we will use later.

**Definition 3.2.8** (see [81, Definition X.3]). Given functors

$$\mathscr{M} \xrightarrow{F} \mathscr{A}$$
$$K \downarrow$$
$$\mathscr{C}$$

a **left Kan extension** of $F$ along $K$ consists of a functor $L = \mathrm{Lan}_K F\colon \mathscr{C} \to \mathscr{A}$ with a bijection

$$[\mathscr{C}, \mathscr{A}](\mathrm{Lan}_K F, S) \overset{\rho_S}{\cong} [\mathscr{M}, \mathscr{A}](F, SK)$$

natural in $S$. Likewise, a **right Kan extension** of $F$ along $K$ is a functor $R$ with a bijection $[\mathscr{M}, \mathscr{A}](SK, F) \cong [\mathscr{C}, \mathscr{A}](S, R)$ natural in $S$.

To spell out the definition in detail, there is a natural transformation $\eta$ from $F$ to $LK$ by mapping the identity on $L$ using $\rho$, and for any functor $S \colon \mathscr{C} \to \mathscr{A}$ with a natural transformation $\mu$ from $F$ to $SK$ there exists a unique natural transformation $\sigma$ from $L$ to $S$ satisfying

$$
\begin{array}{ccc}
F & \xrightarrow{\ \eta\ } & LK \\
& {\scriptstyle \mu} \searrow & \downarrow {\scriptstyle \sigma K} \\
& & SK
\end{array}
$$

which follows by tracing the identity of the following diagram

$$
\begin{array}{ccc}
[\mathscr{C}, \mathscr{A}](L, S) & \xrightarrow{\ \cong\ } & [\mathscr{M}, \mathscr{A}](F, SK) \\
{\scriptstyle \sigma \circ -} \uparrow & & \uparrow {\scriptstyle \sigma K \circ -} \\
[\mathscr{C}, \mathscr{A}](L, L) & \xrightarrow[\ \cong\ ]{} & [\mathscr{M}, \mathscr{A}](F, LK)
\end{array}
$$

and the natural transformation $\sigma$ is equal to $\rho^{-1}(\mu)$.

Fixing the functor $K$ in the above definition, if for every functor $F$ a left Kan extension of $F$ along $K$ exists, then the left Kan extension *is* a left adjoint to the precomposition functor of $K$ by the general adjunction criteria [81, Theorem IV.1.2].

The computation of a left Kan extension can be reduced to a colimit provided that there are enough colimits. This kind of Kan extension is called *pointwise*. However, we prefer to define it by a preservation property:

**Definition 3.2.9** (see [81, Definition X.5]). Given a left Kan extension $(L, \eta \colon F \dashrightarrow LK)$ of $F$ along $K$ and a functor $G \colon \mathscr{A} \to \mathscr{D}$, we say that $G$ **preserves** the left Kan extension $(L, \eta)$ if $(GL, G\eta)$ is a left Kan extension of $GF$ along $K$.

**Definition 3.2.10** (see [81, Definition X.5]). Given functors $\mathscr{A} \xleftarrow{F} \mathscr{M} \xrightarrow{K} \mathscr{C}$ where $\mathscr{A}$ is locally small, a left Kan extension $(L, \eta \colon F \dashrightarrow LK)$ of $F$ along $K$ is called **pointwise** if it is preserved by all representable functors $G \cong \mathscr{A}(-, a)$.

**Theorem 3.2.11.** *Let $\mathscr{A} \xleftarrow{F} \mathscr{M} \xrightarrow{K} \mathscr{C}$ be functors where $\mathscr{A}$ is locally small and $(K\downarrow c)$ denote the comma category from $K$ to $c$. The following statements hold:*

1. *(see [81, Corollary X.5.4]) $(L, \eta \colon F \dashrightarrow LK)$ is a pointwise left Kan extension of $F$ along $K$ if and only if for any $a \in \mathscr{A}$ and $c \in \mathscr{C}$, the function*

$$
\mathscr{A}(Lc, a) \longrightarrow [\mathscr{M}^{\mathrm{op}}, \mathsf{Set}](\mathscr{C}(K-, c), \mathscr{A}(F-, a)) \tag{3.3}
$$

*which maps any $(Lc \xrightarrow{g} a)$ to the natural transformation*

$$\mathscr{C}(Km, c) \xrightarrow{L_{Km,c}} \mathscr{A}(LKm, Lc) \xrightarrow{g \circ (-) \circ \eta_m} \mathscr{A}(Fm, a)$$

*for each component $m \in \mathscr{M}$ is a bijection.*[4]

2. *(see [81, Lemma X.5&Theorem X.3.1]) $(L, \eta \colon F \dashrightarrow LK)$ is a pointwise left Kan extension of $F$ along $K$ if $Lc$ is the colimit of*

$$(K \downarrow c) \xrightarrow{P^c} \mathscr{M} \xrightarrow{F} \mathscr{A} \tag{3.4}$$

*with a limiting cocone $\left( Fm \xrightarrow{Lf \circ \eta_m} Lc \right)_{f \colon Km \to c}$ for all objects $c \in \mathscr{C}$ where $P^c$ is the functor projecting a morphism $(Km \to c)$ to $m$. Conversely, if the colimit of (3.4) exists with a limiting cocone $\lambda$ for any $c$, then the left Kan extension of $F$ along $K$ is given by*

$$Lc := \operatorname{Colim} FP^c, \quad Lg \colon \operatorname{Colim} FP^c \to \operatorname{Colim} FP^d \quad and \quad \eta_m := \lambda_{id_{Km}}$$

*where $Lg$ is the unique morphism commuting with the limiting cocones.*

3. *(see [81, Exercise X.4.1].) Suppose that for any $m, m' \in \mathscr{M}$ and $c \in \mathscr{C}$ the copowers $\mathscr{C}(Km', c) \cdot Fm$ exist. Then, a pointwise left Kan extension of $F$ along $K$ exists if and only if the following coend*

$$\int^m \mathscr{C}(Km, c) \cdot Fm \tag{3.5}$$

*exists for every $c \in \mathscr{C}$ where $\mathscr{C}(K-, c) \cdot F-$ is a bifunctor from $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to $\mathscr{A}$.*

*Proof.* Each statement can be found in the cited theorems, so we only sketch the coend version. Using (3.3), it suffices to show there is an isomorphism

$$\mathscr{A}\left( \int^m \mathscr{C}(Km, c) \cdot Fm, a \right) \cong [\mathscr{M}^{\mathrm{op}}, \mathsf{Set}](\mathscr{C}(K-, c), \mathscr{A}(F-, a))$$

---

[4] For the interested reader, this shows that $Lc$ is a colimit of $F$ weighted by $\mathscr{C}(K-, c)$.

natural in $a$. This follows from a simple computation:

$$\mathscr{A}\left(\int^m \mathscr{C}(Km, c) \cdot Fm, a\right)$$

$$\cong \int_m \mathscr{A}(\mathscr{C}(Km, c) \cdot Fm, a) \qquad \{\text{Hom maps coends to ends}\}$$

$$\cong \int_m \mathsf{Set}(\mathscr{C}(Km, c), \mathscr{A}(Fm, a)) \qquad \{\text{by the universal property of copowers}\}$$

$$\cong [\mathscr{M}^{\mathrm{op}}, \mathsf{Set}](\mathscr{C}(K-, c), \mathscr{A}(F-, a)) \qquad \{\text{by Example 3.2.4}\}$$

$$\square$$

In (3.3), we use the so-called **canonical functor** (also known as *nerve functor, restricted Yoneda embedding*) $\widetilde{K}: \mathscr{C} \to [\mathscr{M}^{\mathrm{op}}, \mathsf{Set}]$ of $K$ which is defined by

$$\widetilde{K}c = \mathscr{C}(K-, c) \quad \text{and} \quad \widetilde{K}f = f \circ -: \widetilde{K}c \to \widetilde{K}d \tag{3.6}$$

for $f: c \to d$. In (3.4), the projection $P: (K{\downarrow}c) \to \mathscr{M}$ is regarded as a diagram and it is called the **canonical diagram of $c$ with respect to $K$**; whenever $K$ is an inclusion functor from a subcategory $\mathscr{C}'$ we call $P$ the **canonical diagram of $c$ in $\mathscr{C}'$**. Moreover, if an object $c$ with all morphisms $Km \to c$ is a colimit of the canonical diagram, then we call $c$ the **canonical colimit** with respect to $K$.

**Example 3.2.12.** The finitary powerset functor is a left Kan extension of the powerset functor:

$$\mathcal{P}_\omega = \mathrm{Lan}_J \mathcal{P}J \quad \text{with} \quad \eta_n = id_n: \mathcal{P}Jn = \mathcal{P}n \to \mathcal{P}_\omega J = \mathcal{P}n$$

where $J$ is the full inclusion from the category $\mathsf{Set}_\omega$ of finite sets to $\mathsf{Set}$. Since $\mathsf{Set}$ is cocomplete, the colimit of (3.4) always exists and it is not hard to see that the finitary powerset of $X$ is a colimit of

$$(J{\downarrow}X) \xrightarrow{P^c} \mathsf{Set}_\omega \xrightarrow{\mathcal{P} \circ J} \mathsf{Set}.$$

Whenever the colimits of $(K{\downarrow}c) \to \mathscr{M} \to \mathscr{A}$ exist for every $c$, the left Kan extension is always pointwise; however, the converse is not true, see [31, Exercise 3.9.7] for a counterexample; a Kan extension of $F$ along $K$ is not necessarily an *extension* of $F$ in the sense that $\mathrm{Lan}_K F \circ K$ is not necessarily isomorphic to $F$.

**Corollary 3.2.13** (see [81, Corollary X.3.2-4]). *Let $\mathscr{A} \xleftarrow{F} \mathscr{M} \xrightarrow{K} \mathscr{C}$ be functors. The following statements hold:*

1. *If $\mathscr{M}$ is small and $\mathscr{A}$ cocomplete, then the precomposition functor with $K$ has a left adjoint, i.e. the left Kan extension of $F$ along $K$ exists.*

2. *If the functor $K$ is full and faithful, then the natural transformation $\eta\colon F\xrightarrow{\cdot} LK$ of a left Kan extension $(L,\eta)$ is an isomorphism $F \cong LK$.*

3. *In particular, if $\mathscr{M}$ is a full, small subcategory of a cocomplete category $\mathscr{C}$ and $K$ is the full inclusion, then there exists a functor $L$ such that*

$$F = LK \quad \text{with} \quad id_F\colon F\xrightarrow{\cdot}F$$

*is a left Kan extension extending $F$.*

We close this part with cocompletion of any small category.

**Lemma 3.2.14.** *For every small category $\mathscr{M}$ and a functor $K\colon \mathscr{M} \to \mathscr{C}$ to a cocomplete category $\mathscr{C}$, the canonical functor $\widetilde{K}\colon \mathscr{C} \to [\mathscr{M}^{\mathrm{op}},\mathsf{Set}]$ is a right adjoint to the left Kan extension of $K$ along the Yoneda embedding $\mathcal{Y}$.*



*Proof.* For any functor $H\colon \mathscr{M}^{\mathrm{op}} \to \mathsf{Set}$, we have the following natural isomorphisms

$$
\begin{aligned}
&\mathscr{C}\Big(\big(\mathrm{Lan}_{\mathcal{Y}}K\big)H,d\Big) \\
&\cong \mathscr{C}\Big(\int^{c} Hc \cdot Kc, d\Big) && \{\text{see below}\} \\
&\cong \int_{c} \mathscr{C}(Hc \cdot Kc, d) && \{\mathrm{Hom}(-,d) \text{ preserves coends}\} \\
&\cong \int_{c} \mathsf{Set}(Hc, \mathscr{C}(Kc,d)) \\
&\cong [\mathscr{M}^{\mathrm{op}},\mathsf{Set}](H, \mathscr{C}(K-,d)) && \{\text{by example 3.2.4}\}
\end{aligned}
$$

where $\mathrm{Lan}_{\mathcal{Y}}KH \cong \int^{c} Hc \cdot Kc$ by the coend formula of the left Kan extension and the Yoneda Lemma as well as a few simple computations. $\qquad\square$

Every left adjoint preserves colimits and the category $[\mathscr{M}^{\mathrm{op}},\mathsf{Set}]$ of contravariant Set-valued functors is cocomplete, so $\mathrm{Lan}_{\mathcal{Y}}K$ is uniquely determined by representables as we know that every $H\colon \mathscr{M}^{\mathrm{op}} \to \mathsf{Set}$ is a colimit of representables by Proposition 3.2.7. Thus, the category $[\mathscr{M}^{\mathrm{op}},\mathsf{Set}]$ is the *free cocompletion* of $\mathscr{M}$.

## 3.3   Density and its Presentation

One example that uses the canonical diagram is the notion of density:

**Definition 3.3.1** (see [81, Definition X.6]). A functor $K\colon \mathscr{M} \to \mathscr{C}$ is **dense** if the identity functor $\mathcal{I}$ of $\mathscr{C}$ with the identity natural transformation $id_K\colon K \dashrightarrow K$ is the pointwise *left* Kan extension of $K$ along $K$. A subcategory of $\mathscr{C}$ is *dense* if the inclusion functor is dense.

Considering the fact that $c \cong d$ iff $\mathscr{C}(-,c) \cong \mathscr{C}(-,d)$, i.e. every object is determined by morphisms into it, a dense functor sharpens the idea by the following:

**Proposition 3.3.2** ([81]). *Let $K\colon \mathscr{M} \to \mathscr{C}$ be a functor. The following statements are equivalent:*

1. *$K$ is dense.*

2. *The canonical functor $\widetilde{K}$ is full and faithful.*

3. *Every object $c$ in $\mathscr{C}$ is the canonical colimit with respect to $K$.*

*Proof.* By definition, $K$ is dense if and only if the identity functor $\mathcal{I}$ with $id_K$ is the pointwise left Kan extension $\mathsf{Lan}_K K$. Thus, the above two equivalent characterisations are applications of Theorem 3.2.11:

1. By the first statement of Theorem 3.2.11, the identity functor with $id_K$ is a pointwise left Kan extension of $K$ along $K$ if and only if the function defined in (3.3)

$$\mathscr{A}(c,a) \longrightarrow [\mathscr{M}^{\mathrm{op}}, \mathsf{Set}](\mathscr{C}(K-,c), \mathscr{C}(K-,a))$$

   is a bijection. By definition, this function maps $g\colon c \to a$ to a natural transformation $\mathscr{C}(K-,c) \to \mathscr{C}(K-,a)$ defined for each component $m$ by

$$(\alpha\colon Km \to c) \mapsto (g \circ (\mathcal{I}\alpha) \circ id) = g \circ \alpha,$$

   and this natural transformation is precisely $\widetilde{K}g$. Thus, the bijectivity means that $\widetilde{K}$ is full and faithful.

2. By the second statement of Theorem 3.2.11, the identity functor with $id_K$ is a pointwise left Kan extension of $K$ along $K$ if and only if the functor

$$(K \downarrow c) \xrightarrow{P^c} \mathscr{M} \xrightarrow{K} \mathscr{C}$$

has the limiting cocone $\left(Km \xrightarrow{f} c\right)$ for every $c \in \mathscr{C}$, but it is simply the definition of canonical colimit with respect to $K$.

$\square$

It is now easy to see that the category consisting of only the singleton set is dense in Set. Considering the canonical colimits, we wonder if we can replace canonical diagrams with more convenient diagrams when $K$ is full and faithful.

**Definition 3.3.3.** Let $\Phi = (G_\gamma \colon \mathscr{K}_\gamma \to \mathscr{C})_{\gamma \in \Gamma}$ be a collection of diagrams in $\mathscr{C}$ such that a colimit of each $G_\gamma$ exists. For any full subcategory $\mathscr{A}$ of $\mathscr{C}$, the **(colimit) closure** under $\Phi$, denoted $\overline{\mathscr{A}}$, of $\mathscr{A}$ is the smallest isomorphism-closed full subcategory containing $\mathscr{A}$ such that for any diagram $G_\gamma$ lying in $\overline{\mathscr{A}}$ its colimit is also in $\overline{\mathscr{A}}$.

The closure can be constructed by transfinite induction:

**Lemma 3.3.4** (see [61, Section 3.5]). *The closure of a full subcategory $\mathscr{A}$ of $\mathscr{C}$ under a collection of diagrams exists if $\mathscr{C}$ has colimits for these diagrams.*

**Definition 3.3.5.** A **density presentation** of a full and faithful functor $K \colon \mathscr{M} \to \mathscr{C}$ is a collection $\Phi = (G_\gamma \colon \mathscr{K}_\gamma \to \mathscr{C})_{\gamma \in \Gamma}$ of diagrams in $\mathscr{C}$ satisfying that

1. each diagram $G_\gamma$ has a colimit;

2. $\mathscr{C}$ is the closure of $\mathscr{M}$ under $\Phi$[5];

3. every colimit of $G_\gamma$ is preserved by the restricted Yoneda embedding $\widetilde{K}$.

For every full, faithful, and dense functor $K$, there is a **canonical density presentation** of $K$ formed by canonical diagrams of objects in $\mathscr{C}$ along $K$.

Colimits preserved by the restricted Yoneda embedding $\widetilde{K}$ of some functor $K$ are of particular interest, since it implies that any morphism from some $Km$ to some object $c$ factors through some morphism $Km \to G_i$. To see this, consider a colimit $(G_i \xrightarrow{\alpha_i} c)_{i \in \mathscr{I}}$ preserved by $\widetilde{K}$. The natural transformation

$$\mathscr{C}(K, \alpha) \colon \widetilde{K}G \dashrightarrow \widetilde{K}c = \mathscr{C}(K-, c)$$

---

[5] Since $K$ is a full and faithful functor, $\mathscr{M}$ is equivalent to a full subcategory of $\mathscr{C}$. To be precise, the isomorphism-closed full subcategory generated by the image of $K$.

is a colimit of $\widetilde{K}G$. In particular, every $\mathscr{C}(Km, c)$ is a colimit of a diagram $\mathscr{C}(Km, G-)$ in Set, since (co)limits of functors are computed pointwise. Therefore, every morphism $Km \to c$ factors through some $\alpha_i \colon G_i \to c$. Moreover, such an $\alpha_i$ is uniquely determined up to a *zigzag* of morphisms by the colimit construction in Set. In detail, the canonical colimit in Set is given by the quotient of

$$\dot{\bigcup_{i \in \mathscr{I}}} \mathscr{C}(Km, G_i)$$

subject to the smallest equivalence relation containing a relation defined by $f \sim g$ if $g$ factors through $f$

$$
\begin{array}{ccc}
Km & \xrightarrow{\ g\ } & G_j \\
f \downarrow & \nearrow Gh & \\
G_i & &
\end{array}
$$

for some $h \colon i \to j$. Morphisms are equal precisely if there exists a finite sequence $(h_j)_{j \in n}$ of morphisms such that the diagram

$$
\begin{array}{ccccccc}
Km & = \cdots = & Km & = \cdots = & Km \\
f \downarrow & & \downarrow & & \downarrow g \\
G_i \xrightarrow{Gh_0} \cdots \xleftarrow{Gh_{j-1}} & G_k & \xrightarrow{Gh_j} \cdots \xrightarrow{Gh_{n-1}} & G_j,
\end{array}
$$

commutes, namely, a zigzag of morphisms connecting $f$ and $g$.

**Theorem 3.3.6.** *Given a full and faithful functor $K \colon \mathscr{M} \to \mathscr{C}$, the following are equivalent:*

1. *$K$ is dense.*

2. *A density presentation $\Phi = (G_\gamma \colon \mathscr{K}_\gamma \to \mathscr{C})_{\gamma \in \Gamma}$ of $K$ exists.*

*If $\mathscr{M}$ is small, then each diagram $G_\gamma$ can be chosen to be small.*

*Proof.* Since $K$ is dense, there is a canonical density presentation of $K$.

On the other hand, suppose that there is a density presentation $\Phi$ of $K$. Let $\mathscr{B}$ be the isomorphism-closed full subcategory of $\mathscr{C}$ such that for every object $c \in \mathscr{B}$ the mapping

$$\mathscr{C}(c, d) \xrightarrow{\widetilde{K}_{c,d}} \mathrm{Hom}(\widetilde{K}c, \widetilde{K}d)$$

is a bijection. The essential image of $\mathscr{M}$ under $K$, i.e. the isomorphism-closed category spanned by $Km$, for $m \in \mathscr{M}$, is a full subcategory of $\mathscr{B}$:

$$
\begin{aligned}
&\mathscr{C}(Km,d) \\
&\cong \mathrm{Hom}(\mathscr{M}(-,m),\mathscr{C}(K-,d)) &&\{\text{by Yoneda Lemma}\} \\
&\cong \mathrm{Hom}(\mathscr{C}(K-,Km),\mathscr{C}(K-,d)) &&\{\text{K is full and faithful}\} \\
&= \mathrm{Hom}(\widetilde{K}Km,\widetilde{K}d) &&\{\text{by definition}\}
\end{aligned}
$$

which maps any $g\colon Km \to d$ to a natural transformation $\alpha\colon \mathscr{C}(K-,Km) \dashrightarrow \mathscr{C}(K-,d)$ defined by

$$(f\colon Kc \to Km) \mapsto Kc \xrightarrow{f} Km \xrightarrow{g} d$$

for each component, so the bijection is exactly $\widetilde{K}_{c,d}$.

Consider any diagram $G \in \Phi$ which is not only a diagram in $\mathscr{C}$ but also a diagram in $\mathscr{B}$. It suffices to show that its colimit is also in $\mathscr{B}$, so by the closure property $\mathscr{B}$ must be $\mathscr{C}$ and $\widetilde{K}$ is full and faithful, i.e. $K$ is dense: For any colimit $c$ of $G$, the following natural isomorphisms

$$
\begin{aligned}
&\mathrm{Hom}(c,d) \\
&\cong \varprojlim_{k} \mathrm{Hom}(Gk,d) &&\{c \text{ is a colimit of } G\} \\
&\cong \varprojlim_{k} \mathrm{Hom}(\widetilde{K}Gk,\widetilde{K}d) &&\{\text{G is a diagram in } \mathscr{B}\} \\
&\cong \mathrm{Hom}(\varinjlim_{k} \widetilde{K}Gk,\widetilde{K}d) &&\{\text{Hom-functor maps colimits to limits}\} \\
&\cong \mathrm{Hom}(\widetilde{K}c,\widetilde{K}d) &&\{\widetilde{K} \text{ preserves } \mathrm{Colim}\, G\}
\end{aligned}
$$

show that $c$ is in $\mathscr{B}$ by definition. $\qquad\square$

**Example 3.3.7.** The category of finite subsets is dense in Set. Note that every set is a directed union of its finite subsets, and to show this to be a density presentation, it suffices to show that the canonical functor $\widetilde{J}$ of the inclusion functor $J$ preserves directed unions of finite subsets. Again, it suffices to show that every function $f\colon n \to X$ for $n \in \omega$ factors through some finite subset of $X$, and it is trivial by the standard factorisation of $f$.

Correspondingly, we say that a functor $L\colon \mathscr{C} \to \mathscr{D}$ is **determined by** a category $\mathscr{M}$ **along** a full, faithful, and dense functor $K\colon \mathscr{M} \to \mathscr{C}$ if $(L, id\colon LK \to LK)$ is the left Kan extension $L = \mathrm{Lan}_K LK$ of $LK$ along $K$.

By a similar argument to Theorem 3.3.6, we have further the following facts:

**Theorem 3.3.8** (see [61, Theorem 5.29]). *Let $K\colon \mathcal{M} \to \mathcal{C}$ be a full, faithful, and dense functor. The following properties of a functor $L\colon \mathcal{C} \to \mathcal{D}$ are equivalent:*

1. *The identity $id\colon LK \overset{\cdot}{\longrightarrow} LK$ is the unit of the pointwise left Kan extension $L$ of $LK$ along $K$.*

2. *$L$ preserves every colimit which is preserved by $\widetilde{K}$.*

3. *For any density presentation $\Phi$ of $K$, each colimit of a diagram in $\Phi$ is preserved by $L$.*

4. *There is a density presentation $\Phi$ of $K$ such that each colimit of a diagram in $\Phi$ is preserved by $L$.*

**Example 3.3.9.** Let $J\colon \mathrm{Set}_\omega \hookrightarrow \mathrm{Set}$ be the full inclusion functor. Instead of the canonical colimits as in Example 3.2.12, the finitary powerset functor $\mathcal{P}_\omega$ is the pointwise left Kan extension of $\mathcal{P}J$ along $J$ by the above theorem since $\mathcal{P}_\omega$ preserves directed unions, and $J$ is full and faithful.

*Remark* 3.3.10. Note that $\mathrm{Set}_\omega$ is not small, but for our purposes it can be replaced with the category of natural numbers and functions between them. For simplicity, we denote it also by $\mathrm{Set}_\omega$.

## 3.4 Locally Presentable Categories and Accessibility

In the previous subsection, we showed that the category of finite ordinals is a dense subcategory of Set; in this section, we generalise this situation to so-called *locally presentable categories*, introduced by P. Gabriel and F. Ulmer [44], where every object is a filtered colimit from a set of presentable objects. For elementary proofs and details, we refer to [10, 14].

Recall that in [81, Chapter IX] a small category is called *filtered* if every finite subcategory has a cocone. A special case is a directed poset regarded as a category. Filteredness can be described in another way: a category $\mathcal{D}$ is filtered if and only if $\mathcal{D}$-colimits *commute with* finite limits in Set, see [14, 59]. That is, $\mathcal{D}$ is filtered if for every diagram $F\colon \mathcal{D} \times \mathcal{J} \to \mathrm{Set}$ where $\mathcal{J}$ is finite, the canonical morphism

$$\operatorname*{Colim}_{d \in \mathcal{D}} \operatorname*{Lim}_{j \in \mathcal{J}} F(d, j) \longrightarrow \operatorname*{Lim}_{j \in \mathcal{J}} \operatorname*{Colim}_{d \in \mathcal{D}} F(d, j) \tag{3.7}$$

is an isomorphism.

For any regular cardinal $\lambda$, a category is called $\lambda$-**small** if it has fewer than $\lambda$ many morphisms, and is called $\lambda$-**complete** if it has all limits of $\lambda$-small diagrams. In particular, an $\aleph_0$-small category is called *finite*.

**Proposition 3.4.1.** *Let $\mathscr{D}$ be a small category and $\lambda$ a regular cardinal. The following statements are equivalent:*

1. *Every $\lambda$-small subcategory of $\mathscr{D}$ has a compatible cocone in $\mathscr{D}$.*

2. *Every $\mathscr{D}$-colimit commutes with $\lambda$-limits.*

*Any small category $\mathscr{D}$ satisfying one of the above statements is called $\lambda$-**filtered**; a $\lambda$-**filtered colimit** is a colimit of a $\lambda$-filtered diagram.*

## 3.4.1 Presentable Object

**Definition 3.4.2** (see [10, Definition 1.13 & 1.21, Remark 1.21]). Let $\lambda$ be a regular cardinal. An object $k$ in a locally small category $\mathscr{K}$ is $\lambda$-**presentable** if the representable functor

$$\mathrm{Hom}(k,-)\colon \mathscr{K} \to \mathsf{Set}$$

preserves $\lambda$-filtered colimits. In particular, an object is **finitely presentable** if it is $\aleph_0$-presentable.

For any category $\mathscr{K}$ and regular cardinal $\lambda$, the notations

$$\mathscr{K}_\lambda \quad \text{or} \quad \mathsf{Pres}_\lambda \mathscr{K}$$

denote any small full subcategory of $\mathcal{K}$ equivalent to the category of $\lambda$-presentable objects by abuse of notation.

**Proposition 3.4.3** (see [10, Lemma 1.6]). *For any regular cardinal $\lambda$, a $\lambda$-small colimit of $\lambda$-presentable objects is $\lambda$-presentable.*

*Proof.* Let $\mathscr{K}$ be a locally $\lambda$-presentable category, $D\colon \mathscr{I} \to \mathscr{K}$ a $\lambda$-small diagram in $\mathscr{K}$ consisting of $\lambda$-presentable objects, and $F$ a $\lambda$-filtered diagram in $\mathscr{K}$. The

following isomorphisms

$$
\begin{aligned}
&\mathrm{Hom}(\underset{i\in\mathscr{I}}{\mathrm{Colim}}\, D_i, \underset{j\in\mathscr{J}}{\mathrm{Colim}}\, E_j) \\
&\cong \underset{i\in\mathscr{I}}{\mathrm{Lim}}\,\mathrm{Hom}(D_i, \underset{j\in\mathscr{J}}{\mathrm{Colim}}\, E_j) && \{\,\mathrm{Hom}(-,a)\ \text{preserves colimits}\,\} \\
&\cong \underset{i\in\mathscr{I}}{\mathrm{Lim}}\,\underset{j\in\mathscr{J}}{\mathrm{Colim}}\,\mathrm{Hom}(D_i, E_j) && \{\,D_i\ \text{is } \lambda\text{-presentable}\,\} \\
&\cong \underset{j\in\mathscr{J}}{\mathrm{Colim}}\,\underset{i\in\mathscr{I}}{\mathrm{Lim}}\,\mathrm{Hom}(D_i, E_j) && \{\,\text{by Proposition 3.4.1}\,\} \\
&\cong \underset{j\in\mathscr{J}}{\mathrm{Colim}}\,\mathrm{Hom}(\underset{i\in\mathscr{I}}{\mathrm{Colim}}\, D_i, E_j) && \{\,\mathrm{Hom}(-,a)\ \text{preserves colimits}\,\}
\end{aligned}
$$

shows that any $\lambda$-colimit of $\lambda$-presentable objects is $\lambda$-presentable. $\qquad\square$

## 3.4.2 Locally Presentable Categories

**Proposition 3.4.4** (see [10, Remark 1.19]). *Let $\mathscr{K}$ be a locally small and cocomplete category and $\lambda$ a regular cardinal. The following are equivalent:*

1. *There exists a full small subcategory $\mathscr{A}$ of $\mathscr{K}$ consisting of $\lambda$-presentable objects such that every object $k \in \mathscr{K}$ is a $\lambda$-filtered colimit of objects from $\mathscr{A}$.*

2. *Every object $k \in \mathscr{K}$ is a $\lambda$-filtered colimit of objects from $\mathscr{K}_\lambda$ and there exists a set of representatives up to isomorphism.*

*Any category $\mathcal{K}$ satisfying one of the above statements is called **locally $\lambda$-presentable**; a **locally presentable** category is locally $\kappa$-presentable for some regular cardinal $\kappa$.*

Therefore, a locally $\lambda$-presentable category $\mathscr{K}$ consists of a category $\mathscr{K}$ and a set $\mathrm{Pres}_\lambda \mathscr{K} = \mathscr{K}_\lambda$ of $\lambda$-presentable objects generating $\mathscr{K}$.[6]

**Lemma 3.4.5.** *For every full inclusion $J \colon \mathscr{K}_\lambda \to \mathscr{K}$, the canonical functor $\widetilde{J}$ preserves any $\lambda$-filtered colimit.*

*Proof.* Let $\tau \colon G \dashrightarrow c$ be a $\lambda$-filtered colimit for some diagram $G$. For every $\lambda$-presentable object $k$,

$$
\mathscr{K}(k, G) \dashrightarrow \mathscr{K}(k, c)
$$

is a colimit by definition. Since colimits of functors are computed pointwise, the functor $\widetilde{J}$ preserves every $\lambda$-filtered colimit. $\qquad\square$

---

[6]The term "generating" is not ambiguous; indeed, a category is locally presentable if and only if it is cocomplete and has a *strong generator* formed by $\lambda$-presentable objects, see [10, Theorem 1.11].

Using the density presentation consisting of filtered diagrams, we have following corollaries by Theorem 3.3.6:

**Corollary 3.4.6.** *Given a locally $\lambda$-presentable category $\mathscr{K}$, the small subcategory $\mathrm{Pres}_\lambda \mathscr{K}$ is dense in $\mathscr{K}$.*

Recall that a functor $J$ is dense iff the canonical functor $\widetilde{J}$ is full and faithful.

**Corollary 3.4.7.** *Every locally $\lambda$-presentable category $\mathscr{K}$ is equivalent to a reflective subcategory of $[\mathscr{A}^{\mathrm{op}}, \mathsf{Set}]$ where $\mathscr{A} = \mathscr{K}_\lambda$. In particular, every locally $\lambda$-presentable category is complete.*

*Proof.* Let $J \colon \mathscr{K}_\lambda \to \mathscr{K}$ be the full inclusion. By Lemma 3.2.14, the canonical functor

$$\widetilde{J} \colon \mathscr{K} \to [\mathscr{K}_\lambda^{\mathrm{op}}, \mathsf{Set}]$$

is a right adjoint to $\mathrm{Lan}_y J$. By Corollary 3.4.6, $\widetilde{J}$ is full and faithful, so $\mathscr{K}$ is equivalent to a reflective subcategory of $[\mathscr{K}_\lambda^{\mathrm{op}}, \mathsf{Set}]$. $\qquad\square$

**Example 3.4.8.** Every variety of single-sorted algebras is locally finitely presentable. In detail, a variety $\mathscr{A}$ of single-sorted algebras is precisely a category $\mathscr{A}$ with a monadic and finitary functor $U \colon \mathscr{A} \to \mathsf{Set}$ with the free algebra construction as the left adjoint by [32, Proposition 4.6.2]. Moreover, finitely presentable objects in $\mathscr{A}$ are precisely finitely generated algebras by Proposition 3.8.14 *op. cit.*, and every object in $\mathscr{A}$ is a filtered colimit of finitely generated algebras by Proposition 3.8.12 *op. cit.* As a result, the category of Boolean algebras, distributive lattices, and modal algebras are all locally finitely presentable.

### 3.4.3 Finitary and Accessible Functors

As objects in a locally presentable category are formed by $\lambda$-filtered colimits, it would be of interest to consider the following functors:

**Definition 3.4.9.** Given a regular cardinal $\lambda$, a functor $F$ is called $\lambda$-**accessible** if $F$ preserves $\lambda$-filtered colimits. In particular, an $\aleph_0$-accessible functor is called **finitary**.

The full subcategory consisting of $\lambda$-accessible functors is denoted by $\mathrm{Acc}_\lambda[\mathscr{K}, \mathscr{L}]$ and particularly the full subcategory consisting of finitary functors is denoted by $\mathrm{Fin}[\mathscr{K}, \mathscr{L}]$.

Given a locally $\lambda$-presentable category $\mathcal{K}$, the collection of $\lambda$-filtered diagrams is a density presentation of $J\colon \mathcal{K}_\lambda \to \mathcal{K}$, so by Theorem 3.3.8 we have the following corollary:

**Corollary 3.4.10.** *Let $\lambda$ be a regular cardinal, $\mathcal{K}$ with a set $\mathcal{K}_\lambda \xrightarrow{J} \mathcal{J}$ of $\lambda$-presentable objects generating $\mathcal{K}$, and $\mathcal{L}$ locally $\lambda$-presentable categories. There is an equivalence*

$$\mathrm{Lan}_J\colon [\mathcal{K}_\lambda, \mathcal{L}] \xrightarrow{\cong} \mathrm{Acc}_\lambda[\mathcal{K}, \mathcal{L}]$$

*with a pseudo-inverse $(-) \circ J$. That is, $F$ is of the form*

$$F = \mathrm{Lan}_J FJ$$

*if and only if $F$ is $\lambda$-accessible.*

In particular, every functor $F$ between locally presentable categories has an accessible coreflection defined by $\mathrm{Lan}_J FJ$. For example, the finitary powerset functor $\mathcal{P}_\omega$ is given by the finitary coreflection of the powerset functor $\mathcal{P}$.

**Corollary 3.4.11.** *Let $\mathcal{K}$ be a locally $\lambda$-presentable category. The following statements hold:*

1. *Every $\lambda$-accessible endofunctor $F$ is of the form*

$$\int^{a \in \mathcal{K}_\lambda} \mathcal{K}(Ja, -) \cdot Fa$$

   *where $\mathcal{K}_\lambda$ is the set of $\lambda$-presentable objects generating $\mathcal{K}$ and $J$ the full inclusion from $\mathcal{K}_\lambda$ to $\mathcal{K}$.*

2. *Particularly, every finitary $\mathsf{Set}$ endofunctor $T\colon \mathsf{Set} \to \mathsf{Set}$ can be written as a coequaliser:*

$$\coprod Tm \times X^n \rightrightarrows \coprod Tn \times X^n \twoheadrightarrow TX.$$

*Proof.* Every finitary Set endofunctor has the coend formula and the coend formula can be written as the coequaliser by Theorem 3.2.6. Furthermore, the copower $\mathsf{Set}(n, -) \cdot Tn$ is isomorphic to $Tn \times X^n$, so the second statement follows. $\square$

**Corollary 3.4.12.** *For any $\mathsf{Set}$ functor $T$, $T$ is finitary in the sense of Definition 3.4.9 if and only if $T$ is finitary in the sense of (2.6)*

*Proof.* The collection of directed families of finite subsets and the collection of filtered diagrams in Set are both density presentations of $\mathsf{Set}_\omega \hookrightarrow \mathsf{Set}$, so by Theorem 3.3.8 it follows. $\qquad\square$

### 3.4.4   Properties of Locally Presentable Categories

Recall that every reflective subcategory, which is full by definition, is closed under limits of its super-category, so by Corollary 3.4.7 it follows that

**Corollary 3.4.13.** *Every locally presentable category is well-powered, i.e. each object has only a set of subobjects.*

*Proof.* It follows from the fact that the category $[\mathscr{K}_\lambda, \mathsf{Set}]$ is complete and well-powered. $\qquad\square$

**Theorem 3.4.14** (see [10, Theorem 1.66]). *Given locally presentable categories $\mathscr{K}$ and $\mathscr{L}$, the following statements hold:*

1. *A functor $L \colon \mathscr{K} \to \mathscr{L}$ has a right adjoint if and only if $L$ preserves colimits.*

2. *A functor $R \colon \mathscr{L} \to \mathscr{K}$ has a left adjoint if and only if $R$ preserves limits and is accessible.*

*Proof Sketch.* Consider the Freyd Adjoint Functor Theorem. The solution set condition is satisfied by the set of presentable objects. $\qquad\square$

The proof of the following Theorem 3.4.15 is rather technical, see [10, Section 1.D] for details:

**Theorem 3.4.15** (see [10, Theorem 1.58]). *Every locally presentable category is well-copowered, i.e. every object has a set of quotients.*

**Corollary 3.4.16.** *Every locally presentable category has the* (StrongEpi, Mono)*- and the* (Epi, StrongMono)*-factorisation system.*

*Proof.* By Theorem 3.1.7 and its dual. $\qquad\square$

# 3.5 Foundations

In category theory, various foundations may be able to accommodate category of small categories, category of *all* small sets, and so on. Set-theoretically, a *universe* is used where the set theory does not contain the notion of *proper class*:

**Definition 3.5.1.** A **Grothendieck universe** is a set $U$ satisfying

| | |
|---|---|
| **Transitivity** | $x \in U$ implies $x \subseteq U$; |
| **Closure under Operations** | for $u \in U$ and $v \in U$, the union $u \cup v$, the pair $\langle u, v \rangle$, the product $u \times v$, the powerset set $\mathcal{P}u$, the union $\bigcup u$ are all in $U$; |
| **Infinity** | the set $\omega$ of all finite ordinals is in $U$. |
| **Replacement** | for any surjective function $f : a \to b$ such that $a \in U$ and $b \subset U$, then $b \in U$. |

Every universe is clearly a model of ZFC set theory, i.e. axioms in ZFC set theory are interpretable in the given universal set. Even the first infinite cardinal $\omega$ is a model of ZFC without the Axiom of Infinity. Under a universe $U$, a set $u$ is $U$-**small** if $u \in U$; $u$ is $U$-**large** if it is a subset of $U$. By transitivity, every $U$-small set is $U$-large. Then, we say a set is $U$-**superlarge** if it is not $U$-large, e.g. $\{U\}$. A category is $U$-large if the collection of objects is a $U$-large set; and it is $U$-superlarge if the collection of objects is not $U$-large.

We give an explicit definition of the category of sets:

**Definition 3.5.2.** Given a Grothendieck universe $U$, the category $\mathsf{Set}_U = \mathsf{Set}$ of $U$-small sets consists of

**objects**: the large set $U$ as the collection of objects;

**morphisms**: the large set

$$\mathrm{mor} := \{(X, Y, f) \mid X \in U, Y \in U, f \text{ is a functional relation from } X \text{ to } Y\}$$

and the identity $(X, X, id_X)$ for every set $X \in U$ is the identity function $id_X$ with its domain and codomain.

**domain and comain**: the domain function

$$\mathrm{dom}\colon \mathrm{mor} \to U \quad \text{is defined by} \quad (X, Y, f) \mapsto X$$

and similarly for the codomain function $\mathrm{cod}\colon \mathrm{mor} \to U$;

**compositions**: the composition of morphisms are the usual composition of functions.

The category $\mathsf{Set}$ of $U$-small sets with functions between $U$-small sets is $U$-large. On the other hand, the category $\mathsf{SET}(= \mathsf{SET}_U)$ used in [3] of classes consisting of $U$-large sets with functions is *not* large but superlarge since the collection $\mathcal{P}U$ of $U$-large sets is not a subset of $U$.

We may characterise universes by cardinalities as well:

**Definition 3.5.3.** A cardinal $\kappa$ is called **(strongly) inaccessible** if it is a strong limit, i.e. $2^\lambda < \kappa$ for every $\lambda < \kappa$, larger than $\aleph_0$, and is regular.

Every set can be classified by its rank using transfinite induction in the **von Neumann universe**. For any ordinal $\alpha$ define the following sets:

1. $V_0 = \emptyset$;

2. $V_{\alpha+1} = \mathcal{P}V_\alpha$;

3. $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ if $\alpha$ is a limit ordinal.

Given this hierarchy, we can characterise any Grothendieck universe precisely by an inaccessible cardinal:

**Theorem 3.5.4** (see [108]). *$U$ is a Grothendieck universe if and only if $U = V_\alpha$ for some inaccessible cardinal. Moreover, the cardinality of a Grothendieck is equal to the index $\alpha$.*

Also, *op. cit.*, the existences of universes and inaccessible cardinals are equivalent:

1. For every cardinal $\beta$, there exists an inaccessible cardinal $\alpha$ such that $\beta < \alpha$

2. (*Axiom of Universes*) For every set $x$, there is a Grothendieck universe $U$ such that $x \in U$.

Note that any one of the above statements is not provable in ZFC and is independent of ZFC, see [55, Theorem 12.12]. By postulating the Axiom of Universes, every set $x$ has a universe $U$ of discourse and thus a category $\mathsf{Set}_U$ where $x$ belongs to.

### 3.5.1 Set-based *v.s.* $\lambda$-accessibility

As claimed in Remark 2.1.19, we clarify that $\lambda$-accessible functors and set-based functors are all the same. Some discussions also appear in [8, 17], but it is still nice to mention it on the ground of the foundation issue.

**Definition 3.5.5** (see [3]). Given a universe $U$, a SET endofunctor $T$ is called **set-based** if for every $U$-large set $X$ and $x \in TX$ there exists a $U$-small $X_0 \subseteq X$ such that $x_0 \in TX_0$ and $x = (T\iota)x_0$ where $\iota\colon X_0 \hookrightarrow X$ is the inclusion function.

Given a functor $T$, it is trivial to see that $T$ is set-based if and only if

$$TX = \bigcup \{ T\iota[TX_0] \mid \iota\colon X_0 \subseteq_U X \}$$

where $X_0 \subseteq_U X$ means that $X_0$ is a $U$-small subset of $X$. Note that $X_0 \subseteq_U X$ if and only if the cardinality of $X_0$ is less than the cardinality of the universe $U$.

By Corollary 3.4.12 and Theorem 3.5.4, the equivalence then follows:

**Proposition 3.5.6.** *For any* SET *functor* $T$, *$T$ is set-based if and only if $T$ is $\alpha$-accessible as a functor into the category* $Set_{U'}$ *where $\alpha$ is the cardinality of the given Grothendieck universe $U$ and $U'$ is any universe containing $U$.*

The set-based condition itself is also superfluous:

**Theorem 3.5.7** (see [8, Theorem 2.2]). *Every endofunctor of* SET *is set-based.*

For any functor $T\colon \mathrm{Set} \to \mathrm{SET}$, we extend $T$ to a set-based SET functor continuously by the left Kan extension:

$$
\begin{array}{ccc}
\mathrm{Set} & \xrightarrow{\ J\ } & \mathrm{SET} \\
 & {}_{T}\searrow & \big\downarrow {\scriptstyle \mathrm{Lan}_J T} \\
 & & \mathrm{SET,}
\end{array}
$$

but, given that every endofunctor of SET is set-based, the extension is unique, up to isomorphism. In particular, every endofunctor of Set has a unique extension to SET.

*Remark* 3.5.8. Regarding the final coalgebra construction given by P. Aczel [3], the result asserts that the final coalgebra exists for any Set functor, in SET, without any assumption. By enlarging the universe of discourse, we may simply assume the existence of a final coalgebra for any Set functor under the Axiom of Universes.

### 3.5.2 Preservation of Inclusions

The preservation of inclusions and the preservation of injections are two common assumptions for Set endofunctors in practice, but they differ fundamentally. One classical example distinguishing these two notions is the Hom-functor:

**Example 3.5.9.** For every set $X$, the Hom-functor $\mathrm{Hom}(X, -)$ does not preserve inclusions only injections. By the very definition of category given in [81] or Definition 3.5.2, every morphism in Set has a domain and codomain. Thus $\mathrm{Hom}(X, Y)$ is disjoint from $\mathrm{Hom}(X, Z)$ whenever $Y \neq Z$. On the other hand, given a monomorphism $j \colon Y \to Z$, the post-composition $j_* \colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ by $j$ is injective simply by $j$ being mono.

Notice the preservation of inclusions depends on the definition of category. In the context of ZFC set theory with the Axiom of Universes, an alternative definition of category is given in [31]:

**Definition 3.5.10.** A **category** $\mathscr{C}$ consists of a collection $\mathrm{ob}\mathscr{C}$ of objects; and for every two objects $x, y$ in $\mathrm{ob}\mathscr{C}$ there is a set $\mathscr{C}(x, y)$ as the collection of morphisms from $x$ to $y$. Moreover, there is an identity $id_x \in \mathscr{C}(x, x)$ for every object $x$ and a composition function

$$\circ \colon \mathscr{C}(x, y) \times \mathscr{C}(y, z) \to \mathscr{C}(x, z)$$

subject to the unit law and the associativity law. Note that $\mathscr{C}(x, y)$ and $\mathscr{C}(x', y')$ are not assumed to be disjoint when $x \neq x'$ or $y \neq y'$.

Then, we can define a category of sets with *untyped functions*:

**Definition 3.5.11** (*cf.* Definition 3.5.2). Given a Grothendieck universe $U$, a category $\mathsf{Set}'_U$ consists of $U$ as the collection of objects; and for every two sets $X, Y \in U$, the following set

$$\mathsf{Set}'_U(X, Y) := \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is functional} \}$$

as the collection of morphisms from $X$ to $Y$. The identity morphism $id_X$ is the diagonal relation and the composition of morphisms $\circ \colon \mathsf{Set}'_U(X, Y) \times \mathsf{Set}'_U(Y, Z) \to \mathsf{Set}'_U(X, Z)$ is the composition of relations.

The category $\mathsf{Set}''_U$ of sets with typed functions is defined similarly by replacing $\mathsf{Set}'_U(X, Y)$ with the collection of typed functions:

$$\mathsf{Set}''_U(X, Y) := \{ (X, Y, f) \in \{X\} \times \{Y\} \times \mathcal{P}(X \times Y) \mid f \text{ is functional} \}.$$

There is a functor $I' \colon \mathsf{Set}''_U \to \mathsf{Set}'_U$ which is an identity on objects and maps each typed functions $(X, Y, f \colon X \to Y)$ to its graph $f$. Conversely, we can map each untyped function $f \in \mathsf{Set}'_U(X, Y)$ to its typed function. It follows that $\mathsf{Set}'_U$ is isomorphic to $\mathsf{Set}''_U$, unsurprisingly. However, the hom-functor preserves inclusions in the untyped version:

**Example 3.5.12.** In $\mathsf{Set}'_U$ the hom-functor preserves inclusions. By the Axiom of Extensionality, two sets $X$ and $Y$ are equal if and only if

$$x \in X \iff x \in Y$$

for every $x$. Therefore, given any inclusion $j \colon Y \subseteq Z$ and a function $f \colon X \to Y$, the composite $j \circ f$ is *equal* to $f$, so $f$ is in $\mathsf{Hom}(X, Z)$ and $j_* \colon \mathsf{Hom}(X, Y) \subseteq \mathsf{Hom}(X, Z)$.

To sum up, the preservation of inclusions is not a property stable under another equivalent definition of category; it depends on the very definition of the category of sets; and it is not invariant under natural isomorphisms by Theorem 2.1.25.

On the other hand, the preservation of injections is independent of the definition of category (of sets) and invariant under natural isomorphisms. Moreover, injections are precisely monomorphisms in $\mathsf{Set}$, and the preservation of injections can be equally phrased in the context of an elementary topos.

## 3.6 Concreteness

### 3.6.1 Concrete Categories

First, a **concrete category** is a category $\mathscr{C}$ with a faithful functor $U \colon \mathscr{C} \to \mathsf{Set}$. Moreover, there are a number of examples where every object is determined by a single object.

**Definition 3.6.1.** A concrete category $(\mathscr{C}, U \colon \mathscr{C} \to \mathsf{Set})$ is **representable** if the forgetful functor $U$ is representable, i.e. $U$ is naturally isomorphic to $\mathsf{Hom}(X_0, -)$ for some $X_0$.

**Example 3.6.2.**    1. For the category $\mathsf{Set}$, the category $\mathsf{Pos}$ of posets, the category $\mathsf{Top}$ of topological spaces, and the category $\mathsf{Meas}$ of measurable spaces, every object $X$ is bijective with the collection of morphisms from the singleton $1$ to $X$.

2. For the category of Boolean algebras, distributive lattices, lattices, and semi-lattices, the underlying set of any object $A$ is bijective with the hom-set $\mathsf{Hom}(F1, A)$ where $F$ is the left adjoint to the forgetful functor, i.e. the free functor.

3. In general, every concrete category $\mathscr{A}$ whose forgetful functor has a left adjoint $F$ is representable, since for every object $A \in \mathscr{A}$, we have the following isomorphisms

$$\mathscr{A}(F1, A) \cong \mathsf{Set}(1, UA) \cong UA$$

natural in $A$ where the first isomorphism follows from the adjunction and the second follows from the representable forgetful functor of $\mathsf{Set}$. Namely, the forgetful functor of $\mathscr{A}$ is representable by $F1$. Particularly, every variety of (single-sorted) finitary algebras is, of course, concrete and representable.

**Proposition 3.6.3** (see [6, Proposition 7.37 & 7.44]). *Every representable faithful functor reflects monomorphisms and epimorphisms; and it also preserves monomorphisms.*

**Proposition 3.6.4.** *For every representable concrete category $(\mathscr{X}, U)$, the representing object of $U$ is a generator, i.e. given any two morphisms $f, g \colon X \to Y$, if $fx = gx$ for every $x \colon X_0 \to X$ then $f = g$.*

Given this property, we call every morphism from the representing object to an object $X$ an **element** of $X$, and using this property we show that there is a concrete category which is not representable.

**Example 3.6.5.** Some categories have a non-representable faithful functor to $\mathsf{Set}$:

1. The obvious forgetful functor for coalgebras is generally *not* representable. For example, the category of image-finite Kripke frames has a terminal object, so there exists only one morphism from the representing object. It implies that there is only one element in the final $\mathcal{P}_\omega$-coalgebra, so every two elements in any Kripke frame are bisimilar, clearly a contradiction.

2. The product category $\mathsf{Set}^2$ of $\mathsf{Set}$ has a faithful functor $U$ to $\mathsf{Set}$ defined as follows

$$\langle X_1, X_2 \rangle \mapsto X_1 + X_2 \quad \text{and} \quad \langle f_1, f_2 \rangle \mapsto f_1 + f_2.$$

The forgetful functor is obviously faithful since $f_1 + f_2 = g_1 + g_2$ if and only if $f_1 = g_1$ and $f_2 = g_2$. However, it is not representable, and it suffices to show there is no generator for $\mathsf{Set}^2$. Consider a pair $\langle Y, \emptyset \rangle$ where $Y$ is a non-empty set. The only one function with codomain $\emptyset$ is the empty graph, i.e. $\emptyset$ itself. It

follows that there is no pair of functions from $\langle A, B \rangle$ to $\langle Y, \emptyset \rangle$ for any nonempty set $B$. Therefore, to have

$$\mathsf{Hom}(\langle A, B \rangle, \langle Y, \emptyset \rangle) \cong Y \,\dot{\cup}\, \emptyset = Y$$

the set $B$ must be empty and $A = 1$. However, it follows that $\mathsf{Hom}(\langle 1, \emptyset \rangle, \langle \emptyset, Y \rangle) = \emptyset \not\cong Y$. Thus, the forgetful functor $U$ is not representable.

## 3.6.2 Concrete Dualities

Given two categories $\mathscr{X}$ and $\mathscr{A}$, we say that $\mathscr{X}$ is **dual** to $\mathscr{A}$ if there exists a dual adjunction on the right, that is, an isomorphism

$$\mathscr{A}(a, Px) \cong \mathscr{X}(x, Sa)$$

natural in $x$ and $a$ for some contravariant functors $P \colon \mathscr{X} \to \mathscr{A}$ and $S \colon \mathscr{A} \to \mathscr{X}$. Covariantly, every dual adjunction is an adjunction $P \dashv S \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{X}$ with a natural isomorphism $\mathscr{A}^{\mathrm{op}}(Px, a) \cong \mathscr{X}(x, Sa)$.

**Proposition 3.6.6** (see [93]). *Given a dual adjunction $P \dashv S \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{X}$ between representable concrete categories $(\mathscr{X}, |{-}|)$ and $(\mathscr{A}, |{-}|)$ whose forgetful functors are represented by $X_0$ and $A_0$ respectively, the following statements hold:*

1. *The underlying sets of $PX_0$ and $SA_0$ are isomorphic, i.e. $|PX_0| \cong |SA_0|$;*

2. *Contravariant functors $P$ and $S$ are given by homming into $PX_0$ and $SA_0$, i.e. for each $X \in \mathscr{X}$ and $A \in \mathscr{A}$ we have*

$$|PX| \cong \mathscr{X}(X, \Omega_{\mathscr{X}}) \quad and \quad |SA| \cong \mathscr{A}(A, \Omega_{\mathscr{A}})$$

*where $\Omega_{\mathscr{X}} = SA_0$ and $\Omega_{\mathscr{A}} = PX_0$ respectively.*

*Proof.* It is simple to show by definition and the dual adjunction. $\square$

Since the underlying sets of $\Omega_{\mathscr{A}}$ and $\Omega_{\mathscr{X}}$ are isomorphic, we simply denote them by $\Omega$.

# Chapter 4

# Coalgebras and Algebras

In this chapter, we focus on categorical properties of coalgebras based on endofunctors of general categories instead of Set. We cover a few topics related to the *coinduction principle*, e.g. bisimilarity (also known as *behavioural equivalence*), final coalgebras and minimisation, in line with coalgebraic logic which describes coalgebras up to behavioural equivalence.

In the second part, we turn to the transfinite construction for the situation where the category of (co)algebras for a (co)free (co)monad over an endofunctor $L$ coincides with the category of (co)algebras for the endofunctor $L$.

In the end, we discuss the categorical perspective of varieties of algebras, and give a few density presentations towards the class of endofunctors which are presentable by operations and equations.

## 4.1  Coalgebras

Instead of any particular category, we formulate properties in an axiomatic way with a few results requiring that the category is locally $\lambda$-presentable.

**Definition 4.1.1.** A $T$-**coalgebra** for an endofunctor $T \colon \mathscr{X} \to \mathscr{X}$ consists of an object $x \in \mathscr{X}$ and a morphism $x \xrightarrow{\xi} Tx$. Given $T$-coalgebras $\langle x, \xi \rangle$ and $\langle y, \gamma \rangle$, a **coalgebra homomorphism** $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$ is a morphism $f$ in $\mathscr{X}$ satisfying $Tf \circ \xi = \gamma \circ f$.

> In this section, $T$ always denotes an endofunctor of $\mathscr{X}$.

Similar to Set coalgebras, the collection of $T$-coalgebras with coalgebra homomorphisms forms a category, denoted by $\mathscr{X}_T$. The forgetful functor defined $\langle x, \xi \rangle \mapsto x$ is usually denoted by $U_T$ or $U$; it is clearly *faithful*.

**Proposition 4.1.2.** *The forgetful functor* $U \colon \mathscr{X}_T \to \mathscr{X}$ *is conservative, i.e.* $U$ *reflects isomorphisms.*

*Proof.* Assume we are given a coalgebra homomorphism $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$ whose underlying morphism is an isomorphism with the inverse $f^{-1}$. It suffices to show that $f^{-1}$ is a coalgebra homomorphism:

$$(Tf \circ \xi) \circ f^{-1} = (\gamma \circ f) \circ f^{-1} = \gamma \qquad \{f \text{ is a coalgebra homomorphism}\}$$
$$\implies \xi \circ f^{-1} = Tf^{-1} \circ \gamma \qquad\qquad\quad \{\text{by applying } Tf^{-1}\}$$

so $f^{-1}$ is the inverse to $f$ in $\mathscr{X}_T$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

### 4.1.1 Limits and Colimits of Coalgebras

Similar to the category of algebras, colimits are inherited from the underlying category, and depending on the type functor limits may be inherited:

**Proposition 4.1.3.** *The following statements hold:*

1. *The forgetful functor* $U \colon \mathscr{X}_T \to \mathscr{X}$ *creates colimits. In particular,* $U$ *preserves colimits of a diagram* $D$ *if* $\mathscr{X}$ *has colimits of* $UD$.

2. *The forgetful functor* $U \colon \mathscr{X}_T \to \mathscr{X}$ *creates the limit of a diagram* $D$ *(resp. weakly) whenever* $T$ *preserves the limit of* $UD$ *(resp. weakly). In particular,* $U$ *creates* $U$-*absolute limits.*

*Proof.* The creation of colimits follows from a similar argument to that in Proposition 2.1.4. We sketch the creation of limits for the sake of completeness. Given a diagram $D \colon \mathscr{I} \to \mathscr{X}_T$ in the category of coalgebras for $T$, consider the limit of $UD$ in $\mathscr{X}$:

where the cone $\left( T(\operatorname{Lim} x_i) \xrightarrow{T\pi_i} T x_i \right)$ is limiting as $T$ preserves it by assumption. Therefore, given the cone defined by

$$\left( \operatorname{Lim} x_i \xrightarrow{\pi_i} x_i \xrightarrow{\xi_i} T x_i \right)$$

there is a unique morphism $\xi \colon \operatorname{Lim} x_i \to T(\operatorname{Lim} x_i)$, i.e. a $T$-coalgebra with carrier $\operatorname{Lim} x_i$. It is now routine to check that $\left( \operatorname{Lim} x_i \xrightarrow{\xi} T(\operatorname{Lim} X_i) \right)$ is a limit of $D$.

In particular, if $\left( U D_i \xrightarrow{\mu_i} x \right)_{i \in \mathscr{I}}$ is a $U$-absolute limit, then $T$ preserves it by definition so $U$ creates this limit. $\qquad\square$

It is easy to see that every faithful functor reflects epimorphisms and monomorphisms. Moreover, the pair of identities $(id_Y, id_Y)$ is the cokernel[1] of an epimorphism $f \colon X \to Y$. Thus, we have the following corollary:

**Corollary 4.1.4.** *The forgetful functor* $U \colon \mathscr{X}_T \to \mathscr{X}$ *reflects epimorphisms and monomorphisms; it preserves epimorphisms, if* $\mathscr{X}$ *has cokernels.*

## 4.1.2 Comonadicity

The comonadicity of $U \colon \mathscr{X}_T \to \mathscr{X}$ implies the existence of a final $T$-coalgebra, if $\mathscr{X}$ has a terminal object. Recall that by Beck's Monadicity Theorem [81, Theorem VI.7.1] a functor $V \colon \mathscr{C} \to \mathscr{D}$ is monadic if and only if $V$ has a left adjoint and $V$ creates coequalisers of $V$-split pairs. Dually, a functor $U$ is *comonadic* if and only if $U$ has a right adjoint and $U$ creates equalisers of $U$-split pairs. However, since $U$-split pairs are $U$-absolute equalisers, $T$ must preserve them. Hence by Proposition 4.1.3 the forgetful creates equalisers of $U$-split pairs. To sum up, we have the following equivalent characterisations:

**Proposition 4.1.5.** *The following statements are equivalent:*

1. *The forgetful functor* $U \colon \mathscr{X}_T \to \mathscr{X}$ *is comonadic.*

2. *The forgetful functor has a right adjoint.*

---

[1] Given a morphism $f$, a pair $(p_1, p_2)$ of morphisms is a cokernel pair of $f$ is $(p_1, p_2)$ is a pushout of $(f, f)$.

### 4.1.3 Generalised Products of Coalgebras

In the following, we generalise the product of Kripke frames [43] or so-called *generalized product* [96], parametric in any strong endofunctor on a monoidal category.

**Definition 4.1.6.** Given Kripke frames $\langle X_i, R_i \rangle$, for $i = 1, 2$, the **product of Kripke frames** $\langle X_1, R_1 \rangle$ and $\langle X_2, R_2 \rangle$ consists of two Kripke frames

$$R_1^\vee := \{ (\langle x, z \rangle, \langle y, z \rangle) \mid x \ R_1 \ y \} \quad \text{and} \quad R_2^\vee := \{ (\langle x, y \rangle, \langle x, z \rangle) \mid y \ R_2 \ z \}$$

on the Cartesian product $X_1 \times X_2$.

The product of Kripke frames is used to study multi-modal logic, a modal logic with two modalities associated with two different Kripke structure on the Cartesian product $X_1 \times X_2$.

Observe that the product of Kripke frames gives rise to a coalgebra for $\mathcal{P} \times \mathcal{P}$ given by the universal property

$$\begin{array}{ccc}
 & & \mathcal{P}(X_1 \times X_2) \\
 & \nearrow^{f_{R_1^\vee}} & \uparrow \pi_1 \\
X_1 \times X_2 & \dashrightarrow & \mathcal{P}(X_1 \times X_2) \times \mathcal{P}(X_1 \times X_2) \\
 & \searrow_{f_{R_2^\vee}} & \downarrow \pi_2 \\
 & & \mathcal{P}(X_1 \times X_2)
\end{array} \qquad (4.1)$$

since $\mathcal{P}(X_1 \times X_2) \times \mathcal{P}(X_1 \times X_2)$ is equal to $(\mathcal{P} \times \mathcal{P})(X_1 \times X_2)$ where $f_{R_i^\vee}$ is the corresponding $\mathcal{P}$-coalgebra for the Kripke frame $\langle X_1 \times X_2, R_i^\vee \rangle$.

However, the product of Kripke frames is *not* categorically a product. Recall the definition of a strong functor:

**Definition 4.1.7** (see [64]). Let $(\mathscr{X}, \otimes, I)$ be a monoidal category with an associator $\alpha_{x,y,z} \colon (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z)$, a left unitor $\lambda_x \colon I \otimes x \xrightarrow{\cong} x$, and a right unitor $\rho_x \colon x \otimes I \xrightarrow{\cong} x$. An endofunctor $T$ of $\mathscr{X}$ is called **strong** if there exists a morphism, called *tensorial strength*,

$$t_{x,y} \colon x \otimes Ty \to T(x \otimes y)$$

natural in $x$ and $y$ satisfying the unit law and the associative law, i.e.

$$\begin{array}{ccc}
I \otimes Tx & \xrightarrow{t_{I,x}} & T(I \otimes x) \\
& {\scriptstyle \lambda_{Tx}} \searrow & \downarrow {\scriptstyle T\lambda_x} \\
& & Tx
\end{array}
\qquad \text{and}$$

$$\begin{array}{ccc}
& (x \otimes y) \otimes Tz \xrightarrow{\alpha_{x,y,Tz}} x \otimes (y \otimes Tz) & \\
{\scriptstyle t_{x \otimes y, z}} \swarrow & & \searrow {\scriptstyle x \otimes t_{y,z}} \\
T((x \otimes y) \otimes z) & & x \otimes T(y \otimes z) \\
{\scriptstyle T\alpha_{x,y,z}} \searrow & & \swarrow {\scriptstyle t_{x,y \otimes z}} \\
& T(x \otimes (y \otimes z)) &
\end{array}$$

commute for every $x, y$ and $z$.

**Proposition 4.1.8.** *For every strong endofunctor $T$ of a monoidal category $(\mathscr{X}, \otimes, I)$ with a tensorial strength, the operation defined by*

$$x \otimes_T \langle y, \gamma \rangle = x \otimes y \xrightarrow{x \otimes \gamma} x \otimes Ty \xrightarrow{t_{x,y}} T(x \otimes y)$$

*for any object $x$ and a coalgebra $\langle y, \gamma \rangle$; and $f \otimes_T g = f \otimes g$ for any morphism $f$ and a coalgebra homomorphism $g$ is a functor from $\mathscr{X} \times \mathscr{X}_T$ to $\mathscr{X}_T$.*[2]

*Proof.* It suffices to check that $g \otimes_T \langle x, \xi \rangle$ is a $T$-coalgebra homomorphism for any morphism $g \colon y \to z$ and coalgebra $\langle x, \xi \rangle$, and $x \otimes_T g$ is a coalgebra homomorphism for any object $x$ and coalgebra homomorphism $g$ so the functoriality of $\otimes_T$ is inherited from the functoriality of $\otimes$. As for $x \otimes_T g$ consider the following diagram

$$\begin{array}{ccccc}
& & \overset{x \otimes_T \langle y, \gamma \rangle}{\xrightarrow{\hspace{6cm}}} & & \\
x \otimes y & \xrightarrow{x \otimes \gamma} & x \otimes Ty & \xrightarrow{t_{x,y}} & T(x \otimes y) \\
{\scriptstyle x \otimes g} \downarrow & & \downarrow {\scriptstyle x \otimes Tg} & & \downarrow {\scriptstyle T(x \otimes g)} \\
x \otimes z & \xrightarrow[x \otimes \zeta]{} & x \otimes Tz & \xrightarrow[t_{x,z}]{} & T(x \otimes z) \\
& & \underset{x \otimes_T \langle z, \zeta \rangle}{\xrightarrow{\hspace{6cm}}} & &
\end{array}$$

where the left square commutes by the functoriality of $x \otimes -$ and the right square commutes by the naturality of $t_{x,-}$, so $g \otimes_T \langle x, \xi \rangle$ is a coalgebra homomorphism. As

---

[2] This proposition essentially says that the category $(\mathscr{X}, \otimes, I)$ has a monoidal action $\otimes_T$, see [54], on the category of coalgebras for any strong endofunctor $T$ of $\mathscr{X}$.

for $g \otimes \langle x, \xi \rangle$, consider the following diagram

$$
\begin{array}{ccccc}
& & y \otimes_T \langle x, \xi \rangle & & \\
& y \otimes x \xrightarrow{y \otimes \xi} y \otimes Tx \xrightarrow{t_{y,x}} T(y \otimes x) & & \\
g \otimes x \downarrow & g \otimes Tx \downarrow & & \downarrow T(g \otimes x) \\
& z \otimes x \xrightarrow[z \otimes \xi]{} z \otimes Tx \xrightarrow[t_{z,x}]{} T(z \otimes x) & & \\
& & z \otimes_T \langle x, \xi \rangle & &
\end{array}
$$

where the right square commutes by naturality of $t_{-,x}$ and the left square commutes by the functoriality of $\otimes$. $\qquad\square$

**Theorem 4.1.9.** *Let $(\mathscr{X}, \otimes, I)$ be a monoidal category with the left unitor $\lambda_y : I \otimes y \cong y$ and $T$ a strong functor with a tensorial strength $t_{x,y} : x \otimes Ty \to T(x \otimes y)$. For any morphism $\left( x \xrightarrow{h} I \right)$ to the unit $I$, the composite*

$$
\left( x \otimes y \xrightarrow{h \otimes y} I \otimes y \xrightarrow{\lambda_y} y \right) : (x \otimes_T \langle y, \gamma \rangle) \to \langle y, \gamma \rangle
$$

*is a $T$-coalgebra homomorphism for any object $x$ and $T$-coalgebra $\langle y, \gamma \rangle$.*

*Proof.* Consider the following diagram:

$$
\begin{array}{ccccc}
& & x \otimes_T \langle y, \gamma \rangle & & \\
x \otimes y & \xrightarrow{x \otimes \gamma} & x \otimes Ty & \xrightarrow{t_{x,y}} & T(x \otimes y) \\
h \otimes y \downarrow & & h \otimes Ty \downarrow & & \downarrow T(h \otimes y) \\
I \otimes y & \xrightarrow{I \otimes \gamma} & I \otimes Ty & \xrightarrow{t_{I,y}} & T(I \otimes y) \quad T(\lambda_y \circ h \otimes y) \\
\lambda_y \downarrow & & \lambda_{Ty} \downarrow & & \downarrow T\lambda_y \\
y & \xrightarrow{\gamma} & Ty & = & Ty \leftarrow
\end{array}
$$

where the upper-left square commutes by the functoriality of $\otimes$; the upper-right and the lower-left square commute by naturality; and the lower-right square commutes by the unit law of the strength. $\qquad\square$

If the monoidal category is symmetric, i.e. there exists a natural isomorphism $x \otimes y \cong y \otimes x$ satisfying the coherence condition, then the tensorial strength also has a *left strength* $t'_{x,y} : Tx \otimes y \to T(x \otimes y)$ natural in $x$ and $y$. Similarly, we may extend $\otimes$ to a functor $_T\otimes : \mathscr{X}_T \times \mathscr{X} \to \mathscr{X}_T$ using the left strength.

**Proposition 4.1.10.** *Let $\mathscr{X}$ be a category with binary products and a terminal object $\mathbf{1}$. The following statements hold:*

1. *Given an object $x$, and a coalgebra $\left(y \xrightarrow{\gamma} Ty\right)$ for a strong endofunctor $T$ of $(\mathscr{X}, \times, \mathbf{1})$, the projection $\pi_y \colon x \times y \to y$ is a coalgebra homomorphism, i.e. the diagram*

$$
\begin{array}{ccc}
x \times y & \xrightarrow{\ x \times_T \gamma\ } & T(x \times y) \\
{\scriptstyle \pi_y} \downarrow & & \downarrow {\scriptstyle T\pi_y} \\
y & \xrightarrow{\ \gamma\ } & Ty
\end{array}
$$

   *commutes.*

2. *Particularly, given strong endofunctors $T_i$ of $(\mathscr{X}, \times, \mathbf{1})$ and $T_i$-coalgebras $\langle X_i, \xi_i \rangle$ for $i = 1, 2$, the projection $\pi_i \colon x_1 \times x_2 \to x_i$ is a $T_i$-coalgebra homomorphism, i.e.*

$$
\begin{array}{ccc}
x_1 \times x_2 & \xrightarrow{\ X_{T_1} \times \xi_1\ } & T_1(x_1 \times x_2) \\
{\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle T_1 \pi_2} \\
x_1 & \xrightarrow{\ \xi_1\ } & T_1 x_1
\end{array}
\qquad and \qquad
\begin{array}{ccc}
x_1 \times x_2 & \xrightarrow{\ X \times_{T_2} \xi_2\ } & T_2(x_1 \times x_2) \\
{\scriptstyle \pi_2} \downarrow & & \downarrow {\scriptstyle T_2 \pi_2} \\
x_2 & \xrightarrow{\ \xi_2\ } & T_2 x_2
\end{array}
$$

   *commute.*

*Proof.* Note that $(\mathscr{X}, \times, \mathbf{1})$ forms a symmetric monoidal category.

1. By assumption, $\mathbf{1}$ is a terminal object, so every object $x$ has a unique morphism to $\mathbf{1}$, denoted $h$. Moreover, the projection map $\pi_y \colon x \times y$ may be realised as the composite

$$
x \times y \xrightarrow{\ h \times y\ } \mathbf{1} \times y \xrightarrow{\ \lambda_y\ } y
$$

   where $\lambda_y$ is the obvious isomorphism. Now, apply Theorem 4.1.9.

2. Since the tensor product $\times$ is symmetric, we have functors $\otimes_{T_2} \colon \mathscr{X} \times \mathscr{X}_{T_2} \to \mathscr{X}$ and $_{T_1}\!\otimes \colon \mathscr{X}_{T_1} \times \mathscr{X} \to \mathscr{X}$ given by $T_2$ and $T_1$ respectively. Then, use the first statement with $_{T_1}\!\otimes$ and $\otimes_{T_2}$ repeatedly.

$\square$

Every Set endofunctor has a *unique* tensorial strength with respect to the Cartesian closed structure by the composite:

$$
\begin{array}{ccc}
X \times TY & \xrightarrow{\hspace{4cm}} & T(X \times Y) \\
{\scriptstyle \eta \times id}\downarrow & & \uparrow{\scriptstyle ev} \\
\mathsf{Hom}(Y, X \times Y) \times TY & \xrightarrow[T_{Y,X \times Y} \times id]{} & \mathsf{Hom}(TY, T(X \times Y)) \times TY
\end{array}
$$

where *ev* and $\eta$ is the counit and the unit of the adjunction $\mathsf{Hom}(X \times Y, Z) \cong \mathsf{Hom}(X, \mathsf{Hom}(Y, Z))$, and $T_{Y,X \times Y}$ is the functor acting on the hom-set $\mathsf{Hom}(Y, X \times Y)$.[3]

Thus, we find that the product of Kripke frames is an application of Proposition 4.1.10:

**Corollary 4.1.11.** *Given two coalgebras* $\langle X_i, \xi_i \colon X_i \to TX_i \rangle$ *on* Set, *there is a coalgebra for* $T_1 \times T_2$ *with the carrier* $X_1 \times X_2$ *such that the projection* $\pi_i \colon X_1 \times X_2 \to X_i$ *is a* $T_i$*-coalgebra homomorphism for* $i = 1, 2$.

*Proof.* Using Proposition 4.1.10, we have two coalgebras for $T_1$ and $T_2$ with $X_1 \times X_2$ as the carrier respectively, and they amount to a coalgebra for $T_1 \times T_2$ by universal property in a similar way to (4.1).

$\square$

*Remark* 4.1.12. This construction on Set is discussed with the coalgebraic hybrid modal logic in [96], but it is in fact works for any strong functor with respect to the monoidal structure given by finite products.

### 4.1.4 Factorisation Systems on Coalgebras

One of the most useful perspectives of coalgebras is its power of formalising bisimilarity parametric in endofunctors $T$. As we have seen in Chapter 2, the classical notion of bisimilarity can be formalised in more than one way. The formalisation chosen impacts on the later development of the theory. In particular, we are interested in a notion dependent on a 'factorisation system' as the recent development on minimisation [15] suggests. This also covers the classic results about Set coalgebras using the standard (Surjection, Injection)-factorisation system, e.g. [46].

**Definition 4.1.13.** Given a right $(\mathcal{E}, \mathcal{M})$-factorisation system on the category of $T$-coalgebras, we define the following notions:

---

[3] Indeed, every Set functor determines a strength uniquely, see [64].

1. an $\mathcal{E}$-**minimal coalgebra** with respect to $(\mathcal{E}, \mathcal{M})$ is a coalgebra $\langle x, \xi \rangle$ without proper $\mathcal{E}$-quotient;

2. the **behavioural equivalence with respect to** $(\mathcal{E}, \mathcal{M})$ of a coalgebra $\langle x, \xi \rangle$ is the kernel in $\mathscr{X}$ of the greatest $\mathcal{E}$-quotient map of $\langle x, \xi \rangle$. Note that the greatest $\mathcal{E}$-quotient object must be $\mathcal{E}$-minimal.

3. $\mathcal{E}$-**minimisation** is the reflector $\nabla$ from $\mathscr{X}_T$ to the subcategory of $\mathcal{E}$-minimal coalgebras, if exists.

*Remark* 4.1.14. By Proposition 2.1.27, a Set coalgebra is *minimal* if and only if it has no proper quotient. Hence, we take the latter characterisation as our general definition. Moreover, the kernel of the reflection given by minimisation in Theorem 2.1.28 is behavioural equivalence by Corollary 2.1.29. A stronger condition on the factorisation system is needed to have a *coinduction principle*, see Section 4.1.5.

Given a functor $U$ and a collection $\mathcal{M}$ of morphisms, the notation $U^{-1}\mathcal{M}$ denotes the collection of morphisms whose image is in $\mathcal{M}$.

**Proposition 4.1.15.** *Given an $(\mathcal{E}, \mathcal{M})$-factorisation system on $\mathscr{X}$, the category of $T$-coalgebras has the $(U^{-1}\mathcal{E}, U^{-1}\mathcal{M})$-factorisation system, if $T$ preserves $\mathcal{M}$-morphisms.*

*Proof.* By Proposition 4.1.2, it is trivial to see that $U^{-1}\mathcal{E}$ and $U^{-1}\mathcal{M}$ are closed under isomorphisms and compositions.

Assume that $T$ preserves $\mathcal{M}$-morphisms. For every a $T$-coalgebra homomorphism $f : \langle x, \xi \rangle \rightarrow \langle y, \gamma \rangle$, the underlying morphism is factored into a composite

$$x \xrightarrow{f} y = x \xrightarrow{e} fx \xrightarrow{m} y$$

for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and by the diagonalisation property there is a unique morphism from $fx$ to $T(fx)$ as follow

$$
\begin{array}{ccccc}
x & \xrightarrow{\ \ e\ \ } & fx & \xrightarrow{\ \ m\ \ } & y \\
{\scriptstyle \xi}\downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} \\
Tx & \xrightarrow[Te]{} & T(fx) & \xrightarrow[Tm]{} & Ty.
\end{array}
$$

Therefore, every coalgebra homomorphism has an $(\mathcal{E}, \mathcal{M})$-factorisation. To see the diagonalisation property, considering the following diagram
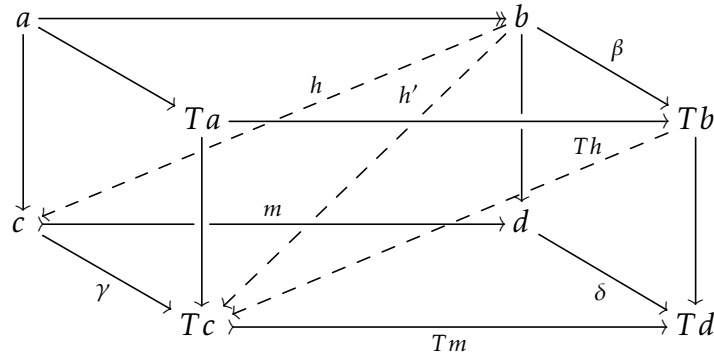


where $h$ and $h'$ are the unique morphisms filling the square in the underlying category. It remains to show that $h$ is a coalgebra homomorphism, and we simply need to verify that $\gamma \circ h$ and $Th \circ \beta$ also fill the diagonal from $b$ to $Tc$ by diagram chasing. Then, by the uniqueness of $h'$, it follows that $h$ is a coalgebra homomorphism.     □

**Corollary 4.1.16.** *Let $(\mathcal{E}, \mathcal{M})$ be a right factorisation system on a cocomplete and $\mathcal{E}$-cowellpowered category $\mathscr{X}$, and $T$ an endofunctor of $\mathscr{X}$. If $T$ preserves $\mathcal{M}$-morphisms, then $\mathscr{X}_T$ admits $(U^{-1}\mathcal{E})$-minimisation.*

*Proof.* This follows from Proposition 3.1.10.     □

**Example 4.1.17.** Minimal coalgebras with respect to a factorisation system given by an $\mathcal{M}$-morphism preserving functor are discussed in various places in the literature:

1. Minimal coalgebras considered in [46] are essentially minimal coalgebras with respect to the $(U^{-1}\mathsf{Surjections}, U^{-1}\mathsf{Injections})$-factorisation, since every $\mathsf{Set}$ endofunctor preserves injections with a non-empty domain and $\mathsf{Set}$ supports the standard factorisation system consisting of surjections and injections.[4]

2. The minimisation given in [15] is another construction of the reflective subcategory without using the cointersection of $\mathcal{E}$-quotients, connecting it to the known minimisation algorithms for automata.

---

[4] To use the corollary directly, consider that the category of $T'$-coalgebras is isomorphic to $\mathscr{X}_T$ for an injection-preserving functor $T'$ which is naturally isomorphic to $T$ on non-empty sets by Proposition 2.1.4.

## 4.1.5 Minimality and Behavioural Equivalence

In general, behavioural equivalence with respect to a right $(\mathcal{E}, \mathcal{M})$-factorisation system on Set does not generalise behavioural equivalence for Set in the sense of Proposition 2.1.27:

**Example 4.1.18.** Consider the trivial *right* factorisation system (isomorphism, morphism) on the category of $\mathcal{P}$-coalgebras. Every $\mathcal{P}$-coalgebra is itself the greatest $\mathcal{E}$-quotient, since every $\mathcal{E}$-map is an isomorphism. Thus, every behavioural equivalence of a $\mathcal{P}$-coalgebra $\langle X, \xi \rangle$ with respect to this factorisation system is simply the kernel of the identity map on $\langle X, \xi \rangle$, that is, equality on the carrier. Obviously, any two bisimilar elements are not necessarily equal.

To remedy the insufficiency for describing behavioural equivalence (or bisimilarity), we require that every coalgebra $\mathcal{M}$-homomorphism is also monic in the underlying category. This extra requirement is sufficient for us to obtain *the coinduction principle*:

**Theorem 4.1.19** (The Coinduction Principle). *Let $\mathcal{X}$ be a category with pullbacks and a right $(\mathcal{E}, \mathcal{M})$-factorisation system on $\mathcal{X}_T$ such that $U\mathcal{M} \subseteq \mathsf{Mono}\mathcal{X}$. Given a $T$-coalgebra $\langle x, \xi \rangle$, the behavioural equivalence of $\langle x, \xi \rangle$, if exists, is the greatest kernel among morphisms from $\langle x, \xi \rangle$.*

*Proof.* Let $r_\xi \colon \langle x, \xi \rangle \to \nabla \langle x, \xi \rangle$ be a greatest $\mathcal{E}$-quotient. For any coalgebra homomorphism $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$, consider the following diagram in the underlying category:
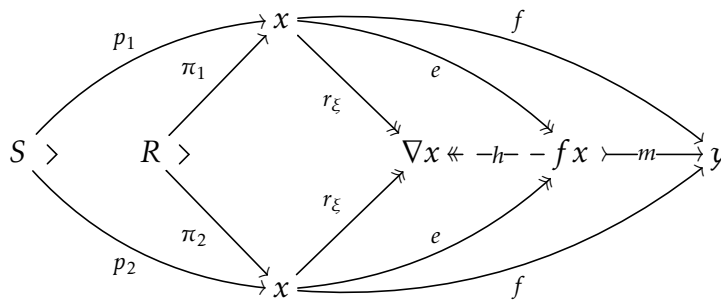
where $f$ is factored as $f = me$, $(\pi_1, \pi_2)$ is the kernel pair of $r_\xi$, and $(p_1, p_2)$ is the kernel pair of $f$. By construction, we derive $r_\xi \circ p_1 = r_\xi \circ p_2$ from the kernel pair of $f$:

$$
\begin{aligned}
& f p_1 = f p_2 \\
\Longleftrightarrow\; & (me)p_1 = (me)p_2 && \{\text{by factorisation}\} \\
\Longrightarrow\; & e p_1 = e p_2 && \{m \text{ is monic in } \mathscr{X}\} \\
\Longrightarrow\; & h e p_1 = h e p_2 && \{\text{by post-composing with } h\} \\
\Longrightarrow\; & r_\xi p_1 = r_\xi p_2 && \{\text{by the greatest quotient } \nabla x\}
\end{aligned}
$$

so there is a unique morphism $i$ from $S$ to $R$ such that $p_1 = \pi_1 i$ and $p_2 = \pi_2 i$. $\quad\square$

**Corollary 4.1.20.** *For any* Set*-coalgebra $\langle x, \xi \rangle$, the behavioural equivalence with respect to the standard $(U^{-1}\text{Surjections}, U^{-1}\text{Injections})$-factorisation system is the behavioural equivalence in the sense of Definition 2.1.5.*

*Proof.* Recall Proposition 2.1.7. $\quad\square$

Let us close this subsection by giving a condition which guarantees the existence of a *small* final coalgebra, making use of Proposition 3.1.12 and Proposition 4.1.15:

**Corollary 4.1.21.** *Let $(\mathcal{E}, \mathcal{M})$ be a proper factorisation system on a cocomplete and $\mathcal{M}$-well-powered category $\mathscr{X}$, and $T$ an $\mathcal{M}$-preserving endofunctor of $\mathscr{X}$. The following are equivalent:*

1. *A final coalgebra exists.*

2. *A complete lattice, regarded as a category, equivalent to the reflective subcategory of $(U^{-1}\mathcal{E})$-minimal $T$-coalgebras exists.*

*Remark* 4.1.22. To summarise, the notion of behavioural equivalence is parametric in right $(\mathcal{E}, \mathcal{M})$-factorisation systems and it has the coinduction principle if $\mathcal{M}$-morphisms are monic in the underlying category. One might be justified to conclude that the use of a final coalgebra for behavioural equivalence is rather heavy.

## 4.1.6 Coalgebras of Accessible Functors

Given an accessible endofunctor $T$ of a locally presentable category, the category of $T$-coalgebras is well-behaved by the following theorem:[5]

---

[5] The original statement in [9, Theorem 4.2] uses an *accessible category* with $\omega$-chains. Here, we simplify the situation by using a locally presentable category and the fact that the forgetful functor creates colimits.

**Theorem 4.1.23** ([9, Theorem 4.2]). *Let $\mathcal{K}$ be a locally $\lambda$-presentable category for an uncountable regular cardinal $\lambda$. Then, for every $\lambda$-accessible endofunctor $T$ of $\mathcal{K}$*

1. *a coalgebra is $\lambda$-presentable in $\mathcal{K}_T$ if and only if its underlying object is $\lambda$-presentable in $\mathcal{K}$ and*

2. *the category $\mathcal{K}_T$ is $\lambda$-accessible.*

**Corollary 4.1.24.** *Given a locally presentable category $\mathcal{K}$ and an accessible functor $T$, the forgetful functor $U \colon \mathcal{K}_T \to \mathcal{K}$ has a right adjoint $G$. In particular, the final coalgebra exists as the value of $G$ on a final object in $\mathcal{K}$.*

*Proof.* According to Theorem 3.4.14, a functor between locally presentable categories has a right adjoint if and only if it preserves colimits. Since $\mathcal{K}$ is cocomplete, the forgetful functor preserves all colimits and thus this statement follows. $\square$

The right adjoint is isomorphic to the functor defined by

$$Gx := \mathrm{Colim}\,((U{\downarrow}x) \to \mathcal{K}) = \int^{\langle y, \gamma \rangle} \mathcal{K}(y, x) \cdot \langle y, \gamma \rangle \tag{4.2}$$

for every $x \in \mathcal{K}$ by the formal criterion of the existence of an adjoint [81, Theorem 2]. Using the Special Adjoint Functor Theorem, the final coalgebra can be regarded as a coequaliser of the coproduct of the $\lambda$-presentable coalgebras. This follows in the same spirit as [3, 19].

Another useful fact related to minimisation is the existence of the reflective subcategory of minimal coalgebras:

**Proposition 4.1.25.** *For any accessible endofunctor $T$ of a locally presentable category $\mathcal{K}$, the category $\mathcal{K}_T$ admits $\mathsf{Epi}$-minimisation.*

*Proof.* Every locally presentable category has the $(\mathsf{Epi}, \mathsf{StrongMono})$-factorisation system by Corollary 3.4.16 and every factorisation system determines a reflective subcategory by Proposition 3.1.8. $\square$

## 4.2 Transfinite Induction

The usual techniques for constructing a final coalgebra, an initial algebra, or free algebras are actually based the same categorical construction. In [60], M. Kelly

provides a comprehensive study of transfinite induction for *well-pointed* endofunctors and reduces other situations such as *pointed* endofunctors and (mere) endofunctors to the well-pointed case. In this section, we present the situation for mere endofunctors for later use in detail.

## 4.2.1 The Free Algebra Sequence

We denote the category of *small* ordinals, i.e. ordinals which are small sets, by Ord. Formally, Ord is an inaccessible cardinal and also an ordinal, so we can apply transfinite induction on objects of Ord.

**Definition 4.2.1** (see [60]). Given an endofunctor $L$ of a cocomplete category $\mathscr{A}$, the **free algebra sequence** $\widehat{L}\colon \mathrm{Ord} \to [\mathscr{A}, \mathscr{A}]$ is defined by mutual transfinite recursion on objects and morphisms, where $\widehat{L}\kappa$ and $\widehat{L}(\kappa \xrightarrow{f} \sigma)$ are written as $L_\kappa$ and $f_\sigma^\kappa$ for convenience:

**Objects**: define objects $L_\kappa$ by induction on $\kappa$:

   **Zero ordinal**: $L_0 := \mathcal{I}$.

   **Successor ordinals**: $L_{\kappa+1} := \mathcal{I} + LL_\kappa$ for every $\kappa \in \mathrm{Ord}$.

   **Limit ordinals**: for a limit ordinal $\lambda$, define $L_\lambda$ to be a colimit of the morphisms

$$\{ f_\sigma^\kappa \colon L_\kappa \to L_\sigma \mid \kappa \leq \sigma < \lambda \}$$

   where the $f_\sigma^\kappa$ are defined as follows:

**Morphisms**: for each ordinal $\kappa \leq \sigma < \lambda$, define $f_\sigma^\kappa \colon L_\kappa \to L_\sigma$ by induction on $\kappa$ omitting the case $f_\kappa^\kappa = id$:

   **Zero ordinal**: define
$$f_{\sigma+1}^0 = \mathrm{inl} \colon L_0 \to L_{\sigma+1}$$

   to be the injection; for a limit ordinal $\alpha < \lambda$ define $f_\alpha^0$ to be the morphism from $L_0$ to $L_\alpha$ in the limiting cocone.

   **Successor ordinals**: define

$$f_{\sigma+1}^{\kappa+1} := id + L f_\sigma^\kappa \colon \mathcal{I} + LL_\kappa \to \mathcal{I} + LL_\sigma$$

   for $\kappa < \sigma$; for a limit ordinal $\alpha < \lambda$ define $f_\alpha^{\kappa+1}$ to be the morphism from $L_{\kappa+1}$ to $L_\alpha$ in the limiting cocone of $L_\alpha$.

**Limit ordinals**: for a limit ordinal $\alpha < \lambda$ and $\alpha < \sigma < \lambda$, define

$$f_\sigma^\alpha : L_\alpha \dashrightarrow L_\sigma$$

to be the mediating morphism from the colimit $L_\alpha$ to the vertex of the cocone $(f_\sigma^\kappa)_{\kappa < \alpha}$ given by induction hypothesis.

Note that strictly speaking it is necessary to show that $\widehat{L}$ is a functor from Ord, i.e. $f_\lambda^\kappa \circ f_\kappa^\sigma = f_\lambda^\sigma$. For convenience, the transfinite sequence $\widehat{L}$ on an object $a$ is denoted by $\widehat{L}a$ by currying.

The free algebra sequence on an object $a \in \mathscr{A}$ may be considered as the following informal expression

$$a + L(a + L(a + \cdots)).$$

We say that the sequence *stabilises* on an object $a \in \mathscr{A}$ if the connecting morphism $f_{\kappa+1}^\kappa : L_\kappa a \to L_{\kappa+1} a$ is an isomorphism for some ordinal $\kappa$.

**Definition 4.2.2.** An $L$-algebra $\alpha : La \to a$ is **free** over some object $x$ if it is the initial object in the comma category $(x \downarrow U)$ for the forgetful functor $U : \mathscr{A}^L \to \mathscr{A}$.

That is, there exists a morphism $i : x \to a$ injecting $x$ into $a$ such that every $L$-algebra $\beta : Lb \to b$ with a morphism $f : x \to b$ has a unique $L$-algebra homomorphism $\overline{f} : \langle a, \alpha \rangle \to \langle b, \beta \rangle$ such that $f = \overline{f} \circ i$.

**Theorem 4.2.3** (see [60]). *Given an endofunctor $L$ of a cocomplete category $\mathscr{A}$ and an object $a \in \mathscr{A}$, if the free algebra sequence of $L$ over $a$ stabilises at $\kappa$ with the inverse*

$$[i, \alpha] : a + LL_\kappa a \to L_\kappa a,$$

*then the morphism $\alpha : LL_\kappa a \to L_\kappa a$ is the free $L$-algebra over $a$ with $i : a \to L_\kappa a$ injecting $a$ into $L_\kappa a$.*

*Proof.* Let $\left( Lb \xrightarrow{\beta} b \right)$ be a $L$-algebra with a morphism $g : a \to b$. First we construct a cocone

$$(g_\kappa : L_\kappa a \to b)_{\kappa \in \mathrm{Ord}}$$

by transfinite induction as follows.

**basic step**: define $g_0 = g$;

**successor ordinal** $\kappa + 1$: define $g_{\kappa+1}$ as the composite $[id_b, \beta] \circ (g + Lg_\kappa)$ in the following diagram

$$
\begin{array}{ccc}
L_\kappa a & \xrightarrow{\;f^\kappa_{\kappa+1}\;} & a + LL_\kappa a \\
\scriptstyle g_\kappa \downarrow & & \downarrow \scriptstyle g+Lg_\kappa \\
b & \xleftarrow{\;[id_b,\beta]\;} & b + Lb
\end{array}
$$

where $[id_b, \beta]$ is the mediating morphism given by $id_b \colon b \to b$ and $\beta \colon Lb \to b$;

**limit ordinal** $\lambda$: define $g_\lambda$ as the meditating morphism from the colimit $L_\lambda a$ of $(L_\kappa a)_{\kappa<\lambda}$ to $b$ given by

$$
\left( L_\kappa \xrightarrow{\;g_\kappa\;} b \right)_{\kappa<\lambda}
$$

which is a cocone of $(L_\kappa a)_{\kappa<\lambda}$ by construction.

Assume that the sequence $(L_\kappa a)_{\kappa \in \mathsf{Ord}}$ stabilises at some ordinal $\gamma$ with $[i, \alpha] \colon a + LL_\gamma a \to L_\gamma a$ as the inverse of $f^\gamma_{\gamma+1}$. Then, the diagram

$$
\begin{array}{ccc}
L_\gamma a & \xrightarrow{\;f^\gamma_{\gamma+1}\;} & a + LL_\gamma a \\
\scriptstyle g_\gamma \downarrow & & \downarrow \scriptstyle g+Lg_\gamma \\
b & \xleftarrow{\;[id_b,\beta]\;} & b + Lb
\end{array}
\qquad \text{commutes so does} \qquad
\begin{array}{ccc}
L_\gamma a & \xleftarrow{\;[i,\alpha]\;} & a + LL_\gamma a \\
\scriptstyle g_\gamma \downarrow & & \downarrow \scriptstyle g+Lg_\gamma \\
b & \xleftarrow{\;[id_b,\beta]\;} & b + Lb.
\end{array}
$$

But, the diagram on the right simply says that $g_\gamma$ is an $L$-algebra homomorphism from $\langle L_\gamma a, \alpha \rangle$ to $\langle b, \beta \rangle$ and $g = g_\gamma \circ i$. Moreover, $f^0_\gamma = i$ by construction. Now it remains to show uniqueness.

Let $h \colon \langle L_\gamma a, \alpha \rangle \to \langle b, \beta \rangle$ be another $L$-algebra homomorphism with $g = h \circ i$. We show that $g_\kappa = h \circ f^\kappa_\gamma$ for each ordinal $\kappa$ by transfinite induction so that it follows

$$
g_\gamma = h \circ f^\gamma_\gamma = h \circ id = h.
$$

For the basic step, $g = h \circ i = h \circ f^0_\gamma$ holds trivially as $f^0_\gamma = i$; and for limit ordinals it is also trivial. For any ordinal $\kappa$, note that $g_{\kappa+1} = [id_b, \beta] \circ (g + Lg_\kappa)$ by definition and consider the diagram

$$
\begin{array}{ccc}
 & a + LL_\kappa a & \\
\scriptstyle f^{\kappa+1}_\gamma \nearrow & \downarrow \scriptstyle id+Lf^\kappa_\gamma & \\
L_\gamma a \xleftarrow{\;[i,\alpha]\;} a + LL_\gamma a & & \Big\} \scriptstyle g+Lg_\kappa \\
\scriptstyle h \downarrow & \downarrow \scriptstyle g+Lh & \\
b \xleftarrow{\;[id,\beta]\;} b + Lb & &
\end{array}
$$

where $id + Lf_\gamma^\kappa = f_{\gamma+1}^{\kappa+1}$; the right square commutes by induction hypothesis $g_\kappa = h \circ f_\gamma^\kappa$; and the lower square commutes by assumption. The upper triangle also commutes, because $f_{\gamma+1}^\gamma \circ f_\gamma^{\kappa+1} = f_{\gamma+1}^{\kappa+1}$ and $[i, \alpha]$ is the inverse of $f_{\gamma+1}^\gamma$. Therefore, $g_{\kappa+1} = h \circ f_\gamma^{\kappa+1}$. $\qquad\qquad\square$

**Example 4.2.4.** An **initial sequence** of an endofunctor $L$ is the free algebra sequence $\widehat{L0}$ over an initial object $\mathbf{0}$.

**Example 4.2.5.** Let $\mathscr{X}$ be a complete category and $T$ an endofunctor of $\mathscr{X}$. A **final sequence** of $T$ is a free algebra sequence of $T^{\mathrm{op}} \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{X}^{\mathrm{op}}$ over the terminal object $\mathbf{1}$ of $\mathscr{X}$, i.e. a final sequence is $\widehat{T^{\mathrm{op}}\mathbf{1}}$. By $\mathbf{1} \times x \cong x$ for any object $x$, the sequence is the usual final sequence defined in [19, 109]:

$$\mathbf{1} \xleftarrow{\ \ f_1^0\ \ } T\mathbf{1} \xleftarrow{\ \ f_2^1\ \ } T^2\mathbf{1} \longleftarrow \cdots$$

where $f_1^0$ is the unique morphism to the terminal object. If the final sequence stabilises at $\lambda$, the inverse of the isomorphism $f_{\lambda+1}^\lambda \colon T^{\lambda+1}\mathbf{1} \to T^\lambda\mathbf{1}$ is a final $T$-coalgebra.

**Proposition 4.2.6.** *Given a finitary endofunctor $L$ on a cocomplete category $\mathscr{A}$, the free algebra sequence of $L$ over any object $a \in \mathscr{A}$ stabilises at $\omega$, i.e. the first infinite limit ordinal.*

*Proof.* It suffices to show that the next object $L_{\omega+1}a$ is a colimit of the sequence of first $\omega$ objects, so by construction it is isomorphic to $L_\omega a$ with the mediating morphism as the isomorphism.

By assumption, $\left(LLf_\omega^i \colon LL_i a \to LL_\omega a\right)_{i \in \omega}$ is a colimit, so by the Parameter Theorem [81, Theorem V.3.1], the following cocone is a colimit of $(L_{i+1}a)_{i \in \omega}$

$$\left(a + LL_i a \xrightarrow{\ id_a + LLf_\omega^i = f_{\omega+1}^{i+1}\ } a + LL_\omega a\right)_{i \in \omega}.$$

It remains to show that $f_{\omega+1}^0 = f_{\omega+1}^1 \circ f_1^0$ with the limiting cocone $(f_{\omega+1}^{i+1})_{i \in \omega}$ is a colimit of $(L_i a)_{i \in \omega}$. However, given any cocone $(\sigma_i)_{i \in \omega}$ from $(L_i a)_{i \in \omega}$ to some $b$, it is easy to see that $\sigma_0 = h \circ f_{\omega+1}^0$ by diagram chasing

where $h$ is the mediating morphism from the colimit $L_{\omega+1}a$ and $\sigma_{i+1} = h \circ f_{\omega+1}^{i+1}$ for $i \in \omega$ by construction. Thus $L_{\omega+1}a$ is a colimit of the first $\omega$-objects in the sequence. $\qquad\square$

Similarly, the free algebra sequence of any $\lambda$-accessible endofunctor stabilises at $\lambda$ using the same argument.

**Corollary 4.2.7.** *For any $\lambda$-accessible functor $L$ of a cocomplete category $\mathscr{A}$, the forgetful functor $U^L \colon \mathscr{A}^L \to \mathscr{A}$ has a left adjoint. Moreover, it is monadic.*

*Proof.* Define a mapping $F_0$ on objects by assigning to $a$ the $\lambda$-th object $L_\lambda a$ with the injecting map $i \colon a \to L_\lambda a$ as the unit. By [81, Theorem 2], there is a left adjoint $F$ to $U$ with $Fa = F_0 a$. By the dual of Proposition 4.1.5, $U^L$ is also monadic. $\qquad\square$

*Remark* 4.2.8. By the dual of Proposition 4.1.5, the forgetful functor $U \colon \mathscr{A}^L \to \mathscr{A}$ is monadic for any $\lambda$-accessible functor $L$. Moreover, the induced monad is a free monad over $L$. However, given a monadic and $\lambda$-accessible functor[6] $U \colon \mathscr{A} \to \mathsf{Set}$, the composite

$$\left( \mathscr{A}^L \xrightarrow{U_L} \mathscr{A} \xrightarrow{U} \mathsf{Set} \right)$$

does not need to be monadic.

## 4.2.2 Minimisation via Transfinite Sequence

For any endofunctor $T$ of **Set**, not every interesting final sequence stabilises at the first non-empty limit ordinal $\omega$ or stabilises at all—there might be no final coalgebra, as the example of the covariant powerset functor shows. On the other hand, minimal coalgebras always exist.

Given a right $(\mathcal{E}, \mathcal{M})$-factorisation system on a complete and $\mathcal{E}$-cowellpowered category $\mathscr{X}$, we can further construct minimal coalgebras from the final sequence:

**Theorem 4.2.9** (see [15]). *Given a right $(\mathcal{E}, \mathcal{M})$-factorisation system on a complete category $\mathscr{X}$ which is $\mathcal{E}$-cowellpowered, and an $\mathcal{M}$-preserving endofunctor $T$, then $\mathscr{X}_T$ admits $(U^{-1}\mathcal{E})$-minimisation.*

So far, we have collected several conditions for minimisation, i.e. Proposition 3.1.8, Proposition 3.1.10, Corollary 4.1.16, Proposition 4.1.25, and Theorem 4.2.9, using either colimits or limits; with or without a terminal object, $\mathcal{M}$-preservation, and $\mathcal{M}$-wellpoweredness or $\mathcal{E}$-cowellpoweredness.

---

[6] That is, $\mathscr{A}$ is a variety of single-sorted algebras.

# 4.3   Algebras

For the use of equational logics, we characterise the isomorphism between the category of (co)algebras for a functor $L$ and the category of (co)algebras for the (co)monad $\mathbb{L}$ induced by $L$, also continuing Section 4.1.2 and the free algebra construction.

Then, we discuss categorical properties of varieties of $(\Sigma, E)$-algebras. In addition to Example 3.4.8, we characterise every algebra as a *sifted colimit* of finitely generated *free algebras*. The class of *finitely based* endofunctors on a variety, which are functors preserving sifted colimits, is introduced.

According to Remark 4.2.8, the composite of finitary monadic functors is not necessarily monadic. On the other hand, if an endofunctor $L$ of a variety is finitely based, then the previous composite is monadic, see Theorem 4.3.35.

*Remark* 4.3.1. The notion of *finitely based* functor was first introduced by Velebil and Kurz [104] in the context of enriched category theory. In spite of the generality developed in *op. cit.*, our discussion is restricted to varieties of single-sorted algebras with an eye towards applications in Chapter 5. Another approach via the presentation of a functor is given in [30] and they coincide in varieties.


## 4.3.1   Free Monads

By Remark 4.2.8, one might expect the converse, namely the category of algebras of a given free monad over an endofunctor $L$ is isomorphic to the category of $L$-algebras, but this is not true as pointed out by Barr [60, Section 22.2].

**Definition 4.3.2.** Given an endofunctor $L$ of some category $\mathscr{A}$, a monad $\mathbb{M} = (M, \mu, \eta)$ on $\mathscr{A}$ with a natural transformation $(\theta\colon L \overset{\cdot}{\to} M)$ is **algebraically-free** over $L$ if every $L$-algebra $(La \overset{\alpha}{\to} a)$ factors though $\theta_a$ via a unique $\mathbb{M}$-algebra $Ma \overset{\bar{\alpha}}{\to} a$ and every $L$-algebra homomorphism $f\colon (a, \bar{\alpha}) \to (b, \bar{\beta})$ is also an $\mathbb{M}$-algebra homomorphism, i.e.

$$
\begin{array}{ccc}
La \xrightarrow{\theta_a} Ma & & \\
\quad \searrow_{\alpha} \quad \Big\downarrow{\exists!\,\bar{\alpha}} & \text{and} & \\
\qquad\qquad a & &
\end{array}
$$

commutes.

That is, the functor from the category of $\mathbb{M}$-algebras to the category of $L$-algebras, given by $\theta$, is an isomorphism. Correspondingly, we have **cofree comomad** and **coalgebraically-cofree comonad** as duals to *free monad* and *algebraically-free monad*.[7]

**Proposition 4.3.3.** *Let* $L \colon \mathscr{A} \to \mathscr{A}$ *be an endofunctor on* $\mathscr{A}$. *The following statements are equivalent:*

1. *There exists an algebraically-free monad over* $L$.

2. *The forgetful functor* $\mathscr{A}^L \to \mathscr{A}$ *has a left adjoint.*

*Proof.* For any algebraically-free monad $\mathbb{M}$ over $L$, the forgetful functor $\mathscr{A}^L \to \mathscr{A}$ has a left adjoint as it is isomorphic to the category of $\mathbb{M}$-algebras.

Conversely, suppose that the forgetful functor $U \colon \mathscr{A}^L \to \mathscr{A}$ has a left adjoint $F$ with $\eta$ as the unit and $\epsilon$ as the counit. Let $Fa \colon LUFa \to UFa$ denote the free algebra over $a$. The composite

$$\theta_a \colon La \xrightarrow{L\eta_a} LUFa \xrightarrow{Fa} UFa$$

defines a natural transformation from $L$ to $UF$. By Proposition 4.1.5, $U$ is monadic, so the comparison functor $K \colon \mathscr{A}^L \to \mathscr{A}^{UF}$ defined by

$$K(La \xrightarrow{\alpha} a) = UFa \xrightarrow{U\epsilon_\alpha} a \quad \text{and} \quad Kf = f$$

is an isomorphism. We show that the pre-composition functor $\theta^* \colon \mathscr{A}^{UF} \to \mathscr{A}^L$ is the left inverse of $K$, so the induced monad $(UF, U\epsilon F, \eta)$ is algebraically-free over $L$. Given an $L$-algebra $(La \xrightarrow{\alpha} a)$, consider its image under $\theta^* \circ K$ in the following diagram



where the upper rectangle commutes by the triangular identities, the left square commutes by $U\alpha = a$; and the right square commutes since $\epsilon_\alpha$ is an $L$-algebra homomorphism from $FU\alpha$ to $\alpha$. To sum up, $\alpha = (\theta^* \circ K)\alpha$, so the statement follows. $\square$

---

[7] Algebraically free monads coincide with free monads in a complete and locally small category, see [60].

## 4.3.2 Algebras of Finitary Functors over Varieties

To motivate the subsequent subsections, we walk through algebras of a finitary endofunctor of some variety informally. First, let $\mathscr{A}$ be a variety of single-sorted algebras with a forgetful functor $U \colon \mathscr{A} \to \mathsf{Set}$ and a left adjoint $F$ to it, and $\mathscr{A}_\omega$ the full subcategory consisting of finitely generated algebras.

Given a finitary endofunctor $L$ of $\mathscr{A}$, consider the coend formula of $L$ for an $L$-algebra

$$\alpha \colon \int^{K \in \mathscr{A}_\omega} \mathscr{A}(K, A) \cdot LK \to A,$$

which corresponds uniquely to a family of mappings

$$\left( \mathscr{A}(K, A) \xrightarrow{\alpha_K} \mathscr{A}(LK, A) \right)_{K \in \mathscr{A}_\omega}$$

indexed by $\mathscr{A}_\omega$ because of the following one-to-one correspondence

$$\mathsf{Hom}\!\left( \int^K \mathscr{A}(K, A) \cdot LK, A \right) \cong \int_K \mathsf{Hom}(\mathscr{A}(K, A) \cdot LK, A) \cong \int_K \mathsf{Set}(\mathscr{A}(K, A), \mathscr{A}(LK, A)).$$

Every morphism $\sigma$ from the free algebra $F1$ to $LK$ defines a *$K$-ary* operation[8] $\overline{\sigma}$ by

$$\frac{\dfrac{K \xrightarrow{a} A}{F1 \xrightarrow{\alpha_K(a) \circ \sigma} A}}{1 \xrightarrow{\overline{\sigma}(a)} UA}$$

mapping each morphism $a \colon K \to A$ to an element of $A$. Note that such a morphism $\sigma$ may be identified with an element of $LK$.

Thus, an $L$-algebra precisely consists of a carrier $A$ and $K$-ary operations

$$\left( \mathscr{A}(K, A) \xrightarrow{\overline{\sigma}} UA \right)_{\sigma \in LK}$$

satisfying certain equations for each $K \in \mathscr{A}_\omega$ and $\sigma \in LK$. For $K = Fn$, a $K$-ary operation gives rise to an $n$-ary operation $UA^n \to UA$, in the sense of universal algebra, by the bijection $\mathscr{A}(Fn, A) \cong \mathsf{Set}(n, UA)$. Moreover, each $L$-algebra homomorphism $f \colon \langle A, \alpha \rangle \to \langle B, \beta \rangle$ is precisely an operation-preserving morphism.

---

[8] It is an *operation* in the sense of Linton's categorical algebra [79] and in particular Kelly and Power's notion [62]. Also see [7] for a recent study for finitary functors on locally finitely presentable categories.

*Remark* 4.3.4. In some situations, every finitely generated object is free, e.g. finite-dimensional vector spaces, finite sets, and every finitely generated Boolean algebra, except the trivial Boolean algebra 1 [76]. In that case, every $K$-ary operation boils down to an $n$-ary operation $UA^n \to UA$ by the adjunction $\mathscr{A}(Fn, A) \cong \mathsf{Set}(n, UA)$. However, it does not hold in general.

### 4.3.3 Varieties of Algebras

**Density Presentation for Varieties of Algebras**

**Definition 4.3.5.** Similar to filtered categories defined in (3.7), a category $\mathscr{S}$ is **sifted** if for every functor $F \colon \mathscr{S} \times \mathscr{J} \to \mathsf{Set}$ where $\mathscr{J}$ is finite and discrete, the canonical morphism

$$\underset{s \in \mathscr{S}}{\mathrm{Colim}} \prod_{j \in \mathscr{J}} F(s, j) \to \prod_{j \in \mathscr{J}} \underset{s \in \mathscr{S}}{\mathrm{Colim}} F(s, j) \tag{4.3}$$

is an isomorphism. A **sifted colimit** is a colimit of a sifted diagram.

**Theorem 4.3.6** (see [11, Theorem 2.15]). *A small category $\mathscr{D}$ is sifted if and only if it is not empty and for every pair $\langle a, b \rangle$ of objects, the comma category $(\langle a, b \rangle \downarrow \Delta)$ for the diagonal functor $\Delta \colon \mathscr{D} \to \mathscr{D} \times \mathscr{D}$ is (non-empty and) connected.*

**Example 4.3.7.** Every filtered colimit and *reflexive coequaliser* is sifted. A **reflexive coequaliser** is a coequaliser of a pair of morphisms with a common section $s$, i.e. it is a colimit of the diagram

$$a \underset{g}{\overset{f}{\underset{\longleftarrow s \longrightarrow}{\rightrightarrows}}} b \circlearrowleft id$$

where $fs = gs = id$. Every colimit of a filtered diagram commutes with finite limits in Set including finite products, so filtered colimits are sifted. Reflexive coequalisers are also examples, as it is easy to verify by Theorem 4.3.6.

**Example 4.3.8.** For every monadic functor $U \colon \mathscr{A} \to \mathscr{X}$, the canonical presentation of an object $A \in \mathscr{A}$

$$FUFUA \xrightarrow[\ FU\epsilon_A\ ]{\overset{\epsilon_{FUA}}{\longrightarrow}} FUA \xrightarrow{\ \epsilon_A\ } A \tag{4.4}$$

is a reflexive coequaliser with a common section $F\eta UA$ by the triangular identities.

**Lemma 4.3.9.** *Every finitary and monadic functor $U \colon \mathscr{A} \to \mathsf{Set}$ creates sifted colimits.*

*Proof.* Let $H\colon \mathscr{J} \to \mathscr{A}$ be a sifted diagram and $(\tau_i\colon UH_i \to C)_{i\in\mathscr{J}}$ a colimit of $UH$. Without loss of generality, we assume that $C$ is a quotient of the disjoint union $UH_i$. For any $n$-ary operation $\sigma$, the canonical morphism $h\colon \mathrm{Colim}\,(UH_i)^n \to C^n$ is an isomorphism by definition. Thus, we can define an $n$-ary operation $\sigma_C$ on $C$ by

$$
\begin{array}{ccc}
\mathrm{Colim}\,(UH_i)^n & \xleftarrow{\;h^{-1}\;} & C^n \\
{\scriptstyle \mathrm{Colim}(\sigma_{H_i})}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma_C} \\
\mathrm{Colim}\,(UH_i) & \xrightarrow[\;\;id\;\;]{} & C.
\end{array}
$$

To see that each $\tau_i$ is a homomorphism, let $\sigma$ be an $n$-ary operation and consider the diagram



where $\mu$ is the limiting cocone of $UH^n$. By diagram chasing, we know that the function $\tau_i$ is an $\mathscr{A}$-homomorphism. Since $\mathrm{Colim}\,(UH_i)^n$ is a quotient of the disjoint union $(UH_i)^n$ so that every $n$-tuple $([a_i])_{i\in n}$ of $C$ must be mapped by some $\tau_i^n$. Hence $\sigma_C$ is uniquely defined, because $\tau_i$ is a homomorphism. Therefore, we have shown that there is a unique cocone $\tau'$ of $H$ satisfying $U\tau' = \tau$.

Finally, we show that $\tau'$ is indeed a limiting cocone. Let $(\lambda_i\colon H_i \to A)_i$ be a cocone and $f\colon UC \to UA$ a mediating *function* for $U\lambda$. To check that $f$ is a homomorphism, it suffices to check that $\sigma_A \circ \lambda_i^n = \lambda_i \circ \sigma_{H_i}$ for each $\sigma$ because of the isomorphism $h\colon \mathrm{Colim}(UH_i)^n \to C^n$, but this is just the assumption. $\qquad\square$

**Lemma 4.3.10.** *Given a finitary and monadic functor* $U\colon \mathscr{A} \to \mathsf{Set}$ *with a left adjoint* $F$, *the* $\mathsf{Set}$*-valued functor*

$$\mathscr{A}(Fn, -)\colon \mathscr{A} \to \mathsf{Set}$$

*preserves sifted colimits for every* $n \in \omega$.

*Proof.* By adjunction, every set $\mathscr{A}(Fn, A)$ is bijective with $\mathsf{Set}(n, UA)$ natural in $A$. The forgetful functor $U$ creates sifted colimits, so $U$ preserves them. Then, it suffices to show that $\mathsf{Set}(n, -)$ preserves sifted colimits in $\mathsf{Set}$ for each finite number $n$. Now, given a sifted colimit $(H_i \xrightarrow{\tau_i} C)_{i \in \mathscr{D}}$, we observe that $\mathsf{Set}(n, H_i) = H_i^n$, so it follows that $H_i^n \to C^n$ is also a sifted colimit by the definition of being sifted. $\qquad\square$

Given a variety $\mathscr{A}$ of algebras, let $\mathscr{A}_\omega^f$ and $J : \mathscr{A}_\omega^f \hookrightarrow \mathscr{A}$ denote the full subcategory consisting of $Fn$, for $n \in \omega$, and the embedding to $\mathscr{A}$, respectively.

**Theorem 4.3.11** (see [104, Lemma 3.5]). *Given a finitary and monadic functor $U : \mathscr{A} \to \mathsf{Set}$ with a left adjoint $F : \mathsf{Set} \to \mathscr{A}$, the collection of filtered diagrams in $\mathscr{A}$ and canonical presentations is a density presentation of the embedding $J : \mathscr{A}_\omega^f \to \mathscr{A}$.*

*Proof.* Every variety of algebras is complete and cocomplete, so every colimit $(\sigma_i : A_i \to C)$ of a diagram in any of the above collections exists. Hence it remains to show that the closure of $\mathscr{A}_\omega^f$ is $\mathscr{A}$ itself and every colimit $(\sigma_i : A_i \to C)$ is preserved by the restricted Yoneda embedding $\widetilde{J}$ defined in (3.6).

We observe that for every object $A \in \mathscr{A}$ the canonical presentation (4.4) is a coequaliser by monadicity, so every object $A$ can be constructed as a coequaliser of free algebras. Moreover, every free algebra $FX$ is a filtered colimit of $Fn$ in $\mathscr{A}_\omega^f$ since $F$ as a left adjoint preserves every colimit and every set $X$ is a filtered colimit of finite sets. Thus, the closure of $\mathscr{A}_\omega^f$ under filtered colimits and canonical presentations is $\mathscr{A}$.

Since canonical presentations are reflexive coequalisers (see Example 4.3.8) and filtered colimits are sifted (by definition), the restricted Yoneda embedding $\widehat{J}$ preserves them by Lemma 4.3.10. $\qquad\square$

In addition to the above presentation given in [104], we provide the following extra presentations:

**Theorem 4.3.12.** *Given a variety $\mathscr{A}$ of algebras, the following collections of diagrams are density presentations of $J : \mathscr{A}_\omega^f \to \mathscr{A}$:*

1. *The collection of filtered diagrams and $U$-split pairs.*

2. *The collection of filtered diagrams and reflexive pairs in $\mathscr{A}$.*

3. *The collection of sifted diagrams in $\mathscr{A}$.*

*Proof.* By Theorem 4.3.11, we know that the closure of $\mathscr{A}_\omega^f$ under filtered colimits and canonical presentations is $\mathscr{A}$. By Example 4.3.7 and Example 4.3.8, filtered colimits are sifted, and canonical presentations are also sifted, specifically reflexive coequalisers. In addition, every canonical presentation is a $U$-split coequaliser, since $\eta_{UA}$ and $\eta_{UFUA}$ split the canonical presentation:

$$UFUFUA \xleftarrow{\;\eta_{UFUA}\;} UFUA \xleftarrow{\;\eta_{UA}\;} UA.$$

Hence, the closure of $\mathscr{A}_\omega^f$ under either

1. sifted colimits;

2. filtered diagrams and reflexive pairs; or

3. filtered diagrams and $U$-split pairs,

is the variety $\mathscr{A}$.

It remains to show that the restricted Yoneda embedding $\widetilde{J}$ preserves them.

**sifted colimits & filtered colimits with reflexive coequalisers**: By Lemma 4.3.10, $\widehat{J}$ preserves sifted colimits including filtered colimits and reflexive coequalisers.

**filtered colimits with $U$-split coequalisers**: Every $U$-split pair is a $U$-absolute coequaliser by [81, Corollary VI.6], i.e. the image under $U$ is an absolute colimit. Thus, every $U$-split coequaliser is preserved by $\mathsf{Set}(n, -) \circ U$.

$\square$

**Properties of a Variety of Algebras**

**Lemma 4.3.13.** *The following statements hold:*

1. *If a coequaliser has a kernel pair, then it is the coequaliser of its kernel pair;*

2. *Every regular epimorphism in a category with kernel pairs is a reflexive coequaliser of its kernel pair.*

3. *The regular epimorphisms in a variety of algebras are precisely surjective homomorphisms.*

*Proof.*   1. Let $e\colon X \to Y$ be a coequaliser of morphisms $f, g\colon S \to X$. Consider the kernel pair of $e$ in the following diagram
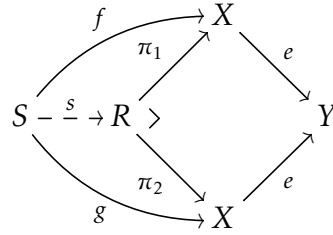


where $s$ is the mediating morphism since $ef = eg$ by definition. Given a morphism $h$ with $h\pi_1 = h\pi_2$, it follows that

$$h\pi_1 = h\pi_2$$
$$\implies h\pi_1 s = h\pi_1 s$$
$$\iff hf = hg, \qquad \{\pi_1 s = f \text{ and } \pi_2 s = g \text{ by construction}\}$$

so by the coequaliser $(f, g)$ of $e$ there is a unique morphism $j$ with $h = je$ satisfying the universal property of coequaliser.

2. Every regular epimorphism $e$ is a coequaliser of some pair of morphisms, so $f$ is the coequaliser of its kernel pair. Then, consider the following diagram
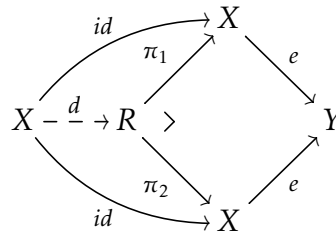


and the outer diagram commutes trivially. It follows that there exists a mediating morphism $d\colon X \to R$ such that $\pi_1 \circ d = \pi_2 \circ d = id$.

3. Given an algebra $A$ of a variety $\mathscr{A}$, the kernel pair of a morphism $f$ is exactly the projections $\pi_1, \pi_2\colon R \rightrightarrows A$ of the *congruence* defined by

$$\ker(f) := \{(a_1, a_2) \in A \times A \mid fa_1 = fa_2\}.$$

It is not hard to check that $\ker(f)$ is an algebra considering the preservation of operations by homomorphisms, so it is a reflexive pair.[9] It follows that a regular

---

[9]The common section of projections is the diagonal function $\Delta(a) = (a, a)$ for each $a \in A$.

epimorphism $e \colon X \to Y$ is in fact the quotient of $X$ under the congruence, i.e. $Y \cong X/R$, so $e$ is surjective.

Conversely, every homomorphism $f \colon X \to Y$ is the composite of a coequaliser of its kernel and an injective homomorphism

$$X \xrightarrow{\; f \;} f[X]$$
$$f \searrow \quad \downarrow id|_{f[X]}$$
$$Y$$

where $f[X] \cong X/\ker(f)$ by the First Isomorphism Theorem in universal algebra, see [34, Section 6] for details. Thus, every surjective homomorphism is a regular epimorphism.

$\square$

### 4.3.4 Equationally Presentable Functors and Monads

By Theorem 3.3.8 and the density presentations given in Theorem 4.3.11 and Theorem 4.3.12, we immediately have the following equivalent characterisations:

**Corollary 4.3.14.** *Given a monadic and finitary functor* $U \colon \mathscr{A} \to \mathsf{Set}$ *and an endofunctor* $L$ *of* $\mathscr{A}$, *the following statements are equivalent:*

1. *$L$ is finitary and preserves canonical presentations.*

2. *$L$ preserves sifted colimits.*

3. *$L$ is finitary and preserves reflexive coequalisers.*

4. *$L$ is finitary and preserves $U$-split coequalisers.*

5. *$L$ is the left Kan extension of $LJ$ along $J \colon \mathscr{A}_\omega^f \to \mathscr{A}$.*

A functor is called **finitely based** if it satisfies one of the above statements. A **finitely based monad** is a monad $(T, \mu^T, \eta^T)$ whose functor part is finitely based. The class of finitely based endofunctors of a variety $\mathscr{A}$ is denoted by

$$\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$$

which is the full subcategory of the category of finitary endofunctors of $\mathscr{A}$.

*Remark* 4.3.15. The equivalent characterisations of functors which preserve filtered colimits and canonical presentations, and sifted-colimit preserving functors may be found in the work of equationally-presentable functors in [30, 76, 104]. The third can be found in [12, Theorem 2.1] assuming that the category is finitely cocomplete, but we simplify the argument using the density presentation. The fourth appears to be original.

**Corollary 4.3.16**. *Every finitary endofunctor of* Set *is finitely based.*

*Proof.* Every reflexive coequaliser in Set is a surjection having a right inverse using the Axiom of Choice, so it is preserved by any functor. $\square$

**Definition 4.3.17**. Given a finitary and monadic functor $U \colon \mathscr{A} \to$ Set with a left adjoint and a finitely based endofunctor $L$ of $\mathscr{A}$, the object $LFn$ is called the **object of $n$-ary operations of** $L$.

From the analysis in Section 4.3.2, we know that elements $\sigma \in LFn$ give rise to $n$-ary operations $\overline{\sigma} \colon UA^n \to UA$ over the carrier set $UA$.

**Properties of Finitely Based Functors**

Since finitely based functors are defined by the preservation of sifted colimits, it is easy to see the following:

**Proposition 4.3.18**. *The composite of two finitely based endofunctor is finitely based.*

By Lemma 4.3.13, we also know that

**Corollary 4.3.19**. *Every finitely based functor preserves surjections.*

The condition of finitary surjection-preservation is strictly weaker:

**Example 4.3.20** (*cf.* [104, Example 3.12]). Not every finitary functor which preserves surjections is finitely based. Consider the category Ab of abelian groups. Every abelian group $A = (A, 0, +)$ is the direct sum[10] of its torsion subgroup $A_T$ defined by

$$A_T := \{ a \in A \mid na = 0 \text{ for some natural number } n \},$$

---

[10] In the category of abelian groups, direct sums are coproducts.

and the quotient $A^* := A/A_T$. The quotient $A^*$ is torsion-free, i.e. every element $a$ of $A^*$ except the identity satisfies $na \neq 0$ for every $n \in \omega$. For every group homomorphism $\varphi \colon A \to B$, there is an obvious map $\varphi \colon A^* \to B^*$ defined by

$$(a + A_T) \mapsto (\varphi(a) + B_T)$$

for every $a \in A$. It follows that the construction of the torsion-free subgroup is functorial, denoted by $I^* \colon \mathsf{Ab} \to \mathsf{Ab}$. Now, it is not hard to see that $I^*$ is finitary and preserves surjections. Nevertheless, every finitely generated free Abelian group is simply a torsion-free group $\mathbb{Z}^n$ for some $n \in \omega$, and thus $(\mathbb{Z}^n)^* = \mathbb{Z}^n$ The left Kan extension of $I^*$ along the subcategory of finitely generated free Abelian groups is just the identity. It follows that $I^*$ is not finitely based.

By Corollary 3.4.10, the equivalence of categories follows

$$\mathrm{Lan}_J \colon [\mathscr{A}_\omega, \mathscr{A}] \xrightarrow{\cong} \mathsf{Fin}[\mathscr{A}, \mathscr{A}] \tag{4.5}$$

with a pseudo-inverse $(-) \circ J$ where $\mathscr{A}_\omega$ is the full subcategory consisting of finitely generated algebras and $J \colon \mathscr{A}_\omega \to \mathscr{A}$ is the embedding. Similarly, there is an equivalence

$$\mathrm{Lan}_{J'} \colon [\mathscr{A}_\omega^f, \mathscr{A}] \xrightarrow{\cong} \mathsf{FinB}[\mathscr{A}, \mathscr{A}] \tag{4.6}$$

where $\mathscr{A}_\omega^f$ is a full subcategory of $\mathscr{A}$ on finitely generated free algebras and $J' \colon \mathscr{A}_\omega^f \to \mathscr{A}$ is the embedding. These two equivalences of categories are particularly important and convenient for the subsequent discussion.

**Proposition 4.3.21** (see [104, Lemma 3.13]). *Given a variety $\mathscr{A}$ of algebras, the category of finitely based endofunctors of $\mathscr{A}$ is a coreflective subcategory of the category of finitary endofunctors of $\mathscr{A}$, i.e. the inclusion*

$$J \colon \mathsf{FinB}[\mathscr{A}, \mathscr{A}] \longrightarrow \mathsf{Fin}[\mathscr{A}, \mathscr{A}]$$

*has a right adjoint.*

*Proof.* For any functor $L \colon \mathscr{A}_\omega^f \to \mathscr{A}$, the left Kan extension of $L$ along the inclusion $J' \colon \mathscr{A}_\omega^f \to \mathscr{A}_\omega$ always exists, i.e. an adjunction

$$(-) \circ J' \vdash \mathrm{Lan}_{J'} \colon [\mathscr{A}_\omega^f, \mathscr{A}] \longrightarrow [\mathscr{A}_\omega, \mathscr{A}]$$

by Corollary 3.2.13, because $\mathscr{A}_\omega^f$ is small and $\mathscr{A}$ is cocomplete. $\qquad\square$

Note that we have a chain of coreflective subcategories

$$\mathsf{FinB}[\mathscr{A},\mathscr{A}] \xrightarrow[\top]{\longleftarrow} \mathsf{Fin}[\mathscr{A},\mathscr{A}] \xrightarrow[\top]{\longleftarrow} [\mathscr{A},\mathscr{A}] \tag{4.7}$$

given by left Kan extensions, so finitely based (resp. finitary) endofunctors of $\mathscr{A}$ are closed under colimits of $[\mathscr{A},\mathscr{A}]$.

**Proposition 4.3.22.** *The free monad over a finitely based endofunctor is finitely based.*

*Proof.* From the free algebra sequence, every functor $(L_i)_{i<\omega}$ is finitely based and by Proposition 4.2.6, the free monad is the $\omega$-th object in the free algebra sequence, i.e. a colimit of $(L_i)_{i<\omega}$, so the free monad is finitely based by Proposition 4.3.21. $\qquad\square$

**Freely Generated Finitely Based Functors**

**Proposition 4.3.23** (see [104, Proposition 3.15]). *Given a finitary and monadic functor $U\colon \mathscr{A} \to \mathsf{Set}$ with a left adjoint $F$, the functor*

$$[F,U] := U(-)F\colon \mathsf{FinB}[\mathscr{A},\mathscr{A}] \longrightarrow \mathsf{Fin}[\mathsf{Set},\mathsf{Set}]$$

*is finitary and monadic.*

We first show that $[F,U]$ is finitary; using Beck's Monadicity Theorem, we then show that $[F,U]$ has a left adjoint and creates coequalisers of $[F,U]$-split pairs.

*Proof.* Colimits of $[\mathscr{A},\mathscr{A}]$ are computed pointwise and $\mathsf{FinB}[\mathscr{A},\mathscr{A}]$ is its coreflexive subcategory by Proposition 4.3.21, so colimits of $\mathsf{FinB}[\mathscr{A},\mathscr{A}]$ are also computed pointwise. It follows that $(\tau_i F\colon L_i F \dashrightarrow LF)_{i\in\mathscr{D}}$ is a colimit whenever $(\tau_i\colon L_i \dashrightarrow L)_i$ is a colimit in $\mathsf{FinB}[\mathscr{A},\mathscr{A}]$. By assumption $U$ being finitary, $(U\tau_i F\colon UL_i F \dashrightarrow ULF)_i$ is a filtered colimit, provided that the diagram is filtered.

Let $F'\colon \mathsf{Set}_\omega \to \mathscr{A}_\omega^f$ be the restriction of $F$. For each functor $H\colon \mathsf{Set}_\omega \to \mathsf{Set}$, the left Kan extension $\mathrm{Lan}_{F'}FH$ always exists, since $\mathscr{A}$ is cocomplete and $\mathscr{A}_\omega^f$ is small. Therefore, we have isomorphisms

$$[\mathscr{A}_\omega^f,\mathscr{A}](\mathrm{Lan}_{F'}FH,L) \cong [\mathsf{Set}_\omega,\mathscr{A}](FH,LF') \tag{4.8}$$

$$\cong [\mathsf{Set}_\omega,\mathsf{Set}](H,ULF') \tag{4.9}$$

natural in $H$ and $L$.

Given a $[F, U]$-split pair $\tau_1, \tau_2 \colon L_1 \rightrightarrows L_2$ of natural transformations for functors $L_1, L_2 \colon \mathscr{A}_\omega^f \to \mathscr{A}$, the split coequaliser

$$UL_1 Fn \xrightarrow[U\tau_2 Fn]{U\tau_1 Fn} UL_2 Fn \longrightarrow X_n \qquad (4.10)$$

exists for each $n \in \omega$. By the monadicity of $U$, there exists a coequaliser

$$L_1 Fn \xrightarrow[\tau_2 Fn]{\tau_1 Fn} L_2 Fn \longrightarrow A_n$$

whose image under $U$ is (4.10), so the coequaliser of $\tau_1$ and $\tau_2$ exists in $[\mathscr{A}_\omega^f, \mathscr{A}]$. It follows that the functor $[F, U]$ is monadic. □

*Notation* 4.3.24. Given a finitary functor $H \colon \mathsf{Set} \to \mathsf{Set}$, the resulting finitely based endofunctor in Proposition 4.3.23 is denoted by $\widehat{H} \colon \mathscr{A} \to \mathscr{A}$.

The resulting finitely based functor $\widehat{H}$ on $\mathscr{A}_\omega^f$ for a finitary functor $H \colon \mathsf{Set} \to \mathsf{Set}$ is given explicitly as follows:[11]

$$
\begin{aligned}
&(\mathrm{Lan}'_F FH) F'm \\
&\cong \int^n \mathscr{A}_\omega^f (F'n, F'm) \cdot FHn && \{\text{by Theorem 3.2.11}\} \\
&\cong \int^n \mathsf{Set}(n, UFm) \cdot FHn && \{\text{by the full inclusion } \mathscr{A}_\omega^f \hookrightarrow \mathscr{A} \text{ and } F \dashv U\} \\
&\cong F\left( \int^n \mathsf{Set}(n, UFm) \cdot Hn \right) && \{\text{the left adjoint } F \text{ preserves coends}\} \\
&\cong F\left( \int^n Hn \cdot \mathsf{Set}(n, UFm) \right) && \{\text{copower in } \mathsf{Set} \text{ is commutative}\} \\
&\cong FHUFm. && \{\text{by Proposition 3.2.7}\}
\end{aligned}
$$

Since $\widehat{H}$ preserves sifted colimits and every algebra is the sifted colimit of its canonical diagram in $\mathscr{A}_\omega^f$ by Theorem 4.3.11, the resulting value of $\widehat{H}$ on any $A \in \mathscr{A}$ is precisely $FHUA$.

**Example 4.3.25.** For a finitary functor $T$, the syntax of the logic of the cover modality is a finitely based functor $FTU$ by Example 2.3.25.

---

[11] The computation appears in [104, Remark 3.16], but the last step is not explained in detail.

**Proposition 4.3.26.** *Given the adjunction* $\mathsf{FinB}[\mathscr{A},\mathscr{A}](\widehat{H},L) \cong \mathsf{Fin}[\mathsf{Set},\mathsf{Set}](H,ULF)$, *the unit* $\eta_H \colon H \dashrightarrow U\widehat{H}F$ *on finite sets is given explicitly as follows:*

$$(\eta_H)_n \colon a \mapsto (UFHi_n \circ i_{Hn})(a) \tag{4.11}$$

*for* $a \in Hn$ *where* $i \colon \mathcal{I} \to UF$ *is the unit for the adjunction* $F \dashv U \colon \mathscr{A} \to \mathsf{Set}$.

*Proof.* The unit $\eta'_m \colon FHm \to \widehat{H}F'm$ of (4.8) of the coend formula is given by

$$FHm \xrightarrow{\iota_{id_{Fm}}} \mathscr{A}^f_\omega(F'm,F'm) \cdot FHm$$

where $\iota_{id_{Fm}}$ is the canonical injection, and $w$ is the universal wedge. The isomorphism between the coend $\int^n \mathsf{Set}(n,UFm) \cdot Hn$ and $HUFm$ is given by

$$(f,x) \mapsto Hf(x).$$

for $\left(n \xrightarrow{f} UFm\right)$ and $x \in Hn$. By (4.9), the resulting unit is then given in (4.11). $\qquad\square$

For every variety of $(\Sigma,E)$-algebras, the set $\Sigma$ of operations may be viewed as an assignment, called **finitary signature**, from the set $\aleph_0$ of natural numbers to the set $\Sigma n$ of $n$-ary operations, i.e. a functor

$$\Sigma \colon \aleph_0 \to \mathsf{Set}$$

where $\aleph_0$ is regarded as a discrete category. Each signature gives rise to a **polynomial functor** $H_\Sigma$ of $\Sigma$ defined by

$$H_\Sigma := \coprod_{n \in \omega} \Sigma_n \times (-)^n, \tag{4.12}$$

and an $H_\Sigma$-algebra consists of a family of operations $\sigma \colon X^n \to X$ for each $\sigma \in \Sigma_n$. The polynomial functor $H_\Sigma$ is in fact a left Kan extension of $\Sigma$ along the inclusion $E \colon \aleph_0 \to \mathsf{Set}$ as we can see from the coend formula

$$(\mathrm{Lan}_E \Sigma)X = \int^{\aleph_0} \mathsf{Set}(-,X) \cdot \Sigma n \cong \coprod_{n \in \omega} X^n \cdot \Sigma n$$

for each set $X$ by Theorem 3.2.11. Therefore, there is a right adjoint to the construction given by pre-composition with $E$. From the coend formula, the unit is given as follows:

**Proposition 4.3.27.** *Given the adjunction* $\mathsf{Fin}[\mathsf{Set}, \mathsf{Set}](H_\Sigma, H) \cong [\aleph_0, \mathsf{Set}](\Sigma, HE)$ *where* $E \colon \aleph_0 \to \mathsf{Set}$, *the unit* $\eta_\Sigma \colon \Sigma \to H_\Sigma E$ *is given explicitly as follows:*

$$(\eta_\Sigma)n \colon \sigma \in \Sigma n \mapsto \langle \sigma, (0, \ldots, n-1) \rangle.$$

**Example 4.3.28.** Given a set $X$, define a finitary signature $\Sigma_X$ by

$$(\Sigma_X)n = \begin{cases} X & \text{if } n = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

The corresponding polynomial functor is simply a constant functor with value $X$. Indeed, this mapping is a full embedding $\mathsf{Set} \hookrightarrow [\aleph_0, \mathsf{Set}]$.

**Theorem 4.3.29** (see [104, Theorem 3.18]). *Given a finitary and monadic functor* $U \colon \mathscr{A} \to \mathsf{Set}$ *with a left adjoint* $F$, *the composite*

$$\mathsf{FinB}[\mathscr{A}, \mathscr{A}] \xrightarrow{\;[F,U]\;} \mathsf{Fin}[\mathsf{Set}, \mathsf{Set}] \xrightarrow{\;[E,\mathsf{Set}]\;} [\aleph_0, \mathsf{Set}]$$

*is finitary and monadic where* $[E, \mathsf{Set}]$ *is the precomposition with the inclusion* $E \colon \aleph_0 \to \mathsf{Set}_\omega$.

*Proof.* The functor $[E, \mathsf{Set}] = (-) \circ E$ is finitary since colimits in $\mathsf{Fin}[\mathscr{A}, \mathscr{A}]$ as a coreflective subcategory of $[\mathscr{A}, \mathscr{A}]$ are computed pointwise, so the restriction to $\aleph_0$ is also a colimit as well as the composite $(-) \circ E$ with $[F, U]$.

The composite of adjunctions is again an adjunction, so it remains to show that the composite is monadic.

Using the equivalences (4.5) and (4.6), it is equivalent to show the composite

$$[\mathscr{A}_\omega^f, \mathscr{A}] \xrightarrow{\;[F,U]\;} [\mathsf{Set}_\omega, \mathsf{Set}] \xrightarrow{\;(-)\circ E\;} [\aleph_0, \mathsf{Set}]$$

is monadic, but this follows from the same argument as the proof in Proposition 4.3.23. $\qquad\square$

**Example 4.3.30.** Given a set $X$, by the embedding $\mathsf{Set} \hookrightarrow [\aleph_0, \mathsf{Set}]$ defined as in Example 4.3.28, the resulting finitely based functor of $X$ is the constant functor with value $FX$.

**Example 4.3.31.** In Example 2.3.25, we showed that every set $\Lambda$ of predicate liftings gives rise to an abstract logic whose syntax functor is

$$F\left(\bigsqcup \Lambda_n \times (-)^n\right) U = F H_\Lambda U = \widehat{H_\Lambda}$$

which is finitely based by Theorem 4.3.29 where $\Lambda_n$ is the set of $n$-ary predicate liftings. That is, we regard the set $\Lambda$ as a signature by mapping every natural number $n$ to the set of $n$-ary predicate liftings.

*Remark* 4.3.32. The monadic functor from $\mathsf{FinB}[\mathscr{A},\mathscr{A}]$ to $[\aleph_0, \mathsf{Set}]$ amounts to saying that every finitely based endofunctor has a canonical presentation as a coequaliser of freely generated finitely based endofunctors

$$\widehat{H_\Gamma} \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} \widehat{H_\Sigma} \longrightarrow L \tag{4.13}$$

since every finitely based endofunctor $L$ has a canonical presentation by the monadicity of $[F, U]$ where $\Gamma$ and $\Sigma$ are finitary signatures given explicitly by

$$\Sigma := ULFE \quad \text{and} \quad \Gamma := U\widehat{H_\Sigma}FE$$

where $E \colon \aleph_0 \to \mathsf{Set}_\omega$ is the inclusion.

*Remark* 4.3.33. By Theorem 4.3.29, any map from $\widehat{H_\Gamma}$ to $\widehat{H_\Sigma}$ corresponds uniquely to the transpose $\Gamma \to U\widehat{H_\Sigma}FE$; so the pair of parallel morphisms in (4.13) boils down to a family of equations

$$\iota_n \colon \mathcal{E}_n \subseteq U\widehat{H_\Sigma}Fn \times U\widehat{H_\Sigma}Fn \tag{4.14}$$

for each $n \in \omega$, where $Fn$ serves as the object of $n$-many variables; $U\widehat{H_\Sigma}Fn$ is the set of **rank-1 terms** of $n$-many variables; and a pair of rank-1 terms in $\mathcal{E}_n$ is a **rank-1 equation** valid in $L$. Conversely, let $\mathcal{E}$ be a family of rank-1 equations. By the projections $\pi_1, \pi_2 \colon (U\widehat{H_\Sigma}FE)^2 \rightrightarrows U\widehat{H_\Sigma}FE$, $\mathcal{E}$ gives two maps $\pi_1\iota, \pi_2\iota \colon \mathcal{E} \rightrightarrows U\widehat{H_\Sigma}F$, so $\mathcal{E}$ is bijective with a pair of natural transformations between finitely based endofunctors $(\pi_1\iota)^*, (\pi_2\iota)^* \colon \widehat{H_\mathcal{E}} \rightrightarrows \widehat{H_\Sigma}$ by the monadic adjunction. For convenience, write $t \sim_n t'$ for $(t, t') \in (U\widehat{H_\Sigma}Fn)^2$, so a set $\mathcal{E}$ of equations for $\Sigma$ is described by a signature $\mathcal{E} \colon n \mapsto \{\ldots, t \sim_n t', \ldots, \}$.

A rank-1 term $t \in U\widehat{H_\Sigma}Fn = UFH_\Sigma UFn$ for some finitary signature $\Sigma$ is to be understood as a term in $\mathscr{A}$ consisting of at most one layer of operations in $\Sigma$ at terms

of $n$-variables:

$$\underbrace{U \quad F \quad H_\Sigma \quad \overbrace{U \quad Fn}^{\Sigma\text{-operations}}}_{\substack{\text{rank-1 terms}}}$$
$$\underbrace{\phantom{U \quad F \quad H_\Sigma \quad U \quad Fn}}_{\substack{n\text{-ary terms}}}$$

and see the following for a concrete example:

**Example 4.3.34.** In Definition 2.2.8, the endofunctor $\mathbb{M}$ is presented by a signature $\Sigma \colon \aleph_0 \to \mathsf{Set}$ and a family $\mathcal{E}$ of equations defined by

$$\Sigma_n := \begin{cases} \{\blacklozenge\} & \text{if } n = 1 \\ \emptyset & \text{otherwise;} \end{cases} \quad \text{and} \quad \mathcal{E}_n := \begin{cases} \{\blacklozenge \bot \sim_0 \bot\} & \text{if } n = 0 \\ \{\blacklozenge(0 \vee 1) \sim_2 \blacklozenge 0 \vee \blacklozenge 1\} & \text{if } n = 2 \\ \emptyset & \text{otherwise,} \end{cases}$$

respectively, where each term in $\mathcal{E}$ is given step by step in following way:

| 0 | (none) | 2 | 0, 1 |
|:---:|:---:|:---:|:---|
| $UF0$ | $\bot$ | $UF2$ | $0 \vee 1, 0, 1$ |
| $H_\Sigma UF0 = UF0$ | $\blacklozenge\bot$ | $H_\Sigma UF2 = UF2$ | $\blacklozenge(0 \vee 1), \blacklozenge 0, \blacklozenge 1$ |
| $UFUF0$ | $\blacklozenge\bot, \bot$ | $UFUF2$ | $\blacklozenge(0 \vee 1), \blacklozenge 0 \vee \blacklozenge 1$ |

**Algebras of Finitely Based Functors**

Recall Corollary 4.2.7 that the category of $L$-algebras of a finitary endofunctor $L$ of $\mathscr{A}$ is monadic over $\mathscr{A}$. Continuing Section 4.3.2, we show that every algebra $LA \to A$ of a finitely based functor on a variety $\mathscr{A}$ is indeed an extension of the carrier algebra $A$, e.g. every modal algebra is an extension of Boolean algebra.

**Theorem 4.3.35** (*cf.* [104, Theorem 4.1]). *Let $U \colon \mathscr{A} \to \mathsf{Set}$ be a monadic and finitary functor, $L$ a finitely based endofunctor of $\mathscr{A}$, and $U_L \colon \mathscr{A}^L \to \mathscr{A}$ the forgetful functor. The composite $U\,U_L$ is monadic and finitary.*

*Proof.* By the dual of Proposition 4.1.3, $U_L$ creates filtered colimits. Hence $U_L$ is finitary, and so is the composite $U \circ U_L$.

By the induction construction, $U_L$ has a left adjoint since $L$ preserves $\omega$-chains. It follows that the composite has a left adjoint.

To prove the monadicity, we use the Crude Monadicity Theorem: the right adjoint $U U_L$ is monadic if it reflects isomorphisms, $\mathscr{A}^L$ has and $U U_L$ preserves reflexive

coequalisers. By the dual of Proposition 4.1.2, the forgetful functor $U_L$ reflects isomorphisms so that the composite also reflects isomorphisms. The category $\mathscr{A}$ is cocomplete, so it has all reflexive coequalisers. By definition, $L$ preserves reflexive coequalisers. Hence, $U_L$ creates reflexive coequalisers by the dual of Proposition 4.1.3. In particular, $\mathscr{A}^L$ has and $U_L$ preserves reflexive coequalisers. Finally, $U U_L$ preserves reflexive coequalisers as $U$ also does. $\qquad\square$

**Exactness of the Category of Finitely Based Endofunctors**

Previously, we have seen that every finitely based endofunctor has a canonical presentation and every presentation consisting of rank-1 equations defines a finitely based endofunctor. In this subsection, we study *congruences* for finitely based endofunctors via regularity and exactness. See [32, Chapter 2] for general discussions.

**Definition 4.3.36.** Given a category with pullbacks, an **equivalence relation** on an object $x$ is a pair of jointly monic morphisms $r_1, r_2 \colon R \rightrightarrows x$ such that there exist

**(reflexivity)** a morphism $\delta \colon x \to R$ with $r_i \circ \delta = id$, for $i = 1, 2$;

**(symmetry)** a morphism $\sigma \colon R \to R$ with $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$;

**(transitivity)** a morphism $\tau \colon R \times_x R \to R$ with $r_i \circ \tau = r_i \circ \rho_i$, for $i = 1, 2$, where $(\rho_1, \rho_2)$ is the pullback of $(r_1, r_2)$ as depicted in the diagram

$$
\begin{array}{ccc}
R \times_x R & \xrightarrow{\ \rho_2\ } & R \\
{\scriptstyle \rho_1}\big\downarrow & & \big\downarrow{\scriptstyle r_1} \\
R & \xrightarrow[\ r_2\ ]{} & x.
\end{array}
$$

An equivalence relation $(R, r_1, r_2)$ is **effective** if the coequaliser, denoted $e$, of $(r_1, r_2)$ exists and $e$ has $(r_1, r_2)$ as its kernel pair.

**Proposition 4.3.37.** *In a category with coequalisers, the kernel pair of any morphism is an effective equivalence relation.*

For example, every set-theoretic equivalence relation $R \subseteq X \times X$ with projections $\pi_1, \pi_2 \colon R \rightrightarrows X$ is an equivalence relation. The reflexivity morphism $\delta$ and the symmetry morphism $\sigma$ are rather obvious, and the transitivity morphism $\tau \colon R \times_A R \to R$ is given by $(\langle x, y \rangle, \langle y, z \rangle) \mapsto \langle x, z \rangle$. Further, equivalence relations in Set are all effective. The coequaliser, of course, can be given by the canonical projection $e$ to the

quotient $X/R$, and the condition that $(r_1, r_2)$ is a kernel pair of $e$ is equivalent to that $[x] = [y]$ if and only if $x \, R \, y$.

Similarly, a *congruence* $R$ on a $(\Sigma, \mathcal{E})$-algebra $A$, which is an equivalence relation $R$ on its carrier and is also a subalgebra of $A \times A$, is an equivalence relation in the category of $(\Sigma, \mathcal{E})$-algebras. Moreover, every congruence is also effective.

**Definition 4.3.38** (see [32]). A category is **regular** if *a*) every morphism has a kernel pair; *b*) every kernel pair has a coequaliser; *c*) the pullback of a regular epimorphism along any morphism exists and is also a regular epimorphism. Moreover, a regular category is **exact** if every equivalence relation is effective.

For example, Set and any variety of algebras are exact; on the other hand, the category of topological spaces is not regular.

**Theorem 4.3.39** (see [32, Section 2.1]). *Every regular category has the* $(\mathsf{RegEpi}, \mathsf{Mono})$-*factorisation system.*

*Proof Sketch.* Given any morphism $f : X \to Y$, consider its kernel pair $(u, v)$ and the coequaliser $e$ of $(u, v)$:

$$R \mathrel{\substack{\xrightarrow{u} \\ \xrightarrow{v}}} X \xrightarrow{f} Y$$
$$e \downarrow \quad \nearrow^{i}$$
$$X/R$$

Since $f u = f v$ by construction, there exists a unique morphism $i : X/R \to Y$ with $f = i \circ e$. To check that $i$ is a monomorphism, we use the requirement that the pullback of a regular epimorphism along any morphism exists and is also a regular epimorphism. A complete proof can be found in [32, Theorem 2.1.3]. □

**Proposition 4.3.40** (see [32, Example 2.4.7]). *For every small category $\mathscr{I}$ and a regular (resp. exact) category $\mathscr{C}$, the functor category $[\mathscr{I}, \mathscr{C}]$ is regular (resp. exact).*

By the equivalence (4.6), it follows:

**Corollary 4.3.41.** *Given a variety $\mathscr{A}$, the category $\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$ of finitely based functors is exact. In particular, $\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$ has the $(\mathsf{RegEpi}, \mathsf{Mono})$-factorisation system and every equivalence relation in $\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$ is effective.*

*Remark* 4.3.42. By (4.6), an equivalence relation on a finitely based functor corresponds to a family of congruence relations involving $n$-many variables indexed by $n \in \omega$ subject to substitution of variables.

**Example 4.3.43.** Define two natural transformation $\pi^i \colon \mathcal{K}_{F1} \Rightarrow \mathbb{M}A$ for each component $A$ on the generator $* \in 1$ by

$$\pi_A^1(*) = \blacklozenge\top \quad \text{and} \quad \pi_A^2(*) = \top,$$

that is, the generator $*$ represents the equation $\blacklozenge\top = \top$. The functor $\overline{\mathbb{M}}$ defined by

$$\overline{\mathbb{M}}A = \mathsf{BA}\langle\, \{\blacklozenge a\}_{a \in A} \mid \blacklozenge\bot = \bot;\, \blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b;\, \blacklozenge\top = \top \,\rangle$$

with the canonical projection is the coequaliser of $(\pi^1, \pi^2)$, *c.f.* Definition 2.2.8. Note that the constant functor $\mathcal{K}_{F1}$ is generated by the signature consisting of only one nullary operation and the equation $\blacklozenge\top = \top$ is a rank-1 equation involving no variables.

To sum up, every coequaliser of $u, v \colon R \Rightarrow L$ intuitively amounts to adding equations to $L$.

# Chapter 5

# Coalgebraic Logics via Duality

We introduce a category consisting of interpretations of modalities, called *one-step semantics* using a contravariant functor $P \colon \mathscr{X} \to \mathscr{A}$. The categorical constructions in this category generalise the fusion of modal logics as coproduct, (some) many-sorted modal logics as tensor product, product of logics of predicate liftings, previously discussed in [36, 37, 70, 98] in a set-level framework and [75] in a categorical setting using Stone duality. Moreover, the combination of logics of predicate liftings and the logic of the cover modality is possible in this context, since they are all objects in this category.

By indexing over the category of endofunctors $T$ of $\mathscr{X}$, we obtain the category of coalgebraic logics for $T$-coalgebras. An abstract analysis of this category shows that there is always a *full equational logic* for $T$-coalgebras given by a dual adjunction on the right. We characterise this full equational logic in terms of (generalised) predicate liftings.

Further, the category of one-step semantics gives birth to the category consisting of multi-step semantics. A free construction over a one-step semantics is given in Section 5.3. This category outlines a possible formulation of multi-step coalgebraic logic using monads and comonads.

In the end, we show how to use one-step semantics as a coalgebraic logic in a point-free style. We show the adequacy property and the Hennessy-Milner property under a suitable condition, and conclude by a few instances showing that this framework properly generalises other existing approaches.

Note that we do not assume that the dual adjunction is a concrete duality or a logical connection in the sense of [77, 93]. To study equational coalgebraic logics, the category

$\mathscr{A}$ is assumed to be a variety but nothing more. In fact, it is possible to allow other structures such as quasi-varieties or enriched algebras. See Chapter 6 for future work.

Throughout this chapter, we always assume that $P \colon \mathscr{X} \to \mathscr{A}$ is a contravariant functor. Do not confuse $P$ with the covariant powerset functor $\mathcal{P}$.

## 5.1   One-Step Semantics

**Definition 5.1.1.** A **one-step semantics over** $P$ (or simply **one-step semantics**) is a triple $(L, T, \delta)$ consisting of

- an endofunctor $T \colon \mathscr{X} \to \mathscr{X}$, called the *type of one-step semantics*;

- an endofunctor $L \colon \mathscr{A} \to \mathscr{A}$, called the *syntax of modalities*;

- a natural transformation $\delta \colon LP \dot{\longrightarrow} PT$, called the *interpretation of modalities.*

A $T$-**logic** is a one-step semantics whose type functor is $T$.

Given type functors $T_1$ and $T_2$, every natural transformation $\nu \colon T_2 \dot{\longrightarrow} T_1$ converts a $T_1$-logic to a $T_2$-logic:

$$
\begin{array}{ccc}
LP & \xrightarrow{\ \delta\ } & PT_1 \\
 & \searrow & \downarrow{\scriptstyle P\nu} \\
 & & PT_2
\end{array}
$$

and it defines a functor from $\mathscr{X}_{T_2}$ to $\mathscr{X}_{T_1}$.

**Example 5.1.2.** For predicate liftings of a Set endofunctor $T$, a *natural relation* [90] for $T$ is a natural transformation $\nu \colon T \to \mathcal{P}$. A natural relation induces an interpretation for $T$ by composing the standard interpretation for $\mathcal{P}$ using the syntax of modal logic, i.e. modal operators $\Box$ and $\Diamond$ so that classical modal logic is able to describe $T$-coalgebras. A natural relation corresponds to intersection-preserving predicate liftings, i.e. continuous predicate liftings. See [97] for more details.

**Example 5.1.3.** Given a subfunctor $\nu \colon T' \hookrightarrow T$ of some endofunctor $T \colon \mathsf{Set} \to \mathsf{Set}$, every $T$-logic is converted to a $T'$-logic in a natural way by composing with $P\nu$. For example, the finitary powerset functor is a subfunctor of the full powerset functor, and every predicate lifting for $\mathcal{P}$ is also a predicate lifting for the finitary powerset functor. Subfunctors are of interest as they inherit expressiveness.

**Definition 5.1.4** (see [63, 74]). Given $T$-logics $(L_1, \delta_1)$ and $(L_2, \delta_2)$, a **translation** from $(L_1, \delta_1)$ to $(L_2, \delta_2)$ is a natural transformation $\tau \colon L_1 \overset{\cdot}{\longrightarrow} L_2$ such that the diagram

$$
\begin{array}{ccc}
L_1 P & \overset{\delta_1}{\longrightarrow} & PT \\
\tau P \downarrow & \nearrow{\scriptstyle \delta_2} & \\
L_2 P & &
\end{array}
$$

commutes.

**Example 5.1.5.** In classical modal logic, the possibility $\Diamond$ and necessity $\Box$ modalities can be defined by Moss' cover modality $\nabla$ by setting:

$$\Box\varphi \coloneqq \nabla\{\varphi\} \vee \nabla\emptyset \quad \text{and} \quad \Diamond\varphi \coloneqq \nabla\{\varphi, \top\}$$

for every proposition $\varphi \in \mathcal{M}_{\mathcal{P}}$, and by (2.19) it follows that

$$
\begin{aligned}
\nabla\{\varphi\} \vee \nabla\emptyset &= (\Box\varphi \wedge \Diamond\varphi) \vee (\Box\bot \wedge \top) \\
&= (\Box\varphi \wedge \Diamond\varphi) \vee \Box\bot
\end{aligned}
$$

and $\nabla\{\varphi, \top\} = \Box(\varphi \vee \top) \wedge \Diamond\varphi \wedge \Diamond\top$. The translation defines a natural transformation $\tau$ from $\mathbb{M}$ to $F\mathcal{P}_\omega U$ such that the interpretation is invariant under this translation, i.e. the diagram

$$
\begin{array}{ccc}
\mathbb{M}\mathcal{Q}X & \overset{\tau_{\mathcal{Q}X}}{\longrightarrow} & F\mathcal{P}_\omega U(\mathcal{Q}X) \\
{\scriptstyle \delta_X}\searrow & & \swarrow{\scriptstyle \overline{\nabla}_X} \\
& \mathcal{Q}TX &
\end{array}
$$

commutes where $\mathbb{M}$ and the natural transformation $\delta \colon \mathbb{M}\mathcal{Q} \overset{\cdot}{\longrightarrow} \mathcal{Q}T$ are given in Definition 2.2.8 and Proposition 2.2.13; the natural transformation $\overline{\nabla}$ is given in Example 2.3.25.

**Example 5.1.6.** On the other hand, the cover modality for $\mathcal{P}$ can also be defined by the usual modal operators:

$$\nabla\alpha \coloneqq \Box\bigvee\alpha \wedge \bigwedge\Diamond\alpha \quad \text{for} \quad \alpha \subseteq_\omega \mathcal{M}\mathcal{L}$$

where $\mathcal{M}\mathcal{L}$ is the language of finitary modal logic and $\Diamond\alpha \coloneqq \{\Diamond\varphi \mid \varphi \in \alpha\}$. It also defines a natural transformation from $F\mathcal{P}U$ to $\mathbb{M}$ and the interpretation is invariant under this translation. Kurz and Leal study (one-step) translations between Moss' cover modality and (singleton) predicate liftings in [74].

As we observed, there are two types of morphisms for one-step semantics: one is a morphism between types of one-step semantics fixing the syntax; and the other is a translation. Putting these morphisms together, we obtain the following category:

**Definition 5.1.7.** Let the pre-composition and post-composition with $P$ be denoted by

$$P^*\colon [\mathscr{A},\mathscr{A}] \to [\mathscr{X},\mathscr{A}] \quad \text{and} \quad P_*\colon [\mathscr{X},\mathscr{X}] \to [\mathscr{X},\mathscr{A}]$$

respectively. Define the following categories:

1. the comma category $(P^*{\downarrow}P_*)$ is called the **category of one-step semantics** of $P$, denoted $\mathsf{CoLog}^P$ (or simply $\mathsf{CoLog}$ if there is no ambiguity);

2. each fibre $(P^*{\downarrow}PT)$ over some endofunctor $T$ is the **category of $T$-logics**, denoted $\mathsf{CoLog}^P_T$ (or simply $\mathsf{CoLog}_T$).

The category of one-step semantics has two projection functors:

$$
\begin{array}{ccc}
 & \mathsf{CoLog} & \\
{\scriptstyle U_L}\swarrow & & \searrow{\scriptstyle U_R} \\
[\mathscr{A},\mathscr{A}] & & [\mathscr{X},\mathscr{X}]
\end{array}
$$

defined by mapping $(L,T,\delta\colon LP{\longrightarrow}PT)$ to its syntax $L$ and its type $T$ respectively.

We give a 'trivial' example which plays an important role in composition.

**Example 5.1.8.** Every category of one-step semantics has an identity one-step semantics consisting of $(\mathcal{I},\mathcal{I},id_P)$.

## 5.1.1 Properties of Categories of One-Step Semantics

**Liftings**

It is well-known that every one-step semantics provides a complex $L$-algebra construction for $T$-coalgebras:

**Proposition 5.1.9.** *Every one-step semantics* $(L,T,\delta)$ *defines a lifting* $P^\delta\colon \mathscr{X}_T \to \mathscr{A}^L$ *of $P$ along the forgetful functors of $L$-algebras and $T$-coalgebras by mapping*

$$\langle x,\xi\rangle \; to \; (Px, P\xi \circ \delta_x) \quad and \quad f\colon \langle y,\gamma\rangle \to \langle x,\xi\rangle \; to \; Pf$$

*Proof.* By the naturality of $\delta$ and the functoriality of $P$ the following diagram

$$
\begin{array}{ccccc}
LPx & \xrightarrow{\delta_x} & PTx & \xrightarrow{P\xi} & Px \\
\scriptstyle{LPf}\downarrow & & \scriptstyle{PTf}\downarrow & & \scriptstyle{Pf}\downarrow \\
LPy & \xrightarrow{\delta_y} & PTy & \xrightarrow{P\gamma} & Py
\end{array}
$$

commutes, namely, $Pf$ is an algebra homomorphism. Now, the statement follows easily from the functoriality of $P$. $\square$

**Proposition 5.1.10.** *Every morphism $(\tau, \nu)$ of one-step semantics from $(L, T, \delta)$ to $(M, V, \theta)$ defines functors*

$$
\nu_* \colon \mathscr{X}_V \to \mathscr{X}_T \quad \text{and} \quad \tau^* \colon \mathscr{A}^M \to \mathscr{A}^L
$$

*by $\langle x, \xi \rangle \mapsto \langle x, \nu_x \circ \xi \rangle$ and $\langle a, \alpha \rangle \mapsto \langle a, \alpha \circ \tau_a \rangle$. Moreover, $P^\theta$ is also a lifting of $P^\delta$ along $\nu_*$ and $\tau^*$.*

*Proof.* A morphism $(\tau, \nu)$ of one-step semantics satisfies $P\nu \circ \delta = \theta \circ \tau P$. For every $V$-coalgebra $\langle x, \xi \rangle$, we have the following commutative diagram

$$
\begin{array}{ccccc}
LPx & \xrightarrow{\delta_x} & PTx & \xrightarrow{P(\nu_x \circ \xi)} & Px \\
\scriptstyle{\tau PX}\downarrow & & \scriptstyle{P\nu_x}\downarrow & & \|\ \\
MPx & \xrightarrow{\theta_x} & PVx & \xrightarrow{P\xi} & Px,
\end{array}
$$

so the statement follows. $\square$

## Coreflections

**Proposition 5.1.11.** *Given a coreflective subcategory $\mathscr{B}$ of $[\mathscr{A}, \mathscr{A}]$, the pullback of the inclusion $J$ along the forgetful functor $U_L$*

$$
\begin{array}{ccc}
\mathsf{CoLog} \times_{\mathscr{A}} \mathscr{B} & \xhookrightarrow{J'} & \mathsf{CoLog} \\
\downarrow & & \downarrow{\scriptstyle U_L} \\
\mathscr{B} & \xhookrightarrow{J} & [\mathscr{A}, \mathscr{A}].
\end{array}
$$

*is also coreflective.*

*Proof.* Let $r$ be the coreflector of $J: \mathcal{B} \to [\mathcal{A}, \mathcal{A}]$ and $(\rho_L: r(L) \dashrightarrow L)_L$ be the coreflection. Given a semantics $\delta: LP \dashrightarrow PT$, the composite with the coreflection $\rho_L: r(L) \dashrightarrow L$ is an object $(\delta \circ \rho_L P): r(L)P \dashrightarrow PT$ in the pullback with a morphism $(\rho_L, id)$ by construction. Given another semantics $(L', T', \delta': L'P \to PT')$ with $L' \in \mathcal{B}$ and a morphism $(\tau, \nu): \delta' \to \delta$, there is a unique natural transformation $\overline{\tau}: L' \to r(L)$ satisfying $\tau = \rho_L \circ \overline{\tau}$ and $(\overline{\tau}, \nu)$ is a morphism from $\delta'$ to $\delta \circ \rho_L P$:

$$
\begin{array}{ccccc}
r(L)P & \xrightarrow{\ \rho_L P\ } & LP & \xrightarrow{\ \delta\ } & PT \\
& \nwarrow_{\overline{\tau}P} & \big\uparrow{\tau P} & & \big\uparrow{P\nu} \\
& & L'P & \xrightarrow{\ \delta'\ } & PT',
\end{array}
$$

so the morphism $(\overline{\tau}, \nu)$ is the unique morphism satisfying $(\tau, \nu) = (\rho_L, id) \circ (\overline{\tau}, \nu)$. $\qquad\square$

**Corollary 5.1.12.** *Let $\mathcal{A}$ be a variety of algebras (resp. locally finitely presentable category). The full subcategory consisting of one-step semantics $(L, T, \delta)$ with a finitely based functor (resp. finitary) $L$ is a coreflective subcategory of* CoLog.

A one-step semantics $(L, T, \delta)$ is **finitary** if $L$ is finitary on a locally finitely presentable category and it is (finitary) **equational** if $L$ is finitely based on a variety of algebras. For convenience, ECoLog denotes the full subcategory consisting of equational one-step semantics and FCoLog denotes the full subcategory consisting of one-step finitary semantics and thus ECoLog $\subseteq$ FCoLog.

## 5.1.2   Colimits and Limits in CoLog

Colimits of functors may be calculated pointwise provided the colimits on each object exist. We call such a colimit *pointwise*. Recall that there is an adjunction $[\mathcal{I}, [\mathcal{C}, \mathcal{A}]] \cong [\mathcal{I} \times \mathcal{C}, \mathcal{A}]$, so every diagram in a functor category $[\mathcal{C}, \mathcal{A}]$ is a family of diagrams in the codomain category $\mathcal{A}$. Most limits and colimits in CoLog$^P$ are computed pointwise.

**Example 5.1.13.** Suppose that $\mathcal{A}$ has an initial object $\mathbf{0}$ and $\mathcal{X}$ has a terminal object $\mathbf{1}$. The trivial one-step semantics $(\mathcal{K}_{\mathbf{0}}, \mathcal{K}_{\mathbf{1}}, \mathbf{0}P = \mathbf{0} \to P\mathbf{1})$ is an initial object since by the initiality the following diagram always commutes

$$
\begin{array}{ccccc}
\mathbf{0} & \mathbf{0} \dashrightarrow P\mathbf{1} & & \mathbf{1} \\
\big\downarrow & \big\downarrow \qquad \big\downarrow{P!} & & \big\uparrow{!} \\
\mathbf{0} & LP \xrightarrow{\ \delta\ } PT & & T
\end{array}
$$

in $\mathrm{CoLog}^P$ for any contravariant functor $P \colon \mathscr{X} \to \mathscr{A}$.

**Colimits**

**Lemma 5.1.14.** *Given any functor $P \colon \mathscr{X} \to \mathscr{A}$, the induced pre-composition functor*

$$P^* \colon [\mathscr{A}, \mathscr{A}] \to [\mathscr{X}, \mathscr{A}]$$

*preserves pointwise (co)limits.*

*Proof.* Given a pointwise colimit $(\gamma_i \colon L_i \to L)_{i \in \mathscr{I}}$ in the functor category $[\mathscr{A}, \mathscr{A}]$, the collection of morphisms $\left( L_i P x \xrightarrow{\gamma_i P x} L P x \right)$ is a colimit since $\gamma$ is pointwise. It follows that $(\gamma_i P \colon L_i P \to L P)_{i \in \mathscr{I}}$ is a colimit. The argument for pointwise limits is similar. $\qquad\square$

**Theorem 5.1.15.** *The pair $(U_L, U_R)$ of projection of $\mathrm{CoLog}^P$ creates pointwise colimits.*

*Proof.* Let $D \colon \mathscr{I} \to \mathrm{CoLog}$ be a diagram with $D_i$ denoted by $(L_i, T_i, \delta_i \colon L_i P \dashrightarrow P T_i)$, $U_L D(i \xrightarrow{f} j) \coloneqq \tau f$ and $U_R D(i \xrightarrow{f} j) \coloneqq \nu f$. Suppose that $U_L D$ and $(U_R D)^{\mathrm{op}}$ have

$$\text{a colimit } (\iota_i \colon L_i \dashrightarrow \mathrm{Colim}\, L_i) \quad \text{and} \quad \text{a limit } (\pi_i \colon \mathrm{Lim}\, T_i \dashrightarrow T_i)$$

which are pointwise in $[\mathscr{A}, \mathscr{A}]$ and $[\mathscr{X}, \mathscr{X}]$ respectively. By the preservation of pointwise colimits of $P^*$, there is a unique morphism $(\mathrm{Colim}\,\delta) \colon (\mathrm{Colim}\, L_i) P \dashrightarrow P(\mathrm{Lim}\, T_i)$ in the following diagram



since $(P\pi_i \circ \delta_i \colon L_i P \to P \mathrm{Lim}\, T_i)_{i \in \mathscr{I}}$ is a cocone from $U_L D$ to $P \mathrm{Lim}\, T_i$. It remains to show that $(\mathrm{Lim}\, T_i, \mathrm{Colim}\, L_i, \mathrm{Colim}\,\delta)$ is a colimit in $\mathrm{CoLog}$.

Let $(\gamma_i)_{i \in \mathscr{I}}$ be a cocone of $D$ to some one-step semantics $(LP \xrightarrow{\cdot} PT)$ which gives a cocone for $U_L D$ and a cone for $(U_R D)^{\mathrm{op}}$ respectively via projections. Let $U_L \gamma_i = \tau_i$ and $U_R \gamma_i = \nu_i$ for each $i$. Thus, there exists a unique pair of natural transformations

$$\tau \colon \mathrm{Colim}\, L_i \xrightarrow{\cdot} L \quad \text{and} \quad \nu \colon T \xrightarrow{\cdot} \mathrm{Lim}\, T_i$$

such that

$$\tau_i = \tau \circ \iota_i \quad \text{and} \quad \nu_i = \pi_i \circ \nu.$$

We only need to show that the pair $(\tau, \nu)$ is a morphism in $\mathsf{CoLog}_T$, and it suffices to show that, for each $i \in \mathscr{I}$ in the following diagram



we have $(P\nu \circ \mathrm{Colim}\, \delta_i) \circ \iota_i P = (\delta \circ \tau P) \circ \iota_i P$. This follows by diagram chasing

$$
\begin{aligned}
&P\nu \circ \mathrm{Colim}\, \delta_i \circ \iota_i P \\
&= (P\nu \circ P\pi_i) \circ \delta_i && \{\text{by the morphism } (\iota_i, \pi_i) \text{ in } \mathsf{CoLog}\} \\
&= P\nu_i \circ \delta_i && \{\text{by } \nu_i = \pi_i \circ \nu \text{ and } P \text{ contravariant}\} \\
&= \delta \circ \tau_i P && \{\text{by the morphism } (\tau_i, \nu_i) \text{ in } \mathsf{CoLog}\} \\
&= \delta \circ \tau P \circ \iota_i P && \{\text{by } \tau_i = \tau \circ \iota_i\}.
\end{aligned}
$$

$\square$

**Corollary 5.1.16.** *Assume that $\mathscr{A}$ and $\mathscr{X}$ are cocomplete and complete respectively. Then, the category $\mathsf{CoLog}^P$ is cocomplete.*

In particular, the categories of one-step semantics for

$$2^- \colon \mathsf{Set} \to \mathsf{Set} \quad \text{and} \quad \mathcal{Q} \colon \mathsf{Set} \to \mathsf{BA}$$

are all cocomplete. By Corollary 5.1.12, equational one-step semantics are closed under colimits in $\mathsf{CoLog}$. Thus, the notion of coproducts in $\mathsf{CoLog}$ generalises the fusion of logics of predicate liftings:

**Example 5.1.17** (Coproducts). The *fusion* of modal logics is a coproduct of logics, in CoLog, induced by predicate liftings. Consider $\mathcal{Q}\colon \mathsf{Set} \to \mathsf{BA}$ and the free adjunction $F \dashv U\colon \mathsf{BA} \to \mathsf{Set}$. As in Example 2.3.25 and Example 4.3.31, every set $\Lambda$ of predicate liftings defines a one-step semantics consisting of

$$L_\Lambda := FH_\Lambda U \quad \text{and} \quad \delta^\Lambda\colon L_\Lambda\mathcal{Q} \dashrightarrow \mathcal{Q}T$$

as the syntax and the interpretation. Similarly, given an $I$-indexed set $\Lambda_i$ of collections of predicate liftings, the coproduct of induced one-step semantics $(L_i, T_i, \delta_i)_{i \in I}$ is a one-step semantics for $T := \prod_{i \in I} T_i$ consisting of the syntax

$$L := \bigsqcup_{i \in I} FH_{\Lambda_i} U \cong F \bigsqcup_{i \in I} H_{\Lambda_i} U,$$

and the interpretation $\delta\colon L\mathcal{Q} \dashrightarrow \mathcal{Q}T$ defined for each set $X$ on the generators by

$$[\lambda](S_j)_{j \in n} \mapsto \left(\pi_i^{-1} \circ \lambda\right)(S_j)_{j \in n}$$

for $\lambda \in \Lambda_{i,n}$ and a family of subsets $(S_j \subseteq X)_{j \in n}$ where $\pi_i\colon \prod T_i \to T_i$ is the $i$-th projection. The resulting logic is the minimal logic containing each $\Lambda_i$ and extending every coalgebraic logic $(L_i, T_i, \delta_i)$ conservatively. See [37] for details in the situation of predicate liftings in Set.

**Example 5.1.18.** Product of modal logics [43] and generalised product of coalgebraic logics [96] are in fact a two-step construction:

1. As we have see in Chapter 4, the generalised product of $(x_1 \xrightarrow{\xi_1} T_1 x_1)$ and $(x_2 \xrightarrow{\xi_2} T_2 x_2)$ is a coalgebra for $T_1 \times T_2$ in a category with finite products.

2. Given two families $\Lambda_1$ and $\Lambda_2$ of predicate liftings for $T_1$ and $T_2$ respectively, the coproduct of the equational one-step semantics determined by $\Lambda_1$ and $\Lambda_2$ is an equational one-step semantics of type $T_1 \times T_2$ with the syntax $F\left(H_{\Lambda_1} U + H_{\Lambda_2} U\right)$.

Then, the generalised product of coalgebraic logics of predicate liftings is a coalgebraic logics for a pair of coalgebras of type $T_1$ and $T_2$ using the above two construction.

**Example 5.1.19** (Coequalisers). Coalgebras of the non-empty powerset functor $\mathcal{P}_{\neq\emptyset}$ are simply Kripke frames with a non-empty set of successors for each world/state. A one-step semantics for this type $\mathcal{P}_{\neq\emptyset}$ can be obtained as a coequaliser of the one-step semantics for normal modal logic $(\mathbb{M}, \mathcal{P}, \delta)$ by adding the equation $\blacklozenge\top = \top$ as follows. First, we shall see $\mathcal{P}_{\neq\emptyset}$ and $\mathbb{M}$ subject to $\blacklozenge\top = \top$ can be constructed as an equaliser and a coequaliser, respectively:

1. Define two natural transformations $\rho_X^i \colon \mathcal{P}X \rightrightarrows 2$ for each component $X$ by

$$\rho_X^1(S) = \begin{cases} 1 & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset; \end{cases} \quad \text{and} \quad \rho_X^2(S) = 1,$$

and $p^2$ is clearly natural and $p^1$ is natural since $\mathcal{P}f$ maps (non-)empty sets to (non-)empty sets. The equaliser of $p^1$ and $p^2$ for each component $X$ is precisely the collection of non-empty subsets:

$$\{S \subseteq X \mid p_X^1(S) = p_X^2(S) = 1\} = \mathcal{P}_{\neq \emptyset}X,$$

that is, $\mathcal{P}_{\neq \emptyset} \hookrightarrow \mathcal{P}$ is the equaliser of $(p^1, p^2)$.

2. The endofunctor $\overline{\mathbb{M}}$ defined by

$$\overline{\mathbb{M}}A = \mathsf{BA}\langle \{\blacklozenge a\}_{a \in A} \mid \blacklozenge\bot = \bot; \blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b; \blacklozenge\top = \top \rangle$$

for each component $A$ with the canonical projection is a coequaliser of $\mathbb{M}$ with $\pi^i \colon \mathcal{K}_{F1} \rightrightarrows \mathbb{M}$ given in Example 4.3.43.

Let $h \colon \mathcal{K}_{F1} \to \mathcal{Q}2$ be the isomorphism mapping the generator to $\{1\}$, so $(\mathcal{K}_{F1}, \mathcal{K}_2, h)$ is a one-step semantics. We can see that $(\pi^i, \rho^i)_{i=1,2}$ are morphisms $(\mathcal{K}_{F1}, \mathcal{K}_2, h) \to (\mathbb{M}, \mathcal{P}, \delta)$ in $\mathsf{ECoLog}$, by checking commutativity on the generator

$$
\begin{aligned}
&\delta_X(\blacklozenge X) && &&\delta_X(\top) \\
&= \{S \subseteq X \mid S \cap X \neq \emptyset\} && &&= \mathcal{P}X \\
&= \{S \subseteq X \mid \rho_X^1(S) = 1\} &&\text{and} &&= \{S \subseteq X \mid \rho_X^2(S) = 1\} \\
&= (2^{\rho_X^1} \circ h)(*) && &&= (2^{\rho_X^2} \circ h)(*).
\end{aligned}
$$

Hence, the pair $(\pi^i, \rho^i)_{i=1,2}$ has a coequaliser

by Theorem 5.1.15 where $j \colon \mathcal{P}_{\neq \emptyset} \hookrightarrow \mathcal{P}$ is the inclusion. By construction, this interpretation maps each generator $\blacklozenge U$ on $\overline{\mathbb{M}} \mathcal{Q} X$ to the set $\Diamond_X U = \{ S \subseteq X \mid S \cap U \neq \emptyset \}$ for $U \subseteq X$, as expected.

Note that ECoLog is closed under colimits of CoLog by the coreflection, so the above examples of colimits are also colimits in ECoLog.

**Limits in** CoLog

A (unary) predicate lifting of a binary coproduct $T_1 + T_2$ can be given separately by a (unary) predicate lifting of $T_1$ and $T_2$ respectively. It can be seen from the characterisation given in Lemma 2.3.9:

$$\mathsf{Set}((T_1 + T_2)2, 2) = \mathsf{Set}(T_1 2 + T_2 2, 2) \cong \mathsf{Set}(T_1 2, 2) \times \mathsf{Set}(T_2 2, 2).$$

In fact, not only predicate liftings but also syntaxes can be combined in a uniform way, and the corresponding one-step semantics is a product in CoLog.

**Lemma 5.1.20.** *Given a dual adjunction $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, the post-composition functor*

$$P_* \colon [\mathscr{X}^{\mathrm{op}}, \mathscr{X}^{\mathrm{op}}] \to [\mathscr{X}^{\mathrm{op}}, \mathscr{A}]$$

*preserves pointwise limits. In particular, $P$ maps a pointwise colimit of endofunctors of $\mathscr{X}$ to a pointwise limit in $[\mathscr{X}^{\mathrm{op}}, \mathscr{A}]$.*

*Proof.* It follows from the fact that every right adjoint preserves limits and every endofunctor of $\mathscr{X}^{\mathrm{op}}$ is an opposite of some endofunctor of $\mathscr{X}$. $\qquad \square$

**Theorem 5.1.21.** *Given a dual adjunction $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, the pair $(U_L, U_R)$ of projections of $\mathsf{CoLog}^P$ creates pointwise limits.*

*Proof.* Let $D \colon \mathscr{I} \to \mathsf{CoLog}^P$ be a diagram such that $U_L D$ and $(U_R D)^{\mathrm{op}}$ have a pointwise limit and a pointwise colimit

$$(\pi_i \colon \mathsf{Lim}\, L_i \overset{\cdot}{\to} L_i) \quad \text{and} \quad (\iota_i \colon T_i \overset{\cdot}{\to} \mathsf{Colim}\, T_i)$$

in $[\mathscr{A}, \mathscr{A}]$ and $[\mathscr{X}, \mathscr{X}]$ respectively. The construction is dual to Theorem 5.1.15 and the resulting one-step semantic of type $\mathsf{Colim}\, T_i$ is given by the unique morphism $\delta$

as follows:

$$
\begin{array}{ccc}
L_i P & \xrightarrow{\quad \delta_i \quad} & P T_i \\
{\scriptstyle \pi_i P} \uparrow & & \uparrow {\scriptstyle P \iota_i} \\
\mathrm{Lim}\, L_i P & \dashrightarrow[\delta = \mathrm{Lim}\, \delta_i] & \mathrm{Lim}\, P T_i \cong P(\mathrm{Colim}\, T_i).
\end{array}
$$

$\square$

**Proposition 5.1.22.** *Given a dual adjunction* $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, *the following statements are true:*

1. *Finite products in* ECoLog *coincide with products in* CoLog.

2. *Finite limits in* FCoLog *coincide with finite limits in* CoLog.

*Proof.* 1. Let $J$ be the full embedding $\mathscr{A}_\omega^f \hookrightarrow \mathscr{A}$, $L_i$ a finitely based functor for $i = 1, 2$. By (4.3), the value of the left Kan extension $\mathrm{Lan}_J(L_1 \times L_2)J$ on any object $A \in \mathscr{A}$ is a sifted colimit of $L_1 Fn \times L_2 Fn$ indexed by the comma category $(J {\downarrow} A)$. By definition, every sifted colimit commutes with finite products in Set. The forgetful functor $U \colon \mathscr{A} \to$ Set creates sifted colimits and limits, and also reflects isomorphism, so the sifted colimit also commutes with finite products in $\mathscr{A}$. It follows that $\mathrm{Lan}_J(L_1 \times L_2) \cong L_1 \times L_2$. By Theorem 5.1.21, the statement follows.

2. It follows similarly.

$\square$

**Example 5.1.23.** The one-step semantics for finitely-branching Kripke frames with termination, i.e. coalgebras of $1 + \mathcal{P}_\omega$, can be given by the coalgebraic logic of the *cover modality* for $\mathcal{P}$ and a predicate lifting for 1. The combination of these two approaches is possible, since their syntax functors are all finitely based by Example 4.3.25 and Example 4.3.31.

*Open Problem* 5.1.24. Generally speaking, it is not clear what limits of equational one-step semantics are. From the coreflection and the creation of limits, the category of equational one-step semantics is complete. However, the limit of syntaxes is constructed as $\mathrm{Lan}_J(\mathrm{Lim}_i L_i)J$ for the full inclusion $J \colon \mathscr{A}_\omega^f \hookrightarrow \mathscr{A}$, so the syntax is not the pointwise limit unless the index category is finite and discrete.

### 5.1.3   Composition of Logics as Tensor Products in CoLog

A construction for coalgebraic logics of type $T_1 \circ T_2$ built from coalgebraic logics of type $T_1$ and $T_2$ respectively appears in the work of Cîrstea [36]:

**Example 5.1.25.** A predicate lifting for $\mathcal{P}(A \times \mathcal{I})$, the type of Kripke frames indexed by $A$, can be obtained by a predicate lifting for $A \times \mathcal{I}$ and $\mathcal{P}$ respectively. In Example 2.3.6, each $\bar{a} \colon U\mathcal{Q} \to U\mathcal{Q}(A \times \mathcal{I})$ is a predicate lifting for $A \times \mathcal{I}$; and $\lozenge \colon U\mathcal{Q} \to U\mathcal{Q}\mathcal{P}$ is a predicate lifting for $\mathcal{P}$. By the naturality, we also have a natural transformation $\lozenge_{A \times -} \colon 2^{A \times -} \to 2^{\mathcal{P}(A \times -)}$, and by composition we obtain a predicate lifting for $\mathcal{P}(A \times \mathcal{I})$:

$$2^{-} \xrightarrow{\bar{a}} 2^{A \times -} \xrightarrow{\lozenge_{A \times -}} 2^{\mathcal{P}(A \times -)},$$

and the resulting predicate lifting is the usual interpretation of $\langle a \rangle$ in multi-modal logic.

To characterise the composition of one-step semantics, we describe one-step semantics in 2-cells.[1] An object in CoLog is a 2-cell $\left( \downarrow \overrightarrow{\underset{\delta}{\Leftarrow}} \downarrow \right)$ and a morphism $(\tau, \nu) \colon \delta_1 \to \delta_2$ in CoLog amounts to an equality between 2-cells:

$$\left| \overrightarrow{\underset{\delta_2}{\Leftarrow}} \left( \overleftarrow{\underset{\tau}{\Leftarrow}} \right) \right| = \left( \overleftarrow{\underset{\nu}{\Leftarrow}} \right) \overrightarrow{\underset{\delta_1}{\Leftarrow}} \right| \tag{5.1}$$

as the morphism $(\tau, \nu)$ defines a one-step semantics with type $T_2$ and syntax $L_1$ by the commutativity $\delta_2 \circ \tau P = P\nu \circ \delta_1$. The projection functors $(U_L, U_R)$ may be displayed by 'projecting' edges:

$$(U_L, U_R) \colon \left( \left| \overrightarrow{\underset{\delta_2}{\Leftarrow}} \left( \overleftarrow{\underset{\tau}{\Leftarrow}} \right) \right| = \left( \overleftarrow{\underset{\nu}{\Leftarrow}} \right) \overrightarrow{\underset{\delta_1}{\Leftarrow}} \right| \right) \mapsto \left( \left( \overleftarrow{\underset{\tau}{\Leftarrow}} \right) \, , \, \left( \overleftarrow{\underset{\nu}{\Leftarrow}} \right) \right).$$

Given two one-step semantics $(L_i, T_i, \delta_i)$ for $i = 1, 2$, define the **composite** $\delta_1 \otimes \delta_2$ by the pasting diagram

$$
\begin{array}{ccccc}
\xleftarrow{\;\;L_1\;\;} & & \xleftarrow{\;\;L_2\;\;} & & \xleftarrow{\;\;L_1\;\;} \quad \xleftarrow{\;\;L_2\;\;} \\
\Big\uparrow \; \underset{\delta_1}{\Leftarrow} \; \Big\uparrow & \otimes & \Big\uparrow \; \underset{\delta_2}{\Leftarrow} \; \Big\uparrow & := & \Big\uparrow \; \underset{\delta_1}{\Leftarrow} \; \Big\uparrow \quad \Big\uparrow \; \underset{\delta_2}{\Leftarrow} \; \Big\uparrow \\
\xleftarrow{\;\;T_1\;\;} & & \xleftarrow{\;\;T_2\;\;} & & \xleftarrow{\;\;T_1\;\;} \quad \xleftarrow{\;\;T_2\;\;}
\end{array}
$$

---

[1] To read a 2-cell diagram, see [81, Section XII.3-4] for reference.

i.e. $\delta_1 \otimes \delta_2 = \delta_1 T_2 \circ L_1 \delta_2$ is a one-step semantics of type $T_2 T_1$ with the syntax $L_2 L_1$. In fact, such a composition defines a *tensor product* on CoLog:

**Lemma 5.1.26.** *The composition $\otimes$ of one-step semantics, mapping each pair of morphisms*

$$(\tau_o, \nu_o) \colon (L_1, T_1, \delta_1) \to (L_3, T_3, \delta_3) \quad \text{and} \quad (\tau_e, \nu_e) \colon (L_2, T_2, \delta_2) \to (L_4, T_4, \delta_4)$$

*to $(\tau_o, \nu_o) \otimes (\tau_e, \nu_e) \colon \delta_1 \otimes \delta_2 \to \delta_3 \otimes \delta_4$ defined by*

$$\left( \underset{L_4}{\overset{L_2}{\Downarrow \tau_e}} \; \underset{L_3}{\overset{L_1}{\Downarrow \tau_o}} \; , \; \underset{T_4^{\mathrm{op}}}{\overset{T_2^{\mathrm{op}}}{\Downarrow \nu_e}} \; \underset{T_3^{\mathrm{op}}}{\overset{T_1^{\mathrm{op}}}{\Downarrow \nu_o}} \right)$$

*i.e. the horizontal composites $\tau_o \tau_e$ and $\nu_o \nu_e$, is a bifunctor.*

*Proof.* First, we need to check that $(\tau_o, \nu_o) \otimes (\tau_e, \nu_o)$ is a morphism in CoLog; it follows readily from the following pasting diagrams

$$\begin{array}{c} \overset{\Leftarrow}{\nu_e} \; \Downarrow \delta_2 \\ \overset{\Leftarrow}{\tau_o} \; \Downarrow \delta_1 \end{array} \quad = \quad \begin{array}{c} \Downarrow \delta_4 \; \overset{\Leftarrow}{\tau_e} \\ \Downarrow \delta_3 \; \overset{\Leftarrow}{\nu_o} \end{array}$$

since $(\tau_e, \nu_e)$, as a morphism in CoLog, i.e. (5.1), implies that the upper pasting diagrams are equal. The argument for the lower pasting diagrams is similar. By the interchange law for 2-cells of natural transformations, the order of composition is irrelevant.

By the equality of the pasting diagrams, it suffices to consider one side of it and it is trivial to show that $\otimes$ preserves identities and compositions. E.g.

**identity:** for any two semantics $(L, T, \delta)$ and $(L', T', \delta')$ the horizontal composition of identities is identity: $\quad \underset{L'}{\overset{L'}{\Downarrow id}} \; \underset{L}{\overset{L}{\Downarrow id}} \quad = \quad \underset{LL'}{\overset{LL'}{\Downarrow id}} \quad .$

**composition:** for morphisms $\left\{ (L_i, T_i, \delta_i) \xrightarrow{(\tau_i, \nu_i)} (L_i', T_i', \delta_i') \right\}_{i=1,2,3}$ it follows from the associativity of horizontal composition:

$$\underset{L_3'}{\overset{L_3}{\Downarrow \tau_3}} \; \underset{L_1' L_2'}{\overset{L_1 L_2}{\Downarrow \tau_1 \tau_2}} \quad = \quad \underset{L_2' L_3'}{\overset{L_2 L_3}{\Downarrow \tau_2 \tau_3}} \; \underset{L_1'}{\overset{L_1}{\Downarrow \tau_1}} \quad = \quad \underset{L_1' L_2' L_3'}{\overset{L_1 L_2 L_3}{\Downarrow \tau}}$$

$\square$

**Proposition 5.1.27.** *The composition $\otimes$ of one-step semantics with the identity semantics $(\mathcal{I}, \mathcal{I}, id_P)$ defines a strict monoidal structure on $\mathsf{CoLog}^P$.*

*Proof.* Associator, left unitor, and right unitor all follows from the associative law and the unit law of composition of morphisms. E.g. the right unitor is given by $\delta \otimes id_P = \delta\mathcal{I} \circ Lid = \delta \circ id = \delta$. $\square$

The full subcategory of equational one-step semantics is closed under compositions by Proposition 4.3.18:

**Corollary 5.1.28.** *Any composite of two equational (resp. finitary) one-step semantics is (resp. finitary) equational.*

**Example 5.1.29.** Continuing Example 5.1.25, we consider their corresponding one-step semantics $(FU, \Diamond^*)$ and $(F(A \cdot U), \lambda)$ where $\Diamond^*$ is the transpose of $\Diamond$ and $\lambda$ is the transpose of

$$[\bar{a}]_{a \in A} \colon A \cdot U\mathcal{Q} \dashrightarrow U\mathcal{Q}T.$$

Then, the composition of one-step semantics of type $\mathcal{P}(A \times \mathcal{I})$ consists of

$$FUF(A \cdot U) \quad \text{and} \quad FUF(A \cdot U)\mathcal{Q} \xrightarrow{FU\lambda} FU\mathcal{Q}(A \times \mathcal{I}) \xrightarrow{\Diamond A \times \mathcal{I}} \mathcal{Q}\mathcal{P}(A \times \mathcal{I})$$

as the syntax and the interpretation, i.e. an equational one-step semantics for labelled Kripke frames.

## 5.1.4   The Mate Operation of One-Step Semantics

The 2-cell perspective of one-step semantics with a dual adjunction $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ gives us a way of defining a transpose by pasting the unit and counit. In fact, the transpose provides the same information, but the use of transpose will ease the study of the expressiveness problem later.

In this subsection, we assume that $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ is a dual adjunction with the unit $\eta \colon \mathcal{I} \dashrightarrow PS$ in $\mathscr{A}$ and the counit $\epsilon^{\mathrm{op}} \colon SP \dashrightarrow \mathcal{I}$ in $\mathscr{X}^{\mathrm{op}}$. The counit may be presented by $\epsilon \colon \mathcal{I} \dashrightarrow SP$ for simplicity.

**Definition 5.1.30.** The **mate** $\delta^*$ of a one-step semantics $(L, T, \delta)$ is a natural transformation from $TS$ to $SL$ defined by the following pasting diagram

$$
\begin{array}{c}
\end{array}
$$

in the opposite of $\mathscr{X}$, that is, $\delta^* = SL\eta \circ S\delta S \circ \epsilon TS$.

Note that the construction $(-)^*$ maps objects in $\mathsf{CoLog} = (P^* \!\downarrow\! P_*)$ to objects in the comma category $(S^* \!\downarrow\! S_*)$ where $S^*$ and $S_*$ are the pre-composition and the post-composition with $S$ as before. We will show that $(S^* \!\downarrow\! S_*)$ is a category dually isomorphic to $\mathsf{CoLog}$, so we denote it by $\mathsf{CoLog}^*$. To see this, define the mate $\theta^*$ of an object $\theta$ in $\mathsf{CoLog}^*$ by pasting the unit and the counit:

$$
\begin{array}{c}
\end{array}
$$

which is a natural transformation from $PT$ to $LP$ in $\mathscr{A}^{\mathrm{op}}$.

**Lemma 5.1.31.** *Mate correspondence on objects of* $\mathsf{CoLog}$ *is a bijection.*

*Proof.* By computing $(\delta^*)^*$ of $(\delta\colon LP \dashrightarrow PT)$ directly, we have

$$
\begin{array}{c}
\end{array}
$$

where the last diagram is equal to $\delta$ by the triangle identities. $\qquad\square$

**Proposition 5.1.32.** *The category* $\mathsf{CoLog}^{\mathrm{op}}$ *is isomorphic to* $\mathsf{CoLog}^*$.

*Proof.* By the previous lemma, it suffices to show that for every morphism $(\tau, \nu)\colon (L_1, T_1, \delta_1) \to (L_2, T_2,$
in $\mathsf{CoLog}$, the pair $(\nu, \tau)$ is a morphism from $(L_2, T_2, \delta_2^*)$ to $(L_1, T_1, \delta_1^*)$ in $\mathsf{CoLog}^*$.[2]

---
[2] The direction is reversed.

Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
& & & & \delta_1^* & & \\
T_1 S & \xrightarrow{\epsilon T_1 S} & SPT_1 S & \xrightarrow{S\delta_1 S} & SL_1 PS & \xrightarrow{SL_1\eta} & SL_1 \\
{\scriptstyle \nu S}\Big\uparrow & & {\scriptstyle SP\nu S}\Big\uparrow & & {\scriptstyle S\tau PS}\Big\uparrow & & \Big\uparrow{\scriptstyle S\tau} \\
T_2 S & \xrightarrow[\epsilon T_2 S]{} & SPT_2 S & \xrightarrow[S\delta_2 S]{} & SL_2 PS & \xrightarrow[SL_2\eta]{} & SL_2 \\
& & & & \delta_2^* & &
\end{array}
$$

where the upper and lower rectangles commute by definition, and the remaining diagrams commute by naturality. Hence, $(\nu, \tau)$ is a morphism in $\mathsf{CoLog}^*$. □

The isomorphism then gives the simple fact that the mate of a (pointwise) colimit of one-step semantics is a (pointwise) limit. Likewise, the composition of one-step semantics can be computed in the transpose form. Define a bifunctor $\oplus$ in $\mathsf{CoLog}^*$ by the pasting diagrams:



that is, $\theta_1 \oplus \theta_2 = \theta_1 L_2 \circ T_1 \theta_2$ is a natural transformation from $T_1 T_2 S$ to $SL_1 L_2$. This operation $\oplus$ with the triple $(\mathcal{I}, \mathcal{I}, id)$ of identities defines a monoidal structure on $\mathsf{CoLog}^*$.

**Proposition 5.1.33.** *For any two one-step semantics* $(L_1, T_1, \delta_1)$ *and* $(L_2, T_2, \delta_2)$, *the following equation holds*

$$(\delta_1 \otimes \delta_2)^* = \delta_1^* \oplus \delta_2^*. \tag{5.2}$$

*In particular,* $(\delta_1 \otimes \delta_2)^* = \delta_1^* L_2 \circ T_1 \delta_2^*$.

*Proof.* By the triangle identities, the following pasting diagrams (in $\mathscr{X}^{\mathrm{op}}$)



are equal where the left pasting diagram presents $(\delta_1 \otimes \delta_2)^*$ and the right diagram presents $\delta_1^* \oplus \delta_2^*$, so the statement follows. □

### 5.1.5  One-Step Semantics for $T$-Coalgebras

We proceed with the investigation of the category of $T$-logics, i.e. the collection of one-step semantics over a fixed type functor $T\colon \mathscr{X} \to \mathscr{X}$.

> Throughout this subsection, $T$ always denotes an endofunctor of $\mathscr{X}$, and as before $P\colon \mathscr{X} \to \mathscr{A}$ is a contravariant functor.

**Proposition 5.1.34.** *The forgetful functor $U\colon \mathsf{CoLog}_T \to [\mathscr{A}, \mathscr{A}]$ reflects isomorphisms.*

*Proof.* Given a translation $\tau\colon (L_1, \delta_1) \to (L_2, \delta_2)$ such that $\tau$ is an isomorphism in $[\mathscr{A}, \mathscr{A}]$, we have $\delta_2 = \delta_1 \circ \tau^{-1}P$, i.e. the inverse $\tau^{-1}$ is a translation, since each component $\tau_A\colon L_1 A \to L_2 A$ is an isomorphism. Thus, $\tau$ is an isomorphism in $\mathsf{CoLog}_T$ with the inverse $\tau^{-1}$. $\qquad\square$

**Colimits in $\mathsf{CoLog}_T$**

**Theorem 5.1.35.** *The forgetful functor $U\colon \mathsf{CoLog}_T \to [\mathscr{A}, \mathscr{A}]$ creates pointwise colimits. In particular, $U$ reflects and preserves colimits which exist in $[\mathscr{A}, \mathscr{A}]$.*

*Proof.* Let $D\colon \mathscr{I} \to \mathsf{CoLog}_T$ be a diagram in $\mathsf{CoLog}_T$ where every $D_i$ is denoted by $(L_i, \delta_i)$, and a pointwise colimit $(\gamma_i\colon L_i \dashrightarrow L)_{i \in \mathscr{I}}$ of $UD$ in the category of endofunctors of $\mathscr{A}$. By Lemma 5.1.14, $(\gamma_i P\colon L_i P \dashrightarrow LP)_{i \in \mathscr{I}}$ is a colimit of $P^*UD$. Consider the following diagram



where the outer triangle commutes, i.e. $(\delta_i\colon L_i P \dashrightarrow PT)$ is a cocone from $P^*UD$ to $PT$ since $Df$ is a translation (Definition 5.1.4); the inner triangle commutes because $(\gamma_i P\colon L_i P \to LP)_{i \in \mathscr{I}}$ is a limiting cocone. It follows that there exists a unique natural transformation $\delta$ satisfying that $\delta_i = \delta \circ \gamma_i P$, i.e. a translation. Thus, we have a cocone

$$\left( (L_i, \delta_i) \xrightarrow{\gamma_i} (L, \delta) \right)_{i \in \mathscr{I}}$$

in $\mathsf{CoLog}_T$. It remains to show that $(\gamma_i)_{i \in \mathscr{I}}$ is a colimit of $D$.

Given a cocone $\alpha$ from $D$ to some $(L', \delta')$ in $\mathsf{CoLog}_T$, we want to show that there is a unique translation from $(L, \delta)$ to $(L', \delta')$. There is a unique natural transformation $\tau$ from $L$ to $L'$ satisfying $\alpha_i = \tau \circ \gamma_i$ since $\alpha$ is a cocone from $UD$ to $L'$. To show that $\tau$ is a translation, i.e. $\delta = \delta' \circ \tau P$, it suffices to show that $\delta' \circ \tau P \circ \gamma_i P = \delta_i$. By the following diagram

$$
\begin{array}{ccccc}
 & & \delta_i & & \\
L_iP & \xrightarrow{\gamma_iP} & LP & \xrightarrow{\delta} & PT \\
 & \alpha_iP \searrow & \downarrow \tau P & \nearrow \delta' & \\
 & & L'P & &
\end{array}
$$

where the outer triangle commutes by assumption ($\alpha$ is a cocone in $\mathsf{CoLog}_T$); the inner triangle on the left commutes since $\alpha_i = \tau \circ \gamma_i$ for each $i$, we have $(\delta' \circ \tau P) \circ \gamma_i P = \delta_i$ for each $i$. By the construction of $\delta$, $\delta' \circ \tau P$ must be $\delta$, i.e. $\tau$ is a translation from $(L, \delta)$ to $(L', \delta')$. It follows that the forgetful functor creates pointwise colimits. $\square$

**Corollary 5.1.36.** *The category $\mathsf{CoLog}_T$ is cocomplete if $\mathscr{A}$ is cocomplete.*

**Proposition 5.1.37.** *For every coreflective subcategory $\mathscr{C}$ of the category of endofunctors of $\mathscr{A}$, the pullback of the coreflective inclusion $R \vdash i \colon \mathscr{C} \hookrightarrow [\mathscr{A}, \mathscr{A}]$ along the forgetful functor is also coreflective, i.e.*

$$
\begin{array}{ccc}
\mathscr{C} \times \mathsf{CoLog}_T & \underset{\pi_1}{\overset{\top}{\rightleftarrows}} & \mathsf{CoLog}_T \\
\pi_2 \downarrow \quad \lrcorner & & \downarrow U \\
\mathscr{C} & \underset{i}{\overset{R}{\underset{\top}{\rightleftarrows}}} & [\mathscr{A}, \mathscr{A}]
\end{array}
$$

*Proof.* Note that the pullback category consists of $T$-logics with syntax in $\mathscr{C}$. Let $R$ be the coreflector of $i$, i.e the right adjoint of the inclusion $i$. Define a functor from $\mathsf{CoLog}_T$ to the pullback by mapping each $T$-logic $(L, \delta)$ to a $T$-logic $(RL, \delta \circ \rho P)$ where $\rho$ is the coreflection of $L$. For any translation $\tau \colon (L', \delta') \to (L, \delta)$ for some $L' \in \mathscr{C}$, $\tau$ factors through $\rho$ via a unique natural transformation $\overline{\tau}$, so it suffices to show that $\overline{\tau}$ is a translation, i.e. the diagram

$$
\begin{array}{ccc}
(RL)P & \xrightarrow{\rho P} & LP \\
\overline{\tau}P \nwarrow \searrow & & \nearrow \tau P \\
\delta \circ \rho P \searrow & L'P & \swarrow \delta \\
 & \downarrow \delta' & \\
 & PT &
\end{array}
$$

commutes. By diagram chasing, it follows easily. □

In particular, it follows that any pullback of a coreflective inclusion is closed under pointwise colimits of $[\mathscr{A}, \mathscr{A}]$ and has any type of limits which $\mathsf{CoLog}_T$ has.

**Terminal Object in** $\mathsf{CoLog}_T$ **via Duality**

If the functor $P\colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ is a dual adjoint, then the category of $T$-logics has a 'maximal' $T$-logic into which every other $T$-logic can be translated. It may be viewed as the 'most expressive' $T$-logic as we will show that expressiveness is stable under translation.

**Theorem 5.1.38.** *Given a dual adjunction $S \dashv P\colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, there is a terminal object*

$$(PTS, PT\epsilon\colon PTSP \dot{\longrightarrow} PT)$$

*in* $\mathsf{CoLog}_T$ *where* $\epsilon\colon \mathcal{I} \to SP$ *is the counit.*

*Proof.* Let $\eta\colon \mathcal{I} \dot{\longrightarrow} SP$ be a unit of the dual adjunction. For every $T$-logic $(L, \delta\colon LP \dot{\longrightarrow} PT)$, there exists a natural transformation $\tau\colon L \dot{\longrightarrow} PTS$ given by



that is $\tau = \delta S \circ L\eta$. To see that $\tau$ is a translation consider the following diagram:



The left triangle commutes by the triangle identity of adjunctions, and the right square commutes by the naturality of the interpretation $\delta$.

To see uniqueness, assume that $\tau'\colon L \dashrightarrow PTS$ is another translation from $(L,\delta)$ to the full logic $(PTS, PT\epsilon)$. Consider the following diagram:

$$
\begin{array}{ccc}
L & \xrightarrow{\ \tau'\ } & PTS \\
{\scriptstyle L\eta}\big\downarrow & {\scriptstyle PTS\eta}\big\downarrow & \\
LPS & \xrightarrow[\tau'PS]{} PTSPS \xrightarrow[PT\epsilon S]{} & PTS
\end{array}
$$

with $id$ and $\delta S$.

The left square commutes by the naturality of $\tau'$, the right triangle is the triangle identities of adjunctions, and the lower edge is the definition of a translation. Therefore, $\tau' = \delta S \circ L\eta$, i.e. the uniqueness of translation follows. $\qquad\square$

Such a terminal object $(PTS, PT\epsilon)$ in $\mathsf{CoLog}_T$ is called the **full ($T$-)logic**.

## 5.2  Equational Coalgebraic Logics

Recall that every collection of predicate liftings for an endofunctor $T$ of $\mathsf{Set}$ introduces a $T$-logic as well as the cover modality. Moreover, their syntax functors are *finitely based*. Any $T$-logic consisting of a finitely based functor is a coalgebraic logic which can be presented concretely and algebraically. We now aim to characterise this class of logics.

**Definition 5.2.1.** Given a variety $\mathscr{A}$, a $T$-logic $(L,\delta)$, where $L$ is finitely based, is called **equational**.

Denote the category of equational $T$-logics by $\mathsf{ECoLog}_T$; by definition it is the pullback of the following diagram:

$$
\begin{array}{ccc}
\mathsf{ECoLog}_T & \underset{\pi_1}{\overset{\top}{\rightleftarrows}} & \mathsf{CoLog}_T \\
{\scriptstyle \pi_2}\big\downarrow & {\scriptstyle \lrcorner} & \big\downarrow{\scriptstyle U} \\
\mathsf{FinB}[\mathscr{A},\mathscr{A}] & \underset{i}{\overset{\top}{\rightleftarrows}} & [\mathscr{A},\mathscr{A}]
\end{array}
$$

By (4.7), Proposition 5.1.37, and Corollary 5.1.36, we obtain the following:

**Corollary 5.2.2.** *The category of equational logics is a coreflective subcategory of* $\mathsf{CoLog}_T$. *In particular,* $\mathsf{ECoLog}_T$ *is closed under colimits of* $\mathsf{CoLog}_T$ *and is cocomplete.*

The following examples of equational logics are the main building blocks of equational logics:

**Definition 5.2.3.** Let $U\colon \mathscr{A} \to \mathsf{Set}$ be a functor with a left adjoint $F$. Then, we define the following:

1. A natural transformation $\lambda$ from $(UP)^n$ to $UPT$ for some set $n \in \mathsf{Set}$ is called an $n$-**ary predicate lifting** for $T$. A **finitary predicate lifting** is a predicate lifting of a finitary arity.

2. The $n$-**ary unimodal logic** $(L, \delta^\lambda)$ of a $n$-ary predicate lifting $\lambda$ is a $T$-logic consisting of

$$L := FU^n \quad \text{and the transpose of} \quad \lambda\colon (UP)^n \overset{\cdot}{\longrightarrow} UPT$$

as the syntax and the interpretation, respectively.

3. Given a set of finitary predicate liftings $\Lambda$, the **logic of predicate liftings** $(L, \delta^\Lambda)$ consists of

$$L := \widehat{H_\Lambda} \quad \text{and the transpose of} \quad [\lambda]_{\lambda \in \Lambda}\colon \coprod_{n \in \omega} \Lambda_n \cdot (UP)^n \overset{\cdot}{\longrightarrow} UPT$$

as the syntax and the interpretation, respectively, where $\Lambda_n$ denotes the collection of $n$-ary predicate liftings.

**A Characterisation of Equational Coalgebraic Logic**

In Chapter 2, we have seen that the logic of the cover modality gives rise to a set of predicate liftings by the presentation of the type functor $T$. We would like to point out further that every equational coalgebraic logic is precisely a logic of predicate liftings subject to an axiomatisation.

*Remark* 5.2.4. Let $U\colon \mathscr{A} \to \mathsf{Set}$ be a finitary and monadic functor with a left adjoint $F$. By Theorem 5.1.35 and Corollary 5.2.2, the following characterisations are evident:

1. Every equational $T$-logic with $FU^n$ as its syntax is precisely a unimodal $T$-logic.

2. Every equational $T$-logic with $\widehat{H_\Sigma} = FH_\Sigma U$ as its syntax is precisely a logic of predicate liftings.

3. Moreover, every logic of predicate liftings for $T$ is a coproduct of unimodal $T$-logics, since the polynomial functor $H_\Sigma$ is a coproduct of copowers $\Sigma_n \cdot (-)^n$ indexed by $n \in \omega$.

By Theorem 4.3.29 and Theorem 5.1.35, we immediately have the following general characterisation of every equational logic:

**Corollary 5.2.5.** *Every equational $T$-logic is a coequaliser of a logic of predicate liftings for $T$.*

## 5.2.1 Translations between Equational Logics

Every equational logic is a coequaliser of a logic of predicate liftings, so we look at translations from a logic of predicate liftings to some equational logic first and parallel morphisms in $\mathsf{ECoLog}_T$, i.e. two parallel translations, later.

As a logic of predicate liftings is a coproduct of unimodal logics, any morphism from a coproduct boils down to a family of morphisms from each component. To see how translation works in detail, it suffices to look at a translation from a unimodal logic to an equational logic $(L, \delta)$.

Given a unimodal logic induced by a predicate lifting $\lambda$, every translation $\tau \colon (FU^n, \delta^\lambda) \to (L, \delta)$ corresponds to a natural transformation $\tau \colon U^n \dashrightarrow UL$ in $\mathsf{Set}$ by restriction and it satisfies the commutative diagram

$$
\begin{array}{ccc}
(UPx)^n & \xrightarrow{\;\tau Px\;} & ULPx \\
& \searrow{\lambda} & \downarrow{U\delta_x} \\
& & UPTx
\end{array}
$$

as given in Example 5.1.5. That is, for each algebra $A \in \mathscr{A}$, the translation $\tau$ maps $a \in (UA)^n$ to an equivalence class of *rank-1* sentences[3] of $\mathscr{A}$, i.e. an element in $ULA$. The translation must respect the interpretation of the predicate lifting $\lambda$ as well.

---

[3] $LA$ is a quotient of $\widehat{H_\Sigma}A$ for $\Sigma := ULF$.

## 5.2.2 Equations Valid under Interpretation and Complete Axiomatisation

Continuing Remark 4.3.33, an equation for a logic of predicate liftings is similar, but it has to remain valid under the given *interpretation*: Let $\Lambda$ be a set of finitary predicate liftings and $(\widehat{H_\Lambda}, \delta^\Lambda)$ the corresponding equational logic. Then the composite

$$U\widehat{H_\Lambda}Fn \xrightarrow{U\widehat{H_\Lambda}a} U\widehat{H_\Lambda}Px \xrightarrow{U\delta_x^\Lambda} UPTx$$

for any $(Fn \xrightarrow{a} Px)$ gives an interpretation of terms, i.e. elements in $U\widehat{H_\Lambda}Fn$, under $\Lambda$. We denote the composite by $[\![-]\!]_{\Lambda,x}^a$ or simply $[\![-]\!]_x^a$ whenever confusion is unlikely.

**Definition 5.2.6.** Given a set $\Lambda$ of predicate liftings for $T$, a rank-1 equation $t \sim_n t'$ of $\Lambda$ is **valid under the interpretation** of $\Lambda$ if $[\![t]\!]_x^a = [\![t']\!]_x^a$ for any $x$ and $n$-tuple $a = (a_i)_{i \in n}$ of $Px$.

**Lemma 5.2.7.** *Let $\Lambda$ be a set of predicate liftings for $T$, and $E$ a family of rank-1 equations of the signature $\Lambda$. Then, every equation $t \sim t'$ in $E$ is valid under interpretation of $\Lambda$ if and only if the diagram*

$$(5.3)$$

$$En \xrightarrow{(\eta E)n} U\widehat{H_E}Fn \underset{U(\pi_2\iota)^*Fn}{\overset{U(\pi_1\iota)^*Fn}{\rightrightarrows}} U\widehat{H_\Sigma}Fn$$

$$\downarrow U\widehat{H_\Sigma}a$$

$$U\widehat{H_\Sigma}Px \xrightarrow{U\delta_x^\Lambda} UPTx$$

*commutes for every $n$, $x$, and $(Fn \xrightarrow{a} Px)$.*

*Proof.* Note that $[F, U] = U(-)F$ is a right adjoint. For any $f \colon E \dashrightarrow U\widehat{H_\Sigma}F$ the following diagram

$$\begin{array}{c} E \\ {\scriptstyle \eta E} \downarrow \quad \searrow {\scriptstyle f} \\ U\widehat{H_E}F \xrightarrow[U f^* F]{} U\widehat{H_\Sigma}F \end{array}$$

commutes, since $U f^* F \circ \eta E$ is the transpose of $f^*$. Let $\pi_1\iota, \pi_2\iota \colon E \rightrightarrows U\widehat{H_E}F$ be the corresponding projections of $E$. Replacing $f$ with $\pi_i\iota$, it follows by a direct computation. $\qquad\square$

**Theorem 5.2.8.** *Let $\Lambda$ be a set of predicate liftings for $T$, $E$ a family of rank-1 equations of the signature $\Lambda$, and $(\pi_i\iota)^*$ the transposes of $\pi_i\iota$ for $i = 1, 2$. The following statements are equivalent:*

1. *Every equation* $t \sim t' \in E$ *is valid under the interpretation of* $\Lambda$.

2. *The following diagram commutes*

$$\widehat{H_E}P \underset{(\pi_2\iota)^*P}{\overset{(\pi_1\iota)^*P}{\rightrightarrows}} \widehat{H_\Lambda}P \xrightarrow{\delta^\Lambda} PT. \tag{5.4}$$

*Proof.* Let $a$ be a morphism $Fn \to Px$. Then, we have the following diagram:

$$
\begin{array}{ccc}
En \xrightarrow{(\eta E)n} U\widehat{H_E}Fn \underset{U(\pi_2\iota)^*Fn}{\overset{U(\pi_1\iota)^*Fn}{\rightrightarrows}} U\widehat{H_\Lambda}Fn \\
\Big\downarrow {\scriptstyle U\widehat{H_E}a} \qquad\qquad \Big\downarrow {\scriptstyle U\widehat{H_\Lambda}a} \\
U\widehat{H_E}Px \underset{U(\pi_2\iota)^*Px}{\overset{U(\pi_1\iota)^*Px}{\rightrightarrows}} U\widehat{H_\Lambda}Px \xrightarrow{U\delta_x^\Lambda} UPTx
\end{array}
$$

where $\eta_E$ is the unit of the monadic adjunction in Theorem 4.3.29, and the two squares commute by naturality. If the lower fork commutes, then (5.3) holds by diagram chasing.

Conversely, since finitely based functors are determined on the full subcategory $\mathscr{A}_\omega^f$ spanned by $Fn$'s, it suffices to show that for every $n$ and $Fn \xrightarrow{a} Px$ the following diagram

$$\widehat{H_E}Fn \rightrightarrows \widehat{H_\Lambda}Fn \to \widehat{H_\Lambda}Px \to PTx$$

commutes. By assumption, every equation $t \sim_n t' \in En$ leads to the identical values $[\![t]\!]_x^a = [\![t']\!]_x^a$ under the interpretation, so the generated equations in $\widehat{H_E}Fn$ are also valid. $\square$

Intuitively, a coequaliser in $\mathsf{ECoLog}_T$ adds equations valid under the given interpretation.

**Theorem 5.2.9.** *For any endofunctor $T$, the category $\mathsf{ECoLog}_T$ has $(\mathsf{RegEpi}, U^{-1}\mathsf{Mono})$-factorisation system where $U$ is the forgetful functor $U: \mathsf{ECoLog}_T \to \mathsf{FinB}[\mathscr{A}, \mathscr{A}]$.*

For convenience, we adopt another equivalent definition of factorisation system: the pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms forms a factorisation system if every morphism $f$ has a unique factorisation $f = m \circ e$, for $e \in \mathcal{E}$ and $m \in \mathcal{M}$, up to unique isomorphism; $\mathcal{E}$ and $\mathcal{M}$ contain isomorphisms and are closed under composition.

*Proof.* Consider a morphism $\tau\colon (L_1, \delta_1) \to (L_2, \delta_2)$ in $\mathsf{CoLog}_T$ and its factorisation, by Corollary 4.3.41, in the diagram:

$$
\begin{array}{c}
L_1 P \\
\Big\downarrow{\scriptstyle eP} \quad \searrow^{\delta_1} \\
LP \xrightarrow{\ \delta\ } PT \\
\Big\downarrow{\scriptstyle iP} \quad \nearrow_{\delta_2} \\
L_2 P,
\end{array}
$$

and define $\delta := \delta_2 \circ iP \colon LP \dashrightarrow PT$. We can see that $i$ is a translation from $(L, \delta)$ to $(L_2, \delta_2)$ by construction, and $e$ is a translation from $(L_1, \delta_1)$ to $(L, \delta)$ by diagram chasing. Thus, $\tau$ admits a $(U^{-1}\mathsf{RegEpi}, U^{-1}\mathsf{Mono})$-factorisation.

Suppose that $\tau$ has another factorisation $(L_1, \delta_1) \xrightarrow{v} (L', \delta') \xrightarrow{u} (L_2, \delta_2)$ in $\mathsf{CoLog}_T$ such that $u$ is a monomorphism in $\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$ and $v$ is a regular epimorphism in $\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$. By unique factorisation, there exists a unique isomorphism $f\colon L' \to L$ satisfying $f \circ v = e$ and $u = i \circ f$. By Proposition 5.1.34 and the preservation of isomorphisms by any functor, the unique factorisation up to isomorphism follows from that the unique isomorphism is a translation $f\colon (L, \delta) \to (L', \delta')$: Consider the diagram

$$
\begin{array}{c}
L'P \\
{\scriptstyle vP}\nearrow \quad \Big\downarrow{\scriptstyle fP} \quad \searrow^{\delta'} \\
L_1 P \xrightarrow[eP]{} LP \xrightarrow[\delta]{} PT \\
\underbrace{\qquad\qquad\qquad\qquad}_{\delta_1}
\end{array}
$$

where every sub-diagram, except the right triangle, commutes by assumption. Since $\delta' \circ vP = (\delta \circ fP) \circ vP$ by diagram chasing and $vP$ is an epimorphism, it follows that $\delta' = \delta \circ fP$, i.e. the isomorphism $f$ is a translation from $(L', \delta')$ to $(L, \delta)$.

Note that the forgetful functor $U$ preserves colimits by Theorem 5.1.35, so every regular epimorphism in $\mathsf{ECoLog}_T$ is also a regular epimorphism in $\mathsf{FinB}[\mathscr{A}, \mathscr{A}]$. Thus, we have completed the proof. $\qquad\square$

**Definition 5.2.10.** We say that an equational logic has a **complete axiomatisation** if it has no proper regular quotient, i.e. every regular epimorphism from it is an isomorphism.

*Remark* 5.2.11. That is, an equational logic $(L, \delta)$ has a complete axiomatisation if the syntax $L$ contains every rank-1 equation valid under the interpretation $\delta$ but nothing more.

### 5.2.3 Coalgebraic Modalities via Dualities

So far, we have not yet established the existence of an equational logic. In this subsection, we show its existence and the existence of a full equational logic and give a characterisation of full equational logic.

⊘ In this subsection, we assume a dual adjunction $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, so $P$ is not only a contravariant functor but also a dual adjoint. As before, $T$ denotes an endofunctor of $\mathscr{X}$.

**The Full Equational Logic via Duality**

We observe that a terminal object in $\mathsf{ECoLog}_T$ exists by Theorem 5.1.38 and Corollary 5.2.2, and is completely determined by $PTSFn$ by Corollary 4.3.14:

**Corollary 5.2.12** (*cf.* [63, Theorem 5.3]). *Given an algebraic category $\mathscr{A}$, $\mathsf{ECoLog}_T$ has a terminal object consisting of*

$$L := \mathrm{Lan}_J PTSJ \quad and \quad \delta \colon LP \xrightarrow{\rho P} PTSP \xrightarrow{PT\epsilon} PT \tag{5.5}$$

*where $J$ is the full embedding of the subcategory $\mathscr{A}_\omega^f$ of $\mathscr{A}$ on finitely generated free objects, $\rho$ is the coreflection $\mathrm{Lan}_J PTSJ \dashrightarrow PTS$ and $\epsilon$ is the counit of the dual adjunction.*

We call the terminal object in $\mathsf{ECoLog}_T$ the **full equational ($T$-)logic**.

By Example 4.3.25, the logic of the cover modality is categorically a logic of predicate liftings subject to some axioms and it has nothing to do with the interpretation of modalities but the syntax alone. We also obtain a unique translation to the full equational logic immediately:

**Corollary 5.2.13** (*cf.* [74, Theorem 4.27]). *For the contravariant powerset functors $Q \colon \mathsf{Set} \to \mathsf{BA}$ and $2^- \colon \mathsf{Set} \to \mathsf{Set}$, the logics of the cover modality for any weak pullback preserving functors has a unique translation to the full equational logic.*

**Objects of Finitary Predicate Liftings**

In Lemma 2.3.9, the collection of $n$-ary predicate liftings for a Set functor $T$ is in bijection with the powerset of $T2^n$ using the Yoneda Lemma and the contravariant powerset functor $\mathcal{Q}\colon \mathsf{Set} \to \mathsf{BA}$ being representable. However, an adjunction $F \dashv U\colon \mathscr{A} \to \mathsf{Set}$ suffices to give a characterisation of the object of predicate liftings:

**Lemma 5.2.14.** *Let* $U\colon \mathscr{A} \to \mathsf{Set}$ *be a functor with a left adjoint* $F$. *The following statements hold:*

1. *(**Object of** $n$-**ary Predicate Liftings**). There is a one-to-one correspondence*

$$[-]\colon UPTSFn \overset{\cong}{\to} \mathsf{Nat}(UP^n, UPT)$$

   *for any set* $n$ *and any endofunctor* $T$ *of* $\mathscr{X}$.

2. *(**Substitution of Variables**). The correspondence is natural in* $T$ *and* $n$ *in the Kleisli category of the induced monad* $UF$ *by* $F \dashv U$. *That is, as for the naturality in* $n$, *for any morphism* $f\colon n \to UFm$ *the diagram*

$$
\begin{array}{ccc}
UPTSFn & \xrightarrow{\ [-]\ } & \mathsf{Nat}(UP^n, UPT) \\
{\scriptstyle UPTS\bar{f}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{f}^*} \\
UPTSFm & \xrightarrow[\ [-]\ ]{} & \mathsf{Nat}(UP^m, UPT)
\end{array}
$$

   *commutes where* $\bar{f}\colon Fn \to Fm$ *is the transpose of* $f$ *and* $\hat{f}^*$ *is the pre-composition with* $(UP^m \xrightarrow{\hat{f}} UP^n)$ *defined by*

$$(m \xrightarrow{a} UP) \mapsto (n \xrightarrow{f} UFm \xrightarrow{Ua} UFUP \xrightarrow{U\epsilon P} UP).$$

*Proof.* By the Yoneda Lemma, we have the correspondence

$$UPTX \cong \mathsf{Nat}(\mathscr{X}(-, X), UPT)$$

natural in $X$ and $T$. Restricting $X$ to $SFn$, it follows that $UPTSFn$ is in bijection with

$$
\begin{aligned}
&\mathsf{Nat}(\mathscr{A}(Fn, P-), UPT) &&\{\text{by the adjunction } S \dashv P\colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}\} \\
\cong\ &\mathsf{Nat}(UP^n, UPT). &&\{\text{by the adjunction } F \dashv U\colon \mathscr{A} \to \mathsf{Set}\}
\end{aligned}
$$

Every morphism from $Fn$ to $Fm$ is determined uniquely by a function $n$ to $UFm$, so the second statement follows. □

Note that the set $n$ does not need to be finite.

**Corollary 5.2.15**. *Assuming that $P$ has a dual adjoint, the full subcategory consisting of unimodal logics is small.*

*Remark* 5.2.16. For convenience, we introduce the bracket notation for the correspondence so that $[\lambda]$ denotes the induced predicate lifting of $\lambda$ with domain $\mathscr{A}(FX, Px)$ or $UP^X$. Given an element $\lambda \in UPTSFX$, the induced predicate lifting with domain $\mathscr{A}(FX, Px)$ is given explicitly by

$$[\lambda]_x(a) = UPTa^*(\lambda) \tag{5.6}$$

for any $(FX \xrightarrow{a} Px)$ where $a^* = Sa \circ \epsilon_x$ is the transpose of $a$ for the dual adjunction with counit $\epsilon_x \colon x \to SPx$.

**A Characterisation of the Full Equational Logic**

We show that the full equational logic can be understood as the logic of all finitary predicate liftings subject to a complete axiomatisation.

By Corollary 5.2.5, we know that the full equational logic $(\mathrm{Lan}_J PTSJ, PT\epsilon \circ \rho P)$ is a coequaliser of the logic of predicate liftings consisting of

$$F\left(\bigsqcup_{n\in\omega} UPTSFn \times U^n\right) \quad \text{and} \quad (PT\epsilon \circ \rho P) \circ e \tag{5.7}$$

as the syntax and the interpretation, respectively, where $e$ is the counit of $\mathrm{Lan}_J PTSJ$ given by the monadic adjunction in Theorem 4.3.29. We call it **the logic of all (finitary) predicate liftings**, denoted by $(L^\Lambda, \delta^\Lambda)$ if confusion is unlikely.

By Lemma 5.2.14, it is possible to form a coproduct of all unimodal logics, and it is characterised explicitly as follows:

**Lemma 5.2.17**. *The logic of all predicate liftings for $T$ is a coproduct of all unimodal $T$-logics.*

*Proof.* It is easy to see that the syntax of the logic of all predicate liftings is the coproduct of the syntaxes of all unimodal logics by reshuffling the order, i.e.

$$\coprod_{\substack{n \in \omega, \\ \lambda \in UPTSFn}} FU^n \cong F \coprod_{n, \lambda} U^n \qquad \{F \text{ preserves coproducts}\}$$

$$\cong F \coprod_n \coprod_\lambda U^n \qquad \{\text{the order of coproducts is irrelevant}\}$$

$$\cong F \coprod_n UPTSFn \cdot U^n \qquad \{\text{by the definition of copower}\}$$

$$\cong F \coprod_n UPTSFn \times U^n.$$

As for the interpretation, we compute the value for every $\lambda$ in $UPTSFn$ and $n$-tuple $(Fn \xrightarrow{a} Px)$ through the interpretation. Let $J$ be the full embedding $\mathscr{A}_\omega^f \hookrightarrow \mathscr{A}$. By Corollary 3.2.13, there is a left Kan extension $L = \mathrm{Lan}_J PTSJ$ with the identity $id\colon L \xrightarrow{\cdot} L$ as the unit such that $PTSJ = LJ$, that is, $PTSFn = LFn$ for every $n$.

By Theorem 3.2.11, we have

$$\mathscr{A}(LPx, PTx) \cong [(\mathscr{A}_\omega^f)^{\mathrm{op}}, \mathsf{Set}](\mathscr{A}(J-, Px), \mathscr{A}(PTSJ-, PTx)),$$

so the interpretation of the full equational logic $(PT\epsilon_x \circ \rho Px)$ gives the natural transformation $[-_2]_x'(-_1)\colon \mathscr{A}(J-, Px) \xrightarrow{\cdot} \mathscr{A}(PTSJ-, PTx)$ defined for each component $Fn$ by

$$\left(Fn \xrightarrow{a} Px\right) \mapsto PTSFn \xrightarrow{id} LFn = PTSFn \xrightarrow{La} LPx \xrightarrow{\rho Px} PTSPx \xrightarrow{PT\epsilon_x} PTx.$$

The morphism $\rho Px \circ La$ is equal to $PTSa$, since $\rho$ is the coreflection of $PTS$. For each $(Fn \xrightarrow{a} Px)$, the resulting map $U[-]_x'(a)\colon UPTSFn \to UPTx$ is precisely

$$U[-]_x'(a) = UPT\epsilon \circ UPTSa = UPT(Sa \circ \epsilon) = [-]_x(a)$$

where $[-]_x(a)$ is given in (5.6). Hence, it follows that every element

$$(\lambda, a) \quad \text{in} \quad UPTSFn \times \mathscr{A}(Fn, Px)$$

is interpreted as $[\lambda](a)$. Now, it is clear that the logic of all finitary predicate liftings is the coproduct of all unimodal logics. □

**Lemma 5.2.18.** *Every translation from the full equational $T$-logic is a split monomorphism. In particular, the full equational $T$-logic has a complete axiomatisation.*

*Proof.* Recall that every morphism from a terminal object $t$ is a split monomorphism, because the identity is the only endomorphism of it: $id \overset{\curvearrowright}{\phantom{.}} t \overset{f}{\underset{!x}{\rightleftarrows}} x$ . Moreover, an epimorphism $f$ which is also a split monomorphism must be invertible.[4] $\qquad\square$

**Theorem 5.2.19.** *An equational $T$-logic $(L, \delta)$ is full if and only if*

1. *every unimodal $T$-logic has a translation to $(L, \delta)$ and,*

2. *$(L, \delta)$ has a complete axiomatisation.*

Recall that $\mathsf{ECoLog}_T$ has $(\mathsf{RegEpi}, U^{-1}\mathsf{Mono})$-factorisation system by Theorem 5.2.9.

*Proof.* By finality, every unimodal logic has a translation to the full equational logic. By Lemma 5.2.18, it has a complete axiomatisation.

Conversely, if every unimodal logic has a translation to $(L, \delta)$, then there exists a translation $\tau$ from the logic of all predicate liftings $(L^\Lambda, \delta^\Lambda)$ to $(L, \delta)$. Consider the pushout $i_1$ of the regular epimorphism $e$ from $(L^\Lambda, \delta^\Lambda)$ to $(\mathrm{Lan}JPTSJ, PT\epsilon \circ \rho P)$ along the translation $\tau$ in the following diagram

$$
\begin{array}{ccc}
(L^\Lambda, \delta^\Lambda) & \longrightarrow\!\!\!\rightarrow & (\mathrm{Lan}_J PTSJ, PT\epsilon\rho P) \\
\Big\downarrow & \overset{!}{\nearrow} & \Big\downarrow {\scriptstyle i_2} \\
(L, \delta) & \underset{i_1}{\longrightarrow\!\!\!\rightarrow} & (L', \delta').
\end{array}
$$

The morphism $i_1$ is a regular epimorphism, because every pushout preserves $\mathcal{E}$-morphisms w.r.t. any $(\mathcal{E}, \mathcal{M})$-factorisation system. However, it implies that $i_1$ is an isomorphism by assumption. Hence, the composite $(! \circ i_1^{-1})$ is a split epimorphism, since $(! \circ i_1^{-1}) \circ i_2 = id$ by the finality of the full equational logic. Again, by the assumption that $(L, \delta)$ has no proper regular quotient, the unique map is an isomorphism, so the statement follows. $\qquad\square$

By the characterisation, a full equational logic $(L, \delta)$ may be presented efficiently, if we can show that every finitary predicate lifting can be translated to it and it is subject to a complete axiomatisation:

**Example 5.2.20.** The equational logic $(\mathbb{M}, \delta)$ for the powerset functor given in Proposition 2.2.13 along with the standard interpretation of $\Diamond$ is a full equational logic.

---

[4]Let $g$ be the right inverse of $f$. We have $gf = id$ by assumption and also $fgf = f$ by composing $f$ on both sides. It follows that $fg = id$ by $f$ being epic.

It is known that every predicate lifting for the powerset functor can be translated to a rank-1 sentence using $\Diamond$ and Boolean connectives only. It is also well-known that normal modal logic is complete with respect to the class of Kripke frames, so that every rank-1 equation valid under the usual Kripke semantics can be deduced from

$$\Diamond \bot = \bot \quad \text{and} \quad \Diamond(a \vee b) = \Diamond a \vee \Diamond b.$$

Thus, $(\mathbb{M}, \delta)$ has a complete axiomatisation by Remark 5.2.11.

**Example 5.2.21.** On the other hand, consider the non-empty powerset functor $\mathcal{P}_{\neq \emptyset}$ with the inclusion $j \colon \mathcal{P}_{\neq \emptyset} \hookrightarrow \mathcal{P}$ again, which is in Example 5.1.19. The equational logic $(\mathbb{M}, 2^j \circ \delta)$ does not have a complete axiomatisation with respect to the non-empty powerset functor $\mathcal{P}_{\neq \emptyset}$, since $(\overline{\mathbb{M}}, \overline{\delta})$ is a proper regular quotient of $(\mathbb{M}, 2^j \circ \delta)$. Similarly, $(\overline{\mathbb{M}}, \overline{\delta})$ has a complete axiomatisation, since it is complete with respect to the class of Kripke frames with a non-empty set of successors.

**Predicate Liftings, Concretely**

Suppose that the category $\mathcal{X}$ has a representable forgetful functor $U \colon \mathcal{X} \to \mathsf{Set}$, then we also have a characterisation of predicate liftings in the same form of Lemma 2.3.9:

**Proposition 5.2.22.** *Let $U \colon \mathcal{A} \to \mathsf{Set}$ be a functor with left adjoint $F$ and $(\mathcal{X}, U)$ a representable concrete category. Then,*

1. *every predicate lifting is precisely a natural transformation $\mathcal{X}(-, \Omega)^n \overset{\cdot}{\to} \mathcal{X}(T-, \Omega)$ up to isomorphism.*

2. *there is a bijection $|PTSFn| \cong \mathsf{Nat}(\mathcal{X}(-, \Omega)^n, \mathcal{X}(T-, \Omega))$ for each set $n \in \mathsf{Set}$;*

3. *if the category $\mathcal{X}$ has products, then for each set $n \in \mathsf{Set}$, the underlying set of $PTSFn$ is bijective with $\mathcal{X}(T\Omega^n, \Omega)$*

*where $\Omega$ is the dualising object of $S \dashv P \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{A}$.*

*Proof.* By Proposition 3.6.6, we have $UP^n \cong \mathcal{X}(-, \Omega)^n$ and $UPT \cong \mathcal{X}(T-, \Omega)$, and by Lemma 5.2.14 the second statement follows.

Suppose that $\mathcal{X}$ has finite products. It suffices to show that $SFn \cong \Omega^n$, so it follows that $|PTSFn| \cong |PT\Omega^n| \cong \mathcal{X}(T\Omega^n, \Omega)$ by the Yoneda Lemma. Consider the following

isomorphisms natural in $X$,

$$\mathscr{X}(X, SFn) \cong \mathscr{A}(Fn, PX) \qquad \{\text{by the dual adjunction}\}$$
$$\cong \mathscr{X}(X, \Omega)^n \qquad \{\text{by Proposition 3.6.6}\}$$
$$\cong \mathscr{X}(X, \Omega^n) \qquad \{\text{by the universal property of product}\}$$

so $SFn \cong \Omega^n$ by the Yoneda Lemma. $\qquad\qquad\square$

To end this section, we demonstrate the use of Lemma 5.2.14 with the contravariant powerset functor $\mathcal{Q}$, emphasising that the correspondence is not merely a *set-theoretical* translation but also *algebraic* translation:

**Example 5.2.23.** The dual operator $\square$ of possibility $\lozenge$ was shown to be a predicate lifting in Example 2.3.6, and this fact can be deduced from Lemma 5.2.14. Let $S: \mathsf{BA} \to \mathsf{Set}$ be the ultrafilter functor, i.e. a dual adjoint to $\mathcal{Q}$. Consider the function $f: 1 \to UF1$ defined by

$$f: * \mapsto \neg*$$

and its corresponding homomorphism $\bar{f}: F1 \to F1$ on the free Boolean algebra $F1$. The $\lozenge$ operator is mapped to a predicate lifting $(\lozenge \circ \neg)$. Then, we take the complement of $(\lozenge \circ \neg)$ in $\mathcal{Q}\mathcal{P}SF1$ and obtain the predicate lifting $\square$ given by

$$\{\{\bot, \top\}, \{\top\}\} \xmapsto{\ (-)^{\complement}\ } \{\emptyset, \{\bot\}\} \xmapsto{\ \mathcal{P}\bar{f}\ } \{\emptyset, \{\top\}\}.$$

Note that the substitution is given by pre-composing with $UPTS\bar{f}$, so the order of above applications is important.

## 5.3 Multi-Step Semantics

The monoidal structure $(\mathsf{CoLog}, \otimes, id)$, see Proposition 5.1.27, allows us to introduce monoids in $\mathsf{CoLog}$ along with the composition of logics, and we tentatively suggest it as a framework for multi-step coalgebraic modal logic. Since this monoidal structure is very similar to the monoidal category of endofunctors, we may expect that every monoid in $(\mathsf{CoLog}, \otimes, id)$ is monad-like.

**Definition 5.3.1.** Considering the monoidal category $\mathsf{CoLog} = (\mathsf{CoLog}, \otimes, id)$, we define the following:

1. Every monoid object in $(\mathsf{CoLog}, \otimes, id)$ is called a **multi-step semantics**.

2. The category of monoids in $(\mathsf{CoLog}, \otimes, id)$ is called the **category of multi-step semantics**, denoted $\infty\mathsf{CoLog}$.

**Proposition 5.3.2.** *The forgetful functor $U_L \colon (\mathsf{CoLog}, \otimes, id) \to (\mathsf{End}(\mathscr{A}), \circ, I)$ is a strict monoidal functor and similarly for $U_R$.*

*Proof.* By definition. $\qquad\square$

**Corollary 5.3.3.** *Every multi-step semantics consists of precisely a monad $\mathbb{L} = (L, \mu^L, \eta^L)$ on $\mathscr{A}$, a comonad $\mathbb{T} = (T, \mu^T, \eta^T)$ on $\mathscr{X}$ and an interpretation $\delta \colon LP \to TP$ preserving the multiplication and the unit*

$$
\begin{array}{ccc}
L^2 P \xrightarrow{\delta \otimes \delta} PT^2 & & P \xrightarrow{\ id\ } P \\
{\scriptstyle \mu^L P}\downarrow \qquad \downarrow {\scriptstyle P\mu^T} & and & {\scriptstyle \eta^L P}\downarrow \qquad \downarrow {\scriptstyle P\eta^T} \\
LP \xrightarrow{\ \delta\ } PT & & LP \xrightarrow{\ \delta\ } PT,
\end{array}
\tag{5.8}
$$

*i.e. $(\mu^L, \mu^T)$ and $(\eta^L, \eta^T)$ are morphisms in $\mathsf{CoLog}$.*

**Corollary 5.3.4.** *Every morphism $(\tau, \nu) \colon (\delta, \mu, \eta) \to (\delta', \mu', \eta')$ in $\infty\mathsf{CoLog}$ consists of precisely a monad morphism and a comonad morphism*

$$
\tau \colon \mathbb{L} \to \mathbb{L}' \quad and \quad \nu \colon \mathbb{T} \to \mathbb{T}'
$$

*respectively such that $(\tau, \nu)$ is also a morphism in $\mathsf{CoLog}$, where $\mathbb{L}$ and $\mathbb{T}$ are a monad and a comonad given by $(\delta, \mu, \eta)$ and similarly for $\mathbb{L}'$ and $\mathbb{T}'$.*

We call (5.8) the *homomorphism condition* for multi-step logics. Intuitively, the multi-step semantics is consistent with evaluation.

*Notation* 5.3.5. For every multi-step semantics $(\delta, \mu, \eta)$, the corresponding monad and comonad are denoted by $\mathbb{L} = (L, \mu^L, \eta^L)$ and $\mathbb{T} = (T, \mu^T, \eta^T)$ respectively. By the previous corollaries, a multi-step semantics is denoted by $(\mathbb{L}, \mathbb{T}, \delta)$.

Similarly, we say that a multi-step semantics $(\mathbb{L}, \mathbb{T}, \delta)$ is **equational** if the functor part $L$ is finitely based, and call it **finitary** if $L$ is finitary.

### 5.3.1 Interpreting Multi-Step Semantics

Similar to one-step semantics, every multi-step semantics $(\mathbb{L}, \mathbb{T}, \delta)$ introduces a contravariant functor from the category $\mathscr{X}_{\mathbb{T}}$ of coalgebras of the comonad $\mathbb{T}$ to the category $\mathscr{A}^{\mathbb{L}}$ of algebras of the monad $\mathbb{L}$:

**Proposition 5.3.6.** *Every multi-step semantics* $(\mathbb{L}, \mathbb{T}, \delta)$ *defines a lifting of* $P$ *along the forgetful functors:*

$$
\begin{array}{ccc}
\mathscr{X}_{\mathbb{T}} & \xrightarrow{P^{\delta}} & \mathscr{A}^{\mathbb{L}} \\
U^T \downarrow & & \downarrow U^L \\
\mathscr{X} & \xrightarrow{P} & \mathscr{A}
\end{array}
$$

*by* $P^{\delta} \colon (x, \xi) \mapsto P\xi \circ \delta_x$

*Proof.* Given a $\mathbb{T}$-coalgebra $(x, \xi)$, we verify that $P^{\delta}(x, \xi)$ is an $\mathbb{L}$-algebra by diagram chasing:

**the unit law**: The following diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\quad id \quad} & \\
Px & \xrightarrow{P\eta^T} PTx \xrightarrow{P\xi} & Px \\
 & \eta^L_{Px} \searrow \quad \uparrow \delta_x \quad \nearrow P^{\delta}(\xi) & \\
 & LPx & 
\end{array}
$$

commutes, because the right triangle commutes by definition, the upper rectangle commutes by the counit law, and the left triangle commutes by the preservation of unit.

**the associative law**: Consider the diagram

$$
\begin{array}{ccccc}
 & & L(P^{\delta}\xi) & & \\
L^2 Px & \xrightarrow{L\delta_x} & LPTx & \xrightarrow{LP\xi} & LPx \\
 & (\delta \otimes \delta)_x \searrow & \downarrow \delta_{Tx} & & \downarrow \delta_x \\
\mu^L \downarrow & & PT^2 x & \xrightarrow{PT\xi} & PTx & \quad P^{\delta}\xi \\
 & & P\mu^T_x \downarrow & & \downarrow P\xi \\
LPx & \xrightarrow{\delta_x} & PTx & \xrightarrow{P\xi} & Px \\
 & & P^{\delta}\xi & & 
\end{array}
$$

where the triangle inside commutes by the definition of $\otimes$. Now, notice that the remaining squares

$$
\begin{array}{ccc}
LPTx \xrightarrow{LP\xi} LPx & PT^2x \xrightarrow{PT\xi} PTx & L^2Px \xrightarrow{(\delta\otimes\delta)_x} PT^2x \\
\delta_{Tx}\downarrow \quad\quad \downarrow\delta_x & P\mu_x^T\downarrow \quad\quad \downarrow P\xi & \mu^L\downarrow \quad\quad \downarrow P\mu_x^T \\
PT^2x \xrightarrow{PT\xi} PTx & PTx \xrightarrow{P\xi} Px & LPx \xrightarrow{\delta_x} PTx
\end{array}
$$

and and

commute by naturality of $\delta$, the co-associative law of $\xi$, and the preservation of multiplication.

For every coalgebra morphism $f \colon (x,\xi) \to (y,\gamma)$, set $P^\delta(f) = Pf$ and it is an $\mathbb{L}$-algebra morphism using the same argument as in Proposition 5.1.9. $\qquad\square$

**Proposition 5.3.7.** *Every morphism $(\tau,\nu)$ of multi-step semantics from $(\mathbb{L},\mathbb{T},\delta)$ to $(\mathbb{M},\mathbb{V},\theta)$, defines functors*

$$
\nu_* \colon \mathscr{X}_{\mathbb{V}} \to \mathscr{X}_{\mathbb{T}} \quad\text{and}\quad \tau^* \colon \mathscr{A}^{\mathbb{M}} \to \mathscr{A}^{\mathbb{L}}
$$

*satisfying $P^\theta \circ \nu_* = \tau^* \circ P^\delta$ as in Proposition 5.1.10.*

## 5.3.2 Freely Generated Multi-Step Semantics

Given a one-step semantics $\delta \colon LP \dashrightarrow PT$ with a morphism $(\tau,\nu)$ from the identity logic to $\delta$, we may iterate the composition of $\delta$ with itself to obtain an interpretation up to $n$-steps as far as we want by applying $(\delta \otimes -)$ inductively:[5]

$$
\begin{array}{ccccccccc}
P & \longrightarrow & LP & \longrightarrow & L^2P & \longrightarrow & \cdots & \longrightarrow & L^{i+1}P & \longrightarrow & \cdots \\
id\downarrow & & \delta\downarrow & & \delta\otimes\delta\downarrow & & & & \downarrow\delta\otimes\delta^i & & \\
P & \longrightarrow & PT & \longrightarrow & PT^2 & \longrightarrow & \cdots & \longrightarrow & PT^{i+1} & \longrightarrow & \cdots
\end{array}
$$

where $\delta^0 = id, \delta^{i+1} = \delta \otimes \delta^i$ and $\delta^\kappa = \mathrm{Colim}_{i<\kappa}\,\delta^i$ for any limit ordinal $\kappa$.

The sequence stabilises whenever both of the upper sequence and the lower sequence stabilise. The fixed-points $L^\kappa, T^\lambda$ of the upper and lower sequences are the (co)free (co)monad of $L$ and $T$ respectively. The resulting natural transformation $\delta^\mu$, where $\mu = \max(\kappa,\lambda)$, from $L^\kappa P$ to $PT^\lambda$ is the interpretation containing every step of the interpretation. In the following, we extend the process to functors $L$ and $T$ such that forgetful functors $U^L$ and $U^T$ have a left and right adjoint, respectively, without induction.

---

[5] For 'mere' one-step semantics, apply the free algebra sequence of $(\delta \otimes -)$ instead.

**Lemma 5.3.8** (see [32, Lemma 4.5.1]). *Given a comonad $\mathbb{T} = (T, \mu^T, \eta^T)$ on $\mathscr{X}$, and a monad $\mathbb{L} = (L, \mu^L, \eta^L)$ on $\mathscr{A}$, liftings $\overline{P}$ of $P$ along the forgetful functors*

$$
\begin{array}{ccc}
\mathscr{X}^{\mathbb{T}} & \xrightarrow{\overline{P}} & \mathscr{A}^{\mathbb{L}} \\
U^T \left( \vdash \right) F^T & & U^L \left( \vdash \right) F^L \\
\mathscr{X} & \xrightarrow{\quad P \quad} & \mathscr{A}
\end{array}
\tag{5.9}
$$

*are bijective with natural transformation $\delta: LP \dot{\longrightarrow} PT$ satisfying* (5.8).

*Proof Sketch.* In (5.9), $F^L \dashv G^L$ denotes the adjunction induced[6] by the monad $\mathbb{L}$; and similarly for $F^T \dashv G^T$. Define a natural transformation $id^*: F^L P \dot{\longrightarrow} \overline{P} F^T$ by the pasting diagram:

$$
\begin{array}{ccccc}
\mathscr{X}^{\mathrm{op}} & \xrightarrow{F^T} & (\mathscr{X}^{\mathrm{op}})^{\mathbb{T}^{\mathrm{op}}} & \xrightarrow{\overline{P}} & \mathscr{A}^{\mathbb{L}} \\
& (\eta^T)^{\mathrm{op}} & \downarrow U^T & = & U^L \downarrow \quad \epsilon^L \quad \mathcal{I} \\
\mathcal{I} & & \mathscr{X}^{\mathrm{op}} & \xrightarrow{P} & \mathscr{A} \xrightarrow{F^L} \mathscr{A}^{\mathbb{L}}
\end{array}
$$

where $\epsilon^L: F^L U^L \dot{\longrightarrow} \mathcal{I}$ is the counit defined by $\epsilon^L(La \xrightarrow{\alpha} a) = \alpha: (La, \mu^L) \to (a, \alpha)$ for every $\mathbb{L}$-algebra $\alpha$. The desired natural transformation $\delta = \delta^{\overline{P}}: LP \dot{\longrightarrow} PT$ is then obtained by the pasting diagram

$$
\begin{array}{ccc}
\mathscr{X}^{\mathrm{op}} & \xrightarrow{\quad\quad P \quad\quad} & \mathscr{A} \\
T \downarrow \quad = \quad F^T \searrow \quad \Downarrow_{id^*} \quad F^L \nearrow \quad = \quad \downarrow L \\
\quad \quad (\mathscr{X}^{\mathrm{op}})^{\mathbb{T}^{\mathrm{op}}} \xrightarrow{\overline{P}} \mathscr{A}^{\mathbb{L}} \\
U^T \nearrow \quad = \quad U^L \searrow \\
\mathscr{X}^{\mathrm{op}} & \xrightarrow{\quad\quad P \quad\quad} & \mathscr{A}
\end{array}
$$

that is, $\delta = U^L id^* = U\left( \epsilon^L \overline{P} F^T \circ F^L P(\eta^T)^{\mathrm{op}} \right) = U^L \epsilon^L \overline{P} F^T \circ U^L F^L P(\eta^T)^{\mathrm{op}}$. The resulting natural transformation morphism on a component $x$ is the composite

$$
LPx \xrightarrow{LP\eta^T x} LPTx \xrightarrow{\overline{P}(\mu_x^T)^{\mathrm{op}}} PTx.
\tag{5.10}
$$

For verification, we refer to [32, Lemma 4.5.1]. $\qquad\qquad\square$

**Lemma 5.3.9.** *For every pair of a monad and comonad morphisms $\tau: \mathbb{L} \to \mathbb{M}$ and $\nu: \mathbb{V} \to \mathbb{T}$ and two multi-step semantics $(\mathbb{T}, \mathbb{L}, \delta)$ and $(\mathbb{V}, \mathbb{M}, \theta)$, the following diagram*

---

[6]See [81, Section VI.2]) for details.

*commutes*

$$
\begin{array}{ccc}
\mathscr{X}_{\mathbb{T}} & \xrightarrow{P^{\delta}} & \mathscr{A}^{\mathbb{L}} \\
{\scriptstyle \nu_{*}}\big\uparrow & & \big\uparrow{\scriptstyle \tau^{*}} \\
\mathscr{X}_{\mathbb{V}} & \xrightarrow[P^{\theta}]{} & \mathscr{A}^{\mathbb{M}}
\end{array}
$$

*if and only if* $(\tau, \nu)$ *is a morphism from* $(\mathbb{L}, \mathbb{T}, \delta)$ *to* $(\mathbb{M}, \mathbb{V}, \theta)$ *where* $\tau^{*}$ *and* $\nu_{*}$ *are defined in Proposition 5.3.7.*

*Proof.* By Proposition 5.3.7, it suffices to check necessary condition. However, this is trivial by Lemma 5.3.8 since $P^{\delta}\nu_{*}$ and $\tau^{*} \circ P^{\theta}$ given by $P\nu \circ \delta$ and $\theta \circ \tau P$ are equal. $\qquad\square$

**Theorem 5.3.10.** *Let* $T$ *and* $L$ *be endofunctors of* $\mathscr{X}$ *and* $\mathscr{A}$ *respectively. If the forgetful functors* $U^{T}\colon \mathscr{X}_{T} \to \mathscr{X}$ *and* $U^{L}\colon \mathscr{A}^{L} \to \mathscr{A}$ *have a right adjoint and a left adjoint, respectively, then every one-step semantics* $(L, T, \delta)$ *has a free multi-step semantics* $(\mathbb{L}, \mathbb{T}, \delta)$ *over it defined in* (5.11) *below.*

A *free multi-step semantics* $(\mathbb{L}, \mathbb{T}, \delta)$ on a one-step semantics $(L, T, \delta)$ is a free object along the forgetful functor from $\infty\mathsf{CoLog}$ to $\mathsf{CoLog}$.

*Proof.* By Proposition 4.3.3 and the assumption, $L$ has an algebraically-free monad $\mathbb{L}$ and by the same reasoning $T$ has a coalgebraically-cofree comonad $\mathbb{T}$. By coalgebraical cofreeness, there is a natural transformation $\nu\colon \mathbb{T} \to T$ such that the functor $\nu_{*}\colon \mathscr{X}^{\mathbb{T}} \to \mathscr{X}^{T}$ is an isomorphism and similarly there is a natural transformation $\tau\colon L \to \mathbb{L}$ such that $\tau^{*}$ is an isomorphism.

Also, $\delta$ defines a lifting $P^{\delta}$ of $P$ along forgetful functors. Therefore, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
& & & & \overline{P} & & & & \\
& & & \overbrace{\hspace{7cm}} & & & & & \\
\mathscr{X}_{\mathbb{T}} & \xrightarrow{\nu_{*}} & \mathscr{X}_{T} & \xrightarrow{P^{\delta}} & \mathscr{A}^{L} & \xrightarrow{\tau^{*}} & \mathscr{A}^{\mathbb{L}} & & \\
{\scriptstyle G}\big(\vdash\big){\scriptstyle U^{T}} & & & & & & {\scriptstyle U^{L}}\big(\vdash\big){\scriptstyle F} & & \\
\mathscr{X} & = & \mathscr{X} & \xrightarrow[P]{} & \mathscr{A} & = & \mathscr{A}. & &
\end{array}
$$

It is clear that $\overline{P} = (\tau^{*})^{-1} P^{\delta} \nu_{*}$ is a lifting of $P$. Such a lifting $\overline{P}$ corresponds uniquely to a family of morphism $\delta_{x}$ defined by

$$
\mathbb{L}Px \xrightarrow{\ \mathbb{L}P\eta_{x}^{T}\ } \mathbb{L}P\mathbb{T}x \xrightarrow{\ \overline{P}\mu_{x}^{T}\ } P\mathbb{T}x \tag{5.11}
$$

natural in $x$ by (5.10) and $\overline{P}\mu_x^T$ is equal to the unique $\mathbb{L}$-algebra $(\mathbb{L}P\mathbb{T}x \xrightarrow{\alpha_x} P\mathbb{T}x)$ making the following diagram commutes

$$
\begin{array}{ccc}
LP\mathbb{T}x & \xrightarrow{\delta\mathbb{T}x} PT\mathbb{T}x & \xrightarrow{Pv\mathbb{T}x} P\mathbb{T}\mathbb{T}x \\
{\scriptstyle \tau P\mathbb{T}x}\downarrow & & \downarrow{\scriptstyle P\mu_x} \\
\mathbb{L}P\mathbb{T}x & \dashrightarrow{\quad\alpha_x\quad} & P\mathbb{T}x.
\end{array}
$$

We show that $(\tau, v)$ is an *injection* with the universal property:

$(\tau, v)$ **forms a morphism**: To show that $\delta$ is injected into $\delta$ via $(\tau, v)$, observe that the outer square of the diagram

$$
\begin{array}{ccc}
LP & \xrightarrow{\quad\delta\quad} & PT \\
{\scriptstyle id}\downarrow & & \\
LP & \xrightarrow{LP\eta^T} LP\mathbb{T} & \downarrow{\scriptstyle Pv} \\
{\scriptstyle \tau P}\downarrow & \downarrow{\scriptstyle \tau P\mathbb{T}} & \\
\mathbb{L}P & \xrightarrow{\mathbb{L}P\eta^T} \mathbb{L}P\mathbb{T} \xrightarrow{\alpha} & P\mathbb{T}
\end{array}
$$

commutes. It remains to show that the intermediate diagram commutes, since the innermost square commutes by the naturality of $\tau$. By the very definition of $\alpha$, the intermediate diagram is reduced to

$$
\begin{array}{ccc}
LP & \xrightarrow{\delta} & PT \\
{\scriptstyle LP\eta^T}\downarrow & & \downarrow{\scriptstyle PT\eta^T} \\
LP\mathbb{T} & \xrightarrow{\delta\mathbb{T}} & PT\mathbb{T} \\
 & & \downarrow{\scriptstyle Pv\mathbb{T}} \quad {\scriptstyle Pv} \\
{\scriptstyle \tau P\mathbb{T}}\downarrow & & P\mathbb{T}^2 \\
 & & \downarrow{\scriptstyle P\mu^T} \\
\mathbb{L}P\mathbb{T} & \xrightarrow{\alpha} & P\mathbb{T}
\end{array}
\tag{5.12}
$$

where the upper square and the lower square commute by naturality and definition respectively.

The right rectangle remains. Consider the following diagram

$$
\begin{array}{ccc}
\mathbb{T} & \xrightarrow{\mu^T} \mathbb{T}^2 & \xrightarrow{v\mathbb{T}} T\mathbb{T} \\
 & {\scriptstyle id}\searrow \quad \downarrow{\scriptstyle \mathbb{T}\eta^T} & \downarrow{\scriptstyle T\eta^T} \\
 & \mathbb{T} \xrightarrow{\quad v\quad} & T
\end{array}
$$

where the left triangle commutes by the co-associative law of $\mathbb{T}$; and the right square commutes by the naturality of $\nu$. The application of the contravariant functor $P$ shows that the right rectangle of (5.12) commutes.

**Factorisation through** $(\tau, \nu)$: Suppose that there is a one-step semantics morphism $(\sigma, \lambda) \colon (L, T, \delta) \to (\mathbb{M}, \mathbb{V}, \boldsymbol{\theta})$. By (co)freeness of $\mathbb{L}$ (and $\mathbb{T}$), there exists unique monad and comonad morphisms

$$\overline{\sigma} \colon \mathbb{L} \to \mathbb{M} \quad \text{and} \quad \overline{\lambda} \colon \mathbb{V} \to \mathbb{T}$$

satisfying $\sigma = \overline{\sigma} \circ \tau$ and $\lambda = \nu \circ \overline{\lambda}$ respectively. Displaying the situation by the following diagram

$$
\begin{array}{ccc}
\mathscr{X}_{\mathbb{V}} & \xrightarrow{P^{\theta}} & \mathscr{A}^{\mathbb{M}} \\
\overline{\lambda}_{*} \downarrow & & \downarrow \overline{\sigma}^{*} \\
\mathscr{X}_{\mathbb{T}} & \xrightarrow{\overline{P} = P^{\delta}} & \mathscr{A}^{\mathbb{L}} \\
\nu_{*} \downarrow & & \downarrow \tau^{*} \\
\mathscr{X}_{T} & \xrightarrow{P^{\delta}} & \mathscr{A}^{L}
\end{array}
$$

with outer arrows $\lambda_{*}$ on the left and $\sigma^{*}$ on the right

we have $\lambda_{*} = \nu_{*} \circ \overline{\lambda}_{*}$ and $\sigma^{*} = \tau^{*} \circ \overline{\sigma}^{*}$ by construction; $\sigma^{*} \circ P^{\theta} = P^{\delta} \circ \lambda_{*}$ by assumption and Lemma 5.3.9; $\tau^{*} \circ \overline{P} = P^{\delta} \circ \nu_{*}$ by construction. Since $\nu_{*}$ and $\tau^{*}$ are isomorphisms, it follows that $\overline{P} \circ \overline{\lambda}_{*} = \overline{\sigma}^{*} \circ P^{\theta}$ by diagram chasing. By Lemma 5.3.9, $(\overline{\tau}, \overline{\nu})$ is a morphism from $(\mathbb{L}, \mathbb{T}, \delta)$ to $(\mathbb{M}, \mathbb{V}, \boldsymbol{\theta})$.

$\square$

By construction, we can invoke Proposition 4.3.22:

**Corollary 5.3.11.** *The free multi-step semantics of a one-step equational semantics is equational.*

## 5.4 Logical Properties

In this section, we demonstrate the use of one-step semantics as the building block of modalities for coalgebraic modal logic.

Throughout this section, we assume that $L$ is an endofunctor of $\mathscr{A}$ and $T$ an endofunctor of $\mathscr{X}$. As before, $P \colon \mathscr{X} \to \mathscr{A}$ is a contravariant functor.

## 5.4.1 Language derived from a Syntax

Let $\Psi$ be a set of atomic propositions. Every Kripke model of $\Psi$ consists of a Kripke frame $\langle X, \xi\colon X \to \mathcal{P}X \rangle$ and a valuation $v\colon \Psi \to 2^X$ assigning to every element in $\Psi$ a subset of $X$. This gives the following fragment of the satisfaction relation

$$x \models \psi \quad \text{if and only if} \quad x \in v(\psi)$$

for every $\psi \in \Psi$. Equivalently, valuations $\Psi \to 2^X$ are in bijection with Boolean algebra homomorphisms $F\Psi \to \mathcal{Q}X$ where $\mathcal{Q}X$ is the algebra of powerset with $U\mathcal{Q}X = 2^X$.

The satisfaction relation $\models$ is completely determined by the valuation $v$. This is because the language of modal logic with respect to $\Psi$ is, in fact, a free modal algebra over the free Boolean algebra $F\Psi$.

In general, we can replace $F\Psi$ with any Boolean algebra $A$. By Example 4.3.34, we know that the modal construction $\mathbb{M}\colon \mathsf{BA} \to \mathsf{BA}$ is finitary, so there is a left adjoint to the forgetful functor $\mathsf{BA}^{\mathbb{M}} \to \mathsf{BA}$. Then, a free modal algebra over $A$ exists and it serves as a language over $A$. Then, we define:

**Definition 5.4.1.** A **language** of a syntax $L$ over an $\mathscr{A}$-object $\Psi$ (of atomic propositions) is a free $L$-algebra over $\Psi$, i.e. an $L$-algebra $\langle \Phi, \iota\colon L\Phi \to \Phi \rangle$ with an isomorphism

$$\mathscr{A}(\Psi, U\langle a, \alpha \rangle) \cong \mathscr{A}^L(\langle \Phi, \iota \rangle, \langle a, \alpha \rangle)$$

natural in $\langle a, \alpha \rangle$ where $U\colon \mathscr{A}^L \to \mathscr{A}$ is the forgetful functor.

We define $(L, \Psi)$-**model** as an object of $(\Psi \downarrow U)$ and we see that a language of $L$ over $\Psi$ may also be seen as an initial $(L, \Psi)$-model. If confusion is unlikely, we omit the object $\Psi$.

Every $L$-algebra $\langle a, \alpha \rangle$ with a morphism from $\Psi$ to $a$ is an object of the comma category $(\Psi \downarrow U)$.

**Lemma 5.4.2.** *Suppose that $\mathscr{A}$ has finite coproducts. The comma category $(\Psi \downarrow U)$ is isomorphic to the category of $(L + \Psi)$-algebras.*

*Proof.* Every $(L + \Psi)$-algebra is a morphism from the coproduct $(La + \Psi)$ to $a$, so it precisely consists of an $L$-algebra $(La \xrightarrow{\alpha} a)$ with a morphism $(\Psi \xrightarrow{v} a)$, i.e. an

object $(\alpha, v)$ in $(\Psi \downarrow U)$. Every $(L + \Psi)$-algebra homomorphism $f \colon \left((L+\Psi)a \xrightarrow{\alpha} a\right) \to \left((L+\Psi)b \xrightarrow{\beta} b\right)$ satisfies the commutative diagram

$$
\begin{array}{ccc}
La + \Psi & \xrightarrow{\;[\alpha, v_a]\;} & a \\
{\scriptstyle Lf + id_\Psi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Lb + \Psi & \xrightarrow[\;[\beta, v_b]\;]{} & b
\end{array}
$$

so it is precisely an $L$-algebra homomorphism with $f \circ v_a = v_b$. $f$ is a morphism in $(\Psi \downarrow U)$. $\qquad\square$

**Corollary 5.4.3.** *Suppose that $\mathscr{A}$ has finite coproducts. An $L$-algebra $\langle \Phi, \iota \rangle$ with a morphism $v \colon \Psi \to \Phi$ is a language if and only if the $(L + \Psi)$-algebra*

$$
L\Phi + \Psi \xrightarrow{\;[\iota, v]\;} \Phi
$$

*is initial.*

*Notation* 5.4.4. We denote the unique morphism from the initial $(L+\Psi)$-algebra to some $(L+\Psi)$-algebra $\langle a, \alpha \rangle$ as well as from the initial object in the comma category $(\Psi \downarrow U)$ by $[\![-]\!]_\alpha^{L,\Psi}$ or just $[\![-]\!]_\alpha$ and call it the **interpretation of a language**.

**Language of Equational Coalgebraic Logic**

In modal logic, a language is over a *set* $\Psi$ of atomic propositions instead of an arbitrary object. Thus, we say that an $L$-algebra $\langle \Phi, \iota \rangle$ is a language of $L$ over a *set* $\Psi$ if $\langle \Phi, \iota \rangle$ is a language over the *free object* $F\Psi$ of $\Psi$. The usual inductive definition corresponds to the following:

**Proposition 5.4.5.** *Given a monadic and finitary functor $U \colon \mathscr{A} \to \mathsf{Set}$ with a left adjoint $F$ and a finitely based functor $L \colon \mathscr{A} \to \mathscr{A}$, the language of $L$ over any set $\Psi$ exists and can be constructed via the free-algebra sequence. Moreover, the category of $(L + F\Psi)$-algebras is monadic over $\mathsf{Set}$.*

*Proof.* By Corollary 5.4.3, Corollary 4.2.7, and Theorem 4.3.35, it suffices to show that $(L + F\Psi)$ is finitely based, but this is easy: The coproduct commutes with any colimit, the functor $L$ and the constant functor of $F\Psi$ preserve sifted colimits, and so does $(L + F\Psi)$. $\qquad\square$

**Example 5.4.6.** The language of normal modal logic with a set $\Psi$ of atomic propositions subject to normality is generated by the initial sequence $(\mathbb{M} + F\Psi)$ where $\mathbb{M}$ is the functor introduced in Definition 2.2.8.

## 5.4.2 Adequacy

In this section, we assume that $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ is a dual adjunction, and $(L, \delta)$ is a $T$-logic.

Recall that a Kripke model consists of a Kripke frame and a valuation, so we define a $(T, \Psi)$-**model** as a $T$-coalgebra $\langle x, \xi \rangle$ with a morphism $(\Psi \xrightarrow{v} Px)$ from some object $\Psi$ in $\mathscr{A}$. In the absence of $\Psi$, $(T, \Psi)$-model simply refers to a $T$-coalgebra.

Every $T$-coalgebra can be turned into an $L$-algebra by a $T$-logic $(L, \delta)$, so every $(T, \Psi)$-model gives rise to a $(L, \Psi)$-model by the lifting $P^\delta \colon \mathscr{X}_T \to \mathscr{A}^L$.

**Definition 5.4.7.** Let $\langle \Phi, \iota \rangle$ be a language of $(L, \delta)$ over an object $\Psi$. The **theory map** $th_\xi$ of a $T$-model $\langle x, \xi, v \rangle$ is the transpose of its language interpretation $[\![-]\!] \colon \Phi \to Px$. The **logical equivalence** $R_\xi$ **with respect to** $\langle \Phi, \iota \rangle$ of $\langle x, \xi, v \rangle$ is the kernel in $\mathscr{X}$ of the theory map:

$$
\begin{array}{ccc}
R_\xi & \xrightarrow{\ \pi_1\ } & x \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle th_\xi} \\
x & \xrightarrow[\ th_\xi\ ]{} & S\Phi.
\end{array}
$$

**Example 5.4.8.** The satisfaction relation given in (2.32) is captured by the theory map, i.e.

$$th(x) = \{\, \varphi \in \Phi \mid x \models \varphi \,\}.$$

Two elements $x, y$ are logically equivalent if and only if they satisfy the same set of formulae, that is, $th(x) = th(y)$.

**Lemma 5.4.9.** *For any coalgebra homomorphism $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$, the theory map $th_\xi$ of $\langle x, \xi \rangle$ is a composite*

$$th_\xi = th_\gamma \circ f.$$

*Proof.* Let $\langle \Phi, \iota \rangle$ be an initial $L$-algebra. By initiality, the following diagram

$$\langle \Phi, \iota \rangle \xrightarrow{\;\llbracket - \rrbracket_{(P^\delta \xi)}\;} P^\delta \langle y, \gamma \rangle$$

with $\llbracket - \rrbracket_{(P^\delta \gamma)}$ diagonal down to $P^\delta \langle x, \xi \rangle$ and $Pf$ down on the right.

commutes. By the dual adjunction, we have

$$x \xrightarrow{\eta_x} SPx \xrightarrow{S\llbracket - \rrbracket} S\Phi$$

with $f$ down left, $SPf$ in the middle, $S\llbracket - \rrbracket$ diagonal, and $y \xrightarrow{\eta_y} SPy$ on the bottom.

where $\eta \colon \mathcal{I} \longrightarrow SP$ is the unit, so $th_\xi = th_\gamma \circ f$. $\square$

In the classical modal logic, the *adequacy property* means that every pair of bisimilar elements are logically equivalent. From this it follows that logical equivalence is invariant under $p$-morphisms (equivalently, $\mathcal{P}$-coalgebra homomorphisms). In the abstract framework, these two properties are translated to suitable commutative diagrams as follows:

**Lemma 5.4.10.** *For any coalgebra homomorphism $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$, the logical equivalence $R_\xi$ with respect to $\langle x, \xi \rangle$ is invariant under $f$ and the kernel of $f$ is contained in $R_\xi$.*

*Proof.* Given a coalgebra homomorphism $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$, the theory map of $\xi$ is a composite $th_\xi = th_\gamma \circ f$ by Lemma 5.4.9. Let $R_\xi$ and $R_\gamma$ be kernels of the theory maps $th_\xi$ and $th_\gamma$ respectively in the following diagram

By construction, we have $th_\gamma \circ (f \circ \pi_1) = th_\gamma \circ (f \circ \pi_2)$, so there is a unique morphism from the logical equivalence $R_\xi$ to $R_\gamma$.

Further, let $B$ be the kernel of $f$ in the following diagram:

$$
\tag{5.13}
$$

By diagram chasing, it is easy to see that there exists a mediating morphism from $B$ to $R_\xi$. □

**Proposition 5.4.11** (The Adequacy Property). *Let $\langle x, \xi, v_x \rangle$ and $\langle y, \gamma, v_y \rangle$ be $(T, \Psi)$-models, and $f \colon \langle x, \xi \rangle \to \langle y, \gamma \rangle$ a coalgebra homomorphism. The logical equivalence $R_\xi$ with respect to $\langle x, \xi, v_x \rangle$ is invariant under $f$ and the kernel of $f$ is contained in $R_\xi$, provided that $v_x = Pf \circ v_y$.*

*Proof.* For a coalgebra homomorphism $f$ with $v_x = Pf \circ v_y$, $Pf$ is a morphism in the comma category $(\Psi \downarrow U)$, so it follows that $[\![-]\!]_{P^\delta \xi} = Pf \circ [\![-]\!]_{P^\delta \gamma}$. Using the same argument as in Lemma 5.4.10, it follows. □

*Remark* 5.4.12. In classical modal logic, the requirement $v_x = Pf \circ v_y$ says that the worlds $x$ and $f(x)$ satisfy the same set of atomic propositions. The unique morphism from $R_\xi$ to $R_\gamma$ is given by mapping $(x, y)$ to $(fx, fy)$ in Set using the canonical pullback. Hence, the theorem shows that the pair $(fx, fy)$ are modally equivalent if $(x, y)$ are. Also, behaviourally equivalent elements are modally equivalent.

*Remark* 5.4.13. Since the difference between $(T, \Psi)$-models and $T$-coalgebras are minor, we focus on $T$-coalgebras in the subsequent discussion.

## 5.4.3   Expressiveness

As in the previous subsection, we assume that $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ is a dual adjunction and $(L, , T, \delta)$ is a one-step semantics.

The Hennessy-Milner property, also known as *expressiveness*, states that every two logically equivalent elements are also bisimilar. We give an abstract version as follows:

**Definition 5.4.14.** A one-step semantics $(L, T, \delta)$ is called **expressive** if for any $T$-coalgebra $\langle x, \xi \rangle$, the logical equivalence $R_\xi$ of $\langle x, \xi \rangle$ is contained in the kernel $B_f$ of some coalgebra homomorphism $f : \langle x, \xi \rangle \to \langle y, \gamma \rangle$, i.e. there is a morphism $h$ from $R_\xi$ to $B_f$ such that $\pi_i = \pi_i' \circ h$ for $i = 1, 2$:

$$\tag{5.14}$$

Note that we do not assume any $(\mathcal{E}, \mathcal{M})$-factorisation system or minimisation. However, in the presence of minimisation, this reduces to the usual definition:

**Proposition 5.4.15.** *Given the conditions as in Theorem 4.1.19, a one-step semantics $(L, T, \delta)$ is expressive if and only if for any $T$-coalgebra $\langle x, \xi \rangle$ the logical equivalence $R_\xi$ is contained in the behavioural equivalence.*

*Proof.* If $(L, T, \delta)$ is expressive, then by the Coinduction Principle (Theorem 4.1.19) the statement follows immediately. The converse is trivial by definition. $\square$

**Translations Preserve Expressiveness**

Given a translation $\tau$ from an expressive $T$-logic $(L^e, \delta^e)$ to a $T$-logic $(L, \delta)$, by the very definition of it, the theory map w.r.t $(L^e, \delta^e)$ factors through the theory map w.r.t. $(L, \delta)$. To see this, let $\langle x, \xi \rangle$ be a $T$-coalgebra and consider the following diagram

where $\Phi$ and $\Phi^e$ are languages for $L$ and $L^e$, respectively, $[\![-]\!]$ denotes the unique morphism from the initial $L$-algebra over $\Phi$, and similarly for $[\![-]\!]^e$. By the definition

of translation, the lower left square commutes and $\delta_x^e = \delta_x \circ \tau_{Px}$. The remaining diagrams commute by initiality. Therefore, it follows that the theory map w.r.t. $(L^e, \delta^e)$ of $\langle x, \xi \rangle$ factors as

$$th_\xi^e = S[\![-]\!]_\Phi^e \circ th_\xi \tag{5.15}$$

where $[\![-]\!]_\Phi^e$ is independent of $\langle x, \xi \rangle$.

Informally, if any two elements $x$ and $y$ are logically equivalent w.r.t. $(L, \delta)$, then they must be logically equivalent w.r.t. $(L^e, \delta^e)$. By the expressiveness of $(L^e, \delta^e)$, it follows that $(L, \delta)$ is also expressive. It also holds for any one-step semantics:

**Theorem 5.4.16.** *Let $\tau \colon (L^e, \delta^e) \to (L, \delta)$ be a translation of $T$-logics. Then, $(L, \delta)$ is expressive if $(L^e, \delta^e)$ is expressive.*

*Proof.* It suffices to show that for any coalgebra $\langle x, \xi \rangle$, the logical equivalence $R_\xi$ of $\langle x, \xi \rangle$ w.r.t. $(L, \delta)$ is always contained in the logical equivalence $S_\xi$ w.r.t. $(L^e, \delta^e)$. By (5.15) and chasing the following diagram



there is a mediating map from $R_\xi$ to $S_\xi$ satisfying $\pi_i = \pi_i' \circ h$ for $i = 1, 2$, so the statement follows. $\qquad\square$

Recall that the full (resp. equational) $T$-logic is a terminal object in $\mathsf{CoLog}_T$ (resp. $\mathsf{ECoLog}_T$).

**Corollary 5.4.17.** *There exists an expressive (resp. equational) $T$-logic if and only if the full (resp. equational) logic is expressive.*

## One-Step Expressiveness

As argued by Klin informally [63, Definition 4.1], expressiveness holds if the theory map $th_\xi$ of $\langle x, \xi \rangle$ factors through a monomorphism in $\mathscr{X}$ after a coalgebra homomorphism $\langle x, \xi \rangle \xrightarrow{f} \langle x', \xi' \rangle$. We justify his definition formally in our framework and further show that the monomorphism is indeed the theory map of $\langle x', \xi' \rangle$.

**Lemma 5.4.18** (see [53, Theorem 4], also *cf.* [63, Theorem 4.2]). *Let* $\langle \Phi, \iota \rangle$ *be a language of* $L$ *and* $(\mathcal{E}, \mathcal{M})$ *a factorisation system on* $\mathcal{X}$. *If the mate* $\delta_\Phi^*\colon TS\Phi \to SL\Phi$ *(Definition 5.1.30) is an* $\mathcal{M}$-*morphism and* $T$ *preserves* $\mathcal{M}$-*morphisms, then the theory map* $th_\xi$ *of a coalgebra* $\langle x, \xi \rangle$ *factors through an coalgebra epimorphism* $e\colon \langle x, \xi \rangle \to \langle x', \xi' \rangle$:

$$
\begin{array}{ccccc}
 & & \overset{th_\xi}{\xrightarrow{\hspace{3cm}}} & & \\
x & \overset{e}{\twoheadrightarrow} & x' & \overset{m}{\rightarrowtail} & S\Phi \\
\Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \xi'} & & \Big\downarrow{\scriptstyle S\iota} \\
 & & & & SL\Phi \\
 & & & & \Big\uparrow{\scriptstyle \delta_\Phi^*} \\
Tx & \underset{Te}{\rightarrow} & Tx' & \underset{Tm}{\rightarrowtail} & TS\Phi
\end{array}
\tag{5.16}
$$

*Proof.* The outer diagram of (5.16) commutes by computing the transpose of the following diagram

$$
\begin{array}{ccc}
\Phi & \overset{\cong}{\longleftarrow} & L\Phi \\
\Big\downarrow{\scriptstyle [\![-]\!]} & & \Big\downarrow{\scriptstyle L[\![-]\!]} \\
Px & \underset{P\xi}{\longleftarrow} PTx & \underset{\delta_x}{\longleftarrow} LPx
\end{array}
$$

and a detailed argument can be found in [63, Theorem 4.2]. Then, by the $(\mathcal{E}, \mathcal{M})$-factorisation and the $\mathcal{M}$-preserving functor $T$, the theory map $th_\xi$ factors and $Tm$ is also an $\mathcal{M}$-morphism. Since $\mathcal{M}$ is closed under composition, $\delta_\Phi^* \circ Tm$ is an $\mathcal{M}$-morphism. By the diagonal fill-in property, there exists a unique morphism from $x'$ to $Tx'$ such that the left and right squares in (5.16) commute. $\qquad\square$

**Theorem 5.4.19.** *Suppose that a)* $\mathcal{X}$ *has a proper* $(\mathcal{E}, \mathcal{M})$-*factorisation system; b)* $T$ *preserves* $\mathcal{M}$-*morphisms; c) the mate of* $\delta\colon LP \dashrightarrow PT$ *is pointwise in* $\mathcal{M}$. *Then, for any coalgebra* $\langle x, \xi \rangle$, *the following statements hold:*

1. *The theory map* $th_\xi$ *is a composite of a coalgebra epimorphism* $e\colon \langle x, \xi \rangle \to \langle x', \xi' \rangle$ *and the theory map of* $\langle x', \xi' \rangle$.

2. *The pullback of* $e$ *is contained in the logical equivalence of* $\langle x, \xi \rangle$, *i.e.* $(L, \delta)$ *is expressive.*

*Proof.*    1. By Lemma 5.4.18, the theory map of $\langle x, \xi \rangle$ factors through a coalgebra $\mathcal{E}$-morphism $e\colon \langle x, \xi \rangle \twoheadrightarrow \langle x', \xi' \rangle$ and an $\mathcal{M}$-morphism $m\colon x' \rightarrowtail S\Phi$. It follows

that $th_\xi = th_\gamma \circ e$ by Lemma 5.4.9. Moreover, since $th_\xi = m \circ e$ and by assumption $e$ is epic, it follows that $th_\gamma = m$.

2. Let $R_\xi$ be the logical equivalence of $\langle x, \xi \rangle$ and $B$ the kernel of $e$. Consider the following commutative diagram:



By diagram chasing and the monomorphism $th_{\xi'}$, it is clear that there exists a mediating morphism from $R_\xi$ to the pullback $B$.

$\square$

**Definition 5.4.20.** A one-step semantics is called **one-step expressive** if it satisfies the conditions in Theorem 5.4.19.

As we have seen on page 143, a coequaliser of logics of predicate liftings does not add any new modalities but equations, so it should not add any expressive power.

**Proposition 5.4.21.** *Suppose that $\mathscr{X}$ has a proper $(\mathcal{E}, \mathcal{M})$-factorisation system and $T$ preserves $\mathcal{M}$-morphisms. Then, for any pointwise coequaliser $\tau \colon (L_1, \delta_1) \to (L_2, \delta_2)$ in $\mathsf{CoLog}_T$, $(L_1, \delta_1)$ and $(L_2, \delta_2)$ are both one-step expressive if the mate of $\delta_2$ is a pointwise $\mathcal{M}$-morphism.*

*Proof.* By the isomorphism $\mathsf{CoLog}^* \cong \mathsf{CoLog}^{\mathrm{op}}$, we have $\delta_1^* = S\tau \circ \delta_2^*$. By Theorem 5.4.19, it suffices to show that $S\tau \colon SL_2 \to SL_1$ is a pointwise $\mathcal{M}$-morphism, since $\mathcal{M}$ is closed under composition.

By the dual adjunction $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, $S$ maps colimits to limits, so the natural transformation $S\tau \colon SL_2 \to SL_1$ is a pointwise regular monomorphism. By Proposition 3.1.6 and the fact that every regular monomorphism is extremal, it follows that $S\tau$ is a pointwise $\mathcal{M}$-morphism. $\square$

As we have characterised, every equational $T$-logic is in fact a coequaliser of logics of predicate liftings shown in Section 5.2. From this, we can easily see the expressive limit of equational logics in terms of predicate liftings.

**Corollary 5.4.22.** *Suppose that $\mathscr{X}$ has a proper $(\mathcal{E}, \mathcal{M})$-factorisation system and $T$ preserves $\mathcal{M}$-morphisms. The expressiveness of the logic of all finitary predicate liftings is equivalent to the existence of an expressive equational $T$-logic.*

A similar analysis shows that subfunctors also inherits one-step expressiveness:

**Proposition 5.4.23.** *Suppose that $\mathscr{X}$ has a proper $(\mathcal{E}, \mathcal{M})$-factorisation system and $T$ preserves $\mathcal{M}$-morphisms. Then, for any one-step expressive semantics $(L, T, \delta)$ and a pointwise $\mathcal{M}$-natural transformation $\nu\colon T' \dot{\longrightarrow} T$, the one-step semantics*

$$LP \xrightarrow{\delta} PT \xrightarrow{P\nu} PT'$$

*is one-step expressive.*

*Proof.* First we show that $T'$ preserves $\mathcal{M}$-morphisms: Let $m\colon X \rightarrowtail Y$ be an $\mathcal{M}$-morphism and consider the diagram

$$
\begin{array}{ccc}
T'X & \xrightarrow{T'f} & T'Y \\
\scriptstyle{\nu_X}\big\downarrow & & \big\downarrow\scriptstyle{\nu_Y} \\
TX & \underset{Tf}{\rightarrowtail} & TY.
\end{array}
$$

Since $\mathcal{M}$ is closed under composition, the morphism $Tf \circ \nu_X$ is an $\mathcal{M}$-morphism. By commutativity and assumption, the morphism $\nu_Y \circ T'f$ and $\nu_Y$ are $\mathcal{M}$-morphisms, so $T'f$ is also an $\mathcal{M}$-morphism by the left cancellation law. It shows that $T'$ preserves $\mathcal{M}$-morphisms.

The mate of $P\nu \circ \delta$ is equal to $\nu S \circ \delta^*$ by the isomorphism $\mathsf{CoLog}^* \cong \mathsf{CoLog}^{\mathrm{op}}$. Thus, the composite is a pointwise $\mathcal{M}$-morphism since $\nu$ and $\delta^*$ are pointwise $\mathcal{M}$-morphisms by assumption. $\qquad\square$

**Expressiveness for Strongly Locally Presentable Categories**

For a finitary functor $T\colon \mathscr{X} \to \mathscr{X}$, we improve Klin's expressiveness condition [63, Theorem 4.4] to the full equational $T$-logic instead of the full finitary $T$-logic. For the difference between finitary $T$-logics and equational $T$-logics, see Section 4.3.2. First, we recall the condition given by Klin:

**Definition 5.4.24** (see [5]). A locally finitely presentable category is **strongly locally finitely presentable** if for any cofiltered limit $(x \xrightarrow{\sigma_i} x_i)$ and any monomorphism $f : y \to x$ with $y$ a finitely presentable object, $\sigma_i \circ f$ is monic for some $i$.

For example, Set, Pos and $\mathsf{Vec}_K$ are strongly locally finitely presentable. On the other hand, the category Ab and the category of Un of unary algebras, i.e. sets with a unary operation are not strongly locally finitely presentable. See [5] for details.

**Theorem 5.4.25** (See [63, Theorem 4.4]). *Assume that a)* $\mathscr{X}$ *is strongly locally finitely presentable; b)* $\mathscr{A}$ *is locally finitely presentable; and c)* $\epsilon : \mathcal{I} \dashrightarrow SP$ *is pointwise monic. If* $T$ *is finitary and preserves monomorphisms, then the mate* $\delta^*$ *of the* full finitary logic *is pointwise monic.*

In [63], the theorem is established in two steps. First, it is shown that if

$$\{TSf : TSA \to TSA_i\}_{f \in (\mathscr{A}_\omega \downarrow A)} \tag{5.17}$$

is jointly monic, where $\mathscr{A}_\omega$ is the category of finitely presentable objects, then the mate $\delta^*$ is pointwise monic. Second, the family (5.17) is shown to be jointly monic by the strong local presentability.

Similarly, we can prove an equational version by the observation that every finitely presentable object in a variety $\mathscr{A}$ is precisely a coequaliser of some $Fn$. Since the dual adjoint $S$ maps colimits to limits, the coequaliser becomes a monomorphism $SA \rightarrowtail SFn$.

**Lemma 5.4.26.** *Let* $A$ *be an object in a variety of algebras. Assume that* $T$ *preserves monomorphisms. The following collection of morphisms is jointly monic*

$$\{TSg : TSA \to TSFn\}_{g \in (\mathscr{A}_\omega^f \downarrow A)}$$

*if and only if the family* (5.17) *is jointly monic where* $\mathscr{A}_\omega^f$ *is the subcategory generated by* $Fn$, *for* $n \in \omega$.

Therefore we can show expressiveness for the full equational $T$-logic using the same argument as in [63, Theorem 4.4]:

**Theorem 5.4.27.** *Let* $\mathscr{X}$ *be a strongly locally presentable category,* $\mathscr{A}$ *a variety of algebras, and* $T : \mathscr{X} \to \mathscr{X}$ *a finitary and monomorphism-preserving functor. If the counit* $\epsilon : \mathcal{I} \dashrightarrow SP$ *is pointwise monic, then the full equational* $T$*-logic is one-step expressive.*

### 5.4.4 Modularity of Expressiveness

As shown by Cîrstea [36], the expressiveness property as well as completeness is stable under fusion, composition, product, and other constructions of coalgebraic logics of Set predicate liftings. In this subsection, we work out the cases of composition and colimits in CoLog, and leave remaining constructions for future work.

> We assume that $S \dashv P \colon \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ is a dual adjunction. Further, the category $\mathscr{X}$ has a proper $(\mathcal{E}, \mathcal{M})$-factorisation system, so a $T$-logic $(L, \delta)$ is one-step expressive if $T$ preserves $\mathcal{M}$-morphisms and the mate $\delta^*$ is an $\mathcal{M}$-morphism.

**Identity One-Step Semantics**

As we said in the very beginning, the identity one-step semantics $(\mathcal{I}, \mathcal{I}, id_P)$ is important, because it is exactly the placeholder in an informal expression such as $\mathcal{P} \times (-)$.

**Proposition 5.4.28.** *The identity one-step semantics $(\mathcal{I}, \mathcal{I}, id_P)$ is one-step expressive.*

*Proof.* The identity functor, of course, preserves $\mathcal{M}$-morphisms, and the class $\mathcal{M}$ contains isomorphisms, including the identity. $\qquad\square$

**Composition of One-Step Expressive Semantics**

Recall that by Proposition 5.1.33, we have the equality $(\delta_1 \otimes \delta_2)^* = \delta_1^* \oplus \delta_2^*$.

**Theorem 5.4.29.** *Given two one-step semantics $(L_i, T_i, \delta_i)$ for $i = 1, 2$, their composition*

$$\delta_1 \otimes \delta_2 \colon L_1 L_2 P \dashrightarrow P T_1 T_2$$

*is one-step expressive, if both $(L_1, T_1, \delta_1)$ and $(L_2, T_2, \delta_2)$ are one-step expressive.*

*Proof.* By (5.2), the mate of $\delta_1 \otimes \delta_2$ is equal to $\delta_1^* L_2 \circ T_1 \delta_2^*$. By assumption, $T_i$ preserves $\mathcal{M}$-morphisms and $\delta_i^*$ is a pointwise $\mathcal{M}$-morphism for $i = 1, 2$. Hence the composite $\delta_1^* L_2 \circ T_1 \delta_2^*$ is a pointwise $\mathcal{M}$-morphism and $T_1 T_2$ preserves $\mathcal{M}$-morphisms. $\qquad\square$

**Colimits of One-Step Expressive Semantics**

For colimits of one-step expressive semantics, we have to take care of pointwise colimits as before. Under the isomorphism between $\mathsf{CoLog}^{\mathrm{op}}$ and $\mathsf{CoLog}^*$, colimits are mapped to limits. Using the dual adjoint $S$, a pointwise limit of a diagram $D\colon \mathscr{I} \to \mathsf{CoLog}^*$ is indeed a family of limits

$$\left((\operatorname{Lim} T_i S a) \xrightarrow{(\operatorname{Lim} D)a} (\operatorname{Lim} S L_i a)\right)_{i \in \mathscr{I}} \tag{5.18}$$

in the arrow category $\mathscr{X}^{\to}$ for each $a \in \mathscr{A}$. Thus, we conclude the following:

**Theorem 5.4.30.** *Let $D\colon \mathscr{I} \to \mathsf{CoLog}$ be a diagram. If every $(L_i P \xrightarrow{D_i} P T_i)_{i \in \mathscr{I}}$ is one-step expressive and a pointwise colimit of $D$ exists, then the colimit of $D$ is one-step expressive.*

*Proof.* The colimit of $D$ is a one-step semantics of type $\operatorname{Lim}_i T_i$, so we first show that $\operatorname{Lim}_i T_i$ preserves $\mathcal{M}$-morphisms. By assumption, each $T_i$ preserves $\mathcal{M}$-morphisms and by Proposition 3.1.4, $\mathcal{M}$-morphisms are closed under limits in $\mathscr{X}^{\to}$, so $\operatorname{Lim} T_i$ preserves $\mathcal{M}$-morphisms.

Next we show that the mate of $(\operatorname{Colim} D)$ is a pointwise $\mathcal{M}$-morphism. Every $D_i^*$ is a pointwise $\mathcal{M}$-morphism by assumption, that is, $D_i^* a\colon T_i S a \to S L_i a$ is an $\mathcal{M}$-morphism for any $a \in \mathscr{A}$, so the limit of $D_i^* a$ in the arrow category $\mathscr{X}^{\to}$ is also an $\mathcal{M}$-morphism. It follows that $\operatorname{Lim}(D_i^*) \cong (\operatorname{Colim} D_i)^*$ is a pointwise $\mathcal{M}$-morphism since $\mathcal{M}$ contains isomorphisms. $\qquad\square$

Note that for Set functors, the hom-functor $\operatorname{Hom}(A, -)$ is naturally isomorphic to an $A$-fold product $\prod_A \mathcal{I}$. Thus, the above theorem also includes this case. We conclude with an application of modularity.

## 5.4.5 Examples of Modular Constructions

To demonstrate the modularity, we exhibit a few examples used in the literature. We begin with a well-known example of an expressive modal logic, which fails to apply the general expressiveness theorem for finitary functors:

**Example 5.4.31.** We show that $I$-labelled image-finite branching transition systems, i.e. coalgebras for the type

$$\mathcal{P}_\omega(-)^I \cong \prod_I \mathcal{P}_\omega(-)$$

has a one-step expressive equational semantics. Note that $\mathcal{P}_\omega(-)^I$ is not a finitary functor unless $I$ is finite, so we cannot apply Theorem 5.4.27 directly. However, by Theorem 5.4.30, we only need to give a one-step expressive semantics for $\mathcal{P}_\omega$ and it is well-known that normal modal logic is equational and is one-step expressive for $\mathcal{P}_\omega$. Therefore, the copower by $I$ of the one-step expressive semantics $(\mathbb{M}, \delta)$ is one-step expressive. The syntax functor $\coprod_I \mathbb{M}$ is isomorphic to the functor defined by

$$\mathbb{M}^I A := \mathsf{BA}\langle \{\blacklozenge_i a\}_{i \in I, a \in A} \mid \blacklozenge_i \bot = \bot, \blacklozenge_i(a \vee b) = \blacklozenge_i a \vee \blacklozenge_i b \rangle,$$

and concretely the language can be generated by the syntax

$$\varphi := \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond_i \varphi \quad (i \in I)$$

i.e. the usual multi-modal logic.

### Segala Systems

The non-deterministic feature can also be combined with probabilistic distributions in a number of ways. For example, a non-deterministic transition may lead to a probabilistic transition as follows:

**Definition 5.4.32** (see [98, 99]). A **simple Segala system** over a set $I$ of actions is a coalgebra of the type $\mathcal{P}(\mathcal{D}(-))^I$ where $\mathcal{D}$ is the probability distribution functor given in Example 2.1.3. That is, a simple Segala system on a state space $X \in \mathsf{Set}$ is a function

$$\xi \colon X \to (I \to \mathcal{P}(\mathcal{D}X)).$$

By Example 5.4.31, a one-step semantics for Segala systems over a set $I$ of actions can be constructed as the composition of a one-step semantics of type $\mathcal{P}(-)^I$ and a one-step semantics of type $\mathcal{D}$. Example 2.3.6 gives a set of predicate liftings for $\mathcal{D}$ so a one-step semantics. Thus, we obtain a logic as follows:

**Example 5.4.33.** Let $(\mathbb{M}^D, \delta^D)$ denote the one-step semantics of the distribution functor induced by the predicate liftings given in Example 2.3.6 and $(\mathbb{M}^I, \delta^I)$ the normal multi-modal logic. The composition $(\mathbb{M}^I, \mathcal{P}(-)^I, \delta^I) \otimes (\mathbb{M}^D, \mathcal{D}, \delta^D)$ is a one-step semantics for Segala systems. Concretely, its language can be generated by a

two-sorted syntax

$$\varphi := \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond_i \psi \quad (i \in I)$$
$$\psi := \top \mid \psi \wedge \psi \mid \neg\psi \mid [p]\varphi \quad (p \in [0,1])$$

similar to the syntax given in [98, Section 2].[7]

The one-step expressiveness of $(\mathbb{M}^D, \mathcal{D}, \delta^D)$ is shown in [36, 53], so it follows that the collection of image-finite Segala systems systems has the expressiveness:

**Proposition 5.4.34.** *The composition of $(\mathbb{M}^I, \mathcal{P}_\omega(-)^I, \delta^I)$ and $(\mathbb{M}^D, \mathcal{D}, \delta^D)$ is one-step expressive for coalgebras of the functor $\mathcal{P}_\omega(\mathcal{D})^I$, viz. image-finite Segala systems.*

**Alternating Systems**

**Definition 5.4.35** (see [48, 98]). An **alternating system** over a set $I$ of actions is a coalgebra for the functor $\mathcal{D} + \mathcal{P}^I$, i.e. a function $\xi \colon X \to \mathcal{D}X + \mathcal{P}X^I$.

**Example 5.4.36.** A one-step semantics for alternating systems is then given by a product of $(\mathbb{M}^D, \mathcal{D}, \delta^D)$ and $(\mathbb{M}^I, \mathcal{P}^I, \delta^I)$, and the resulting syntax functor is $\mathbb{M}^D \times \mathbb{M}^I$. Abstractly, the initial sequence of $(\mathbb{M}^D \times \mathbb{M}^I)$ generates its language at $\omega$-th object, and by the commutativity (4.3), the $\omega$-th object is isomorphic to the product of the $\omega$-th objects in the $\mathbb{M}^I$-sequence and $\mathbb{M}^D$-sequence respectively. Concretely, its language $\mathcal{L}$ is generated by a 3-sorted syntax:

$$\varphi := \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond_i \varphi \quad (i \in I)$$
$$\psi := \top \mid \psi \wedge \psi \mid \neg\psi \mid [p]\psi \quad (p \in [0,1])$$
$$\mathcal{L} \ni \chi := (\varphi, \psi)$$

where the third rule $\chi := (\varphi, \psi)$ replaces the third rule of the 3-sorted syntax given in [98]:

$$\chi := \top \mid \chi \wedge \chi \mid \neg\chi \mid \varphi + \psi.$$

This does not affect the logic because each of cases can be translated to $(\top, \top)$, $(\varphi \wedge \varphi, \psi \wedge \psi)$, $(\neg\varphi, \neg\psi)$, and $(\varphi, \psi)$ in $\mathcal{L}$ respectively.

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[7]The probability modality in [98] is restricted to rational probabilities, i.e. $[p]\phi$ for $p \in [0,1] \cap \mathbb{Q}$.

### 5.4.6 Jónsson-Tarski Duality

For a dual adjunction $S \dashv P : \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$, a **Jónsson-Tarski duality** for an endofunctor $L$ (or a monad $\mathbb{L}$) of $\mathscr{A}$ to an endofunctor $T$ (or a comonad $\mathbb{T}$) of $\mathscr{X}$ is a dual adjunction $\overline{S} \dashv \overline{P} : (\mathscr{X}^{\mathrm{op}})_{T^{\mathrm{op}}} \to \mathscr{A}_L$ where $\overline{P}$ is a lifting of $P$ along forgetful functors of $\mathscr{A}^L$ and $\mathscr{X}_T$, i.e.

$$
\begin{array}{ccc}
\mathscr{X}_T & \xrightleftharpoons[\overline{S}]{\overline{P}} & \mathscr{A}^L \\[4pt]
U_{\mathscr{X}} \downarrow & & \downarrow U_{\mathscr{A}} \\[4pt]
\mathscr{X} & \xrightleftharpoons[S]{P} & \mathscr{A}
\end{array}
\tag{5.19}
$$

If the lifting $\overline{P}$ is given by a semantics $(L, T, \delta)$, it is called *Jónsson-Tarski duality for $(L, T, \delta)$*.

⚠ We assume that $S \dashv P : \mathscr{X}^{\mathrm{op}} \to \mathscr{A}$ is a dual adjunction in this subsection.

#### Jónsson-Tarski Duality for Multi-Step Semantics

Jónsson-Tarski duality for multi-step semantics is in fact a simple application of the adjoint lifting theorem by Johnstone [56].

**Theorem 5.4.37** ([56], also see [32, Section 4.5]). *Let $\mathbb{L}$ be a monad on a category $\mathscr{A}$ and $\mathbb{T}$ a comonad on a category $\mathscr{X}$. Suppose that $\mathscr{X}_{\mathbb{T}}$ has coreflexive equalisers, then every lifting $\overline{P}$ along forgetful functors has a dual adjoint on the right, i.e. a Jónsson-Tarski duality.*

Note that coreflexive equalisers are reflexive coequalisers in the opposite category.

The collection of liftings is in one-to-one correspondence with the collection of multi-step semantics by Lemma 5.3.8, and it shows that every multi-step semantics has a Jónsson-Tarski duality provided that the category of $\mathbb{T}$-coalgebras has coreflexive equalisers. Also note that the condition is independent of the choice of $\delta$.

*Remark* 5.4.38. Suppose that $\mathscr{X}$ is complete. Then, the existence of coreflexive equalisers is equivalent to the completeness of the category $\mathscr{X}_{\mathbb{T}}$, see [80].

**Jónsson-Tarski Duality for One-Step Semantics**

**Proposition 5.4.39.** *Let $L$ and $T$ be endofunctors of $\mathscr{A}$ and $\mathscr{X}$ respectively. Suppose that the forgetful functor $U^T\colon \mathscr{X}_T \to \mathscr{X}$ (resp. $U^L\colon \mathscr{A}^L \to \mathscr{A}$) has a right (resp. left) adjoint and $\mathscr{X}_T$ has coreflexive equalisers. Then every one-step semantics $(L,T,\delta)$ has a Jónsson-Tarski duality.*

*Proof.* The forgetful functors $U^L$ and $U^T$ are monadic and comonadic, respectively, by Proposition 4.1.5. The statement follows by Theorem 5.4.37. $\qquad\square$

It follows that every one-step equational (or finitary) semantics of an $\lambda$-accessible type on a locally presentable category has such a duality:

**Corollary 5.4.40.** *Let $\mathscr{X}$ and $\mathscr{A}$ be locally presentable categories, and $L$ and $T$ are accessible endofunctors of $\mathscr{A}$ and $\mathscr{X}$ respectively. Then every one-step semantics $(L,T,\delta)$ has a Jónsson-Tarski duality.*

*Proof.* By Corollary 4.2.7, the forgetful functor $\mathscr{A}^L \to \mathscr{A}$ has a left adjoint. By Corollary 4.1.24, the forgetful functor $\mathscr{X}_T \to \mathscr{X}$ has a right adjoint and $\mathscr{X}_T$ is complete. Thus, the statement follows from the previous proposition. $\qquad\square$

# Chapter 6

# Future Work

## 6.1  More about CoLog

The definition of CoLog is fairly general and a few restrictions might be necessary to exploit its logical use in depth. For example, we have shown in Proposition 5.1.22 that finite products of equational one-step semantics are computed pointwise, and this does not hold in general.

**Logical Connection**  A logical connection is a dual adjunction with a pair of dualising objects, as discussed in [20, 77, 93]. In this setting, the dualising object serves as an object of truth values, but it is not clear whether the definition of CoLog should be strengthened in a similar fashion and whether we would have more properties.

**More Constructions**  We only discussed colimits, limits, and compositions of one-step semantics. It is not clear if CoLog inherits other constructions from its underlying categories of endofunctors. For example, it would be useful if CoLog had a closed monoidal structure.

**Order-Enrichment**  In particular, the collection of theories is naturally a partially ordered set, and elements in a coalgebra might be ordered by satisfaction relation. Also, the positive Kripke frames can be modelled by coalgebras for the convex powerset endofunctor of Pos. This line of research is recently investigated in [16, 25,

58]. The author believes the picture would be clearer if CoLog were enriched over a monoidal category instead of Pos.

**Characterisations**   A few notions are formulated in high-level descriptions, and some of them are not characterised explicitly in detail. For example, the one-step expressiveness condition generalises the separating property. However, it is not clear what a separating condition should be in a point-free style. We notice that the mate $\delta^*$ on a language $\langle \Phi, \iota \rangle$, which is constructed by the free algebra sequence, is a map of type

$$TS\Phi \xrightarrow{\delta^*_\Phi} SL\Phi \xrightarrow{\cong} S\Phi$$

and $S\Phi$ is a cofiltered limit of the following sequence

$$S\Psi \leftarrow S\Psi \times SL\Psi \leftarrow S \times SL(\Psi + L\Psi) \leftarrow \dots$$

since $S \colon \mathscr{A} \to \mathscr{X}$ maps colimits to limits. Assuming that $\mathscr{X}$ is strongly locally presentable, then this implies that there exists a morphism $S\sigma_i \colon S\Phi \to SL_i$ such that $S\sigma_i \circ \delta^*_\Phi$ is monic if $\delta^*_\Phi$ is. A further characterisation may explain and link the classical notion.

**Modularity of Expressiveness**   We have shown that the expressiveness property is stable under colimits and compositions. It would be useful in practice if we could work out more constructions including limits of one-step semantics.

**Completeness**   So far, we did not discuss the completeness property with respect to one-step semantics in the general setting. It is shown in [65] and [75] that a one-step semantics is complete if the interpretation is injective for Stone dualities **BA** and Stone (resp. Set).

**Multi-Step Semantics**   The category of multi-step semantics is only a first step towards multi-step coalgebraic modal logic. We expect that it will aid the future development in this direction.

## 6.2 A Classification of Coalgebraic Logics

The notion of finitely based functor was introduced originally for finitary adjunctions of descent type in the enriched context. Therefore, it would be interesting to classify different types of coalgebraic modal logics in line with this setting.

In addition, quasi-varieties are in somewhere between locally presentable categories and varieties. It would be natural if logical axioms are given by implications instead of equations.

## 6.3 Conclusion

We hope that this framework will help experts and beginners alike understand coalgebraic modal logics in a systematic approach, as we have seen that the categorical structure unifies and generalises coalgebraic modal logic far beyond from the original setting—Set.

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