Mobile impurities in one-dimensional quantum liquids

by

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Abstract

For a long time the Luttinger liquid has served as the standard low energy description of gapless one-dimensional quantum systems. More recently, using an effective mobile impurity model has allowed progress to be made beyond the linearity of the spectrum. In this thesis we show that the depleton model of mobile impurities can be extended to become equivalent to the effective impurity model and calculate the dynamic response functions. We also use exact Bethe ansatz solutions to confirm predictions of the effective model to demonstrate the reliability of both the depleton model and the response functions.

CHAPTER 1

INTRODUCTION

Systems that are restricted to a single spatial dimension can have very different physics to those of higher dimensions. The one-dimensionality of the system can be both a help and a hindrance. Some of the established methods from higher dimensions fail to work in one-dimension, but there are other methods that, it seems, are only applicable in one-dimension. The main theme of this thesis is to use one of the helpful properties, exact solubility of certain models, to test whether some more widely applicable methods do give a good description of the physics we aim to understand. Along the way we shall see that the exactly soluble models are interesting in their own right.

Understanding the behaviour of interacting quantum many-body systems is a significant challenge, we will begin with a brief discussion of what can be done in higher dimensions. The low energy properties of electrons in a normal metal are described by Fermi liquid theory[31], to which the concept of a quasi-particle is central. Quasi-particles are long lived excitations used to describe the low energy behaviour. In normal metals the quasi-particles are similar to free electrons, they can be labelled by their momenta and their energy is then given by the dispersion relation. The amplitude of an electron tunneling into the system with momentum and energy $\epsilon(q, \omega)$ is proportional to the spectral function $A(q, \omega)$. The spectral function for a non-interacting system is $A(q, \omega) \propto \delta(q - \omega(q))$, when interactions are added back in

perturbatively this delta function peak widens to become a Lorentzian. The majority of the spectral weight remains in a small region around the non-interacting peak indicating the similarity of the free particles and quasi-particles. Turning our attention to bosonic particles we can also see examples of the perturbative addition of interactions leading to long lived quasi-particle excitations. The most famous example of this is the solution of the weakly interacting Bose gas [4]. In this case the quasi-particles describe the excitations above the condensate ground function. Here the quasi-particles differ substantially from the a free particle, but the method of perturbatively adding interaction is the same.

Attempts to apply the same scheme in one-dimension failed because the perturbation theory leads to divergences. Tomonaga [38] was first to find a way to avoid these problems by taking the non-linear dispersion and linearising around the Fermi points. This linearisation simplifies the problem and Tomonaga was able to map it onto that of free bosons, the bosonic excitations being density fluctuations of the underlying fermionic system. A great deal of progress has been made in calculating the properties of the linearised system and the names Luttinger[26], Mattis and Lieb[28], Dzyaloshinskii and Larkin[9] feature heavily. Another major breakthrough was achieved by Haldane [14] who showed that the non-linearity of the dispersion often did not affect the correlation functions over long distances. This allowed many more models to be treated in the same way including bosonic systems and spin chains, leading to the concept of a Luttinger liquid, with many different one-dimensional systems showing the same universal behaviour.

In contrast to Fermi liquids, when attempting to add the non-linear terms back in perturbatively instead of broadening the delta function into a Lorentzian the degeneracy caused by linearising the spectrum plagues the perturbation theory. Adding the non-linearity was achieved by realising a connection with another problem, the Fermi edge singularity[27][31][35]. The models consist of an impurity coupled to a Luttinger liquid and allows calculation of edge singularities for the non-linear Luttinger liquid [17][5][7]. This is the link between mo-

bile impurities and correlation functions in one-dimension. The exponents describing the edge singularities have been linked with the dispersion of the impurity [17] and the finite size spectra through results from conformal field theory. In some exactly soluble models the finite size corrections can be calculated exactly and using this method the exact exponents of the singularities have been found [7] [10] [32]. The use of exact solutions to test the reliability of the phenomenology is the main theme of this thesis.

Mobile impurities play an important part in the understanding of the low energy behaviour of one-dimensional systems beyond the linear approximation, they have also be studied in their own right. Impurity physics has been around for a long time, in condensed matter there are famous problems such as the Kondo effect [20]and mobile impurities in Helium. More recently developments in ultra-cold atoms have brought experimental systems in low dimensional regimes [30]. Mobile impurities can be studied in ultra-cold atoms by mixing atomic species, 'flipping' some atoms into a different hyperfine state or adding ions to the system. In each case it is even possible to apply an external force exclusively to the impurities. The dynamics of an impurity in one-dimension can show unusual behaviour; for example the dispersion is periodic. This happens because in a system with density n, the background superfluid can absorb an extra $2\pi n$ momentum with no energy cost in the thermodynamic limit. This periodicity in the dispersion has consequences, using the so called depleton theory to study the dynamics of an impurity it has been shown that Bloch oscillations are possible in the absence of a lattice [34].

In this thesis we will show that the depleton theory can be extended to calculate the edge singularities, the depleton method will make it easy to see connections between calculations which are less apparent in other methods. Following that we use exact calculations to test both the application of the depleton model to describe the interaction with phonons by calculating the backscattering amplitude and the edge singularity theory by checking that the suggested relationships between the dispersion and the Fermi surface shifts are exact

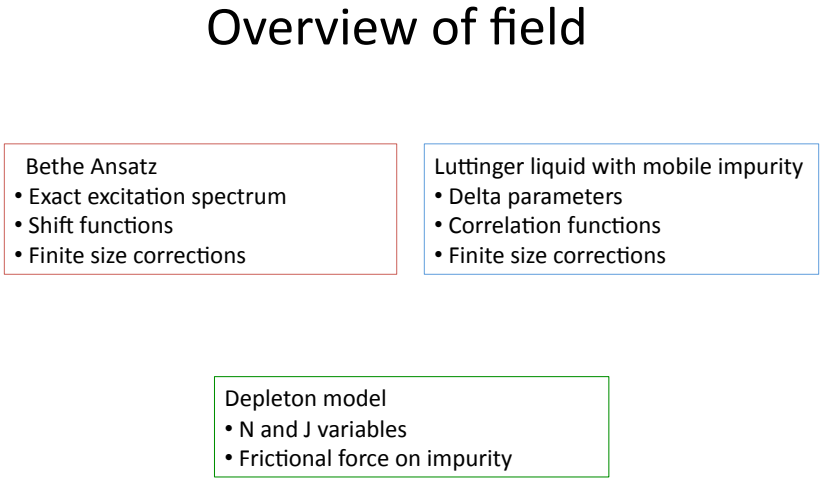

Figure 1.1: A sketch of the field

identities.

Before explaining the structure of the thesis we will give a brief overview of the field. The results of the thesis will connect different areas of research. The areas can be grouped into three broad blocks. These three blocks are shown in Fig. 1, they are the Bethe ansatz techniques, the depleton approach and the mobile impurity coupled to a Luttinger liquid. Under each group title are the most important, for our purposes, aspects of the research.

For the Bethe ansatz we have the exact excitation spectra which we will calculate, the shift functions which will be central to our calculations and the finite size corrections to the spectra. We will find new exact relations between the derivatives of the excitation spectra and the shift functions. These confirm that the shift functions are closely related to the delta parameters in the way that had been previously predicted by comparing the finite size corrections. These exact identities are a new addition to the field.

The N and J parameters in the depleton approach and the delta parameters are clearly closely related. This can be seen by comparing their definitions. This observation leads us to a dramatically simpler understanding of the connection between the different correlation functions that we will see how to calculate. Using our new method calculation of one correlation function can be easily extended to a family of related correlation functions.

Finally we will calculate the frictional force, which is defined in terms of derivatives of the energy spectra, for the exactly soluble models. We find that it vanishes exactly in all three cases, this is understandable due to the integrability of the models. These three results are the most important parts of the work. We will now explain the structure of the thesis where we explain these results and how they fit into previous work.

We will start by introducing the required background theory, chapter two covers the topics of Bethe ansatz, bosonisation and conformal symmetry. The single component contact interacting Bose gas or Lieb-Liniger [25] model will be treated in detail. We derive the Bethe ansatz equations and then show how to find the ground state and classify the zero temperature excitations. These excitations come in two types, particle like and hole like as in free fermionic systems. As well as the zero temperature excitations we will meet the shift function and see some of it's unexpected properties. These unexpected properties of the shift function will be crucial to our exact calculations later on. We will also describe the two component version called the Yang-Gaudin model [39][11] after the original derivation. The multi-component case can be treated using the the algebraic Bethe ansatz, but we omit this as despite being a very elegant approach we do not use it for our results. We choose instead to quote the Bethe equations. From the Bethe equations we again find the zero temperature excitations, a new excitation called the spinon appears. The ground state is the same as the one component case, the spinon is an excitation consisting of changing the 'colour' of one of the particles. The spinon is important because it is much more like an impurity, it is distinguishable from the other particles and has quadratic dispersion at small momenta.

Also contained in chapter two is an introduction to bosonisation. A phenomenological derivation of the standard Luttinger liquid Hamiltonian is given and it's justification is discussed. We will need to calculate some correlation functions of the Luttinger liquid and we choose to use the conformal symmetry approach [13], this allows us to efficiently calculate the results we want for correlation functions of exponents of the fields. These are exactly the form of correlation functions which will arise later on. The conformal symmetry approach is convenient because we can show the relationship between the finite size energy spectrum and correlation functions.

Chapter three discusses the depleton theory [34], which we have already mentioned. We will see the separation of scales allows the impurity to be thought of as a delta function in it's interaction with phonons. These phase shifts N and J will turn out to be very natural variables especially in the Bethe ansatz language. Important results quoted in this section include the relation of the shifts in Luttinger liquid fields to the dispersion of the impurity, which follows from Galilean invariance and the universal coupling of the impurity to the background. By the end of the chapter we will have the full Lagrangian for an impurity interacting with a background Luttinger liquid describing the phononic degrees of freedom of the problem. In the next chapter we will extend the depleton theory to allow calculation of the edge singularities.

As we have just stated, in chapter four we will extend the depleton theory to cover the calculation of edge singularities. First, however, we will see how they have been calculated using a unitary operator to remove the interaction term. This procedure is very similar to the method used in the Fermi edge calculations and was motivated by this. Once we have seen what has been done we show how it relates to the depleton model. This is the first of our own results and we will see how to calculate the edge singularities from the depleton model. We will show that the results are the same as those previously derived, a benefit of our approach is that connections between different correlation functions are much more

easily apparent in the N and J notation.

Chapter five is where we start to use the exact solutions of the Lieb-Liniger and Yang-Gaudin models to test the depleton theory of mobile impurities in one-dimension. We have already seen that the exponents describing the edge singularities are related to the finite size spectra, it is possible to calculate these finite size spectra for the Bethe ansatz soluble models. The calculation required is to find the corrections to the energy for a large but finite system in the presence of an excitation. We will show the results and compare with the known finite size spectra for a shifted Luttinger liquid. The calculations themselves are long and difficult and are included in the appendix A, we will find that the finite size corrections are given in terms of the shifts in the Fermi surface position. We now have the exponents expressed in two different ways; first we have the N and J defined in terms of the dispersion of the impurity and secondly we have them in terms of the Fermi surface shifts from the finite size analysis. A natural question to ask is, "are these two approaches consistent?" The answer to this question is the first of our main exact results. We will show that the suggested relationships between the derivatives of the dispersion and the Fermi surface shifts are exact identities in the Lieb-Liniger and Yang-Gaudin systems. Having shown that the depleton theory passes it's first test we move on to the next one and calculate the backscattering amplitude exactly for the particle, hole and spinon. Here we see again that the N and J variables are very natural, the backscattering amplitude becomes much simpler in this language. We expect that the backscattering amplitude should vanish for the integrable cases and indeed the amplitude is exactly zero for the particle, hole and spinon.

CHAPTER 2

THE BAG OF TRICKS

In this chapter we shall meet some of the techniques available to a physicist working on 1D systems. Specifically, they are the Bethe ansatz methods, bosonisation and conformal field theory. This list is far from complete and we will make no attempt to provide a full description of even this incomplete list. We will instead pick and choose just the ideas that we will use later on. This will result in us scratching the surface of the Bethe ansatz methods and seeing even less of bosonisation and conformal field theory.

These methods are very different in their approach and in what we can learn by using them. We shall see that when they are applied in combination their effectiveness can be greater than the sum of their parts. The Bethe ansatz refers to a group of ideas and techniques all based on an exact microscopic approach. This means that we start with a microscopic Hamiltonian and can analytically determine some properties exactly, for example the ground state wave function, spectrum of excitations and thermodynamics. A downside is that the method is complicated and it is usually very difficult, if not impossible, to calculate even the simplest correlation functions. A bigger problem is that only a very special set of Hamiltonians can be solved at all. Fortunately, ultra cold atoms experiments provide examples where some of these models can be created and studied.

In contrast to Bethe ansatz, bosonisation can be applied widely to one-dimensional gases,

lattice systems, spin chains and others. It does not depend on the microscopic properties of the model, but the universal properties instead. This is a huge advantage, however there are downsides too. The main issue is whether we can be confident that effective model has captured the physics we are interested in. Clearly experimental verification is the most important measure, but we can do some checks more easily. If we can find examples where we can apply both the effective theory and the exact theory then we can check the reliability of the effective theory. This is where the combination of approaches benefits us. We can use the Bethe ansatz to give us confidence in more widely applicable, but less rigorously justified theory.

Some may be concerned that since the Bethe ansatz models are somehow special, equivalence with more generic theory may not be important or may even be misleading. The important points here are; that the more widely applicable theories are not capable of describing the exact solubility and that the exactly soluble models are often limits or particular parameter space points of natural physical models. Therefore we should expect that the widely applicable models fit with the exact at the special soluble points. We should not, however, extrapolate properties of the exact systems into the non soluble regions of parameter space without great caution.

2.1 Bethe ansatz

Bethe ansatz methods are a collection of techniques for solving some 1D quantum Hamiltonians exactly. They originate from Bethe's solution of the Heisenberg magnet over eighty years ago[3]. Since then further problems have been solved; some of the more notable being the contact interacting Bose gas[25] and the Hubbard model[24]. Also the techniques themselves have been extended to calculate more properties of the models. Usually it is possible to find the ground state wave function, spectrum of elementary excitations and many thermody-

namic quantities with relative ease[40]. The calculation of correlation functions, however, is often extremely difficult. Another severe limitation of Bethe ansatz methods is that only a small number of models can be solved this way and usually any perturbation or sometimes even adjustment of the parameters prevents solution.

2.1.1 One component and Lieb-Liniger

We will start with the simplest case of identical particles, either bosons or spinless fermions interacting via a pair potential. Choosing units such that $\hbar = 1$ and $m = 1/2$ the Hamiltonian is,

$$H = \sum_i \partial_{x_i}^2 + \sum_{i<j} V(x_i - x_j). \tag{2.1}$$

Due to conservation of momentum and energy in 1D when two particles with the same mass scatter, the outgoing momenta are simply a rearrangement of the ingoing momenta and the asymptotic wave function changes by a phase shift only. Another peculiarity of 1D is that if we know the wave function in the region $R : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq L$ then we can reconstruct the whole wave function by using the statistics. With this in mind we make Bethe's ansatz that the wave function in R is of the form:

$$\Psi^R(\mathbf{x}) = \sum_P a(P) \exp(i \sum_j \lambda_{P_j} x_j). \tag{2.2}$$

Here $P = (P_1, P_2, ..., P_N)$ is a permutation of $(1, 2, ..., N)$, $a(P)$ is a coeffiecient depending on this permutation. The λ variables play the role of momentum in the region R. They are not the momenta of the particles and are called the quasimomenta instead. The $a(P)$ depend only on the permutation and not how the scattering occurs, for this to be consistent all scattering events must be equivalent to a sequence of two body scattering events and all sequences leading to the same permutation must be equivalent. Fortunately there exist some

pair potentials where this is true. It may seem unlikely that this approach will be exact for long range pair potentials as we will only be in the asymptotic region at low density. Surprisingly the result is often exact in the thermodynamic limit. More importantly for the examples we will be interested in, a zero range potential is always in the asymptotic region. It is these contact interactions that will will deal with. The simplest non trivial example is the repulsive contact interacting Bose gas called the Lieb-Liniger model. Spinless fermions with contact interactions are equivalent to non interacting fermions. The Lieb-Liniger model of identical bosons with repulsive contact interactions in 1D is one of the most important Bethe ansatz soluble models, the ground state and elementary excitations at zero temperature were found by Lieb and Liniger in 1967 [25], the thermodynamics of the model were first investigated by Yang and Yang [40]. Since then many more results have been found including some correlation functions . We will only be interested in the zero temperature solutions. The Hamiltonian is

$$H_{LL} = -\sum_{i=1}^{N} \partial_{x_i}^2 + 2c \sum_{i>j} \delta(x_i - x_j). \tag{2.3}$$

If we consider the region $\tilde{R} : 0 \leq x_1 < x_2 < \cdots < x_N \leq L$ i.e. exclude the parts of R where the particles interact then the Hamiltonian becomes the free Hamiltonian

$$H_{LL}^{\tilde{R}} = -\sum_{i=1}^{N} \partial_{x_i}^2, \tag{2.4}$$

with the boundary conditions

$$\partial_{x_i}\psi - \partial_{x_j}\psi = c\psi \qquad x_i = x_j. \tag{2.5}$$

This is completely equivalent to solving (2.3) in $\mathbb{R}^N$.

Let's see how we go about constructing the wave function in practice, we will use the method Lieb and Liniger presented [25]. The are other ways of doing this that are more

powerful, such as the Fermi-Bose mapping [22], but they do not demonstrate the structure we have described as clearly. We begin by considering the first terms in the sum over permutations and keeping in mind that the asymptotic wave functions will differ by a phase we write:

$$\begin{aligned}\Psi \propto &\exp i(\lambda_1 x_1 + \lambda_2 x_2 + \dots) - \exp(-i\theta(\lambda_1, \lambda_2)) \exp i(\lambda_2 x_1 + \lambda_1 x_2 + \dots) \\ &+ \sum_{P'} a(P') \exp(i \sum_j \lambda_{P_j} x_j). \end{aligned} \tag{2.6}$$

Where P' are the other permutations excluding the one we have written explicitly. The next step is to enforce the boundary conditions (2.5), or more carefully the one boundary condition at $x_1 = x_2 = y$. After canceling like terms this leads to:

$$i(\lambda_1 - \lambda_2)(1 + \exp(-i\theta)) = c(1 - \exp(-i\theta)), \tag{2.7}$$

solving this for θ gives:

$$\theta(\lambda_1, \lambda_2) = 2 \arctan\left(\frac{\lambda_1 - \lambda_2}{c}\right). \tag{2.8}$$

Having solved for this single permutation we can use the consistency relation to find the full answer, because any sequence of pair permutations leading to the same overall permutation must give the same answer we can use any. Lieb and Liniger suggest writing out the numbers one to n and underneath writing the permutation and drawing a line from each number in the top row to where it appears in the lower. From this diagram we now multiply by the factor $\exp(-i\theta(\lambda_i, \lambda_j))$ for each crossing of i and j, once we have got all of the factors this generates the overall factor $a(P)$.

Now if we add the normalisation and extend the wave function by symmetry to $\mathbb{R}^N$ then

we get,

$$\Psi = \left\{ N! \prod_{j>k} ((\lambda_j - \lambda_k)^2 + c^2) \right\}^{-\frac{1}{2}} \tag{2.9}$$

$$\times \sum_P (-1)^P \prod_{1 \le j < k \le N} [\lambda_{P_j} - \lambda_{P_k} - sgn(x_j - x_k) ic] \exp\left(i \sum_k \lambda_{P_k} x_k\right). \tag{2.10}$$

We should note that the wave function is symmetric in x variables but antisymmetric in the λ variables. A simple limit of the system is the $c \to \infty$ where (2.5) become the same as fermionic statistics and the wave function is equal to that of non interacting fermions in the region R. The energy eigenstates are also simultaneously eigenstates of the total momentum and number operators. The eigenvalues are given by,

$$E = \sum_j \lambda_j^2, \tag{2.11}$$

$$P = \sum_j \lambda_j^1, \tag{2.12}$$

$$N = \sum_j \lambda_j^0. \tag{2.13}$$

If we consider the system with periodic boundary conditions then,

$$\Psi(x_1, x_2, \ldots, x_j, \ldots, x_N) = \Psi(x_1, x_2, \ldots, x_j + L, \ldots, x_N). \tag{2.14}$$

This requirement constrains the allowed values of the λ variables to be solutions of,

$$\exp(i\lambda_j L) = -\prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \qquad j = 1, \ldots, N. \tag{2.15}$$

These are the Bethe equations for the Lieb-Liniger model. Solving these equations allows us to write down the wave function for a system with periodic boundary conditions, however the

explicit form of the wave function is rather unwieldy and usually not useful for calculating other quantities. We will see how we can calculate many properties of the system without going via the wave function.

The properties of a given state are determined by the solution of (2.15), taking the logarithm we get,

$$L\lambda_j + \sum_k \theta(\lambda_j - \lambda_k) = 2\pi n_j \qquad j = 1, \ldots, N. \tag{2.16}$$

$$\theta(x) = 2\arctan\left(\frac{x}{c}\right) \tag{2.17}$$

Where the n_j are integers which specify the eigenstate, i.e. a choice of the n_j uniquely fixes the λ_j. As θ is a monotonically increasing function we can see that if $n_j > n_k$ then $\lambda_j > \lambda_k$. We should think of the n_j as living on an integer valued lattice where double occupancy is not allowed. The spacing between the λ variables will increase when the spacing between the corresponding n variables does but it also depends on the position of the other λ as well as the interaction strength c.

The integer lattice provides us with an easy way to find the ground state of the system, the energy is $E = \sum_j \lambda_j^2$ so if we choose the n_j symmetrically about zero and as densely as possible we will get the lowest energy. The ground state is therefore the Fermi sea in the n lattice, this lattice will be very useful when considering elementary excitations too. The ground state λ can now be calculated by solving (2.17) with the ground state n. We note that because (2.15) couples all λ together any excitation above the ground state will affect all λ.

Will will now consider the system in the thermodynamic limit,

$$N \mapsto \infty, \quad L \mapsto \infty, \quad \frac{N}{L} \mapsto n. \tag{2.18}$$

When we do this we go from having a set of λ_j to a density $\rho(\lambda)$ which is defined as,

$$\rho(\lambda_j) = \lim \frac{1}{L(\lambda_{j+1} - \lambda_j)}. \tag{2.19}$$

In the thermodynamic limit the ground state Bethe equations become,

$$\rho(\lambda) - \int_{-q}^{q} d\mu \frac{K(\lambda, \mu)}{2\pi} \rho(\mu) = \frac{1}{2\pi}. \tag{2.20}$$

Where $K(x, y) = 2c/(c^2 + (x - y)^2)$ and q is the highest filled quasimomentum state. The density of the system is given by,

$$n(q) = \int_{-q}^{q} \rho(\nu) d\nu. \tag{2.21}$$

Many of the functions we deal with will be described by integrals of the same form as (2.20). It will sometimes be advantageous to write the convolution with a kernel in a more compact operator style.

$$[\hat{K} f](x) = \int_{-q}^{q} d\mu K(x, \mu) f(\mu). \tag{2.22}$$

We will in some cases suppress the dependences even further and write $\hat{K} f$, it should be noted that if the function being operated on is a function of more than one variable then the convolution is always over the first variable. In this compact operator form the ground state equation (2.20) becomes,

$$\rho(\lambda) - \frac{1}{2\pi} [\hat{K} \rho](\lambda) = \frac{1}{2\pi} \tag{2.23}$$

or in the most concise notation,

$$\left(1 - \frac{\hat{K}}{2\pi}\right) \rho = \frac{1}{2\pi}. \tag{2.24}$$

Now that we have taken the thermodynamic limit we will choose to work in the grand

canonical ensemble. We introduce the chemical potential, h, and the ground state energy is,

$$E_{GS} = L \int_{-q}^{q} d\mu \rho(\mu)(\mu^2 - h). \tag{2.25}$$

Now we can consider the elementary excitations above the ground state. They come in two varieties, particle like and hole like. The particle excitations consist of an extra particle with quasimomentum $|\lambda_p| > q$ being added, hole excitations consist of the removal of a particle with quasi-momentum $|\lambda_h| < q$. The momentum and energy of these excitations can be thought of as made up of two parts. The first part is the bare momentum and energy of the excitation, for the particle they are λ_p and $\lambda_p^2 - h$ respectively. The second part is the dressed momentum and energy which comes from shifting the quasi-momentum of the other particles. This occurs because all of the quasi-momenta are coupled through (2.20). To calculate these shifts we start with (2.15), in the ground state we have,

$$L\lambda_j + \sum_k \theta(\lambda_j - \lambda_k) = 2\pi \left(j - \frac{N+1}{2} \right) \tag{2.26}$$

If we add an extra particle with quasi-momentum λ_p to the system then the new quasi-momenta $\tilde{\lambda}$ are defined by,

$$L\tilde{\lambda}_j + \sum_k \theta(\tilde{\lambda}_j - \tilde{\lambda}_k) + \theta(\tilde{\lambda}_j - \lambda_p) = 2\pi \left(j - \frac{N+1}{2} \right) - \pi. \tag{2.27}$$

Subtracting (2.27) from (2.26) and discarding terms of order $O(1/L)$ we get,

$$L(\lambda_j - \tilde{\lambda}_j) - \theta(\lambda_j - \lambda_p) + (\lambda_j - \tilde{\lambda}_j) \sum_k K(\lambda_j, \lambda_k) - \sum_k K(\lambda_j, \lambda_k)(\lambda_k - \tilde{\lambda}_k) = \pi. \tag{2.28}$$

We will now introduce the shift function F_B, defined as,

$$F_B \equiv \frac{\lambda_j - \tilde{\lambda}_j}{\lambda_{j+1} - \lambda_j}. \tag{2.29}$$

We are using a different definition to the standard one, usually there is a minus sign in case of the hole. In our definition the shift function is always interpreted in the same way but each type of excitation has it's own shift function. As we are about to see the hole shift function has an extra minus sign. Taking the thermodynamic limit of (2.28) we get,

$$F_B(\lambda|\lambda_p) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi} F_B(\mu|\lambda_p) = \frac{\pi + \theta(\lambda - \lambda_p)}{2\pi}. \tag{2.30}$$

For the hole excitation we remove a particle and the shift function is,

$$F_B(\lambda|\lambda_h) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi} F_B(\mu|\lambda_h) = -\frac{\pi + \theta(\lambda - \lambda_h)}{2\pi}. \tag{2.31}$$

We note here that the definitions of the shift function differ from those of Korepin *et. al.* [22] which is defined as

$$F(\lambda|\nu) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi} F(\mu|\nu) = \frac{\theta(\lambda - \nu)}{2\pi}. \tag{2.32}$$

This shift function does not satisfy the correct boundary conditions. However, it will be useful for doing calculations because it's symmetries are simpler and it is straight forwardly related to the F_B functions by,

$$F_B(\lambda|\nu) = \pm \left\{ F(\lambda|\nu) + \pi\rho(\lambda) \right\}. \tag{2.33}$$

Where the $\pm$ is for particles and holes respectively. Useful properties of the shift function are $F(x,y) = -F(-x,-y)$ and a rather surprising identity found by Slavnov[21][36], which

will be essential for many of the calculations we will do, it is:

$$F(x,y) + F(y,x) = F(x,q)F(y,q) - F(x,-q)F(y,-q). \tag{2.34}$$

To find a non-linear symmetry relation for a function defined by a linear integral operator seems very unusual indeed.

We now take a small aside to introduce a new function that will be used in many of our calculations from now on. R is defined by,

$$R(\lambda,\nu) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi} R(\mu,\nu) = \frac{K(\lambda,\nu)}{2\pi}. \tag{2.35}$$

Which can be written in operator form in two different ways,

$$R = \left(1 - \frac{\hat{K}}{2\pi}\right)^{-1} \frac{K}{2\pi}, \tag{2.36}$$

$$R = \left(1 + \hat{R}\right) \frac{K}{2\pi}. \tag{2.37}$$

From which we can see that,

$$\left(1 - \frac{\hat{K}}{2\pi}\right)^{-1} = \left(1 + \hat{R}\right). \tag{2.38}$$

Other properties of $R(x,y)$ that we will need are $R(x,y) = R(-x,-y)$ which is fairly clear from (A.23) and $R(x,y) = R(y,x)$ which is less clear and is proved in [36].

Now that we have defined the shift function we can write the expressions for the momen-

tum ΔP and energy ΔE of the elementary excitations, they are,

$$\Delta P = \lambda - \int_{-q}^{q} d\mu F_B(\mu|\lambda) \tag{2.39}$$

$$\Delta E = \lambda^2 - h - 2\int_{-q}^{q} d\mu F_B(\mu|\lambda)\mu, \tag{2.40}$$

$$= \lambda^2 - h - 2\int_{-q}^{q} d\mu F(\mu|\lambda)\mu. \tag{2.41}$$

We will now show that $\Delta P(\lambda_p) = k(\lambda_p)$ where,

$$k(\lambda_p) = \lambda_p - \pi n + \int_{-q}^{q} d\mu \theta(\lambda_p - \mu)\rho(\mu). \tag{2.42}$$

First we need the result,

$$2\pi\rho(\lambda) = \left(1 - \frac{\hat{K}}{2\pi}\right)^{-1} 1, \tag{2.43}$$

$$= (1 + \hat{R})1, \tag{2.44}$$

$$2\pi\rho(\lambda) - 1 = \int_{-q}^{q} d\mu R(\mu, \lambda) = \int_{-q}^{q} d\mu K(\mu, \lambda)\rho(\mu). \tag{2.45}$$

We will also write (2.32) as,

$$F(\nu|\lambda) = \left(1 - \frac{\hat{K}}{2\pi}\right)^{-1} \frac{\theta(\nu - \lambda)}{2\pi}, \tag{2.46}$$

$$= \left(1 - \frac{\hat{K}}{2\pi}\right)^{-1} - \int^{\lambda} dy \frac{K(\nu, y)}{2\pi}, \tag{2.47}$$

$$= -\int^{\lambda} dy R(\nu, y). \tag{2.48}$$

We can now evaluate the integral in (2.39):

$$\int_{-q}^{q} d\mu F_B(\mu|\lambda) = n\pi - \int_{-q}^{q} d\mu \int^{\lambda} dy R(\mu, y), \tag{2.49}$$

$$= n\pi - \int^{\lambda} dy \int_{-q}^{q} d\mu K(\mu, y)\rho(mu), \tag{2.50}$$

$$= n\pi - \int_{-q}^{q} d\mu \theta(\lambda - \mu)\rho(\mu). \tag{2.51}$$

Substituting this into (2.39) gives (2.42). For hole like excitations the momentum is $-k(\lambda)$.

We will now show that $\Delta E(\lambda) = \epsilon(\lambda)$ where,

$$\epsilon(\lambda) - \int_{-q}^{q} d\mu \frac{K(\lambda - \mu)}{2\pi} \epsilon(\mu) = \lambda^2 - h, \tag{2.52}$$

where $\epsilon(q) = \epsilon(-q) = 0$. This can be written as,

$$\epsilon(\lambda) = (1 + \hat{R})(\lambda^2 - h) = \lambda^2 - h + \int_{-q}^{q} d\mu R(\lambda, \mu)(\mu^2 - h). \tag{2.53}$$

Comparison with (2.40) shows that the statement $\Delta E(\lambda) = \epsilon(\lambda)$ is equivalent to,

$$\int_{-q}^{q} d\mu R(\lambda, \mu)(\mu^2 - h) = -2 \int_{-q}^{q} d\mu F(\mu|\lambda)\mu. \tag{2.54}$$

The first integral on the right hand side is,

$$\int_{-q}^{q} d\mu R(\lambda, \mu)\mu^2 = - \int_{-q}^{q} d\mu \partial_\mu F(\lambda, \mu)\mu^2. \tag{2.55}$$

Integrating by parts and using Slavnov's identity (5.14) we get,

$$\int_{-q}^{q} d\mu R(\lambda,\mu)\mu^2 = 2\int_{-q}^{q} d\mu F(\lambda|\mu)\mu - q^2(F(\lambda|q) - F(\lambda|-q)), \tag{2.56}$$

$$= -2\int_{-q}^{q} d\mu F(\mu|\lambda)\mu - q^2(F(\lambda|q) - F(\lambda|-q)) + 2(F(\lambda|q) - F(\lambda|-q))\int_{-q}^{q} d\mu F(\mu|q)\mu. \tag{2.57}$$

The second integral is,

$$-h\int_{-q}^{q} d\mu R(\lambda,\mu) = h\int_{-q}^{q} d\mu \partial_\mu F(\lambda,\mu) = h(F(\lambda|q) - F(\lambda|-q)). \tag{2.58}$$

The extra terms violating the identity are,

$$-(F(\lambda|q) - F(\lambda|-q))\left(q^2 - h - 2\int_{-q}^{q} d\mu F(\mu|q)\mu\right). \tag{2.59}$$

This is proportional to $\Delta E(q)$ and h is chosen such that $\Delta E(q) = 0$ and our identity is proved. Again for a hole the energy of the excitation is $-\epsilon(\lambda)$, it is still positive however. We should also note that the spinon excitation is quadratic at small quasi-momentum and is the softest mode in the system.

In this section we have solved the Lieb-Liniger model finding the ground state and elementary excitations at zero temperature. The equations for the energy and momentum of the excitations as well as the shift functions will be needed for later calculations. The symmetry of the shift function described by Slavnov's identity will also be needed. Next we will consider a two component boson system with contact interactions, we will need more sophistication techniques to deal with it and will have a new type of excitation, the spinon.

2.1.2 Two components and Yang-Gaudin

We have just seen that the consistency for single component Bethe ansatz are met by the contact interacting gas. What happens if we have two species of boson? This problem was first solved by Yang[39] and Gaudin [11], the technique is called nested Bethe ansatz. The consistency relationship will clearly be more complicated. Before discussing these consistency relations we need to deal with a technical point.

We should also mention that the related topic of two species contact interacting fermions was studied earlier by McGuire [29]. This problem is simpler because the particles only interact with the other species, this is not the case with bosons.

When identical particles scatter from one another we do not need to be concerned with distinguishing which particle is which after the collision, as they are identical. If we have more than one component however, say red particles and blue particles, then if two unlike particles scatter there can be transmission or reflection. If the red particle begins on the left and transmission occurs then it ends on the right, if it reflects it stay on the left. It turns out to be helpful to think of the colour of the particles as a quantum number that can be exchanged, or not, during scattering. Now that the colour of the particles is not fixed we have a choice when labeling them. If before the scattering, particle one is on the left and two is on the right, then after the scattering we have two choices as to how we label the particles. Either the particle on the left or right is particle one. The choice of the particle on the right being number one is called the transmission diagonal representation Fig.[2.2] and in this representation reflection corresponds to an exchange of colour quantum numbers and transmission does not. Alternatively we can choose particle one to always be on the left, this is called the reflection diagonal representation Fig.[2.1]. In the reflection diagonal representation transmission corresponds to the exchange of colour quantum numbers.

To discuss the consistency relations we will use the reflection diagonal representation, we

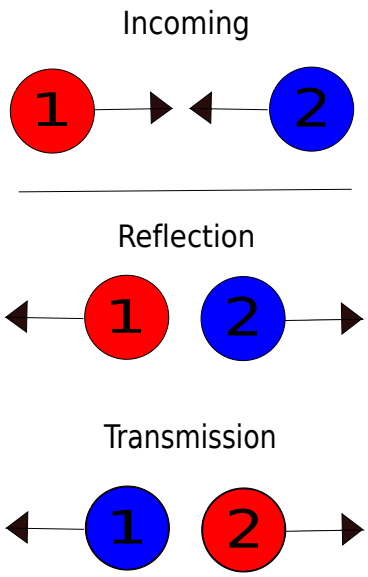

Figure 2.1: The reflection diagonal representation of 2 component scattering

introduce the scattering matrix:

$$S(\lambda) = \begin{pmatrix} \Theta & 0 & 0 & 0 \\ 0 & R & T & 0 \\ 0 & T & R & 0 \\ 0 & 0 & 0 & \Theta \end{pmatrix}, \tag{2.60}$$

where we use the same notation as Sutherland [37]. These scattering matrices describe the scattering of adjacent particles, so S_1 scatters the first and second particles, S_2 scatters the second and third and so on. In a Galilean invariant system they can only depend on the difference of the quasi-momenta, so the scattering matrix for the first pair of particles having quasi-momenta λ_1 and λ_2 is written $S_1(\lambda_1 - \lambda_2)$. We find the consistency relations by demanding that the scattering matrices obey the same relations that the generators of the

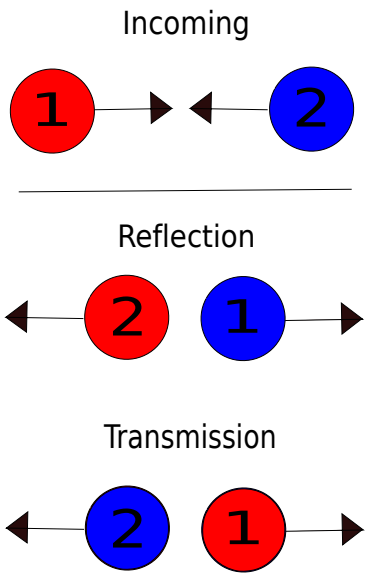

Figure 2.2: The transmission diagonal representation of 2 component scattering

permutation group. The generators of the permutation group, α_j satisfy [37]:

$$(\alpha_j \alpha_k)^{n(j,k)} = I, \tag{2.61}$$

where,

$$n(j,k) = \begin{cases} 1, & |j-k| = 0 \\ 3, & |j-k| = 1 \\ 2, & |j-k| > 1. \end{cases} \tag{2.62}$$

Let's see what this implies for our scattering matrices, the first case gives $S_1(\lambda_1, \lambda_2)S_1(\lambda_2, \lambda_1) = I$ which tells us that the matrices are unitary which we expect. The third case tells us that the scattering matrices commute for separated particles, this again we would expect. The second case is the interesting one, it can be written $S_1 S_2 S_1 = S_2 S_1 S_2$ or including the dependencies

explicitly:

$$S_1(\lambda_2,\lambda_3)S_2(\lambda_1,\lambda_3)S_1(\lambda_1,\lambda_2) = S_2(\lambda_1,\lambda_2)S_1(\lambda_1,\lambda_3)S_2(\lambda_2,\lambda_3) \tag{2.63}$$

These are the famous Yang-Baxter equations and are the consistency relations we have been looking for.

To find the Bethe equations in the two component case we impose periodic boundary conditions, to do this we move to the transmission diagonal representation and the equation is:

$$e^{i\lambda_j L} S_{j,j-1} \dots S_{j,1} S_{j,N} \dots S_{j,j+1} \Psi = \Psi \tag{2.64}$$

The solution to this can be found using the algebraic Bethe ansatz technique which is very clearly explained in [37] and very thoroughly explained in [22]. We will not go into any detail here, but we will need the Bethe equations for the case of the Yang-Gaudin model with a single particle of the "blue" kind in a sea of "red" particles[41]:

$$L\lambda_j + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j + \theta(2\lambda_j - 2\xi) + \pi. \tag{2.65}$$

The n_j are the integer lattice that we described in the single component case, we now extend this picture to include the "blue" particle. We put the particles down on the integer lattice as before and then select one of these particles to be the "blue" one. The quasimomenta of this particle is ξ, note that we have not added a particle to the system just made one of them distinguishable from the rest, this will be important.

The ground state of the two component system is the same as the single component case, so changing the colour of one of the particles in the ground state is an excitation. This type of excitation we will call a spinon, it is sometimes called a magnon. The shift function,

momentum and energy of this excitation are found by the same process we used for the particle and hole excitations. The results are [41]:

$$\bar{F}(\lambda|\xi) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi} \bar{F}(\mu|\xi) = -\frac{\pi + \theta(2\lambda - 2\xi)}{2\pi} \tag{2.66}$$

$$k(\xi) = \int_{-q}^{q} d\mu \rho(\mu)(\pi + \theta(2\mu - 2\xi)), \tag{2.67}$$

$$\omega(\xi) = -\frac{1}{\pi} \int_{-q}^{q} d\mu \epsilon(\mu) K(2\mu - 2\xi). \tag{2.68}$$

Here ϵ and ρ are the energy and quasimomenta density defined in the single component case. These are the equations that describe the "spinon excitation", they are similar in form to the equations for the particle and hole excitations with a couple of notable exceptions. Firstly there are no bare terms in the momentum or the energy, this is because we have not added a new particle. All we have done is modify the way one of the particles affects the others. Additionally we see some extra factors of two appearing in the arguments of some functions. These will not cause us too much trouble but will force us to do our calculations separately for the spinon, where for the particle and hole it is easier to do one calculation and then use symmetries to get the result for the other excitation.

This brings the Bethe ansatz introduction to an end, we have seen how the Lieb-Liniger model is solved and now have the equations we want for this model and the Yang-Gaudin single spinon excitation case. Later we will use these equations to do much more complicated calculations, the techniques of resolvent operators, integration by parts and the Slavnov identity will continue to be our main tools though. In the following section and next two chapters we will review and then extend the theory of mobile impurities in one-dimension. These sections will not be very rigorous, but the will be much more widely applicable. We would like to check that the theory is reliable and we will return to the Bethe ansatz to do this later.

2.2 Bosonisation and conformal field theory

We now move away from exact results in favour of some phenomenology, we will see how we can use bosonisation to find a low energy effective theory for the phononic degrees of freedom in the one-dimension Bose gas. The effective theory is equivalent to a massless bosonic field and has a special symmetry which we will exploit, the conformal symmetry. This will allow us to calculate the types of correlation function we require in our later calculations. We will also see the restrictions that this symmetry imposes on the correlation functions. The universal description of gapless one-dimensional systems with linear dispersion is called the Luttinger liquid. We start with an introduction.

2.2.1 Luttinger liquids

Many one-dimensional models with a linear gapless spectrum belong to the same universality class, members of this class are known as Luttinger liquids. When we say two models are in the same universality class, we mean that much of their low energy behaviour can be described by the same effective theory, even though the models differ in their microscopic detail and even their statistics. Bosonisation is a method that allows us to find a suitable low energy effective theory. The name bosonisation is an artifact of the study of fermionic one dimensional systems where the collective excitations are described by bosonic fields. It is a little misleading as it gives the impression that it is not suitable for bosonic systems. This is not the case, in fact a strength of bosonisation is that fermions and bosons can be treated on the same footing. The differences in the statistics will show up in the correlation functions. In this section we will see how to derive the low energy effective theory, see how we can use the Bethe ansatz results to fix the parameters for exact models and motivate the following section on conformal field theories.

We begin with an introduction to bosonisation using the notation and results in the

excellent review by Cazalilla[6]. A one dimensional bosonic system can be described using second quantised field operators obeying the bosonic commutation relations $[\Psi(x), \Psi^\dagger(x')] = \delta(x - x')$. If we write the operators in polar form,

$$\Psi^\dagger(x) = \sqrt{\rho(x)} e^{i\phi(x)}. \tag{2.69}$$

Then the commutation relations of the original field operators imply that,

$$e^{i\phi(x')} \rho(x) e^{-i\phi(x')} - \rho(x) = \delta(x - x'). \tag{2.70}$$

Keeping in mind that we are looking for a low energy effective theory we split the fields into fast and slow parts,

$$\rho(x) = \rho_<(x) + \rho_>(x), \tag{2.71}$$

$$\phi(x) = \phi_<(x) + \phi_>(x). \tag{2.72}$$

The slow fields are the part of the fields that have fourier components below a cut off in energy and frequency. The cut-offs cannot generally be determined exactly and only the order is known. At low energies the density fluctuates around the mean value ρ_0 and it is useful to define a new operator by $\rho_<(x) = \rho_0 + \Pi(x)$. Introducing another new field $\Xi(x)$ related to the density by $\frac{1}{\pi}\partial_x \Xi(x) = \rho_0 + \Pi(x)$ we can think of $\Xi(x)$ as a field that takes the value $n\pi$ at the position of the nth particle. Using this we rewrite the density,

$$\rho(x) = \sum_i \delta(x - x_i) = \partial_x \Xi(x) \sum_{n=-\infty}^{\infty} \delta(\Xi(x) - n\pi). \tag{2.73}$$

Where x_i is the position operator of the ith particle. Using the Poisson summation formula,

$$\sum_{j=-\infty}^{\infty} f(j) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dz f(z) e^{2k\pi i z}, \tag{2.74}$$

we can write the density in the desired form,

$$\rho(x) = \frac{1}{\pi} \partial_x \Xi(x) \sum_{n=-\infty}^{\infty} e^{2ni\Xi(x)}. \tag{2.75}$$

The $n = 0$ term is just the slowly varying field we have described. The $n = \pm 1$ terms correspond to excitations with momentum $k \approx \pm 2\pi\rho_0$, these excitations involve a particle being excited from one Fermi surface to the other. This is a low energy excitation despite it not having low momentum.

In our new representation the field operators are:

$$\Psi^\dagger(x) \sim (\rho_0 + \Pi(x))^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{2ni\Xi(x)} e^{-i\phi(x)}. \tag{2.76}$$

We can construct the fermionic field operator Ψ_F by Jordon-Wigner transformation,

$$\Psi_F(x) = \Psi(x) e^{i\Xi(x)}. \tag{2.77}$$

In order to find an effective low energy theory we take the general Hamiltonian for a system of particles in a one-dimensional box length L interacting via pair potential and substitute the leading order terms in our expansions. The general Hamiltonian is,

$$H = \frac{\hbar^2}{2m} \int_0^L dx \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + \frac{1}{2} \int_0^L dx dy V(x-y) \rho(x) \rho(y). \tag{2.78}$$

Substituting in (2.75) and (2.76) the Hamiltonian takes the form,

$$H_{LL} = \frac{\hbar}{2\pi} \int_0^L dx \left(\nu_J (\partial_x \phi(x))^2 + \nu_N (\partial_x \Xi(x) - \pi\rho_0)^2 \right). \tag{2.79}$$

The parameters ν_N and ν_J are the phase and density stiffness respectively. In bosonisation there are almost as many conventions for choice of fields and parameters as there are articles. We will choose to use $\theta(x) = \Xi(x) - x\pi\rho_0$ and parameters $c = \sqrt{\nu_N \nu_J}$ and $K = \sqrt{\nu_J/\nu_N}$. c is the speed of sound and K is the Luttinger parameter which describes the strength of quantum fluctuations. In terms of these fields and parameters the Hamiltonian is,

$$H_{LL} = \frac{\hbar c}{2\pi} \int_0^L dx \left(K (\partial_x \phi(x))^2 + \frac{1}{K} (\partial_x \theta(x))^2 \right). \tag{2.80}$$

The parameters in the model can be fixed either experimentally, numerically or in some cases by comparison with exact solutions.

Next we will ask what happens if we apply periodic boundary conditions to the system. In terms of the original fields the boundary conditions are $\Psi^\dagger(x) = \Psi^\dagger(x + L)$, applying this to (2.76) we find,

$$\Xi(x + L) = \Xi(x) + \tilde{N}\pi \tag{2.81}$$

$$\phi(x + L) = \phi(x) + \tilde{J}\pi. \tag{2.82}$$

Where $\tilde{N}$ is the number operator and $\tilde{J}$ is an operator with even integer eigenvalues. Both of these operators commute with the Hamiltonian and therefore they are conserved. These eigenvalues label the distinct topological states the system can be in and the tilde is used to distinguish them from the N and J variables we will meet in the depleton theory. The similarity in the notation is deliberate.

We will now switch to the Lagrangian representation to discuss the correlation functions

of the model. Making the usual Legendre transformation to the Lagrangian and integrating over time we get the action:

$$S = \frac{1}{2\pi} \int dx dt \left[-\partial_x \theta \partial_t \phi - \frac{c}{K} (\partial_x \theta(x))^2 - cK (\partial_x \phi(x))^2 \right]. \tag{2.83}$$

We have set $\hbar = 1$ for convenience and will now Wick transform to imaginary time $\tau = it$, the action is:

$$S = \frac{1}{2\pi} \int dx d\tau \left[-i\partial_x \theta \partial_\tau \phi - \frac{c}{K} (\partial_x \theta(x))^2 - cK (\partial_x \phi(x))^2 \right]. \tag{2.84}$$

Having got to the low energy theory we want, we could use standard techniques to calculate the correlation functions [12], but we choose to use another method. The method of choice is to use the conformal symmetry to find the correlation functions and also connect the exponents in the singularities tot the finite size correction to the energy. This will allow us the match quantities from the Bethe ansatz with the parameters in our effective impurity model.

2.2.2 Correlation functions of a Gaussian model

We want to understand how to calculate correlation functions of the low energy model, we will start by calculating the correlation functions of a massless bosonic field Θ. The reason we do this is that it leads to the simplest calculation of correlation functions and can be related to (2.83) very easily. It will also allow us to see how the properties of the correlation functions of the massless bosonic field correspond to the properties of our low energy model. To begin with we will follow the development in Gogolin, Nersesyan and Tsvelik [13]; the

action is:

$$S = \frac{1}{2}\int_R dx d\tau \left[c^{-1}(\partial_\tau \Theta)^2 + c(\partial_x \Theta)^2\right]. \tag{2.85}$$

R can be an arbitrary region of the (x, τ) plane, but to begin with this is just the usual $R : 0 < x < L, 0 < \tau < \beta$ with β the inverse temperature which is infinite in all cases we consider. To calculate correlation functions we continue in the usual way and define the generating functional:

$$Z[\eta] = \int_R D\Theta e^{iS+\eta(\mathbf{x})\Theta(\mathbf{x})}. \tag{2.86}$$

This integral can be done by Fourier transforming the field and η and shifting the field to remove the linear term leaving a Gaussian integral. The result is:

$$Z[\eta]/Z[0] = \exp\left[\frac{1}{2}\eta(\xi)G(\xi,\xi')\eta(\xi')\right]. \tag{2.87}$$

Here the Green's function is defined as,

$$-(c^{-1}\partial_\tau^2 + c\partial_x^2)G(x,\tau;x',\tau') = \delta(x-x')\delta(\tau-\tau'). \tag{2.88}$$

The result is easily generalised to arbitrary R by simply substituting the corresponding Green's function. It is very useful to introduce the complex co-ordinates $z = \tau + ix/c$ and $\bar{z} = \tau + ix/c$ at this point. When we are considering a large system of size L we find the Green's function is:

$$G(z,\bar{z}) = \frac{1}{4\pi}\ln\left(\frac{L}{z\bar{z}+a^2}\right), \tag{2.89}$$

a is a short distance cut-off which is required to stop ultra-violet divergences appearing, as we will see in a moment. The correlation functions we will be interested in later on are those of exponents of bosonic fields. These are particularly easy to construct from our generating functional, if we choose:

$$\eta(\xi) = i\sum_{j=1}^{N} \beta_j \delta(\xi - \xi_j), \tag{2.90}$$

then using (2.87) we get:

$$< \exp(i\beta_1\Theta(\xi_1)) \dots \exp(i\beta_N\Theta(\xi_N)) >= \exp\left[-\sum_{j<k} \beta_j\beta_k G(\xi_j,\xi_k)\right] \exp\left[-\frac{1}{2}\sum_j \beta_j^2 G(\xi_j,\xi_j)\right]. \tag{2.91}$$

It should be clarified that the β_j are not related to the inverse temperature. Substituting (2.88) we find that:

$$< \exp(i\beta_1\Theta(\xi_1)) \dots \exp(i\beta_N\Theta(\xi_N)) >= \prod_{j<k} \left(\frac{z_{jk}\bar{z}_{jk}}{a^2}\right)^{(\beta_j\beta_k/4\pi)} \left(\frac{L}{a}\right)^{-(\sum_j \beta_j)/4\pi}. \tag{2.92}$$

In all of the cases we deal with $\sum_j \beta_j = 0$ so the L/a term is one, we also notice that the correlation function can be factorised leading to:

$$< \exp(i\beta_1\Theta(\xi_1)) \dots \exp(i\beta_N\Theta(\xi_N)) >= \prod_{j<k} \left(\frac{z_{jk}}{a}\right)^{(\beta_j\beta_k/4\pi)} \prod_{j<k} \left(\frac{\bar{z}_{jk}}{a}\right)^{(\beta_j\beta_k/4\pi)}. \tag{2.93}$$

This factorisation may seem innocuous, but combined with the knowledge that the bosonic exponents form a basis it tells us that when we compute correlation functions we can treat the analytic and anti-analytic parts of the fields completely independently. Or order to make this explicit we split the Θ field into it's analytic and anti-analytic parts $\Theta(z,\bar{z}) = \chi_+(z) + \chi_-(\bar{z})$.

We define the field dual to Θ by:

$$\Phi(z,\bar{z}) = \chi_+(z) - \chi_-(\bar{z}), \tag{2.94}$$

Now we are ready to explain the connection with (2.83), the fields Θ and Φ are scaled versions of θ and ϕ:

$$\frac{\theta}{\pi} = \sqrt{\frac{K}{2\pi}}\Theta, \tag{2.95}$$

$$\phi = \sqrt{\frac{\pi}{2K}}\Phi. \tag{2.96}$$

The actions are identical to each other once this identification is made.

We want to calculate exponents of fields using the factorisation and therefore define the analytic and anti-analytic bosonic exponents:

$$A(\beta,z) \equiv \exp\left[\frac{i}{2}\beta(\Phi(z,\bar{z}) + \Theta(z,\bar{z}))\right] = \exp(i\beta\chi_+(z)), \tag{2.97}$$

$$\bar{A}(\bar{\beta},\bar{z}) \equiv \exp\left[\frac{i}{2}\bar{\beta}(\Phi(z,\bar{z}) - \Theta(z,\bar{z}))\right] = \exp(i\bar{\beta}\chi_-(\bar{z})). \tag{2.98}$$

Which make calculation of the pair correlation function easy, it is:

$$\begin{aligned} &< A(\beta,z_1)\bar{A}(\bar{\beta},\bar{z}_1)A(-\beta,z_2)\bar{A}(-\bar{\beta},\bar{z}_2) > \\ &= (z_{12})^{-\beta^2/4\pi}(\bar{z}_{12})^{-\bar{\beta}^2/4\pi} = \frac{1}{|z_{12}|^{2d}}\left(\frac{z_{12}}{\bar{z}_{12}}\right)^S. \end{aligned} \tag{2.99}$$

We have introduced some new notation here d and S are called the scaling dimension and

the conformal spin respectively, they can be written as:

$$d = \frac{1}{8\pi}(\beta^2 + \bar{\beta}^2) = \Delta + \bar{\Delta}, \tag{2.100}$$

$$S = \frac{1}{8\pi}(\beta^2 - \bar{\beta}^2) = \Delta - \bar{\Delta}. \tag{2.101}$$

Δ and $\bar{\Delta}$ are called the conformal dimensions. More interesting than the new notation is noticing that the branch cut singularities in (2.99) will only cancel out to give a uniquely defined correlation function if:

$$\beta^2/4\pi - \bar{\beta}^2/4\pi = 2S = integer. \tag{2.102}$$

This means that the only uniquely defined, and therefore physical, fields have either integer or half integer conformal spin. This requirement can also be written in terms of coefficients of the θ/π and ϕ fields. If we start with the exponent:

$$\exp i \left(J\frac{\theta}{\pi} + N\phi \right), \tag{2.103}$$

we can calculate $2S$ using the rescaling (2.96) . We find that the requirement for a uniquely defined function is:

$$\frac{NJ}{2\pi} = integer. \tag{2.104}$$

We are currently unsure how to interpret this, or even whether it is significant. We would like to better understand the implications of the analytic structure of the correlation function.

In this section we have seen how to calculate correlation functions of a Gaussian model when the region R is the entire plane. The pair correlation function will be very important

later on.

$$< A(\beta, z_1)\bar{A}(\bar{\beta}, \bar{z}_1)A(-\beta, z_2)\bar{A}(-\bar{\beta}, \bar{z}_2) > \tag{2.105}$$

$$= (z_{12})^{-2\Delta}(\bar{z}_{12})^{-2\bar{\Delta}} \tag{2.106}$$

The exponents are:

$$\Delta = \beta^2/8\pi, \tag{2.107}$$

$$\bar{\Delta} = \bar{\beta}^2/8\pi. \tag{2.108}$$

In the next section we will see how to use the conformal symmetry to write the conformal dimensions in terms of finite size contributions to the energy. Later we will use the Bethe ansatz to calculate these conformal dimensions for the integrable models we have discussed.

2.2.3 Conformal symmetry

At the beginning of the last section we wrote the action for the Gaussian model defined on an arbitrary region R of the (x, τ) plane. We then continued assuming that we meant a large rectangle. However, (2.91) is valid for any R provided we substitute the correct Green's function, there is a trick for doing this. If we know the mapping $z(\xi)$ that takes the region R to the entire plane then it can be shown that [13]:

$$G(\xi_1, \xi_2) = -\frac{1}{2\pi}\ln|z(\xi_1) - z(\xi_2)| - \frac{1}{4\pi}\ln(|\partial_{\xi_1} z(\xi_1)\partial_{\xi_2} z(\xi_2)|). \tag{2.109}$$

Substituting this into (2.91) we can write the analytic part of the pair correlation function, to get the full correlation function we just multiply by the anti-analytic part:

$$D(\xi_1,\xi_2)=\frac{1}{(z(\xi_1)-z(\xi_2))^{2\Delta}}(\partial_{\xi_1}z(\xi_1)\partial_{\xi_2}z(\xi_2))^{\Delta}. \tag{2.110}$$

We want to relate the correlation functions in an infinite system $R:-\infty<x<\infty;-\infty<\tau<\infty$ to those in a finite system $R:0<x<L;-\infty<\tau<\infty$ where both are at $T=0$. The relevant mapping that takes us from the infinite plane to the infinite strip is:

$$z(\xi)=\exp(2\pi\xi/L), \tag{2.111}$$

$$\bar{z}(\bar{\xi})=\exp(2\pi\bar{\xi}/L). \tag{2.112}$$

Using this we can write the full pair correlation function on the infinite strip:

$$D(\xi_1,\xi_2)\bar{D}(\xi_1,\xi_2)=\left(\frac{\pi/L}{\sinh[\pi(\xi_1-\xi_2)/L]}\right)^{2\Delta}\left(\frac{\pi/L}{\sinh[\pi(\bar{\xi}_1-\bar{\xi}_2)/L]}\right)^{2\bar{\Delta}}. \tag{2.113}$$

If we take the limit that the separation is large then (2.113) becomes:

$$D(\xi_1,\xi_2)\bar{D}(\xi_1,\xi_2)=\left(\frac{\pi}{L}\right)^{\Delta+\bar{\Delta}}\exp\left[-\frac{2\pi}{L}(\xi_1-\xi_2)\Delta\right]\exp\left[-\frac{2\pi}{L}(\bar{\xi}_1-\bar{\xi}_2)\bar{\Delta}\right]. \tag{2.114}$$

Using $\xi_1=\tau_1+ix_1$ and $\xi_1=\tau_2+ix_2$ we can find the dependence on

$$D(\xi_1,\xi_2)\bar{D}(\xi_1,\xi_2)=\left(\frac{\pi}{L}\right)^{\Delta+\bar{\Delta}}\exp\left[-\frac{2\pi}{L}\tau_{12}(\Delta+\bar{\Delta})\right]\exp\left[-\frac{2\pi}{L}ix_{12}(\Delta-\bar{\Delta})\right]. \tag{2.115}$$

To connect the conformal dimensions to the finite size energy corrections we write the

correlation function in the Lehmann representation:

$$D(\tau_{12}, x_{12})\bar{D}(\tau_{12}, x_{12}) = \sum_n |<0| \exp\left[\exp(i\beta\chi(0) + i\bar{\beta}\bar{\chi}(0))\right] |n>|^2 \exp[-E_n\tau_{12} - iP_n x_{12}], \tag{2.116}$$

where E_n and P_n are the energy and momentum of the state $|n>$. The main point here isn't the precise relationship between the exponents, but that the finite size corrections are functions of the conformal dimensions of the operators. In practice we will compare the finite size corrections of different models and match parameters between the two. This is how we will first connect the Bethe ansatz equations with the parameters of the effective theory.

We now know how to calculate certain correlation functions of the Luttinger liquid, we have also seen that the finite size contributions to the energy can be related to these via the conformal transformation. In the next chapters we will see why it is these correlation functions that are needed to calculate the spectral function and dynamic structure factor near the edges.

CHAPTER 3

MOBILE IMPURITIES AND THE DEPLETON

In the previous chapter we saw how bosonisation and conformal field theory can be used to describe the low energy dynamics of one-dimensional systems. In this chapter we will be concerned with a mobile impurity moving through the system. The depleton model of *Schecter et. al.* [34] describes how the treat the interaction between the impurity and phonons in the system, this chapter is essentially a review of the part of this work we will use. The depleton is a quasiparticle made up of the impurity and the non-linear distortion of the background fluid that accompanies it. Key to understanding the interaction between the depleton and the phononic excitations is the separation of length scales. The relevant length scale over which the non-linear distortion falls to zero is the healing length of the background liquid $\xi = 1/mc$ which is much less than the wavelength of the phonons $\sim 1/\rho_0$. This large separation of scales means that we can treat the depleton as point like in it's interactions with the phonons. The interaction of a general point like scatterer with the phonons is dependent on only two phase shifts in the phononic fields which can be parameterised by N and J the number of particles removed from the background and the phase drop across the scatterer.

We begin with the hydrodynamic description of the system [33], the Lagrangian is:

$$L_0(\mu, n) = \int dx(\mu n - e_0(n)), \tag{3.1}$$

Figure 3.1: Impurity propagating in local environment [34]

where μ is the chemical potential, n is the density and e_0 is the energy density. At equilibrium the density is fixed by the chemical potential and we can write the grand canonical potential,

$$\Omega(\mu) = -L_0(\mu, n(\mu)). \tag{3.2}$$

Introducing an impurity moving with $V = \dot{X}$ into the system and transforming to the co-moving frame we see that the impurity will experience a modified chemical potential and superfluid velocity. Using the Galilean invariance we find these to be:

$$j' = -nV \quad \mu' = \mu + mV^2/2. \tag{3.3}$$

The grand canonical potential is affected by the impurity, in the co-moving frame the extra

contribution due to the impurity is:

$$\Omega'_d(j', \mu') = E'_d - \mu' N_d, \tag{3.4}$$

where the energy E'_d in the co-moving frame can be related to the lab frame by $E'_d = E_d - P_d V + m N_d V^2/2$. This leads to the expression for the Lagrangian of the depleton:

$$L_d(V, n) = P_d V - E_d + \mu N_d = -E'_d + \mu' N_d = -\Omega'_d(j', \mu'). \tag{3.5}$$

3.1 N and J

We now introduce the N and J variables as the linear response to variations of μ' and j',

$$d\Omega'_d(j', \mu') = J dj' + N d\mu', \quad J = \partial_{j'}\Omega', \quad N = \partial_{\mu'}\Omega'. \tag{3.6}$$

In equilibrium the N and J are fixed by μ' and j', but in the depleton model they become independent variables. This, we will see shortly, will be how we describe the interaction with phonons. To use this idea we Legendre transform to:

$$H_d(J, N) = \Omega'_d - j' J - \mu' N, \tag{3.7}$$

and then write the Lagrangian for the impurity as:

$$L = \frac{1}{2} M V^2 - j' J - \mu' N - H_d(J, N). \tag{3.8}$$

If we now express j' and μ' in terms of V and n then we get the Lagrangian in the form we want:

$$L = \frac{1}{2}(M - mN)V^2 + nVJ - \mu N - H_d(J, N). \tag{3.9}$$

The momentum can be defined by the standard formula:

$$P = \partial_V L = (M - mN)V + nJ, \tag{3.10}$$

which we rearrange for the velocity,

$$V(P, J, N) = \frac{P - nJ}{M - mN}. \tag{3.11}$$

Combining (3.9),(3.10) and (3.11) we get to the Hamiltonian expressed in the variables we want:

$$H(P, J, N) = \frac{1}{2}\frac{(P - nJ)^2}{M - mN} + \mu N + H_d(J, N) \tag{3.12}$$

As we have already stated, in the equilibrium case the N and J are fixed, we want to find these values for a given momentum and density of the system so that we can relate the equilibrium Hamiltonian to the dispersion of the impurity,

$$H(P, J_0(P, n), N_0(P, n)) = \frac{1}{2}\frac{(P - nJ_0)^2}{M - mN_0} + \mu N_0 + H_d(J_0, N_0) = E(P, n). \tag{3.13}$$

The velocity can be written in terms of the dispersion as $V_0 = \partial_P E(P, n)$ which gives us the first of the equation we need to fix the equilibrium values N_0 and J_0:

$$V_0(P, n) = \frac{P - nJ_0}{M - mN_0}. \tag{3.14}$$

The other equation comes from differentiating (3.12) with respect to n, this gives us:

$$\partial_n E(P, n) = \partial_n \mu N_0 - V_0(P, n) J_0. \tag{3.15}$$

These equations define the equilibrium values N_0 and J_0 in terms of the derivatives of the dispersion of the impurity, explicitly these relationships are:

$$N_0 = \frac{1}{m(c^2 - V^2)} \left(V(P - MV) + n\partial_n E\right) \tag{3.16}$$

$$J_0 = \frac{1}{m(c^2 - V^2)} \left(\frac{mc^2}{n}(P - MV) + mV\partial_n E\right). \tag{3.17}$$

Where we have written V we mean $\partial_P E$ and we have used $\partial_n \mu = mc^2/n$. As we will use these equilibrium N and J we will drop the subscripts from now on.

3.2 Interaction with phonons

We also want to understand how the impurity will interact with the phononic degrees of freedom. We will use the approach of Popov [33] and describe the dynamics by slowly varying density $\rho = \partial_x \phi/m$ and velocity $u = \partial_x \theta/\pi$ fields, along with a modification of the chemical potential given by:

$$\mu \to \mu - \partial_t \phi - \frac{mu^2}{2}. \tag{3.18}$$

Moving to the co-moving frame the j' and mu' that the impurity experiences are:

$$\mu' = \mu - \partial_t\phi - \frac{mu^2}{2} + \frac{m(V-u)^2}{2} = \mu + \frac{MV^2}{2} - (\partial_t\phi + V\partial_x\phi), \tag{3.19}$$

$$j' = -(n+\rho)(V-u) = -nV - \frac{1}{\pi}(\partial_t\theta + V\partial_x\theta). \tag{3.20}$$

We have used the continuity equation $\partial_t\theta = -\pi(n+\rho)u$ to rewrite the j' term. The extra parts of the local chemical potential and local super current can both be written as full time derivatives of the θ and ϕ fields at the position of the impurity as,

$$\mu' = \mu + \frac{MV^2}{2} - \frac{d}{dt}\phi(X,t), \tag{3.21}$$

$$j' = -nV - \frac{1}{\pi}\frac{d}{dt}\theta(X,t). \tag{3.22}$$

Substituting these into (3.9) we can see that the interaction part of the Lagrangian that we generate is,

$$L_{int} = N\frac{d}{dt}\phi(X,t) + \frac{J}{\pi}\frac{d}{dt}\theta(X,t). \tag{3.23}$$

We will use this expression to two different ways, sometimes we will use it to tell us the source terms that should appear in the phononic action due to a fixed trajectory of an impurity. Later we will see that this will allow efficient calculation of some dynamic correlation functions in one-dimensional systems. Alternatively we can think of N and J as dynamic variables conjugate to the phononic fields at the position of the impurity. This is the approach used by *Schecter et al.* [34] to study the dynamics and Bloch oscillation of impurities. Combining (3.23) and (3.12) we can write the full Lagrangian describing the mobile impurity interacting with the phonons:

$$L_{total} = P\dot{X} - H(P,J,N) + L_{int} + L_{phonons}. \tag{3.24}$$

This is the expression we will use to calculate the correlation functions in the next chapter.

3.3 Frictional force

At finite temperature the fluctuations of the background can cause a viscous friction force, the form of this frictional force has been calculated [34]. It is given by:

$$\dot{P} = F_{friction} = -\frac{16\pi^3}{15}|\Gamma_{\pm}|^2\frac{T^4}{c^2}\left(\frac{c^2+V^2}{c^2-V^2}\right)V. \tag{3.25}$$

The amplitude $\Gamma_{\pm}$ contains the information about the impurity dispersion it is defined as:

$$\Gamma_{\pm}(P,n) = -\frac{1}{c}\left[\frac{M}{m}\partial_P J + J\partial_P N - N\partial_P J + \partial_n N\right]. \tag{3.26}$$

We expect that the integrable excitations should not decay via phonons and because of this we expect that for the integrable cases $\Gamma_{\pm} = 0$. This was shown for the grey soliton, which is an excitation of the the classical integrable Gross-Pitaevskii equation. We will show that $\Gamma_{\pm}$ exactly vanishes for both particle and hole excitations of the Lieb-Liniger model and the spinon in the Yang-Gaudin case.

The depleton model has been used to describe the dynamics of mobile impurities in one-dimension. In the next chapter we show how to use the depleton model to calculate dynamic correlation functions. The back scattering amplitude Γ will reappear later too. We will use the Bethe ansatz equations to exactly calculate Γ as defined (3.26) starting from the exact equations for the dispersion.

CHAPTER 4

CORRELATION FUNCTIONS NEAR THE EDGE

So far when we have considered mobile impurities in one-dimensional systems we have had in mind that the relevant experiments would involve individual excitations traveling through the system. There is at least one other reason for studying mobile impurities and that is they can be used to construct effective theories to calculate dynamic response functions. This area of application is related to the so called X-ray edge problem and has borrowed many ideas from this work. The approach used by many people is to write down an effective mobile impurity Hamiltonian coupled to a Luttinger liquid[16] and using the method of Schotte and Schotte [35] remove the interaction term with a unitary transformation. The parameters of the model can be related to the dispersion of the impurity in Galilean invariant systems[16][2] and to the finite size scaling of the spectrum using conformal field theory[41]. Using the conformal field theory results the singularities at the edge of support have been calculated for the Lieb-Liniger [15] and Yang-Gaudin[41] systems. In this chapter we will extend the depleton theory to calculate these dynamic response functions and find the same results as have been found before. We will see that the connection between the different correlation functions can be understood naturally in the language of the depleton.

The dynamic response functions we are interested in are the dynamic structure factor

(DSF) and the spectral function. The DSF, defined as:

$$S(q,\omega) = \int dxdxe^{-iqx}e^{i\omega t} < \rho(x,t), \rho(0,0) >, \tag{4.1}$$

and is the linear response due to an external field that couples to the density. The spectral function $A(q,\omega) = -\frac{1}{\pi}\Im G(q,\omega)$ with,

$$G(q,\omega) = -i \int dxdte^{-iqx}e^{i\omega t} < T(\psi(x,t)\psi^{\dagger}(0,0)) >, \tag{4.2}$$

is proportional to the probability of a particle with energy ω and momentum q tunneling into the system. The aim is to calculate the singularities in the dynamic response functions they will be of the form:

$$S(q,\omega), A(q,\omega) \sim const + \left| \frac{1}{\omega \pm \epsilon_{1(2)}(q)} \right|^{\mu} \tag{4.3}$$

With different values of μ for different correlation functions and for each type of excitation. We will use μ_1 for the particle edge in the DSF and μ_+ for the particle edge in the spectral functions as examples.

4.1 The X-ray edge approach

The X-ray edge approach to the problem is to start with an effective model for the system[17], this is a high energy particle interacting with a phononic background. Already we can see

similarities with the depleton theory. The Hamiltonian is $H = H_0 + H_d + H_{int}$ where,

$$H_0 = \frac{c}{2\pi} \int dx \left[K(\nabla\theta)^2 + \frac{1}{K}(\nabla\phi)^2 \right], \tag{4.4}$$

$$H_d = \int dx d^\dagger(x) \left[\epsilon(k) - iv_d \frac{\partial}{\partial x} \right] d(x), \tag{4.5}$$

$$H_{int} = \int dx \left(V_R \nabla \frac{\theta - \phi}{2\pi} - V_L \nabla \frac{\theta + \phi}{2\pi} \right) d(x) d^\dagger(x). \tag{4.6}$$

Where $d^\dagger(x)$ creates a particle with momentum k and velocity v_d. In direct analogy with the Fermi edge case the interaction term can be removed by unitary transformation. The H_{int} term is removed by the unitary transformation $U^\dagger H U$, with

$$U^\dagger = e^{-i \int dx \left(\frac{\delta_+(k)}{2\pi} [\tilde{\theta}(x) - \tilde{\phi}(x)] - \frac{\delta_-(k)}{2\pi} [\tilde{\theta}(x) + \tilde{\phi}(x)] \right) d(x) d^\dagger(x)}.$$

Here $\phi = \tilde{\phi}\sqrt{K}$ and $\theta = \tilde{\theta}/\sqrt{K}$ and the $\delta_\pm$ are related to $V_{L,R}$ by

$$\begin{aligned} (V_L - V_R)/\sqrt{K} &= -\delta_-(v_d + v) + \delta_+(v_d - v), \\ (V_L + V_R)\sqrt{K} &= -\delta_-(v_d + v) - \delta_+(v_d - v). \end{aligned} \tag{4.7}$$

By considering a uniform density variation and then separately a uniform current variation Imambekov and Glazman [16] were able to find a model independent relation between the $\delta_\pm$ and derivatives of the dispersion, it is

$$\frac{\delta_\pm(k)}{2\pi} = \frac{\frac{1}{\sqrt{K}} \left(\frac{k}{m} - \frac{\partial\epsilon(k)}{\partial k} \right) \pm \sqrt{K} \left(\frac{1}{\pi} \frac{\partial\epsilon(k)}{\partial n} + \frac{v}{K} \right)}{2 \left(\pm \frac{\partial\epsilon(k)}{\partial k} - v \right)}. \tag{4.8}$$

Equation (4.8) can be split into two parts, one of which does not depend on the details of the impurity.

$$\frac{\delta_{\pm}(k)}{2\pi} = \frac{\frac{1}{\sqrt{K}}\left(\frac{k}{m}\right) \pm \sqrt{K}\left(\frac{1}{\pi}\frac{\partial\epsilon(k)}{\partial n}\right)}{2\left(\pm\frac{\partial\epsilon(k)}{\partial k} - v\right)} \mp \frac{1}{2\sqrt{K}}, \tag{4.9}$$

Kamenev and Glazman [19] have derived similar equations for the spinor Bose liquid.

Once the interaction term has been removed using (4.7) the singularities can be calculated using standard techniques [12], the examples we give are:

$$\mu_1 = 1 - \frac{1}{2}\left(\frac{1}{\sqrt{K}} + \frac{\delta_+ - \delta_-}{2\pi}\right)^2 - \frac{1}{2}\left(\frac{\delta_+ + \delta_-}{2\pi}\right)^2, \tag{4.10}$$

$$\mu_+ = 1 - \frac{1}{2}\left(\frac{\delta_+ - \delta_-}{2\pi}\right)^2 - \frac{1}{2}\left(\frac{\delta_+ + \delta_-}{2\pi}\right)^2. \tag{4.11}$$

They are similar in form but differ by $1/\sqrt{K}$, the depleton variables make it very easy to see the reasons for the extra terms that appear .

4.2 The depleton approach

Now we will extend the depleton theory to calculate the results found using the unitary transformation to remove the interaction term. If ω and q are much larger than other energy and momentum scales in the problem then we can look for a saddle point that will give us the leading contribution to the DSF and spectral function. Here we will follow the work of Iordanskii and Pitaevskii closely[18]. In the functional integral representation we can write the DSF as:

$$S(q,\omega) = \int dX dT D(\rho) D(\phi) e^{-iqX} e^{i\omega T} \rho(X,T)\rho(0,0) e^{iS}, \tag{4.12}$$

where S in the exponential is general action for a Bose field,

$$S = \int dxdt\rho\partial_t\phi - \int dtH(\rho,\phi). \tag{4.13}$$

ϕ is canonically conjugate to ρ. We split the integration over $\rho(x,t')$ and $\phi(x,t')$ into three parts,

$$I. \quad -\infty < t < 0 \tag{4.14}$$

$$II. \quad 0 < t < T \tag{4.15}$$

$$III. \quad T < t < \infty. \tag{4.16}$$

The fields ρ, ϕ are zero at infinity and because of this the Hamiltonian is zero on the I and III sections of the action. We now look for the saddle point by varying the exponential in the integral, first we vary with respect to T which gives,

$$i\omega + i\partial_T S_{II} + i\partial_T S_{III} = 0. \tag{4.17}$$

Using $\partial_t S_{III} = H_{III} = 0$ we find $-\partial_t S_{II} = H_{II} = \omega$ we find that the energy is equal to ω on part II of the trajectory, an expected result. Next a variation of $\rho(0,0)$ and $\rho(x,t)$ a carried out,

$$\rho(x,t) \to \rho_0(x+a,t), \quad \rho(x,0) \to \rho_0(x+b,0). \tag{4.18}$$

Where a and b are small constants, it was shown[18] that this variation will tell us that the momentum is equal to q on part II of the contour. The main contribution to the DSF is given by a trajectory with energy and momentum ω and q.

We have seen that the DSF and spectral function are dominated by configurations with

fixed energy and momentum. There is a special case when ω and q are such that they lie very close to the particle or hole excitation spectra; then the configuration must consist of the particle or hole excitation plus some low energy excitations. We can use this idea to calculate the DSF and spectral function near the excitation spectra by choosing a high energy trajectory satisfying the energy and momentum conditions and using the depleton model to couple it to the low energy states. The trajectory we want is that of a high energy excitation that has momentum q and appears at $(0,0)$ and is removed at (X,T). Using this trajectory and the depleton Langrangian we arrive at the action:

$$S = qX - E(q)T + \frac{J(q)}{\pi}\dot{\theta}(X,T) + N\dot{\phi}(X,T) + \int dt L_{ph}. \tag{4.19}$$

In fixing the trajectory of the high energy excitation we have also fixed the $\Phi(q)$and $N(q)$ coupling parameters and the relationship $X = VT$ with $V = \partial_k E(k)$ the velocity of the excitation. The full time derivative of the $\phi(X,T)$ and $\theta(X,T)$ can be broken into chiral components:

$$\frac{\theta(X,T)}{\pi} = \sqrt{\frac{K}{\pi}}(\chi_+(X,T) + \chi_-(X,T)), \tag{4.20}$$

$$\phi(X,T) = \sqrt{\frac{\pi}{K}}(\chi_+(X,T) - \chi_-(X,T)). \tag{4.21}$$

The interaction term then becomes:

$$\begin{aligned}\frac{\Phi(q)}{\pi}\dot{\theta}(X,T) + N\dot{\phi}(X,T) =& \frac{\Phi(q)}{\pi}\sqrt{\frac{K}{\pi}}(\dot{\chi}_+(X,T) + \dot{\chi}_-(X,T)) \\ &+ N\dot{\phi}(X,T)\sqrt{\frac{\pi}{K}}(\dot{\chi}_+(X,T) - \dot{\chi}_-(X,T)).\end{aligned} \tag{4.22}$$

We can now use the property of the chiral fields $\chi_\pm(x,t) = \chi_\pm(x \mp ct)$ and calculate the full time derivative for the field evaluated at $X = VT$, we get $\dot{\chi}_\pm(X,T) = (V \mp c)\partial_x \chi_\pm(X,T)$.

Substituting this in we get the same form of interaction as the Fermi edge case. We can identify the parameters in each case by comparing coefficients or by comparing the definitions from the dispersion, either way we get:

$$N = -\sqrt{K}\frac{\delta_+ - \delta_-}{2\pi} \tag{4.23}$$

$$\frac{J}{\pi} = -\sqrt{\frac{1}{K}}\frac{\delta_+ + \delta_-}{2\pi}. \tag{4.24}$$

The exponent μ can be written in terms of the N and J variables:

$$\mu = 1 - \frac{1}{2K}N^2 - \frac{K}{2}\left(\frac{J}{\pi}\right)^2. \tag{4.25}$$

We have calculated the singularity in the DSF and the spectral function, but we have a problem; the divergence is apparently the same for both cases. We need to understand what has gone wrong. The difference we are looking for appears because the operators appearing in the DSF does not change the number of particles in the system but the field operator in the Green's function does. We can now see that when we introduce the source terms we should consider whether the operator moves us to a different topological sector of the low energy model or not. We could also choose ω and q such that q is not in the region $0 < q < 2\pi n$ then the excitation is moving the system into a different topological sector corresponding to some amount of flux through the system.

Let's go through some examples to illustrate the idea. The exponent μ_+ for the particle edge in the spectral function is correct because the particle excitation changes the number of particles by one and so does the operator in the correlation function. However μ_1 for the same edge in the DSF does need correcting because the density operator does not change

particle number. In this case the exponent is:

$$\mu_1 = 1 - \frac{1}{2K}(N-1)^2 - \frac{K}{2}\left(\frac{J}{\pi}\right)^2, \tag{4.26}$$

where we reduce N by one to correct for the discrepancy. The advantage of this method is that the original formula is always the same we just need to find the right correction terms. If we want the exponent for the DSF at the hole edge then we should add one to N. The general method is this; the edge that you are interested in defines the type of excitation the N and J are calculated from. You then have to compare the number of particles added by that type of excitation with the number of particles added by the operator in the correlation function and correct for any difference by changing N.

The spectrum is periodic in momentum, if we are outside the region $0 < q < 2\pi n$ then the exponent will change. This effect has been dealt with in the Fermi edge approach [17], in this case the depleton language explains the connection between the different topological sectors. All you have to do is calculate how many multiples of $2\pi n$ have been added and add the same number of multiples of 2π onto J. For example, if we want the particle edge exponent in the region $2\pi n < q < 4\pi n$ then it is:

$$\mu_1 = 1 - \frac{1}{2K}(N-1)^2 - \frac{K}{2}\left(\frac{J}{\pi} + 2\right)^2, \tag{4.27}$$

where we have the same correction to N as previously and have added 2π to J. These extra terms are precisely the terms that appear in the X-ray edge approach that we mentioned earlier. Using the depleton variables makes it easy to see how the exponents are connected and why.

4.3 Concluding remarks

This chapter explains the relationship between the depleton theory and the mobile impurity model. We have seen that the mobile impurity model can be derived from the depleton model. More importantly we have seen that the language of the depleton model makes connections between different correlation functions transparent. The simplicity of the connections between the different exponents is due to the underlying symmetry and gauge invariance of the problem. Because of the gauge invariance of the model the shifts due to the impurity and changes in the topological sector can be treated equivalently. This leads to a simple recipe for finding the correlation functions near an excitation spectrum. We should use the dispersion associated with that impurity to calculate the N and J variables and then correct for any disparity between the number of particles and flux added to the system by the impurity and the operator in the correlation function.

CHAPTER 5

EXACT RESULTS

5.1 Introduction

In this chapter we will see how we can use the Bethe ansatz to give us confidence in the depleton theory of mobile impurities and dynamic correlation functions. We do this by taking equalities from the effective and in general non-integrable theory and showing they hold exactly for the integrable cases. The first case we deal with are the N and J variables that appear in the exponents of the divergence near the single particle excitations. We have defined N and J in terms of derivatives of the dispersion, we also saw how they should be connected with the finite size spectra from the conformal symmetry. We will calculate the finite size spectra of the integrable models and by matching the results see that the N and J can be written in terms of the Bethe ansatz shift functions evaluated at the Fermi surface. This method was pioneered by Periera *et al.* [32] and complements and argument given by Cheianov and Pustilnik[7] and Imambekov and Glazman[16]. This means we have two different definitions of the same quantities and suggests that there should be a way of expressing the derivatives of the energy in terms of the Fermi surface shifts. We will take the expressions for the dispersion from the Bethe ansatz and show that the proposed relationship with the Fermi surface shifts is an exact identity. This should give us great confidence in the

effective theory, even when the model we apply it to is non-integrable.

Another quantity that is derived in the depleton model is the amplitude for phonon interaction $\Gamma_{\pm}$. Similarly to the case of the grey soliton we would expect that $\Gamma_{\pm} = 0$ for excitations of the integrable models. $\Gamma_{\pm}$ is defined in terms of derivatives of the N and J functions, again in the integrable cases we can calculate these exactly. We will see that they, rather non-trivially, vanish as expected. This gives us confidence that the interaction between the depleton and phonons can be trusted.

We would like to jump straight to the calculation of the derivatives of the energy, because these are the quantities that turn up in our description of the coupling to phonons. It turns out that to do these calculations requires a lot of knowledge about the shift functions and the derivatives of the shift functions we begin by listing the partial derivatives with respect to all natural variables of $F(\nu, \mu)$ and $\rho(\lambda)$ for reference. We list just the derivatives of the F rather than F_B functions because they are simpler and we can reconstruct those for F_B easily. In most cases it is better to use the F functions throughout the calculation and swap back to the correct shift function at the end.

$$\partial_\nu F(\nu, \lambda) = R(\nu, \lambda) - F(q, \lambda)R(\nu, q) + F(-q, \lambda)R(\nu, -q), \tag{5.1}$$

$$\partial_\lambda F(\nu, \lambda) = -R(\nu, \lambda), \tag{5.2}$$

$$\partial_q F(\nu, \lambda) = F(q, \lambda)R(\nu, q) + F(-q, \lambda)R(\nu, -q), \tag{5.3}$$

$$\partial_\lambda \rho(\lambda) = -\rho(q)(R(\lambda, q) - R(\lambda, -q)), \tag{5.4}$$

$$\partial_q \rho(\lambda) = \rho(q)(R(\lambda, q) + R(\lambda, -q)). \tag{5.5}$$

We will now derive these results. First up is the derivative of F with respect to it's first argument $\partial_\nu F(\nu, \lambda)$. We first use the symmetry of the kernel, $\partial_\nu K(\mu, \nu) = -\partial_\mu K(\mu, \nu)$ and

then integrate by parts. We will use these steps on many occasions.

$$\begin{aligned}
\partial_\nu F(\nu,\lambda) - \int_{-q}^{q} d\mu \partial_\nu K(\mu,\nu) F(\mu,\lambda) =& \frac{K(\nu,\lambda)}{2\pi} \\
\partial_\nu F(\nu,\lambda) + \int_{-q}^{q} d\mu \partial_\mu K(\mu,\nu) F(\mu,\lambda) =& \frac{K(\nu,\lambda)}{2\pi} \\
\partial_\nu F(\nu,\lambda) - \int_{-q}^{q} d\mu K(\mu,\nu) \partial_\mu F(\mu,\lambda) =& \frac{K(\nu,\lambda)}{2\pi} \\
& - \frac{K(\nu,q)}{2\pi} F(q,\lambda) \\
& + \frac{K(\nu,-q)}{2\pi} F(-q,\lambda). \qquad (5.6)
\end{aligned}$$

We then use the linearity and compare with the definitions of R and F to get,

$$\partial_\nu F(\nu,\lambda) = R(\nu,\lambda) - F(q,\lambda) R(\nu,q) + F(-q,\lambda) R(\nu,-q). \qquad (5.7)$$

The derivative with respect to the second variable is more straight forward:

$$\begin{aligned}
\partial_\lambda F(\nu,\lambda) - \int_{-q}^{q} d\mu K(\mu,\nu) \partial_\lambda F(\mu,\lambda) &= -\frac{K(\nu,\lambda)}{2\pi} \\
\partial_\lambda F(\nu,\lambda) &= -R(\nu,\lambda). \qquad (5.8)
\end{aligned}$$

To calculate $\partial_q F(\nu,\lambda)$ we use $\partial_q \int_{-q}^{q} dx f(x) = f(q) + f(-q)$ to expand,

$$\begin{aligned}
\partial_q F(\nu,\lambda) - \int_{-q}^{q} d\mu K(\mu,\nu) \partial_q F(\mu,\lambda) &= \frac{K(\nu,q)}{2\pi} F(q,\lambda) + \frac{K(\nu,-q)}{2\pi} F(-q,\lambda) \\
\partial_\lambda F(\nu,\lambda) &= R(\nu,q) F(q,\lambda) + R(\nu,-q) F(-q,\lambda). \qquad (5.9)
\end{aligned}$$

The derivatives of ρ can be calculated in a similar fashion.

$$\begin{aligned}
\partial_\lambda \rho(\lambda) - \int_{-q}^{q} d\mu \partial_\lambda K(\mu,\lambda)\rho(\mu) =& 0 \\
\partial_\lambda \rho(\lambda) + \int_{-q}^{q} d\mu \partial_\mu K(\mu,\lambda)\rho(\mu) =& 0 \\
\partial_\lambda \rho(\lambda) - \int_{-q}^{q} d\mu K(\mu,\lambda)\partial_\mu \rho(\mu) =& -\frac{K(q,\lambda)}{2\pi}\rho(q) \\
& + \frac{K(-q,\lambda)}{2\pi}\rho(-q).
\end{aligned} \tag{5.10}$$

$$\partial_\lambda \rho(\lambda) = -\rho(q)\left(R(\lambda,q) - R(\lambda.-q)\right). \tag{5.11}$$

$$\begin{aligned}
\partial_q \rho(\lambda) - \int_{-q}^{q} d\mu K(\mu,\lambda)\partial_q \rho(\mu) =& \frac{K(q,\lambda)}{2\pi}\rho(q) \\
& + \frac{K(-q,\lambda)}{2\pi}\rho(-q).
\end{aligned} \tag{5.12}$$

$$\partial_q \rho(\lambda) = \rho(q)\left(R(\lambda,q) + R(\lambda,-q)\right). \tag{5.13}$$

In addition to these results we will also need the results due to Korepin and Slavnov [21] [36] that give properties of the F function. We will list them again here for convenience:

$$F(\nu,\lambda) = -F(\lambda,\nu) + F(\nu,q)F(\lambda,q) - F(\nu,-q)F(\lambda,-q). \tag{5.14}$$

We will also use

$$2\pi\rho(\lambda) = 1 + F(\lambda,-q) - F(\lambda,q), \tag{5.15}$$

and

$$\frac{1}{2\pi\rho(q)} = 1 - F(q,-q) - F(q,q). \tag{5.16}$$

These results can be combined to give,

$$F(\lambda,q) - F(\lambda,-q) = -2\pi\rho(q)[F(q,\lambda) - F(-q,\lambda)], \tag{5.17}$$

and

$$F(\lambda,q) + F(\lambda,-q) = -\frac{1}{2\pi\rho(q)}[F(q,\lambda) + F(-q,\lambda)]. \tag{5.18}$$

These last two relationships suggest there is something special about the sum and difference of the Fermi surface shifts, this is indeed the case and we shall see that they are linked to the N and J variables.

5.2 Finite size corrections in the Bose gas

We have seen already that the correlation functions of systems which have conformal symmetry are related to the finite size correction to the energy. Now we will use this to find the correspondence between the N and J variables that have been defined in terms of derivatives of the dispersion and some quantities that are known from the Bethe ansatz, specifically the shifts of the Fermi surfaces due to the excitation. The idea is to calculate the finite size corrections exactly in the presence of the integrable excitations and match these with finite size corrections to a shifted Luttinger liquid. This will give the correspondence between the two approaches and allow us to find the exponents for the divergence in the dynamic correlation functions in terms of the Fermi surface shifts. We are cheating slightly when we

do this because previously we found the exponents expressed as the finite size corrections to energy and momentum. However the approach we will use is equivalent and easier to do. This matching leads us to ask whether this correspondence between the N and J and the Bethe ansatz quantities is exact. This is precisely what we will tackle in the next section.

We begin with the finite size corrections in the case of a shifted Luttinger liquid these were given [32], the correction to the energy is:

$$\Delta E = \frac{2\pi c}{L}\left[\frac{1}{4K}(\Delta N - n_{imp})^2 + K(D - d_{imp})^2\right]. \tag{5.19}$$

Here we have used the same notaion as Korepin, ΔN is the change in particle number due to an excitation we will see that it can be identifies with the σ we introduced as the ratio of masses in the impurity model. D is the number of particles scattered round the back of the ring, this is related to the calculation of the correlation functions in different topological sectors by adding 2π to the J variable. In [32] the n_{imp} and d_{imp} are given in terms of the shifts of chiral fields:

$$n_{imp} = -\sqrt{K}\frac{\delta_+ - \delta_-}{2\pi}, \tag{5.20}$$

$$2d_{imp} = -\frac{1}{\sqrt{K}}\frac{\delta_+ + \delta_-}{2\pi}. \tag{5.21}$$

We have already seen how these variables are related to the depleton N and J, so we can identify that:

$$N = n_{imp}, \tag{5.22}$$

$$\frac{J}{\pi} = 2d_{imp}, \tag{5.23}$$

substituting these into (5.19) we get:

$$\Delta E = \frac{2\pi c}{L}\left[\frac{1}{4K}(\Delta N - N)^2 + K\left(D - \frac{J}{2\pi}\right)^2\right]. \tag{5.24}$$

Here we see the seperation into the parts due to changing topological sector ΔN and D and the parts due to scattering of the particles at the Fermi surface. This seperation is what we exploited in the last chapter to see the connection between correlation functions for the particle and holes and those for $q \to q + 2\pi n$.

To find the terms from the Bethe ansatz solutions that correspond to the N and J variables we must calculate the spectrum of a large but finite system that contains one of the integrable excitations. This will give us the desired correspondence between shifted Luttinger liquids and integrable models. The technique[8] [10] is to write the energy in the finite system by using the Euler-Maclaurin formula for approximating sums ny integrals. The Euler-Maclaurin formula is:

$$\frac{1}{L}\sum_{n=n_1}^{n_2} f\left(\frac{n}{L}\right) = \int_{\frac{n_-}{L}}^{\frac{n_+}{L}} dx f(x) + \frac{1}{24L^2}\left(f'\left(\frac{n_-}{L}\right) - f'\left(\frac{n_+}{L}\right)\right) + \ldots. \tag{5.25}$$

You then have to carefully expand all of the quantities in the expression for the energy keeping track of the order in $1/L$ at each stage. The calculation is tricky and long, the particle and hole case was done by [32] and the results for the spinon case are in [41] but without the derivation. My version of the calculation for all three types of excitation that we have been dealing with are given in detail in A.

5.3 Some exact results for the 1D Bose gas

This section contains the results we have been working towards. By matching our definitions of N and J from derivatives of the dispersion and finite size analysis we can predict a

relationship between the shift of the Fermi surfaces and the derivatives of the dispersion. Here we show that these are indeed exact identities in the Bethe ansatz soluble models we have discussed. We also have the definition of the back scattering amplitude in the depleton model which we calculate for each of the integrable excitations.

5.3.1 Inverting partial derivatives

We would like to relate the derivatives of energy with respect to momentum P and density n to the scattering phases at the fermi surface. However, all of the equations we have for the energy are in terms of the Bethe ansatz variables λ and q. In this section we will write ∂_P and ∂_n in terms of ∂_λ and ∂_q.

$$\begin{pmatrix} \partial_q \\ \partial_\lambda \end{pmatrix} = \begin{pmatrix} \partial_q n & \partial_q P \\ 0 & \partial_\lambda P \end{pmatrix} \begin{pmatrix} \partial_n \\ \partial_P \end{pmatrix} \tag{5.26}$$

We will begin by calculating $\partial_q n$. We get,

$$\partial_q n(q) = \rho(q) + \rho(-q) + \int_{-q}^{q} d\nu \partial_q \rho(\nu) \tag{5.27}$$

$$= 2\rho(q) + \int_{-q}^{q} d\nu \partial_q \rho(\nu). \tag{5.28}$$

Using (5.5) we get,

$$\partial_q n(q) = 2\rho(q) + \rho(q) \int_{-q}^{q} d\mu [R(\mu, q) + R(\mu, -q)] \tag{5.29}$$

$$= 2\rho(q)[1 + \int_{-q}^{q} d\mu R(\mu, q)] \tag{5.30}$$

$$= 2\rho(q)[1 + F(q, \lambda) - F(-q, \lambda)] \tag{5.31}$$

$$= 4\pi\rho^2(q). \tag{5.32}$$

Next up is $\partial_q P$, we begin by differentiating the momentum,

$$\partial_q P = -\int_{-q}^{q} d\mu \partial_q F_B(\mu, \lambda) - [F_B(q, \lambda) + F_B(-q, \lambda)]. \tag{5.33}$$

Using (5.3) the first integral is,

$$\begin{aligned}
\int_{-q}^{q} d\mu \partial_q F_B(\mu, \lambda) &= \int_{-q}^{q} d\mu [R(\mu, q) F_B(q, \lambda) + R(\mu, -q) F_B(-q, \lambda)] && (5.34)\\
&= [F_B(q, \lambda) + F_B(-q, \lambda)] \int_{-q}^{q} d\mu R(\mu, q) && (5.35)\\
&= [F_B(q, \lambda) + F_B(-q, \lambda)](F(q, -q) - F(q, q)) && (5.36)\\
&= [F_B(q, \lambda) + F_B(-q, \lambda)](2\pi\rho(q) - 1). && (5.37)
\end{aligned}$$

Substituting back gives,

$$\partial_q P(\lambda) = -2\pi\rho(q) \left(F_B(q, \lambda) + F_B(-q, \lambda)\right). \tag{5.38}$$

Lastly we need $\partial_\lambda P$, differentiating we get,

$$\begin{aligned}
\pm\partial_\lambda P(\lambda) &= 1 + \int_{-q}^{q} d\mu \partial_\lambda \theta(\lambda - \mu)\rho(\mu) && (5.39)\\
&= 1 + \int_{-q}^{q} d\mu K(\lambda, \mu)\rho(\mu) && (5.40)\\
&= 2\pi\rho(\lambda). && (5.41)
\end{aligned}$$

Putting this all together leads to:

$$\begin{pmatrix} \partial_q \\ \partial_\lambda \end{pmatrix} = \begin{pmatrix} 4\pi\rho(q)^2 & -2\pi\rho(q)[F_B(q, \lambda) + F_B(-q, \lambda)] \\ 0 & 2\pi\rho(\lambda) \end{pmatrix} \begin{pmatrix} \partial_n \\ \partial_P \end{pmatrix}, \tag{5.42}$$

which can be inverted to give,

$$\begin{pmatrix} \partial_n \\ \partial_P \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi\rho(q)^2} & \frac{F_B(q,\lambda)+F_B(-q,\lambda)}{2\pi\rho(\lambda)} \\ 0 & \frac{1}{2\pi\rho(\lambda)} \end{pmatrix} \begin{pmatrix} \partial_q \\ \partial_\lambda \end{pmatrix}. \tag{5.43}$$

We can now go back and forth between (P, n) and (λ, q), which will allow us to take our expressions from the depleton model and relate them to the definitions of energy, momentum etc. from the Bethe ansatz.

5.3.2 Derivatives of the Lieb-Liniger dispersion

In this section we will calculate the derivatives of the dispersion from the Bethe ansatz equations and then convert them to derivatives with respect to momentum and density and express them in terms of the scattering phases. We begin by treating the particle like excitation. To calculate $\partial_q\epsilon(\lambda, q)$ we differentiate (2.52) and get,

$$\partial_q\epsilon(\lambda, q) - \int_{-q}^{q} d\mu \frac{K(\lambda-\mu)}{2\pi}\partial_q\epsilon(\mu, q) = -\partial_q h. \tag{5.44}$$

Where the boundary terms are zero because $\epsilon(q, q) = \epsilon(-q, q) = 0$. Comparing with (2.20) gives,

$$\partial_q\epsilon(\lambda, q) = -2\pi\rho(\lambda)\partial_q h. \tag{5.45}$$

Rewriting this in terms of F and introducing the sound velocity $v = \partial_q h$ and the Luttinger parameter $K = 4\pi^2\rho^2(q)$ we have

$$\partial_q\epsilon(\lambda, q) = -v\left(1 + \sqrt{K}[F_B(q, \lambda) - F_B(-q, \lambda)]\right). \tag{5.46}$$

Next we calculate $\partial_\lambda \epsilon(\lambda, q)$. We begin by differentiating (2.52) to get,

$$\partial_\lambda \epsilon(\lambda, q) - \int_{-q}^{q} d\mu \frac{\partial_\lambda K(\lambda - \mu)}{2\pi} \epsilon(\mu, q) = 2\lambda. \tag{5.47}$$

Using $\partial_\lambda K(\lambda - \mu) = -\partial_\mu K(\lambda - \mu)$ and integrating by parts we get,

$$\partial_\lambda \epsilon(\lambda, q) - \int_{-q}^{q} d\mu \frac{K(\lambda - \mu)}{2\pi} \partial_\mu \epsilon(\mu, q) = 2\lambda. \tag{5.48}$$

Which we can write as,

$$\left(1 - \frac{\hat{K}}{2\pi}\right) \partial_\lambda \epsilon(\lambda, q) = 2\lambda, \tag{5.49}$$

or equivalently:

$$\partial_\lambda \epsilon(\lambda, q) = (1 + \hat{R}) 2\lambda. \tag{5.50}$$

Explicitly the integral is:

$$\partial_\lambda \epsilon(\lambda, q) - 2\lambda = \hat{R}[2\lambda] = 2 \int_{-q}^{q} d\mu R(\mu, \lambda) \mu \tag{5.51}$$

To evaluate this we first rewrite (2.32) as,

$$\left(1 - \frac{\hat{K}}{2\pi}\right) F(\nu, \lambda) = -\int^{\lambda} d\mu \frac{K(\nu - \mu)}{2\pi}. \tag{5.52}$$

Which leads to,

$$F(\nu, \lambda) = -\int^{\lambda} d\mu \left(1 - \frac{\hat{K}}{2\pi}\right)^{-1} \frac{K(\nu - \mu)}{2\pi} = -\int^{\lambda} d\mu R(\nu, \mu). \tag{5.53}$$

Now integrating by parts in (A.37) gives,

$$\partial_\lambda \epsilon(\lambda, q) - 2\lambda = -2q(F(\lambda, q) + F(\lambda, -q)) + 2\int_{-q}^{q} d\mu F(\lambda, \mu). \tag{5.54}$$

We now use the Slavnov identity (5.14),

$$\begin{aligned}\int_{-q}^{q} d\mu F(\lambda, \mu) = &- \int_{-q}^{q} d\mu F(\mu, \lambda) && (5.55)\\ &+ F(\lambda, q)\int_{-q}^{q} d\mu F(\mu, q) && (5.56)\\ &- F(\lambda, -q)\int_{-q}^{q} d\mu F(\mu, -q).\end{aligned}$$

Substituting the momentum we get,

$$\begin{aligned}\int_{-q}^{q} d\mu F(\lambda, \mu) = &- (\lambda - \pi n - P(\lambda)) && (5.57)\\ &+ F(\lambda, q)\,(q - \pi n - P(q)) && (5.58)\\ &- F(\lambda, -q)\,(-q - \pi n - P(-q))\,.\end{aligned}$$

We now have,

$$\begin{aligned}\partial_\lambda \epsilon(\lambda, q) = &- 2P(\lambda) + 2\pi n\,(1 - F(\lambda, q) + F(\lambda, -q)) && (5.59)\\ &- 2F(\lambda, q)P(q) + 2F(\lambda, -q)P(-q)\end{aligned}$$

Using $P(q) = 0$ and $P(-q) = -2\pi n$ this gives,

$$\partial_\lambda \epsilon(\lambda, q) = +\,2P(\lambda) + 2\pi n\,(1 - F(\lambda, q) - F(\lambda, -q))\,. \tag{5.60}$$

Using the Slavnov identities and changing F to F_B we get

$$\partial_\lambda \epsilon(\lambda, q) = 2P(\lambda) + v\sqrt{K}\left(F_B(q,\lambda) + F_B(-q,\lambda)\right). \tag{5.61}$$

We can relate the derivatives with respect to λ and q to those w.r.t P and n. Rewriting (5.42) in terms of the Luttinger liquid parameter K and the sound velocity v we get, for the particle,

$$\begin{pmatrix} \partial_q \\ \partial_\lambda \end{pmatrix} = \begin{pmatrix} \frac{K}{\pi} & -\sqrt{K}[F_B(q,\lambda) + F_B(-q,\lambda)] \\ 0 & 1 + \sqrt{K}[F_B(q,\lambda) - F_B(-q,\lambda)] \end{pmatrix} \begin{pmatrix} \partial_n \\ \partial_P \end{pmatrix}. \tag{5.62}$$

This gives us

$$\partial_q \epsilon = -v\left(1 + \sqrt{K}[F_B(q,\lambda) - F_B(-q,\lambda)]\right) \tag{5.63}$$

$$= \frac{K}{\pi}\partial_n \epsilon - \sqrt{K}[F_B(q,\lambda) + F_B(-q,\lambda)]\partial_P \epsilon, \tag{5.64}$$

and

$$\partial_\lambda \epsilon = 2P(\lambda) + c\sqrt{K}\left(F_B(q,\lambda) + F(-q,\lambda)\right) \tag{5.65}$$

$$= \left(1 + \sqrt{K}[F_B(q,\lambda) - F_B(-q,\lambda)]\right)\partial_P \epsilon. \tag{5.66}$$

If we compare this to the definitions of N and J remembering that for the particle $\sigma = 1$:

$$\partial_P E(P,n)(\sigma - N) = 2P - cK\frac{J}{\pi},$$

$$\partial_n E(P,n) = \frac{c\pi}{K}N - J\partial_P E(P,n), \tag{5.67}$$

then we can identify N and J as:

$$\begin{aligned} N &= -\sqrt{K}\left(F_B(q|\lambda) - F_B(-q|\lambda)\right), \\ \frac{J}{\pi} &= -\frac{1}{\sqrt{K}}\left(F_B(q|\lambda) + F_B(-q|\lambda)\right). \end{aligned} \tag{5.68}$$

This is precisely the relationship that has been suggested using the results from other approaches. We have found it by the most direct and only rigorous method. Using some of the identities for F we can also write N as:

$$N = F(\lambda|q) - F(\lambda|-q) = \pm(1 - 2\pi\rho(\lambda)) \tag{5.69}$$

Using $V = \partial_P \epsilon$ and rearranging we find,

$$\begin{pmatrix} V - v & V + v \\ -V + v & V + v \end{pmatrix} \begin{pmatrix} F_B(q,\lambda) \\ F_B(-q,\lambda) \end{pmatrix} = \begin{pmatrix} \sqrt{K}\left(\frac{\partial_n \epsilon}{\pi} + \frac{c}{K}\right) \\ -\frac{1}{\sqrt{K}}(2P - V) \end{pmatrix}. \tag{5.70}$$

Inverting this to find the F_B functions we get,

$$F_B(q,\lambda) = \frac{1}{2}\frac{1}{V - c}\left\{\sqrt{K}\left(\frac{\partial_n \epsilon}{\pi} + \frac{c}{K}\right) + \frac{1}{\sqrt{K}}(2P - V)\right\}, \tag{5.71}$$

$$F_B(-q,\lambda) = \frac{1}{2}\frac{1}{V + c}\left\{\sqrt{K}\left(\frac{\partial_n \epsilon}{\pi} + \frac{c}{K}\right) - \frac{1}{\sqrt{K}}(2P - V)\right\}. \tag{5.72}$$

Which match exactly the definitions of $\delta_\pm$ in the work of Glazman and Imambekov [17]

These results are for the particle excitation, we can get the results for the hole easily by

using the mapping:

$$\begin{aligned} F_B &\to -F_B, \\ \epsilon &\to -\epsilon, \\ P &\to -P. \end{aligned} \tag{5.73}$$

Which transforms the equations for the particle to those for the hole. Applying this to (5.72) gives the equations for the Fermi shifts in the hole case.

$$F_B(q,\lambda) = \frac{1}{2}\frac{1}{V-v}\left\{\sqrt{K}\left(\frac{\partial_n \epsilon}{\pi} - \frac{v}{K}\right) + \frac{1}{\sqrt{K}}(2P+V)\right\}, \tag{5.74}$$

$$F_B(-q,\lambda) = \frac{1}{2}\frac{1}{V+v}\left\{\sqrt{K}\left(\frac{\partial_n \epsilon}{\pi} - \frac{v}{K}\right) - \frac{1}{\sqrt{K}}(2P+V)\right\}. \tag{5.75}$$

5.3.3 Derivatives of the Yang-Gaudin dispersion

We now want to do the same calculation for the spinon excitation in the Yang-Gaudin model. We will proceed using similar techniques but will find that the calculation is much more complicated. We begin in the same way by finding the transformation from derivatives in k, n to those in ξ, q. Differentiating the spinon momentum we find:

$$\begin{aligned} \partial_\xi k(\xi) &= \int_{-q}^{q} d\lambda \rho(\lambda) \partial_\xi \theta(2\lambda - 2\xi), \\ &= -\int_{-q}^{q} d\lambda \rho(\lambda) \partial_\lambda \theta(2\lambda - 2\xi), \\ &= \int_{-q}^{q} d\lambda \partial_\lambda \rho(\lambda) \theta(2\lambda - 2\xi) - \rho(q)[\theta(2q - 2\xi) + \theta(2q + 2\xi)], \\ &= -\rho(q) \int_{-q}^{q} d\lambda (R(\lambda, q) - R(\lambda, -q))\theta(2\lambda - 2\xi) - \rho(q)[\theta(2q - 2\xi) - \theta(-2q - 2\xi)]. \end{aligned} \tag{5.76}$$

Careful inspection suggests that we should write this as:

$$\begin{aligned}\partial_\xi k(\xi) &= 2\pi\rho(q)(1+\hat{R})\left(-\frac{\theta(2q-2\xi)}{2\pi}+\frac{\theta(-2q-2\xi)}{2\pi}\right),\\ &= 2\pi\rho(q)\left(\bar{F}(q,\xi)-\bar{F}(-q,\xi)\right).\end{aligned} \tag{5.77}$$

Next we compute $\partial_q k(\xi)$:

$$\partial_q k(\xi) = \pi\partial_q n + \int_{-q}^{q} d\lambda \partial_q \rho(\lambda)\theta(2\lambda-2\xi) + \rho(q)[\theta(2q-2\xi)+\theta(-2q-2\xi)]. \tag{5.78}$$

Expanding the $\partial_q\rho(\lambda) = \rho(q)(R(\lambda,q)+R(\lambda,-q))$ this time we have:

$$\begin{aligned}\partial_q k(\xi) &= \pi\partial_q n + 2\pi\rho(q)(1+\hat{R})\left(\frac{\theta(2q-2\xi)}{2\pi}+\frac{\theta(-2q-2\xi)}{2\pi}\right),\\ &= 4\pi^2\rho(q)^2 + 2\pi\rho(q)(1+\hat{R})\left(\frac{\theta(2q-2\xi)}{2\pi}+\frac{\theta(-2q-2\xi)}{2\pi}-2\pi\rho(q)\right),\\ &= -2\pi\rho(q)\left(\bar{F}(q,\xi)+\bar{F}(-q,\xi)\right).\end{aligned} \tag{5.79}$$

We are now in position to write the transformation:

$$\begin{pmatrix}\partial_q\\ \partial_\xi\end{pmatrix} = \begin{pmatrix}4\pi\rho(q)^2 & -2\pi\rho(q)[\bar{F}(q,\xi)+\bar{F}(-q,\xi)]\\ 0 & 2\pi\rho(q)\left(\bar{F}(q,\xi)-\bar{F}(-q,\xi)\right)\end{pmatrix}\begin{pmatrix}\partial_n\\ \partial_k\end{pmatrix}, \tag{5.80}$$

The next step is to calculate the derivatives of the spinon dispersion. The dispersion is:

$$\omega(\xi) = -\frac{1}{\pi}\int_{-q}^{q} d\lambda\epsilon(\lambda)K(2\lambda,2\xi). \tag{5.81}$$

First we take the derivative with the respect to ξ:

$$\begin{aligned}\partial_\xi \omega(\xi) &= -\frac{1}{\pi}\int_{-q}^{q} d\lambda \epsilon(\lambda)\partial_\xi K(2\lambda, 2\xi), \\ &= -\frac{1}{\pi}\int_{-q}^{q} d\lambda \partial_\lambda \epsilon(\lambda) K(2\lambda, 2\xi), \\ &= -\frac{1}{2\pi}\int_{-q}^{q} d\lambda \left(2P(\lambda) + 2\pi n[1 - F(\lambda, q) - F(\lambda, -q)]\right) \partial_\lambda \theta(2\lambda - 2\xi). \end{aligned} \tag{5.82}$$

Here we have integrated by parts remembering that the boundary terms vanish because $\epsilon(\pm q) = 0$ and substituted $\partial_\lambda \epsilon(\lambda)$ (5.60). The first part of this integral is of the form:

$$\begin{aligned}&\int_{-q}^{q} d\lambda P(\lambda)\partial_\lambda \theta(2\lambda - 2\xi), \\ &= -\int_{-q}^{q} d\lambda \partial_\lambda P(\lambda)\theta(2\lambda - 2\xi) + P(q)\theta(2q - 2\xi) + P(-q)\theta(2q + 2\xi), \\ &= -\int_{-q}^{q} d\lambda 2\pi\rho(\lambda)\theta(2\lambda - 2\xi) - 2\pi n\theta(2q + 2\xi), \\ &= -2\pi[k(\xi) - \pi n] - 2\pi n\theta(2q + 2\xi). \end{aligned} \tag{5.83}$$

The other part of the integral is:

$$\int_{-q}^{q} d\lambda [1 - F(\lambda, q) - F(\lambda, -q)]\partial_\lambda \theta(2\lambda - 2\xi). \tag{5.84}$$

Integrating by parts gives:

$$\begin{aligned}&\int_{-q}^{q} d\lambda [1 - F(\lambda, q) - F(\lambda, -q)]\partial_\lambda \theta(2\lambda - 2\xi) \\ &= \int_{-q}^{q} d\lambda [\partial_\lambda F(\lambda, q) + \partial_\lambda F(\lambda, -q)]\theta(2\lambda - 2\xi) \\ &+ [1 - F(q, q) - F(q, -q)]\theta(2q - 2\xi) \\ &- [1 + F(q, q) + F(q, -q)]\theta(-2q - 2\xi). \end{aligned} \tag{5.85}$$

We know how to expand the derivatives of F and get:

$$\begin{aligned}
&\int_{-q}^{q} d\lambda \left\{R(\lambda,q) - F(q,q)R(\lambda,-q) - F(q,-q)R(\lambda,-q)\right\}\theta(2\lambda-2\xi)\\
+&\int_{-q}^{q} d\lambda \left\{R(\lambda,-q) - F(q,-q)R(\lambda,q) - F(q,q)R(\lambda,-q)\right\}\theta(2\lambda-2\xi)\\
&+[1-F(q,q)-F(q,-q)]\theta(2q-2\xi)\\
&-[1+F(q,q)+F(q,-q)]\theta(-2q-2\xi). \qquad (5.86)
\end{aligned}$$

Changing the sign on the dummy variable in both terms containing $R(\lambda,-q)$ and using the symmetry $R(x,y) = R(-x,-y)$:

$$\begin{aligned}
&\int_{-q}^{q} d\lambda \left\{R(\lambda,q) - F(q,q)R(\lambda,-q) - F(q,-q)R(\lambda,q)\right\}\theta(2\lambda-2\xi)\\
+&\int_{-q}^{q} d\lambda \left\{R(\lambda,-q) - F(q,-q)R(\lambda,-q) - F(q,q)R(\lambda,-q)\right\}\theta(2\lambda-2\xi)\\
&+[1-F(q,q)-F(q,-q)]\theta(2q-2\xi)\\
&+[1-F(q,q)-F(q,-q)]\theta(-2q-2\xi)+2\theta(2q+2\xi). \qquad (5.87)
\end{aligned}$$

We can now take out a factor of $1-F(q,q)-F(q,-q) = 1/2\pi\rho(q)$ and use the operator notation:

$$\begin{aligned}
&\frac{1}{2\pi\rho(q)}\left\{(1+\hat{R})[\theta(2q-2\xi)+\theta(-2q-2\xi)]\right\}+2\theta(2q+2\xi)\\
&=2\theta(2q+2\xi)-\frac{1}{\rho(q)}[\bar{F}(q|\xi)+\bar{F}(-q|\xi)]+2\pi. \qquad (5.88)
\end{aligned}$$

Combining (5.83) and (5.88) and canceling terms we find that the derivative is:

$$\partial_\xi\omega(\xi) = 2k(\xi)+\frac{n}{\rho(q)}[\bar{F}(q|\xi)+\bar{F}(-q|\xi)]. \qquad (5.89)$$

The last derivative we require is $\partial_q\omega(\xi)$, happily this is less work than the last one. We begin in the same way by substituting and integrating by parts:

$$\begin{aligned}\partial_q\omega(\xi) &= -\frac{1}{\pi}\int_{-q}^{q} d\lambda \partial_q\epsilon(\lambda)K(2\lambda,2\xi),\\ &= -\frac{1}{2\pi}\int_{-q}^{q} d\lambda\left(-2\pi\rho(\lambda)der_q h\right)\partial_\lambda\theta(2\lambda-2\xi)\\ &= \partial_q h\left(-\int_{-q}^{q} d\lambda\partial_\lambda\rho(\lambda)\theta(2\lambda-2\xi)+\rho(q)(\theta(2q-2\xi)-\theta(-2q-2\xi))\right).\end{aligned} \tag{5.90}$$

Now substituting $\partial_\lambda\rho(\lambda)$ from (5.4):

$$\begin{aligned}\partial_q\omega(\xi) &= \partial_q h\rho(q)\left(\theta(2q-2\xi)-\theta(-2q-2\xi)+\int_{-q}^{q} d\lambda(R(\lambda,q)-R(\lambda,-q))\theta(2\lambda-2\xi)\right),\\ &= \partial_q h\rho(q)\left((1+\hat{R})[\theta(2q-2\xi)-\theta(-2q-2\xi)]\right),\\ &= -\partial_q h 2\pi\rho(q)[\bar{F}(q|\xi)-\bar{F}(-q|\xi)].\end{aligned} \tag{5.91}$$

We can now use $K = 4\pi^2\rho(q)^2 = 2\pi n/c$ to remove factors of $\rho(q)$ and n, the partial derivatives become:

$$\begin{pmatrix}\partial_q\\ \partial_\xi\end{pmatrix} = \begin{pmatrix}K/\pi & -\sqrt{K}[\bar{F}(q,\xi)+\bar{F}(-q,\xi)]\\ 0 & \sqrt{K}[\bar{F}(q,\xi)-\bar{F}(-q,\xi)]\end{pmatrix}\begin{pmatrix}\partial_n\\ \partial_k\end{pmatrix}. \tag{5.92}$$

Using this transformation we find:

$$\begin{aligned}\partial_q\omega &= -c\sqrt{K}[\bar{F}(q,\xi)-\bar{F}(-q,\xi)] = K\frac{\partial_n\omega}{\pi}-\sqrt{K}[\bar{F}(q,\xi)+\bar{F}(-q,\xi)]V,\\ \partial_\xi\omega &= 2k + c\sqrt{K}[\bar{F}(q,\xi)+\bar{F}(-q,\xi)] = \sqrt{K}[\bar{F}(q,\xi)-\bar{F}(-q,\xi)]V\end{aligned} \tag{5.93}$$

Again this allows us to match with the depleton variables, this time $\sigma = 0$ because we are dealing with the spinon which does not change the particle number. As expected we find our

result to be:

$$N = -\sqrt{K}\left(\bar{F}(q,\xi) - \bar{F}(-q,\xi)\right) \tag{5.94}$$

$$\frac{J}{\pi} = -\frac{1}{\sqrt{K}}\left(\bar{F}(q,\xi) + \bar{F}(-q,\xi)\right). \tag{5.95}$$

We note here that the formulae for N and J are virtually the same in each case, this is because the changes in particle number have already been taken into account in the different shift functions. When we use these exact values of N and J to calculate correlation functions we must correct for disparities in exactly the same way as previously.

5.3.4 Calculation of $\Gamma_{\pm}$

As we have described already the depleton model can be used to describe the interaction of an impurity with phonons in one-dimension and the backscattering amplitude is the part of the friction force which depends on the details of the impurity. The amplitude can be calculated in terms of the derivatives of the N and J variables. We will now see that this amplitude is simplified significantly when written in terms of the Bethe ansatz variables λ and q. The amplitude $\Gamma_{\pm}$ in the (P,n) representation is:

$$-c\Gamma_{\pm} = (\sigma - N)\partial_P J + J\partial_P N + \partial_n N. \tag{5.96}$$

We can combine the results for the Jacobian matrix of each type of interaction into:

$$\begin{pmatrix} \partial_q \\ \partial_\lambda \end{pmatrix} = \begin{pmatrix} \frac{K}{\pi} & \frac{K}{\pi}J \\ 0 & \sigma - N \end{pmatrix} \begin{pmatrix} \partial_n \\ \partial_k \end{pmatrix}, \tag{5.97}$$

and immediately we see that writing Γ in terms of λ and q derivatives will simplify it considerably. We get,

$$-\frac{c\Gamma_{\pm}}{\pi} = \frac{\partial_\lambda J}{\pi} + \frac{\partial_q N}{K}. \tag{5.98}$$

The simplicity of the expression in these variables is striking. It seems that the Bethe ansatz variables λ and q are in some sense the natural variables to use when describing the interaction between the depleton N and J variables with the phononic background. This is quite unexpected.

The next step, of course, is to calculate this quantity beginning with the particle/hole case. The first term can be calculated using (5.6) and (5.8):

$$\begin{aligned}\frac{\partial_\lambda J}{\pi} &= -\frac{1}{\sqrt{K}}[\partial_\lambda F(q|\lambda) + \partial_\lambda F(-q|\lambda)] \\ &= \pm\frac{1}{\sqrt{K}}[R(q,\lambda) + R(-q,\lambda)].\end{aligned} \tag{5.99}$$

The best way to calculate the other term is to use $N(\lambda) = \pm(1 - 2\pi\rho(\lambda))$,

$$\partial_q N = \mp 2\pi\partial_q\rho(\lambda) = \mp 2\pi\rho(q)[R(q,\lambda) + R(-q,\lambda)]. \tag{5.100}$$

Substituting these into (5.98) we find that $\Gamma_{\pm} = 0$. This is what we hoped to find, it may seem that this is a trivial result as the particle and hole excitations are energy eigenstates of the Hamiltonian, however we should remember that the depleton model doesn't 'know' this. The only input is the dispersion, so encoded in the form of the integrable dispersion is the cancelation of the impurity-phonon interactions. It has been shown [23] that in the weakly interacting limit the hole dispersion is the same as that of the grey soliton in the Gross-Pitaevskii equation. So we can see that this result is equivalent to the vanishing amplitude for the grey soliton in the weakly interacting limit.

Finally we need to calculate this amplitude for the spinon. The fastest way to do this is to notice that the only difference will be factors of two multiplying all terms, this is because of the definitions of the shift functions. It can of course also be done by directly calculation the derivatives of the spinon shift function. In either case we find that $\Gamma_{\pm} = 0$ for the spinon too.

5.4 Conclusions

This chapter contains the main results of this thesis. We set out to use the Bethe ansatz to test the depleton theory of mobile impurities and the calculation of dynamic response functions using an effective impurity model. We began the chapter by presenting the finite size corrections to the Lieb-Liniger and Yang-Gaudin models. Using results from conformal field theory we know that the finite size corrections to the energy are the conformal dimensions of the operators, this gives us the N and J variables in terms of the shift functions from the integrable models. These N and J are also defined in terms of derivatives of the dispersion in the depleton model. Combining these two definitions suggests that the shift functions can be written in terms of derivatives of the dispersion. Beginning with the Bethe ansatz form of the dispersion we calculate the derivatives with respect to the momentum and density exactly, we find that the relationship is precisely what was expected confirming that our model is reliable.

Another quantity that we have come across in the depleton model is the back scattering amplitude. For the integrable excitations we expect there to be no viscous friction force and therefore the back scattering amplitude should be zero. Again we take the definitions of the dispersion from the Bethe ansatz and calculate this quantity. it is found to be exactly zero for the three Bethe ansatz excitations we consider.

CHAPTER 6

CONCLUSIONS

Haldane [14] showed that the long range correlation functions of many one-dimensional models are not strongly affected by linearising the dispersion. Since then the Luttinger liquid has been used as a universal effective theory for gapless one-dimensional systems. The challenge of how to include marginal terms that lift the degeneracy associated with the linear spectrum was not straight forward to overcome. Progress was made once the connection with the Fermi edge problem was discovered [1], this non-linear Luttinger liquid has connected the study of mobile impurities to the calculation of correlation functions in these systems.

The main theme of this thesis has been to use exact results from Bethe ansatz solutions to test the phenomenological theory of the depleton model and non-linear Luttinger liquid. These theories are much more widely applicable than the exact results, but as they are developed completely independently of the Bethe ansatz the results gives us confidence in the theory as a whole. This allows us to use the Bethe ansatz in a way that gets around the usual problem of it's narrow range of application. Having said this, the structure of the Bethe ansatz equations for the Lieb-Liniger and Yang-Gaudin is very similar. We would expect that other Bethe ansatz soluble models with a shift function would have analogous relationships between the Fermi surface shifts and derivatives of the dispersion.

We began by setting the scene of one-dimensional quantum systems by introducing boson-

isation and conformal field theory along with the Bethe ansatz. The Bethe ansatz techniques allowed us to calculate the ground state and zero temperature excitations of the contact interacting repulsive Bose gas. There are other models that can be solved using the same techniques and have similar structure in terms of the equations. We calculated the energy and momentum of the elementary excitations of the Lieb-Liniger and Yang-Gaudin models. We also introduced the shift functions which describe the effect of the impurity on the rest of the system. These shift functions are important, especially the shifts of the Fermi surfaces, when considering the correlation functions of the system.

Complimentary to the exact Bethe ansatz approach is the idea of a low energy effective theory. As we have mentioned already, the Luttinger liquid has been considered the go to effective theory for gapless models in one-dimension. One of the disadvantages of the exact solutions is the extreme difficulty when trying to calculate correlation functions. The effective theory is useful because the correlation functions can be calculated easily, you must be very wary of whether the results are applicable though. We saw how using the conformal invariance of the model some of the correlation functions can be found.

Progress beyond the linear spectrum approximation has been made using the mobile impurity model [17]. In this model an impurity is coupled to the Luttinger liquid in a very general way allowing the main feature of certain dynamic response functions to be calculated. A major part of this thesis is to show the very close connections between this model and the depleton model [34] used to study mobile impurities in their own right. We include a review of the depleton model which describes how to derive the coupling between the impurity and phononic degrees of freedom. In the process of doing this we introduce variables N and J which are the number of particles expelled from the background and the phase dropped across the impurity. These variables turn out to be very natural and simplify the connections between different correlation functions considerably. In addition to the interaction between the impurity and the phonons the depleton model can also be used to calculate the friction

force on the impurity. The corresponding amplitude for the frictional process can be written in terms of the derivatives of the N and J variables, later in the thesis we calculate these exactly.

The conformal invariance of the effective model allows us to match the N and J variables with the finite size spectra of the models. This is particularly useful for the integrable models because we can calculate these spectra exactly, although with some difficulty. This leads us to identify the N and J variables with the difference and sum of the the shift functions at the Fermi points. If the effective model is good than this relationship should be true. We test this in the most direct way we can: staring from the definitions in terms of the dispersion relation we calculate the N and J variables exactly in terms of the Fermi point shifts. We find exact identities suggesting that the model is indeed reliable. In addition to this the identities are new and interesting from the point of view of exact solutions.

Integral to this calculation is finding expressions for the derivatives of the dispersion, we then turn our attention to the amplitude for friction which contains exactly these terms. On closer inspection the amplitude takes a very elegant form when written in terms of the derivatives of the depleton N and J variables with respect to the Bethe ansatz quasi-momentum and Fermi momentum. We have also calculated these amplitudes exactly, they vanish as would be expected for the integrable models. This shows that the lack of interaction with phonons is encoded in the dispersion relation of the integrable excitations.

The extension of the depleton model to calculate dynamic response functions, the exact relationships between the dispersion and the and shift functions and the vanishing of the friction amplitude are the main results of the thesis. However there are many other questions that remain unanswered. The most obvious extension is probably to look for similar exact identities in other integrable models. As the structure of the integral equations is similar across several models we expect that similar identities may be found. Another question that we mentioned in this thesis is to do with the analytic structure of the correlation functions.

The requirement that the correlation functions are uniquely defined appears to be broken. We think this may have connections to the Friedel sum rule, but this is at a very early stage of investigation.

APPENDIX A

FINITE SIZE CORRECTIONS

In these notes we will follow Woynarovich[8] as well as Essler[10] and Pereira et al[32] and calculate the finite size corrections to the spectrum of the 1D Bose gas in the presence of an extra excitation above the standard low energy current and density excitations. These results will allow us to find the relationship between the phenomenological shifts in the Luttinger liquid Hamiltonian and the Bethe ansatz shift functions.

A.1 Finite size expansion

In the ground state the counting function for the Lieb-Liniger gas is

$$z(\lambda) = \lambda + \frac{1}{L}\sum_j \theta(\lambda - \lambda_j), \tag{A.1}$$

the Bethe equations are

$$z(\lambda_j) = \frac{2\pi I_j}{L}, \tag{A.2}$$

and the root densites are given by,

$$2\pi\rho(\lambda) = \partial_\lambda z(\lambda). \tag{A.3}$$

We are interested in calculating the finite size corrections in the presence of an excitation. In this scenario the counting function becomes

$$z(\lambda) = \lambda + \frac{1}{L}\sum_j \theta(\lambda - \lambda_j) + \frac{1}{L}\Phi(\lambda, \xi), \tag{A.4}$$

where $\Phi(\lambda, \xi) = \pm\theta(\lambda - \xi)$ for particle/hole excitations and $\Phi(\lambda, \xi) = -\theta(2\lambda - 2\xi)$ for the spinon. The Bethe equations also change,

$$z(\lambda_j) = \frac{2\pi I_j \pm \pi}{L} \tag{A.5}$$

depending on the type of excitation.

We begin with the part of the calculation that is independent of Φ, we now use the Euler Maclaurin formula,

$$\frac{1}{L}\sum_{n=n_1}^{n_2} f\left(\frac{n}{L}\right) = \int_{\frac{n_-}{L}}^{\frac{n_+}{L}} dx f(x) + \frac{1}{24L^2}\left(f'\left(\frac{n_-}{L}\right) - f'\left(\frac{n_+}{L}\right)\right) + \ldots, \tag{A.6}$$

where

$$n_+ = n_2 + \frac{1}{2}, \quad n_- = n_1 - \frac{1}{2}, \tag{A.7}$$

to expand the root density and get

$$\rho(\lambda) = \frac{1}{2\pi} + \int_{Q_-}^{Q_+} d\mu \frac{K(\lambda-\mu)}{2\pi}\rho(\mu) + \frac{\Phi'(\lambda,\xi)}{2\pi L} + \frac{1}{24L^2}\left(\frac{K'(\lambda-Q_+)}{2\pi\rho(Q_+)} - \frac{K'(\lambda-Q_-)}{2\pi\rho(Q_-)}\right) + \ldots, \tag{A.8}$$

where ξ is the particle excitation quasimomentum. We now expand $\rho(\lambda)$ up to order $1/L^2$ in the form

$$\rho(\lambda) = \rho_0(\lambda) + \frac{1}{L}\rho_1(\lambda) + \frac{1}{24L^2}\left(\frac{\rho_+(\lambda)}{\rho(Q_+)} - \frac{\rho_-(\lambda)}{\rho(Q_-)}\right). \tag{A.9}$$

The integral equations that the expansion terms satisfy are,

$$\rho_0(\lambda) - \int_{Q_-}^{Q_+} d\mu \frac{K(\lambda,\mu)}{2\pi}\rho_0(\mu) = \frac{1}{2\pi} \tag{A.10}$$

$$\rho_1(\lambda) - \int_{Q_-}^{Q_+} d\mu \frac{K(\lambda,\mu)}{2\pi}\rho_1(\mu) = \frac{\Phi'(\lambda,\xi)}{2\pi} \tag{A.11}$$

$$\rho_\pm(\lambda) - \int_{Q_-}^{Q_+} d\mu \frac{K(\lambda,\mu)}{2\pi}\rho_\pm(\mu) = \frac{K'(\lambda-Q_\pm)}{2\pi}. \tag{A.12}$$

We note that these terms all contain terms of order $1/L$ due to the integration limits,

$$Q_+ = q + \delta_+ \tag{A.13}$$

$$Q_- = -q + \delta_-. \tag{A.14}$$

The energy of the system is

$$E = \sum_j \epsilon_0(\lambda_j) + \epsilon_0(\xi), \tag{A.15}$$

which can be expanded using the Euler Maclaurin formula to get,

$$E = L\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho(\mu) + \epsilon_0(\xi) - \frac{1}{24L}\left(\frac{\epsilon_0'((Q_+)}{\rho(Q_+)} - \frac{\epsilon_0'(Q_-)}{\rho(Q_-)}\right). \tag{A.16}$$

Substituting in our expansion for ρ gives,

$$\begin{aligned} E =& L\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho_0(\mu) + \int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho_1(\mu) \\ &+ \frac{1}{24L}\left(\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\frac{\rho_+(\mu)}{\rho(Q_+)} - \int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\frac{\rho_-(\mu)}{\rho(Q_-)}\right) \\ &+ \epsilon_0(\xi) - \frac{1}{24L}\left(\frac{\epsilon_0'((Q_+)}{\rho(Q_+)} - \frac{\epsilon_0'(Q_-)}{\rho(Q_-)}\right). \end{aligned} \tag{A.17}$$

We begin by dealing with the parts of the energy that are explicitly of order $1/L$, they are

$$\begin{aligned} &\frac{1}{24L}\left(\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\frac{\rho_+(\mu)}{\rho(Q_+)} - \int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\frac{\rho_-(\mu)}{\rho(Q_-)}\right) \\ &- \frac{1}{24L}\left(\frac{\epsilon_0'((Q_+)}{\rho(Q_+)} - \frac{\epsilon_0'(Q_-)}{\rho(Q_-)}\right). \end{aligned} \tag{A.18}$$

As we are not interested in terms of higher order we can substitute the zeroth order expression for $Q_\pm$ and ρ giving,

$$\begin{aligned} &\frac{1}{24L}\left(\int_{-q}^{q} \epsilon_0(\mu)\frac{\rho_+(\mu)}{\rho(q)} - \int_{-q}^{q} \epsilon_0(\mu)\frac{\rho_-(\mu)}{\rho(q)}\right) \\ &- \frac{1}{24L}\left(\frac{\epsilon_0'((q)}{\rho(q)} - \frac{\epsilon_0'(-q)}{\rho(q)}\right) \end{aligned} \tag{A.19}$$

and the zeroth order equations for ρ and $\rho_\pm$ are,

$$\rho(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho(\mu) = \frac{1}{2\pi} \tag{A.20}$$

$$\rho_\pm(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_\pm(\mu) = \frac{K'(\lambda \mp q)}{2\pi}. \tag{A.21}$$

We need to evaluate the integral,

$$\int_{-q}^{q} d\mu \rho_\pm(\mu)\mu^2, \tag{A.22}$$

to do this we introduce the function $R(\lambda,\nu)$ defined by,

$$R(\lambda,\nu) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}R(\mu,\nu) = \frac{K(\lambda-\nu)}{2\pi} \tag{A.23}$$

and take the derivative,

$$\partial_\lambda R(\lambda,\nu) - \int_{-q}^{q} \partial_\lambda \frac{K(\lambda,\mu)}{2\pi}R(\mu,\nu) = \frac{K'(\lambda-\nu)}{2\pi} \tag{A.24}$$

$$\partial_\lambda R(\lambda,\nu) + \int_{-q}^{q} \partial_\mu \frac{K(\lambda,\mu)}{2\pi}R(\mu,\nu) = \frac{K'(\lambda-\nu)}{2\pi} \tag{A.25}$$

integrating by parts we get,

$$\begin{aligned}\partial_\lambda R(\lambda,\nu) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\partial_\mu R(\mu,\nu) =& \frac{K'(\lambda-\nu)}{2\pi} \\ &- \frac{K(\lambda-q)}{2\pi}R(q,\nu) \\ &+ \frac{K(\lambda+q)}{2\pi}R(-q,\nu).\end{aligned} \tag{A.26}$$

Now comparing (A.21), (A.23) and (A.26) we find,

$$\rho_{\pm}(\lambda) = \partial_{\lambda} R(\lambda, \pm q) + R(\lambda, q)R(q, \pm q) - R(\lambda, -q)R(-q, \pm q). \tag{A.27}$$

Going back to (A.22),

$$\begin{aligned}\int_{-q}^{q} d\mu \rho_{\pm} \mu^2 = &\int_{-q}^{q} d\mu \partial_{\mu} R(\mu, \pm q)\mu^2 \\ &+ R(q, \pm q) \int_{-q}^{q} d\mu R(\mu, q)\mu^2 \\ &- R(-q, \pm q) \int_{-q}^{q} d\mu R(\mu, -q)\mu^2 \qquad \text{(A.28)} \\ = &\int_{-q}^{q} d\mu \partial_{\mu} R(\mu, \pm q)\mu^2 \\ &+ (R(q, \pm q) - R(q, \mp q)) \int_{-q}^{q} d\mu R(\mu, q)\mu^2. \qquad \text{(A.29)}\end{aligned}$$

The first integral can be calculated by integrating by parts and using $R(q, q) - R(q, -q) = -\rho'(q)/\rho(q)$.

$$\begin{aligned}\int_{-q}^{q} d\mu \partial_{\mu} R(\mu, \pm q)\mu^2 &= \left[R(\mu, \pm q)\mu^2\right]_{-q}^{q} - 2\int_{-q}^{q} d\mu R(\mu, \pm q)\mu \qquad \text{(A.30)} \\ &= \mp \frac{\epsilon_0(q)\rho'(q)}{\rho(q)} \pm \left(\epsilon_0'(q) - \epsilon'(q)\right). \qquad \text{(A.31)}\end{aligned}$$

The other integral is,

$$\begin{aligned}\int_{-q}^{q} d\mu R(\mu, \pm q)\mu^2 &= -\int_{-q}^{q} d\mu \partial_{\mu} F(\pm q, \mu)\mu^2 \qquad \text{(A.32)} \\ &= -\left[F(\pm q, \mu)\mu^2\right]_{-q}^{q} + 2\int_{-q}^{q} d\mu F(\pm q, \mu)\mu \qquad \text{(A.33)}\end{aligned}$$

Using the Slavnov identity,

$$\begin{aligned}
2\int_{-q}^{q} d\mu F(\pm q,\mu)\mu = &-2\int_{-q}^{q} d\mu F(\mu,\pm q)\mu \\
&+2F(\pm q,q)\int_{-q}^{q} d\mu F(\mu,q)\mu \\
&-2F(\pm q,-q)\int_{-q}^{q} d\mu F(\mu,-q)\mu \qquad \text{(A.34)}\\
&=2\left(-1+F(\pm q,q)-F(\pm q,-q)\right)\int_{-q}^{q} d\mu F(\mu,q)\mu \qquad \text{(A.35)}\\
&=2\pi\rho(q)(\epsilon(q)-\epsilon_0(q)), \qquad \text{(A.36)}
\end{aligned}$$

which leads to

$$\int_{-q}^{q} d\mu R(\mu,\pm q)\mu^2 = \epsilon_0(q)(2\pi\rho(q)-1)+2\pi\rho(q)(\epsilon(q)-\epsilon_0(q)) = -\epsilon_0(q). \qquad \text{(A.37)}$$

Finally we get,

$$\int_{-q}^{q} d\mu \rho_{\pm}(\mu)\mu^2 = \pm\left(\epsilon_0'(q)-\epsilon'(q)\right). \qquad \text{(A.38)}$$

Using this we find that the terms in (A.18) become simply $-\pi v/L$ where $v=\epsilon'(q)/2\pi\rho(q)$. This allows us to write the expression for the energy as,

$$E = L\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho_0(\mu)+\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho_1(\mu)+\epsilon_0(\xi)-\frac{\pi v}{6L}. \qquad \text{(A.39)}$$

We will now calculate the first integral, we begin by expanding in $\delta_\pm$.

$$\begin{aligned}\int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho_0(\mu) = &\int_{-q}^{q} \epsilon_0(\mu)\rho_0(\mu) \\ &+ \delta_+\epsilon_0(q)\rho_0(q) - \delta_-\epsilon_0(q)\rho_0(-q) \\ &+ \frac{\delta_+^2}{2}\left(\epsilon_0'(q)\rho_0(q) + \epsilon_0(q)\rho_0'(q)\right) \\ &- \frac{\delta_-^2}{2}\left(\epsilon_0'(-q)\rho_0(-q) + \epsilon_0(q)\rho_0'(-q)\right)\end{aligned} \tag{A.40}$$

Remembering that there is also $1/L$ dependence in ρ_0 we expand that in $\delta_\pm$ to get

$$\begin{aligned}\rho_0(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_0(\mu) = &\frac{1}{2\pi} \\ &+ \delta_+\frac{K(\lambda-q)}{2\pi}\rho_0(q) - \delta_-\frac{K(\lambda+q)}{2\pi}\rho_0(-q) \\ &+ \frac{\delta_+^2}{2}\left(-\frac{K'(\lambda-q)}{2\pi}\rho_0(q) + \frac{K(\lambda-q)}{2\pi}\rho_0'(q)\right) \\ &- \frac{\delta_-^2}{2}\left(-\frac{K'(\lambda+q)}{2\pi}\rho_0(-q) + \frac{K(\lambda+q)}{2\pi}\rho_0'(-q)\right).\end{aligned} \tag{A.41}$$

Now expanding $\rho_0(q)$,

$$\rho_0(\lambda) = \rho_{0,0}(\lambda) + \frac{1}{L}\rho_{0,1}(\lambda) + \frac{1}{L^2}\rho_{0,2}(\lambda). \tag{A.42}$$

The equations satified by the terms in the expansion are,

$$\rho_{0,0}(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_{0,0}(\mu) = \frac{1}{2\pi}, \tag{A.43}$$

$$\rho_{0,1}(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_{0,1}(\mu) = L\delta_+ \frac{K(\lambda-q)}{2\pi}\rho_{0,0}(q) - L\delta_- \frac{K(\lambda+q)}{2\pi}\rho_{0,0}(q), \tag{A.44}$$

$$\begin{aligned}\rho_{0,2}(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_{0,2}(\mu) =& L\delta_+ \frac{K(\lambda-q)}{2\pi}\rho_{0,1}(q) - L\delta_- \frac{K(\lambda+q)}{2\pi}\rho_{0,1}(-q) \\ &+ \frac{(L\delta_+)^2}{2}\left(-\frac{K'(\lambda-q)}{2\pi}\rho_{0,0}(q) + \frac{K(\lambda-q)}{2\pi}\rho'_{0,0}(q)\right) \\ &+ \frac{(L\delta_+)^2}{2}\left(\frac{K'(\lambda+q)}{2\pi}\rho_{0,0}(q) + \frac{K(\lambda+q)}{2\pi}\rho'_{0,0}(q)\right).\end{aligned} \tag{A.45}$$

The zeroth order contribution to (A.40) is,

$$\int_{-q}^{q} \epsilon_0(\mu)\rho_{0,0}(\mu) \tag{A.46}$$

which is just the energy density of the ground state e_{GS}. The first order contribution is zero because the dressed energy has roots at the integration boundaries. The second order contribution is,

$$\begin{aligned}\int_{-q}^{q} \epsilon_0(\mu)\rho_{0,2}(\mu) &+ L\delta_+\epsilon_0(q)\rho_{0,1}(q) - L\delta_-\epsilon_0(q)\rho_{0,1}(-q) \\ &+ \frac{(L\delta_+)^2}{2}\left(\epsilon'_0(q)\rho_{0,0}(q) + \epsilon_0(q)\rho'_{0,0}(q)\right) \\ &+ \frac{(L\delta_-)^2}{2}\left(\epsilon'_0(q)\rho_{0,0}(-q) + \epsilon_0(q)\rho'_{0,0}(q)\right).\end{aligned} \tag{A.47}$$

$\rho_{0,1}$ and $\rho_{0,2}$ can be written as,

$$\begin{aligned}\rho_{0,1} =& L\delta_+ R(\lambda,q)\rho_{0,0}(q) - L\delta_- R(\lambda,-q)\rho_{0,0}(q), && \text{(A.48)}\\ \rho_{0,2} =& L\delta_+ R(\lambda,q)\rho_{0,1}(q) - L\delta_- R(\lambda,-q)\rho_{0,1}(-q)\\ &+ \frac{(L\delta_+)^2}{2}\left(-\rho_+(\lambda)\rho_{0,0}(q) + R(\lambda,q)\rho'_{0,0}(q)\right)\\ &+ \frac{(L\delta_+)^2}{2}\left(\rho_-(\lambda)\rho_{0,0}(q) + R(\lambda,-q)\rho'_{0,0}(q)\right). && \text{(A.49)}\end{aligned}$$

Using (A.37) and (A.38) we find,

$$\begin{aligned}\int_{-q}^{q}\epsilon_0(\mu)\rho_{0,2}(\mu) =& - L\delta_+\epsilon_0(q)\rho_{0,1}(q) + L\delta_-\epsilon_0(q)\rho_{0,1}(-q)\\ &+ \frac{(L\delta_+)^2}{2}\left(-(\epsilon'_0(q)-\epsilon'(q))\rho_{0,0}(q) - \epsilon_0(q)\rho'_{0,0}(q)\right)\\ &+ \frac{(L\delta_+)^2}{2}\left(-(\epsilon'_0(q)-\epsilon'(q))\rho_{0,0}(q) - \epsilon_0(q)\rho'_{0,0}(q)\right). && \text{(A.50)}\end{aligned}$$

The entire second order contribution then reduces to,

$$\frac{1}{2}\epsilon'(q)\left((L\delta_+)^2+(L\delta_-)^2\right) = \pi v\rho^2(q)\left[(L\delta_+)^2+(L\delta_-)^2\right]. \qquad \text{(A.51)}$$

Which means that the energy can now be written as,

$$E = Le_{GS} + \int_{Q_-}^{Q_+} d\mu\epsilon_0(\mu)\rho_1(\mu) + \epsilon_0(\xi) - \frac{\pi v}{6L} + \frac{\pi v\rho^2(q)}{L}\left[(L\delta_+)^2+(L\delta_-)^2\right]. \qquad \text{(A.52)}$$

We note that so far in the calculation we have not used the form of Φ, we now have to.

A.1.1 Particle/hole

The final integral is

$$\int_{Q_-}^{Q_+} d\mu \epsilon_0(\mu)\rho_1(\mu) = \int_{-q}^{q} d\mu \epsilon_0(\mu)\rho_1(\mu) + \delta_+\rho_1(q)\epsilon_0(q) - \delta_-\rho_1(-q)\epsilon_0(q). \tag{A.53}$$

We expand ρ_1 as before,

$$\rho_1(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_1(\mu) = \frac{K(\lambda-\xi)}{2\pi} + \delta_+\rho_1(q)\frac{K(\lambda-q)}{2\pi} - \delta_-\rho_1(-q)\frac{K(\lambda+q)}{2\pi}. \tag{A.54}$$

Expanding ρ_1 in $1/L$ gives,

$$\rho_1(\lambda) = \rho_{1,0}(\lambda) + \frac{1}{L}\rho_{1,1}(\lambda) \tag{A.55}$$

where the terms satisfy,

$$\rho_{1,0}(\lambda) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi}\rho_{1,0}(\mu) = \frac{K(\lambda-\xi)}{2\pi} \tag{A.56}$$

$$\rho_{1,1}(\lambda) - \int_{-q}^{q} d\mu \frac{K(\lambda,\mu)}{2\pi}\rho_{1,1}(\mu) = L\delta_+\rho_{1,0}(q)\frac{K(\lambda-q)}{2\pi} - L\delta_-\rho_{1,0}(-q)\frac{K(\lambda+q)}{2\pi}. \tag{A.57}$$

The zeroth order term is,

$$\begin{aligned}\int_{-q}^{q} d\mu R(\xi,\mu)\mu^2 =& -\int_{-q}^{q} d\mu \partial_\mu F(\xi,\mu)\mu^2 \\ =& -\left[F(\xi,\mu)\mu^2\right]_{-q}^{q} + 2\int_{-q}^{q} d\mu F(\xi,\mu)\mu \\ =& -\epsilon_0(q)[F(\xi,q)-F(\xi,-q)] - 2\int_{-q}^{q} d\mu F(\mu,\xi)\mu \\ &+ [F(\xi,q)-F(\xi,-q)]2\int_{-q}^{q} d\mu F(\mu,q)\mu \\ =& -\epsilon_0(q)[F(\xi,q)-F(\xi,-q)] + \epsilon(\xi) - \epsilon_0(\xi) \\ &+ [F(\xi,q)-F(\xi,-q)](\epsilon_0(q)-\epsilon(q)) \\ =& \epsilon(\xi)-\epsilon_0(\xi). \end{aligned} \tag{A.58}$$

The first order terms sum to zero and the energy is,

$$E_{particle} = Le_{GS} + \epsilon(\xi) - \frac{\pi v}{6L} + \frac{\pi v \rho^2(q)}{L}\left[(L\delta_+)^2 + (L\delta_-)^2\right]. \tag{A.59}$$

The hole excitation differs only in the sign of $K(\lambda-\xi)$ and the bare energy. This results in

$$E_{hole} = Le_{GS} - \epsilon(\xi) - \frac{\pi v}{6L} + \frac{\pi v \rho^2(q)}{L}\left[(L\delta_+)^2 + (L\delta_-)^2\right] \tag{A.60}$$

for the energy.

A.1.2 Spinon

The final integral is still

$$\int_{Q_-}^{Q_+} d\mu \epsilon_0(\mu)\rho_1(\mu) = \int_{-q}^{q} d\mu \epsilon_0(\mu)\rho_1(\mu) + \delta_+\rho_1(q)\epsilon_0(q) - \delta_-\rho_1(-q)\epsilon_0(q). \tag{A.61}$$

ρ_1 is now,

$$\rho_1(\lambda) - \int_{-q}^{q} \frac{K(\lambda,\mu)}{2\pi}\rho_1(\mu) = -2\frac{K(2\lambda-2\xi)}{2\pi} + \delta_+\rho_1(q)\frac{K(\lambda-q)}{2\pi} - \delta_-\rho_1(-q)\frac{K(\lambda+q)}{2\pi}. \tag{A.62}$$

Expanding ρ_1 as before we now have

$$\rho_{1,0}(\lambda,\xi) - \int_{-q}^{q} d\mu\frac{K(\lambda,\mu)}{2\pi}\rho_{1,0}(\mu,\xi) = -2\frac{K(2\lambda-2\xi)}{2\pi} \tag{A.63}$$

$$\rho_{1,1}(\lambda) - \int_{-q}^{q} d\mu\frac{K(\lambda,\mu)}{2\pi}\rho_{1,1}(\mu) = L\delta_+\rho_{1,0}(q)\frac{K(\lambda-q)}{2\pi} - L\delta_-\rho_{1,0}(-q)\frac{K(\lambda+q)}{2\pi}. \tag{A.64}$$

The first order terms are again zero. The shift function for the spinon excitation is

$$\bar{F}(\lambda,\xi) - \int_{-q}^{q} d\mu\frac{K(\lambda,\mu)}{2\pi}\bar{F}(\mu,\xi) = -\frac{\pi+\theta(2\lambda-2\xi)}{2\pi} \tag{A.65}$$

and the derivative is

$$\begin{aligned}\partial_\lambda\bar{F}(\lambda,\xi) - \int_{-q}^{q} d\mu\frac{K(\lambda,\mu)}{2\pi}\partial_\mu\bar{F}(\mu,\xi) = &-2\frac{K(2\lambda-2\xi)}{2\pi} \\ &- \frac{K(\lambda-q)}{2\pi}\bar{F}(q,\xi) \\ &+ \frac{K(\lambda+q)}{2\pi}\bar{F}(-q,\xi).\end{aligned} \tag{A.66}$$

Comparing (A.63) with (A.66) we find,

$$\rho_{1,0}(\lambda,\xi) = \partial_\lambda\bar{F}(\lambda,\xi) + \bar{F}(q,\xi)R(\lambda,q) - \bar{F}(-q,\xi)R(\lambda,-q). \tag{A.67}$$

The integral is

$$\begin{aligned}\int_{-q}^{q} d\mu\rho_{1,0}(\mu,\xi)\mu^2 &= \int_{-q}^{q} d\mu\partial_\mu\bar{F}(\mu,\xi)\mu^2 \\ &\quad + \left[\bar{F}(q,\xi)-\bar{F}(-q,\xi)\right]\int_{-q}^{q} d\mu R(q,\mu)\mu^2 \\ &= -2\int_{-q}^{q} d\mu\bar{F}(\mu,\xi)\mu + \left[\bar{F}(q,\xi)-\bar{F}(-q,\xi)\right]q^2 \\ -\left[\bar{F}(q,\xi)-\bar{F}(-q,\xi)\right]\epsilon_0(q) & \\ &= -2\int_{-q}^{q} d\mu\bar{F}(\mu,\xi)\mu = \omega\xi, \end{aligned} \tag{A.68}$$

where $\omega\xi$ is the energy of the spinon above the ground state. The energy is

$$E_{spinon} = Le_{GS} + \omega(\xi) - \frac{\pi v}{6L} + \frac{\pi v\rho^2(q)}{L}\left[(L\delta_+)^2 + (L\delta_-)^2\right]. \tag{A.69}$$

A.2 Calculating $\delta_\pm$

Now that we have the finite size correction to the energy in terms of the change of the integral boundaries we should calculate what the change of boundaries are. We do this by considering the density of and current through the system. These calculation are specific to the type of excitation in the system.

A.2.1 Particle/hole

The density of the system is,

$$\frac{N}{L} = n + \frac{\Delta N}{L} = \frac{I_+ - I_-}{L} = \int_{Q_-}^{Q_+} d\mu\rho(\mu). \tag{A.70}$$

expanding the integral in $\delta\pm$ and equating terms of order $1/L$ we find,

$$\begin{aligned}\frac{\Delta N}{L} =&\frac{1}{L}\int_{-q}^{q} d\mu(\rho_{0,1}(\mu)+\rho_{1,0}(\mu)) + \delta_+\rho_{0,0}(q) - \delta_-\rho_{0,0}(q)\\ =&\delta_+\rho_{0,0}(q)\int_{-q}^{q} d\mu R(q,\mu) - \delta_-\rho_{0,0}(q)\int_{-q}^{q} d\mu R(q,\mu)\\ &+\delta_+\rho_{0,0}(q) - \delta_-\rho_{0,0}(q) + \frac{1}{L}\int_{-q}^{q} d\mu R(\xi,\mu)\\ =&(\delta_+\rho_{0,0}(q) - \delta_-\rho_{0,0}(q))[-F(q,q)+F(q,-q)+1] + \frac{1}{L}\int_{-q}^{q} d\mu R(\xi,\mu)\\ =&\rho(q)Z(q)(\delta_+ - \delta_-) + \frac{n_{imp}}{L}, \end{aligned} \tag{A.71}$$

where

$$n_{imp} = \int_{-q}^{q} d\mu R(\xi,\mu) = -[F(\xi,q) - F(\xi,-q)]. \tag{A.72}$$

If we consider an excitation where D particles are moved from one Fermi point to the other we have,

$$\frac{2D}{L} = \frac{I_+ + I_-}{L} = \int_{Q_-}^{-\infty} d\mu\rho(\mu) - \int_{\infty}^{Q_+} d\mu\rho(\mu). \tag{A.73}$$

Expanding as previously we find,

$$\begin{aligned}\frac{2D}{L} =&\delta_+\rho_{0,0}(q) + \delta_-\rho_{0,0}(q)\\ &+\frac{1}{L}\int_{-\infty}^{-q} d\mu[\rho_{0,1}(\mu)+\rho_{1,0}(\mu)]\\ &-\frac{1}{L}\int_{q}^{\infty} d\mu[\rho_{0,1}(\mu)+\rho_{1,0}(\mu)]\\ &=[\delta_+\rho_{0,0}(q)+\delta_-\rho_{0,0}(q)][1-F(q,q)-F(q,-q)] + \frac{2d_{imp}}{L}\\ &=[\delta_+\rho_{0,0}(q)+\delta_-\rho_{0,0}(q)]Z^{-1}(q) + \frac{2d_{imp}}{L}, \end{aligned} \tag{A.74}$$

where

$$2d_{imp} = \int_{-\infty}^{-q} d\mu R(\xi,\mu) - \int_{q}^{\infty} d\mu R(\xi,\mu) = -[F(\xi,q) + F(\xi,-q)]. \tag{A.75}$$

The finite size spectrum is,

$$E = Le_{GS} + \epsilon(\xi) - \frac{\pi v}{6L} + \frac{2\pi v}{L}\left[\frac{\Delta N - n_{imp}}{2Z}\right]^2 + \frac{2\pi v}{L} Z^2 \left(D - d_{imp}\right)^2 \tag{A.76}$$

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