NILPOTENT INJECTORS IN FINITE GROUPS

by

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Abstract

We prove that the odd nilpotent injectors (a certain type of maximal nilpotent subgroup) of a minimal simple group are all conjugate, extending the result from soluble groups. We also prove conjugacy in $\mathrm{GU}_3(q)$ and $\mathrm{SU}_3(q)$. In a minimal counterexample to the conjecture that the odd nilpotent injectors of an arbitrary finite group are all conjugate we show that there must be a component, which cannot be of type A_n except possibly $3 \cdot A_6$ or $3 \cdot A_7$. Finally, we produce a partial result on minimal simple groups for a more general type of nilpotent injector.

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Contents

Chapter 1

Introduction

Nilpotent injectors are a type of maximal nilpotent subgroup which were first defined in [15]. It is proved by Mann in [16] that they form a single conjugacy class in groups G which satisfy $C_G(F(G)) \leqslant F(G)$. The first step in proving this theorem is to prove that they exist in those groups, as the definition does not guarantee it. In [13] Bialostocki gives an alternative definition which *does* guarantee their existence, and, at least in the groups considered by Mann, is equivalent to the original. Bialostocki's definition is as follows: Let G be a finite group. Then

$$\begin{aligned}
&d_2(G) = \max \{|A| \mid A \leqslant G \text{ is nilpotent of class at most } 2\};\\
&\mathcal{A}_2(G) = \{A \leqslant G \mid A \text{ is nilpotent of class at most 2 and } |A| = d_2(G)\};\\
&\mathrm{Max}_{\mathcal{N}}(G) = \{I \leqslant G \mid I \text{ is maximal nilpotent}\};\\
&\mathcal{NI}(G) = \{I \in \mathrm{Max}_{\mathcal{N}}(G) \mid I \text{ contains an element of } \mathcal{A}_2(G)\}.
\end{aligned}$$

A nilpotent injector is defined to be an element of $\mathcal{NI}(G)$. In [13] Bialostocki makes the conjecture that $\mathcal{NI}(G)$ forms a single conjugacy class in any finite group G. He verifies it for the Symmetric groups in the same paper and for the Alternating groups in [14]. Other groups for which the conjecture holds are the General Linear Groups (Sheu [17]) and groups in which every local subgroup L satisfies $C_L(F(L)) \leqslant F(L)$ (Flavell [4]). However, the conjecture turns out to be false, and a counterexample is provided in [18, p47].

It is clear that there are very many examples of groups in which arbitrary maximal nilpotent subgroups are not conjugate (any non-nilpotent group in fact). Nilpotent injectors differ from such subgroups because they contain elements of $\mathcal{A}_2(G)$. One effect of having such a subgroup is that nilpotent injectors contain every nilpotent subgroup that they normalize. The following theorem of Glauberman is fundamental in establishing that fact:

Theorem 1.1 *(Glauberman)[7, Theorem B, p470] Let G be a finite group and let $A \in \mathcal{A}_2(G)$. Suppose $B \leqslant G$ is nilpotent and normalized by A. Then AB is nilpotent.*

In a different paper from the same journal, Arad and Glauberman prove a similar result:

Theorem 1.2 *(Arad, Glauberman)[6, Proposition 1, p313] Let G be a finite group and let $A \leqslant G$ have maximum possible order subject to being abelian of odd order. Suppose $B \leqslant G$ is nilpotent of odd order normalized by A. Then AB is nilpotent.*

The proof can be found in Chapter 3 (Theorem 3.5). We are led into making the following definition:

Let G be a finite group. Then

$$d_{\mathcal{O}}(G) = \max\{|A| \mid A \leqslant G \text{ is abelian of odd order}\};$$

$$\mathcal{A}_{\mathcal{O}}(G) = \{A \leqslant G \mid A \text{ is abelian and } |A| = d_{\mathcal{O}}(G)\};$$

$$\mathrm{Max}_{\mathcal{NO}}(G) = \{I \leqslant G \mid I \text{ is maximal subject to being nilpotent of odd order}\};$$

$$\mathcal{NI}_{\mathcal{O}}(G) = \{I \in \mathrm{Max}_{\mathcal{NO}}(G) \mid I \text{ contains an element of } \mathcal{A}_{\mathcal{O}}(G)\}.$$

Arad and Glauberman's Theorem, together with the results proved on nilpotent injectors, suggest that elements of $\mathcal{NI}_{\mathcal{O}}(G)$, which we call **odd nilpotent injectors**, may be conjugate in various classes of groups. Further motivation to study $\mathcal{NI}_{\mathcal{O}}(G)$ is given by Bender in [11] where he observes that if G is soluble and $A \in \mathcal{A}_{\mathcal{O}}(G)$ then any 2-subgroup normalized by A is contained in $\mathcal{O}_2(G)$ (see Theorem 3.19), and says, "I have a feeling that abelian $2'$-subgroups of maximal order could be interesting objects in arbitrary finite groups".

The majority of this thesis is devoted to the study of odd nilpotent injectors. In Chapter 4 we prove that they form a single conjugacy class inside minimal simple groups, a special case of the groups considered by Flavell in [4]. In Chapter 6 we calculate the elements of $\mathcal{A}_{\mathcal{O}}(\mathrm{GU}_3(q))$ and $\mathcal{A}_{\mathcal{O}}(\mathrm{SU}_3(q))$ explicitly and prove that $\mathcal{NI}_{\mathcal{O}}(\mathrm{GU}_3(q))$ and $\mathcal{NI}_{\mathcal{O}}(\mathrm{SU}_3(q))$ form single conjugacy classes. In Chapter 7 we consider a minimal counterexample to the conjecture that the odd nilpotent injectors of an arbitrary finite group are all conjugate and show that such a group must have a component which cannot be of alternating type, except possibly $3 \cdot A_6$ or $3 \cdot A_7$.

Chapter 5 does not concern odd nilpotent injectors. When Bialostocki conjectured in [13] that nilpotent injectors always formed a single conjugacy class, he also made a more general conjecture. First we need a generalization of $\mathcal{NI}(G)$.

Let G be a finite group and let π be a set of primes. Then

$$\begin{aligned}
d_{2,\pi}(G) &= \max\{|A| \mid A \leqslant G \text{ is a nilpotent } \pi\text{-subgroup of class at most } 2\};\\
\mathcal{A}_{2,\pi}(G) &= \{A \leqslant G \mid A \text{ is nilpotent of class at most 2 and } |A| = d_{2,\pi}(G)\};\\
\mathrm{Max}_{\mathcal{N}\pi}(G) &= \{I \leqslant G \mid I \text{ is maximal subject to being a nilpotent } \pi\text{-subgroup}\};\\
\mathcal{NI}_\pi(G) &= \{I \in \mathrm{Max}_{\mathcal{N}\pi}(G) \mid I \text{ contains an element of } \mathcal{A}_{2,\pi}(G)\}.
\end{aligned}$$

Elements of $\mathcal{NI}_\pi(G)$ are called π-nilpotent injectors. Bialostocki's conjecture was that $\mathcal{NI}_\pi(G)$ always formed a single conjugacy class. Of course, the fact that this is not true of $\mathcal{NI}(G)$ makes the conjecture false. In Chapter 5 we make some progress towards proving the conjecture for minimal simple groups.

Chapter 2

Preliminaries

All groups and vector spaces considered in this thesis are finite.

We often use the letters G and p without explicitly defining them. Whenever this happens G is to be taken as an arbitrary group and p as an arbitrary prime.

A result which is either already known or assumed to be so will be called a lemma or a theorem. A result which is thought to be original will be called a proposition or a theorem, and the two types of theorem will be distinguished by a reference being given in the former case.

2.1 Notation and Assumed Results

Let H be a subgroup of G and let σ and τ each be a set of primes. We use the following notation, most of which is standard:

$\pi(G)$ = the set of prime divisors of $|G|$;

$Max(G)$ = the set of maximal subgroups of G;

$\mathcal{O}_\sigma(G)$ = the largest normal σ-subgroup of G;

$\mathcal{O}(G) = \mathcal{O}_{2'}(G)$;

$\mathcal{O}_{\sigma,\tau}(G)$ = the inverse image of $\mathcal{O}_\tau(G/\mathcal{O}_\sigma(G))$ in G;

$F(G)$ = the Fitting subgroup of G;

$F_\sigma(G) = \mathcal{O}_\sigma(F(G))$;

$F_{\sigma,\tau}(G)$ = the inverse image of $F_\tau(G/\mathcal{O}_\sigma(G))$ in G;

$\Phi(G)$ = the Frattini subgroup of G;

$\Omega(G)$ = the group generated by the elements of order p for a p-group G;

$\mathcal{N}_G(H,\sigma)$ = the set of all σ-subgroups of G which are normalized by H;

$\mathcal{N}^*_G(H,\sigma)$ = the set of maximal elements of $\mathcal{N}_G(H,\sigma)$ with respect to inclusion.

We remark that if σ (or τ) consists of a single prime then we write that prime in place of σ (or τ).

Given $N \trianglelefteq G$ we use the bar notation for the quotient group G/N. That is, we set $\overline{G} = G/N$, meaning that for any $H \leqslant G$ or any $g \in G$ we let $\overline{H}$ be the group HN/N and $\overline{g}$ be the coset Ng. When we make a statement of the form "let $\overline{H} \leqslant \overline{G}$", we are not defining a subgroup H of G, just a subgroup $\overline{H}$ of $\overline{G}$. By the Correspondence Theorem there is a unique subgroup H of G containing N which maps onto $\overline{H}$. This will be called **the inverse image of $\overline{H}$ in G**. More generally, if $K \leqslant G$ and $\overline{H} \leqslant \overline{K}$ then the group $H \cap K$ is called **the inverse image of $\overline{H}$ in K**, where H is the inverse image of $\overline{H}$ in G. Corollary 2.3 shows that both inverse images map onto $\overline{H}$, justifying the definition.

Lemma 2.1 *[2, 1.14, p6] Let A, B and C be subgroups of a group G. Suppose $A \leqslant C$. Then $AB \cap C = A(B \cap C)$.*

Lemma 2.2 *Let $N \trianglelefteq G$ and set $\overline{G} = G/N$. Let $H, K \leqslant G$ and suppose N is contained in H or K. Then $\overline{H \cap K} = \overline{H} \cap \overline{K}$.*

Proof. Without loss of generality assume $N \leqslant H$. We have $\overline{H \cap K} \leqslant \overline{H} \cap \overline{K}$. Furthermore, Lemma 2.1 allows us to make the following calculation: $|\overline{H \cap K}| = |(H \cap K)N|/|N| = |H \cap KN|/|N| = |H||KN|/|N||HKN| = |H/N||KN/N|/|HK/N| = |\overline{H}||\overline{K}|/|\overline{HK}| = |\overline{H} \cap \overline{K}|$. □

Corollary 2.3 *Let $N \trianglelefteq G$ and set $\overline{G} = G/N$. Let $\overline{H} \leqslant \overline{G}$ and let $K \leqslant G$ such that $\overline{H} \leqslant \overline{K}$. Let H be the inverse image of $\overline{H}$ in G and let H_1 be the inverse image of $\overline{H}$ in K. Then $\overline{H} = \overline{H_1}$.*

Proof. Immediate from the lemma because H contains N and $\overline{H} \leqslant \overline{K}$. $\square$

Lemma 2.4 *(Frattini Argument) [3, 3.1.4, p58] Let G act on a set X and suppose $H \leqslant G$ acts transitively on X. Then $G = Stab_G(x)H$ for any $x \in X$.*

Lemma 2.5 *[1, 3.2.2, p64] An abelian group with a faithful irreducible representation is cyclic.*

Theorem 2.6 *[1, 5.1.1, p173] $G = \langle \Phi(G), x_1, ..., x_n \rangle$ if and only if $G = \langle x_1, ..., x_n \rangle$.*

Lemma 2.7 *[1, 5.1.3, p174] Let P be a p-group. Then $P/\Phi(P)$ is elementary abelian.*

Theorem 2.8 *(Burnside) [1, 5.1.4, p174] Let P be a p-group and ψ a p'-automorphism of P. If $[\psi, P/\Phi(P)] = 1$ then $[\psi, P] = 1$.*

Theorem 2.9 *[1, 5.2.3, p177] Let G act coprimely on the abelian group A. Then $A = C_A(G) \times [A, G]$. In particular, if G acts coprimely on a vector space V then $V = C_V(G) \oplus [V, G]$.*

Theorem 2.10 *[3, 8.2.7, p187] Let G act coprimely on a group H and suppose at least one of G and H is soluble. Then $H = [H, G]C_H(G)$.*

Lemma 2.11 *(Thompson $P \times Q$ Lemma) [1, 5.3.4, p179] Let P be a p-group and let Q be a p'-group such that $P \times Q$ acts on the p-group G. If $[C_G(P), Q] = 1$ then $[G, Q] = 1$.*

Lemma 2.12 *[1, 5.4.10, p199] [4, 2.11, p411] Let p be an odd prime and let P be a p-group.*

(i) *If P does not contain a noncyclic abelian normal subgroup then P is cyclic;*

(ii) *If $Q \trianglelefteq P$ contains noncyclic abelian normal subgroups then one of them is normal in P.*

Theorem 2.13 *[1, 6.1.3, p218] Let G be soluble. Then $C_G(F(G)) \leqslant F(G)$.*

Theorem 2.14 *(Schur-Zassenhaus Theorem) [1, 6.2.1, p221] Let $N \trianglelefteq G$ such that $|N|$ and $|G/N|$ are coprime. Then there exists a complement to N in G. Moreover, if N or G/N is soluble then all such complements are conjugate.*

Theorem 2.15 *[1, 6.2.2(i), p224] Let the group H act coprimely on G, and suppose at least one of G and H is soluble. Then for any prime p there exists a Sylow p-subgroup of G that is left invariant by H.*

Definition 2.16 *Let G and H be as in the above theorem. Then we denote the set of H-invariant Sylow p-subgroups of G by $\mathrm{Syl}_p(G; H)$.*

Theorem 2.17 *[1, 6.2.4, p225] Let the noncyclic abelian π-group A act on the π'-group G. Then $G = \langle C_G(a) | a \in A^{\#} \rangle$.*

Theorem 2.18 *[3, 8.3.4(c), p193] Let $V \cong \mathbb{Z}_p \times \mathbb{Z}_p$ act coprimely on G. Then $[G, V] = \langle [C_G(v), V] | v \in V^{\#} \rangle$.*

Theorem 2.19 *[3, 8.2.7, p187] Let H be a group acting coprimely on G and suppose at least one of H and G is soluble. Then $[G, H, H] = [G, H]$.*

Lemma 2.20 *[3, 8.2.2(a), p184] Let H be a group acting on G and let N be a normal H-invariant subgroup of G. Suppose that $|H|$ and $|N|$ are coprime and that at least one of H and N is soluble. Set $\overline{G} = G/N$. Then $C_{\overline{G}}(H) = \overline{C_G(H)}$.*

Definition 2.21 *Let $n \in \mathbb{N}$ and let π be a set of primes. Then $(n)_\pi$ denotes the largest π-factor of n. We drop the brackets when considering the order of a group, i.e., we write $|G|_\pi$ instead of $(|G|)_\pi$.*

Definition 2.22 *Let π be a set of primes. A subgroup H of G is a* ***Hall π-subgroup*** *if $|H| = |G|_\pi$. We denote the set of Hall π-subgroups by $\mathrm{Hall}_\pi(G)$.*

Theorem 2.23 *(Hall's Theorem) [1, 6.4.1, p231] Let G be soluble and let π be a set of primes. Then*

(i) *G contains a Hall π-subgroup;*

(ii) *the Hall π-subgroups of G are all conjugate;*

(iii) *every π-subgroup of G is contained in a Hall π-subgroup of G.*

Lemma 2.24 *(Goldschmidt) [2, 31.15, p159] Let G be soluble and let P be a p-subgroup of G. Then $\mathcal{O}_{p'}(N_G(P)) \leqslant \mathcal{O}_{p'}(G)$.*

2.2 General Results

Lemma 2.25 *Let $D \cong \mathbb{Z}_p^3$ act on the nontrivial p'-groups X and Y. Then there exists $d \in D^\#$ such that $C_X(d) \neq 1 \neq C_Y(d)$.*

Proof. Choose a counterexample with $|X| + |Y|$ as small as possible. Then D acts irreducibly on X. By Theorem 2.15 we see that X is a q-group for some prime q, and it follows that X is elementary abelian because $\Omega(Z(X)) \neq 1$ is D-invariant. So $\overline{D} = D/C_D(X)$ has a faithful irreducible representation, which implies that $\overline{D}$ is cyclic by Lemma 2.5. Thus $\overline{D} \cong \mathbb{Z}_p$ and $C_D(X) \cong \mathbb{Z}_p^2$. Similarly, $C_D(Y) \cong \mathbb{Z}_p^2$, so we must have $|C_D(X) \cap C_D(Y)| \geqslant p$. $\square$

Lemma 2.26 *Let P be a p-group which contains a noncyclic abelian normal subgroup. Then P has a normal subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.*

Proof. Let $A \trianglelefteq P$ be noncyclic abelian. Then $\Omega(A) \trianglelefteq P$ is also noncyclic abelian. Let $B \leqslant \Omega(A) \cap Z(P)$ have order p and set $\overline{P} = P/B$. Then $1 \neq \overline{\Omega(A)} \trianglelefteq \overline{P}$, so we can take $\overline{C} \leqslant \overline{\Omega(A)} \cap Z(\overline{P})$ of order p. Let C be the inverse image of $\overline{C}$ in P. Then $C \trianglelefteq P$ and $|C| = p^2$. Since $C \leqslant \Omega(A)$ it must be the case that $C \cong \mathbb{Z}_p \times \mathbb{Z}_p$. $\square$

Lemma 2.27 *Let P be a p-group acting on the cyclic p'-group G. Assume p is larger than any prime divisor of $|G|$. Then P acts trivially on G.*

Proof. Suppose not, so that $[P, G] \neq 1$. Pick q such that $[P, \mathcal{O}_q(G)] \neq 1$. Then Theorem 2.8 implies that

$$[P, \mathcal{O}_q(G)/\Phi(\mathcal{O}_q(G))] \neq 1.$$

Since G is cyclic we have $\mathcal{O}_q(G)/\Phi(\mathcal{O}_q(G)) \cong \mathbb{Z}_q$ by Lemma 2.7. But then p divides $|\mathrm{Aut}(\mathbb{Z}_q)| = q - 1$, a contradiction since $p > q$. $\square$

Lemma 2.28 *Let H be a group acting coprimely on the soluble group K. If $[H, F(K)] = 1$ then $[H, K] = 1$.*

Proof. Let $G = H \ltimes K$. Then since $H \leqslant C_G(F(K))$ and $C_G(F(K)) \trianglelefteq G$ we get $[K, H] \leqslant K \cap C_G(F(K)) = C_K(F(K)) \leqslant F(K)$. Hence $[K, H, H] \leqslant [F(K), H] = 1$, giving $[K, H] = 1$ by Lemma 2.19. $\square$

Lemma 2.29 *(Bender) Let M and H be maximal subgroups of the simple group G. Suppose that M and H are soluble and that $F(M) \leqslant H$ and $F(H) \leqslant M$. Then $M = H$ or there exists a prime p such that $F(M) = \mathcal{O}_p(M)$ and $F(H) = \mathcal{O}_p(H)$.*

Proof. Let p and q be distinct primes. Then $[\mathcal{O}_p(M), \mathcal{O}_q(H)] \leqslant \mathcal{O}_p(M) \cap \mathcal{O}_q(H) = 1$. Therefore $\pi(F(M)) \subseteq \pi(F(H))$, since if p is in $\pi(F(M))$ but not in $\pi(F(H))$ then $1 \neq \mathcal{O}_p(M) \leqslant C_H(F(H)) \leqslant F(H)$, a contradiction. Similarly $\pi(F(H)) \subseteq \pi(F(M))$, so

$$\pi(F(M)) = \pi(F(H)).$$

Let $p \in \pi(F(M))$. Then $F(\mathcal{O}_{p'}(H)) \leqslant F(H)$ because $\mathcal{O}_{p'}(H) \trianglelefteq H$, and it follows that $F(\mathcal{O}_{p'}(H)) = F_{p'}(H)$. Since $[\mathcal{O}_p(M), F_{p'}(H)] \leqslant \mathcal{O}_p(M) \cap F_{p'}(H) = 1$ we get $[\mathcal{O}_p(M), F(\mathcal{O}_{p'}(H))] = 1$. Lemma 2.28 then yields

$$[\mathcal{O}_p(M), \mathcal{O}_{p'}(H)] = 1,$$

and so $\mathcal{O}_{p'}(H) \leqslant N_G(\mathcal{O}_p(M)) = M$. Hence $\mathcal{O}_{p'}(H) \leqslant N_M(\mathcal{O}_p(H))$. Moreover, $\mathcal{O}_{p'}(H)$ is normal in $N_M(\mathcal{O}_p(H))$ since $N_M(\mathcal{O}_p(H)) \leqslant N_G(\mathcal{O}_p(H)) = H$, therefore $\mathcal{O}_{p'}(H) \leqslant \mathcal{O}_{p'}(N_M(\mathcal{O}_p(H)))$. By Goldschmidt's Lemma (2.24) we have $\mathcal{O}_{p'}(N_M(\mathcal{O}_p(H))) \leqslant \mathcal{O}_{p'}(M)$, and we conclude that

$$\mathcal{O}_{p'}(H) \leqslant \mathcal{O}_{p'}(M).$$

We now repeat the argument with the roles of M and H switched to deduce that $\mathcal{O}_{p'}(H) = \mathcal{O}_{p'}(M)$. The result follows, because either $\mathcal{O}_{p'}(M) \neq 1$, in which case $H = N_G(\mathcal{O}_{p'}(H)) = N_G(\mathcal{O}_{p'}(M)) = M$ or $\mathcal{O}_{p'}(M) = 1$ and p was the only prime in $\pi(F(M)) = \pi(F(H))$ $\square$

Lemma 2.30 *(Bender) Assume the hypothesis of Lemma 2.29, except replace the condition $F(H) \leqslant M$ with $\pi(F(H)) \subseteq \pi(F(M))$. Then the conclusion still holds.*

Proof. Let $q \in \pi(F(H))$ and let p be a prime distinct from q. Then $\mathcal{O}_q(M) \neq 1$, so $C_{\mathcal{O}_q(H)}(\mathcal{O}_q(M)) \leqslant N_G(\mathcal{O}_q(M)) = M$. This implies that

$$[C_{\mathcal{O}_q(H)}(\mathcal{O}_q(M)), \mathcal{O}_p(M)] \leqslant \mathcal{O}_p(M) \cap \mathcal{O}_q(H) = 1,$$

and Thompson's $P \times Q$ Lemma gives

$$[\mathcal{O}_q(H), \mathcal{O}_p(M)] = 1. \tag{†}$$

It follows that $\pi(F(H)) = \pi(F(M))$, because if $p \in \pi(F(M)) \backslash \pi(F(H))$ then $1 \neq$

$\mathcal{O}_p(M) \leqslant C_H(F(H)) \leqslant F(H)$.

If $|\pi(F(H))| = 1$ then we are done, so we may assume $|\pi(F(H))| \geqslant 2$. Then by (†) we have $\mathcal{O}_q(H) \leqslant N_G(F_{q'}(M)) = M$ and $F_{q'}(H) \leqslant N_G(\mathcal{O}_q(M)) = M$. Hence $F(H) \leqslant M$ and Lemma 2.29 provides the result. □

Theorem 2.31 *(Bender) Let G be a simple group and let M and H be maximal subgroups of G. Suppose M and H are soluble and let $\pi = \pi(F(M))$. Assume $\pi(F(H)) \subseteq \pi$ and $X \leqslant F(M) \cap H$ such that $C_{F(M)}(X) \leqslant X$. Then $M = H$ or there exists a prime p such that $F(H) = \mathcal{O}_p(H)$ and $F(M) = \mathcal{O}_p(M)$.*

Proof. If $|\pi| = 1$ then we are done because $|\pi(F(H))| \geqslant 1$, so we may take distinct primes $p \in \pi$ and $q \in \pi(F(H))$. Since

$$Z(F(M)) \leqslant C_{F(M)}(X) \leqslant X \leqslant H,$$

we see that $Z(\mathcal{O}_p(M)) \times Z(\mathcal{O}_q(M))$ acts on $\mathcal{O}_q(H)$. We have $C_{\mathcal{O}_q(H)}(Z(\mathcal{O}_q(M))) \leqslant N_G(Z(\mathcal{O}_q(M))) = M$, so

$$\begin{aligned}[][C_{\mathcal{O}_q(H)}(Z(\mathcal{O}_q(M))), Z(\mathcal{O}_p(M))] &\leqslant [M, \mathcal{O}_p(M)] \cap [\mathcal{O}_q(H), Z(F(M))] \\ &\leqslant \mathcal{O}_p(M) \cap \mathcal{O}_q(H) = 1.\end{aligned}$$

Thompson's $P \times Q$ Lemma then gives

$$[\mathcal{O}_q(H), Z(\mathcal{O}_p(M))] = 1. \qquad (\dagger)$$

We deduce that $\pi(F(H)) = \pi$, because (†) holds for every prime $q \in \pi(F(H))$ distinct from p, so if $p \notin \pi(F(H))$ then $Z(\mathcal{O}_p(M)) \leqslant C_H(F(H)) \leqslant F(H)$, a contradiction.

It now follows from (†) and the fact that $|\pi| \geqslant 2$ that $\mathcal{O}_p(H) \leqslant N_G(Z(F_{p'}(M))) = M$ and $F_{p'}(H) \leqslant N_G(Z(\mathcal{O}_p(M))) = M$. Hence $F(H) \leqslant M$, and because $\pi = \pi(F(H))$ the previous lemma applies (with the roles of H and M switched). $\square$

Lemma 2.32 *Let $H, K \leqslant G$ have odd order and suppose the images of H and K inside $G/\mathcal{O}_2(G)$ are conjugate. Then H and K are conjugate in G.*

Proof. By hypothesis there exists $g \in G$ such that $H^g\mathcal{O}_2(G) = K\mathcal{O}_2(G)$. It follows from the theorem of Schur-Zassenhaus that H^g is conjugate to K since they are both complements to $\mathcal{O}_2(G)$ in $K\mathcal{O}_2(G)$, and $\mathcal{O}_2(G)$ is soluble. $\square$

A standard application of the next two results is where G is soluble and $F = F(G)$. We use a more general hypothesis in order to apply Proposition 2.34 to prove Theorem 3.11.

Lemma 2.33 *Let $H, K \leqslant G$ be nilpotent and suppose $F \leqslant H \cap K$ such that $C_G(F) \leqslant F$ and $F \trianglelefteq G$. Then for any distinct primes p and q we have $[\mathcal{O}_p(H), \mathcal{O}_q(K)] = 1$.*

Proof. Let p and q be distinct primes. Then

$$\mathcal{O}_p(H) \leqslant C_G(\mathcal{O}_{p'}(F)) \trianglelefteq G$$

and

$$\mathcal{O}_q(K) \leqslant C_G(\mathcal{O}_{q'}(F)) \trianglelefteq G,$$

so we get

$$[\mathcal{O}_p(H), \mathcal{O}_q(K)] \leqslant C_G(\mathcal{O}_{p'}(F)) \cap C_G(\mathcal{O}_{q'}(F)) = C_G(F) = Z(F).$$

So $\mathcal{O}_q(K)$ normalizes $Z(F)\mathcal{O}_p(H)$. Since $Z(F)\mathcal{O}_p(H)$ is nilpotent it follows that

$$\mathcal{O}_q(K) \text{ normalizes } \mathcal{O}_p(Z(F))\mathcal{O}_p(H),$$

as this is the unique Sylow p-subgroup of $Z(F)\mathcal{O}_p(H)$. By symmetry $\mathcal{O}_p(H)$ normalizes $\mathcal{O}_q(Z(F))\mathcal{O}_q(K)$. Hence

$$\begin{aligned}[\mathcal{O}_p(H), \mathcal{O}_q(K)] &\leqslant [\mathcal{O}_p(H), \mathcal{O}_q(Z(F))\mathcal{O}_q(K)] \cap [\mathcal{O}_p(Z(F))\mathcal{O}_p(H), \mathcal{O}_q(K)] \\ &\leqslant \mathcal{O}_q(Z(F))\mathcal{O}_q(K) \cap \mathcal{O}_p(Z(F))\mathcal{O}_p(H) = 1.\end{aligned}$$

and we are done. $\square$

Proposition 2.34 *Let $I, J \in Max_{\mathcal{N}\mathcal{O}}(G)$ and suppose $F \leqslant I \cap J$ such that $C_G(F) \leqslant F$ and $F \trianglelefteq G$. Then I and J are conjugate.*

Proof. First note that $\pi(F) = \pi(I)$, since if p is in $\pi(I)$ but not in $\pi(F)$ then $1 \neq \mathcal{O}_p(I) \leqslant C_G(F) \leqslant F$, contradicting the choice of p. Set $\pi = \pi(I) = \pi(F) = \pi(J)$ and choose $p \in \pi$. Then $[\mathcal{O}_p(I), \mathcal{O}_{p'}(J)] = 1$ by Lemma 2.33.
We now let

$$C(\sigma) = C_G(\mathcal{O}_\sigma(I)) \cap C_G(\mathcal{O}_\sigma(J))$$

for any set of primes σ, and observe that, by what we have just argued, $\mathcal{O}_p(I) \leqslant C(p')$. By symmetry we also have $\mathcal{O}_p(J) \leqslant C(p')$. The maximality of I implies

$$\mathcal{O}_p(I) \in \mathrm{Syl}_p(C(p'))$$

as $C(p') \leqslant C_G(\mathcal{O}_{p'}(I))$. Also $\mathcal{O}_p(J) \in \mathrm{Syl}_p(C(p'))$.
Let $\pi = \{p_1, ..., p_r\}$ so that $I = \mathcal{O}_{p_1}(I) \times ... \times \mathcal{O}_{p_r}(I)$ and $J = \mathcal{O}_{p_1}(J) \times ... \times \mathcal{O}_{p_r}(J)$. Then there exists an element of $C(p_1')$ conjugating $\mathcal{O}_{p_1}(I)$ to $\mathcal{O}_{p_1}(J)$ whilst fixing $\mathcal{O}_{p_1'}(I)$ and $\mathcal{O}_{p_1'}(J)$. So

$$\mathcal{O}_{p_1}(I) \times \mathcal{O}_{p_2}(I) \times ... \times \mathcal{O}_{p_r}(I)$$

is conjugate to

$$\mathcal{O}_{p_1}(J) \times \mathcal{O}_{p_2}(I) \times ... \times \mathcal{O}_{p_r}(I).$$

Similarly, there exists an element of $C(p_2')$ which conjugates $\mathcal{O}_{p_2}(I)$ to $\mathcal{O}_{p_2}(J)$ whilst fixing $\mathcal{O}_{p_2'}(I)$ and $\mathcal{O}_{p_2'}(J)$, so

$$\mathcal{O}_{p_1}(J) \times \mathcal{O}_{p_2}(I) \times ... \times \mathcal{O}_{p_r}(I)$$

is conjugate to

$$\mathcal{O}_{p_1}(J) \times \mathcal{O}_{p_2}(J) \times ... \times \mathcal{O}_{p_r}(I).$$

Continue the argument to conclude that I is conjugate to J. $\square$

Chapter 3

Odd Nilpotent Injectors

Throughout this chapter and the next whenever we have a group G we use π to denote the set of prime divisors of $d_{\mathcal{O}}(G)$.

3.1 Arbitrary Groups

The first two propositions are obvious.

Proposition 3.1 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and let $B \leqslant G$ be of odd order. Then $C_B(A) \leqslant A$.*

Proposition 3.2 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and $I \in \mathcal{NI}_{\mathcal{O}}(G)$. If $A \leqslant H \leqslant G$ then $A \in \mathcal{A}_{\mathcal{O}}(H)$ and if $I \leqslant H \leqslant G$ then $I \in \mathcal{NI}_{\mathcal{O}}(H)$.*

Proposition 3.3 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Then $\pi = \pi(A) = \pi(I)$ for any $I \in \mathcal{NI}_{\mathcal{O}}(G)$.*

Proof. Since elements of $\mathcal{A}_{\mathcal{O}}(G)$ all have the same order it suffices to assume $A \leqslant I$. The result then follows from Proposition 3.1 since I is nilpotent of odd order. $\square$

Proposition 3.4 *Let H be a group. Then*

$$(i)\quad \mathcal{A}_{\mathcal{O}}(G \times H) = \{A_G \times A_H \mid A_G \in \mathcal{A}_{\mathcal{O}}(G),\ A_H \in \mathcal{A}_{\mathcal{O}}(H)\};$$

$$(ii)\quad \mathcal{NI}_{\mathcal{O}}(G \times H) = \{I_G \times I_H \mid I_G \in \mathcal{NI}_{\mathcal{O}}(G),\ I_H \in \mathcal{NI}_{\mathcal{O}}(H)\}.$$

Proof. Let $\phi_G : G \times H \longrightarrow G$ and $\phi_H : G \times H \longrightarrow H$ be the projection maps and let $A \in \mathcal{A}_{\mathcal{O}}(G \times H)$. Then $A\phi_G \times A\phi_H$ is abelian of odd order and $A \leqslant A\phi_G \times A\phi_H$. So $A = A\phi_G \times A\phi_H$ and (i) follows because clearly $d_{\mathcal{O}}(G)d_{\mathcal{O}}(H) \leqslant d_{\mathcal{O}}(G \times H)$. The proof of (ii) is similar. □

Theorem 3.5 *(Arad, Glauberman) [6, Proposition 1, p313] Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and let $B \leqslant G$ be nilpotent of odd order normalized by A. Then AB is nilpotent.*

Theorem 3.6 *Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$. Then I contains every nilpotent subgroup of G of odd order that it normalizes.*

Proof. Suppose false and let $B \leqslant G$ be nilpotent of odd order normalized by I but with $B \nleqslant I$. Choose B to be minimal with this property. If B is not a p-group then for some prime p we have $B = \mathcal{O}_p(B) \times \mathcal{O}_{p'}(B)$ with $1 < \mathcal{O}_p(B), \mathcal{O}_{p'}(B) < B$. Since both direct factors are normalized by I, the minimality of B implies $\mathcal{O}_p(B), \mathcal{O}_{p'}(B) \leqslant I$, giving $B \leqslant I$. Since $B \nleqslant I$ we deduce that B is a p-group for some prime p.
Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ with $A \leqslant I$. By Theorem 3.5 we see that AB is nilpotent, so $C_B(\mathcal{O}_{p'}(A)) = B$. This observation, together with Proposition 3.1 and the fact that $\mathcal{O}_p(A) \leqslant \mathcal{O}_p(I)$, yields

$$C_B(\mathcal{O}_p(I)) \leqslant C_B(\mathcal{O}_p(A)) = C_B(A) \leqslant A.$$

Thus $C_B(\mathcal{O}_p(I)) \leqslant I$, so $[C_B(\mathcal{O}_p(I)), \mathcal{O}_{p'}(I)] = 1$. The $P \times Q$ Lemma implies that

$$[B, \mathcal{O}_{p'}(I)] = 1.$$

Hence IB is nilpotent, and the maximality of I gives $B \leqslant I$. This contradiction completes the proof. □

Proposition 3.7 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and let $I \in Max_{\mathcal{NO}}(G)$. Set $\overline{G} = G/\mathcal{O}_2(G)$. Then*

$$(i) \quad \overline{A} \in \mathcal{A}_{\mathcal{O}}(\overline{G});$$
$$(ii) \quad \overline{I} \in Max_{\mathcal{NO}}(\overline{G}).$$

In particular, if $I \in \mathcal{NI}_{\mathcal{O}}(G)$ then $\overline{I} \in \mathcal{NI}_{\mathcal{O}}(\overline{G})$.

Proof. (i) Since A has odd order we see that $\overline{A} \cong A$, so $\overline{A}$ is abelian of odd order. Suppose $\overline{A} \notin \mathcal{A}_{\mathcal{O}}(\overline{G})$. Then there exists $\overline{A_1} \in \mathcal{A}_{\mathcal{O}}(\overline{G})$ such that

$$|\overline{A_1}| > |\overline{A}|.$$

Let A_1 be the inverse image of $\overline{A_1}$ in G. Then since $\mathcal{O}_2(G)$ is a normal Sylow 2-subgroup of A_1, the Schur-Zassenhaus theorem implies the existence of a complement C to $\mathcal{O}_2(G)$ in A_1. We have $C \cong \overline{C} = \overline{A_1}$, so C is abelian of odd order, and

$$|C| = |\overline{A_1}| > |\overline{A}| = |A|,$$

contradicting the maximality of $|A|$.
(ii) Suppose $\overline{I} \notin \mathrm{Max}_{\mathcal{NO}}(\overline{G})$. Then $\overline{I} < \overline{I_1} \in \mathrm{Max}_{\mathcal{NO}}(\overline{G})$. Let I_1 be the inverse image of $\overline{I_1}$ in G. The Schur-Zassenhaus theorem again gives us a complement K to $\mathcal{O}_2(G)$ in I_1, and by Hall's Theorem we may assume $I \leqslant K$. Again we have $K \cong \overline{K} = \overline{I_1}$. So K is nilpotent of odd order with $I < K$. This contradicts the maximality of I. $\square$

We define the set $\mathcal{N}_{\mathcal{O}}(G)$ to be the set of subgroups of G which are nilpotent of odd order and contain every nilpotent subgroup of odd order that they normalize. So $\mathcal{NI}_{\mathcal{O}}(G) \subseteq \mathcal{N}_{\mathcal{O}}(G)$.
The next two results are applied only in the last chapter on components. Observe that Theorem 3.9 is a statement about $\mathcal{N}_{\mathcal{O}}(G)$ rather than $\mathcal{NI}_{\mathcal{O}}(G)$. The reason for this is

that if $I \in \mathcal{N}_{\mathcal{O}}(G)$ then $I/\mathcal{O}(Z(G)) \in \mathcal{N}_{\mathcal{O}}(G/\mathcal{O}(Z(G)))$, whereas the same cannot be said of $\mathcal{NI}_{\mathcal{O}}(G)$. Following this theorem we make no further mention of $\mathcal{N}_{\mathcal{O}}(G)$.

Lemma 3.8 *Let $Z \leqslant Z(G)$ and set $\overline{G} = G/Z$.*

- *(i) If $I \in Max_{\mathcal{NO}}(G)$ then $\overline{I} \in Max_{\mathcal{NO}}(\overline{G})$;*
- *(ii) If $\overline{I} \in Max_{\mathcal{NO}}(\overline{G})$ then $I \in Max_{\mathcal{NO}}(G)$ where I is a Hall $2'$-subgroup of the inverse image of $\overline{I}$ in G;*
- *(iii) If $I \in \mathcal{N}_{\mathcal{O}}(G)$ then $\overline{I} \in \mathcal{N}_{\mathcal{O}}(\overline{G})$;*
- *(iv) If $\overline{I} \in \mathcal{N}_{\mathcal{O}}(\overline{G})$ then $I \in \mathcal{N}_{\mathcal{O}}(G)$ where I is a Hall $2'$-subgroup of the inverse image of $\overline{I}$ in G.*

Proof. A central extension of a nilpotent group is nilpotent. □

Theorem 3.9 *[5, I.4.3, p20] Let $I \in \mathcal{N}_{\mathcal{O}}(G)$ and let $H \trianglelefteq G$. Then $I \cap H \in Max_{\mathcal{NO}}(H)$.*

Proof. Let G be a counterexample in which $|G| + |H|$ is minimized. So $G = HI$.
Suppose $Z = \mathcal{O}(Z(G)) \neq 1$ and set $\overline{G} = G/Z$. Then $\overline{I} \in \mathcal{N}_{\mathcal{O}}(\overline{G})$ by Lemma 3.8(iii), and the minimality of $|G|$ implies

$$\overline{I} \cap \overline{H} \in \mathrm{Max}_{\mathcal{NO}}(\overline{H}).$$

Now, since $Z \leqslant I$ Lemma 2.2 implies $\overline{I \cap H} = \overline{I} \cap \overline{H}$, so $\overline{I \cap H} \in \mathrm{Max}_{\mathcal{NO}}(\overline{H})$. The inverse image of $\overline{I \cap H}$ in G is $(I \cap H)Z$, which is nilpotent of odd order. So

$$(I \cap H)Z \in \mathrm{Max}_{\mathcal{NO}}(HZ) \qquad (\dagger)$$

by Lemma 3.8(ii). Let $I \cap H < K \in \mathrm{Max}_{\mathcal{NO}}(H)$. Then $(I \cap H)Z \leqslant KZ$ and $KZ \leqslant HZ$ is nilpotent of odd order, so $(I \cap H)Z = KZ$ by (†). This implies $|I \cap H||Z|/|I \cap H \cap Z| = |K||Z|/|K \cap Z|$, which gives $|I \cap H|/|H \cap Z| = |K|/|K \cap Z|$. Since $I \cap H < K$ it must be the case that $|H \cap Z| < |K \cap Z|$. But $K \cap Z \leqslant H \cap Z$, a contradiction. Hence

$$\mathcal{O}(Z(G)) = 1.$$

If $\mathcal{O}(Z(H)) \neq 1$ then since $\mathcal{O}(Z(H)) \trianglelefteq I$ we get $1 \neq \mathcal{O}(Z(H)) \cap Z(I) \leqslant \mathcal{O}(Z(G)) = 1$. Thus $\mathcal{O}(Z(H)) = 1$.

Suppose $|I|$ and $|H|$ are coprime. Then for each prime $p \in \pi(H)$ there exists an I-invariant Sylow p-subgroup P of H by Theorem 2.15. If $p \neq 2$ then by definition of I we have $P \leqslant I$, so we deduce that $\pi(H) = \{2\}$. It is clear that $I \cap H \in \mathrm{Max}_{\mathcal{NO}}(H)$ in this case. Therefore we can pick a prime

$$q \in \pi(I) \cap \pi(H).$$

Now, $\mathcal{O}_q(I)$ permutes the Sylow q-subgroups of H, and since $|\mathrm{Syl}_q(H)| \equiv 1 \bmod q$ there exists an $\mathcal{O}_q(I)$-invariant Sylow q-subgroup Q of H. Then $C_Q(\mathcal{O}_q(I)) \neq 1$, in particular

$$C = C_H(Z(\mathcal{O}_q(I))) \neq 1.$$

We see that I normalizes C, and because $\mathcal{O}(Z(G)) = 1$ it must be the case that $IC < G$. The minimality of G now gives $I \cap C \in \mathrm{Max}_{\mathcal{NO}}(C)$. As C contains elements of odd order we get $I \cap C \neq 1$, giving $I \cap H \neq 1$.

Suppose $I \cap H$ is not normal in H and consider $IN_H(I \cap H)$. By the minimality of $|H|$ we see that $I \cap N_H(I \cap H) \in \mathrm{Max}_{\mathcal{NO}}(N_H(I \cap H))$. However, $I \cap N_H(I \cap H) = I \cap H$, giving $I \cap H \in \mathrm{Max}_{\mathcal{NO}}(N_H(I \cap H))$ and $I \cap H \in \mathrm{Max}_{\mathcal{NO}}(H)$, a contradiction. Thus

$$I \cap H \trianglelefteq H.$$

By definition of I we have $F = \mathcal{O}(F(H)) \leqslant I$, therefore $I \cap H = F$.

Let $F < J \in \mathrm{Max}_{\mathcal{NO}}(H)$ and let p be such that $\mathcal{O}_p(F) < \mathcal{O}_p(J)$. Then

$$C_F(\mathcal{O}_{p'}(F)) = \mathcal{O}_p(F) Z(\mathcal{O}_{p'}(F)) < \mathcal{O}_p(J) Z(\mathcal{O}_{p'}(F)). \tag{$\dagger$}$$

Suppose $\mathcal{O}_{p'}(F) \neq 1$. We see that I normalizes $C_H(\mathcal{O}_{p'}(F))$, and since $\mathcal{O}(Z(H)) = 1$ it must be the case that $C_H(\mathcal{O}_{p'}(F)) < H$. So by the minimality of $|H|$ we have $I \cap C_H(\mathcal{O}_{p'}(F)) \in \mathrm{Max}_{\mathcal{NO}}(C_H(\mathcal{O}_{p'}(F)))$. Since $I \cap C_H(\mathcal{O}_{p'}(F)) = C_F(\mathcal{O}_{p'}(F))$ we get $C_F(\mathcal{O}_{p'}(F)) \in \mathrm{Max}_{\mathcal{NO}}(C_H(\mathcal{O}_{p'}(F)))$, contradicting ($\dagger$). Thus $\mathcal{O}_{p'}(F) = 1$ and

$$F = \mathcal{O}_p(H).$$

By the same argument as before, the minimality of H gives us $I \cap C_H(F) \in \mathrm{Max}_{\mathcal{NO}}(C_H(F))$, and clearly $I \cap C_H(F) = Z(F)$. This implies that

$$Z(F) = C_H(F),$$

for if $h \in C_H(F) \backslash Z(F)$ then $Z(F) < \langle Z(F), h \rangle \leqslant C_H(F)$, and $\langle Z(F), h \rangle$ is nilpotent of odd order.

Now,

$$[\mathcal{O}_{p'}(I) Z(F), H] \leqslant [C_G(F), H] \leqslant H \cap C_G(F) = C_H(F) = Z(F), \tag{$*$}$$

so H normalizes $\mathcal{O}_{p'}(I) Z(F)$. Moreover, H normalizes $\mathcal{O}_{p'}(\mathcal{O}_{p'}(I) Z(F)) = \mathcal{O}_{p'}(I)$. Together with ($*$) this yields

$$[\mathcal{O}_{p'}(I), H] \leqslant Z(F) \cap \mathcal{O}_{p'}(I) = 1.$$

Since $G = HI$ we conclude that $Z(\mathcal{O}_{p'}(I)) \leqslant \mathcal{O}(Z(G)) = 1$ and I is a p-group. Therefore

$I \in \mathrm{Syl}_p(G)$, and $I \cap H = F \in \mathrm{Syl}_p(H)$ because $H \trianglelefteq G$. This contradicts the fact that $F < \mathcal{O}_p(J) \leqslant H$. $\square$

Proposition 3.10 *Let $H \leqslant G$ have odd order and let $C = C_G(H)$. Then $C = Z(H)\mathcal{O}_2(C)$ if and only if H contains every element of odd order that it centralizes.*

Proof. The left to right implication follows from the fact that $Z(H)\mathcal{O}_2(C)$ is a direct product. For the other implication, by hypothesis we see that $Z(H) \in \mathrm{Hall}_{2'}(C)$, so for any $S \in \mathrm{Syl}_2(C)$ we have $C = Z(H)S$. Finally observe that $S = \mathcal{O}_2(C)$ because $[S, Z(H)] = 1$. $\square$

The following result is based upon a theorem of Lausch [10].

Theorem 3.11 *Let $I, J \in \mathrm{Max}_{\mathcal{NO}}(G)$, let $C = C_G(I \cap J)$ and suppose $C = Z(I \cap J)\mathcal{O}_2(C)$. Then I and J are conjugate.*

Proof. Let G be a counterexample of minimal order and pick I and J to contradict the theorem whilst maximizing $|I \cap J|$. Since $I \neq J$ we have $I \cap J < N_I(I \cap J)$ and $I \cap J < N_J(I \cap J)$. Suppose $N = N_G(I \cap J) < G$. Choose $I_1, J_1 \in \mathrm{Max}_{\mathcal{NO}}(N)$ with $N_I(I \cap J) \leqslant I_1$ and $N_J(I \cap J) \leqslant J_1$. If $x \in C_N(I_1 \cap J_1)$ has odd order then $x \in C_G(I \cap J)$ giving

$$x \in I \cap J \leqslant I_1 \cap J_1,$$

and Proposition 3.10 implies that I_1, J_1 and N satisfy the hypothesis of the theorem. So by the minimality of $|G|$, there exists $n \in N$ such that $I_1^n = J_1$. Let $I_2, J_2 \in \mathrm{Max}_{\mathcal{NO}}(G)$ with $I_1 \leqslant I_2$ and $J_1 \leqslant J_2$. Then

$$I \cap J < N_I(I \cap J) \leqslant I \cap I_2$$

and

$$I \cap J < N_J(I \cap J) \leqslant J \cap J_2.$$

The choice of I and J implies that I is conjugate to I_2 and J is conjugate to J_2. (†)
Now, $N_I(I \cap J) \leqslant I_1 \leqslant I_2$, so

$$N_I(I \cap J)^n \leqslant I_1^n \leqslant I_2^n.$$

Since $I_1^n = J_1$ we also have $N_I(I \cap J)^n \leqslant J_1 \leqslant J_2$. This gives

$$N_I(I \cap J)^n \leqslant I_2^n \cap J_2,$$

and again by choice of I and J we deduce that I_2^n is conjugate to J_2. Together with (†), this implies that I is conjugate to J, a contradiction. Hence $I \cap J \trianglelefteq G$.
Now assume $\mathcal{O}_2(G) \neq 1$ and set $\overline{G} = G/\mathcal{O}_2(G)$. Then $\overline{I}, \overline{J} \in \mathrm{Max}_{\mathcal{NO}}(\overline{G})$ by Proposition 3.7. Let $\overline{x} \in C_{\overline{G}}(\overline{I} \cap \overline{J})$. We have $\overline{I \cap J} \leqslant \overline{I} \cap \overline{J}$, so $C_{\overline{G}}(\overline{I} \cap \overline{J}) \leqslant C_{\overline{G}}(\overline{I \cap J})$, and by Lemma 2.20 also $C_{\overline{G}}(\overline{I \cap J}) = \overline{C_G(I \cap J)}$. So

$$\overline{x} \in \overline{C_G(I \cap J)} = \overline{Z(I \cap J)\mathcal{O}_2(C)} = \overline{Z(I \cap J)},$$

the last equality because $\mathcal{O}_2(C) = \mathcal{O}_2(G)$ as $I \cap J \trianglelefteq G$. Therefore $\overline{I} \cap \overline{J}$ contains its own centralizer inside $\overline{G}$, and the minimality of $|G|$ implies that $\overline{I}$ is conjugate to $\overline{J}$. Lemma 2.32 then provides a contradiction. So $\mathcal{O}_2(G) = 1$ and the hypothesis of Proposition 2.34 is satisfied with $F = I \cap J$. This final contradiction completes the proof. □

Corollary 3.12 *If $A \in \mathcal{A}_{\mathcal{O}}(G)$ and $A \leqslant I, J \in \mathcal{NI}_{\mathcal{O}}(G)$ then I and J are conjugate.*

Proof. Let $x \in C_G(I \cap J)$ have odd order. Then $x \in C_G(A)$ implies $x \in A \leqslant I \cap J$. So $I \cap J$ contains every element of odd order that it centralizes, and Proposition 3.10 completes the hypothesis of the previous theorem. □

We only ever apply the next corollary for $I_1, J_1 \in \mathcal{NI}_{\mathcal{O}}(\langle A, B\rangle)$

Corollary 3.13 *Let $A, B \in \mathcal{A}_{\mathcal{O}}(G)$ and let $I, J \in \mathcal{NI}_{\mathcal{O}}(G)$ such that $A \leqslant I$ and $B \leqslant J$. Let $I_1, J_1 \leqslant \langle A, B\rangle$ be nilpotent of odd order containing A and B respectively and assume I_1 and J_1 are conjugate. Then I and J are conjugate.*

Proof. Let $g \in G$ such that $I_1 = J_1^g$. Let $I_1 \leqslant I_2 \in \mathcal{NI}_{\mathcal{O}}(G)$. Then $B^g \leqslant I_1 \leqslant I_2$. Corollary 3.12 implies that I_2 is conjugate to both I and J^g. $\square$

Lemma 3.14 *Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$ and suppose $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq I$. Let $A \in \mathcal{A}_{\mathcal{O}}(I)$. Then $C_A(V)V \in \mathcal{A}_{\mathcal{O}}(I)$. In particular, V is contained in an element of $\mathcal{A}_{\mathcal{O}}(I)$.*

Proof. If $V \leqslant A$ then $C_A(V)V = A$ and the lemma holds, so suppose $V \nleqslant A$. Since A contains every element of odd order that it centralizes we see that $[V, A] \neq 1$, so $[V, \mathcal{O}_p(A)] \neq 1$. Hence $\mathcal{O}_p(A)/C_{\mathcal{O}_p(A)}(V)$ acts faithfully and nontrivially on V, giving

$$\mathcal{O}_p(A)/C_{\mathcal{O}_p(A)}(V) \cong \mathbb{Z}_p$$

because a Sylow p-subgroup of $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p)$ has order p. Then $V \nleqslant A$ implies $|C_{\mathcal{O}_p(A)}(V)| < |C_{\mathcal{O}_p(A)}(V)V|$, forcing $C_{\mathcal{O}_p(A)}(V)V\mathcal{O}_{p'}(A) \in \mathcal{A}_{\mathcal{O}}(I)$. We finally note that $C_{\mathcal{O}_p(A)}(V)\mathcal{O}_{p'}(A) = C_A(V)$. $\square$

3.2 Soluble Groups

Theorem 3.15 *Suppose one of the following holds:*

(i) G *is soluble*;

(ii) $C_G(F(G)) \leqslant F(G)$ *and* $\mathcal{O}_2(G) = 1$.

Then $\mathcal{NI}_{\mathcal{O}}(G)$ is a single conjugacy class of subgroups.

Proof. We may assume $\mathcal{O}_2(G) = 1$ in both cases by Lemma 2.32 and Proposition 3.7. Let $I, J \in \mathcal{NI}_{\mathcal{O}}(G)$. Then $F(G) \leqslant I \cap J$ by Theorem 3.6, and the result follows from Proposition 2.34, taking $F = F(G)$. $\square$

Lemma 3.16 *(Mann) [16] Let G be soluble and suppose $\mathcal{O}_2(G) = 1$. For each $p \in \pi(G)$ pick $S_p \in \mathrm{Syl}_p(C_G(F_{p'}(G)))$. Then $\langle S_p | p \in \pi(G)\rangle$ is the direct product of the groups S_p and $\langle S_p | p \in \pi(G)\rangle \in \mathcal{NI}_{\mathcal{O}}(G)$. Moreover, every element of $\mathcal{NI}_{\mathcal{O}}(G)$ is of this form.*

Proof. If $|\pi(G)| = 1$ then the result is clear, so assume $|\pi(G)| \geqslant 2$ and take distinct primes $p, q \in \pi(G)$. Let $P \in \mathrm{Syl}_p(C_G(F_{p'}(G)))$ and $Q \in \mathrm{Syl}_q(C_G(F_{q'}(G)))$. We have

$$F(G) \leqslant PF_{p'}(G) \cap QF_{q'}(G)$$

with both $PF_{p'}(G)$ and $QF_{q'}(G)$ nilpotent. Then $[P, Q] = 1$ by Lemma 2.33, taking $F = F(G)$. This proves that $\langle S_p | p \in \pi(G)\rangle$ is the direct product of the groups S_p as claimed. Now, let $I \in \mathcal{NI}_{\mathcal{O}}(G)$. By Theorem 3.6 we have $F(G) \leqslant I$, so $\mathcal{O}_p(I) \leqslant C_G(F_{p'}(G))$. Let $\mathcal{O}_p(I) \leqslant P \in \mathrm{Syl}_p(C_G(F_{p'}(G)))$. Then $PF_{p'}(G)$ is nilpotent of odd order, so we may take

$$PF_{p'}(G) \leqslant J \in \mathrm{Max}_{\mathcal{NO}}(G).$$

Proposition 2.34 implies that I is conjugate to J, so $\mathcal{O}_p(I)$ is isomorphic to $\mathcal{O}_p(J)$. Since $\mathcal{O}_p(I) \leqslant P \leqslant \mathcal{O}_p(J)$ it follows that $\mathcal{O}_p(I) = P$. Thus the elements of $\mathcal{NI}_{\mathcal{O}}(G)$ have the required form, and the statement that $\langle S_p | p \in \pi(G)\rangle \in \mathcal{NI}_{\mathcal{O}}(G)$ follows. $\square$

Corollary 3.17 *(Mann) [16] Let G be soluble. If $I \in \mathcal{NI}_{\mathcal{O}}(G)$ and $\mathcal{O}_p(I) \leqslant P \in \mathrm{Syl}_p(G)$ then $\mathcal{O}_p(I) \trianglelefteq P$.*

Proof. Since P is isomorphic to its image inside $G/\mathcal{O}_2(G)$, it suffices to prove the result in the case $\mathcal{O}_2(G) = 1$. In this case $\mathcal{O}_p(I) \in \mathrm{Syl}_p(C_G(F_{p'}(G)))$ by Lemma 3.16, then $C_G(F_{p'}(G)) \trianglelefteq G$ implies $\mathcal{O}_p(I) = P \cap C_G(F_{p'}(G))$, giving $\mathcal{O}_p(I) \trianglelefteq P$. $\square$

Theorem 3.18 *(Thompson-Bender) [9, 1.12, p9] Let G be soluble and suppose $A \leqslant G$ is an abelian p-subgroup for some odd prime p. Suppose further that A contains every p-element of its centralizer. If $Q \leqslant G$ is a p'-subgroup normalized by A then $Q \leqslant \mathcal{O}_{p'}(G)$.*

Theorem 3.19 *(Bender) Let G be soluble and let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Then $\text{И}^*_G(A,2) = \{\mathcal{O}_2(G)\}$.*

Proof. By Proposition 3.7(i) we may assume $\mathcal{O}_2(G) = 1$. Let $R \in \text{И}_G(A,2)$ and let $p \in \pi$. We see that $\mathcal{O}_p(A)$ normalizes R and $\mathcal{O}_p(A)R$ normalizes $\mathcal{O}_p(G)$, so we can define a subgroup

$$K = \mathcal{O}_p(A)R\mathcal{O}_p(G).$$

Let $P \in \mathrm{Syl}_p(C_K(\mathcal{O}_p(A)))$. Then $\mathcal{O}_p(A) \leqslant P$ and since $P \leqslant K$ we get the factorization

$$P = \mathcal{O}_p(A)(P \cap R\mathcal{O}_p(G))$$

by Lemma 2.1. It follows from the fact that $\mathcal{O}_p(G)$ is the unique Sylow p-subgroup of $R\mathcal{O}_p(G)$ that $P \cap R\mathcal{O}_p(G) = P \cap \mathcal{O}_p(G)$, hence

$$P = \mathcal{O}_p(A)(P \cap \mathcal{O}_p(G)) = \mathcal{O}_p(A)C_{\mathcal{O}_p(G)}(\mathcal{O}_p(A)).$$

Theorem 3.5 implies that $AF(G)$ is nilpotent, so

$$C_{\mathcal{O}_p(G)}(\mathcal{O}_p(A)) = C_{\mathcal{O}_p(G)}(A) \leqslant A.$$

We conclude that $P = \mathcal{O}_p(A)$ and $\mathcal{O}_p(A)$ contains every p-element of its centralizer inside K. This enables us to apply Theorem 3.18, which gives $R \leqslant \mathcal{O}_{p'}(K)$ and $[R, \mathcal{O}_p(G)] \leqslant \mathcal{O}_{p'}(K) \cap \mathcal{O}_p(G) = 1$. Since p was arbitrary, $R \leqslant C_G(F(G)) \leqslant F(G)$ giving $R = 1$. $\square$

Lemma 3.20 *If G is soluble then $G = N_G(I_0)\mathcal{O}_2(G)$ for any Hall $2'$-subgroup I_0 of the inverse image of $F(G/\mathcal{O}_2(G))$ in G. Moreover, if $I \in \mathcal{NI}_{\mathcal{O}}(G)$ then I contains such a subgroup I_0 which is normal in I and $\pi(I_0) = \pi$.*

Proof. Set $\overline{G} = G/\mathcal{O}_2(G)$. The inverse image of $F(\overline{G})$ in G contains $\mathcal{O}_2(G)$ as a normal Sylow 2-subgroup, so it has a complement I_0 to $\mathcal{O}_2(G)$ by the Schur-Zassenhaus Theorem. Since $F(\overline{G}) \trianglelefteq \overline{G}$ we get

$$I_0\mathcal{O}_2(G) \trianglelefteq G,$$

and a Frattini argument yields $G = N_G(I_0)\mathcal{O}_2(G)$. Now, by Proposition 3.7 and Theorem 3.6 we have $F(\overline{G}) \leqslant \overline{I}$, so $I_0 \leqslant I\mathcal{O}_2(G)$ and by Hall's Theorem I contains a conjugate of I_0. This conjugate is again a Hall $2'$-subgroup of the inverse image of $F(\overline{G})$ in G so the required factorization of G still holds. The last statement follows from the fact that

$$\pi(I_0) = \pi(F(\overline{G})) = \pi(\overline{I}) = \pi,$$

with the second equality holding because $\overline{I}$ is nilpotent containing $F(\overline{G})$ and $\overline{G}$ is soluble. Finally, I_0 is normal in I because $[I_0, I] \leqslant I_0\mathcal{O}_2(G) \cap I = I_0(\mathcal{O}_2(G) \cap I) = I_0$. $\square$

Chapter 4

Minimal Simple Groups

4.1 Odd Nilpotent Injectors

Definition 4.1 *A minimal simple group is a nonabelian simple group all of whose proper subgroups are soluble*

Definition 4.2 *Let E be an elementary abelian p-group. The* ***rank*** *of E, denoted $m(E)$, is the number of direct factors $\mathbb{Z}_p$ of E. More formally, $m(E) = \log_p |E|$. The rank $m(G)$ of an arbitrary group G is defined to be the largest rank amongst all the elementary abelian subgroups of G.*

For the rest of this chapter we assume that G is a minimal simple group. An outline of Flavell's proof that $\mathcal{NI}(G)$ is a single conjugacy class ([4]) is as follows: Let $I \in \mathcal{NI}(G)$. Then I is contained in a unique maximal subgroup L. Suppose I defies the Theorem. Then

- there exists a subgroup $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq I$;
- if $C_G(v) \leqslant L$ for every $v \in V^\#$ then a contradiction follows;
- if there exists $v \in V^\#$ such that $C_G(v) \nleqslant L$ then a contradiction follows.

The last step is by far the hardest. Fortunately in our situation the problem does not arise because if $A \in \mathcal{A}_{\mathcal{O}}(G)$ and $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq A$ then $A \leqslant C_A(v)$ for every $v \in V^{\#}$. The same cannot be said of $\mathcal{A}_2(G)$ because elements of $\mathcal{A}_2(G)$ need not be abelian. On the other hand, an element of $\mathcal{NI}_{\mathcal{O}}(G)$ is not easily shown to be contained in a unique maximal subgroup, whereas it follows easily from Theorem 2.31 that an element of $\mathcal{NI}(G)$ is.
Our proof that $\mathcal{NI}_{\mathcal{O}}(G)$ is a single conjugacy class goes as follows: Suppose $I \in \mathcal{NI}_{\mathcal{O}}(G)$ contradicts the Theorem. Then

- I contains a normal subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$;
- some element of $\mathcal{A}_{\mathcal{O}}(I)$ has rank at least 2;
- if no element of $\mathcal{A}_{\mathcal{O}}(I)$ has rank at least 3 then some $A \in \mathcal{A}_{\mathcal{O}}(I)$ satisfies $|\mathsf{M}^*_G(A, 2)| = 1$;
- if $A \in \mathcal{A}_{\mathcal{O}}(I)$ has rank at least 3 then $|\mathsf{M}^*_G(A, 2)| = 1$;
- I is contained in a unique maximal subgroup and a contradiction follows.

Points 3 and 4 require Thompson-transitivity arguments and Point 5 uses results of Bender on maximal subgroups.

Proposition 4.3 *Let $A, B \in \mathcal{A}_{\mathcal{O}}(G)$ with $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(G)$ and $B \leqslant J \in \mathcal{NI}_{\mathcal{O}}(G)$. Suppose $A, B \leqslant H < G$. Then I and J are conjugate.*

Proof. This follows from Corollary 3.13 because H is soluble. □

Proposition 4.4 *Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$. Suppose I is contained in a maximal subgroup L of G with $\mathcal{O}_2(L) = 1$ and $F(L)$ cyclic. Then I is conjugate to every element of $\mathcal{NI}_{\mathcal{O}}(G)$.*

Proof. Let $p = \max \pi$ and let $\mathcal{O}_p(I) \leqslant P \in \mathrm{Syl}_p(L)$. As $F(L)$ is cyclic, $L/C_L(F(L)) = L/Z(F(L))$ is abelian and $L' \leqslant Z(F(L))$. Since $P' \leqslant L' \leqslant Z(F(L))$ and subgroups of cyclic groups are characteristic we get P' char $Z(F(L))$. Hence

$$P' \text{ char } L.$$

Suppose $P' = 1$. Then P is abelian, so $[P, \mathcal{O}_p(L)] = 1$. We also have $[P, F_{p'}(L)] = 1$ by Lemma 2.27, whence $P \leqslant C_L(F(L)) \leqslant F(L)$ and $P = \mathcal{O}_p(L)$. Therefore $N_G(P) = L$ and $P \in \mathrm{Syl}_p(G)$.

If $P' \neq 1$ then $N_G(P) \leqslant N_G(P') = L$, so in all cases

$$P \in \mathrm{Syl}_p(G).$$

Let $J \in \mathcal{NI}_{\mathcal{O}}(G)$. By Sylow's Theorem we may assume $\mathcal{O}_p(J) \leqslant P$. Then $\mathcal{O}_p(J)$ centralizes $F_{p'}(L)$ by Lemma 2.27. Now, since $\mathcal{O}_2(L) = 1$ Lemma 3.16 implies that

$$\mathcal{O}_p(I) \in \mathrm{Syl}_p(C_L(F_{p'}(L))),$$

and we again apply Sylow's Theorem to assume that $\mathcal{O}_p(J) \leqslant \mathcal{O}_p(I)$. It follows that for any $A \in \mathcal{A}_{\mathcal{O}}(I)$ and $B \in \mathcal{A}_{\mathcal{O}}(J)$ we have

$$[\mathcal{O}_p(B), \mathcal{O}_{p'}(A)] = 1.$$

Therefore $A^* = \mathcal{O}_p(B)\mathcal{O}_{p'}(A)$ is in $\mathcal{A}_{\mathcal{O}}(G)$ with $A^* \leqslant I$. Now observe that $A^*, B \leqslant N_G(\mathcal{O}_p(B)) < G$ and apply Proposition 4.3. $\square$

Corollary 4.5 *Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$. Suppose I is contained in a maximal subgroup L with $\mathcal{O}_2(L) = 1$. If there does not exist a subgroup $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq \mathcal{O}_p(L)$ for some prime p then I is conjugate to every element of $\mathcal{NI}_{\mathcal{O}}(G)$.*

Proof. Lemma 2.12(i) implies that $F(L)$ is cyclic. Now apply the previous result. $\square$

Lemma 4.6 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and suppose $1 \neq R \in \text{И}^*_G(A,2)$. Then $R = \mathcal{O}_2(M)$ for some $M \in Max(G)$ containing A.*

Proof. Since G is simple, $AR \neq G$. Let $AR \leqslant M \in Max(G)$. Then $R \leqslant \mathcal{O}_2(M)$ by Theorem 3.19, and since R is maximal in $\text{И}_G(A,2)$ we get $R = \mathcal{O}_2(M)$. □

Lemma 4.7 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and let $R, S \in \text{И}^*_G(A,2)$. Suppose $R \cap S \neq 1$. Then $R = S$.*

Proof. Suppose false and pick R and S to contradict the lemma whilst maximizing $|R \cap S|$. Since R and S are 2-groups we must have $R \cap S < N_R(R \cap S)$ and $R \cap S < N_S(R \cap S)$. Let $N = N_G(R \cap S)$ and note that $A \leqslant N < G$. Theorem 3.19 then implies that $\text{И}^*_N(A,2) = \{\mathcal{O}_2(N)\}$, so

$$N_R(R \cap S), N_S(R \cap S) \leqslant \mathcal{O}_2(N).$$

Let $\mathcal{O}_2(N) \leqslant R_1 \in \text{И}^*_G(A,2)$. Then $R_1 \cap R \geqslant N_R(R \cap S)$, so $|R_1 \cap R| \geqslant |N_R(R \cap S)| > |R \cap S|$, and the choice of R and S implies

$$R_1 = R.$$

Similarly, $|R_1 \cap S| \geqslant |N_S(R \cap S)| > |R \cap S|$, hence $S = R_1 = R$. □

Lemma 4.8 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and let $R, S \in \text{И}^*_G(A,2)$. Suppose $a \in A^{\#}$ such that $C_R(a) \neq 1 \neq C_S(a)$. Then $R = S$.*

Proof. Let $C = C_G(a)$ and note that $A \leqslant C < G$. Theorem 3.19 implies that $\text{И}^*_C(A,2) = \{\mathcal{O}_2(C)\}$, so $C_R(a), C_S(a) \leqslant \mathcal{O}_2(C)$. Let $\mathcal{O}_2(C) \leqslant R_1 \in \text{И}^*_G(A,2)$. Then

$$R \cap R_1 \geqslant C_R(a) \neq 1 \text{ and } S \cap R_1 \geqslant C_S(a) \neq 1,$$

and Lemma 4.7 yields $R = R_1 = S$. □

4.2 The Rank 1 Case

Proposition 4.9 *Suppose $I \in \mathcal{NI}_{\mathcal{O}}(G)$ is cyclic. Then I is conjugate to every element of $\mathcal{NI}_{\mathcal{O}}(G)$.*

Proof. First we argue that for any $H < G$ containing I we have

$$H = N_H(I)\mathcal{O}_2(H). \tag{†}$$

Set $\overline{H} = H/\mathcal{O}_2(H)$. Then $F(\overline{H}) \leqslant \overline{I}$, and as $\overline{I}$ is abelian and H is soluble, $F(\overline{H}) = \overline{I}$. So $\overline{I} \trianglelefteq \overline{H}$ and $I\mathcal{O}_2(H) \trianglelefteq H$. A Frattini argument now yields (†).

Let $p = \max \pi$ and let $\mathcal{O}_p(I) \leqslant P \in \mathrm{Syl}_p(N_G(I))$. We have $[P, \mathcal{O}_{p'}(I)] = 1$ by Lemma 2.27 since I is cyclic. Then as $\mathcal{O}_p(I) \in \mathrm{Syl}_p(C_G(\mathcal{O}_{p'}(I)))$ we get

$$P = \mathcal{O}_p(I).$$

Let $N = N_G(\mathcal{O}_p(I))$. Then $N = N_N(I)\mathcal{O}_2(N)$ by (†). In fact

$$N = N_G(I)\mathcal{O}_2(N)$$

because $N_G(I) \leqslant N_G(\mathcal{O}_p(I)) = N$. So a Sylow p-subgroup of $N_G(I)$ is a Sylow p-subgroup of N, and we deduce that $\mathcal{O}_p(I) \in \mathrm{Syl}_p(N)$. Hence $\mathcal{O}_p(I) \in \mathrm{Syl}_p(G)$, and we may now take any $J \in \mathcal{NI}_{\mathcal{O}}(G)$ and assume that $\mathcal{O}_p(J) \leqslant \mathcal{O}_p(I)$ by Sylow's Theorem. In fact we must have $\mathcal{O}_p(J) = \mathcal{O}_p(I)$ because $|\mathcal{O}_p(J)|$ is at least $|\mathcal{O}_p(I)|$ as $I \in \mathcal{A}_{\mathcal{O}}(G)$. Thus $I, J \leqslant N_G(\mathcal{O}_p(I)) < G$ and Theorem 3.15 provides the result. □

Corollary 4.10 *Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$ and suppose every element of $\mathcal{A}_{\mathcal{O}}(I)$ is cyclic. Then I is conjugate to every element of $\mathcal{NI}_{\mathcal{O}}(G)$.*

Proof. Apply Lemmas 3.14 and 2.12(i) to deduce that I is cyclic. Then Proposition 4.9 completes the proof. $\square$

4.3 The Rank 2 Case

Throughout this section we assume that no element of $\mathcal{A}_{\mathcal{O}}(I)$ has rank greater than 2 and that at least one has rank 2. So there exists $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq I$ by Lemmas 2.12(i) and 2.26. The aim of this section is to show that one of the following holds:

- I is conjugate to every element of $\mathcal{NI}_{\mathcal{O}}(G)$;
- there exists such a V for which $C_G(V)$ has even order;
- V does not normalize a nontrivial 2-subgroup of G.

Note that we can find some $A \in \mathcal{A}_{\mathcal{O}}(I)$ containing V by Lemma 3.14.

Proposition 4.11 *$V \leqslant \mathcal{O}_{2,p}(C_G(v))$ for every $v \in V^{\#}$.*

Proof. Let $V \leqslant A \in \mathcal{A}_{\mathcal{O}}(I)$. We first show that V char $C_I(V)$. Clearly

$$V \leqslant \Omega = \Omega(Z(\mathcal{O}_p(C_I(V)))),$$

and by Proposition 3.1 we have $Z(C_I(V)) \leqslant A$ as $A \leqslant C_I(V)$. So $\Omega \leqslant A$. Thus $m(\Omega) \leqslant 2$ and since V and Ω are both elementary abelian we get

$$V = \Omega \text{ char } C_I(V)$$

as claimed.

Let $v \in V^{\#}$ and let $C = C_G(v)$. Since $A \leqslant C_I(V) \leqslant C$ we may take $K \in \mathcal{NI}_{\mathcal{O}}(C)$ containing $C_I(V)$. We must have $|I : C_I(V)| = 1$ or p as $\mathrm{Aut}(V) \cong \mathrm{GL}_2(p)$, and it then follows from Corollary 3.12 that $|K : C_I(V)| = 1$ or p as K is conjugate to a subgroup of

I. Hence $C_I(V) \trianglelefteq K$, so

$$V \trianglelefteq K$$

by the first paragraph.

Our aim now is to show that in the quotient group $\widetilde{C} = C/\mathcal{O}_2(C)$ we have $\widetilde{V} \leqslant \mathcal{O}_p(\widetilde{C})$. If we were to do this we would begin by defining another quotient group $\overline{\widetilde{C}}$ of $\widetilde{C}$. Rather than use this confusing notation we will simply assume that $\mathcal{O}_2(C) = 1$.

Set $\overline{C} = C/\langle v \rangle$. Since $\mathcal{O}_2(C) = 1$ we have $F(C) \leqslant K$, so $F(C)$ normalizes V. Therefore $\overline{F(C)}$ normalizes $\overline{V}$. Now, $F(\overline{C}) = \overline{F(C)}$ because $\langle v \rangle \leqslant Z(C)$, so $F(\overline{C})$ normalizes $\overline{V}$ and we deduce that

$$[F(\overline{C}), \overline{V}] \leqslant F(\overline{C}) \cap \overline{V}.$$

As $|\overline{V}| = p$ we have either $F(\overline{C}) \cap \overline{V} = \overline{V}$ or $F(\overline{C}) \cap \overline{V} = 1$. Both cases lead to the conclusion that $\overline{V} \leqslant F(\overline{C})$, this being the case if the latter statement holds because $C_{\overline{C}}(F(\overline{C})) \leqslant F(\overline{C})$. Hence $\overline{V} \leqslant \mathcal{O}_p(\overline{C}) = \overline{\mathcal{O}_p(C)}$, and we conclude that $V \leqslant \mathcal{O}_p(C)$ since $\langle v \rangle \leqslant \mathcal{O}_p(C)$. $\square$

Proposition 4.12 *Let $V \leqslant H < G$ and let $Q \in \text{И}_H(V, 2)$. Then $[Q, V] \leqslant \mathcal{O}_2(H)$.*

Proof. Instead of working in $H/\mathcal{O}_2(H)$ we assume that $\mathcal{O}_2(H) = 1$ and show $[Q, V] = 1$. Theorem 2.18 implies

$$[Q, V] = \langle [C_Q(v), V] | v \in V^\# \rangle$$

and for any $v \in V^\#$ we have $[C_Q(v), V] \leqslant Q \cap \mathcal{O}_{2,p}(C_G(v)) \leqslant \mathcal{O}_2(C_G(v))$ by Proposition 4.11. So $[C_Q(v), V] \leqslant \mathcal{O}_2(C_G(v)) \cap H \leqslant \mathcal{O}_2(C_H(v))$. Goldschmidt's Lemma (2.24) allows us to deduce that $\mathcal{O}_2(C_H(v)) \leqslant \mathcal{O}_{p'}(H)$ because $\mathcal{O}_2(C_H(v)) \leqslant \mathcal{O}_{p'}(C_H(v))$, therefore $[C_Q(v), V] \leqslant \mathcal{O}_{p'}(H)$, giving

$$[Q, V] \leqslant \mathcal{O}_{p'}(H).$$

Now, $[C_{F_{p'}(H)}(v), V] \leqslant F_{p'}(H) \cap \mathcal{O}_{2,p}(C_G(v)) = 1$, so by another application of Theorem

2.18, as above, $[F_{p'}(H), V] = 1$. Then since $F_{p'}(H) = F(\mathcal{O}_{p'}(H))$ we get $[F(\mathcal{O}_{p'}(H)), V] = 1$, and Lemma 2.28 yields

$$[\mathcal{O}_{p'}(H), V] = 1.$$

Hence $[Q, V, V] \leqslant [\mathcal{O}_{p'}(H), V] = 1$ and Theorem 2.19 implies $[Q, V] = 1$. This concludes the proof. □

Lemma 4.13 *Let $Q_1, Q_2 \in \mathcal{U}^*_G(V, 2)$. Suppose $Q_1 \cap Q_2 \neq 1$. Then there exists $c \in C_G(V)$ such that $Q_1^c = Q_2$.*

Proof. Suppose false and pick Q_1 and Q_2 to maximize $|Q_1 \cap Q_2|$. Let $R = Q_1 \cap Q_2$ and let $N = N_G(R) < G$. We have $R < Q_1$, so

$$R < N_{Q_1}(R).$$

Similarly $R < N_{Q_2}(R)$. Let $R_1, R_2 \in \mathcal{U}^*_N(V, 2)$ and $S_1, S_2 \in \mathcal{U}^*_G(V, 2)$ such that $N_{Q_1}(R) \leqslant R_1 \leqslant S_1$ and $N_{Q_2}(R) \leqslant R_2 \leqslant S_2$. Since $R < N_{Q_1}(R) \leqslant Q_1 \cap S_1$, the choice of Q_1 and Q_2 implies that there exists $c_1 \in C_G(V)$ such that

$$Q_1^{c_1} = S_1.$$

Similarly, there exists $c_2 \in C_G(V)$ such that $Q_2^{c_2} = S_2$. Now, Theorem 2.10 gives us the factorization $R_1 = C_{R_1}(V)[R_1, V]$, so by the previous proposition $R_1 \leqslant C_{R_1}(V)\mathcal{O}_2(N)$. In fact

$$R_1 = C_{R_1}(V)\mathcal{O}_2(N)$$

because $C_{R_1}(V)\mathcal{O}_2(N)$ is a 2-subgroup of N normalized by V. Again, we also have $R_2 = C_{R_2}(V)\mathcal{O}_2(N)$. Next we show that $C_{R_1}(V) \in \mathrm{Syl}_2(C_N(V))$. Let $C_{R_1}(V) \leqslant T \in \mathrm{Syl}_2(C_N(V))$. Then $R_1 \leqslant T\mathcal{O}_2(N)$, and again since $T\mathcal{O}_2(N)$ is a 2-group normalized by

V we must have $R_1 = T\mathcal{O}_2(N)$. Thus $T \leqslant C_{R_1}(V)$ giving $C_{R_1}(V) = T$. So

$$C_{R_1}(V), C_{R_2}(V) \in \mathrm{Syl}_2(C_N(V)).$$

Let $n \in C_N(V)$ such that $C_{R_1}(V)^n = C_{R_2}(V)$. Then

$$R_1^n = C_{R_1}(V)^n \mathcal{O}_2(N) = C_{R_2}(V)\mathcal{O}_2(N) = R_2.$$

Now,

$$R^n < N_{Q_1}(R)^n \leqslant R_1^n = R_2 \leqslant S_2,$$

and also

$$R^n < N_{Q_1}(R)^n \leqslant R_1^n \leqslant S_1^n.$$

The choice of Q_1 and Q_2 implies that there exists $c_3 \in C_G(V)$ such that $S_1^{nc_3} = S_2$. Hence

$$Q_1^{c_1 n c_3 c_2^{-1}} = S_1^{n c_3 c_2^{-1}} = S_2^{c_2^{-1}} = Q_2,$$

and $c_1 n c_3 c_2^{-1} \in C_G(V)$. $\square$

Proposition 4.14 *Let $Q \in \mathcal{U}_G^*(V, 2)$ and suppose $v \in V^{\#}$ such that $[C_Q(v), V] \neq 1$. Then $[C_Q(v), V] = [\mathcal{O}_2(C_G(v)), V]$.*

Proof. By Proposition 4.11 we have $[C_Q(v), V] \leqslant \mathcal{O}_{2,p}(C_G(v)) \cap Q \leqslant \mathcal{O}_2(C_G(v))$, so

$$[C_Q(v), V] = [C_Q(v), V, V] \leqslant [\mathcal{O}_2(C_G(v)), V].$$

Let $\mathcal{O}_2(C_G(v)) \leqslant Q_1 \in \mathcal{U}_G^*(V, 2)$. Then since $1 \neq [C_Q(v), V] \leqslant Q \cap Q_1$, Lemma 4.13 applies and we can find $c \in C_G(V)$ such that $Q_1^c = Q$. Now, $C_G(V) \leqslant C_G(v)$, so $C_G(V)$

normalizes $\mathcal{O}_2(C_G(v))$ and

$$\mathcal{O}_2(C_G(v)) = \mathcal{O}_2(C_G(v))^c \leqslant Q_1^c = Q.$$

From this we deduce that $\mathcal{O}_2(C_G(v)) \leqslant Q \cap C_G(v) = C_Q(v)$, giving

$$[\mathcal{O}_2(C_G(v)), V] \leqslant [C_Q(v), V]$$

and completing the proof. $\square$

Corollary 4.15 *Let $Q \in \text{И}^*_G(V, 2)$. Then $C_G(V)$ normalizes $[Q, V]$. In particular, A normalizes $[Q, V]$ for any $A \in \mathcal{A}_{\mathcal{O}}(I)$ containing V.*

Proof. Theorem 2.18 gives

$$\begin{aligned} [Q, V] &= \langle [C_Q(v), V] | v \in V^{\#} \rangle \\ &= \langle [C_Q(v), V] | v \in V^{\#} \text{ and } [C_Q(v), V] \neq 1 \rangle \\ &= \langle [\mathcal{O}_2(C_G(v)), V] | v \in V^{\#} \text{ and } [C_Q(v), V] \neq 1 \rangle. \end{aligned}$$

This is clearly normalized by $C_G(V)$. $\square$

Proposition 4.16 *If $C_G(V)$ has even order then $C_G(V)$ is transitive on $\text{И}^*_G(V, 2)$ and for every $A \in \mathcal{A}_{\mathcal{O}}(I)$ containing V we have $|\text{И}^*_G(A, 2)| = 1$.*

Proof. Let $1 \neq S \in \mathrm{Syl}_2(C_G(V))$ and let $S \leqslant Q \in \text{И}^*_G(V, 2)$. We show that any other element of $\text{И}^*_G(V, 2)$ is conjugate to Q under $C_G(V)$.

Let $Q_1 \in \text{И}^*_G(V, 2)$. If $Q_1 \leqslant C_G(V)$ then Q_1 is conjugate to S under $C_G(V)$ and it must be the case that $S = Q$ since otherwise Q_1 would not be maximal inside $\text{И}_G(V, 2)$. So Q_1 is conjugate to Q in this case. Now assume $[Q_1, V] \neq 1$. Then there exists $v \in V^{\#}$ such that

$$[C_{Q_1}(v), V] \neq 1.$$

Now, $S \leqslant C_G(V) \leqslant C_G(v)$, so we may take

$$S\mathcal{O}_2(C_G(v)) \leqslant Q_2 \in \text{И}^*_G(V,2).$$

Then $1 \neq S \leqslant Q \cap Q_2$, and Q is conjugate to Q_2 under $C_G(V)$ by Lemma 4.13. On the other hand, Proposition 4.11 implies $[C_{Q_1}(v), V] \leqslant \mathcal{O}_{2,p}(C_G(v)) \cap Q_1 \leqslant \mathcal{O}_2(C_G(v)) \leqslant Q_2$, so $1 \neq [C_{Q_1}(v), V] \leqslant Q_1 \cap Q_2$ and Q_1 is conjugate to Q_2 under $C_G(V)$. This proves the first statement.

Let $V \leqslant A \in \mathcal{A}_{\mathcal{O}}(I)$ and let $R_1, R_2 \in \text{И}^*_G(A,2)$. Then we can choose $Q_1, Q_2 \in \text{И}^*_G(V,2)$ such that $R_1 \leqslant Q_1$ and $R_2 \leqslant Q_2$. By Lemma 4.8 and Theorem 2.17 we may assume $[R_1, V] \neq 1 \neq [R_2, V]$. Now, A normalizes $[Q_1, V]$ by Corollary 4.15, so we can take $[Q_1, V] \leqslant S_1 \in \text{И}^*_G(A,2)$. Then

$$1 \neq [R_1, V] \leqslant [Q_1, V] \cap R_1 \leqslant S_1 \cap R_1$$

and Lemma 4.7 yields $R_1 = S_1$. So $[Q_1, V] \leqslant R_1$, and similarly $[Q_2, V] \leqslant R_2$. We now pick $c \in C_G(V)$ conjugating Q_1 into Q_2 and apply Corollary 4.15 to deduce that

$$[Q_1, V] = [Q_1, V]^c = [Q_1^c, V] = [Q_2, V].$$

This implies that $1 \neq [Q_1, V] \leqslant R_1 \cap R_2$, and therefore $R_1 = R_2$. $\square$

Proposition 4.17 *Let $V \leqslant A \in \mathcal{A}_{\mathcal{O}}(I)$ and suppose $C_G(V)$ has odd order. Then $\text{И}^*_G(V,2) = \text{И}^*_G(A,2)$ and $C_G(V)$ normalizes each element of $\text{И}^*_G(V,2)$. Furthermore, if $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$ where q is a prime different from p and $C_G(W)$ has odd order then $\text{И}^*_G(V,2) = \text{И}^*_G(W,2)$.*

Proof. Let $Q \in \text{И}^*_G(V,2)$. The factorization $Q = [Q,V]C_Q(V)$ implies $Q = [Q,V]$ since $C_G(V)$ has odd order, so A normalizes Q by Corollary 4.15. This allows us to deduce that

$\text{И}^*_G(V,2) = \text{И}^*_G(A,2)$ because $V \leqslant A$. To show that $C_G(V)$ normalizes Q, let $g \in C_G(V)$. Then $Q, Q^g \in \text{И}^*_G(V,2)$. We may assume $Q \neq 1$, so there exists $v \in V^\#$ such that $C_Q(v) \neq 1$. Then also $C_{Q^g}(v) \neq 1$. Hence $Q = Q^g$ by Lemma 4.8 since $\text{И}^*_G(V,2) = \text{И}^*_G(A,2)$.
The final statement now follows because we can choose A to contain W as well as V by Lemma 3.14. $\square$

For the next two results we introduce the following hypothesis:

(H) The centralizer of every subgroup of the form $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$ has odd order and $\text{И}^*_G(V,2) \neq \{1\}$.

We will show that if (H) holds then I is conjugate to every element of $\mathcal{NI}_{\mathcal{O}}(G)$. By Proposition 4.16 this will achieve the goal for this section as set out at the beginning. Let И denote $\text{И}^*_G(V,2)$.

Proposition 4.18 *Assume (H). Let q be a prime distinct from p. Suppose $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$. Then there exists a maximal subgroup M of G such that $I \leqslant M$ and* $\mathrm{Syl}_p(M) \cap \mathrm{Syl}_p(G) \neq \emptyset$ *or* $\mathrm{Syl}_q(M) \cap \mathrm{Syl}_q(G) \neq \emptyset$.

Proof. Suppose no such maximal subgroup exists. Amongst all maximal subgroups containing I pick one which maximizes the sum of the orders of a Sylow p-subgroup and a Sylow q-subgroup. Call this maximal subgroup M. Without loss of generality we can assume $V \trianglelefteq P \in \mathrm{Syl}_p(M)$ and $W \trianglelefteq Q \in \mathrm{Syl}_q(M)$ by Lemma 2.12(ii) and Corollary 3.17. So P permutes the elements of И. Since $\text{И} = \text{И}^*_G(A,2)$, we know that И has at most $p+1$ elements by Lemma 4.8 and the fact that V contains exactly $p+1$ subgroups of order p. So there are three possibilities for the lengths of the orbits of И under the action of P. The possibilities are:

(i) all 1;

(ii) p and 1;

(iii) p.

Now, by the previous proposition, $\text{И} = \text{И}_G^*(W, 2)$, so Q acts on И as well. Thus we also know that И has at most $q + 1$ elements, as we can find $B \in \mathcal{A}_{\mathcal{O}}(I)$ containing W, and $\text{И}_G^*(W, 2) = \text{И}_G^*(B, 2)$. Again, there are three possibilities for the lengths of the orbits of И under Q:

(I) all 1;

(II) q and 1;

(III) q.

So there are nine cases here, but we need only consider six of them as we can freely interchange p and q.

Case (i):

In this case $\text{И} = \text{И}_G^*(P, 2)$ since each element of И is normalized by P and $V \leqslant P$. Therefore $N_G(P)$ permutes the elements of И. Both $N_G(V)$ and $N_G(W)$ permute the elements of И for the same reason, and we cannot have $\langle N_G(P), N_G(V), N_G(W)\rangle = G$ because V is in the kernel of the action. So

$$\langle N_G(P), N_G(V), N_G(W)\rangle \leqslant N \in Max(G).$$

We see that $P \notin \mathrm{Syl}_p(N)$ as $P \notin \mathrm{Syl}_p(G)$, and this gives us a contradiction to the choice of M after observing that $I, Q \leqslant N_G(W) \leqslant N$.

Case (ii), (II):

This gives an immediate contradiction as $p+1 = |\text{И}| = q+1$ and $p \neq q$.

Case (ii), (III):

The contradiction in this case is because $p+1 = |\text{И}| = q$ and p and q are distinct odd primes.

Case (iii), (III):

As before, $p = |\text{И}| = q$ and $p \neq q$. $\square$

Proposition 4.19 *Assume (H). Then there exists $A \in \mathcal{A}_{\mathcal{O}}(I)$ containing V and $M \in Max(G)$ such that $A \leqslant M$ and $P \in \mathrm{Syl}_p(M) \cap \mathrm{Syl}_p(G)$. Moreover, $V \trianglelefteq P$.*

Proof. Suppose that for some prime $q \neq p$ there exists $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$. Then the result holds by the previous proposition, after potentially replacing V with W. So we may suppose no such q exists, in which case

$$\mathcal{O}_{p'}(I) \text{ is cyclic.}$$

Assume no such pair A, M exists. Amongst all elements of $\mathcal{A}_{\mathcal{O}}(I)$ which contain such a subgroup V and amongst all maximal subgroups of G choose $A \in \mathcal{A}_{\mathcal{O}}(I)$ and $M \in Max(G)$ so that $A \leqslant M$ and so that the order of a Sylow p-subgroup of M is maximized. We show that without loss of generality V is normal in a Sylow p-subgroup of M. Let $A \leqslant I_1 \in \mathcal{NI}_{\mathcal{O}}(M)$ and let $\mathcal{O}_p(I_1) \leqslant P \in \mathrm{Syl}_p(M)$. Then $\mathcal{O}_p(I_1) \trianglelefteq P$ by Corollary 3.17. Since $V \leqslant \mathcal{O}_p(I_1)$ we see that $\mathcal{O}_p(I_1)$ is noncyclic, then Lemma 2.12(ii) implies that we can find $\mathbb{Z}_p \times \mathbb{Z}_p \cong V_1 \trianglelefteq \mathcal{O}_p(I_1)$ such that $V_1 \trianglelefteq P$, and there exists $A_1 \in \mathcal{A}_{\mathcal{O}}(I_1)$ containing V_1. Let $I_1 \leqslant I_2 \in \mathcal{NI}_{\mathcal{O}}(G)$. Then I_2 is conjugate to I because $A \leqslant I \cap I_2$. Moreover, A_1 and M (with respect to I_2) satisfy the conditions that we originally imposed on A and M

(with respect to I). Therefore we can replace I by I_2, in which case we may assume that $V \trianglelefteq P$.

As before, P acts on И and the same three cases arise for the lengths of the orbits.

Case(i):

A simplified version of the argument from Case(i) of the previous proposition deals with this.

Case(ii):

It may be the case that $\mathcal{O}_2(M) = 1$, so we begin by showing we may suppose $\mathcal{O}_2(M) \neq 1$ and $\text{И}^*_G(P, 2) = \{\mathcal{O}_2(M)\}$. First, recall that by Lemma 4.6 the element of И normalized by P is of the form $\mathcal{O}_2(N)$ for some $N \in Max(G)$ containing A . We have $\langle A, P\rangle \leqslant N_G(\mathcal{O}_2(N)) = N$, and so we can replace M by N if necessary to get $\mathcal{O}_2(M) \in \text{И}$. We remark that this does not invalidate our assumption that $V \trianglelefteq P$. Next, as $V \leqslant P$ we see that

$$\mathcal{O}_2(M) \in \text{И}^*_G(P, 2).$$

Now let R be any element of $\text{И}^*_G(P, 2)$ and let $R \leqslant S \in \text{И}$. Then for any $x \in P$ we have $S^x \in \text{И}$ because $P \leqslant N_G(V)$, and since P normalizes R we get $1 \neq R \leqslant S \cap S^x$, giving $S = S^x$ by Lemma 4.7. Thus P normalizes S, and we deduce that $R = S$ and $R \in \text{И}$. As $\mathcal{O}_2(M)$ is the unique element of И normalized by P we conclude that $R = \mathcal{O}_2(M)$ and

$$\text{И}^*_G(P, 2) = \{\mathcal{O}_2(M)\}.$$

Now, let $P < P_1 \in \mathrm{Syl}_p(G)$. Then $P < N_{P_1}(P)$, and $N_{P_1}(P)$ permutes the elements of $\text{И}^*_G(P, 2)$. But $\text{И}^*_G(P, 2)$ has a unique element, $\mathcal{O}_2(M)$, which implies that $N_{P_1}(P) \leqslant N_G(\mathcal{O}_2(M)) = M$, a contradiction to the choice of M.

Case(iii):

In this case P does not normalize any element of И, so

$$\mathcal{O}_2(M) = 1$$

and $F_{p'}(M) \leqslant \mathcal{O}_{p'}(I_1)$. Since $I_1 \leqslant I$ we see that $\mathcal{O}_{p'}(I_1)$ is cyclic by the first paragraph, so also $F_{p'}(M)$ is cyclic. Therefore $P/C_P(F_{p'}(M))$ is abelian and

$$P' \leqslant C_P(F_{p'}(M)).$$

Lemma 3.16 then implies that $[P', \mathcal{O}_{p'}(I_1)] = 1$, and in particular, $\mathcal{O}_{p'}(A)$ normalizes P'. As $\mathcal{O}_p(A) \leqslant P$, we deduce that $A \leqslant N_G(P')$. Let $P < P_1 \in \mathrm{Syl}_p(G)$. Then $P < N_{P_1}(P)$ and $P' \trianglelefteq N_{P_1}(P)$. So

$$\langle A, N_{P_1}(P) \rangle \leqslant N_G(P').$$

If $P' \neq 1$ this contradicts the choice of M, and if $P' = 1$ then P commutes with V, so P cannot induce an orbit of length p on И. This is because for any $Q \in \text{И}$ there exists $v \in V^{\#}$ such that $C_Q(v) \neq 1$, and if $[P, V] = 1$ then for any $x \in P$ we have $C_{Q^x}(v) \neq 1$, giving $Q = Q^x$ by Lemma 4.8. This eliminates the final case.

The assumption that $V \trianglelefteq P$ can now be made without loss of generality by a familiar argument. $\square$

The following theorem completes the goals of this section. The hypothesis is satisfied by assuming (H) and applying Proposition 4.19. We avoid assuming (H) explicitly in the hypothesis because we use the result to prove Proposition 4.21, in which we do not assume (H).

Theorem 4.20 *Suppose $V \leqslant A \in \mathcal{A}_{\mathcal{O}}(I)$ and $A \leqslant M \in Max(G)$ such that $V \trianglelefteq P \in \mathrm{Syl}_p(M) \cap \mathrm{Syl}_p(G)$. Then every element of $\mathcal{NI}_{\mathcal{O}}(G)$ is conjugate to I.*

Proof. Let $J \in \mathcal{NI}_{\mathcal{O}}(G)$ and let $B \in \mathcal{A}_{\mathcal{O}}(J)$. By Sylow's Theorem we may assume $\mathcal{O}_p(J) \leqslant P$, so $\mathcal{O}_p(J)$ normalizes V. Now, $|\mathcal{O}_p(J)| \geqslant |\mathcal{O}_p(B)| = |\mathcal{O}_p(A)| \geqslant |V| = p^2$, and since a Sylow p-subgroup of $\mathrm{Aut}(V)$ has order p we get $C_{\mathcal{O}_p(J)}(V) \neq 1$ and

$$\langle J, V\rangle \leqslant N_G(C_{\mathcal{O}_p(J)}(V)) < G.$$

Let $X = \langle J, V\rangle$. Then $[F_{2,p'}(X), V] = \langle [C_{F_{2,p'}(X)}(v), V] | v \in V^{\#}\rangle$ by Theorem 2.18, and Proposition 4.11 implies

$$[C_{F_{2,p'}(X)}(v), V] \leqslant F_{2,p'}(X) \cap \mathcal{O}_{2,p}(C_G(v)) \leqslant \mathcal{O}_2(X)$$

for every $v \in V^{\#}$. Thus

$$[F_{2,p'}(X), V] \leqslant \mathcal{O}_2(X)$$

and in the quotient group $\overline{X} = X/\mathcal{O}_2(X)$ we get $[F_{p'}(\overline{X}), \overline{V}] = 1$. On the other hand, $J \in \mathcal{NI}_{\mathcal{O}}(X)$ implies $\overline{J} \in \mathcal{NI}_{\mathcal{O}}(\overline{X})$, yielding $[\mathcal{O}_{p'}(\overline{J}), \overline{V}] = 1$ by Lemma 3.16. Therefore we can define a subgroup

$$Y = \mathcal{O}_{p'}(J) V \mathcal{O}_2(X)$$

of X. By Hall's Theorem we can take $J \leqslant H \in \mathrm{Hall}_{2'}(X)$ and find $x \in X$ such that $V^x \leqslant H$. Since $\mathcal{O}_p(J)$ normalizes Y we have $Y \trianglelefteq X$, which implies $H \cap Y \in \mathrm{Hall}_{2'}(Y)$. Therefore $H \cap Y$ maps isomorphically onto $\overline{H \cap Y} = \overline{Y}$, which is nilpotent because $[\mathcal{O}_{p'}(\overline{J}), \overline{V}] = 1$. Hence $H \cap Y$ is nilpotent. We have $V^x \leqslant H \cap Y$ with $V^x \cong V \in \mathrm{Syl}_p(Y)$, so we deduce that $V^x \in \mathrm{Syl}_p(H \cap Y)$, i.e, $V^x = \mathcal{O}_p(H \cap Y)$. The fact that $H \cap Y \trianglelefteq H$ then implies that $V^x \trianglelefteq H$, in particular J normalizes V^x. Remembering that $V \trianglelefteq I$, we conclude that $\langle I, J^{x^{-1}}\rangle \leqslant N_G(V) < G$ and Theorem 3.15 completes the proof. □

The final result of this section is used to deal with a scenario that arises in the next section.

Proposition 4.21 *Suppose $C_G(V)$ has even order and I is contained in precisely two maximal subgroups L and M of G satisfying $\mathcal{O}_2(L) = 1 \neq \mathcal{O}_2(M)$. Assume some element of $\mathcal{NI}_{\mathcal{O}}(G)$ is not conjugate to I. Then there exists $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$ such that $W \leqslant F(L)$ and $|И^*_G(B, 2)| = 1$ for some $B \in \mathcal{A}_{\mathcal{O}}(I)$ containing W.*

Proof. Suppose false. Proposition 4.4 implies that $F(L)$ is noncyclic, so there exists $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$ such that $W \leqslant F(L)$. Let $\mathcal{O}_q(I) \leqslant Q \in \mathrm{Syl}_q(L)$. Then since $\mathcal{O}_q(L) \trianglelefteq Q$ we can choose W so that $W \trianglelefteq Q$. Let $W \leqslant B \in \mathcal{A}_{\mathcal{O}}(I)$. We have $|И^*_G(B, 2)| > 1$, in particular $C_G(W)$ has odd order by Proposition 4.16. Then Proposition 4.17 gives $И^*_G(W, 2) = И^*_G(B, 2)$, so

$$\mathcal{O}_2(M) \in И^*_G(W, 2)$$

by Theorem 3.19.

Let $A = C_B(V)V$. Then $A \in \mathcal{A}_{\mathcal{O}}(I)$ by Lemma 3.14. If $q \neq p$ then we have $W \leqslant C_B(V)$, giving $W \leqslant A$, and another application of Proposition 4.16 yields $|И^*_G(A, 2)| = 1$ because $V \leqslant A$. Since we have assumed the result is false we deduce that $q = p$, and for convenience of notation we will change the name of Q to P. Now, A normalizes W, so A permutes the elements of $И^*_G(W, 2)$. We have seen that $\mathcal{O}_2(M) \in И^*_G(W, 2)$ and since $|И^*_G(A, 2)| = 1$ and $A \leqslant M$, it must be the case that $И^*_G(A, 2) = \{\mathcal{O}_2(M)\}$. Therefore A does not normalize any of the other elements of $И^*_G(W, 2)$. Since $A = C_B(V)V$ and B normalizes each element of $И^*_G(W, 2)$, the action of A on $И^*_G(W, 2)$ is completely determined by V. In particular, any orbit of length greater than 1 has length a power of p, so

$$|И^*_G(W, 2)| = np + 1$$

for some $n \in \mathbb{N}$. Now, we picked W so that $W \trianglelefteq P$, so P permutes the elements of $И^*_G(W, 2)$

as well. Since $V \leqslant P$, we deduce that P normalizes a unique element of $\text{И}^*_G(W,2)$ because the same is true of A and V controls the action of A on $\text{И}^*_G(W,2)$. The element must be $\mathcal{O}_2(M)$. So by an argument that can be found in case(ii) of Proposition 4.19,

$$\text{И}^*_G(P,2) = \{\mathcal{O}_2(M)\}.$$

We now let $P \leqslant P_1 \in \mathrm{Syl}_p(G)$ and take successive normalizers to conclude that $\text{И}^*_G(P_1,2) = \{\mathcal{O}_2(M)\}$. Hence $P_1 \leqslant M$. Since $\mathcal{O}_p(I) \trianglelefteq P_1$, we can pick $\mathbb{Z}_p \times \mathbb{Z}_p \cong V_1 \trianglelefteq I$ such that $V_1 \trianglelefteq P_1$. Now apply Theorem 4.20 to obtain a contradiction. $\square$

4.4 The Remaining Cases

From what we have proved so far, we may assume that every element of $\mathcal{NI}_{\mathcal{O}}(G)$ contains a normal subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$ and let $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq I$. Due to the results of the previous two sections there are three cases that we must consider. We define three hypotheses:

- (H1) No element of $\mathcal{A}_{\mathcal{O}}(I)$ has rank greater than 2 and $C_G(V)$ has even order;
- (H2) V does not normalize a nontrivial 2-subgroup of G;
- (H3) Some element of $\mathcal{A}_{\mathcal{O}}(I)$ has rank at least 3.

Lemma 4.22 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and suppose $m(A) \geqslant 3$. Then $|\text{И}^*_G(A,2)| = 1$.*

Proof. Let $R, S \in \text{И}^*_G(A,2)$. We may assume both R and S are nontrivial, so by Lemma 2.25 there exists $a \in A^\#$ such that $C_R(a) \neq 1 \neq C_S(a)$. Hence $R = S$ by Lemma 4.8. $\square$

Proposition 4.23 *If there exist elements of $\mathcal{A}_{\mathcal{O}}(I)$ of rank at least 3 then V is contained in one of them.*

Proof. Let $A \in \mathcal{A}_{\mathcal{O}}(I)$ have rank at least 3 and let $\mathbb{Z}_q^3 \cong D \leqslant A$. If $q \neq p$ then $D \leqslant C_A(V)$ and $C_A(V)V \in \mathcal{A}_{\mathcal{O}}(I)$ by Lemma 3.14. If $q = p$ then $|C_D(V)| \geqslant p^2$ because $\mathrm{Aut}(V) \cong \mathrm{GL}_2(p)$, implying that $C_D(V)V$ is elementary abelian of order at least p^3. As before, $C_A(V)V \in \mathcal{A}_{\mathcal{O}}(G)$, and $C_D(V)V \leqslant C_A(V)V$. $\square$

From now on we let A be an element of $\mathcal{A}_{\mathcal{O}}(I)$ which satisfies $|\text{И}^*_G(A,2)| = 1$. Under (H1) every element of $\mathcal{A}_{\mathcal{O}}(I)$ containing V satisfies this property (see Proposition 4.16), and under (H3) we apply Lemma 4.22. We also make the assumption that $|\pi| \geqslant 2$. This will do us no harm since we wish to prove that $\mathcal{NI}_{\mathcal{O}}(G)$ forms a single conjugacy class. If it were the case that $|\pi| = 1$ then the elements of $\mathcal{NI}_{\mathcal{O}}(G)$ would be Sylow p-subgroups for some prime p and would be conjugate by Sylow's Theorem.

Lemma 4.24 *If $A \leqslant M \in Max(G)$ with $\mathcal{O}_2(M) \neq 1$ then $\text{И}^*_G(A,2) = \{\mathcal{O}_2(M)\}$.*

Proof. Let $\mathcal{O}_2(M) \leqslant R \in \text{И}^*_G(A,2)$ and suppose $\mathcal{O}_2(M) \neq R$. Then

$$\mathcal{O}_2(M) < N_R(\mathcal{O}_2(M)) \leqslant N_G(\mathcal{O}_2(M)) = M.$$

This contradicts the fact that $\text{И}^*_M(A,2) = \{\mathcal{O}_2(M)\}$. So $\mathcal{O}_2(M) \in \text{И}^*_G(A,2)$ and the result follows because $|\text{И}^*_G(A,2)| = 1$. $\square$

Proposition 4.25 *Let $A \leqslant L, M \in Max(G)$. Then $L = M$ if one of the following conditions holds:*

(*i*) $\mathcal{O}_2(L) = 1 = \mathcal{O}_2(M)$ *and* $I \leqslant L \cap M$;

(*ii*) $\mathcal{O}_2(L) \neq 1 \neq \mathcal{O}_2(M)$.

Proof. In case (ii) the previous lemma implies $\text{И}^*_G(A,2) = \{\mathcal{O}_2(L)\} = \{\mathcal{O}_2(M)\}$, so $\mathcal{O}_2(L) = \mathcal{O}_2(M)$ and $L = N_G(\mathcal{O}_2(L)) = N_G(\mathcal{O}_2(M)) = M$. In case (i) Theorem 3.6 implies that $F(L) \leqslant I \leqslant M$ and $F(M) \leqslant I \leqslant L$ and Lemma 2.29 provides the conclusion because $|\pi| \geqslant 2$. $\square$

Proposition 4.26 *Suppose I is contained in a maximal subgroup L of G with $\mathcal{O}_2(L) = 1$. Assume there exists $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$ such that $W \trianglelefteq \mathcal{O}_q(L)$ and $|\text{И}^*_G(B, 2)| = 1$ for some $B \in \mathcal{A}_{\mathcal{O}}(I)$ containing W. Then L is the unique maximal subgroup of G containing I.*

Proof. Suppose false and let $I \leqslant M \in Max(G)$ with $M \neq L$. Then $\mathcal{O}_2(M) \neq 1$ by Proposition 4.25. Let $w \in W^{\#}$ and let $C_G(w) \leqslant H \in Max(G)$. Then $C_{F(L)}(w) \leqslant F(L) \cap H$ and $C_{F(L)}(C_{F(L)}(w)) \leqslant C_{F(L)}(w)$. Thus Theorem 2.31 applies with $X = C_{F(L)}(w)$ and we deduce that one of the following holds:

(i) $L = H$;

(ii) $\pi(F(H)) \nsubseteq \pi(F(L))$;

(iii) $F(L) = \mathcal{O}_r(L)$ and $F(H) = \mathcal{O}_r(H)$ for some prime r.

Case (iii) cannot hold because $|\pi(F(L))| = |\pi| \geqslant 2$. Suppose (ii) holds. Observe $B \leqslant C_G(w)$, so we can let $B \leqslant I_H \in \mathcal{NI}_{\mathcal{O}}(H)$. If $\mathcal{O}_2(H) = 1$ then $F(H) \leqslant I_H$ and

$$\pi(F(H)) = \pi(I_H) = \pi(B) = \pi = \pi(F(L)).$$

This contradicts our assumption that (ii) holds, so $\mathcal{O}_2(H) \neq 1$. Then Proposition 4.25 implies $H = M$. We conclude that for every $w \in W^{\#}$ we have

$$C_G(w) \leqslant L \text{ or } C_G(w) \leqslant M. \tag{†}$$

Now, take any $l \in L$. Since $F(L) \leqslant I \leqslant M$ we see that $F(L)$ normalizes $\mathcal{O}_2(M)$, so also $F(L)$ normalizes $\mathcal{O}_2(M)^l$. Then

$$\mathcal{O}_2(M)^l = \langle C_{\mathcal{O}_2(M)^l}(w) | w \in W^{\#} \rangle.$$

By Theorem 3.19 we have $\mathcal{O}_2(M) \cap L = 1$ because $\mathcal{O}_2(M) \cap L \in \text{И}_L(A, 2) = \{1\}$, and it follows from (†) that whenever $C_{\mathcal{O}_2(M)^l}(w) \neq 1$ we must have $C_{\mathcal{O}_2(M)^l}(w) \leqslant M$, because otherwise $1 \neq C_{\mathcal{O}_2(M)^l}(w) \leqslant \mathcal{O}_2(M)^l \cap L = (\mathcal{O}_2(M) \cap L)^l = 1$. Hence

$$\mathcal{O}_2(M)^l \leqslant M \text{ for every } l \in L.$$

In particular, if we take $k, l \in L$ then $\mathcal{O}_2(M)^{kl^{-1}}$ normalizes $\mathcal{O}_2(M)$, so $\mathcal{O}_2(M)^k$ normalizes $\mathcal{O}_2(M)^l$. Thus $\langle \mathcal{O}_2(M)^L \rangle$ is a 2-group, which is contained in M and is normalized by I, implying that $\langle \mathcal{O}_2(M)^L \rangle \leqslant \mathcal{O}_2(M)$ and $L \leqslant N_G(\mathcal{O}_2(M)) = M$. This is a contradiction because L and M are distinct maximal subgroups. $\square$

Proposition 4.27 *Let $A \leqslant L \in Max(G)$ and let $A \leqslant I_1 \in \mathcal{NI}_{\mathcal{O}}(L)$. Then $I_1 \in \mathcal{NI}_{\mathcal{O}}(G)$.*

Proof. It suffices to show that $N_G(I_1) \leqslant L$. Let $g \in N_G(I_1)$. Then $I_1 \leqslant L, L^g$ and $I_1 \in \mathcal{NI}_{\mathcal{O}}(L^g)$. If $\mathcal{O}_2(L) = 1$ then $F(L) \leqslant I_1 \leqslant L^g$ and $F(L^g) \leqslant I_1 \leqslant L$ which implies $L = L^g$ by Lemma 2.29 since $|\pi(F(L))| = |\pi| \geqslant 2$. If $\mathcal{O}_2(L) \neq 1$ then $\mathcal{U}^*_G(A, 2) = \{\mathcal{O}_2(L)\} = \{\mathcal{O}_2(L^g)\}$ by Proposition 4.25 and $L = L^g$. $\square$

Proposition 4.28 *Suppose I is contained in a unique maximal subgroup L of G. Then L is the unique maximal subgroup containing A.*

Proof. Let $A \leqslant M \in Max(G)$. We split into four cases.

Case 1: $\mathcal{O}_2(L) = 1 = \mathcal{O}_2(M)$.

The argument comes from [4, 3.3, p412]. Let $Z = Z(I)$ and observe that $Z \leqslant A \leqslant M$, so $Z = \mathcal{O}_p(Z) \times \mathcal{O}_{p'}(Z)$ normalizes $\mathcal{O}_p(M)$ for any $p \in \pi$. Since $I \leqslant C_G(\mathcal{O}_p(Z)), C_G(\mathcal{O}_{p'}(Z))$ we have

$$C_G(\mathcal{O}_p(Z)), C_G(\mathcal{O}_{p'}(Z)) \leqslant L,$$

and as L is soluble $Z \leqslant F(L)$, giving $\mathcal{O}_{p'}(Z) \leqslant \mathcal{O}_{p'}(L)$. These observations enable the following calculation:

$$\begin{aligned}
[C_{\mathcal{O}_p(M)}(\mathcal{O}_p(Z)), \mathcal{O}_{p'}(Z)] &\leqslant [L, \mathcal{O}_{p'}(L)] \cap [\mathcal{O}_p(M), M] \\
&\leqslant \mathcal{O}_{p'}(L) \cap \mathcal{O}_p(M) \\
&= 1.
\end{aligned}$$

Hence $[\mathcal{O}_p(M), \mathcal{O}_{p'}(Z)] = 1$ by the $P \times Q$ Lemma. So

$$\mathcal{O}_p(M) \leqslant C_G(\mathcal{O}_{p'}(Z)) \leqslant L,$$

and as p was arbitrary we get $F(M) \leqslant L$. This implies $M = L$ by Lemma 2.30.

Case 2: $\mathcal{O}_2(L) \neq 1 \neq \mathcal{O}_2(M)$.
Follows from 4.25 since $A \leqslant L, M$.

Case 3: $\mathcal{O}_2(L) \neq 1 = \mathcal{O}_2(M)$.
Let $A \leqslant I_1 \in \mathcal{NI}_{\mathcal{O}}(M)$. Then $I_1 \in \mathcal{NI}_{\mathcal{O}}(G)$ by Proposition 4.27, so there exists $g \in G$ such that

$$I_1 = I^g$$

by Corollary 3.12. We note that M is the unique maximal subgroup containing I_1, because if $N \in Max(G)$ with $I_1 \leqslant N$ then $I = I_1^{g^{-1}} \leqslant M^{g^{-1}} \cap N^{g^{-1}}$, giving $M^{g^{-1}} = N^{g^{-1}}$ and $M = N$. Now, $I_1^{g^{-1}} = I$ normalizes $\mathcal{O}_2(L) \neq 1$, so I_1 normalizes $\mathcal{O}_2(L)^g \neq 1$. Since G is simple we get $I_1\mathcal{O}_2(L)^g < G$, and it follows that

$$\mathcal{O}_2(L)^g \leqslant M.$$

Thus $1 \neq \mathcal{O}_2(L)^g \in \text{И}_M(A,2) = \{1\}$, a contradiction.

Case 4: $\mathcal{O}_2(L) = 1 \neq \mathcal{O}_2(M)$.

Similar to case 3. □

Proposition 4.29 *If some element of $\mathcal{NI}_{\mathcal{O}}(G)$ is not conjugate to I then A is contained in a unique maximal subgroup of G.*

Proof. Proposition 4.25 implies that I is contained in at most two maximal subgroups L and M of G satisfying $\mathcal{O}_2(L) = 1 \neq \mathcal{O}_2(M)$. Suppose both exist. Then we need not consider (H2). We wish to apply Proposition 4.26, so we need a subgroup W satisfying the hypothesis. Under (H3), by 4.22 and 4.23 we can assume $V = W$ if we can find $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq I$ such that $W \trianglelefteq F(L)$. It is clear that we can find such a W by Corollary 4.5 and Lemma 2.12(ii). Under (H1) we see that W exists by Proposition 4.21. Thus we apply Proposition 4.26 to conclude that I is contained in a unique maximal subgroup of G. Proposition 4.28 then provides the result. □

Proposition 4.30 *Suppose A is contained in a unique maximal subgroup L of G. If $\mathcal{O}_p(A) \leqslant P \in \mathrm{Syl}_p(L)$ then $P \in \mathrm{Syl}_p(G)$.*

Proof. It is enough to show that $N_G(P) \leqslant L$. Suppose not and let $n \in N_G(P)\backslash L$. Then $\mathcal{O}_p(A) \leqslant P \leqslant L^n$, so $\mathcal{O}_p(A)$ normalizes $\mathcal{O}_{p'}(L^n)$, and we have

$$\mathcal{O}_{p'}(L^n) = \langle C_{\mathcal{O}_{p'}(L^n)}(a) | a \in A^{\#} \rangle.$$

It follows that $\mathcal{O}_{p'}(L^n) \leqslant \langle C_G(a) | a \in A^{\#} \rangle \leqslant L$ since $A \leqslant C_G(a)$ for every $a \in A^{\#}$.

Set $\overline{L} = L/\mathcal{O}_{p'}(L)$. Note that $\mathcal{O}_p(\overline{L})$ normalizes $\overline{\mathcal{O}_{p'}(L^n)}$ since P normalizes $\mathcal{O}_{p'}(L^n)$ and $\overline{P} \in \mathrm{Syl}_p(\overline{L})$. So

$$[\mathcal{O}_p(\overline{L}), \overline{\mathcal{O}_{p'}(L^n)}] \leqslant \mathcal{O}_p(\overline{L}) \cap \overline{\mathcal{O}_{p'}(L^n)} = 1.$$

We now observe that $F(\overline{L}) = \mathcal{O}_p(\overline{L})$ and deduce that

$$\overline{\mathcal{O}_{p'}(L^n)} \leqslant C_{\overline{L}}(F(\overline{L})) \leqslant F(\overline{L}) = \mathcal{O}_p(\overline{L}),$$

giving $\overline{\mathcal{O}_{p'}(L^n)} = 1$ and $\mathcal{O}_{p'}(L^n) \leqslant \mathcal{O}_{p'}(L)$. Since L and L^n are isomorphic we get $\mathcal{O}_{p'}(L^n) = \mathcal{O}_{p'}(L)$. It remains to check that $\mathcal{O}_{p'}(L) \neq 1$ to obtain a contradiction. This holds because if $\mathcal{O}_2(L) = 1$ then $\pi(F(L)) = \pi$ and $|\pi| \geqslant 2$. $\square$

Theorem 4.31 *$\mathcal{NI}_{\mathcal{O}}(G)$ forms a single conjugacy class.*

Proof. Suppose false and pick $J \in \mathcal{NI}_{\mathcal{O}}(G)$ not conjugate to I. Then I and J both satisfy the properties stated at the beginning of this section. In particular, there exists $\mathbb{Z}_q \times \mathbb{Z}_q \cong W \trianglelefteq J$, and for some $B \in \mathcal{A}_{\mathcal{O}}(J)$ containing W we have $|\text{И}^*_G(B,2)| = 1$.
The previous result implies that A and B are each contained in a unique maximal subgroup of G. Call these subgroups L and M respectively and let $\mathcal{O}_q(J) \leqslant Q \in \mathrm{Syl}_q(M)$. Then $\mathcal{O}_q(J) \trianglelefteq Q$ by Corollary 3.17. We wish to assume that $W \trianglelefteq Q$. We can certainly find $\mathbb{Z}_q \times \mathbb{Z}_q \cong W_1 \trianglelefteq J$ such that $W_1 \trianglelefteq Q$ by Lemma 2.12(ii). We may assume $W = W_1$ if there exists $B_1 \in \mathcal{A}_{\mathcal{O}}(J)$ containing W_1 such that $|\text{И}^*_G(B_1,2)| = 1$. If $\mathcal{A}_{\mathcal{O}}(J)$ contains an element of rank at least 3 then this condition is satisfied by 4.22 and 4.23. If not then no element of $\mathcal{A}_{\mathcal{O}}(J)$ has rank greater than 2 and Theorem 4.20 contradicts the fact that we are working in a counterexample. So we assume $W \trianglelefteq Q$. Next we apply Proposition 4.30 and Sylow's Theorem to get $Q \in \mathrm{Syl}_q(G)$ and then $\mathcal{O}_q(I) \leqslant Q$ without loss of generality. So $\mathcal{O}_q(I)$ normalizes W. Since a Sylow q-subgroup of $\mathrm{Aut}(W)$ has order q we deduce that

$$C_{\mathcal{O}_q(I)}(W) \neq 1.$$

Thus $\langle A, W\rangle \leqslant N_G(C_{\mathcal{O}_q(I)}(W)) < G$ and we conclude that

$$W \leqslant L.$$

Now, for every $w \in W^{\#}$ we have $B \leqslant C_G(w)$, so $C_G(w) \leqslant M$, and it follows that

$$F_{q'}(L) = \langle C_{F_{q'}(L)}(w) | w \in W^{\#} \rangle \leqslant M.$$

We finally observe that $\mathcal{O}_q(L) \leqslant \mathcal{O}_q(I) \leqslant M$, giving

$$F(L) \leqslant M.$$

Now, as in Lemma 3.20 we have the factorization

$$L = N_L(I_0)\mathcal{O}_2(L)$$

where I_0 is a Hall $2'$-subgroup of the inverse image of $F(L/\mathcal{O}_2(L))$ in L and $I_0 \trianglelefteq I$. We see that $N_L(I_0)$ contains a Sylow q-subgroup of L, so there exists $l \in L$ such that $W^l \leqslant N_L(I_0)$. In fact we can choose $l \in \mathcal{O}_2(L)$. Then

$$\mathcal{O}_{q'}(I_0) = \langle C_{\mathcal{O}_{q'}(I_0)}(w^l) | w \in W^{\#} \rangle \leqslant M^l$$

since M^l is the unique maximal subgroup containing B^l and $B^l \leqslant C_G(w^l)$ for every $w \in W^{\#}$. Our choice of l implies that $M^l = M$, so $\mathcal{O}_{q'}(I_0) \leqslant M$. We also have $\mathcal{O}_q(I_0) \leqslant \mathcal{O}_q(I) \leqslant M$, giving

$$I_0 \leqslant M.$$

Another application of Lemma 3.20 yields $M = N_M(J_0)\mathcal{O}_2(M)$, again where J_0 is a Hall $2'$-subgroup of the inverse image of $F(M/\mathcal{O}_2(M))$ in M and $J_0 \trianglelefteq J$. Without loss of generality we can assume

$$I_0 \leqslant N_M(J_0)$$

by Hall's Theorem. Let $r, t \in \pi$ be distinct, and consider the action of $\mathcal{O}_r(I_0) \times \mathcal{O}_t(I_0)$

on $\mathcal{O}_r(J_0)$. Since I is contained in $N_G(\mathcal{O}_r(I_0))$, we must have $N_G(\mathcal{O}_r(I_0)) \leqslant L$, therefore $C_{\mathcal{O}_r(J_0)}(\mathcal{O}_r(I_0)) \leqslant L$. Hence

$$\begin{aligned}[C_{\mathcal{O}_r(J_0)}(\mathcal{O}_r(I_0)), \mathcal{O}_t(I_0)] &\leqslant [L, \mathcal{O}_t(I_0)\mathcal{O}_2(L)] \cap \mathcal{O}_r(J_0) \\ &\leqslant \mathcal{O}_t(I_0)\mathcal{O}_2(L) \cap \mathcal{O}_r(J_0) \\ &= 1.\end{aligned}$$

Thompson's $P \times Q$ Lemma then implies

$$[\mathcal{O}_r(J_0), \mathcal{O}_t(I_0)] = 1. \tag{†}$$

So $\mathcal{O}_r(J_0) \leqslant N_G(\mathcal{O}_t(I_0)) \leqslant L$, and in fact since r was chosen arbitrarily,

$$J_0 \leqslant L.$$

Set $\overline{L} = L/\mathcal{O}_2(L)$. Then by (†) we have $[\overline{\mathcal{O}_r(J_0)}, \overline{\mathcal{O}_t(I_0)}] = 1$. Since this holds for any distinct primes r and t and since $\overline{\mathcal{O}_t(I_0)} = \mathcal{O}_t(\overline{L})$ we get

$$[\overline{\mathcal{O}_r(J_0)}, F_{r'}(\overline{L})] = 1.$$

Thus $\overline{J_0}F(\overline{L})$ is nilpotent. By Theorem 3.6 we see that $\overline{I} = \overline{I}F(\overline{L})$ is also nilpotent, and Lemma 2.33 implies that

$$[\overline{\mathcal{O}_r(I)}, \overline{\mathcal{O}_t(J_0)}] = 1. \tag{$*$}$$

Consider the subgroup $K = \mathcal{O}_r(I)\mathcal{O}_t(J_0)\mathcal{O}_2(L)$ of L. Let $\mathcal{O}_r(I) \leqslant H \in \mathrm{Hall}_{\{r,t\}}(K)$. From ($*$) we have $\mathcal{O}_t(J_0)\mathcal{O}_2(L) \trianglelefteq K$, so

$$\mathcal{O}_r(I) \text{ normalizes } H \cap \mathcal{O}_t(J_0)\mathcal{O}_2(L).$$

Therefore we can apply Theorem 2.15 to deduce that $\mathcal{O}_r(I)$ normalizes a Sylow t-subgroup of $H \cap \mathcal{O}_t(J_0)\mathcal{O}_2(L)$. Since $\mathcal{O}_t(J_0) \in \mathrm{Syl}_t(H \cap \mathcal{O}_t(J_0)\mathcal{O}_2(L))$, there exists $l \in \mathcal{O}_2(L)$ such that $\mathcal{O}_r(I)^l$ normalizes $\mathcal{O}_t(J_0)$. Hence $\mathcal{O}_r(I)^l \leqslant N_G(\mathcal{O}_t(J_0)) \leqslant M$, and as $l \in \mathcal{O}_2(L) \leqslant M$ we deduce that

$$\mathcal{O}_r(I) \leqslant M.$$

Repeating the argument for every prime $r \in \pi$ allows us to conclude that $I \leqslant M$, and Theorem 3.15 completes the proof. $\square$

Chapter 5

π-Nilpotent Injectors

We verify that the rank 1 case of the previous chapter holds for π-nilpotent injectors. To do this we first need analogues of many of the results from chapters 3 and 4. Most of the results hold by making superficial changes such as replacing "odd" with "π" or "$\mathcal{A}_{\mathcal{O}}(G)$" with "$\mathcal{A}_{2,\pi}(G)$" etc. We present proofs for those results which have nontrivial differences. A result which is an analogue from one of the chapters mentioned previously is labelled by its own number plus the number of the result to which it is an analogue.

Throughout let π be a set of primes. If σ is the set of prime divisors of $d_{2,\pi}(G)$ then clearly $\sigma \subseteq \pi$. We adopt the convention of removing any redundant primes from π, i.e, if $\sigma \subset \pi$ then we replace π with σ. Thus we can always assume $\sigma = \pi$.

5.1 Arbitrary Groups and Soluble Groups

Theorem 5.1 *(3.5) (Glauberman) Let $A \in \mathcal{A}_{2,\pi}(G)$. Suppose $B \leqslant G$ is a nilpotent π-subgroup normalized by A. Then AB is nilpotent.*

Proof. Adapt the proof of [7, Theorem B, p470]. □

Theorem 5.2 *(Thompson) [1, Theorem 3.11, p185] Let P be a p-group. Then P has a* *subgroup C such that*

(i) *C char P;*

(ii) *C has nilpotence class at most 2;*

(iii) *a nontrivial p'-automorphism of P induces a nontrivial automorphism of C.*

Definition 5.3 *Let P be a p-group. Then the subgroup C from Theorem 5.2 is called a* ***critical subgroup*** *of P.*

Theorem 5.4 *(3.6) (Bender) [4, 1.5, p408] Let $I \in \mathcal{NI}_\pi(G)$. Then I contains every nilpotent π-subgroup that it normalizes.*

Proof. Suppose false and let $B \leqslant G$ be nilpotent and normalized by I but with $B \nleqslant I$. Choose B to be minimal with this property. Then B is a p-group. Also

$$[B, \mathcal{O}_{p'}(I)] \neq 1$$

because IB is not nilpotent.

If $C_B(\mathcal{O}_p(I)) < B$ then by the minimality of B we have $C_B(\mathcal{O}_p(I)) \leqslant I$. Therefore $[C_B(\mathcal{O}_p(I)), \mathcal{O}_{p'}(I)] = 1$ and the $P \times Q$-Lemma gives $[B, \mathcal{O}_{p'}(I)] = 1$, a contradiction. So

$$[B, \mathcal{O}_p(I)] = 1.$$

Let B_0 be a critical subgroup of B. If $B_0 < B$ then again the minimality of B gives $B_0 \leqslant I$ and $[B_0, \mathcal{O}_{p'}(I)] = 1$. This contradicts Theorem 5.2(iii) because $[B, \mathcal{O}_{p'}(I)] \neq 1$. So $B_0 = B$ and B has class at most 2.

Let $A \in \mathcal{A}_{2,\pi}(G)$ with $A \leqslant I$. By Lemma 5.1 we see that AB is nilpotent, so $[B, \mathcal{O}_{p'}(A)] = 1$. Hence $[B, A] = 1$ and AB is a nilpotent π group of class at most 2. By definition of A

this implies $B \leqslant A \leqslant I$, a contradiction. $\square$

Lemma 5.5 *(3.14) [4, 2.7, p410] Let $I \in \mathcal{NI}_\pi(G)$ and suppose $\mathbb{Z}_p \times \mathbb{Z}_p \cong V \trianglelefteq I$. Then $V \leqslant A$ for every $A \in \mathcal{A}_{2,\pi}(I)$.*

Proof. We show that AV has class at most 2. By using commutator relations it suffices to show $\langle A', V', [A, V] \rangle \leqslant Z(AV)$. We have $A/C_A(V) \leqslant \mathrm{Aut}(V) \cong \mathrm{GL}_2(p)$, so

$$|A/C_A(V)| = 1 \text{ or } p$$

because $[\mathcal{O}_{p'}(A), V] = 1$. Therefore $A' \leqslant C_A(V)$, giving $A' \leqslant C_A(V) \cap Z(A) \leqslant Z(AV)$. Clearly $V' = 1$ so we are left with $[A, V]$. Since I is nilpotent we have $[A, V] < V$. If $[A, V] = 1$ then $V \leqslant A$ and we are done, so we can assume $[A, V] \cong \mathbb{Z}_p$. Therefore $[A, V] \cap Z(AV) \neq 1$ because $[A, V] \trianglelefteq AV$, giving $[A, V] \leqslant Z(AV)$. $\square$

The next result to consider is Theorem 3.19. The analogue requires some extra work.

Lemma 5.6 *Let $M, N \trianglelefteq G$. Then $|G/M \cap N| \leqslant |G/M||G/N|$.*

Proof. Use the Second Isomorphism Theorem. $\square$

Theorem 5.7 *(Glauberman) [7, Proposition 1, p470] Let $V \neq 1$ be an elementary abelian p-group. Suppose A is a nilpotent p'-group of class at most 2 acting faithfully on V. Then $|A| < |V|$.*

Corollary 5.8 *Let A be a nilpotent group of class at most 2 acting faithfully and coprimely on the nilpotent group $G \neq 1$. Then $|A| < |G|$.*

Proof. Let G be a minimal counterexample. Suppose G does not have prime power order and let $p \in \pi(G)$. Then by the minimality of G we have $|A/C_A(\mathcal{O}_p(G))| < |\mathcal{O}_p(G)|$ and

$|A/C_A(\mathcal{O}_{p'}(G))| < |\mathcal{O}_{p'}(G)|$. We see that

$$C_A(\mathcal{O}_p(G)) \cap C_A(\mathcal{O}_{p'}(G)) \leqslant C_A(G) = 1,$$

therefore

$$\begin{aligned} |A| &= |A/C_A(\mathcal{O}_p(G)) \cap C_A(\mathcal{O}_{p'}(G))| \\ &\leqslant |A/C_A(\mathcal{O}_p(G))||A/C_A(\mathcal{O}_{p'}(G))| \\ &< |\mathcal{O}_p(G)||\mathcal{O}_{p'}(G)| = |G| \end{aligned}$$

by Lemma 5.6. This contradiction implies that G is a p-group for some prime p. By Burnside's Theorem (2.8) A acts faithfully on $G/\Phi(G)$, and since $G/\Phi(G)$ is elementary abelian, Glauberman's Theorem implies $|A| < |G/\Phi(G)| \leqslant |G|$. □

The proof of the next theorem is influenced by [9, Theorem 1.11, p8].

Theorem 5.9 *Let G be soluble and suppose $A \leqslant G$ is a p-subgroup of nilpotence class at most 2 which has maximal order amongst all such p-subgroups. Then for any $R \in$ $\mathit{\text{И}}_G(A, p')$ we have $R \leqslant \mathcal{O}_{p'}(G)$.*

Proof. Let G be a minimal counterexample. We first argue that $\mathcal{O}_{p'}(G) = 1$. Suppose not and set $\overline{G} = G/\mathcal{O}_{p'}(G)$. If $\overline{A}$ does not satisfy the hypothesis of the lemma then we can find a p-subgroup $\overline{B} \leqslant \overline{G}$ of nilpotence class at most 2 such that $|\overline{A}| < |\overline{B}|$. But then a Sylow p-subgroup of the inverse image of $\overline{B}$ in G is isomorphic to $\overline{B}$, contradicting the maximality of $|A|$. So by the minimality of G we have $\overline{R} \leqslant \mathcal{O}_{p'}(\overline{G}) = 1$. This contradicts the fact that G is a counterexample. Hence

$$\mathcal{O}_{p'}(G) = 1.$$

Let T be a minimal A-invariant subgroup of R. Observe that because R is soluble, T is elementary abelian. We show that $G = \mathcal{O}_p(G)TA$. Let $H = \mathcal{O}_p(G)TA$ and suppose

$H < G$. Then by the minimality of G we have $T \leqslant \mathcal{O}_{p'}(H)$. However, this implies $[T, \mathcal{O}_p(G)] \leqslant \mathcal{O}_{p'}(H) \cap \mathcal{O}_p(G) = 1$, giving $T \leqslant C_G(\mathcal{O}_p(G)) \leqslant \mathcal{O}_p(G)$, a contradiction. So

$$G = \mathcal{O}_p(G)TA.$$

Let $Q = C_{\mathcal{O}_p(G)}(C_A(T))$. Considering the action of $C_A(T) \times C_T(Q)$ on $\mathcal{O}_p(G)$, clearly $[C_{\mathcal{O}_p(G)}(C_A(T)), C_T(Q)] = 1$, and the $P \times Q$ Lemma gives

$$[\mathcal{O}_p(G), C_T(Q)] = 1.$$

Thus $C_T(Q) \leqslant C_G(\mathcal{O}_p(G)) \leqslant \mathcal{O}_p(G)$, whence

$$C_T(Q) = 1.$$

Now let C be a critical subgroup of Q. Since the action of T on Q is faithful, the action of T on C is also faithful. We observe that $[C \cap A, T] \leqslant [C, T] \cap [A, T] \leqslant C \cap T = 1$, so

$$C \cap A = C \cap C_A(T).$$

By the definition of Q we have $[C, C_A(T)] = 1$, thus $C \cap A \trianglelefteq C$. Set

$$\overline{C} = C/C \cap A.$$

This quotient is T-invariant, and looking at the action of $C_T(\overline{C})$ on C we get $[C_T(\overline{C}), C] \leqslant C \cap A$. Therefore

$$[C, C_T(\overline{C}), C_T(\overline{C})] \leqslant [C \cap A, C_T(\overline{C})] \leqslant C \cap T = 1,$$

giving $[C, C_T(\overline{C})] = 1$. So $C_T(\overline{C}) \leqslant C_T(C) = 1$ and the action of T on $\overline{C}$ is faithful. If $\overline{C} = 1$ then clearly this implies $T = 1$ and $|T| = |C/C \cap A|$. If $\overline{C} \neq 1$ then we may apply Corollary 5.8 to the action of T on $\overline{C}$ to get $|T| < |C/C \cap A|$. In both cases,

$$|T| \leqslant |C/C \cap A|.$$

We also apply Corollary 5.8 to the action of $A/C_A(T)$ on T, yielding

$$|A/C_A(T)| < |T|.$$

Combining these inequalities gives

$$|A| < |C_A(T)||C|/|C \cap A| = |C_A(T)C||C_A(T) \cap C|/|C \cap A|.$$

We have already seen that $C \cap A = C \cap C_A(T)$, so we deduce that

$$|A| < |C_A(T)C|.$$

However, both $C_A(T)$ and C have class at most 2, and since $[C_A(T), C] = 1$ this contradicts the maximality of $|A|$. $\square$

Corollary 5.10 *(3.19) Let G be soluble and let $A \in \mathcal{A}_{2,\pi}(G)$. Then $\mathsf{И}^*_G(A, \pi') = \{\mathcal{O}_{\pi'}(G)\}$.*

Proof. We may assume $\mathcal{O}_{\pi'}(G) = 1$. Let $R \in \mathsf{И}_G(A, \pi')$ and let $p \in \pi$. Let

$$K = \mathcal{O}_p(A)R\mathcal{O}_p(G).$$

Theorem 5.1 tells us that $AF(G)$ is nilpotent, in particular

$$[\mathcal{O}_p(G)\mathcal{O}_p(A), \mathcal{O}_{p'}(A)] = 1.$$

Then since $\mathcal{O}_p(A)\mathcal{O}_p(G) \in \mathrm{Syl}_p(K)$, the definition of $\mathcal{A}_{2,\pi}(G)$ and an application of Sylow's Theorem allows us to deduce that $\mathcal{O}_p(A)$ has maximal order amongst all p-subgroups of K of class at most 2. Hence the previous theorem applies and $R \leqslant \mathcal{O}_{p'}(K)$. Therefore

$$[R, \mathcal{O}_p(G)] \leqslant \mathcal{O}_{p'}(K) \cap \mathcal{O}_p(G) = 1.$$

As p was arbitrary, $R \leqslant C_G(F(G)) \leqslant F(G)$ yielding $R = 1$. $\square$

5.2 Minimal Simple Groups and The Rank 1 Case

Definition 5.11 *Let $n \in \mathbb{N}$. The* ***Dihedral*** *group of order 2^n is*

$$D_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, x^y = x^{-1} \rangle.$$

The ***Semidihedral*** *group of order 2^n is*

$$Sd_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, x^y = x^{2^{n-2}-1} \rangle.$$

The ***Generalized Quaternion*** *group of order 2^n is*

$$Q_{2^n} = \langle x, y | x^{2^{n-1}} = y^4 = 1, x^{2^{n-2}} = y^2, x^y = x^{-1} \rangle.$$

Lemma 5.12 *[1, 5.4.10, p199] Let P be a 2-group. Suppose every abelian normal subgroup of P is cyclic. Then P is cyclic, dihedral of order at least 2^4, semidihedral of order at least 2^4, or generalized quaternion of order at least 2^3.*

Definition 5.13 *A* ***quasicyclic*** *group is a nilpotent group G in which $\mathcal{O}(G)$ is cyclic and $\mathcal{O}_2(G)$ is cyclic, dihedral, semidihedral or generalized quaternion.*

We remark that by [1, 5.4.10(i), p199], which appears here as Lemmas 2.12(i) and 5.12, the quasicyclic groups are precisely the nilpotent groups G in which every abelian normal subgroup of $\mathcal{O}(G)$ is cyclic and one of the following holds:

- every abelian normal subgroup of $\mathcal{O}_2(G)$ is cyclic;
- $\mathcal{O}_2(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
- $\mathcal{O}_2(G) \cong D_8$.

Lemma 5.14 *[1, 5.4.3(ii), p.191] Let G be a quasicyclic 2-group. Then every subgroup of G is quasicyclic and $G/\Phi(G)$ is isomorphic to a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

In what follows ϕ denotes the Euler totient function.

Lemma 5.15 *Let G be a noncyclic 2-group in which every abelian normal subgroup is cyclic. Then $\mathrm{Aut}(G)$ is a 2-group or $G = Q_8$ and $\mathrm{Aut}(G)$ has order 24.*

Proof. The proof is similar for each of the three classes (dihedral, semidihedral and generalized quaternion), so we will consider only one case: the generalized quaternion groups. Let $G = Q_{2^n}$ and assume $n \geqslant 4$. By using the relation $x^i y = yx^{-i}$, each element of G can be written in the form x^i or $x^i y$ for some $i \in \{1, ..., 2^{n-1}\}$. The order of x^i is $2^{n-1}/\mathrm{hcf}(i, 2^{n-1})$, and the order of $x^i y$ is 4 because

$$(x^i y)^2 = x^i y(x^i y) = x^i y(yx^{-i}) = x^i y^2 x^{-i} = y^2,$$

using the fact that $Z(G) = \langle y^2 \rangle$. Now let $\theta \in \mathrm{Aut}(G)$. The action of θ is completely determined by its action on the generators x and y. Since x has order $2^{n-1} > 4$, it must be the case that

$$x\theta = x^k \text{ where } \mathrm{hcf}(k, 2^{n-1}) = 1.$$

The number of choices for k is $\phi(2^{n-1}) = 2^{n-2}$. We cannot have $y\theta \in \langle x\theta \rangle$ as θ is a bijection, so

$$y\theta = x^l y \text{ for some } l \in \{1, ..., 2^{n-1}\}$$

and the number of choices for $y\theta$ is 2^{n-1}. The choices for $x\theta$ and $y\theta$ mentioned so far satisfy the first relation. We verify that they also satisfy the other two. First, we observe that $(x^k)^{2^{n-2}}$ has order 2, which implies $(x^k)^{2^{n-2}} = y^2$ because y^2 is the unique element of G of order 2. Thus,

$$(x\theta)^{2^{n-2}} = (x^k)^{2^{n-2}} = y^2 = (x^l y)^2 = (y\theta)^2.$$

Finally,

$$(x\theta)^{y\theta} = (x^k)^{x^l y} = (x^k)^y = (x^y)^k = (x^{-1})^k = (x^k)^{-1} = (x\theta)^{-1}.$$

We conclude that

$$|\mathrm{Aut}(G)| = 2^{n-2} \cdot 2^{n-1} = 2^{2n-3}.$$

It remains to assume $n = 3$ so that $G = Q_8$. In this case x has order 4, so $x\theta$ can be any of the six elements of order 4. The order of $y\theta$ is also 4, and since we cannot have $y\theta \in \langle x\theta \rangle$, there are 4 choices for $y\theta$. The second relation is automatically satisfied because $(x\theta)^2$ and $(y\theta)^2$ both have order 2. For the last relation, observe that $\langle x\theta \rangle \trianglelefteq G$, so either $(x\theta)^{y\theta} = (x\theta)^{-1}$ or $(x\theta)^{y\theta} = (x\theta)$. The latter cannot happen because G is nonabelian. $\square$

Proposition 5.16 *(4.4) Let $I \in \mathcal{NI}_\pi(G)$ and let $I \leqslant L \in Max(G)$. Suppose $\mathcal{O}_{\pi'}(L) = 1$ and every abelian normal subgroup of $F(L)$ is cyclic. Then I is conjugate to every element of $\mathcal{NI}_\pi(G)$.*

Proof. If $F(L)$ is cyclic then the argument from Proposition 4.4 does the job. So we may assume $F(L)$ is noncyclic. By Sylow's Theorem we may also assume $|\pi| \geqslant 2$. Let $p = \max \pi$ and let $\mathcal{O}_p(I) \leqslant P \in \mathrm{Syl}_p(L)$. Pick $J \in \mathcal{NI}_\pi(G)$ and $A, B \in \mathcal{A}_{2,\pi}(G)$ such that $A \leqslant I$ and $B \leqslant J$.

Suppose $[P, F_{p'}(L)] = 1$. Then $\mathcal{O}_p(I) = P$ by Lemma 3.16. We have $Z(\mathcal{O}_p(I)) \leqslant C_L(F(L)) \leqslant F(L)$, so $Z(\mathcal{O}_p(I)) \leqslant \mathcal{O}_p(L)$, which is cyclic since $F(L)$ is quasicyclic and $p \neq 2$. Subgroups of cyclic groups are characteristic, so

$$Z(\mathcal{O}_p(I)) \trianglelefteq L.$$

This implies $N_G(\mathcal{O}_p(I)) \leqslant N_G(Z(\mathcal{O}_p(I))) = L$, giving $\mathcal{O}_p(I) \in \mathrm{Syl}_p(G)$. Without loss of generality, $\mathcal{O}_p(J) \leqslant \mathcal{O}_p(I)$ by Sylow's Theorem. Then

$$A^* = \mathcal{O}_p(B)\mathcal{O}_{p'}(A) \in \mathcal{A}_{2,\pi}(G).$$

So $\langle A, A^* \rangle \leqslant N_G(\mathcal{O}_{p'}(A)) < G$ and $\langle A^*, B \rangle \leqslant N_G(\mathcal{O}_p(B)) < G$, which implies that I is conjugate to J by Proposition 4.3.

So we can suppose $[P, F_{p'}(L)] \neq 1$. Then we can find $q \in \pi \backslash \{p\}$ such that $[P, \mathcal{O}_q(L)] \neq 1$, and Lemma 2.27 gives $q = 2$. Since P acts nontrivially on $\mathcal{O}_2(L)$, Lemma 5.15 yields

$$\mathcal{O}_2(L) \cong Q_8 \text{ and } p = 3.$$

Let $\mathcal{O}_2(I) \leqslant S \in \mathrm{Syl}_2(L)$. We wish to show that $S \in \mathrm{Syl}_2(G)$. We have $Z = Z(\mathcal{O}_2(I)) \leqslant C_L(F(L)) \leqslant F(L)$, so $Z \leqslant Z(\mathcal{O}_2(L))$. Since $Z \neq 1$ and $Z(Q_8)$ has order 2 we deduce that

$$Z = Z(\mathcal{O}_2(L)) \cong \mathbb{Z}_2.$$

After observing that $F_{2'}(L) = \mathcal{O}_3(L)$ we split into two cases as before. First suppose $[S, \mathcal{O}_3(L)] = 1$. Then $\mathcal{O}_2(I) = S$. Therefore $N_G(\mathcal{O}_2(I)) \leqslant N_G(Z) = L$ and $\mathcal{O}_2(I) \in \mathrm{Syl}_2(G)$. Apply the same argument as before.

We may now suppose $[S, \mathcal{O}_3(L)] \neq 1$. Then $[S, \mathcal{O}_3(L)/\Phi(\mathcal{O}_3(L))] \neq 1$, and since $\mathcal{O}_3(L)$

is cyclic and $[\mathcal{O}_2(I), \mathcal{O}_3(L)] = 1$ we get $|S : \mathcal{O}_2(I)| = 2$, using the fact that $\mathcal{O}_2(I) \in \mathrm{Syl}_2(C_L(F_{2'}(L)))$. So

$$S' \leqslant \mathcal{O}_2(I).$$

Now, $Z \trianglelefteq S$ implies $1 \neq Z \cap Z(S) \leqslant Z$. So $Z \cap Z(S) = Z$ and

$$Z \leqslant Z(S).$$

If $Z(S) \leqslant \mathcal{O}_2(I)$ then $Z(S) = Z$ and Z char S, in which case $N_G(S) \leqslant N_G(Z) = L$ and $S \in \mathrm{Syl}_2(G)$. So we may assume $Z(S) \nleqslant \mathcal{O}_2(I)$. Therefore

$$S = \mathcal{O}_2(I)Z(S).$$

We use the formula $|S| = |\mathcal{O}_2(I)||Z(S)|/|\mathcal{O}_2(I) \cap Z(S)|$ to deduce that

$$2 = |S/\mathcal{O}_2(I)| = |Z(S)/\mathcal{O}_2(I) \cap Z(S)| = |Z(S)/Z|.$$

Since $|Z| = 2$ we get $|Z(S)| = 4$. We assumed that $Z(S) \nleqslant \mathcal{O}_2(I)$, so $Z(S) \nleqslant S'$. Hence $1 < S' \cap Z(S) < Z(S)$, giving $|S' \cap Z(S)| = 2$. It follows that $S' \cap Z(S) = Z$ because $S' \cap Z(S) \leqslant Z$. So Z char S, implying $N_G(S) \leqslant N_G(Z) = L$ and

$$S \in \mathrm{Syl}_2(G)$$

as desired. By Sylow's Theorem we may assume $\mathcal{O}_2(J) \leqslant S$.

Let $x \in \mathcal{O}_2(I)$ and suppose conjugation by x induces an inner automorphism on $\mathcal{O}_2(L)$. Then conjugation by x on $\mathcal{O}_2(L)$ is equivalent to conjugation by some $l \in \mathcal{O}_2(L)$. So

$$xl^{-1} \in C_L(\mathcal{O}_2(L)).$$

Since $xl^{-1} \in \mathcal{O}_2(I)$ we see that in fact $xl^{-1} \in C_L(F(L)) = Z(F(L))$, implying

$$x \in Z(\mathcal{O}_2(L))\mathcal{O}_2(L) = \mathcal{O}_2(L).$$

Now, $\mathcal{O}_2(I)/C_{\mathcal{O}_2(I)}(\mathcal{O}_2(L))$ acts as a group of automorphisms on $\mathcal{O}_2(L) \cong Q_8$, and since the inner automorphism group of Q_8 has index 2 inside a Sylow 2-subgroup of $\mathrm{Aut}(Q_8)$ we conclude that at least half the elements of $\mathcal{O}_2(I)$ induce inner automorphisms. Since all such elements are contained in $\mathcal{O}_2(L)$ it follows that

$$|\mathcal{O}_2(I) : \mathcal{O}_2(L)| \leqslant 2 \text{ and } |S : \mathcal{O}_2(L)| \leqslant 4.$$

Now, we have $|\mathcal{O}_2(J)| \geqslant 8$ because $\mathcal{O}_2(L) \leqslant \mathcal{O}_2(I)$ and $\mathcal{O}_2(L)$ has class 2. Therefore,

$$\mathcal{O}_2(L) \cap \mathcal{O}_2(J) \neq 1.$$

We have Z char $\mathcal{O}_2(L) \cap \mathcal{O}_2(J)$ because Z is the unique subgroup of order 2 inside $\mathcal{O}_2(L)$. Since $\mathcal{O}_2(L) \cap \mathcal{O}_2(J) \trianglelefteq \mathcal{O}_2(J)$ we deduce that $\langle I, J \rangle \leqslant N_G(Z) < G$. This completes the proof. □

It may not be the case that an analogue of Lemma 4.7 holds since the proof is dependent upon elements of $\mathcal{V}_G(A, \pi')$ being nilpotent. The proof would work if one were to consider the special case $\pi = r'$ for some prime r. Lemma 4.8 may also not hold because elements of $\mathcal{A}_{2,\pi}(G)$ need not be abelian.

We split the proof of the analogue of Proposition 4.9 into two parts for convenience of exposition.

Proposition 5.17 *(4.9) Let $I \in \mathcal{NI}_\pi(G)$. Suppose I is noncyclic, every abelian normal subgroup of I is cyclic, and $|\pi| \geqslant 2$. Let $p = max\ \pi$. Assume $I \leqslant L, M \in Max(G)$ with*

$|L|_p$ *and* $|M|_2$ *maximal subject to this constraint. Then one of the following holds:*

(i) $\mathcal{O}_p(I) \in \mathrm{Syl}_p(L) \cap \mathrm{Syl}_p(G)$;

(ii) $\mathcal{O}_2(I) \in \mathrm{Syl}_2(M) \cap \mathrm{Syl}_2(G)$;

(iii) $\mathcal{O}_2(I) \leqslant S \in \mathrm{Syl}_2(M) \cap \mathrm{Syl}_2(G)$ *and* $|S : \mathcal{O}_2(I)| = 2$.

Proof. Set

$$\overline{L} = L/\mathcal{O}_{\pi'}(L)$$

and let $\mathcal{O}_p(I) \leqslant P \in \mathrm{Syl}_p(L)$. So $\mathcal{O}_p(\overline{I}) \leqslant \overline{P} \in \mathrm{Syl}_p(\overline{L})$. Suppose $[\overline{P}, F_{p'}(\overline{L})] = 1$. Then $\mathcal{O}_p(\overline{I}) = \overline{P}$ and $\mathcal{O}_p(I) = P$ because both map isomorphically onto their images. Since $I \leqslant N_G(\mathcal{O}_p(I))$ we get $\mathcal{O}_p(I) \in \mathrm{Syl}_p(G)$ by the maximality of $|L|_p$. Therefore (i) holds. Now suppose $[\overline{P}, F_{p'}(\overline{L})] \neq 1$. Then we can find $q \in \pi\backslash\{p\}$ such that $[\overline{P}, \mathcal{O}_q(\overline{L})] \neq 1$ and by Lemma 2.27 it must be the case that $q = 2$. Since $\mathcal{O}_2(\overline{I})$ is quasicyclic we apply Lemma 5.14 to deduce that $\mathcal{O}_2(\overline{L})/\Phi(\mathcal{O}_2(\overline{L}))$ is isomorphic to a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The fact that $[\overline{P}, \mathcal{O}_2(\overline{L})/\Phi(\mathcal{O}_2(\overline{L}))] \neq 1$ implies

$$p = 3.$$

because $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathrm{S}_3$. Set

$$\widetilde{M} = M/\mathcal{O}_{\pi'}(M)$$

and let $\mathcal{O}_2(I) \leqslant S \in \mathrm{Syl}_2(M)$ so that $\mathcal{O}_2(\widetilde{I}) \leqslant \widetilde{S} \in \mathrm{Syl}_2(\widetilde{M})$. Since $\mathcal{O}_2(I)$ is nonabelian,

$$S' \neq 1.$$

After observing that $F_{2'}(\widetilde{M}) = \mathcal{O}_3(\widetilde{M})$ we split into two cases as before. First suppose

$[\widetilde{S}, \mathcal{O}_3(\widetilde{M})] = 1$. Then $\mathcal{O}_2(\widetilde{I}) = \widetilde{S}$ and $\mathcal{O}_2(I) = S$. We get $\mathcal{O}_2(I) \in \mathrm{Syl}_2(G)$ by the maximality of $|M|_2$. Thus (ii) holds.

Now suppose $[\widetilde{S}, \mathcal{O}_3(\widetilde{M})] \neq 1$. Then $[\widetilde{S}, \mathcal{O}_3(\widetilde{M})/\Phi(\mathcal{O}_3(\widetilde{M}))] \neq 1$, and since $\mathcal{O}_3(\widetilde{M})$ is cyclic and $[\mathcal{O}_2(\widetilde{I}), \mathcal{O}_3(\widetilde{M})] = 1$ we get $|\widetilde{S} : \mathcal{O}_2(\widetilde{I})| = 2$ because $\mathcal{O}_2(\widetilde{I}) \in \mathrm{Syl}_2(C_{\widetilde{M}}(F_{2'}(\widetilde{M})))$. So

$$|S : \mathcal{O}_2(I)| = 2 \text{ and } S' \leqslant \mathcal{O}_2(I).$$

Thus $1 \neq S' \trianglelefteq I$, giving $\langle I, N_G(S)\rangle \leqslant N_G(S') < G$. This implies $S \in \mathrm{Syl}_2(G)$ by the maximality of $|M|_2$, and (iii) holds. □

Proposition 5.18 *(4.9) Let $I \in \mathcal{NI}_\pi(G)$. Suppose every abelian normal subgroup of I is cyclic. Then I is conjugate to every element of $\mathcal{NI}_\pi(G)$.*

Proof. As in Proposition 5.16 we may assume I is noncyclic and $|\pi| \geqslant 2$. Let $p = \max \pi$ and let $I \leqslant L, M \in Max(G)$ such that $|L|_p$ and $|M|_2$ are maximized. By the previous proposition we have the following cases:

(i) $\mathcal{O}_p(I) \in \mathrm{Syl}_p(L) \cap \mathrm{Syl}_p(G)$;

(ii) $\mathcal{O}_2(I) \in \mathrm{Syl}_2(M) \cap \mathrm{Syl}_2(G)$;

(iii) $\mathcal{O}_2(I) \leqslant S \in \mathrm{Syl}_2(M) \cap \mathrm{Syl}_2(G)$ and $|S : \mathcal{O}_2(I)| = 2$.

Let $J \in \mathcal{NI}_\pi(G)$, let $B \in \mathcal{A}_{2,\pi}(J)$ and let $A \in \mathcal{A}_{2,\pi}(I)$. If case (i) holds we may suppose $\mathcal{O}_p(J) \leqslant \mathcal{O}_p(I)$ without loss of generality. Then $A^* = \mathcal{O}_p(B)\mathcal{O}_{p'}(A) \in \mathcal{A}_{2,\pi}(G)$. So $\langle A, A^*\rangle \leqslant N_G(\mathcal{O}_{p'}(A)) < G$ and $\langle A^*, B\rangle \leqslant N_G(\mathcal{O}_p(B)) < G$, which implies that I is conjugate to J.

The same argument applies to case (ii) so we are left with case (iii). Again without loss of generality

$$\mathcal{O}_2(J) \leqslant S.$$

Now, $\mathcal{O}_2(I)$ is quasicyclic so contains a cyclic subgroup at index 2. Therefore

$$|\mathcal{O}_2(I) : \mathcal{O}_2(A)| \leqslant 2. \tag{1}$$

Since

$$|\mathcal{O}_2(J)| \geqslant |\mathcal{O}_2(B)| = |\mathcal{O}_2(A)| \tag{2}$$

we must have

$$|S : \mathcal{O}_2(J)| \leqslant 4. \tag{3}$$

If $\mathcal{O}_2(J) \leqslant \mathcal{O}_2(I)$ then our argument for case (i) applies, so we can assume $\mathcal{O}_2(J) \nleqslant \mathcal{O}_2(I)$, which implies $S = \mathcal{O}_2(I)\mathcal{O}_2(J)$ and

$$|\mathcal{O}_2(J) : \mathcal{O}_2(I) \cap \mathcal{O}_2(J)| = 2. \tag{4}$$

Groups of orders 2 and 4 do not satisfy the hypothesis imposed on $\mathcal{O}_2(I)$, so we have $|\mathcal{O}_2(I)| \geqslant 8$. Therefore $|\mathcal{O}_2(J)| \geqslant 4$ by (1) and (2), giving

$$\mathcal{O}_2(I) \cap \mathcal{O}_2(J) \neq 1$$

by (4). If $\mathcal{O}_2(J) \trianglelefteq S$ then $\langle I, J \rangle \leqslant N_G(\mathcal{O}_2(I) \cap \mathcal{O}_2(J)) < G$ and I is conjugate to J. So we may assume $\mathcal{O}_2(J)$ is not normal in S. Hence

$$|S : \mathcal{O}_2(J)| = 4$$

by (3). We deduce from this that $\mathcal{O}_2(I) > \mathcal{O}_2(A)$ by (2). So

$$|\mathcal{O}_2(I) : \mathcal{O}_2(A)| = 2.$$

As we have already observed, $\mathcal{O}_2(I)$ contains a cyclic subgroup at index 2, so we can choose $\mathcal{O}_2(A)$ to be cyclic. Since $\mathcal{O}_2(I) > \mathcal{O}_2(A)$ and groups of order 8 have class at most 2 we must have $|\mathcal{O}_2(I)| \geqslant 16$, so $|\mathcal{O}_2(J)| \geqslant 8$ by (2). Then (4) implies $|\mathcal{O}_2(I) \cap \mathcal{O}_2(J)| \geqslant 4$. Therefore

$$\mathcal{O}_2(A) \cap \mathcal{O}_2(J) \neq 1.$$

Now, $\mathcal{O}_2(I) \trianglelefteq S$ implies $Z(S) \cap \mathcal{O}_2(I) \neq 1$. Let $Z \leqslant Z(S) \cap \mathcal{O}_2(I)$ have order 2. We have

$$Z \leqslant Z(\mathcal{O}_2(I)) \leqslant \mathcal{O}_2(A),$$

and since $\mathcal{O}_2(A)$ is cyclic, Z is the unique subgroup of $\mathcal{O}_2(A)$ of order 2. Hence $Z \leqslant \mathcal{O}_2(A) \cap \mathcal{O}_2(J)$, and $\langle A, J \rangle \leqslant N_G(Z) < G$. Proposition 4.3 provides the result. $\square$

Chapter 6

$\mathrm{GU}_3(q)$

Throughout this chapter let q be a power of p and let $F = \mathrm{GF}(q^2)$. We will always take γ to be a generator of the multiplicative group F^*, which is well known to be cyclic.

6.1 Some properties of GF(q^2)

The map from $F \longrightarrow F$ taking $\lambda \mapsto \lambda^q$ is the unique automorphism of F of order 2. We denote λ^q by $\overline{\lambda}$. Most of the results in this section concern this automorphism.

Definition 6.1 *The map $tr : GF(q^k) \longrightarrow GF(q)$ taking $x \mapsto x + x^q + ... + x^{q^{k-1}}$ is the* ***trace*** *map.*

It is easily verified that tr is GF(q)-linear.

Definition 6.2 *The subset $\{\lambda \in F \mid \overline{\lambda} = \lambda\}$ of F is the* ***fixed subfield*** *of the automorphism.*

The fixed subfield of F is indeed a subfield and has order q. We will call this subfield GF(q), which we may do so without ambiguity as F contains a unique subfield of order q. We will not use GF(q) in any other sense from now on.

The following lemma provides a useful characterization of the fixed subfield.

Lemma 6.3 *GF(q)* $= \{\mu\overline{\mu} \mid \mu \in F\}$.

Proof. Let $\lambda \in \mathrm{GF}(q)^*$. Then $\lambda^{q-1} = 1$ gives

$$\lambda \in \langle \gamma^{q+1} \rangle = \langle \gamma\overline{\gamma} \rangle$$

as this is the unique subgroup of F^* of order $q-1$. We see that $\langle \gamma\overline{\gamma} \rangle = \{\mu\overline{\mu} \mid \mu \in F^*\}$ since γ generates F^*, so $\mathrm{GF}(q)^* \subseteq \{\mu\overline{\mu} \mid \mu \in F^*\}$. We clearly have inclusion in the other direction, so we get

$$\mathrm{GF}(q)^* = \{\mu\overline{\mu} \mid \mu \in F^*\}.$$

This implies the result. $\square$

Lemma 6.4 *For any $\mu \in F^*$ there are precisely $q+1$ elements $\alpha \in F^*$ such that $\mu\overline{\mu} = \alpha\overline{\alpha}$.*

Proof. We have $\mu\overline{\mu} = \alpha\overline{\alpha}$ if and only if $\mu\alpha^{-1}\overline{\mu\alpha^{-1}} = 1$. Since multiplication by α^{-1} induces a bijection on F^* it suffices to count the number of solutions to the equation $x^{q+1} = 1$. Clearly there are $q+1$ solutions. $\square$

Lemma 6.5 *The kernel of the trace map from F into GF(q) has order q. In particular, for any $\lambda \in$ GF(q) there are precisely q elements $\mu \in F$ such that $\mu + \overline{\mu} = \lambda$.*

Proof. For $\mu \in F$ we have $tr(\mu) = 0$ if and only if $\overline{\mu} = -\mu$. If q is odd then the elements of $\mathrm{GF}(q)^*$ do not satisfy this condition and, for example, $\gamma^{\frac{1}{2}(q+1)}$ does. If q is even then the elements of $\mathrm{GF}(q)$ satisfy the condition and the other elements do not. We conclude that the kernel has order q as it is a $\mathrm{GF}(q)$-subspace of F and we have shown that it is not 0 or F. The final statement is because the elements of $\mathrm{GF}(q)$ are each mapped onto by a coset of the kernel. $\square$

This result can be generalized as follows:

Proposition 6.6 *For any $\beta \in F^*$ let $\phi_\beta : F \longrightarrow GF(q)$ be the map taking α to $\beta\overline{\alpha} + \alpha\overline{\beta}$. Then ϕ_β is GF(q)-linear and Ker(ϕ_β) has order q. In particular, for any $\lambda \in GF(q)$ there are precisely q elements $\alpha \in F$ such that $\beta\overline{\alpha} + \alpha\overline{\beta} = \lambda$.*

Proof. It is clear that ϕ_β is GF(q)-linear. Let $\alpha \in F$ and let $\delta = \alpha\overline{\beta}$. Then $\beta\overline{\alpha} + \alpha\overline{\beta} = 0$ if and only if $\delta + \overline{\delta} = 0$. Since multiplication by α induces a bijection of F, the result follows from the previous Lemma. □

Lemma 6.7 *Let $\beta \in F^*$. Then there are precisely $q - 1$ elements $\alpha \in F^*$ such that $\alpha^{q-1} = \beta^{q-1}$.*

Proof. Let $\beta = \gamma^r$ and $\alpha = \gamma^s$. Then $\alpha^{q-1} = \beta^{q-1}$ if and only if $q + 1$ divides $r - s$. Modulo $q^2 - 1$ there are $q - 1$ values of s which satisfy this condition. □

6.2 Unitary Spaces

Definition 6.8 *Let V be a vector space over F. A* ***hermitian form*** *on V is a map $(\; , \;) : V \times V \longrightarrow F$ satisfying the following conditions:*

$$\begin{aligned} (u + v, w) &= (u, w) + (v, w); \\ (\lambda u, v) &= \lambda(u, v); \\ (u, v) &= \overline{(v, u)}; \end{aligned}$$

for all $u, v, w \in V$, $\lambda \in F$.

The form is ***degenerate*** *if there exists $0 \neq v \in V$ such that $(u, v) = 0$ for all $u \in V$. If no such v exists then the form is* ***nondegenerate***.

Definition 6.9 *A* ***unitary form*** *is a nondegenerate hermitian form. A* ***unitary space*** *is a vector space endowed with a unitary form.*

Unless explicitly stated otherwise we will always denote a unitary form by $(\ , \ )$.

Lemma 6.10 *Let V be a unitary space over F. Then V has an orthonormal basis.*

Proof. We first argue that there exists $v \in V$ such that $(v, v) \neq 0$, and then that without loss of generality $(v, v) = 1$.

Suppose $(v, v) = 0$ for all $v \in V$. Then for any $v, w \in V$ we have $0 = (v + w, v + w) = (v, v) + (v, w) + (w, v) + (w, w) = (v, w) + (w, v)$, so

$$(v, w) = -(w, v). \tag{$\dagger$}$$

Now pick $\mu \in F \backslash \mathrm{GF}(q)$ so that $\mu \neq \overline{\mu}$. Then

$$\begin{aligned} 0 &= (v + \mu w, v + \mu w) \\ &= (v, \mu w) + (\mu w, v) \\ &= \overline{\mu}(v, w) + \mu(w, v) \end{aligned}$$

giving $\overline{\mu}(v, w) = -\mu(w, v)$. From ($\dagger$) we have $-\mu(w, v) = \mu(v, w)$, so we get $\overline{\mu}(v, w) = \mu(v, w)$. The choice of μ implies that $(v, w) = 0$, and since v and w were arbitrary this contradicts the nondegeneracy of V. Thus we may pick $v \in V$ such that $(v, v) = \alpha \neq 0$. Since $(v, v) = \overline{(v, v)}$ we have $\alpha \in \langle \gamma^{q+1} \rangle$. Let $\alpha = \gamma^{r(q+1)}$. Then

$$\begin{aligned} (\gamma^{-rq} v, \gamma^{-rq} v) &= \gamma^{-rq} \overline{\gamma^{-rq}} (v, v) \\ &= \gamma^{-rq} (\gamma^{-rq})^q \gamma^{rq+r} \\ &= \gamma^{-rq-rq^2+rq+r} \\ &= \gamma^{-rq^2+r} \\ &= \gamma^{-r(q^2-1)} \\ &= (\gamma^{q^2-1})^{-r} \\ &= 1. \end{aligned}$$

So without loss of generality we can assume $(v, v) = 1$. Since $V = \langle v \rangle \oplus v^{\perp}$ we see that $v^{\perp}$ is nondegenerate, so by induction on $\dim V$ we get an orthonormal basis for $v^{\perp}$. This basis together with v gives the required basis for V. $\square$

Definition 6.11 *Let V be an n-dimensional vector space with a form $(\ ,\ )$. If $n = 2m$ is even then a **hyperbolic basis** for V is a basis $u_1, u_2, ..., u_m,\ v_1, v_2, ..., v_m$ such that*

$$\begin{aligned} (u_i, u_j) = (v_i, v_j) &= 0 \text{ for all } i, j; \\ (u_i, v_j) &= 0 \text{ if } i \neq j; \\ (u_i, v_j) &= 1 \text{ if } i = j. \end{aligned}$$

If $n = 2m + 1$ is odd then the basis is required to contain an additional vector d such that

$$\begin{aligned} (d, u_i) = (d, v_i) &= 0 \text{ for all } i; \\ (d, d) &= 1. \end{aligned}$$

Lemma 6.12 *Let V be a unitary space over F. Then V has a hyperbolic basis.*

Proof. Let $n = \dim V$ and let $v_1, v_2, ..., v_n$ be an orthonormal basis for V. The case $n = 1$ is clear. Suppose $n = 2$. We first find a vector u such that $(u, u) = 0$. Pick $\alpha \in F$ such that $\alpha\overline{\alpha} = -1$. Then

$$\begin{aligned} (v_1 + \alpha v_2, v_1 + \alpha v_2) &= (v_1, v_1) + (v_1, \alpha v_2) + (\alpha v_2, v_1) + (\alpha v_2, \alpha v_2) \\ &= 1 + \alpha\overline{\alpha} \\ &= 0. \end{aligned}$$

Now choose $w \in V \backslash u^{\perp}$. Then $(u, \overline{(u, w)^{-1}} w) = 1$, so without loss of generality $(u, w) = 1$. Let $\beta = (w, w) \in \mathrm{GF}(q)$. By Lemma 6.5 we can find $\lambda \in F$ such that $\lambda + \overline{\lambda} = -\beta$. Set $v = \lambda u + w$. Then $(u, v) = 1$ and

$$\begin{aligned}(v,v) &= \lambda(u,w)+\overline{\lambda}(w,u)+(w,w)\\ &= \lambda+\overline{\lambda}-\lambda-\overline{\lambda}\\ &= 0.\end{aligned}$$

Thus u, v is a hyperbolic basis for V in this case.

Now suppose $n \geqslant 2$. Pair off the basis vectors $v_1, ..., v_n$, with one left over if n is odd. Each two dimensional subspace formed in this way is a unitary space and so has a hyperbolic basis. This proves the lemma. □

6.3 Unitary Groups

Let V be an n-dimensional unitary space over F. The subgroup of $\mathrm{GL}_n(q^2)$ consisting of those transformations which preserve the unitary form is the **General Unitary Group** of V. It is denoted $\mathrm{GU}(V)$ or $\mathrm{GU}_n(q)$. We remark that the underlying field has order q^2 and not q. The **Special Unitary Group** of V is $\mathrm{SU}_n(q) = \mathrm{GU}_n(q) \cap \mathrm{SL}_n(q^2)$.

Lemma 6.13 *Let $n \geqslant 2$. Then*

(i) $GU_1(q) \cong \mathbb{Z}_{q+1}$;

(ii) $GU_n(q)$ *contains a subgroup isomorphic to* $GU_1(q)^n$;

(iii) $SU_n(q)$ *contains a subgroup isomorphic to* $GU_1(q)^{n-1}$.

Proof. (i) Since elements of $\mathrm{GU}_1(q)$ are scalars we have

$$\begin{aligned}\mathrm{GU}_1(q) &= \{\lambda \in F^* \mid (\lambda u, \lambda v) = (u,v) \text{ for all } u, v \in V\}\\ &= \{\lambda \in F^* \mid \lambda\overline{\lambda} = 1\}\\ &= \langle \gamma^{q-1}\rangle\\ &\cong \mathbb{Z}_{q+1}.\end{aligned}$$

(ii) Let $v_1, ..., v_n$ be an orthogonal basis for $V = V_n(q^2)$. Then each $\langle v_i\rangle$ is nondegenerate and $\mathrm{GU}_1(q)^n$ acts on V as a subgroup of $\mathrm{GU}_n(q)$.

(iii) The subgroup of $\mathrm{GU}_n(q)$ from part (ii) can be represented by a diagonal matrix. All but one of the entries can be freely chosen and the other must ensure that the determinant is 1. $\square$

Lemma 6.14 *The order of $GU_n(q)$ is*

$$\begin{cases} q^{n(n-1)/2}(q+1)(q^2-1)(q^3+1)(q^4-1)\cdots(q^n+1) \text{ if } q \text{ odd;} \\ q^{n(n-1)/2}(q+1)(q^2-1)(q^3+1)(q^4-1)\cdots(q^n-1) \text{ if } q \text{ even.} \end{cases}$$

Proof. We follow the procedure described in [12]. Let z_n and y_n denote the number of nonzero vectors in $V = V_n(q^2)$ of norm 0 and 1 respectively. Note that we could replace 1 here with any nonzero element of $\mathrm{GF}(q)^*$, as each arises the same number of times. We prove by induction on n that z_n is given by

$$z_n = \begin{cases} (q^n+1)(q^{n-1}-1) \text{ if } n \text{ is odd;} \\ (q^n-1)(q^{n-1}+1) \text{ if } n \text{ is even.} \end{cases}$$

First we provide an inductive formula for z_n. Let $v_1, ..., v_n$ be an orthonormal basis for V and suppose $\alpha_1 v_1 + ... + \alpha_n v_n$ has norm 0, i.e, $\alpha_1\overline{\alpha_1} + ... + \alpha_n\overline{\alpha_n} = 0$. If $\alpha_1\overline{\alpha_1} = 0$ then there are z_{n-1} values that $\alpha_2, ..., \alpha_n$ can take, and if $\alpha_1\overline{\alpha_1} \neq 0$ then there are y_{n-1}. In the latter case, since there are $q-1$ choices for $\alpha_1\overline{\alpha_1}$ and $q+1$ choices for α_1 we get

$$z_n = z_{n-1} + (q^2-1)y_{n-1}.$$

This gives

$$y_{n-1} = (z_n - z_{n-1})/(q^2-1). \qquad (*)$$

We also have

$$|V^*| = q^{2n} - 1 = (q-1)y_n + z_n. \tag{$**$}$$

Replacing y_n with the expression given by $(*)$ yields

$$z_{n+1} = (q^{2n} - 1)(q+1) - z_n q. \tag{$\dagger$}$$

Now suppose $n > 1$ is even. We calculate z_{n+1} using ($\dagger$) and assuming $z_n = (q^n - 1)(q^{n-1} + 1)$ by induction.

$$\begin{aligned}
z_{n+1} &= (q^{2n} - 1)(q+1) - z_n q \\
&= (q^{2n} - 1)(q+1) - (q^n - 1)(q^{n-1} + 1)q \\
&= (q^n - 1)((q^n + 1)(q+1) - (q^{n-1} + 1)q) \\
&= (q^n - 1)(q^{n+1} + q^n + q + 1 - q^n - q) \\
&= (q^n - 1)(q^{n+1} + 1).
\end{aligned}$$

The calculation is similar if n is odd and we conclude that the formula is correct. Substituting this formula into $(**)$ gives a formula for y_n:

$$y_n = \begin{cases} q^{n-1}(q^n + 1) \text{ if } n \text{ is odd;} \\ q^{n-1}(q^n - 1) \text{ if } n \text{ is even.} \end{cases}$$

The order of $\mathrm{GU}_n(q)$ is equal to the number of ordered orthonormal bases for V, which is $y_n y_{n-1} \cdots y_1$. Evaluating this product yields the result. $\square$

Lemma 6.15 *Let $m \in \mathbb{N}$ be odd and let V be an n-dimensional unitary space over GF(q^{2m}) with unitary form (,). Let W be the set V regarded as an nm-dimensional vector space over GF(q^2). Define $[\ ,\ ] : W \times W \longrightarrow$ GF(q^2) by $[u, v] := tr((u, v))$. Then $[\ ,\ ]$ is a unitary form on W.*

Proof. Let $u, v, w \in W$, $\lambda \in \mathrm{GF}(q^2)$. Then

$$\begin{aligned}
[u+v, w] &= tr((u+v, w)) \\
&= tr((u, w) + (v, w)) \\
&= tr((u, w)) + tr((v, w)) \\
&= [u, w] + [v, w]
\end{aligned}$$

and

$$\begin{aligned}
[\lambda u, v] &= tr((\lambda u, v)) \\
&= tr(\lambda (u, v)) \\
&= \lambda tr((u, v)) \\
&= \lambda [u, v].
\end{aligned}$$

For the next calculation we emphasize the fact that $(\ ,\ )$ is a unitary form over $\mathrm{GF}(q^{2m})$, and so the associated "bar" automorphism is $x \mapsto x^{q^m}$. We also note that the seventh equality makes sense because m is odd.

$$
\begin{aligned}
\overline{[u,v]} &= [u,v]^q \\
&= tr((u,v))^q \\
&= ((u,v) + (u,v)^{q^2} + ... + (u,v)^{q^{2(m-1)}})^q \\
&= (u,v)^q + (u,v)^{q^3} + ... + (u,v)^{q^{2m-1}} \\
&= ((v,u)^{q^m})^q + ((v,u)^{q^m})^{q^3} + ... + ((v,u)^{q^m})^{q^{2m-1}} \\
&= (v,u)^{q^{m+1}} + (v,u)^{q^{m+3}} + ... + (v,u)^{q^{3m-1}} \\
&= (v,u)^{q^{m+1}} + ... + (v,u)^{q^{2m-2}} + (v,u)^{q^{2m}} + (v,u)^{q^{2m+2}} + ... + (v,u)^{q^{3m-1}} \\
&= (v,u)^{q^{m+1}} + ... + (v,u)^{q^{2m-2}} + (v,u)^{q^{2m}-1}(v,u) + ((v,u)^{q^{2m}-1})^{q^2}(v,u)^{q^2} \\
&\quad +... + ((v,u)^{q^{2m}-1})^{q^{m-1}}(v,u)^{q^{m-1}} \\
&= (v,u)^{q^{m+1}} + ... + (v,u)^{q^{2m-2}} + (v,u) + (v,u)^{q^2} + ... + (v,u)^{q^{m-1}} \\
&= tr((v,u)) \\
&= [v,u].
\end{aligned}
$$

It remains to show that $[\ ,\ ]$ is nondegenerate. Let $u \in W^*$. Since $(\ ,\ )$ is nondegenerate we can pick $w \in V$ such that $(u,w) = 1$. This implies that the map $(u,\ ) : V \longrightarrow \mathrm{GF}(q^{2m})$ taking $x \mapsto (u,x)$ is onto. Indeed, if $\alpha \in \mathrm{GF}(q^{2m})$ then

$$
\begin{aligned}
(u, \alpha^{q^m} w) &= (\alpha^{q^m})^{q^m}(u,w) \\
&= \alpha^{q^{2m}} \\
&= \alpha.
\end{aligned}
$$

Now, suppose $[u,v] = 0$ for all $v \in W$. Then $tr((u,v)) = 0$ for all $v \in V$. Together with the fact that the map described above is onto this implies that tr is identically zero. In other words, the polynomial

$$X^{q^{2(m-1)}} + ... + X^{q^2} + X$$

has q^{2m} roots in $\mathrm{GF}(q^{2m})$. But this polynomial can have at most $q^{2(m-1)}$ roots. Hence $[\ ,\ ]$ is nondegenerate and is a unitary form on W. $\square$

Lemma 6.16 *If m is odd then $GU_{nm}(q)$ contains a subgroup isomorphic to $GU_n(q^m)$. In particular, $GU_m(q)$ contains a cyclic subgroup of order q^m+1 which arises from multiplication by elements of $GF(q^{2m})$.*

Proof. Let $\mathrm{GU}_n(q^m)$ act on its natural module V preserving the unitary form $(\ ,\ )$. Define $[\ ,\ ]$ on V as in the previous lemma so that $\mathrm{GU}_{nm}(q)$ acts on V with respect to $[\ ,\ ]$. By definition of $[\ ,\ ]$, the inclusion map from $\mathrm{GU}_n(q^m)$ to $\mathrm{GL}_{nm}(q^2)$ is a monomorphism which maps into $\mathrm{GU}_{nm}(q)$. The final statement now follows from Lemma 6.13(i). $\square$

For the next lemma we assume knowledge of the Galois Group of a field extension and of minimal polynomials.

Lemma 6.17 *Let V be an n-dimensional vector space over the field of order q and identify V with $GF(q^n)$ as a 1-dimensional vector space over itself. Define $\phi : V \longrightarrow V$ by $v \mapsto av$ where a is a generator for the multiplicative group $GF(q^n)^*$. Then $\det_{GF(q)} \phi = a^{q^n-1/q-1}$.*

Proof. Let $\min_V(\phi)$ denote the minimal polynomial of ϕ as an element of $\mathrm{End}_{\mathrm{GF}(q)}V$. This polynomial has coefficients in $\mathrm{GF}(q) \subseteq \mathrm{GF}(q^n)$ and has ϕ as a root. So writing a for the scalar transformation aI, the polynomial $X-a$ divides $\min_V(\phi)$, since this is the minimal polynomial of ϕ as an element of $\mathrm{End}_{\mathrm{GF}(q^n)}\mathrm{GF}(q^n)$. Hence a is a root of $\min_V(\phi)$. We see that the image of a under any element of $\mathrm{Gal}(\mathrm{GF}(q^n) : \mathrm{GF}(q))$ is also a root of $\min_V(\phi)$, and so $a, a^q, a^{q^2}, ..., a^{q^{n-1}}$ are all distinct roots of $\min_V(\phi)$. Since $\min_V(\phi)$ has degree at most n, we conclude that these are precisely the roots of $\min_V(\phi)$. The determinant of ϕ is the absolute value of its characteristic polynomial evaluated at 0, and since we have just seen that the characteristic polynomial is equal to the minimal polynomial we deduce that $\det_{\mathrm{GF}(q)} \phi = a \cdot a^q \cdots a^{q^{n-1}} = a^{1+q+...+q^{n-1}} = a^{q^n-1/q-1}$. $\square$

The following result is only needed in the last section on $\mathrm{SU}_3(q)$.

Lemma 6.18 *Let ω generate the multiplicative group $GF(q^6)^*$. Identify $\langle \omega^{q^3-1} \rangle$ with the cyclic subgroup of $GU_3(q)$ of order q^3+1 as in Lemma 6.16. Then $\langle \omega^{q^3-1} \rangle \cap SU_3(q) \cong \mathbb{Z}_{q^2-q+1}$.*

Proof. Let $\theta = \omega^{q^3-1}$. By the previous lemma, $\det \theta = \theta^{q^6-1/q^2-1} = \theta^{q^4+q^2+1}$. So

$$\det \theta^r = \theta^{r(q^4+q^2+1)}.$$

Then $\det \theta^r = 1$ if and only if q^3+1 divides $r(q^4+q^2+1)$. Both q^3+1 and q^4+q^2+1 have a factor of q^2-q+1 so after cancelling we get $\det \theta^r = 1$ if and only if $q+1$ divides $r(q^2+q+1)$. Since $q^2+q+1 = (q+1)^2 - q$ we see that q^2+q+1 and $q+1$ are coprime, and we deduce finally that

$$\det \theta^r = 1 \text{ if and only if } q+1 \text{ divides } r.$$

Hence $\langle \theta \rangle \cap \mathrm{SU}_3(q) = \langle \theta^{q+1} \rangle \cong \mathbb{Z}_{q^2-q+1}$. $\square$

Lemma 6.19 *$GU_2(q)$ has a cyclic subgroup of order q^2-1.*

Proof. Let u, v be a hyperbolic basis for $V = V_2(q^2)$. Define $\theta : V \longrightarrow V$ by

$$\begin{aligned} u\theta &= \gamma u; \\ v\theta &= \gamma^{q^2-q-1} v. \end{aligned}$$

Clearly θ is an invertible linear transformation of V of order q^2-1, so it suffices to check that θ is an isometry. We need only check it preserves the form on basis elements.

$$
\begin{aligned}
(u\theta, u\theta) &= (\gamma u, \gamma u) = 0;\\
(v\theta, v\theta) &= (\gamma^{q^2-q-1} v, \gamma^{q^2-q-1} v) = 0;\\
(u\theta, v\theta) &= (\gamma u, \gamma^{q^2-q-1} v)\\
&= \gamma\overline{\gamma^{q^2-q-1}}(u, v)\\
&= \gamma^{q^3-q^2-q+1}\\
&= (\gamma^{q^2-1})^{q-1}\\
&= 1.
\end{aligned}
$$

This completes the proof. □

6.4 Numerical Results

The following results will be needed in the last section.

Lemma 6.20 *If* $q \in \mathbb{N}$ *then*

$$
\begin{aligned}
(i) &\quad (q^2-1)_{2'} < (q^3+1)_{2'};\\
(ii) &\quad q(q^2-1)_{2'} < q^2(q+1)_{2'};\\
(iii) &\quad (q^3+1)_{2'} < q^2(q+1)_{2'} \ \textit{if}\ q \geqslant 2;\\
(iv) &\quad (q^3+1) < (q+1)^3;\\
(v) &\quad ((q^2-1)(q+1)) < (q+1)^3.
\end{aligned}
$$

Proof. (i) We have $0 < (q-1)^2 + 1$, which implies

$$q - 1 < q^2 - q + 1.$$

Since $q^2 - q + 1$ is odd we get $(q-1)_{2'} < (q^2-q+1)_{2'}$, hence $(q^2-1)_{2'} < (q^3+1)_{2'}$.

(ii) Cancel by $q(q+1)_{2'}$.

(iii) Cancel by $(q+1)_{2'}$.

(iv) Obvious.

(v) Obvious.

Lemma 6.21 *If $3 \leqslant q \in \mathbb{N}$ is odd then*

$$\begin{aligned} (i) \quad & (q+1)_{2'}^3 < (q^3+1)_{2'}; \\ (ii) \quad & ((q^2-1)(q+1))_{2'} < (q^3+1)_{2'}; \\ (iii) \quad & q(q+1)_{2'}^2 < q^2(q+1)_{2'}. \end{aligned}$$

Proof. (i) We have $0 < 3(q-1)^2$ which implies $(q+1)^2 < 4(q^2-q+1)$ and

$$\frac{1}{4}(q+1)^2 < q^2 - q + 1.$$

Now, q is odd, so $(q+1)^2$ is divisible by 4, and as q^2-q+1 is odd we get $(q+1)_{2'}^2 < (q^2-q+1)_{2'}$. Hence $(q+1)_{2'}^3 < (q^3+1)_{2'}$.

(ii) Similar to (i). First, $0 < 3(q-1)^2 + 2q + 2$ implies $q^2 - 1 < 4(q^2-q+1)$. Now divide by 4, take the odd parts and multiply by $(q+1)_{2'}$.

(iii) Cancel by $q(q+1)_{2'}$ then use the fact that q is odd. □

Lemma 6.22 *Let $2 \leqslant q \in \mathbb{N}$.*

(i) *If q is even or $q \equiv 1 \bmod 4$ then $(q^2-1)_{2'} < (q+1)_{2'}^2$;*

(ii) *If $q \equiv 3 \bmod 4$ then $(q+1)_{2'}^2 < (q^2-1)_{2'}$ unless $q = 3$ in which case they're equal;*

(iii) *$(q+1)^2$ divides q^3+1 if and only if $q = 2$.*

Proof. (i) Obvious if q is even. In the other case let $q = 4n+1$ for some $n \in \mathbb{N}$. Then $2n^2+n < 4n^2+4n+1$, and since $4n^2+4n+1$ is odd we get

$$(2n^2+n)_{2'} < (4n^2+4n+1)_{2'}.$$

As we have taken the odd parts of these numbers we are free to multiply by powers of 2. Thus

$$(8(2n^2+n))_{2'} < (4(4n^2+4n+1))_{2'}.$$

Then $((4n+1)^2-1)_{2'} < ((4n+1)^2+2(4n+1)+1)_{2'}$, i.e, $(q^2-1)_{2'} < (q+1)^2_{2'}$.
(ii) Let $q = 4n+3$ for some $n \in \mathbb{N}$. Then $2n+1$ is odd and greater than $n+1$, so $(n+1)_{2'} < (2n+1)_{2'}$. Now multiply the left side by $(16(n+1))_{2'}$ and the right side by $(8(n+1))_{2'}$ to get the result.
(iii) After cancelling by $q+1$ the question is reduced to considering whether $q+1$ divides q^2-q+1. Since $q^2-q+1 = (q-2)(q+1)+3$ we see that $q+1$ divides q^2-q+1 if and only if $q+1 = 3$. $\square$

6.5 Sylow p-subgroups of $\mathrm{GU}_3(q)$

Throughout this section and the next we let $G = \mathrm{GU}_3(q)$ acting on $V = V_3(q^2)$. Let u, d, v be a hyperbolic basis for V as described in Definition 6.11. All matrices from now on are with respect to such a basis, with the basis elements in the order given.

The following are necessary and sufficient conditions for a matrix $A = (a_{ij})$ to be a unitary matrix:

$$
\begin{aligned}
(I) \quad & a_{11}\overline{a_{13}} + a_{12}\overline{a_{12}} + a_{13}\overline{a_{11}} = 0; \\
(II) \quad & a_{11}\overline{a_{23}} + a_{12}\overline{a_{22}} + a_{13}\overline{a_{21}} = 0; \\
(III) \quad & a_{11}\overline{a_{33}} + a_{12}\overline{a_{32}} + a_{13}\overline{a_{31}} = 1; \\
(IV) \quad & a_{21}\overline{a_{23}} + a_{22}\overline{a_{22}} + a_{23}\overline{a_{21}} = 1; \\
(V) \quad & a_{21}\overline{a_{33}} + a_{22}\overline{a_{32}} + a_{23}\overline{a_{31}} = 0; \\
(VI) \quad & a_{31}\overline{a_{33}} + a_{32}\overline{a_{32}} + a_{33}\overline{a_{31}} = 0.
\end{aligned}
$$

We fix this numbering as we will need to refer to these properties several times later.

Lemma 6.23 *The upper triangular matrices in G form a Sylow p-subgroup of G.*

Proof. Let $A = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \in \mathrm{GU}_3(q)$. The conditions (I) - (VI) translate into

$$
\begin{aligned}
\text{(I)} \quad & a\overline{a} = -(\overline{b} + b); \\
\text{(II)} \quad & c = -\overline{a}.
\end{aligned}
$$

We count the number of choices for b and a. If $b = 0$ then $a = 0$. If $b \neq 0$ and $\overline{b} + b = 0$ then $a = 0$, and since $b \in F^*$ there are $q - 1$ choices for b by Lemma 6.5.

If $b \neq 0$ and $\overline{b} + b \neq 0$ then there are $q^2 - q$ choices for b and $q + 1$ choices for a by Lemma 6.4 since $\overline{b} + b \in \mathrm{GF}(q)$.

Hence we calculate the number of such matrices A to be

$$1 + (q - 1) + (q^2 - q)(q + 1) = q^3.$$

This is the order of a Sylow p-subgroup of G. □

Lemma 6.24 *Let* $P = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & -\overline{a} \\ & & 1 \end{pmatrix} \mid a\overline{a} + b + \overline{b} = 0 \right\} \in \mathrm{Syl}_p(G)$.

Then $Z(P) = \left\{ \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \mid b + \overline{b} = 0 \right\}$ *and* $|Z(P)| = q$.

Proof. Let $X = \begin{pmatrix} 1 & a & b \\ & 1 & -\overline{a} \\ & & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & c & d \\ & 1 & -\overline{c} \\ & & 1 \end{pmatrix} \in P$.

Then $XY = YX$ if and only if

$$a\overline{c} = c\overline{a}. \tag{$\dagger$}$$

If $a \neq 0$ then there are q elements c which satisfy ($\dagger$). This is because if $c \neq 0$ then ($\dagger$) becomes $a^{q-1} = c^{q-1}$, and there are $q - 1$ choices for c by Lemma 6.7. In particular, c may not be arbitrary. Since every element of F arises in the "c" position of some element of P, we deduce that $X \notin Z(P)$ whenever $a \neq 0$. Since we clearly have $X \in Z(P)$ when $a = 0$, we conclude that

$$Z(P) = \left\{ \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \mid b + \overline{b} = 0 \right\}.$$

Lemma 6.5 implies $|Z(P)| = q$. □

Lemma 6.25 $Z(G) = \{\lambda I \mid \lambda\overline{\lambda} = 1\}$ *and* $|Z(G)| = q + 1$.

Proof. Let $A \in Z(G)$ and let $b \in F^*$ such that $b + \overline{b} = 0$. Then A has to commute with both

$$\begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & & \\ & 1 & \\ b & & 1 \end{pmatrix}$$

and it follows that A is a diagonal matrix. It is now easily verified that A must be a scalar matrix, for example by multiplying with an arbitrary upper triangular matrix. The scalar matrices in G are precisely those described in the hypothesis, and they are in the centre. There are $q+1$ of them by Lemma 6.4. □

Proposition 6.26 *Let $P \in \mathrm{Syl}_p(G)$ and let $X \in P \backslash Z(P)$. Then $C_G(X)$ is abelian of order $q^2(q+1)$ and $C_G(X) = C_P(X)Z(G)$.*

Proof. It suffices to prove the first statement since the second will then follow from Lemma 6.25.

Without loss of generality $P = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & -\overline{a} \\ & & 1 \end{pmatrix} \mid a\overline{a} + b + \overline{b} = 0 \right\}$.

Let $X = \begin{pmatrix} 1 & a & b \\ & 1 & -\overline{a} \\ & & 1 \end{pmatrix}$.

Then $a \neq 0$ since $X \notin Z(P)$. Let

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \in C_G(X).$$

We observe that X has precisely one eigenvalue (namely 1) with eigenspace $\langle v \rangle$. So $C_G(X)$ must fix both $\langle v \rangle$ and $v^\perp = \langle d, v \rangle$. Therefore C is an upper triangular matrix, i.e, $c_{21} = c_{31} = c_{32} = 0$. Recall that C satisfies the conditions (I) to (VI) from the beginning of this section. By explicitly calculating XC and CX and comparing matrix entries we obtain a further three restrictions on C:

$$\begin{aligned}(VII) \quad & c_{11}a = ac_{22};\\ (VIII) \quad & c_{11}b - c_{12}\overline{a} = ac_{23} + bc_{33};\\ (IX) \quad & -\overline{a}c_{22} = -\overline{a}c_{33}.\end{aligned}$$

By (VII) and (IX) we have $c_{11} = c_{22} = c_{33}$ because $a \neq 0$. Then (VIII) becomes $-c_{12}\overline{a} = ac_{23}$. Substituting these values into the twelve equations eliminates eight of them. The remaining four are:

$$\begin{aligned}(I) \quad & c_{11}\overline{c_{13}} + c_{12}\overline{c_{12}} + c_{13}\overline{c_{11}} = 0;\\ (II) \quad & c_{11}\overline{c_{23}} + c_{12}\overline{c_{11}} = 0;\\ (III) \quad & c_{11}\overline{c_{11}} = 1;\\ (VIII) \quad & ac_{23} = -c_{12}\overline{a}.\end{aligned}$$

We now have enough information to show that $C_G(X)$ is abelian. Let

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ & d_{22} & d_{23} \\ & & d_{33} \end{pmatrix} \in C_G(X).$$

Again we explicitly calculate CD and DC to see that $CD = DC$ if and only if

$$c_{12}d_{23} = d_{12}c_{23}. \qquad (\dagger)$$

If $c_{12} = 0$ or $d_{12} = 0$ then ($\dagger$) is satisfied by (VIII) remembering that the conditions hold for D as well as C, so we can assume that both c_{12} and d_{12} are nonzero. In this case ($\dagger$)

becomes

$$d_{23}d_{12}^{-1} = c_{23}c_{12}^{-1}.$$

This holds because (VIII) implies that both are equal to $-\overline{a}a^{-1}$. Hence $C_G(X)$ is abelian. To calculate the order of $C_G(X)$ we count the number of choices for the elements which appear in the four equations. The elements in question are c_{11}, c_{13}, c_{12} and c_{23}.

First suppose $c_{12} = 0$. Then $c_{23} = 0$ by (VIII), there are $q+1$ choices for c_{11} by (III) and Lemma 6.4, and there are q choices for c_{13} by (I) and Proposition 6.6.

Now assume $c_{12} \neq 0$. Again there are $q+1$ choices for c_{11}. Rearranging (VIII) gives $c_{23} = -c_{12}\overline{a}a^{-1}$, and we substitute this into (II) to get

$$c_{11}\overline{-c_{12}\overline{a}a^{-1}} + c_{12}\overline{c_{11}} = 0.$$

Since both c_{11} and c_{12} are nonzero this may be rearranged to give

$$a^{q-1} = (c_{11}^{-1}c_{12})^{q-1}.$$

Lemma 6.7 then implies that there are $q-1$ values that $c_{11}^{-1}c_{12}$ can take. Therefore for each choice of c_{11} there are $q-1$ choices for c_{12}. As before there are q choices for c_{13}. We finally deduce that

$$|C_G(X)| = (q+1)q + (q+1)(q-1)q = (q+1)q^2$$

and we are done. $\square$

Corollary 6.27 *If $A \in \mathcal{A}_{\mathcal{O}}(G)$ is a p-group then $|A| = q^2$.*

6.6 Odd Nilpotent Injectors in $\mathbf{GU}_3(q)$

We prove that the odd nilpotent injectors in G are all conjugate. The proof goes as follows: If q is even then

- elements of $\mathcal{A}_{\mathcal{O}}(G)$ are direct products of three copies of elements of $\mathcal{A}_{\mathcal{O}}(\mathrm{GU}_1(q))$;
- $\mathcal{A}_{\mathcal{O}}(G)$ forms a single conjugacy class;
- $\mathcal{NI}_{\mathcal{O}}(G)$ forms a single conjugacy class by Corollary 3.12.

If q is odd then for $A \in \mathcal{A}_{\mathcal{O}}(G)$ and $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(G)$

- $|A|_p > q$;
- A contains a p-element X not in the centre of a Sylow p-subgroup;
- $A = \mathcal{O}(C_G(X))$;
- $I = P\mathcal{O}(Z(G))$ for some $P \in \mathrm{Syl}_p(G)$.

We begin with some preliminaries.

Lemma 6.28 *Let G be a group acting coprimely on a vector space V and preserving a form $(\ ,\ )$ on V. Then $V = [V, G] \oplus^{\perp} C_V(G)$.*

Proof. By Lemma 2.9 we have $V = [V, G] \oplus C_V(G)$. Let $-v + v^g \in [V, G]$ and $w \in C_V(G)$. Then

$$\begin{aligned}(-v + v^g, w) &= (-v, w) + (v^g, w) \\ &= -(v, w) + (v^g, w^g) \\ &= -(v, w) + (v, w) \\ &= 0\end{aligned}$$

and the lemma is proved. $\square$

Lemma 6.29 *(Huppert) [8, 4, p149] If n is odd then every cyclic irreducible subgroup of $GU_n(q)$ has order dividing q^n+1. If n is even then $GU_n(q)$ contains no cyclic irreducible subgroups.*

Theorem 6.30 *(Maschke's Theorem) [1, 3.3.1, p66] If G acts coprimely on a vector space V then V is a direct sum of G-invariant irreducible submodules.*

Lemma 6.31 *[2, 19.3(4), p77] Let V be a vector space with a form. If $U \leqslant V$ is a totally isotropic subspace then $\dim U \leqslant \frac{1}{2}\dim V$.*

Proposition 6.32 *Let $A \leqslant GU_2(q)$ be an abelian p'-group acting on the natural module W. Then one of the following holds:*

(i) *A acts faithfully on a 1-dimensional subspace and $|A|$ divides q^2-1;*

(ii) *$W = W_1 \oplus^{\perp} W_2$ where $\dim W_i = 1$ and each W_i is A-invariant. In particular, $A \leqslant GU(W_1) \times GU(W_2)$ and $|A|$ divides $(q+1)^2$.*

Proof. If A is irreducible then A is cyclic by Lemma 2.5. This contradicts Lemma 6.29. So A acts reducibly on W. Let U be a 1-dimensional A-invariant subspace of W. If A acts faithfully on U then $A \leqslant \mathrm{GL}_1(q^2)$ and $|A|$ divides q^2-1.
Suppose now that A acts nonfaithfully on U. Then $A_1 = C_A(U) \neq 1$. Lemma 6.28 implies that

$$W = C_W(A_1) \oplus^{\perp} [W, A_1],$$

and by definition of A_1 we have $U \subseteq C_W(A_1)$. It follows that both $C_W(A_1)$ and $[W, A_1]$ are 1-dimensional since $C_W(A_1) \neq W$. They are also both A-invariant because A is abelian, and they are nondegenerate. So $A \leqslant \mathrm{GU}_1(q) \times \mathrm{GU}_1(q)$ and $|A|$ divides $(q+1)^2$ by Lemma 6.13. □

Corollary 6.33 *Let $A \in \mathcal{A}_{\mathcal{O}}(GU_2(q))$. Suppose A is a p'-group. Then*

$$|A| = \begin{cases} (q^2-1)_{2'} & \text{if } q \equiv 3 \mod 4; \\ (q+1)^2_{2'} & \text{otherwise.} \end{cases}$$

Proof. We have $|A| \leqslant max\{(q^2-1)_{2'}, (q+1)^2_{2'}\}$. Since abelian subgroups of both orders exist (Lemmas 6.13 and 6.19) we need only check which is larger. See Lemma 6.22. □

Proposition 6.34 *Let $A \leqslant G$ be an abelian p'-group. Then one of the following holds:*

(i) *A is irreducible and $|A|$ divides q^3+1;*

(ii) *A acts faithfully on a 1-dimensional subspace and $|A|$ divides q^2-1;*

(iii) *$V = V_1 \oplus^{\perp} V_2$ where $\dim V_i = i$ and each V_i is A-invariant. In particular, $A \leqslant GU(V_1) \times GU(V_2)$ and $|A|$ divides $(q^2-1)(q+1)$ or $(q+1)^3$.*

Proof. As in Proposition 6.32, if A acts irreducibly on V then A is cyclic. Lemma 6.29 then implies that

$$|A| \text{ divides } q^3+1.$$

Suppose now that A acts reducibly on V. By Maschke's Theorem (6.30) there exists a 1-dimensional A-invariant subspace U. If A acts faithfully on U then $A \leqslant \mathrm{GL}_1(q^2)$ and

$$|A| \text{ divides } q^2-1.$$

Suppose A acts nonfaithfully on U. Then $A_1 = C_A(U) \neq 1$. Applying Lemma 6.28 we have

$$V = C_V(A_1) \oplus^{\perp} [V, A_1],$$

and both direct summands are A-invariant because A is abelian. By definition of A_1 we see that $U \subseteq C_V(A_1)$, and since $C_V(A_1) \neq V$ this implies that $C_V(A_1)$ has dimension 1 or 2. So one of $C_V(A_1)$ and $[V, A_1]$ has dimension 2 and the other has dimension 1. Hence

$$A \leqslant \mathrm{GU}_2(q) \times \mathrm{GU}_1(q).$$

By Proposition 6.32 and Lemma 6.13 we see that $|A|$ divides $(q^2-1)(q+1)$ or $(q+1)^3$.□

Corollary 6.35 *Let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Suppose A is a p'-group. Then*

$$|A| = \begin{cases} (q^3+1)_{2'} & \textit{if } q \textit{ is odd;} \\ (q+1)^3 & \textit{if } q \textit{ is even.} \end{cases}$$

Proof. We have $|A| \leqslant max\{(q^3+1)_{2'}, ((q^2-1)(q+1))_{2'}, (q+1)^3_{2'}\}$. The largest of these is $(q^3+1)_{2'}$ when q is odd and $(q+1)^3_{2'}$ when q is even (see Lemmas 6.20 and 6.21). Abelian subgroups of both orders exist by Lemmas 6.16 and 6.13. □

We now prove the main theorem when q is even.

Proposition 6.36 *If q is even then $\mathcal{NI}_{\mathcal{O}}(G)$ is a single conjugacy class of subgroups.*

Proof. Let $A, B \in \mathcal{A}_{\mathcal{O}}(G)$ with $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(G)$ and $B \leqslant J \in \mathcal{NI}_{\mathcal{O}}(G)$. Then A is a p'-group and $|A| = (q+1)^3$. By Proposition 6.34 we have $V = U_1 \oplus^\perp U_2$ and $A = C_1 \times C_2$ where $\dim U_i = i$ and $C_i \leqslant \mathrm{GU}(U_i)$. Now, $|\mathrm{GU}_2(q)| = q(q-1)(q+1)^2$ and $q+1$ is coprime to $q(q-1)$ because q is even. Therefore $|C_2| = (q+1)^2$ and $|C_1| = q+1$. Applying Corollary 6.32 to the action of C_2 on U_2 then gives

$$A = A_1 \times A_2 \times A_3 \text{ and } V = V_1 \oplus^\perp V_2 \oplus^\perp V_3$$

where $\dim V_i = 1$ and each A_i acts as $\mathrm{GU}_1(q)$ on V_i and centralizes V_j for $j \neq i$. Similarly, $B = B_1 \times B_2 \times B_3$ acting on $W_1 \oplus^\perp W_2 \oplus^\perp W_3 = V$ satisfying the same properties. For each V_i and W_i choose vectors v_i and w_i of norm 1 so that v_1, v_2, v_3 and w_1, w_2, w_3 form orthonormal bases for V. There exists an element $g \in G$ such that

$$v_i^g = w_i \text{ for each } i.$$

Then A^g acts on $W_1 \oplus^\perp W_2 \oplus^\perp W_3$ as a subgroup of $\mathrm{GU}_1(q)^3$, and since $|A| = |B| = |\mathrm{GU}_1(q)^3|$ we deduce that $A^g = B$. Now apply Corollary 3.12 to conclude that I^g is conjugate to J. $\square$

Proposition 6.37 *Suppose q is odd and let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Then $A = \mathcal{O}(C_G(X))$ for some p-element $X \in G$ which is not in the centre of a Sylow p-subgroup of G, and $|A| = q^2(q+1)_{2'}$.*

Proof. We first assume A is neither a p-group nor a p'-group and show that it has the required form in this case. Let

$$W = C_V(\mathcal{O}_p(A)).$$

Since V has characteristic p we see that $W \neq 1$, and as we have assumed $\mathcal{O}_p(A) \neq 1$ we deduce that $1 \neq W \neq V$. So W has dimension 1 or 2. Applying Thompson's $P \times Q$ Lemma to the action of $\mathcal{O}_p(A) \times C_{\mathcal{O}_{p'}(A)}(W)$ on V implies that

$$C_{\mathcal{O}_{p'}(A)}(W) = 1.$$

Thus $\mathcal{O}_{p'}(A)$ acts faithfully on W. The next few paragraphs are similar to part of Proposition 6.34. If $\dim W = 1$ then

$$\mathcal{O}_{p'}(A) \leqslant \mathrm{GL}_1(q^2) \cong \mathbb{Z}_{q^2-1}. \tag{1}$$

Suppose $\dim W = 2$. Then $\dim W > \frac{1}{2} \dim V$, so W is not totally isotropic by Lemma

6.31, i.e, $W \cap W^\perp \neq W$. If $\mathcal{O}_{p'}(A)$ acts irreducibly on W then $\mathcal{O}_{p'}(A)$ is cyclic and W is nondegenerate since $W \cap W^\perp$ is an $\mathcal{O}_{p'}(A)$-invariant proper subspace of W. However, $\mathrm{GU}_2(q)$ contains no cyclic irreducible subgroups by Lemma 6.29. Thus $\mathcal{O}_{p'}(A)$ leaves a 1-dimensional subspace U of W invariant. If $\mathcal{O}_{p'}(A)$ acts faithfully on U then

$$\mathcal{O}_{p'}(A) \leqslant \mathrm{GL}_1(q^2) \cong \mathbb{Z}_{q^2-1}. \tag{2}$$

Suppose now that $\mathcal{O}_{p'}(A)$ acts nonfaithfully on U. Let $A_1 = C_{\mathcal{O}_{p'}(A)}(U) \neq 1$. Then $W = [W, A_1] \oplus^\perp C_W(A_1)$ and both subspaces are 1-dimensional, nondegenerate and $\mathcal{O}_{p'}(A)$-invariant. So

$$\mathcal{O}_{p'}(A) \leqslant \mathrm{GU}_1(q)^2 \cong \mathbb{Z}_{q+1}^2. \tag{3}$$

We conclude that $|\mathcal{O}_{p'}(A)| \leqslant max\{(q^2-1)_{2'}, (q+1)_{2'}^2\}$. Now, if $|\mathcal{O}_p(A)| \leqslant q$ then

$$|A| \leqslant max\{q(q^2-1)_{2'}, q(q+1)_{2'}^2\}.$$

Lemmas 6.20(ii) and 6.21(iii) compare these numbers with $q^2(q+1)_{2'}$. Both are smaller. Since an abelian subgroup of order $q^2(q+1)_{2'}$ exists (see Proposition 6.26) we must therefore have

$$|\mathcal{O}_p(A)| > q.$$

We have seen that $|Z(P)| = q$ for any $P \in \mathrm{Syl}_p(G)$, so it follows that A contains a p-element X which is not in the centre of a Sylow p-subgroup of G. Thus $A \leqslant C_G(X)$. Proposition 6.26 then implies that A has the required form.

It remains to check that there is no p-group or p'-group which is abelian of odd order larger than $|A|$. The orders of any such groups are bounded by Corollarys 6.35 and 6.27. The bounds are $(q^3+1)_{2'}$ and q^2 respectively. A comparison of the former with $q^2(q+1)_{2'}$ is made in Lemma 6.20(iii), and the latter is clearly not bigger than $q^2(q+1)_{2'}$. In all cases, $q^2(q+1)_{2'}$ is the largest. $\square$

Theorem 6.38 *$\mathcal{NI}_{\mathcal{O}}(G)$ is a single conjugacy class of subgroups.*

Proof. By Proposition 6.36 we may assume q is odd. Let $A \in \mathcal{A}_{\mathcal{O}}(G)$ and $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(G)$. Then

$$A = \mathcal{O}(C_G(X)) = \mathcal{O}(C_P(X)Z(G))$$

for some $X \in P \in \mathrm{Syl}_p(G)$ with $X \notin Z(P)$. Since $X \in A$ is a p-element we must have $\mathcal{O}_{p'}(I) \leqslant C_G(X)$. We then use the fact that I has odd order and $C_G(X)$ is abelian to deduce that $\mathcal{O}_{p'}(I) \leqslant \mathcal{O}(C_G(X)) = A$. Therefore

$$\mathcal{O}_{p'}(I) = \mathcal{O}_{p'}(A).$$

Now, $\mathcal{O}_{p'}(A) = \mathcal{O}(Z(G))$, so $[\mathcal{O}_{p'}(I), P] = 1$. This forces

$$I = \mathcal{O}(Z(G))P.$$

Similarly, any $J \in \mathcal{NI}_{\mathcal{O}}(G)$ satisfies $J = \mathcal{O}(Z(G))Q$ for some $Q \in \mathrm{Syl}_p(G)$, and I and J are conjugate by Sylow's Theorem. $\square$

6.7 $\mathbf{SU}_3(q)$

We now consider the Special Unitary Group $\mathrm{SU}_3(q)$. We still denote $\mathrm{GU}_3(q)$ by G and we let $S = \mathrm{SU}_3(q)$.

Lemma 6.39 *(Schur's Lemma) [2, 12.4, p38] Let F be a field and V an irreducible FG-module. Suppose G has order coprime to the characteristic of F. Then End(V) is a division ring.*

Proposition 6.40 *Suppose q is odd and let $B \in \mathcal{A}_{\mathcal{O}}(S)$. Then $B = A \cap S$ for some $A \in \mathcal{A}_{\mathcal{O}}(G)$.*

Proof. Let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Then $A = C_P(X)\mathcal{O}(Z(G))$ for some $X \in P \in \mathrm{Syl}_p(G)$ such that $X \notin Z(P)$. Because $P \leqslant S$ we have $A = (A \cap S)\mathcal{O}(Z(G))$. Now, $|A \cap S| \leqslant |B|$ by definition of B. Together with the observations that $S \cap \mathcal{O}(Z(G)) \leqslant B$ and $\mathcal{O}(Z(G)) \leqslant A$ this allows us to calculate

$$\begin{aligned}
|A| &= |(A \cap S)\mathcal{O}(Z(G))| \\
&= |A \cap S||\mathcal{O}(Z(G))|/|A \cap S \cap \mathcal{O}(Z(G))| \\
&= |A \cap S||\mathcal{O}(Z(G))|/|S \cap \mathcal{O}(Z(G))| \\
&\leqslant |B||\mathcal{O}(Z(G))|/|S \cap \mathcal{O}(Z(G))| \\
&= |B||\mathcal{O}(Z(G))|/|B \cap \mathcal{O}(Z(G))| \\
&= |B\mathcal{O}(Z(G))|.
\end{aligned}$$

Therefore $B\mathcal{O}(Z(G)) \in \mathcal{A}_{\mathcal{O}}(G)$ and $B\mathcal{O}(Z(G)) \cap S = B(\mathcal{O}(Z(G)) \cap S) = B$. □

Proposition 6.41 *Suppose $q \neq 2$ is even and let $B \in \mathcal{A}_{\mathcal{O}}(S)$. Then $B = A \cap S$ for some $A \in \mathcal{A}_{\mathcal{O}}(G)$.*

Proof. By Proposition 6.34 one of the following holds:

- (i) B is irreducible and $|B|$ divides $q^3 + 1$;
- (ii) B acts faithfully on a 1-dimensional subspace of V and $|B|$ divides $q^2 - 1$;
- (iii) $V = V_1 \oplus^{\perp} V_2$ where $\dim V_i = i$ and each V_i is B-invariant. In particular, $B \leqslant \mathrm{GU}(V_1) \times \mathrm{GU}(V_2)$ and $|B|$ divides $(q^2 - 1)(q + 1)$ or $(q + 1)^3$.

Suppose (i) holds. Let K be the subring of $\mathrm{End}(V)$ generated by B. Then K is a division ring by Schur's Lemma (6.39), and since B is abelian, K is a field. So we can view V as a vector space over K. Any K-subspace of V is B-invariant because $B \subseteq K$, and so the irreducibility of B implies that V is 1-dimensional over K. Thus $|K| = |V| = q^6$ and we must have

$$K = \mathrm{GF}(q^6).$$

So B arises as a subgroup of $\mathrm{GF}(q^6)^*$. Hence $|B| \leqslant q^2 - q + 1$ by Lemma 6.18.

Now suppose (iii) holds. Then elements of B look like

$$\begin{pmatrix} c & & \\ & d_{11} & d_{12} \\ & d_{21} & d_{22} \end{pmatrix}$$

where c is a scalar from $\mathrm{GU}_1(q)$ and $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \in \mathrm{GU}_2(q)$. To maximize the order of B it is clear that the matrices D should form an element of $\mathcal{A}_{\mathcal{O}}(\mathrm{GU}_2(q))$ and each c should be defined by $c = \det D^{-1}$. So $|B| = (q+1)^2$ in this case. Thus

$$|B| \leqslant max\{q^2 - q + 1, q^2 - 1, (q+1)^2\}.$$

These are all odd and the largest of them is $(q+1)^2$. An abelian subgroup of order $(q+1)^2$ exists in S by Lemma 6.13. Therefore

$$|B| = (q+1)^2.$$

Now, $(q+1)^2$ does not divide $q^2 - 1$, and $(q+1)^2$ divides $q^3 + 1$ if and only if $q = 2$. Hence we are in case (iii). By allowing the element c mentioned earlier to take the value of any scalar from $\mathrm{GU}_1(q)$ we get a subgroup A of G which is abelian of order $(q+1)^3$, i.e, $A \in \mathcal{A}_{\mathcal{O}}(G)$. Moreover, $A \cap S = B$. $\square$

Theorem 6.42 *$\mathcal{NI}_{\mathcal{O}}(S)$ is a single conjugacy class in S.*

Proof. Let $A \in \mathcal{A}_{\mathcal{O}}(S)$ and let $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(S)$. Suppose q is odd. Then $A = C_P(X)(\mathcal{O}(Z(G)) \cap S)$ for some $X \in P \in \mathrm{Syl}_p(G)$. By the arguments of Theorem 6.38 we deduce that $I = P(\mathcal{O}(Z(G)) \cap S)$ and the result follows by Sylow's Theorem.

Now suppose $q \neq 2$ is even. Let $B \in \mathcal{A}_{\mathcal{O}}(S)$. Then $A = A_1 \cap S$ and $B = B_1 \cap S$ for some $A_1, B_1 \in \mathcal{A}_{\mathcal{O}}(G)$. In a similar fashion to Proposition 6.36, we can find orthonormal bases v_1, v_2, v_3 and w_1, w_2, w_3 such that elements of A look like

$$\begin{pmatrix} a & & \\ & b & \\ & & (ab)^{-1} \end{pmatrix}$$

with respect to v_1, v_2, v_3 and elements of B look like

$$\begin{pmatrix} c & & \\ & d & \\ & & (cd)^{-1} \end{pmatrix}$$

with respect to w_1, w_2, w_3, where a, b, c and d are scalars from $\mathrm{GU}_1(q)$. Consider the subgroups of $\mathrm{GU}_2(q)$ obtained by removing the third row and column from the matrices which form A and B. There subgroups act as $\mathrm{GU}_1(q) \times \mathrm{GU}_1(q)$ with respect to the appropriate orthonormal bases, so there exists $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \mathrm{GU}_2(q)$ conjugating one into the other. Let $\begin{pmatrix} a & \\ & b \end{pmatrix}$ be an element obtained from A in this manner and let $\begin{pmatrix} c & \\ & d \end{pmatrix}$ be its conjugate under g. Then both of these matrices have the same determinant, i.e, $ab = cd$. It follows that the matrix $g_1 = \begin{pmatrix} g_{11} & g_{12} & \\ g_{21} & g_{22} & \\ & & * \end{pmatrix}$ conjugates A into B, where $*$ can take any value from $\mathrm{GU}_1(q)$. Choosing $*$ to be the inverse of the determinant of g yields an element g_1 of $\mathrm{SU}_3(q)$ conjugating A into B. The result follows. Finally, if $q = 2$ then $\mathcal{NI}_{\mathcal{O}}(\mathrm{SU}_3(2)) = \mathrm{Syl}_3(\mathrm{SU}_3(2))$ because $|\mathrm{GU}_3(2)| = 2^3 \cdot 3^4$. $\square$

Chapter 7

Groups With Components Of Alternating Type

We investigate a minimal counterexample to the conjecture that the odd nilpotent injectors of any group are all conjugate. We show that such a group necessarily has components and we prove that if there is a component of alternating type then it must be a triple cover of A_6 or A_7.

We begin with a review of some results on components.

Definition 7.1 *A* ***quasisimple*** *group is a perfect group K such that $K/Z(K)$ is simple. A* ***component*** *of G is a subnormal quasisimple subgroup. The* ***type*** *of a component is the simple group $K/Z(K)$. We denote the set of components of G by Comp(G). The group generated by the components of G is denoted $E(G)$. The* ***Generalized Fitting Subgroup*** *of G is $F^*(G) = F(G)E(G)$.*

Lemma 7.2 *[2, 31.7, p158] [2, 31.12, p159] Let Comp$(G) = \{K_1, ..., K_m\}$. Set $\overline{E(G)} =$*

$E(G)/Z(E(G))$. Then

$$\begin{aligned} &(i) \quad [K_i, K_j] = 1 \text{ if } i \neq j; \\ &(ii) \quad [F(G), E(G)] = 1; \\ &(iii) \quad \overline{E(G)} = \overline{K_1} \times ... \times \overline{K_m}. \end{aligned}$$

Theorem 7.3 *[2, 31.13, p159] $C_G(F^*(G)) \leqslant F^*(G)$.*

Lemma 7.4 *If G acts on the perfect group K then $C_G(K) = C_G(K/Z(K))$. In particular, G acts faithfully on K if and only if G acts faithfully on $K/Z(K)$.*

Proof. Set $\overline{K} = K/Z(K)$. Then $[C_G(\overline{K}), K] \leqslant Z(K)$, so $[C_G(\overline{K}), K, K] = 1$. The Three Subgroups Lemma implies $[K, K, C_G(\overline{K})] = 1$, and since K is perfect we get $C_G(\overline{K}) \leqslant C_G(K)$. The other inclusion is trivial. □

Definition 7.5 *Let $n \in \mathbb{N}$. If there exists a perfect group K satisfying $K/Z(K) \cong G$ and $Z(K) \cong \mathbb{Z}_n$ then K is denoted by $n \cdot G$. If $n = 2$ or 3 then K is called a double cover or a triple cover of G respectively.*

Lemma 7.6 *The quasisimple groups of type A_n are:*

$$\begin{aligned} A_n & \quad \text{for } n \geqslant 5; \\ 2 \cdot A_n & \quad \text{for } n \geqslant 5; \\ 3 \cdot A_6&; \\ 3 \cdot A_7&. \end{aligned}$$

Proof. See [12, 2.7, p27-30]. □

7.1 S_n and A_n

Lemma 7.7 *Let X be a nonempty set and assume $A \in \mathcal{A}_{\mathcal{O}}(Sym(X))$ has orbits $X_1, ..., X_k$ on X. Then $A \cong A_1 \times ... \times A_k$ where $A_i \in \mathcal{A}_{\mathcal{O}}(Sym(X_i))$ for all i.*

Proof. For each i let $\phi_i : A \longrightarrow Sym(X_i)$ be the natural homomorphism. So $A \leqslant A\phi_1 \times \ldots \times A\phi_k$. By the maximality of $|A|$ we get $A = A\phi_1 \times \ldots \times A\phi_k$ and $A\phi_i \in \mathcal{A}_{\mathcal{O}}(Sym(X_i))$ for all i. $\square$

Proposition 7.8 *Let $n \geqslant 3$ and let $A \in \mathcal{A}_{\mathcal{O}}(S_n)$. Write $n = 3t + r$ where $r = 0, 1$ or 2. Then A is a direct product of groups of prime order and*

$$|A| = \begin{cases} 3^t & \text{if } r = 0 \text{ or } 1; \\ 3^{t-1} \cdot 5 & \text{if } r = 2. \end{cases}$$

Proof. Let $X = \{1, ..., n\}$ and let $X_1, ..., X_k$ be the orbits of X under A. Then

$$A = A_1 \times \ldots \times A_k$$

where $A_i \in \mathcal{A}_{\mathcal{O}}(Sym(X_i))$ for all i by Lemma 7.7. Since A_i is abelian and transitive on X_i, we have $|A_i| = |X_i|$ for each i. Now fix i and let $|X_i| = n_i$. Write $n_i = 3t_i + r_i$ where $r_i = 0, 1$ or 2. We see that $Sym(X_i)$ contains an abelian subgroup of order 3^{t_i} which is a direct product of cyclic groups of order 3, so

$$|A_i| = n_i \geqslant 3^{t_i}.$$

Therefore $3^{t_i} \leqslant 3t_i + r_i \leqslant 3t_i + 3 = 3(t_i + 1)$ and $3^{t_i - 1} \leqslant t_i + 1$. This implies $t_i \leqslant 2$. If $t_i = 2$ then $8 \geqslant n_i \geqslant 3^{t_i} = 9$, a contradiction. So $t_i \leqslant 1$ and

$$|A_i| \leqslant 5.$$

Now, by the Orbit Stabilizer Theorem each orbit has odd length, so the lengths are either 1,3 or 5. Suppose we have two orbits X_i and X_j.

If $|X_i| = |X_j| = 5$ then $|A_i \times A_j| = 25$. However, $Sym(X_i \cup X_j)$ contains an abelian subgroup of order $3^3 = 27$. So there is at most one orbit of length 5.
If $|X_i| = 1$ and $|X_j| = 5$ then $|A_i \times A_j| = 5$. Again, $Sym(X_i \cup X_j)$ contains an abelian subgroup of order $3^2 = 9$. So there cannot be orbits of lengths 1 and 5 at the same time.
If $|X_i| = |X_j| = 1$ then there is no orbit of length 5, and since $n \geqslant 3$ there must be another orbit X_l, of length 1 or 3. If $|X_l| = 1$ then $|A_i \times A_j \times A_l| = 1$ and if $|X_l| = 3$ then $|A_i \times A_j \times A_l| = 3$. Neither case is possible because $Sym(X_i \cup X_j \cup X_l)$ contains an abelian subgroup of order 3 in the first case and 5 in the second. Hence there is at most one orbit of length not equal to 3, and that orbit has length 1 or 5.
If $r = 0$ then every orbit has length 3. If $r = 1$ then every orbit has length 3 except one which has length 1. If $r = 2$ then every orbit has length 3 except one which has length 5. The result follows. $\square$

Theorem 7.9 $\mathcal{NI}_{\mathcal{O}}(S_n)$ *is a single conjugacy class.*

Proof. Let $A \in \mathcal{A}_{\mathcal{O}}(S_n)$ and let $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(S_n)$. Write $n = 3t + r$ where $r = 0, 1$ or 2. Suppose $r = 0$ or 1. Then

$$A \cong \langle (1, 2, 3) \rangle \times ... \times \langle (3t - 2, 3t - 1, 3t) \rangle.$$

Assume there exists $p \in \pi(I) \backslash \{3\}$ and let $b \in \mathcal{O}_p(I)$. Then b commutes with A, so it fixes each 3-cycle. Moreover, the action of b on each 3-cycle is either trivial or has order 3. Since $p \neq 3$, we see that $b = 1$. So $I \in \mathrm{Syl}_3(S_n)$ and the theorem holds.
If $n = 5$ then $\mathcal{NI}_{\mathcal{O}}(S_5) = \mathrm{Syl}_5(S_5)$ and we are done. So suppose $r = 2$ and $n > 5$. Then

$$A = A_1 \times A_2$$

where $A_1 = \langle (1, 2, 3, 4, 5) \rangle$ and $A_2 = \langle (6, 7, 8) \rangle \times ... \times \langle (3t, 3t + 1, 3t + 2) \rangle$. The above argument shows that $\pi(I) = \{3,5\}$. Let $x \in \mathcal{O}_5(I)$. Then x commutes with A_2, so $x \in Sym(\{1,2,3,4,5\})$, again by the same argument as above. This implies $\mathcal{O}_5(I) \in \mathrm{Syl}_5(Sym(\{1,2,3,4,5\}))$. Similarly, $\mathcal{O}_3(I) \in \mathrm{Syl}_3(Sym(\{6,7,...,3t+2\}))$ and the result follows by Sylow's Theorem. $\square$

Theorem 7.10 *$\mathcal{A}_{\mathcal{O}}(A_n) = \mathcal{A}_{\mathcal{O}}(S_n)$ and $\mathcal{NI}_{\mathcal{O}}(A_n) = \mathcal{NI}_{\mathcal{O}}(S_n)$. Moreover, $\mathcal{NI}_{\mathcal{O}}(A_n)$ is a single conjugacy class in A_n.*

Proof. The first two statements are immediate because $|S_n : A_n| = 2$. For the last statement, follow the proof of the previous theorem and change "*Sym*" to "*Alt*". $\square$

7.2 $\mathcal{NI}_{\mathcal{O}}(G)$ and Components

First we have a corollary of Theorem 3.9.

Corollary 7.11 *Let $I \in \mathcal{NI}_{\mathcal{O}}(G)$. Then $I \cap E(G) = \langle I \cap K | K \in Comp(G) \rangle$ and for every $K \in Comp(G)$ we have $I \cap K \in Max_{\mathcal{NO}}(K)$.*

Proof. Let $E = E(G)$ and $Z = Z(E)$. Set $\overline{E} = E/Z$. Let $\mathrm{Comp}(G) = \{K_1, ..., K_n\}$ and for each i let $\pi_i : \overline{E} \longrightarrow \overline{K_i}$ be the projection map. Since $I \cap E \in \mathrm{Max}_{\mathcal{NO}}(E)$ it follows that $\overline{I \cap E} \in \mathrm{Max}_{\mathcal{NO}}(\overline{E})$ by Lemma 3.8. We have $\overline{I \cap E} \leqslant (\overline{I \cap E})\pi_1 \times ... \times (\overline{I \cap E})\pi_n$, and since this direct product is nilpotent of odd order in $\overline{E}$ we get

$$\overline{I \cap E} = (\overline{I \cap E})\pi_1 \times ... \times (\overline{I \cap E})\pi_n.$$

For each i let L_i be the inverse image of $(\overline{I \cap E})\pi_i$ in K_i. Since $[L_i, Z] = 1$ we see that L_i is nilpotent, so we can pick $J_i \in \mathrm{Hall}_{2'}(L_i)$. Moreover, as $(\overline{I \cap E})\pi_i$ has odd order we get $\overline{J_i} = \overline{L_i}$. So

$$\overline{I \cap E} = \overline{J_1} \cdots \overline{J_n}.$$

Taking inverse images gives $(I \cap E)Z = J_1 \cdots J_n Z$. Now, $\mathcal{O}(Z) \leqslant I$ and $\mathcal{O}(Z) \leqslant J_i$ for each i, so in fact

$$(I \cap E)Z = (I \cap E)\mathcal{O}_2(Z) = J_1 \cdots J_n \mathcal{O}_2(Z).$$

Then clearly $(I \cap E)\mathcal{O}_2(Z) = (I \cap E) \times \mathcal{O}_2(Z) = (J_1 \cdots J_n) \times \mathcal{O}_2(Z)$, giving $I \cap E = J_1 \cdots J_n$. The first statement follows because $J_i \leqslant I \cap K_i$ for each i and $I \cap E \in \mathrm{Max}_{\mathcal{NO}}(E)$. The second statement is now straightforward. □

The rest of this section is devoted to proving that elements of $\mathcal{A}_{\mathcal{O}}(G)$ normalize components of type A_n, except possibly the triple covers of A_6 and A_7. However, only the last result (Theorem 7.16) actually contains any mention of the type of a component.

Let G be a group in which the components $K = K_1, K_2 ..., K_m$ of G are all conjugate under $A \in \mathcal{A}_{\mathcal{O}}(G)$. Assume $m \geqslant 3$. Let $A \leqslant I \in \mathcal{NI}_{\mathcal{O}}(G)$ and observe that

$$N_I(K) = N_I(K_i)$$

and

$$N_G(K) = N_G(K_i)$$

for all i. Let $E = E(G)$ and set

$$\overline{G} = G/C_G(E).$$

Then we can identify $\overline{G}$ with a subgroup of

$$\mathrm{Aut}(E) = S_m \ltimes (\mathrm{Aut}(K_1) \times ... \times \mathrm{Aut}(K_m)).$$

Under this identification $N_{\overline{G}}(K)$ is a subgroup of $\mathrm{Aut}(K_1) \times ... \times \mathrm{Aut}(K_m)$ because, as before, $N_{\overline{G}}(K)$ normalizes each K_i. Define π_i to be the projection map from $N_{\overline{G}}(K)$ into

$\mathrm{Aut}(K_i)$.

We let (H) be the hypothesis that G is a group satisfying the conditions and with the notation described above.

Results 7.12 to 7.15 are poached from [5, II.7].

Lemma 7.12 *Assume (H). Then*

$$
\begin{aligned}
(i)\quad & \overline{I \cap E} = \overline{I} \cap \overline{E} = \overline{I \cap K_1} \times ... \times \overline{I \cap K_m}; \\
(ii)\quad & \overline{I \cap K_i} = \overline{I} \cap \overline{K_i} = (\overline{I \cap E})\pi_i \in Max_{\mathcal{NO}}(\overline{K_i}) \text{ for all } i; \\
(iii)\quad & \overline{N_I(K)}\pi_i \cap \overline{K_i} = \overline{I} \cap \overline{K_i} \text{ for all } i.
\end{aligned}
$$

Proof. Theorem 3.9 implies $I \cap E \in \mathrm{Max}_{\mathcal{NO}}(E)$. Then since $C_G(E) \cap E \leqslant Z(E)$ we get $\overline{I \cap E} \in \mathrm{Max}_{\mathcal{NO}}(\overline{E})$ by Lemma 3.8. It follows that $\overline{I \cap E} = \overline{I} \cap \overline{E}$ because $\overline{I \cap E} \leqslant \overline{I} \cap \overline{E}$. Similarly, $\overline{I \cap K_i} = \overline{I} \cap \overline{K_i}$ because $I \cap K_i \in \mathrm{Max}_{\mathcal{NO}}(K_i)$ for all i, from Corollary 7.11. The same lemma gives $\overline{I \cap E} = \overline{I \cap K_1} \times ... \times \overline{I \cap K_m}$ and (i) is proven. Moreover, $(\overline{I \cap E})\pi_i = \overline{I \cap K_i}$ for all i, proving (ii).
For (iii) we have $I \cap E \leqslant N_I(K)$, so $(\overline{I \cap E})\pi_i \leqslant (\overline{N_I(K)})\pi_i$. Then (ii) gives $\overline{I} \cap \overline{K_i} \leqslant (\overline{N_I(K)})\pi_i \cap \overline{K_i}$. Since $\overline{I} \cap \overline{K_i} \in \mathrm{Max}_{\mathcal{NO}}(\overline{K_i})$ and $(\overline{N_I(K)})\pi_i$ is a homomorphic image of $N_I(K)$, which is nilpotent of odd order, the result follows. $\square$

Lemma 7.13 *Assume (H). Then* $\overline{N_A(K)} \cap \ker \pi_i = 1$ *for all* i.

Proof. Let $\overline{g} \in \overline{N_A(K)} \cap \ker \pi_1$. For each i we can find $a \in A$ such that $K^a = K_i$. Since A is abelian we get $\overline{g} = \overline{g}^{\overline{a}} \in (\overline{N_A(K)} \cap \ker \pi_1)^{\overline{a}} = \overline{N_A(K)} \cap \ker \pi_i$. So $\overline{g} \in \bigcap_i \ker \pi_i = 1$. $\square$

Lemma 7.14 *Assume (H). Then* $|\overline{A}| = m|\overline{N_A(K)}\pi_i|$ *for all* i.

Proof. We have

$$\begin{aligned} |\overline{A} : \overline{N_A(K)}| &= |AC_G(E)/C_G(E) : N_A(K)C_G(E)/C_G(E)| \\ &= |A/A \cap C_G(E) : N_A(K)/N_A(K) \cap C_G(E)| \\ &= |A : N_A(K)| \end{aligned}$$

because $A \cap C_G(E) = N_A(K) \cap C_G(E)$. Now, $A/N_A(K)$ is an abelian group acting faithfully and transitively on the set $\mathrm{Comp}(G)$, so $|A : N_A(K)| = |\mathrm{Comp}(G)| = m$. Finally,

$$\begin{aligned} |\overline{N_A(K)}| &= |\overline{N_A(K)}\pi_i||\overline{N_A(K)} \cap \ker \pi_i| \\ &= |\overline{N_A(K)}\pi_i|. \end{aligned}$$

by the previous lemma. Since $|\overline{A}| = |\overline{A} : \overline{N_A(K)}||\overline{N_A(K)}|$ the result follows. □

Lemma 7.15 *Assume (H). Suppose $\overline{C_1} \leqslant \overline{K_1}$ is such that its inverse image in K_1 contains an abelian subgroup D_1 of odd order satisfying $|\overline{C_1} : \overline{D_1}| \leqslant k$. Then $|\overline{C_1}|^m \leqslant k^m|\overline{A}|$.*

Proof. Since A is transitive on $K_1, ..., K_m$, for each i we can find $\overline{C_i} \leqslant \overline{K_i}$ such that $\overline{C_i} \cong \overline{C_1}$ and the inverse image of $\overline{C_i}$ in K_i contains a subgroup D_i isomorphic to D_1 satisfying $|\overline{C_i} : \overline{D_i}| \leqslant k$. Let

$$\overline{C} = \langle \overline{C_1}, ..., \overline{C_m} \rangle = \overline{C_1} \times ... \times \overline{C_m}.$$

Then $|\overline{C}| = |\overline{C_1}|^m$. Let $D = \langle D_1, ..., D_m \rangle$. The fact that $|\overline{C_i} : \overline{D_i}| \leqslant k$ for each i implies $|\overline{C} : \overline{D}| \leqslant k^m$. Let $A^* = C_A(E)D$. Since each D_i is abelian of odd order and E is a central product of its components we see that D is abelian of odd order, hence A^* is also.

Therefore $|A^*| \leqslant |A|$. Now,

$$\begin{aligned} |A^*| &= |C_A(E)||D|/|C_A(E) \cap D| \\ &\geqslant |C_A(E)||D|/|C_G(E) \cap D| \\ &= |C_A(E)||\overline{D}| \\ &\geqslant |C_A(E)||\overline{C}|/k^m. \end{aligned}$$

So $|C_A(E)||\overline{C}| \leqslant k^m|A^*| \leqslant k^m|A| = k^m|\overline{A}||A \cap C_G(E)| = k^m|\overline{A}||C_A(E)|$. Hence $|\overline{C_1}|^m = |\overline{C}| \leqslant k^m|\overline{A}|$. □

Theorem 7.16 *Suppose G has a component K of type A_n which is not a triple cover of A_6 or A_7. Let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Then A normalizes K.*

Proof. Suppose false and let G be a minimal counterexample. Then G has at least 3 components and $G = AE(G)$. Let Ω be the orbit of Comp(G) containing K under the action of A. If Ω is smaller than Comp(G) then $A\langle\Omega\rangle < AE(G)$ and the minimality of G gives us a contradiction. So A acts transitively on Comp(G) and hypothesis (H) is satisfied.

Now, $\overline{B} = \overline{N_A(K)\pi_1}$ is a subgroup of Aut(K), and by Lemma 7.4 we can also regard it as a subgroup of Aut($\overline{K}$). Since $\overline{K} \cong A_n$ we have $|\mathrm{Aut}(\overline{K}) : \mathrm{Inn}(\overline{K})| = 2$ or 4 by [12, p18-19]. As $\overline{B}$ has odd order we deduce that $\overline{B} \leqslant \mathrm{Inn}(\overline{K}) = \overline{K}$.

We have $Z(K) \preccurlyeq \mathbb{Z}_2$, so the inverse image of $\overline{B}$ in K, being nilpotent, contains a Hall $2'$-subgroup which maps isomorphically onto $\overline{B}$. Then applying Lemma 7.15 with $k = 1$ yields $|\overline{B}|^m \leqslant |\overline{A}|$. On the other hand, $|\overline{A}| = m|\overline{B}|$ by Lemma 7.14. So

$$|\overline{B}|^{m-1} \leqslant m.$$

Since $\overline{B}$ has odd order, this is a mockery unless $\overline{B} = 1$. Therefore $|\overline{A}| = m$. Now pick a subgroup $\overline{C} \leqslant \overline{K}$ of order 3. As before, Lemma 7.15 gives us $|\overline{C}|^m = 3^m \leqslant |\overline{A}| = m$, a contradiction. $\square$

Proposition 7.17 *Suppose G has a component K isomorphic to $3 \cdot A_6$ or $3 \cdot A_7$. Let $A \in \mathcal{A}_{\mathcal{O}}(G)$. Then in the action of A on Comp(G) the orbit containing K has length at most 3.*

Proof. Suppose false and let G be a minimal counterexample. Then G has at least 5 components, $G = AE(G)$ and A acts transitively on $\mathrm{Comp}(G)$. In particular, hypothesis (H) is satisfied. As in the previous theorem, $\overline{B} = \overline{N_A(K)\pi_1} \leqslant \mathrm{Inn}(\overline{K}) = \overline{K}$. Let B be the inverse image of $\overline{B}$ in K. Since $\overline{B}$ is abelian of odd order we must have $|\overline{B}| \in \{1, 3, 3^2, 5, 7\}$. Therefore $|B| \in \{1, 3, 3^2, 3^3, 5, 3.5, 7, 3.7\}$. Groups with those orders each contain an abelian subgroup of index at most 3. Hence Lemma 7.15 applies with $k = 3$ and we conclude that $|\overline{B}|^m \leqslant 3^m|\overline{A}|$. On the other hand, $|\overline{A}| = m|\overline{B}|$ by Lemma 7.14. So

$$|\overline{B}|^{m-1} \leqslant m3^m.$$

If $|\overline{B}| \geqslant 11$ then this is a contradiction, so $|\overline{B}| \leqslant 9$. Therefore $|\overline{A}| \leqslant 9m$. Now pick a subgroup $\overline{C} \leqslant \overline{K}$ of order 9. Again Lemma 7.15 applies with $k = 3$, so $|\overline{C}|^m \leqslant 3^m|\overline{A}|$, i.e, $9^m \leqslant 3^m|\overline{A}| \leqslant 3^m 9m$, implying $3^{m-2} \leqslant m$. This contradicts the fact that $m \geqslant 5$. $\square$

7.3 A Minimal Counterexample

Let G be a minimal counterexample to the conjecture that the odd nilpotent injectors are all conjugate in any group. Let $I, J \in \mathcal{NI}_{\mathcal{O}}(G)$ such that I is not conjugate to J and let $A, B \in \mathcal{A}_{\mathcal{O}}(G)$ with $A \leqslant I$ and $B \leqslant J$.

Lemma 7.18 *$G = \langle A, B\rangle$ and $\mathcal{O}_2(G) = 1$. Moreover, G has a component.*

Proof. $G = \langle A, B\rangle$ by the minimality of G and Corollary 3.13. If $\mathcal{O}_2(G) \neq 1$ then the conjecture holds in $G/\mathcal{O}_2(G)$ by the minimality of G, and Lemma 2.32 provides the conclusion. So $\mathcal{O}_2(G) = 1$. If G has no components then $C_G(F(G)) \leqslant F(G)$ and Theorem 3.15 applies. $\square$

Lemma 7.19 *Let $K \in Comp(G)$. If K is of alternating type then K is simple or K is a triple cover of A_6 or A_7. If K is simple then $K \trianglelefteq G$.*

Proof. For the first statement, see Lemma 7.6 and note that $\mathcal{O}_2(Z(E(G))) \leqslant \mathcal{O}_2(G) = 1$. The second follows from Theorem 7.16 because $G = \langle A, B \rangle$. $\square$

Lemma 7.20 *Let $K \in Comp(G)$ and suppose K is of type A_n but is not a triple cover of A_6 or A_7. Set $\overline{G} = G/K$. Then $\overline{A} \in \mathcal{A}_{\mathcal{O}}(\overline{G})$ and $\overline{I} \in \mathcal{NI}_{\mathcal{O}}(\overline{G})$.*

Proof. Suppose $\overline{A} \notin \mathcal{A}_{\mathcal{O}}(\overline{G})$. Let $\overline{D} \in \mathcal{A}_{\mathcal{O}}(\overline{G})$ and let D be the inverse image of $\overline{D}$ in G. Observe that

$$C_D(K) \cap K = Z(K) = 1.$$

Now, $D/C_D(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$, which is either S_n or contains S_n at index 2. Therefore $|D/C_D(K)|$ divides $4|K|$. On the other hand, $|KC_D(K)/C_D(K)| = |K/K \cap C_D(K)| = |K|$ divides $|D/C_D(K)|$ because $KC_D(K) \leqslant D$. So

$$|D/C_D(K)| = |K| \text{ or } 2|K| \text{ or } 4|K|.$$

If $|D/C_D(K)| = 2|K|$ then $|D| = 2|K||C_D(K)| = 2|KC_D(K)|$, implying $|D : KC_D(K)| = 2$ and $|\overline{D} : \overline{C_D(K)}| = 2$, a contradiction because $\overline{D}$ has odd order. Similarly we cannot have $|D/C_D(K)| = 4|K|$ and we conclude that $|D/C_D(K)| = |K|$ and

$$D = C_D(K) \times K.$$

Thus $C_D(K) \cong \overline{C_D(K)} = \overline{D}$, which is abelian of odd order, and $|C_D(K)| = |\overline{D}| > |\overline{A}| =$

$|A|/|A \cap K|$. Therefore $|C_D(K)||A \cap K| > |A|$, giving

$$|C_D(K)(A \cap K)||A \cap K \cap C_D(K)| = |C_D(K)(A \cap K)| > |A|.$$

This contradiction implies $\overline{A} \in \mathcal{A}_{\mathcal{O}}(\overline{G})$.

Suppose $\overline{I} \notin \mathcal{NI}_{\mathcal{O}}(\overline{G})$. Let $\overline{I} < \overline{H} \in \mathcal{NI}_{\mathcal{O}}(\overline{G})$ and let H be the inverse image of $\overline{H}$ in G. An identical argument to that presented above allows us to deduce that

$$H = C_H(K) \times K.$$

Now, $I \leqslant H$ and I is maximal subject to being nilpotent of odd order, which implies that $I = I_1 \times I_2$ where I_1 and I_2 are maximal subject to being nilpotent of odd order in $C_H(K)$ and K respectively. However, $C_H(K) \cong \overline{C_H(K)} = \overline{H}$, which is nilpotent of odd order, implying that $I_1 = C_H(K)$. Therefore $\overline{I} = \overline{I_1} = \overline{C_H(K)} = \overline{H}$, a contradiction. $\square$

Lemma 7.21 *Let $K \in Comp(G)$ and suppose K is of type A_n but is not a triple cover of A_6 or A_7. Set $\overline{G} = G/K$. If $\overline{I}$ and $\overline{J}$ are conjugate in $\overline{G}$ then I and J are conjugate in G.*

Proof. By hypothesis we have $IK = J^g K$ for some $g \in G$. Without loss of generality we may assume $J^g = J$. If $IK < G$ then I and J are conjugate by minimality of G. So

$$G = IK.$$

Now, $A/C_A(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$ of odd order, and since the inner automorphism group of K is normal of index 2 or 4 in $\mathrm{Aut}(K)$ we see that $A/C_A(K)$ induces inner automorphisms. So there is an abelian subgroup $D \leqslant K$ of order $|A/C_A(K)|$ which is centralized by $A/C_A(K)$, and hence by A. Thus $D \leqslant A$. Since $|DC_A(K)| =$

$|D||C_A(K)| = |A/C_A(K)||C_A(K)| = |A|$, it follows that

$$A = DC_A(K).$$

We also note that $D \in \mathcal{A}_{\mathcal{O}}(K)$, since if not then a subgroup $D_1 \in \mathcal{A}_{\mathcal{O}}(K)$ would yield a group $D_1C_A(K)$ abelian of odd order and larger than A. Similarly, there exists $E \in \mathcal{A}_{\mathcal{O}}(K)$ such that

$$B = EC_B(K).$$

So $\langle A, B\rangle \leqslant KC_G(K)$. Since $G = \langle A, B\rangle$ we get

$$G = KC_G(K) = K \times C_G(K).$$

It follows that $C_A(K), C_B(K) \in \mathcal{A}_{\mathcal{O}}(C_G(K))$. Lemma 3.4 and the minimality of G now provide the result, unless it is the case that $G = K$ or $G = C_G(K)$. The latter cannot happen because K is a component, and the former case is Theorem 7.10. $\square$

Theorem 7.22 *Let $K \in Comp(G)$ and suppose K is of type A_n. Then K is a triple cover of A_6 or A_7.*

Proof. Suppose not. Set $\overline{G} = G/K$. Then $\overline{I}, \overline{J} \in \mathcal{NI}_{\mathcal{O}}(\overline{G})$ by Lemma 7.20, and $\overline{I}$ is conjugate to $\overline{J}$ by the minimality of G. Lemma 7.21 then implies that I is conjugate to J. $\square$

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