THE BEHAVIOUR OF THE RIEMANN ZETA-FUNCTION IN THE CRITICAL STRIP

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1. INTRODUCTION

In making an investigation into the distribution of prime numbers, it is natural to seek as a starting point some transformation between an expression in which the prime numbers explicitly occur, and one in which they do not.

Such a transformation was first given by Euler in 1737 (Commentationes Acad. Sci. Imp.Petropolitane IX p.160-188) who proved that

1.1 $$\prod_{n=1}^{\infty}(1-p_n^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$$

in which p_n denotes the n^{th} prime number and where both the series and product converge if s (supposed real in Euler's work) has real part greater than unity.

Riemann, in his investigations on the distribution of primes (Über die Anzahl der Primzahlen unter einer gegebenen Grosse. Berliner Monatsber, 1889) took this transformation as the basis of his work, and was naturally led to a discussion of the function, $\zeta(s)$, of the complex variable $s = \sigma + it$, σ, t being real, defined when $\sigma > 1$ by the series in 1.1

In order to define $\zeta(s)$ for all values of σ Riemann obtained the integral form

1.2 $$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

where $\sigma > 1$, and then proved that the right hand side of this expression is merely a particular case of the contour integral,

1.21 $$\frac{-\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz$$

where the path of integration starts from 'infinity' on the real axis, encircles the origin in the positive direction without enclosing any of the points $z = \pm 2\pi n i$ $(n = 1, 2, 3 \cdots\cdots)$, which are poles of the integral, and returns to the starting point.

Now the above contour integral is a one-valued analytic function of s for all values of s, hence the only possible singularities of the expression 1.21 are at the points $s = 1, 2, 3 \cdots$ which are the simple poles of $\Gamma(1-s)$. But for $s = 2, 3, 4, \cdots$ this expression reduces to 1.2 which is certainly finite at these points, so that the only singularity of 1.21 is at $s = 1$.

Hence we obtain the representation

1.3 $$\zeta(s) = \frac{-\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

which continues $\zeta(s)$ over the entire s -plane, except at the point s = 1 where it possesses a simple pole.

Working from this expression, Riemann proved, firstly, that $(s-1)\zeta(s)$ and $\zeta(s) - \frac{1}{s-1}$ are integral transcendental functions, and secondly, that the function $\Gamma(\frac{1}{2}s)\,\pi^{-\frac{1}{2}s}\,\zeta(s)$ remains unchanged when s is replaced by $1-s$.

This last remark gives the classical functional equation

1.4 $$\zeta(1-s) = 2(2\pi)^{-s} \cos \tfrac{1}{2}s\pi\, \Gamma(s)\, \zeta(s)$$

which affords a series representation of $\zeta(s)$ when $\sigma < 0$, and from which it is evident that, in order to study the behaviour of the Zeta function, it is only necessary to consider $\sigma \geq \frac{1}{2}$.

Since the product $\prod_{n=1}^{\infty} (1 - p_n^{-s})$ converges when $\sigma > 1$ it follows from 1.1 that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$ and consequently, by 1.4, no zeros in the half-plane $\sigma < 0$ excepting the obviously trivial ones at $s = -2, -4, -6, \cdots$. Hence all the non-trivial zeros lie in that strip of the s -plane, defined by $0 \leq \sigma \leq 1$; it is this region which is called the 'CRITICAL STRIP', whilst the line $\sigma = \frac{1}{2}$ may be conveniently called the 'CRITICAL LINE'.

It is at this stage in Riemann's memoir that his analysis loses rigour, and he asserts without proof that :-

(1) If we put $s = \frac{1}{2} + it$, t being regarded as a complex variable, and set
$$\Xi(t) = \tfrac{1}{2}\pi^{-\frac{1}{2}s} s(s-1)\,\Gamma(\tfrac{1}{2}s)\,\zeta(s).$$
so that the functional equation is expressed by $\Xi(t) = \Xi(-t)$,

then $\Xi(t)$ considered as a function of t^2 is, in modern language, an integral function of genus zero.

(II) The number of zeros of $\zeta(s)$ in the region defined by $0 \leq \sigma \leq 1$, $T \geq t > 0$ is approximately equal to

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$$

(III) It is probable that all the non-trivial zeros of $\zeta(s)$ lie on the critical line, or, what is the same thing, all the zeros of $\Xi(t)$ are real.

Of these assertions, the first to be proved was (I), the proof being incidental in a theory of integral functions discussed by Hadamard in 1893 ('Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann' J.de Math. (4) 9. 1893)

Hence if $a_1, a_2, a_3, \cdots$ are the zeros of $\Xi(t)$ with positive real part, then

1.5 $$\Xi(t) = \Xi(0) \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{a_n^2}\right)$$

the product converging absolutely for all values of t.

Using this result von Mangoldt investigated the second assertion, and in his first memoir on this subject (Crelles J.114 1895) proved that the number of zeros of $\zeta(s)$ in the region defined in (II) is of the form

1.6 $$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + R(T)$$

where R (T) is of the same order of magnitude as $\log^2 T$. In his second memoir ('Math. Ann. 60 1905) von Mangoldt reduced the error term to the magnitude of $\log T$ which still remains the best form.

Although Hadamard (Bull. de la Soc. Math. de France 24. 1896) and De la Vallée Poussin (Ann. de la Soc.Sc.de Bruxelles 20 (2) 1896) independently proved that no zeros of $\zeta(s)$ lie on the line $\sigma = 1$ (and therefore on the line $\sigma = 0$) the third assertion of Riemann, now known simply as 'Riemann's Hypothesis', is still unproved.

Other contributions of an important nature were made by Jensen, Lerch, Stieltjes, Mellin and Landau, and the behaviour of $\zeta(s)$ outside the critical strip was well explored by the time

Landau's important treatise '<u>Handbuch der Lehre von der Verteilung der Primzahlen</u>' (Vol. 1 & 2 Leipzig 1909) was published.

The analysis of $\zeta(s)$ inside the critical strip has since proved to be one of the greatest problems of modern mathematics, and although steady progress has been made, yet the real nature of $\zeta(s)$ inside this region remains undisclosed.

This analysis forms the subject of this thesis, and an account of all the important results contributed since 1909 is given; this period being selected since it allows us to take Landau's treatise as a reference work (we will, in fact, refer to it as H.) and complete those parts concerning the Zeta-function.

The notation to be employed will closely follow Landau. Thus the symbols O and o will be as defined in H p.59, with one important addition. If we meet with an inequality of the type $|F(t)| < \mathfrak{f}(c)\,\varphi(t)$ and write $F(t) = O\{\varphi(t)\}$ then the constant implied by the O would be a function of c, whilst if we write $F(t) = O\left(\mathfrak{f}(c)\,\varphi(t)\right)$ then we should infer that the above inequality is satisfied for some choice of $\mathfrak{f}(c)$.

As usual, $\Re(s)$ and $\Im(s)$ denote the real and imaginary parts of s respectively.

We make one important alteration in the notation of Landau. In order to avoid the tedious introduction of a set of constants $c_1, c_2, c_3 \cdots c_n$ into an analysis involving a considerable amount of inequality work (see e.g. H. p.171 et seq.) we shall follow Borel in using one constant c throughout the analysis in which it first occurs. A constant of this type - i.e. a constant which may roughly be described as an 'absorbing constant' - will, in order to prevent any ambiguity, be referred to as a Borel constant.

2. APPLICATION OF THE DIOPHANTINE APPROXIMATION.

The use of Diophantine methods in the study of the Zeta-function first occurs in a memoir by Bohr and Landau in 1910 (Über das Verhalten von $\zeta(s)$ und $\zeta_K(s)$ in der Nähe der Geraden $\sigma = 1$. Gött. Nach 1910) in which a theorem originally due to Dirichlet (Berliner Sitzungsber 1842) is used to prove the unboundedness of $\zeta(s)$ in the neighbourhood of the line $\sigma = 1$. This theorem asserts that, given N real numbers $\alpha_1, \alpha_2, \cdots \alpha_N$, and a positive integer q, it is possible to find N integers $x_1, x_2, \cdots x_N$, and a t in the range $1 \leq t \leq q^N$ so that

$$|t\alpha_n - x_n| < \frac{1}{q} \qquad (n = 1, 2, \cdots N)$$

It will be useful to give the first application of this theorem. Let δ be a positive number then :-

$$|\zeta(1+\delta+it)| \geq \left|\sum_1^N n^{-1-\delta-it}\right| - \left|\sum_{N+1}^\infty n^{-1-\delta-it}\right| > \left|\sum_1^N n^{-1-\delta} e^{-it\log n}\right| - \sum_{N+1}^\infty n^{-1-\delta}$$

$$> \sum_1^N \cos(-t\log n)\, n^{-1-\delta} - \sum_{N+1}^\infty n^{-1-\delta}$$

Now by Dirichlet's theorem, we can find t_0 in the range $1 \leq t_0 \leq 6^N$, so that

$$|-t_0 \log n - 2x_n\pi| \leq \tfrac{1}{3}\pi \qquad (n = 1, 2, \cdots, N.)$$

Hence

$$\cos(-t_0 \log n) > \tfrac{1}{2} \qquad (n = 1, 2, \cdots, N; \quad 1 \leq t_0 \leq 6^N)$$

Consequently

$$|\zeta(1+\delta+it_0)| > \tfrac{1}{2}\sum_1^N n^{-1-\delta} - \sum_{N+1}^\infty n^{-1-\delta} > \tfrac{1}{2}\int_1^N z^{-1-\delta}\,dz - \int_N^\infty z^{-1-\delta}\,dz$$

$$\therefore \quad |\zeta(1+\delta+it_0)| > \frac{1}{2\delta}\left(1 - \frac{3}{N^\delta}\right) \qquad (1 \leq t_0 \leq 6^N)$$

Take $N = [6^{1/\delta}]$, where $[x]$ will always mean the integer nearest x, then

$$|\zeta(1+\delta+it_0)| > \frac{1}{\delta} \qquad (1 \leq t_0 \leq 6^N)$$

Since δ can be made arbitrarily small, it follows that $\zeta(s)$ is unbounded in the neighbourhood of $\sigma = 1$

Further, since $t_0 \leq 6^{[6^{1/\delta}]}$, we have

$$\log\log t_0 < \log\log 6 + \frac{1}{\delta}\log 6 < \frac{c}{\delta}$$

which gives the result :-

THEOREM 1. It is possible to find a constant K such that

$$|\zeta(s)| > K \log\log t$$

for $\sigma > 1$ and $t > \tau > e^e$.

Dirichlet's theorem is essentially contained ina result given by Kronecker in 1884 ('Werke' Vol 3, p.49) which will be taken as the foundation of the work in this section and may be stated as follows.

KRONECKER'S THEOREM Suppose $\lambda_1, \lambda_2, \cdots \lambda_N$, are N linearly independent real numbers (i.e. no relation of the type $a_1\lambda_1 + a_2\lambda_2 + \cdots + a_N\lambda_N = a_{N+1}$ where $a_1, a_2, \cdots a_N$, are integers, not all zero, holds between $\lambda_1, \lambda_2, \cdots \lambda_N$). Let $\varphi_1, \varphi_2, \cdots \varphi_N$, be N arbitrarily chosen, real numbers, and ϵ is positive and arbitrarily small. Then N integers $h_1, h_2, \cdots h_N$, and a real number t exist such that

$$|t\lambda_n - \varphi_n - h_n| < \epsilon \qquad (n = 1, 2, \cdots N)$$

Geometrically speaking, the theorem asserts that the set of points obtained from the set $(t\lambda_1, t\lambda_2, \cdots\cdots t\lambda_N)$ in N- dimensional space by reduction of co-ordinates modulo 1 lie everywhere dense in the N- dimensional unit-cube.

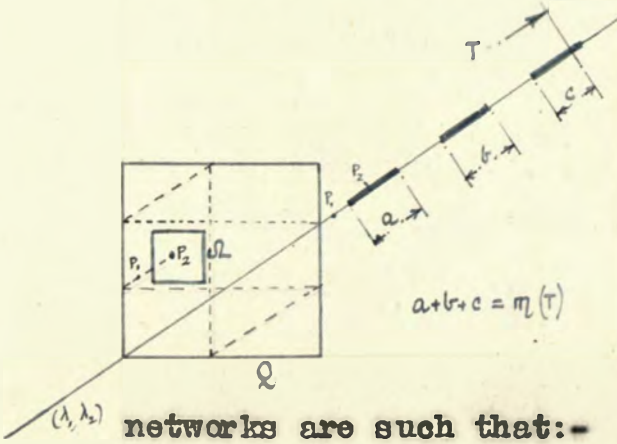
For the proof of this theorem we refer to the Proc. Lond. Math. Soc. 1922 where two particularly elegant proofs by Lettenmeyer and Bohr, based on geometric and analytic ideas respectively, will be found.

Before proceeding with our discussion of the Zeta-function, it will be advantageous to deduce the following COROLLARY TO KRONECKER'S THEOREM.

Let $\lambda_1, \lambda_2, \cdots \lambda_N$ denote, as before, a set of N linearly independent real numbers. Let there be given in the N- dimensional unit-cube $Q: 0 \le \eta_n < 1 \quad (n = 1, 2, \cdots N.)$, a cube Ω oriented parallel to the axes of the $\eta_1, \eta_2 \cdots \eta_N$ space ; $\bar{\eta}_n \le \eta_n \le \bar{\eta}_n + d$ with side of length $d < 1$. Let $M = M(\Omega, T)$ denote the set of all values of t in the interval $0 \le t \le T$ for which the point P_t obtained from the point $(t\lambda_1, t\lambda_2, \cdots t\lambda_N)$ by reduction of its co-ordinates modulo 1 belongs to the cube Ω. Then for every $T > 0$ the set M consists of a finite number of intervals, and, if $m(T)$ denotes the sum of the lengths of these intervals, then

$$\lim_{T \to \infty} \frac{m(T)}{T} = d^N.$$

We will only prove for the two-dimensional case; the extension to the general case will be obvious.



The statement that for $T>0$, M consists of a finite number of intervals is evident. The unit square Q is covered in two different ways with a network of equal sized squares of side δ (ultimately to be made small) oriented parallel to the axes. These two networks are such that:-

<u>For first type</u> : Centres of subsquares must all lie in Q and have no common point. Let ℓ be the number of these subsquares lying <u>wholly</u> in Q and λ the number lying <u>wholly</u> in Ω

<u>For second type</u>: Each point of Q lies inside at least one subsquare. Let L be the number of these subsquares which contain at least one point of Q and Λ the number containing at least one point of Ω

By Kronecker's theorem the points in Q lie everywhere dense, hence δ, the length of side of both types of subsquares, can be chosen so that for a given positive ϵ,

2.1
$$d^2 - \tfrac{1}{2}\epsilon < \frac{\lambda}{L} \le \frac{\Lambda}{\ell} < d^2 + \tfrac{1}{2}\epsilon$$

Let $m'_\delta(T)$ denote the sum of lengths of intervals of t in $0 \le t \le T$ for which the pt. obtained from the pt. $(t\lambda_1, t\lambda_2, \cdots t\lambda_N)$ by reduction of its co-ordinates modulo 1 lies in a subsquare of first type. Similarly define $m''_\delta(T)$ with respect to subsquares of second type.

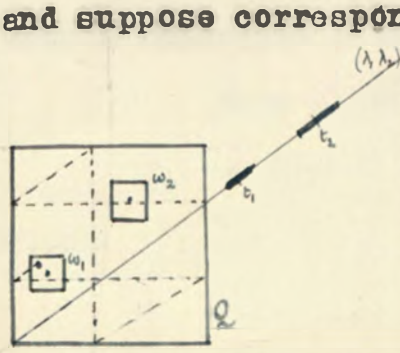
Then evidently

$$\sum_\lambda m'_\delta(T) \le m(T) \le \sum_\Lambda m''_\delta(T) \quad ; \quad \sum_\ell m'_\delta(T) \le T \le \sum_L m''_\delta(T)$$

so that, by division, we have

2.2
$$\frac{\sum_\lambda m'_\delta(T)}{\sum_L m''_\delta(T)} \le \frac{m(T)}{T} \le \frac{\sum_\Lambda m''_\delta(T)}{\sum_\ell m'_\delta(T)}$$

Let ω_1 and ω_2 be two subsquares of first type with centres at P_{t_1} and P_{t_2} and suppose corresponding values of m' are m'_{ω_1} and m'_{ω_2}



The point reduced from $(t\lambda_1, t\lambda_2)$ falls in ω_1 if, and only if, the point $\{(t-t_1+t_2)\lambda_1, (t-t_1+t_2)\lambda_2\}$ lies in ω_2; t being sufficiently large.

Hence $|m'_{\omega_1}(T) - m'_{\omega_2}(T)| < C$, where C is a constant

Evidently $\lim_{T\to\infty} m'_{\omega_1}(T) = \infty$, so that

2.3 $$\lim_{T\to\infty} \frac{m'_{\omega_1}(T)}{m'_{\omega_2}(T)} = 1$$

An analogous result will, of course, hold for the second type of subsquare. Equations 2.1 to 2.3 show that it is possible to find $T > T_0$, T_0 depending on ϵ, so that

$$\frac{\sum_\lambda m'_\delta(T)}{\sum_L m''_\delta(T)} > \frac{\lambda}{L} - \tfrac{1}{2}\epsilon, \quad \text{and} \quad \frac{\sum_\Lambda m''_\delta(T)}{\sum_l m'_\delta(T)} < \frac{\Lambda}{l} + \tfrac{1}{2}\epsilon$$

whence

$$d^2 - \epsilon < \frac{m(T)}{T} < d^2 + \epsilon.$$

which proves the corollary. This corollary naturally leads to the following theorem.

THEOREM 2. Let Ω be a region in the N-dimensional unit-cube Q, measurable in the sense of Jordan. Let its content be P.
Let $M = M(\Omega, T)$ be the set of values of t lying between 0 and T for which the point P_t reduced from the point $(t\lambda_1, \cdots t\lambda_N)$ falls in Ω; and denote by $m_o(T)$ and $m_i(T)$ the outer and inner measure of the set M.

Then $$\lim_{T\to\infty} \frac{m_o(T)}{T} = \lim_{T\to\infty} \frac{m_i(T)}{T} = P$$

Take the case $N=2$. Since Ω is measurable, Q can be covered with a network of subsquares of side δ oriented parallel to sides of Q, such that, if N_1 is the number of subsquares lying wholly in Ω, and N_2 the number which contain at least one point of Ω, then $N_1\delta^2 > P - \epsilon$, and $N_2\delta^2 < P + \epsilon$, δ depending on the positive ~~ϵ~~ ϵ.

Since $m_i(T) \geq \sum_{N_1} m_\delta(T)$, $m_o(T) \leq \sum_{N_2} m_\delta(T)$, and by the previous corollary, $\lim_{T\to\infty} \frac{m_\delta(T)}{T} = \delta^2$, the theorem easily follows.

We now enter into a discussion of the set of values which $\zeta(s)$ takes when s describes a vertical line $\sigma = \sigma_0$ in the complex plane.

From Euler's product form 1.1, we have, for $\sigma_0 > 1$,

$$\log \zeta(\sigma_0 + it) = -\sum_{n=1}^{\infty} \log(1 - p_n^{-\sigma_0 - it}) = -\sum_{n=1}^{\infty} \log(1 + p_n^{-\sigma_0} e^{i\mu_n})$$

where $\mu_n = \pi - t\log p_n$.

Let $M(\sigma_0)$ be the set of values which the more general form

$$F(\varphi_1, \varphi_2, \cdots \varphi_n, \cdots) = -\sum_{n=1}^{\infty} \log(1+p_n^{-\sigma_0} e^{i\varphi_n}) \qquad (\varphi_n \text{ real})$$

takes when $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ independently run through all real values or, what comes to the same thing, all real values in the range $0 \leq \varphi_n < 2\pi$.

Using Kronecker's theorem, it is possible to shew that the set of values of $\log \zeta(\sigma_0+it)$ $(-\infty<t<\infty)$ is included in the set $M(\sigma_0)$, so that we are led to a discussion of $M(\sigma_0)$.

When φ_n varies from 0 (inclusive) to 2π (exclusive), the number $1+p_n^{-\sigma_0} e^{i\varphi_n}$ describes a circle, so that the point $\log(1+p_n^{-\sigma_0} e^{i\varphi_n})$ describes a convex curve in the complex plane. Consequently, a determination of the set $M(\sigma_0)$ necessitates the 'addition' of an infinite number of convex curves. Bohr ('Om Addition af uendelig mange konvekse Kurver' Oversigt over Danske Vidensk.Selsk.Forhand 1913) has studied this problem in detail, and has obtained results, which, from some points of view, have quite a definite character. For example, he has found that if $\sigma_0 > 1$ then the set $M(\sigma_0)$ of points in the complex plane form a finite region, D, bounded either by a convex curve, or by two convex curves, one being interior to the other. From this it follows that the set of values of $\log \zeta(\sigma_0+it)$ $(\sigma_0>1, -\infty<t<\infty)$, is always dense in D.
(Bohr. 'Comptes Rendus' 154. 1912, p.1080)

It would appear that this method, depending as it does on Euler's product, is limited to $\sigma_0 > 1$. However, Bohr and Courant (Crelle J.114 1914) have generalised the method so that it is applicable for $\sigma_0 > \frac{1}{2}$. We proceed with their investigation. Let us write

2.4
$$\zeta(s) = R_N(s) \prod_{n=1}^{N} (1-p_n^{-s})^{-1}$$

so that $R_N(s) = \prod_{N+1}^{\infty} (1-p_n^{-s})^{-1} = \zeta(s) \prod_{n=1}^{N} (1-p_n^{-s})$; and first discuss $R_N(s)$.

Consider the function

$$\mathfrak{S}_N(s) = (R_N(s)-1)(1-2^{1-s}) = \sum_{n=1}^{\infty} \frac{b_n^{(N)}}{n^s}.$$

Since the series $\zeta(s)(1-2^{1-s}) = 1-2^{-s}+3^{-s}-\cdots$ converges for $\sigma > 0$ it is easy to see that $b_n^{(N)} = 0$ for $n < p_{N+1}$, and $|b_n^{(N)}| \leq 2$ for $n \geq p_{N+1}$. and by known theorems (H. p.778)

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\mathcal{G}_N(\sigma_0+it)|^2 dt = \sum_{1}^{\infty} |b_n^{(N)}|^2 n^{-2\sigma_0} = \alpha_N \quad (\text{say})$$

where $\sigma > \frac{1}{2}$ and $\alpha_N > 0$

Hence

2.5 $$\int_{-T}^{T} |\mathcal{G}_N(\sigma_0+it)|^2 dt < 3\alpha_N T \qquad (T \geq T_0 > 1;\ \sigma_0 > 1)$$

From a previous remark relative to value of $|b_n^{(N)}|$ for $n < p_{N+1}$ and $n \geq p_{N+1}$, it is obvious that for a fixed σ_0 in the interval $\frac{1}{2} < \sigma_0 \leq 1$ we have

.6 $$\lim_{N\to\infty} \alpha_N = 0$$

If $\frac{1}{2} < \sigma_0 < 1$ then on the line σ_0, $|1-2^{1-s}| \geq c > 0$, hence

$$\int_{-T}^{T} |R_N(\sigma_0+it)-1|^2 dt < \frac{3\alpha_N T}{c^2}, \qquad \text{using 2.5}.$$

Let $\mu(T) =$ sum of lengths of intervals of T between $-T$ and $+T$ for which $|R_N(\sigma_0+it)-1| \geq \epsilon$, where ϵ is arbitrarily small and positive, then this inequality gives

$$\mu(T) < \frac{3\alpha_N T}{c^2\epsilon^2}$$ so that, by 2.6, it is possible to choose $N_0 = N_0(\sigma_0, \epsilon, \delta)$ such that, for δ given arbitrarily small and positive

2.7 $$\mu(T) < \delta T \qquad (N > N_0;\ T > T_0)$$

On the line $\sigma_0 = 1$ we have the zeros of $1-2^{1-s}$ (at the points $t = \frac{2\pi i\nu}{\log 2}$, $\nu = \pm 1, \pm 2, \cdots$) to contend with. Enclose each of these zeros in an interval of length $\delta_1 = \delta_1(\delta)$, δ_1 being ~~so~~ chosen so that sum of lengths of these intervals between $-T$ and $+T$ is smaller than $\frac{1}{2}\delta T$, $T > 0$ and sufficiently large

On the set of intervals remaining, which we denote by $I(T, \delta)$ we certainly have $|1-2^{1-s}| > c > 0$, where c is a constant, independent of T, and also

$$\int_{-T}^{T} |\mathcal{G}_N(\sigma_0+it)|^2 dt < 3\alpha_N T,$$

where the dash denotes that integration is through the range $I(T,\delta)$

$$\therefore \quad {\int_{-T}^{T}}' |R_N(\sigma_0+it)-1|^2\,dt < \frac{3\alpha_N T}{c^2}$$

The length of sub-intervals of $I(T,\delta)$ in which $|R_N(\sigma_0+it)-1| \geq \epsilon$ is therefore smaller than $\frac{3\alpha_N T}{c^2\epsilon^2}$

Choose N_0 so that, for $N > N_0$, $\frac{3\alpha_N}{c^2\epsilon^2} < \frac{1}{2}\delta$

Then $\quad \mu(T) < \frac{1}{2}\delta T + \frac{1}{2}\delta T = \delta T$

which completes the proof of the following theorem.

<u>THEOREM 3.</u> Let σ_0 be a fixed number in the interval $\frac{1}{2} < \sigma_0 \leq 1$ and let ϵ and δ be arbitrary small positive numbers. Then it is possible to find $N_0 = N_0(\sigma_0, \epsilon, \delta)$ such that the sum $\mu(T)$ of the lengths of the intervals of the real variable t between $-T$ and $+T$ for which $|R_N(\sigma_0+it)-1| \geq \epsilon$ is smaller than δT, N being $>N_0$ and $T > T_0(N)$.

We now investigate the value of

$$\prod_1^N (1-p_n^{-s})^{-1}$$

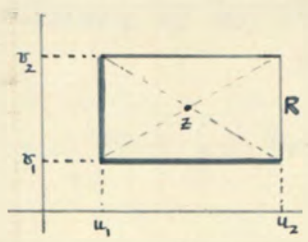
or, what comes to the same thing,

$$\sum_1^N \log(1-p_n^{-s}).$$

Let us consider the more general expression $S_N(\eta_1,\eta_2\cdots\eta_N) = \sum_1^N \log(1-r_n e^{2\pi i\eta_n})$,

where N is a positive integer, $0<r_n<1$ $(n=1,2,\cdots N)$, and $\eta_1 \cdots \eta_N$ are N real variables. Let Σ_N denote the set of points $S_N(\eta_1,\eta_2,\cdots\eta_N)$ on the complex plane $z = S_N$ obtained when $\eta_1\,\eta_2\cdots\eta_N$ independently take all values in the interval $0 \leq \eta_n < 1$. It is easy to see that when the η's run through these values, the point $\log(1-r_n e^{2\pi i\eta_n})$ describes a convex curve Γ_n, so that Σ_N is the set of points formed by the 'addition' of the N convex curves $\Gamma_1, \Gamma_2, \cdots \Gamma_N$. Let R denote a rectangle drawn in the S_N plane with centre at z and oriented parallel to axes $z = u+iv$ and so defining a set of points $u_1 \leq u < u_2$, $v_1 \leq v < v_2$.

Let N be fixed and $\Omega = \Omega_N$ be the set of points in the N-dimensional unit-cube Q $(0 \leq \eta_n < 1)$ in $\eta_1\,\eta_2\cdots\eta_N$ space, such that the point $S_N(\eta_1,\eta_2,\cdots\eta_N)$ lies in the rectangle R. The following four theorems will be fundamental:-



(I) The set Ω_N is measurable. Its measure (in Jordan's sense) will be denoted by $P_R^{(N)}(z)$ and expresses the probability that the point S_N lies in R.

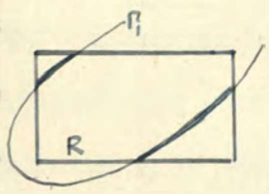
(II) When the rectangle R moves continuously with same dimensions and orientation, then $P_R^{(N)}(z)$ is a continuous function of the centre z.

(III) $P_R^{(N)}(z) > 0$ when, and only when, at least one point of Σ_N lies in R. For all other values of z, $P_R^{(N)}(z) = 0$

(IV) For every positive ϵ there exists a positive δ such that, for all values of z, $P_R^{(N)}(z) < \epsilon$ only when the area of R is less than δ

We sketch a proof of these theorems, using induction.

For the case $N=1$ the set Σ_1 coincides with the single convex curve Γ_1.



The curve Γ_1 is divided into a finite number of intervals by the rectangle R, each of which defines a corresponding interval in the set Ω_1 of values of η_1. Since these intervals change continuously as R moves continuously, (I) and (II) are evidently true.

The third assertion is evident and (IV) follows without difficulty.

It is now only necessary to show that if the theorems are true for case N, then they are true for case $N+1$.

Let R be a fixed rectangle with centre z_0. Let $\bar\eta_{N+1}$ be a fixed value in the interval $0 \leq \eta_{N+1} < 1$ and denote by $\phi(\bar\eta_{N+1})$ the set of points in the N-dimensional unit-cube $0 \leq \eta_n < 1$ $(n = 1, 2, \cdots N)$ such that $S_N(\eta_1, \eta_2, \cdots \eta_N)$ falls in R.

Since
$$S_{N+1}(\eta_1, \eta_2, \cdots \eta_N, \bar\eta_{N+1}) = S_N(\eta_1 \cdots \eta_N) + \log(1 - r_{N+1} e^{2\pi i \bar\eta_{N+1}})$$
we see that $\phi(\bar\eta_{N+1})$ is the set of points $(\eta_1 \cdots \eta_N)$ for which $S_N(\eta_1 \cdots \eta_N)$ lies in the rectangle R with centre at $z_0 - \log(1 - r_{N+1} e^{2\pi i \bar\eta_{N+1}})$.

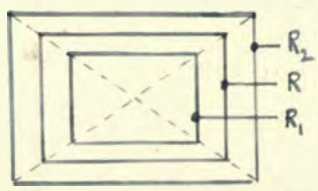
Hence, by hypothesis, $\phi(\bar\eta_{N+1})$ is measurable and has measure $P_R^{(N)}\{z_0 - \log(1 - r_{N+1} e^{2\pi i \bar\eta_{N+1}})\}$.

We now shew that the set $\Omega_{N+1} = \Omega_{N+1}(z_0)$ is measurable, and that its measure $P_R^{(N+1)}(z_0)$ agrees with the integral:-

$$J = \int_0^1 P_R^{(N)}\{z_0 - \log(1 - r_{N+1} e^{2\pi i \bar\eta_{N+1}})\} \, d\bar\eta_{N+1}$$

which has a sense, since $P_R^{(N)}(z)$ is, by hypothesis, continuous.

If A_o and A_i are the outer and inner measures of Ω_{N+1} we have to show that for every positive ϵ, $A_o < J+\epsilon$ and $A_i > J-\epsilon$, then since $A_o \geqslant A_i$ it follows that $A_o = A_i = J$



Let R_1 and R_2 be two rectangles drawn with same centre as the rectangle R, and defined relative to R, as shewn in the diagram.

By (IV) we can make R_1 and R_2 so near to R, that

2.8 $$P_{R_1}^{(N)} > P_R^{(N)} - \tfrac{1}{2}\epsilon, \quad \text{and} \quad P_{R_2}^{(N)} < P_R^{(N)} + \tfrac{1}{2}\epsilon$$

Divide the integration range $0 \leq \bar{\eta}_{N+1} \leq 1$ into sub-intervals of length δ so that

2.9 $$\left| J - \delta \sum_{\delta} P_R^{(N)}\left\{z_0 - \log\left(1 - r_{N+1} e^{2\pi i \bar{\eta}_{N+1}}\right)\right\}\right| < \tfrac{1}{2}\epsilon$$

and for each point η_{N+1} in the sub-interval $(\bar{\eta}_{N+1}, \bar{\eta}_{N+1}+\delta)$ the corresponding rectangle R with centre $z_0 - \log\left(1 - r_{N+1} e^{2\pi i \eta_{N+1}}\right)$ lies entirely in the rectangle R_2 associated with $\bar{\eta}_{N+1}$ and entirely outside the rectangle R_1 associated with $\bar{\eta}_{N+1}$.

Hence the inner and outer measure of those parts of Ω_{N+1} for which η_{N+1} belongs to the interval $(\bar{\eta}_{N+1}, \bar{\eta}_{N+1}+\delta)$ are at least equal to $\delta P_{R_1}^{(N)}$, or at most, equal to $\delta P_{R_2}^{(N)}$ respectively, where R_1 and R_2 are associated with $\bar{\eta}_{N+1}$.

Hence $$A_o \leq \delta \sum_{\delta} P_{R_2}^{(N)} \quad \text{and} \quad A_i \geq \delta \sum_{\delta} P_{R_1}^{(N)}$$

so that, using 2.8,

$$A_o \leq \delta \sum_{\delta} P_R^{(N)} + \tfrac{1}{2}\epsilon \quad \text{and} \quad A_i \geq \delta \sum_{\delta} P_R^{(N)} - \tfrac{1}{2}\epsilon$$

Therefore using 2.9, $A_o < J+\epsilon$ and $A_i > J-\epsilon$ so that

2.10 $$J = P_R^{(N+1)}(z_0)$$

By definition of J it follows that $P_R^{(N+1)}(z)$ is continuous, (II); and if $P_R^{(N)}(z) < \epsilon$ for all z so is $P_R^{(N+1)}(z)$, (IV). The remaining theorem (III) is deducible by evident arguments using 2.10. Having now obtained the fundamental properties of $P_R^{(N)}$ we prove the following theorem.

THEOREM 4. Let the sequence $0 < r_1, r_2, \cdots r_n, \cdots\cdots, < 1$, be such that $\sum r_n$ diverges, whilst $\sum r_n^2$ converges. Let R_0 be a fixed rectangle in the z-plane with centre at z_0, and oriented parallel to axes. Then a positive number $k = k(R_0, z_0)$ exists such that

$$P_{R_0}^{(N)}(z_0) > k \qquad \text{for N sufficiently large.}$$

Let $S_{N,M}(\eta_{N+1}, \eta_{N+2}, \cdots \eta_M) = \sum_{n=N+1}^{M} \log(1 - r_n e^{2\pi i \eta_n}) \quad (M > N)$, and let $P_R^{(N,M)}(z)$ be the measure of the set of points in $\eta_{N+1} \cdots \eta_M$ space lying in the unit-cube $0 \leq \eta_n < 1 \quad (n = N+1, N+2, \cdots M)$ for which $S_{N,M}(\eta_{N+1} \cdots \eta_M)$ lies in R of centre z

It is evident that $P_R^{(N,M)}(z)$ will have the properties (I) to (IV) attributed to $P_R^{(N)}(z)$.

Since

$$|S_{N,M}(\eta_{N+1} \cdots \eta_M)|^2 = \sum_{N+1}^{M} \log(1 - r_n e^{2\pi i \eta_n}) \cdot \sum_{N+1}^{M} \log(1 - r_n e^{-2\pi i \eta_n}) = \sum_{n=N+1}^{M}\sum_{m=1}^{\infty} \frac{r_n^m e^{2\pi i \eta_n m}}{m} \cdot \sum_{n=N+1}^{M}\sum_{m=1}^{\infty} \frac{r_n^m e^{-2\pi i \eta_n m}}{m},$$

it follows that

$$J = \int_0^1\int_0^1 \cdots \int_0^1 |S_{N,M}(\eta_{N+1} \cdots \eta_M)|^2 d\eta_{N+1} \cdots d\eta_M = \sum_{n=N+1}^{M}\sum_{m=1}^{\infty} \frac{r_n^{2m}}{m^2} < \sum_{n=1}^{\infty} r_n^2 \sum_1^{\infty} \frac{1}{m^2} < A,$$

~~S~~since $\sum r_n^2$ converges, A, being a constant dependent only on the r's.

Divide the z-plane into a network of congruent 'axes-parallel' rectangles R, and let

$P' = \sum' P_R^{(N,M)}$, where the dash denotes summation over all the rectangle R, possessing at least one point common to the circle $|z| \leq \sqrt{2A}$, and $P'' = \sum'' P_R^{(N,M)}$, the summation being carried over all the remaining rectangles and, of course, is the measure of the set Ω'' of points $(\eta_{N+1} \cdots \eta_M)$ lying in the unit-cube Q $(0 \leq \eta_n < 1,\ n = N+1 \cdots M)$ for which $S_{N,M}$ lies in any rectangle R having no point in common with the above circle.

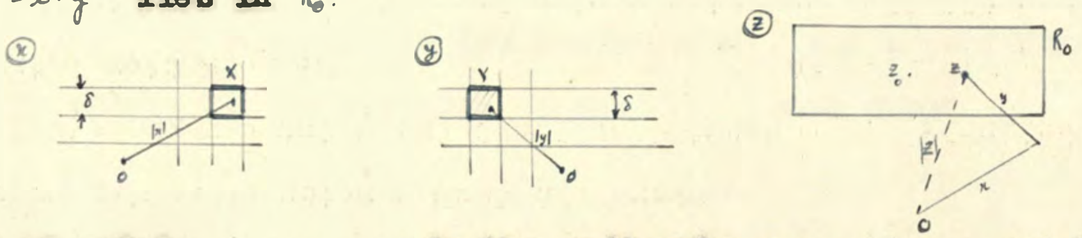
We have

$$A > \int_Q |S_{N,M}|^2 d\eta_{N+1} \cdots d\eta_M \geq \int_{\Omega''} |S_{N,M}|^2 d\eta_{N+1} \cdots d\eta_M \geq 2A \int_{\Omega''} d\eta_{N+1} \cdots d\eta_M = 2A P''.$$

2.11 Hence $P'' < \frac{1}{2}$ so that $P' > \frac{1}{2}$, — since $P' + P''$ is the contents of unit-cube. Now let $N_0 = N_0(R_0, z_0)$ be so chosen that the set $\sum_{N_0}$ lies inside the circle $|z| \leq \sqrt{2A} + B$, where A is defined above, and B is the greatest distance of a point on the edge of the rectangle R_0 (centre z_0) from the origin. Let $N_1 > N_0$. Represent on three planes the complex variables

$$x = S_{N_0} \; ; \quad y = S_{N_0, N} \; ; \quad z = S_N = x + y .$$

Divide the x and y planes into a network of congruent squares of side δ, where δ is any fixed number less than one third of the smallest side of R_0. It is clear that to each square Y in the y-plane then corresponds at least one square X in the x-plane, such that, if y be arbitrarily chosen in Y and x arbitrarily chosen in X then $z = x + y$ lies in R_0.



The set of points $(\eta_1 \cdots \eta_{N_0})$ in the unit-cube $0 \le \eta_n < 1$ $(n = 1 \cdots N_0)$ for which S_{N_0} lies in the square of X is measurable; let its measure be $P_X^{(N_0)}$. The set of points $(\eta_{N_0+1} \cdots \eta_N)$ in the unit-cube $0 \le \eta_n < 1$ $(n = N_0+1, \cdots N)$ for which $S_{N_0,N}$ lies in the square Y is measurable; let its measure be $P_Y^{(N_0,N)}$. $\therefore$ The set of points $(\eta_1 \cdots \eta_N)$ in the unit-cube $0 \le \eta_n < 1$ $(n = 1 \cdots N)$ for which S_{N_0} and $S_{N_0,N}$ simultaneously lie in X and Y respectively, is measurable and has measure $P_X^{(N_0)} \cdot P_Y^{(N_0,N)}$.

Hence

2.12 $$P_{R_0}^{(N)}(z_0) \ge \sum P_X^{(N_0)} \cdot P_Y^{(N_0,N)}$$

where the summation is taken over all the squares Y (and of course, the corresponding set of squares X).

Since $P > 0$ this inequality persists if the sum is taken over all the squares Y which have at least one point in common with the circle $|y| \le \sqrt{2A}$. For such squares, Y, the corresponding squares X must lie inside the fixed circle $|x| \le \sqrt{2A} + B$ (since $z = x + y$ lies inside R_0). Owing to our choice of N_0 there will be a finite number of squares X lying in the circle $|x| \le \sqrt{2A} + B$ each possessing at least one point of Σ_{N_0} inside it. Hence by (III) page 12 each of the corresponding probabilities $P_X^{(N_0)}$ is positive, so that a positive k exists, independent of N, but dependent on N_0 ~~i.e., on~~ i.e. on R_0 and z_0, such that for all the squares Y of this type

$$P_X^{(N_0)} > 2k$$

so that from 2.12 $P^{(N)}_{R_0}(z_0) > 2k \sum_Y P_Y^{(N_0,N)}$

where the summation is taken over all squares Y which contain at least one point common to the circle $|y| \leq \sqrt{2A}$. But by 2.11, this sum is $> \frac{1}{2}$ hence $P^{(N)}_{R_0}(z_0) > k$; and the theorem is proved.

We are now in a position to prove the following theorem relative to the Zeta-function.

THEOREM 5. Let σ_0 be a number in the interval $\frac{1}{2} < \sigma_0 \leq 1$. Then the values of $\zeta(s)$ on the line $\sigma = \sigma_0$ lie everywhere dense in complex ζ-plane.

That is to say, for a given complex number 'a' $\neq 0$ and a positive ϵ, arbitrarily chosen, it is possible to find t_0 such that

$$|\zeta(\sigma_0 + it_0) - a| < \epsilon, \qquad t_0 = t_0(\sigma_0, a, \epsilon).$$

Choose a positive $\epsilon_1 = \epsilon_1(\epsilon, a)$ so that, from the inequalities $|x-a| < \epsilon_1$, $|y-1| < \epsilon_1$ follows $|xy - a| < \epsilon$. Now determine $\epsilon_2 > 0$ so that the inequality $|\log x - \log a| < \epsilon_2$ is consistent with $|x-a| < \epsilon_1$.

Associate $\prod_1^N (1 - p_n^{-s})^{-1}$ and $R_N(s)$ for x and y respectively, and we see that our theorem will be true if we can determine N and t_0 so that

$$\begin{cases} \left|-\log \prod_1^N \left(1 - p_n^{-\sigma_0 - it_0}\right) - \log a\right| < \epsilon_2 & a \neq 0 \\ \\ |R_N(\sigma_0 + it_0) - 1| < \epsilon_1 & \end{cases}$$

2.13

Write $p_n^{-\sigma_0} = r_n$ and $\lambda_n = -\log p_n / 2\pi$ so that

$$S_N(\sigma_0 + it) = \sum_1^N \log(1 - p_n^{-\sigma_0 - it}) = \sum_1^N \log(1 - r_n e^{2\pi i t \lambda_n})$$

$\therefore$ To prove the first of 2.13 is equivalent to proving that N and t_0 can be determined so that

$$|S_N(\sigma_0 + it_0) + \log a| < \epsilon_2 \qquad (a \neq 0)$$

Let R_0 be a fixed rectangle in the z-plane, oriented parallel to the axes, and lying wholly in the circle $|z + \log a| < \epsilon_2$.

Since $\frac{1}{2} < \sigma_0 \leq 1$ and $r_n = p_n^{-\sigma_0}$ the series $\sum r_n$ diverges and $\sum r_n^2$ converges. Let $S_N(\eta_1 \cdots \eta_N) = \sum_1^N \log(1 - r_n e^{2\pi i \eta_n})$ then Theorem 4 indicates the existence of a positive number $k = k(R_0, r_1, r_2 \cdots) = k(a, \epsilon, \sigma_0)$ and an integer $N_0 = N_0(a, \epsilon, \sigma_0)$ such that for all $N > N_0$ the probability, $P_{R_0}^{(N)}$, that the point $S_N(\eta_1, \cdots \eta_N)$ lies in R_0 is greater than k, i.e.,

2.14 $$P_{R_0}^{(N)} > k \qquad (N > N_0)$$

The necessary and sufficient condition that the point $S_N(\sigma_0 + it)$ belongs to the rectangle R_0 is that the point obtained from the point $(t\lambda_1, \cdots t\lambda_N)$ by reduction of its co-ordinates to Mod.1, belongs to the set Ω, whose measure is the above $P_{R_0}^{(N)}$.

Since $\lambda_n = -\log p_n / 2\pi$ the set $\lambda_1 \cdots \lambda_N$ are linearly independent and we may use Theorem 2.

Let $M = M(T) = M^{(N)}(T)$ be the set of values of t between 0 and T for which the point $(t\lambda_1 \cdots \cdots t\lambda_N)$ by reduction of its co-ordinates to mod.1 belongs to Ω. Then $M(T)$ is equally the set of values of t between 0 and T for which the point $S_N(\sigma_0 + it)$ falls in R_0 so that, $M(T)$ consists of a finite number of intervals, is measurable, and its measure $m(T) = m^{(N)}(T)$ is the sum of the lengths of these intervals. Consequently by Theorem 2

$$\lim_{T\to\infty} \frac{m^{(N)}(T)}{T} = P_{R_0}^{(N)}$$

so that, associating $S_N(\sigma_0 + it)$ with $S_N(\eta_1, \cdots \eta_N)$ and using 2.14

2.15 $$\lim_{T\to\infty} \frac{m^{(N)}(T)}{T} > k \qquad (N > N_0)$$

Now consider $R_N(\sigma_0 + it)$. Applying Theorem 3 with $\epsilon = \epsilon_1$ and $\delta = \frac{1}{2}k$ we have, the sum of the lengths, $\mu(T)$, of the finite number of intervals $L(T)$ of values of t between $-T$ and $+T$ for which $|R_N(\sigma_0 + it) - 1| \geq \epsilon_1$ is smaller than $\frac{1}{2}kT$, for T sufficiently large.

$\therefore$ The content $\mu^*(T)$ of the subset $L^*(T)$ of points in $L(T)$ for which $t > 0$ is certainly smaller than $\frac{1}{2}kT$

Choose $T > T_0$ so that $\frac{m^{(N)}(T)}{T} > \frac{1}{2}k$ which is possible by 2.15. For these values of T and N_0 determined, we have simultaneously

$$m^{(N)}(T) > \tfrac{1}{2}kT \qquad \text{and} \qquad \mu^*(T) < \tfrac{1}{2}kT$$

Hence, there is at least on value of t, say t_0, in the interval $(0, T)$

belonging to the set $M^{(N)}(T)$ and <u>not to the</u> set $L^{*}(T)$ so that

$$|R_N(\sigma_0 + it_0) - 1| < \epsilon_1$$

and $$|S_N(\sigma_0 + it_0) + \log a| < \epsilon_2 ,$$ by definition of the rectangle R.

Consequently it is possible to find t_0 such that

$$|\zeta(\sigma_0 + it_0) - a| < \epsilon \qquad a \neq 0.$$

This theorem is valid for values of $\zeta(\sigma + it)$ on the fixed line $\sigma = \sigma_0$; if we could prove that the same theorem holds in the neighbourhood of the line $\sigma = \sigma_0$ then we should obtain immediately, the following result given by Bohr ('<u>Comptes Rendes</u>' 158. 1914) :-

> The function $\zeta(s)$ take every value, except zero, infinitely often in the strip $\frac{1}{2} < \sigma < 1$.

This theorem had already been established by Bohr and Landau in 1913 ('<u>Math.Ann</u>' 74, 1913) assuming the truth of Riemann's hypothesis, using analytic methods based on an extension of the Picard theorem concerning integral functions.

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3. THE ORDER OF [redacted] $\zeta(s)$ ON A VERTICAL LINE

If it is possible to determine a finite number $\mu = \mu(\sigma)$ such that for ϵ positive and arbitrarily small, and $\sigma = \sigma_1$

$$f(\sigma+it) = O\left(|t|^{\mu+\epsilon}\right), \quad f(\sigma+it) \neq O\left(|t|^{\mu-\epsilon}\right),$$

then $f(\sigma+it)$ is said to be of finite order for $\sigma = \sigma_1$.
It is only necessary to consider the discussion of convergency of Dirichlet series and the contour integration of functions involving such series, to realise that the determination of the order function associated with a given Dirichlet series is a problem of the greatest importance.

Properties of this order function, $\mu(\sigma)$, and problems of an allied nature were first investigated by Lindelöf and Phragmén ('Acta Math.' 31. 1908) and the following beautiful theorem is of fundamental importance.

THE PHRAGMÉN-LINDELÖF THEOREM (H. p.849)

If $f(s)$ is regular and of finite order for $\sigma_1 \le \sigma \le \sigma_2$, $f(s) = O\left(|t|^{\alpha}\right)$ for $\sigma = \sigma_1$ and $O\left(|t|^{\beta}\right)$ for $\sigma = \sigma_2$, then $f(s) = O\left(|t|^{\nu(\sigma)}\right)$ uniformly for $\sigma_1 \le \sigma \le \sigma_2$, $\nu(\sigma)$ being the linear function which assumes the values α and β for $\sigma = \sigma_1$ and $\sigma = \sigma_2$. In particular, if $f(s)$ is of finite order for $\sigma_1 \le \sigma \le \sigma_2$ and bounded on the lines $\sigma = \sigma_1$, $\sigma = \sigma_2$, then it is bounded in the whole strip between them.
In the case of the Zeta-function, it is trivial that

$$\mu(\sigma) = 0 \qquad \text{for } \sigma \ge 1$$

and equally trivial, using the functional equation, that

$$\mu(\sigma) = \tfrac{1}{2} - \sigma \qquad \text{for } \sigma \le 0$$

so that, by the Phragmen-Lindelöf theorem

3.1 $$\mu(\sigma) \le \nu(\sigma) = \tfrac{1}{2} - \sigma \qquad \text{for } 0 \le \sigma \le 1.$$

Mellin in 1900, and Landau in 1905 had successively given $\mu(\sigma) \le 1-\sigma$ and $\mu(\sigma) \le \tfrac{3}{4}(1-\sigma)$ so that the above result due to Lindelöf ('Bull. Sc. Math.' (2) 32 I 1908) is certainly a big improvement, although it is doubtless very crude - in fact, the actual determination of the order function associated with a given Dirichlet series is a problem of the greatest difficulty.

Another theorem which will be of use in the sequel concerns the maximum value of the modulus of an analytic function. Let $f(s)$ be regular for $|s| \leq \rho$, and let $M(r)$ denote the maximum value of $|f(s)|$ on the circle $|s| = r < \rho$, then it is known that $M(r)$ is a steadily increasing function of r and is also a convex function of $\log r$. This last result - which was given independently by Blumenthal ('Jahresbericht' V.16. p.97), Faber ('Math.Ann.' 63, p.549) and Hadamard ('Bull.de la Soc.Math.de France' 24 (1896) - usually known as HADAMARD'S THREE-CIRCLE THEOREM-may be written:-

$$\log M(r) \leq \frac{\log (r_2/r)}{\log (r_2/r_1)} \log M(r_1) + \frac{\log (r/r_1)}{\log (r_2/r_1)} \log M(r_2).$$

if $0 < r_1 \leq r \leq r_2 < \rho$.

It is apparent that of particular interest is the study of the order of $\zeta(s)$ on this line $\sigma = 1$: we proceed to discuss this case first.

Working from the trivial result $\rho(1) = 1$ Mellin (1900) first gave $\zeta(1+it) = O(\log t)$ - we need only consider positive values of t - and this result was sharpened by Hardy & Littlewood, in 1912, who obtained

$$\zeta(1+it) = o(\log t).$$

We are thus led to seek for a finer result, i.e. to seek for a function which increases slower than $\log t$ but which approximates to the growth of $\zeta(1+it)$. We will prove:-

THEOREM 6 $\quad \zeta(1+it) \neq o(\log\log t)$.

Consider the function

$$f(s) = \frac{\zeta(s)}{\log\log s} \cdot \frac{s}{s+\gamma},$$

where γ is a positive constant, defined in the region $t \geq t_0 = e^e$, $1 \leq \sigma \leq 2$. Assume theorem to be false so that $|\zeta(s)| < c \log\log t$ (where c is to be a Borel constant) on the line $\sigma = 1$ $(t > t_0)$, giving, since $|s| < |s+\gamma|$ and $|\log\log s| > \log\log t$

$$|f(s)| < c \qquad \text{for} \qquad \sigma = 1,\ t > t_0$$

It is trivial that

$$|f(s)| < c \qquad \text{for } \sigma = 2, \ t > t_0$$

In the region in which $f(s)$ is defined, $\zeta(s) = O(t)$ hence

$$f(s) = O\left\{\frac{t}{\log\log t} \cdot \frac{t}{t}\right\} = O(t) \qquad \{1 \leq \sigma \leq 2, \ t > t_0\}$$

These three results, using the Phragmén-Lindelöf Theorem, shew that $f(s)$ is bounded in the region in which it is defined. Hence

$$|\zeta(s)| \leq c \left|\frac{s+r}{s}\right| |\log\log s| \qquad (1 \leq \sigma \leq 2, \ t > t_0)$$

which, in virtue of the obvious fact $\lim\limits_{t\to\infty} |\log\log s| / \log\log t = 1$, gives

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(s)|}{\log\log t} \leq c \qquad \text{for } 1 \leq \sigma \leq 2, \ t > t_0 .$$

This result directly contradicts Theorem 1, so that we must have

$$\zeta(1+it) \geq \kappa \log\log t, \qquad \kappa \text{ being the constant of Th. 1}$$

i.e. $\zeta(1+it)$ is sometimes of order as great as $\log\log t$.

Littlewood (Proc. Lond. Math. Soc.' 1921) has recently considerably refined this result, giving, but without proof

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} \geq e^{\gamma} = 1.78\cdots$$

whilst, if Riemann's hypothesis is true,

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} \leq 2e^{\gamma},$$

where γ is Euler's constant.

It is seen from this work that the amplitude of the oscillatory function $|\zeta(1+it)|$ tends to infinity with extreme slowness as t increases indefinitely. From numerical results obtained by a method to be explained later (see 7.0 p.58) it appears that the amplitude of the function $|\zeta(1+it)|/\log\log t$ fairly quickly becomes definite and then converges slowly to a definite value certainly in the neighbourhood of 2. For example, a crest of the curve $|\zeta(1+it)|$ occurs at the point $t = 35.5$ at which point $|\zeta(1+it)| = 1.97$ and the ratio $\frac{|\zeta(1+it)|}{\log\log t} = 1.55$.

It seems highly probable that the truth is expressed by

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} = e^{\gamma}.$$

Another result of a deep character has recently been obtained by Wehl ('Math. Zeits.' X 1921) viz.,

3.2 $$\zeta(1+it) = O\left(\frac{\log t}{\log\log t}\right)$$

but we omit the proof, which is long and intricate and does not appear to contain any new method which may be of use in further developments of the subject.

We now obtain results by application of Hadamard's theorem - a step first taken by Littlewood in 1912 ('Comptes Rendus' 154).

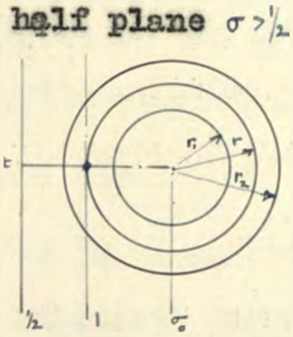
THEOREM 7 If Riemann's hypothesis is true, then

$$\zeta(1+it) = O(\log\log t . \log\log\log t)$$

and

$$\frac{1}{\zeta(1+it)} = O(\log\log t . \log\log\log t)$$

If Riemann's hypothesis is true, then log $\zeta(s)$ is analytic in the half plane $\sigma > \frac{1}{2}$ except at the point $s = 1$, so that Hadamard's theorem is applicable to the function $f(s) = \log \zeta(s)$ taking



$r_2 = \rho < 1$, $r = \frac{\rho}{2}$, $r_1 = \frac{\rho}{2} - \frac{1}{\log_n t . \log_m t}$

where $\log_n t$ denotes the n^{th} logarithm of t.

Choose $t > t_0$ so that $\frac{1}{\log_n t \log_m t} < \frac{\rho}{4}$, then $\rho/2 > r_1 > \rho/4$.

It is trivial that

3.31 $$M(r_2) = O(\log t).$$

If $\sigma > 1$ then $\log \zeta(s) < \log\left(1 + \int_1^\infty u^{-s}\,ds\right) < \log\frac{1}{s-1} + 1$

Choose $\sigma_0 = 1 + \frac{1}{2}r$, then

3.32 $$M(r_1) < \log \zeta\left(1 + \frac{1}{\log_n t \log_m t}\right) < \log_{n+1} t + \log_{m+1} t + 1$$

It is evident that

3.33 $$\frac{\log r/r_1}{\log r_2/r_1} < c \log\frac{r}{r_1} < c\,\frac{r-r_1}{r_1} < \frac{c}{\rho \log_n t \log_m t}, \quad \text{and} \quad \frac{\log r_2/r}{\log r_2/r_1} < 1,$$

c being a Borel constant.

Substituting 3.31, 3.32 and 3.33 into Hadamard's theorem, we obtain

$$M(r) < (\log t)^{\frac{c}{\rho \log_n t \log_m t}} . (\log_{n+1} t + \log_{m+1} t + 1)$$

Hence, taking $n=2$,

$$M(r) < e^{c\frac{\log\log t}{\log_n t \log_m t}}\left(\log_{n+1} t + \log_{m+1} t + 1\right) = e^{\frac{c}{\log_m t}}\left(\log_3 t + \log_{m+1} t + 1\right)$$

$$< \left(\log_3 t + \log_{m+1} t + 1\right)\left(1 + \frac{c}{\log_m t}\right).$$

Finally, taking $m=3$ and t sufficiently large

$$M(r) < \log_3 t + \log_4 t + c$$

$\therefore$ Since $\log|\zeta(1+it)| = |\Re \log\zeta(1+it)| \leq |f(1)| < M(r)$ we have

$$-\log_3 t - \log_4 t - c < \log|\zeta(1+it)| < \log_3 t + \log_4 t + c$$

giving the theorem in question.

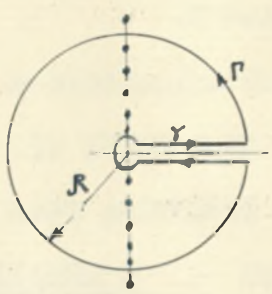
Hadamard and de la Vallée Poussin independently proved that no zeros lie on the line $\sigma=1$ (see H. p.321) so that, since the circle r_2 may be drawn arbitrarily near to the line $\sigma=1$, this theorem really assumes much less than the truth of Riemann's hypothesis.

We now proceed with a discussion of $\zeta(s)$ for values of s lying in the critical strip. For values of s lying outside the critical strip, we have the reprsentations

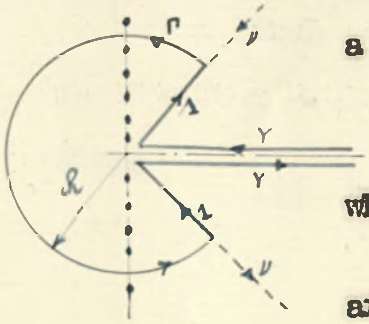
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\sigma>1), \qquad \zeta(s) = 2(2\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1} \quad (\sigma<0),$$

and it seems natural to enquire whether it is possible to obtain any compromise between these two results which will persist throughout the critical strip.

Riemann's method of obtaining the classical functional equation was to integrate the function $\frac{(-z)^{s-1}}{e^z-1}$ around the contour shewn in the diagram, and then make $R\to\infty$. The integral around Γ vanishes



whilst the integral along the straight lines γ tends to the value given by the integral in 1.3. Inside the contour we have simple poles at $z=2\pi n i$ $(n=\pm1,\pm2,\cdots)$ and the residues at these poles contribute the sum $\sum n^{s-1}$. If now we give R a fixed value, say sufficient to introduce x terms of the series $\sum n^{s-1}$, and take the contour as shewn in second diagram, it would seem possible to obtain



a representation of the form

$$\zeta(s) = \sum_1^{\infty} n^{-s} = \chi\sum_{n<x} n^{s-1} + R$$

where $\chi = 2(2\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)$,

and R represents an order term arising from the integral taken around Γ and the lines Λ.

The form of this result suggest that it may be possible to take, say, the first y terms of the series $\sum n^{-s}$ and absorb the remainder in the order term R, choosing y suitably, (e.g. depending on x) so that the error term is of the best possible form.

In order to effect this we could write

$$\frac{(-z)^{s-1}}{e^{z}-1} = (-z)^{s-1}\sum_{n\leq y} e^{-nz} + \frac{e^{-(y+1)z}(-z)^{s-1}}{e^{z}-1}$$

and integrate the first term along the path $\gamma\Lambda\Gamma$ and the second along $\nu\Gamma\nu$ which, of course, is possible and evidently preferable. The first of these steps would give

$$\sum_{n\leq y} n^{-s} + \sum_{n\leq y}\left\{\int_{\Gamma} + \int_{\Lambda}\right\}(-z)^{s-1}e^{-nz}\,dz$$

in which occurs the integral $\int_{\Lambda}$, which does not converge in the ordinary sense if $\sigma\leq 0$. This last difficulty is easily overcome by the introduction of Hadamard's notion of a 'generalised integral' ('Acta Math.' 31. 1908 p.339)

The second step will involve no special difficulty if we make suitable choice of y, taking in fact

$$2\pi xy = |t|.$$

and a simple contribution to the error term is obtained.

Working in this way, we see that it is possible to arrive at a result of the desired nature, and which has been called by Hardy and Littlewood, the APPROXIMATE FUNCTIONAL EQUATION.

Investigations of this nature have been made by these mathematicians in two recent papers ('Math. Zeits.' X; 'Proc.Lond.Math.Soc.' 21. 1922) and we content ourselves with the full statement of their most polished result.

THEOREM 8 Suppose that H and K are two positive constants, and that $s=\sigma+it$ with $|\sigma|\leq H$.

Let $\chi = 2(2\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)$, and $2\pi xy = |t|$

Then

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi\sum_{n<y} n^{s-1} + R$$

Where

(I) $R = O(x^{-\sigma}) + O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma})$ if $x>K$, $y>K$.

(II) $R = O\left(\dfrac{x^{1-\sigma}}{|t|^{1/2} + x\,\delta(y)}\right)$ if $x > k,\ y > K$ and $x \geq y$

(III) $R = O\left(\dfrac{|t|^{1/2-\sigma}\,y^{\sigma}}{|t|^{1/2} + y\,\delta(x)}\right)$ if $x > K,\ y > K$ and $y \geq x$

Where $\delta(z)$ denotes the absolute value of the difference between z and the integer nearest z, and the constants implied by the O's depend only on H and K.

In the first of the previously mentioned memoirs, the investigations were of a rather clumsy nature, being based upon the equality

$$\zeta(s) = \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + 2(2\pi)^{s-1} \sum_{n=1}^{\infty} n^{s-1} \int_{2\pi n x}^{\infty} u^{-s} \cos u \, du$$

— valid when $\sigma > 0$, $s \neq 1$, and $x > 0$, which, if integral, necessitates the replacement of the last term in the first sum by $\frac{1}{2} x^{-s}$ — and a discussion of $\int_{2\pi n x}^{\infty} u^{-s} \cos u \, du$ for different values of n.

The result obtained was a crude form of Theorem 8 (I), being valid only for a fixed σ lying between 0 and 1 and, further, containing an unnecessary factor log $|t|$ in the two order terms of R.

The second memoir contrasts remarkably with the first - at least as far as the approximate functional equation is concerned - and Theorem 8 is deduced by extremely rigorous analysis from the obvious equality,

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{1}{2i}\int_{L_1} (\cot \pi z - i) z^{-s} dz - \frac{1}{2i}\int_{L_2} (\cot \pi z - i) z^{-s} dz - \frac{x^{1-s}}{1-s}$$

where x is not an integer, and L_1 and L_2 are the straight lines drawn from the point x on the real axis, and at angles $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$ to direction of real axis. Although this genesis of Theorem 8 is apparently different from that sketched above, the analysis is of exactly the same nature.

Suppose $0 \leq \sigma \leq H$ and always regard t as being positive them by Stirling's theorem,

$$\chi = \left(\frac{t}{2\pi e}\right)^{-it} \left(\frac{t}{2\pi}\right)^{1/2-\sigma} e^{\frac{1}{4}\pi i} \left\{1 + O(1/t)\right\}$$

so that, since

$$O\left(t^{-1/2-\sigma} \sum_{n<y} n^{\sigma-1}\right) = O\left(t^{-1/2-\sigma} y^{\sigma} \log y\right) = O\left(x^{-\sigma}\right)$$

theorem 8 gives:-

THEOREM 9 If $2\pi xy = t$, x and y both positive, then

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi' \sum_{n<y} n^{s-1} + O(x^{-\sigma}) + O(t^{1/2-\sigma} y^{\sigma-1})$$

where

$$\chi' = \left(\frac{t}{2\pi e}\right)^{it} \left(\frac{t}{2\pi}\right)^{1/2-\sigma}$$

uniformly for $0 \leq \sigma \leq H$, $t > 0$

Putting $x = y = \sqrt{\frac{t}{2\pi}}$, and using $\sum_{n<x} n^{-1/2-it} = O(x^{1/2})$
it is easy to find

3.4 $$\zeta(1/2 + it) = O(t^{1/4})$$

Let us now consider the order of $\zeta(s)$ on a vertical line lying in the critical strip. Lindelöf's result, 3.1, remains the best in existence, although Wehl's result 3.2 may be used to effect a minor improvement in the theorem

$$\zeta(s) = O\{t^{1/2(1-\sigma)} \log t\} \qquad \text{uniformly for } 0 \leq \sigma \leq 1$$

given in H § 240.

THEOREM 10 If Riemann's hypothesis is true, then

$$\log \zeta(\sigma + it) = O\{(\log t)^{2(1-\sigma)+\epsilon}\}$$

uniformly for $1/2 + \delta \leq \sigma \leq 1$; ϵ and δ being two arbitrarily small, positive and independent numbers.

Choose δ and take $r_1 = \sigma_0 - (1 + 1/2\delta)$, $r_3 = \sigma_0 - 1/2(1+\delta)$, $r = \sigma_0 - \sigma$, $f(s) = \log \zeta(s)$ (which is analytic inside these circles if Riemann's hypothesis is true and t sufficiently large) in Hadamard's theorem. Known theorems give $M(r_1) = O(1)$, $M(r_2) = O(\log t)$
Choosing $\sigma_0 = 1/2 t$, so that the pole $s=1$ is avoided,

$$\lim_{t\to\infty} \frac{\log r/r_1}{\log r_2/r_1} = 2(1-\sigma), \qquad \text{hence by Hadamard,}$$

$$M(r) = O\{(\log t)^{2(1-\sigma)+\epsilon}\}$$

and the theorem follows as in Theorem 4

THEOREM 11. If Riemann's hypothesis is true, then a constant a exists such that

$$\log \zeta(\sigma + it) \neq O\{(\log t)^{a(1-\sigma)}\} \qquad (1/2 < \sigma < 1)$$

We use the following corollary to Hadamard's theorem - using same notation. Let $r_2 < 2r_1$, then for each value $M^{x} \geq 1$, $M^{x} \geq M(r_1)$, and $M^{x} \leq M(r)$ we have

3.5
$$M(r_2) \geq \left\{ \frac{M(r)}{M^{x}} \right\}^{\frac{1}{2}\cdot\frac{r_2-r_1}{r-r_1}}$$

For, since $M^{x} \geq M(r_1)$ we have, from Hadamard's theorem,

$$\{M(r)\}^{\log r_2/r_1} \leq \{M^{x}\}^{\log r_2/r}\{M(r_2)\}^{\log r/r_1}$$

$\therefore$, Since, $M^{x} \geq 1$, $\quad M(r_2) \geq \left\{\frac{M(r)}{M^{x}}\right\}^{\log r_2/r_1 : \log r/r_1}$

Since $r_2 < 2r_1$, $\frac{r_2-r_1}{r_1} < 1$ consequently $\log r_2/r_1 > \frac{1}{2}\cdot\frac{r_2-r_1}{r_1}$

Also $\log\frac{r}{r_1} = \log\left(1+\frac{r-r_1}{r_1}\right) < \frac{r-r_1}{r_1}$

so that corollary is true if $M^{x} < M(r)$.

Consider the function $f(s) = \frac{\zeta'(s)}{\zeta(s)}$

From Theorem 1, it is an obvious deduction that a constant c_1 exists such that

3.51
$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| > c_1 \log\log t \qquad \text{for } \sigma > 1, \quad t > t_0$$

The function $\frac{\zeta'}{\zeta}(s)$ possesses a pole of the first order at $s=1$, and it is possible to find a constant c_2 such that

3.52
$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| < \frac{c_2}{\sigma-1} \qquad \text{for } \sigma > 1$$

Write $c_3 = c_2/c_1$ and apply 3.5 with $f(s) = \frac{\zeta'(s)}{\zeta(s)}$, taking

$$r_1 = \sigma_0 - \left(1+\frac{2c_3}{\log\log t}\right), \quad r = \sigma_0 - \left(1+\frac{c_3}{\log\log t}\right), \quad r_2 = \sigma_0 - \sigma \quad (\tfrac{1}{2} < \sigma < 1)$$

where $s_0 = \sigma_0 + it$ ~~is~~ is centre of these three circles, $(\sigma_0 > 1)$

Take $M^{x} = \frac{1}{2} c_1 \log\log t$

For t sufficiently large we have

(I) $0 < r_1 < r < r_2 < 2r_1$ and $M^{x} > 1$; (II) Choosing σ_0 suitably, say $= \frac{1}{2}t$, the pole at $s=1$ is avoided and, assuming truth of Riemann's hypothesis, $\frac{\zeta'(s)}{\zeta(s)}$ is analytic inside the three circles; (III) in, and on the inner circle each point has abscissae $\geq 1 + \frac{2c_3}{\log\log t}$ and consequently by 3.52

$$M(r_1) \leq \tfrac{1}{2} c_1 \log\log t = M^{x}.$$

(IV) every point on the circle $|s-s_0| = r$ has abscissae > 1, hence, by 3.51

$$M(r) > c_1 \log\log t = 2M^{x} > M^{x}$$

These four arguments justify the application of 3.5, which gives, since $M_2/M^x > 2$, $\frac{r_2 - r_1}{r - r_1} > \frac{1-\sigma}{2c_3}$,

$$M(r_2) > 2^{(1-\sigma)\log\log t/4c_3} > (\log t)^{a(1-\sigma)}$$

Hence for $\frac{1}{2} < \sigma < 1$ we have

3.53 $$\left|\frac{\zeta'(s)}{\zeta(s)}\right| > (\log t)^{a(1-\sigma)} \qquad t > t_0.$$

By Cauchy's theorem $\left|\frac{\zeta'}{\zeta}(s)\right| \leq \frac{\text{Max. } |\log\zeta(s)| \text{ on } |s - s_0| = r}{r}$

hence, by taking r sufficiently small, we have using 3.53

$$|\log\zeta(s)| > r(\log t)^{a(1-\sigma)}$$

which, of course, proves theorem in question.

If we define a function $\nu(\sigma)$ for $\frac{1}{2} < \sigma < 1$ so that

$$\log\zeta(s) = O\{(\log t)^{\nu(\sigma)+\epsilon}\}, \quad \log\zeta(s) \neq O\{(\log t)^{\nu(\sigma)-\epsilon}\}$$

where ϵ is positive and arbitrarily small, then, if Riemann's hypothesis is true, Theorems 10 and 11 give

$$a(1-\sigma) \leq \nu(\sigma) \leq 2(1-\sigma) \qquad (\tfrac{1}{2} < \sigma < 1)$$

Using Hadamard's theorem it is possible to shew that $\nu(\sigma)$ is a convex function.

A more precise form of Theorem 10 may be obtained by arguments of almost the same nature as those used in the proof of Theorem 7, and was given by Littlewood in 1912.
A proof may be constructed on the lines given in the proof of Theorem 7.

THEOREM 12 If Riemann's hypothesis is true, then

$$\zeta(s) = O\left\{\left(\frac{\log t . \log\log t}{\log\log\log t}\right)^{2(1-\sigma)} \log\log\log t\right\}$$

uniformly for $\frac{1}{2} + \frac{\delta}{\log\log t} \leq \sigma \leq 1$, δ being any positive number.
If we put $\epsilon = \delta$ into Theorem 10, a trivial reduction gives

3.6 $$\zeta(s) = O(t^{\epsilon}) \quad \text{for } \sigma > \tfrac{1}{2}$$

and consequently, by the [redacted] functional equation,

3.61 $$\zeta(s) = O\left(t^{\frac{1}{2}-\sigma+\epsilon}\right) \quad \text{for} \quad \sigma < \tfrac{1}{2},$$

a fact which had been conjectured by Lindelöf in 1908. It follows from continuity that

3.62 $$\zeta(\tfrac{1}{2}+it) = O(t^{\epsilon})$$

and moreover, it can easily be shewn, by use of the Phragmén-Lindelöf theorem (see e.g. G.H.Hardy - M.Riesz, Cambridge Math.Tract No.18, 1918, Theorem 15), that 3.62 implies 3.6. Hence LINDELÖF'S HYPOTHESIS is most simply expressed by the assertion

$$\zeta(\tfrac{1}{2}+it) = O(t^{\epsilon})$$

for every positive ϵ.

We have seen from Theorem 10 that :-

THEOREM 13 If Riemann's hypothesis is true, so is Lindelöf's; hence any deduction from the latter is, a fortiori, valid as a deduction from the former.

4. MEAN VALUE THEOREMS

We have already remarked that the hypothesis of Lindelöf, which is most simply expressed by the assertion that

4.11 $$|\zeta(\tfrac{1}{2}+it)| = O(|t|^{\epsilon})$$

For every positive ϵ, is a consequence of the more drastic assertion of Riemann.

If 4.11 is true, then it is trivial that

4.12 $$\frac{1}{T}\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = O(T^{\epsilon})$$

for every positive integer k, but it is not evident that the converse holds. If 4.12 is true, then

4.13 $$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt < aT^{b_1}$$

where a depends on k, and b_1, and all b's which follow, are absolute constants

If 4.11 is false, then a set of number t_n can be found, such that t_n tends to infinity with n and

4.14 $$|\zeta(\tfrac{1}{2}+it_n)| > b_2 t_n^{b_3}$$

Now $$|\zeta(\tfrac{1}{2}+it) - \zeta(\tfrac{1}{2}+it_n)| = \left|\int_{t_n}^{t} \zeta'(\tfrac{1}{2}+iu)\,du\right|$$

$< 2b_4 |t-t_n|\, t_n^{b_5}$ using a known theorem

$< 2b_4 < \tfrac{1}{2} b_2 t_n^{b_3}$

if $|t-t_n| \leq t_n^{-b_5}$, and n sufficiently large.

Hence, using 4.14

$$|\zeta(\tfrac{1}{2}+it)| > \tfrac{1}{2} b_2 t_n^{b_3} \qquad (|t-t_n| \leq t_n^{-b_5})$$

Take $T = \tfrac{2}{3} t_n$ so that, n being sufficiently large, the interval $(t_n - t_n^{-b_5},\ t_n + t_n^{-b_5})$ lies in $(T, 2T)$, then

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt > \int_{t_n - t_n^{-b_5}}^{t_n + t_n^{-b_5}} (\tfrac{1}{2} b_2 t_n^{b_3})^{2k}\,dt = 2(\tfrac{1}{2}b_2)^{2k}\, t_n^{2kb_3 - b_5}$$

which contradicts 4.13. Therefore:-

THEOREM 14 A necessary and sufficient condition for the truth of Lindelöf's hypothesis is that

$$\frac{1}{T}\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = O(T^{\epsilon}),$$

for every positive integer k, ϵ being positive and arbitrarily small. Other deductions from Lindelöf's hypothesis have been given by Hardy and Littlewood in Proc.Roy.Soc.Lond. June 1923.

A general mean value theorem due to Schnee and Landau (H.p.826) gives, as a particular case,

$$\int_{-T}^{T} |\zeta(\sigma+it)|^2 \, dt \sim 2\zeta(2\sigma)T, \qquad \text{for } \sigma > 1/2,$$

and consequently, using the functional equation

$$\int_{-T}^{T} |\zeta(\sigma+it)|^2 \, dt \sim (2\pi)^{2\sigma-1}\zeta(2-2\sigma)\frac{T^{2-2\sigma}}{2-2\sigma} \qquad \text{for } \sigma < 1/2.$$

To complete this result and, incidentally, to discuss a particular case of 4.13, it is necessary to evaluate

$$\int_{-T}^{T} |\zeta(1/2+it)|^2 \, dt.$$

This evaluation has been performed in two distinct ways by Hardy and Littlewood, the first depended on very intricate analysis of an indirect character ('Acta Math.' 41. 1918), and the second was a direct discussion based on the approximate functional equation, theorem 9. The result is, that

$$\int_{-T}^{T} |\zeta(1/2+it)|^2 \, dt \sim 2T\log T$$

and 4.12 is evidently true for $k=1$.

In a later paper ('Proc.London Math.Soc.' 21 1921 p.39) Hardy and Littlewood have discussed the value of

$$\int_{-T}^{T} |\zeta(\sigma+it)|^4 \, dt$$

and, in particular, have arrived at the theorem:-

THEOREM 15 $$\int_{-T}^{T} |\zeta(1/2+it)|^4 \, dt = O\{T(\log T)^4\}$$

We give the analysis resulting in this theorem; the proof of the previously mentioned result concerning $\int_{-T}^{T} |\zeta(1/2+it)|^2 \, dt$ can be similarly made up.

Taking $x = y = \sqrt{\frac{t}{2\pi}} = \tau$ in theorem 9, it is easy to find

$$|\zeta(1/2+it)|^4 \le A\left|\sum_{n<\tau} n^{-1/2-it}\right|^4 + A\left|\chi'\sum_{n<\tau} n^{-1/2+it}\right|^4 + O(t^{-1})$$

in which A is a Borel constant.

It follows without difficulty that $\chi' = O(1)$, so that it is sufficient for the truth of the above theorem, to prove that

$$I_1 = \int_0^T \Big|\sum_{n<\tau} n^{-1/2-it}\Big|^4 dt = O\{T(\log T)^4\},$$

and

$$I_2 = \int_0^T \Big|\sum_{n<\tau} n^{-1/2+it}\Big|^4 dt = O\{T(\log T)^4\}.$$

We proceed to discuss I_1; I_2 may be similarly investigated. Invert the order of integration and summation, remembering that τ is a function of t. Terms in (l,m,n,p) will occur, if τ is larger than l^2, m^2, n^2 and p^2. Hence if $\mathcal{T} = \sqrt{\frac{T}{2\pi}}$, and $T_1 = T_1(l,m,n,p) = 2\pi \max.(l^2, m^2, n^2, p^2)$ we have

$$I_1 = \sum_{l,m,n,p<\mathcal{T}} (lmnp)^{-1/2} \int_{T_1}^{T} \left(\frac{np}{lm}\right)^{it} dt$$

$$= O\Big\{T\sum_{lm=np}^{\mathcal{T}} (lmnp)^{-1/2}\Big\} + O\Big(\sum_{lm\neq np}^{\mathcal{T}} \frac{(lmnp)^{-1/2}}{\left|\log\frac{np}{lm}\right|}\Big).$$

Let, as usual, $d(\nu)$ represent the number of divisors of ν, then the number of positive integral solutions of $lm=np=\nu$ is not greater than $\{d(\nu)\}^2$. Hence

$$\sum_{lm=np}^{\mathcal{T}} (lmnp)^{-1/2} < \sum_{\nu<AT} \frac{\{d(\nu)\}^2}{\nu} = O\Big\{\log T \sum_{\nu<AT}\{d(\nu)\}^2\Big\},$$ by partial summation

Similarly

$$\sum_{lm\neq np}^{\mathcal{T}} \frac{(lmnp)^{-1/2}}{\left|\log\frac{np}{lm}\right|} \leq \sum_{\substack{\mu,\nu<\mathcal{T}^2\\ \mu\neq\nu}} \frac{d(\mu)\,d(\nu)}{(\mu\nu)^{1/2}\log\left(\frac{\mu}{\nu}\right)} = \Sigma_1 + \Sigma_2$$

where Σ_1 contains terms for which $\left|\frac{\mu}{\nu}-1\right| \leq 1/2$, and Σ_2 the remainder In Σ_2, $\left|\log\frac{\mu}{\nu}\right| > A$ hence

$$\Sigma_2 = O\Big(\sum_{\mu<AT} \frac{d(\mu)}{\mu^{1/2}}\Big)^2 = O\{T(\log T)^4\},$$ by partial summation and use of Dirichlet's theorem $\sum_1^n d(\nu) \sim n\log n$. (H. p.665)

$$\Sigma_1 \leq \sum_{\mu<\mathcal{T}^2}\sum_{0<|r|<\frac12\mu} \frac{d(\mu)\,d(\mu+r)}{\{\mu(\mu+r)\}^{1/2}\left|\log\left(1+\frac{r}{\mu}\right)\right|} = O\Big\{\sum_{\mu<AT}\sum_{0<|r|<\frac12\mu} \frac{d(\mu)\,d(\mu+r)}{\mu}\cdot\frac{\mu}{|r|}\Big\}$$

$$= O\Big\{\sum_{0<|r|<AT}\frac{1}{|r|}\sum_{\mu<AT} d(\mu)\,d(\mu+r)\Big\} = O\Big\{\log T\Big[\sum_1^T\{d(\mu)\}^2\cdot\sum_1^T\{d(\mu+r)\}^2\Big]^{1/2}\Big\}$$

$$= O\Big\{\log T\sum_{n<AT}\{d(n)\}^2\Big\}.$$

Hence collecting results we have

4.4 $$I_1 = O\Big\{\log T\sum_{n<AT}\{d(n)\}^2\Big\} + O\{T(\log T)^4\}.$$

and it remains to discuss

$$\sum_1^n \{d(\nu)\}^2.$$

A general theorem due to Ramanujan ('Messenger of Math.' 45. 1916) gives us a corollary

$$\frac{\{\zeta(s)\}^4}{\zeta(2s)} = \sum_1^\infty \frac{\{d(n)\}^2}{n^s} \qquad (\sigma > 1)$$

which may easily be verified by use of Euler's product.

A slight modification of Theorem 46, p.825 H, shews that, if $a_1\, a_2 \cdots$ be an infinite sequence of numbers, such that $|a_n| = O(n^{\beta-1})$ $\beta > 0$, and $F(s) = \sum_1^\infty a_n n^{-s}$, then

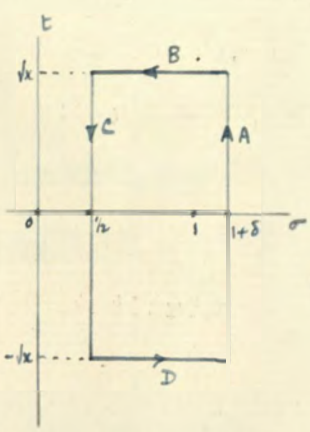
$$\left| \sum_{n<x} a_n - \frac{1}{2\pi i}\int_{\alpha - i\omega}^{\alpha + i\omega} \frac{x^s}{s} F(s)\, ds \right| < \frac{K x^\alpha}{\omega}$$

$\alpha > \beta$, $\omega > 0$, and K is a constant independent of x and ω

Hence

4.5 $$\sum_{n<x} \{d(n)\}^2 = \frac{1}{2\pi i}\int_{1+\delta-i\sqrt{x}}^{1+\delta+i\sqrt{x}} \frac{x^s}{s}\,\frac{\zeta^4(s)}{\zeta(2s)}\, ds + O(x^{\frac{1}{2}+\epsilon})$$

where δ and ϵ are arbitrarily chosen positive quantities.



Taking the contour shewn, we have

$$\frac{1}{2\pi i}\int_A \frac{x^s}{s}\cdot\frac{\zeta^4(s)}{\zeta(2s)}\, ds = \text{Residue at } s=1,\ - \frac{1}{2\pi i}\left\{\int_B + \int_D + \int_C\right\} \frac{x^s}{s}\cdot\frac{\zeta^4(s)}{\zeta(2s)}\, ds.$$

The residue at $s=1$ is coefficient of q^{-1} in expansion of

$$\frac{x^{1+q}}{1+q}\cdot\frac{\zeta^4(1+q)}{\zeta(2+2q)} = \frac{x^{1+q}}{(1+q)\,\zeta(2)}\left\{\frac{1}{q^4} + \frac{c_1}{q^3} + \frac{c_2}{q^2} + \cdots\right\}\left\{1 - 2q\zeta'(2) + \cdots\right\}$$

$$= \frac{x}{\zeta(2)}\left\{\frac{1}{q^4} + \frac{a}{q^3} + \frac{b}{q^2} + \cdots\right\}\left\{1 + q\log x + q^2\log^2 x + \cdots\right\}$$

Hence residue is $O(x\log^3 x)$

It is easy to see that the contributions from the integrals taken around B, C and D are much less important than this term, so we find

4.6 $$\sum_{n<x} d^2(n) = O(x\log^3 x)$$

∴ From 4.4, we have

$$I_1 = O\{T(\log T)^3\cdot\log T\} + O\{T\log^4 T\}$$

$$= O\{T(\log T)^4\}.$$

which proves the theorem.

Using this theorem it is possible to considerably refine the result 4.6 (B.M.Wilson Proc Lond. Math. Soc. 21. 1921) Returning to 4.5 and evaluating the integral along the line $\sigma = \frac{1}{2}$, using this theorem 15, we obtain a contribution of the same order of magnitude as the integrals taken along B and D., i.e. $O(x^{1/2+\epsilon})$ where ϵ is arbitrarily small and positive. Evaluating the residue at $s=1$ more precisely, we find

$$\sum_{n\le x}\{d(n)\}^2 = x\left\{\frac{1}{\pi^2}(\log x)^3 + \left(\frac{12\gamma-3}{\pi^2} - \frac{36\,\zeta'(2)}{\pi^4}\right)(\log x)^2 + C\log x + D\right\} + O(x^{1/2+\epsilon}),$$

where γ is Euler's constant, and C and D are absolute constants.

Thus

$$\sum_{n\le x}\{d(n)\}^2 \sim \frac{x\log^3 x}{\pi^2}$$

This result leads us to suspect that

$$\int_{-T}^{T}\left|\zeta\left(\tfrac12+it\right)\right|^4 dt \sim \frac{T(\log T)^4}{\pi^2}$$

but this result does not follow by the methods of Theorem 15.

A verification of this result has been obtained by Ingham, ('Proc. Lond. Math. Soc.' April 1923) who states that his proof is based upon an "approximate functional equation" for the square of $\zeta(s)$ due to Hardy and Littlewood, but as yet unpublished.

Another mean value theorem, but of a slightly different nature, may be conveniently given here.

THEOREM 16 If $\chi(s) = \sum_{2}^{\infty}\frac{(-)^{n-1}n^{-s}}{\log n}$ then

$$\int_{-T}^{T}\left|\chi\left(\tfrac12+it\right)\right|^2 dt \sim 2T\sum_{n=2}^{\infty}\frac{1}{n(\log n)^2}.$$

Let us write

4.7 $$\chi(s) = \sum_{n\le x}\frac{n^{-s}(-)^{n-1}}{\log n} + \sum_{n>x}\frac{(-)^{n-1}n^{-s}}{\log n}$$

Now

4.71 $$\sum_{n>x}\frac{(-)^{n-1}n^{-s}}{\log n} = \sum_{n>x}\left(\frac{1}{\log n}-\frac{1}{\log n+1}\right)\sum_{x<\nu\le n}(-)^{\nu-1}\nu^{-s} = \sum_{n>x}O\left(\frac{1}{n\log^2 n}\right)\sum_{x<\nu\le n}(-)^{\nu-1}\nu^{-s}.$$

Further

$$\sum_{n\le x}(-)^{n-1}n^{-s} = \sum_{n\le x}n^{-s} - 2^{1-s}\sum_{n\le x}n^{-s} + O(x^{-\sigma}) = (1-2^{1-s})\sum_{n\le x}n^{-s} + O(x^{-\sigma}),\quad \sigma>0$$

From the approximate functional equation, Theorem 9 (III), we obtain, if $t<\frac{2\pi x}{c}$ where $c>0$, so that $y<1$ and the χ term disappears,

$$\sum_{n\le x}n^{-s} + O(x^{-\sigma}) = \zeta(s)$$

Hence

$$\sum_{n\le x}(-)^{n-1}n^{-s} = (1-2^{1-s})\zeta(s) + O(x^{-\sigma})$$

so that

$$\sum_{x<n}(-)^{n-1}n^{-s} = O(x^{-\sigma}).$$

$\therefore$ From 4.71 and 4.7

$$\chi(s) = \sum_{n \le x} \frac{(-)^{n-1} n^{-s}}{\log n} + O(x^{-\sigma}) \qquad \left\{ t < \frac{2\pi x}{c},\ c > 1,\ \sigma > 0 \right\}$$

$$\therefore \quad \chi(\tfrac{1}{2} + it) = \sum_{n < AT} \frac{(-)^{n-1} n^{-it}}{\sqrt{n}\,\log n} + O(t^{-\frac{1}{2}}) = \Theta + O(t^{-\frac{1}{2}})$$ say, where A

is an absolute constant.

4.72 $\therefore$
$$\int_{-T}^{T} |\chi(\tfrac{1}{2} + it)|^2 dt = \int_{-T}^{T} |\Theta|^2 dt + O\left\{ \int_{-T}^{T} |\Theta|^2 dt \right\}^{\frac{1}{2}} + O(1)$$

Now

4.73
$$\int_{-T}^{T} |\Theta|^2 dt = \sum_{m,n}^{AT} \int_{-T}^{T} \frac{(-)^{n-1} n^{-it}}{\sqrt{n}\,\log n} \cdot \frac{(-)^{m-1} m^{it}}{\sqrt{m}\,\log m} = \int_{-T}^{T} \sum \frac{1}{n \log^2 n}\, dt + \sum_{m \ne n}^{AT} \frac{(-)^{m+n}}{\sqrt{mn}\,\log m \log n} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt.$$

We have

$$\sum_{m \ne n}^{AT} \frac{(-)^{m+n}}{\sqrt{mn}\,\log m \log n} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt = \sum \frac{(-)^{m+n}}{\sqrt{mn}\,\log m \log n \left|\log \frac{m}{n}\right|} = \sum_{m < \frac{1}{2} n}^{AT} + \sum_{\frac{1}{2} n \le m \le \frac{3}{2} n}^{AT} + \sum_{\frac{3}{2} n < m}^{AT}$$

For the first of these sums, we have

$$O\left(\sum_{m,n=1}^{AT} \frac{1}{\sqrt{mn}\,\log m \log n} \right) = O\left(\sum_{1}^{T} \frac{1}{\sqrt{n}\,\log T} \right)^2 = O\left(\frac{T}{\log^2 T} \right)$$

Similarly for the third sum.

In the second sum write $m = n + k$, $|k| \le \frac{1}{2} n$, $\frac{1}{\log\left|\frac{m}{n}\right|} = O\left(\frac{n}{k}\right)$,

then this sum is

$$O\left\{ \sum_{n=1}^{T} \sum_{k=1}^{\frac{1}{2} n} \frac{1}{n(n+k) \log n \log \overline{n+k}} \cdot \frac{n}{k} \right\} = O\left\{ \frac{1}{\log^2 T} \cdot \sum_{n=1}^{T} \sum_{k=1}^{T} \frac{1}{k} \right\} = O\left(\frac{T}{\log T} \right)$$

$\therefore$ Collecting these results and returning to 4.73, we have

$$\int_{-T}^{T} |\Theta|^2 dt = 2T \sum_{n < AT} \frac{1}{n \log^2 n} + O\left(\frac{T}{\log T} \right)$$

$\therefore$ From 4.72

$$\frac{1}{2T} \int_{-T}^{T} |\chi(\tfrac{1}{2} + it)|^2 dt \sim \sum_{2}^{\infty} \frac{1}{n (\log n)^2}$$

and the theorem is proved.

We conclude this section with a mean value theorem of a more general nature, which will be of great use in the sequel. Let $f(s)$ be defined by the series
$$f(s) = \sum_{1}^{\infty} c_n n^{-s}$$
where $c_1, c_2, \cdots c_n, \cdots$ is a sequence of numbers, each of which is of absolute value less than one, and such that their sum converges. Thus if we write
$$S(x) = \sum_{n \le x} c_n$$
then $S(x)$ is bounded, so that
$$S(x) < S,$$ where S is a positive number.

The series defining $f(s)$ evidently converges for $\sigma > 0$.

THEOREM 17   Let $0<h\le \frac12$ and $\sigma \ge \frac12 + h$, $T>8$.
Then, if $c_n = 0$ for $n \le N$, where N is an integer ≥ 3.
we have

$$\int_{-T}^{T} |b(s)|^2\,dt < \frac{TN^{-2h}}{h} + \frac{44}{h^2}(ST)^{1-2h}(\log ST)^2 .$$

From the identity

$$\sum_{m=1}^{q}\frac{\bar{c}_m}{m^s}\sum_{n=m+1}^{\infty}\frac{c_n}{n^s} + \sum_{m=1}^{q}\frac{c_m}{m^s}\sum_{n=m+1}^{\infty}\frac{\bar{c}_n}{n^s} + \sum_{n=1}^{q}\frac{c_n\,\bar{c}_n}{n^s\,n^s} = \sum_{m=1}^{q}\frac{\bar{c}_m}{m^s}\sum_{n=1}^{q}\frac{c_n}{n^s} + \sum_{m=1}^{q}\frac{\bar{c}_m}{m^s}\sum_{n=q+1}^{\infty}\frac{c_n}{n^s} + \sum_{n=1}^{q}\frac{c_n}{n^s}\sum_{m=q+1}^{\infty}\frac{\bar{c}_m}{m^s}$$

where $\bar{c}_n$ is the conjugate of c_n, it is obvious that

$$|b(s)|^2 = \sum_{1}^{\infty}\frac{|c_n|^2}{n^{2\sigma}} + 2\,\Re\sum_{m=1}^{\infty}\frac{\bar{c}_m}{m^{\sigma-it}}\sum_{n=m+1}^{\infty}\frac{c_n}{n^{\sigma+it}}$$

It is not difficult to shew that both of these series converge uniformly if $\sigma > \frac12$

Hence

4.8 $$\int_{-T}^{T}|b(s)|^2\,dt = 2T\sum_{1}^{\infty}\frac{|c_n|^2}{n^{2\sigma}} + 2\,\Re\sum_{m=1}^{\infty}\frac{\bar{c}_m}{m^{\sigma}}\int_{-T}^{T} m^{it}\sum_{m+1}^{\infty}\frac{c_n}{n^{\sigma+it}}\,dt \qquad (\sigma > \tfrac12)$$

Consider the value, with $y \ge m+1$ and integral, of

$$K(m) = \int_{-T}^{T} m^{it}\sum_{y}^{\infty}\frac{c_n}{n^{\sigma+it}}\,dt .$$

Since $S(x) = \sum_{n\le x} c_n$ we have, by partial summation,

$$K(m) = -\frac{S(y-1)}{y^{\sigma}}\int_{-T}^{T}\left(\frac{m}{y}\right)^{ti}dt + \int_{y}^{\infty}\frac{S(u)\,du}{u^{\sigma+1}}\int_{-T}^{T}(\sigma+it)\left(\frac{m}{n}\right)^{it}dt$$

By partial integration, we easily find

$$\left|\int_{-T}^{T}(\sigma+it)\left(\frac{m}{n}\right)^{it}dt\right| \le \frac{2(\sigma+T)+2T}{\log n/m}$$

so that, if $T > 2\sigma > 1$, $y \ge m+1$, we have

$$|K(m)| < \frac{2\,S(T)}{y^{\sigma}} + 5ST\int_{y}^{\infty}\frac{du}{u^{\sigma+1}\log\frac{u}{y-1}}$$

This last integral is certainly less than

$$\frac{1}{y^{\sigma}}\int_{y}^{e(y-1)}\frac{du}{u\log\frac{u}{y-1}} + \int_{y}^{\infty}\frac{du}{u^{\sigma+1}} < -\frac{\log 1/y}{y^{\sigma}} + \frac{1}{\sigma y^{\sigma}} < \frac{\log y + 2}{y^{\sigma}}$$

which gives

4.81 $$|K(m)| < 29\,\frac{ST}{y^{\sigma}}\log y \qquad (T > 2\sigma > 1,\ y \ge m+1)$$

We now discuss the third term in 4.8, introducing the conditions

$$0 < h \le \tfrac12, \quad \sigma \ge \tfrac12 + h.$$

SUPPOSE $\sigma \le 4$.

$$\sum_{1}^{\infty}\frac{\bar{c}_m}{m^{\sigma}}\int_{-T}^{T} m^{ti}\sum_{m+1}^{\infty}\frac{c_n}{n^{\sigma+it}} = \sum_{1}^{\infty}\frac{\bar{c}_m}{m^{\sigma}}K(m) = \sum_{m<[ST]} + \sum_{n=[ST]} + \sum_{m>[ST]} = \Sigma_1 + \Sigma_2 + \Sigma_3$$

For $y > [ST]$ put $y = m+1$ in 4.81 and we find

$$K(m) < 29\,\frac{ST\log m}{m^{1/2+h}}$$

Therefore

4.82 $$\sum\nolimits_3' < 29\, ST \sum_{m > [ST]} \frac{\log m}{m^{\frac12 + h + \sigma}} < 29\, ST \sum_{m>[ST]} \frac{\log m}{m^{1+2h}}$$

For $y = [ST]$, put $y = [ST]+1$ in 4.81 and we find

4.83 $$K(m) < 29\, (ST)^{\frac12 - h} \log (ST)$$

For $m \leq [ST]$ we have, since $|c_n| < 1$,

$$\left| \int_{-T}^{T} m^{ti} \sum_{m+1}^{[ST]} \frac{c_n}{n^{\sigma+it}}\, dt \right| = \left| \sum_{m+1}^{[ST]} \frac{c_n}{n^{\sigma}} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt \right| \leq 2 \sum_{m+1}^{[ST]} \frac{(ST)^{\frac12 - h}}{n \log \frac{n}{m}}$$

$$< 2(ST)^{\frac12-h}\left\{ \frac{1}{(m+1)\log \frac{m+1}{m}} + \int_{m+1}^{ST} \frac{du}{u \log \frac{u}{m}} \right\} < 2(ST)^{\frac12-h}\left\{ \frac{1}{m\cdot\frac{1}{m}} + \log\log \frac{ST}{m} - \log\log\left(1+\frac{1}{m}\right) \right\}$$

$$< 2(ST)^{\frac12-h}\left\{ 2 + \log \frac{ST}{m} - \log \frac{1}{2m} \right\} < 2(ST)^{\frac12-h}\left\{ 2 + \log 2 + \log ST \right\}$$

4.84 $$< 5\, (ST)^{\frac12-h} \log ST.$$

Hence combining 4.82, 4.83 and 4.84, we have

$$\left| \sum_{1}^{\infty} \frac{\bar{c}_m}{m^{\sigma}} K(m) \right| < 34\, (ST)^{\frac12-h} \log ST \sum_{1}^{[ST]} m^{-\frac12-h} + 29\, ST \sum_{m>[AT]} \log m\; m^{-1-2h}$$

$$< 34\, (ST)^{1-2h} \log ST \sum_{1}^{[ST]} \frac{1}{m} + 29\, ST \left\{ \frac{\log ST}{(ST)^{1+2h}} + \int_{ST}^{\infty} \frac{\log u}{u^{1+2h}}\, du \right\}$$

$$< \frac{34}{h^2} (ST)^{1-2h} \log ST \cdot \frac{3}{2} \log ST + 29\, ST \left\{ \frac{\frac12 \log^2 ST}{4h^2 (ST)^{2h}} + \frac{\frac12 \log^2 ST}{4h^2 (ST)^{2h}} + \frac{\frac14 \log^2 ST}{4h^2 (ST)^{2h}} \right\}$$

the last step being obtained by partial integration and introduction of h's into denominators.

4.85 Hence $$\left| \sum_{1}^{\infty} \frac{\bar{c}_m}{m^{\sigma}} K(m) \right| < \frac{1}{h^2}\left\{ (ST)^{1-2h} \log^2 ST \right\}\left\{ \frac{102}{8} + \frac{145}{16} \right\} < \frac{22}{h^2} (ST)^{1-2h} \log^2 ST.$$

If $\sigma > 4$ $$|K(m)| < \sum_{m+1}^{\infty} \frac{2}{n^4 \log n/m} < 4m \sum_{m+1}^{\infty} \frac{1}{n^4} < 8m,$$

and the above result is easily seen to hold.

Now introduce the condition that $c_n = 0$ for $n \leq N$ $(N \geq 3)$ into equation 4.8

The first term is
$$\leq 2T \sum_{N+1}^{\infty} \frac{1}{n^{1+2h}} < \frac{TN^{-2h}}{h}$$

and the second term is, by 4.85, less than
$$\frac{44}{h^2} (ST)^{1-2h} (\log ST)^2$$

so that the theorem is justified.

5. THE ZEROS OF [redacted] $\zeta(s)$. (THE RIEMANN-MANGOLDT FORM)

An analysis of the distribution of the zeros of an analytic function is considerably helped by a fundamental theorem due to Jensen ('Acta Math.'t XXII 1899) of which we give a statement, and certain deductions.

JENSEN'S THEOREM. Let $f(z)$ be analytic inside the circle $|z| \leq r$ and suppose it to possess zeros at $a_1 a_2 \cdots a_m \cdots$ of which the first n lie inside the previously defined circle. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(z)| \, d\varphi \text{ on the circle} = \log \left| \frac{f(0)\, r^n}{a_1 a_2 \cdots a_n} \right|$$

where $z = re^{i\varphi}$ on the circle, and where each zero is counted as many times as is indicated by its multiplicity.

If $M(r)$ is the maximum value of $|f(z)|$ on the circle $|z| = r$, then evidently

$$M(r) \geq \left| \frac{f(0)\, r^n}{a_1 \cdots a_n} \right|$$

COROLLARY. The above inequality persists for all values of n i.e. including zeros lying outside the circle $|z| = r$.

Let $a_1 a_2 \cdots a_m$ be the first m zeros. Then

(I) If $m > n$, $0 < r < |a_m|$, and

$$\left| \frac{f(0)\, r^m}{a_1 \cdots a_m} \right| = \left| \frac{f(0)\, r^n}{a_1 \cdots a_n} \right| \frac{r}{a_{n+1}} \cdots \frac{r}{a_m} < \left| \frac{f(0)\, r^n}{a_1 \cdots a_n} \right| \leq M(r)$$

(II) If $m < n$ then

$$M(r) \geq \left| \frac{f(0)\, r^n}{a_1 \cdots a_n} \right| \geq \left| \frac{f(0)\, r^m}{a_1 \cdots a_m} \right| \frac{r}{a_{m+1}} \cdots \frac{r}{a_n} \geq \left| \frac{f(0)\, r^m}{a_1 \cdots a_m} \right|$$

(III) If $m = n$ the inequalities are the same

Let $g(\varphi)$ be a continuous function defined in the interval $(0 \div 2\pi)$ then, since the geometric mean is less than, or at most equal to, the arithmetic mean, we have

$$\sqrt[n]{e^{g(0)} e^{g(2\pi/n)} \cdots e^{g(n-1 \cdot \frac{2\pi}{n})}} \leq \frac{1}{n}\left\{ e^{g(0)} + e^{g(\frac{2\pi}{n})} + \cdots \right\}$$

$$\therefore \quad \exp \frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\, d\varphi \leq \frac{1}{2\pi}\int_0^{2\pi} e^{g(\varphi)}\, d\varphi$$

Take $g(\varphi) = \log |f(z)|^2$ where $z = re^{i\varphi}$, then

$$\exp \frac{1}{2\pi}\int_0^{2\pi} \log |f(z)|^2 \, d\varphi \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 \, d\varphi$$

Let $f(z) = \sum_0^\infty \alpha_n z^n$, analytic for $|z| \leq r$, and $\alpha_0 \neq 0$

then $\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 d\varphi = \sum_0^\infty |\alpha_n|^2 r^{2n}$, so that preceding equality gives

$$\sum_0^\infty |\alpha_n|^2 r^{2n} \geq \exp. \frac{1}{2\pi}\int_0^{2\pi} \log |f(z)|^2 d\varphi .$$

Let $0 < \rho < r$, and Q the number of zeros, $a_1 \ldots a_n \ldots$, of $f(z)$ lying in the circle $|z| \leq \rho$ then by the above corollary

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(z)|^2 d\varphi \geq 2 \log \frac{|\alpha_0| r^Q}{a_1 a_2 \cdots a_Q}$$

∴ we have

$$\sum_0^\infty |\alpha_n|^2 r^{2n} \geq \frac{|\alpha_0|^2 r^{2Q}}{a_1^2 a_2^2 \cdots a_Q^2} .$$

giving Petrovitch's theorem (Landau - '__Bull. de la Soc. Math. de France__' 33. 1905)

$$\frac{1}{\rho^{2Q}} \leq \frac{1}{|\alpha_0|^2} \sum_0^\infty |\alpha_n|^2$$

Without loss of generality take $\frac{r+\rho}{2} = 1$, then

$$\begin{aligned}(r-\rho)Q &= (1-\rho)2Q < \left(\tfrac{1}{\rho}\right)^{2Q} - 1 \leq \frac{1}{|\alpha_0|^2}\sum_0^\infty |\alpha_n|^2 \\ &\leq \frac{1}{|\alpha_0|^2}\sum |\alpha_n|^2 \frac{r^{2n+2}-1}{r-1} \cdot \frac{1}{2n+2} \leq \frac{1}{|\alpha_0|^2 (r-\rho)} \sum_0^\infty |\alpha_n|^2 \frac{r^{2n+2}}{2n+2} \\ &= \frac{2}{|\alpha_0|^2 (r-\rho)} \int_0^r R \sum_0^\infty |\alpha_n|^2 R^{2n} dR \leq \frac{1}{\pi |\alpha_0|^2 (r-\rho)} \iint_{|z|\leq r} |f(z)-1|^2 du\,dv\end{aligned}$$

where $u + iv = z$

We have thus obtained the theorem:-

__THEOREM 18__ Let $f(z) = \sum_0^\infty \alpha_n z^n$ be analytic for $|z| \leq r$. Suppose $\alpha_0 \neq 0$ and $0 < \rho < r$, and let Q be the number of zeros of $f(z)$ in the region $|z| \leq \rho$. Then

$$Q \leq \frac{1}{\pi |\alpha_0|^2 (r-\rho)^2} \iint_{|z|\leq r} |f(z)-1|^2 du\,dv \qquad (z = u + iv)$$

We are now in a position to investigate fairly thoroughly the distribution of the zeros of the Zeta-function, and in this section we will consider only the development of the RIEMANN-MANGOLDT form 1.6

Let $N(T)$ denote the number of zeros of the Zeta-function, whose imaginary parts lie between 0 and T, and $N(\sigma, T)$ the number which, in addition, have real parts greater than σ.
As stated in the introduction, Riemann proved that $N(1,T) = 0$ and made the hypothesis $N(\frac{1}{2},T) = 0$, further, without proof he gave the asymptotic formula, 1.6, for $N(T)$.

The method used by von Mangoldt in proving Riemann's form for $N(T)$ depended on Hadamard's product form, and consequently involved a discussion of the series
$$\sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$
where ρ is a complex zero of $\zeta(s)$

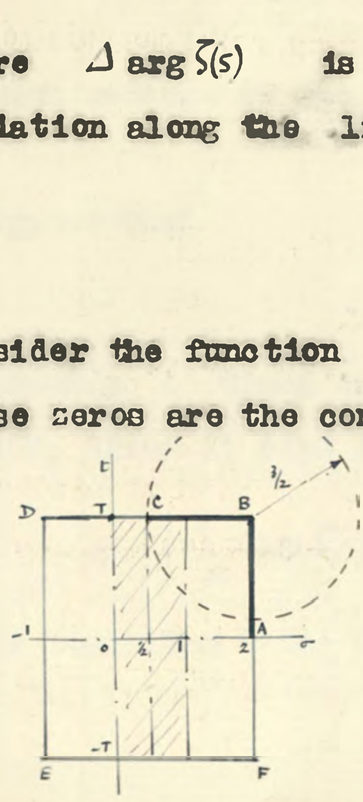
Considerable improvements in the details of von Mangoldt's analysis were made by Landau in 1909, and the exposition given in the 'Handbuch' § 91, remains the most refined proof based on the product form.

Backlund in 1914 ('Comptes Rendus' 158) succeeded, by a very simple application of Jensen's theorem, in avoiding a discussion of the above series, and it seems difficult to realise that the method was not previously used.

We will first prove a general theorem and then devote ourselves to a discussion of the error term which appears.

THEOREM 19
$$N(T) = \frac{T}{2\pi} \operatorname{Log} \frac{T}{2\pi} - \frac{T}{2\pi} + R(T)$$
where
$$R(T) = \frac{7}{8} + O\left(\frac{1}{T}\right) + \frac{1}{\pi} \Delta \arg \zeta(s)$$
where $\Delta \arg \zeta(s)$ is the variation in $\arg \zeta(s)$ obtained by continuous variation along the lines $\sigma = 2$ to $s = 2 + iT$ and $2 + iT$ to $s = \frac{1}{2} + iT$.

Consider the function $\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{1}{2}s} \Gamma(\frac{1}{2}s) \zeta(s)$
whose zeros are the complex zeros of $\zeta(s)$. The number of the zeros of $\xi(s)$ inside the rectangular contour A B D E F, defined in diagram, is given by



$$\frac{1}{2\pi i} \int_{BDEF} \frac{\xi'(s)}{\xi(s)} ds,$$
which, in virtue of the functional equation

$\xi(s) = \xi(1-s)$, is equivalent to

$$\frac{2}{\pi i}\int_{ABC} \frac{\xi'(s)}{\xi(s)}\, ds = \frac{2}{\pi} \underset{ABC}{\Delta} \arg \xi(s)$$

where $\underset{ABC}{\Delta} \arg \xi(s)$ denotes the variation of arg $\xi(s)$ when s describes the contour A B C in the positive sense.

Since $s(s-1)\Gamma(\frac{1}{2}s)$ and $s(s-1)\zeta(s)$ are finite at $s=0$ and $s=1$ respectively, it follows that all the zeroes of $\xi(s)$ inside A B D E F are zeros of $\zeta(s)$ and consequently - by the funtional equation - there are $2N(T)$ of them .

$$\therefore \quad N(T) = \frac{1}{\pi} \underset{ABC}{\Delta} \arg \xi(s)$$

Now, from the definition of $\xi(s)$, we have

$$\Delta \arg \xi(s) = \Delta \arg s + \Delta \arg s-1 + \Delta \arg \pi^{-\frac{1}{2}s} + \Delta \arg \Gamma(\tfrac{1}{2}s) + \Delta \arg \zeta(s)$$

It is evident that, $\underset{ABC}{\Delta} \arg s + \underset{ABC}{\Delta} \arg s-1 = \pi$

Since $\arg z = \Im \log z$, we have

$$\underset{ABC}{\Delta} \arg \pi^{-\frac{1}{2}s} = \Im \log \pi^{-(\frac{1}{4}+\frac{1}{2}iT)} = -\frac{T}{2} \log \pi$$

Since $\arg \Gamma(\frac{1}{2}s) = 0$ at $s=2$, $\underset{ABC}{\Delta} \arg \Gamma(\frac{1}{2}s) = \arg \Gamma(\frac{1}{4}+\frac{1}{2}iT)$ and, using Stirling's formula,

$$\arg \Gamma(\tfrac{1}{4}+\tfrac{1}{2}iT) = \Im \log \Gamma(\tfrac{1}{4}+\tfrac{1}{2}iT) = \Im\left\{(\tfrac{1}{2}iT-\tfrac{1}{4})\log \tfrac{1}{2}iT - \tfrac{1}{2}iT + O(\tfrac{1}{T})\right\}$$

$$= \frac{T}{2}\log\frac{T}{2} - \frac{T}{2} - \frac{\pi}{8} + O(\tfrac{1}{T})$$

$\therefore$ Collecting results

$$N(T) = \frac{1}{\pi}\left\{\pi + \frac{T}{2}\log\frac{T}{2} - \frac{T}{2}\log\pi - \frac{\pi}{8} - \frac{T}{2} + O(\tfrac{1}{T}) + \underset{ABC}{\Delta} \arg \zeta(s)\right\}$$

which reduces to the theorem required.

Suppose that

$$\Re\, \zeta(s) = \tfrac{1}{2}\left\{\zeta(\sigma+it) + \zeta(\sigma-it)\right\}$$

vanishes l times on the line B C in the diagram on previous page, then, since it does not vanish on the line AB, we have

$$\left|\underset{ABC}{\Delta} \arg \zeta(s)\right| = \Im \log \zeta(\tfrac{1}{2}+it) < (l+1)\pi$$

the logarithm being defined by continuous variation.

Draw a circle of radius $r = \frac{3}{2}$ with centre at $B \equiv 2 + iT$, and let there be n zeros of the function

$$f(s) = \tfrac{1}{2}\left\{\zeta(s+iT) + \zeta(s-iT)\right\} = \mathcal{R}\,\zeta(\sigma+iT) \quad \text{for} \quad s = \sigma,$$

inside this circle. Since $l \leq n$ we have

$$\left| \Delta_{ABC} \arg \zeta(s) \right| < (n+1)\pi$$

By known theorems

$$\log M(r) = \log \operatorname{Max} \left| f(2 + \tfrac{3}{2} e^{i\varphi}) \right| \quad (0 \leq \varphi < 2\pi) \quad = O(\log T)$$

The value of $f(s)$ at centre of circle is $\mathcal{R}\,\zeta(2+iT) > A$, where A is a constant.

∴ By Jensen's theorem

$$n < A \log M(r) = O(\log T)$$

consequently $\Delta_{ABC} \arg \zeta(s) = O(\log T)$ and we obtain Mangoldt's result

THEOREM 20 $$R(T) = O(\log T)$$

If Lindelöf's hypothesis is correct. then we may replace the O by o - a replacement first given by Bohr (Bohr, Landau and Littlewood, 'Bull. Acad. Belg. '15 1913) under the more restricted condition that Riemann's hypothesis is true.

Since Lindelof's hypothesis is assumed true, then

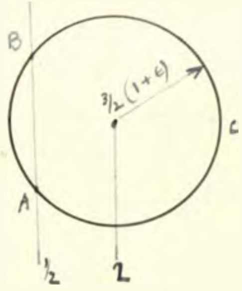
$$\zeta(s) = O(t^{\epsilon^2}) \quad \text{for } \sigma \geq \tfrac{1}{2},$$

and, by the functional equation,

$$\zeta(s) = O(t^{\frac{1}{2} - \sigma + \epsilon^2}) \quad \text{for } \sigma \leq \tfrac{1}{2}.$$

As in proof of von Mangold's formulae, we find an upper bound for n inside a circle of radius $r = \frac{3}{2}(1+\epsilon)$ and centre at $s = 2$, n being the number of zeros of

$$f(s) = \tfrac{1}{2}\left\{\zeta(s+iT) + \zeta(s-iT)\right\}$$



By Jensen's theorem, taking $\epsilon < \frac{1}{3}$ and $T > t_0 + 2$

$$n \operatorname{Log}(1+\epsilon) < \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Log}\left|f(2+re^{i\varphi})\right| d\varphi - \operatorname{Log}|f(2)|$$

$$\therefore n < \frac{1}{2\pi \operatorname{Log}(1+\epsilon)} \int_0^{2\pi} \operatorname{Log}\left|f(2+re^{i\varphi})\right| d\varphi + \frac{2}{\epsilon}$$

On the arc ACB of the circle using Lindelof's hypothesis,

$$\operatorname{Log}\left|f(2+re^{i\varphi})\right| < \operatorname{Log}(T+2)^{\epsilon^2} = \epsilon^2 \operatorname{Log}(T+2)$$

On the arc ADC of the circle

$$\operatorname{Log}\left|f(2+re^{i\varphi})\right| < \operatorname{Log}(T+2)^{\frac{1}{2}-\sigma+\epsilon^2} \leq \epsilon\left(\tfrac{3}{2}+\epsilon\right)\operatorname{Log}(T+2)$$

The lengths of arcs ADC and BCA are respectively less than $K\sqrt{\epsilon}$ and 3π

Hence

$$n < \frac{1}{\epsilon}\left\{K\sqrt{\epsilon}\,.\,\epsilon\left(\tfrac{3}{2}+\epsilon\right) + 3\pi\epsilon^2\right\}\operatorname{Log}(T+2) + \tfrac{1}{2}\epsilon \qquad \text{(K absolute)}$$

$$< K\sqrt{\epsilon}\operatorname{Log} T, \qquad \text{if } T \text{ is sufficiently large.}$$

Therefore $n = o(\log t)$ and we obtain:-

THEOREM 21 If Lindelöf's hypothesis is true, then

$$R(T) = o(\log T)$$

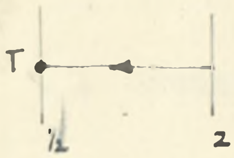
THEOREM 22 If Riemann's hypothesis is true, it is possible to find an absolute positive constant c such that

$$R(T) \neq O\{(\log T)^c\}$$

or, more precisely,

$$\lim_{T\to\infty}\inf \frac{R(T)}{(\log T)^c} = -\infty, \qquad \lim_{T\to\infty}\sup \frac{R(T)}{(\log T)^c} = +\infty.$$

We know that $R(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iT) + O(1)$ where $\zeta(\frac{1}{2}+iT)$ is obtained by continuous variation along the line $t=T$ from $\sigma=2$ to $\sigma=\frac{1}{2}$.



Suppose the theorem is false, then it will be possible to choose a positive number c, so that

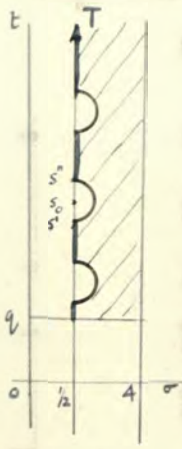
$$R(T) < K(\log T)^c \quad \text{or} \quad -R(T) < K(\log T)^c$$

where K is a positive Borel constant, and T sufficiently large.

$\therefore$ $\arg\zeta(\frac{1}{2}+iT) < K(\log T)^c$, and $-\arg\zeta(\frac{1}{2}+iT) < K(\log T)^c$

Suppose we find arg $\zeta(\tfrac{1}{2}+iT)$ by variation along a vertical line $\sigma=\tfrac{1}{2}$, then the above inequality will persist if we define $\arg\zeta(\tfrac{1}{2}+iT_0) = \lim\limits_{\epsilon\to 0}\left\{\arg\zeta(\tfrac{1}{2}+i(T_0+\epsilon)) + \arg\zeta(\tfrac{1}{2}+i(T_0-\epsilon))\right\}$
where $\tfrac{1}{2}+iT_0$ is a zero of $\zeta(s)$.

Hence for $T>0$ we have



$$\arg\zeta(\tfrac{1}{2}+iT) < K\{\log(T+2)\}^c, \qquad -\arg\zeta(\tfrac{1}{2}+iT) < K\{\log(T+2)\}^c$$

Choose an absolute constant q smaller than ordinate of the first zero, and lying in the interval $(0,2)$.
Around each zero on the line $\sigma=\tfrac{1}{2}$, $t>q$ draw half circles lying to the right of the line, and of radius smaller than 1/6, and sufficiently small to avoid cutting each other.
Suppose $s_0=\tfrac{1}{2}+iT_0$ is a zero, and let s'' and s' be the two points in which the half circle around s_0 touches the line $\sigma=\tfrac{1}{2}$.

Then $\arg\zeta(s) < \arg\zeta(s'')+1$ or $\arg\zeta(s) < \arg\zeta(s')-1$

on each half circle.

Let G denote the region shaded in the diagram and in which, since Riemann's hypothesis is supposed true, $\log\zeta(s)$ is analytic.
On the left hand border - including half circles - we have

$$\arg\zeta(s) < K\{\log(t+2+\tfrac{2}{6})\}^c \quad \text{or} \quad -\arg\zeta(s) < K\{\log(t+2+\tfrac{2}{6})\}^c$$

i.e. $\Im\log\zeta(s) < K\{\log(t+2)\}^c$ or $-\Im\log\zeta(s) < K\{\log(t+2)\}^c$

On the right hand and bottom borders of G, it is known that

$$|\Im\log\zeta(s)| < K$$

∴ Along the entire contour of G,

5.1 $$\Im\log\zeta(s) < K\{\log(t+2)\}^c \quad \text{or} \quad -\Im\log\zeta(s) < K\{\log(t+2)\}^c$$

For $\sigma\geq\tfrac{1}{2}$ we know that, if t is not the ordinate of a zero

$$\Im\log\zeta(s) = O(\log t)$$

∴ Assuming Riemann's hypothesis

5.2 $$\Im\log\zeta(s) = O(\log t) \qquad \text{uniformly in } G$$

Write

$$g(s) = e^{-i\log\zeta(s)} \qquad \text{or} \qquad g(s) = e^{i\log\zeta(s)}$$

Then on the contour of G, using 4.1

$$g(s) < e^{K \log^c (t+2)}$$

and inside G, using 5.2

$$g(s) < e^{K \log (t+2)}$$

On and inside G, we have uniformly

$$\log s = \log t + O(1) \quad \therefore \ (\log s)^c = (\log t)^c + O\{(\log t)^{c-1}\}$$

and so

$$2\{\log(t+2)\}^c + K > \mathcal{R}(\log s)^c > \tfrac{1}{2}\{\log(t+2)\}^c - K$$

Write
$$H(s) = g(s)\, e^{-2K(\log s)^c}$$

then on contour of G,

$$|H(s)| < e^{K\{\log(t+2)\}^c - K\{\log(t+2)\}^c + K} < K$$

and inside G, $|H(s)| < e^{K\log(t+2) - K\{\log(t+2)\}^c + K} = O(t^K)$

Hence by the Phragmén-Lindelöf theorem, H(s) is bounded inside G

i.e. $|H(s)| < K$ in G.

$\therefore$ $|g(s)| < K e^{K\log^c(t+2) + K}$ in G.

$\therefore$ $\mathcal{R} \log g(s) < K \log^c (t+2)$ inside G.

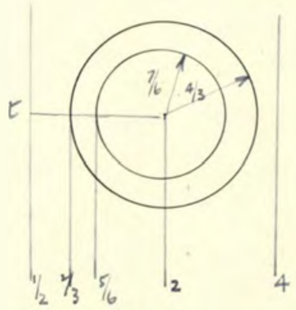
$\therefore$ In particular, for $2/3 \leq \sigma \leq 4$, $t \geq 2$

We now make use of CARATHEODORY'S THEOREM :- (H. p. 299)

If $F(s)$ is analytic inside and on the circle $|s-s_0|<r$,
$0<\rho<r$, A = maximum value of $\mathcal{R}F(s)$ on $|s-s_0|=r$,
then

$$F(s) \text{ on } |s-s_0|=\rho \leq \frac{2r}{r-\rho}|F(s_0)| + 2A\frac{\rho}{r-\rho}.$$

Take $\log \zeta(s) = F(s)$



$r = 4/3$, $\rho = 7/6$ and $s_0 = 2 + it$

Then we have, for $t \geq 10/3$, $5/6 \leq \sigma < 2$

$$|\log \zeta(s)| \leq \log \zeta(2) + \log \zeta(2) \frac{5/2}{1/6} + K\{\log(t + 2 + 4/3)\}^c \frac{7/6}{1/6}$$
$$< K(\log t)^c$$

$\therefore$ in the half plane $\sigma \geq 5/6$, $t \geq 2$ we have

$$|\log \zeta(s)| < K(\log t)^c$$

which directly contradicts THEOREM 11, and therefore our theorem must be true.

Theorem 21 was proved by making use of Lindelöf's assertion concerning the order of $\zeta(s)$ in the critical strip, but we have in Theorem 12, a far deeper result which we should expect to give a better form of Theorem 21, but under the assumption that Riemann's hypothesis is true. Cramér in fact, has performed this analysis ('Math. Zeits.' 2. 1918) and found that, if Riemann's hypothesis is true, then

$$R(T) = O\left\{\log T \sqrt{\frac{\log\log\log T}{\log\log t}}\right\}$$

but he failed to realise that a much better result could be obtained by using Bohr's method of proving Theorem 21, assuming Riemann's hypothesis, being apparently misguided by the simplicity of his deduction of this latter theorem. Theorem 12 states that, if Riemann's hypothesis is true, then

$\log \zeta(s) = O\left\{\left(\frac{\log t \log\log t}{\log\log\log t}\right)^{2(1-\sigma)} \log\log\log t\right\}$ uniformly in σ , for $\frac{1}{2} + \frac{\delta}{\log\log t} \leq \sigma \leq 1$

hence, with reference to a well known theorem, we have for

$$\frac{1}{2} + \frac{\log\log\log t}{\log\log t} \leq \sigma \leq 1$$

$$\log \zeta(s) = O\left\{\log t \cdot \log\log t \cdot e^{-2\frac{\log\log\log t}{\log\log t}(\log\log t + \log\log\log t + \log\log\log\log t)}\right\}$$

5.31 $$= O\left\{\frac{\log t}{\log\log t}\right\}$$

Applying Cauchy's inequality to a circle of radius $\frac{1}{2}\frac{\log\log\log t}{\log\log t}$.

we have, for $\sigma = \frac{1}{2} + 2\log\log\log t/\log\log t$

$$\frac{\zeta'(s)}{\zeta(s)} = O\left\{\frac{\log t}{\log\log t}\cdot\frac{\log\log t}{\log\log\log t}\right\} = O(\log t)$$

It is known (H. p.316) that

5.3 $$\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) = \frac{\zeta'(s)}{\zeta(s)} - b + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s-1)}{\Gamma(\frac{1}{2}s-1)}$$

where ρ is a complex zero of $\zeta(s)$, and b is an absolute constant

∴ For $\sigma = \frac{1}{2} + 2\log\log\log t/\log\log t$ we have

$$\sum_\rho \Re\left\{\frac{1}{s-\rho}+\frac{1}{\rho}\right\} = O(\log t)$$

∴ Writing $\rho = \frac{1}{2}+i\gamma$, Riemann's hypothesis being true, we find

$$\sum_{t<\gamma\leq t+2\frac{\log\log\log t}{\log\log t}} \frac{(\sigma-\frac{1}{2})}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} = O(\log t)$$

The number of terms in this series is $N\left\{t+2\frac{\log\log\log t}{\log\log t}\right\} - N(T)$, each of which is greater than

$$\frac{2\log\log\log t/\log\log t}{(2\log\log\log t/\log\log t)^2+(2\log\log\log t/\log\log t)^2} = \frac{\log\log t}{4\log\log\log t}$$

Hence

$$N\left\{t+2\frac{\log\log\log t}{\log\log t}\right\} - N(t) = O\left\{\log t\cdot\frac{\log\log\log t}{\log\log t}\right\}$$

Let T be different from the ordinate of any zero, and $> e^e$ (ie $\log\log\log t > 0$) Divide the interval $T-1<t<T+1$ symmetrically about the point T into sub-intervals, each of length $\delta = \delta(T) = \log\log\log T/\log\log T$. From the last equation we see that the number of γ's in each of these sub-intervals is $O(\delta\log T)$, therefore, if $\sum_0$ denotes summation for ρ with $T-1<\gamma<T+1$, we have

$$\left|\Im\int_{\frac{1}{2}+\delta+Ti}^{\frac{1}{2}+Ti}\sum_0\frac{1}{s-\rho}\,ds\right| \leq \sum_0\left|\Im\int\frac{ds}{s-\rho}\right| = \sum_0\tan^{-1}\frac{\delta}{|T-\gamma|} \leq \sum_0 \mathrm{Min.}\left(\frac{\pi}{2},\frac{\delta}{|T-\gamma|}\right)$$

Hence

5.4 $$\Im\int_{\frac{1}{2}+\delta+Ti}^{\frac{1}{2}+Ti}\sum_0\frac{1}{s-\rho}\,ds = O\left\{\delta\log T\left(\frac{\pi}{2}+\frac{\delta}{\delta}+\frac{\delta}{2\delta}+\cdots+\frac{\delta}{[\frac{1}{\delta}]\delta}\right)\right\} = O\left\{\delta\log\frac{1}{\delta}\log T\right\}$$

since $|T-\gamma| \geq 0, \delta, 2\delta, \cdots$.

From 5.3, we have (H. p.316), uniformly for $\frac{1}{2}\leq\sigma\leq 1$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_0\frac{1}{s-\rho} + O(\log t)$$

$\therefore$ Using 5.4

$$\tilde{\mathfrak{I}} \int_{\frac{1}{2}+\delta+iT}^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)}\, ds = O\left(\delta \log \frac{1}{\delta} . \log T\right) = O\left\{ \log T . \frac{(\log\log\log T)^2}{\log\log T} \right\}$$

From 5.31

$$\tilde{\mathfrak{I}} \int_{2+Ti}^{\frac{1}{2}+\delta+Ti} \frac{\zeta'(s)}{\zeta(s)} = \tilde{\mathfrak{I}}\left\{ \log \zeta(\tfrac{1}{2}+\delta+Ti)\right\} - \tilde{\mathfrak{I}}\left\{\log \zeta(2+Ti)\right\} = O\left\{\log T . \frac{1}{\log\log T}\right\}$$

Hence

$$R(T) = O(1) + \frac{1}{\pi} \tilde{\mathfrak{I}} \int_{2+Ti}^{\frac{1}{2}+Ti} \frac{\zeta'(s)}{\zeta(s)}\, ds = O(1) + \frac{1}{\pi}\left\{ \tilde{\mathfrak{I}} \int_{\frac{1}{2}+\delta+Ti}^{\frac{1}{2}+Ti} + \tilde{\mathfrak{I}} \int_{\frac{1}{2}+Ti}^{2+Ti} \right\} \frac{\zeta'(s)}{\zeta(s)}\, ds$$

$$= O(1) + O\left\{ \log T . \frac{(\log\log\log T)^2}{\log\log T}\right\} + O\left\{\frac{\log T}{\log\log T}\right\}$$

and we obtain Landau's theorem:-

<u>THEOREM 23</u> If Riemann's hypothesis is true, then

$$R(T) = O\left\{ \log T . \frac{(\log\log\log T)^2}{\log\log T}\right\}$$

The mean value of $R(T)$ has been investigated by Cramér ('<u>Math. Zeits</u>'.4. 1919) and Littlewood ('<u>Proc.Lond.Math.Soc</u>.' 1921) For example, Cramér gives a proof that

$$\int_0^T R(t)\, dt = \tfrac{7}{8}T + O(T^{\epsilon})$$

where ϵ is any positive number, regardless of Riemann's hypothesis; the results given by Littlewood are of a finer nature, but, unfortunately, are not accompanied with proofs.

6. THE ZEROS OF $\zeta(s)$. II DISTRIBUTION OF THE ZEROS.

In default of producing a direct proof of Riemann's hypothesis we naturally endeavour to reduce $N(\sigma, T)$ for $\sigma > \frac{1}{2}$ to an order as small as possible and raise $N_0(T)$ — which we will use to denote the number of zeros on the critical line, with ordinates t in the range $0 < t < T$ — to an order which approximates as near as possible to the Riemann-Mangoldt form.

The most elegant theorem we posses relative to the order of $N(\sigma, T)$ was given by Carlson in 1920 ('Arkiv for Mat., Astr.Och.Fys' No.15) and depends on two theorems already noted - Theorems 17 and 18.

In order to produce a function to which Theorem 17 is applicable, we consider

$$(1 - 2^{1-s})\left\{\zeta(s)\sum_1^N \mu(n) n^{-s} - 1\right\} = \sum_1^\infty e_n n^{-s},$$

where $\sigma > 1$, N is an integer greater than 3, and $\mu(n)$ is Möbius' function. (H. p.567) The coefficient e_n will, of course, also depend on N.

Since $(1 - 2^{1-s})\zeta(s) = \sum_1^\infty (-)^{n-1} n^{-s}$ we have

6.1 $$\sum_1^\infty e_n n^{-s} = \sum_1^\infty \frac{(-)^{n-1}}{n^s} \sum_1^N \frac{\mu(n)}{n^s} - 1 + \frac{2}{2^s}$$ convergent for $\sigma > 0$

Further, since $\frac{1}{\zeta(s)} = \sum_1^\infty \mu(n) n^{-s} \quad (\sigma > 1)$ (H. p.576), we have

6.2 $$\sum_1^\infty \frac{e_n}{n^s} = (1 - 2^{1-s})\sum_1^\infty n^{-s}\left\{\sum_1^N \mu(n) n^{-s} - \sum_1^\infty \mu(n) n^{-s}\right\} = -\sum_1^\infty (-)^{n-1} n^{-s} \sum_{N+1}^\infty \mu(n) n^{-s},$$

from which it is evident that

$$e_n = 0 \qquad \text{for} \quad n \leq N.$$

From 6.1, for $n > 2$

$$|e_n| = \left|\sum_{\substack{mq = n \\ q \leq N}} (-)^{m-1} \mu(q)\right| \leq \sum_{q / N} 1 < C N^{\frac{1}{2}\epsilon}$$

where C depends on ϵ and is greater than 1 (and will be taken as a Borel constant) and ϵ is positive and arbitrarily small.

For $x > N$

$$\sum_{n \leq x} \frac{e_n}{n^{\frac{1}{2}\epsilon}} = \left|\sum_{\substack{3 \leq mq \leq x \\ q \leq N}} \frac{(-)^{m-1}}{m^{\frac{1}{2}\epsilon}} \frac{\mu(q)}{q^{\frac{1}{2}\epsilon}}\right| = \left|\sum_{\substack{q \leq x \\ q \leq N}} \frac{\mu(q)}{q^{\frac{1}{2}\epsilon}} \sum_{3/q \leq m \leq x/q} \frac{(-)^{m-1}}{m^{\frac{1}{2}\epsilon}}\right| \leq \sum_{q \leq N} 1 \leq N$$

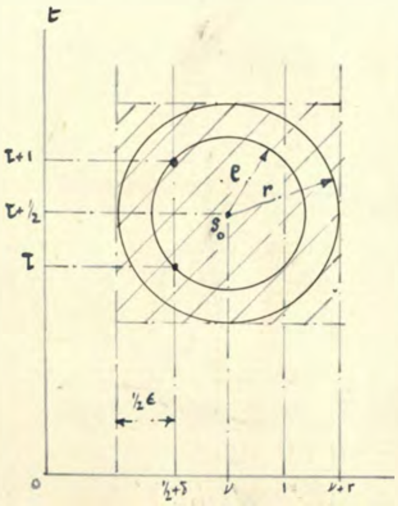
We now apply the mean value theorem 17, taking :-

$$c_n = \frac{e_n}{C n^{\frac{1}{2}\epsilon}}, \quad \delta = N, \quad h = \delta - \epsilon,$$

where δ is a positive small number, independent of ϵ, such that $0 < \epsilon < \delta < \tfrac12$. Since $h < \tfrac12$, we have, for $\sigma \geq \tfrac12 + (\delta-\epsilon) + \tfrac12\epsilon = \tfrac12 + \delta - \tfrac12\epsilon$, $T > 8$

6.3 $$\int_{-T}^{T} |1-2^{1-s}|^2 \left|\zeta(s)\sum_{1}^{N}\frac{\mu(n)}{n^s}-1\right|^2 dt < C\left\{\frac{TN^{-2h}}{h} + \frac{44}{h^2}(NT)^{1-2h}\log^2(NT)\right\} < C\left\{TN^{-2\delta} + (NT)^{1-2\delta}\right\}(NT)^{3\epsilon}.$$

It is to be observed that C has now become dependent on δ as well as on ϵ, but remains a Borel constant as far as T and N are concerned. We discuss the integral appearing in 6.3, by means of Theorem 18.



Choose $\nu = \nu(\delta,\epsilon)$ so that $r = r(\delta,\epsilon) = \nu - (\tfrac12 + \delta - \tfrac12\epsilon)$ is integral and greater than $\rho = \rho(\delta,\epsilon) = \left\{(\nu - \tfrac12 - \delta)^2 + \tfrac14\right\}^{\frac12}$. Take $s_0 = \nu + i(T + \tfrac12)$ as centre of two circles of radii r and ρ respectively, so that the circle of radius ρ will pass through the points $\tfrac12 + \delta + Ti$ and $\tfrac12 + \delta + (T+1)i$. By Theorem 18 and trivial reasoning, we find that the number of zeros of $\zeta(s)$ in the shaded region $(T > r - \tfrac12)$

is
$$\leq \frac{4}{\pi(r-\rho)^2}\int_{\frac12+\delta-\frac12\epsilon}^{\nu+r} d\sigma \int_{T+\frac12-r}^{T+\frac12+r} \left|\zeta(s)\sum_{1}^{N}\frac{\mu(n)}{n^s}-1\right|^2 dt$$

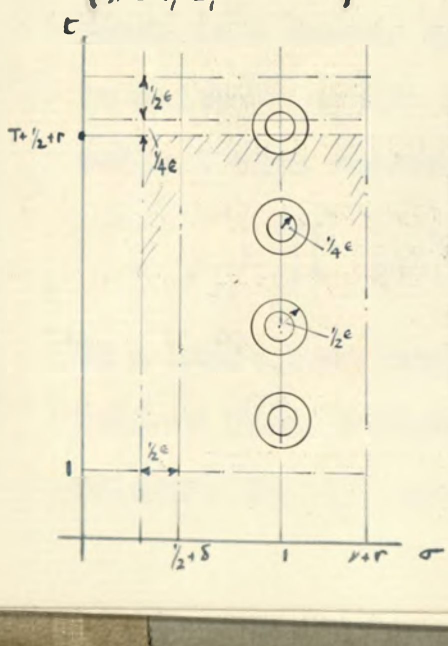
Hence the number of zeros in the region $\sigma \geq \tfrac12 + \delta$, $r+1 \leq t \leq T$, is
$$\leq \frac{4}{\pi(r-\rho)^2}\cdot 2r \int_{\frac12+\delta-\frac12\epsilon}^{\nu+r} d\sigma \int_{1}^{T+\frac12+r} \left|\zeta(s)\sum_{1}^{N}\frac{\mu(n)}{n^s}-1\right|^2 dt$$

Consequently

6.4 $$N(\tfrac12+\delta, T) < C + C\int_{\frac12+\delta-\frac12\epsilon}^{\nu+r} d\sigma \int_{1}^{T+\frac12+r} \left|\zeta(s)\sum_{1}^{N}\frac{\mu(n)}{n^s}-1\right|^2 dt.$$

From 6.3, since r and ν only depend on δ and ϵ, we have
$$\int_{\frac12+\delta-\frac12\epsilon}^{\nu+r} d\sigma \int_{1}^{T+\frac12+r} |1-2^{1-s}|^2\left|\zeta(s)\sum_{1}^{N}\frac{\mu(n)}{n^s}-1\right|^2 dt < C(NT)^{3\epsilon}\left\{TN^{-2\delta} + (NT)^{1-2\delta}\right\}$$

Now $(1-2^{1-s})$ has zeros on the line $\sigma = 1$, with ordinates $t = \dfrac{2\lambda\pi}{\log 2}$. $(\lambda = 1, 2, 3 \cdots)$



Enclose these zeros with circles of radius $\tfrac14\epsilon$. Let the region of the above integration be called H(T) and let D(T) denote the region H(T) less the circles, and portions of circles, which enclose those zeros of $(1 - 2^{1-s})$ which lie in, or cut H. Outside the circles $|1 - 2^{1-s}| > k = k(\delta,\epsilon)$. Hence for $T > r+1$, we have :—

6.5 $$\iint_{D(T)} \left|\zeta(s)\sum_1^N \mu(n)n^{-s}-1\right|^2 d\sigma\,dt < \frac{c}{k^2}(NT)^{3\epsilon}\left\{TN^{-2\delta}+(TN)^{1-2\delta}\right\} < c(NT)^{3\epsilon}\left\{TN^{-2\delta}+(TN)^{1-2\delta}\right\},$$

Inside the circles $|s-v_\lambda| = \frac{1}{2}\epsilon$, where v_λ is a zero of $(1-2^{1-s})$, the function $\zeta(s)\sum_1^N \mu(n)n^{-s}-1$ is regular, and we easily see that

$$\iint_{|s-v_\lambda|\le\frac{1}{4}\epsilon} \left|\zeta(s)\sum_1^N \mu(n)n^{-s}-1\right|^2 d\sigma\,dt \le \iint_{\frac{1}{4}\epsilon\le|s-v_\lambda|\le\frac{1}{2}\epsilon} \left|\zeta(s)\sum_1^N \mu(n)n^{-s}-1\right|^2 d\sigma\,dt$$

In order to sum for all circles which may possibly intersect H(T) it is necessary take all values of λ in the range

$$0 \le \frac{2\pi\lambda}{\log 2} \le T+\tfrac{1}{2}+r+\tfrac{1}{4}\epsilon,$$

Denote a sum for these values of λ by $\sum_\lambda'$, then

6.6 $$\sum_\lambda{}' \iint_{|s-v_\lambda|\le\frac{1}{4}\epsilon} \left|\zeta(s)\sum_1^N \frac{\mu(n)}{n^s}-1\right|^2 d\sigma\,dt \le \sum_\lambda{}' \iint_{\frac{1}{4}\epsilon\le|s-v_\lambda|\le\frac{1}{2}\epsilon} \le \iint_{D(T+\frac{3}{4}\epsilon)} \left|\zeta(s)\sum_1^N \frac{\mu(n)}{n^s}-1\right|^2 d\sigma\,dt < c(NT)^{3\epsilon}\left\{TN^{-2\delta}+(TN)^{1-2\delta}\right\}$$

Equations 6.4, 6.5 and 6.6 give

$$N(\tfrac{1}{2}+\delta, T) < c\,(NT)^{3\epsilon}\left\{TN^{-2\delta}+(NT)^{1-2\delta}\right\}$$

Choose $N=[T^{2\delta}]$, then for $T > T_0(\delta,\epsilon)$ we have

$$N(\tfrac{1}{2}+\delta,T) < cT^{(1+2\delta)3\epsilon}\,T^{1-4\delta^2} < cT^{1-4\delta^2+6\epsilon} < T^{1-4\delta^2+7\epsilon}$$

Hence, by a slight change in notation, we obtain Carlson's result:-

THEOREM 24 For a fixed $\sigma>\frac{1}{2}$, and $T > T_0(\epsilon,\delta)$ we have

$$N(\sigma,T) = O\left\{T^{1-4(\sigma-\frac{1}{2})^2+\epsilon}\right\}$$

where ϵ is arbitrarily small and positive.

+

By comparison with Mangoldt's form for N(T). we infer that the majority of the zeros of the Zeta-function lie in the neighbourhood of the critical line.

The germ of the method employed in the above proof is to be found in a memoir by Bohr and Landau, ('Palermo Rendiconti' 37. 1914) in which $N(\sigma,T)=O(T)$ for $\sigma>\frac{1}{2}$, was deduced by use of a theorem less refined than theorem 18, namely:-

$$Q < K\iint_{|z|\le r} |b(z)|^2\,du\,dv,$$ using the same notation.

In a subsequent memoir ('Comptes Rendus' 158. 1914) Bohr and Landau refined their methods and obtained $N(\sigma,T)=o(T)$, but although the theorem relative to Q was improved so as to include the double integral of $|b(z)-1|^2$

the essential point - the vanishing of the initial terms of $f(z) - 1$ when expanded in the form $\sum c_n n^{-z}$, if $f(z)$ is suitably chosen - was missed.

It now remains to consider the number of zeros of $\zeta(s)$ which actually lie on the critical line, i.e. a discussion of $N_0(T)$.

That some zeros lie on the critical line, was first established by Gram in 1895 who, by numerical methods, detected the presence of three zeros of form $\frac{1}{2} + it$, with positive values of t.

Gram continued his numerical investigations of $\zeta(\frac{1}{2} + it)$ (using the Euler-Maclaurin sum formula) in a memoir of 1903 ('Acta Math.' 27) and gave the ordinates of the first fifteen zeros lying on the critical line.

Backlund has continued the numerical determination of the zeros; his latest result ('Comptes Rendus' 158. 1914) gives the first seventy-nine zeros of $\zeta(s)$ all of which he finds lie on the critical line with ordinates in the range $0 < t < 200$. It is, of course, only necessary to consider the zeros with positive imaginary part.

That an infinite number of zeros lie on the critical line was first proved by Hardy in 1914 ('Comptes Rendus' 158) in a memoir which marks, as far as the zeros of $\zeta(s)$ are concerned, the greatest step forward since the appearance of Riemann's memoir. Working from the Mellin integral

$$e^{-y} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)\, y^{-u}\, du \qquad \{\Re(y) > 0,\ k > 0\}$$

taking $y = \pi e^{i\alpha}\ (-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi)$ and setting

$$e^{-y} = e^{-\pi\cos\alpha - i\pi\sin\alpha} = e^{\pi i\tau} = q = \rho e^{i\varphi},$$

Hardy deduced that

6.5 $$\int_0^\infty \frac{e^{\alpha t} + e^{-\alpha t}}{\frac{1}{4} + t^2}\, t^{2n}\, \Xi(t)\, dt = \frac{(-)^n \pi}{4^{2n}} \cos\frac{\alpha}{4} - \left(\frac{d}{d\alpha}\right)^{2n}\left\{\tfrac{1}{2}\pi e^{\frac{1}{4}i\alpha}\, \vartheta_3(0|\tau)\right\}$$

and then, by making $\alpha \to \frac{1}{2}\pi$, so that $q \to -1$ following a tangent to radius $\varphi = \pi$, and using the transformation $\vartheta_3(0|\tau) = (-i\tau)^{-\frac{1}{2}}\vartheta_3(0|\tau')$, where $\tau\tau' = -1$, the last term vanishes and we obtain

6.6 $$\int_0^\infty \frac{\cosh \frac{1}{4}\pi t}{t^2 + \frac{1}{4}}\, t^{2n}\, \Xi(t)\, dt = \frac{(-)^n \pi}{2^{2n}} \cos\frac{\pi}{8}.$$

From this, Hardy proves that, by taking n large, there is no number t_0 such that $\Xi(t)$ is of fixed sign when $t > t_0$, and thence $\zeta(s)$ has an infinity of zero on the line $\sigma = \frac{1}{2}$.

Landau ('Math.Ann.76. 1915') considerably improved the method of proof by shewing that the analysis from 6.5l to 6.6 inclusive could be replaced by using the elementary fact that $\vartheta_3(0/\tau) = O\left\{\sqrt{\frac{1}{1-|q|}}\right\}$ further, he deduced that $\zeta(s)$ has at least one zero on the line joining $\frac{1}{2}+\tau i$ (exclusive) to $\frac{1}{2}+\tau^{e} i$ (inclusive), from which can be obtained

$$\lim_{T\to\infty} \frac{N_0(T)}{\log\log T} > 0$$

Hardy and Littlewood considerably improved this result in 1918 by giving $N_0(T) > K T^{3/4-\delta}$ but the best results we posses were given by these writers in 1921 ('Math.Zeits' 10) in what is possibly the deepest contribution yet made to the theory of the Zeta-function.

LEMMA 1.

If $\frac{1}{2} < k < 1$, $-k < \mathfrak{R}(a) < 1-k$, $-k < \mathfrak{R}(b) < 1-k$, $\mathfrak{R}(y) > 0$,

and y^{-2s} has its principal value, then

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(s+a)\Gamma(s+b)\zeta(2s+2a)\zeta(2s+2b)\, y^{-2s}\, ds = i\pi^{\frac{1}{2}\nu\pi i}\, y^{a+b} \sum_{n=1}^{\infty} c_n H_\nu^{(1)}(2iny),$$

where $\nu = a-b$, $H_\nu^{(1)}$ is Hankel's cylinder function, and

$$c_n = n^{b-a}\sigma_{2(a-b)}(n) = n^{-\nu}\sigma_{2\nu}(n) = n^{\nu}\sigma_{-2\nu}(n),$$

$\sigma_r(n)$ being the sum of the r^{th} powers of the division of n.

Since

$$\zeta(2s+2a)\zeta(2s+2b) = \sum_{m,n} \frac{1}{(mn)^{2s} m^{2a} n^{2b}} = \sum_{1}^{\infty} \frac{d_n}{n^{2s}},$$

where

$$d_n = \sum_{d/n} d^{-2a}\left(\frac{n}{d}\right)^{-2b} = n^{-2a}\sigma_{2\nu}(n) = n^{-a-b} c_n$$

the integrand is $\Gamma(s+a)\Gamma(s+b)\, y^{-2s} \sum d_n n^{-s}$, so that it is only necessary to show that

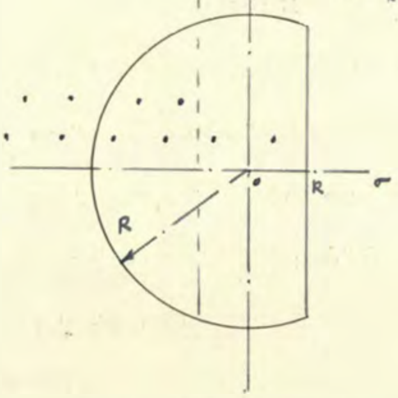
$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(s+a)\Gamma(s+b)(ny)^{-2s}\, ds = i\pi (ny)^{a+b} e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(2iny)$$

or, what is the same thing,

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(s+a)\Gamma(s+b)\, y^{-2s}\, ds = i\pi\, y^{a+b} e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(2iy).$$

The integrand has poles at $s=-n-{b \atop a}$ $(n=0,1,2,\cdots)$

Taking the contour shewn, and making $R\to\infty$, we easily find



$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} = \sum_0^\infty \frac{(-)^n\Gamma(b-a-n)}{n!}\, y^{2n+2a} + \sum_0^\infty \frac{(-)^n\Gamma(a-b-n)}{n!}\, y^{2n+2b}$$

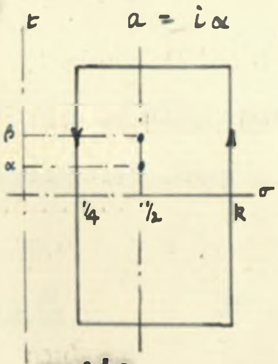
$$= \frac{\pi y^{a+b}}{\sin\pi(a-b)}\left\{\sum_0^\infty \frac{y^{2n+b-a}(-)^n}{n!\,\Gamma(1-a+b+n)} - \sum_0^\infty \frac{(-)^n y^{2n+a-b}}{n!\,\Gamma(1-b+a+n)}\right\}$$

$$= -\frac{\pi y^{a+b}e^{\frac12\nu\pi i}}{\sin\pi(a-b)}\left\{e^{-\nu\pi i}\sum_0^\infty \frac{(-)^n (iy)^{2n+b-a}}{\Gamma(1-a+b+n)} - \sum_0^\infty \frac{(-)^n (iy)^{2n+a-b}}{\Gamma(1-b+a+n)}\right\}$$

$$= -\frac{\pi y^{a+b}e^{\frac12\nu\pi i}}{\sin\nu\pi}\left\{e^{-\nu\pi i}J_\nu(2iy) - J_{-\nu}(2iy)\right\} = i\pi y^{a+b}e^{\frac12\nu\pi i}H_\nu^{(1)}(2iy).$$

Suppose that a and b are pure imaginaries, e.g. let

6.7 $a=i\alpha$, $b=i\beta$, with $\alpha\neq\beta$, and $0\le {\alpha \atop \beta}\le H$.



Deform the contour as shown in the diagram - the justification is simple. This introduces two poles, at $s=\frac12-{a \atop b}$, the sum of their residues being

$$\tfrac12\pi\left\{\Gamma(\tfrac12+b-a)\zeta(1+2b-2a)y^{-1+2a} + \Gamma(\tfrac12+a-b)\zeta(1+2a-2b)y^{-1+2b}\right\} = O\{b(H)\}$$

Write

$$\frac{1}{2\pi i}\int_{\frac14-i\infty}^{\frac14+i\infty}\Gamma(s+a)\Gamma(s+b)\zeta(2s+2a)\zeta(2s+2b)y^{-2s} = \frac{1}{2\pi i}\left\{\int_{\frac14-i\infty}^{\frac14-iH}+\int_{\frac14-iH}^{\frac14}+\int_{\frac14}^{\frac14+i\infty}\right\}$$

The second of these integrals is evidently $O\{b(H)\}$

Since $|\Gamma(\frac14+i(t+\alpha))| < A|t+\alpha|^{-\frac14}e^{-\frac12\pi|t+\alpha|} < A|t+H|^{-\frac14}e^{-\frac12\pi|t+H|}$, (A is a Borel constan

and $|\zeta(\frac12+2i(t+\alpha))| < A\{|t|^{\frac14}+1\}$ (see 3.5), and similarly for the factors involving β, and, further, $|y^{-2s}| < Ae^{2\theta t} < Ae^{-\frac12\pi|t|}$ ~~and~~

6.8 if we take $y=\pi e^{i\theta}=\pi e^{i(\frac12\pi-\epsilon)}$ (ϵ to be made eventually zero),

We find that the first of the above integrals is

$$< A\int_{-\infty}^{-H}|t+H|^{-\frac12}\left\{|t|^{\frac14}+1\right\}^2 e^{-\pi|t+H|-\frac12\pi|t|}\,dt < b(H)$$

Hence, these results give,

6.9
$$\frac{1}{2\pi i}\int_{\frac14}^{\frac14+i\infty}\Gamma(s+a)\Gamma(s+b)\zeta(2s+2a)\zeta(2s+2b)y^{-2s}\,ds = i\pi y^{a+b}e^{\frac12\nu\pi i}\sum_1^\infty c_n H_\nu^{(1)}(2iny) + O\{b(H)\}$$

LEMMA 2 With same notation and definitions 6.7 and 6.8, we have

$$\sum_1^\infty c_n H_\nu^{(1)}(2iny) = A\,\frac{e^{\frac12\pi(\alpha-\beta)}}{i}\left\{\sum_1^\infty c_n\frac{e^{-2\pi n\epsilon}}{\sqrt{n}} + O(1)\right\}$$

It is known that

$$H_{\nu}^{(1)}(2iny) = \frac{A}{\sqrt{niy}}\, e^{-2ny - \frac{1}{4}\pi i(2\nu+1)}\left\{1 + O\left(\tfrac{1}{n}\right)\right\}$$

, (G.N.Watson, '<u>A treatise on Bessel Functions</u>' p.198), hence with notation of the lemma,

$$H_{\nu}^{(1)}(2iny) = \frac{A}{i}\,\frac{e^{\frac{1}{2}\pi(\alpha-\beta)}}{\sqrt{n}}\, e^{-2n\pi\epsilon}\, e^{O(n\epsilon^2)}\left\{1 + O\left(\tfrac{1}{n}\right)\right\}$$

It is easy to justify the replacing of $e^{O(n\epsilon^2)}$ by $1 + O(n\epsilon^2)$,
For, if $n\epsilon^2 < A$ then, $e^{An\epsilon^2} < 1 + An\epsilon^2$
and for $n\epsilon^2 \geq A$

$$\sum_{n\epsilon^2 \geq A} |c_n|\,\frac{e^{-2\pi n\epsilon + O(n\epsilon^2)}}{\sqrt{n}} < \sum_{n\epsilon^2 \geq A} n\, e^{-\pi n\epsilon} < A\, e^{-A/\epsilon}.$$

Hence,

$$\sum_{1}^{\infty} c_n H_{\nu}^{(1)}(2iny) = \frac{A e^{\frac{1}{2}\pi(\alpha-\beta)}}{i}\left\{\sum_{1}^{\infty} c_n \frac{e^{-2\pi n\epsilon}}{\sqrt{n}}\left(1 + O\left(\tfrac{1}{n}\right) + O(n\epsilon^2)\right)\right\}$$

Since

$$c_n = n^{i(\beta-\alpha)}\sigma_{2i(\alpha-\beta)}(n) = O\{d(n)\} = O\left(n^{1/4}\right)$$

it is wasily seen that the last two terms in the above, are $O(1)$;
hence the lemma follows.

<u>LEMMA 3</u> If

$$b_n = \int_0^H\int_0^H c_n\, d\alpha\, d\beta \qquad (0<\alpha<H,\ 0<\beta<H)$$

then

$$\sum_{1}^{m} b_n = O(Hm).$$

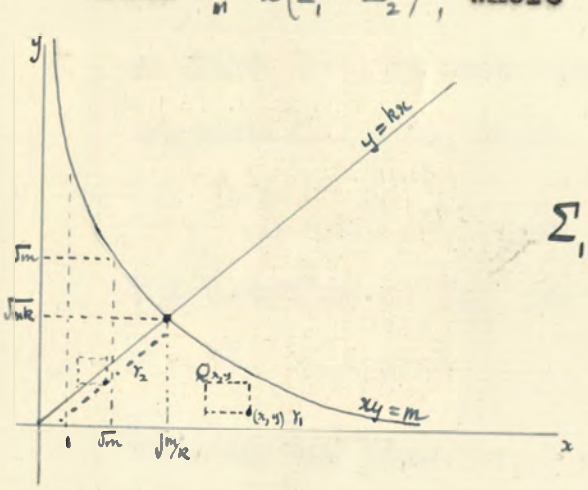
we have

$$b_n = \int_0^H\int_0^H n^{i(\alpha-\beta)}\sum_{d/n} d^{2i(\beta-\alpha)}\, d\alpha\, d\beta = \sum_{d/n}\int_0^H\int_0^H \left(\frac{n}{d^2}\right)^{i(\alpha-\beta)} d\alpha\, d\beta = A\sum_{d/n}\left\{\frac{\sin \frac{1}{2}H\log n/d^2}{\log n/d^2}\right\}^2,$$

so that

$$\sum_{1}^{m} b_n = A\sum_{\substack{x,y>0\\ xy\leq m}}\left\{\frac{\sin\frac{1}{2}H\log x/y}{\log x/y}\right\}^2 = A\, S_m.$$

Write $S_m = 2(\Sigma_1 + \Sigma_2)$, where Σ_1 is defined by $0 < kx \leq y \leq x$, $xy \leq m$,
and Σ_2 is defined by $0 < y \leq kx$, $xy \leq m$ $(\frac{1}{2} < k < 1)$



In Σ_1, using $\left(\frac{\sin Hu}{Hu}\right)^2 < H^2$, we have

$$\Sigma_1 \leq H^2\left\{\sum_{1\leq x\leq\sqrt{m}}\left\{\overline{1-k}\,x + 1\right\} + \sum_{\sqrt{m}\leq x\leq\sqrt{m/k}}\left(\frac{m}{x} - kx + 1\right)\right\}$$

$$= O\{H^2 m(1-k)\} + O(H^2\sqrt{m}) + O\left\{H^2 m \log\frac{1}{k}\right\}$$

$$= O\left\{H^2 m\log\frac{1}{k}\right\} + O(H^2\sqrt{m}),$$ when $m \to \infty$.

To evaluate Σ_2, divide its terms into two groups thus:- To every point (x, y) associate a square $Q_{x,y}$, of which two opposite corners are

are (x,y) and $(x-1,y+1)$, then the FIRST CLASS, γ_1, will contain all terms for which the associated square does not cross the line $y=kx$, and the SECOND CLASS, γ_2, will contain terms whose associated square does cross this line. If (x,y) belongs to γ_1 then,

$$\frac{1}{\{\log \frac{x}{y}\}^2} = \iint_{Q_{xy}} \frac{d\xi\, d\eta}{(\log \xi/\eta)^2}, \qquad \text{so that}$$

$$\sum_{\gamma_1} < \iint \frac{d\xi\, d\eta}{(\log \xi/\eta)^2},$$

where the region of integration is defined by $0 \le \eta \le k\xi$, $\xi \le m$, $\xi(\eta-1) \le m$, and, a fortiori, defined by $0 \le \eta \le k\xi$, $\xi\eta \le 2m$.

Transforming to polar co-ordinates, we find

$$\sum_{\gamma_1} < 2m \int_0^{\tan^{-1}k} \frac{d\theta}{\cos\theta \sin\theta \{\log \tan\theta\}^2} = 2m \int_0^k \frac{dt}{t(\log t)^2} = O\left\{\frac{m}{\log \frac{1}{k}}\right\}.$$

Evidently $\sum_{\gamma_2} = O(H^2\sqrt{m})$ since the number of terms in γ_2 is less than A $\times$ length of line joining origin to the point $\left(\sqrt{\frac{m}{k}}, \sqrt{mk}\right)$.

Hence, collecting results, we have

$$\sum_1^m b_m = O\left\{H^2 m \log \frac{1}{k}\right\} + O\left\{H^2\sqrt{m}\right\} + O\left\{\frac{m}{\log 1/k}\right\}$$

Take $k = 1 - \frac{1}{H}$ and we obtain

$$\sum_1^m b_n = O\{Hm\}$$

We are now in a position to prove the following theorem:—

THEOREM 25 There exists a constant $K>0$, and a T_0 such that

$$N_0(T) > KT \qquad \text{for } T > T_0.$$

Let us write

$$s = \tfrac{1}{2} + it,$$

$$\Gamma(\tfrac{1}{2}s)\zeta(s) = \frac{2\pi^{\frac{1}{2}s}\,\Xi(t)}{s(s-1)} = 2\pi^{1/4} t^{1/4} e^{-\frac{1}{4}\pi t} \pi^{\frac{1}{2}it} X(t),$$

so that $X(t)$ is real for real values of t.

Suppose $t>0$ then, using Stirling's theorem,

$$\zeta(s) = -\left(\frac{2}{\pi}\right)^{1/4} e^{\frac{1}{8}\pi i} (2\pi e)^{\frac{1}{2}it} e^{-\frac{1}{2}it\log t} X(t)\left\{1 + O\left(\frac{1}{t}\right)\right\}$$

Use notation of 6.7 and 6.8, i.e $a = i\alpha$, $b = i\beta$, $\alpha \neq \beta$, $0 < {\alpha \atop \beta} < H$,

$$y = \pi e^{i\theta} = \pi e^{i(\frac{1}{2}\pi - \epsilon)} \qquad (\epsilon \to 0)$$

then on the line $s = \frac{1}{4} + it$ we have,

$$\Gamma(s+a)\zeta(2s+2a) = A(t+\alpha)^{-1/4} e^{-\frac{1}{2}\pi(t+\alpha)} \pi^{i(t+\alpha)} X(2t+2\alpha)\left\{1 + O\left(\frac{1}{t}\right)\right\}$$

$$= A\, t^{-1/4} e^{-\frac{1}{2}\pi(t+\alpha)} \pi^{i(t+\alpha)} X(2t+2\alpha)\left\{1 + O(1/t)\right\},$$

and similarly for $\Gamma(s+b)\zeta(2s+2b)$.

Also $$y^{-2s} = \pi^{-\frac{1}{2}-2it} e^{-\frac{1}{4}(\pi-2\epsilon)i} e^{(\pi-2\epsilon)t}$$

so that 6.9, Lemma 1, may be written

$$A e^{-\frac{1}{4}\pi i} e^{-\frac{1}{2}\pi(\alpha+\beta)} \pi^{i(\alpha+\beta)} \{1+O(\epsilon)\} \int_0^\infty t^{\frac{1}{2}} X(2t+2\alpha) X(2t+2\beta) e^{-2\epsilon t} \{1+O(\tfrac{H}{t})\}\, dt$$

$$= i\pi y^{a+b} e^{\frac{1}{2}\nu\pi i} \sum_1^\infty c_n H_\nu^{(1)}(2iny) + O\{b(H)\}$$

It is easily seen that $X(2t+2\alpha)X(2t+2\beta) = O\{b(H)t^{2/3}\}$

and $y^{-2s} = O\{e^{(\pi-2\epsilon)t}\}$, so that the error term in the above integral contributes $O\{b(H)\epsilon^{-1/6}\}$

Modify the right hand side of the above equation by using Lemma 2, and $i\pi e^{\frac{1}{2}\nu\pi i} y^{a+b} = A i \pi^{i(\alpha+\beta)} e^{-\pi\alpha}\{1+O(\epsilon)\}$.

Some simple reduction gives

$$\int_0^\infty t^{-\frac{1}{2}} X(2t+2\alpha)X(2t+2\beta) e^{-2\epsilon t}\, dt = A\sum_1^\infty c_n \frac{e^{-2n\pi\epsilon}}{\sqrt{n}} + O\{b(H)\epsilon^{-1/6}\}$$

Integrate with respect to α and β from 0 to H, and we have, writing $I = I(t,H) = \int_t^{t+H} X(u)\, du$, and using Lemma 3,

$$\int_0^\infty t^{-\frac{1}{2}} e^{-2\epsilon t} \{I(2t,2H)\}^2 dt = O\{\tfrac{H}{\sqrt{\epsilon}}\}$$

Hence

$$\int_0^\infty e^{-2\epsilon t} I^2 dt = \int_0^\infty e^{-\epsilon t} t^{\frac{1}{2}} . t^{-\frac{1}{2}} e^{-\epsilon t} I^2 dt = O(\tfrac{1}{\sqrt{\epsilon}})\int_0^\infty I^2 t^{-\frac{1}{2}} e^{-\epsilon t} dt = O(\tfrac{H}{\epsilon})$$

Consequently

6.91 $$\int_0^T \{I(t,H)\}^2 dt < e\int_0^T e^{-\frac{t}{T}} I^2 dt < e\int_0^\infty e^{-\frac{t}{T}} I^2 dt < AHT.$$

Now let $$\underset{\sim}{I} = \underset{\sim}{I}(t,H) = \int_t^{t+H} |X(u)|\, du,$$

then if $\underset{\sim}{I} = |I|$ there is no zero of $X(u)$ in the interval $(t, t+H)$

If $\eta(s) = (1-2^{1-s})\zeta(s) = 1 + \sum_2^\infty \frac{(-)^{n-1}}{n^s}$, and $s = \frac{1}{2}+it$, then

$$|X(t)| > B\left|\frac{\eta(s)}{1-2^{1-s}}\right| > B|\eta(s)|, \qquad \text{so that}$$

$$\underset{\sim}{I} > B\Re\int_t^{t+H} \eta(s)\, ds = BH + B\Re\left\{i\sum_2^\infty \frac{(-)^{n-1}}{n^{\frac{1}{2}+i(t+H)}\log n} - i\sum_2^\infty \frac{(-)^{n-1}}{n^{\frac{1}{2}+it}\log n}\right\}.$$

6.92 Hence

$$\underset{\sim}{I} > BH + \psi, \quad \text{say}.$$

Let $\mathfrak{I}$ denote the interval $(T, 2T)$ and $\mathfrak{U}$ the subset of $\mathfrak{I}$ in which $\underset{\sim}{I} < \frac{1}{2}BH$, B being the constant of 6.92

Then in $\mathfrak{U}$ we must have

$$|\psi| > \tfrac{1}{2}BH.$$

Now in notation of Theorem 16, it is evident that

$$\psi = O\left\{|\chi(\tfrac{1}{2}+i(t+H))| + |\chi(\tfrac{1}{2}+it)|\right\},$$

and it follows from this Theorem that,

$$\int_T^{2T} |\psi|^2\,dt < CT.$$

Let $m\mathfrak{U}$ denote the measure of $\mathfrak{U}$ then, $\tfrac{1}{4}A^2 H\, m\mathfrak{U} < CT$.

6.93 Therefore $m\mathfrak{U} < \epsilon_H T$, where $\epsilon_H \to 0$ as $H \to \infty$.

Hence $\underset{\sim}{I} > \frac{1}{2}BH$ if t lies in $\mathfrak{I}$ and H is sufficiently large, except perhaps in a subset $\mathfrak{U}$ of $\mathfrak{I}$ whose measure is less than $\epsilon_H T$.

But from 6.91

$$\int_T^{2T} I^2\,dt < AHT.$$

Hence if $|I| > BH$ in a subset $\mathfrak{V}$ of $\mathfrak{I}$ of measure $m\mathfrak{V}$, we have

6.94
$$m\mathfrak{V} < \epsilon_H T.$$

Comparing these results we see that $|I| < \underset{\sim}{I}$ through all $\mathfrak{I}$ except a subset $\mathfrak{W}$ of measure less than $\epsilon_H T$.

Divide $\mathfrak{I}$ into $\left[\frac{T}{2H}\right]$ pairs of abutting intervals γ_1, γ_2, each except the last γ_2 of length H, and each γ_2 lying immediately to the right of the corresponding γ_1. Then either γ_1 or γ_2 contains a zero of $X(t)$, unless γ_1 consists entirely of points of $\mathfrak{W}$.

If the second case occurs for n γ_1's, we have $nH < \epsilon_H T$

$$\therefore \quad n < \frac{\epsilon_H}{H} T.$$

Hence there are in $\mathfrak{I}$ at least $(\tfrac{1}{2} - \epsilon_H)\frac{T}{H} > T/4H$ zeros of $X(t)$, (H being sufficiently large) and consequently, since the zeros of $\zeta(\tfrac{1}{2}+it)$ have ordinates given by zeros of $X(t)$, the theorem is proved.

An even deeper theorem given by Hardy and Littlewood in the same memoir, and deduced by analogous methods, but using the approximate functional equation, is:-

THEOREM 24 Let $U = T^a$, where $a > \frac{1}{2}$. Then $K = K(a) > 0$ exists such that

$$N_0(T+U) - N_0(T) > KU \quad \text{for} \quad T > T_0 = T_0(a)$$

Extremely interesting results are disclosed when numerical determinations of $\zeta(\frac{1}{2}+it)$ are made. The most convenient means for performing these calculations is to utilise the Euler-Maclaurin sum Formulae, which gives, without much difficulty (See Lindelof, '<u>Sur une formule sommatoire generale</u>'. '<u>Acta Math.</u>' 27. 1903)

7.0 $$\zeta(s) = \sum_1^N n^{-s} - \tfrac{1}{2} N^{-s} + \frac{N^{1-s}}{1-s} + \sum_1^K T_n + R_K$$

where
$$T_n = (-1)^{n+1} \frac{B_n}{2n} \cdot \frac{s(s+1)\cdots(s+2n-2)}{(2n-1)!} n^{-s+1-2n}$$

B_n being the n^{th} Bernoullian number

and
$$|R_K| < |s+2K+1| \left(\frac{1}{N+2K+1} + \frac{1}{2N} \right) |T_{K+1}| .$$

Further help, both from a theoretical and numerical point of view, can be obtained from the following analysis due to Gram ('<u>Act Math.</u>' 27 1903). Let the real and imaginary parts of $\zeta(\frac{1}{2}+it)$ be denoted by $A(t)$ and $B(t)$ respectively, so that we may write

7.1 $$\zeta(\tfrac{1}{2}+it) = A(t) + i B(t) = M e^{i\varphi(t)}$$

Using the functional equation, we find
$$e^{-2i\varphi(t)} = \frac{\zeta(\frac{1}{2}-it)}{\zeta(\frac{1}{2}+it)} = 2^{\frac{1}{2}-it} \pi^{-\frac{1}{2}-it} \cos\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\pi i t\right) \Gamma\left(\tfrac{1}{2}+it\right)$$

hence
$$-2i\varphi(t) = -ti \log 2\pi + \tfrac{1}{2} \log \frac{\cos(\frac{1}{4}\pi + \frac{1}{2}\pi i t)}{\cos(\frac{1}{4}\pi - \frac{1}{2}\pi i t)} + \tfrac{1}{2} \log \frac{\Gamma(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}-it)} .$$

Reduction of this equality eventually gives
$$-2\varphi(t) = \frac{t}{2} \log\left(t^2 + \tfrac{1}{4}\right) - t(1 + \log 2\pi) + \tan^{-1} e^{-\pi t} - \frac{\pi}{4} - \frac{4t}{12(4t^2+1)} + O\left(\frac{1}{t}\right)$$

or, more approximately

7.2 $$-\frac{\varphi(t)}{\pi} = \frac{t}{2\pi}\left(\log \frac{t}{2\pi} - 1\right) - \frac{1}{8} + O\left(\frac{1}{t}\right) .$$

Let ρ denote a zero of $\zeta(\frac{1}{2}+it)$ i.e. a zero of M, and consequently

a value of t which makes $A(t)$ and $B(t)$ simultaneously vanish.
Let α and β respectively denote the remaining zeros of $A(t)$ and $B(t)$, which are evidently given by $\cos\varphi(t) = 0$ i.e. $\pm\varphi(t) = \pi(n+\frac{1}{2})$ and $\sin\varphi(t) = 0$, i.e. $n\pi = \pm\varphi(t)$ so that, approximately

7.3
$$\frac{\alpha}{2\pi}\left(\log\frac{\alpha}{2\pi} - 1\right) = \frac{1}{8} + \frac{1}{2}\cdot 2n+1$$
$$\frac{\beta}{2\pi}\left(\log\frac{\beta}{2\pi} - 1\right) = \frac{1}{8} + n. \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

It is evident that all the zeros, α and β, are real, and it is simple to obtain the following approximate values for the first few

$\alpha_1 \simeq$ 1	$\alpha_6 =$ 33.6	$\beta_1 =$ 3.5	$\beta_6 =$ 31.8
$\alpha_2 =$ 14.7	$\alpha_7 =$ 37.3	$\beta_2 =$ 9.6	$\beta_7 =$ 35.6
$\alpha_3 =$ 20.7	$\alpha_8 =$ 40.6	$\beta_3 =$ 17.8	$\beta_8 =$ 39.0
$\alpha_4 =$ 25.5	$\alpha_9 =$ 44.0	$\beta_4 =$ 23.2	$\beta_9 =$ 42.4
$\alpha_5 =$ 29.7	$\alpha_{10} =$ 47.1	$\beta_5 =$ 27.7	$\beta_{10} =$ 45.6

In order to obtain the zeros of $\zeta(\frac{1}{2}+it)$ it is necessary to separate the real and imaginary parts, $A(t)$ and $B(t)$ using 7.0, then calculate these functions for, say, $t = 1, 2, 3 \cdots$ . . . . and select those values of t which appear to be in the neighbourhood of the values required. The zeros may then be obtained by interpolation, and if the ordinate obtained does not coincide with a value of α and β previously determined, we may be certain of having obtained a true zero of $\zeta(\frac{1}{2}+it)$. The actual process is simple, but the calculations involved are extremely laborious.

Working in this way Gram has obtained the position of the first fifteen zeros as follows:-

$\rho_1 =$ 14.1347	$\rho_4 =$ 30.4249	$\rho_7 =$ 40.9187	$\rho_{10} =$ 49.7738	$\rho_{13} =$ 59.4
$\rho_2 =$ 21.0220	$\rho_5 =$ 32.9350	$\rho_8 =$ 43.3271	$\rho_{11} =$ 52.8	$\rho_{14} =$ 61.0
$\rho_3 =$ 25.0108	$\rho_6 =$ 37.5862	$\rho_9 =$ 48.0051	$\rho_{12} =$ 56.4	$\rho_{15} =$ 65.0

but the first ten are given correct to six decimal places.

The graph of $\zeta(\frac{1}{2}+it)$ for t ranging from 10 to 50 which is appended, has been constructed by making use of Gram's figures above, the values of α and β previously given, some ordinates given by Lindelöf in his memoir already cited, and other ordinates determined by using

7.0 with $N=10$ and $K=1$; neglecting the remainder, which is sufficiently accurate for the purpose required.

A first glance at this graph reveals the striking fact that $\Re\zeta(\frac{1}{2}+it)$ is seldom negative. This, no doubt, is due to the fact that $\Re\zeta(\frac{1}{2}+it)$ is essentially $1+2^{-\frac{1}{2}}\cos t\log 2+3^{-\frac{1}{2}}\cos t\log 3+\dots+N^{-\frac{1}{2}}\cos t\log N$, where N can be taken small when t is in the range of our graph, and, in this case, the initial term, $+1$, exerts a prepondering influence over the remaining. It is not clear whether this influence adjusts itself as t increases - in fact, if Riemann's hypothesis is true, it seems probable that it does not.

A more careful analysis of the graph shews that the zeros of $\zeta(\frac{1}{2}+it)$ are separated by the zeros of B(t), but not the zeros of A(t). This is extremely important since it can be easily seen from 7.3 that the number of zeros of B(t) lying between 10 and T is approximately $\frac{T}{2\pi}\left(\log\frac{T}{2\pi}-1\right)+\frac{7}{8}$, which agrees with von Mangoldt's form, so that a great step forward would be to prove that this interlacing of the zeros of $\zeta(\frac{1}{2}+it)$ and $\Im\zeta(\frac{1}{2}+it)$ persists for all t.

Doubtless in attempting any analysis towards this end, we shall encounter exactly the same difficulties as have already baffled all efforts, but it would be interesting to continue the numerical work up to $T=200$, all the zeros of $\zeta(\frac{1}{2}+it)$ in this range being known.

From 7.1 we have

$$A(t)+iB(t) = e^{2i\varphi}\{A(t)-iB(t)\}$$

By differentiating with respect to t, and successively putting $t=\alpha$ (so that $A=0$, $B\neq 0$, and $e^{2i\varphi}=-1$) and $t=\beta$ (so that $A\neq 0$, $B=0$, and $e^{2i\varphi}=+1$) we obtain

$$A'(\alpha) = -\varphi'(\alpha)B(\alpha), \quad \text{and} \quad B'(\beta) = \varphi'(\beta)A(\beta).$$

Now for $t>2\pi$, $\varphi'(t)$ is always negative, so that we have

$A'(\alpha)$ has always the same sign as $B(\alpha)$

$B'(\beta)$ has always the opposite sign to $A(\beta)$

Let β_ν and $\beta_{\nu+1}$ denote two consecutive zeros of B(t); then, if $A(\beta_\nu)$ and $A(\beta_{\nu+1})$ have same sign, so do $B'(\beta_\nu)$ and $B'(\beta_{\nu+1})$. But since $B(\beta_\nu) = B(\beta_{\nu+1}) = 0$ it must happen that B(t) vanishes at least once in this interval.

For the zeros of B(t) in the interval 10 to 50, we see from the graph that A(β) is always positive, hence the interlacing of the zeros previously mentioned, persists if A(t) is positive for all values of t given by B(t) = 0. This remark allow us to assert that the number of zeros of $\zeta(\frac{1}{2}+it)$ in the range $10 < t < T$ is approximately equal to $\frac{T}{2\pi}\left\{\log\frac{T}{2\pi}-1\right\}+\frac{7}{8}$, provided that $\Re\,\zeta(\frac{1}{2}+it)$ is positive for all values of t given by $\Im\,\zeta(\frac{1}{2}+it)=0$.

Upon glancing through the ordinates of the zeros of $\zeta(\frac{1}{2}+it)$ we at once note their extreme irregularity, and Gram remarked that it is very probable that these roots are intimately connected with the prime numbers. This connection could almost be inferred from Riemann's formula,

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \cdots = Li(x) - \sum_{\rho}\{Li(x^{\rho}) + Li(x^{1-\rho})\} + \int_x^{\infty}\frac{dt}{(t^2-1)t\log t} - \log 2,$$

where $\pi(x)$ denotes the number of primes $\leq x$, $Li(x)$ denotes the logarithmic integral $\int_0^x \frac{dt}{\log t}$, and ρ denotes a complex zero of $\zeta(s)$, the summation running through all values of ρ with positive imaginary part - but this formula may only be an 'accidental' property of the zeros of $\zeta(s)$ in the same way as the area of a circle, πr^2, may be called an 'accidental' property of the number π.

A much more convincing result is obtained when we consider the function defined as follows.

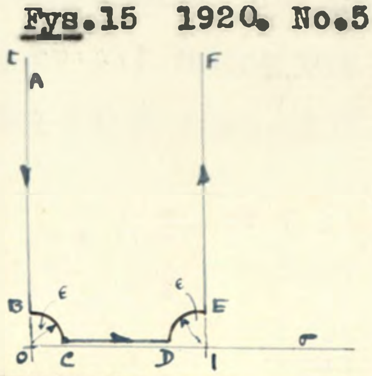
Let $\rho = \eta + i\gamma$, be a zero of the Zeta-function $(0<\eta<1,\ |\gamma|>12)$ and $z = x+iy\ (y>0)$ then define a function $V(z)$ thus:-

$$V(z) = \sum_{\gamma>0} e^{\rho z}$$

where the summation involves all zeros with positive ordinates.

The series evidently converge absolutely.

This function has been very thoroughly analysed by Cramér ('Studien über die Nullstellen der Riemann'schen Zeta-funktion' Math.Zeitschr.4 1919, 'Some theorems concerning prime numbers' Arkiv f.Mat.,Astr.och Fys.15 1920. No.5), and we proceed with his discussion .



Let Γ represent the indented contour shewn in the diagram, then it is not difficult to rigorously prove that

$$2\pi i\, V(z) = \int_{\Gamma} e^{sz}\frac{\zeta'(s)}{\zeta(s)}\,ds.$$

Integrate by parts, and the integrated part vanishes (using $\log\zeta(1+it) = O(\log^2 t)$ H.p.327) and we have

$$2\pi i\, V(z) = -z\int_{\Gamma} e^{sz}\log\zeta(s)\,ds.$$

In order to define the many valued function $\log\zeta(s)$, cut the s-plane from $s=1$ to $s=-\infty$ so that in any closed region of the plane, each branch of $\log\zeta(s)$ is one-valued. Select that branch which is defined by the series $\sum_{p,m}\frac{1}{mp^{ms}}$ for $\sigma>0$, p being a prime number.

It is easy to see that $\log\zeta(s)$ has imaginary part $-\pi i$ above, and πi below, the cut from D to C.

Let us write $\quad 2\pi i\, V(z) = z\int_{CBA} - z\int_{CD} - z\int_{DEF} e^{sz}\log\zeta(s)\,ds.$

where the contours are described in a direction agreeing with the sequence of letters defining them.

Using the functional equation,

$$z\int_{CBA} e^{sz}\log\zeta(s)\,ds = z\int_{CBA} e^{sz}\left\{s\log 2\pi - \log\pi + \log\sin\tfrac{1}{2}s\pi + \log\Gamma(1-s) + \log\zeta(1-s)\right\}ds.$$

At the point C, $\log\zeta(s)$ has imaginary part $-\pi i$, and the corresponding point $(1-s)$ lies below the cut, so that, at the same point C, $\Im\log\zeta(1-s)=\pi i$. All the terms in the integrand on the right are real for $s=\epsilon$ except the last, so that the preceding form must be corrected to give

$$z\int_{CBA} e^{sz}\log\zeta(s)\,ds = z\int_{CBA} e^{sz}\left\{s\log 2\pi - \log\pi + \log\sin\tfrac{1}{2}s\pi + \log\Gamma(1-s) + \log\zeta(1-s) - 2\pi i\right\}ds.$$

Making ϵ, the radius of the indentations, tend to zero, we find

7.4 $$2\pi i\, V(z) = \begin{cases} -z\int_1^{1+i\infty} e^{sz}\log\zeta(s)\,ds + z\int_0^{i\infty} e^{sz}\log\zeta(1-s) - z(\log\pi + 2\pi i)\int_0^{i\infty} e^{sz}\,ds + z\log 2\pi\int_0^{i\infty} s e^{sz}\,ds \\ -z\int_0^1 e^{sz}\log\zeta(s)\,ds + z\int_0^{i\infty} e^{sz}\log\sin\tfrac{1}{2}s\pi\,ds + z\int_0^{i\infty} e^{sz}\log\Gamma(1-s)\,ds, \end{cases}$$

where, in addition to the convention regarding $\log\zeta(s)$, we must have $\log\Gamma(1-s)$ real for $s=0$, and (since $\log\sin\frac{1}{2}s\pi$ is real at C, and consequently has imaginary part $\frac{1}{2}\pi i$ at B) $\log\sin\frac{1}{2}s\pi$ has fixed imaginary part $\frac{1}{2}\pi i$ along the line AB.

The first term of 7.4 is

7.5 $$-z\lim_{\epsilon\to 0}\int_{1+i\epsilon}^{1+i\infty} e^{sz}\sum_{p,m}\frac{1}{mp^{ms}}\,ds = ze^{z}\lim_{\epsilon\to 0}\sum_{p,m}\frac{1}{mp^{m(1+i\epsilon)}(z-\log p^m)} = ze^{z}\sum_{p,m}\frac{1}{mp^m(z-\log p^m)},$$

the change in order of integration and summation being made by Dini's theorem.

By writing $\sin \tfrac{1}{2} s\pi$ in exponentials and simple reduction, we find

7.6 $$z\int_0^{i\infty} e^{sz} \log \sin \tfrac{1}{2} s\pi \, ds = C + \log 2 + \tfrac{1}{2}\pi i\left(\frac{1}{z} - 1\right) + \psi\left(\frac{z}{\pi i}\right)$$

where C is Euler's constant, and ψ is the logarithmic derivative of the Gamma function.

Using Legendre's integral for $\psi(z)$ (Whittaker & Watson '<u>Modern Analysis</u>' p.260. Ex.16) we find

7.7 $$z\int_0^{i\infty} e^{sz} \log \Gamma(1-s)\, ds = -e^{z}\int_1^{\infty} \psi(s) e^{-sz}\, ds = \frac{C}{z} - \frac{1}{z}\int_0^{\infty} \frac{t}{e^t - 1} \cdot \frac{dt}{t+z} \quad \text{if } \Re(z) = x > 0.$$

Substituting 7.5, 7.6, and 7.7 into 7.4, gives

7.8 $$\begin{aligned} V(z) = {} & \frac{z e^{z}}{2\pi i}\sum_{p,m} \frac{1}{mp^m(z - \log p^m)} - \frac{z}{2\pi i}\sum_{p,m} \frac{1}{mp^m(z + \log p^m)} + \left(\tfrac{1}{4} + \frac{C + \log 2\pi}{2\pi i}\right)\left(1 + \frac{1}{z}\right) + \frac{1}{2\pi i}\psi\left(\frac{z}{\pi i}\right) \\ & + \tfrac{1}{2} e^{z} - \frac{z}{2\pi i}\int_0^1 e^{sz} \log|\zeta(s)|\, ds - \frac{1}{2\pi i z}\int_0^{\infty} \frac{t}{e^t - 1} \cdot \frac{dt}{t+z}. \end{aligned}$$

The last integral converges in any closed region of the z-plane which does not contain the origin or any negative real number, hence

$$\int_0^{\infty} \frac{t}{e^t - 1} \cdot \frac{dt}{t+z} = \int_0^{\infty e^{i\alpha}} \frac{t}{e^t - 1} \cdot \frac{dt}{t+z}, \quad \text{where } -\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi < \arg z < \tfrac{1}{2}\pi.$$

The last integral has the vector arg $z = \pi + \alpha$ as singular line and, taking α arbitrarily near to $\tfrac{1}{2}\pi$, we see, referring to 7.8, that if the z-plane is cut along the negative part of the imaginary axis then $V(z)$ is meromorphic and has singularities at $z = \pm \log p^m$ - where is any prime number - which are simple poles.

From this it follows that the series $\sum_\rho e^{\rho z}$, where z is a complex quantity with positive real and imaginary parts, and summation runs through all the zeros of $\zeta(s)$ with positive imaginary part, diverges at all of the points

$$z = \pm \log p^m$$

It appears from this that the connection between the distribution of the zeros of $\zeta(s)$ and the distribution of primes must be of an extremely subtle nature.

If we assume the truth of Riemann's hypothesis a very perplexing result may be obtained from 7.8

Take $\rho = \tfrac{1}{2} + ri$ substitute into 7.8, and extract the real part of $V(z)$.

This gives without difficulty

$$\lim_{y\to 0}\sum_{\gamma>0} e^{-\gamma y}\cos\gamma x = e^{-\frac{1}{2}x}\left\{\frac{1}{x}\left(1+\frac{1}{x}\right)+\frac{1}{2\pi}\tilde{S}\,\psi\left(\frac{x}{\pi i}\right)+\frac{1}{2}e^{x}\right\}$$

It is not difficult to find

$$\frac{1}{2\pi}\tilde{S}\,\psi\left(\frac{x}{\pi i}\right) = -\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2x}+\frac{1}{e^{2x}-1}\right)$$

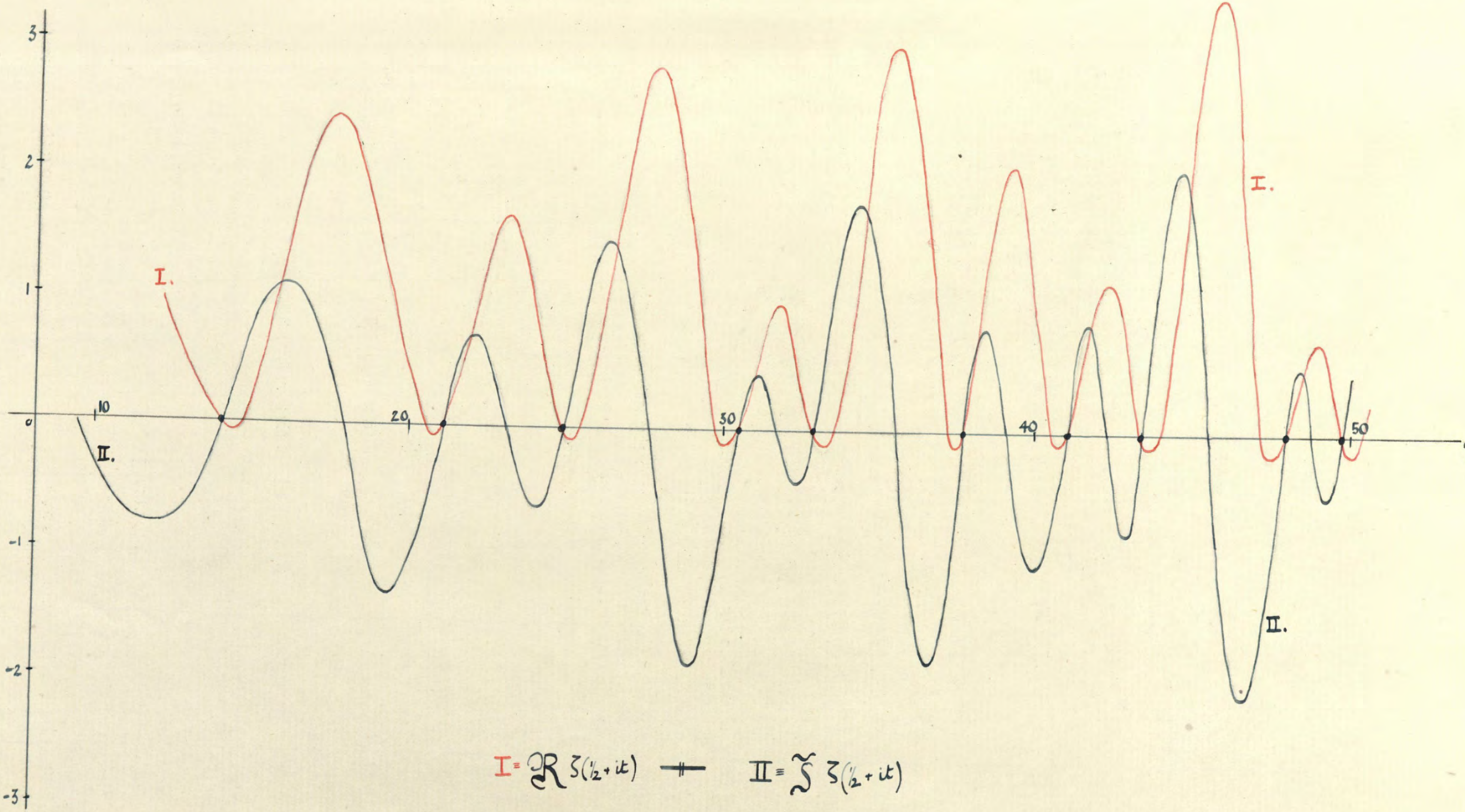
so that simple reduction gives

$$\lim_{y\to 0}\sum_{\gamma>0} e^{-\gamma y}\cos\gamma x = \frac{e^{\frac{3}{2}x}-e^{-\frac{3}{2}x}-e^{-\frac{1}{2}x}}{2\left(e^{x}-e^{-x}\right)} \qquad (x>0)$$

If $x<0$ some of the preceding arguments may easily be modified to give

$$\lim_{y\to 0}\sum_{\gamma>0} e^{-\gamma y}\cos\gamma x = \frac{e^{\frac{3}{2}x}-e^{-\frac{3}{2}x}+e^{+\frac{1}{2}x}}{2\left(e^{x}-e^{-x}\right)}.$$

This result, which is remarkable since the prime numbers do not explicitly occur, was given by Cramér in the already mentioned memoir, but had previously been obtained - although in a less acute form - by Mellin ('Ann. Acad. Sci. Fenn.' A.10. 1911, No.11).



I = $\Re\,\zeta(\frac{1}{2}+it)$ —+— II = $\Im\,\zeta(\frac{1}{2}+it)$

THE ZETA-FUNCTION ON THE CRITICAL LINE.