NOVEL ASPECTS OF 3D $\mathcal{N} = 2$ CHERN–SIMONS–MATTER THEORIES

by

OSAMA K. F. KHLAIF

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Abstract

In this thesis, we explore new results we recently obtained about the infrared physics of 3d $\mathcal{N} = 2$ SQCD with a unitary gauge group, in particular in the presence of a non-zero Fayet–Iliopoulos (FI) parameter and with generic values of the Chern–Simons levels. We study the 3d gauged linear sigma model (GLSM) (also known as 3d A-model) approach to the computation of the 3d $\mathcal{N} = 2$ twisted chiral ring of half-BPS lines. We analyse the moduli space of supersymmetric vacua in the theory, and we study its dependence on the Chern–Simons levels and the sign of the FI parameter. For particular values of the Chern–Simons levels, the twisted chiral ring has a neat interpretation in terms of the quantum K-theory (QK) of the complex Grassmannian variety. We propose a new set of line defects of the 3d gauge theory, dubbed Grothendieck lines, which represent equivariant Schubert classes in the QK ring. In particular, we show that the double Grothendieck polynomials, which represent the equivariant Chern characters of the Schubert classes, arise physically as Witten indices of certain quiver supersymmetric quantum mechanics. We also explain two distinct ways to compute K-theoretic enumerative invariants using the 3d GLSM approach. Moreover, we study infrared dualities associated with these 3d SQCDs. We use our techniques and analysis to test these dualities and, for some cases, we give a geometric interpretation for these dualities.

Publications

This thesis is based on the following list publications [1, 2, 3, 4]:

[1] Twisted indices, Bethe ideals and 3d $\mathcal{N} = 2$ infrared dualities ,
[JHEP 05 (2023) 148], [hep-th/2301.10753],
in collaboration with Cyril Closset.

[2] On the Witten index of 3d $\mathcal{N} = 2$ unitary SQCD with general CS levels,
[SciPost Phys. 15, 085 (2023)], [hep-th/2305.00534],
in collaboration with Cyril Closset.

[3] Grothendieck lines in 3d SQCD and the quantum K-theory of the Grassmannian,
[JHEP 12 (2023) 082], [hep-th/2309.06980],
in collaboration with Cyril Closset.

[4] New results on 3d $\mathcal{N} = 2$ SQCD and its 3d GLSM interpretation,
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in collaboration with Cyril Closset.

Other Publications and Preprints

Other publications and preprints during my PhD at the University of Birmingham which will not be discussed in the thesis [5, 6, 7]:

[1] Virasoro Constraint for Uglov Matrix Model,
[JHEP 04 (2020) 029], [hep-th/2201.06839],
in collaboration with Taro Kimura.

[2] One-form symmetries and the 3d $\mathcal{N} = 2$ A-model: Topologically twisted indices and CS theories,
[SciPost Phys. 18 (2025) 066], [hep-th/2405.18141],
in collaboration with Cyril Closset and Elias Furrer.

[3] The 3d A-model and generalised symmetries, Part I: bosonic Chern–Simons theories,
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In Loving Memory of My Grandfather

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CONTENTS

CHAPTER 1

INTRODUCTION

The main question in theoretical physics is to formulate a quantum theory of gravity. The most prominent and successful candidate in answering this question so far has been (super)string theory [8, 9]. The fundamental ingredient in this theory is a one-dimensional string that moves around in ten-dimensional spacetime M, tracing a two-dimensional worldsheet Σ. In this sense, the path integrals of string amplitudes take into account all possible embeddings of Σ in M. To connect this theory to our four-dimensional world, one needs to compactify six of the spacetime directions by taking $M = \mathbb{R}^{1,3} \times X$.

During the past three decades, viewing string theory as the study of maps from Σ to a six-dimensional space X has turned out to be very fruitful in different areas in mathematics, most significantly in enumerative geometry. In a particular subsector of the theory – namely the topological strings –, the string scattering amplitudes give rise to a new type of enumerative invariants associated with the (almost-complex/Kähler) targets X. Later on, this led to the development of what now is referred to as the *Gromov–Witten theory* [10, 11, 12].

From a field theoretic perspective, conformal invariance is imposed on the worldsheet theory. This is particular for the case where X is a Calabi–Yau 3-fold (CY3). These 2d CFTs can be UV completed by 2d gauged linear sigma models (GLSMs). The CY3 target X is realised as the Higgs branch of the 2d gauge theory [13]. Any target variety X that can be realised by geometric invariant theory (GIT) can be engineered as a 2d GLSMs. In [11, 14], Witten showed that there is a topological sector of the theory – referred to as the A-model – where the correlation functions count holomorphic maps from Σ to X. This corresponds to the topological strings. To this end, a certain amount of supersymmetry must be imposed on the 2d theory. For instance, for the target to be Kähler, one needs to consider 2d GLSMs with $\mathcal{N} = (2, 2)$ supersymmetry. Meanwhile, $\mathcal{N} = (0, 2)$ supersymmetry is enough to obtain the less constrained almost-complex manifolds. In our work here, we will focus on the earlier case. For recent advancements in the latter one see the nice review by Sharpe [15].

More recently, a K-theoretic uplift of Gromov–Witten invariants was proposed by Givental [16] and Lee [17]. From a supersymmetric quantum field theory perspective, these correspond to considering three-dimensional GLSMs rather than two-dimensional ones living on geometries of the form $\Sigma \times S^1$ [18, 19, 20]. The amount of supersymmetry that needs to be imposed in this case is $\mathcal{N} = 2$. The topological sector of these theories is usually referred to as the 3d A-model, which is a 2d A-model with an infinite number of massive Kaluza–Klein (KK) modes living on the circle fibre.

One of the main players affecting the dynamics of these 3d gauge theories is the Chern–Simons interaction terms. Three-dimensional Chern–Simons theory has been a major influencer in different areas in mathematical physics. One of the earliest contributions appeared in the seminal work of Witten [21] where he computed invariants of knots as correlation functions of Wilson lines in pure Chern–Simons theory. Later on, he went in another direction to show that Chern–Simons theory can be viewed as a model for an open string field theory [22]. These ideas inspired many advances in the study of topological string theory and Gromov–Witten theory – e.g. the work of Gopakumar and Vafa [23, 24], where they connected the partition function of Chern–Simons theory on the three-sphere with the Gromov–Witten theory on the resolved conifold.

In a general 3d gauge theory, the gauge coupling g_{3d}^2 has a mass dimension 1. This renders the theory strongly coupled in the infrared, and thus more difficult to analyse using the conventional Feynman perturbation techniques. As we will explain later, one way to survive these obstacles could be through dualities. For 3d $\mathcal{N} = 2$ supersymmetric gauge theories, there is a plethora of dualities relating a strongly-coupled field theory on one side with a weakly-coupled one on the other side. From an algebraic geometry perspective, special cases of these dualities lead to equivalences between different geometric spaces. Identifying the partition functions of the dual theories amounts to identifying enumerative invariants of these different geometries.

Exploring some of these aspects will be our main goal in this thesis. In this chapter, we give an overview of 2d and 3d GLSMs and their connection with Gromov–Witten theory and its K-theoretic uplift. We will also give a brief discussion of the types of 3d dualities we will be interested in throughout our work.

1.1 2d $\mathcal{N} = (2, 2)$ supersymmetric gauge theories

This section focuses on 2d $\mathcal{N} = (2, 2)$ GLSMs and their topological twisting on a genus-g closed Riemann surface Σ_g. After reviewing some aspects of the associated effective topological field theory, we discuss the connection with Gromov–Witten (GW) theory, which is the intersection theory on the moduli space of holomorphic maps from Σ_g to an algebraic variety X. In the last part of the section, we discuss the correspondence between 2d GLSMs and quantum cohomology for the case $\Sigma_g = \mathbb{P}^1$.

1.1.1 2d $\mathcal{N} = (2,2)$ SUSY algebra on Euclidean $\mathbb{R}^2$

In this subsection, we give a brief review of the building blocks of $\mathcal{N} = (2,2)$ supersymmetric gauge theories on Euclidean $\mathbb{R}^2$. For more details, see section 2 of [25] or chapter 12 of [26]. These theories have four supercharges, and they can be obtained by dimensional reduction of $\mathcal{N} = 1$ in 4d or $\mathcal{N} = 2$ in 3d. Our primary focus in this work will be on the connection with the latter case.

Denoting the supercharges by $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$, the $\mathcal{N} = (2,2)$ algebra is given by:

$$\begin{aligned} \{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} &= 2P_z \ , & \{\mathcal{Q}_-, \overline{\mathcal{Q}}_-\} &= -2P_{\overline{z}} \ , \\ \{\mathcal{Q}_-, \overline{\mathcal{Q}}_+\} &= -i\widetilde{Z} \ , & \{\mathcal{Q}_+, \overline{\mathcal{Q}}_-\} &= i\widetilde{Z}^\dagger \ . \end{aligned} \tag{1.1.1}$$

with the other anticommutators being trivial.[1] Here, $\widetilde{Z}$ is a complex central charge and $\widetilde{Z}^\dagger$ its complex conjugate. Here we are taking the complex coordinates $(z, \overline{z})$ on $\mathbb{R}^2$ which are related to usual (x^0, x^1) by $z \equiv x^0 + ix^1$ and $\overline{z} \equiv x^0 - ix^1$. In (1.1.1), P_z and $P_{\overline{z}}$ are the momenta in the holomorphic and anti-holomorphic directions respectively. Meanwhile, the $\pm$ indices of the supercharges indicate their spins under $\mathrm{Spin}(2) \cong U(1)_{\mathrm{E}}$.

The algebra (1.1.1) has two automorphisms which are usually referred to as axial R-symmetry $U(1)_A$ and vector R-symmetry $U(1)_V$. The R-charges of $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$ are shown in table 1.1. These two symmetries will play a major role in twisting the 2d $\mathcal{N} = (2,2)$ theories on a curved 2d spacetime Σ_g and also in the GLSM construction as we will discuss later in this chapter.

2d $\mathcal{N} = (2,2)$ field theory content

Let us now take G to be a simply-connected or unitary gauge group. To write down a supersymmetric Lagrangian for a 2d gauge theory, we utilise the different possible supersymmetric multiplets that we get from studying the representation theory of the algebra (1.1.1). We have the following three types of supersymmetric multiples:

[1] This assumption can be relaxed to include other central charges in the algebra, but this will not be directly related to our discussion here.

	$U(1)_V$	$U(1)_A$	$U(1)_{\rm E}$	$U(1)^A_{\rm E'}$	$U(1)^B_{\rm E'}$
$\mathcal{Q}_-$	-1	$+1$	$+1$	0	$+2$
$\overline{\mathcal{Q}}_+$	$+1$	$+1$	-1	0	0
$\overline{\mathcal{Q}}_-$	$+1$	-1	$+1$	$+2$	0
$\mathcal{Q}_+$	-1	-1	-1	-2	-2

Table 1.1: Charge assignments for the 2d $\mathcal{N} = (2, 2)$ supercharges under the axial and vector R-symmetries in the first two columns. In the last three columns are the charges of these operators under the Euclidean symmetries before and after topological twisting.

- *Vector multiplet* $\mathcal{V}$. This multiplet contains information about the gauge connection A_μ associated with the principal G-bundle P. More explicitly, this multiplet has the following data:

$$\mathcal{V} = (\sigma + iA_t,\ A_\mu,\ \lambda_\pm,\ D)\ , \tag{1.1.2}$$

  where σ is a real scalar, $\lambda_\pm$ are the gauginos – the supersymmetric partners of the gauge field A_μ – and D is an auxiliary real scalar field. The reason we wrote complex scalar component as $\sigma + iA_t$ is due to its relation with 3d $\mathcal{N} = 2$ as we review in the next section. For shortness, we will denote this field by $\widetilde{\sigma}$ below. All these fields live in the adjoint representation of the gauge group G.

- *Chiral multiplet* Φ. This type of multiplets is used to couple gauge theory with matter fields. For a representation space $V_{\mathfrak{R}}$ associated with the G-representation $\mathfrak{R}$, we have the multiplet $= (\phi, \psi_\pm, F)$. Here ϕ is a complex scalar, $\psi_\pm$ are Dirac fermions and F is an auxiliary complex scalar field. Chiral multiplets are characterised by the requirement that:

$$[\overline{\mathcal{Q}}_+, \Phi] = [\overline{\mathcal{Q}}_-, \Phi] = 0\ . \tag{1.1.3}$$

  Associated with Φ, one can also have an anti-chiral superfield $\widetilde{\Phi}$, which has field content analogous to Φ but in the conjugate representation $\overline{\mathfrak{R}}$.

- *Twisted chiral multiplet* Σ. Another way of writing the field contents of the vector multiplet (1.1.2) is by using the twisted chiral multiplet Σ. This multiplet contains the field strength $F_{z\overline{z}} = \partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z + [A_z, A_{\overline{z}}]$ rather than the gauge connection A_μ. Similar to the chiral multiplet case (1.1.3), a twisted chiral multiplet satisfies:

$$[\overline{\mathcal{Q}}_+, \Sigma] = [\mathcal{Q}_-, \Sigma] = 0\ . \tag{1.1.4}$$

A comment on the anomalies of the R-symmetries. In [13], Witten argued that, although the vector $U(1)_V$ R-symmetry is non-anomalous when going to the quantum level of the 2d supersymmetric field theory, there is a gauge anomaly associated with the axial $U(1)_A$ R-symmetry measured by the sum of the charges of the matter fields under the gauge group. Later in our discussion, we will comment on the geometric interpretation of this condition from the sigma model perspective.

Semi-classical description of the field theory

Using the above three ingredients, one can write down the explicit form of the supersymmetric Lagrangian of the theory of interest[1]. As usual, when studying quantum field theory, we are interested in computing the moduli space of the supersymmetric vacua of the theory. The classical vacua can be obtained by minimising the scalar potential $U(\widetilde{\sigma}, \phi)$ of the theory, which can be read off straightforwardly from the Lagrangian.

For simplicity of notation, let us take the gauge group to be $U(1)$ and consider M chiral matter multiplets Φ_α of $U(1)$ charges q_α. In this case, we get:

$$U(\widetilde{\sigma}, \phi) = \frac{e^2}{2} D^2 + \sum_{\alpha=1}^{M} |F_\alpha|^2 + 2|\widetilde{\sigma}|^2 \sum_{\alpha=1}^{M} q_\alpha^2 |\phi_\alpha|^2 \ . \tag{1.1.5}$$

Here e is the $U(1)$ gauge coupling.

Since D and F_α are auxiliary fields, one can integrate them out using their equations of motion:

$$\begin{aligned} D &= -e^2 \left(\sum_{\alpha=1}^{M} q_\alpha |\phi_\alpha|^2 - \widetilde{\xi} \right) \ , \\ F_\alpha &= \frac{\partial W}{\partial \phi_\alpha} \ , \qquad \alpha = 1, \cdots, M \ . \end{aligned} \tag{1.1.6}$$

For the first equation, $\widetilde{\xi}$ is the 2d real Fayet-Iliouplos (FI) parameter associated with the $U(1)$ twisted chiral multiplet. This can be complexified using the 2d theta angle term $\frac{\theta}{2\pi} \int F_{z\bar{z}}$ to give us $\widetilde{\tau} \equiv i\widetilde{\xi} + \frac{\theta}{2\pi} \in \mathbb{C}$. More generally, with each $U(1)_a$ factor of the gauge group G we can associate a 2d FI parameter $\widetilde{\xi}^a$.

Meanwhile, in the second equation in (1.1.6), we introduced the function W. This is the superpotential which is a gauge-invariant quasi-homogenous function of the scalar fields ϕ. W must have charge 2 under the $U(1)_V$ vector R-symmetry and is invariant

[1] In our work, we are mainly focusing on Lagrangian field theories, that is, theories that have a Lagrangian description.

under the $U(1)_A$ axial one. This leads to its quasi- homogeneity – taking $q_{V,\alpha}$ to be the $U(1)_V$ charge of the scalar field ϕ_α, we have $W(\lambda^{q_{V,\alpha}}\phi_\alpha) = \lambda^2 W(\phi_\alpha)$.

Substituting (1.1.6) back into potential (1.1.5), we see that since it has squared terms, the minima are given by:

$$\begin{aligned} &\sum_{\alpha=1}^{M} q_\alpha |\phi_\alpha|^2 - \widetilde{\xi} = 0 \ , \\ &q_\alpha \, \widetilde{\sigma} \, \phi_\alpha = 0 \ , \qquad \alpha = 1, \cdots, M \ , \\ &\frac{\partial W}{\partial \phi_\alpha} = 0 \ , \qquad \alpha = 1, \cdots, M \ . \end{aligned} \tag{1.1.7}$$

Solving these equations, up to a gauge transformation, in terms of $\widetilde{\sigma}$ and ϕ gives us the classical supersymmetric vacua of the theory. These come into three forms: Coulomb branch vacua (where $\widetilde{\sigma}$ obtains a VEV and $\phi_\alpha = 0$), Higgs branch vacua (where $\widetilde{\sigma} = 0$ and ϕ_α obtain VEVs) and *hybrid* vacua, which is a mixture between the two earlier cases[1].

For non-abelian gauge theories. The discussion of the Abelian case above can be generalised to the case where the gauge group G is non-abelian. The scalar potential $U(\widetilde{\sigma}, \phi)$ (1.1.5) becomes:

$$U(\widetilde{\sigma}, \phi) = \frac{1}{2e^2} \mathrm{Tr}[\widetilde{\sigma}, \widetilde{\sigma}^\dagger]^2 + \frac{e^2}{2} \sum_{a,b=1}^{\mathrm{rk}(G)} (D^a{}_b)^2 + \sum_{\alpha=1}^{M} |F_\alpha|^2 + 2 \sum_{\alpha=1}^{M} |(\widetilde{\sigma} + \widetilde{m}) \cdot \phi_\alpha|^2 \ , \tag{1.1.8}$$

with rk(G) being the rank of the gauge group[2]. Comparing this with (1.1.5) in the abelian case, we now have the first term, which involves the commutator of the complex adjoint scalar with its complex conjugate. The new vacuum equations associated with this term state that this bracket trivialises over the moduli space of SUSY vacua. Thus, one can diagonalise it using the action of the gauge group:

$$\widetilde{\sigma} = \mathrm{diag}(\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_{\mathrm{rk}(G)}) \ . \tag{1.1.9}$$

Effectively, at a generic point on the classical Coulomb branch, this breaks the gauge group G down to its maximal torus $U(1)^{\mathrm{rk}(G)}$. Parts of the full gauge symmetry can be restored in regions where some of the eigenvalues of $\widetilde{\sigma}$ coincide. For example, at the origin of the Coulomb branch $\widetilde{\sigma} = 0$, the entire gauge symmetry G will be restored.

[1]The last type of vacua does not appear in the $U(1)$ gauge theory example we are considering here, rather, it can appear for the cases where G is of a higher rank. For more details, see section 3.2 below for the 3d $\mathcal{N} = 2$ SQCD case.

[2]In this thesis we will use the notation rk(G) and rank(G) for the rank of the group G interchangeably.

With this in mind, let us come back to the scalar potential (1.1.8) to explain the notation. The D-term equations of motion are of the form:[1]

$$D^a{}_b = \mu^a{}_b(\phi) - \tilde{\xi}\delta^a{}_b \ , \qquad \mu^a{}_b(\phi) \equiv \sum_{\alpha=1}^{M} \phi^A_\alpha \ (q^a_{\alpha,b})_A{}^B \ \phi^\dagger_{\alpha,B} \ , \tag{1.1.10}$$

for $a, b = 1, \cdots, \mathrm{rk}(G)$. Here $(q^a_{\alpha,b})_A{}^B$ are the charges associated with the G-representation $\mathfrak{R}_\alpha$ with the indices $A, B = 1, \cdots, \dim(\mathfrak{R}_\alpha)$. The sum over repeated indices is assumed. One can write the expressions for these charges explicitly in terms of the generators of the Lie algebra in both the adjoint and $\mathfrak{R}_\alpha$ representations. For example, one finds that, for the fundamental and anti-fundamental representations of $G = U(N_c)$, these take the form:

$$(q^a_{\overline{\square},b})_A{}^B = -(q^a_{\square,b})_A{}^B = -\delta^a{}_A\delta_b{}^B \ , \tag{1.1.11}$$

keeping in mind that, in this particular example, $\dim(\square) = \dim(\overline{\square}) = \mathrm{rk}(G)$.

Moreover, note that in the last term in (1.1.8), we introduced the complex matrix of *twisted masses* $\widetilde{m}$. The geometric significance of these will be more clear later in this section. For now, we define it as a VEV for the complex scalar of a background vector multiplet $\mathcal{V}_F$ for the flavour symmetry group G_F that we can weakly couple our supersymmetric gauge theory with. In writing the last term in (1.1.8), '$\cdot$' stands for the actions of the scalar fields $\widetilde{\sigma}$ and $\widetilde{m}$ on ϕ_α depending on its representation $\mathfrak{R}_\alpha \times \mathfrak{R}_{F,\alpha}$ under $G \times G_F$.

Now, minimising the scalar potential (1.1.8) we get the following equations for the semi-classical supersymmetric vacua of the theory:

$$\begin{aligned} &\mu^a{}_b(\phi) = \tilde{\xi}\delta^a{}_b \ , && a, b = 1, \cdots, \mathrm{rk}(G) \ , \\ &(\widetilde{\sigma} + \widetilde{m}) \cdot \phi_\alpha = 0 \ , && \alpha = 1, \cdots, M \ , \\ &\frac{\partial W}{\partial \phi_\alpha} = 0 \ , && \alpha = 1, \cdots, M \ , \end{aligned} \tag{1.1.12}$$

up to a gauge transformation. Indeed, these specialise to (1.1.7) for the case $G = U(1)$. In the next section, we will discuss the connection between this construction and GIT quotients of complex projective varieties realised as the Higgs branches of these 2d gauge theories.

[1] For simplicity of notation and presentation, here we are assuming that there is only one FI parameter. This will be the case for the theories we are studying in this thesis.

1.1.2 Topological twisting and twisted indices

In our discussion in the previous section, we only considered the cases where our 2d supersymmetric field theory is living on flat Euclidean space $\mathbb{C}$. But what we would like to do – as we motivated in the introduction of this chapter – is to study these theories on a curved Riemann surface Σ_g. We will focus on the case where this is a closed manifold. For more discussion, see the original papers of Witten on the topic [11, 14] and [27] for more recent discussion from the perspective of rigid supersymmetry.

The supersymmetry transformations are written in terms of the parameters $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ associated with the supersymmetry generators $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$. These parameters are spinorial in nature, that is, they are global (anti-)holomorphic sections of $\mathcal{K}^{\frac{1}{2}}$ or $\overline{\mathcal{K}}^{\frac{1}{2}}$ with $\mathcal{K}$ and $\overline{\mathcal{K}}$ being the canonical line bundle and its dual respectively. This causes no issues when formulating the theory on 2d flat spaces like $\mathbb{C}$ or $\Sigma_1 = T^2$, since in these cases $\mathcal{K}$ is trivial and one can choose $\mathcal{K}^{\frac{1}{2}}$ to be trivial as well, thus, the four SUSY parameters are well-defined, i.e. 2d $\mathcal{N} = (2,2)$ supersymmetric Lagrangians can be written for these two surfaces.

With that being said, one will run into some obstructions when trying to write a consistent $\mathcal{N} = (2,2)$ supersymmetric field theory on a genus $g > 1$ Riemann surface [11, 14] [1]. One way to see how this happens is to consider the naive $\mathcal{N} = (2,2)$ SUSY deformation of the Lagrangian on Σ_g. One finds that for the action to be supersymmetric, one needs the SUSY parameters to be covariantly constants:

$$\nabla_\mu \epsilon_\pm = \nabla_\mu \bar{\epsilon}_\pm = 0 \;, \qquad \mu = z, \bar{z} \;. \tag{1.1.13}$$

where ∇_μ is the covariant derivative on the Riemann surface taking into account the spin connection ω_μ.

A global solution to these equations cannot always be found on Σ_g, but one can find a way about it and try to save as much supersymmetry as possible on Σ_g. This is the idea of *twisting* the supersymmetric theory. Roughly speaking, the idea is the following. The SUSY parameters of the original theory are spinors with respect to the Euclidean symmetry $U(1)_\mathrm{E}$. One can instead take advantage of the other two symmetries (line bundles) that we have in the theory, the R-symmetries. As we show in table 1.1, the parameters $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ carry charges under these symmetries, therefore, what we can do is to redefine what we mean by the Euclidean spacetime symmetry $U(1)_\mathrm{E}$ by *twisting* it

[1] For the case where the 2d surface is a Riemann sphere $\mathbb{P}^1$, it turns out that one can fully place the 2d $\mathcal{N} = (2,2)$ on it. We will not consider this case in our work here since – as was pioneered by Morrison and Plesser for the abelian case [28] – all the information we are interested in is contained in the twisted version of these theories. For a discussion on the general case, see [29, 30]. For the connection between the two versions of the theory, see [31].

with one of these symmetries and thus have new charges of these charges under the new Euclidean symmetry which we will denote by $U(1)_{\mathrm{E}'}$.

As a result of this, we will see that the remaining fermionic symmetries will be generated by scalar operators rather than spinorial ones. Moreover, these generators will be BRST-like operators in the sense that they square to zero. This will lead us to study the de Rham complexes and the cohomology rings of these operators leading to very fruitful mathematical connections. For a very recent review on twisted supersymmetric theories, see the 2021 TASI lectures [32].

Topologically twisted field theory

Denoting the R-symmetry that we choose to twist $U(1)_{\mathrm{E}}$ with by $U(1)_R$, this process amounts to modifying the constraints (1.1.13) by deforming the covariant derivative to include also a background gauge field $A_\mu^{(R)}$ for the $U(1)_R$ line bundle. This results in a new possibility of solving the new equations and therefore finding an amount of supersymmetry that can be preserved on Σ_g. We will focus on two special cases for $U(1)_R$ which are either $U(1)_R = U(1)_V$ or $U(1)_R = U(1)_A$ which lead to the A-twist and B-twist respectively. As we will discuss momentarily, one of the interesting aspects of these two twists is that the resulting theories are topological field theories in the sense that correlation functions of the 'physical' operators do not depend on the metric on Σ_g nor on the complex structure. Therefore, these two twists are referred to as *topological twists* of the 2d theory[1].

For instance, denoting the generators of $U(1)_{\mathrm{E}}$ and $U(1)_R$ respectively by $\mathcal{Q}_{\mathrm{E}}$ and $\mathcal{Q}_R$, the charge operator $\mathcal{Q}_{\mathrm{E}'}$ of the new Lorentz group $U(1)_{\mathrm{E}'}$ is simply given by $\mathcal{Q}_{\mathrm{E}'} = \mathcal{Q}_{\mathrm{E}} + \mathcal{Q}_R$, i.e. we take $U(1)_{\mathrm{E}'}$ to be the diagonal of $U(1)_{\mathrm{E}} \times U(1)_R$. The charges of the four operators $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$ before and after the twisting are summarised in table 1.1 above. For example, we can see in the A-twist we obtain the scalar fermionic symmetry generator $\mathcal{Q}_A \equiv \mathcal{Q}_- + \overline{\mathcal{Q}}_+$. Meanwhile, in the B-twist we get the scalar operator $\mathcal{Q}_B \equiv \overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_-$. Using the algebra (1.1.1), it is a straightforward exercise to see that the symmetry operator $\mathcal{Q}$ in either case is a nilpotent operator in the sense that $\mathcal{Q}^2 = 0$.

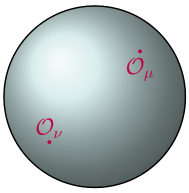
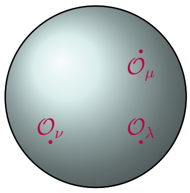
Physical operators and chiral rings. In the twisted theory, we define the physical operators to be those invariant under the fermionic symmetry $\mathcal{Q}$ – i.e. $[\mathcal{Q}, \mathcal{O}] = 0$. Recalling the definitions of the chiral and twisted chiral operators given in (1.1.3) and (1.1.4), we observe that an operator $\mathcal{O}$ is a physical operator in the A-twisted theory if it is a twisted chiral operator and if it is a chiral operator in the B-twist case.

[1] There is one more type of twist that one can perform for the 2d $\mathcal{N} = (2,2)$ theory which is called the *half-twist* or the *holomorphic twist* [11, 33, 34]. In this case, the correlation functions of the theory depend only on the holomorphic coordinates on the Riemann surface hence the name.

Moreover, since $\mathcal{Q}$ is nilpotent, the physical operator $\mathcal{O}$ is defined up to $\mathcal{Q}$-exact terms. Thus, the set of physical operators $\mathcal{R}_{\mathcal{Q}} := \{\mathcal{O}_\mu\}_{\mu\in I}$[1] defines the $\mathcal{Q}$-cohomology ring with some indexing set I. For reasons that are clear from the discussion above, we will refer to this cohomology ring as the *(twisted) chiral ring* in the (A-) B-twist.

As an algebra, one can define a product on $\mathcal{R}_{\mathcal{Q}}$. This is the fusion of the operators when bringing them close to each other on Σ_g. To understand the nature of this fusion, let us first make the following remark concerning the stress-energy tensor of the 2d theory. Twisting the Euclidean symmetry by $U(1)_R$ amounts to redefining the original stress-energy tensor by shifting it with terms that depend on the conserved Noether current of $U(1)_R$. As a result, one can show that the new stress-energy tensor is actually $\mathcal{Q}$-exact. Following the arguments of Witten [35], this observation can be applied to show that the correlation functions of the physical operators do not depend on the metric nor the complex structure on Σ_g. In particular, these correlators do not depend on the points $z_* \in \Sigma_g$ where the operators are inserted. In this sense, our twisted 2d theory is a 2d TQFT[2].

As a result, when studying the fusion of two (twisted) chiral ring operators – which we will denote by $\mathcal{O}_\mu \star \mathcal{O}_\nu$ – we will not get any dependence on the positions of these operators on Σ_g let alone any singular behaviour as is usual when studying OPEs in 2d CFTs. To explicitly write down the fusion rules of $\mathcal{R}_{\mathcal{Q}}$, one only needs two types of correlation functions to consider, namely the genus-0 two and three-point functions:

$\eta_{\mu\nu}(\mathcal{O}) := \Big\langle$  $\Big\rangle$, $\quad C_{\mu\nu\lambda}(\mathcal{O}) := \Big\langle$  $\Big\rangle$. (1.1.14)

We will refer to the two-point correlator $\eta_{\mu\nu}(\mathcal{O})$ as *the topological metric*[3] and the three-point correlators $C_{\mu\nu\lambda}(\mathcal{O})$ as *the structure constants*. We are also indicating the dependence of these operators on the basis we choose of the ring $\mathcal{R}_{\mathcal{Q}}$.

In terms of these two components, the product of two chiral ring operators is given explicitly by:

$$\mathcal{O}_\mu \star \mathcal{O}_\nu = C_{\mu\nu}{}^{\lambda}(\mathcal{O})\mathcal{O}_\lambda + [\mathcal{Q}, \cdots] \ , \tag{1.1.15}$$

[1]Later in the discussion we will be mainly focusing on the A-twist we will drop the A index and denote it simply by $\mathcal{R}$ or $\mathcal{R}_{\text{2d}}$ to distinguish it from the one we get in the 3d A-model as we discuss in chapter 2.

[2]Actually it is a cohomological topological field theory (Coh-TFT) [36] since correlation functions are topological up to $\mathcal{Q}$-exact terms.

[3]From the equivalence between 2d TQFTs and Frobenius algebras (see [37] for an overview and [38] for a more detailed exposition), the ring $\mathcal{R}_{\mathcal{Q}}$ is a Frobenius algebra with $\eta_{\mu\nu}$ being the associated Frobenius pairing.

where the sum over repeated indices is assumed. Here, we are assuming that the topological metric is non-degenerate, and we defined:

$$C_{\mu\nu}{}^{\lambda}(\mathcal{O}) := C_{\mu\nu\delta}(\mathcal{O})\eta^{\delta\lambda}(\mathcal{O}) \ , \qquad \text{with} \quad \eta_{\mu\lambda}(\mathcal{O})\eta^{\lambda\nu}(\mathcal{O}) = \delta^{\nu}_{\mu} \ . \tag{1.1.16}$$

2d partition functions

Let us now go back to our 2d $\mathcal{N} = (2,2)$ supersymmetric gauge theories that we discussed in the previous section. We are interested in studying these theories on a closed Riemann surface Σ_g of genus g where we choose to perform the A-twist. But before doing all that, let us first point out that, when studying the IR behaviour of the theory with massive matter fields, one can show that integrating out these massive modes gives rise to an *effective twisted superpotential* $\widetilde{\mathcal{W}}(\tilde{\sigma}, \widetilde{m})$ that depends on the complex scalar field $\tilde{\sigma}$ of the gauge vector multiplet and twisted masses $\widetilde{m} \in \mathbb{C}$. This superpotential receives only 1-loop contributions, and it is exact for non-renormalisation reasons [13, 28].

More explicitly, for a theory with M matter multiplets Φ_α, the effective twisted superpotential takes the form :

$$\begin{aligned}\widetilde{\mathcal{W}}(\tilde{\sigma}, \widetilde{m}) = &-\frac{1}{2\pi i}\sum_{\alpha=1}^{M}\sum_{(\rho_\alpha,\rho_{F,\alpha})}(\rho_\alpha(\tilde{\sigma}) + \rho_{F,\alpha}(\widetilde{m}))(\log(\rho_\alpha(\tilde{\sigma}) + \rho_{F,\alpha}(\widetilde{m})) - 1)\\ &-\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha(\tilde{\sigma}) + \tilde{\tau}(\tilde{\sigma}) \ ,\end{aligned} \tag{1.1.17}$$

where the first sum is over the weights of the representation $\mathfrak{R}_\alpha \times \mathfrak{R}_{F,\alpha}$ of the multiplet Φ_α under $G \times G_F$. Meanwhile, the second term is the contribution of the gauge W-bosons with Δ_+ being the set of positive roots associated with the algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Recall here that $\tilde{\tau}$ is the complexified 2d FI parameter.

Placing the theory on Σ_g. Now let us place this supersymmetric gauge theory on Σ_g with a topological A-twist. Effectively, the Lagrangian of the resulting topological theory is given by [19]:

$$\int_{\Sigma_g} d^2z\sqrt{g}\left(-2F^a_{z\bar{z}}\frac{\partial\widetilde{\mathcal{W}}}{\partial\tilde{\sigma}_a} + \frac{\partial^2\widetilde{\mathcal{W}}}{\partial\tilde{\sigma}_a\partial\tilde{\sigma}_b}\overline{\lambda}^a \wedge \lambda^b + \frac{i}{2}\widetilde{\Omega}(\tilde{\sigma}, \widetilde{m})R\right) \ . \tag{1.1.18}$$

In writing this expression, we used the fact that in the effective description, the gauge group G is broken to its maximal torus (1.1.9) where the indices $a, b = 1, \cdots, \mathrm{rk}(G)$. The last term comes from coupling the theory to a curved background which gives the *effective* *dilaton* $\widetilde{\Omega}(\tilde{\sigma}, \widetilde{m})$ with R here being the Ricci scalar of Σ_g, $\frac{1}{4\pi}\int_{\Sigma_g} R = 2-2g$. The explicit form of this potential is not important for our discussion here. We will give the form of its 3d version in section 5.1 instead. The main point here is that the low-energy dynamics of the 2d TQFT are completely encoded in $\widetilde{\mathcal{W}}(\tilde{\sigma}, \widetilde{m})$ and $\widetilde{\Omega}(\tilde{\sigma}, \widetilde{m})$.

Correlation function of physical operators. As an example, one can exactly work out the correlation function of the physical operators $\mathcal{O}_\mu$ in the twisted chiral ring $\mathcal{R}_{\text{2d}}$. Using the effective action (1.1.18) above, we get [39, 19]:

$$\langle \mathcal{O}_\mu(\tilde{\sigma}, \widetilde{m}) \rangle_{\Sigma_g} = \sum_{\tilde{\sigma}_* \in \mathcal{S}_{\text{BE}}^{\text{2d}}} \mathcal{O}_\mu(\tilde{\sigma}_*, \widetilde{m}) \mathcal{H}(\tilde{\sigma}_*, \widetilde{m})^{g-1} \ , \tag{1.1.19}$$

where we are summing over the set of Bethe vacua $\mathcal{S}_{\text{BE}}^{\text{2d}}$ which are the supersymmetric vacua of the 2d theory:

$$\mathcal{S}_{\text{BE}}^{2d} := \{\tilde{\sigma}_* \mid e^{2\pi i \frac{\partial \widetilde{\mathcal{W}}}{\partial \tilde{\sigma}_*}} = 1 \ , \ \forall a = 1, \cdots, \text{rk}(G) \ , \ w(\tilde{\sigma}) \neq \tilde{\sigma} \ , \ \forall w \in W_G\}/W_G \ , \tag{1.1.20}$$

with W_G being the Weyl group associated with G. Moreover, in (1.1.19) we introduced the *handle-gluing operator* $\mathcal{H}(\tilde{\sigma}, \widetilde{m})$ which is defined as:

$$\mathcal{H}(\tilde{\sigma}, \widetilde{m}) := e^{2\pi i \widetilde{\Omega}} \det_{a,b} \frac{\partial^2 \widetilde{\mathcal{W}}}{\partial \tilde{\sigma}_a \partial \tilde{\sigma}_b} \ . \tag{1.1.21}$$

As a direct application of the result (1.1.19), one can explicitly compute the topological metric $\eta_{\mu\nu}(\mathcal{O})$ and the structure constants $C_{\mu\nu\lambda}(\mathcal{O})$ associated with the 2d TQFT as defined in (1.1.14) and hence work out the ring relations of $\mathcal{R}_{\text{2d}}$ as in (1.1.15).

1.1.3 2d GLSMs and GW theory

The class of supersymmetric gauge theories that we are interested in our work are referred to as *gauged linear sigma models* (GLSMs) [13]. For a recent overview of the developments in this area, see [40]. From an effective field theoretic perspective, these theories are viewed as UV completion of *non-linear sigma models* (NLSM) in the IR. Let us recall here that a 2d NLSM living on Σ is a theory describing maps from Σ to some target geometry X. The target X could be a smooth projective variety or a Deligne–Mumford (or Artinian) stack. Here, we will consider X to be of the first type.

From a more mathematical perspective, the GLSM associated with an algebraic variety X gives us a physical realisation of the *geometric invariant theory* (GIT) construction of

X or any complete intersections or hypersurfaces inside X. To see how this works, let us first give a brief review of how the GIT construction works.

Let us take the affine complex space $V \cong \mathbb{C}^n$ which is acted upon linearly by the $\mathbb{C}$-reductive group $G_{\mathbb{C}}$[1] – for our purposes here, we will take $G_{\mathbb{C}} = \bigtimes_{I=1}^{n_G} \mathrm{GL}(N_I)$. What we would like to do is to study the space of $G_{\mathbb{C}}$-orbits of V, $V/G_{\mathbb{C}}$. Depending on the nature of the $G_{\mathbb{C}}$-action, there are cases where the topological space $X/G_{\mathbb{C}}$ contains non-closed orbits, obstructing it from being a complex variety. GIT is a prescription to cure this simply by getting rid of the points in V that would lead to these non-closed orbits upon taking the quotient.

The prescription involves introducing a new piece of data that we denote by θ that would instruct us on how to remove these *unstable* points from V. This is a character of the symmetry group – it is a group homomorphism $\theta : G_{\mathbb{C}} \longrightarrow \mathbb{C}^{\times}$. In terms of this character, one can define the *semistable* locus $V_{\theta}^{\mathrm{ss}} \subseteq V$ such that the quotient by G gives a smooth complex variety $X = V /\!\!/_{\theta}\, G_{\mathbb{C}} \equiv V_{\theta}^{\mathrm{ss}}/G_{\mathbb{C}}$[2].

From the GLSM perspective, the complex algebraic variety X is none other than the Higgs branch of the 2d theory with gauge group G – whose complexification is $G_{\mathbb{C}}$. The dimension of the affine space V is determined by the number of chiral matter multiplets Φ_{α} in the theory and the G-representations $\mathfrak{R}_{\alpha}$. More explicitly, $V \cong \mathbb{C}^{\dim(\mathfrak{R})}$ with $\mathfrak{R} \equiv \bigoplus_{\alpha=1}^{M} \mathfrak{R}_{\alpha}$. Moreover, the stability condition θ and hence the semistable locus is determined by the 2d real FI parameter and the D-term equations (1.1.12) where $\mu : V \longrightarrow \mathfrak{g}^*$ is the moment map associated with the G-action on V. Taking into account the F*-term* equations coming from the superpotential W, the target variety is then given by the Kähler quotient:

$$X = \mu^{-1}(\widetilde{\xi})/G \,\cap\, dW(0)^{-1} \equiv \{\phi \in V \mid \partial_{\phi} W = 0 \;, \quad \mu(\phi^{\dagger}, \phi) = \widetilde{\xi}\}/G \;, \tag{1.1.22}$$

As was explained by Luty and Taylor [41], this description of the moduli space of vacua is equivalent to the GIT quotient:

$$X = V /\!\!/_{\widetilde{\xi}}\, G_{\mathbb{C}} \,\cap\, dW(0)^{-1} \equiv \{\phi \in V \mid \partial_{\phi} W = 0\} /\!\!/_{\widetilde{\xi}}\, G_{\mathbb{C}} \;. \tag{1.1.23}$$

Let us consider the following two examples, which will appear frequently in our work:

[1]By linear action here we simply mean that $v \mapsto g \cdot v$, $\forall v \in V$ and $\forall g \in G_{\mathbb{C}}$.

[2]Here we are assuming that the action of G on the semistable locus V_{θ}^{ss} is trivial (i.e. all the points in V_{θ}^{ss} have trivial $G_{\mathbb{C}}$-stabilisers). This assumption leads to the fact that the topological space X that we end up with is indeed a smooth complex variety. In the more general cases where there are points with non-trivial but finite stabilisers, X will be a smooth Deligne–Mumford stack and a smooth Artinian stack in the infinite stabilisers case.

Figure 1.1: Quiver representation of 2d $\mathcal{N} = (2,2)$ GLSMs with targets $\mathbb{P}^{n_f-1}$ on the left and $\mathrm{Gr}(N_c, n_f)$ on the right. The circle represents the gauge group, and the square represents the flavour symmetry. The arrows connecting the two are the chiral matter multiplets charged under both symmetry groups.

Example I: the projective space $\mathbb{P}^{n_f-1}$. As an example, let us consider 2d $\mathcal{N} = (2,2)$ gauge theory coupled to $M = n_f$ matter multiplets with positive unit charges under $U(1)$. Let us also take $W = 0$. In this case, the D-term equations (1.1.7) give us:

$$\sum_{\alpha=1}^{n_f} |\phi_\alpha|^2 = \widetilde{\xi} \ , \tag{1.1.24}$$

which, for $\widetilde{\xi} > 0$ ensures that the point $(\phi_1, \cdots, \phi_{n_f}) = (0, \cdots, 0) \in \mathbb{C}^{n_f}$ is deleted leaving us with the semistable locus $\mathbb{C}^{n_f} - \{0\}$ under the $U(1)$ action. Thus, the Higgs branch of the theory is given by:

$$X = \mathbb{C}^{n_f} /\!\!/_{\widetilde{\xi}>0} \mathbb{C}^* = \left(\mathbb{C}^{n_f} - \{0\}\right) / \mathbb{C}^* \cong \mathbb{P}^{n_f-1} \ . \tag{1.1.25}$$

The 2d gauge theory is a simple example of the quiver gauge theory. The quiver associated with it is given in figure 1.1 above.

Example II: the Grassmannian variety Gr(N_c, n_f). As a generalisation of the example above, let us now take the gauge group $G = U(N_c)$ and $M = n_f$ matter multiplets in the fundamental representation of $U(N_c)$. Again, let us take $W = 0$. In this case, the D-term equations (1.1.12) are of the form:

$$\sum_{\alpha=1}^{n_f} \phi\phi^\dagger = \widetilde{\xi} \ , \tag{1.1.26}$$

where $\phi \equiv (\phi_1, \cdots, \phi_{n_f}) \in V \equiv \mathbb{C}^{N_c \times n_f}$. For $\widetilde{\xi} > 0$ this equation states that the semistable locus $V^{\mathrm{ss}}_{\widetilde{\xi}}$ consists of $N_c \times n_f$ matrices of full rank. Therefore, taking the quotient by the $\mathrm{GL}(N_c)$ action gives us a basis for N_c-dimensional subspace of $\mathbb{C}^{n_f}$:

$$X = V /\!\!/_{\widetilde{\xi}>0} U(N_c) = V^{\mathrm{ss}}_{\widetilde{\xi}>0} / \mathrm{GL}(N_c) = \mathrm{Gr}(N_c, n_f) \ . \tag{1.1.27}$$

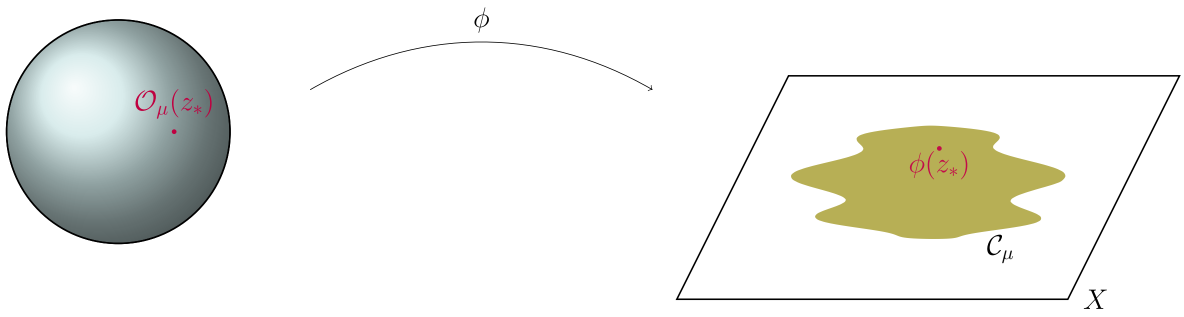


Figure 1.2: Insertion of the physical operator $\mathcal{O}_\mu$ at $z_* \in \mathbb{P}^1$ restricts the sigma model to the maps that send z_* to lie inside the homology cycle $\mathcal{C}_\mu \in \mathrm{H}_\bullet(X)$ associated with the physical operator in the $\mathcal{Q}_A$-cohomology ring.

Similar to the previous example, this gauge theory is a quiver gauge theory represented by the quiver given in figure 1.1 above.

A more general case that one can consider is the partial flag variety. We will make some comments on the GIT construction of this type of variety in chapter 6.

Intersection theory on the moduli space of holomorphic maps

Now that we have reviewed the two main pieces of the game, 2d GLSMs and topological A-twist, let us now look at the scenario where they are both together. That is, let us look at topological A-twisted 2d GLSMs on a genus-g closed Riemann surface Σ_g. Recall from the above discussion that a 2d GLSM consists of the following data:

$$(G, V, \rho_V, W) \tag{1.1.28}$$

with ρ_V giving (the weights of) the action of the gauge group G on the representation space V spanned by the matter fields ϕ_α. Meanwhile, the superpotential $W \in \mathrm{Sym}V^*$ is a polynomial on the space V^* dual to V. To simplify the discussion, let us assume that it is vanishing.

It turns out that upon topologically A-twisting this theory on Σ_g, the $\mathcal{Q}_A$-cohomology ring corresponds to the de Rham cohomology of the target variety $X = V /\!/_{\tilde{\xi}} G$ [11, 14]. More explicitly, a physical operator $\mathcal{O}_\mu \in \mathcal{R}_{\mathrm{2d}}$ of $U(1)_A$ charge $p+q$ and $U(1)_V$ charge $-p+q$ is associated with a (p,q)-differential form $\omega_\mu \in H^{p,q}(X)$. Using Poincaré duality, one can associate with the operator $\mathcal{O}_\mu$ a homology element $\mathcal{C}_\mu \in \mathrm{H}_\bullet(X)$ where the differential form ω_μ is supported. Another way of saying this is that inserting the operator $\mathcal{O}_\mu$ at a point $z_* \in \Sigma_g$ restricts the maps of the NLSM to those that send the point $z_\star$ to a point lying inside $\mathcal{C}_\mu$. This is depicted in figure 1.2 for the Riemann sphere case $\Sigma_g = \mathbb{P}^1$.

Moduli space of holomorphic maps.

Fixing a complex structure on the Riemann surface Σ_g, let us now consider the correlation function of a set of n physical operators $\mathcal{O}_{\mu_i}$ inserted at $z_i \in \Sigma_g$. The path integral of the 2d sigma model sums over all possible configurations including the degree of the map $\phi : \Sigma_g \longrightarrow X$ – this is defined as the homology class $\beta \equiv \phi_*[\Sigma_g] \in \mathrm{H}_2(X, \mathbb{Z})$. Therefore, one can decompose the path integral as a sum over contributions from different sectors of maps of different degrees:

$$\langle \mathcal{O}_{\mu_1}(z_1) \cdots \mathcal{O}_{\mu_n}(z_n) \rangle_{\Sigma_{g,n}} = \sum_{\beta \in \mathrm{H}_2(X,\mathbb{Z})} \langle \mathcal{O}_{\mu_1}(z_1) \cdots \mathcal{O}_{\mu_n}(z_n) \rangle^{\beta}_{\Sigma_{g,n}} \ , \tag{1.1.29}$$

where we included the index n to $\Sigma_{g,n}$ to indicate that there are n marked points on it where the operators are inserted. The terms appearing on the r.h.s are given by:

$$\langle \mathcal{O}_{\mu_1}(z_1) \cdots \mathcal{O}_{\mu_n}(z_n) \rangle^{\beta}_{\Sigma_{g,n}} = \int_{\phi_*[\Sigma_g]=\beta} [d\phi][d\Psi] e^{-S_{\mathrm{NLSM}}[\phi,\Psi]} \mathcal{O}_{\mu_1} \cdots \mathcal{O}_{\mu_n} \ , \tag{1.1.30}$$

where Ψ stands for all fermionic degrees of freedom we have in the theory.

This path integral can be worked out explicitly and exactly via equivariant localisation with respect to the fermionic scalar operator $\mathcal{Q}_A$. The $\mathcal{Q}_A$ fixed locus turns out to be the space of holomorphic maps $\phi : \Sigma_{g,n} \longrightarrow X$, $\partial_{\bar{z}} \phi = 0$. These are usually referred to as *world-sheet instantons* since they minimise the action S_{NLSM} of the topological sigma model [11]:[1]

$$S_{\mathrm{NLSM}} \left[\partial_{\bar{z}} \phi = 0 \right] = \int_{\Sigma_g} \phi^* \Omega_X \equiv \Omega_X \cdot \beta \ , \tag{1.1.31}$$

where Ω_X is the Kähler form on X. The class $[\Omega_X] \in \mathrm{H}^{1,1}(X)$ is proportional to the FI parameter $\tilde{\xi}$ of the GLSM, as can be seen from the D-term equations in 1.1.12.

Denoting the space of these holomorphic maps by $\mathcal{M}_{g,n}(X, \beta)$, equivariant localisation leads to the following result[2]:

$$\langle \mathcal{O}_{\mu_1}(z_1) \cdots \mathcal{O}_{\mu_n}(z_n) \rangle^{\beta}_{\Sigma_{g,n}} = e^{-\Omega_X \cdot \beta} \int_{\mathcal{M}_{g,n}(X,\beta)} \mathrm{ev}_1^*(\omega_{\mu_1}) \wedge \cdots \wedge \mathrm{ev}_n^*(\omega_{\mu_n}) \ , \tag{1.1.32}$$

with *the evaluation map* $\mathrm{ev}_i : \mathcal{M}_{g,n}(X, \beta) \longrightarrow X$ sends the holomorphic map ϕ to its

[1] The topological sigma model we are considering here, besides being independent of the metric on Σ_g as we argued earlier, does not depend on the complex structure on Σ_g nor that of X. It only depends on the Kähler class of X.

[2] Generically, the moduli space $\mathcal{M}_{g,n}(X, \beta)$ is infinite and one is required to compactify it to be able to define its homology class for this integral to make sense. As we will discuss in the next section, the GLSM construction offers us a particular choice of compactification, which is referred to as the *quasi-maps compactification*.

evaluation $\phi(z_i) \in X$ at the i-th marked point $z_i \in \Sigma_g$. This induces the pullback map on the cohomologies $\mathrm{ev}_i^* : \mathrm{H}^\bullet(X) \longrightarrow \mathrm{H}^\bullet(\mathcal{M}_{g,n}(X,\beta))$. In this sense, the correlation functions of the sigma model in the A-twist count holomorphic maps from Σ_g to the target X with extra constraints on where each marked point in Σ_g lands in X.

Further comments on the final result (1.1.32)**.** Before we move on in our discussion, let us mention the following important point that is left implicit in writing (1.1.32). At the beginning of our discussion here, we fixed the complex structure (equivalently, the Riemannian metric) on the worldsheet $\Sigma_{g,n}$ and derived the worldsheet instanton equations with respect to that complex structure. But, when integrating over the moduli space of holomorphic maps $\mathcal{M}_{g,n}(X,\beta)$, one needs to integrate over the moduli space of equivalence classes of complex structures on $\Sigma_{g,n}$. This is usually denoted by $\mathcal{M}_{g,n}$.

From the physics perspective, this amounts to coupling the 2d supersymmetric gauge theory to 2d topological gravity à la Witten [11, 12]. Upon this coupling, one can compute the GW invariants using the holomorphic anomaly equations [42, 43]. This is beyond the scope of our work and we will not discuss it here. We are mainly interested in the case $g = 0$ where there is only one equivalence class of complex structures.

Selection rules. As we mentioned earlier, the vector $U(1)_V$ R-symmetry is non-anomalous at the quantum level. This demands the invariance of the path integral (1.1.30) under $U(1)_V$,

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \dim_{\mathbb{C}} \mathcal{M}_{g,n}(X,\beta) \ . \tag{1.1.33}$$

The second equality follows from (1.1.32) which states that the correlation function $\langle \mathcal{O}_{\mu_1} \cdots \mathcal{O}_{\mu_n} \rangle_{\Sigma_{g,n}}$ receives contributions from a finite number of homology classes $\beta \in \mathrm{H}_2(X,\beta)$.

On the other hand, the axial anomaly helps us compute the dimension of the moduli space [14]:

$$\dim_{\mathbb{C}} \mathcal{M}_{g,n}(X,\beta) = c_1(X)\cdot\beta + \dim_{\mathbb{C}} X(1-g) \ , \tag{1.1.34}$$

with $c_1(X)\cdot\beta = \int_{\Sigma_g} \phi^* c_1(X)$.

What about the 2d B-model? One can follow the same arguments for the B-twist scenario. In that case, one finds that the path integral localises on constant maps from Σ_g to X, and the correlation functions end up computing period integrals on X. But in the B-model, there is a catch. Due to the axial anomaly, the constraint for the B-model to be well-defined at the quantum level is for the sum of charges of the matter fields to be trivial. From the NLSM perspective, this translates into the Calabi-Yau (CY) condition on X, namely $c_1(X) = 0$ [13].

(Small) quantum cohomology of X

Let us now focus on the case $g = 0$. This will be our main case of study in this thesis[1]. The genus-0 GW invariants of the target X (1.1.32) give rise to a deformation of the standard de Rham cohomology ring $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ to what is usually referred to as the (equivariant) *small quantum cohomology* $\mathrm{QH}^\bullet_{\mathrm{eq}}(X)$. The equivariance here is with respect to the maximal torus of the isometry group of the target X. The ring relations are of the form:

$$\omega_\mu \star \omega_\nu = \sum_{\beta \in \mathrm{H}_2(X,\mathbb{Z})} q_{2\mathrm{d}}^\beta \, {C^{(\beta)}_{\mu\nu}}^\lambda(\omega) \, \omega_\lambda \ , \tag{1.1.35}$$

where we chose $\{\omega_\mu\}_{\mu \in I}$ to be a basis of $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$. The parameter $q_{2\mathrm{d}}$ is to keep track of the degree of the map, and it is defined such that $q_{2\mathrm{d}}^\beta \equiv e^{-\Omega_X \cdot \beta}$. To define the contribution ${C^{(\beta)}_{\mu\nu}}^\lambda(\omega)$ of the degree-β holomorphic maps to the quantum product, let us first define the following two elements:

$$\eta_{\mu\nu}(\omega) \equiv \sum_{\beta \in \mathrm{H}_2(X,\mathbb{Z})} q_{2\mathrm{d}}^\beta \eta^{(\beta)}_{\mu\nu}(\omega) \ , \qquad C_{\mu\nu\lambda}(\omega) \equiv \sum_{\beta \in \mathrm{H}_2(X,\mathbb{Z})} q_{2\mathrm{d}}^\beta C^{(\beta)}_{\mu\nu\lambda}(\omega) \ , \tag{1.1.36}$$

where the intersection numbers $\eta^{(\beta)}_{\mu\nu}(\omega)$ and $C^{(\beta)}_{\mu\nu\lambda}(\omega)$ are given explicitly by:

$$\begin{aligned} \eta^{(\beta)}_{\mu\nu}(\omega) &\equiv \int_{\mathcal{M}_{0,2}(X,\beta)} \mathrm{ev}_1^*(\omega_\mu) \wedge \mathrm{ev}_2^*(\omega_\nu) \ , \\ C^{(\beta)}_{\mu\nu\lambda}(\omega) &\equiv \int_{\mathcal{M}_{0,3}(X,\beta)} \mathrm{ev}_1^*(\omega_\mu) \wedge \mathrm{ev}_2^*(\omega_\nu) \wedge \mathrm{ev}_3^*(\omega_\lambda) \ . \end{aligned} \tag{1.1.37}$$

The matrix $\eta_{\mu\nu}(\omega)$ is an invertible matrix with inverse $\eta^{\nu\rho}(\omega)$. With this in mind, we define:

$${C_{\mu\nu}}^\lambda(\omega) = C_{\mu\nu\rho}(\omega)\eta^{\rho\lambda}(\omega) \ , \qquad {C_{\mu\nu}}^\lambda(\omega) = \sum_{\beta \in \mathrm{H}_2(X,\mathbb{Z})} q_{2\mathrm{d}}^\beta {C^{(\beta)}_{\mu\nu}}^\lambda(\omega) \ . \tag{1.1.38}$$

It has been shown that the quantum product is both commutative and associative [12, 46] – see also [47, 48] for more concrete mathematical discussion in terms of Frobenius manifolds.

2d GLSM/quantum cohomology correspondence. These definitions along with the

[1] On $\mathbb{P}^1$, we only have one class of complex structures. Therefore, when viewing the path integrals we are computing here as (topological) string amplitudes, the integral over the moduli space of complex structures is trivial in this case. For $g \geq 1$ one instead still needs to couple the 2d sigma model to 2d topological gravity [11, 44, 45] to take into account all possible classes of complex structures on Σ_g [12].

identification (1.1.32) lead us to the conclusion that the 2d twisted chiral ring $\mathcal{R}_{\rm 2d}$ is equivalent to the quantum cohomology ring (rather than the classical de Rham cohomology as stated earlier) of the target variety X. The Frobenius pairing and structure constants of $\mathrm{QH}^\bullet_{\rm eq}(X)$ defined in (1.1.36) are to be identified with the topological metric and structure constants of $\mathcal{R}_{\rm 2d}$ defined in (1.1.14):

$$\begin{aligned}\eta_{\mu\nu}(\omega) &= \sum_{\widetilde{\sigma}_*\in\mathcal{S}^{\rm 2d}_{\rm BE}} \mathcal{H}(\widetilde{\sigma}_*,\widetilde{m})^{-1}\mathcal{O}_\mu(\widetilde{\sigma}_*,\widetilde{m})\mathcal{O}_\nu(\widetilde{\sigma}_*,\widetilde{m})\ , \\ C_{\mu\nu\lambda}(\omega) &= \sum_{\widetilde{\sigma}_*\in\mathcal{S}^{\rm 2d}_{\rm BE}} \mathcal{H}(\widetilde{\sigma}_*,\widetilde{m})^{-1}\mathcal{O}_\mu(\widetilde{\sigma}_*,\widetilde{m})\mathcal{O}_\nu(\widetilde{\sigma}_*,\widetilde{m})\mathcal{O}_\lambda(\widetilde{\sigma}_*,\widetilde{m})\ .\end{aligned} \tag{1.1.39}$$

Here, we used the explicit formula (1.1.19) for the correlation functions in the topological A-model in terms of a sum over the Bethe vacua. We are also assuming that $\{\mathcal{O}_\mu\}_{\mu\in I}$ is the basis of $\mathcal{R}_{\rm 2d}$ that corresponds to $\{\omega_\mu\}_{\mu\in I}$. See, for example, [31] for an explicit calculation of these correlation functions as Jeffery–Kirwan (JK) residues obtained via Coulomb branch localisation techniques.

In (1.1.39), we introduced the explicit dependence on the 2d complex twisted masses $\widetilde{m}$. From the geometry perspective, these correspond to the equivariant parameters associated with the isometry group of X. Said differently, the isometry group is realised as the global flavour symmetry group rotating the chiral matter fields defining X.

1.2 3d $\mathcal{N}=2$ supersymmetric gauge theory

In this section we discuss some aspects of 3d $\mathcal{N}=2$ supersymmetric gauge theories on 3-manifolds. We start with the flat cases, the Euclidean space $\mathbb{R}^3$ and the real 3-torus T^3. We discuss aspects of the moduli space of supersymmetric vacua and 3d infrared dualities for special class of gauge theories.

Later in the section we give a brief review on the prescription of placing these 3d theories on curved geometries. This gives us the 3d A-models which are 3d uplifts of the 2d A-model we studied in the previous section. Focusing on 3d GLSMs on $\mathbb{P}^1\times S^1_\beta$, we study the connection with quantum K-theory of the target space X and how to realise the moduli space of quasimaps – a compactification of the moduli space of holomorphic maps introduced above – from reducing the 3d theory to an $\mathcal{N}=2$ supersymmetric quantum mechanics on S^1_β. In the end, we make some comments on the case $X=\mathrm{Gr}(N_c,n_f)$, which will be our case of interest in this work.

1.2.1 3d $\mathcal{N}=2$ gauge theory on $\mathbb{R}^3$

Similar to the discussion we had in the previous section about 2d $\mathcal{N}=(2,2)$ supersymmetry on $\mathbb{R}^2$, let us now consider 3d $\mathcal{N}=2$ supersymmetric gauge theories on $\mathbb{R}^3$. Let us take the coordinates to be $(z,\overline{z},t)$. The algebra is equivalent to (1.1.1) by dimensional reduction of the 3d theory along the 'time' direction t. More explicitly, the algebra in this case is given by:

$$\begin{aligned} \{\mathcal{Q}_+,\overline{\mathcal{Q}}_+\} &= 2P_z \ , & \{\mathcal{Q}_-,\overline{\mathcal{Q}}_-\} &= -2P_{\overline{z}} \ , \\ \{\mathcal{Q}_-,\overline{\mathcal{Q}}_+\} &= -i(Z+iP_t) \ , & \{\mathcal{Q}_+,\overline{\mathcal{Q}}_-\} &= i(Z-iP_t) \ . \end{aligned} \tag{1.2.1}$$

with P_t being the momentum along the t-direction and Z is 3d real central charge. Note that $\widetilde{Z}\equiv Z+iP_t$ is the 2d complex central charge of the corresponding 2d $\mathcal{N}=(2,2)$ algebra (1.1.1).

With this equivalence in mind, the representations of this algebra are the same as those of the 2d $\mathcal{N}=(2,2)$ ones discussed in the section above. The main difference is that, for the gauge group G, the scalar for the gauge vector multiplet σ is now a real adjoint scalar.

Semi-classical description of the field theory

Let us now study the low-energy dynamics of the 3d $\mathcal{N}=2$ supersymmetric gauge theory. As before, take the gauge group G to be simply-connected or a union of unitary groups and couple the theory with M chiral matter multiplets Φ_α in representation $\mathfrak{R}_\alpha$ of G. Writing down the explicit form of the Lagrangian of this theory, one finds that the scalar potential is of the form (1.1.8):

$$U(\sigma,\phi)=\frac{1}{2e^2}\,\mathrm{Tr}[\sigma,\sigma^\dagger]^2+\frac{e^2}{2}\sum_{a,b=1}^{\mathrm{rk}(G)}(D^a{}_b)^2+\sum_{\alpha=1}^{M}|F_\alpha|^2+2\sum_{\alpha=1}^{M}|(\sigma+m)\cdot\phi_\alpha|^2 \ . \tag{1.2.2}$$

The difference now, as mentioned above, is that the scalar components of the 3d vector multiplets are real rather than complex. For instance, the real mass matrix m is a VEV for the scalar of a 3d background vector multiplet associated with the global flavour symmetry G_F of the system.

The other difference – which leads to more rich structures in 3d – are Chern–Simons terms. These take the following supersymmetric form:

$$\frac{K}{4\pi}\int_{\mathcal{M}_3} \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A-\overline{\lambda}\lambda+2D\sigma\right) \ . \tag{1.2.3}$$

Here, we are indicating that this can be defined on any 3-manifold $\mathcal{M}_3$.

The last term in (1.2.3) modifies the D-term equations (1.1.10) as follows:

$$D^a{}_b = \mu^a{}_b(\phi) - \frac{\delta^a{}_b}{2\pi}(\xi + k\sigma_a) + f^a{}_b(\sigma, m) \ , \tag{1.2.4}$$

for $a, b = 1, \cdots \mathrm{rk}(G)$. The moment map $\mu^a{}_b(\phi)$ takes the same form as the 2d one (1.1.10) where now it involves the 3d scalars ϕ_α. The term $f^a{}_b(\sigma, m)$ contains one-loop shifts to the effective CS levels and FI parameter [49]. These shifts are associated with the relation between the CS level K appearing in (1.2.3) and the k appearing explicitly in (1.2.4). The details of this connection will be discussed in subsection 2.2.2 below. We will give the explicit form for $f^a{}_b(\sigma, m)$ later for our theories of interest.

Putting all the pieces together, the equations describing the semi-classical SUSY vacua of the 3d gauge theory are of the form:

$$\begin{aligned}
&\mu^a{}_b(\phi) = \xi\delta^a{}_b \ , && a, b = 1, \cdots, \mathrm{rk}(G) \ , \\
&(\sigma + m)\cdot\phi_\alpha = 0 \ , && \alpha = 1, \cdots, M \ , \\
&\frac{\partial W}{\partial\phi_\alpha} = 0 \ , && \alpha = 1, \cdots, M \ .
\end{aligned} \tag{1.2.5}$$

In 3d, one can also have a topological symmetry $U(1)_J$ associated with each $U(1)$ factor in the gauge group G. Taking $F_{\mathrm{top}\,\mu\nu}$ to be the field strength associated with the gauge $U(1)$ factor, the Noether current of $U(1)_J$ is $J^{\mathrm{top}}_\mu = \epsilon_{\mu\nu\rho}F^{\nu\rho}_{\mathrm{top}}$. The conservation law of J^{top}_μ follows directly from the Bianchi identity associated with $F_{\mathrm{top}\,\mu\nu}$. The 3d real FI parameter ξ that we introduced in (1.2.4) can be viewed as a VEV for the real scalar σ_J living in a background vector multiplet $\mathcal{V}_J$ for the topological symmetry $U(1)_J$. The form of the SUSY vacua solving the equations (1.2.5) depends on the sign of ξ, but the number of these vacua does not.

This independence can be seen as follows. Let us look at the reduced 3d theory on $\mathbb{R}^2 \times S^1$. As we will discuss in the next chapter, in this case, the real 3d FI parameter ξ gets complexified by the holonomy of A^J_3 – the component of the topological $U(1)_J$ background gauge field along the S^1 direction. Therefore, any phase transitions become of complex dimension, which one can avoid [13, 49].

Counting the moduli space of SUSY vacua: 3d Witten index

The moduli space of supersymmetric vacua of the 3d supersymmetric gauge theory can be worked out explicitly by solving the equations (1.2.5). These are semi-classical solutions in the sense that they do not account for potential (strongly-coupled) quantum effects in some regimes. We will comment on this point more presently.

One of the main physical quantities that one computes when studying the moduli space of vacua is the 3d *Witten index* that we denote by $\mathbf{I}_{\mathrm{W}}$. This was first defined by Witten [50] and worked out for 3d pure Chern–Simons theories with $G = SU(N)_k$ and studied (relatively) more recently by Intriligator and Seiberg [49] for the 3d theories with gauge group $U(1)_k$ and matter multiples charged under it. This index can be computed by placing the 3d theory on a (flat) real 3-torus T^3 meanwhile imposing a periodic boundary condition for the fermionic degrees of freedom along the 'time' circle:

$$\mathbf{I}_{\mathrm{W}} = \mathrm{Tr}_{\mathscr{H}_{T^2}}(-1)^F \ , \tag{1.2.6}$$

where F is the fermion number operator and the trace is taken over the Hilbert space $\mathscr{H}_{T^2}$ associated with a 2d space slice T^2.

Moduli space of vacua. Let us look at the possible forms of the solutions to (1.2.5). For our purpose, it will be more useful to consider particular limits on the masses such that the index remains well-defined – as we will see in chapter 3, it is useful to take all the masses to be vanishing. By doing this, we will encounter three types of (semi-classical) vacua:

(i) The *Higgs vacua* are compact moduli spaces $\mathcal{M}_H$, including the case of discrete vacua. Any such compact branch $\mathcal{M}_H$ contributes to the Witten index through its Euler characteristic [51],

$$\chi(\mathcal{M}_H) \leq \mathbf{I}_{\mathrm{W}} \ . \tag{1.2.7}$$

In particular, Higgs vacua will be realised as GIT quotients by the gauge symmetry similar to what we had in the 2d GLSMs story in (1.1.22). In this sense, the 3d gauge theory will be a 3d GLSM (i.e. a 3d uplift of a 2d GLSM) with target variety $\mathcal{M}_H$. Generically, this moduli space will be given by:

$$\mathcal{M}_H = V_0 /\!\!/_{\xi}\, G_0 \ . \tag{1.2.8}$$

with $V_0 \subseteq V \cong \mathbb{C}^{\dim(\mathfrak{R})}$ spanned by non-vanishing scalar fields ϕ and $G_0 \subseteq G$ the unbroken part of the full gauge symmetry G.

(ii) The *topological vacua* consist of pure $\mathcal{N}=2$ supersymmetric Chern–Simons theories; this is equivalent to having a 3d topological quantum field theory (TQFT) in the infrared, as the gauginos are massive. Such TQFT sectors contribute to the Witten index through the non-zero index of the $\mathcal{N}=2$ pure CS theory,

$$\mathbf{I}_{\mathrm{W}}[\mathrm{TQFT}] \leq \mathbf{I}_{\mathrm{W}} \, . \tag{1.2.9}$$

Topological vacua arise at fixed non-zero values of the fields σ_a in (1.2.5) so that all chiral multiplets ϕ are massive and can be integrated out, leaving behind an effective pure CS theory.

(iii) The *Coulomb vacua* are semi-classical vacua that open up where (part of) the non-abelian gauge symmetry is restored (with vanishing effective CS levels). In this case, we have continuous solutions for $\sigma_a \neq 0$, and the semi-classical analysis is usually not reliable – some strong-coupling effect may modify the picture entirely [52].[1] Hence, when such vacua arise, we will have to make some *ad-hoc* conjectures about the corresponding contribution to the index:

$$\mathbf{I}_{\mathrm{W}}[\text{strongly-coupled vacua}] \leq \mathbf{I}_{\mathrm{W}} \, . \tag{1.2.10}$$

Moreover, in general, we may have *hybrid vacua*, where some of these possibilities arise simultaneously. In particular, for the theories we will study in chapter 3, we will find many *Higgs-topological vacua*. These are simply cases where there is a residual TQFT at every point on the Higgs branch, which we may view as a trivial fibration TQFT $\to \mathcal{M}_{\mathrm{hybrid}} \to \mathcal{M}_H$.[2] We denote such vacua by $\mathcal{M}_H \times$ TQFT. Their contribution to the Witten index is simply the product of the geometric and TQFT contributions,

$$\mathbf{I}_{\mathrm{W}}[\mathcal{M}_{\mathrm{hybrid}}] = \chi(\mathcal{M}_H)\mathbf{I}_{\mathrm{W}}[\mathrm{TQFT}] \leq \mathbf{I}_{\mathrm{W}} \, . \tag{1.2.11}$$

[1] The semi-classical analysis is valid far away in the Coulomb branch – in the regimes with large $|\sigma|$. In regions near the origin of the Coulomb branch, it is expected that quantum effects will dominate [52, 53, 54]. This amounts to generating effective superpotentials that deform the geometry of the total moduli space of vacua in that regime. Therefore, one needs to keep track of these deformations when computing the Witten index for instance. This is beyond the scope of this thesis and we hope that we will report on this more in the future.

[2] In a general gauge theory, we could have a non-trivial fibration because different subgroups of the gauge group might survive at different points on the Higgs branch. Here, at each solution for the σ's and ϕ's, we have a standard Higgs mechanism, and the TQFT arises from gauge fields that do not couple at all to the chiral multiplets that obtain a VEV; hence, the fibration is trivial.

We will discuss all these cases more thoroughly in section 3.2 for unitary 3d SQCD theories.

1.2.2 3d infrared dualities

One of the fascinating aspects appearing in some quantum field theories is that they come in infrared dual pairs. Roughly speaking, one would start with two distinct QFTs in the UV and follow their behaviour along the renormalisation group (RG) flow. If they are in the same dual pair, it will turn out that they both flow to the same fixed point in the IR. For instance, there are cases where one of the theories in the dual pairs is weakly-coupled in regimes where the other theory is strongly-coupled which helps in better understanding the dynamics of the latter in these regimes.

In our work, we will focus on a special class of 3d supersymmetric Chern–Simons–matter theories that we denote by $\mathrm{SQCD}[N_c, k, l, n_f, n_a]$. This is defined as a 3d $\mathcal{N} = 2$ $U(N_c)_{k,k+lN_c}$ gauge theory with n_f fundamental chiral multiplets and n_a antifundamental chiral multiplets. Importantly, we allow for generic Chern–Simons levels k and $(k+lN_c)N_c$ for the $SU(N)$ and $U(1)$ factors of $U(N_c) \cong (SU(N_c) \times U(1))/\mathbb{Z}_{N_c}$.

These theories enjoy infrared (Seiberg-like) dualities [55, 56, 57, 58]. Such dualities have been extensively studied in recent years [59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83], and they can even be related to non-supersymmetric 'bosonisation' dualities in 3d –see *e.g.* [84, 85, 86, 87, 88, 89, 90, 91].

The precise form of the dual gauge theory description of unitary SQCD depends non-trivially on the CS levels k and l. For $l = 0$, we have the famous Aharony duality (for $k = 0$, $n_f = n_a$) [56], the Giveon–Kutasov duality (for $k \neq 0$, $n_f = n_a$) [57], and some 'chiral' Seiberg-like dualities (for $n_f \neq n_a$) [58]. Very recently, these dualities were generalised to the case $l \neq 0$ by Nii [76] (for $k \neq 0$, $n_f = n_a$) and by Amariti and Rota [92] (in the other cases). All these new dualities can be obtained from the standard $l = 0$ dualities by a suitable application of the Kapustin–Strassler–Witten (KSW) $\mathrm{SL}(2, \mathbb{Z})$ action on 3d $\mathcal{N} = 2$ field theories [93, 94], as first pointed out in [92]. In chapter 4, we revisit this derivation and clarify some subtle aspects of it, especially as it pertains to the various Chern–Simons contact terms, which must be determined in order to specify any 3d duality fully.

There is a variety of ways by which one can test 3d IR dualities. The two methods we will follow in this work are by matching 3d Witten indices (1.2.6)[1] and by explicitly matching the moduli spaces of vacua on both sides using known geometric equivalences and dualities for 3d TQFTs. We will study this in the next two chapters. For now, let us give a general overview of the form of the dual theories of 3d unitary SQCD.

[1]Or more generally, 3d twisted indices which we will define in the next subsection.

Unitary SQCD with $l = 0$

Unitary SQCD with $l = 0$ with gauge group $U(N_c)_k$ has an infrared-dual description in terms of a $U(N_c^D)_{-k}$ gauge group, with a particular matter content that depends on the parameters k, n_f and n_a. There are four distinct cases to consider [58]:

(i) **Aharony dual.** For $k = 0$, $n_f = n_a \equiv N_f$, we have the $U(N_f - N_c)_0$ dual description, known as the Aharony dual [53]. The dual description involves N_f^2 gauge-invariant chiral multiplets (the 'mesons' of the 'electric' SQCD description), as well as two additional singlets charged under the topological symmetry (the 'monopoles'[1] of the electric description).

(ii) **Minimally chiral case.** For $k \neq 0$ and $|k| > |k_c|$, we have a $U(N_c^D)_{-k}$ description with $n_f n_a$ mesons and no monopole singlets. We call these the 'minimally chiral' theories. In the non-chiral case, $n_f = n_a \equiv N_f$, we have a $U(N_f + |k| - N_c)_{-k}$ dual gauge group, and this is known as the Giveon–Kutasov duality [57].

(iii) **Marginally chiral case.** For $k \neq 0$ and $|k| = |k_c|$, we have a $U(N_c^D)_{-k}$ description with $n_f n_a$ mesons and one monopole singlet. We call these the 'marginally chiral' theories.

(iv) **Maximally chiral case.** For $|k| < |k_c|$, we have a $U(n_f - N_c)_{-k}$ or $U(n_a - N_c)_{-k}$ description if $n_f > n_a$ or $n_a > n_f$, respectively, with $n_f n_a$ mesons and no monopole singlets. We call these the 'maximally chiral' theories.

Unitary SQCD for general l.

In the general case with arbitrary $l \in \mathbb{Z}$, the computation of the Witten index becomes more involved, as we will discuss momentarily. The theory also has a very interesting 'magnetic' dual description with a product unitary gauge group [76, 92]:

$$G^D = U(N_c^D) \times U(1) \ , \tag{1.2.12}$$

with N_c^D defined in (4.0.2), if $|k| \geq |k_c|$ (if $|k| < |k_c|$, the dual gauge group remains $U(N_c^D)$). There are again four cases to consider, as for $l = 0$. We will study these dualities in more detail in section 4.2. Here, let us only summarise their key features:

[1] Recall that these are disorder operators in the sense that their insertion at a point leads to a singular behaviour of some of the fields in the theory around that point. For a 3d theory with gauge group G of rank rk$G$, these are classified by $\pi_2(G/U(1)^{\text{rk}G}) \cong \mathbb{Z}^{\text{rk}G}$. They are also charged under the topological symmetry that we discussed above. See [53] for more details.

(i) **Amariti–Rota dual.** For $k=0$, $n_f = n_a \equiv N_f$, we have a dual gauge theory:

$$\underbrace{U(N_f - N_c)_{0,0} \times U(1)_l}_{0} \ , \tag{1.2.13}$$

which was first derived in [92]. The line connecting the gauge factors denotes a (vanishing) mixed CS level, $k_{12} = 0$. Importantly, the theory also has matter fields charged under both gauge factors.

(ii) **Minimally chiral case.** For $k \neq 0$ and $|k| > |k_c|$, we have a dual gauge theory:

$$\underbrace{U\Big(|k| + \frac{1}{2}(n_f + n_a) - N_c\Big)_{-k,\,-k+\text{sign}(k) N_c^D} \times U(1)_{l+\text{sign}(k)}}_{\text{sign}(k)} \ , \tag{1.2.14}$$

and there is no matter charged under the $U(1)_{l\pm 1}$ factor. It only couples to the $U(N_c^D)$ sector through the mixed CS level $k_{12} = \text{sign}(k) = \pm 1$.

(iii) **Marginally chiral.** For $k \neq 0$ and $|k| = |k_c|$, we have a dual description:

$$\underbrace{U(\max(n_f, n_a) - N_c)_{-k,\,-k+\frac{1}{2}\text{sign}(k) N_c^D} \times U(1)_{l+\frac{1}{2}\text{sign}(k)}}_{\frac{1}{2}\text{sign}(k)} \ , \tag{1.2.15}$$

and there is some matter charged under both gauge groups.

(iv) **Maximally chiral case.** For $|k| < |k_c|$, we have a dual gauge theory:

$$U(\max(n_f, n_a) - N_c)_{-k,\,-k+l N_c^D} \ , \tag{1.2.16}$$

similarly to the $l=0$ case. The details of this duality and its derivation will be discussed in subsection 4.2.6.

When $l=0$, these dualities can be reduced to the previous unitary dualities. The knowledge of the dual description immediately yields some non-trivial information. For instance, whenever $N_c^D < 0$ the Witten index vanishes, and therefore supersymmetry could be broken. The limiting cases of the dualities for which $N_c^D = 0$ also give us 's-confining' phases (an IR description in terms of chiral multiplets only) if $l=0$ or if $|k| < |k_c|$.

1.2.3 3d $\mathcal{N}=2$ on a curved geometry and the 3d A-model

In the previous section, we reviewed aspects of 3d $\mathcal{N}=2$ supersymmetric gauge theories living on flat Euclidean space $\mathbb{R}^3$. In the same spirit as what we did in subsection 1.1.2 for 2d $\mathcal{N}=(2,2)$ theories, to connect these 3d theories to enumerative geometry, we would like to place them on a curved 3-manifold $\mathcal{M}_3$ – in particular, we will be mainly interested in the case $\mathcal{M}_3 = \Sigma_g \times S^1_\beta$ with Σ_g being a closed genus-g Riemann surface and S^1_β is a circle of size β.

The obstruction to placing a 3d $\mathcal{N}=2$ theories on arbitrary $\mathcal{M}_3$ results from the fact that the associated non-flat Riemannian metric $g_{\mu\nu}$ might break supersymmetry. Assuming that our 3d theories have an exact (non-anomalous) $U(1)_R$ R-symmetry, there is a prescription for how one can couple the theory with the curved spacetime metric and obtain a *curved-space rigid supersymmetry.* We will not review this here but we will mention the main results for the theories we are interested in. The reader is invited to consult [95, 96, 97, 98, 99] and references therein for an introduction and discussion.

In the end, it turns out that a 3-manifold $\mathcal{M}_3$ with metric $g_{\mu\nu}$ admits 3d $\mathcal{N}=2$ (or part of it) depending on the number of independent solutions to the following set of *generalised Killing spinor equations*:[1]

$$\begin{aligned}\left(\nabla_\mu - iA^{(R)}_\mu\right)\zeta &= -\frac{1}{2}H\gamma_\mu\zeta + \frac{i}{2}V_\mu\zeta - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta\ ,\\ \left(\nabla_\mu + iA^{(R)}_\mu\right)\widetilde{\zeta} &= -\frac{1}{2}H\gamma_\mu\widetilde{\zeta} - \frac{i}{2}V_\mu\widetilde{\zeta} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta}\ .\end{aligned} \tag{1.2.18}$$

Here $A^{(R)}_\mu$ is a background gauge field associated with the $U(1)_R$ line bundle and H and V_μ are background auxiliary fields with the constraint that $\nabla_\mu V^\mu = 0$. It has been argued in [97, 101] that a solution to these equations exists if and only if the 3-manifold $\mathcal{M}_3$ admits a *transversely holomorphic foliation* (THF). Therefore, a classification of possible curved backgrounds where one can place 3d $\mathcal{N}=2$ supersymmetric field theories is given by the classification of THFs of 3-manifolds [102, 103, 104].

[1] Following [100], the Pauli matrices γ_μ are given by:

$$(\gamma^\mu)_\alpha{}^\beta = \left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ ,\ \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\ ,\ \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\right\}\ , \tag{1.2.17}$$

with $\alpha, \beta = \mp$.

3d A-model on Seifert geometry

An example of these THFs is the Seifert manifold. This is an S^1-fibration over a 2d orbifold $\hat{\Sigma}$[1]:

$$S^1 \longrightarrow \mathcal{M}_3 \longrightarrow \hat{\Sigma}_g \ . \tag{1.2.19}$$

For example, one can consider the trivial S^1_β-fibration over the smooth Riemann surface Σ_g. This is will be out main case of interest in this work.[2]

3d $\mathcal{N}=2$ supersymmetric gauge theories on Seifert 3-manifolds $\mathcal{M}_3$ were studied extensively by Closset, Kim and Willett in [105]. The generalised Killing equations (1.2.18) in this case simplify into a form analogous to the one we had for A-twisted 2d $\mathcal{N}=(2,2)$ in subsection 1.1.2 – that is these geometries preserve only two supercharges, $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$, of the original $\mathcal{N}=2$ algebra. For this reason, this model is referred to as the 3d A-model.

2d TQFT vs SUSY localisation. As usual in QFTs, we are interested in computing correlation functions of 'physical' operators in the theory[3]. This is usually done through saddle point approximation and using Feynman diagrams. In supersymmetric field theories, it turns out that one can compute such correlation functions exactly using supersymmetric localisation, where the saddle-point (1-loop) approximation becomes exact due to supersymmetry. There is a vast literature on the principle and the fruits of supersymmetric localisation. See for example [106] for an overview in different dimensions and [107, 108, 109, 110, 111] and the review [112] for the best studied examples in 3d.

These exact results are typically given in terms of an ordinary integral over the Cartan subalgebra of the gauge algebra or of some complexification thereof. This leaves one with the still-challenging task of evaluating that integral explicitly.

An alternative approach uses the observation made above, upon which the computation of many observables is reduced to a problem in an auxiliary 2d $\mathcal{N}=(2,2)$ field theory subjected to the topological A-twist. For the case $\mathcal{M}_3 = \Sigma_g \times S^1_\beta$, the associated partition function captures the Witten index [51] of the 3d $\mathcal{N}=2$ gauge theory on a

[1] An orbifold point on a Riemann surface is such that a local neighbourhood of this point looks like a subset of $\mathbb{C}/\mathbb{Z}_q$ for some $q \in \mathbb{Z}$ rather than $\mathbb{C}$.

[2] We are interested in studying 3d GLSMs – a 3d uplift of the 2d GLSMs we studied in the previous section. Our interest in this special case of Seifert manifolds has to do with the direct enumerative geometry interpretation of the partition functions (rather, 3d twisted indices) in these cases in terms of K-theoretic Gromov–Witten invariants as we will explain below. Even though supersymmetric partition functions are computable for the general Seifert case [105] via supersymmetric localisation, it is still not clear to us what the new data means from an enumerative geometric perspective.

[3] The term physical depends on the theory considered. For instance, for the supersymmetric theories on Seifert geometry we are discussing here, we are interested in computing correlation functions of half-BPS operators preserving the leftover two supercharges.

Riemann surface of genus g [19, 113, 114, 115]:

$$Z_{\Sigma_g \times S^1_\beta}(y) = \mathrm{Tr}_{\Sigma_g}\Big((-1)^F y^{Q_F}\Big) \ . \tag{1.2.20}$$

A special case of this is when $\Sigma_g = T^2$, the real 2-torus. In this case, this computes what we will refer to as the 3d *flavoured Witten index* (1.2.6) which is an integer.

1.2.4 3d GLSMs and quantum K-theory of quasi-maps

In this section, we give a general overview of A-twisted 3d GLSMs with a view towards the computation of K-theoretic invariants[1]. While our main example in this work will be X the complex Grassmanian (1.1.27) (see chapter 5 for more details), we will start here with a more general discussion.

Consider a 3d $\mathcal{N}=2$ supersymmetric gauge theory with gauge group G and chiral multiplets Φ in the representation $\mathfrak{R}$ of G, and let $V \cong \mathbb{C}^{\dim(\mathfrak{R})}$ denote the vector space in which the chiral-multiplet scalars ϕ are valued.[2] As pointed out earlier in this section, the 3d $\mathcal{N}=2$ theory generally contains non-trivial Chern–Simons (CS) interactions, which will be important later on. We shall assume that the theory admits non-trivial Fayet–Iliopoulos (FI) parameters, denoted collectively by ξ, and that, for some choice of ξ, the Higgs branch $X \equiv \mathcal{M}_H$ is a compact Kähler manifold obtained as the GIT quotient $X \cong V/\!\!/_\xi\, G$.

The assumption that it be compact is a strong one, but necessary for the kind of enumerative geometry that we are interested in. Finally, let us further assume that the Chern–Simons interactions are such that the Higgs branch captures the full-flavoured Witten index (1.2.6) of the theory (at fixed ξ), namely the 3d Witten index is equal to the topological Euler characteristic of X:

$$\mathbf{I}_{\mathrm{W}} = \chi(X) \ . \tag{1.2.21}$$

We will refer to these choices of the CS levels as the geometric window of the theory where no contributions are coming from the topological vacua that we introduced in subsection 1.2.1 above.

[1]The following arguments are based on discussions with Cyril Closset and Heeyeon Kim.

[2]Note that $\mathfrak{R}$ is generally reducible.

The A-twist on $\mathbb{P}^1 \times S^1_\beta$ and localisation to quasimaps

We choose explicit coordinates $z, \bar{z}, t$ on $\mathbb{P}^1 \times S^1_\beta$. For concreteness, we may choose the round metric:

$$ds^2 = \beta^2 dt^2 + \frac{4dz\, d\bar{z}}{(1+z\bar{z})^2} \ . \tag{1.2.22}$$

As discussed in subsection 1.2.3, one can put any 3d $\mathcal{N} = 2$ supersymmetric field theory with a $U(1)_R$ symmetry on this background by performing a topological A-twist on $\mathbb{P}^1$.

Consider a 3d GLSM of the type considered above. In order to discuss the 3d path integral in terms of the enumerative geometry of X, we consider the following supersymmetric Lagrangian on $\mathbb{P}^1 \times S^1_\beta$:

$$\mathscr{L} = \frac{1}{e^2}\left(\mathscr{L}_{\text{YM}} + \mathscr{L}_{\text{def}}\right) + \frac{1}{g^2}\mathscr{L}_{\text{matter}} + \mathscr{L}_{\text{CS}} \ . \tag{1.2.23}$$

Here, g^2 and e^2 are arbitrary non-negative parameters, which we will take to zero in a scaling limit such that $e^2/g^2 \to 0$ as well. This is allowed because both the Yang-Mills and matter (chiral multiplet) Lagrangians are $\mathcal{Q}$-exact [100]. Crucially, in addition to the standard Yang-Mills and matter Lagrangians, we introduced a $\mathcal{Q}$-exact deformation term:

$$\mathscr{L}_{\text{def}} = (D - 2iF_{z\bar{z}})\Big(\mu(\phi^\dagger, \phi) - \tau\Big) + \text{fermions} \ , \tag{1.2.24}$$

where $\tau \in \mathbb{R}$ is a 'fake' FI parameter that will allow us to localise onto the Higgs branch [31, 116]. Note that the Chern–Simons terms $\mathscr{L}_{\text{CS}}$ in (1.2.23) include the ordinary 3d FI parameters ξ, and that they are not $\mathcal{Q}$-exact.

This deformation term modifies the equations of motion for the auxiliary real scalar D (1.2.4) to read:

$$D = \mu - \tau + \frac{e^2}{g^2}\mu - \frac{e^2}{2\pi}\xi_{\text{eff}}(\sigma) \ . \tag{1.2.25}$$

Here, the effective FI parameter reads:

$$\xi_{\text{eff}}(\sigma) = \xi + K\sigma + \cdots \ , \tag{1.2.26}$$

schematically, where K denotes the bare CS terms and the ellipsis denotes one-loop corrections that depend on the value of σ in a given vacuum, exactly as in flat space where we denoted them by $f(\sigma, m)$ in (1.2.4).

In general, one has to be careful about the scaling limit $e^2 \to 0$ when considering this localisation scheme, because the product $e^2\xi_{\text{eff}}$ in (1.2.25) might stay finite as we send

$e^2 \to 0$. For a 3d GLSM onto X with CS levels in the geometric window, as we assumed here, we can safely localise to the naive equation:

$$D = \mu - \tau \cong \phi\phi^\dagger - \tau \ , \tag{1.2.27}$$

where we give a schematic expression for the moment map. On general ground, the path integral of the 3d gauge theory on $\mathbb{P}^1 \times S^1_\beta$ localises onto the intersection of the equations of motion and of the off-shell supersymmetry equations. The latter read:

$$D + *F = 0 \ , \qquad D_{\bar{z}}\phi = 0 \ , \qquad (D_t - \sigma - m)\phi = 0 \ , \tag{1.2.28}$$

and we can show that the real scalar σ must be constant [31]. Here, m denotes real mass terms for flavour symmetries, F denotes the curvature 2-form of a principal $G_{\mathbb{C}}$-bundle $\mathcal{P} \to \mathbb{P}^1$ with a hermitian connection A, and $*F = -2iF_{z\bar{z}}$ is the Hodge-star dual of F on $\mathbb{P}^1$. After the A-twist, the 3d scalar ϕ of R-charge $r \in \mathbb{Z}$ becomes a section of $\mathcal{K}^{\frac{r}{2}} \otimes E$, where $\mathcal{K} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ is the canonical line bundle on the sphere and E is the associated vector bundle in the representation $\mathfrak{R}$ of G:

$$\phi \in \Gamma\left(\mathcal{K}^{\frac{r}{2}} \otimes E\right) \ , \qquad E \equiv \mathcal{P} \times_G V \ . \tag{1.2.29}$$

Note, however, that ϕ also depends on the imaginary time t. We also have non-trivial flat connections along S^1_β for both the gauge fields and the background gauge fields for flavour symmetries, which we denote by A_t and A_t^F, schematically. Thus, given our localisation scheme that imposes (1.2.27), we conclude that we localise onto the following equations:

$$\mu(\phi) + *F = \tau \ , \qquad D_{\bar{z}}\phi = 0 \ , \qquad (D_t - \sigma - m)\phi = 0 \ . \tag{1.2.30}$$

For future reference, let us define the dimensionless complex parameters:

$$u \equiv i\beta\sigma - \beta A_t \ , \qquad \nu \equiv i\beta m - \beta A_t^F \ , \tag{1.2.31}$$

so that the last equation in (1.2.30) reads:

$$(-i\partial_t + u + \nu)\phi = 0 \ . \tag{1.2.32}$$

Expanding ϕ in Fourier modes, $\phi = \sum_{n\in\mathbb{Z}} \phi_n(z,\bar{z})e^{int}$, we have the equations

$$(n + u + \nu)\phi_n = 0 \ , \tag{1.2.33}$$

for every n. Furthermore, the modes ϕ_n with $n \neq 0$ are massive and decouple from the low-energy dynamics. We are left with the zero-modes ϕ_0 (which we will still denote by ϕ, by a slight abuse of notation), with the constraint $(u + \nu)\phi = 0$. Then, the two first equations in (1.2.30) are standard vortex equations on $\mathbb{P}^1$. Assuming that $\nu = 0$, for now, we will obtain the generic vortex solutions if and only if $\sigma = 0$. (As per usual, $\nu \neq 0$ corresponds to some equivariant deformation of X.)

From vortices to quasimaps. From the above discussion, we expect that the 3d $\mathcal{N} = 2$ supersymmetric path integral localises onto the space of solutions to the vortex equations on the Riemann sphere. This moduli space (or moduli stack) naturally decomposes into moduli spaces of non-abelian vortices of fixed degree β[1] – The degree is defined of the vortex as the magnetic flux of the gauge group G through the Riemann sphere. One can argue that the lattice of these magnetic fluxes can be identified with the homology group $\mathrm{H}_2(X,\mathbb{Z})$ [116]. This identification will become more apparent later on in chapter 5 for the case $G = U(N_c)$. These vortices have a core (wherein the magnetic flux is concentrated) of size of order $1/\tau$. Since we can choose τ as we wish, we will take the limit $\tau \to \infty$, in which case the magnetic flux becomes concentrated at points $z_\star$ at which the chiral scalar vanishes, $\phi(z_\star) = 0$. These points $z_\star$ are called the *base points*. Let $B \subset \mathbb{P}^1$ denote the set of base points associated with a given magnetic flux F (that is, for a given principal $G_\mathbb{C}$-bundle $\mathcal{P}$). Since $F = 0$ away from the base points, we see that the BPS solution ϕ provides us with a holomorphic map to X except at a finite number of points:

$$\phi \, : \, \mathbb{P}^1 - B \longrightarrow X \ . \tag{1.2.34}$$

Mathematically, the data of such a pair $(\mathcal{P}, \phi)$ is essentially the data of an (ε=0^+)-stable quasimap [117, 118, 119] – see also [120] for an overview and application to the Grassmannian case.[2] Indeed, it is a well-known fact that quasimaps provide us with a natural mathematical compactification of the moduli space of homomorphic maps into the GIT quotient X, and that this compactification arises physically from the A-twist of

[1]Note that we are using the same symbol for the size of the S^1 fibres in $\mathbb{P}^1 \times S^1_\beta$ and the degree of the vortex and the degree of holomorphic maps. We hope this will cause no confusion as each should be understood from the context where they are used.

[2]Our 'fake' FI parameter $\tau > 0$ can be related to the notion of ε-stability in the definition of quasimaps, with $\varepsilon = \frac{1}{\tau}$ the size of the vortices, at least in the limit $\tau \to \infty$. Strictly speaking, in the present case, the relevant moduli space is the so-called quasimap graph space.

the GLSM [28].

Localisation to $\mathcal{N}=2$ SQM and K-theoretic GW invariants

While 2d GLSMs localise to quasimap moduli space:

$$\mathrm{QMap}_{0,n}(X) = \bigoplus_{\beta\in \mathrm{H}_2(X,\mathbb{Z})} \mathrm{QMap}_{0,n}(X,\beta) \ , \tag{1.2.35}$$

of maps from $\mathbb{P}^1$ with n marked points[1] to the target variety X, one expects that the 3d GLSM localises to a $\mathcal{N}=2$ supersymmetric quantum mechanics (SQM) onto $\mathrm{QMap}_{0,n}(X)$. The Hamiltonian of this quantum mechanics simply governs the evolution of the low-energy fluctuations around $\mathrm{QMap}_{0,n}(X)$ in the τ-direction. Then, by standard arguments [51, 121, 122, 123], one expects that the path integral of 3d A-twisted GLSM into X – i.e. the 3d twisted index defined in (1.2.20) specialised to $g=0$ – gives us the holomorphic Euler characteristic of the moduli space, schematically:

$$Z_{\mathbb{P}^1\times S^1} = \sum_{\beta\in \mathrm{H}_2(X,\mathbb{Z})} q^\beta\, \chi(\mathrm{QMap}_{0,n}(X,\beta), \mathcal{O}^{\mathrm{vir}}) \ , \tag{1.2.36}$$

with $\mathcal{O}^{\mathrm{vir}}$ is the (virtual) structure sheaf on the quasimap moduli space in the sense of [17]. Here, q is the exponentiated physical FI parameter, $q\sim e^{-\xi}$, and we choose τ and ξ to be aligned in some appropriate sense.

3d GLSMs/quantum K-theory correspondence. Recall that in the 2d story we had in the previous section, A-twisting the theory on $\mathbb{P}^1$ gives us the 2d twisted chiral ring $\mathcal{R}_{\mathrm{2d}}$ of $\mathcal{Q}$-closed point operators up to $\mathcal{Q}$-exact terms. A similar scenario happens in 3d A-models on $\mathbb{P}^1\times S^1_\beta$, where we get 3d twisted chiral rings $\mathcal{R}_{\mathrm{3d}}$ consisting of half-BPS line operators wrapping the S^1_β fibre at some points $z_*\in\mathbb{P}^1$.

For the case where the 3d gauge theory is a GLSM with target X, elements of the twisted chiral ring $\mathcal{R}_{\mathrm{3d}}$ flow to coherent sheaves on X which gives a correspondence between $\mathcal{R}_{\mathrm{3d}}$ and the (equivariant) K-theory ring $\mathrm{K}_T(X)$ [124, 18, 20]. This is depicted in figure 5.1 below. In fact, the precise statement is that $\mathcal{R}_{\mathrm{3d}}$ is isomorphic to the quantum

[1]Note here that these marked points are different from the base points that we mentioned earlier. The meaning of the marked points will be clear momentarily.

K-theory ring $\mathrm{QK}_T(X)$ of the target X as defined in [16, 125, 17, 126]:[1]

$$\langle \mathscr{L}_{\mu_1} \cdots \mathscr{L}_{\mu_n} \rangle_{\mathbb{P}^1 \times S^1_\beta} = \chi_T(\mathrm{QMap}_{0,n}(X, \beta), \mathcal{O}^{\mathrm{vir}} \otimes \mathrm{ev}_1^*(\mathcal{E}_{\mu_1}) \otimes \cdots \otimes \mathrm{ev}_n^*(\mathcal{E}_{\mu_n})) \ , \tag{1.2.37}$$

where here $\mathcal{E}_{\mu_i} \in \mathrm{K}(X)$ is the sheaf associated with the half-BPS line $\mathscr{L}_{\mu_i}$ and we pull it back to the quasimap moduli space using the evaluation map $\mathrm{ev}_i : \mathrm{QMap}_{0,n}(X, \beta) \to X$.

Since the 3d A-model is a 2d TQFT, the twisted chiral ring $\mathcal{R}_{\mathrm{3d}}$, and hence the quantum K-theory ring $\mathrm{QK}_T(X)$, have the structure of a Frobenius algebra with the pairing (topological metric) and the structure constants given by:

$$\begin{aligned} \eta_{\mu\nu}(\mathscr{L}) &:= \langle \mathscr{L}_\mu \mathscr{L}_\nu \rangle_{\mathbb{P}^1 \times S^1_\beta} = \sum_{\beta \in \mathrm{H}_2(X, \mathbb{Z})} q^\beta \eta^{(\beta)}_{\mu\nu}(\mathscr{L}) \ , \\ \mathcal{N}_{\mu\nu\lambda}(\mathscr{L}) &:= \langle \mathscr{L}_\mu \mathscr{L}_\nu \mathscr{L}_\lambda \rangle_{\mathbb{P}^1 \times S^1_\beta} = \sum_{\beta \in \mathrm{H}_2(X, \mathbb{Z})} q^\beta \mathcal{N}^{(\beta)}_{\mu\nu\lambda}(\mathscr{L}) \ , \end{aligned} \tag{1.2.38}$$

where here we took $\{\mathscr{L}_\mu\}_{\mu \in I}$ to be a basis for the ring $\mathcal{R}_{\mathrm{3d}}$ with the corresponding basis for $\mathrm{K}_T(X)$ being $\{\mathcal{E}_\mu\}_{\mu \in I}$. Moreover, the K-theoretic GW invariants appearing on the r.h.s. above are given by:

$$\begin{aligned} \eta^{(\beta)}_{\mu\nu}(\mathscr{L}) &:= \chi_T(\mathrm{QMap}_{0,2}(X, \beta), \mathcal{O}^{\mathrm{vir}} \otimes \mathrm{ev}_1^*(\mathcal{E}_\mu) \otimes \mathrm{ev}_2^*(\mathcal{E}_\nu)) \ , \\ \mathcal{N}^{(\beta)}_{\mu\nu\lambda}(\mathscr{L}) &:= \chi_T(\mathrm{QMap}_{0,3}(X, \beta), \mathcal{O}^{\mathrm{vir}} \otimes \mathrm{ev}_1^*(\mathcal{E}_\mu) \otimes \mathrm{ev}_2^*(\mathcal{E}_\nu) \otimes \mathrm{ev}_3^*(\mathcal{E}_\lambda)) \ . \end{aligned} \tag{1.2.39}$$

In chapter 2 we will review aspects and results from 3d A-model introduced earlier in this section that will allow us to explicitly and exactly compute the 3d correlation functions in (1.2.38). We will also give an algebro-geometric algorithm that will make the computations easily performed.

The ring structure on $\mathcal{R}_{\mathrm{3d}}$ is given by:

$$\mathscr{L}_\mu \star \mathscr{L}_\nu = \mathcal{N}_{\mu\nu}{}^\lambda(\mathscr{L}) \mathscr{L}_\lambda \ , \tag{1.2.40}$$

where,

$$\mathcal{N}_{\mu\nu}{}^\lambda(\mathscr{L}) = \mathcal{N}_{\mu\nu\rho}(\mathscr{L}) \eta^{\rho\lambda}(\mathscr{L}) \ , \qquad \eta^{\rho\lambda}(\mathscr{L}) \eta_{\lambda\nu}(\mathscr{L}) = \delta^\rho{}_\nu \ , \tag{1.2.41}$$

where repeated indices are summed over. In this sense, the quantum K-theory ring of the

[1] In the geometric window, the Chern–Simons level could give rise to line bundles over the moduli space of quasimaps. We are suppressing this here in our discussion. We will come back to this point later in chapter 6.

target X is a deformation of the usual K-theory ring where the multiplication is given by the tensor product of sheaves. One recovers the classical ring relations of $\mathrm{K}_T(X)$ by looking at the degree-0 terms in the q-expansion of the coefficients $\mathcal{N}_{\mu\nu}{}^{\lambda}$.

It is worth mentioning that there is another approach to quantum K-theory through the study of the partition function of the 3d $\mathcal{N}=2$ theory on a $D^2 \times S^1$ [20, 127], where D^2 is topologically a disk, which more closely parallels the approach by Givental [128]. See also [129, 130] for more discussion from the physics perspective.

Quantum K-theory of the complex Grassmannian. In our work we will focus on the case $X = \mathrm{Gr}(N_c, n_f)$ the complex Grassmannian (1.1.27). We realise this as the Higgs branch of a unitary 3d SQCD. For more details, see chapter 5. The quantum K-theory ring of the Grassmannian was first worked out by Buch and Mihalcea in [131]. From a supersymmetric field theory perspective, it was computed in [132, 131, 133, 134, 135, 136, 137, 138] by studying the algebra of supersymmetric Wilson lines in these theories. The novelty of our work in chapter 5 is that we construct a different class of line defects that flow in the IR to the standard basis of $\mathrm{K}_T(X)$ given by the Schubert cells.

The moduli space of quasimaps was constructed in terms of the so-called quotient schemes in [139, 140]. For an introduction, see [141], and for more recent discussion on this, see [142, 143, 144]. It was argued by Donagi and Sharpe in [145] that this quotient scheme can be realised from the 2d GLSM perspective – see also [146]. It turns out that one is required to use more advanced tools of GIT where the quotient is taken with respect to a non-reductive symmetry group [147, 148]. It remains to be understood how to realise this quasimap moduli space as the Higgs branch of the supersymmetric quantum mechanics obtained by reducing the 3d SQCD on the Riemann sphere [149, 116, 150].

1.3 Plan of the thesis

The thesis is organised as follows:

- In chapter 2, we start with reviewing aspects of 3d A-model on $\mathcal{M}_3 = \Sigma_g \times S^1_\beta$ with gauge group G which is simply-connected or a product of unitary groups. This is done in section 2.1, which also appeared as appendix A of [6] with some edits in the notation to be compatible with the rest of the thesis.

  In section 2.2, we specialise to theories with the gauge group being a product of unitary Lie groups. After deriving some explicit results for the 3d twisted indices of these theories, we introduce an algorithm using techniques from algebraic geometry that computes these indices explicitly. This section is based on section 2 of [1].

- In chapter 3 we start with section 3.1 in which we define the class of 3d unitary SQCD theories that we will study in this work and we fix the conventions of all possible Chern–Simons levels. We show in examples the power of the algorithm that we develop in section 2.2 in computing twisted indices and making conjectures for 3d Witten indices. This section is based on section 3 of [1].

  In section 3.2, we look at the semi-classical description of the unitary 3d SQCDs of interest. We analyse the moduli space of supersymmetric vacua for the case with $n_a = 0$, and we give a complete formula for the Witten index for these theories. We also derive a recurrence relation that allows us to work out the Witten index for the cases with $n_a \neq 0$. This section is based on section 3 of [2]. This section is accompanied by the first part of appendix A, which is based on subsection 2.1 of [2].

- In chapter 4 we study some infrared dualities associated with 3d unitary SQCD. In section 4.1 we start with the case with the Chern–Simons level $l = 0$ where we review some well-known dualities associated with this case from and we fix all possible CS levels on both sides of these dualities. This section is based on section 4 of [1]. This section is accompanied with appendix B which is based on appendix A of [1].

  In section 4.2 we look at the case with $l \neq 0$. We start by reviewing the Kapustin–Strassler–Witten $\mathrm{SL}(2,\mathbb{Z})$ transformations and show how they can be used to derive new dualities for these SQCDs from those with $l = 0$. We also fix the CS levels on both sides for the dualities to work. This section is based on section 5 of [1].

  In section 4.3, we look at the dualities for cases with $n_a = 0$. We follow the same semi-classical analysis for the moduli space of vacua for the dual theories and show explicit matching of the SUSY vacua on both sides of the dualities. This section is based on section 5 of [2]. This section is accompanied by the last two sections of appendix A, which are based on section 2.2 and appendix A of [2], respectively. It is also accompanied by appendix C, which is based on appendix B of [2].

- In chapter 5, we study the quantum K-theory ring of the complex Grassmannian variety $\mathrm{Gr}(N_c, n_f)$. After discussing the associated 3d A-model in section 5.1, in section 5.2.2, we construct a new set of line defects in the UV that flow to the Schubert classes in $\mathrm{K}(\mathrm{Gr}(N_c, n_f)$. Later in the chapter, we also argue that one can use the algorithm developed in chapter 2 to work out the quantum ring relations. In section 5.4, we discuss the 2d limit of our construction. We show that, in this limit, our construction reduces to point-defects that flow to the Schubert classes in the cohomology ring of the Grassmannian.

This chapter is accompanied with appendices D, E and F. All these are based on [3, 4].

- In chapter 6, we give a summary of our work and discuss some future directions.

CHAPTER 2

3D A-MODEL ON $\Sigma_g \times S^1_\beta$ AND BETHE IDEALS

In this chapter we look at 3d $\mathcal{N} = 2$ supersymmetric gauge theories on the background $\Sigma_g \times S^1_\beta$ with a partial A-twist along the genus-g closed Riemann surface Σ_g. In section 2.1, we give a lightning review of the essential ingredients to compute the partition functions (or rather the twisted indices) of these theories.

Focusing on the case where the gauge group G is a product of unitary Lie groups, in section 2.2 we develop an algorithm from computational algebraic geometry to explicitly work out these twisted indices. In the same section, we also review the 3d parity anomaly, where we set up some of the conventions that we will follow in later chapters.

The algorithm we develop in this chapter will be applied in chapter 4 in testing IR dualities. Moreover, we will revisit these algorithms in chapter 5 to compute the quantum K-theory of the complex Grassmannian manifold.

2.1 3d A-model on $\Sigma_g \times S^1_\beta$: a lightning review

In this subsection, we review some aspects of the 3d A-model for 3d $\mathcal{N} = 2$ Chern–Simons–matter gauge theory $\mathcal{T}_G$ with a gauge group G, a product of simply connected compact Lie groups and of unitary gauge groups. We refer to [151, 100] for further background and explanations.

2.1.1 The Coulomb branch parameters

The main player in this discussion is the 3d classical Coulomb branch parameter, which we denote by u. To define this variable, we put our 3d theory on $\mathbb{R}^2 \times S^1_\beta$. Effectively, this gives us a 2d $\mathcal{N} = (2, 2)$ theory with an infinite number of massive Kaluza–Klein (KK) modes that carry momentum along the compactified dimension S^1_β with radius β. In the 2d $\mathcal{N}=(2,2)$ language, the vector multiplet is repackaged into a twisted chiral multiplet whose lowest component is a complex scalar u. This dimensionless scalar is defined by combining the real scalar σ with the 3d gauge field along the S^1-direction, A_t:

$$u = \beta(i\sigma + A_t) \in \mathfrak{t}/\Lambda^G_{\rm mw} \ . \tag{2.1.1}$$

Here, $\mathfrak{t}$ is the Cartan subalgebra of G, and $\Lambda^G_{\rm mw}$ is the magnetic weight lattice. Choosing $\{e^a\}_{a=1}^{{\rm rank}(G)}$ to be an integral basis for the magnetic weight lattice $\Lambda^G_{\rm mw}$, we can expand u as follows:

$$u = u_a e^a \ , \tag{2.1.2}$$

where the sum over the index $a = 1, \cdots, {\rm rank}(G)$ is assumed. This basis is integral in the sense that we have:

$$\rho(u) = \rho^a u_a \ , \qquad\qquad \rho^a \equiv \rho(e^a) \in \mathbb{Z} \ , \tag{2.1.3}$$

for any electric weight $\rho \in \Lambda^G_{\rm w}$. Under large gauge transformations, the Coulomb branch parameters u_a transform as:

$$u_a \sim u_a + n_a \ , \qquad n_a \in \mathbb{Z} \ . \tag{2.1.4}$$

Therefore, it is sometimes useful to introduce the single-valued parameters:

$$x_a \equiv e^{2\pi i u_a} \ , \qquad a = 1, \cdots, {\rm rank}(G) \ . \tag{2.1.5}$$

One can play a similar game for any flavour symmetry group G_F that might be present in the theory. In this case, we consider a 3d $\mathcal{N}=2$ background vector multiplet with real scalar m^F and gauge field A^F, and we define the 2d twisted masses:

$$\nu \equiv \beta(im^F + A^F_t) \in \mathfrak{t}^F/\Lambda^{G_F}_{\rm mw} \ , \tag{2.1.6}$$

where $\mathfrak{t}^F$ is the Cartan subalgebra of G_F.

As in the case of the gauge group G, we can pick an integral basis $\{e^\alpha_F\}_{\alpha=1}^{{\rm rank}(G_F)}$ for the flavour magnetic weight lattice $\Lambda^{G_F}_{\rm mw}$, so that:

$$\nu = \nu_\alpha e^\alpha_F \ , \tag{2.1.7}$$

where $\alpha = 1, \cdots, \text{rank}(G_F)$ runs over a maximal torus of the flavour group.

2.1.2 The effective twisted superpotential and the effective dilaton

The 2d $\mathcal{N} = (2,2)$ low energy effective description on Σ_g is controlled by the so-called effective twisted superpotential $\mathcal{W}(u,\nu)$ and by the effective dilaton $\Omega(u,\nu)$, which one obtains upon integrating out the massive charged chiral multiplets on $\Sigma_g \times S^1_\beta$ [19].

Effective twisted superpotential. The twisted superpotential receives contributions from the CS action and from the 3d $\mathcal{N} = 2$ chiral multiplets. It has the following general form:

$$\mathcal{W}(u,\nu) = \mathcal{W}_{\text{matter}}(u,\nu) + \mathcal{W}_{\text{CS}}(u,\nu) \ . \tag{2.1.8}$$

The matter contribution reads:[1]

$$\mathcal{W}_{\text{matter}}(u,\nu) \equiv \frac{1}{(2\pi i)^2} \sum_{(\rho,\rho_F)\in\mathfrak{R}\times\mathfrak{R}_F} \text{Li}_2\left(e^{2\pi i(\rho(u)+\rho_F(\nu))}\right) \ , \tag{2.1.10}$$

where $\mathfrak{R}\times\mathfrak{R}_F$ is the gauge and flavour representation of the 3d $\mathcal{N} = 2$ chiral multiplets Φ under $G \times G_F$, and $(\rho,\rho_F) \in \Lambda^G_{\text{w}} \times \Lambda^{G_F}_{\text{w}}$ are the corresponding weights. The CS contributions are schematically given by:

$$\mathcal{W}_{\text{CS}}(u) = \frac{1}{2}\sum_{a,b} K_{ab}\left(u_a u_b + \delta_{ab} u_a\right) \ , \tag{2.1.11}$$

where K_{ab} denote the effective UV CS levels associated with the gauge group G in the so-called $U(1)_{-\frac{1}{2}}$ quantization [100].[2] The expression (2.1.11) is the contribution from the gauge CS terms, but the flavour CS levels [152] contribute similarly. In subsection 2.2.1, we will give more explicit formulas for these contributions in the case G is a product of unitary gauge groups.

Effective dilaton. The effective dilaton is a holomorphic function that couples u to the

[1] Following [151], we define the dilogarithm such that:

$$\frac{\partial}{\partial x}\text{Li}_2(x) = -\frac{\log(1-x)}{x} \ . \tag{2.1.9}$$

[2] For more details, see subsection 2.2.2 below.

curvature of Σ_g [27, 19]. For our gauge theory, it reads:

$$\Omega(u,\nu) = \Omega_{\text{CS}}(u,\nu) + \Omega_{\text{matter}}(u,\nu) + \Omega_{\text{W-boson}}(u) \ . \tag{2.1.12}$$

The CS contribution involves (mixed) CS levels for the $U(1)_R$ symmetry [151]:

$$\Omega_{\text{CS}}(u,\nu) = K_{RG} \sum_{a=1}^{\text{rank}(G)} u_a + \sum_{\alpha=1}^{\text{rank}(G_F)} K_{R\alpha}\nu_\alpha + \frac{1}{2}K_{RR} \ . \tag{2.1.13}$$

The matter contribution reads:

$$\Omega_{\text{matter}}(u,\nu) = -\frac{1}{2\pi i} \sum_{(\rho,\rho_F)\in\mathfrak{R}\times\mathfrak{R}_F} (r_{\rho_F}-1)\log\left(1-e^{2\pi i(\rho(u)+\rho_F(\nu))}\right) \ , \tag{2.1.14}$$

where r_{ρ_F} denote the R-charges of the chiral multiplets. Finally, the W-bosons contribute to the effective dilaton potential as:[1]

$$\Omega_{\text{W-bosons}}(u) = -\frac{1}{2\pi i}\sum_{\alpha\in\Delta}\log\left(1-e^{2\pi i\alpha(u)}\right) \ , \tag{2.1.15}$$

where the sum is over the roots of the gauge group G.

2.1.3 The 3d topologically twisted index

The topologically twisted index for a 3d $\mathcal{N}=2$ gauge theory with gauge group G can be computed as a trace over certain operators in the 3d A-model [19, 115]. All these operators $\mathcal{O}$ are given 'off-shell' as function $\mathcal{O}(u,v)$ of the gauge and flavour parameters u and ν. They diagonalise the Bethe vacua, and their 'on-shell' values at solutions of the Bethe equations are denoted by $\mathcal{O}(\hat{u},v)$:

$$\mathcal{O}|\hat{u}\rangle = \mathcal{O}(\hat{u},v)|\hat{u}\rangle \ . \tag{2.1.16}$$

The meaning of the states $|\hat{u}\rangle$ will be clear momentarily.

The Bethe equations themselves are written in terms of the gauge flux operators:

[1] Note that we are using α to denote the roots of the algebra as well as to index the flavour symmetry parameters as in (2.1.7). We hope that there will be no confusion.

$$\Pi_a(u,\nu) = \exp\left(2\pi i \frac{\partial \mathcal{W}(u,\nu)}{\partial u_a}\right) , \qquad a = 1, \cdots, \text{rank}(G) . \tag{2.1.17}$$

By definition, the gauge flux operators are trivial on-shell, $\Pi_a(\hat{u}) = 1$. The set of Bethe vacua is defined as:

$$\mathcal{S}_{\text{BE}} \equiv \left\{\hat{u} \in \mathfrak{t}/\Lambda^G_{\text{mw}} : \, \Pi_a(\hat{u},\nu) = 1 \, , \forall a \quad \text{and} \quad w \cdot \hat{u} \neq \hat{u} \, , \forall w \in W_G\right\} / W_G . \tag{2.1.18}$$

Here, we need to exclude putative solutions that have a non-trivial stabiliser for the action of the Weyl group W_G [153], and we then identify all solutions related by the Weyl symmetry.

Given a non-trivial flavour symmetry group G_F,[1] we similarly define the flavour flux operator:

$$\Pi_{F,\,\alpha}(u,\nu) = \exp\left(2\pi i \frac{\partial \mathcal{W}(u,\nu)}{\partial \nu_\alpha}\right) , \qquad \alpha = 1, \cdots, \text{rank}(G_F) . \tag{2.1.19}$$

Finally, the most important operator is the handle-gluing operator $\mathcal{H}(u,\nu)$ which is given by:

$$\mathcal{H}(u,\nu) = \exp\left(2\pi i \Omega(u,\nu)\right) \det_{1\leq a,b\leq \text{rank}(G)} \left(\frac{\partial^2 \mathcal{W}(u,\nu)}{\partial u_a \partial u_b}\right) . \tag{2.1.20}$$

The insertion of this operator on Σ_g has the effect of adding a handle, thus increasing the genus of the Riemann surface [19].

The 3d flavoured twisted index from the A-model. The 3d flavoured twisted index is the Witten index defined as a trace over the Hilbert space of the theory compactified on Σ_g with a topological A-twist and with fugacities $y_\alpha \equiv e^{2\pi i \nu_\alpha}$ and background magnetic fluxes $\mathfrak{n}$ for the flavour symmetry:

$$Z_{\Sigma_g \times S^1} = \text{Tr}_{\mathscr{H}_{\Sigma_g;\mathfrak{n}}} \left((-1)^F \prod_{\alpha=1}^{\text{rank}(G_F)} y_\alpha^{Q^F_\alpha}\right) , \tag{2.1.21}$$

where $\mathscr{H}_{\Sigma_g;\mathfrak{n}}$ is the Hilbert space on the Riemann surface in the presence of background fluxes and Q^F_α are the flavour charge operators. The twisted index above can be computed in the A-model formalism as a trace over $\mathscr{H}_{S^1} \cong \text{Span}_{\mathbb{C}}\left\{ |\hat{u}\rangle \,\middle|\, \hat{u} \in \mathcal{S}_{\text{BE}}\right\}$, the ground-state

[1] Or rather a flavour symmetry algebra; we may assume that the fundamental group of G_F is a free abelian group for our purposes here.

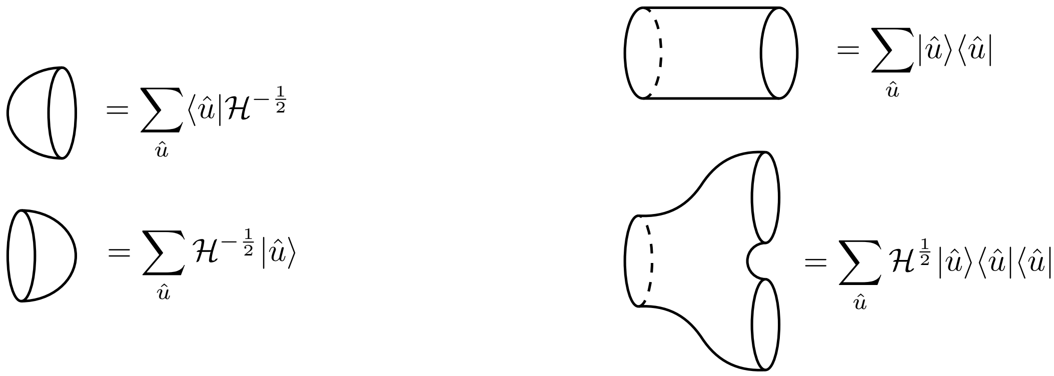


Figure 2.1: Operators in 2d TQFT corresponding to the cap, cylinder, and pair of pants. We can think of $\mathcal{H}^{\frac{1}{2}}$ as a formal square root of the handle-gluing operator $\mathcal{H}$, keeping in mind that we always obtain integer powers of $\mathcal{H}$ when computing observables on a closed Σ_g. For simplicity, here we are taking the flavour fluxes to be vanishing.

Hilbert space spanned by the Bethe vacua:

$$Z_{\Sigma_g \times S^1_\beta} = \mathrm{Tr}_{\mathscr{H}_{S^1}} \left(\mathcal{H}^{g-1} \Pi_F^{\mathfrak{n}} \right) \ . \tag{2.1.22}$$

This follows from viewing the 3d A-model as a 2d TQFT. From this perspective, the twisted index (2.1.22) can be computed by basic surgery operations on the Riemann surface. The three basic ingredients are the cap, the cylinder and the pair of pants, to which the TQFT functor assigns states as summarised in figure 2.1.[1]

For the G gauge theory, this 2d TQFT formula takes the explicit form [115]:

$$Z_{\Sigma_g \times S^1}(\nu)_{\mathfrak{m},\,\mathfrak{n}} = \sum_{\hat{u} \in \mathcal{S}_{\mathrm{BE}}} \mathcal{H}^{g-1}(\hat{u}, \nu) \Pi(\hat{u}, \nu)^{\mathfrak{m}} \Pi_F(\hat{u}, \nu)^{\mathfrak{n}} \ . \tag{2.1.23}$$

where we evaluate (2.1.19) and (2.1.20) at the Bethe vacua. Note that we use the shorthand notation $\Pi_F^{\mathfrak{n}} \equiv \prod_\alpha \Pi_{F,\,\alpha}^{\mathfrak{n}_\alpha}$.

[1] This figure is taken from the work [6] in collaboration with Cyril Closset and Elias Furrer. We thank Elias for drawing this.

2.2 Twisted indices of unitary gauge theories from Bethe ideals

In this section, we discuss any 3d $\mathcal{N} = 2$ supersymmetric gauge theory with a gauge group:

$$G = \prod_{I=1}^{n_G} U(N_I) \ , \tag{2.2.1}$$

with chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of G. For any such theory, we explain how to compute the twisted index (2.1.23) exactly, using the computational algebraic geometry approach of [154]. For each $U(N)$ gauge group, we keep track of the Chern–Simons levels K and $K + LN$, corresponding to the two factors in $U(N) \cong (SU(N) \times U(1))/\mathbb{Z}_N$. For $A = A_\mu dx^\mu$ a $U(N)$ gauge field, we have:

$$i\frac{K}{4\pi}\int \mathrm{tr}\left(A \wedge dA - \frac{2i}{3}A^3\right) + i\frac{L}{4\pi}\int \mathrm{tr}(A) \wedge d\,\mathrm{tr}(A) \ , \tag{2.2.2}$$

with the trace in the fundamental representation, plus the standard supersymmetric completion. Setting $k = K$ and $l = L$, this Chern–Simons theory is generally denoted by [155]:

$$U(N)_{k,\, k+lN} \ , \tag{2.2.3}$$

and we sometimes use the notation $U(N)_k$ for the special case $l = 0$. In this normalisation, the overall $U(1) \subset U(N)$ gauge field $\frac{1}{N}\mathrm{tr}(A)$ has an abelian CS level:

$$k_{U(1)} = N(k + lN) \ . \tag{2.2.4}$$

Let us already note that, in the presence of charged fermions, the levels k, l appearing in (2.2.3) are not exactly the bare CS levels K, L that appear in (2.2.2) due to certain one-loop shifts, as we will review momentarily.

2.2.1 Twisted index and Bethe vacua

After compactifying the 3d theory on a circle, let us consider the Coulomb branch of the effective 2d $\mathcal{N} = (2, 2)$ Kaluza–Klein (KK) theory, which is spanned by the dimensionless gauge parameters (2.1.1):

$$u_{a_I} = \beta A_{t,a_I} + i\beta\sigma_{a_I} \ , \tag{2.2.5}$$

where the first term corresponds to the holonomy of the 3d abelian gauge field along the circle of radius β, and σ_{a_I} are the 3d vector-multiplet scalars; here we use the index $a_I = 1, \cdots, N_I$ for a given $U(N_I)$ factor. The low-energy abelian gauge group is the maximal torus subgroup:

$$\prod_{I=1}^{n_G} \prod_{a_I=1}^{N_I} U(1)_{a_I} \subseteq G \ , \tag{2.2.6}$$

of rank $r_G \equiv \text{rank}(G) = \sum_{I=1}^{n_G} N_I$. To compute the twisted index (2.1.23) for this theory, we first work out the explicit forms of the effective twisted superpotential $\mathcal{W}(u, v)$ (2.1.8) and effective dilaton $\Omega(u, v)$ (2.1.12). Here we chose a maximal torus of the flavour group G_F, whose rank is denoted by $r_F \equiv \text{rank}(G_F)$:

$$\prod_{\alpha=1}^{r_F} U(1)_\alpha \subseteq G_F \ . \tag{2.2.7}$$

For each $U(N_I)$ gauge group, we also have a residual gauge symmetry S_{N_I}, the Weyl group of $U(N_I)$, which acts as permutations of the N_I parameters x_{a_I} (at any fixed I).

Effective twisted superpotential. The full low-energy twisted superpotential of the 3d gauge theory on a circle is given by the sum of some 'matter' and Chern–Simons contributions:

$$\mathcal{W} = \mathcal{W}_{\text{matter}} + \mathcal{W}_{\text{CS},GG} + \mathcal{W}_{\text{CS},GF} + \mathcal{W}_{\text{CS},FF} \ , \tag{2.2.8}$$

with

$$\begin{aligned}
\mathcal{W}_{\text{matter}} &= \frac{1}{(2\pi i)^2} \sum_i \sum_{\rho_i \in \mathfrak{R}_i} \text{Li}_2(x^{\rho_i} y^{\rho_{F,i}}) \ , \\
\mathcal{W}_{\text{CS},GG} &= \sum_{I=1}^{n_G} \left(\frac{K_I}{2} \sum_{a_I=1}^{N_I} (u_{a_I}^2 + u_{a_I}) + \frac{L_I}{2} \left(\left(\sum_{a_I=1}^{N_I} u_{a_I} \right)^2 + \sum_{a_I=1}^{N_I} u_{a_I} \right) \right) \\
&\quad + \sum_{I>J} K_{IJ} \left(\sum_{a_I=1}^{N_I} u_{a_I} \right) \left(\sum_{a_J=1}^{N_J} u_{a_J} \right) \ , \\
\mathcal{W}_{\text{CS},GF} &= \sum_{I=1}^{n_G} \sum_{\alpha=1}^{r_F} K_{\alpha I} \nu_\alpha \left(\sum_{a_I=1}^{N_I} u_{a_I} \right) \ , \\
\mathcal{W}_{\text{CS},FF} &= \sum_{\alpha=1}^{r_F} \frac{K_\alpha}{2} (\nu_\alpha^2 + \nu_\alpha) + \sum_{\alpha>\beta} K_{\alpha\beta} \nu_\alpha \nu_\beta + \frac{1}{24} K_g \ .
\end{aligned} \tag{2.2.9}$$

Here, we have $x^{\rho_i} \equiv e^{2\pi i \rho_i(u)}$, for any weight ρ_i of the representation $\mathfrak{R}_i$ of G under which the chiral multiplet Φ_i transforms. We also defined $y^{\rho_{F,i}} \equiv e^{2\pi i \rho_{F,i}(\nu)}$, where $\rho_{F,i}(\nu) = \sum_\alpha \rho_{F,i}^\alpha \nu_\alpha$, in terms of the $U(1)_\alpha$ flavour charges $\rho_{F,i}^\alpha = Q_F^\alpha[\Phi_i]$. In addition, the 'gauge-

gauge' Chern–Simons terms $\mathcal{W}_{\mathrm{CS},GG}$ are the bare Chern–Simons terms for the gauge group, including mixed CS terms between distinct $U(N_I)$ factors, and similarly for the mixed gauge-flavour CS terms $\mathcal{W}_{\mathrm{CS},GF}$ and the pure flavour CS terms $\mathcal{W}_{\mathrm{CS},FF}$, where we included the gravitational CS level K_g [156]. Note that $K_{IJ} = K_{JI}$ and $K_{\alpha\beta} = K_{\beta\alpha}$. The Fayet–Iliopoulos (FI) terms for each $U(N_I)$ factor appear, in this formalism, as part of $\mathcal{W}_{\mathrm{CS},GF}$, as a mixed term between gauge and topological symmetries. (We usually use the notation $\nu_\alpha = \tau$ for a topological symmetry. Note that, due to the non-zero gauge CS levels, only a subset of the naive n_G topological currents will be independent of the other flavour currents, in general.) Let us also note that it is important to distinguish between 'bare CS levels' (denoted here by capital letters K or L) and 'effective CS levels' in the UV gauge theory (which we denote by k and l) – see [152, 105] and section 2.2.2 below.

Effective dilaton. Similarly, the effective dilaton potential (2.1.12) reads:

$$\Omega = \Omega_{\text{matter}} + \Omega_{\text{CS}} \ , \tag{2.2.10}$$

where:

$$\begin{aligned} \Omega_{\text{matter}} &= -\frac{1}{2\pi i}\sum_i \sum_{\rho_i\in\mathfrak{R}_i} (r_i - 1)\log(1 - x^{\rho_i} y^{\rho_{F,i}}) \\ &\quad - \frac{1}{2\pi i}\sum_{I=1}^{n_G} \sum_{\substack{a_I, b_I \\ a_I\neq b_I}} \log(1 - x_{a_I} x_{I,b_I}^{-1}) \ , \\ \Omega_{\text{CS}} &= \sum_{I=1}^{n_G} K_{RI} \sum_{a_I=1}^{N_I} u_{a_I} + \sum_{\alpha=1}^{r_F} K_{R\alpha}\nu_\alpha + \frac{1}{2}K_{RR} \ . \end{aligned} \tag{2.2.11}$$

As in (2.2.9), we denote the bare CS levels by a capital K.

Flux operators and handle-gluing operators. Given $\mathcal{W}$ and Ω, one finds the gauge and flavour flux operators defined in (2.1.17) and (2.1.19) respectively:

$$\Pi_{a_I}(x, y) \equiv \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial u_{a_I}}\right) \ , \qquad \Pi_{F,\,\alpha}(x, y) \equiv \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial \nu_\alpha}\right) \ , \tag{2.2.12}$$

respectively, and the handle-gluing operator (2.1.20):

$$\mathcal{H}(x, y) = e^{2\pi i\Omega} \times \det_{a_I, b_J}\left(\frac{\partial^2 \mathcal{W}}{\partial u_{a_I}\partial u_{b_J}}\right) \ . \tag{2.2.13}$$

More explicitly, we have:

$$\Pi_{F,\,\alpha}(x,y) = \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \left(\frac{1}{1 - x^{\rho_i} y^{\rho_{F,i}}}\right)^{\rho^\alpha_{F,i}} \prod_{I=1}^{n_G} \left(\prod_{a_I=1}^{N_I} x_{a_I}\right)^{K_{\alpha I}} \prod_{\alpha=1}^{r_F} (-y_\alpha)^{K_\alpha} \prod_{\beta \neq \alpha} y_\beta^{K_{\alpha\beta}} \;, \tag{2.2.14}$$

for the flavour flux operators, and:

$$\begin{aligned} \mathcal{H}(x,y) \;&= \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \left(\frac{1}{1 - x^{\rho_i} y^{\rho_{F,i}}}\right)^{r_i - 1} \prod_{I=1}^{n_G} \left(\prod_{a_I \neq b_I} \frac{1}{1 - x_{a_I} x_{b_I}^{-1}} \left(\prod_{a_I=1}^{N_I} x_{a_I}\right)^{K_{RI}} \right) \\ &\quad \times \prod_{\alpha=1}^{r_F} y_\alpha^{K_{R\alpha}} \, (-1)^{K_{RR}} \, \det(\mathbf{H}) \;, \end{aligned} \tag{2.2.15}$$

for the handle-gluing operator, with the $r_G \times r_G$ Hessian matrix:

$$\mathbf{H}_{a_I, b_J} = \sum_i \rho_i^{a_I} \rho_i^{b_J} \, \frac{x^{\rho_i} y^{\rho_{F,i}}}{1 - x^{\rho_i} y^{\rho_{F,i}}} \;+\; \delta_{IJ} \left(\delta_{a_I b_J} K_I + L_I\right) + K_{IJ} \;, \tag{2.2.16}$$

where $K_{IJ} = 0$ if $I = J$. Importantly, (2.2.14) and (2.2.15) are rational functions of the variables x_{a_I} and y_α.

The Bethe equations. The gauge flux operators for our unitary gauge theories read:

$$\begin{aligned} \Pi_{a_I}(x,y) \;&= \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \left(\frac{1}{1 - x^{\rho_i} y^{\rho_{F,i}}}\right)^{\rho_i^{a_I}} (-x_{a_I})^{K_I} \, (-1)^{L_I} \left(\prod_{b_I=1}^{N_I} x_{b_I}\right)^{L_I} \\ &\quad \times \prod_{J \neq I} \left(\prod_{b_J=1}^{N_J} x_{b_J}\right)^{K_{IJ}} \prod_{\alpha=1}^{r_F} y_\alpha^{K_{\alpha I}} \;. \end{aligned} \tag{2.2.17}$$

As explained in (2.1.18), the Bethe vacua of the theory are solutions to the following set of equations:

$$\Pi_{a_I}(x,y) = 1 \;, \quad \forall I, a_I \;, \tag{2.2.18}$$

which are acted on freely by the Weyl group [153].[1] These Bethe equations can be seen as a coupled system of r_G polynomial equations in the r_G variables x. Each complete Weyl orbit of allowable solutions gives a particular vacuum:

[1]Fixed points of the Weyl group correspond to would-be 2d vacua with a partially restored non-abelian gauge symmetry. This semi-classical analysis receives quantum correction, and it was convincingly argued in [153] that such vacua do not contribute – *i.e.* they are lifted by strong-coupling effects.

$$\mathcal{S}_{\rm BE} = \left\{ \hat{x} = (\hat{x}_{a_I}) \;\middle|\; \Pi_{a_I}(\hat{x}, y) = 1 \;, \;\; \text{and} \;\; \hat{x}_{a_I} \neq \hat{x}_{b_I} \;, \forall a_I \neq b_I \;, \;\; \text{for each } I \right\} \Big/ W_G \;. \tag{2.2.19}$$

Here $W_G = S_{N_1} \times \cdots \times S_{N_m}$.

Twisted index as a sum over Bethe vacua. The flavoured Witten index [49] of a 3d $\mathcal{N} = 2$ theory is computed by its partition function on the torus with vanishing background fluxes:

$$\mathbf{I}_{\rm W} = Z_{T^2 \times S^1}(y)_{\mathfrak{n}=0} = {\rm Tr}_{\mathscr{H}_{T^2}} \left((-1)^F \prod_\alpha y_\alpha^{Q_F^\alpha} \right) \;. \tag{2.2.20}$$

On a flat torus, supersymmetry prevents any dependence on y_α, which only acts as an infrared regulator, and the index is an integer which equals the number of Bethe vacua:

$$\mathbf{I}_{\rm W} = |\mathcal{S}_{\rm BE}| \;. \tag{2.2.21}$$

The topologically twisted index (2.1.23) on Σ_g is then a natural generalisation of the 3d Witten index. For our theory here, it takes the following form:

$$Z_{\Sigma_g \times S^1}(y)_{\mathfrak{n}} = \sum_{\hat{x} \in \mathcal{S}_{\rm BE}} \mathcal{H}(\hat{x}, y)^{g-1} \prod_{\alpha=1}^{r_F} \Pi_{F,\,\alpha}(\hat{x}, y)^{\mathfrak{n}_\alpha} \;. \tag{2.2.22}$$

This is the formula which we would like to evaluate as explicitly as possible in this work.

Example: $U(1)_k$ theory with n_f fundamentals. As a very simple example to illustrate the above formalism, consider a $U(1)_k$ gauge theory with n_f charged chiral multiplets of charge 1 and R-charge r (with the constraint $k + \frac{n_f}{2} \in \mathbb{Z}$). Here, the UV Chern–Simons level is equal to k and therefore, the bare CS level is

$$K = k + \frac{n_f}{2} \;, \tag{2.2.23}$$

as we review in the next subsection. We also have an $SU(n_f)$ flavour symmetry with fugacities y_i, $i = 1, \cdots, n_f$, with $\prod_{i=1}^{n_f} y_i = 1$, and let us say that the n_f flavours transform in the antifundamental of $SU(n_f)$. The twisted superpotential then reads:

$$\mathcal{W} = \frac{1}{(2\pi i)^2} \sum_{i=1}^{n_f} {\rm Li}_2(x y_i^{-1}) + \frac{K}{2}(u^2 + u) + \tau u \;, \tag{2.2.24}$$

where we have chosen to set all bare CS terms for the non-gauge symmetries to zero except for a mixed CS term $K_{GT} = 1$ between the $U(1)$ gauge symmetry and the topological symmetry $U(1)_T$ (this gives the FI term, with τ the FI parameter)[1]. There is a single Bethe equation,

$$\Pi = \frac{(-x)^K q}{\prod_{i=1}^{n_f}(1 - x y_i^{-1})} = 1 \qquad \Leftrightarrow \qquad P(x) \equiv \prod_{i=1}^{n_f}(y_i - x) - (-1)^K x^K q = 0 \ , \tag{2.2.25}$$

with the notation $q = e^{2\pi i \tau}$. There are thus $\max(n_f, K)$ Bethe vacua, corresponding to the roots $\hat{x}$ of this polynomial $P(x)$. The handle-gluing operator reads

$$\mathcal{H}(x, y) = \prod_{i=1}^{n_f}(y_i - x)^{1-r} \left(K + \sum_{i=1}^{n_f} \frac{x}{y_i - x} \right) \ . \tag{2.2.26}$$

For instance, let us choose $k = 0$, $n_f = 2$ and $r = 0$, with $(y_1, y_2) = (y, y^{-1})$. The Bethe vacua then correspond to the two solutions:

$$\hat{x}_\pm = \frac{1 - qy + y^2 \pm \sqrt{(1 - qy + y^2)^2 - 4y^2}}{2y} \ , \tag{2.2.27}$$

and plugging into

$$Z_{\Sigma_g \times S^1}(y, q)_0 = \mathcal{H}(\hat{x}_-)^{g-1} + \mathcal{H}(\hat{x}_+)^{g-1} \ , \qquad \mathcal{H}(x) = 1 - x^2, \tag{2.2.28}$$

with $\mathfrak{n} = 0$, one finds

$$Z_{S^2 \times S^1} = 1 \ , \qquad Z_{T^2 \times S^1} = 2 \ , \qquad Z_{\Sigma_2 \times S^1} = 2 - y^2 - y^{-2} + 2q(y + y^{-1}) - q^2 \ , \tag{2.2.29}$$

for $g = 0, 1, 2$. Given its definition as an index, it is clear that the final result for $Z_{\Sigma_g \times S^1}(y, q)$ must be a rational function of the flavour fugacities, as is indeed the case here.

[1]Note that this is the complexified 2d FI parameter whose imaginary part is the 3d real FI parameter. This is not to be confused with the fake real FI parameter we introduced earlier in (1.2.24) in the previous chapter.

2.2.2 Parity anomaly, CS contact terms and bare CS levels

When quantising the various Dirac fermions present in our 3d gauge theories, it is important to specify how we deal with the corresponding parity anomalies [157, 158, 159]. Recall that a 'parity anomaly' in three space-time dimensions is a mixed anomaly between 3d parity and (background) gauge invariance. (Here 'parity' is really time reversal symmetry in Euclidean signature.) We choose to quantise all the fermions that appear in the free UV description in a gauge-preserving manner, hence generally breaking parity. In this, we follow exactly the conventions explained in Appendix A of [105].

Let us denote the abelianised symmetries $U(1)_{\mathbf{a}}$ (where the index $\mathbf{a} = (a_I, \alpha)$ runs over gauge and flavour indices, considering a maximal torus of $G \times G_F$ for simplicity) under which chiral multiplets Φ_i have charges $\rho_i^{\mathbf{a}}$. For each chiral multiplet, we use the '$U(1)_{-\frac{1}{2}}$ quantization', which corresponds to having the CS contact terms $\kappa_{\mathbf{ab}} = -\frac{1}{2}\rho^{\mathbf{a}}\rho^{\mathbf{b}}$ – in particular, for a chiral multiplet coupled to a single $U(1)$ with charge 1, we would generate a CS contact term $\kappa = -\frac{1}{2}$, hence the name. We then have the total 'matter' one-loop contributions:

$$\kappa^{\Phi}_{\mathbf{ab}} = -\frac{1}{2}\sum_i \sum_{\rho_i \in \mathfrak{R}} \rho_i^{\mathbf{a}} \rho_i^{\mathbf{b}} \ , \tag{2.2.30}$$

with $\rho_i^{\mathbf{a}} = (\rho_i^{a_I}, \rho_{F,i}^{\alpha})$, noting that the sum $\sum_{\rho_i \in \mathfrak{R}_i}$ only runs over the gauge (non-flavour) weights. A similar discussion holds for non-abelian groups.

We should also consider the Chern–Simons contact terms involving the R-symmetry current and its superpartners [156, 152]. For the gauginos in vector multiplets, we choose the 'symmetric' quantisation for any pair of non-zero roots $\{\boldsymbol{\alpha}, -\boldsymbol{\alpha}\}$, so that the CS contact terms for the gauge symmetry are not shifted. On the other hand, it is most convenient to choose a '$U(1)_{\frac{1}{2}}$ quantisation' for the $U(1)_R$ symmetry and for the gravitational CS contact term, which contribute $\delta\kappa_{RR} = \frac{1}{2}\dim(G)$ and $\delta\kappa_g = \dim(G)$, respectively [105]. In summary, we have the UV 'matter' contributions (2.2.30) and

$$\begin{aligned}
\kappa^{\Phi}_{\mathbf{a}R} &= -\frac{1}{2}\sum_i \sum_{\rho_i \in \mathfrak{R}_i} \rho_i^{\mathbf{a}}(r_i - 1) \ , \\
\kappa^{\Phi}_{RR} &= -\frac{1}{2}\sum_i \dim(\mathfrak{R}_i)(r_i - 1)^2 + \frac{1}{2}\dim(G) \ , \\
\kappa^{\Phi}_{g} &= -\sum_i \dim(\mathfrak{R}_i) + \dim(G) \ ,
\end{aligned} \tag{2.2.31}$$

for the gauge-R, RR and gravitational contact terms, respectively.

To define the gauge theory in the UV, we need to specify all the 'Chern–Simons terms' for the gauge and flavour symmetries. This is equivalent to specifying the value of the CS contact terms $\kappa \equiv \kappa^\Phi + K$, namely:

$$
\begin{aligned}
&\kappa_{\mathbf{ab}} = \kappa^\Phi_{\mathbf{ab}} + K_{\mathbf{ab}} \;, \qquad && \kappa_{\mathbf{a}R} = \kappa^\Phi_{\mathbf{a}R} + K_{\mathbf{a}R} \;, \\
&\kappa_{RR} = \kappa^\Phi_{RR} + K_{RR} \;, \qquad && \kappa_g = \kappa^\Phi_g + K_g \;,
\end{aligned}
\tag{2.2.32}
$$

where κ^Φ denotes the matter (and gaugino) contributions (2.2.30)-(2.2.31), and K denotes the 'bare CS levels' which appear in the classical Lagrangian. These are the same bare CS levels that appear in (2.2.9) and (2.2.11). Note that, while our choice of quantisation is conventional, the quantities κ truly define the UV theory. All the bare CS levels K are integer-quantised, as required by gauge invariance.

Let us emphasise that we denote by κ the CS contact terms as computed in the free UV theory, before turning on the gauge couplings. As the gauge theory flows to the strongly-coupled infrared, the CS contact terms κ for *non-gauge* symmetries become non-trivial observables, $\kappa(\mu)$, the parity-odd contributions to two-point functions of conserved currents [152]. These observables can be computed at the IR fixed point, in principle, by studying the S^3 partition function as a function of the background fields [156, 97].

Chern–Simons levels. In keeping with common convention, we shall denote by:

$$
k \equiv \kappa = \kappa^\Phi + K, \tag{2.2.33}
$$

the CS contact terms for the gauge groups in the UV, which we also call the (UV effective) Chern–Simons levels.[1] Note that k, unlike K, can be half-integers. For each $U(N_I)$ gauge group, we have the levels k and l defined as in (2.2.3), with:

$$
k_I = \kappa^\Phi_I + K_I \;, \qquad l_I = \kappa^\Phi_{U(1),I} + L_I \;. \tag{2.2.34}
$$

We can also have mixed CS levels between different $U(N_I)$ factors, with:

$$
k_{IJ} = \kappa^\Phi_{IJ} + K_{IJ} \;, \qquad I \neq J \;. \tag{2.2.35}
$$

Decomposing the representations $\mathfrak{R}_i$ (restricted to $U(N_I)$ as appropriate) into $SU(N_I)$

[1] In most of the supersymmetric literature, k is simply called 'the Chern–Simons level', and we will use this phrase when no confusion is possible. For our purposes, however, it is important to clearly distinguish between the effective CS levels in the UV (k) and the bare levels (K) that correspond to terms in the classical Lagrangian. This is necessary in order to remove any ambiguity (including any sign ambiguity) in the computation of the twisted index, and of all supersymmetric partition functions for that matter [100].

representations $\hat{\mathfrak{R}}_i$ together with $U(1)$ charges $Q_{I,i}$, we have the one-loop contributions:

$$\begin{aligned}
\kappa_I^\Phi &= -\frac{1}{2}\sum_i T(\hat{\mathfrak{R}}_i) \ , \\
\kappa_{IJ}^\Phi &= -\frac{1}{2}\sum_i \frac{\dim(\mathfrak{R}_i)}{N_I N_J} Q_{I,i}Q_{J,i} \ , \\
\kappa_{U(1),I}^\Phi &= -\frac{1}{2}\sum_i \frac{1}{N_I^2}\Big(\dim(\mathfrak{R}_i)\, Q_{I,i}^2 - N_I\, T(\hat{\mathfrak{R}}_i)\Big) \ ,
\end{aligned} \tag{2.2.36}$$

where the quadratic index $T(\hat{\mathfrak{R}}_i)$ for $SU(N_I)$ is normalised so that $T(\text{fund}) = 1$, and $Q_I(\text{fund}) = 1$ for the fundamental representation of $U(N_I)$.[1] Therefore $T(\text{adj}) = 2N_I$ and $Q_I(\text{adj}) = 0$ (hence $\delta\kappa_{U(1),I}^\Phi = 1$) for the adjoint representation of $U(N_I)$. For later purpose, let us also mention that $T(\det) = 0$ and $Q_I(\det) = N_I$ (hence $\delta\kappa_{U(1),I}^\Phi = -\frac{1}{2}$) for the 1-dimensional determinant representation.

2.2.3 Bethe ideal and companion matrix method

While the sum-over-Bethe-vacua formula (2.2.22) is elegant and simple-looking, to evaluate it explicitly seems to require rather complicated algebraic manipulations, wherein we would first solve the algebraic equations (2.2.18) in the r_G variables x, and then plug the solutions back into the summand appearing in (2.2.22). Of course, this is not actually doable except in the very simplest cases, such as for the example (2.2.28).

In turns out, however, that it is not necessary to explicitly find the solutions $\hat{x}$ in order to compute the twisted index. Instead, one can use powerful computational algebraic geometry methods [154, 160], as we now review.

Companion matrix method. First, consider any ideal $\mathcal{I}$ of a polynomial ring in n variables over a field $\mathbb{K}$, $\mathcal{I} \subset \mathbb{K}[x_1, \cdots, x_n]$. Concretely, it will be generated by a set of m polynomials $P_i(x)$,

$$\mathcal{I} = (P) = (P_1, \cdots, P_m) \ . \tag{2.2.37}$$

We also define the associated quotient ring $\mathcal{R}$ and the algebraic variety $\mathcal{V}$:

$$\mathcal{R} = \frac{\mathbb{K}[x_1, \cdots, x_n]}{\mathcal{I}} \ , \qquad\qquad \mathcal{V} = \mathrm{Z}(\mathcal{I}) \cong \operatorname{Spec}\mathcal{R} \ . \tag{2.2.38}$$

[1] Given the weights $\rho_i = (\rho_i^{a_I})$ of the $U(N_I)$ representation $\mathfrak{R}_i$, we have $Q_{I,i} = \sum_{a=1}^{N_I} \rho_i^{a_I}$.

The variety $\mathcal{V}$ is the set of solutions to the coupled equations:

$$P_i(x_1, \cdots, x_n) = 0 \ , \qquad\qquad i = 1, \cdots, m \ . \tag{2.2.39}$$

We assume that this variety is *zero-dimensional* – that is, it consists of discrete points $\hat{x} \in \mathbb{K}^n$. This is equivalent to $\mathcal{R}$ being finite as an abelian group, so let us denote by

$$d_{\mathcal{R}} = |\mathcal{R}| \ , \tag{2.2.40}$$

the number of discrete solutions. Now, consider two polynomials, $Q_1, Q_2 \in \mathbb{K}[x_1, \cdots, x_n]$. We are interested in the quantity

$$Z(Q_1/Q_2) \equiv \sum_{\hat{x}\in\mathcal{V}} \frac{Q_1(\hat{x})}{Q_2(\hat{x})} \ , \tag{2.2.41}$$

assuming that $Q_2 \notin \mathcal{I}$. This can be computed as follows. First, one needs to pick an ordering $\prec$ for the monomials of the polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$. Then, any polynomial Q has a unique leading term, $\mathrm{LT}(Q)$, with respect to that ordering. We then choose a Gröbner basis for the ideal $\mathcal{I}$ with respect to the ordering $\prec$, which consist of some m' elements:

$$\mathcal{G}(\mathcal{I}) = \{g_1, \cdots, g_{m'}\} \ . \tag{2.2.42}$$

A Gröbner basis $\mathcal{G}(\mathcal{I})$ is a generating set for $\mathcal{I}$ such that the leading term of any $P \in \mathcal{I}$ is proportional to the leading term of some $g_i \in \mathcal{G}(\mathcal{I})$, namely $\frac{\mathrm{LT}(P)}{\mathrm{LT}(g_i)} \in \mathbb{K}$. This then allows one to reduce any polynomial $Q \in \mathbb{K}[x_1, \cdots, x_n]$ along $\mathcal{I}$,

$$Q(x) = \sum_{i=1}^{m'} c_i\, g_i(x) \ + R(x) \ , \qquad\qquad c_i \in \mathbb{K} \ , \tag{2.2.43}$$

where $R(x)$ is called the remainder. Let $[Q] \in \mathcal{R}$ denote the equivalence class of any polynomial $Q \in \mathbb{K}[x_1, \cdots, x_n]$ in the quotient ring $\mathcal{R} = \mathbb{K}[x_1, \cdots, x_n]/\mathcal{I}$. Given a Gröbner basis, the remainder R in (2.2.43) is unique, and thus provides a useful representative of $[Q] = [R]$. Given two polynomials Q_1, Q_2, we have that $[Q_1] = [Q_2]$ if and only if $R_1 = R_2$, and in particular $[Q] = 0$ ($Q \in \mathcal{I}$) if and only if $R = 0$.

The determination of Gröbner bases can be done on a computer using standard algorithms, implemented most easily using Singular [161]. Given a Gröbner basis (2.2.42), we also obtain a canonical $\mathbb{K}$-basis for the quotient ring $\mathcal{R}$:

$$\mathcal{B}(\mathcal{R}) = \{\mathfrak{e}_1, \cdots, \mathfrak{e}_{d_\mathcal{R}}\} . \tag{2.2.44}$$

Then, to any polynomial $Q \in \mathbb{K}[x_1, \cdots, x_n]$, we can associate a $d_\mathcal{R} \times d_\mathcal{R}$ *companion matrix* $\mathfrak{M}_Q$ valued in $\mathbb{K}$, which is defined by:

$$[Q][\mathfrak{e}_s] = \sum_{r=1}^{d_\mathcal{R}} (\mathfrak{M}_Q)_{sr} [\mathfrak{e}_r] . \tag{2.2.45}$$

One can show that the companion matrix respects the ring structure, with the product given by matrix multiplication:

$$\mathfrak{M}_{Q_1+Q_2} = \mathfrak{M}_{Q_1} + \mathfrak{M}_{Q_2} , \qquad \mathfrak{M}_{Q_1 Q_2} = \mathfrak{M}_{Q_1} \mathfrak{M}_{Q_2} . \tag{2.2.46}$$

We can further generalise this construction to define the companion matrix of rational functions $\frac{Q_1}{Q_2} \in \mathbb{K}(x_1, \cdots, x_n)$ using the matrix inverse of the denominator:

$$\mathfrak{M}_{Q_1/Q_2} = \mathfrak{M}_{Q_1} (\mathfrak{M}_{Q_2})^{-1} . \tag{2.2.47}$$

Finally, the key result is that the quantity (2.2.41) is simply given by the trace of the corresponding companion matrix:

$$Z(Q_1/Q_2) = \mathrm{Tr}\left(\mathfrak{M}_{Q_1/Q_2}\right) . \tag{2.2.48}$$

In fact, the eigenvalues $\lambda_1, \cdots, \lambda_{d_\mathcal{R}} \in \mathbb{K}$ of the companion matrix $\mathfrak{M}_Q$ are exactly equal to Q evaluated on the variety $\mathcal{V}$ [160], namely $\lambda_s = Q(\hat{x}_s)$ for some ordering of the solutions $\hat{x}_s$, $s = 1, \cdots, d_\mathcal{R}$, to (2.2.39).

Application to abelian gauge theories. The companion matrix method is directly applicable to abelian gauge theories, namely for $N_I = 1$, $\forall I$. The assumption that the variety $\mathcal{V}$ is zero-dimensional is essentially an assumption that we can lift any non-compact branch of the 3d moduli space of vacua by generic mass deformations – this is generally possible only if the theory has enough flavour currents. In that case, the Bethe vacua are determined entirely by the conditions:

$$\Pi_I(x) \equiv \frac{p_{I,1}(x)}{p_{I,2}(x)} = 1 \qquad \Leftrightarrow \qquad P_I(x) \equiv p_{I,1}(x) - p_{I,2}(x) = 0 \ , \qquad I = 1, \cdots, n_G \ . \tag{2.2.49}$$

The polynomials $P_I \in \mathbb{K}[x_1, \cdots, x_{n_G}]$ generate the *Bethe ideal*, $\mathcal{I}_{\rm BE}$. Here, $\mathbb{K}$ is taken to be

$$\mathbb{K} = \mathbb{Z}(y_1, \cdots, y_{r_F}) \ , \tag{2.2.50}$$

the field of fractions in the flavour fugacities. We can then compute the twisted index (2.2.22) in terms of companion matrices, as:

$$Z_{\Sigma_g \times S^1}(y)_{\mathfrak{n}} = \mathrm{Tr}\left((\mathfrak{M}_{\mathcal{H}})^{g-1} \prod_{\alpha=1}^{r_F} (\mathfrak{M}_{\Pi_{F,\,\alpha}})^{\mathfrak{n}_\alpha} \right) \ . \tag{2.2.51}$$

Simple abelian example: Let us illustrate the procedure with the example (2.2.25)-(2.2.26) with $K = 1$, $n_f = 2$ and $r = 0$. In that case, the Gröbner basis is simply $\{g_1\} = \{x^2 y + x\,(qy - y^2 - 1) + y\}$ and the $\mathbb{K}$-basis is $\{\mathfrak{e}_1, \mathfrak{e}_2\} = \{x, 1\}$. Then, the companion matrix for the handle-gluing operator is

$$\mathfrak{M}_{\mathcal{H}} = \begin{pmatrix} -q^2 - y^2 - y^{-2} + 2q(y + y^{-1}) & q - y - y^{-1} \\ -q + y + y^{-1} & 2 \end{pmatrix} \ , \tag{2.2.52}$$

which reproduces (2.2.29). We can similarly compute the companion matrix for the $SU(2) \times U(1)_T$ flavour flux operators:

$$\mathfrak{M}_{\Pi_{(y)}} = q^{-1} \begin{pmatrix} qy^{-2} + y^{-1} - y^{-3} & 1 - y^{-2} \\ -1 + y^{-2} & q + y^{-1} - y \end{pmatrix} \ , \qquad \mathfrak{M}_{\Pi_{(\tau)}} = \begin{pmatrix} -q + y + y^{-1} & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{2.2.53}$$

Bethe ideal for the non-abelian theory. Given the gauge group $G = \prod_{I=1}^{n_G} U(N_I)$, the Bethe vacua are given by (2.2.19). Let us write the Bethe equations in terms of polynomials in the variables $x = (x_{a_I})$ over the field (2.2.50), as in the abelian case:

$$\Pi_{a_I}(x) \equiv \frac{p_{a_I,1}(x)}{p_{a_I,2}(x)} = 1 \ , \qquad P_{a_I}(x) \equiv p_{a_I,1}(x) - p_{a_I,2}(x) = 0 \ . \tag{2.2.54}$$

Due to gauge invariance, the Bethe equations and all the flavour flux and handle gluing operators are symmetric under S_{N_I}, the permutation of the variables x_{a_I} for each I. Hence, we can restrict our attention to Weyl-symmetric polynomials. In particular, we have:

$$P_{a_I} \in \mathbb{K}[x]^{W_G} \ . \tag{2.2.55}$$

To discard the spurious solutions to the Bethe equations $P_{a_I}(x) = 0$, which correspond to $\hat{x}_{a_I} = \hat{x}_{b_I}$ for $a_I \neq b_I$ at fixed I, we use a symmetrisation trick [154]. Let us define the polynomials:

$$\widehat{P}_{a_I b_I}(x) \equiv \frac{P_{a_I} - P_{b_I}}{x_{a_I} - x_{b_I}} \ , \qquad a_I > b_I \ , \tag{2.2.56}$$

for $I = 1, \cdots, n_G$. The Bethe ideal in the x_{a_I} variables is given by the ideal generated by the polynomials P and $\widehat{P}$:

$$\mathcal{I}^{(x)}_{\rm BE} = (P, \widehat{P}) \subset \mathbb{K}[x]^{W_G} \ . \tag{2.2.57}$$

To obtain the Bethe vacua, we should gather the solutions into Weyl orbits, which have size $|W_G| = \prod_{I=1}^{n_G} N_I!$. To avoid this large redundancy, it is convenient to introduce new variables s_{I,I_a} defined as the symmetric polynomials in x_{a_I} (at fixed I). More precisely, let us introduce the polynomials:

$$\widehat{S}_{a_I}(x,s) \equiv S_{I,a_I}(x) - s_{I,a_I} \ , \tag{2.2.58}$$

which are linear in s_{a_I}. Here, at fixed I, $S_{I,a_I}(x) = S_a(x)$ is the n-th elementary symmetric polynomial in N variables x_b ($b = 1, \cdots, N = N_I$):

$$S_a(x_1, \cdots, x_N) \ = \sum_{1\leq b_1 < \cdots < b_a \leq N} x_{b_1} \cdots x_{b_a} \ . \tag{2.2.59}$$

Let us also introduce some auxiliary variables w_I and define:

$$\widehat{W}_I \equiv w_I s_{I,N_I} - 1 \ , \tag{2.2.60}$$

so that imposing $\widehat{W}_I = 0$ implies $s_{I,N_I} = \prod_{a_I=1}^{N_I} x_{a_I} \neq 0$, thus removing any spurious solutions on which some x_{a_I} variables would vanish (such solutions would be located at infinity on the 3d Coulomb branch). Starting with the extended ideal:

$$\mathcal{I}^{(x,w,s)}_{\rm BE} = (P, \widehat{P}, \widehat{S}, \widehat{W}) \subset \mathbb{K}[x,s,w] \ , \tag{2.2.61}$$

we can reduce it to an ideal in $\mathbb{K}[s]$, eliminating x_{I_a} (and w_I) from the description by an appropriate choice of monomial ordering:

$$\mathcal{I}_{\rm BE}^{(s)} = \mathcal{I}_{\rm BE}^{(x,w,s)}\Big|_{\rm reduce} \,. \tag{2.2.62}$$

We also write any rational operator $\mathcal{O}(x)$ on the 2d Coulomb branch in terms of the s_{a_I} variables:

$$\mathcal{O}(x) \equiv \frac{Q_1(x)}{Q_2(x)}\,, \quad Q_1, Q_2 \in \mathbb{K}[x] \qquad \longrightarrow \qquad \mathcal{O}(s) \equiv \frac{\widetilde{Q}_1(s)}{\widetilde{Q}_2(s)}\,, \quad \widetilde{Q}_1, \widetilde{Q}_2 \in \mathbb{K}[s]\,. \tag{2.2.63}$$

Let us denote the quotient ring and the variety relative to the Bethe ideal by:

$$\mathcal{R}_{\rm BE}^{(s)} = \mathbb{K}[s]\Big/\mathcal{I}_{\rm BE}^{(s)}\,, \qquad\qquad \mathcal{V}_{\rm BE} \cong \operatorname{Spec}\mathcal{R}_{\rm BE}^{(s)}\,. \tag{2.2.64}$$

We assume that the 'Bethe variety' $\mathcal{V}_{\rm BE}$ is zero-dimensional, so that the number of points in $\mathcal{V}_{\rm BE} \subset \mathbb{K}^{r_G}$ is equal to the number of Bethe vacua:

$$|\mathcal{V}_{\rm BE}| = d_{\mathcal{R}_{\rm BE}} = |\mathcal{S}_{\rm BE}|\,\,. \tag{2.2.65}$$

Finally, we choose a Gröbner basis $\mathcal{G}\Big(\mathcal{I}_{\rm BE}^{(s)}\Big)$, so that we can define the companion matrix $\mathfrak{M}_{\mathcal{O}}$ of any rational operator $\mathcal{O}(s)$. We can then compute the twisted index exactly as in (2.2.51). More generally, the expectation value of any rational operator $\mathcal{O}$ on Σ_g is given by:

$$\langle\mathcal{O}\rangle_{\Sigma_g\times S^1} = \operatorname{Tr}\Big((\mathfrak{M}_{\mathcal{H}})^{g-1}\,\mathfrak{M}_{\mathcal{O}}\Big)\,\,. \tag{2.2.66}$$

It is interesting to note that not every 3d A-model observable is rational. In particular, the Seifert fibering operators defined in [151, 105] are not rational in x_{a_I} – instead, they are locally holomorphic functions in the variables $u_{a_I} = \frac{1}{2\pi i}\log(x_{a_I})$. It would be very interesting, but likely quite challenging, to extend the methods of this chapter to include fibering operators, perhaps using ideas from [162].

CHAPTER 3

3D $\mathcal{N} = 2$ SQCD: FLAVOURED WITTEN INDEX AND MODULI SPACE OF VACUA

In this chapter, we focus on a special case of the formalism developed in subsection 2.2.1, that is unitary SQCD$[N_c, k, l, n_f, n_a]$. This is a 3d $\mathcal{N} = 2$ supersymmetric gauge theory with a single unitary gauge group $U(N_c)$ coupled to n_f fundamental and n_a antifundamental chiral multiplets:

$$U(N_c)_{k,k+lN_c} \ , \ (n_f \, \square \ , \ n_a \, \overline{\square}) \ , \tag{3.0.1}$$

with $k + \frac{1}{2}(n_f + n_a) \in \mathbb{Z}$. Interestingly, these theories admit infrared-dual descriptions akin to Seiberg dualities [163]. The case $l = 0$ is well understood [53, 56, 57, 58], and the general case with $l \neq 0$ has been addressed very recently in the literature [76, 92]. We study these dualities in detail in the next chapter. In this chapter, we analyse the vacuum structure and compute $\mathbf{I}_{\mathrm{W}}[N_c, k, l, n_f, n_a]$; the Witten index for this theory.

In section 3.1 we study the SQCD theory from the perspective of the A-model. After fully defining the theory following the conventions of subsection 2.2.2, we employ the algorithm developed in subsection 2.2.3 to compute the twisted indices of these theories[2] and we conjecture formulas for the Witten indices of some of the cases based on numerical evidence.

In section 3.2, we give an explicit analysis of the vacuum structure of these theories for the case $n_a = 0$ by looking at the semiclassical equations of the theory on Euclidean $\mathbb{R}$ following [49]. We derive a complete formula for the moduli space of vacua and the Witten index for this case. For $n_a \neq 0$, we give a physical derivation to a recurrence relation that computes the Witten index for any SQCD$[N_c, k, l, n_f, n_a]$.

[2] In practice, we face limitations are due to computing power: to find Gröbner basis for large and complicated ideals can be prohibitive on a laptop computer (especially for $\mathbb{C}(y)$-valued polynomials with many distinct y_α's).

	$U(N_c)$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$
Φ_α	$\square$	$\overline{\square}$	$\mathbf{1}$	1	0	r
$\widetilde{\Phi}^\beta$	$\overline{\square}$	$\mathbf{1}$	$\square$	1	0	r

Table 3.1: Charge assignments for 3d SQCD$[N_c, k, l, n_f, n_a]$.

3.1 Twisted indices for unitary SQCD

In this section, we will discuss the 3d SQCD theory defined above from the perspective of the 3d A-model. After fixing the CS levels as part of the definition of the theory following subsection 2.2.2, we use the companion matrix algorithm of subsection 2.2.3 to explicitly work out the twisted index of these theories in some examples. Based on numerical evidence, we conjecture explicit formulas for the Witten index in some special cases. We leave a more concrete derivation and generalisation of these formulas for section 3.2.

3.1.1 Defining 3d SQCD: Flavour symmetry and Chern–Simons contact terms

To fully define the 'electric' theory (3.0.1), we need to specify all the Chern–Simons levels, including the Chern–Simons contact terms for the flavour symmetry, as reviewed in section 2.2.2. The theory has a flavour symmetry:[1]

$$G_F = SU(n_f) \times SU(n_a) \times U(1)_A \times U(1)_T \ , \tag{3.1.1}$$

and a $U(1)_R$ symmetry under which all (anti)fundamental chiral multiplets are assigned R-charge r, assuming that $n_a n_f > 0$. We will assume that $r \in \mathbb{Z}$, so that the theory can be coupled to $\Sigma_g \times S^1$ with the A-twist [151].[2] If either n_f or n_a vanishes, we lose the axial symmetry $U(1)_A$, which rotates both fundamental and antifundamental chiral multiplets with the same phase.

Denoting by Φ_α, $\alpha = 1, \cdots, n_f$, and $\widetilde{\Phi}^\beta$, $\beta = 1, \cdots, n_a$, the fundamental and antifundamental chiral multiplets, we have the charge assignment shown in table 3.1. No

[1] Perhaps up to a discrete quotient. Here we make no claim about the exact global form of the flavour symmetry group, which could depend in subtle ways on the CS levels – see *e.g.* [164, 165] for related discussions.

[2] We can choose any $r \in \mathbb{R}$ in the UV, and the choice $r \in \mathbb{Z}$ allows us to define the 3d A-model on curved space. Whenever the theory flows to a 3d $\mathcal{N} = 2$ SCFT in the IR, there also exists a dynamically determined superconformal R-charge, R_{SCFT}, which can be computed by F-maximisation [109, 156].

fundamental field is charged under the topological symmetry, $U(1)_T$. The charged objects are the (bare) monopole operators of minimal magnetic flux, $\mathfrak{T}^\pm$, which carry topological charge ± 1. The monopole operators carry an electric charge:

$$Q_0[\mathfrak{T}^\pm] = \pm(k + lN_c) - \frac{1}{2}(n_f - n_a) \ . \tag{3.1.2}$$

under the $U(1) \subset U(N_c)$, and they also transform in the representation

$$\hat{\mathfrak{R}}[\mathfrak{T}^\pm] = \mathrm{Sym}^{\pm k - \frac{1}{2}(n_f - n_a)}(\square) \tag{3.1.3}$$

of $SU(N_c)$ [166, 167]. The 3d classical Coulomb branch is then lifted by the (effective) CS interactions, in general. We have a 3d quantum Coulomb branch only when $\mathfrak{T}^+$ and/or $\mathfrak{T}^-$ are gauge-invariant, in which case their VEVs span the Coulomb branch.

To fully specify the gauge theory, we not only need to specify the $U(N_c)$ CS levels k and l, but also any potential mixed CS level between $U(1) \subset U(N_c)$ and the abelian flavour symmetries. Let us first note that we have the one-loop contributions:

$$\kappa^\Phi_{GG} = -\frac{1}{2}(n_f + n_a) \ , \qquad \kappa^\Phi_{GA} = -\frac{1}{2}(n_f - n_a) \ , \qquad \kappa^\Phi_{GR} = -\frac{1}{2}(n_f - n_a)(r-1) \ , \tag{3.1.4}$$

and $\kappa^\Phi_{GT} = 0$, to the gauge (GG) and gauge-flavour (GF) CS contact terms (with $\kappa^\Phi_I = \kappa_{GG}$, in the conventions of section 2.2.2). We then have:

$$K = k + \frac{1}{2}(n_f + n_a) \ , \qquad L = l \ , \tag{3.1.5}$$

for the bare CS levels for the gauge symmetry. We also choose the bare levels[1]:

$$\begin{aligned} K_{GA} &= \begin{cases} \Theta(-k)(n_f - n_a) \ , & \text{if} \quad |k| \geq \frac{1}{2}|n_f - n_a| \ , \\ \mathrm{sign}(n_f - n_a)(\frac{1}{2}|n_f - n_a| - k) \ , & \text{if} \quad |k| < \frac{1}{2}|n_f - n_a| \ , \end{cases} \\ K_{GR} &= K_{GA}(r-1) \ , \end{aligned} \tag{3.1.6}$$

with $\Theta(x)$ the Heaviside step function,[2] and:

$$K_{GT} = 1 \ , \tag{3.1.7}$$

[1]The motivation for this choice will be clear when we test 3d IR dualities in chapter 4.
[2]Here defined as $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x \leq 0$.

which corresponds to a standard FI term. Finally, we take all the flavour bare CS levels to vanish:

$$K_{SU(n_f)} = K_{SU(n_a)} = K_{AA} = K_{TT} = K_{AT} = K_{RA} = K_{RT} = K_{RR} = K_g = 0 \ . \qquad (3.1.8)$$

We then have $\kappa = \kappa^\Phi$ for the flavour symmetry contact terms in the UV, with:

$$\begin{aligned}
&\kappa^\Phi_{SU(n_f)} = -\frac{1}{2}N_c \ , && \kappa^\Phi_{SU(n_a)} = -\frac{1}{2}N_c \ , \\
&\kappa^\Phi_{AA} = -\frac{1}{2}(n_f+n_a)N_c \ , && \kappa^\Phi_{TT} = 0 \ , \\
&\kappa^\Phi_{RA} = -\frac{1}{2}(n_f+n_a)N_c(r-1) \ , && \kappa^\Phi_{RT} = 0 \ , \\
&\kappa^\Phi_{RR} = -\frac{1}{2}(n_f+n_a)N_c(r-1)^2 + \frac{1}{2}N_c^2 \ , && \kappa^\Phi_{AT} = 0 \ , \\
&\kappa^\Phi_{g} = -\frac{1}{2}(n_f+n_a)N_c + N_c^2 \ .
\end{aligned} \qquad (3.1.9)$$

Note that the parametrisation of the R-symmetry through the arbitrary R-charge $r = R[\Phi_\alpha] = R[\widetilde{\Phi}^\beta]$ is somewhat redundant, since one can always mix $U(1)_R$ with the axial symmetry $U(1)_A$, as $R \to R + \Delta r\, Q_A$ [53]. By considering the minimal coupling to background gauge fields, one finds that a shift $r \to r + \Delta r$ leads to the following shifts of the bare CS levels:

$$\begin{aligned}
&K_{RA} \to K_{RA} + \Delta r\, K_{AA} \ , \\
&K_{RI} \to K_{RI} + \Delta r\, K_{AI} \ \ (I \neq A) \ , \\
&K_{RR} \to K_{RR} + 2\Delta r\, K_{RA} + (\Delta r)^2\, K_{AA} \ ,
\end{aligned} \qquad (3.1.10)$$

and similarly for the CS contact terms themselves.

Bethe equations for SQCD. Let us introduce the flavour parameters:

$$y_\alpha \ \ (\alpha = 1, \cdots, n_f) \ , \quad \prod_{\alpha=1}^{n_f} y_\alpha = 1 \ , \qquad \widetilde{y}_\beta \ \ (\beta = 1, \cdots, n_a) \ , \quad \prod_{\beta=1}^{n_a} \widetilde{y}_\beta = 1 \ , \quad (3.1.11)$$

for $SU(n_f) \times SU(n_a)$, as well as $y_A = e^{2\pi i \nu_A}$ for $U(1)_A$ and $q = e^{2\pi i \tau}$ for $U(1)_T$. The Bethe equations (2.2.18) for SQCD$[N_c, k, l, n_f, n_a]$ can be written as:

$$\prod_{\alpha=1}^{n_f}(y_\alpha - x_a y_A) - (-1)^{l+k+\frac{1}{2}(n_f+n_a)} q\, y_A^{K_{GA}} \prod_{\beta=1}^{n_a}(x_a - \widetilde{y}_\beta y_A)\; x_a^{k+\frac{1}{2}(n_f-n_a)} \left(\prod_{b=1}^{N_c} x_b\right)^l = 0 \;, \tag{3.1.12}$$

for $a = 1, \cdots, N_c$. For $l = 0$, we have N_c decoupled equations, but in general, we have a coupled system of N_c equations in N_c variables x_a. Then, the Gröbner basis techniques described in section 2.2.3 become particularly useful.

3.1.2 Twisted index for $U(N_c)_{0,lN_c}$, N_f SQCD

Consider the $U(N_c)_{0,\,lN_c}$ gauge theory with N_f pairs of fundamental and antifundamental matter. We find that the Witten index of this theory is given by:

$$\mathbf{I}_{\mathrm{W}}[N_c, 0, l, N_f, N_f] = \frac{N_f + |l| N_c}{N_f} \binom{N_f}{N_c} \;, \tag{3.1.13}$$

For $l = 0$, this is a well-known result which follows from the fact that the vacua are in one-to-one correspondence with the sets of N_c distinct roots of a degree-N_f polynomial [115]. For $l \neq 0$, we proceed as follows. By considering certain limits in the space of flavour fugacities, as done in [168] in a similar context[1], one can show that the number of Bethe vacua satisfies the recursion relation:

$$\mathbf{I}_{\mathrm{W}}[N_c, 0, l, N_f, N_f] = \mathbf{I}_{\mathrm{W}}[N_c, 0, l, N_f - 1, N_f - 1] + \mathbf{I}_{\mathrm{W}}[N_c - 1, 0, l, N_f - 1, N_f - 1] \;. \tag{3.1.14}$$

Furthermore, one can show that:

$$\mathbf{I}_{\mathrm{W}}[1, 0, l, N_f, N_f] = N_f + |l| \;, \qquad \mathbf{I}_{\mathrm{W}}[N_f, 0, l, N_f, N_f] = 1 + |l| \;. \tag{3.1.15}$$

The first equality follows from the fact that, for $N_c = 1$, we have an abelian gauge theory $U(1)_l$ with N_f pairs of electrons of charge ± 1, whose index was computed in [49]. The second equality follows from the fact that, for $N_c = N_f$, the Amariti–Rota dual (as will be discussed in section 4.2.3 below) is a $U(1)_l$ theory with one flavour pair. With these initial conditions, one can solve the recursion relation (3.1.14) to obtain (3.1.13).

Twisted indices: a few examples. Let us now consider a few simple examples of

[1] A more concrete derivation based on mass deformations of the theory will be given in subsection 3.2.3 below.

twisted indices computed using the companion matrix method of subsection 2.2.3. First, consider the $U(2)_0$ theory ($l = 0$) with $N_f = 2$ flavours and $r = 0$. There is a single Bethe vacuum, and the index is given by:

$$Z_{\Sigma_g \times S^1}^{U(2)_0, N_f=2}(y_A, \chi, \widetilde{\chi}) = \left(1 - \chi\widetilde{\chi} y_A^2 + (\chi^2 + \widetilde{\chi}^2 - 2) y_A^4 - \chi\widetilde{\chi} y_A^6 + y_A^8\right)^{g-1} \equiv Z_{4M} \ . \quad (3.1.16)$$

As we will discuss in chapter 4 below, this theory is dual to four free chiral multiplets, the mesons ${M_\alpha}^\beta \equiv \Phi_\alpha \widetilde{\Phi}^\beta$, and this is apparent in the index. Here we introduced the characters $\chi = y_1 + y_2$ and $\widetilde{\chi} = \widetilde{y}_1 + \widetilde{y}_2$ for the $SU(2) \times SU(2)$ flavour symmetry.

As another example, let us consider the abelian theory $U(1)_l$ with CS level l and $N_f = 1$ flavour. This theory has $|l| + 1$ Bethe vacua. On the sphere ($g = 0$), we have:

$$Z_{S^2 \times S^1}^{U(1)_l, N_f=1}(y_A, q) = \frac{1}{1 - y_A^2} \ , \quad (3.1.17)$$

for any l. At genus $g = 1$, we have the Witten index, $Z_{T^3} = |l| + 1$. This is in agreement with our discussion below (3.1.15) concerning $\mathbf{I}_{\mathrm{W}}[N_f, 0, l, N_f, N_f]$. At genus $g = 2$, we find:

$$Z_{\Sigma_2 \times S^1}^{U(1)_l, N_f=1}(y_A, q) = \begin{cases} 1 - 2l - l^2 - (l-1)^2 y_A^2 - \delta_{l,-1}\ (q + q^{-1}) \ , & \text{if } \ l < 0 \\ (1+l)^2 + (l^2 - 2l - 1) y_A^2 + \delta_{l,1}\ (q + q^{-1})\, y_A^2 \ , & \text{if } \ l \geq 0 \ , \end{cases} \quad (3.1.18)$$

and similar formulas can be worked out for any $g > 2$. In particular, we observe that the index terminates (thus, there is a finite number of states on any Σ_g with $g > 0$) and that states charged under the topological symmetry (weighted by q) only appear for $0 < |l| \leq g - 1$.

Next, consider $U(2)_{0,2l}$ with $N_f = 2$ and $r = 0$, a theory with $|l| + 1$ Bethe vacua. By explicit computation, one can check that the index factorises as:

$$Z_{\Sigma_g \times S^1}^{U(2)_{0,2l}, N_f=2}(y_A, \chi, \widetilde{\chi}) = (1 - y_A^4)^{1-g}\ Z_{4M}\ Z_{\Sigma_g \times S^1}^{U(1)_l, N_f=1}(y_A^2, q) \ , \quad (3.1.19)$$

with Z_{4M} as in (3.1.16). This can be precisely explained in terms of the Amariti–Rota duality, to be discussed in detail in section 4.2.3. In particular, the prefactor $(1 - y_A^4)^{1-g}$ should be written as $(1 - y_A^{-4})^{1-g} \times y_A^{-4(g-1)} \times (-1)^{g-1}$, corresponding to a chiral multiplet of $U(1)_A$ charge -4 and to some flavour CS levels $K_{RA} = -4$ and $K_{RR} = 1$.

3.1.3 The Witten index for $n_f = n_a$ and $k \neq 0$

Consider the theory with $n_f = n_a \equiv N_f$ flavours and $k \neq 0$. Consider first the case $N_f = 0$, which is the $\mathcal{N} = 2$ supersymmetric Chern–Simons theory $U(N_c)_{k,\, k+lN_c}$. We claim that the Witten index of this theory is given by:

$$\mathbf{I}_{\mathrm{W}}[N_c, k, l, 0, 0] = \frac{|k + lN_c|}{|k|} \binom{|k|}{N_c} , \tag{3.1.20}$$

for $|k| \geq N_c$, with the understanding that the index vanishes for $k < N_c$ (supersymmetry is broken in that case [50, 49]). This can be understood from the fact that:

$$U(N_c)_{k,\, k+lN_c} \cong \frac{SU(N_c)_k \times U(1)_{N_c(k+lN_c)}}{\mathbb{Z}_{N_c}} . \tag{3.1.21}$$

Setting $x_a = \widetilde{x}_a x_0$ with the constraint $\prod_{a=1}^{N_c} \widetilde{x}_a = 1$, the Bethe equations for the pure 3d $\mathcal{N} = 2$ CS theory read:

$$\widetilde{x}_a^k = (-1)^{k+l} q^{-1} x_0^{-(k+lN_c)} , \qquad a = 1, \cdots, N_c , \tag{3.1.22}$$

which implies that x_0 is proportional to a $N_c(k + lN_c)$-th root of unity. Plugging back this solution into (3.1.22), we have the Bethe equations for the $SU(N_c)_k$ supersymmetric CS theory. Taking into account the redundancy in our parametrisation $x_a = \widetilde{x}_a x_0$ and the overall $\mathbb{Z}_{N_c}$ quotient in (3.1.21), one obtains (3.1.20) written as:

$$\mathbf{I}_{\mathrm{W}}[N_c, k, l, 0, 0] = \frac{|k + lN_c|}{N_c} \times \mathbf{I}_{\mathrm{W}}[SU(N_c)_k] , \qquad \mathbf{I}_{\mathrm{W}}[SU(N_c)_k] = \binom{|k| - 1}{N_c - 1} , \tag{3.1.23}$$

using the $SU(N_c)_k$ Witten index computed in [50, 169].

For any $N_f > 0$, we have the recursion relation:

$$\mathbf{I}_{\mathrm{W}}[N_c, k, l, N_f, N_f] = \mathbf{I}_{\mathrm{W}}[N_c, k, l, N_f - 1, N_f - 1] + \mathbf{I}_{\mathrm{W}}[N_c - 1, k, l, N_f - 1, N_f - 1] , \tag{3.1.24}$$

similarly to (3.1.14). Thus, given the result (3.1.20), we can compute the Witten index recursively. For instance, one easily finds:

$$\mathbf{I}_{\mathrm{W}}[N_c, k, l, N_f, N_f] = \frac{N_f + k + lN_c}{N_f + k} \binom{N_f + k}{N_c} , \quad \text{if } k > 0 \text{ and } l \geq 0 , \tag{3.1.25}$$

$k\backslash l$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0	12	11	10	9	8	7	**6**	**6**	**6**	7	8	9	10	11	12
1	17	15	13	11	9	7	**6**	**6**	**6**	7	9	11	13	15	17
2	21	18	15	12	9	**6**	**6**	**6**	9	12	15	18	21	24	27
3	24	20	16	12	8	**6**	**6**	10	14	18	22	26	30	34	38
4	26	21	16	11	**6**	**6**	10	15	20	25	30	35	40	45	50
5	27	21	15	9	**6**	10	15	21	27	33	39	45	51	57	63
6	27	20	13	**6**	10	14	21	28	35	42	49	56	63	70	77
7	26	18	10	10	14	20	28	36	44	52	60	68	76	84	92
8	24	15	10	14	18	27	36	45	54	63	72	81	90	99	108
9	21	15	14	18	25	35	45	55	65	75	85	95	105	115	125
10	21	14	18	22	33	44	55	66	77	88	99	110	121	132	143

Table 3.2: Witten index for $U(2)_{k,k+2l}$ with $n_f = 4$ fundamentals, for some values of k, l. The values with $k < 0$ can be obtained by parity (that is, the index for some (k, l) is the same as the index for $(-k, -l)$). The cases with the 'geometric value' $\mathbf{I}_{\rm W} = 6$ are given in bold.

where we also assumed that $N_c \leq N_f + k$. For general values of k and l, the recursive definition (3.1.24) (with the boundary condition (3.1.20)) gives us the explicit formula:

$$\mathbf{I}_{\rm W}[N_c, k, l, N_f, N_f] = \sum_{j=0}^{N_f} \frac{|k + l(N_c - j)|}{|k|} \binom{N_f}{j} \binom{|k|}{N_c - j} . \tag{3.1.26}$$

We checked numerically, in a large number of examples, that the index so obtained matches the number of Bethe vacua computed by Gröbner basis methods. In section 3.2, we give a more physical derivation of the Witten index $\mathbf{I}_{\rm W}[N_c, k, l, n_f, n_a]$ generalising the above result for the case $n_f \neq n_a$.

3.1.4 Twisted index for chiral theories

Finally, let us briefly discuss the case of chiral theories. For any SQCD theory with $n_f n_a > 0$, we have a recursion relation (see section 3.2 below):

$$\mathbf{I}_{\rm W}[N_c, k, l, n_f, n_a] = \mathbf{I}_{\rm W}[N_c, k, l, n_f - 1, n_a - 1] + \mathbf{I}_{\rm W}[N_c - 1, k, l, n_f - 1, n_a - 1] . \tag{3.1.27}$$

$k\backslash l$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$\frac{1}{2}$	21	19	17	15	13	11	**10**	**10**	**10**	11	12	13	14	15	17
$\frac{3}{2}$	25	22	19	16	13	**10**	**10**	**10**	**10**	12	15	18	21	24	27
$\frac{5}{2}$	28	24	20	16	12	**10**	**10**	**10**	14	18	22	26	30	34	38
$\frac{7}{2}$	30	25	20	15	**10**	**10**	**10**	15	20	25	30	35	40	45	50
$\frac{9}{2}$	31	25	19	13	**10**	**10**	15	21	27	33	39	45	51	57	63
$\frac{11}{2}$	31	24	17	**10**	**10**	15	21	28	35	42	49	56	63	70	77
$\frac{13}{2}$	30	22	14	**10**	15	20	28	36	44	52	60	68	76	84	92
$\frac{15}{2}$	28	19	**10**	15	20	27	36	45	54	63	72	81	90	99	108
$\frac{17}{2}$	25	15	15	20	25	35	45	55	65	75	85	95	105	115	125
$\frac{19}{2}$	21	15	20	25	33	44	55	66	77	88	99	110	121	132	143
$\frac{21}{2}$	21	20	25	30	42	54	66	78	90	102	114	126	138	150	162

Table 3.3: Witten index for $U(2)_{k,k+2l}$ with $n_f = 5$ fundamentals, for some values of k, l. The cases with the 'geometric value' $\mathbf{I}_{\mathrm{W}} = 10$ are given in bold.

Assuming that $n_f > n_a$ without loss of generality, the question is then to find the Witten index for $n_a = 0$,

$$\mathbf{I}_{\mathrm{W}}[N_c, k, l, n_f, 0] \; . \tag{3.1.28}$$

The computation of the index (3.1.28) for general k and l is part of a rather rich story, which will be addressed more thoroughly in the next section. Here, let us just mention that we can easily compute it using Gröbner basis methods, at least for small enough values of the parameters N_c, n_f and k, l. Some examples are displayed in tables 3.2 and 3.3, where we computed the index for $U(N_c)_{k,k+lN_c}$ with n_f fundamentals for $(N_c, n_f) = (2, 4)$ and $(2, 5)$. Let us comment on the fact that, for some values of the parameters, this index takes the 'geometric value' which saturates the bound:

$$\mathbf{I}_{\mathrm{W}}[N_c, k, l, n_f, 0] \;\geq\; \mathbf{I}_{\mathrm{W}}^{\mathrm{Higgs}} = \chi(\mathrm{Gr}(N_c, n_f)) = \binom{n_f}{N_c} \; . \tag{3.1.29}$$

This lowest value of the index, $\mathbf{I}_{\mathrm{W}}^{\mathrm{Higgs}}$, is the contribution from the Higgs branch of the gauge theory in a phase with vanishing masses and a positive FI parameter. That Higgs branch is the Grassmannian $\mathrm{Gr}(N_c, n_f)$, and the Witten index of this geometric phase is given by its Euler characteristic. For general values of k, l, there are also a number of additional contributions to the index from topological and mixed topological-geometric vacua, similarly to the abelian cases considered in [49].

More examples of twisted indices: Grassmannian theories. Let us consider the genus-0 index for the $\mathrm{Gr}(2,4)$ 'Grassmannian theories', defined as the $U(2)_{k,k+2l}$, $n_f = 4$ theories with CS levels such that $\mathbf{I}_{\mathrm{W}} = 6$. There are 15 such theories. Setting the $SU(4)$ flavour fugacities to $y_i = 1$ for simplicity, the $S^2 \times S^1$ twisted index of these theories is given by:

$$Z_{S^2\times S^1}^{U(2)_{k,k+2l},\, n_f=4} = \begin{cases} 0\ , & \text{if } (k,l) \in \{(0,-1),(0,0),(0,1),(1,-1),(1,0),(1,1),(5,-4)\}\ , \\ 1\ , & \text{if } (k,l) \in \{(2,-1),(2,0),(3,-2),(3,-1),(4,-2),(5,-3)\}\ , \\ \frac{1}{1-q^2}\ , & \text{if } (k,l) \in \{(2,-2),(4,-3)\}\ . \end{cases} \tag{3.1.30}$$

We can similarly compute the higher-genus index. For instance, for this $\mathrm{Gr}(2,4)$ theory with $(k,l) = (2,-1)$ (and $y_i = 1$), we find:

$$Z_{\Sigma_g\times S^1}^{U(2)_{2,0},\, n_f=4} = \begin{cases} 1\ , & \text{for } g=0\ , \\ 6\ , & \text{for } g=1\ , \\ 24q - 30q^2 + 2q^3\ , & \text{for } g=2\ , \\ 288q^2 - 1256q^3 + 1188q^4 - 216q^5 + 4q^6\ , & \text{for } g=3\ , \\ 1920q^3 - 672q^4 + 31896q^5 - 28944q^6 + 5832q^7 + 24q^8 - 40q^9 & \text{for } g=4\ , \\ 16896q^4 + 428032q^5 - 157280q^6 - 1098432q^7 + 975760q^8\ , & \\ -243968q^9 + 10464q^{10} + 1984q^{11} - 112q^{12}\ , & \text{for } g=5\ , \end{cases} \tag{3.1.31}$$

for the first few values of g. It would be very interesting to understand these and many similar results for the twisted index from an explicit quantisation of the theory on Σ_g with the A-twist, similarly to [170, 171].

3.2 Witten index and semi-classical vacua of unitary SQCD

In this section, we further comment on the Witten index of the $\mathcal{N} = 2$ SQCD theory that we studied in the previous section. We show that the Witten index for an arbitrary number of fundamental and antifundamental chiral multiples can be derived from the

Witten index for SQCD with only fundamental multiplets, which we also study thoroughly in this section.

3.2.1 Semiclassical vacuum moduli space

As mentioned earlier, the flavoured Witten index (2.2.20) of any 3d $\mathcal{N}=2$ supersymmetric gauge theory captures the number of vacua of that theory deformed by real masses $m = -\frac{1}{2\pi}\log|y|$ associated with the flavour symmetry. Whenever the mass deformation is generic enough to lift all non-compact branches of the vacuum moduli space, the Witten index is well-defined. Moreover, it is independent of the specific mass deformation chosen [49].

Let ϕ^a_α and $\widetilde{\phi}^\beta_a$ denote the fundamental and antifundamental scalars in the chiral multiplets of SQCD$[N_c, k, l, n_f, n_a]$. With generic mass deformations (including the real FI parameter ξ) and upon diagonalising the real adjoint scalar, $\sigma \to \text{diag}(\sigma_a)$, the semi-classical vacuum equations read (1.2.5):

$$\begin{aligned}
&(\sigma_a - m_\alpha)\phi^a_\alpha = 0 \ , && \alpha = 1,\cdots,n_f \ , \\
&(-\sigma_a + \widetilde{m}_\beta)\widetilde{\phi}^\beta_a = 0 \ , && \beta = 1,\cdots,n_a \ , \\
&\sum_{\alpha=1}^{n_f} \phi^{\dagger\alpha}_a \phi^b_\alpha - \sum_{\beta=1}^{n_a} \widetilde{\phi}^{\dagger b}_\beta \widetilde{\phi}^\beta_a = \frac{\delta_a{}^b}{2\pi} F_a(\sigma;\xi,m) \ ,
\end{aligned} \tag{3.2.1}$$

with $a = 1,\cdots,N_c$ (and no summation implied). The piecewise-linear functions F_a read:

$$F_a(\sigma;\xi,m) = \xi + k\sigma_a + l\sum_{b=1}^{N_c}\sigma_b + \frac{1}{2}\sum_{\alpha=1}^{n_f}|\sigma_a - m_\alpha| - \frac{1}{2}\sum_{\beta=1}^{n_a}|-\sigma_a + \widetilde{m}_\beta| \ . \tag{3.2.2}$$

These include the contribution from one-loop shifts to the effective CS levels and FI parameter [49]. Here we use a convenient notation where m_α, $\widetilde{m}_\beta$ are $U(n_f)\times U(n_a)$ parameters, which includes part of the gauge symmetry – the actual flavour symmetry being $U(n_f)\times U(n_a)/U(1) \cong SU(n_f)\times SU(n_a)\times U(1)_A$, plus the topological symmetry $U(1)_T$.[1]

[1]Here, we are not being particularly careful about the global form of the flavour group. In the case $n_a = 0$, on which we focus in this work, the full $SU(n_f)$ group acts as a symmetry.

3.2.2 $U(1)_k$ with n_f chiral multiplets of charge $+1$: a review

Before tackling the non-abelian case SQCD$[N_c, k, l, n_f, 0]$, it will be useful to review the computation of the index for a $U(1)_k$ gauge theory coupled to n_f chiral multiplets of charges $+1$ [49]. Recall from the discussion in subsection 2.2.2 that the CS level k is quantised as $k + \frac{n_f}{2} \in \mathbb{Z}$. We consider the case with zero mass for the chiral multiplet, but we assume that the FI parameter is non-zero. In that case, the semi-classical vacuum equations (3.2.1) and (3.2.2) become:

$$\begin{aligned} &\sigma\phi_\alpha = 0 \; , && \alpha = 1, \cdots, n_f \; , \\ &\sum_{\alpha=1}^{n_f} \phi_\alpha^\dagger \phi_\alpha = \frac{1}{2\pi} F(\sigma) \; , && F(\sigma) \equiv \xi + k\sigma + \frac{n_f}{2} |\sigma| \; . \end{aligned} \tag{3.2.3}$$

Higgs vacua: The first equation in (3.2.3) implies that Higgs vacua may only arise at the origin of the would-be Coulomb branch ($\sigma = 0$). Then, the Higgs branch is governed by the standard D-term equation:

$$\sum_{\alpha=1}^{n_f} \phi_\alpha^\dagger \phi_\alpha = \xi \; . \tag{3.2.4}$$

There are no solutions for $\xi < 0$, so let us assume that $\xi > 0$, for now. Then, upon quotienting by the $U(1)$ gauge group, we have a standard Kähler quotient description of the projective space $\mathbb{CP}^{n_f-1}$. Its contribution to the total Witten index is thus:

$$\mathbf{I}_{\mathrm{W,I}}[1, k, 0, n_f, 0] = \chi\left(\mathbb{CP}^{n_f-1}\right) = n_f \; . \tag{3.2.5}$$

In anticipation of the next section, we call this a solution of *type I*. Note that this solution exists for any value of k.

Topological vacua: These arise if $\phi = 0$ and $\sigma \neq 0$, in which case we need to find non-trivial solutions to the equation:

$$F(\sigma) = \xi + \left(k + \mathrm{sign}(\sigma)\frac{n_f}{2}\right)\sigma = 0 \; , \tag{3.2.6}$$

We shall call such solutions the *type III* solutions.[1] Let us start with the non-marginal case, that is, with $|k| \neq \frac{n_f}{2}$. Assuming that $\xi > 0$, we have two solutions (depending on

[1]In the non-abelian case, we will also find Higgs-topological vacua, which will be the *type II* solutions.

the sign of σ):

$$\begin{cases} \sigma^{+} = -\frac{\xi}{k+\frac{n_f}{2}} > 0 \ , & \text{iff } \ k + \frac{n_f}{2} < 0 \ , \\ \sigma^{-} = -\frac{\xi}{k-\frac{n_f}{2}} < 0 \ , & \text{iff } \ k - \frac{n_f}{2} > 0 \ . \end{cases} \tag{3.2.7}$$

We will use the notation $\sigma^{\pm}$ for solutions for σ such that $\sigma^{+} > 0$ and $\sigma^{-} < 0$, respectively. We then have the following effective abelian CS theories:

$$\mathcal{M}_{\rm III}^{\xi>0}[1, k, 0, n_f, 0] = \Theta\left(-k - \frac{n_f}{2}\right) U(1)_{k+\frac{n_f}{2}} \oplus \Theta\left(k - \frac{n_f}{2}\right) U(1)_{k-\frac{n_f}{2}} \ . \tag{3.2.8}$$

Here and in the following, we find it convenient to use the Heaviside step function $\Theta(x)$ defined around (3.1.7) to keep track of constraints on the parameters. (In the present case, we have a TQFT contribution $U(1)_{k\pm\frac{n_f}{2}}$ for $k < -\frac{n_f}{2}$ and $k > \frac{n_f}{2}$, respectively, and no TQFT if $|k| < \frac{n_f}{2}$. For $|k| = \frac{n_f}{2}$, there is a naive $U(1)_0$ contribution, which is lifted by the FI term.) As discussed in (A.1.8), the Witten index for the $U(1)_K$ CS theory is equal to $|K|$, hence the vacua (3.2.8) contribute:

$$\mathbf{I}_{\rm W,\, III}^{\xi>0}[1, k, 0, n_f, 0] = \begin{cases} |k| - \frac{n_f}{2} \ , & \text{if } |k| > \frac{n_f}{2} \ , \\ 0 \ , & \text{if } |k| < \frac{n_f}{2} \ , \end{cases} \tag{3.2.9}$$

to the total index of the abelian theory. We must also carefully treat the marginal cases, $k = \pm\frac{n_f}{2}$, in which case we find the solutions:

$$\sigma^{\pm} = \mp \frac{\xi}{n_f} \ , \qquad \text{iff } \ \xi < 0 \ . \tag{3.2.10}$$

Hence, there are no topological solutions for $\xi > 0$.

The full Witten index. Adding the contributions (3.2.5) and (3.2.9) in the regime $\xi > 0$, one finds:

$$\mathbf{I}_{\rm W}[1, k, 0, n_f, 0] = \begin{cases} |k| + \frac{n_f}{2} \ , & \text{if } |k| \geq \frac{n_f}{2} \ , \\ n_f \ , & \text{if } |k| < \frac{n_f}{2} \ . \end{cases} \tag{3.2.11}$$

As already mentioned, the index is independent of the mass deformation as long as all non-compact directions of the moduli space are lifted. If we now consider the case $\xi < 0$,

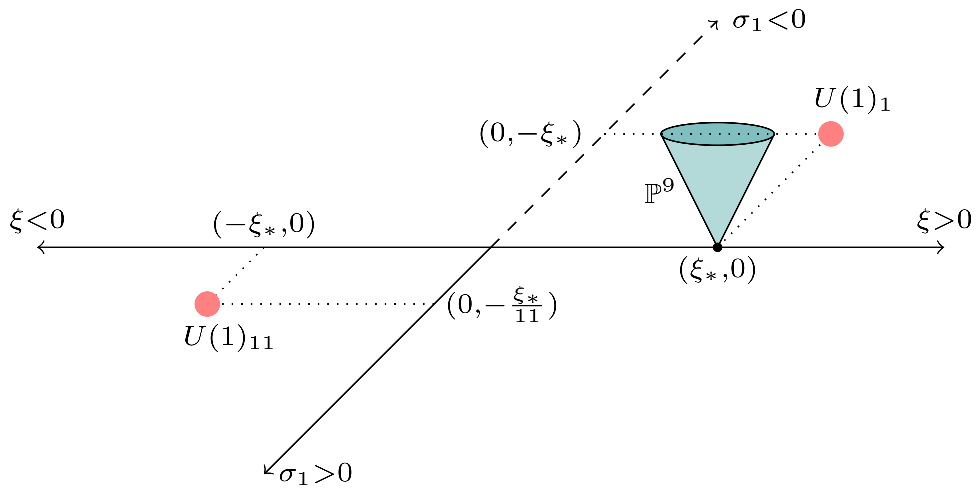


Figure 3.1: Moduli space of vacua of $U(1)_6$ theory coupled with 10 matter multiplets of charge +1. The total number of SUSY vacua on both sides of the FI line is the same and equals 11.

there are no Higgs vacua, but we find the following topological vacua:

$$\begin{cases} \sigma^+ = -\frac{\xi}{k+\frac{n_f}{2}} > 0 \ , & \text{iff } k + \frac{n_f}{2} > 0 \ , \\ \sigma^- = -\frac{\xi}{k-\frac{n_f}{2}} < 0 \ , & \text{iff } k - \frac{n_f}{2} < 0 \ , \end{cases} \tag{3.2.12}$$

as well as the vacua (3.2.10) in the marginal cases. Hence, we have:

$$\mathcal{M}_{\text{III}}^{\xi<0}[1,k,0,n_f,0] = \Theta\left(k+\frac{n_f}{2}\right) U(1)_{k+\frac{n_f}{2}} \oplus \Theta\left(-k+\frac{n_f}{2}\right) U(1)_{k-\frac{n_f}{2}} \ , \tag{3.2.13}$$

which indeed reproduces (3.2.11). As an example of this analysis, see figure 3.1 for the moduli space of vacua of the $U(1)_6$ theory with 10 multiplets with charges +1 and for both regimes of the FI parameter ξ.

3.2.3 Recursion relation for the flavoured Witten index

Let us now consider the Witten index of SQCD$[N_c, k, l, n_f, n_a]$ with $n_f \geq n_a$, without loss of generality.[1] The index with $n_f = n_a$ is given by (3.1.26), hence we shall focus on the 'chiral' case $n_f > n_a$ (that is, $k_c \equiv \frac{1}{2}(n_f - n_a) > 0$). We will show that the Witten index

[1] This also gives us the result for $n_a > n_f$ since $\mathbf{I}_{\text{W}}[N_c, k, l, n_f, n_a] = \mathbf{I}_{\text{W}}[N_c, k, l, n_a, n_f]$.

satisfies the recursion relation:

$$\mathbf{I}_{\mathrm{W}}[N_c,k,l,n_f,n_a]=\mathbf{I}_{\mathrm{W}}[N_c,k,l,n_f-1,n_a-1]+\mathbf{I}_{\mathrm{W}}[N_c-1,k,l,n_f-1,n_a-1]\ . \quad (3.2.14)$$

Recall that this relation was conjectured numerically earlier in (3.1.27).

Derivation. Assuming that $n_a>0$, consider the $U(1)_{n_f}\times U(1)_{n_a}\subset U(n_f)\times U(n_a)$ flavour symmetry under which the chiral multiplets ϕ_{n_f} and $\widetilde{\phi}^{n_a}$ are charged, and let us turn on a large mass m for the diagonal $U(1)\subset U(1)_{n_f}\times U(1)_{n_a}$. Then, the vacuum equations (3.2.1) and (3.2.2) take the form:

$$\begin{aligned}
&(\sigma_a-m)\,\phi^a_{n_f}=0\ ,\\
&(-\sigma_a+m)\,\widetilde{\phi}^{n_a}_a=0\ \ ,\\
&\sigma_a\,\phi^a_\alpha=0\ ,\quad \alpha=1,\cdots,n_f-1\ ,\\
&\sigma_a\,\widetilde{\phi}^\beta_a=0\ ,\quad \beta=1,\cdots,n_a-1\ ,\\
&\sum_{\alpha=1}^{n_f}\phi^{\dagger\alpha}_a\phi^b_\alpha-\sum_{\beta=1}^{n_a}\widetilde{\phi}^{\dagger b}_\beta\widetilde{\phi}^\beta_a=\frac{\delta_a{}^b}{2\pi}F_a(\sigma)\ ,
\end{aligned} \quad (3.2.15)$$

with:

$$F_a(\sigma,m)\equiv\xi+(k+\mathrm{sign}(\sigma_a)k_c)\,\sigma_a+l\sum_{b=1}^{N_c}\sigma_b\ . \quad (3.2.16)$$

More precisely, we assume that $|m_\alpha|,|\widetilde{m}_\beta|\ll|m|$ for $\alpha\neq n_f$, $\beta\neq n_a$, so we may ignore those small masses for simplicity of notation. In this limit, the solutions decompose into two disjoint families. Firstly, we have the solutions with $|\sigma_a|\ll|m|$, in which case the vacuum equations correspond to the low-energy effective field theory:

$$U(N_c)_{k,k+lN_c}\quad\text{coupled with}\quad(n_f-1)\,\square\oplus\ (n_a-1)\,\overline{\square}\ . \quad (3.2.17)$$

Indeed, the large real mass m allows us to integrate out the flavour pair ϕ_{n_f}, $\widetilde{\phi}^{n_a}$, and the CS levels do not change in that particular limit. Secondly, we should consider potential solutions with $\sigma_{N_c}\approx m$. In that limit, we Higgs the gauge group to:

$$U(N_c)\longrightarrow U(\underbrace{N_c-1)_{k,k+lN_c}\times U(1}_{l})_{k_{\mathrm{eff}}}\ , \quad (3.2.18)$$

At low energy, there are still n_f-1 fundamental chiral multiplets and n_a-1 antifundamental chiral multiplets charged under the $U(N_c-1)$ factor, while the chiral multiplets $\phi^a_{n_f}$, $\widetilde{\phi}^{n_a}_a$ ($a < N_c$) charged under $U(N_c-1)$ are integrated out as above. On the other hand, the fields $\phi^{N_c}_{n_f}$, $\widetilde{\phi}^{n_a}_{N_c}$ remain light, thus contributing one flavour (*i.e.* two chiral multiplets of charge ± 1) to the abelian sector. Moreover, integrating out all the massive fields leads to a shift of the abelian CS level according to:

$$k_{\text{eff}} = k + l + \text{sign}(\sigma_{N_c}) k_c \ . \tag{3.2.19}$$

We have $\sigma_{N_c} = m + \delta$, with δ very small by assumption. Naively, we might think that the $U(1)$ sector in (3.2.18) would contribute non-trivially to the Witten index, as the Witten index of a $U(1)_{k_{\text{eff}}}$ theory with one flavour is $1 + |k_{\text{eff}}|$. However, this includes $|k_{\text{eff}}|$ topological vacua which would arise at parametrically large values of δ; this would violate our assumption and therefore we should not count those putative vacua [49]. As far as computing the Witten index is concerned, therefore, the second class of solutions is isomorphic to the number of vacua for the partially Higgsed theory:

$$U(N_c-1)_{k,k+lN_c} \quad \text{coupled with} \quad (n_f - 1)\,\square \oplus \ (n_a - 1)\,\overline{\square} \ . \tag{3.2.20}$$

In this way, we just derived the advertised recursion relation (3.1.27) in the previous section.

Witten index: the general case. Using the recursion relation (3.2.14) and assuming that $n_f > n_a$, we find the following explicit expression for the index:

$$\mathbf{I}_{\text{W}}[N_c, k, l, n_f, n_a] = \sum_{\beta=0}^{n_a} \binom{n_a}{\beta} \mathbf{I}_{\text{W}}[N_c - \beta, k, l, n_f - n_a, 0] \ . \tag{3.2.21}$$

Thus, to compute $\mathbf{I}_{\text{W}}[N_c, k, l, n_f, n_a]$ in general, all we have left to do is to explicitly compute $\mathbf{I}_{\text{W}}[N_c, k, l, n_f, 0]$, namely the Witten index for 'chiral' SQCD with only fundamental matter. This is the theory we will study in the rest of this section.

3.2.4 Witten index for $U(N_c)_{k,k+lN_c}$ with n_f fundamentals

Given the previous discussion, we now focus on the $U(N_c)_{k,k+lN_c}$ gauge theory with n_f fundamental chiral multiplets. Setting the masses to zero and turning on a non-zero FI term, the semi-classical vacuum equations (3.2.1)-(3.2.2) reduce to:

$$\begin{aligned} &\sigma_a\,\phi^a_\alpha = 0 ~, && \alpha = 1,\cdots, n_f ~, \\ &\sum_{\alpha=1}^{n_f} \phi^{\dagger\alpha}_a \phi^b_\alpha = \frac{\delta_a^{\ b}}{2\pi} F_a(\sigma) ~, && F_a(\sigma) = \xi + k\sigma_a + l\sum_{b=1}^{N_c}\sigma_b + \frac{n_f}{2}|\sigma_a| ~, \end{aligned} \tag{3.2.22}$$

with $a = 1, \cdots, N_c$. In the abelian case ($N_c = 1$), the solutions were discussed in section 3.2.2. As we will now show, the structure of the vacuum with $\xi \neq 0$ in the non-abelian case is rather intricate, especially for non-zero values of l. Let us first explain the general structure of the solution before discussing it in more detail – to be pedagogical, we will first analyse the $U(2)$ theory before giving the general $U(N_c)$ result later in this subsection

Higgs vacuum (*Type I*): the complex Grassmannian. The simplest solutions are for $\sigma_a = 0$, $\forall a$, which we call the *Type I* vacua. This corresponds to the solution to the D-term relation:

$$\sum_{\alpha=1}^{n_f} \phi^{\dagger\alpha}_a \phi^b_\alpha = \frac{\delta_a^{\ b}}{2\pi}\xi ~, \tag{3.2.23}$$

modulo $U(N_c)$ gauge transformations. For $\xi > 0$, this famously gives us the complex Grassmannian manifold $\mathrm{Gr}(N_c, n_f)$ (see *e.g.* [25]), while there are no solution for $\xi < 0$. We write this vacuum and its contribution to the Witten index as:

$$\mathcal{M}_{\mathrm{I}} = \Theta(\xi)\,\mathrm{Gr}(N_c, n_f) ~, \qquad \mathbf{I}_{\mathrm{W,I}} = \Theta(\xi)\binom{n_f}{N_c} ~. \tag{3.2.24}$$

For general values of the CS levels k, l, there will be many more topological and Higgs-topological vacua, as well as some strongly-coupled vacua. As already mentioned in (3.1.29), the geometric contribution (3.2.24) provides a lower bound for the Witten index. This is because all other vacua that we find (at $\xi > 0$) are bosonic, as we will see, so they all contribute to the index with a positive sign.

Higgs-topological (*Type II*) and topological vacua (*Type III*). All such vacua arise as solutions with $\sigma_a \neq 0$ for at least some σ_a's. Due to the residual gauge transformations that permutes the σ's, we only need to specify how many σ's are zero and how many are non-zero. We encounter the following four possibilities:

Type II,a: What we call Type II vacua are the solutions such that some but not all σ's vanish. We then pick $\sigma_a = 0$ for $a = 1, \cdots, N_c - p$, and $\sigma_{a'} \neq 0$ for $a' = N_c - p + 1, \cdots, N_c$, for $0 < p < N_c$. In the *Type II,a* case, we choose the $\sigma_{a'}$'s to all have the same sign. We then obtain a hybrid Higgs-topological vacuum $\mathrm{Gr}(N_c - p) \times U(p)$, where $U(p)$ denotes some pure CS theory with gauge group

$U(p)$.

Type II,b: These are the Type II vacua such that, out of $p+q>0$ non-zero σ's, we choose p of them to be positive and q of them to be negative, which would lead to a Higgs-topological vacuum $\text{Gr}(N_c-p-q)\times U(p)\times U(q)$. In the end, it will turn out that there are no such vacua in our theory. We should note that some of these 'Higgs-topological vacua' are actually ordinary Higgs vacua – this occurs whenever the TQFT sector is (dual to) a trivial theory with a single state.

Type III,a: What we call Type III vacua are topological vacua, which arise when all the σ's are non-zero. *Type III,a* vacua corresponds to choosing all σ_a's to have the same sign, in which case we obtain an effective $U(N_c)$ CS theory.

Type III,b: For these vacua, we choose p of the σ's to be positive, and N_c-p of them to be negative, which then leaves us with an effective $U(p)\times U(N_c-p)$ CS theory.

Strongly-coupled vacua (*Type IV*). Finally, we have to address the logical possibility that there might exist strongly-coupled vacua at small values of σ. By comparing our computation to the Bethe-vacua counting of section 3.1, we find that such vacua arise whenever there exist *Coulomb vacua* in the semi-classical analysis (see section 3.2.1). Each such semi-classical Coulomb branch corresponds to setting N_c-p σ's to zero, and having an effective CS level 0 for some $SU(p)\subset U(p)\subseteq U(N_c)$ unhiggsed subgroup that may appear at some values of the σ's. The effective 3d $\mathcal{N}=2$ $SU(p)_0$ gauge theory without matter does not have any stable vacuum [52]. However, we expect that, in those cases, there exists additional, strongly-coupled supersymmetric vacua that survive near the origin of the Coulomb branch (which is otherwise lifted non-perturbatively). These 'true Type IV' vacua are not captured by our analysis, but we are nonetheless able to derive their contribution to the index in various ways (in particular by computing the index for either sign of the FI parameter; see also section 4.3.3 for the dual perspective).

$U(2)_{k,k+2l}$ with n_f fundamentals

Let us first consider the $U(2)_{k,k+2l}$ gauge theory with n_f fundamentals. We wish to solve the vacuum equations:

$$\begin{aligned} &\sigma_a\,\phi^a_\alpha=0\ , && a=1,2\ ,\quad \alpha=1,\cdots,n_f\ ,\\ &\sum_{\alpha=1}^{n_f}\phi^{\dagger\alpha}_a\phi^b_\alpha=\frac{\delta_a{}^b}{2\pi}F_a(\sigma)\ , && F_a(\sigma)=\xi+k\sigma_a+l(\sigma_1+\sigma_2)+\frac{n_f}{2}|\sigma_a|\ . \end{aligned} \tag{3.2.25}$$

Let us now consider all possible solutions to these equations. The **Type I** solution is as discussed above – we have a Grassmannian manifold $\text{Gr}(2, n_f)$ if $\xi > 0$, which contributes:

$$\mathbf{I}_{\text{W, I}}[2, k, l, n_f, 0] = \Theta(\xi) \binom{n_f}{2} = \Theta(\xi) \, \frac{n_f(n_f - 1)}{2} \ . \tag{3.2.26}$$

There are also many possible Higgs-topological and topological vacua, as well as potential strongly-coupled vacua if $|k| = \frac{n_f}{2}$, as we will now discuss.

Type II vacua. If we assume that $\sigma_1 = 0$ and $\sigma_2 \neq 0$ in (3.2.25), we have to solve the equations:

$$\begin{aligned} 2\pi \sum_{\alpha=1}^{n_f} \phi_1^{\dagger\alpha} \phi_\alpha^1 &= F_1(\sigma) = \xi + l\sigma_2 > 0 \ , \\ F_2(\sigma) &= \xi + \left(k + l + \text{sign}(\sigma_2)\frac{n_f}{2}\right) \sigma_2 = 0 \ . \end{aligned} \tag{3.2.27}$$

Note the inequality $F_1(\sigma) > 0$, which is necessary for the Higgs-branch $\mathbb{CP}^{n_f-1}$ to exists, while the solutions to $F_2(\sigma) = 0$ correspond to TQFTs ($U(1)$ CS theories). Let us first observe that we cannot obtain any solution in the case $k + l + \text{sign}(\sigma_1)\frac{n_f}{2} = 0$, since $\xi \neq 0$ by assumption. (Similar cases will appear repeatedly in the following analysis, and we will not discuss them explicitly.) Now, at non-zero values on the real line σ_2, we have two solutions analogous to (3.2.7), namely:

$$\begin{cases} \sigma_2^+ = -\frac{\xi}{k+l+\frac{n_f}{2}} > 0 \ , & \text{iff } \text{sign}(\xi)\left(k + l + \frac{n_f}{2}\right) < 0 \text{ and } k + \frac{n_f}{2} < 0 \ , \\ \sigma_2^- = -\frac{\xi}{k+l-\frac{n_f}{2}} < 0 \ , & \text{iff } \text{sign}(\xi)\left(k + l - \frac{n_f}{2}\right) > 0 \text{ and } k - \frac{n_f}{2} > 0 \ , \end{cases} \tag{3.2.28}$$

where the other inequalities above arise from demanding the volume of the Higgs branch to be positive.

This gives us the second component of the moduli space of vacua of the $U(2)$ theory:

$$\begin{aligned} \mathcal{M}_{\text{II}}[2, k, l, n_f, 0] \ &= \Theta\left(-k - \frac{n_f}{2}\right) \Theta\left(\xi\left(-k - l - \frac{n_f}{2}\right)\right) \mathbb{CP}^{n_f-1} \times U(1)_{k+l+\frac{n_f}{2}} \\ &\oplus \Theta\left(k - \frac{n_f}{2}\right) \Theta\left(\xi\left(k + l - \frac{n_f}{2}\right)\right) \mathbb{CP}^{n_f-1} \times U(1)_{k+l-\frac{n_f}{2}} \ , \end{aligned} \tag{3.2.29}$$

which contributes to the Witten index as:

$$\begin{aligned} \mathbf{I}_{\text{W, II}}[2, k, l, n_f, 0] \ &= \Theta\left(-k - \frac{n_f}{2}\right) \Theta\left(\xi\left(-k - l - \frac{n_f}{2}\right)\right) n_f \left|k + l + \frac{n_f}{2}\right| \\ &+ \Theta\left(k - \frac{n_f}{2}\right) \Theta\left(\xi\left(k + l - \frac{n_f}{2}\right)\right) n_f \left|k + l - \frac{n_f}{2}\right| \ . \end{aligned} \tag{3.2.30}$$

Type III vacua. Next, we consider the topological vacua. For the Type III,a solutions, we choose σ_1, σ_2 to be of the same sign. It then turns out that $\sigma_1 = \sigma_2$, and we find the two solutions:

$$\begin{cases} \sigma_1 = \sigma_2 = \sigma^+ \equiv -\frac{\xi}{k+2l+\frac{n_f}{2}} > 0 \ , & \text{if} \ \ \text{sign}(\xi)\left(k+2l+\frac{n_f}{2}\right) < 0 \ , \\ \sigma_1 = \sigma_2 = \sigma^- \equiv -\frac{\xi}{k+2l-\frac{n_f}{2}} < 0 \ , & \text{if} \ \ \text{sign}(\xi)\left(k+2l-\frac{n_f}{2}\right) > 0 \ , \end{cases} \tag{3.2.31}$$

which correspond to the following topological vacua:

$$\begin{aligned} \mathcal{M}_{\text{III,a}}[2,k,l,n_f,0] =& \Theta\left(\xi\left(-k-2l-\frac{n_f}{2}\right)\right) U(2)_{k+\frac{n_f}{2},k+\frac{n_f}{2}+2l} \\ & \oplus \Theta\left(\xi\left(k+2l-\frac{n_f}{2}\right)\right) U(2)_{k-\frac{n_f}{2},k-\frac{n_f}{2}+2l} \ . \end{aligned} \tag{3.2.32}$$

These vacua contribute to the Witten index according to (A.1.5), namely:

$$\begin{aligned} \mathbf{I}_{\text{W, III,a}}[2,k,l,n_f,0] =& \Theta\left(\xi\left(-k-2l-\frac{n_f}{2}\right)\right) \mathbf{I}_{\text{W}}\left[\ 2 \quad l \ \middle| \ k+\frac{n_f}{2}\ \right] \\ & + \Theta\left(\xi\left(k+2l-\frac{n_f}{2}\right)\right) \mathbf{I}_{\text{W}}\left[\ 2 \quad l \ \middle| \ k-\frac{n_f}{2}\ \right] \ , \end{aligned} \tag{3.2.33}$$

where we used the notation introduced in (A.1.11). We may also have Type III,b vacua with $\sigma_1 > 0$ and $\sigma_2 < 0$, in which case the vacuum equations (3.2.25) reduce to:

$$\begin{aligned} F(\sigma_1) &= \xi + \left(k+\frac{n_f}{2}\right)\sigma_1 + l(\sigma_1+\sigma_2) = 0 \ , \\ F(\sigma_1) &= \xi + \left(k-\frac{n_f}{2}\right)\sigma_1 + l(\sigma_1+\sigma_2) = 0 \ , \end{aligned} \tag{3.2.34}$$

which have a unique solution:

$$\begin{aligned} \sigma_1 &= -\xi\frac{k-\frac{n_f}{2}}{\left(k+\frac{n_f}{2}\right)\left(k-\frac{n_f}{2}\right)+2kl} > 0 \ , \\ \sigma_2 &= -\xi\frac{k+\frac{n_f}{2}}{\left(k+\frac{n_f}{2}\right)\left(k-\frac{n_f}{2}\right)+2kl} < 0 \ . \end{aligned} \tag{3.2.35}$$

The inequalities constraining the appearance of this solution can be simplified to:

$$|k| < \frac{n_f}{2} \ , \qquad \xi\left(k^2+2kl-\frac{1}{4}n_f^2\right) > 0 \ . \tag{3.2.36}$$

Thus we have the vacua:

$$\begin{aligned}&\mathcal{M}_{\mathrm{III,b}}[2,k,l,n_f,0] = \\ &\quad\Theta\left(\frac{n_f}{2}-|k|\right)\Theta\left(\xi\left(k^2+2kl-\frac{1}{4}n_f^2\right)\right)\underbrace{U(1)_{k+l+\frac{n_f}{2}}\times U(1)_{k+l-\frac{n_f}{2}}}_{l}\ .\end{aligned} \tag{3.2.37}$$

Using the result (A.1.10) for abelian CS theories, we have the index contribution:

$$\begin{aligned}\mathbf{I}_{\mathrm{W,\,III,b}}\left[2,k,l,n_f,0\right] = \Theta\left(\frac{n_f}{2}-|k|\right)\Theta\left(\xi\left(k^2+2kl-\frac{1}{4}n_f^2\right)\right)&\\ \times\,\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} 1 & 0 & k+l+\frac{n_f}{2} & l \\ 1 & 0 & l & k+l-\frac{n_f}{2}\end{array}\right]&\ .\end{aligned} \tag{3.2.38}$$

Type IV vacua. Finally, we have to be careful about the 'marginal case' $|k| = \frac{n_f}{2}$, in which case we can have continuous Coulomb branch solutions. Consider first the case $k = \frac{n_f}{2}$. We have a *Type IV,a* solution corresponding to $F_1(\sigma) = F_2(\sigma) = 0$ with $\sigma_1 < 0$, $\sigma_2 < 0$:

$$\sigma_1+\sigma_2 = -\frac{\xi}{l} < 0 \qquad \text{iff}\ \ \xi l > 0\ . \tag{3.2.39}$$

An analogous solution exists in the case $k = -\frac{n_f}{2}$ if $\xi l < 0$. In either case, we find some continuous '$SU(2)$ Coulomb branch' spanned by $\sigma_1 - \sigma_2 \in \mathbb{R}$. This would naively render the Witten index ill-defined. Here, however, we effectively have an $SU(2)_0$ gauge theory at low energy, thus we expect this Coulomb branch to be lifted non-perturbatively [52]. There remains the possibility that some strongly-coupled vacua could survive near $\sigma_1 = \sigma_2 = 0$, and we will thus make a conjecture for their contribution to the Witten index.

In fact, for any fixed CS level $l \in \mathbb{Z}$, the Type IV vacua only arise for one sign of ξ. Hence, at any given value of the CS levels, we can simply consider the appropriate sign for ξ in order to compute the Witten index in terms of the vacua of Type I, II and III only. Since the index should be the same for either sign of ξ, this allows us to derive the necessary contribution from Type IV vacua:

$$\mathbf{I}_{\mathrm{W,\,IV}}\left[2,k,l,n_f,0\right] = \delta_{k,\frac{n_f}{2}}\,\Theta(\xi l)\,|l|(n_f-1) + \delta_{k,-\frac{n_f}{2}}\,\Theta(-\xi l)\,|l|(n_f-1)\ . \tag{3.2.40}$$

On the other hand, we can only speculate on the nature of the corresponding vacua $\mathcal{M}_{\mathrm{IV}}$ – see also the discussion in section 4.3.3.

The full Witten index. Summing up all the above contributions, the final answer

$k\backslash l$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0	12	11	10	9	8	7	**6**	**6**	**6**	7	8	9	10	11	12
1	17	15	13	11	9	7	**6**	**6**	**6**	7	9	11	13	15	17
2	21	18	15	12	9	**6**	**6**	**6**	6 +3	6 +6	6 +9	6 +12	6 +15	6 +18	6 +21
3	24	20	16	12	8	**6**	**6**	10	14	18	22	26	30	34	38
4	26	21	16	11	**6**	**6**	10	15	20	25	30	35	40	45	50
5	27	21	15	9	**6**	10	15	21	27	33	39	45	51	57	63
6	27	20	13	**6**	10	14	21	28	35	42	49	56	63	70	77
7	26	18	10	10	14	20	28	36	44	52	60	68	76	84	92
8	24	15	10	14	18	27	36	45	54	63	72	81	90	99	108
9	21	15	14	18	25	35	45	55	65	75	85	95	105	115	125
10	21	14	18	22	33	44	55	66	77	88	99	110	121	132	143

Table 3.4: Witten index for $U(2)_{k,k+2l}$ with $n_f = 4$ fundamentals, for some values of k, l. The case with the minimal value $\mathbf{I}_\mathrm{W} = 6$ are given in bold. The contributions from Type IV vacua when $\xi > 0$ are shown in red.

reads:

$$\begin{aligned} \mathbf{I}_\mathrm{W}\left[2,k,l,n_f,0\right] \;&=\; \mathbf{I}_{\mathrm{W,\,I}}\left[2,k,l,n_f,0\right] + \mathbf{I}_{\mathrm{W,\,II}}\left[2,k,l,n_f,0\right] \\ &\quad + \mathbf{I}_{\mathrm{W,\,III,a}}\left[2,k,l,n_f,0\right] + \mathbf{I}_{\mathrm{W,\,III,b}}\left[2,k,l,n_f,0\right] \\ &\quad + \mathbf{I}_{\mathrm{W,\,IV}}\left[2,k,l,n_f,0\right] \;. \end{aligned} \tag{3.2.41}$$

This explicit formula can be compared to the numerical counting of Bethe vacua using Gröbner basis methods as we did in subsection 3.1.4, and we find perfect agreement. As an example, consider the $U(2)_{k,k+2l}$ theory with 4 fundamentals. Picking $\xi > 0$, the Type I vacua contributes $\chi(\mathrm{Gr}(2,4)) = 6$ and we then have a number of Type II, III, IV vacua, as well as some Type IV vacua for $k = 2$, as indicated in the table 3.4. This exactly reproduces table 3.2 above.

Preliminary comments on the phase transition at $\xi = 0$. While the index is the same for $\xi > 0$ and $\xi < 0$, the structure of the vacuum changes in intricate ways as we change the sign of the FI parameter. This is an interesting 3d analogue of the 2d CY/LG correspondence [13]. From our analysis above, we have an explicit form of the full vacuum moduli space $\mathcal{M}$ for each sign of ξ (except when Type IV vacua arise). We show some examples of this in table 3.5. For instance, looking at the first line with $(k,l) = (0,10)$, for $\xi > 0$ we have a Higgs and a topological vacuum, which contribute to the Witten

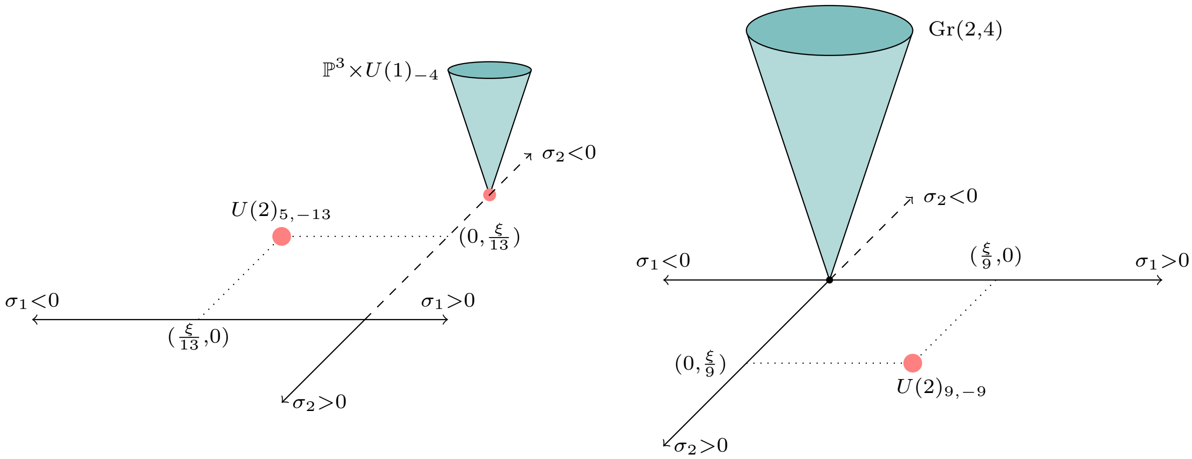


Figure 3.2: Moduli space of vacua of 3d $\mathcal{N}=2$ SQCD$[2,7,-9,4,0]$. LEFT: the case for negative FI parameter ξ. RIGHT: the case for positive ξ. The number of vacua matches on both sides and equals 42.

index as:

$$\mathbf{I}_{\rm W}\left[\mathrm{Gr}(2,4)\right]+\mathbf{I}_{\rm W}\left[U(2)_{-2,18}\right]=6+9=15\ , \tag{3.2.42}$$

while the $\xi<0$ phase consists of two topological vacua:

$$\mathbf{I}_{\rm W}\left[U(2)_{2,22}\right]+\mathbf{I}_{\rm W}\left[U\underbrace{(1)_{12}\times U(1)}_{10}{}_{8}\right]=11+4=15\ . \tag{3.2.43}$$

Note also that, for general values of (k,l), we can have hybrid Higgs-topological vacua for either sign of ξ, while the pure Higgs branch only exists for $\xi>0$. As we comment in chapter 6, understanding this phase structure is still a work in progress.

$U(N_c)_{k,k+lN_c}$ with n_f fundamentals

Let us now discuss the complete solutions to (3.2.22) in the general case. Following the general discussion above, the **Type I** solution is given by (3.2.24). Let us now discuss all the other classes of solutions.

Type II vacua. Let us take the first N_c-p σ_a's to be vanishing, and the last p to be non-zero. In this case, we have the following set of equations (3.2.22):

k	l	$\xi > 0$ phase	$\xi < 0$ phase
0	10	$\mathrm{Gr}(2,4) \oplus U(2)_{-2,18}$	$U(2)_{2,22} \oplus \underbrace{U(1)_{12} \times U(1)_8}_{10}$
1	3	$\mathrm{Gr}(2,4) \oplus \underbrace{U(1)_6 \times U(1)_2}_{3}$	$U(2)_{3,9}$
3	-2	$\mathrm{Gr}(2,4)$	$\mathbb{CP}^3 \times U(1)_{-1} \oplus U(2)_{5,1}$
4	7	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_9 \oplus U(2)_{2,16}$	$U(2)_{6,20}$
5	-6	$\mathrm{Gr}(2,4) \oplus U(2)_{7,-5}$	$\mathbb{CP}^3 \times U(1)_{-3} \oplus U(2)_{3,-9}$
6	-4	$\mathrm{Gr}(2,4)$	$U(2)_{4,-4}$
7	-9	$\mathrm{Gr}(2,4) \oplus U(2)_{9,-9}$	$\mathbb{CP}^3 \times U(1)_{-4} \oplus U(2)_{5,-13}$
8	8	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{14} \oplus U(2)_{6,22}$	$U(2)_{10,26}$
9	10	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{17} \oplus U(2)_{7,77}$	$U(2)_{11,31}$
10	5	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{13} \oplus U(2)_{8,18}$	$U(2)_{12,22}$

Table 3.5: Moduli spaces of vacua for $U(2)$ theory coupled with 4 fundamental multiplets and different values of the levels k and l. We include both phases of ξ, the positive and negative one.

$$
\begin{aligned}
&\sum_{\alpha=1}^{n_f} \phi_a^{\dagger\alpha}\phi_\alpha^b = \frac{\delta_a^b}{2\pi}\left(\xi + l \sum_{a'=N_c-p+1}^{N_c} \sigma_{a'}\right) , \qquad a,b = 1,\cdots,N_c-p ,\\
&F_{a'}(\sigma) = \xi + \left(k + \frac{n_f}{2}\right)\sigma_{a'} + l \sum_{b'=N_c-p+1}^{N_c} \sigma_{b'} = 0 , \quad a' = N_c-p+1,\cdots,N_c .
\end{aligned} \tag{3.2.44}
$$

Let us first assume that all the non-vanishing σ's are positive and that $k + \frac{n_f}{2} \neq 0$. In that case, the second line in (3.2.44) implies that all the non-vanishing σ's are equal:

$$
\sigma_{N_c-p+1} = \sigma_{N_c-p+2} = \cdots = \sigma_{N_c} \equiv \sigma^+ , \tag{3.2.45}
$$

so that (3.2.44) reduces to:

$$
\begin{aligned}
&\sum_{\alpha=1}^{n_f} \phi_a^{\dagger\alpha}\phi_\alpha^b = \frac{\delta_a^b}{2\pi}\left(\xi + pl\sigma^+\right) , \qquad a,b = 1,\cdots,N_c-p ,\\
&F_+(\sigma) \equiv \xi + \left(k + pl + \frac{n_f}{2}\right)\sigma^+ = 0 .
\end{aligned} \tag{3.2.46}
$$

We then have a single Higgs-topological vacuum with the solution:

$$\sigma^+ = -\frac{\xi}{k+pl+\frac{n_f}{2}} > 0 \ , \qquad \text{iff} \ \ \text{sign}(\xi)\left(k+pl+\frac{n_f}{2}\right) < 0 \text{ and } k+\frac{n_f}{2} < 0 \ , \quad (3.2.47)$$

corresponding to a low-energy effective CS theory $U(p)_{k+\frac{n_f}{2},k+\frac{n_f}{2}+pl}$ at every point on the Higgs branch $\text{Gr}(N_c-p,n_f)$ that arises from solving the first set of equations in (3.2.46). The last inequality in (3.2.47) comes from requiring that the volume of the Grassmannian manifold is positive.

There is also a similar solution where all the non-vanishing eigenvalues are chosen to be negative (they are then equal as long as $k-\frac{n_f}{2}\neq 0$), with:

$$\sigma^- = -\frac{\xi}{k+pl-\frac{n_f}{2}} < 0 \ , \qquad \text{iff} \ \text{sign}(\xi)\left(k+pl-\frac{n_f}{2}\right) > 0 \text{ and } k-\frac{n_f}{2} > 0 \ . \quad (3.2.48)$$

In summary, choosing all possible values of p, we have the following Type II,a vacua:

$$\begin{aligned}&\mathcal{M}_{\text{II}}\left[N_c,k,l,n_f,0\right] = \\ &\qquad \Theta\left(-k-\frac{n_f}{2}\right)\bigoplus_{p=1}^{N_c-1}\Theta\left(\xi\left(-k-pl-\frac{n_f}{2}\right)\right)\text{Gr}(N_c-p,n_f)\times U(p)_{k+\frac{n_f}{2},k+\frac{n_f}{2}+pl} \\ &\qquad \oplus\Theta\left(k-\frac{n_f}{2}\right)\bigoplus_{p=1}^{N_c-1}\Theta\left(\xi\left(k+pl-\frac{n_f}{2}\right)\right)\text{Gr}(N_c-p,n_f)\times U(p)_{k-\frac{n_f}{2},k-\frac{n_f}{2}+pl} \ .\end{aligned} \quad (3.2.49)$$

They contribute to the index as:

$$\begin{aligned}&\mathbf{I}_{\text{W},\text{II}}\left[N_c,k,l,n_f,0\right] = \\ &\quad \Theta\left(-k-\frac{n_f}{2}\right)\sum_{p=1}^{N_c-1}\Theta\left(\xi\left(-k-pl-\frac{n_f}{2}\right)\right)\binom{n_f}{N_c-p}\mathbf{I}_{\text{W}}\left[\begin{array}{cc|c} p & l & k+\frac{n_f}{2}\end{array}\right] \\ &\quad +\Theta\left(k-\frac{n_f}{2}\right)\sum_{p=1}^{N_c-1}\Theta\left(\xi\left(k+pl-\frac{n_f}{2}\right)\right)\binom{n_f}{N_c-p}\mathbf{I}_{\text{W}}\left[\begin{array}{cc|c} p & l & k-\frac{n_f}{2}\end{array}\right] \ .\end{aligned} \quad (3.2.50)$$

As we mentioned at the beginning of this section, one should also consider the possibility of having some of the non-vanishing σ's to be positive and the other being negative. In this case, we would get Type II,b vacua $\text{Gr}(N_c-p-q,n_f)\times U(p)\times U(q)$. It turns out, however, that in this case, the conditions for the TQFTs to appear and for the Grassmannian variety to be of positive size are not mutually compatible, thus, there are no such vacua.

Type III vacua. Taking all the σ's to be non-zero, we obtain various topological vacua. For instance, if all the σ_a's are assumed to be positive, then from (3.2.22) we have:

$$F_a(\sigma) = \xi + \left(k + \frac{n_f}{2}\right)\sigma_a + l\sum_{b=1}^{N_c}\sigma_b = 0 \ , \quad a = 1, \cdots, N_c \ . \tag{3.2.51}$$

Any such solution has $\sigma_1 = \cdots = \sigma_{N_c} \equiv \sigma^+$ (assuming $k + \frac{n_f}{2} \neq 0$), and we find:

$$\sigma^+ = -\frac{\xi}{k + lN_c + \frac{n_f}{2}} > 0 \ , \qquad \text{iff} \ \ \text{sign}(\xi)\left(k + lN_c + \frac{n_f}{2}\right) < 0 \ . \tag{3.2.52}$$

Similarly, for $\sigma_a < 0$ (and assuming $k - \frac{n_f}{2} \neq 0$), we obtain a solution:

$$\sigma^- = -\frac{\xi}{k + lN_c - \frac{n_f}{2}} \ , \qquad \text{iff} \ \ \text{sign}(\xi)\left(k + lN_c - \frac{n_f}{2}\right) > 0 \ . \tag{3.2.53}$$

These two solutions exhaust the Type III,a vacua, which are given by:

$$\begin{aligned} \mathcal{M}_{\text{III,a}}[N_c, k, l, n_f, 0] \ &= \ \Theta\left(\xi\left(-k - lN_c - \frac{n_f}{2}\right)\right) U(N_c)_{k+\frac{n_f}{2}, k+\frac{n_f}{2}+lN_c} \\ &\oplus \ \Theta\left(\xi\left(k + lN_c - \frac{n_f}{2}\right)\right) U(N_c)_{k-\frac{n_f}{2}, k-\frac{n_f}{2}+lN_c} \ , \end{aligned} \tag{3.2.54}$$

which contribute to the Witten index as:

$$\begin{aligned} \mathbf{I}_{\text{W, III,a}}[N_c, k, l, n_f, 0] \ &= \ \Theta\left(\xi\left(-k - lN_c - \frac{n_f}{2}\right)\right) \mathbf{I}_{\text{W}}\left[\begin{array}{cc|c} N_c & l & k + \frac{n_f}{2} \end{array}\right] \\ &+ \ \Theta\left(\xi\left(k + lN_c - \frac{n_f}{2}\right)\right) \mathbf{I}_{\text{W}}\left[\begin{array}{cc|c} N_c & l & k - \frac{n_f}{2} \end{array}\right] \ , \end{aligned} \tag{3.2.55}$$

As for the Type III,b solutions, they are the solutions to the following equations:

$$\begin{aligned} &F_a(\sigma) = \xi + \left(k + \frac{n_f}{2}\right)\sigma_a + l\sum_{b=1}^{N_c}\sigma_b = 0 \ , \quad a = 1, \cdots, p \ , \\ &F_{a'}(\sigma) = \xi + \left(k - \frac{n_f}{2}\right)\sigma_{a'} + l\sum_{b}^{N_c}\sigma_b = 0 \ , \quad a' = p+1, \cdots, N_c \ , \end{aligned} \tag{3.2.56}$$

where we took the first p σ's to be of positive sign and the rest to be negative. It again follows that $\sigma_a = \sigma^+$ and $\sigma_{a'} = \sigma^-$ (assuming $|k| \neq \frac{n_f}{2}$), so that (3.2.56) simplifies to:

$$
\begin{aligned}
&\xi + \left(k + pl + \frac{n_f}{2}\right)\sigma^+ + (N_c - p)l\sigma^- = 0 \ , \\
&\xi + \left(k + (N_c - p)l - \frac{n_f}{2}\right)\sigma^- + pl\sigma^+ = 0 \ ,
\end{aligned}
\tag{3.2.57}
$$

and we have a unique solution:

$$
\sigma^+ = -\frac{\xi\left(k - \frac{n_f}{2}\right)}{\mathcal{L}(p, N_c, k, l, n_f)} > 0 \qquad \text{and} \qquad \sigma^- = -\frac{\xi\left(k + \frac{n_f}{2}\right)}{\mathcal{L}(p, N_c, k, l, n_f)} < 0 \ , \tag{3.2.58}
$$

where we defined the quantity:

$$
\mathcal{L}(p, N_c, k, l, n_f) \equiv \left(k + pl + \frac{n_f}{2}\right)\left(k + (N_c - p)l - \frac{n_f}{2}\right) - p(N_c - p)l^2 \ . \tag{3.2.59}
$$

Then, the corresponding Type III,b topological vacua are:

$$
\begin{aligned}
\mathcal{M}_{\text{III,b}}[N_c, k, l, n_f, 0] = \Theta\left(\frac{n_f}{2} - |k|\right) \bigoplus_{p=1}^{N_c-1} \Theta\left(\xi\mathcal{L}(p, N_c, k, l, n_f)\right) \\
\underbrace{U(p)_{k+\frac{n_f}{2}, k+\frac{n_f}{2}+pl} \times U(N_c - p)}_{l}{}_{k-\frac{n_f}{2}, k-\frac{n_f}{2}+(N_c-p)l} \ ,
\end{aligned}
\tag{3.2.60}
$$

and their contributions to the index read:

$$
\begin{aligned}
&\mathbf{I}_{\text{W, III,b}}\left[N_c, k, l, n_f, 0\right] = \\
&\qquad \Theta\left(\tfrac{n_f}{2} - |k|\right) \sum_{p=1}^{N_c-1} \Theta\left(\xi\mathcal{L}(p, N_c, k, l, n_f)\right) \mathbf{I}_{\text{W}} \left[\begin{array}{cc|cc} p & l & k + \frac{n_f}{2} & l \\ N_c - p & l & l & k - \frac{n_f}{2} \end{array}\right] ,
\end{aligned}
\tag{3.2.61}
$$

in terms of the CS index (A.1.12).

Type IV vacua. When $|k| = \frac{n_f}{2}$, semi-classical Coulomb-branch directions open up, rendering our analysis non-reliable. We expect that the actual 'Type IV' vacua are strongly-coupled vacua. As we discussed around (3.2.40) in the $U(2)$ theory case, we can always calculate the contribution of these (conjectured) vacua to the total Witten index by comparing to the Witten index computed with the opposite sign of FI parameter (which does not have any Type IV contributions). In this way, we find the following contributions, in

$k\backslash l$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}$	30	28	26	24	22	**21**	**21**	**21**	**21**	24	27	30	33
$\frac{3}{2}$	34	30	26	23	22	**21**	**21**	**21**	**21**	23	27	31	35
$\frac{5}{2}$	45	40	35	30	25	**21**	**21**	**21**	**21**	**21**	26	31	36
$\frac{7}{2}$	90	75	60	45	30	**21**	**21**	21 +15	21 +30	21 +45	21 +60	21 +75	21 +90
$\frac{9}{2}$	175	140	105	70	35	**21**	56	91	126	161	196	231	266
$\frac{11}{2}$	315	245	175	105	35	56	126	196	266	336	406	476	546
$\frac{13}{2}$	525	399	273	147	56	126	252	378	504	630	756	882	1008
$\frac{15}{2}$	819	609	399	224	91	252	462	672	882	1092	1302	1512	1722
$\frac{17}{2}$	1209	879	584	289	196	462	792	1122	1452	1782	2112	2442	2772
$\frac{19}{2}$	1704	1244	784	324	336	792	1287	1782	2277	2772	3267	3762	4257
$\frac{21}{2}$	2344	1664	984	409	616	1287	2002	2717	3432	4147	4862	5577	6292

Table 3.6: Witten index for $U(5)_{k,k+5l}$ with $n_f = 7$ fundamentals, for some values of k, l. The cases with the minimal value $\mathbf{I}_{\rm W} = 21$ are given in bold. The contributions from Type IV vacua when $\xi > 0$ are shown in red.

general:

$$\mathbf{I}_{\rm W,IV}[N_c, k, l, n_f, 0] = \delta_{k,\frac{n_f}{2}}\Theta(\xi l)|l|\binom{n_f-1}{N_c-1} + \delta_{k,-\frac{n_f}{2}}\Theta(-\xi l)|l|\binom{n_f-1}{N_c-1} \,. \tag{3.2.62}$$

It would be interesting to better understand the physics of these strongly-coupled vacua, but this is left for future work.

The full Witten index. Putting all the above contributions together, we have now computed the full Witten index:

$$\begin{aligned}\mathbf{I}_{\rm W}\left[N_c, k, l, n_f, 0\right] =& \mathbf{I}_{\rm W,I}\left[N_c, k, l, n_f, 0\right] + \mathbf{I}_{\rm W,II}\left[N_c, k, l, n_f, 0\right] \\ &+ \mathbf{I}_{\rm W,III,a}\left[N_c, k, l, n_f, 0\right] + \mathbf{I}_{\rm W,III,b}\left[N_c, k, l, n_f, 0\right] \\ &+ \mathbf{I}_{\rm W,IV}\left[N_c, k, l, n_f, 0\right] \ .\end{aligned} \tag{3.2.63}$$

This result can be compared to the Bethe vacua counting, and one finds perfect agreement for all the cases that we checked.[1]

[1] Our explicit formula gives us the index for any choice of the parameters, while the Bethe vacua counting is limited, in practice, to relatively low values of the parameters due to the slowness of Gröbner bases algorithms.

k	l	$\xi > 0$ phase	$\xi < 0$ phase
$\frac{1}{2}$	8	$\mathrm{Gr}(5,7) \oplus \underbrace{U(2)_{4,36} \times U(3)}_{8}{}_{-3,21}$	$\underbrace{U(3)_{4,28} \times U(2)}_{8}{}_{-3,13} \oplus \underbrace{U(4)_{4,36} \times U(1)}_{8}{}_{5}$
$\frac{3}{2}$	3	$\mathrm{Gr}(5,7) \oplus \underbrace{U(3)_{5,14} \times U(2)}_{3}{}_{-2,4}$	$U(5)_{5,20} \oplus \underbrace{U(4)_{5,17} \times U(1)}_{3}{}_{1}$
$\frac{5}{2}$	2	$\mathrm{Gr}(5,7)$	$U(5)_{6,16} \oplus \underbrace{U(4)_{6,14} \times U(1)}_{2}{}_{1}$
$\frac{9}{2}$	5	$\mathrm{Gr}(5,7) \oplus \mathrm{Gr}(4,7) \times U(1)_6$	$U(5)_{8,33}$
$\frac{11}{2}$	-3	$\mathrm{Gr}(5,7) \oplus U(5)_{9,-6}$	$\mathrm{Gr}(4,7) \times U(1)_{-1} \oplus \mathrm{Gr}(3,7) \times U(2)_{2,-4}$

Table 3.7: Moduli spaces of vacua for $U(5)_{k,k+5l}$ with $n_f = 7$ fundamental chiral multiplets, at some values of k and l and for either sign of ξ.

Example: $U(5)$ **gauge theory with** $n_f = 7$**.** Similarly to our discussion of the $U(2)$ theory with 4 fundamentals in subsection 3.2.4, let us compute the index explicitly in this example – this is shown in table 3.6. We can also consider the explicit form of the vacua for this theory with either choice of sign for ξ, for any given value of the CS levels k and l, as shown in table 3.7. For example, we see that in the case $(k,l) = (\frac{5}{2}, 2)$, we have a pure Higgs branch in the positive-ξ region and topological vacua on the other side. The index matches on both sides according to:

$$\mathbf{I}_{\mathrm{W}}\left[U(5)_{6,16}\right] + \mathbf{I}_{\mathrm{W}}\left[\underbrace{U(4)_{6,14} \times U(1)}_{2}{}_{1}\right] = 16 + 5 = 21 = \chi\left(\mathrm{Gr}(5,7)\right) \ . \tag{3.2.64}$$

Similar considerations hold for all the other examples shown, and examples of arbitrary complexity can be generated using Mathematica. For more examples, see appendix C.

CHAPTER 4

SEIBERG-LIKE DUALITIES FOR 3D SQCD

In this chapter, we study in detail the IR dualities associated with 3d unitary SQCD that we overviewed in subsection 1.2.2. We start with the case for $l = 0$ in section 4.1 where we revisit the different possible dual theories depending on the CS level k and its relation to the 'chirality' parameter:[2]

$$k_c \equiv \frac{n_f - n_a}{2} \ . \tag{4.0.1}$$

The 'dual rank' is given by:

$$N_c^D \equiv \begin{cases} \frac{1}{2}(n_f + n_a) + |k| - N_c \ , & \text{if} \quad |k| \geq |k_c| \ , \\ \max(n_f, n_a) - N_c \ , & \text{if} \quad |k| \leq |k_c| \ . \end{cases} \tag{4.0.2}$$

We fix all possible mixed CS levels on both sides of the dualities by matching the 3d twisted indices using the techniques and results of the previous two chapters. This section is accompanied by appendix B, where we show explicitly how one can derive these dualities via mass deformations starting with Aharony duality.

Meanwhile, in section 4.2, we study the more general case with $l \in \mathbb{Z}$. As we will see, the dual gauge group is more intricate in this case compared to the $l = 0$ one. We argue that these dualities can be derived from the ones with $l = 0$ using the KSW $\mathrm{SL}(2, \mathbb{Z})$ action [93, 94] which we also review in detail for our supersymmetric setting.

Using the semi-classical analysis performed in section 3.2 for $n_a = 0$ case, in section 4.3 we specialise in the case with $n_a = 0$. We extend our semi-classical analysis performed in 3.2 to the other possible dual theories and show the explicit matching of the super-

[2] Using some slight abuse of terminology and following [58], we call the 3d SQCD theory 'chiral' if $n_f \neq n_a$. This is because the 4d analogue of that theory would be chiral in the usual sense. (One similarly talks about the 3d $\mathcal{N} = 2$ chiral multiplet, it being the dimensional reduction of the 4d $\mathcal{N} = 1$ chiral multiplet.)

	$U(N_c^D)$	$SU(N_f)$	$SU(N_f)$	$U(1)_A$	$U(1)_T$	$U(1)_R$
φ_β	$\square$	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$
$\widetilde{\varphi}^\alpha$	$\overline{\square}$	$\square$	$\mathbf{1}$	-1	0	$1-r$
${M_\alpha}^\beta$	$\mathbf{1}$	$\overline{\square}$	$\square$	2	0	$2r$
$\mathfrak{T}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	1	r_T
$\mathfrak{T}^-$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	-1	r_T

Table 4.1: Field content of the Aharony dual theory, with $N_c^D = N_f - N_c$ and r_T given in (4.1.2).

symmetric vacua on both sides. This subsection is accompanied by appendix A where we provide dualities concerning 3d pure Chern–Simons theories needed for this matching of the vacua.

4.1 Infrared dualities for $U(N_c)_k$ SQCD, revisited

In this section we review well-known IR dualities for SQCD$[N_c, k, 0, n_f, n_a]$. We start the discussion with Aharony duality where $k = 0$ and $n_f = n_a = N_f$ and we show how one can RG flow to the other Seiberg-like dualities via mass deformations. More details on this deformation are moved to appendix B.

4.1.1 Aharony duality ($k = l = 0$, $n_f = n_a$)

If we set $k = l = 0$ and $n_f = n_a \equiv N_f$, we have the famous Aharony duality [56]:

$$\boxed{U(N_c)_0\ ,\ N_f(\square\oplus\overline{\square}) \qquad \longleftrightarrow \qquad U(N_f-N_c)_0\ ,\ N_f(\square\oplus\overline{\square})\ ,\ \Big({M_\alpha}^\beta,\ \mathfrak{T}_+,\ \mathfrak{T}_-\Big)}\ . \tag{4.1.1}$$

This can be viewed as the most fundamental infrared duality for $U(N_c)$ SQCD, in the sense that all the other dualities summarised in the introduction above can be derived from Aharony duality using appropriate limits [58, 92]. The matter content and the charges of the Aharony dual theory are shown in table 4.1. We have dual flavours φ_β and $\widetilde{\varphi}^\alpha$ in the fundamental and antifundamental of $U(N_f - N_c)$, respectively. The gauge singlets ${M_\alpha}^\beta$ and $\mathfrak{T}^\pm$ are identified with the gauge-invariant 'mesons' ${M_\alpha}^\beta = \phi_\alpha\widetilde{\phi}^\beta$ and with the monopole operators, respectively, in the electric description. The monopoles have R-charge:

$$r_T = -N_f(r-1) - N_c + 1\ . \tag{4.1.2}$$

The gauge singlets are coupled to the gauge sector through the superpotential $W = \widetilde{\varphi}^\alpha M_\alpha{}^\beta \varphi_\beta + \mathfrak{T}^+ t_+ + \mathfrak{T}^- t_-$, where $t_\pm$ are the gauge-invariant monopole operators of the dual theory.

It is clear that we have $K = \frac{1}{2}(n_f + n_a)$ and $L = 0$ for the $U(N_c^D)$ bare CS levels. Moreover, given our conventions for the electric theory that we introduced in subsection 3.1.1, the magnetic theory must have non-vanishing bare CS levels for the flavour symmetry. We have:

$$K^{(\mathrm{Ah})}_{SU(N_f)} = N_f - N_c \ , \qquad \widetilde{K}^{(\mathrm{Ah})}_{SU(N_f)} = N_f - N_c \ , \tag{4.1.3}$$

for the $SU(N_f) \times SU(N_f)$ flavour symmetry and:

$$\begin{aligned}
K^{(\mathrm{Ah})}_{TT} &= 1 \ , \\
K^{(\mathrm{Ah})}_{AA} &= 4N_f^2 - 2N_c N_f \ , \\
K^{(\mathrm{Ah})}_{AR} &= 2N_f^2 + \left(4N_f^2 - 2N_c N_f\right)(r-1) \ , \\
K^{(\mathrm{Ah})}_{RR} &= N_c^2 + N_f^2 + 4N_f^2(r-1) + \left(4N_f^2 - 2N_c N_f\right)(r-1)^2 \ , \\
K^{(\mathrm{Ah})}_{g} &= 2N_f(N_f - N_c) + 2 \ ,
\end{aligned} \tag{4.1.4}$$

for the abelian flavour symmetry (as well as for the gravitational CS contact term) [105]. All other flavour and gauge-flavour levels vanish except for:

$$K^{(\mathrm{Ah})}_{GT} = -1 \ , \tag{4.1.5}$$

which is the statement that the FI parameter changes sign under the duality – equivalently, the topological currents of the $U(N_c)$ and $U(N_c^D)$ dual gauge groups are identified up to a sign.

In the limiting case $N_f = N_c$, the dual theory consists of a linear σ-model with $N_f^2 + 2$ chiral multiplets $M_\alpha{}^\beta$, $\mathfrak{T}^\pm$ which are coupled through the superpotential $W = \mathfrak{T}^+\mathfrak{T}^- \det(M)$ [53]. Finally, for $N_f < N_c$, we either have a quantum-deformed moduli space (for $N_f = N_c - 1$) or supersymmetry breaking (for $N_f < N_c - 1$) [56].

	$U(N_c^D)$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$	condition
φ_β	$\square$	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$	
$\widetilde{\varphi}^\alpha$	$\overline{\square}$	$\square$	$\mathbf{1}$	-1	0	$1-r$	
${M_\alpha}^\beta$	$\mathbf{1}$	$\overline{\square}$	$\square$	2	0	$2r$	
$\mathfrak{T}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	1	r_T	$k = \frac{1}{2}(n_f - n_a)$
$\mathfrak{T}^-$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	-1	r_T	$k = -\frac{1}{2}(n_f - n_a)$

Table 4.2: Field content of the infrared dual of unitary SQCD with $l = 0$. The gauge singlets $\mathfrak{T}^\pm$ only appear in the marginally chiral case (or in the Aharony dual), as indicated. Here $N_f \equiv N_c + N_c^D$.

4.1.2 Minimally chiral duality with $l = 0$

Next, let us consider the case $k \neq 0$ with $l = 0$ and with the 'minimally chiral' condition $|k| > |k_c|$. We have a duality [57, 58]:

$$\boxed{U(N_c)_k \ , \ \Big(n_f\, \square, n_a\, \overline{\square}\Big) \qquad \longleftrightarrow \qquad U(N_c^D)_{-k} \ , \ \Big(n_a\, \square, n_f\, \overline{\square}\Big) \ , \ \Big({M_\alpha}^\beta\Big) \ .} \tag{4.1.6}$$

with:

$$N_c^D = \frac{1}{2}(n_f + n_a) + |k| - N_c \ . \tag{4.1.7}$$

Now the singlet sector only consists of the 'mesons' ${M_\alpha}^\beta$, with the standard superpotential $W = \widetilde{\varphi}^\alpha {M_\alpha}^\beta \varphi_\beta$ as in 4d Seiberg duality [163]. The matter content is shown in table 4.2. There are also mixed gauge-flavour bare CS levels:

$$K_{GT} = -1 \ , \qquad K_{GA} = \Theta(k)(n_f - n_a) \ , \qquad K_{GR} = \Theta(k)(n_f - n_a)(r - 1) \ . \tag{4.1.8}$$

Finally and importantly, we have the bare flavour CS levels shown in table 4.3. Note that the bare CS levels depend on the sign of k, and that changing the sign of k does not simply change the sign of the bare CS levels. This is because of our parity-violating conventions, of course (see subsection 2.2.2). For any given theory with the '$U(1)_{-\frac{1}{2}}$ quantisation' convention, a parity transformation changes the sign of the physical contact terms $\kappa = \kappa^\Phi + K$ as $\kappa \to -\kappa$ but the UV 'matter' contribution κ^Φ remains the same by convention, as defined in (2.2.30)-(2.2.31), hence a parity transformation changes the bare CS levels according to:

$$\mathrm{P} \ : \ K \to -K - 2\kappa^\Phi \ . \tag{4.1.9}$$

	$k > \|k_c\|$	$k \leq -\|k_c\|$
$K_{SU(n_f)}$	N_c^D	$-N_c + n_a$
$K_{SU(n_a)}$	N_c^D	$-N_c + n_f$
K_{AA}	$(n_f + n_a)N_c^D$	$4n_f n_a - (n_f + n_a)N_c$
K_{TT}	-1	1
K_{AT}	0	0
K_{RA}	$(n_f + n_a)N_c^D r$	$(n_a + n_f)N_c - 2n_f n_a$ $+(4n_f n_a - (n_f + n_a)N_c)r$
K_{RT}	0	0
K_{RR}	$(-N_c^D + (n_f + n_a)r^2)N_c^D$	$(N_c - n_f)(N_c - n_a)$ $+r^2(4n_f n_a - (n_f + n_a)N_c)$ $+2r((n_a + n_f)N_c - 2n_f n_a)$
K_g	$(n_f + n_a - 2k)N_c^D - 2$	$2n_f n_a - (n_f + n_a + 2k)N_c + 2$

Table 4.3: Flavour bare CS levels for the minimally-chiral $U(N_c^D)_{-k}$ gauge theory ($l = 0$), as well as for the marginally chiral case with $k = -|k_c| < 0$.

Now, if we consider the fact that we chose $K_F = 0$ for all the flavour bare CS levels in the 'electric' theory (3.1.8), $K_F^{(e)} = 0$, irrespective of the sign of k, we find that, in the magnetic theory:

$$K_F^{(m)}\Big|_{k \to -k} = -K_F^{(m)} - 2\kappa_F^{\Phi(m)} + 2\kappa_F^{\Phi(e)} \; , \tag{4.1.10}$$

where $\kappa_F^{\Phi(e)}$ and $\kappa_F^{\Phi(m)}$ denote the matter contributions in the electric and magnetic theories, respectively. This gives us the relation between the two columns in table 4.3.[1]

This minimally chiral duality, including all the flavour CS levels, can be derived from the Aharony duality by integrating out flavours [58], as we review in appendix B.

4.1.3 Marginally chiral duality with $l = 0$

Consider now the 'marginally chiral' cases with $|k| = |k_c|$ and $l = 0$, with k_c defined in (4.0.1). In this case, the dual gauge theory is coupled to another gauge singlet in addition to the mesons, corresponding to the single gauge-invariant monopole operator $\mathfrak{T}^+$ or $\mathfrak{T}^-$ in the electric theory (for $k = k_c$ or $k = -k_c$, respectively), as indicated in table 4.2. We

[1] For instance, consider K_{AA}. We have $\kappa_{AA}^{\Phi(e)} = -\frac{1}{2}(n_f + n_a)N_c$ according to (3.1.9), and $\kappa_{AA}^{\Phi(m)} = -\frac{1}{2}(n_f + n_a)N_c - 2n_f n_a$ in the magnetic theory (including the contribution from the mesons). Then the relation (4.1.10) is indeed satisfied by the levels given in table 4.3.

	$k=\|k_c\|\ ,\ \ n_f>n_a$	$k=\|k_c\|\ ,\ \ n_a>n_f$
$K_{SU(n_f)}$	N_c^D	N_c^D
$K_{SU(n_a)}$	N_c^D	N_c^D
K_{AA}	$n_f^2+(n_f+n_a)N_c^D$	$n_a^2+(n_f+n_a)N_c^D$
K_{TT}	0	0
K_{AT}	$-n_f$	n_a
K_{RA}	$-N_c^D n_f+r(n_f^2+(n_f+n_a)N_c^D)$	$-N_c^D n_a+r(n_a^2+(n_f+n_a)N_c^D)$
K_{RT}	$N_c^D-rn_f$	$-N_c^D+rn_a$
K_{RR}	$-2rN_c^D n_f$ $+r^2(n_f^2+(n_f+n_a)N_c^D)$	$-2rN_c^D n_a$ $+r^2(n_a^2+(n_f+n_a)N_c^D)$
K_g	$2n_a N_c^D$	$2n_f N_c^D$

Table 4.4: Flavour bare CS levels for the marginally chiral dual with $k=|k_c|>0$ ($l=0$).

then have the duality [58]:

$$\boxed{U(N_c)_k\ ,\ \Big(n_f\,\square, n_a\,\overline{\square}\Big)\qquad\longleftrightarrow\qquad U(N_c^D)_{-k}\ ,\ \Big(n_a\,\square, n_f\,\overline{\square}\Big)\ ,\ \Big({M_\alpha}^\beta, \mathfrak{T}^\epsilon\Big)\ ,} \tag{4.1.11}$$

with a dual superpotential $W=\tilde{\varphi}^\alpha {M_\alpha}^\beta \varphi_\beta+\mathfrak{T}^\epsilon t_\epsilon$ for $\epsilon=\pm$. Note that:

$$N_c^D=\frac{1}{2}(n_f+n_a)+|k|-N_c=\max(n_f,n_a)-N_c\ , \tag{4.1.12}$$

in this case. Note also that the R-charge of $\mathfrak{T}^\epsilon$ is given by:

$$r_T=-(N_c^D+N_c)(r-1)-N_c+1\ . \tag{4.1.13}$$

The mixed gauge-flavour CS levels are the same as in (4.1.8). The flavour bare CS levels have to be carefully determined by real-mass deformation from Aharony duality, like for the minimally chiral case (see appendix B). For $k=|k_c|>0$, those levels are given in table 4.4, while for $k=-|k_c|$ they were given in table 4.3.

4.1.4 Maximally chiral duality with $l=0$

Finally, we have the maximally chiral case when $|k|<|k_c|$ and $l=0$. The general form of the duality is the same as in for minimally chiral case (4.1.6), with the matter content

	$\lvert k\rvert < \lvert k_c\rvert~,~~ n_f > n_a$	$\lvert k\rvert < \lvert k_c\rvert~,~~ n_a > n_f$
$K_{SU(n_f)}$	$k + \frac{1}{2}(n_f + n_a) - N_c$	$n_a - N_c$
$K_{SU(n_a)}$	$n_f - N_c$	$k + \frac{1}{2}(n_f + n_a) - N_c$
K_{AA}	K^+_{AA}	K^-_{AA}
K_{TT}	0	0
K_{AT}	$-n_f$	n_a
K_{RA}	$K^{(0)+}_{RA} + rK^+_{AA}$,	$K^{(0)-}_{RA} + rK^-_{AA}$,
K_{RT}	$N^D_c - rn_f$	$-N^D_c + rn_a$
K_{RR}	$N^D_c(k - k_c) + 2rK^{(0)+}_{RA} + r^2K^+_{AA}$	$N^D_c(k + k_c) + 2rK^{(0)-}_{RA} + r^2K^-_{AA}$
K_g	$2n_aN^D_c$	$2n_fN^D_c$

Table 4.5: Flavour bare CS levels for the maximally chiral dual, $|k| < |k_c|$ ($l = 0$). The explicit forms for K^+_{AA}, K^-_{AA}, $K^{(0)+}_{RA}$ and $K^{(0)-}_{RA}$ are given in (4.1.16).

of table 4.2, but with the dual rank given by:

$$N^D_c = \max(n_f, n_a) - N_c~, \tag{4.1.14}$$

and with the mixed gauge-flavour CS levels:

$$K_{GT} = -1~, \qquad K_{GA} = \mathrm{sign}(k_c)(k+|k_c|)~, \qquad K_{GR} = \mathrm{sign}(k_c)(k+|k_c|)(r-1)~. \tag{4.1.15}$$

Finally, we have the flavour bare CS levels given in table 4.5, with:

$$\begin{aligned} K^+_{AA} &\equiv n_f^2 + 3n_f(k - k_c) + 2N^D_c(n_f - k_c)~, \\ K^-_{AA} &\equiv n_a^2 + 3n_a(k + k_c) + 2N^D_c(n_a + k_c)~, \\ K^{(0)+}_{RA} &\equiv -(k - k_c)(n_f + N^D_c) + n_fN^D_c~, \\ K^{(0)-}_{RA} &\equiv -(k + k_c)(n_a + N^D_c) + n_aN^D_c~. \end{aligned} \tag{4.1.16}$$

One can check the matching of partition functions across these dualities explicitly. Given the precise definition of the dual theories, including all the bare CS levels, one can apply the formalism of chapter 2 to verify that the twisted indices of dual theories exactly agree:

$$Z_{\Sigma_g\times S^1}[\text{SQCD}] = Z_{\Sigma_g\times S^1}[\text{dual SQCD}]~. \tag{4.1.17}$$

The proof of this equality for Aharony duality in the '$U(1)_{-\frac{1}{2}}$ quantisation' was given in [105], building on previous works [113, 114, 115, 151], and the equality of twisted indices for the other SQCD theories with $l = 0$ then follows from standard RG flow arguments. Here, our focus was instead on computing the index on both sides explicitly and the fact that (4.1.17) indeed holds in many examples [1] is a nice check of our formalism. Moreover, for unitary SQCD with $l \neq 0$, which we study in the next section, no general proof of this equality is available so far.

4.1.5 Special cases: abelian dualities

Let us briefly discuss a few special cases with $N_c = 1$ where the dual theory consists of chiral multiplets only. These 'elementary' dualities will be particularly useful in the next section.

The SQED/XYZ duality. Consider the $U(1)_0$ theory with $n_f = n_a = 1$. It has a dual description in terms of three chiral multiplets M, $\mathfrak{T}^{\pm}$ with the superpotential [53]:

$$W = \mathfrak{T}^+ \mathfrak{T}^- M \ , \tag{4.1.18}$$

also known as the XYZ model (with $X = M$, $Y = \mathfrak{T}^+$, $Z = \mathfrak{T}^-$). This can be viewed as a limiting case of Aharony duality (4.1.1). This theory has a flavour symmetry $U(1)_T \times U(1)_A$ and a R-symmetry. In our conventions, we have the following flavour CS contact terms in the dual description:

$$\begin{aligned} &K_{TT} = 1 \ , && K_{TA} = K_{TR} = 0 \ , \\ &K_{AA} = 2 \ , && K_{AR} = 2r \ , \\ &K_{RR} = 2r^2 \ , && K_g = 2 \ . \end{aligned} \tag{4.1.19}$$

as a limiting case of (4.1.4).

The $U(1)_{\pm 1}$ CS theory. Consider the 3d $\mathcal{N} = 2$ CS theory $U(1)_1$ without matter fields. This is the 'almost trivial theory' studied in [94]: it is dual to an invertible theory (*i.e.* a trivial theory with CS contact terms):

$$U(1)_1 \qquad \longleftrightarrow \qquad K_{TT} = -1 \ , \ \ K_{RT} = 0 \ , \ \ K_{RR} = 0 \ , \ \ K_g = -2 \ . \tag{4.1.20}$$

[1] On a laptop computer, we can check the matching of indices across dualities for most gauge theories with rank up to 3 and with the parameters k, l, n_f, n_a small enough.

This is a special case of the Giveon–Kutasov duality [57], and these CS contact terms are obtained by setting $N_c = n_a = n_f = k = 1$ in table 4.3. For the other sign of the Chern–Simons level, we similarly find:

$$U(1)_{-1} \qquad \longleftrightarrow \qquad K_{TT} = 1 \ , \ \ K_{RT} = 0 \ , \ \ K_{RR} = 1 \ , \ \ K_g = 4 \ . \tag{4.1.21}$$

The $U(1)_{\pm\frac{1}{2}}$ theory coupled to one chiral flavour. Let us consider the $U(1)_{\frac{1}{2}}$ theory coupled to one chiral multiplet $\Phi_\pm$ of electric charge ± 1 and R-charge r, with $k = \frac{1}{2}$. This theory has a flavour symmetry $U(1)_T$, and it is dual to a free chiral multiplet $\mathfrak{T}^\pm$ of $U(1)_T$ charge ± 1 and R-charge $1 - r$, with the following CS contact terms:

$$U(1)_{\frac{1}{2}} \ , \ \Phi_\pm \ , \ K_{GR} = 0 \quad \longleftrightarrow \quad \mathfrak{T}^\pm \ , \quad \begin{cases} K_{TT} = 0 \ , & K_{RT} = \mp r \ , \\ K_{RR} = r^2 \ , & K_g = 0 \ . \end{cases} \tag{4.1.22}$$

Note that, in our conventions, we have a bare CS level $K_{GG} = 1$ for the $U(1)$ gauge group on the 'electric' side of the duality. With the opposite sign for the UV CS level, $k = -\frac{1}{2}$, we find instead:

$$U(1)_{-\frac{1}{2}} \ , \ \Phi_\pm \ , \ K_{GR} = \pm(r-1) \quad \longleftrightarrow \quad \mathfrak{T}^\mp \ , \quad \begin{cases} K_{TT} = 1 \ , & K_{RT} = 0 \ , \\ K_{RR} = -r^2 + 2r \ , & K_g = 2 \ . \end{cases} \tag{4.1.23}$$

These well-known dualities [172] are limiting cases of the marginally chiral dualities reviewed in section 4.1.3.

4.2 Infrared dualities for $U(N_c)_{k,k+lN_c}$ SQCD

In this section, we discuss the infrared dualities for unitary SQCD with a general value of l. These dualities were first discovered by Nii for $n_a = n_f$ and $k \neq 0$ [76]. They were further generalised by Amariti and Rota [92], who argued that the dualities with $l \neq 0$ can be easily derived from the $l = 0$ dualities by using Kapustin–Strassler and Witten's $\mathrm{SL}(2, \mathbb{Z})$ action on 3d field theories with abelian symmetries [93, 94]. We elaborate on this construction in the following.

4.2.1 $\mathrm{SL}(2,\mathbb{Z})$ action and the 3d A-model

Let us first discuss the Kapustin–Strassler–Witten $\mathrm{SL}(2,\mathbb{Z})$ transformations in the language of the 3d A-model. We will then rederive all the $l \neq 0$ dualities by an appropriate $\mathrm{SL}(2,\mathbb{Z})$ transformation of the $l = 0$ dualities summarised in the previous section. We again pay particular attention to deriving the exact bare CS levels in all cases.

Let us consider some 3d $\mathcal{N} = 2$ supersymmetric gauge theory $\mathcal{T}$ with a $U(1)_f$ flavour symmetry, and the associated 3d A-model determined by the twisted superpotential $\mathcal{W}(u,\nu)$ and the effective dilaton $\Omega(u,\nu)$, with ν the $U(1)_f$ chemical potential (*i.e.* the 2d twisted mass). Here, u_a denotes gauge parameters for dynamical vector multiplets, and the remaining flavour parameters are left implicit. The $\mathrm{SL}(2,\mathbb{Z})$ action sends $\mathcal{T}$ to another field theory $g[\mathcal{T}]$ for any $g \in \mathrm{SL}(2,\mathbb{Z})$. Let $\mathbf{S}$ and $\mathbf{T}$ denote the two standard generators of $\mathrm{SL}(2,\mathbb{Z})$, with:

$$\mathbf{S}^2 = \mathbf{C} \;, \qquad (\mathbf{ST})^3 = \mathbf{C} \;, \qquad \mathbf{C}^2 = \mathbf{1} \;, \tag{4.2.1}$$

where $\mathbf{C}$ is the central element generating $\mathbb{Z}_2 \subset \mathrm{SL}(2,\mathbb{Z})$. These actions generate new field theories with the same number of $U(1)$ symmetries. They act on $\mathcal{T}$ as follows:

(i) $\mathbf{S} : \mathcal{T} \to \mathbf{S}[\mathcal{T}] \equiv \mathcal{T}/U(1)_f$ corresponds to gauging the abelian symmetry $U(1)_f$ with a 3d $\mathcal{N} = 2$ vector multiplet. The new $U(1)_f$ dynamical field strength $F_f = dA_f$ gives us the conserved current of a new topological symmetry, denoted by $U(1)_{f'}$, which we couple to a background $U(1)_{f'}$ multiplet with a supersymmetric f-f' mixed CS level, also known as a 3d BF term:

$$-\frac{i}{2\pi}\int \left(A_{f'} \wedge F_f + \cdots\right) \;, \tag{4.2.2}$$

where the ellipsis denotes the supersymmetric completion. At the level of the 3d A-model, we rename ν as v to indicate that it is now a gauge parameter, and the new coupling (4.2.2) appears as a quadratic term in the new twisted superpotential:

$$\mathcal{W}(u,\nu) \xrightarrow{\mathbf{S}} \mathcal{W}(u,v) - \nu' v \;, \qquad \Omega(u,\nu) \xrightarrow{\mathbf{S}} \Omega(u,v) \;. \tag{4.2.3}$$

with ν' the $U(1)_{f'}$ parameter. We then simply add a new equation for the $U(1)_f$ gauge symmetry to the Bethe equations:

$$\left\{\Pi_a \equiv e^{2\pi i \frac{\partial \mathcal{W}}{\partial u_a}} = 1\right\} \qquad \xrightarrow{\mathbf{S}} \qquad \left\{\Pi_a = 1 \;,\; \Pi_v \equiv e^{2\pi i\left(\frac{\partial \mathcal{W}}{\partial v} - \nu'\right)} = 1\right\} \;. \tag{4.2.4}$$

(ii) $\mathbf{T}\,:\,\mathcal{T}\to\mathbf{T}[\mathcal{T}]$ corresponds to shifting the $U(1)_f$ CS contact term by 1:

$$\kappa_{ff}\to\kappa_{ff}+1\;, \tag{4.2.5}$$

by adding a level-1 3d $\mathcal{N}=2$ supersymmetric Chern–Simons interaction for the $U(1)_f$ background vector multiplet to the action:

$$S\to S+\frac{i}{4\pi}\int\left(A_f\wedge dA_f+\cdots\right)\;. \tag{4.2.6}$$

In the 3d A-model, we then have:

$$\mathcal{W}(u,\nu)\xrightarrow{\mathbf{T}}\mathcal{W}(u,\nu)+\frac{1}{2}(\nu^2+\nu)\;,\qquad\qquad \Omega(u,\nu)\xrightarrow{\mathbf{T}}\Omega(u,\nu)\;. \tag{4.2.7}$$

(iii) The central element $\mathbf{C}$ acts as a sign flip on the $U(1)_f$ current and its superpartners, which is equivalent to a sign flip of the $U(1)_f$ background vector multiplet. Thus, in the 3d A-model:

$$\mathcal{W}(u,\nu)\xrightarrow{\mathbf{C}}\mathcal{W}(u,-\nu)\;,\qquad\qquad \Omega(u,\nu)\xrightarrow{\mathbf{C}}\Omega(u,-\nu)\;. \tag{4.2.8}$$

It is interesting to verify the $\text{SL}(2,\mathbb{Z})$ relations (4.2.1) directly in the 3d A-model formalism. We use the fact that, when a gauge field A_0 only appears linearly through a 3d BF term,

$$S_0=\frac{i}{2\pi}\sum_{i\neq 0}K_{0i}\int A_0\wedge dA_i\;, \tag{4.2.9}$$

the path integral over A_0 gives us a functional Dirac δ-function [94]:

$$\int[dA_0]e^{-S_0}=\delta\Big(\sum_{i\neq 0}K_{0i}A_i\Big)\;, \tag{4.2.10}$$

and similarly in the 3d $\mathcal{N}=2$ supersymmetric context. Now, consider the action:

$$\mathbf{S}^2\;:\;\mathcal{W}(\nu)\longrightarrow\mathcal{W}(\nu'')=\mathcal{W}(v)-v'v-\nu''v'\;. \tag{4.2.11}$$

Here, the path integral over the v' vector multiplet gives us $\delta(v+\nu'')$, schematically speaking, and therefore, we obtain the original theory with a sign flip of ν, as expected:

$$\mathcal{W}(u,\nu) \overset{\mathbf{S}^2=\mathbf{C}}{\longrightarrow} \mathcal{W}(u,-\nu'') \ , \tag{4.2.12}$$

up to a slight subtlety to be discussed momentarily. To compute $(\mathbf{ST})^3$, note that we have:

$$\mathbf{ST} \ : \ \mathcal{W}(u,\nu) \longrightarrow \mathcal{W}(u,v) + \frac{1}{2}v(v+1) - \nu' v \ , \tag{4.2.13}$$

and therefore:

$$\begin{aligned} \mathcal{W}(u,\nu) \overset{(\mathbf{ST})^3}{\longrightarrow} \mathcal{W}(u,v) + \frac{1}{2}v(v+1) + \frac{1}{2}v'(v'+1) + \frac{1}{2}v''(v''+1) \\ - v'v - v''v' - \nu'''v'' \ . \end{aligned} \tag{4.2.14}$$

After performing a change of variable $v' \to v' + v'' + \nu'''$, integrating out v'' gives us $\delta(v+\nu''')$, and we obtain:

$$\mathcal{W}(u,\nu) \overset{(\mathbf{ST})^3}{\longrightarrow} \mathcal{W}^{\mathcal{T}}(u,-\nu''') + \frac{1}{2}v'(v'+1) + 2\nu'''(\nu'''+v') \ . \tag{4.2.15}$$

The (sign-flipped) original theory is now tensored with a decoupled topological sector, which is an 'almost trivial' theory [94], namely a $U(1)_1$ CS theory. Indeed, the additional Bethe equation for v' in (4.2.15) is decoupled from the other Bethe equations of the full theory, and it has a unique solution.

4.2.2 From $U(N_c)_k$ to $U(N_c)_{k,k+lN_c}$ SQCD

The SL$(2,\mathbb{Z})$ action discussed above allows us to generate a non-zero CS level l starting from a $U(N_c)_k$ gauge theory, as we now explain.

$\mathbf{S}$ and $\mathbf{S}^{-1}$ on 3d $\mathcal{N}=2$ supersymmetric theories. When acting with $\mathbf{S}$ on $\mathcal{T}$, we introduce a new abelian vector multiplet. It contains a single gaugino, which shifts some of CS contact terms in the UV according to:

$$\kappa_{RR} \overset{\mathbf{S}}{\to} \kappa_{RR} + \frac{1}{2} \ , \qquad \kappa_g \overset{\mathbf{S}}{\to} \kappa_g + 1 \ , \tag{4.2.16}$$

in our conventions. Thus, more precisely, the action of $\mathbf{S}^2$ on $\mathcal{T}$ actually gives us:

$$\mathcal{W}(u,\nu) \overset{\mathbf{S}^2=\mathbf{C}}{\longrightarrow} \mathcal{W}(u,-\nu'') + \frac{1}{12} \;, \qquad \Omega(u,\nu) \overset{\mathbf{S}^2=\mathbf{C}}{\longrightarrow} \Omega(u,-\nu'') + \frac{1}{2} \;, \tag{4.2.17}$$

which includes the shifts $K_g \to K_g+2$ and $K_{RR} \to K_{RR}+1$. By a slight abuse of notation, let us then define an inverse operation:

$$\mathbf{S}^{-1} \equiv \delta(K) \circ \mathbf{C} \circ \mathbf{S} \;. \tag{4.2.18}$$

It consists of the naive inverse, $\mathbf{C} \circ \mathbf{S}$, combined with a shift of the bare CS levels:

$$\delta(K) \;:\; K_{RR} \to K_{RR} - 1 \;, \qquad K_g \to K_g - 2 \;, \tag{4.2.19}$$

so that $\mathbf{S}^{-1}\mathbf{S}$ is truly the identity on $\mathcal{T}$.

The $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ action on $U(N_c)_k$ SQCD. Let us now start with $\mathcal{T}$ being SQCD with $l = 0$. We can obtain the $l \neq 0$ theory by acting with $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ on the topological symmetry $U(1)_T$ of the $l = 0$ theory. Indeed, at the level of the A-model, let τ denote the $U(1)_T$ parameter and ν the other flavour parameters. Let us also decompose the gauge parameters u_a as:

$$u_a = \widetilde{u}_a + u_0 \;, \qquad \sum_{a=1}^{N_c} \widetilde{u}_a = 0 \;. \tag{4.2.20}$$

We have that the twisted superpotential is linear in τ:

$$\mathcal{W}(u,\nu,\tau) = \mathcal{W}_0(u,\nu) + \tau N_c u_0 \;, \tag{4.2.21}$$

and that $\Omega = \Omega(u,\nu)$ is τ-independent. That is, all the flavour CS levels $K_{T\alpha}$ and K_{RT} vanish, with τ coupling to the gauge symmetry with $K_{GT} = 1$. To act with $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$, we first render $U(1)_T$ dynamical, relabelling $\tau \to v$, and we introduce w the flavour parameter for the new topological symmetry. We add a level-l for the latter, before gauging it with $\mathbf{S}^{-1}$, and we call the new abelian flavour symmetry $U(1)_T$ again, with a new parameter τ. Thus, we have:

$$\mathcal{W}(u,\nu,\tau) \overset{\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}}{\longrightarrow} \mathcal{W}_0(u,\nu) + vN_c u_0 - wv + \frac{l}{2}w(w+1) + \tau w \;. \tag{4.2.22}$$

The vector multiplet for v only appears linearly, hence, we can integrate it out, which leads to a δ-function constraint $w = N_c u$, and we then obtain precisely the general SQCD theory:

$$\mathcal{W}(u,\nu,\tau) \overset{\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}}{\longrightarrow} \mathcal{W}(u,\nu,\tau) + \frac{l}{2} N_c u_0 (N_c u_0 + 1) \ . \tag{4.2.23}$$

This action only introduced the $l \neq 0$ CS term, and it did not change any of the flavour CS levels thanks to the definition (4.2.18)-(4.2.19).

4.2.3 Amariti–Rota duality ($k = 0$, $n_f = n_a$)

To obtain the dual descriptions of SQCD with generic l, we can simply act with $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ on the dual descriptions reviewed in the previous section. Let us start with the case of $k = 0$ and $n_f = n_a \equiv N_f$. The dual description at $l = 0$ is the Aharony magnetic theory discussed below (4.1.1). At the level of the 3d A-model, the $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ action on the Aharony dual theory gives us:

$$\begin{aligned} \mathcal{W} \ &= \ \mathcal{W}_0 + \frac{1}{(2\pi i)^2}\left(\mathrm{Li}_2(z y_A^{-N_f}) + \mathrm{Li}_2(z^{-1} y_A^{-N_f})\right) - v N_c^D u_0 + \frac{1}{2} v(v+1) - v w \\ &\quad + \frac{l}{2} w(w+1) + \tau w \ , \end{aligned} \tag{4.2.24}$$

where we renamed τ to v (and $q = e^{2\pi i \tau}$ to $z = e^{2\pi i v}$), $\mathcal{W}_0$ is v-independent, and we used the same notation as in (4.2.20) for the dual gauge group $U(N_c^D)$. This new theory contains a subsector that is isomorphic to SQED. Indeed, we have:

$$\begin{aligned} &\mathcal{W} = \mathcal{W}_0 + \frac{l}{2} w(w+1) + \tau w + \mathcal{W}_{\mathrm{SQED}} \ , \\ &\mathcal{W}_{\mathrm{SQED}} \equiv \frac{1}{(2\pi i)^2}\left(\mathrm{Li}_2(z y_A^{-N_f}) + \mathrm{Li}_2(z^{-1} y_A^{-N_f})\right) + \frac{1}{2} v(v+1) + \widetilde{\tau} v \ , \end{aligned} \tag{4.2.25}$$

with $\widetilde{\tau} \equiv -w - N_c^D u_0$. Using the SQCD/$XYZ$ duality reviewed in section 4.1.5, we can integrate out the vector multiplet for v, and we obtain:

$$\begin{aligned} \mathcal{W}_{\mathrm{SQED}} \ \leftrightarrow \ & \frac{1}{(2\pi i)^2}\left(\mathrm{Li}_2(y_A^{-2N_f}) + \mathrm{Li}_2(y_A^{N_f} x_0 x_{(w)}) + \mathrm{Li}_2(y_A^{N_f} x_0^{-1} x_{(w)}^{-1})\right) \\ & + N_f^2 \nu_A(\nu_A + 1) + \frac{1}{2} w(w+1) + \frac{1}{2} N_c^D u_0 (N_c^D u_0 + 1) + N_c^D w u_0 \\ & + \frac{1}{12} \ , \end{aligned} \tag{4.2.26}$$

	$U(N_c^D)$	$U(1)^{(w)}$	$SU(N_f)$	$SU(N_f)$	$U(1)_A$	$U(1)_T$	$U(1)_R$
φ_β	$\square$	0	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$
$\widetilde{\varphi}^\alpha$	$\overline{\square}$	0	$\square$	$\mathbf{1}$	-1	0	$1-r$
${M_\alpha}^\beta$	$\mathbf{1}$	0	$\overline{\square}$	$\square$	2	0	$2r$
$\mathcal{B}_+$	$\det^{+1}$	1	$\mathbf{1}$	$\mathbf{1}$	N_f	0	$-r_T+1$
$\mathcal{B}_-$	$\det^{-1}$	-1	$\mathbf{1}$	$\mathbf{1}$	N_f	0	$-r_T+1$
X	$\mathbf{1}$	0	$\mathbf{1}$	$\mathbf{1}$	$-2N_f$	0	$2r_T$

Table 4.6: Field content of the Amariti–Rota dual theory, with $N_c^D = N_f - N_c$. Here $\det^{\pm 1}$ denotes the one-dimensional representation of $U(N_c^D)$ with weight $\rho = (\pm 1, \cdots, \pm 1)$, and $r_T = -N_f(r-1) - N_c + 1$ like in the Aharony dual theory.

and similarly for the effective dilaton. Here $x_0 = e^{2\pi i u_0}$ and $x_{(w)} = e^{2\pi i w}$. The local application of the duality shifts various flavour CS terms, as dictated by (4.1.19), and one must also take into account the shift (4.2.19). In total, only the K_{AR} and K_{RR} CS contact terms incur a shift $K \to K + \Delta K$ with respect to the Aharony dual theory, with:

$$\Delta K_{AR} = -N_f(2 + 2(r_T - 1)) , \qquad \Delta K_{RR} = 4(r_T - 1) + 2(r_T - 1)^2 + 1 . \qquad (4.2.27)$$

Moreover, the new topological symmetry $U(1)_T$ corresponds to the magnetic flux of the new $U(1)^{(w)}$ gauge group (with vector multiplet w), and the new FI term τ enters as in (4.2.25), thus we have $K_{Tw} = 1$. Proceeding in this way, we obtain the Amariti–Rota duality [92]:

$$\boxed{\begin{array}{ccc} & & U\underbrace{(N_f - N_c)_{0,0} \times U(1)}_{0}{}_l^{(w)} , \\ U(N_c)_{0, l N_c} ,\ N_f(\square \oplus \overline{\square}) & \longleftrightarrow & \\ & & N_f(\square \oplus \overline{\square}) ,\ \left({M_\alpha}^\beta, \mathcal{B}_+, \mathcal{B}_-, X\right) . \end{array}} \qquad (4.2.28)$$

The dual matter fields and their charges are given in table 4.6. In addition to the dual flavours and mesonic fields, we have an additional singlet X as well as the 'baryonic' fields $\mathcal{B}_\pm$ charged under both $U(1)$ factors of the gauge group. The gauge singlets are coupled to the gauge sector through the superpotential:

$$W = \widetilde{\varphi}^\alpha {M_\alpha}^\beta \varphi_\beta + \mathcal{B}_+ \mathcal{B}_- X . \qquad (4.2.29)$$

Note that the $U(1)^{(w)}$ gauge group has an UV effective CS level l, which corresponds to $K_{ww} = l+1$. Similarly, the effective mixed CS level between the two gauge groups vanishes, $k_{G_0 w} = 0$, but we have a bare CS level $K_{G_0 w} = 1$ as shown in (4.2.26). Finally, the flavour bare CS levels for the Amariti–Rota dual are slightly different from the ones of the Aharony dual theory (4.1.4) due to (4.2.27). We have:

$$
\begin{aligned}
K^{(\mathrm{AR})}_{SU(N_f)} &= \widetilde{K}^{(\mathrm{AR})}_{SU(N_f)} = N_c^D \ , \\
K^{(\mathrm{AR})}_{AA} &= 2N_f \left(N_c^D + 2N_f\right) \ , \\
K^{(\mathrm{AR})}_{AR} &= -2N_f \left(2N_c^D + 1\right) + 2N_f \left(N_c^D + 2N_f\right) r \ , \\
K^{(\mathrm{AR})}_{RR} &= 1 + N_c^D(3N_c^D + 4) - 4N_f \left(2N_c^D + 1\right) r + 2N_f \left(N_c^D + 2N_f\right) r^2 \ , \\
K^{(\mathrm{AR})}_{g} &= 2N_f N_c^D + 2 \ ,
\end{aligned}
\tag{4.2.30}
$$

with all other flavour CS levels vanishing.

4.2.4 Minimally chiral duality with general l

Let us now turn on the CS level k. We use the notation $k_c \equiv \frac{1}{2}(n_f - n_a)$ as before, and we first consider the minimally chiral case, namely the case $|k| > |k_c|$ with $k \neq 0$. The dual theory is obtained by an $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ action on the dual theory of section 4.1.2. In the 3d A-model, this gives us:

$$
\mathcal{W} = \mathcal{W}_0 - vN_c^D u_0 - \frac{\mathrm{sign}(k)}{2} v(v+1) - vw + \frac{l}{2} w(w+1) + \tau w \ , \tag{4.2.31}
$$

with the same conventions as in the previous subsection. Now, the subsector involving the gauge field for v is simply the 'almost trivial' CS theory $U(1)_{-\,\mathrm{sign}(k)}$. We use the duality (4.1.20) if $k < 0$, and we use the duality (4.1.21) if $k > 0$. By a straightforward computation, we then derive the following generalised Nii duality [76, 92]:

$$
\boxed{
\begin{array}{ccc}
 & & \underbrace{U(N_c^D)_{-k,\, -k+\mathrm{sign}(k)N_c^D} \times U(1)}_{\mathrm{sign}(k)}{}_{l+\mathrm{sign}(k)} \ , \\
U(N_c)_{k,k+lN_c} \ , \ \left(n_f \,\square, n_a \,\overline{\square}\right) & \longleftrightarrow & \\
 & & \left(n_a \,\square, n_f \,\overline{\square}\right) \ , \ \left(M_\alpha{}^\beta\right) \ ,
\end{array}
}
\tag{4.2.32}
$$

	$U(N_c^D)$	$U(1)^{(w)}$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$	condition
φ_β	$\square$	0	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$	
$\widetilde{\varphi}^\alpha$	$\overline{\square}$	0	$\square$	$\mathbf{1}$	-1	0	$1-r$	
${M_\alpha}^\beta$	$\mathbf{1}$	0	$\overline{\square}$	$\square$	2	0	$2r$	
$\mathcal{B}_+$	$\det^{+1}$	1	$\mathbf{1}$	$\mathbf{1}$	N_f	0	$-r_T+1$	$\lvert k\rvert = k_c$
$\mathcal{B}_-$	$\det^{-1}$	-1	$\mathbf{1}$	$\mathbf{1}$	N_f	0	$-r_T+1$	$\lvert k\rvert = -k_c$

Table 4.7: Field content of the infrared dual of unitary SQCD with $l=0$ and $|k| \geq |k_c|$. The baryonic fields $\mathcal{B}_\pm$ only appear in the marginally chiral case (or in the Amariti–Rota dual), as indicated. Recall that $r_T = -(N_c+N_c^D)(r-1) - N_c + 1$.

with $N_c^D = |k| + \frac{1}{2}(n_f+n_a) - N_c$ and the dual superpotential $W = \widetilde{\varphi}^\alpha {M_\alpha}^\beta \varphi_\beta$.

The matter content of the magnetic theory is given in the upper part of table 4.7. As for the Amariti–Rota duality, the new $U(1)_T$ symmetry only enters through the standard FI term of the $U(1)^{(w)}$ gauge group, and the other mixed gauge-global CS levels (not involving $U(1)_T$) remain the same as in (4.1.8), hence we have:

$$K_{G_0T} = 0\,, \quad K_{wT} = 1\,, \quad K_{GA} = \Theta(k)(n_f - n_a)\,, \quad K_{GR} = \Theta(k)(n_f-n_a)(r-1)\,. \tag{4.2.33}$$

The flavour bare CS levels of the generalised Nii dual are given in table 4.8.

Note that for $l=0$ the second gauge group $U(1)_{l\pm1}$ reduces to $U(1)_{\pm 1}$. This can be eliminated from the description using a local duality (4.1.20)–(4.1.21), thus recovering the dual theory of section 4.1.2.

4.2.5 Marginally chiral duality with general l

Next, we consider the marginally chiral case, $|k| = |k_c|$. It is most convenient to consider the cases with positive and negative k separately.

Marginally chiral case with $k = |k_c|$

For definiteness, let us start by considering the case $k = k_c > 0$ (with $n_f > n_a$). The $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ transformation on the magnetic theory of subsection 4.1.3 gives us:

$$\mathcal{W} = \mathcal{W}_0 + \frac{1}{(2\pi i)^2}\mathrm{Li}_2(y_A^{-n_f} z) - vN_c^D u_0 - n_f v \nu_A - wv + \frac{l}{2}w(w+1) + \tau w\,, \tag{4.2.34}$$

	$k > \|k_c\|$	$k < -\|k_c\|$
$K_{SU(n_f)}$	N_c^D	$-N_c + n_a$
$K_{SU(n_a)}$	N_c^D	$-N_c + n_f$
K_{AA}	$(n_f + n_a)N_c^D$	$4n_f n_a - (n_f + n_a)N_c$
K_{TT}	0	0
K_{AT}	0	0
K_{RA}	$(n_f + n_a)N_c^D r$	$(n_a + n_f)N_c - 2n_f n_a$ $+(4n_f n_a - (n_f + n_a)N_c)r$
K_{RT}	0	0
K_{RR}	$(-N_c^D + (n_f + n_a)r^2)N_c^D$	$(N_c - n_f)(N_c - n_a) - 1$ $+2r((n_a + n_f)N_c - 2n_f n_a)$ $+r^2(4n_f n_a - (n_f + n_a)N_c)$
K_g	$(n_f + n_a - 2k)N_c^D$	$2n_f n_a - (n_f + n_a + 2k)N_c - 2$

Table 4.8: Flavour bare CS levels for the minimally-chiral dual theory with general l.

where again $\mathcal{W}_0$ denotes all the τ-independent terms in twisted superpotential the theory we started with. Renaming $\tau \to v$ upon gauging $U(1)_T$, the v-dependent terms precisely correspond to the electric theory in the elementary duality (4.1.23), namely:

$$\mathcal{W}_{U(1)_{-\frac{1}{2}},\Phi_+} = \frac{1}{(2\pi i)^2}\text{Li}_2(y_A^{-n_f} z) + \tilde{\tau} v \ , \qquad \tilde{\tau} \equiv -N_c^D u_0 - w - n_f v \nu_A \ . \tag{4.2.35}$$

More precisely, we can identify the singlet $\mathfrak{T}^+$ with the field Φ_+ in (4.1.23), with R-charge $r \to r_T$, noting that $K_{RT} = r_T - 1$ in the original $l = 0$ theory. The path integral over the vector multiplet v then gives a dual singlet which we can call $\mathcal{B}_+$, of R-charge $1 - r_T$, with charge $(\det, 1)$ under the remaining gauge group $U(N_c^D) \times U(1)^{(w)}$. We have:

$$\mathcal{W}_{U(1)_{-\frac{1}{2}},\Phi_+} \quad \leftrightarrow \quad \frac{1}{(2\pi i)^2}\text{Li}_2(x_0^{N_c^D} x_{(w)} y_A^{n_f}) + \frac{1}{2}\tilde{\tau}(\tilde{\tau}+1) + \frac{1}{12} \ . \tag{4.2.36}$$

The twisted superpotential of the new dual theory then reads:

$$\begin{aligned}\mathcal{W} = \quad & \mathcal{W}_0 + \frac{1}{(2\pi i)^2}\text{Li}_2(x_0^{N_c^D} x_{(w)} y_A^{n_f}) + \frac{1}{2}N_c^D u_0(N_c^D u_0 + 1) + N_c^D u_0 w \\ & + \frac{l+1}{2}w(w+1) + \tau w - \frac{n_f^2}{2}\nu_A(\nu_A + 1) \ ,\end{aligned} \tag{4.2.37}$$

	$k=\|k_c\|>0$	$k=-\|k_c\|<0$
$K_{SU(n_f)}$	N_c^D	$-N_c+n_a$
$K_{SU(n_a)}$	N_c^D	$-N_c+n_f$
K_{AA}	$(n_f+n_a)N_c^D$	$(n_f+n_a)N_c^D+3n_an_f\equiv K_{AA}^{(0)}$
K_{TT}	0	0
K_{AT}	0	0
K_{RA}	$N_c^D(n_f+n_a)r$	$K_{RA}^{(0)}+rK_{AA}^{(0)}$, $K_{RA}^{(0)}\equiv -N_c^D(n_f+n_a)+2\max(n_f,n_a)^2$ $-\max(n_f,n_a)(N_c^D+n_f+n_a+1)$
K_{RT}	0	0
K_{RR}	$N_c^D(-N_c^D+(n_f+n_a)r^2)$	$N_c^D(2N_c^D+n_f+n_a-2\max(n_f,n_a)+2)$ $2rK_{RA}^{(0)}+r^2K_{AA}^{(0)}$
K_g	$2\max(n_f,n_a)N_c^D$	$2N_c^D(n_f+n_a-\max(n_f,n_a))$

Table 4.9: Flavour bare CS levels for the marginally chiral dual theories, $|k|=|k_c|$, for general l.

and we have a similar transformation of the effective dilaton, as dictated by the dual flavour CS levels of the elementary duality (4.1.23). A careful accounting of the bare CS levels gives us the following non-zero shifts $K\to K+\Delta K$ with respect to the $l=0$ theory:

$$\Delta K_{AA}=-n_f^2\ ,\quad \Delta K_{RA}=(r_T-1)n_f\ ,\quad \Delta K_{RR}=-r_T^2+2r_T-1\ . \tag{4.2.38}$$

A similar computation can be carried out for the case $k=-k_c>0$ (with $n_a>n_f$). This replaces the singlet $\mathfrak{T}_-$ in the original theory with a chiral multiplet $\mathcal{B}_+$ of charge $(\det^{-1},-1)$ under $U(N_c^D)\times U(1)^{(w)}$. The flavour CS level shifts are like in (4.2.38) with n_f replaced by n_a.

In summary, for $k=\epsilon k_c>0$, for $\epsilon=\pm$, we have the duality:

$$\boxed{U(N_c)_{k,k+lN_c}\ ,\ \left(n_f\,\square, n_a\,\overline{\square}\right)\quad\longleftrightarrow\quad \begin{array}{c}\underbrace{U(N_c^D)_{-k,\,-k+\frac{1}{2}N_c^D}\times U(1)^{(w)}_{l+\frac{1}{2}}}_{\frac{1}{2}}\ ,\\[2em] \left(n_a\,\square, n_f\,\overline{\square}\right)\ ,\ \left({M_\alpha}^\beta,\,\mathcal{B}_\epsilon\right)\ ,\end{array}} \tag{4.2.39}$$

with the dual superpotential $W = \widetilde{\varphi}^\alpha M_\alpha{}^\beta \varphi_\beta$. The precise matter content is given in table 4.7. The mixed gauge-flavour CS levels are the same as in (4.2.33), namely:

$$K_{G_0T} = 0 \ , \quad K_{wT} = 1 \ , \quad K_{GA} = (n_f - n_a) \ , \quad K_{GR} = (n_f - n_a)(r-1) \ . \qquad (4.2.40)$$

and the bare flavour CS levels are given on the l.h.s of table 4.9.

Marginally chiral case with $k = -|k_c|$

The dual theories with $k = -|k_c| < 0$ can be derived similarly. In this case, we need to use the elementary duality (4.1.22). We find non-trivial shifts of the gauge-flavour CS levels:

$$\Delta K_{G_0A} = \Delta K_{wA} = \pm \max(n_f, n_a) \ , \qquad \Delta K_{G_0R} = \Delta K_{wA} = \mp r_T \ , \qquad (4.2.41)$$

for $k = \mp k_c < 0$. We also have the flavour CS levels shifts:

$$\begin{aligned} &\Delta K_{AA} = \max(n_f, n_a)^2 \ , \qquad &&\Delta K_{AR} = -r_T \max(n_f, n_a) \ , \\ &\Delta K_{RR} = r_T^2 - 1 \ , &&\Delta K_g = -2 \ , \end{aligned} \qquad (4.2.42)$$

compared to the levels shown on the right-hand-side of table 4.3.

Fixing $k = -\epsilon k_c < 0$, for $\epsilon = \pm$, we then have the duality:

$$\boxed{U(N_c)_{k, k+lN_c} \ , \ \left(n_f \,\square, n_a \,\overline{\square}\right) \qquad \longleftrightarrow \qquad \begin{array}{c} \underbrace{U(N_c^D)_{-k,\, -k-\frac{1}{2}N_c^D} \times U(1)^{(w)}_{l-\frac{1}{2}}}_{-\frac{1}{2}} \ , \\ \\ \left(n_a \,\square, n_f \,\overline{\square}\right) \ , \ \left(M_\alpha{}^\beta, \, \mathcal{B}_\epsilon\right) \ , \end{array}} \qquad (4.2.43)$$

with the superpotential $W = \widetilde{\varphi}^\alpha M_\alpha{}^\beta \varphi_\beta$ and the matter content of table 4.7. We have the mixed gauge-flavour CS levels:

$$
\begin{aligned}
&K_{G_0T} = 0 \ , \\
&K_{wT} = 1 \ , \\
&K_{G_0A} = K_{wA} = \epsilon \max(n_f, n_a) \ , \\
&K_{G_0R} = K_{wA} = -\epsilon\, r_T \ ,
\end{aligned}
\tag{4.2.44}
$$

with r_T defined as in table 4.7. The bare flavour CS levels are given in table 4.9.

4.2.6 Maximally chiral duality with general l

Last but not least, let us consider the maximally chiral duality. Starting from the $l = 0$ dual theory of subsection 4.1.4, the $\mathbf{S}^{-1}\mathbf{T}^l\mathbf{S}$ action gives us:

$$
\begin{aligned}
&\mathcal{W} = \mathcal{W}_0 - vN_c^D u_0 + K_{AT}^{(*)} v\nu_A - vw + \frac{l}{2}w(w+1) + \tau w \ , \\
&\Omega = \Omega + K_{RT}^{(*)} v \ .
\end{aligned}
\tag{4.2.45}
$$

Here, as before, $\mathcal{W}_0$ and Ω_0 contain all the v- and w-independent terms. Here we denote by $K^{(*)}$ the levels given in table 4.5. Note that the vector multiplet for v appears linearly, unlike in the previous cases. Therefore, integrating it out generates a functional δ-function:

$$
\delta\left(W_\mu + \mathrm{tr}(A_\mu) - K_{AT}^{(*)} A_\mu^{(A)} - K_{AR}^{(*)} A_\mu^{(R)}\right) , \tag{4.2.46}
$$

with $\mathrm{tr}(A) = N_c^D A_0$ the $U(1) \subset U(N_c^D)$ gauge field, W_μ the w gauge field, and $A_\mu^{(F)}$ denoting the background gauge fields for a $U(1)_F$ symmetry. Note the appearance of the $U(1)_R$ gauge field because of the mixed topological-R level, $K_{RT}^{(*)} \neq 0$. Eliminating w from the description, we obtain:

$$
\begin{aligned}
&\mathcal{W} = \mathcal{W}_0 - \tau N_c^D u_0 - \Delta K_{GA}\, \nu_A N_c^D u_0 + \frac{l}{2} N_c^D u_0 (N_c^D u_0 + 1) + \Delta K_{AA}\, \nu_A(\nu_A + 1) \ , \\
&\Omega = \Omega_0 + \Delta K_{GR}\, N_c^D u_0 + \Delta K_{RA}\, \nu_A + \frac{1}{2}\Delta K_{RR} \ ,
\end{aligned}
\tag{4.2.47}
$$

	$U(N_c^D)$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$
φ_β	$\square$	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$
$\tilde{\varphi}^\alpha$	$\overline{\square}$	$\square$	$\mathbf{1}$	-1	0	$1-r$
${M_\alpha}^\beta$	$\mathbf{1}$	$\overline{\square}$	$\square$	2	0	$2r$

Table 4.10: Field content for the infrared dual of unitary SQCD with $|k| < |k_c|$.

where we defined:

$$\begin{aligned} &\Delta K_{GA} = -lK_{AT}^{(*)} \; , && \Delta K_{GR} = -lK_{RT}^{(*)} \; , \\ &\Delta K_{AA} = l\left(K_{AT}^{(*)}\right)^2 \; , && \Delta K_{RR} = l\left(K_{RT}^{(*)}\right)^2 \; , \\ &\Delta K_{RA} = lK_{RT}^{(*)}K_{AT}^{(*)} \; . \end{aligned} \tag{4.2.48}$$

In particular, we find that the maximally chiral dual theory has a gauge group $U(N_c^D)_{-k,-k+lN_c^D}$ with $N_c^D = \max(n_f, n_a) - N_c$, with the matter content of table 4.10.

In summary, we have:

$$\boxed{U(N_c)_{k,\,k+lN_c} \; , \; \left(n_f\,\square, n_a\,\overline{\square}\right) \quad \longleftrightarrow \quad U(N_c^D)_{-k,\,-k+lN_c^D} \; , \; \left(n_a\,\square, n_f\,\overline{\square}\right) \; , \; \left({M_\alpha}^\beta\right) \; ,} \tag{4.2.49}$$

with the usual Seiberg-dual superpotential. The dual theory contains the mixed gauge-flavour CS levels:

$$\begin{aligned} &K_{GT} = -1 \; , \\ &K_{GA} = \mathrm{sign}(k_c)\Big(k + |k_c| + l\max(n_f, n_a)\Big) \; , \\ &K_{GR} = \mathrm{sign}(k_c)\Big(\big(k + |k_c| + l\max(n_f, n_a)\big)(r-1) - lN_c\Big) \; , \end{aligned} \tag{4.2.50}$$

and the flavour bare CS levels are given in table 4.11. This dual theory trivially reduces to the dual of section 4.1.4 upon setting $l = 0$.

As a special case of the maximally chiral duality, consider $n_a = 0$ and a CS level $|k| < \frac{n_f}{2}$. We then have the simple-looking duality:

$$\boxed{U(N_c)_{k,\,k+lN_c} \; , \; n_f\,\square \qquad \longleftrightarrow \qquad U(n_f - N_c)_{-k,\,-k+l(n_f-N_c)} \; , \; n_f\,\overline{\square} \; .} \tag{4.2.51}$$

	$\lvert k\rvert < \lvert k_c\rvert$, $n_f > n_a$	$\lvert k\rvert < \lvert k_c\rvert$, $n_a > n_f$
$K_{SU(n_f)}$	$k + \frac{1}{2}(n_f + n_a) - N_c$	$n_a - N_c$
$K_{SU(n_a)}$	$n_f - N_c$	$k + \frac{1}{2}(n_f + n_a) - N_c$
K_{AA}	$K^+_{AA} \equiv (l+1)n_f^2 + 3n_f(k - k_c)$ $+2N_c^D(n_f - k_c)$	$K^-_{AA} \equiv (l+1)n_a^2 + 3n_a(k + k_c)$ $+2N_c^D(n_a + k_c)$
K_{TT}	0	0
K_{AT}	$-n_f$	n_a
K_{RA}	$K^{(0)+}_{RA} + rK^+_{AA}$, $K^{(0)+}_{RA} \equiv -(k - k_c)(n_f + N_c^D)$ $+(l+1)n_f N_c^D$	$K^{(0)-}_{RA} + rK^-_{AA}$, $K^{(0)-}_{RA} \equiv -(k + k_c)(n_a + N_c^D)$ $+(l+1)n_a N_c^D$
K_{RT}	$N_c^D - rn_f$	$-N_c^D + rn_a$
K_{RR}	$N_c^D(k - k_c + lN_c^D)$ $+2rK^{(0)+}_{RA} + r^2K^+_{AA}$	$N_c^D(k + k_c + lN_c^D)$ $+2rK^{(0)-}_{RA} + r^2K^-_{AA}$
K_g	$2n_a N_c^D$	$2n_f N_c^D$

Table 4.11: Flavour bare CS levels for the maximally chiral dual, $|k| < |k_c|$, for general l. Note that, unlike in all the other cases, the flavour CS levels depend on l.

Upon choosing a positive FI parameter, the Higgs branch of the electric theory is given by the complex Grassmannian $\text{Gr}(N_c, n_f)$. This duality has a natural interpretation in terms of the obvious geometric isomorphism of the dual Higgs branches:

$$\text{Gr}(N_c, n_f) \cong \text{Gr}(n_f - N_c, n_f) \ . \tag{4.2.52}$$

In this Higgs phase, certain choices of the levels k, l have an interpretation in terms of the (generalised) quantum K-theory of $\text{Gr}(N_c, n_f)$– see *e.g.* [132, 133, 134, 135, 136, 137]. We will come back to this point in the next chapter.

4.3 3d $\mathcal{N} = 2$ IR dualities: matching the moduli spaces

In section 3.2, we worked out the vacuum moduli space of SQCD$[N_c, k, l, n_f, 0]$ with vanishing $SU(n_f)$ masses but with a non-zero FI parameter. Thus, as an interesting consistency check of our computation, it is natural to ask whether the same result is indeed reproduced in the dual magnetic theories discussed in the previous section. Alternatively, the results of this section can be viewed as new detailed checks of the recently proposed dualities for $l \neq 0$ [76, 92, 1]. Therefore, in this section, we study the vacua of the Seiberg-like dual to $U(N_c)_{k,k+lN_c}$ with n_f fundamentals and $\xi \neq 0$. Note that $k_c = \frac{n_f}{2} > 0$, and we then have the dual rank:

$$N_c^D \equiv \begin{cases} |k| + \frac{n_f}{2} - N_c \ , & \text{if} \quad |k| \geq \frac{n_f}{2} \ , \\ n_f - N_c \ , & \text{if} \quad |k| \leq \frac{n_f}{2} \ . \end{cases} \tag{4.3.1}$$

Let us use the notation $\epsilon \equiv \mathrm{sign}(k)$. We have three cases to consider in turn:

(i) **Minimally-chiral case, for $|k| > \frac{n_f}{2}$.** We then have a magnetic theory with gauge group (4.2.32):

$$\underbrace{U(N_c^D)_{-k,\,-k+\epsilon N_c^D} \times U(1)}_{\epsilon}{}_{l+\epsilon} \ , \tag{4.3.2}$$

and with n_f antifundamental chiral multiplets $\widetilde{\varphi}^\alpha$ coupled to the $U(N_c^D)$ factor. The $U(1)$ and $U(N_c^D)$ factors are coupled together by a mixed CS term at level $\epsilon = \pm 1$.

(ii) **Maximally-chiral case, for $|k| < \frac{n_f}{2}$.** In this case, the magnetic theory is simply (4.2.49):

$$U(N_c^D)_{-k,\,-k+lN_c^D} \ , \tag{4.3.3}$$

coupled to the same n_f antifundamental matter fields.

(ii) **Marginally-chiral case, for $|k| = \frac{n_f}{2}$.** In this last case, we have the dual gauge group (4.2.39) and (4.2.43):

$$\underbrace{U(N_c^D)_{-k,\,-k+\frac{1}{2}\epsilon N_c^D} \times U(1)}_{\frac{1}{2}\epsilon}{}_{l+\frac{1}{2}\epsilon} \ , \tag{4.3.4}$$

and the same n_f antifundamental chiral multiplets, but we also have an additional chiral multiplet charged under both $U(1) \subset U(N_c^D)$ and the second $U(1)$.

For $l = 0$, these dualities can be simplified further, and one recovers the well-known cases studied in [58]. For $|k| > \frac{n_f}{2}$ and $l + \epsilon = 0$, we can integrate out the $U(1)_0$ gauge field in (4.3.2) and we essentially obtain a $SU(N_c^D)_{-k}$ theory with n_f antifundamentals – this interesting special case will have to be considered separately.

4.3.1 The minimally-chiral dual theory – $|k| > \frac{n_f}{2}$

In the case $|k| > \frac{n_f}{2}$, we have the Nii–Amariti–Rota duality [76, 92]:

$$U(N_c)_{k,k+lN_c} \; \oplus n_f \; \Box \quad \longleftrightarrow \quad \underbrace{U(N_c^D)_{-k,\,-k+\epsilon N_c^D} \times U(1)}_{\epsilon}{}_{l+\epsilon} \; \oplus n_f \; \overline{\Box}_0 \; . \tag{4.3.5}$$

To study the matching of the moduli spaces and of the Witten index across this duality, we first need to analyse the semi-classical vacuum equations for the magnetic theory in (4.3.5). To write down these equations, we need to recall that the FI parameter of the electric theory is mapped to the FI parameter of the $U(1)_{l+\epsilon}$ theory. (There is no independent topological symmetry for $U(N_c^D)$ because the mixed CS level is $\epsilon = 1$ or -1.) We then have:

$$\begin{aligned}
&- \sigma_a \widetilde{\varphi}_a^\alpha = 0 \; , \qquad \alpha = 1, \cdots, n_f \; , \qquad a = 1, \cdots, N_c^D \; , \\
&- \sum_{\alpha=1}^{n_f} \widetilde{\varphi}_\alpha^{\dagger a} \widetilde{\varphi}_b^\alpha = \frac{\delta^a{}_b}{2\pi} \left(-k\sigma_a + \epsilon \sum_{b=1}^{N_c^D} \sigma_b - \frac{n_f}{2} |\sigma_a| + \epsilon \widetilde{\sigma} \right) \; , \\
&\xi + \epsilon \sum_{c=1}^{N_c^D} \sigma_c + (l+\epsilon) \widetilde{\sigma} = 0 \; ,
\end{aligned} \tag{4.3.6}$$

where, as in the electric theory, we are taking all matter fields to be massless. Here, σ_b and $\widetilde{\sigma}$ are the real scalars associated with $U(N_c^D)$ and with the $U(1)_{l+\epsilon}$ factor, respectively.

Dual theory for $l + \epsilon \neq 0$

Let us first assume that $l + \epsilon \neq 0$. Then, we can use the last equation in (4.3.6) to eliminate $\widetilde{\sigma}$ from the computation:

$$\widetilde{\sigma} = -\frac{\xi}{l+\epsilon} - \frac{\epsilon}{l+\epsilon} \sum_{b=1}^{N_c^D} \sigma_b \; . \tag{4.3.7}$$

Substituting this back into (4.3.6), we find:

$$
\begin{aligned}
&- \sigma_a \widetilde{\varphi}_a^\alpha = 0 \ , \qquad i = 1, \cdots, n_f \ , \qquad a = 1, \cdots, N_c^D \ , \\
&\sum_{\alpha=1}^{n_f} \widetilde{\varphi}_\alpha^{\dagger a} \widetilde{\varphi}_b^\alpha = \frac{\delta^a{}_b}{2\pi} \frac{\epsilon}{l+\epsilon} F_a(\sigma) \ , \\
&F_a(\sigma) \equiv \xi + \left(\tilde{k} - l + \text{sign}(\sigma_a) \frac{\tilde{n}_f}{2}\right) \sigma_a - l \sum_{b \neq a}^{N_c^D} \sigma_b \ ,
\end{aligned}
\tag{4.3.8}
$$

where we conveniently defined the parameters:

$$
\tilde{k} \equiv \epsilon k \, (l+\epsilon) \ , \qquad \text{and} \qquad \tilde{n}_f \equiv \epsilon n_f (l+\epsilon) \ . \tag{4.3.9}
$$

Note that, while we apparently eliminated the $U(1)_{l+\epsilon}$ factor from the description, this does not mean that it disappears at low energy at this level of the discussion.

We see that the equations (4.3.8) are very similar to (3.2.22); therefore, we can follow the same strategy to solve them. We will also use a similar typology (*Type I^D, II^D, ...*). The way vacua are matched across duality may be quite complicated, as we will see. Of course, we know that Higgs vacua should match to Higgs vacua, topological vacua to topological vacua, and indeed they do, but the infrared duality of the larger SQCD theory descends to a geometric and level/rank duality in each vacuum, of the general form:

$$
\text{Gr}(p, n_f) \times \{\text{TQFT}\} \quad \longleftrightarrow \quad \text{Gr}(n_f - p, n_f) \times \{\text{level/rank-dual TQFT}\} \ . \tag{4.3.10}
$$

The TQFTs on each side are matched precisely through the 3d $\mathcal{N}=2$ level/rank dualities discussed in appendix A.

Type I^D. This is the vacuum with $\sigma_a = 0$, $\forall a$. We are left with the D-term equation:

$$
\sum_{\alpha=1}^{n_f} \widetilde{\varphi}_\alpha^{\dagger a} \widetilde{\varphi}_b^\alpha = \frac{\delta^a{}_b}{2\pi} \frac{\xi}{1+\epsilon l} \ , \tag{4.3.11}
$$

which gives us the Grassmannian $\text{Gr}(N_c^D, n_f)$ if the effective FI term is positive. In addition, (4.3.7) implies that $\tilde{\sigma}$ obtains a non-zero value, and we then have a $U(1)_{l+\epsilon}$ that survives at low energy. Hence, we have a Higgs-topological vacuum:

$$
\mathcal{M}_{\text{I}^D}^{\min}[N_c, k, l, n_f, 0] = \Theta(\xi(1+\epsilon l)) \ \text{Gr}(N_c^D, n_f) \times U(1)_{l+\epsilon} \ , \tag{4.3.12}
$$

which contributes to the Witten index as:

$$\mathbf{I}^{\min}_{\mathrm{W,\,I}^D}[N_c,k,l,n_f,0] = \Theta(\xi(1+\epsilon l)) \ \ |l+\epsilon| \begin{pmatrix} n_f \\ N_c^D \end{pmatrix} \ . \tag{4.3.13}$$

Type II^D. Consider taking $1 \le p \le N_c^D - 1$ of the σ's to be positive while the remaining ones vanish. Then (4.3.8) implies that all the non-zero eigenvalues are equal, and the equations simplify to:

$$\begin{aligned} &-\sigma^+ \widetilde{\varphi}^\alpha_a = 0 \ , \qquad \alpha = 1,\cdots,n_f \ , \qquad\qquad a = 1,\cdots,p \ , \\ &\sum_{\alpha=1}^{n_f} \widetilde{\varphi}^{\dagger b}_\alpha \widetilde{\varphi}^\alpha_b = \frac{1}{1+\epsilon l}\left(\xi - pl\sigma^+\right) > 0 \ , \qquad b = 1,\cdots,N_c^D - p \ , \\ &\xi + \left(\tilde{k} - pl + \frac{\tilde{n}_f}{2}\right)\sigma^+ = 0 \ . \end{aligned} \tag{4.3.14}$$

A hybrid Higgs-topological solution exists in the window:

$$\mathrm{sign}(\xi)\left(pl - \tilde{k} - \frac{\tilde{n}_f}{2}\right) > 0 \ , \qquad \text{and} \qquad k + \frac{n_f}{2} < 0 \ , \tag{4.3.15}$$

where the first condition follows from $\sigma^+ > 0$ and the second follows from the positivity of the size of the Grassmannian Higgs branch. A similar solution exists where we take the p non-vanishing σ's to be negative, in the window:

$$\mathrm{sign}(\xi)\left(\tilde{k} - pl - \frac{\tilde{n}_f}{2}\right) > 0 \ , \qquad \text{and} \qquad k - \frac{n_f}{2} > 0 \ . \tag{4.3.16}$$

The $U(1)_{l+\epsilon}$ also survives at low energy, and we then find the branches of hybrid vacua:

$$\begin{aligned} &\mathcal{M}^{\min}_{\mathrm{II}^D}[N_c,k,l,n_f,0] = \\ &\Theta\left(-k - \frac{n_f}{2}\right) \bigoplus_{p=1}^{N_c^D-1} \Theta\left(\xi\left(-\tilde{k} + pl - \frac{\tilde{n}_f}{2}\right)\right) \mathrm{Gr}(N_c^D - p, n_f) \times \underbrace{U(p)_{-k-\frac{n_f}{2},-k-\frac{n_f}{2}+\epsilon p} \times U(1)}_{\epsilon}{}_{l+\epsilon} \\ &\oplus \Theta\left(k - \frac{n_f}{2}\right) \bigoplus_{p=1}^{N_c^D-1} \Theta\left(\xi\left(\tilde{k} - pl - \frac{\tilde{n}_f}{2}\right)\right) \mathrm{Gr}(N_c^D - p, n_f) \times \underbrace{U(p)_{-k+\frac{n_f}{2},-k+\frac{n_f}{2}+\epsilon p} \times U(1)}_{\epsilon}{}_{l+\epsilon} \ . \end{aligned} \tag{4.3.17}$$

We see that they contribute to the index as:

$$\begin{aligned}
&\mathbf{I}^{\min}_{\mathrm{W,\,II}^D}[N_c,k,l,n_f,0] = \\
&\quad \Theta\left(-k-\frac{n_f}{2}\right)\sum_{p=1}^{N_c^D-1}\Theta\left(\xi\left(-\tilde{k}+pl-\frac{\tilde{n}_f}{2}\right)\right)\binom{n_f}{N_c^D-p}\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} p & \epsilon & -k-\frac{n_f}{2} & \epsilon \\ 1 & 0 & \epsilon & l+\epsilon \end{array}\right] \\
&\quad +\Theta\left(k-\frac{n_f}{2}\right)\sum_{p=1}^{N_c^D-1}\Theta\left(\xi\left(\tilde{k}-pl-\frac{\tilde{n}_f}{2}\right)\right)\binom{n_f}{N_c^D-p}\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} p & \epsilon & -k+\frac{n_f}{2} & \epsilon \\ 1 & 0 & \epsilon & l+\epsilon \end{array}\right],
\end{aligned} \tag{4.3.18}$$

using (A.1.12) again for the TQFT contributions.

Type III^D. The third type of solutions correspond to topological vacua. In the analysis of the electric theory, we had solutions of Type III,a and III,b. In the present case, it turns out that only the analogue of Type III,a exists, where all the eigenvalues σ_a are taken to have the same sign. In the case where they are all positive, we have $\sigma_a = \sigma^+ > 0$ that solves:

$$\xi + \left(\tilde{k} - lN_c^D + \frac{\tilde{n}_f}{2}\right)\sigma^+ = 0\ , \qquad \mathrm{sign}(\xi)\left(\tilde{k} - lN_c^D + \frac{\tilde{n}_f}{2}\right) < 0\ , \tag{4.3.19}$$

which exists only in the window indicated. Similarly, there is a solution $\sigma_a = \sigma^- < 0$. In total, we find the vacua:

$$\begin{aligned}
&\mathcal{M}^{\min}_{\mathrm{III}^D}[N_c,k,l,n_f,0] = \\
&\quad \Theta\left(\xi\left(-\tilde{k}+lN_c^D-\frac{\tilde{n}_f}{2}\right)\right)\underbrace{U(N_c^D)_{-k-\frac{n_f}{2},-k-\frac{n_f}{2}+\epsilon N_c^D}\times U(1)_{l+\epsilon}}_{\epsilon} \\
&\quad \oplus\Theta\left(\xi\left(\tilde{k}-lN_c^D-\frac{\tilde{n}_f}{2}\right)\right)\underbrace{U(N_c^D)_{-k+\frac{n_f}{2},-k+\frac{n_f}{2}+\epsilon N_c^D}\times U(1)_{l+\epsilon}}_{\epsilon}\ ,
\end{aligned} \tag{4.3.20}$$

which contribute to the index as:

$$\begin{aligned}
\mathbf{I}^{\min}_{\mathrm{W,\,III}^D}[N_c,k,l,n_f,0] &= \Theta\left(\xi\left(-\tilde{k}+lN_c^D-\frac{\tilde{n}_f}{2}\right)\right)\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} N_c^D & \epsilon & -k-\frac{n_f}{2} & \epsilon \\ 1 & 0 & \epsilon & l+\epsilon \end{array}\right] \\
&\quad \oplus\Theta\left(\xi\left(\tilde{k}-lN_c^D-\frac{\tilde{n}_f}{2}\right)\right)\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} N_c^D & \epsilon & -k+\frac{n_f}{2} & \epsilon \\ 1 & 0 & \epsilon & l+\epsilon \end{array}\right].
\end{aligned} \tag{4.3.21}$$

N_c	k	l	n_f	Electric Side	Magnetic Side
3	4	10	6	$\mathrm{Gr}(3,6)$ $\oplus\ \mathrm{Gr}(2,6) \times U(1)_{11}$	$\mathrm{Gr}(3,6) \times \underbrace{U(1)_0 \times U(1)_{11}}_{1}$ $\oplus\ \mathrm{Gr}(4,6) \times U(1)_{11}$
5	$\frac{9}{2}$	0	7	$\mathrm{Gr}(5,7)$ $\oplus\ \mathrm{Gr}(4,7) \times U(1)_1$	$\mathrm{Gr}(2,7) \times \underbrace{U(1)_0 \times U(1)_1}_{1}$ $\oplus\ \mathrm{Gr}(3,7) \times U(1)_1$
5	$-\frac{9}{2}$	2	7	$\mathrm{Gr}(5,7)$ $\oplus\ U(5)_{-8,5}$	$\mathrm{Gr}(2,7) \times \underbrace{U(1)_0 \times U(1)_1}_{-1}$ $\oplus\ \underbrace{U(3)_{8,5} \times U(1)_1}_{-1}$
6	5	-4	8	$\mathrm{Gr}(6,8)$ $\oplus\ U(6)_{9,-15}$	$\mathrm{Gr}(2,8) \times \underbrace{U(1)_0 \times U(1)_{-3}}_{1}$ $\oplus\ \underbrace{U(3)_{-9,-6} \times U(1)_{-3}}_{1}$

Table 4.12: Matching moduli spaces of vacua across the minimally-chiral duality with $\xi > 0$.

Full Witten index. One can check that there are no more solutions, so that the full Witten index of the minimally-chiral dual theory takes the form:

$$\begin{aligned} \mathbf{I}_{\mathrm{W}}^{\min}[N_c,k,l,n_f,0] =& \mathbf{I}_{\mathrm{W,\,I}^D}^{\min}[N_c,k,l,n_f,0] + \mathbf{I}_{\mathrm{W,\,II}^D}^{\min}[N_c,k,l,n_f,0] \\ &+ \mathbf{I}_{\mathrm{W,\,III}^D}^{\min}[N_c,k,l,n_f,0]\ . \end{aligned} \tag{4.3.22}$$

Matching across the duality. Using these results, one can check that the moduli spaces match exactly between the electric and magnetic description for either sign of the FI parameter, as predicted by the duality 4.3.5. For instance, in the case $\xi > 0$, the electric theory vacua always include the pure Higgs branch 3.2.24. The dual description is simply through the Grassmannian duality:

$$\mathrm{Gr}(N_c, n_f) \qquad \longleftrightarrow \qquad \mathrm{Gr}(n_f - N_c, n_f)\ . \tag{4.3.23}$$

The left-hand-side Higgs branch arises as a Type II^D vacuum with a TQFT that happens to be trivial (with Witten index $\mathbf{I}_{\mathrm{W}} = 1$), namely the vacua in (4.3.17) with $p = |k| - \frac{n_f}{2}$.

Here, let us simply display this general matching in some examples for $\xi > 0$ (table 4.12) and $\xi < 0$ (table 4.13). More examples are listed in appendix C. For instance, looking at the last case, $(N_c,k,l,n_f) = (6,5,-4,8)$, in table 4.12, we see that the matching

N_c	k	l	n_f	Electric Side	Magnetic Side
3	4	10	6	$U(3)_{7,37}$	$U\underbrace{(4)_{-7,-3} \times U(1)}_{1}{}_{11}$
5	$\frac{9}{2}$	0	7	$U(5)_{8,8}$	$U\underbrace{(3)_{-8,-5} \times U(1)}_{1}{}_{1}$
5	$-\frac{9}{2}$	2	7	$\mathrm{Gr}(4,7) \oplus U(1)_1$	$\mathrm{Gr}(3,7) \oplus U(1)_1$
6	5	-4	8	$\mathrm{Gr}(5,8) \times U(1)_{-3}$	$\mathrm{Gr}(3,8) \times U(1)_{-3}$

Table 4.13: Matching moduli spaces of vacua across the minimally-chiral duality with $\xi < 0$.

of the vacua follows from the dualities:

$$\begin{aligned} \mathrm{Gr}(6,8) \qquad &\longleftrightarrow \qquad \mathrm{Gr}(2,8) \times U\underbrace{(1)_0 \times U(1)}_{1}{}_{-3} \ , \\ U(6)_{9,-15} \qquad &\longleftrightarrow \qquad U\underbrace{(3)_{-9,-6} \times U(1)}_{1}{}_{-3} \ . \end{aligned} \tag{4.3.24}$$

Here, the TQFT on the r.h.s of the first line has a single state, and on the second line, we have an instance of the level/rank duality (A.2.1) (here for $(N,k,l) = (6,9,-4)$). Similar matching holds for every component of the moduli space for every theory.

Dual theory for $l + \epsilon = 0$

We must treat separately the case $|k| > \frac{n_f}{2}$ with $l + \epsilon = 0$. In this case, we have a $U(1)_0$ coupled to the trace of the $U(N_c^D)$ vector multiplet by a BF term, and integrating it out imposes:

$$\mathrm{tr}\left(A^{U(N_c^D)}\right) = -\epsilon A_T \ , \qquad \sum_{b=1}^{N_c^D} \sigma_b = -\epsilon\xi \ . \tag{4.3.25}$$

This gets rid of the $U(1) \subset U(N_c^D)$, and the duality (4.3.5) becomes:

$$U(N_c)_{k,k-\epsilon N_c} \ \oplus n_f \,\square \quad \longleftrightarrow \quad SU(N_c^D)_{-k} \ \oplus n_f \,\overline{\square} \ . \tag{4.3.26}$$

In this formulation, the topological symmetry of the electric theory maps to the baryonic symmetry of the $SU(N_c^D)$ magnetic dual theory. (A closely related non-supersymmetric duality was first discussed in [84].)

This magnetic theory has the same moduli space of vacua as some $SU(n_c)_k$ theory coupled to n_f fundamental chiral multiplets (this is just a CP transformation):

$$SU(n_c)_k \;\oplus n_f \,\square\, . \tag{4.3.27}$$

We leave a more detailed analysis of this theory for the motivated reader. Here, we simply conjecture an explicit formula for its flavoured Witten index:

$$\mathbf{I}_{\rm W}\left[SU(n_c)_k \;\oplus n_f \,\square\right] = \binom{|k| + \frac{n_f}{2} - 1}{n_c - 1} , \qquad\qquad \text{if } \; |k| > \frac{n_f}{2} \, . \tag{4.3.28}$$

Of course, this reduces to the known result (A.1.4) for the pure CS theory if $n_f = 0$. We checked that, for $n_c = N_c^D = |k| + \frac{n_f}{2} - N_c$, this indeed matches with the index computed by the complicated formula (3.2.63) in the electric theory. For $n_c = 2$, the formula (4.3.28) was derived in [49].

4.3.2 The maximally-chiral dual theory – $|k| < \frac{n_f}{2}$

In the maximally-chiral case with $k < \frac{n_f}{2}$, we have the duality [1]:

$$U(N_c)_{k,\, k+lN_c} \;\oplus n_f \,\square \qquad \longleftrightarrow \qquad U(N_c^D)_{-k,\, -k+lN_c^D} \;\oplus n_f \,\overline{\square}\; , \tag{4.3.29}$$

with $N_c^D \equiv n_f - N_c$. The semi-classical equations for the magnetic theory take the form:

$$\begin{aligned} &-\sigma_a \widetilde{\varphi}^\alpha_a = 0 \; , \qquad \alpha = 1, \cdots, n_f \; , \qquad a = 1, \cdots, N_c^D \; , \\ &-\sum_{\alpha=1}^{n_f} \widetilde{\varphi}^{\dagger a}_\alpha \widetilde{\varphi}^\alpha_b = \frac{\delta^a{}_b}{2\pi} \left(-\xi - k\sigma_a - \frac{n_f}{2}|\sigma_a| + l \sum_{c=1}^{N_c^D} \sigma_c \right) \; . \end{aligned} \tag{4.3.30}$$

The sign in front of the FI term is due to the fact that the topological current flips sign across this duality [1]. We then see that the analysis of the solutions will be completely similar to the one for the electric theory, after taking into account that we effectively changed the sign of l relative to k, and that we have to integrate out matter fields according to the effective real mass $-\sigma_a$. Here we denote the types of vacua by I_D, II_D,..., not to be confused with the minimally-chiral case discussed above.

Type I_D vacua: The Higgs branch. Taking all the σ's to vanish, we have the usual

Higgs branch equation that gives us the Grassmannian:

$$\mathcal{M}^{\max}_{\mathrm{I}_D} = \Theta(\xi)\,\mathrm{Gr}(N_c^D, n_f)\ , \qquad\qquad \mathbf{I}^{\max}_{\mathrm{W},\,\mathrm{I}_D} = \Theta(\xi)\binom{n_f}{N_c^D}\ . \tag{4.3.31}$$

Note that this exactly matches the Higgs branch vacuum (3.2.24) in the electric theory, due to the Grassmannian duality $\mathrm{Gr}(N_c, n_f) \cong \mathrm{Gr}(n_f - N_c, n_f)$.

Type II$_D$ vacua. There are no Type II$_D$ vacua in this magnetic theory because of the constraint $|k| \neq \frac{n_f}{2}$, as we can see already from the computation in (3.2.49).

Type III$_D$ vacua. As in the electric theory, we have the Type III,a$_D$ solutions when all the σ's are of the same sign. They give us:

$$\begin{aligned}\mathcal{M}^{\max}_{\mathrm{III,a}_D}[N_c,k,l,n_f,0] &= \Theta\left(\xi\left(-k+N_c^D l-\frac{n_f}{2}\right)\right) U(N_c^D)_{-k-\frac{n_f}{2},-k-\frac{n_f}{2}+N_c^D l}\\ &\oplus\Theta\left(\xi\left(k-N_c^D l-\frac{n_f}{2}\right)\right) U(N_c^D)_{-k+\frac{n_f}{2},-k+\frac{n_f}{2}+N_c^D l}\ ,\end{aligned} \tag{4.3.32}$$

and contribute to the index as:

$$\begin{aligned}\mathbf{I}^{\max}_{\mathrm{W,\,III,a}_D}[N_c,k,l,n_f,0] &= \Theta\left(\xi\left(-k+N_c^D l-\frac{n_f}{2}\right)\right)\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|c} N_c^D & l & -k-\frac{n_f}{2}\end{array}\right]\\ &+\Theta\left(\xi\left(k-N_c^D l-\frac{n_f}{2}\right)\right)\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|c} N_c^D & l & -k+\frac{n_f}{2}\end{array}\right]\ .\end{aligned} \tag{4.3.33}$$

Finally, we can also have Type III,b$_D$ solutions that give us the topological vacua:

$$\begin{aligned}\mathcal{M}^{\max}_{\mathrm{III,b}_D}[N_c,k,l,n_f,0] &= \bigoplus_{p=1}^{N_c^D-1}\Theta\left(\xi\,\mathcal{L}(p,N_c^D,k,-l,n_f)\right)\\ &\underbrace{U(p)_{-k-\frac{n_f}{2},-k-\frac{n_f}{2}+lp}\times U(N_c^D-p)}_{l}{}_{-k+\frac{n_f}{2},-k+\frac{n_f}{2}+l(N_c^D-p)}\ ,\end{aligned} \tag{4.3.34}$$

with the quadratic function $\mathcal{L}$ defined in (3.2.59), which contribute to the index as:

$$\begin{aligned}&\mathbf{I}^{\max}_{\mathrm{W,\,III,b}_D}[N_c,k,l,n_f,0] =\\ &\sum_{p=1}^{N_c^D-1}\Theta\left(\xi\,\mathcal{L}(p,N_c^D,k,-l,n_f)\right)\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} p & l & -k-\frac{n_f}{2} & l\\ N_c^D-p & l & l & -k+\frac{n_f}{2}\end{array}\right].\end{aligned} \tag{4.3.35}$$

Full Witten index and matching across the duality. Adding the above contribu-

N_c	k	l	n_f	Electric Side	Magnetic Side
3	0	10	6	$\mathrm{Gr}(3,6) \oplus U(3)_{-3,27} \oplus \underbrace{U(1)_{13} \times U(2)}_{10}{}_{-3,17}$	$\mathrm{Gr}(3,6) \oplus U(3)_{-3,27} \oplus \underbrace{U(1)_{13} \times U(2)}_{10}{}_{-3,17}$
3	1	3	6	$\mathrm{Gr}(3,6) \oplus \underbrace{U(1)_{7} \times U(2)}_{3}{}_{-2,4}$	$\mathrm{Gr}(3,6) \oplus U(3)_{-4,5}$
3	2	1	6	$\mathrm{Gr}(3,6)$	$\mathrm{Gr}(3,6)$
5	$\frac{5}{2}$	0	7	$\mathrm{Gr}(5,7)$	$\mathrm{Gr}(2,7)$
5	$-\frac{3}{2}$	2	7	$\mathrm{Gr}(5,7) \oplus U(5)_{-5,5}$	$\mathrm{Gr}(2,7) \oplus U(2)_{-2,2}$
6	2	-4	8	$\mathrm{Gr}(6,8) \oplus U(6)_{6,-18} \oplus \underbrace{U(5)_{6,-14} \times U(1)}_{-4}{}_{-6}$	$\mathrm{Gr}(2,8) \oplus U(2)_{2,-6} \oplus \underbrace{U(1)_{-10} \times U(1)}_{-4}{}_{-2}$

Table 4.14: Matching moduli spaces of vacua across the maximally-chiral duality with $\xi > 0$.

tions, the Witten index is given by:

$$\mathbf{I}_{\mathrm{W}}^{\max}[N_c, k, l, n_f, 0] = \mathbf{I}_{\mathrm{W,\,I}_D}^{\max}[N_c, k, l, n_f, 0] + \mathbf{I}_{\mathrm{W,\,III,a}_D}^{\max}[N_c, k, l, n_f, 0] + \mathbf{I}_{\mathrm{W,\,III,b}_D}^{\max}[N_c, k, l, n_f, 0] \; . \tag{4.3.36}$$

It is not complicated to prove that the vacua match one-to-one across the duality. We display some examples of dual vacua for $\xi > 0$ and $\xi < 0$ in table 4.14 and 4.15, respectively; see also appendix C. In particular, note that if $\xi < 0$ we can only have topological vacua, in which case the matching of the vacua follows from the level/rank duality (A.2.10).

4.3.3 The marginally-chiral dual theory – $|k| = \frac{n_f}{2}$

Finally, let us briefly discuss the case $k = \epsilon \frac{n_f}{2}$. We have the marginally-chiral duality [92, 1]:

$$U(N_c)_{\epsilon \frac{n_f}{2}, \epsilon \frac{n_f}{2} + l N_c} \oplus n_f \, \square \longleftrightarrow \underbrace{U(n_f - N_c)_{-\epsilon \frac{n_f}{2}, -\epsilon \frac{n_f}{2} + \frac{1}{2}\epsilon (n_f - N_c)} \times U(1)}_{\frac{1}{2}\epsilon}{}_{l + \frac{1}{2}\epsilon} \oplus n_f \, \overline{\square}_0 \oplus \mathbf{det}_{+1} \; . \tag{4.3.37}$$

N_c	k	l	n_f	Electric Side	Magnetic Side
3	0	10	6	$U(3)_{3,33} \oplus \underbrace{U(1)_7 \times U(2)_{3,23}}_{10}$	$U(3)_{3,33} \oplus \underbrace{U(1)_7 \times U(2)_{3,23}}_{10}$
3	1	3	6	$U(3)_{4,13}$ $\oplus \underbrace{U(2)_{4,10} \times U(1)_1}_{3}$	$\underbrace{U(1)_{-1} \times U(2)_{2,8}}_{3}$ $\oplus \underbrace{U(2)_{-4,2} \times U(1)_5}_{3}$
3	2	1	6	$\underbrace{U(2)_{5,7} \times U(1)_0}_{1} \oplus U(3)_{5,8}$	$U(3)_{-5,-2} \oplus \underbrace{U(2)_{-5,-3} \times U(1)_2}_{1}$
5	$\frac{5}{2}$	0	7	$U(5)_{6,6} \oplus U(4)_{6,6} \times U(1)_{-1}$	$U(1)_{-6} \times U(1)_1 \oplus U(2)_{-6,-6}$
6	2	-4	8	$\underbrace{U(4)_{6,-10} \times U(2)_{-2,-10}}_{-4}$	$U(2)_{-6,-14}$

Table 4.15: Matching moduli spaces of vacua across the maximally-chiral duality with $\xi < 0$.

The dual theory involves the 'baryon' $\mathcal{B}$ that transforms in the determinant representation of $U(N_c^D)$ ($N_c^D \equiv n_f - N_c$) and has charge 1 under the additional $U(1)$ factor. Let us choose $k = \frac{n_f}{2} > 0$ for definiteness. The semiclassical vacuum equations then read:

$$\begin{aligned}
&-\sigma_a \widetilde{\varphi}_a^\alpha = 0 \;, \qquad \alpha = 1, \cdots, n_f \;, \qquad a = 1, \cdots, N_c^D \;, \\
&\left(\sum_{a=1}^{N_c^D} \sigma_a + \widetilde{\sigma} \right) \mathcal{B} = 0 \;, \\
&\delta^a{}_b \, \mathcal{B}^\dagger \mathcal{B} - \sum_{\alpha=1}^{n_f} \widetilde{\varphi}_\alpha^{\dagger a} \widetilde{\varphi}_b^\alpha = \frac{\delta^a{}_b}{2\pi} \left(-\frac{n_f}{2}\sigma_a + \frac{1}{2}\sum_{c=1}^{N_c^D} \sigma_c + \frac{1}{2}\widetilde{\sigma} - \frac{n_f}{2}|\sigma_a| + \frac{1}{2}\left| \sum_{c=1}^{N_c^D} \sigma_c + \widetilde{\sigma} \right| \right) \;, \\
&\mathcal{B}^\dagger \mathcal{B} = \frac{1}{2\pi} \left(\xi + \frac{1}{2}\sum_{c=1}^{N_c^D} \sigma_c + \left(l + \frac{1}{2} \right) \widetilde{\sigma} + \frac{1}{2} \left| \sum_{c=1}^{N_c^D} \sigma_c + \widetilde{\sigma} \right| \right) \;.
\end{aligned} \tag{4.3.38}$$

We will not attempt to solve these equations here. Instead, let us simply consider a very special case for which these equations trivialise.

Special case $N_c^D = 0$

If we choose $n_f = N_c = 2k$ in the electric theory, the dual theory is simply the abelian theory:

$$U(1)_{l+\frac{1}{2}} \oplus \mathcal{B} \;, \tag{4.3.39}$$

which was discussed in section 3.2.2. We have the vacua:

$$\Theta(\xi)\,\mathbb{CP}^0 \;\oplus \Theta(-\xi(l+1))\,U(1)_{l+1} \;\oplus \Theta(\xi l)\,U(1)_l \;, \tag{4.3.40}$$

and the index:

$$\mathbf{I}_{\mathrm{W}}\left[n_f, \frac{n_f}{2}, l, n_f, 0\right] = \frac{1}{2} + \left|l + \frac{1}{2}\right| = \begin{cases} l+1\;, & \text{if } l \geq 0\;, \\ |l|\;, & \text{if } l < 0\;. \end{cases} \tag{4.3.41}$$

Let us compare this to the results of section 3.2.4. One directly sees that the various types of vacua of the electric theory contribute as:

$$\begin{aligned} \mathbf{I}_{\mathrm{W,\,I}}\left[n_f, \frac{n_f}{2}, l, n_f, 0\right] &= \Theta(\xi)\;, \\ \mathbf{I}_{\mathrm{W,\,III,a}}\left[n_f, \frac{n_f}{2}, l, n_f, 0\right] &= \Theta\left(-\xi(l+1)\right)|l+1|\;, \\ \mathbf{I}_{\mathrm{W,\,IV}}\left[n_f, \frac{n_f}{2}, l, n_f, 0\right] &= \Theta\left(\xi l\right)|l|\;, \end{aligned} \tag{4.3.42}$$

with no contribution from the Type II and III,b vacua. Adding all the contributions as in (3.2.63), this matches exactly with (4.3.41). We also see that, for $N_c^D = 0$, the strongly-coupled vacua (Type IV vacua) of the electric theory, whose existence we conjectured in section 3.2, correspond to an ordinary topological vacua in the dual description – namely the $U(1)_l$ TQFT in (4.3.40).

Case $N_c^D > 0$

For $k = \frac{n_f}{2}$ and $N_c = n_f - N_c > 0$, however, we find that the electric 'Type IV' vacua also reappear as 'quantum vacua' in the magnetic description. In the electric description, as discussed in section 3.2.4, we have the vacua:

$$\mathcal{M}^{\mathrm{elec}} = \Theta(\xi)\,\mathrm{Gr}(N_c, n_f) \;\oplus \Theta(-\xi(n_f + lN_c))\,U(N_c)_{n_f, n_f+lN_c} \;\oplus \Theta(\xi l)\,\mathcal{M}_{\mathrm{IV}}\;, \tag{4.3.43}$$

where $\mathcal{M}_{\mathrm{IV}}$ denotes the 'quantum vacua'. Looking at the magnetic description, it is useful to write the equations (4.3.38) in terms of the effective real mass for the 'baryon' field $\mathcal{B}$:

$$\sigma_B \equiv \sum_{a=1}^{N_c^D} \sigma_a \,+ \widetilde{\sigma}\;. \tag{4.3.44}$$

We can then analyse the theory by moving along the σ_B line. The first obvious solution is for $\sigma_B = 0 = \sigma_a$, in which case we have the Higgs vacuum:

$$\Theta(\xi)\,\mathrm{Gr}(N_c^D, n_f) \ , \tag{4.3.45}$$

which obviously matches the first term in (4.3.43). For $\sigma_B > 0$, there is also a single solution with $\sigma_a > 0$ that corresponds to the TQFT:

$$\Theta(-\xi((l+1)n_f - lN_c^D))\, U\underbrace{(N_c^D)_{-n_f,\,-n_f+N_c^D} \times U(1)}_{1}{}_{l+1} \ , \tag{4.3.46}$$

which reproduces exactly the second vacuum in (4.3.43) (after a level/rank duality). Finally, we find that, for any $N_c^D > 0$, there is a continuous Coulomb-branch vacuum for $\sigma_B < 0$ and $\sigma_a < 0$. Assuming $\sigma_B < 0$, we can integrate out $\mathcal{B}$ to obtain a decoupled $U(1)_l$ sector (with the FI parameter ξ). This accounts for the factor $\Theta(\xi)|l|$ of the Witten index of the conjectured Type-IV vacuum:

$$\mathbf{I}_{\mathrm{W,\,IV}} = \Theta(\xi l)\,|l|\binom{n_f-1}{N_c-1} = \Theta(\xi l)\,|l|\binom{n_f-1}{N_c^D} \ , \tag{4.3.47}$$

but we do not have a complete understanding of this vacuum in the magnetic theory.

CHAPTER 5

GROTHENDIECK LINES AND QUANTUM K-THEORY OF THE GRASSMANNIAN

In this chapter, we study the 3d GLSM computation of the equivariant quantum K-theory ring of the complex Grassmannian from the perspective of line defects. As we discussed earlier, the 3d GLSM onto $X = \mathrm{Gr}(N_c, n_f)$ is a circle compactification of the 3d $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $U(N_c)_{k,k+lN_c}$ and n_f fundamental chiral multiplets, for any choice of the Chern–Simons levels (k, l) in the 'geometric window'. For $k = N_c - \frac{n_f}{2}$ and $l = -1$, the twisted chiral ring generated by the half-BPS lines wrapping the circle has been previously identified with the quantum K-theory ring $\mathrm{QK}_T(X)$. We identify new half-BPS line defects in the UV gauge theory, dubbed *Grothendieck lines*, which flow to the structure sheaves of the (equivariant) Schubert varieties of X. They are defined by coupling $\mathcal{N} = 2$ supersymmetric gauged quantum mechanics of quiver type to the 3d GLSM. We explicitly show that the 1d Witten index of the defect worldline reproduces the Chern characters for the Schubert classes, which are written in terms of double Grothendieck polynomials. This gives us a physical realisation of the Schubert-class basis for $\mathrm{QK}_T(X)$. We then use 3d A-model techniques to explicitly compute $\mathrm{QK}_T(X)$ as well as other K-theoretic enumerative invariants, such as the topological metric. We also consider the 2d/0d limit of our 3d/1d construction, which gives us local defects in the 2d GLSM, the *Schubert defects*, that realise equivariant quantum cohomology classes.

This chapter is organised as follows. In section 5.1, we review aspects of the 3d A-model for the 3d $\mathcal{N} = 2$ $U(N_c)_{k,k+lN_c}$ gauge theory coupled to n_f fundamental chiral multiplets. We recall the JK residue formula for the $\mathbb{P}^1 \times S^1_\beta$ correlation functions and show how these expressions simplify in the geometric window. In section 5.2, we explicitly construct the Grothendieck lines as 1d supersymmetric quivers, and we demonstrate that these lines flow to the structure sheaves of the Schubert subvarieties of $X = \mathrm{Gr}(N_c, n_f)$. In section 5.3, we revisit the computation of $\mathrm{QK}_T(X)$ from the GLSM perspective. In section 5.4, we briefly discuss the 2d limit of our results. Various supplementary materials are provided in appendices D, E, and F.

5.1 The 3d A-model onto the Grassmannian

Let us consider the 3d $\mathcal{N}=2$ SQCD$[N_c,k,l,n_f,0]$ theory that we studied in some detail in section 3.2 above. Recall that we have the gauge group $U(N_c)_{k,k+lN_c}$ coupled to n_f fundamental chiral multiplets, with $k+\frac{n_f}{2}\in\mathbb{Z}$ due to the parity anomaly (see subsection 2.2.2 for details). Let us study this theory on $\Sigma\times S^1_\beta$, with the topological A-twist along the Riemann surface Σ. We are interested in the half-BPS line defects that preserve the A-twist supercharges and which must then wrap the S^1_β. Such lines appear as twisted chiral local operators from the perspective of the effective 2d $\mathcal{N}=(2,2)$ supersymmetric theory on Σ. For more details on the 3d A-model, see chapter 2.

The effective twisted superpotential for SQCD$[N_c,k,l,n_f,0]$ reads (2.2.8):

$$\begin{aligned}\mathcal{W} &= \frac{1}{(2\pi i)^2}\sum_{\alpha=1}^{n_f}\sum_{a=1}^{N_c}\mathrm{Li}_2(x_a y_\alpha^{-1})+\tau\sum_{a=1}^{N_c}u_a\\ &\quad+\frac{k+\frac{n_f}{2}}{2}\sum_{a=1}^{N_c}u_a(u_a+1)+\frac{l}{2}\left(\left(\sum_{a=1}^{N_c}u_a\right)^2+\sum_{a=1}^{N_c}u_a\right)\ .\end{aligned}\tag{5.1.1}$$

The Bethe equations (2.2.18),

$$\Pi_a(x,y,q)=1\ ,\tag{5.1.2}$$

determine the 2d vacua of the 3d A-model. The quantity Π_a is the gauge flux operator (2.1.17):

$$\Pi_a(x,y,q)\equiv e^{2\pi i\partial_{u_a}\mathcal{W}}=q(-\det x)^l(-x_a)^{k+\frac{n_f}{2}}\prod_{\alpha=1}^{n_f}\frac{1}{1-x_a y_\alpha^{-1}}\ .\tag{5.1.3}$$

Here, we also introduced the notation $\det x\equiv\prod_{a=1}^{N_c}x_a$.

5.1.1 Correlation functions of half-BPS lines and Frobenius algebra

We wish to compute the correlation functions of half-BPS lines, $\mathscr{L}_\mu$. We will sometimes denote by $\mathscr{L}$ any given product of such lines located at distinct points on Σ:

$$\mathscr{L}=\prod_s\mathscr{L}_{\mu_s}(p_s)\ .\tag{5.1.4}$$

Due to the topological A-twist, correlation functions do not depend on the positions $p_s \in \Sigma$, hence we will often omit these labels in the following. Parallel lines form a ring, $\mathcal{R}_{3\text{d}}$, which is defined physically as the twisted chiral ring obtained by fusion along Σ, with the product:

$$\mathscr{L}_\mu \star \mathscr{L}_\nu = \mathcal{N}_{\mu\nu}{}^{\lambda} \mathscr{L}_\lambda \;, \qquad\qquad \mathcal{N}_{\mu\nu}{}^{\lambda} \in \mathbb{K} \equiv \mathbb{Z}(y, q) \;. \tag{5.1.5}$$

Here, $\{\mathscr{L}_\mu\}$ forms a $\mathbb{K}$-basis of $\mathcal{R}_{3\text{d}}$, and the sum over repeated indices is understood. The structure coefficients are encoded by certain 3-point functions. Moreover, the ring $\mathcal{R}_{3\text{d}}$ is endowed with a Frobenius algebra structure because the A-model is a 2d TQFT on Σ – see [134] for an explicit discussion in the present case. The non-degenerate Frobenius metric η, also known as the topological metric, is simply the two-point function on the Riemann sphere:

$$\eta_{\mu\nu} \equiv \eta(\mathscr{L}_\mu, \mathscr{L}_\nu) = \Big\langle \mathscr{L}_\mu(p_1)\mathscr{L}_\nu(p_2) \Big\rangle_{\mathbb{P}^1 \times S^1} \;. \tag{5.1.6}$$

Let us denote by $\eta^{\mu\nu}$ the inverse metric. Then, the structure constants are obtained from the genus-0 three-point functions according to:

$$\mathcal{N}_{\mu\nu}{}^{\lambda} = \eta^{\lambda\delta}\mathcal{N}_{\mu\nu\delta} \;, \qquad\qquad \mathcal{N}_{\mu\nu\delta} = \Big\langle \mathscr{L}_\mu(p_1)\mathscr{L}_\nu(p_2)\mathscr{L}_\delta(p_3) \Big\rangle_{\mathbb{P}^1 \times S^1} \;. \tag{5.1.7}$$

Here, the 'expectation value' $\langle \cdots \rangle_{\Sigma \times S^1}$ denotes the unnormalised path integral of the 3d $\mathcal{N} = 2$ supersymmetric field theory on $\Sigma \times S^1$ with the A-twist along Σ and with the periodic spin structure on S^1. Note that these observables are valued in the field $\mathbb{K} = \mathbb{Z}(y, q)$ of rational functions with integer coefficients of the 3d flavour parameters y_α and q. The integrality property is related to the fact that we are considering a 3d $\mathcal{N} = 2$ theory on $\Sigma \times S^1$, hence the observables could also be computed, in principle, as traces over Hilbert spaces of certain effective supersymmetric quantum mechanics on S^1 – see *e.g.* [170, 173, 171].

Let us now review two distinct ways by which we will explicitly compute these observables:

The Bethe-vacua formula

From the perspective of the infrared Coulomb-branch theory on Σ, one can compute any correlator on the sphere as a sum over the Bethe vacua [19]:

$$\Big\langle \mathscr{L} \Big\rangle_{\mathbb{P}^1 \times S^1_\beta} = \sum_{\hat{x} \in \mathcal{S}_{\text{BE}}} \mathcal{H}(\hat{x}, y)^{-1} \, \mathscr{L}(\hat{x}, y, q) \;. \tag{5.1.8}$$

Here, $\mathcal{H}$ is the handle-gluing operator (2.1.20). As an explicit rational function in the gauge parameters, it reads:

$$\mathcal{H}(x, y, q) = \det(\mathbf{H})\; e^{2\pi i \Omega} \;, \tag{5.1.9}$$

with $\mathbf{H}$ the Hessian matrix:

$$\mathbf{H}_{ab}(x, y) = \frac{\partial \mathcal{W}}{\partial u_a \partial u_b} = \delta_{ab}\left(k + \frac{n_f}{2} + \sum_{\alpha=1}^{n_f} \frac{x_a y_\alpha^{-1}}{1 - x_a y_\alpha^{-1}}\right) + l \;, \tag{5.1.10}$$

and Ω the effective dilaton potential (2.1.12), which is given by:

$$e^{2\pi i \Omega} = \prod_{\alpha=1}^{n_f} (1 - x_a y_\alpha)^{-r+1} \prod_{\substack{a,b \\ a\neq b}} (1 - x_a x_b^{-1})^{-1} (\det x)^{K_{RG}} \;. \tag{5.1.11}$$

Note the dependence on the R-charge $r \in \mathbb{Z}$ for the chiral multiplets and on the gauge-R bare CS level K_{GR} – we will shortly set $r = 0$ and $K_{GR} = 0$. In writing down (5.1.1) and (5.1.11), we set all bare flavour and R-symmetry CS levels to zero, $K_{FF} = K_{FR} = K_{RR} = 0$, following the conventions of subsection 3.1.1. In (5.1.8), the product of lines, $\mathscr{L}$, is represented by a polynomial $\mathscr{L}(x, y, q)$. These line operators will be studied in more detail in section 5.2.

In principle, the sum over Bethe vacua (5.1.8) can be performed explicitly using computational algebraic geometry methods, as we explained in 2.2.3. In practice, this method relies on Gröbner basis algorithms that quickly become computationally prohibitive as we increase n_f and N_c. Such methods can nonetheless be used to efficiently compute the quantum K-theory ring of the Grassmannian for small-enough values of N_c and n_f, as we will discuss in section 5.3.

The Jeffery–Kirwan residue formula

The other method to compute correlation functions on the sphere is through supersymmetric localisation in the UV theory, which leads to a JK residue formula [113] (see also [114, 115]):

$$\Big\langle \mathscr{L} \Big\rangle_{\mathbb{P}^1 \times S^1_\beta} = \frac{1}{N_c!} \sum_{\mathfrak{m} \in \mathbb{Z}^{N_c}} \sum_{x_* \in \widetilde{\mathfrak{M}}^{\mathfrak{m}}_{\text{sing}}} \underset{x = x_*}{\text{JK-Res}} \left[\mathbf{Q}(x_*), \eta_\xi\right] \mathfrak{I}_\mathfrak{m}[\mathscr{L}](x, y, q) \;. \tag{5.1.12}$$

This formula involves a sum over all $U(N_c)$ magnetic fluxes $\mathfrak{m} = (\mathfrak{m}_a) \in \mathbb{Z}^{N_c}$ through $\mathbb{P}^1$, and a sum over JK residues in each flux sector. The JK residues are taken with respect to the N_c-form:

$$\mathfrak{I}_{\mathfrak{m}}[\mathscr{L}](x, y, q) = (-2\pi i)^{N_c} Z_{\mathfrak{m}}(x, y, q) \mathscr{L}(x, y, q)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{N_c}}{x_{N_c}}\ , \tag{5.1.13}$$

which has codimension-N_c singularities (including at infinity) denoted by $\widetilde{\mathfrak{M}}_{\text{sing}}$. For each magnetic flux $\mathfrak{m} \in \mathbb{Z}^{N_c}$, the factor $Z_{\mathfrak{m}}$ is given in terms of the effective dilaton potential and gauge flux operators as:

$$Z_{\mathfrak{m}}(x, y, q) = e^{-2\pi i \Omega} \prod_{a=1}^{N_c} \Pi_a(x, y, q)^{\mathfrak{m}_a}\ . \tag{5.1.14}$$

To each singularity $x_* \in \widetilde{\mathfrak{M}}_{\text{sing}}$, one assigns a charge vector $\mathbf{Q}(x_*)$ which determines whether or not the singularity contributes non-trivially to the JK residue, given a choice of the auxiliary parameter η_ξ. In appendix D, we further study this JK residue formula for the SQCD$[N_c, k, l, n_f, 0]$ theory we are considering here. In particular, we show that for a certain choice of η_ξ, the sum over singularities x_* is closely related to the sum over 3d vacua that we studied in section 3.2.

5.1.2 The geometric window and the Grassmannian 3d GLSMs

For positive FI parameters and vanishing masses, $\xi > 0$ and $\nu_\alpha = 0$, the SQCD$[N_c, k, l, n_f, 0]$ theory has a pure Higgs branch given by the Grassmannian manifold:

$$\mathcal{M}_{\text{Higgs}} = \text{Gr}(N_c, n_f)\ , \tag{5.1.15}$$

For generic values of k and l, the theory also has a number of additional topological and hybrid vacua, as we discussed in section 3.2 above, which precludes a purely geometric interpretation of the infrared physics. At fixed N_c and n_f with $n_f \geq N_c$, we say that the $U(N_c)$ gauge theory is in the *geometric window* if and only if its Witten index is equal to the Euler characteristic of the Grassmannian:

$$\mathbf{I}_{\text{W}}[N_c, k, l, n_f, 0] = \chi(\text{Gr}(N_c, n_f)) = \binom{n_f}{N_c}\ . \tag{5.1.16}$$

$N_c\backslash n_f$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	11	16	22	29	37	46	56	67	79	92	106	121	137	154	172	191	211	232
3	9	9	11	14	18	22	27	32	38	44	51	58	66	74	83	92	102	112
4	–	12	10	11	12	15	18	21	25	29	33	38	43	48	54	60	66	73
5	–	–	17	12	12	13	14	16	19	22	25	28	32	36	40	44	49	54
6	–	–	–	23	15	13	14	15	16	18	20	23	26	29	32	35	39	43
7	–	–	–	–	30	19	16	15	16	17	18	20	22	24	27	30	33	36
8	–	–	–	–	–	38	23	19	17	17	18	19	20	22	24	26	28	31
9	–	–	–	–	–	–	47	28	22	20	19	19	20	21	22	24	26	28
10	–	–	–	–	–	–	–	57	33	26	23	21	21	21	22	23	24	26
11	–	–	–	–	–	–	–	–	68	39	30	26	24	23	23	23	24	25
12	–	–	–	–	–	–	–	–	–	80	45	34	29	27	25	25	25	25

Table 5.1: Some values of $n_{\rm gw}(N_c, n_f)$, the number of theories in the geometric window with $l < 0$.

The geometric window intersects the subset of theories with $l = 0$ at $|k| \leq \frac{n_f}{2}$. For $l \neq 0$, we find a larger but finite number of theories in the geometric window.[1] Let $n_{\rm gw}(N_c, n_f)$ denote the number of such theories with $l < 0$. Then, the number of distinct theories in the geometric window is:

$$N_{\rm gw}^{\rm tot}(N_c, n_f) = 2n_{\rm gw}(N_c, n_f) + n_f + 1 \ . \tag{5.1.17}$$

Note that the levels (k, l) and $(-k, -l)$ give equivalent theories, hence, there are (at most) $n_{\rm gw} + \lfloor \frac{n_f+1}{2} \rfloor$ inequivalent models in the geometric window. Using the known formula for the Witten index (3.2.63), we can compute $n_{\rm gw}$ by brute force, as shown in table 5.1. For $N_c = 2$ and $N_c = 3$ (and $n_f > N_c$), we find the patterns:

$$\begin{aligned} n_{\rm gw}(2, n_f) &= \frac{n_f(n_f+1)}{2} + 1 + \delta_{n_f,3} \ , \\ n_{\rm gw}(3, n_f) &= \begin{cases} \frac{n_f^2}{4} + 2 + 3\delta_{n_f,4} \ , & \text{if } n_f \text{ is even} \ , \\ \frac{n_f^2}{4} + \frac{7}{4} + \delta_{n_f,5} \ , & \text{if } n_f \text{ is odd} \ . \end{cases} \end{aligned} \tag{5.1.18}$$

We did not find any clear pattern in general, however. It may be useful to further distinguish between theories in the geometric window which are maximally chiral ($|k| < \frac{n_f}{2}$), marginally chiral ($|k| = \frac{n_f}{2}$), or minimally chiral ($|k| > \frac{n_f}{2}$). We denote the number

[1] We exclude the case $N_c = 1$ from this discussion, since l is a redundant parameter in that case. A completely explicit (though unwieldy) formula for $\mathbf{I}_{\rm W}[N_c, k, l, n_f, 0]$ is given in (3.2.63). We do not have a more elegant description of the geometric window at the moment.

of such theories with $l < 0$ by $n_{\rm gw}^+$, $n_{\rm gw}^0$ and $n_{\rm gw}^-$, respectively, with $n_{\rm gw} = n_{\rm gw}^+ + n_{\rm gw}^0 + n_{\rm gw}^-$. The three types of theories are distinguished by their infrared dual description, as we discussed in section 4.3. In the maximally chiral case, we have the duality (4.3.29):

$$U(N_c)_{k,\, k+lN_c} \ , \ n_f \,\square \qquad \longleftrightarrow \qquad U(n_f - N_c)_{-k,\, -k+l(n_f-N_c)} \ , \ n_f \,\overline{\square} \ , \tag{5.1.19}$$

in which case the Higgs branch vacua are easily matched as:

$$\text{Gr}(N_c, n_f) \cong \text{Gr}(n_f - N_c, n_f) \ . \tag{5.1.20}$$

One can also check, in examples, that we have $n_{\rm gw}^+(N_c, n_f) = n_{\rm gw}^+(n_f - N_c, n_f)$, as one would expect from the duality (5.1.19). The dualities for the marginally and maximally chiral theories are more complicated, with the dual gauge group being $U(|k| + \frac{n_f}{2} - N_c) \times U(1)$; in that case, in the classification of section 3.2, the Higgs branch (5.1.15) appears in the dual gauge theory as a hybrid Higgs-topological vacuum such that the would-be TQFT factor has a single state. In this work, we shall focus on the 'electric' SQCD description; it would certainly be interesting to understand better the duality map on defect lines (beyond the well-understood $l = 0$ case [115]).

Grassmannian 3d $\mathcal{N} = 2$ GLSMs

For the massless SQCD$[N_c, k, l, n_f, 0]$ theory with (k, l) in the geometric window, by definition, we have a unique Higgs branch of vacua (5.1.15) when $\xi > 0$. Turning on the $SU(n_f)$ mass parameters, $y_\alpha \neq 1$, along the maximal torus $T \subset U(n_f)$, one achieves a T-equivariant deformation of the geometry, and the Higgs branch collapses to $\chi(\text{Gr}(N_c, n_f))$ massive vacua.

Picking any (k, l) in the geometric window, we then obtain a 3d GLSM which corresponds to the 3d gauge theory on $\Sigma \times S^1_\beta$. Following the 3d renormalisation group flow, it is expected that the gauge theory with $\xi \gg 0$ flows to an infrared 3d NLSM onto $X \equiv \text{Gr}(N_c, n_f)$:

$$\text{3d NLSM} \ : \ \Sigma \times S^1_\beta \longrightarrow X \ . \tag{5.1.21}$$

We will not attempt to precisely define the NLSM infrared phase of our theory, however. Instead, more conservatively, we shall define and study the 3d GLSM as an ordinary 2d GLSM for the effective 2d field theory on Σ at scales $\mu \ll \frac{1}{\beta}$, with β the radius of S^1. The 3d origin of this construction remains apparent in the 2d gauge-theory description through the non-trivial periodicities of the CB scalars u_a and through the explicit form of the effective twisted superpotential and effective dilaton potential (5.1.1) and (5.1.11). Most importantly, the twisted chiral operators are now line defects, as emphasised above. In the next section, we study these line operators in detail. We will come back to the study of the 3d GLSM observables in section 5.3.

5.1.3 $\mathbb{P}^1 \times S^1_\beta$ correlation functions in the geometric window

In the geometric window, the JK residue formula (5.1.12) for the genus-0 correlators simplifies significantly. Then, as we explain in appendix D, the only contributing singularities are "Higgs branch" singularities (where chiral multiplets go massless on the 3d Coulomb branch). Let us write the correlators as a sum over topological sectors:

$$\Big\langle \mathscr{L} \Big\rangle_{\mathbb{P}^1 \times S^1_\beta}(q, y) = \sum_{d=0}^{\infty} q^d \, \mathbf{I}_d[\mathscr{L}](y) \; , \tag{5.1.22}$$

where $d = |\mathfrak{m}| \equiv \sum_{a=1}^{N_c} \mathfrak{m}$ is the magnetic flux for the overall $U(1) \subset U(N_c)$. It corresponds to the degree of the holomorphic map $\phi : \Sigma \to X$ in the infrared NLSM realisation. At each degree, we have the residue formula:

$$\mathbf{I}_d[\mathscr{L}](y) \equiv \sum_{\substack{\mathfrak{m}_a \geq 0 \\ |\mathfrak{m}|=d}} \; \sum_{1 \leq \alpha_1 < \cdots < \alpha_{N_c} \leq n_f} \; \underset{\{x_a = y_{\alpha_a}\}}{\text{Res}} \; \frac{(-1)^{|\mathfrak{m}|(K+l)+N_c} \Delta(x) \mathscr{L}(x, y)}{\prod_{a=1}^{N_c} x_a^{\mathfrak{r}_a} \prod_{\alpha=1}^{n_f} \left(1 - x_a y_\alpha^{-1}\right)^{1+\mathfrak{m}_a}} \; . \tag{5.1.23}$$

Note that we have set the R-charge $r = 0$, which is the natural choice from the GLSM point of view. We also defined the integers

$$\mathfrak{r}_a \equiv N_c + K_{GR} - l|\mathfrak{m}| - K\mathfrak{m}_a \; , \qquad a = 1, \cdots, N_c \; , \tag{5.1.24}$$

at fixed $\mathfrak{m}$, and the Vandermonde determinant:

$$\Delta(x) \equiv \prod_{1 \leq a \neq b \leq N_c} (x_a - x_b) \; . \tag{5.1.25}$$

The expression (5.1.23) is given by a finite sum over all the magnetic fluxes $\mathfrak{m}_a \geq 0$ at fixed degree $d = |\mathfrak{m}|$. Then, in each flux sector $\mathfrak{m} = \{\mathfrak{m}_a\}$, we sum over all the "Higgs-branch" residues at:

$$(x_1, \cdots, x_{N_c}) = (y_{\alpha_1}, \cdots, y_{\alpha_{N_c}}) \; , \qquad 1 \leq \alpha_1 < \cdots < \alpha_{N_c} \leq n_f \; . \tag{5.1.26}$$

We use the S_{N_c} gauge symmetry to order the singularities as indicated, thus cancelling out the $N_c!$ factor in (5.1.12).[1]

The non-equivariant limit

The residue formula (5.1.23) is valid for generic values of the equivariant parameters y_α. The JK residue formula (5.1.12) holds more generally, however. In particular, let us consider the non-equivariant limit, setting $y_\alpha = 1$. Then, we have:

$$\mathbf{I}_d[\mathscr{L}] \equiv \sum_{\substack{\mathfrak{m}_a \geq 0 \\ |\mathfrak{m}|=d}} \operatorname*{Res}_{\{x_a=1\}} \frac{(-1)^{\mathfrak{m}(K+l)+N_c} \Delta(x) \mathscr{L}(x)}{N_c! \prod_{a=1}^{N_c} x_a^{\mathfrak{r}_a} (1-x_a)^{n_f(1+\mathfrak{m}_a)}} \,. \tag{5.1.27}$$

with a single residue at the codimension-N_c singularity $x_a = 1$, in each flux sector. (Note the factor of $1/N_c!$ compared to (5.1.23).)

$U(1)$ example

As a simple example, consider the GLSM onto $\mathbb{P}^{n_f-1}$, which is the $U(1)_k$ theory with n_f charge-1 chiral multiplets ($K \equiv k + \frac{n_f}{2}$), with the constraint $0 \leq K \leq n_f$ so that we are in the geometric window. We also choose $K_{RG} = 0$. Focussing on the non-equivariant limit, the formula (5.1.23) gives us:

$$\begin{aligned} \mathbf{I}_\mathfrak{m}[\mathscr{L}] &= \frac{(-1)^{\mathfrak{m}K+1}}{2\pi i} \oint \frac{dx}{x^{1-K\mathfrak{m}}} \frac{\mathscr{L}(x)}{(1-x)^{n_f(\mathfrak{m}+1)}} \\ &= \frac{(-1)^{n_f(\mathfrak{m}+1)+\mathfrak{m}K+1}}{(n_f(\mathfrak{m}+1)-1)!} \left[\frac{d^{n_f(\mathfrak{m}+1)-1}}{dx^{n_f(\mathfrak{m}+1)-1}} \frac{\mathscr{L}(x)}{x^{1-K\mathfrak{m}}} \right] \Bigg|_{x=1} , \end{aligned} \tag{5.1.28}$$

for the non-negative integer $\mathfrak{m} = d$. In particular, we find that the partition function on the sphere is given by:

$$Z_{\mathbb{P}^1 \times S^1_\beta} = \langle 1 \rangle_{\mathbb{P}^1 \times S^1_\beta} = \begin{cases} 1\,, & \text{if } \ 0 < K \leq n_f\,, \\ \frac{1}{1-q}\,, & \text{if } \ K = 0\,. \end{cases} , \tag{5.1.29}$$

[1] We also used the fact that singularities with $x_a = x_b = y_\alpha$ for some $a \neq b$ have vanishing residue due to the Vandermonde determinant in the numerator.

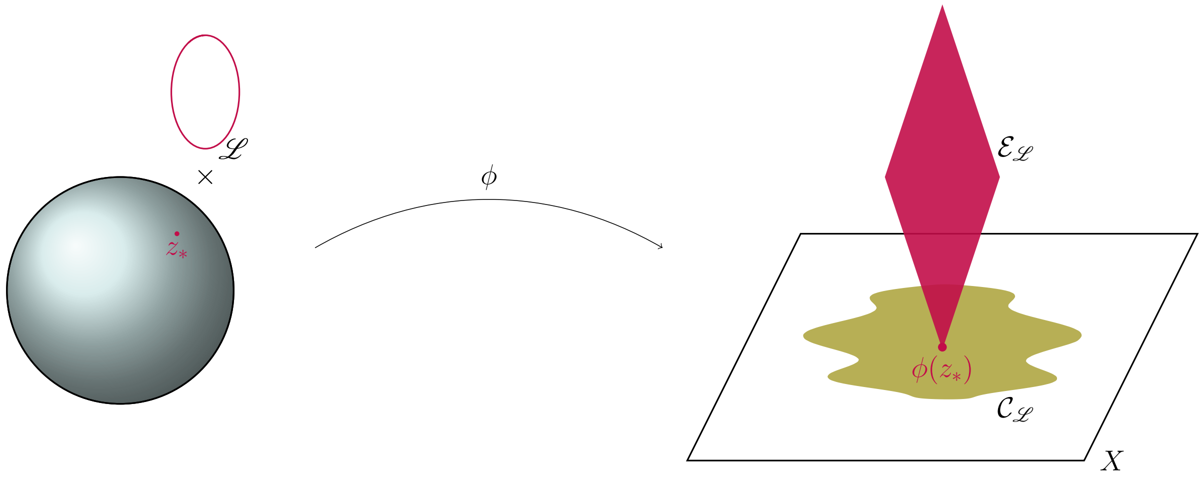


Figure 5.1: The insertion of a half-BPS line defect $\mathscr{L}$ at a point z_* restricts the target of the map ϕ to $\mathcal{C}_{\mathscr{L}} \subseteq X$ at that point and the line defect flows in the IR to the coherent sheaf $\mathcal{E}_{\mathscr{L}} \in \mathrm{K}(X)$.

Indeed, we easily see that $\mathbf{I}_{\mathfrak{m}}[1] = 1$ if $\mathfrak{m} = 0$ or if $K = 0$ (for any $m \geq 0$), while $\mathbf{I}_{\mathfrak{m}}[1] = 0$ when $0 < K \leq n_f$ and $\mathfrak{m} > 0$ because $K\mathfrak{m} - 1 < n_f(\mathfrak{m}+1)$ and $K\mathfrak{m} - 1 \geq 0$ in that case.

5.2 Wilson lines and Grothendieck lines

The 3d uplift of the standard 2d GLSM modifies the target-space interpretation of the twisted chiral operators. While the 2d local operators $\omega \in \mathcal{R}^{2\mathrm{d}}$ represent cohomology classes on the target space X, the 3d line operators wrapping the S^1_β should be interpreted as coherent sheaves on X [174, 122, 123]:

$$\mathscr{L} \in \mathcal{R}_{3\mathrm{d}} \;, \qquad\qquad \mathrm{RG} : \mathscr{L} \longrightarrow \mathcal{E}_{\mathscr{L}} \in \mathrm{coh}(X) \;. \tag{5.2.1}$$

It is expected that the RG flow maps any half-BPS defect line $\mathscr{L}$ defined in the UV 3d $\mathcal{N} = 2$ gauge theory to a coherent sheaf $\mathcal{E}_{\mathscr{L}}$ on the target space X. More precisely, as we will discuss further below, the physical observables only depend on the Grothendieck group of the abelian category $\mathrm{coh}(X)$ – that is, on the K-theory of X (see figure 5.1):

$$\mathscr{L} \;\to\; [\mathcal{E}_{\mathscr{L}}] \in \mathrm{K}(X) \;. \tag{5.2.2}$$

A proper, first-principle understanding of this map would require a better understanding of the 3d NLSM phase itself. We will not try to tackle this problem here. Instead, we will take a pedestrian view of K-theory based on representing sheaves by their Chern character:

$$\mathrm{ch} \,:\, \mathrm{K}(X) \to \mathrm{H}^\bullet(X) \,:\, [\mathcal{E}_{\mathscr{L}}] \mapsto \mathrm{ch}(\mathcal{E}_{\mathscr{L}}) \,. \tag{5.2.3}$$

This is particularly suited to the 3d A-model description, as we will review momentarily. From this perspective, it will become clear that the physical observables defined in subsection 5.1.1 can only depend on the K-theory class of the line $\mathscr{L}$, because they only depend on the Chern characters.

Mathematically, the statement that the 3d uplift of a 2d GLSM corresponds to the K-theory of the target space follows from the observation made by M. Atiyah [175]:[1]

$$\mathrm{K}_T(X) \cong \mathrm{H}^\bullet_{S^1_\beta}(\mathcal{L}X) \,. \tag{5.2.4}$$

Here $\mathcal{L}X$ is the loop space of X – the space of maps from S^1 to the target X. Moreover, $\mathrm{H}^\bullet_{S^1_\beta}$ is the $U(1)$-equivariant cohomology ring whose equivariant parameter $\frac{1}{\beta}$ being the mass of the KK modes.

Historically, Atiyah's work was originally motivated by Witten's work connecting supersymmetric quantum mechanics with index theory [174]. These arguments are analogous to the ones we followed in chapter 1 to interpret the 3d twisted indices as quantum K-theory invariants of X.

A chief motivation to study this 3d GLSM into the Grassmannian is to give a physics derivation of the quantum K-theory ring of $X = \mathrm{Gr}(N_c, n_f)$ [17, 131]. In the recent literature, this was achieved by studying Wilson lines in the UV gauge theory [132, 20, 133, 134]. Under the map (5.2.1), Wilson lines map to locally free sheaves (*i.e.* to complex vector bundles) on X. In this section, we construct a new class of UV line defects, dubbed Grothendieck lines, which flow to sheaves with compact support on subvarieties of X. The Grothendieck lines are instances of vortex loops, as studied *e.g.* in [177, 178, 179, 180]. They give us a much more natural basis to describe the quantum K-theory ring of X, in direct parallel with standard mathematical results [131].

5.2.1 The Coulomb-branch perspective

Recall from chapter 2 that in the 3d A-model description, we deal with the effective field theory on the 2d $\mathcal{N} = (2,2)$ Coulomb branch parametrised by the dimensionless

[1] For more recent discussion on this, see [176]. I would like to thank K. Dedushenko for explaining this point to me.

quantities:

$$u_a = i\beta\sigma_a - a_{0,a} \ , \qquad a_{0,a} \equiv \frac{1}{2\pi}\int_{S^1} A_a \ , \qquad a = 1, \cdots , N_c \ , \tag{5.2.5}$$

where σ_a and A_a denote the real scalar and 1-form gauge field, respectively, in the 3d $\mathcal{N} = 2$ $U(1)_a$ vector multiplet. The single-valued parameters x_a are best interpreted as the holonomies of the half-BPS Wilson loops of unit charge for these abelian vector multiplets:

$$x_a = e^{2\pi i u_a} = \exp\left(-i\int_{S^1}(A_a - i\sigma_a d\psi)\right) \ . \tag{5.2.6}$$

The abelianised theory retains the Weyl group S_{N_c} as a residual gauge symmetry that acts by permutation on the x_a variables.

Given the Grassmannian $X = \mathrm{Gr}(N_c, n_f)$, we have the rank-$N_c$ tautological vector bundle (also called the universal subbundle), $\mathcal{S}$, whose fibre at a point $p \in X$ is the linear subspace $\mathbb{C}^{N_c} \subset \mathbb{C}^{n_f}$ that this point represents:

$$\mathbb{C}^{N_c} \to \mathcal{S} \to X \ . \tag{5.2.7}$$

In the 2d GLSM, the scalars σ_a can be identified with the curvature of the dual bundle $\mathcal{S}$ [25]. By the same token, we can identify $2\pi i u_a$ with the Chern roots of $\mathcal{S}$, so that:

$$\mathrm{ch}(\mathcal{S}) = \sum_{a=1}^{N_c} x_a \ . \tag{5.2.8}$$

More generally, the Chern character of any vector bundle (or coherent sheaf) on X can be written as a symmetric polynomial in the x_a's. Given any UV lines $\mathscr{L}$, the A-model computation of their correlation functions only depends on the symmetric polynomials that give us the Chern classes of the coherent sheaves $\mathcal{E}_{\mathscr{L}}$:

$$\mathscr{L}(x) \equiv \mathrm{ch}(\mathcal{E}_{\mathscr{L}}) \ \in \ \mathrm{K}(X) \ , \tag{5.2.9}$$

schematically. While we have set $y_i = 1$ in this discussion, turning on the flavour fugacities simply corresponds to a T-equivariant deformation, as already mentioned, so that we have:

$$\mathscr{L}(x, y) \ \in \ \mathrm{K}_T(X) \ , \tag{5.2.10}$$

with $\mathscr{L}(x, y)$ representing the equivariant Chern character of some T-equivariant coherent sheaf. In the rest of this section, we identify the UV lines that flow to some important classes of coherent sheaves on the Grassmannian.

5.2.2 Wilson lines and vector bundles

Given any representation $\mathfrak{R}$ of $U(N_c)$, we have the half-BPS Wilson line:

$$W_{\mathfrak{R}} = \mathrm{Tr}_{\mathfrak{R}}\left(Pe^{-i\int_{S^1}(A-i\sigma d\psi)}\right) , \tag{5.2.11}$$

whose classical VEV gives us the Chern character:

$$W_{\mathfrak{R}}(x) \equiv \mathrm{ch}(\mathcal{E}_{W_{\mathfrak{R}}}) = \mathrm{Tr}_{\mathfrak{R}}(x) . \tag{5.2.12}$$

In particular, the fundamental Wilson line, $W_\square$, flows to the tautological line bundle:

$$\mathcal{E}_{W_\square} \cong \mathcal{S} , \qquad\qquad W_\square(x) = \sum_{a=1}^{N_c} x_a . \tag{5.2.13}$$

Recall that an irreducible representation of $U(N_c)$ is specified by a partition $\rho = [\rho_1, \cdots, \rho_{N_c}]$ or, equivalently, by a Young tableau with ρ_a boxes in the a-th row. The Wilson loop in the representation $\mathfrak{R}_\rho$ determined by ρ is then represented in the 3d A-model by the Schur polynomial:

$$W_\rho(x) = \mathrm{ch}(\mathcal{E}_{W_\rho}) = s_\rho(x) . \tag{5.2.14}$$

In fact, one can directly argue that W_ρ flows to the vector bundle $\mathcal{E}_{W_\rho}$ of rank $\dim(\mathfrak{R}_\rho)$ obtained from tensor products of S by using the Schur functor. The basic reason is that the local operators that can end on Wilson lines, which corresponds to sections of $\mathcal{E}_{W_\rho}$, are local operators transforming in the gauge-representation $\mathfrak{R}_\rho$ that are built out of chiral-multiplet operators such as ϕ itself, which maps onto the full target space. To restrict the support of the would-be coherent sheaf $\mathcal{E}_{\mathscr{L}}$ in the infrared to a subvariety of X, we need to use a different construction of the line $\mathscr{L}$, as we now explain.

5.2.3 Grothendieck lines and Schubert classes

We will now construct half-BPS lines that flow to coherent sheaves that are not necessarily locally free. In particular, we would like to construct lines that flow to the Schubert classes – the structure sheaves of Schubert varieties –, denoted by $\mathcal{O}_\lambda$, which form a more convenient basis for the K-theory ring of X. Any coherent sheaf has a free resolution:

$$\cdots \to W_{\mathfrak{R}_2} \to W_{\mathfrak{R}_1} \to \mathscr{L} \to 0 \ , \tag{5.2.15}$$

written here, schematically, in terms of line operators in the UV gauge theory, with the locally-free sheaves constructed from Wilson lines (the representations $\mathfrak{R}_i$ that appear in (5.2.15) are not necessarily reducible). At the level of the Chern characters, we then have:

$$\mathscr{L}(x,y) = \sum_i (-1)^{i+1}\, W_{\mathfrak{R}_i}(x,y) \ . \tag{5.2.16}$$

In principle, to compute any set of observables, we could work out the relevant polynomials $\mathscr{L}(x,y)$ from the knowledge of the free resolutions (5.2.15) – this was done *e.g.* in [135], in some examples. For instance, we have:

$$0 \to \det \mathcal{S} \to \mathcal{O} \to \mathcal{O}_{\square} \to 0 \ , \tag{5.2.17}$$

where $\mathcal{O}_{\square}$ is the codimension-one Schubert variety, to be introduced momentarily. Instead of going down that route, we would prefer to directly define the defect line $\mathscr{L}$ in the UV. This allows us to compute the polynomials $\mathscr{L}(x,y)$ directly from the physics.

Schubert cells and Schubert subvarieties

Before discussing the physical construction, let us briefly review some important geometric facts about the Grassmannian – we refer to [181, 182] for further background.

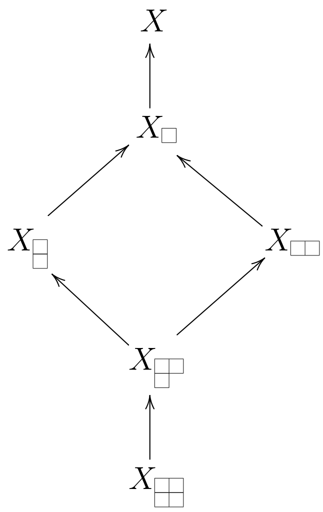
There is a natural action of $\mathrm{GL}(n_f,\mathbb{C})$ on the Grassmannian $X \equiv \mathrm{Gr}(N_c, n_f)$. Indeed, any N_c-plane in $\mathbb{C}^{n_f}$ can be represented by the $N_c \times n_f$ matrix:

$$\phi = (\phi^a{}_\alpha) \ , \qquad a = 1, \cdots, N_c \ , \quad \alpha = 1, \cdots, n_f \ , \tag{5.2.18}$$

modulo the action of the complexified gauge group $\mathrm{GL}(N_c,\mathbb{C})$ from the left, and with the symmetry $\mathrm{GL}(n_f,\mathbb{C})$ acting from the right. We can use the gauge freedom to pick a

$w \in S_4$	λ	ϕ	$\dim(X_\lambda)$
$\{1,2,3,4\}$	$[0,0]$	$\begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \end{pmatrix}$	4
$\{1,3,2,4\}$	$[1,0]$	$\begin{pmatrix} 1 & \star & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix}$	3
$\{2,3,1,4\}$	$[1,1]$	$\begin{pmatrix} 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix}$	2
$\{1,4,2,3\}$	$[2,0]$	$\begin{pmatrix} 1 & \star & \star & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	2
$\{2,4,1,3\}$	$[2,1]$	$\begin{pmatrix} 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	1
$\{3,4,1,2\}$	$[2,2]$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	0

(a) Schubert cells of Gr$(2,4)$.



(b) Hasse diagram of Schubert varieties.

Figure 5.2: LEFT: Schubert cells for Gr$(2,4)$, indexed by permutations or partitions, with the matrix ϕ_λ shown explicitly. RIGHT: The Hasse diagram of inclusion of Schubert varieties inside Gr$(2,4)$, where the arrow $X \to Y$ denotes inclusion of X inside Y.

representative:

$$\phi = \begin{pmatrix} 1 & 0 & \cdots & 0 & \star & \cdots & \star \\ 0 & 1 & \cdots & 0 & \star & \cdots & \star \\ \vdots & & \ddots & & & \ddots & \\ 0 & 0 & \cdots & 1 & \star & \cdots & \star \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{N_c} & \bigstar_{N_c \times (n_f - N_c)} \end{pmatrix} . \tag{5.2.19}$$

Here and in the following, $\mathbb{I}_n$ denotes a $n \times n$ identity matrix and $\bigstar_{n\times m}$ denotes any $n \times m$ matrix with undetermined entries (similarly, $0_{n\times m}$ will denote a matrix with vanishing entries).

The homology of X is generated by the Schubert varieties $X_\lambda \subseteq X$, which are the closure of the Schubert cells, $\mathcal{C}_\lambda$. The Schubert cells correspond to fixed loci under the maximal torus $T \subset \mathrm{GL}(n_f, \mathbb{C})$. Each Schubert cell can be indexed by a choice of N_c integers:

$$I = \{\alpha_1, \cdots, \alpha_{N_c}\} , \qquad \text{such that:} \quad 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{N_c} \le n_f . \tag{5.2.20}$$

Such a set defines a Grassmannian permutation in S_{n_f},

$$w = (I\,J) = (\alpha_1 \cdots \alpha_{N_c}\, \gamma_1 \cdots \gamma_{n_f - N_c}) \ , \qquad \text{with} \quad \gamma_1 < \gamma_2 < \cdots < \gamma_{n_f - N_c} \ , \quad (5.2.21)$$

where the ordered set J is the complement of I inside $\{1, \cdots, n_f\}$. Equivalently, and more conveniently for our purpose, we shall index $\mathcal{C}_\lambda$ by the corresponding partition $\lambda = [\lambda_1, \cdots, \lambda_{N_c}]$ with $\lambda_a \leq n_f - N_c$, which is related to the Grassmannian permutation w by:

$$\lambda = [\alpha_{N_c} - N_c \ , \ \cdots \ , \alpha_1 - 1] \ . \quad (5.2.22)$$

This also corresponds to the Young tableaux that can fit inside a $N_c \times (n_f - N_c)$ rectangle. The Schubert cell $\mathcal{C}_\lambda$ can be represented by a $N_c \times n_f$ matrix ϕ_λ in which the $\alpha_n - 1$ first entries of the n-th row vanish, the α_n-th entry is equal to one, and the entries above this latter entry also vanish. For $I = (1, 2, \cdots, N_c)$, we have the trivial partition, $\lambda = [0, \cdots, 0]$, and the Schubert cell is the full Grassmannian, with ϕ_λ given by (5.2.19). The codimension of the corresponding Schubert variety is given by the length of the partition, $|\lambda| \equiv \sum_{a=1}^{N_c} \lambda_a$, hence the dimension is:

$$\dim(X_\lambda) = N_c(n_f - N_c) - |\lambda| \ . \quad (5.2.23)$$

Finally, the Schubert variety X_λ is also given by the disjoint union of all the 'smaller' Schubert cells:

$$X_\lambda \equiv \overline{\mathcal{C}}_\lambda = \bigsqcup_{\nu \supseteq \lambda} \mathcal{C}_\nu \ , \quad (5.2.24)$$

where it is understood that we consider all partitions ν whose Young tableau contains the Young tableau of λ. As an example, the Schubert cells and Schubert varieties inside $\mathrm{Gr}(2,4)$ are shown in figure 5.2.

The cohomology of the Grassmannian is freely generated (as a vector space) by the cocycles $[X_\lambda]$ Poincaré dual to the Schubert varieties:

$$\mathrm{H}^\bullet(X, \mathbb{Z}) \cong \mathbb{Z}\Big\langle [X_\lambda] \Big\rangle \ . \quad (5.2.25)$$

In particular, the Kähler class of X in $\mathrm{H}^{1,1}(X) \cong \mathrm{H}^2(X, \mathbb{C})$ is proportional to $[X_{[1,0,\cdots,0]}]$. The classical cohomology ring,

$$[X_\lambda] \cup [X_\mu] = \sum_\nu c_{\lambda\mu}{}^\nu \ [X_\nu] \ , \quad (5.2.26)$$

can be worked out using Schubert calculus (or, equivalently, from the representation theory of the symmetric group S_{n_f}). The structure coefficients $c_{\lambda\mu}{}^{\nu} \in \mathbb{Z}$ are known as the Littlewood—Richardson (LR) coefficients. Finally, we denote by $\mathcal{O}_\lambda$ the structure sheave of X_λ. The classical K-theoretic product takes the form:

$$[\mathcal{O}_\lambda] \star [\mathcal{O}_\mu] = \sum_\nu C_{\lambda\mu}{}^{\nu} \, [\mathcal{O}_\nu] \ . \tag{5.2.27}$$

Here, the integers $C_{\lambda\mu}{}^{\nu}$ are the K-theoretic LR coefficients. These coefficients can only be non-vanishing if:

$$|\nu| \geq |\lambda| + |\mu| \ , \tag{5.2.28}$$

with the equality being saturated in the cohomological case (in which case $C_{\lambda\mu}{}^{\nu} = c_{\lambda\mu}{}^{\nu}$).[1] By a slight abuse of notation, we will often denote the K-theory class $[\mathcal{O}_\lambda]$ by $\mathcal{O}_\lambda$.

Line defects and 1d $\mathcal{N} = 2$ supersymmetric gauge theories

We would like to construct the line defects of the 3d $\mathcal{N} = 2$ gauge theory on $\Sigma \times S^1_\beta$ in the UV, which flow to the Schubert classes $\mathcal{O}_\lambda$ in the IR description. These UV defects, denoted by $\mathscr{L}_\lambda$, wrap the S^1_β and are localised at a point $p \in \Sigma$. In the GLSM description, we have the maps:

$$\Phi \, : \, \Sigma \to X \ , \tag{5.2.29}$$

given by the n_f fundamental chiral-multiplet scalars of the UV gauge theory. In particular, we have $\Phi = \phi$ for the constant maps, with ϕ given by (5.2.19) up to gauge transformations. We then need to construct a defect $\mathscr{L}_\lambda$ such that its insertion at the point p restricts the constant maps onto $X_\lambda \subset X$. We schematically write this as:

$$\mathscr{L}_\lambda \Phi(p) : p \to X_\lambda \ . \tag{5.2.30}$$

In practical terms, this means that, in the presence of the defect, ϕ should be restricted to take value in the Schubert cell C_λ, namely $\phi = \phi_\lambda$ as defined in subsection 5.2.3.

This can be achieved very naturally using a standard construction for defects in supersymmetric gauge theories – see *e.g.* [183, 184, 179]. We simply couple the 3d $\mathcal{N} = 2$ gauge theory to a 1d $\mathcal{N} = 2$ gauge theory — a gauged supersymmetric quantum mechanics (SQM) — with $U(N_c)$ flavour symmetry that lives on the line and couples to the 3d gauge fields.

[1] This is true in the non-equivariant case. In the equivariant case, the selection rules are weaker.

Defining the 1d defect. The gauged SQM we choose to consider is a 1d $\mathcal{N}=2$ linear quiver with a 1d gauge group:

$$G_{\rm 1d} = \prod_{l=1}^{n} U(r_l) \ , \tag{5.2.31}$$

with $n \leq n_f$, as shown in figure 5.3. Here, each node represents a 1d gauge group $U(r_l)$. Let us recall that the 1d vector multiplet consists of the 1d fields:

$$\mathcal{V}^{(l)}_{\mathcal{N}=2} = (\sigma^{(l)}, v_t^{(l)}, \lambda^{(l)}, \overline{\lambda}^{(l)}, D^{(l)}) \ , \qquad l = 1, \cdots, n \ , \tag{5.2.32}$$

where $\sigma^{(l)}$ is a real scalar field, $v_t^{(l)}$ is the gauge connection, $\lambda^{(l)}, \overline{\lambda}^{(l)}$ are gauginos, and $D^{(l)}$ is an auxiliary field. The fields (5.2.32) transform in the adjoint representation of $U(r_l)$.

Matter fields of 1d $\mathcal{N}=2$ gauge theories are organised into chiral multiplets and fermi multiplets – see *e.g.* [185] for a detailed account.[1] As shown in figure 5.3, our 1d quiver has bifundamental chiral multiplets connecting the gauge nodes:

$$\varphi_l^{l+1} \ : \ U(r_l) \to U(r_{l+1}) \ , \qquad l = 1, \cdots, n \ , \tag{5.2.33}$$

meaning that φ_l^{l+1} transforms in the fundamental representation of $U(r_l)$ and in the antifundamental representation of $U(r_{l+1})$. The corresponding scalar fields are naturally represented by $r_l \times r_{l+1}$ matrices. From now on, it is understood that $r_0 \equiv 0$ and $r_{n+1} \equiv N_c$. We will also assume that:

$$0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq N_c \ . \tag{5.2.34}$$

We also couple M_l fermi multiplets to the gauge group $U(r_l)$. The fermi multiplet, denoted by $\Lambda^{(l)}_{\alpha^{(l)}}$, transforms in the fundamental of $U(r_l)$ and is charged under some subgroup $U(1) \subset U(1)^{n_f} \subset U(n_f)$ of the 3d flavour symmetry. For each l, the coupling of the M_l fermi multiplets to the $U(n_f)$ flavour group is determined by an indexing set:

$$I_l = \{\alpha_1^{(l)} \ , \ \cdots \ , \ \alpha_{M_l}^{(l)}\} \subseteq \{1 \ , \ 2 \ , \cdots \ , \ n_f - 1\} \ . \tag{5.2.35}$$

We will further assume that $M_l \geq r_{l+1} - r_l$ for all l,[2] and $I_l \cap I_{l'} = \emptyset$ if $l \neq l'$. The fermi multiplets also couple directly to the 3d chiral multiplets ϕ due to the presence of the

[1] These are the straightforward dimensional reduction of the familiar 2d $\mathcal{N}=(0,2)$ supermultiplets [13].

[2] It turns out that this condition and (5.2.34) are necessary in order to preserve supersymmetry in the infrared. This can be understood through studying the Witten index of these theories, as we will briefly mention below.

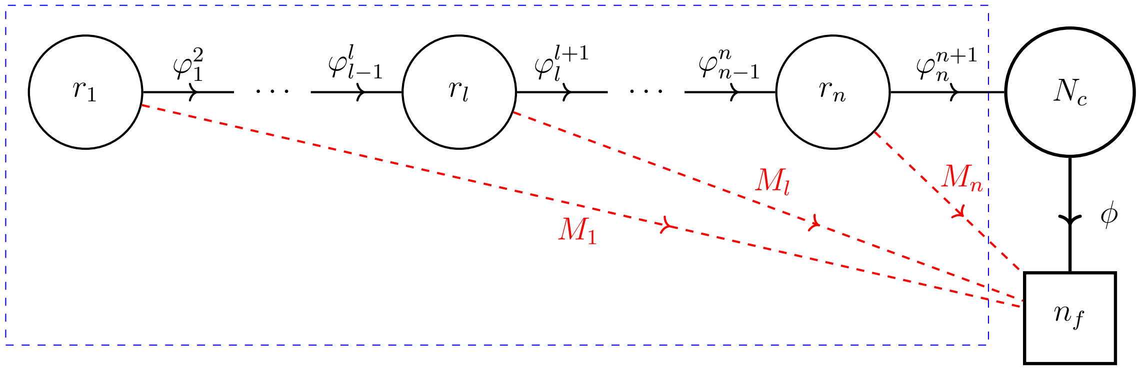


Figure 5.3: 1d $\mathcal{N} = 2$ linear quiver coupled to 3d $\mathcal{N} = 2$ Gr(N_c, n_f) GLSM. The 1d quiver is inscribed inside the dashed blue rectangle. The horizontal solid arrows denote bifundamental 1d $\mathcal{N} = 2$ chiral multiplets, and the dashed red arrows denote the M_l 1d $\mathcal{N} = 2$ fermi multiplets coupled to each $U(r_l)$ gauge group.

following 1d $\mathcal{N} = 2$ superpotential (also known as J-terms):

$$L_J = \int d\theta J^{(l)}_{\alpha^{(l)}}(\varphi;\phi)\Lambda^{(l)}_{\alpha^{(l)}} \,, \qquad J^{(l)}_{\alpha^{(l)}}(\varphi;\phi) \equiv \varphi_l^{l+1}\cdot\varphi_{l+1}^{l+2}\cdots\varphi_{n-1}^{n}\cdot\varphi_n^{n+1}\cdot\phi^{(l)}_{\alpha^{(l)}} \,, \quad (5.2.36)$$

where $\cdot$ denotes matrix multiplication. This is the most natural superpotential consistent with 1d and 3d gauge invariance. Note that the 1d $\mathcal{N} = 2$ quiver gauge theory breaks the $U(n_f)$ flavour symmetry explicitly. This is necessary in order to construct defects that flow to the Schubert varieties. Any choice of Schubert cell similarly breaks GL$(n_f, \mathbb{C})$-covariance.

The line defect defined by this 1d-3d coupled system will be denoted by:

$$\mathscr{L}\begin{bmatrix}\mathbf{r}\\ \mathbf{M}\end{bmatrix} \,, \qquad \text{with} \quad \begin{bmatrix}\mathbf{r}\\ \mathbf{M}\end{bmatrix} \equiv \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ M_1 & M_2 & \cdots & M_n\end{bmatrix} \,. \qquad (5.2.37)$$

Here, $\mathbf{r}$ denotes the ranks of the 1d gauge groups along the quiver, and $\mathbf{M}$ gives us the distribution of fermi multiplets at each quiver node. Without loss of generality, we pick the indexing sets I_l in (5.2.35) as follows:

$$I_l = \left\{1 + \sum_{k=l+1}^{n} M_k,\ 2 + \sum_{k=l+1}^{n} M_k, \cdots,\ M_l + \sum_{l=k+1}^{n} M_k\right\} \,, \qquad l = 1, \cdots, n \,. \quad (5.2.38)$$

That is, we start with $I_n = \{1, \cdots, M_n\} \subset \{\alpha\}_{\alpha=1}^{n_f}$ and we keep coupling the fermi multiplets to the next available $U(1)_\alpha \subset T$ factors as we go towards the left tail of the quiver in figure 5.3. Let us also mention that, to fully specify the 1d quiver (5.2.37), we also need to specify 1d bare Chern–Simons levels κ_l – *i.e.* 1d Wilson lines – for each $U(r_l)$ factor. Here, we set the 1d CS levels to $\kappa_l = 0$ in the "$U(1)_{-\frac{1}{2}}$ quantisation convention", as we will explain below. We will briefly discuss the effect of turning on $\kappa_l \neq 0$ at the end of this section.

The 1d vacuum equations. By assumption, the CS levels of the 3d gauge theory are chosen to be in the geometric window. Then, for positive 3d FI parameter, $\xi > 0$, the theory flows to the Higgs branch $\mathcal{M}_{\text{Higgs}} \cong X$, and the matrix ϕ for the 3d chiral multiplets describe the Grassmannian as in (5.2.19), up to gauge transformation. After coupling this 3d theory to the 1d defect (5.2.37), further constraints need to be imposed at $p \in \Sigma$, which corresponds to imposing the 1d vacuum equations. We expect that this further constrains the map ϕ at the point p to describe a subvariety of the full 3d Higgs branch:

$$\mathrm{V}\begin{bmatrix} \mathbf{r} \\ \\ \mathbf{M} \end{bmatrix} \subseteq \mathrm{Gr}(N_c, n_f) \ . \tag{5.2.39}$$

Indeed, due to the superpotential terms (5.2.36), we have to impose the J-term equations:

$$J^{(l)}_{\alpha^{(l)}}(\varphi) \equiv \varphi_l^{l+1} \cdot \varphi_{l+1}^{l+2} \cdots \varphi_n^{n+1} \cdot \phi^{(l)}_{\alpha^{(l)}} = 0 \ , \qquad l = 1, \cdots, n \ , \quad \alpha^{(l)} = 1, \cdots, M_l \ . \tag{5.2.40}$$

Here, for convenience of notation, we decomposed the $N_c \times n_f$ matrix ϕ into the following blocks:

$$\phi = \left(\ \phi^{(n)} \ \middle| \ \phi^{(n-1)} \ \middle| \ \cdots \ \middle| \ \phi^{(1)} \ \middle| \ \phi^{(0)} \ \right) \ , \tag{5.2.41}$$

where $\phi^{(l)}$ is an $N_c \times M_l$ matrix consisting of the M_l N_c-vectors $\phi^{(l)}_{\alpha^{(l)}}$, for $\alpha^{(l)} \in I_l$, with:

$$M_0 \equiv n_f - \sum_{l=1}^{n} M_l \ . \tag{5.2.42}$$

We also need to impose the 1d D-term equations. Since we are interested in the geometric phase of the 1d theory, let us set to zero the scalars of the 1d gauge vector multiplets, $\sigma^{(l)} = 0$. At each 1d gauge node $U(r_l)$, we then have the constraint:

$$\varphi_l^{l+1} \cdot \varphi_l^{l+1\,\dagger} - \varphi_{l-1}^{l\,\dagger} \cdot \varphi_{l-1}^{l} = \zeta_l \, \mathbb{I}_{r_l} \ , \qquad l = 1, \cdots, n \ , \tag{5.2.43}$$

with the understanding that $r_0 \equiv 0$. Here, ζ_l is the 1d FI parameter associated with $U(1) \subset U(r_l)$, and $\mathbb{I}_{r_l}$ is the identity matrix on $\mathbb{C}^{r_l}$.

Solving for $\phi^{(n)}$. Let us first consider the J-equation (5.2.40) for $l = n$. We have:

$$\left(\varphi_n^{n+1}\right)^{i_n} \cdot \phi^{(n)}_{\alpha^{(n)}} = 0 \ , \qquad \alpha^{(n)} = 1, \cdots, M_n \ , \qquad i_n = 1, \cdots, r_n \ , \tag{5.2.44}$$

which give us r_n equations for each $\phi^{(n)}_{\alpha^{(n)}} \in \mathbb{C}^{N_c}$, $\alpha^{(n)} = 1, \cdots, M_n$. We can then choose the $N_c \times M_n$ matrix $\phi^{(n)}$ to be:

$$\phi^{(n)} = \left(\begin{array}{c|c} \mathbb{I}_{N_c - r_n} & \bigstar_{N_c - r_n,\, M_n - (N_c - r_n)} \\ \hline 0_{r_n,\, N_c - r_n} & 0_{r_n,\, M_n - (N_c - r_n)} \end{array}\right) \ , \tag{5.2.45}$$

with $\bigstar$ denoting the undetermined elements. Here we used the $U(N_c)$ gauge freedom to fix the first $N_c - r_n$ vectors, while the fact that the bottom part of (5.2.45) is vanishing is due to (5.2.44) together with a convenient choice of φ_n^{n+1} that is allowed thanks to the 1d gauge freedom. The number of undetermined entries in (5.2.45) is equal to:

$$\mathrm{F}_n \equiv \left(M_n - (N_c - r_n)\right)(N_c - r_n) \ . \tag{5.2.46}$$

Plugging back the expression (5.2.45) in (5.2.44), we see that the first $(N_c - r_n)$ entries of each row of the matrix $\varphi_n^{N_c}$ must vanish. In addition, we can use the gauge actions on $\varphi_n^{N_c}$ to diagonalise the non-trivial block, so that we have:

$$\varphi_n^{N_c} = \left(\begin{array}{c|c} 0_{r_n,\, N_c - r_n} & \mathrm{C}^{(n)} \end{array}\right) \ , \tag{5.2.47}$$

for $\mathrm{C}^{(n)}$ a diagonal matrix, $\mathrm{C}^{(n)} = \mathrm{diag}(\mathrm{c}_1^{(n)}, \mathrm{c}_2^{(n)}, \cdots, \mathrm{c}_{r_n}^{(n)})$ for $\mathrm{c}_{i_n}^{(n)} \in \mathbb{C}^*$. Note that we assumed that we are on the 1d Higgs branch – namely, that the vacuum expectation value (5.2.47) Higgses $U(r_n)$ entirely. This is enforced by the D-term equations, given an appropriate choice of 1d FI parameters, as we will discuss momentarily.

Solving for $\phi^{(n-1)}$. Before looking at the next node of the 1d quiver, for $l = n-1$, let us first consider the D-term equations (5.2.43) associated with $U(r_n)$, which reads:

$$\varphi_n^{n+1} \cdot \varphi_n^{N_c\,\dagger} - \varphi_{n-1}^{n\,\dagger} \cdot \varphi_{n-1}^{n} = \zeta_n \,\mathbb{I}_{r_n} \ . \tag{5.2.48}$$

This, along with (5.2.47), implies that the r_n columns of the matrix φ^n_{n-1} are orthogonal. Since these columns are vectors in $\mathbb{C}^{r_{n-1}}$ (and given the assumption $r_n \geq r_{n-1}$), we have $r_n - r_{n-1}$ linearly dependent columns. This constrains the matrix φ^n_{n-1} to be of the form:

$$\varphi^n_{n-1} = \left(\begin{array}{c|c} 0_{r_{n-1},\, r_n - r_{n-1}} & \mathrm{C}^{(n-1)} \end{array} \right) , \tag{5.2.49}$$

where $\mathrm{C}^{(n-1)}$ can be gauge-fixed to be a diagonal matrix.

We then consider the J-equations (5.2.40) associated with the fermi multiplets in the fundamental of $U(r_{n-1})$. It reads:

$$\varphi^n_{n-1} \cdot \varphi^{N_c}_n \cdot \phi^{(n-1)}_{\alpha^{(n-1)}} = 0 , \qquad \alpha^{(n-1)} = 1, \cdots , M_{n-1} . \tag{5.2.50}$$

The explicit expressions (5.2.47) and (5.2.49) imply that, for each $\alpha^{(n-1)}$, there are r_{n-1} linear relations amongst the N_c components the vector $\phi^{(n-1)}_{\alpha^{(n-1)}}$. One can further use the leftover 3d gauge symmetry to choose $r_n - r_{n-1}$ vector consistent with the equations. We then find the explicit expression:

$$\phi^{(n-1)} = \left(\begin{array}{c|c} 0_{N_c - r_n,\, r_n - r_{n-1}} & \bigstar_{N_c - r_n,\, M_{n-1} - (r_n - r_{n-1})} \\ \hline \mathbb{I}_{r_n - r_{n-1}} & \bigstar_{r_n - r_{n-1},\, M_{n-1} - (r_n - r_{n-1})} \\ \hline 0_{r_{n-1},\, r_n - r_{n-1}} & 0_{r_{n-1},\, M_{n-1} - (r_n - r_{n-1})} . \end{array} \right) . \tag{5.2.51}$$

The number of free entries is given by:

$$\mathrm{F}_{n-1} \equiv \left(M_{n-1} - (r_n - r_{n-1})\right) (N_c - r_{n-1}) . \tag{5.2.52}$$

Solving for $\phi^{(l)}$. We can follow the same procedure repeatedly for $l = n-1, n-2, \cdots , 1$. After fixing the matrix φ^{l+1}_l using the associated 1d $U(r_l)$ and $U(r_{l+1})$ gauge symmetries, equations (5.2.40) become r_l equations in N_c variables fixing the lower $r_l \times M_l$ part of the block to be trivial. Furthermore, using the leftover 3d gauge symmetry, we can fix

$r_{l+1} - r_l$ vectors of $\phi^{(l)}$. This then leads us to the explicit solution:

$$\phi^{(l)} = \left(\begin{array}{c|c} 0_{N_c - r_{l+1},\, r_{l+1}-r_l} & \bigstar_{N_c - r_{l+1},\, M_l - (r_{l+1}-r_l)} \\ \hline \mathbb{I}_{r_{l+1}-r_l} & \bigstar_{r_{l+1}-r_l,\, M_l-(r_{l+1}-r_l)} \\ \hline 0_{r_l,\, r_{l+1}-r_l} & 0_{r_l,\, M_l-(r_{l+1}-r_l)} \end{array}\right) , \tag{5.2.53}$$

with the following numbers of undetermined entries:

$$\mathrm{F}_l \equiv (M_l - (r_{l+1} - r_l))\,(N_c - r_l) \ , \qquad l = 1, \cdots, n-2 \ . \tag{5.2.54}$$

After gauge fixing, the matrix φ_l^{l+1} takes the form:

$$\varphi_l^{l+1} = \left(\begin{array}{c|c} 0_{r_l,\ r_{l+1}-r_l} & \mathrm{C}^{(l)} \end{array} \right) , \tag{5.2.55}$$

with the diagonal matrix $\mathrm{C}^{(l)} = \mathrm{diag}(\mathrm{c}_1^{(l)}, \mathrm{c}_2^{(l)}, \cdots, \mathrm{c}_{r_l}^{(l)})$, for $\mathrm{c}_{i_l}^{(l)} \in \mathbb{C}^*$.

Solving for $\phi^{(0)}$. For the last block in (5.2.41), denoted by $\phi^{(0)}$, we do not have any constraints coming from the 1d vacuum structure equations. In general, however, we can still fix r_1 vector using the 3d gauge freedom, so that we have:

$$\phi^{(0)} = \left(\begin{array}{c|c} 0_{N_c - r_1,\, r_1} & \bigstar_{N_c - r_1,\, M_0 - r_1} \\ \hline \mathbb{I}_{r_1} & \bigstar_{r_1,\, M_0 - r_1} \end{array}\right) . \tag{5.2.56}$$

Note that the number of undetermined component is given by $\mathrm{F}_0 \equiv N_c\,(M_0 - r_1)$, where M_0 is defined in (5.2.42).

Final result for ϕ. Putting the above results together, we find that the line defect (5.2.37) constrains the matrix ϕ to take the form:

$$\phi = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\mathbb{I}_{N_c-r_n} & \bigstar & 0 & \bigstar & \cdots & 0 & \bigstar & \cdots & 0 & \bigstar & 0 & \bigstar \\ \hline
0 & 0 & \mathbb{I}_{r_n-r_{n-1}} & \bigstar & \cdots & 0 & \bigstar & \cdots & 0 & \bigstar & 0 & \bigstar \\ \hline
0 & 0 & 0 & 0 & \cdots & 0 & \bigstar & \cdots & 0 & \bigstar & 0 & \bigstar \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \cdots & \mathbb{I}_{r_{l+1}-r_l} & \bigstar & \cdots & 0 & \bigstar & 0 & \bigstar \\ \hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \bigstar & 0 & \bigstar \\ \hline
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mathbb{I}_{r_2-r_1} & \bigstar & 0 & \bigstar \\ \hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \mathbb{I}_{r_1} & \bigstar
\end{array}\right) . \tag{5.2.57}$$

This parameterizes the subspace (5.2.39) of $X = \mathrm{Gr}(N_c, n_f)$ of dimension:

$$\dim_{\mathbb{C}} \mathrm{V}\begin{bmatrix} \mathbf{r} \\ \mathbf{M} \end{bmatrix} = \sum_{l=0}^{n} \mathrm{F}_l = \sum_{l=0}^{n} \left(M_l - (r_{l+1} - r_l)\right)(N_c - r_l) \ . \tag{5.2.58}$$

In fact, we see that the matrix (5.2.57) describes a Schubert cell:

$$C_\lambda = \mathrm{V}\begin{bmatrix} \mathbf{r} \\ \mathbf{M} \end{bmatrix} \ , \tag{5.2.59}$$

with the partition (5.2.22) given by:

$$\lambda = \Big[\underbrace{\widetilde{M}_1 + r_1 - N_c}_{r_1} , \cdots , \underbrace{\widetilde{M}_l + r_l - N_c}_{r_l - r_{l-1}} , \cdots , \underbrace{\widetilde{M}_n + r_n - N_c}_{r_n - r_{n-1}} , \underbrace{0}_{N_c - r_n} \Big] , \tag{5.2.60}$$

where $\underbrace{a}_{b}$ means that the integer a repeats b times, and we defined the quantities:

$$\widetilde{M}_l \equiv \sum_{k=l}^{n} M_k \ , \tag{5.2.61}$$

for ease of notation. One can check that (5.2.58) indeed equals (5.2.23).

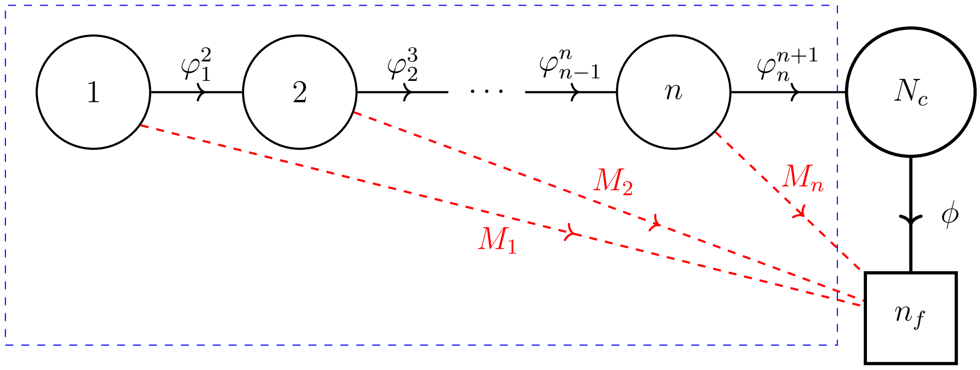


Figure 5.4: Generic Grothendieck defect $\mathscr{L}_\lambda$ with $n \leq N_c$. The numbers of fermi multiplets, M_l, are given in terms of the partition λ as explained in the main text.

Grothendieck lines and Schubert classes

Given a choice of Schubert variety $X_\lambda \subseteq X$, we can construct a line defect (5.2.37) that flows to X_λ in the infrared simply by fixing the parameters r_l and M_l in terms of λ as in (5.2.60). Whenever several possible solutions for $(\mathbf{r}, \mathbf{M})$ exist, the corresponding 1d quivers are related by dualities, as we will briefly discuss below. We then claim that the UV line defect flows to the structure sheaf of X_λ, $\mathcal{O}_\lambda$, by giving us the so-called Schubert class $[\mathcal{O}_\lambda]$ in K-theory. We write this as:

$$\mathscr{L}_\lambda \equiv \mathscr{L}\begin{bmatrix}\mathbf{r}\\ \mathbf{M}\end{bmatrix} \cong \mathcal{O}_\lambda \ . \tag{5.2.62}$$

These line defects thus give us the Grothendieck lines. We will verify this claim by explicit computation of the 1d Witten index.

The generic Grothendieck lines. It is convenient to define a 'generic' Grothendieck line that corresponds to the partition:

$$\lambda = [\lambda_1, \cdots, \lambda_n, \underbrace{0, \cdots, 0}_{N_c - n}] \ , \qquad \lambda_1 \leq n_f - N_c \ , \tag{5.2.63}$$

with generic non-zero λ_a. Then, the relation (5.2.60) gives us:

$$\begin{aligned} & r_l = l \ , && l = 1, \cdots, n \ , \\ & M_l = \lambda_l - \lambda_{l+1} + 1 \ , && l = 1, \cdots, n-1 \ , \\ & M_n = \lambda_n - n + N_c \ . && \end{aligned} \tag{5.2.64}$$

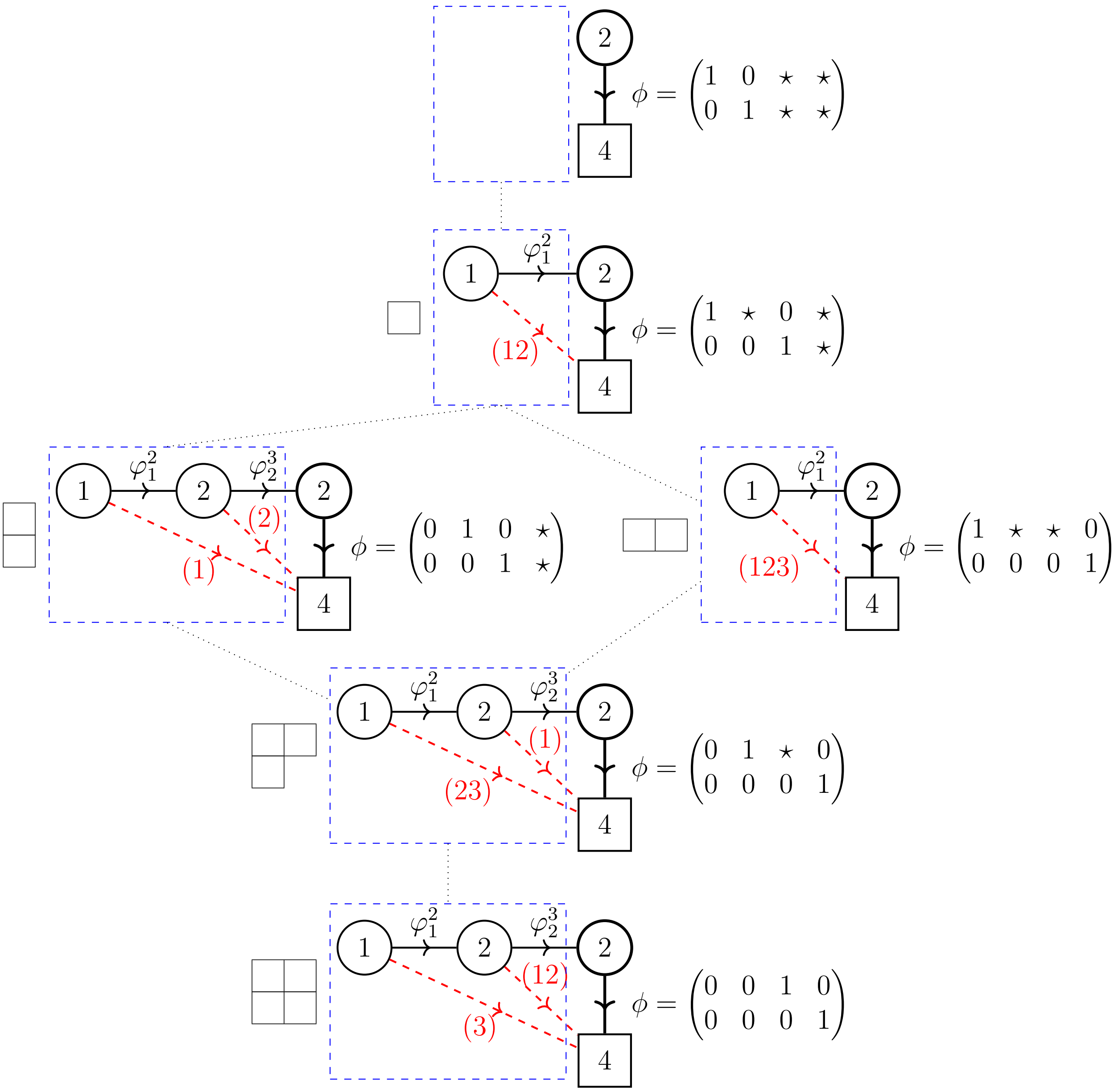


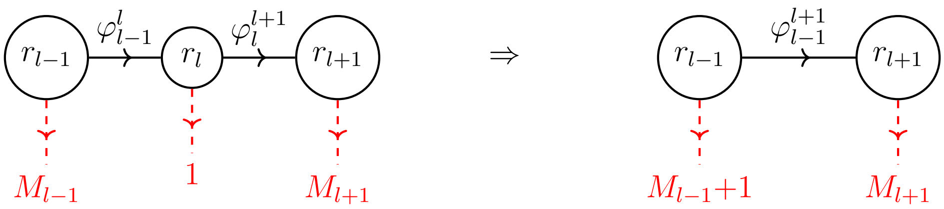
Figure 5.5: Generic Grothendieck defects $\mathscr{L}_\lambda$ for Gr(2, 4). The index set $I_l = (\alpha^{(l)})$ for the fermi multiplets coupling to $U(r_l)$ is displayed next to each red dashed arrow. Note that the 1d quivers for $\lambda = [1, 1]$ and for $\lambda = [2, 2]$ can be simplified by simply removing the $U(1)$ node, as explained in the main text.

This Grothendieck line defect is a 'complete flag' quiver, as displayed in figure 5.4. As an example, all the 1d quivers giving us the Grothendieck lines $\mathscr{L}_\lambda$ for Gr(2, 4) are shown in figure 5.5. One can write down similar Hasse diagrams for any Grassmannian variety – the example of Gr(3, 5) is worked out in appendix F, see figure F.1.

Duality moves. In general, the generic Grothendieck line is not the most efficient presentation of the defect line $\mathscr{L}_\lambda$. Indeed, it is clear from (5.2.60) that the most 'efficient' 1d quiver has n nodes where n is the number of *distinct* non-zero values for λ_a. The quiver simplification can be realised in terms of the following duality move. Whenever we have a node such that:

$$r_l = r_{l+1} - 1 \;, \qquad \text{and} \quad M_l = 1 \;, \tag{5.2.65}$$

we can remove the node, reconnect the $U(r_{l-1})$ and $U(r_{l+1})$ with a chiral multiplet arrow, and shift M_{l-1} to $M_{l-1} + 1$. Pictorially, we have:



(5.2.66)

This is a special example of a Seiberg-like duality for 1d $\mathcal{N} = 2$ gauge theories, applied at the $U(r_l)$ node[1] – here, the dual gauge group is trivial (it "condenses"), leaving us with the new chiral and fermi-multiplet "mesons" that are shown in (5.2.66). Note that the duality move (5.2.66) also holds for $l = 1$ if $r_1 = r_2 - 1$, with the net effect being simply to remove the leftmost node of the quiver.

Example: the point and the line. As an example of the construction above, let us look at the following two special cases: the point and the line cells inside Gr(N_c, n_f). In the first case, we have the partition:

$$\mu_{\text{point}} = [n_f - N_c, \cdots, n_f - N_c] \;. \tag{5.2.67}$$

Following our labelling of fermi multiplets (5.2.38), the index sets for the fermi multiplets

[1] These dualities are generalisations of the Gadde–Gukov–Putrov 2d $\mathcal{N} = (0, 2)$ trialities [186]. In the original work [3], this duality move was stated more generally for the case $r_l = r_{l+1} - M_l$. In a different project of Cyril Closset and James Wynne, they found that this duality is true only for the case stated here. I would like to thank them both for pointing that out.

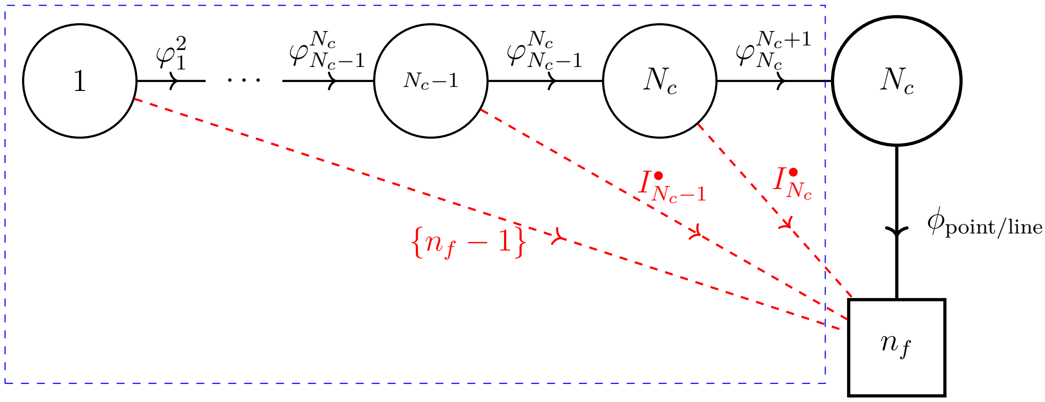


Figure 5.6: Grothendieck defect associated with the point/line Schubert cell inside $\mathrm{Gr}(N_c, n_f)$. Here the bullet in $I^\bullet$ stands for either point or line. For the first case, it is given by (5.2.68) and for the latter it is given by (5.2.71).

are:

$$I_\ell^{\text{point}} = \begin{cases} \{n_f - \ell\} \ , & \ell = 1, \cdots, N_c - 1 \ , \\ \{1, \cdots, n_f - N_c\} \ , & \ell = N_c \ . \end{cases} \tag{5.2.68}$$

Moreover, one can check [3] that the final form of the matter matrix ϕ_{point} is:

$$\phi_{\text{point}} = \left(0_{N_c \times (n_f - N_c)} \mid 1_{N_c \times N_c} \right) \ , \tag{5.2.69}$$

from which we deduce that, indeed, the dimension of the Schubert cell in this case is zero.

In the case of the line cell, we have the following N_c-partition:

$$\mu_{\text{line}} = [n_f - N_c, \cdots n_f - N_c, n_f - N_c - 1] \ . \tag{5.2.70}$$

In this case, the 1d fermi multiplets are indexed via:

$$I_\ell^{\text{line}} = \begin{cases} \{n_f - \ell\} \ , & \ell = 1, \cdots, N_c - 2 \ , \\ \{n_f - N_c, n_f - N_c + 1\} \ , & \ell = N_c - 1 \ , \\ \{1, \cdots, n_f - N_c - 1\} \ , & \ell = N_c \ . \end{cases} \tag{5.2.71}$$

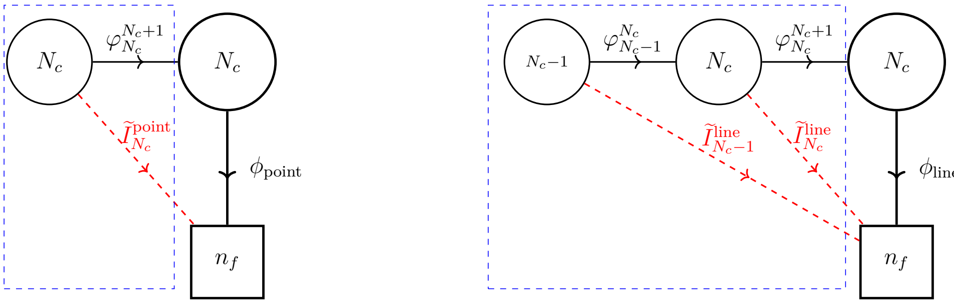


Figure 5.7: Simplified form of the Grothendieck defects associated with the point (left quiver) and line (right quiver) Schubert cells in $\text{Gr}(N_c, n_f)$. These are obtained by applying the 1d triality (5.2.66) to the quivers in figure 5.6. The index sets are given in (5.2.73) for the point and in (5.2.74) for the line.

.

The final form of the matter matrix ϕ_{line} in this case is given by:

$$\phi_{\text{line}} = \left(\begin{array}{c|c|c|c} 0_{1\times(n_f-N_c-1)} & 1 & \star & 0_{1\times(N_c-1)} \\ \hline 0_{(N_c-1)\times(n_f-N_c-1)} & 0_{(N_c-1)\times 1} & 0_{(N_c-1)\times 1} & 1_{(N_c-1)\times(N_c-1)} \end{array}\right) , \tag{5.2.72}$$

which shows that the Schubert cell in this case is one-dimensional. In either case, the 1d quiver coupled to the 3d GLSM is given in figure 5.6 above.

As an application of the duality move (5.2.66), we can simplify the form of the 1d quivers defining the point and line cells in figure 5.6. Starting from the left of the quiver, for instance, we can simply drop all the nodes which have a single fermi multiplet, to obtain the quivers in figure 5.7. For the point-cell, we have the new single index set:

$$\widetilde{I}_{N_c}^{\text{point}} = \{1, \cdots, n_f - N_c\} , \tag{5.2.73}$$

and for the line-cell, we have the two index:

$$\widetilde{I}_{\ell}^{\text{line}} = \begin{cases} \{n_f - N_c, n_f - N_c + 1\} , & \ell = N_c - 1 , \\ \{1, \cdots, n_f - N_c - 1\} , & \ell = N_c . \end{cases} \tag{5.2.74}$$

Chern character and 1d Witten index. Given the above discussion, we expect that any Grothendieck line $\mathscr{L}_\lambda$, defined as the above 1d $\mathcal{N}=2$ quiver coupled to the 3d $U(N_c)$ gauge fields, flows to a coherent sheave with support on $X_\lambda \subset X$. We further claimed in (5.2.62) that $\mathscr{L}_\lambda$ exactly gives us the structure sheaves $\mathcal{O}_\lambda$. To verify this claim at the level of the 3d A-model, we need to understand how the insertion of the line affects the low-energy 2d Coulomb branch description. This can be worked out simply by considering the path integral over the 1d fields along the S^1, which computes the Witten index of the 1d $\mathcal{N}=2$ quiver. We denote this 1d index by $\mathscr{L}_\lambda$ again, by abuse of notation:

$$\mathscr{L}_\lambda(x,y) \equiv \mathcal{I}_{\rm W}^{\rm 1d}\begin{bmatrix} \mathbf{r} \\ \mathbf{M} \end{bmatrix}(x,y) \ . \tag{5.2.75}$$

Note that it depends on the 3d gauge and flavour parameters, x_a and y_α. In the NLSM interpretation, this 1d path integral should give us the Chern character of the coherent sheaf that the line defect flows to. Indeed, we will now check that:

$$\mathscr{L}_\lambda(x,y) = \mathrm{ch}_T(\mathcal{O}_\lambda) \ , \tag{5.2.76}$$

The equivariant Chern character of $\mathcal{O}_\lambda$ is known to be given in terms of the *double Grothendieck polynomial* associated with the partition λ [187, 188, 189]:

$$\mathrm{ch}_T(\mathcal{O}_\lambda) = \mathfrak{G}_\lambda(x,y) \ . \tag{5.2.77}$$

The latter can be written in the following explicit form [190]: [1][2]

$$\mathfrak{G}_\lambda(x,y) = \frac{\det_{1\le a,b\le N_c}\left(x_a^{b-1}\prod_{\alpha=1}^{\lambda_b+N_c-b}(1-x_a y_\alpha^{-1})\right)}{\prod_{1\le a<b\le N_c}(x_b-x_a)} \ . \tag{5.2.78}$$

In the non-equivariant limit, $y\to 1$, one obtains the ordinary Grothendieck polynomials:

$$\mathfrak{G}_\lambda(x) = \frac{\det_{1\le a,b\le N_c}\left(x_a^{b-1}(1-x_a)^{\lambda_b+N_c-b}\right)}{\prod_{1\le a<b\le N_c}(x_b-x_a)} \ , \tag{5.2.79}$$

for the partition $\lambda=[\lambda_1,\cdots,\lambda_n,0,\cdots,0]$. For $\lambda=[1,0,\cdots,0]$, for instance, we have:

$$\mathfrak{G}_\square(x,y) = 1-\det x = \mathrm{ch}(\mathcal{O}_\square) \ , \tag{5.2.80}$$

in agreement with (5.2.17).

[1]We use conventions compatible with our physical realisation. The variables x and b of [190] correspond to $1-x$ and $1-y^{-1}$, respectively, in our variables (and after setting their parameter β to $\beta=-1$).

[2]Note that there is a minus sign difference in the denominator compared with equation (3.69) of the original paper [3]. We thank Zhihao Duan for pointing out this typo.

Example. The double Grothendieck polynomials associated to the Schubert subvarieties of $\mathrm{Gr}(2,4)$ read:

$$
\begin{aligned}
\mathfrak{G}_{(1)}(x,y) &= 1 - \frac{x_1 x_2}{y_1 y_2} \ , \\
\mathfrak{G}_{(1,1)}(x,y) &= 1 - \frac{x_1 + x_2}{y_1} + \frac{x_2 x_1}{y_1^2} \ , \\
\mathfrak{G}_{(2)}(x,y) &= 1 - \frac{x_1 x_2}{y_1 y_2} - \frac{x_1 x_2}{y_1 y_3} - \frac{x_1 x_2}{y_2 y_3} + \frac{x_1^2 x_2 + x_1 x_2^2}{y_1 y_2 y_3} \ , \\
\mathfrak{G}_{(2,1)}(x,y) &= \frac{x_2 x_1^2}{y_1 y_2 y_3} - \frac{x_2^2 x_1^2}{y_1^2 y_2 y_3} - \frac{x_1}{y_1} - \frac{x_2 x_1}{y_2 y_3} + \frac{x_2^2 x_1}{y_1 y_2 y_3} + \frac{x_2 x_1}{y_1^2} - \frac{x_2}{y_1} + 1 \ , \\
\mathfrak{G}_{(2,2)}(x,y) &= \frac{x_1^2}{y_1 y_2} - \frac{x_2 x_1^2}{y_1^2 y_2} - \frac{x_2 x_1^2}{y_1 y_2^2} + \frac{x_2^2 x_1^2}{y_1^2 y_2^2} - \frac{x_1}{y_1} + \frac{2 x_2 x_1}{y_1 y_2} - \frac{x_2^2 x_1}{y_1^2 y_2} - \frac{x_1}{y_2} \\
&\quad + \frac{x_2 x_1}{y_1^2} + \frac{x_2 x_1}{y_2^2} - \frac{x_2^2 x_1}{y_1 y_2^2} - \frac{x_2}{y_1} - \frac{x_2}{y_2} + \frac{x_2^2}{y_1 y_2} + 1 \ .
\end{aligned}
\tag{5.2.81}
$$

In the non-equivariant limit, we get the corresponding Grothendieck polynomials:

$$
\begin{aligned}
\mathfrak{G}_{(1)}(x,y) &= 1 - x_1 x_2 \ , \\
\mathfrak{G}_{(1,1)}(x,y) &= (1 - x_1)(1 - x_2) \ , \\
\mathfrak{G}_{(2)}(x,y) &= 1 + x_1 x_2 \left(x_1 + x_2 - 3 \right) \ , \\
\mathfrak{G}_{(2,1)}(x,y) &= (1 - x_1)\left(1 - x_2\right)\left(1 - x_1 x_2\right) \ , \\
\mathfrak{G}_{(2,2)}(x,y) &= (1 - x_1)^2 (1 - x_2)^2 \ .
\end{aligned}
\tag{5.2.82}
$$

Computing the 1d partition function. The Witten index of any 1d $\mathcal{N} = 2$ gauge theory can be written in terms of the JK residue on the 1d complexified Coulomb branch [185], which is parameterised by complex variables $z \in \mathbb{C}^*$. The 1d Witten index (5.2.75) is then written as a nested contour integral over these 1d gauge variables:[1]

$$
\mathscr{L}_\lambda(x,y) = \oint_{\mathrm{JK}} \left[\prod_{l=1}^{n} \frac{1}{r_l!} \prod_{i_l=1}^{r_l} \frac{-dz_{i_l}^{(l)}}{2\pi i z_{i_l}^{(l)}} \prod_{1 \le i_l \neq j_l \le r_l} \left(1 - \frac{z_{i_l}^{(l)}}{z_{j_l}^{(l)}} \right) \right] \mathrm{Z}^{\mathrm{1d}}_{\mathrm{matter}}(z,x,y) \ , \tag{5.2.83}
$$

[1] Here, in quantising the theory, we appropriately cancelled the 1d parity anomaly with a natural choice of 1d CS levels – see section 5.2.3 below for more details.

where,

$$\mathrm{Z}^{\mathrm{1d}}_{\mathrm{matter}}(z,x,y) \equiv \prod_{l=1}^{n-1}\prod_{i_l=1}^{r_l} \frac{\prod_{\alpha^{(l)}\in I_l}\left(1-\frac{z^{(l)}_{i_l}}{y_{\alpha^{(l)}}}\right)}{\prod_{j_{l+1}=1}^{r_{l+1}}\left(1-\frac{z^{(l)}_{i_l}}{z^{(l+1)}_{j_{l+1}}}\right)} \prod_{i_n=1}^{r_n} \frac{\prod_{\alpha^{(n)}\in I_n}\left(1-\frac{z^{(n)}_{i_n}}{y_{\alpha^{(n)}}}\right)}{\prod_{a=1}^{N_c}\left(1-\frac{z^{(n)}_{i_n}}{x_a}\right)} \ . \tag{5.2.84}$$

The measure factor in (5.2.83) arises from 1d W-bosons for each gauge node along the quiver, and the matter multiplets contribute to (5.2.84), with the numerator and denominator contributions arising from one-loop determinants of the 1d chiral and fermi multiplets, respectively. Note that the coupling to the 3d parameters x_a only arises as the contribution from the N_c fundamental chiral multiplets φ_n^{n+1} of the $U(r_n)$ node.

The JK residue in (5.2.83) is determined by the choice of 1d FI parameters. In order to solve the D-term equations on the 1d "Higgs branch" as above, we choose the ζ_l parameters along the 1d quiver to be all positive. Then, the integration contour corresponds to iteratively picking the poles from the chiral multiplets from the denominator of (5.2.84), and then performing the $U(r_l)$ integrations in increasing order, from $l=1$ to $l=n$.

Explicit computations. For definiteness, let us compute the JK residue (5.2.83) with the 'generic' choice of linear quiver (5.2.64). To isolate the relevant poles, we can rewrite the index as:

$$\mathscr{L}_\lambda(x,y) = (-1)^{\mathfrak{n}(N_c,n)}(\det x)^n \oint \prod_{l=1}^{n}\left[\frac{d^l z^{(l)}\Delta^{(l)}(z)}{l!(2\pi i)^l \det z^{(l)}}\right]\widetilde{\mathrm{Z}}^{\mathrm{1d}}_{\mathrm{matter}}(z,x,y) \ , \tag{5.2.85}$$

where the Vandermonde determinant $\Delta^{(l)}(z)$ is given by:

$$\Delta^{(l)}(z) \equiv \prod_{1\le i_l\neq j_l\le l}\left(z^{(l)}_{i_l}-z^{(l)}_{j_l}\right) \ . \tag{5.2.86}$$

Here we used the obvious shorthand notations $\det x \equiv \prod_{a=1}^{N_c} x_a$ and $\det z^{(l)} \equiv \prod_{i_l=1}^{l} z^{(l)}_{i_l}$, and we introduced a sign factor $\mathfrak{n}(N_c,n) \equiv n(N_c+1)+\sum_{l=1}^{n-1} l^2$. The matter contributions to (5.2.85) reads:

$$\widetilde{\mathrm{Z}}^{\mathrm{1d}}_{\mathrm{matter}}(z,x,y) \equiv \prod_{l=1}^{n-1}\prod_{i_l=1}^{l} \frac{\prod_{\alpha^{(l)}\in I_l}\left(1-\frac{z^{(l)}_{i_l}}{y_{\alpha^{(l)}}}\right)}{\prod_{j_{l+1}=1}^{l+1}\left(z^{(l)}_{i_l}-z^{(l+1)}_{j_{l+1}}\right)} \prod_{i_n=1}^{n} \frac{\prod_{\alpha^{(n)}\in I_n}\left(1-\frac{z^{(n)}_{i_n}}{y_{\alpha^{(n)}}}\right)}{\prod_{a=1}^{N_c}\left(z^{(n)}_{i_n}-x_a\right)} \ . \tag{5.2.87}$$

Abelian 1d quiver. As a warm-up exercice, let us consider the case when the N_c-partition

λ consists of a single row, $\lambda = [\lambda_1, 0, \cdots, 0]$. Then, the linear quiver consist of a single $U(1)$ node, and (5.2.85) gives us:

$$\begin{aligned}\mathscr{L}_{[\lambda_1,0,\cdots,0]}(x,y) &= (-1)^{\mathfrak{n}(N_c,1)}\det x \oint \frac{dz}{2\pi i z}\frac{\prod_{\alpha=1}^{\lambda_1-1+N_c}(1-zy_\alpha^{-1})}{\prod_{a=1}^{N_c}(z-x_a)} \\ &= (-1)^{N_c+1}\det x \sum_{a=1}^{N_c}\frac{\prod_{\alpha=1}^{\lambda_1-1+N_c}(1-x_a y_\alpha^{-1})}{x_a\prod_{b\neq a}(x_a-x_b)} \ ,\end{aligned} \tag{5.2.88}$$

where we picked the poles $z = x_a$ with $a = 1, \cdots, N_c$. A little algebra shows that this expression reproduces the double Grothendieck polynomial (5.2.78) associated with this partition.

The general case. Performing the contour integrals of (5.2.85) recursively, one ends up with the following explicit formula:

$$\mathscr{L}_\lambda(x,y) = (-1)^{\mathfrak{n}(N_c,n)}(\det x)^n \sum_{\mathcal{J}}\prod_{l=1}^{n}\Delta^{(J_l)}(x)\prod_{i_l\in J_l}\frac{\prod_{\alpha^{(l)}\in I_l}\left(1-x_{i_l}y^{-1}_{\alpha^{(l)}}\right)}{x_{i_l}\prod_{\substack{j_{l+1}\in J_{l+1}\\ j_{l+1}\neq i_l}}\left(x_{i_l}-x_{j_{l+1}}\right)} \ , \tag{5.2.89}$$

with the index sets $\mathcal{J}$ defined as:

$$\mathcal{J} = \{J_1,\ J_2,\ \cdots,\ J_n\} \ , \tag{5.2.90}$$

such that:

$$J_1 \subset J_2 \subset \cdots \subset J_n \subseteq \{1,\cdots,N_c\} \ , \qquad |J_l| = l \ . \tag{5.2.91}$$

Here, we introduced the Vandermonde-like factors:

$$\Delta^{(J_l)}(x) \equiv \prod_{\substack{i_l,j_l\in J_l\\ i_l\neq j_l}}(x_{i_l}-x_{j_l}) \ , \qquad l = 1,\cdots,n \ . \tag{5.2.92}$$

The expression (5.2.89) can be further massaged to:

$$\mathscr{L}_\lambda(x,y) = (-1)^{\mathfrak{n}(N_c,n)}\sum_{\mathcal{J}}\prod_{l=1}^{n}\left[\left(\prod_{j_l\in\bar{J}_l}x_{j_l}\right)\prod_{i_l\in J_l}\frac{\prod_{\alpha^{(l)}\in I_l}\left(1-x_{i_l}y^{-1}_{\alpha^{(l)}}\right)}{\prod_{j_{l+1}\in J_{l+1}\setminus J_l}\left(x_{i_l}-x_{j_{l+1}}\right)}\right] \ , \tag{5.2.93}$$

where $\bar{J}_l \equiv \{1, 2, \cdots, N_c\} \setminus J_l$. This expression is actually the cofactor expansion of the following determinant:

$$\mathscr{L}_\lambda(x,y) = \frac{\det_{1\le a,b\le N_c}\left(x_a^{N_c-b}\prod_{l=N_c-b+1}^{N_c}\prod_{\alpha^{(l)}\in I_l}\left(1-x_a y_{\alpha^{(l)}}^{-1}\right)\right)}{\prod_{1\le a<b\le N_c}(x_a-x_b)} = \mathfrak{G}_\lambda(x,y)\ , \qquad (5.2.94)$$

with the index sets I_l defined exactly as in (5.2.38) for $l=1,\cdots,n$, with $I_l=\emptyset$ for $l>n$. It is then easy to see that this is exactly the same expression as in (5.2.78), by redefining $b\to N_c-b+1$. Hence, we have shown that the Witten index of our Grothendieck defect lines precisely reproduces the double Grothendieck polynomial (5.2.78).

Effect of the 1d CS levels

Similarly to how 3d $\mathcal{N}=2$ GLSMs are only fully determined once we specify all the Chern–Simons levels, the 1d $\mathcal{N}=2$ defects discussed above also allow for the presence of one-dimensional Chern–Simons levels, κ_l, associated to the $U(1)\subset U(r_l)$ factors. Equivalently, this corresponds to adding an abelian Wilson line with charge κ_l. Note that these CS levels must be integer-quantised:

$$\kappa_l\in\mathbb{Z}\ . \qquad (5.2.95)$$

We call these the bare CS levels. There are also "effective" 1d CS levels that we chose so that all 1d fermions are in the "$U(1)_{-\frac{1}{2}}$ quantisation", in order to cancel the 1d parity anomaly – see *e.g.* subsection 2.2.2 for a detailed discussion in the 3d context; we pick the same conventions in 1d.

Thus, the general line defect of figure 5.4 should be refined to include this additional data:

$$\mathscr{L}_\lambda^{(\kappa)}\begin{bmatrix} 1 & 2 & \dots & n-1 & n \\ M_1 & M_2 & \dots & M_{n-1} & M_n \\ \kappa_1 & \kappa_2 & \dots & \kappa_{n-1} & \kappa_n \end{bmatrix}\ . \qquad (5.2.96)$$

The addition of these 1d Wilson lines does not affect the structure of the Schubert variety that $\mathscr{L}_\lambda^{(\kappa=0)}$ maps into, hence we expect that there exists a distinct non-locally free sheaf with support on X_λ that (5.2.96) maps into. On the other hand, at the level of the 1d partition function, the inclusion of a Wilson line in the representation $(\mathbf{det})^{\kappa_l}$ of $U(r_l)$ amounts to adding the factor $(-\det z^{(l)})^{\kappa_l}$ to the integrand of the expression (5.2.83) for the Witten index.

Following the same procedure as we did for the case with $\kappa_l=0$, we find that (5.2.93) is generalised to:

$$\mathscr{L}_\lambda^{(\kappa)}(x,y) = (-1)^{\mathfrak{n}(N_c,n,\kappa)} \sum_{\mathcal{J}} \prod_{l=1}^{n} \left[\left(\prod_{j_l \in \bar{J}_l} x_{j_l} \right) \prod_{i_l \in J_l} \frac{x_{i_l}^{\kappa_l} \prod_{\alpha^{(l)} \in I_l} \left(1 - x_{i_l} y_{\alpha^{(l)}}^{-1}\right)}{\prod_{j_{l+1} \in J_{l+1} \setminus J_l} \left(x_{i_l} - x_{j_{l+1}}\right)} \right] , \tag{5.2.97}$$

where $\mathfrak{n}(N_c, n, \kappa) = \mathfrak{n}(N_c, n) + \sum_{l=1}^{n} \kappa_l$, and with the index sets $\mathcal{J}$ as defined in (5.2.90). The expression (5.2.97) can be written more compactly as:

$$\mathscr{L}_\lambda^{(\kappa)}(x,y) = \frac{(-1)^{\sum_{l=1}^{n} \kappa_l} \det_{1\le a,b\le N_c} \left(x_a^{N_c - b + \sum_{c=N_c-b+1}^{N_c} \kappa_c} \prod_{l=N_c-b+1}^{N_c} \prod_{\alpha^{(l)} \in I_l} \left(1 - x_a y_{\alpha^{(l)}}^{-1}\right) \right)}{\prod_{1\le a<b\le N_c} (x_a - x_b)} , \tag{5.2.98}$$

where it is understood that $\kappa_l = 0$ if $l > n$. This obviously reduces to (5.2.78) if $\kappa_l = 0$ for every l.

Example: Twisting the structure sheaf on $\mathbb{P}^{n_f-1}$. As a simple example, consider the $\mathbb{P}^{n_f-1}$ GLSM, corresponding to $N_c = 1$. In this case, we have n_f Schubert varieties indexed by one-row partitions $\lambda = [\lambda]$ of length at most $n_f - 1$. Therefore, the Grothendieck line defects are realised by 1d quivers with a single node at most. Following the discussion above, we find that the characteristic polynomial (5.2.98) associated with the Schubert variety $X_\lambda \subset \mathbb{P}^{n_f-1}$ is given by:

$$\mathscr{L}_\lambda^{\kappa}(x,y) = (-x)^\kappa \prod_{\alpha=1}^{\lambda} (1 - x y_\alpha^{-1}) , \qquad \lambda = 0, \cdots, n_f - 1 , \tag{5.2.99}$$

where $\kappa \in \mathbb{Z}$ is the 1d CS level that we can attach to the 1d $U(1)$ gauge group in the line defect. The interpretation of this defect is that it flows to the structure sheaf on X_λ twisted by the locally-free sheaf $\mathcal{O}(-\kappa)$, namely:

$$\mathscr{L}_\lambda^\kappa \cong \mathcal{O}_\lambda(-\kappa) , \tag{5.2.100}$$

up to a shift functor. It would be interesting to explicitly construct the coherent sheaves corresponding to (5.2.96) for any κ_l when $N_c > 1$. We leave this as a challenge for the interested reader.

5.3 The quantum K-theory of $\mathrm{Gr}(N_c, n_f)$, revisited

In this section, we revisit the GLSM computation of the standard quantum K-theory of the Grassmannian. As first discussed in [133, 134], $\mathrm{QK}(\mathrm{Gr}(N_c, n_f))$ is naturally realised by the $U(N_c)$ gauge theory discussed in section 5.1 for a specific choice of the CS levels, namely:[1]

$$k = N_c - \frac{n_f}{2} \;, \qquad l = -1 \;, \qquad K_{RG} = 0 \;. \tag{5.3.1}$$

We also set $r = 0$ for the R-charge of the chiral multiplets. It is indeed known that the algebra of Wilson loops of this specific 3d $\mathcal{N} = 2$ gauge theory is isomorphic to the quantum K-theory ring of $\mathrm{Gr}(N_c, n_f)$ [133, 134]. Here, we simply wish to insist on the simple fact that we are free to choose any convenient basis for the chiral ring $\mathcal{R}_{\mathrm{3d}}$ of this 3d $\mathcal{N} = 2$ gauge theory compactified on a circle. In particular, we saw in the previous section that the Grothendieck lines $\mathscr{L}_\lambda \cong \mathcal{O}_\lambda$ can be naturally defined as line defects, giving us a physically and mathematically natural basis in terms of Schubert classes. At the level of the 3d A-model, we should simply represent the Schubert classes by the corresponding (double) Grothendieck polynomials, as in (5.2.77).

5.3.1 QK ring from the Bethe ideal

The simplest way to compute the 3d chiral ring, and hence the quantum K-theory ring of X, is to compute the corresponding Bethe ideal as discussed in subsection 2.2.3. Schematically, we have:

$$\mathcal{R}_{\mathrm{3d}} = \frac{\mathbb{K}[x_1, \cdots, x_{N_c}]^{S_{N_c}}}{(\partial\mathcal{W})} \;, \tag{5.3.2}$$

where $(\partial\mathcal{W})$ denotes the algebraic ideal generated by the Bethe equations (5.1.2). This is slightly imprecise for $N_c > 1$, because we need to properly account for the non-abelian sector. This can be done by a standard symmetrisation trick (2.2.56). Let us write the Bethe equations as polynomial equations in the gauge variables $x = \{x_a\}$:

$$P_a(x) = 0 \;, \qquad P \in \mathbb{K}[x] \;, \qquad a = 1, \cdots, N_c \;. \tag{5.3.3}$$

[1] These levels are the ones we would obtain if we start by considering the 3d $\mathcal{N} = 4$ $U(N_c)$ gauge theory with n_f fundamental hypermultiplets, which describes the total space of the cotangent bundle over $\mathrm{Gr}(N_c, n_f)$, and if we then turn on a real mass term that triggers an RG flow which 'integrates out' the non-compact fibres. Note also that choosing $K_{RG} \neq 0$ is simply equivalent to inserting the K-theory class corresponding to the line bundle $(\det S)^{-K_{RG}}$ – this follows from the way this CS level appears in (5.1.11).

We define the symmetrised polynomials:

$$\widehat{P}_{ab} = \frac{P_a - P_b}{x_a - x_b} \in \mathbb{K}[x] \ , \qquad a > b \ , \tag{5.3.4}$$

and the Bethe ideal $\mathcal{I}_{\rm BE}^{(x)} = (P, \widehat{P})$ generated by the polynomials P_a and $\widehat{P}_{ab}$. The next step is to change variables to take care of the residual gauge symmetry S_{N_c} on the Coulomb branch. At this stage, let us introduce the formal variables $\mathcal{O}_\lambda$, which are to be identified with the (double) Grothendieck polynomials $\mathfrak{G}_\lambda(x)$ (for all allowed partitions λ), as well as an additional variable w (2.2.60), and the corresponding polynomials:[1]

$$\widehat{G}_\lambda(x, \mathcal{O}_\lambda) \equiv \mathfrak{G}_\lambda(x) - \mathcal{O}_\lambda \ , \qquad\qquad \widehat{W}(x, w) = w \det x - 1 \ . \tag{5.3.5}$$

This gives us a new Bethe ideal of a larger polynomial ring:

$$\mathcal{I}_{\rm BE}^{(x,w,\mathcal{O})} = (P, \widehat{P}, \widehat{G}, \widehat{W}) \subset \mathbb{K}[x, w, \mathcal{O}] \ . \tag{5.3.6}$$

Since the Grothendieck polynomials are symmetric polynomials in x_a and the Bethe ideal $\mathcal{I}_{\rm BE}^{(x)}$ is S_{N_c}-invariant, we can reduce this larger Bethe ideal to an ideal in terms of the formal variables $\mathcal{O}_\lambda$ only, using the relations $\widehat{G}_\lambda = 0$:

$$\mathcal{I}_{\rm BE}^{(\mathcal{O})} = \mathcal{I}_{\rm BE}^{(x,w,\mathcal{O})}\Big|_{\rm reduce} \subset \mathbb{K}[\mathcal{O}] \ . \tag{5.3.7}$$

This ideal, which we shall dub *the Grothendieck ideal*, can be computed very efficiently using Gröbner basis methods, as explained at length in subsection 2.2.3. In this way, we arrive at a completely gauge-invariant description of the twisted chiral ring (5.3.2), thus giving us an explicit presentation of the (equivariant) quantum K-theory ring directly in terms of the variables $\mathcal{O}_\lambda$:

$$\mathcal{R}_{\rm 3d} \cong {\rm QK}_T(X) \cong \frac{\mathbb{K}[\mathcal{O}]}{\mathcal{I}_{\rm BE}^{(\mathcal{O})}} \ . \tag{5.3.8}$$

The computation of the Grothendieck ideal is completely equivalent to deriving the quantum product structure in terms of the Schubert classes:

$$\mathcal{O}_\mu \star \mathcal{O}_\nu = \mathcal{N}_{\mu\nu}{}^\lambda \, \mathcal{O}_\lambda \ , \qquad\qquad \mathcal{N}_{\mu\nu}{}^\lambda \in \mathbb{K} \ . \tag{5.3.9}$$

[1] The constraint $\widehat{W} = 0$ ensures that non-physical solutions of the Bethe equations such that $x_a = 0$ (for any gauge index a) are disallowed.

Obviously, we could also present the ring $\mathcal{R}_{3\text{d}}$ in terms of any complete set of symmetric polynomials we might be interested in, simply by changing the equations (5.3.5).

Example of $\mathbb{P}^{n_f-1}$

Let us start with the trivial example of the 3d $\mathcal{N}=2$ $U(1)_{-\frac{n_f}{2}}$ gauge theory with n_f chiral multiplets of charge 1. In this case, the 3d twisted chiral ring has a simple description:

$$\mathcal{R}_{3\text{d}} \cong \frac{\mathbb{C}[x]}{\left(\prod_{\alpha=1}^{n_f}(1-xy_\alpha^{-1}-q)\right)} \ , \tag{5.3.10}$$

since we do not need to worry about the effect of the non-abelian gauge symmetry. This ring is generated by a single variable, x, which corresponds to the tautological line bundle $\mathcal{O}(1)$ – this is the Wilson line of unit charge in the gauge theory. The Schubert classes $\mathcal{O}_\lambda$, on the other hand, are indexed by a one-dimensional partition, $\lambda = 1, \cdots, n_f-1$, and are represented by the double Grothendieck polynomials:

$$\mathcal{O}_\lambda = \prod_{\alpha=1}^{\lambda}(1-xy_\alpha^{-1}) \ . \tag{5.3.11}$$

Treating $\mathcal{O}_\lambda$ as a formal variable, the identification (5.3.11) is imposed by the relation $\widehat{G}_\lambda = 0$ in the quotient ring. Indeed, we are interested in writing the ring (5.3.10) as:

$$\mathcal{R}_{3\text{d}} \cong \frac{\mathbb{K}[\mathcal{O}_1, \cdots, \mathcal{O}_{n_f-1}]}{\mathcal{I}_{\text{BE}}^{(\mathcal{O})}} \ . \tag{5.3.12}$$

Of course, this is a redundant parametrisation since one generator would suffice. For instance, we could write the QK ring entirely in terms of the variable:

$$\mathcal{O}_1 = 1 - xy_1^{-1} \qquad \leftrightarrow \qquad x = y_1(1-\mathcal{O}_1) \ , \tag{5.3.13}$$

with the understanding that (5.3.11) gives us an expression for $\mathcal{O}_\lambda$ in terms of products of $\mathcal{O}_1$. The presentation (5.3.12) is most useful if we are interested in working out the product structure, however. We give a few examples below. Let us also note that, in the non-equivariant limit, the QK ring is simply:

$$\mathcal{O}_\lambda \star \mathcal{O}_{\lambda'} = q^{\lfloor \lambda+\lambda' \rfloor}\mathcal{O}_{\lambda+\lambda' \bmod n_f} \ , \tag{5.3.14}$$

for $0 \leq \lambda, \lambda' < n_f$. This follows from the fact that the Bethe equation is simply $(1-x)^{n_f} = q$ when $y_\alpha = 1$.

Equivariant QK ring of $\mathbb{P}^1$. For $n_f = 2$, the only non-trivial product is:

$$\mathcal{O}_1 \star \mathcal{O}_1 = \left(1 - \frac{y_2}{y_1}\right)\mathcal{O}_1 + \frac{y_2}{y_1} q \ , \tag{5.3.15}$$

which reduces to $\mathcal{O}_1^2 = q$ in the non-equivariant limit.

Equivariant QK ring of $\mathbb{P}^2$. For $n_f = 3$, we find:

$$\begin{aligned} \mathcal{O}_1 \star \mathcal{O}_1 &= \left(1 - \frac{y_2}{y_1}\right)\mathcal{O}_1 + \frac{y_2}{y_1}\mathcal{O}_2 \ , \\ \mathcal{O}_1 \star \mathcal{O}_2 &= \left(1 - \frac{y_3}{y_1}\right)\mathcal{O}_2 + \frac{y_3}{y_1} q \ , \\ \mathcal{O}_2 \star \mathcal{O}_2 &= \left(1 - \frac{y_3}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right)\mathcal{O}_2 + \frac{y_3}{y_2} q\mathcal{O}_1 + \left(1 - \frac{y_3}{y_2}\right)\frac{y_3}{y_1} q \ . \end{aligned} \tag{5.3.16}$$

This is in perfect agreement with the results of [131] for the equivariant QK product.

Example of $\mathrm{Gr}(2,4)$

The quantum K-theory of the simplest non-trivial Grassmannian, $\mathrm{Gr}(2,4)$, is obtained from the $U(2)_{0,-2}$ gauge theory with $n_f = 4$ fundamentals. The Bethe equations that give us the QK ring are:

$$P_1 \equiv x_2 \prod_{\alpha=1}^{4}(1 - x_1 y_\alpha^{-1}) + q x_1 = 0 \ , \qquad P_2 \equiv x_1 \prod_{\alpha=1}^{4}(1 - x_2 y_\alpha^{-1}) + q x_2 = 0 \ . \tag{5.3.17}$$

In this case, the minimal set of generators of the twisted chiral ring contains two non-trivial elements. For instance, we can write the ring in terms of the Wilson loops:

$$W_{\square} = x_1 + x_2 \ , \qquad W_{\substack{\square \\ \square}} = x_1 x_2 \ , \tag{5.3.18}$$

and work out the algebra of Wilson loops from there, as in [135]. Here, instead, we directly compute the Grothendieck ideal, which gives us the products displayed in table 5.2. Recall that $(y_1, \cdots, y_4)$ are parameters for the $SU(4)$ flavour symmetry; indeed, the structure constants $\mathcal{N}_{\mu\nu}{}^{\lambda} \in \mathbb{K}$ are invariant under the overall rescaling $(y_\alpha) \to (\lambda y_\alpha)$, which is a gauge transformation.

In the non-equivariant limit, this reduces to:

$$
\begin{aligned}
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{1}} = \mathcal{O}_{\ydiagram{1,1}} + \mathcal{O}_{\ydiagram{2}} - \mathcal{O}_{\ydiagram{2,1}} \,, && \mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2,2}} = q\mathcal{O}_{\ydiagram{2}} \,, \\
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{1,1}} = \mathcal{O}_{\ydiagram{2,1}} \,, && \mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2}} = \mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2}} = \mathcal{O}_{\ydiagram{2,1}} \,, && \mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2,1}} = q\mathcal{O}_{\ydiagram{1}} \,, \\
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2,1}} = \mathcal{O}_{\ydiagram{2,2}} + q - q\mathcal{O}_{\ydiagram{1}} \,, && \mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2,2}} = q\mathcal{O}_{\ydiagram{1,1}} \,, \\
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2,2}} = q\mathcal{O}_{\ydiagram{1}} \,, && \mathcal{O}_{\ydiagram{2,1}} \star \mathcal{O}_{\ydiagram{2,1}} = q\mathcal{O}_{\ydiagram{1,1}} + q\mathcal{O}_{\ydiagram{2}} - q\mathcal{O}_{\ydiagram{2,1}} \,. \\
&\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{1,1}} = \mathcal{O}_{\ydiagram{2,2}} \,, && \mathcal{O}_{\ydiagram{2,1}} \star \mathcal{O}_{\ydiagram{2,2}} = q\mathcal{O}_{\ydiagram{2,1}} \,, \\
&\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2}} = q \,, && \mathcal{O}_{\ydiagram{2,2}} \star \mathcal{O}_{\ydiagram{2,2}} = q^2 \,, \\
&\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2,1}} = q\mathcal{O}_{\ydiagram{1}} \,, &&
\end{aligned}
\tag{5.3.19}
$$

in perfect agreement with [131].[1]

Efficiency of this method. The algorithm that we discussed in this subsection proved to be efficient in computing the quantum ring relations of $\mathrm{QK}(\mathrm{Gr}(N_c, n_f))$ for small values of N_c and n_f. In the following table, we list the durations to perform these calculations in MATHEMATICA for several values of N_c and n_f:[2]

N_c	n_f	Time (sec)
1	3	0.094547
2	4	0.108737
2	5	0.120018
2	6	0.163047
2	7	0.342473
3	4	0.13033
3	5	0.763313

(5.3.20)

And the higher we go in N_c and n_f, the slower it gets. For instance, for $(N_c, n_f) = (3, 6)$ it takes 267.497 seconds.

[1]Our computations in the equivariant case also agree perfectly with the results of [131]. We thank Leonardo Mihalcea for sharing some of their computations with us.

[2]These calculations where preformed on a MacBook Pro, M2, 2022.

5.3.2 Correlations functions and enumerative invariants

In the rest of this section, we further comment on the computation of the two- and three-point functions in this case. We can compute them by two distinct methods, as explained in subsection 5.1.1.

Sum over Bethe vacua and companion-matrix method. In principle, we can evaluate the correlation function of any set of lines by performing the sum over Bethe vacua in (5.1.8). Hence, focusing on the insertion of Grothendieck lines $\mathscr{L}_\lambda \cong \mathcal{O}_\lambda$, we have:

$$\Big\langle \mathcal{O}_\mu \mathcal{O}_\nu \cdots \Big\rangle_{\mathbb{P}^1\times S^1_\beta} = \sum_{\hat{x}\in\mathcal{S}_{\rm BE}} \mathcal{H}(\hat{x},y)^{-1}\, \mathfrak{G}_\mu(\hat{x},y)\mathfrak{G}_\nu(\hat{x},y)\cdots \ , \tag{5.3.21}$$

with $\mathfrak{G}_\mu$ the Grothendieck polynomials, and the handle-gluing operator:

$$\mathcal{H}(x,y) = \det_{1\leq a,b\leq N_c}\left(\delta_{ab}\left(N_c + \sum_{\alpha=1}^{n_f}\frac{x_a y_\alpha^{-1}}{1-x_a y_\alpha^{-1}}\right)-1\right)\prod_{\alpha=1}^{n_f}(1-x_a y_\alpha)\prod_{\substack{a,b\\ a\neq b}}(1-x_a x_b^{-1})^{-1}\ . \tag{5.3.22}$$

The sum (5.3.21) can be performed using the companion matrix method discussed in subsection 2.2.3. This amounts to writing each factor $Q(x)$ in the summand of (5.3.21) as a large square matrix $\mathfrak{M}_Q$ of size $|\mathcal{S}_{\rm BE}|$, in a convenient basis of the quotient ring $\mathcal{R}_{\rm 3d}$. The eigenvalues of $\mathfrak{M}_Q$ are equal to $Q(\hat{x})$, the operator Q evaluated at the Bethe vacua. Hence, the sum over Bethe vacua can be performed by taking the trace over a product of companion matrices without having to solve for the eigenvalues themselves.

JK residue. Specialising the JK residue formula (5.1.22)-(5.1.23) to the CS levels (5.3.1), we have:

$$\begin{aligned}\Big\langle \mathscr{L}\Big\rangle_{\mathbb{P}^1\times S^1_\beta} &= \sum_{d=0}^{\infty} q^d\, \mathbf{I}_d[\mathscr{L}]\ ,\\ \mathbf{I}_d[\mathscr{L}] &= \sum_{\substack{\mathfrak{m}_a\geq 0\\ |\mathfrak{m}|=d}} \frac{(-1)^{N_c}}{N_c!}\oint_{\rm JK}\prod_{a=1}^{N_c}\left[\frac{dx_a}{2\pi i}\frac{(\det x)^{-1}}{\prod_{\alpha=1}^{n_f}(1-x_a y_\alpha^{-1})}\right]\Delta(x)F_\mathfrak{m}(x,y)\mathscr{L}(x,y)\ ,\end{aligned} \tag{5.3.23}$$

where it is understood that the JK residue is a sum over all the residues at the codimension-N_c singularities $\{x_a = y_{\alpha_a}\}$, and we defined the flux-dependent factor:

$$F_\mathfrak{m}(x,y) = \frac{(-1)^{|\mathfrak{m}|(N_c-1)}\prod_{a=1}^{N_c} x_a^{N_c\mathfrak{m}_a}}{(\det x)^{|\mathfrak{m}|}\prod_{a=1}^{N_c}\prod_{\alpha=1}^{n_f}(1-x_a y_\alpha^{-1})^{\mathfrak{m}_a}}\ . \tag{5.3.24}$$

This is a completely explicit formula for the genus-zero degree-d K-theoretic enumerative invariants of the Grassmannian. In the 2d limit (which gives us the quantum cohomology of X), the equivalence between the physical and the mathematical results is rigorously established [146], and similar considerations hold in 3d as well [134, 126].

Obviously, the Bethe-vacua formula (5.3.21) is more powerful in principle, since it gives us the full answer directly as a rational function in q, while the JK residue formula gives us the same answer as a Taylor series in q, in which case we need to compute each term individually. The JK residue formula is nonetheless very practical to compute specific enumerative invariants $\mathbf{I}_d[\mathscr{L}]$ at fixed degree as opposed to the full correlation functions.

Two-point functions and topological metric

In the basis for $\mathrm{QK}_T(X)$ spanned by the Schubert classes, $\{\mathcal{O}_\lambda\}$, the topological metric is given by:

$$\eta_{\mu\nu} = \Big\langle \mathcal{O}_\mu \, \mathcal{O}_\nu \Big\rangle_{\mathbb{P}^1 \times S^1_\beta} , \tag{5.3.25}$$

where we insert the double Grothendieck polynomials in the integrand of (5.3.23) as indicated, plugging in $\mathscr{L}(x,y) = \mathfrak{G}_\mu(x,y)\mathfrak{G}_\nu(x,y)$. Alternatively, we can use the Bethe-vacua formula (5.3.21). Let us consider a few simple examples.

Topological metric for $\mathrm{QK}_T(\mathbb{P}^1)$**.** For the abelian $U(1)_{-1}$ theory with 2 matter multiplets with charge 1, we find the following metric in the non-equivariant limit:

$$\eta_{\mu\nu} = \frac{1}{1-q} \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} , \tag{5.3.26}$$

in the basis $\{1, \mathcal{O}_\square\}$. In the equivariant case, the components are the same as above except for:

$$\eta_{\square,\square} = 1 - \frac{y_2}{y_1} + \frac{q}{1-q} . \tag{5.3.27}$$

Topological metric for $\mathrm{QK}_T(\mathbb{P}^2)$**.** For the abelian theory $U(1)_{-\frac{3}{2}}$ with $n_f = 3$, we find the following y-dependent components for the topological metric:

$$\begin{aligned} \eta_{\square,\square\square} &= 1 - \frac{y_3}{y_1} + \frac{q}{1-q} , \\ \eta_{\square\square,\square\square} &= \left(1 - \frac{y_3}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right) + \frac{q}{1-q} . \end{aligned} \tag{5.3.28}$$

The other components (up to symmetry, $\eta_{\mu\nu} = \eta_{\nu\mu}$) are the same as in the non-equivariant limit, where we have:

$$\eta_{\mu\nu} = \frac{1}{1-q}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & q \\ 1 & q & q \end{pmatrix}, \tag{5.3.29}$$

in the basis $\{1, \mathcal{O}_{\square}, \mathcal{O}_{\square\square}\}$.

Topological metric for $\mathrm{QK}_T(\mathbb{P}^4)$**.** For $U(1)_{-\frac{5}{2}}$ with $n_f = 5$, we find that the components of the topological metric have the explicit form:

$$\begin{aligned} \eta_{1,4} &= 1 - \frac{y_5}{y_1} + \frac{q}{1-q}, \\ \eta_{2,3} &= 1 - \frac{y_4 y_5}{y_1 y_2} + \frac{q}{1-q}, \\ \eta_{2,4} &= \left(1 - \frac{y_5}{y_1}\right)\left(1 - \frac{y_5}{y_2}\right) + \frac{q}{1-q}, \end{aligned} \tag{5.3.30}$$

$$\begin{aligned} \eta_{3,3} &= 1 + \frac{y_4 y_5^2}{y_1 y_2 y_3} - \frac{y_4 y_5}{y_2 y_3} - \frac{y_4 y_5}{y_1 y_3} - \frac{y_4 y_5}{y_1 y_2} + \frac{y_4^2 y_5}{y_1 y_2 y_3} + \frac{q}{1-q}, \\ \eta_{3,4} &= \left(1 - \frac{y_5}{y_1}\right)\left(1 - \frac{y_5}{y_2}\right)\left(1 - \frac{y_5}{y_3}\right) + \frac{q}{1-q}, \\ \eta_{4,4} &= \left(1 - \frac{y_5}{y_1}\right)\left(1 - \frac{y_5}{y_2}\right)\left(1 - \frac{y_5}{y_3}\right)\left(1 - \frac{y_5}{y_4}\right) + \frac{q}{1-q}, \end{aligned} \tag{5.3.31}$$

with the obvious index notation $\mu, \nu = 0, \cdots, 4$. The other components are the same as in the non-equivariant limit, in which case we find:

$$\eta_{\mu\nu} = \frac{1}{1-q}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & q \\ 1 & 1 & 1 & q & q \\ 1 & 1 & q & q & q \\ 1 & q & q & q & q \end{pmatrix}. \tag{5.3.32}$$

Topological metric for $\mathrm{QK}_T(\mathrm{Gr}(2,4))$**.** For the $\mathrm{Gr}(2,4)$ case, the y-dependent components of the topological metric have the following explicit form:

$$
\begin{aligned}
\eta_{\ydiagram{1},\ydiagram{2,2}} &= \left(1-\frac{y_3y_4}{y_1y_2}\right)+\frac{q}{1-q}\,, \\
\eta_{\ydiagram{1,1},\ydiagram{2}} = \eta_{\ydiagram{1,1},\ydiagram{2,1}} = \eta_{\ydiagram{2},\ydiagram{2,1}} &= \left(1-\frac{y_4}{y_1}\right)+\frac{q}{1-q}\,, \\
\eta_{\ydiagram{1,1},\ydiagram{2,2}}(q,y) &= \left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_4}{y_1}\right)+\frac{q}{1-q}\,, \\
\eta_{\ydiagram{2,1},\ydiagram{2,1}} &= \left(1-\frac{y_4}{y_1}\right)^2+\frac{q}{1-q}\,, \\
\eta_{\ydiagram{2,1},\ydiagram{2,2}}(q,y) &= \left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)+\frac{q}{1-q}\,, \\
\eta_{\ydiagram{2},\ydiagram{2,2}}(q,y) &= \left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)+\frac{q}{1-q}\,, \\
\eta_{\ydiagram{2,2},\ydiagram{2,2}}(q,y) &= \left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)+\left(1-\frac{y_3y_4}{y_1y_2}\right)q \\
&\quad +\frac{q^2}{1-q}\,.
\end{aligned}
\tag{5.3.33}
$$

The other components take the same form as in the non-equivariant limit, namely:

$$
\eta_{\mu\nu} = \frac{1}{1-q}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & q \\
1 & 1 & 1 & q & q & q \\
1 & 1 & q & 1 & q & q \\
1 & 1 & q & q & q & q \\
1 & q & q & q & q & q^2
\end{pmatrix}\,,
\tag{5.3.34}
$$

in the basis $\{1, \mathcal{O}_{\ydiagram{1}}, \mathcal{O}_{\ydiagram{1,1}}, \mathcal{O}_{\ydiagram{2}}, \mathcal{O}_{\ydiagram{2,1}}, \mathcal{O}_{\ydiagram{2,2}}\}$.

Three-point functions and structure constants

The knowledge of the correlators implies the knowledge of the ring structure, so the JK residue formula gives us another way to compute the ring $\mathrm{QK}_T(X)$. Let us decompose the ring structure constants as:

$$\mathcal{N}_{\mu\nu}{}^{\lambda} = \sum_{d\geq 0} \mathcal{N}_{\mu\nu}^{(d)\lambda}\, q^d \;, \qquad \mathcal{N}_{\mu\nu}^{(d)\lambda} \in \mathbb{Z}(y) \;. \tag{5.3.35}$$

These quantities can be computed using the JK residue formula (5.3.23), as:

$$\mathcal{N}_{\mu\nu}^{(d)\lambda} = \mathbf{I}_d \left[\mathcal{O}_\mu\, \mathcal{O}_\nu\, \mathcal{O}^{\vee\,\lambda}\right] \;, \tag{5.3.36}$$

where we introduced the dual basis $\{\mathcal{O}^{\vee\,\lambda}\}$, indexed by partitions λ, such that:

$$\left\langle \mathcal{O}_\mu\, \mathcal{O}^{\vee\,\nu} \right\rangle_{\mathbb{P}^1\times S^1_\beta} = \delta_\mu{}^\nu \;. \tag{5.3.37}$$

The dual structure sheaves $\mathcal{O}^{\vee\,\lambda}$ [131] can be realised by dual Grothendieck lines whose 1d Witten indices give us the following *dual double Grothendieck polynomials*:

$$\mathcal{O}^{\vee\,\lambda}(x,y) = \frac{\det x}{\prod_{a=1}^{N_c} y_{n_f - N_c + a - \lambda_a^\vee}}\, \mathcal{O}_{\lambda^\vee}(x, y^D) \;, \tag{5.3.38}$$

where $\lambda^\vee$ is the partition dual to λ:

$$[\lambda_1^\vee, \cdots, \lambda_{N_c}^\vee] = [N_c - n_f - \lambda_{N_c}, \cdots, N_c - n_f - \lambda_1] \;, \tag{5.3.39}$$

and y^D denotes the order-inverted $SU(n_f)$ parameters, $y^D_\alpha = y_{n_f+1-\alpha}$.

Dual double Grothendieck polynomials for $\mathbb{P}^{n_f-1}$. For the Schubert cells of $\mathrm{QK}_T(\mathbb{P}^{n_f-1})$, we have the following dual double Grothendieck polynomials:

$$\mathcal{O}^{\vee\,\lambda}(x,y) \equiv \frac{x}{y_{\lambda+1}} \prod_{\alpha=\lambda+2}^{n_f-1} (1 - x y_\alpha^{-1}) \;, \qquad \lambda = 1, \cdots, n_f - 1 \;. \tag{5.3.40}$$

Dual double Grothendieck polynomials for $\mathrm{Gr}(2,4)$. For the case of $\mathrm{Gr}(2,4)$, the dual double Grothendieck polynomials associated with the dual structure sheaves defined in (5.3.38) read:

$$
\begin{aligned}
\mathcal{O}^{\vee,\ydiagram{1}}(x,y) &= \frac{x_1x_2}{y_1y_3} - \frac{x_1x_2^2}{y_1y_3y_4} - \frac{x_1^2x_2^2}{y_1y_2y_3^2} + \frac{x_1^2x_2^2}{y_1y_3y_4^2} - \frac{x_1^2x_2}{y_1y_3y_4} + \frac{x_1^2x_2^3}{y_1y_2y_3^2y_4} \\
&\quad + \frac{x_1^3x_2^2}{y_1y_2y_3^2y_4} - \frac{x_1^3x_2^3}{y_1y_2y_3^2y_4^2} \ , \\
\mathcal{O}^{\vee,\ydiagram{1,1}}(x,y) &= \frac{x_1x_2}{y_2y_3} - \frac{x_1x_2^2}{y_2y_3y_4} - \frac{x_1^2x_2}{y_2y_3y_4} + \frac{x_1^2x_2^2}{y_2y_3y_4^2} \ , \\
\mathcal{O}^{\vee,\ydiagram{2}}(x,y) &= \frac{x_1x_2}{y_1y_4} - \frac{x_1^2x_2^2}{y_1y_2y_3y_4} - \frac{x_1^2x_2^2}{y_1y_2y_4^2} - \frac{x_1^2x_2^2}{y_1y_3y_4^2} + \frac{x_1^2x_2^3}{y_1y_2y_3y_4^2} + \frac{x_1^3x_2^2}{y_1y_2y_3y_4^2} \ , \\
\mathcal{O}^{\vee,\ydiagram{2,1}}(x,y) &= \frac{x_1x_2}{y_2y_4} - \frac{x_1^2x_2^2}{y_2y_3y_4^2} \ , \\
\mathcal{O}^{\vee,\ydiagram{2,2}}(x,y) &= \frac{x_1x_2}{y_3y_4} \ .
\end{aligned}
\tag{5.3.41}
$$

Structure constants for $\mathrm{QK}_T(\mathrm{Gr}(2,4))$**.** As an example of an application of the JK residue formula (5.3.23), we can compute the structure constants up to some degree d using (5.3.36). Given the explicit expression for the dual double Grothendieck polynomials, this is a straightforward computation (using *e.g.* MATHEMATICA). For instance, computing up to order q^4, we find:

$$
\begin{aligned}
\left\langle \mathcal{O}_{\ydiagram{1}}\, \mathcal{O}_{\ydiagram{1}}\, \mathcal{O}^{\vee\,\ydiagram{1}} \right\rangle_{\mathbb{P}^1\times S^1_\beta} &= 1 - \frac{y_3}{y_2} \ , \\
\left\langle \mathcal{O}_{\ydiagram{1,1}}\, \mathcal{O}_{\ydiagram{2,2}}\, \mathcal{O}^{\vee\,\ydiagram{2}} \right\rangle_{\mathbb{P}^1\times S^1_\beta} &= \frac{y_3}{y_1} q \ , \\
\left\langle \mathcal{O}_{\ydiagram{2,1}}\, \mathcal{O}_{\ydiagram{2,1}}\, \mathcal{O}^{\vee\,\ydiagram{2,1}} \right\rangle_{\mathbb{P}^1\times S^1_\beta} &= \left(1 - \frac{y_2}{y_1}\right)\left(1 - \frac{y_4}{y_1}\right)\left(1 - \frac{y_4}{y_3}\right) - q\frac{y_4}{y_1} \ , \\
\left\langle \mathcal{O}_{\ydiagram{2,2}}\, \mathcal{O}_{\ydiagram{2,2}}\, \mathcal{O}^{\vee\,\ydiagram{1,1}} \right\rangle_{\mathbb{P}^1\times S^1_\beta} &= q\left(1 - \frac{y_3}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right)\frac{y_4}{y_2} \ ,
\end{aligned}
\tag{5.3.42}
$$

which indeed agrees with the results already reported in table 5.2.

Classical limit ($q \to 0$): K-theoretic Littlewood—Richardson coefficients

As a sanity check of our computations, it is interesting to consider the "classical" limit $q \to 0$ (that is, the large-volume limit), in which case the equivariant quantum K-theory $\mathrm{QK}_T(X)$ reduces to the equivariant K-theory $\mathrm{K}_T(X)$, with the ring structure (5.2.27) given in terms of the K-theoretic LR coefficients:

$$C_{\lambda\mu}{}^{\nu} = \mathcal{N}^{(0)\nu}_{\lambda\mu} \ . \tag{5.3.43}$$

The JK-residue formula (5.3.23) gives us an explicit expression for these LR coefficients:

$$C_{\lambda\mu}{}^{\nu} = \frac{(-1)^{N_c}}{N_c!} \oint_{\text{JK}} \prod_{a=1}^{N_c} \left[\frac{dx_a}{2\pi i} \frac{(\det x)^{-1} \prod_{b\neq a}(x_a - x_b)}{\prod_{\alpha=1}^{n_f}(1 - x_a y_\alpha^{-1})} \right] \mathcal{O}_\lambda(x,y)\mathcal{O}_\mu(x,y)\mathcal{O}^{\vee\,\nu}(x,y) \ . \tag{5.3.44}$$

This is the general formula in the equivariant case. For $y_i = 1$, this reduces to:

$$C_{\lambda\mu}{}^{\nu} = \frac{(-1)^{N_c}}{N_c!} \oint_{(x_a=1)} \prod_{a=1}^{N_c} \left[\frac{dx_a}{2\pi i} \frac{\prod_{b\neq a}(x_a - x_b)}{(1 - x_a)^{n_f}} \right] (\det x)^{1-N_c}\, \mathfrak{G}_\lambda(x)\mathfrak{G}_\mu(x)\mathfrak{G}_{\nu^\vee}(x) \ , \tag{5.3.45}$$

in terms of the ordinary Grothendieck polynomials, with a single residue at $\{x_a = 1\}$. We checked in many examples that this formula always returns an integer, as expected. It also appears to agree with the known K-theoretic LR coefficients – see *e.g.* [191, 192]. For instance, one can easily check that:

$$\begin{aligned} &C_{[1,0],[1,0]}{}^{[2,1]} &&= -1 &&\text{for } N_c = 2\ ,\quad n_f = 4\ ,\\ &C_{[2,0,0],[2,1,0]}{}^{[3,2,1]} &&= -2 &&\text{for } N_c = 3\ ,\quad n_f = 9\ ,\\ &C_{[3,2,1,0],[3,2,1,0]}{}^{[5,4,2,2]} &&= -9 &&\text{for } N_c = 4\ ,\quad n_f = 10\ . \end{aligned} \tag{5.3.46}$$

Here, we picked examples with $|\nu| > |\lambda| + |\mu|$, which would vanish in the cohomological limit. It may be worthwhile to obtain direct proof that this residue formula indeed gives us the K-theoretic LR coefficients. We leave this as another challenge for the interested reader.

5.3.3 Generalised QK rings

Finally, let us comment on the generalisation of the above considerations when one chooses other Chern–Simons levels (k, l) in the geometric window. The same computations can obviously be performed in such cases, and it is expected that the more general twisted chiral rings essentially correspond to the level structures of Ruan and Zhang [193]:

$$\mathcal{R}_{\text{3d}}[k, l \text{ in geometric window}] \qquad \overset{?}{\longleftrightarrow} \qquad \text{QK}_T(X) \text{ with level structure}\ . \tag{5.3.47}$$

To the best of our knowledge, the precise map between the physics of the CS levels k, l and the mathematical notions of [193] has not been worked out yet, and we hope to better address this point in future work – see [20, 133, 134] for some relevant past works. Here, we simply point out that we can easily compute the left-hand side of (5.3.47) for any value of k, l. It would be interesting to compare these results to direct enumerative geometry computations of QK rings with non-trivial level structures.

Generalised QK rings for $\mathrm{Gr}(2,4)$

Recall that the ordinary QK ring is given by $\mathcal{R}_{\mathrm{3d}}[0,-1]$, which gives us (5.3.19) in the non-equivariant limit. As an example of a generalised QK ring for $\mathrm{Gr}(2,4)$, consider $\mathcal{R}_{\mathrm{3d}}[1,-1]$, for the gauge theory $U(2)_{1,-1}$ with 4 fundamentals. By direct computation, one can work out its ring structure in the basis of Schubert classes, which reads:

$$
\begin{aligned}
&\mathcal{O}_{(1)}^2 = \mathcal{O}_{(1,1)} + \mathcal{O}_{(2)} - \mathcal{O}_{(2,1)} , && \mathcal{O}_{(1,1)} \star \mathcal{O}_{(2,2)} = -q\mathcal{O}_{(2)} + q\mathcal{O}_{(2,1)} , \\
&\mathcal{O}_{(1)} \star \mathcal{O}_{(1,1)} = \mathcal{O}_{(2,1)} , && \mathcal{O}_{(2)}^2 = -q\mathcal{O}_{(1)} + q\mathcal{O}_{(2)} + \mathcal{O}_{(2,2)} , \\
&\mathcal{O}_{(1)} \star \mathcal{O}_{(2)} = -q + q\mathcal{O}_{(1)} + \mathcal{O}_{(2,1)} , && \mathcal{O}_{(2)} \star \mathcal{O}_{(2,1)} = -q\mathcal{O}_{(1)} + q\mathcal{O}_{(2,1)} , \\
&\mathcal{O}_{(1)} \star \mathcal{O}_{(2,1)} = -q + q\mathcal{O}_{(1)} + \mathcal{O}_{(2,2)} , && \mathcal{O}_{(2)} \star \mathcal{O}_{(2,2)} = -q\mathcal{O}_{(1,1)} + q\mathcal{O}_{(2,2)} , \\
&\mathcal{O}_{(1)} \star \mathcal{O}_{(2,2)} = -q\mathcal{O}_{(1)} + q\mathcal{O}_{(1,1)} , && \mathcal{O}_{(2,1)}^2 = -q\mathcal{O}_{(1,1)} - q\mathcal{O}_{(2)} + 2q\mathcal{O}_{(2,1)} , \\
&\mathcal{O}_{(1,1)}^2 = \mathcal{O}_{(2,2)} , && \mathcal{O}_{(2,1)} \star \mathcal{O}_{(2,2)} = -q\mathcal{O}_{(2,1)} + q\mathcal{O}_{(2,2)} , \\
&\mathcal{O}_{(1,1)} \star \mathcal{O}_{(2)} = -q + q\mathcal{O}_{(1,1)} , && \mathcal{O}_{(2,2)}^2 = q^2 - q^2\mathcal{O}_{(1)} . \\
&\mathcal{O}_{(1,1)} \star \mathcal{O}_{(2,1)} = -q\mathcal{O}_{(1)} + q\mathcal{O}_{(1,1)} , &&
\end{aligned}
\tag{5.3.48}
$$

All the other $\mathrm{Gr}(2,4)$ theories in the geometric window can be worked out similarly. We list a few other non-equivariant rings $\mathcal{R}_{\mathrm{3d}}[k,l]$ for $\mathrm{Gr}(2,4)$ in appendix E.

5.4 Quantum cohomology and Schubert defects in the 2d GLSM

In this section, we briefly discuss the dimensional reduction of the 3d GLSM to the ordinary 2d GLSM onto the Grassmannian. The 2d GLSM of interest is the 2d $\mathcal{N} = (2,2)$

$U(N_c)$ gauge theory with n_f fundamental chiral multiplets. Its twisted chiral ring gives us the quantum cohomology of $X = \text{Gr}(N_c, n_f)$, and turning on the twisted masses m_α corresponds to the T-equivariant deformation:[1]

$$\mathcal{R}^{2\text{d}} \cong \text{QH}^\bullet_T(X) \ . \tag{5.4.1}$$

We can similarly compute genus-zero Gromov–Witten (GW) invariants as correlation functions in the 2d GLSM [25, 31].

It is interesting to see how these 2d quantities emerge from our discussion above by taking the 2d limit of the 3d GLSM. Let us consider the limit $\beta \to 0$ on $\Sigma \times S^1_\beta$, where β is the radius of the S^1. The 3d and 2d variables are related as:

$$x_a = e^{-2\pi\beta\sigma_a} \ , \qquad y_\alpha = e^{-2\pi\beta m_\alpha} \ , \qquad q = (2\pi\beta)^{n_f} q_{2\text{d}} \ , \tag{5.4.2}$$

with σ_a the 2d Coulomb-branch scalars, of mass dimension 1. Note that $q_{2\text{d}}$ has mass dimension n_f due to the non-trivial running of the 2d FI parameters.

5.4.1 Defect point operators and Schubert polynomials

The Poincaré duals of the Schubert varieties, $\omega_\lambda \equiv [X_\lambda]$, are also called the Schubert classes:

$$\omega_\lambda \in \text{H}^{2|\lambda|}(X) \ , \tag{5.4.3}$$

and similarly in the equivariant setting. The equivariant Schubert classes in cohomology can be written as double Schubert polynomials $\mathfrak{S}_\lambda(\sigma, m)$, where $\sigma_a \in \text{H}^2(X)$ correspond to the Chern roots of S – see *e.g.* [194, 195, 196]. The double Schubert polynomial $\mathfrak{S}_\lambda(\sigma, m)$ indexed by the partition λ can be written as [197]

$$\mathfrak{S}_\lambda(\sigma, m) \equiv \frac{\det_{1\leq a,b\leq N_c}\left(\prod_{\alpha=1}^{\lambda_a+N_c-b}(\sigma_b - m_\alpha)\right)}{\prod_{1\leq b<a\leq N_c}(\sigma_a - \sigma_b)} \ . \tag{5.4.4}$$

In the non-equivariant limit, $m_\alpha \to 0$, they reduce to the Schur polynomials:

$$s_\lambda(\sigma) \equiv \frac{\det_{1\leq a,b\leq N_c}\left(\sigma_a^{\lambda_{N_c-b+1}-n_f+N_c}\right)}{\prod_{1\leq b<a\leq N_c}(\sigma_a - \sigma_b)} \ . \tag{5.4.5}$$

[1]Here we use the notation m_α instead of m_i, to diminish clutter in some formulas.

We should insert these polynomials in the 2d A-model computation, as we will review below.

Taking the 2d limit $\beta \to 0$ using the parametrisation (5.4.2), we easily check that the double Grothendieck polynomials (5.2.78) reduce to the double Schubert polynomials (5.4.4), with the scaling:

$$\mathfrak{G}_\lambda(x,y) \to (2\pi\beta)^{|\lambda|}\, \mathfrak{S}_\lambda(\sigma, m) \ , \tag{5.4.6}$$

up to higher-order terms in β. Therefore, we expect that the Schubert polynomials can be obtained from the point defects in 2d that we can obtain by wrapping the Grothendieck lines along the S^1_β.

Schubert point defects and 0d $\mathcal{N} = 2$ quivers

Let us thus consider the 2d GLSM coupled to a defect operator, dubbed *Schubert defect*, defined by coupling the 2d theory to a 0d $\mathcal{N} = 2$ supersymmetric quiver – that is, a supersymmetric matrix model (SMM) at the point $p \in \Sigma$ – see *e.g.* [198, 199] for discussions of such gauged SMMs.

The 0d quiver is defined exactly as in the 3d/1d case. We have the 0d gauge group $G_{0\text{d}} = \prod_{l=1}^n U(r_l)$ with bifundamental chiral matter multiplets connecting each two consecutive nodes of the 0d defect, as well as M_l 0d fundamental fermi multiplets at each $U(r_l)$ node, exactly as in figure 5.3. Similarly to (5.2.37), we denote these defects by:

$$\omega \begin{bmatrix} \mathbf{r} \\ \mathbf{M} \end{bmatrix} \ , \qquad \begin{bmatrix} \mathbf{r} \\ \mathbf{M} \end{bmatrix} \equiv \begin{bmatrix} r_1 & \cdots & r_n \\ M_1 & \cdots & M_n \end{bmatrix} \ . \tag{5.4.7}$$

The "supersymmetric vacuum equations" for this defect theory are similar to the ones of the 1d defect; namely, we still have to solve (5.2.40) and (5.2.43). Therefore, the defect restricts the 2d field ϕ at $p \in \Sigma$ to the Schubert cell given as in (5.2.57).

The defect contributes to the 2d A-model according to its supersymmetric 'partition function', which can be obtained by naive dimensional reduction of the 1d index. This gives us a polynomial in σ_a and m_α, which we denote by:

$$\omega_\lambda(\sigma, m) \equiv Z^{0\text{d}} \begin{bmatrix} r_1 & \cdots & r_n \\ M_1 & \cdots & M_n \end{bmatrix} (\sigma, m) \ . \tag{5.4.8}$$

For definiteness, let us consider the 'generic Schubert defect' defined as in figure 5.4. Then the parameters r_l and M_l are given by (5.2.64). This supersymmetric matrix model can be reduced to a JK residue, similar to the 1d index. One finds:

$$\omega_\lambda(\sigma, m) = \prod_{l=1}^{n} \frac{1}{l!} \oint \frac{d^l s^{(l)}}{(2\pi i)^l}\, \Delta^{(l)}(s)\, \mathrm{Z}^{\mathrm{0d}}_{\mathrm{matter}}(s, \sigma, m) \ , \tag{5.4.9}$$

where $s^{(l)}_{i_l}$ are the components of the adjoint complex scalar that live in the 0d $\mathcal{N}=2$ $U(r_l)$ vector multiplet. The matter contribution is given by:

$$\mathrm{Z}^{\mathrm{0d}}_{\mathrm{matter}}(\sigma, m) = \prod_{l=1}^{n-1} \left(\prod_{i_l=1}^{r_l} \frac{\prod_{\alpha^{(l)}\in I_l} \left(s^{(l)}_{i_l} - m_{\alpha^{(l)}}\right)}{\prod_{j_{l+1}=1}^{l+1} \left(s^{(l)}_{i_l} - s^{(l+1)}_{j_{l+1}}\right)} \right) \prod_{i_n=1}^{r_n} \frac{\prod_{\alpha^{(n)}\in I_n} \left(s^{(n)}_{i_n} - m_{\alpha^{(n)}}\right)}{\prod_{a=1}^{N_c} \left(s^{(n)}_{i_n} - \sigma_a\right)} \ , \tag{5.4.10}$$

and we defined the Vandermonde determinant factor as in (5.2.86):

$$\Delta^{(l)}(s) \equiv \prod_{1\leq i_l \neq j_l \leq l} \left(s^{(l)}_{i_l} - s^{(l)}_{j_l}\right) \ . \tag{5.4.11}$$

The factors appearing in the numerator of (5.4.10) come from the Fermi multiplets at each node of the 0d quiver. These are indexed by the sets I_l defined in (5.2.38). Meanwhile, the factors in the denominator originate from the bifundamental chiral multiplets of the 0d quiver. The contour integrals in (5.4.9) should be performed recursively, starting with the $U(r_1)$ node.

The case of 1-partitions. In the case of the partition $\lambda = [\lambda_1, 0, \cdots, 0]$, the partition function (5.4.9) becomes:

$$\omega_{[\lambda_1,0,\cdots,0]}(\sigma, m) = \oint \frac{ds}{2\pi i} \frac{\prod_{\alpha\in I_1}(s - m_\alpha)}{\prod_{a=1}^{N_c}(s-\sigma_a)} = \sum_{1\leq a\leq N_c} \frac{\prod_{\alpha\in I_1}(\sigma_a - m_\alpha)}{\prod_{b\neq a}^{N_c}(\sigma_a - \sigma_b)} \ . \tag{5.4.12}$$

The generic partition. Jumping ahead to the general case, let us take the partition $\lambda = [\lambda_1, \cdots, \lambda_n, 0, \cdots, 0]$. Doing the contour integrals (5.4.9) recursively, as we did for (5.2.89), we find the following explicit form for the partition function of the matrix model:

$$\begin{aligned} \omega_\lambda(\sigma, m) &= \sum_{\mathcal{J}} \prod_{l=1}^{n} \left[\Delta^{(J_l)}(\sigma) \prod_{i_l\in J_l} \frac{\prod_{\alpha^{(l)}\in I_l}(\sigma_{i_l} - m_{\alpha^{(l)}})}{\prod_{j_{l+1}\in J_{l+1}} \left(\sigma_{i_l} - \sigma_{j_{l+1}}\right)} \right] \ , \\ &= \sum_{\mathcal{J}} \prod_{l=1}^{n} \prod_{i_l\in J_l} \frac{\prod_{\alpha^{(l)}\in I_l}(\sigma_{i_l} - m_{\alpha^{(l)}})}{\prod_{j_{l+1}\in J_{l+1}\setminus J_l} \left(\sigma_{i_l} - \sigma_{j_{l+1}}\right)} \ , \end{aligned} \tag{5.4.13}$$

with the indexing sets $\mathcal{J}$ defined in (5.2.90). The Vandermonde factor appearing in the first line is defined exactly as in (5.2.92). The expression (5.4.13) can be massaged into the determinant formula:

$$\omega_\lambda(\sigma, m) = \frac{\det_{1\leq a,b\leq N_c}\left[\prod_{l=N_c-b+1}^{N_c}\prod_{\alpha^{(l)}\in I_l}\left(\sigma_a - m_{\alpha^{(l)}}\right)\right]}{\prod_{1\leq b<a\leq N_c}\left(\sigma_a-\sigma_b\right)} = \mathfrak{S}_\lambda(\sigma, m)\ , \tag{5.4.14}$$

which is none other than the double Schubert polynomial (5.4.4).

Double Schubert polynomials for $\mathrm{Gr}(2,4)$**.** As an example, let us write down the Schubert polynomials for $\mathrm{Gr}(2,4)$:

$$\begin{aligned}
\mathfrak{S}_{\square}(\sigma,m) &= \sigma_1+\sigma_2-m_1-m_2\ ,\\
\mathfrak{S}_{\substack{\square\\\square}}(\sigma,m) &= \sigma_1\sigma_2 - m_1\sigma_1 - m_1\sigma_2 + m_1^2\ ,\\
\mathfrak{S}_{\square\square}(\sigma,m) &= \sigma_1^2+\sigma_1\sigma_2+\sigma_2^2-(m_1+m_2+m_3)\sigma_1-(m_1+m_2+m_3)\sigma_2\\
&\quad + m_1m_2+m_1m_3+m_2m_3\ ,\\
\mathfrak{S}_{\substack{\square\square\\\square}}(\sigma,m) &= \sigma_1\sigma_2^2+\sigma_1^2\sigma_2+(m_1^2+m_1m_2+m_1m_3)\sigma_1+(m_1^2+m_1m_2+m_1m_3)\sigma_2\\
&\quad - m_1\sigma_1^2-m_1\sigma_2^2-(2m_1+m_2+m_3)\sigma_1\sigma_2-m_1^2m_2-m_1^2m_3\ ,\\
\mathfrak{S}_{\substack{\square\square\\\square\square}}(\sigma,m) &= \sigma_1^2\sigma_2^2-(m_1^2m_2+m_1m_2^2)\sigma_1-(m_1^2m_2+m_1m_2^2)\sigma_2+(m_1+m_2)^2\sigma_1\sigma_2\\
&\quad + m_1m_2\sigma_1^2+m_1m_2\sigma_2^2-(m_1+m_2)\sigma_1^2\sigma_2-(m_1+m_2)\sigma_1\sigma_2^2+m_1^2m_2^2\ .
\end{aligned} \tag{5.4.15}$$

They are indeed obtained as the 2d limit of the Grothendieck polynomials (5.2.81).

5.4.2 Localisation formula on $\mathbb{P}^1$ and GW invariants

Any A-model correlation function of the Grassmannian GLSM on $\mathbb{P}^1$ can be obtained from a JK residue similar to (5.3.23), as derived in [31, 113]:

$$\begin{aligned} \left\langle \omega \right\rangle_{\mathbb{P}^1} &= \sum_{d=0}^{\infty} q_{2\mathrm{d}}^d \, \mathbf{I}_d^{2\mathrm{d}}[\omega] \ , \\ \mathbf{I}_d^{2\mathrm{d}}[\omega] &= \sum_{\substack{\mathfrak{m}_a \geq 0 \\ |\mathfrak{m}|=d}} \frac{(-1)^{d(N_c-1)}}{N_c!} \oint_{\mathrm{JK}} \prod_{a=1}^{N_c} \left[\frac{d\sigma_a}{2\pi i} \frac{1}{\prod_{\alpha=1}^{n_f} (\sigma_a - m_\alpha)^{1+\mathfrak{m}_a}} \right] \Delta(\sigma)\, \omega(x,y) \ , \end{aligned} \tag{5.4.16}$$

where $\Delta(\sigma) = \prod_{a\neq b}(\sigma_a - \sigma_b)$. This formula captures all the genus-0 GW invariants of $\mathrm{Gr}(N_c, n_f)$. The 3d and 2d formulas are related by a naive scaling limit. Indeed, if we assume that the insertion in 3d gives a 2d insertion according to:

$$\mathscr{L} \to (2\pi\beta)^{d_\omega} \omega \ , \tag{5.4.17}$$

where d_ω is the mass-dimension of the homogenous polynomial $\omega = \omega(\sigma)$, and if we further assume that we can commute the limit and the integration, then one finds: [1]

$$\mathbf{I}_d[\mathscr{L}] \to (2\pi\beta)^{-\dim(X) - d\, n_f + d_\omega} \, \mathbf{I}_d^{2\mathrm{d}}[\omega] \ . \tag{5.4.18}$$

Let us write down the quantum cohomology ring in the Schubert basis as:

$$\omega_\lambda \omega_\mu = c_{\lambda\mu}{}^{\nu} \, \omega_\nu \ , \qquad\qquad c_{\lambda\mu}{}^{\nu} = \sum_{d\geq 0} c_{\lambda\mu}^{(d)\nu} \, q_{2\mathrm{d}}^2 \ . \tag{5.4.19}$$

By dimensional analysis, we see that, in the non-equivariant limit $m_\alpha = 0$, only one single degree d can contribute to $c_{\lambda\mu}{}^{\nu}$ in 2d, with:

$$|\lambda| + |\mu| = |\nu| + d\, n_f \ . \tag{5.4.20}$$

In particular, using the fact that $|\lambda^\vee| = \dim(X) - |\lambda|$, we find that the structure ring constants in the Schubert basis can be deduced from the 3d results above whenever (5.4.20)

[1] Of course, integration and $\beta \to 0$ limit do not commute in general, which is why QK-theoretic invariants contain strictly more information than GW invariants. In the computation of the general 3d/1d observables in the small radius limit, one must generally consider the contribution of several 'holonomy saddles' [200], which decomposes the 3d quantities into several 2d observables.

holds true:

$$\mathcal{N}_{\lambda\mu}^{(d)\nu} \to c_{\lambda\mu}^{(d)\nu} \ . \tag{5.4.21}$$

In particular, for $d = 0$ and in the non-equivariant limit, the K-theoretic LR coefficients (5.3.43) that satisfy $|\lambda| + |\mu| = |\nu|$ are equal to the ordinary LR coefficients for the product of Schur polynomials, as expected.

$$
\begin{aligned}
\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{1}} &= \left(1-\frac{y_3}{y_2}\right)\mathcal{O}_{\ydiagram{1}} + \frac{y_3}{y_2}\mathcal{O}_{\ydiagram{1,1}} + \frac{y_3}{y_2}\mathcal{O}_{\ydiagram{2}} - \frac{y_3}{y_2}\mathcal{O}_{\ydiagram{2,1}}\,, \\
\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{1,1}} &= \left(1-\frac{y_3}{y_1}\right)\mathcal{O}_{\ydiagram{1,1}} + \frac{y_3}{y_1}\mathcal{O}_{\ydiagram{2,1}}\,, \\
\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2}} &= \left(1-\frac{y_4}{y_2}\right)\mathcal{O}_{\ydiagram{2}} + \frac{y_4}{y_2}\mathcal{O}_{\ydiagram{2,1}}\,, \\
\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2,1}} &= q\frac{y_4}{y_1} - q\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_4}{y_1}\right)\mathcal{O}_{\ydiagram{2,1}} + \frac{y_4}{y_1}\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2,2}} &= q\frac{y_3y_4}{y_1y_2}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_3y_4}{y_1y_2}\right)\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{1,1}} &= \left(1-\frac{y_2}{y_1}\right)\left(1-\frac{y_3}{y_1}\right)\mathcal{O}_{\ydiagram{1,1}} + \left(1-\frac{y_2}{y_1}\right)\frac{y_3}{y_1}\mathcal{O}_{\ydiagram{2,1}} + \frac{y_2}{y_1}\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2}} &= q\frac{y_4}{y_1} + \left(1-\frac{y_4}{y_1}\right)\mathcal{O}_{\ydiagram{2,1}}\,, \\
\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2,1}} &= q\left(1-\frac{y_2}{y_1}\right)\frac{y_4}{y_1} + q\frac{y_2y_4}{y_1^2}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_2}{y_1}-\frac{y_4}{y_1}+\frac{y_2y_4}{y_2^2}\right)\mathcal{O}_{\ydiagram{2,1}} + \left(\frac{y_2}{y_1}-\frac{y_4}{y_1}\right)\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2,2}} &= q\left(1-\frac{y_3}{y_1}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + q\frac{y_3}{y_1}\mathcal{O}_{\ydiagram{2}} + \left(1-\frac{y_3}{y_1}-\frac{y_4}{y_1}+\frac{y_3y_4}{y_1^2}\right)\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2}} &= \left(1-\frac{y_4}{y_3}\right)\left(1-\frac{y_4}{y_2}\right)\mathcal{O}_{\ydiagram{2}} + \left(1-\frac{y_4}{y_3}\right)\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{2,1}} + \frac{y_4}{y_3}\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2,1}} &= q\left(1-\frac{y_4}{y_3}\right)\frac{y_4}{y_1} + q\frac{y_4^2}{y_1y_3}\mathcal{O}_{\ydiagram{1}} + \left(1-\frac{y_4}{y_3}-\frac{y_4}{y_1}+\frac{y_4^2}{y_1y_3}\right)\mathcal{O}_{\ydiagram{2,1}} + \left(1-\frac{y_4}{y_1}\right)\frac{y_4}{y_3}\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2,2}} &= q\left(1-\frac{y_4}{y_2}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + q\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{1,1}} + \left(1-\frac{y_4}{y_2}-\frac{y_4}{y_1}+\frac{y_4^2}{y_1y_2}\right)\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{2,1}} \star \mathcal{O}_{\ydiagram{2,1}} &= q\left(1-\frac{y_2}{y_1}\right)\left(1-\frac{y_4}{y_3}\right)\frac{y_4}{y_1} + q\left(\frac{y_2}{y_1}-\frac{y_4}{y_1}+\frac{y_4}{y_3}-\frac{y_2y_4}{y_1y_3}\right)\mathcal{O}_{\ydiagram{1}} + q\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1,1}} + q\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{2}} \\
&\quad + \left[\left(1-\frac{y_2}{y_1}\right)\left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_3}\right) - q\frac{y_4}{y_1}\right]\mathcal{O}_{\ydiagram{2,1}} + \left(1-\frac{y_4}{y_1}\right)\left(\frac{y_2}{y_1}-\frac{y_4}{y_1}+\frac{y_4}{y_3}-\frac{y_2y_4}{y_1y_3}\right)\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{2,1}} \star \mathcal{O}_{\ydiagram{2,2}} &= q\left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} + q\left(1-\frac{y_3}{y_1}\right)\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{1,1}} + q\left(\frac{y_3}{y_1}-\frac{y_3y_4}{y_1y_2}\right)\mathcal{O}_{\ydiagram{2}} + q\frac{y_3y_4}{y_1y_2}\mathcal{O}_{\ydiagram{2,1}} \\
&\quad + \left(1-\frac{y_4}{y_2}+\frac{y_4^2}{y_1y_2}-\frac{y_3}{y_1}-\frac{y_4}{y_1}+\frac{y_3y_4}{y_1y_2}-\frac{y_3y_4^2}{y_1^2y_2}+\frac{y_3y_4}{y_1^2}\right)\mathcal{O}_{\ydiagram{2,2}}\,, \\
\mathcal{O}_{\ydiagram{2,2}} \star \mathcal{O}_{\ydiagram{2,2}} &= q^2\frac{y_3y_4}{y_1y_2} + q\left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\left(1-\frac{y_4}{y_2}\right)\frac{y_4}{y_1}\mathcal{O}_{\ydiagram{1}} \\
&\quad + q\left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\frac{y_4}{y_2}\mathcal{O}_{\ydiagram{1,1}} + \left(\frac{y_3}{y_1}-\frac{y_3^2}{y_1y_2}-\frac{y_3y_4}{y_1y_2}+\frac{y_3^2y_4}{y_1y_2^2}\right)\mathcal{O}_{\ydiagram{2}} \\
&\quad + \left(\frac{y_3}{y_2}-\frac{y_3^2y_4}{y_1y_2^2}\right)\mathcal{O}_{\ydiagram{2,1}} + \left(1-\frac{y_3}{y_1}\right)\left(1-\frac{y_3}{y_2}\right)\left(1-\frac{y_4}{y_1}\right)\left(1-\frac{y_4}{y_2}\right)\mathcal{O}_{\ydiagram{2,2}}\,.
\end{aligned}
$$

Table 5.2: The equivariant QK product for $\mathrm{Gr}(2,4)$.

CHAPTER 6

SUMMARY AND DISCUSSION

In this chapter, we give a summary for our work and we pose many research questions for future directions.

3d A-model and Companion Matrix. In this thesis, we studied 3d $\mathcal{N} = 2$ Chern–Simons–matter theories placed on the geometry $\Sigma_g \times S^1_\beta$ with Σ_g being a genus-g closed Riemann surface. We provided an algorithm from algebraic geometry that efficiently computes the 3d twisted indices and correlation functions of half-BPS line operators (e.g. Wilson lines) wrapping the circle fibres.

As discussed in chapter 1, this class of supersymmetric backgrounds is a special case of the more general 3d Seifert manifolds. This case was studied in detail in [105]. It was shown that, besides the usual handle-gluing operator, the supersymmetric partition functions get extra contributions encoding (the non-triviality of) the fibration and the presence of the orbifold points. These come in the form of the so-called *Siefert fibering operators* which are *not* rational functions in the Coulomb branch parameters. Therefore, the algorithm that we developed is not directly applicable. It would be interesting to generalise the our companion matrix algorithm for these cases following [162].

Analysing the moduli space of vauca. Focusing on 3d unitary SQCDs, we analysed the supersymmetric vacua for the theories with gauge group $U(N_c)_{k,k+lN_c}$ coupled with n_f matter multiplets in the fundamental representation of the gauge group. Our analysis generalises earlier results of Witten [50] and Intriligator and Seiberg [49]. We found a very intricate and rich dependence of the structure of these moduli spaces on the Chern–Simons levels k and l. Since the analysis we performed is semi-classical, i.e. valid in the large $|\sigma|$ regimes, it did not take into account the possibility of strongly-coupled supersymmetric vacua being present near the origin of the classical Coulomb branch. These vacua are detected by the appearance of non-compact Coulomb branches. This observation helped us to make a conjecture on their contributions at the quantum level to the 3d Witten index.

We provided some checks using 3d IR dualities and the 3d A-model formalism to support our conjecture. Nonetheless, it would be interesting to have a better understanding of these quantum effects and how they deform the geometry of the moduli space of vacua [52, 53, 54, 49].

3d infrared dualities. Additionally, in this work, we studied a class of IR dualities for the 3d unitary SQCDs of interest. We studied well-known and clarified recently-proposed ones for generic values of Chern–Simons levels k and l. Moreover, we proposed the maximally-chiral duality for nonzero l. We also proposed dualities for pure unitary Chern–Simons theory generalising the well-studied level-rank dualities.

We used the algebro-geometric algorithm and the semi-classical analysis to test these dualities by matching twisted indices and SUSY vacua on both sides. In the theories with only fundamental matter, we found cases where these dualities transcend to geometric equivalences (e.g. the Grassmannian duality). It would be interesting to generalise these observations to other cases. For example, when one adds antifundamental multiplets, too. In the geometric phase, these flow to copies of the universal tautological subbundle of the Grassmannian – see the discussion in [132].

3d gauge theory/quantum K-theory correspondence. Moreover, fixing N_c and n_f, using this semi-classical analysis, we determined the geometric window of the 3d SQCD. We defined this as the set of (k, l) such that the 3d gauge theory is a 3d GLSM with the target being the complex Grassmannian variety $X = \mathrm{Gr}(N_c, n_f)$. This led to a new geometric puzzle. In the 3d GLSM/quantum K-theory correspondence, there is a 'standard' choice of the Chern–Simons level for the ring structure of the 3d twisted chiral ring $\mathcal{R}_{3\mathrm{d}}$ to be isomorphic to $\mathrm{QK}_T(X)$. The question is: What about the ring relations for the other choices of (k, l) in the geometric window? We speculated in our discussion that these should be related to the generalised quantum K-theory rings with level structures that were developed very recently by Ruan and Zhang [193]. Using the Gröbner basis algorithm that we developed earlier, we showed in examples how these ring relations look like. However, the explicit connection between our computations and those of Ruan and Zhang are still to be understood. For more discussion on these level structures, see also [201, 202, 203, 136].

Related to this discussion, we also observed that, although changing the sign of the 3d real FI parameter of the SQCD leaves the Witten index unchanged, it dramatically changes the structure of the moduli space. For instance, we noticed that there are choices of the levels k and l where on one side we have a pure geometric Higgs branch, meanwhile, on the other, we get a pure Chern–Simons theory. One is tempted to speculate a connection between this observation and the Verlinde/Grassmannian correspondence studied by Ruan and Zhang [201] in the presence of level structures. This is also a generalisation of

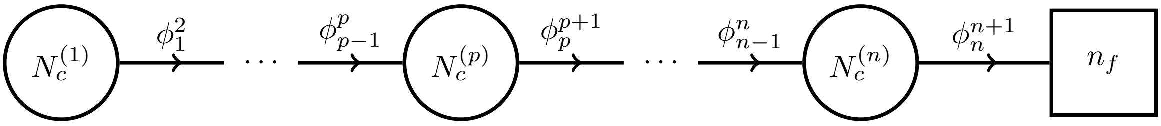


Figure 6.1: 2d $\mathcal{N}=(2,2)$ quiver gauge theory whose target space is a partial flag variety. The data of the partial flag is determined from the ranks of the gauge groups $U(N_c^{(p)})$ and the flavour symmetry group $SU(n_f)$.

the earlier work of Witten [25] in the quantum cohomology case. It would be interesting to explore this point and make the connection more transparent.

Another important direction of our work that we think should be studied more concretely is related to the higher-genus K-theoretic GW invariants. In chapter 1, we pointed out that to study higher-genus cohomological GW invariants, one needs to couple the 2d GLSM to 2d topological gravity. The question is: can this coupling be uplifted to pur 3d story of interest? That is, what is the "3d topological gravity" that one should consider in this case? Moreover, what is the K-theoretic uplift of the holomorphic anomaly equations?

Going more into the quantum K-theory story, we constructed a class of half-BPS defect lines (that we called the Grothendieck lines) in 3d SQCD such that in the IR of the geometric phase, they flow to the Schubert classes of the K-theory ring of the Grassmannian, thus giving a physical realisation of the Schubert-class basis of $\mathrm{QK}_T(X)$. We also showed that in the 2d limit of the 3d theory, this 1d/3d coupled system becomes a 0d/2d system where the point defects flow to Schubert classes in the quantum cohomology ring of X.

It is interesting to note that the (double) Grothendieck polynomials can also be realised as wavefunctions in certain integrable systems [204, 205, 192]. The two perspectives are conjecturally related by the Bethe/gauge correspondence [124, 19]. See *e.g.* [206] for a similar construction of point defects in the 2d $\mathcal{N}=(2,2)$ GLSM for $T^*\mathbb{P}^{n_f-1}$ which correspond to the Bethe wavefunctions in the XXX spin chain.

Recently, our construction has been applied in [207] in their study of Schubert cells in the Lagrangian Grassmannian case; for definitions and discussion from GLSM perspective, see [208]. It was also pointed out in that paper that our construction of the Grothendieck lines is connected to the Kempf-Laksov desingualrisation of the Schubert varieties [209].

An interesting direction for generalising our construction is for the partial flag variety case From the 2d GLSM perspective, these are realised as targets of 2d $\mathcal{N}=(2,2)$ quiver gauge theories with unitary gauge groups as shown in figure 6.1. The CS levels in the 3d uplift can be fixed following similar arguments as we did for the Grassmannian case. That is, we softly break the 3d $\mathcal{N}=4$ theory whose target is the total space of the cotangent bundle of the partial flag and integrate out the massive anti-chiral bifundamentals. For a more recent discussion on the quantum K-theory and quantum cohomology of these varieties, see [210, 211, 212] and references therein.

Generalised symmetry in 3d GLSMs. During the past decade, starting with [213], a reformulation of the concept of symmetry in quantum field theory has been under development. For an introduction to this subject, see the set of lectures [214]. In an ongoing work, we have been exploring and clarifying these ideas from the perspective of the 3d A-model – e.g. see [215, 6, 7] for the study of higher-form symmetries and [216] in the $\mathcal{N} = 4$ case. The hope is that these new concepts will help better understand the 3d A-model and shed new light on the enumerative geometric side of our story following the work of Pantev and Sharpe [217, 218].

These ideas and puzzles are still a work in progress, and we hope to report more on them in the near future.

APPENDIX A

3D $\mathcal{N} = 2$ PURE CHERN–SIMONS THEORIES: WITTEN INDEX AND LEVEL/RANK DUALITIES

In this appendix, we compute the Witten index of $\mathcal{N} = 2$ supersymmetric pure Chern–Simons theories with unitary gauge groups. We also introduce some level/rank dualities for those theories, including an interesting duality for a $U(N) \times U(N')$ theory, which will be useful in later sections.

A.1 Witten index of CS theories with unitary gauge group

Consider the 3d $\mathcal{N} = 2$ Chern–Simons theory:

$$U(N_1)_{k_1,k_1+l_1N_1} \times U(N_2)_{k_2,k_2+l_2N_2} \times \cdots \times U(N_n)_{k_n,k_n+l_nN_n} \ , \tag{A.1.1}$$

where we allow all possible mixed CS levels k_{ij} between different gauge groups, namely:

$$\underbrace{U(N_i)_{k_i,k_i+l_iN_i} \times U(N_j)}_{k_{ij}}{}_{k_j,k_j+l_jN_j} \ , \tag{A.1.2}$$

for $i \neq j$. Let us first recall that we have the following decomposition into $SU(N)$ and $U(1)$ factors:

$$U(N)_{k,k+lN} \cong \frac{SU(N)_k \times U(1)_{N(k+lN)}}{\mathbb{Z}_N} \ . \tag{A.1.3}$$

The $\mathcal{N}=2$ $SU(N)_k$ theory has a Witten index [50, 219, 169]:

$$\mathbf{I}_{\rm W}\left[SU(N)_k\right] = \binom{|k|-1}{N-1} , \tag{A.1.4}$$

while for the $U(N)$ theory we find (3.1.20):

$$\mathbf{I}_{\rm W}\left[U(N)_{k,k+lN}\right] = \frac{|k+lN|}{N}\binom{|k|-1}{N-1} . \tag{A.1.5}$$

The mixed Chern–Simons levels in (A.1.2) only involve the $U(1)$ factors. In the normalisation (A.1.3), this corresponds to:

$$\underbrace{U(1)_{N_i(k_i+l_iN_i)} \times U(1)}_{N_iN_jk_{ij}}{}_{N_j(k_j+l_jN_j)} . \tag{A.1.6}$$

Let us denote by $\boldsymbol{K} \equiv (K_{ij})$ the matrix of CS levels for the abelian sector:

$$K_{ij} = \begin{cases} N_i(k_i + l_iN_i) , & \text{if } i=j , \\ N_iN_jk_{ij} , & \text{if } i \neq j . \end{cases} \tag{A.1.7}$$

A.1.1 Index for the abelian theory

From (A.1.5), a $U(1)_K$ CS theory has a Witten index

$$\mathbf{I}_{\rm W}[1, K] = |K| , \tag{A.1.8}$$

which is, of course, a special case of (A.1.5). For any abelian CS theory with gauge group $U(1)^n$ a matrix of CS levels $\boldsymbol{K}$, the Witten index is the number of Bethe vacua, which is given by the number of solutions to the system of equations (2.2.18):

$$q_i(-x_i)^{K_{ii}} \prod_{j\neq i} x_j^{K_{ij}} = 1 , \qquad i = 1, \cdots, n . \tag{A.1.9}$$

Here q_i are the fugacities for the topological symmetries and x_i are the gauge variables, in the conventions of chapter 2. Then, the Witten index is the absolute value of the determinant of the CS level matrix $\boldsymbol{K}$, which we write as:

$$\mathbf{I}_{\mathrm{W}}\left[\mathbf{1}\,\mathbf{0}\,|\,\boldsymbol{K}\right] = |\det \boldsymbol{K}| \ . \tag{A.1.10}$$

This can be shown by a recursive argument on the number of $U(1)$ factors.

A.1.2 Index for the non-abelian theory.

Let us denote the Witten index of the general unitary CS theory (A.1.1)-(A.1.2) by:

$$\mathbf{I}_{\mathrm{W}}\left[\boldsymbol{N}\,\boldsymbol{l}\,|\,\boldsymbol{k}\right] \equiv \mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cccc} N_1 & l_1 & k_1 & k_{12} & \cdots & k_{1n} \\ N_2 & l_2 & k_{12} & k_2 & \cdots & k_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_n & l_n & k_{1n} & k_{2n} & \cdots & k_n \end{array}\right] , \tag{A.1.11}$$

Then, given the above observations, we find that:

$$\mathbf{I}_{\mathrm{W}}\left[\boldsymbol{N}\,\boldsymbol{l}\,|\,\boldsymbol{k}\right] = |\det \boldsymbol{K}| \prod_{i=1}^{n} \frac{1}{N_i^2}\binom{|k_i|-1}{N_i-1} , \tag{A.1.12}$$

where $\boldsymbol{K}$ is defined as in (A.1.7). Of course, this reduces to (A.1.5) for $n = 1$ and to (A.1.10) in the abelian case.

A.2 Generalised $\mathcal{N} = 2$ level/rank dualities

The $U(N)_{k,k+lN}$ $\mathcal{N} = 2$ CS theory has a dual description [76]:

$$U(N)_{k,k+lN} \qquad \longleftrightarrow \qquad U(\underbrace{|k| - N)_{-k,-k+\epsilon(k-N)} \times U(1)}_{\epsilon}{}_{l+\epsilon} \ , \tag{A.2.1}$$

with $\epsilon \equiv \mathrm{sign}(k)$. Here, the FI parameter on the electric (left-hand) side maps to an FI parameter for the $U(1)_{l+\epsilon}$ factor on the magnetic (right-hand) side. In addition, on the magnetic side, we also have the non-zero CS contact terms:

$$K_{RR} = \begin{cases} -(k-N)^2 \ , & \text{if } \epsilon = 1 \ , \\ N^2 - 1 \ , & \text{if } \epsilon = -1 \ , \end{cases} \qquad K_g = \begin{cases} -2k(k-N) \ , & \text{if } \epsilon = 1 \ , \\ -2kN - 2 \ , & \text{if } \epsilon = -1 \ . \end{cases} \tag{A.2.2}$$

This is a special case of the Nii duality [76], which we further studied in section 4.2. It generalises well-known level/rank dualities for pure CS theories [155], as can be shown by writing these dualities in $\mathcal{N} = 0$ language – for completeness, we discuss this in the next section. Note that for $N = |k|$, the dual description is abelian:

$$U(|k|)_{k,k+l|k|} \qquad \longleftrightarrow \qquad U(1)_{l+\epsilon} \ . \tag{A.2.3}$$

Another interesting special case is for $l = -\epsilon$, in which case the duality can be simplified to:

$$U(N_c)_{k,k+lN} \qquad \longleftrightarrow \qquad SU(|k| - N)_{-k} \ . \tag{A.2.4}$$

Correspondingly, the Witten index (A.1.5) specialises to:

$$\mathbf{I}_{\rm W} \left[N - \epsilon \,|\, k \right] = \binom{|k| - 1}{N^D - 1} = \mathbf{I}_{\rm W} \left[SU(N^D)_{-k} \right] \ , \tag{A.2.5}$$

with $N^D = |k| - N$.

A.2.1 An almost trivial $U(1) \times U(1)$ theory.

Consider the abelian theory:

$$U\underbrace{(1)_{l+\epsilon} \times U(1)}_{l}{}_{l+\epsilon'} \ , \tag{A.2.6}$$

with ϵ, ϵ' equal to 1 or -1, and $l \in \mathbb{Z}$. The Witten index is given by (A.1.10):

$$\mathbf{I}_{\rm W} = |(l+\epsilon)(l+\epsilon') - l^2| = 1 \ , \tag{A.2.7}$$

where the last equality holds if $\epsilon = -\epsilon'$, as we assume in the following. Then we have a unique Bethe vacuum. The Bethe equations (A.1.9) read $q(x')^l(-x)^{l+\epsilon} = 1$,

$q'x^l(-x')^{l+\epsilon'} = 1$, with the unique Bethe vacuum given by:

$$x = (-q)^{l+\epsilon'}(q')^{-l} , \qquad x' = q^{-l}(-q')^{l+\epsilon} . \tag{A.2.8}$$

Here x, x' and q, q' denote the gauge parameter and the topological symmetry fugacities $U(1)_T \times U(1)_{T'}$, respectively. Using the 3d A-model, we can derive (most of) the CS contact terms that remain in the dual description. This gives us the elementary duality:

$$U\underbrace{(1)_{l+\epsilon} \times U(1)}_{l}{}_{l+\epsilon'} \qquad \longleftrightarrow \qquad \begin{cases} K_{TT} = l + \epsilon' \ , \quad K_{T'T'} = l + \epsilon \ , \\ K_{TT'} = -l \ , \qquad K_{RR} = 1 \ , \end{cases} \tag{A.2.9}$$

up to some gravitational CS level K_g which we did not fully determine (see section A.3 below). For $l = 0$, this gives us two decoupled elementary dualities for $U(1)_{\pm 1}$ (see chapter 4).

A.2.2 An $U(N) \times U(N')$ level/rank duality

The level/rank duality (A.2.1) has an interesting generalisation for a gauge group $U(N) \times U(N')$. Setting $\epsilon = \text{sign}(k)$, $\epsilon' = \text{sign}(k')$ and assuming $\epsilon = -\epsilon'$, we find:

$$U\underbrace{(N)_{k,k+lN} \times U(N')}_{l}{}_{k',k'+lN'}$$
$$\longleftrightarrow \quad U\underbrace{(|k| - N)_{-k,-k+l(|k|-N)} \times U(|k'| - N')}_{l}{}_{-k',-k'+l(|k'|-N')} \ , \tag{A.2.10}$$

where the FI parameters of the dual theory are related to the original FI parameters τ, τ' as:

$$\tau_D = -\tau + l\epsilon(\tau - \tau') \ , \qquad \tau'_D = -\tau' + l\epsilon'(\tau' - \tau) \ , \tag{A.2.11}$$

as well the following CS contact terms for the topological symmetries:[1]

$$K_{TT} = l + \epsilon' \ , \qquad K_{T'T'} = l + \epsilon \ , \qquad K_{TT'} = -l \ . \tag{A.2.12}$$

[1] There are also non-zero contact terms K_{RR} and K_g, which we do not keep track of here.

This duality also implies the equality:

$$\mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} N_c & l & k & l \\ N_c' & l & l & k' \end{array}\right] = \mathbf{I}_{\mathrm{W}}\left[\begin{array}{cc|cc} |k|-N_c & l & -k & l \\ |k'|-N_c' & l & l & -k' \end{array}\right] , \tag{A.2.13}$$

if $\epsilon + \epsilon' = 0$. This indeed follows from the explicit expression (A.1.12) for the Witten index.

The duality (A.2.10) can be derived using (A.2.1) and (A.2.9). Indeed, applying the duality (A.2.1) subsequently to each gauge group factor on the electric side of (A.2.10), one finds:

$$\underbrace{U(|k|-N_c)_{-k,-k+\epsilon(|k|-N_c)} \times U(}_{\epsilon}\underbrace{1)_{l+\epsilon} \times U(}_{l}\underbrace{1)_{l+\epsilon'} \times U(|k'|-N_c'}_{\epsilon'})_{-k',-k'+\epsilon'(|k'|-N_c')} , \tag{A.2.14}$$

The $U(1) \times U(1)$ sector can be simplified thanks to the elementary duality (A.2.9). Due to the mixed CS couplings, the effective FI parameters for the $U(1) \times U(1)$ gauge group are:

$$\tau_{\text{eff}} = \tau + \epsilon \sum_{a=1}^{|k|-N} u_a , \qquad \tau'_{\text{eff}} = \tau' + \epsilon' \sum_{a=1}^{|k'|-N'} u'_a , \tag{A.2.15}$$

where u_a, u'_a denote the gauge parameters of the non-abelian factors in (A.2.14). Applying the duality (A.2.9) then leads to (A.2.10). It is simplest to check this using the 3d A-model. Let us denote by u and u' the sums in (A.2.15), so that $\tau_{\text{eff}} = \tau + \epsilon u$ and $\tau'_{\text{eff}} = \tau' + \epsilon' u'$. Then the relevant part of the twisted superpotential reads:[1]

$$\mathcal{W} = \frac{\epsilon}{2}u^2 + \frac{\epsilon'}{2}u'^2 + \mathcal{W}_{\text{magn}}(u, u') , \tag{A.2.16}$$

where the first two terms are the 'l CS levels' for the non-abelian factors in (A.2.14), and $\mathcal{W}_{\text{magn}}(u, u')$ is the superpotential for the abelian factor after applying (A.2.9), which reads:

$$\mathcal{W}_{\text{magn}}(u, u') = \frac{l+\epsilon'}{2}(\tau + \epsilon u)^2 + \frac{l+\epsilon}{2}(\tau' + \epsilon' u')^2 - l(\tau + \epsilon u)(\tau' + \epsilon' u') . \tag{A.2.17}$$

Expanding out (A.2.16) and using $\epsilon\epsilon' = -1$, we recover the CS levels shown in (A.2.10) and (A.2.12).

[1]Here, for simplicity of notation, we omit the terms linear in u, u' that arise from the $U(1)$ CS levels [151].

A.3 $\mathcal{N} = 0$ level/rank dualities

For completeness, here we present the $\mathcal{N} = 0$ (non-supersymmetric) version of the level/rank dualities between Chern–Simons theories discussed in the previous section. They are obtained from the $\mathcal{N} = 2$ dualities by integrating out the gauginos in vector multiplets. Recall that the $\mathcal{N} = 2$ supersymmetric CS interaction includes a term quadratic in the gauginos, which amounts to a real mass:

$$m_\lambda = -\frac{k}{4\pi} \; . \tag{A.3.1}$$

Hence, we have an infrared $\mathcal{N} = 0$ description in which the CS levels are shifted according to the sign of m_λ. More generally, m_λ is a non-trivial mass matrix, which we should then diagonalise. We follow the conventions of subsection 2.2.2 for the $\mathcal{N} = 2$ CS contact terms, except that here we denote by the letters $k, l, \cdots$ the $\mathcal{N} = 2$ CS levels (for either the gauge or flavour symmetries), and by the capital letters $K, L, \cdots$ all the $\mathcal{N} = 0$ CS levels.

A.3.1 Level/rank dual for $U(N)_{K,K+LN}$

Consider the $U(N)_{k,k+lN}$ $\mathcal{N} = 2$ supersymmetric pure CS theory. Without loss of generality, consider:

$$k > 0 \; , \qquad K \equiv k - N \geq 0 \tag{A.3.2}$$

The $\mathcal{N} = 2$ version of the level/rank duality is a specialisation of the Nii duality [76] (see chapter 4 for details):

$$\mathcal{N} = 2 \; : \qquad U(N)_{k,k+lN} \qquad \longleftrightarrow \qquad \underbrace{U(k-N)_{-k,-N} \times U(1)}_{1}{}_{l+1} \tag{A.3.3}$$

with the non-zero $\mathcal{N} = 2$ CS contact terms derived in table 4.8, which amounts to having $\kappa_{RR}^{(e)} = \frac{1}{2}N^2 = \frac{1}{2}\kappa_g^{(e)}$ on the electric side, and $\kappa_{RR}^{(m)} = -\frac{1}{2}(k-N)^2$ and $\kappa_g^{(m)} = N^2 + 1 - k^2$ on the magnetic side. Integrating out the gauginos in the $U(N)$ theory, we shift the CS levels to $K \equiv k - N$ and $L \equiv l + 1$, and $K_{RR}^{(e)} = K_g^{(e)} = 0$ in the IR. On the magnetic side, integrating out the gauginos similarly shifts the CS level for the $SU(K)$ part of the gauge group, so that we have the level/rank duality:

$$\mathcal{N} = 0 \; : \qquad U(N)_{K,K+LN} \qquad \longleftrightarrow \qquad \underbrace{U(K)_{-N,-N} \times U(1)}_{1}{}_{L} \; , \tag{A.3.4}$$

which was first described in [220]. By a straightforward computation, we can check that the $U(1)_R$ symmetry decouples entirely, and we find the relative CS level $K_g = -2kN$. Similarly to [155], we can write this as:

$$\frac{1}{2\pi} A_T \wedge \mathrm{Tr}(F_{U(N)}) \qquad \longleftrightarrow \qquad \frac{1}{2\pi} A_T \wedge F_{U(1)} - 2kN\mathrm{CS}_{\mathrm{grav}} \ , \tag{A.3.5}$$

where we only wrote down terms that depend on background fields. Here the background gauge field A_T couples to the topological symmetry current on either side of the duality, which is $\mathrm{Tr}(F_{U(N)})$ on the electric side, and $F_{U(1)}$ the field strength for the $U(1)$ factor in $U(K) \times U(1)$ on the magnetic side.[1] Moreover, the gravitational Chern–Simons term appearing on the r.h.s has the following explicit form:

$$\int_{M=\partial X} \mathrm{CS}_{\mathrm{grav}} = \frac{1}{192\pi} \int_X \mathrm{tr}(R \wedge R) \ , \tag{A.3.7}$$

where X is a 4-manifold whose boundary is the 3-manifold M on which our CS theory lives.

For $L = \pm 1$, we can use the fact that $U(1)_{\pm 1}$ is an 'almost trivial' theory [94, 155] to simplify (A.3.4), and we arrive at the following dualities:

$$\mathcal{N} = 0 \ : \qquad U(N)_{K,K\pm N} \qquad \longleftrightarrow \qquad U(K)_{-N,-N\mp K} \ , \tag{A.3.8}$$

with:

$$\begin{aligned} \frac{1}{2\pi} A_T \wedge \mathrm{Tr}(F_{U(N)}) \quad \longleftrightarrow \quad \mp \frac{1}{2\pi} A_T \wedge \mathrm{Tr}(F_{U(K)}) \mp \frac{1}{4\pi} A_T \wedge dA_T \\ - (2kN \pm 2)\mathrm{CS}_{\mathrm{grav}} \ . \end{aligned} \tag{A.3.9}$$

For $L = 1$, this is a special case of the Giveon–Kutasov duality (the $\mathcal{N} = 2$ duality $U(N)_k \leftrightarrow U(k-N)_{-k}$) reduced to $\mathcal{N} = 0$, while for $L = -1$ this duality was first derived in [155]. Finally, for $L = 0$, we have $U(1)_0$ theory coupled by a BF term to the $U(K)$

[1] To derive this result starting from the $\mathcal{N} = 2$ version of the duality, one must take into account the fact that the abelian part of the magnetic gauge group is $U(1) \times U(1)$ with a non-trivial mass matrix:

$$M_\lambda = \begin{pmatrix} KN & -K \\ -K & -L \end{pmatrix} \ , \tag{A.3.6}$$

for the abelian gauginos. We use the fact that the two eigenvalues $m_\pm$ are such that $m_+ > 0$ and $m_- < 0$ if $K + LN > 0$, while $m_\pm > 0$ if $K + LN < 0$.

factor in (A.3.4), and the path integral over that gauge field imposes the constraint [94]:

$$\mathrm{Tr}(A^{U(K)}) = -A_T \ , \tag{A.3.10}$$

on the $U(K)$ gauge field, which gives us the standard level/rank duality:

$$\mathcal{N}=0 \ : \qquad U(N)_{K,K} \qquad \longleftrightarrow \qquad SU(K)_{-N} \ . \tag{A.3.11}$$

More precisely, we truly have an $SU(K)$ theory only when the background field A_T is set to zero, while having A_T generic is important to keep track of the $U(1)_T$ symmetry across the duality [155].

A.3.2 Level/rank duality for $U(N) \times U(N')$

Let us also discuss the $\mathcal{N}=0$ version of the duality for the $U(N)\times U(N')$ theory derived in section A.2. First of all, we have the $\mathcal{N}=0$ abelian duality:

$$\mathcal{N}=0 \ : \quad U\underbrace{(1)_{l+1} \times U(1)}_{l}{}_{l-1} \quad \longleftrightarrow \quad \begin{cases} K_{TT} = l-1 \ , & K_{T'T'} = l+1 \ , \\ K_{TT'} = -l \ , & K_g = 2 + c_g \ , \end{cases} \tag{A.3.12}$$

where the magnetic theory is an empty theory with the contact terms as indicated for the background gauge fields A_T and A'_T coupling to the topological symmetry currents F and F'. Here, $c_g \in \mathbb{Z}$ is a constant that we did not determine, although we know that $c_g = 0$ when $l=0$. The $\mathcal{N}=0$ duality (A.3.12) directly follows from (A.2.9), taking into account the fact that the gaugino mass matrix has eigenvalues $m_\pm = -l \pm \sqrt{l^2+1}$. (Requiring that the $U(1)_R$ symmetry consistently decouples then fixes K_{RR} in (A.2.9).)

Finally, we consider the $U(N)_{k,k+lN} \times U(N')_{k',k'+lN'}$ $\mathcal{N}=2$ theory with mixed CS level l. Let us fix $k>0$ and $k'<0$, without loss of generality. Integrating out the gauginos, we obtain the $\mathcal{N}=0$ levels:

$$U\underbrace{(N)_{K,K+(l+1)N} \times U(N')}_{l}{}_{-K',-K'+(l-1)N'} \ , \tag{A.3.13}$$

where we choose $K \geq 0$ and $K' \geq 0$ for simplicity of notation. Then, the $\mathcal{N}=2$

duality (A.2.10) gives us the generalised level/rank duality:

$$\begin{aligned}\mathcal{N}=0\ :\quad & U\underbrace{(N)_{K,K+(l+1)N}\times U(N')}_{l}{}_{-K',-K'+(l-1)N'}\\ & \qquad\longleftrightarrow\quad U\underbrace{(K)_{-N,-N+(l-1)K}\times U(K')}_{l}{}_{N',N'+(l+1)K'}\ .\end{aligned} \tag{A.3.14}$$

Of course, this can also be derived directly in the $\mathcal{N}=0$ context, starting from (A.3.4) and following the same logic as in section A.2.

APPENDIX B

REAL MASS DEFORMATIONS AND THE $l = 0$ DUALITIES

In this appendix, we briefly review the derivation of the minimally, marginally and maximally chiral dualities for $l = 0$, assuming Aharony duality, as originally discussed in [58]. We revisit this analysis here for completeness. This also allows us to explain how to carry out this standard computation in the gauge-symmetry-preserving conventions spelled out in subsection 2.2.2.

B.1 Integrating out massive chiral multiplets

Consider a 3d $\mathcal{N} = 2$ chiral multiplet Φ coupled to $U(1)_I$ vectors multiplets with charges Q_I. We are interested in giving a large real mass $m_0 \in \mathbb{R}$ to Φ, thus integrating it out from the description. In the limit $|m_0| \to \infty$, the UV contact terms are shifted according to:

$$\delta\kappa_{IJ} = \frac{1}{2} Q_I Q_J \, \mathrm{sign}(m_0) \ . \tag{B.1.1}$$

This is easily generalised to any non-abelian symmetry that Φ might be charged under. Integrating out a massive chiral multiplet Φ always shifts the CS contact terms according to:

$$\delta\kappa = -\kappa^{\Phi} \, \mathrm{sign}(m_0) \ , \tag{B.1.2}$$

for any symmetry, where the contributions κ^{Φ} from a single chiral multiplet are defined as in subsection 2.2.2. Importantly, this means that the bare CS levels K are shifted as:

$$\delta K = \begin{cases} 0 \ , & \text{if } m_0 > 0 \ , \\ 2\kappa^{\Phi} \ , & \text{if } m_0 < 0 \ . \end{cases} \tag{B.1.3}$$

Take, for example, a massive chiral multiplet Φ of charge $Q_F \in \mathbb{Z}$ under some $U(1)_F$ symmetry, with a bare CS level K_F. After integrating out Φ, we have the $U(1)_F$ bare CS level $K'_F = K_F$ if $m_0 \to \infty$ and $K'_F = K_F - Q_F^2$ if $m_0 \to -\infty$. Note that the bare Chern–Simons levels so obtained are integer-quantised, as needed by gauge invariance.

B.2 Flowing from Aharony duality

Let us consider the 'electric' theory in Aharony duality, $U(N_c)_0$ with N_f fundamentals and N_f antifundamentals. We can obtain any SQCD$[N_c, k, 0, n_f, n_a]$ (with $l = 0$) by appropriately decoupling flavours.

Let us then consider a particular RG flow:

$$\delta \ : \ \text{SQCD}[N_c, 0, 0, N_f, N_f] \ + \ \delta m_0 \qquad \to \qquad \text{SQCD}[N_c, k, 0, n_f, n_a] \ , \tag{B.2.1}$$

which is triggered by a particular choice of real mass m_0 in the UV theory. This deformation will generate various flavour and mixed gauge-flavour CS levels $K_{GF}^{(e)}$ and $\Delta K_{FF}^{(e)}$, schematically. The $K_{GF}^{(e)}$ levels are part of the definition of the infrared theory. The pure flavour CS levels correspond to local terms, which can be removed at will. Recall that we choose $K_{FF} = 0$ as part of our definition of unitary SQCD (3.1.8).

Because any real mass term is a VEV of a background vector multiplet, we can easily identify the dual mass deformation in the Aharony dual theory reviewed in subsection 4.1.1. By following the corresponding RG flow, we arrive at a dual description for SQCD$[N_c, k, 0, n_f, n_a]$:

$$\delta \ : \ \text{dual SQCD}[N_c, 0, 0, N_f, N_f] \ + \ \delta m_0 \qquad \to \qquad \text{dual SQCD}[N_c, k, 0, n_f, n_a] \ , \tag{B.2.2}$$

In this process, we again generate CS levels $K_{GF}^{(m)}$ and $\Delta K_{FF}^{(m)}$. The bare flavour CS levels shifts are to be added to the levels (4.1.3)-(4.1.4) encountered in the Aharony dual theory. We then compute the magnetic flavour CS levels of dual SQCD as:

$$K_{FF} = K^{(\text{Ah})} + \Delta K_{FF}^{(m)} - \Delta K_{FF}^{(e)} \ , \tag{B.2.3}$$

	$U(N_c)$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$	$U(1)_0$
ϕ_α	$\square$	$\overline{\square}$	$\mathbf{1}$	1	0	r	0
$\widetilde{\phi}^\beta$	$\overline{\square}$	$\mathbf{1}$	$\square$	1	0	r	0
ϕ_i	$\square$	$\overline{\square}$	$\mathbf{1}$	1	0	r	ϵ
$\widetilde{\phi}^j$	$\overline{\square}$	$\mathbf{1}$	$\square$	1	0	r	ϵ

Table B.1: Fields and charges for the mass deformation of the electric theory in the minimally chiral case. Every field charged under $U(1)_0$ is integrated out. Here, we use the flavour indices $\alpha = 1, \cdots, n_f$, $\beta = 1, \cdots, n_a$, $i = 1, \cdots, p$, and $j = 1, \cdots, q$, as well as $\epsilon = \text{sign}(m_0)$. We only keep track of the flavour symmetries that survive in the infrared (and of $U(1)_0$).

where we shifted the 'electric' flavour CS levels generated by the RG flow to the 'magnetic' side of the duality.

B.2.1 Minimally chiral duality: $|k| > |k_c|$

To derive the minimally chiral Seiberg-like duality from Aharony duality, we start with the electric theory SQCD$[N_c, 0, 0, N_f, N_f]$ and we integrate out p fundamental and q antifundamental chiral multiplets with a common real mass m_0, so that we obtain SQCD$[N_c, k, 0, n_f, n_a]$ with:

$$n_f = N_f - p \, , \qquad n_a = N_f - q \, , \qquad k = \frac{1}{2}(p+q)\,\text{sign}(m_0) \, . \tag{B.2.4}$$

Note that $k_c \equiv \frac{1}{2}(n_f - n_a) = \frac{1}{2}(q-p)$ and $|k| = \frac{1}{2}(p+q)$, hence we flow to minimally chiral SQCD, $|k| > |k_c|$, if and only if $p > 0$ and $q > 0$. The light and heavy fields are shown in table B.1. We identify some symmetry $U(1)_0$ along which we deform, with $\epsilon \equiv \text{sign}(m_0) = \text{sign}(k)$. For $k_c \neq 0$, we also need to shift the origin of the real Coulomb branch and of the FI parameter according to:

$$\sigma_a \to \sigma_a + \frac{k_c}{N_f} m_0 \, , \qquad\qquad \xi \to \xi - k_c |m_0| \, . \tag{B.2.5}$$

In other words, the $U(1)_0$ symmetry mixes with the gauge symmetry and with the topological symmetry [58].

	$U(N_c^{\mathrm{D}})$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$	$U(1)_0$
φ_β	$\square$	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$	0
$\widetilde{\varphi}^\alpha$	$\overline{\square}$	$\square$	$\mathbf{1}$	-1	0	$1-r$	0
${M_\alpha}^\beta$	$\mathbf{1}$	$\overline{\square}$	$\square$	2	0	$2r$	0
φ_j	$\square$	$\mathbf{1}$	$\mathbf{1}$	-1	0	$1-r$	$-\epsilon$
$\widetilde{\varphi}^i$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	-1	0	$1-r$	$-\epsilon$
${M_\alpha}^j$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	2	0	$2r$	ϵ
${M_i}^\beta$	$\mathbf{1}$	$\mathbf{1}$	$\square$	2	0	$2r$	ϵ
${M_i}^j$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	2	0	$2r$	2ϵ
$\mathfrak{T}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	1	r_T	$-p$ $(\epsilon=1)$ or q $(\epsilon=-1)$
$\mathfrak{T}^-$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	-1	r_T	$-q$ $(\epsilon=1)$ or p $(\epsilon=-1)$

Table B.2: Fields and charges for the mass deformation of the Aharony dual theory, in the minimally chiral case. Here $\epsilon = \mathrm{sign}(k)$.

On the magnetic side, we consider the Aharony dual gauge theory $U(N_c^D)_0$ with rank:

$$\begin{aligned} N_c^D &= N_f - N_c \\ &= |k| + \frac{1}{2}(n_f + n_a) - N_c \ . \end{aligned} \tag{B.2.6}$$

The $U(1)_0$ charge assignments in the dual theory are shown in table B.2. For p and q strictly positive, all the fields in the bottom rows must be integrated out. The bare CS levels are shifted according to:

$$K_{IJ} \to K_{IJ} - \sum_{\Phi \,|\, Q_0[\Phi]<0} Q_I[\Phi] Q_J[\Phi] \ , \tag{B.2.7}$$

where the sum is over all chiral multiplets with strictly negative $U(1)_0$ charge. To determine the $U(1)_0$ assignment of the gauge singlets $\mathfrak{T}^\pm$, one recalls that they are identified with the monopoles of the electric theory. The latter have an induced $U(1)_0$ charge given by:

$$Q_0\left[\mathfrak{T}^\pm\right] = \mp\frac{1}{2}\sum_\psi Q_G[\psi]\Big|Q_0[\psi]\Big| - \frac{1}{2}\sum_\psi Q_0[\psi] \ , \tag{B.2.8}$$

where Q_G denotes the gauge charge, and we sum over all Dirac fermions in the theory. It is then a straightforward exercise to derive the minimally chiral dual theory spelled out in subsection 4.1.2.

	$U(N_c)$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$	$U(1)_0$
ϕ_α	$\square$	$\overline{\square}$	$\mathbf{1}$	1	0	r	0
$\widetilde{\phi}^\beta$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	1	0	r	0
$\widetilde{\phi}^i$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	1	0	r	1
$\widetilde{\phi}^j$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	1	0	r	-1
ϕ_i	$\square$	$\mathbf{1}$	$\mathbf{1}$	1	0	r	1
ϕ_j	$\square$	$\mathbf{1}$	$\mathbf{1}$	1	0	r	-1

Table B.3: Fields and charges for the mass deformation of the Aharony electric theory, which leads to SQCD with $|k| < |k_c|$, with the massive fields for either $k_c > 0$ (middle two rows) or $k_c < 0$ (bottom two rows). Here $i = 1, \cdots, q$ and $j = 1, \cdots, \widetilde{q}$.

B.2.2 Marginally chiral duality: $|k| = |k_c|$

The marginally chiral cases can be obtained as special limits of the minimally chiral case, when either p or q vanishes:

$$\begin{aligned}
k &= k_c > 0 && \Leftrightarrow && p = 0 \; , \quad \epsilon = 1 \; , \\
k &= -k_c > 0 && \Leftrightarrow && q = 0 \; , \quad \epsilon = 1 \; , \\
k &= k_c < 0 && \Leftrightarrow && p = 0 \; , \quad \epsilon = -1 \; , \\
k &= -k_c < 0 && \Leftrightarrow && q = 0 \; , \quad \epsilon = -1 \; .
\end{aligned} \tag{B.2.9}$$

In those cases, we see from B.2 that either $\mathfrak{T}^+$ or $\mathfrak{T}^-$ survives the mass deformation of the dual theory.

B.2.3 Maximally chiral duality: $|k| < |k_c|$

In the maximally chiral case, we start from the electric side of Aharony duality and integrate fundamental or antifundamentals with opposite masses. Consider first the case $k_c > 0$. In this case, we choose to integrate out q antifundamental with a positive mass, and $\widetilde{q}$ antifundamental with negative mass, so that we obtain:

$$n_f = N_f \; , \qquad n_a = N_f - q - \widetilde{q} \; , \qquad k = \frac{1}{2}(q - \widetilde{q}) \; . \tag{B.2.10}$$

	$U(N_c^D)$	$SU(n_f)$	$SU(n_a)$	$U(1)_A$	$U(1)_T$	$U(1)_R$	$U(1)_0$
φ_β	$\square$	$\mathbf{1}$	$\overline{\square}$	-1	0	$1-r$	0
$\widetilde{\varphi}^\alpha$	$\overline{\square}$	$\square$	$\mathbf{1}$	-1	0	$1-r$	0
${M_\alpha}^\beta$	$\mathbf{1}$	$\overline{\square}$	$\square$	2	0	$2r$	0
φ_i	$\square$	$\mathbf{1}$	$\mathbf{1}$	-1	0	$1-r$	-1
φ_j	$\square$	$\mathbf{1}$	$\mathbf{1}$	-1	0	$1-r$	1
${M_\alpha}^i$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	2	0	$2r$	1
${M_\alpha}^j$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	2	0	$2r$	-1
$\mathfrak{T}_+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	1	r_T	$\widetilde{q}$
$\mathfrak{T}_-$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	-1	r_T	$-q$
$\widetilde{\varphi}^i$	$\square$	$\mathbf{1}$	$\mathbf{1}$	-1	0	$1-r$	-1
$\widetilde{\varphi}^j$	$\square$	$\mathbf{1}$	$\mathbf{1}$	-1	0	$1-r$	1
${M_i}^\beta$	$\mathbf{1}$	$\mathbf{1}$	$\square$	2	0	$2r$	1
${M_j}^\beta$	$\mathbf{1}$	$\mathbf{1}$	$\square$	2	0	$2r$	-1
$\mathfrak{T}_+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	1	r_T	$-q$
$\mathfrak{T}_-$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N_f$	-1	r_T	$\widetilde{q}$

Table B.4: Fields and charges for the mass deformation of the Aharony dual theory, to obtain the maximally chiral case. The middle rows are for $k_c > 0$, and the bottom rows are for $k_c < 0$.

Note that $k_c = \frac{1}{2}(q + \widetilde{q})$ in this case, hence $|k| < k_c$ as expected. We also need to shift the Coulomb branch origin and the FI term according to:

$$\sigma_a \to \sigma_a + \frac{k}{N_f} m_0 \ , \qquad\qquad \xi \to \xi - k_c |m_0| \ . \tag{B.2.11}$$

For $k_c < 0$, we similarly choose to integrate out fundamental multiplets as shown in table B.3. Then we have: .

$$n_f = N_f - q - \widetilde{q} \ , \qquad n_a = N_f \ , \qquad k = \frac{1}{2}(q - \widetilde{q}) \ . \tag{B.2.12}$$

with $k_c = -\frac{1}{2}(q + \widetilde{q})$. In either case, the dual gauge group is $U(N_c^D)_{-k}$ with $N_c^D = \max(n_f, n_a) - N_c$. The details of the dual theory can be worked out from the charge assignment shown in table B.4 for the Aharony dual fields.

APPENDIX C

MATCHING MODULI SPACES OF VACUA: MORE EXPLICIT EXAMPLES

In this appendix, we give some specific examples of our main results for SQCD$[N_c, k, l, n_f, 0]$. In particular, we display the intricate matching of the vacua across the maximally- and minimally-chiral dualities.

C.1 Crossing $\xi = 0$

Here are a few examples of phase transitions from $\xi > 0$ to $\xi < 0$, for various values of the parameters $[N_c, k, l, n_f]$:

N_c	k	l	n_f	$\xi > 0$ phase	$\xi < 0$ phase
3	$\frac{5}{2}$	-5	7	$\mathrm{Gr}(3,7) \oplus U(3)_{6,-9}$	$\underbrace{U(2)_{6,-4} \times U(1)}_{-5}{}_{-6}$
5	$\frac{11}{2}$	0	9	$\mathrm{Gr}(5,9) \oplus \mathrm{Gr}(4,9) \times U(1)_1$	$U(5)_{10,10}$
6	4	-2	10	$\mathrm{Gr}(6,10) \oplus U(6)_{9,-3}$	$\underbrace{U(5)_{9,-1} \times U(1)}_{-2}{}_{-3}$
7	$\frac{3}{2}$	-5	11	$\mathrm{Gr}(7,11) \oplus \underbrace{U(5)_{7,-18} \times U(2)}_{-5}{}_{-4,-14}$ $\oplus\, U(7)_{7,-28} \oplus \underbrace{U(6)_{7,-23} \times U(1)}_{-5}{}_{-9}$	$\underbrace{U(3)_{7,-8} \times U(4)}_{-5}{}_{-4,-24}$ $\oplus\, \underbrace{U(4)_{7,-13} \times U(3)}_{-5}{}_{-4,-19}$
8	5	6	10	$\mathrm{Gr}(8,10) \oplus \mathrm{Gr}(7,10) \times U(1)_7$	$U(8)_{11,59}$

N_c	k	l	n_f	$\xi > 0$ phase	$\xi < 0$ phase
9	8	-7	14	$\mathrm{Gr}(9,14) \oplus U(9)_{15,-48}$	$\mathrm{Gr}(8,14) \times U(1)_{-6}$
10	9	-4	12	$\mathrm{Gr}(10,12) \oplus U(10)_{15,-25}$	$\mathrm{Gr}(9,12) \times U(1)_{-1} \oplus \mathrm{Gr}(8,12) \times U(2)_{3,-5}$ $\oplus\ \mathrm{Gr}(7,12) \times U(3)_{3,-9}$
11	$-\frac{9}{2}$	-4	13	$\mathrm{Gr}(11,13)$	$U(11)_{-11,-55} \oplus U\underbrace{(1)_{-2} \times U(10)}_{-4}{}_{-11,-51}$ $\oplus\ U\underbrace{(2)_{2,-6} \times U(9)}_{-4}{}_{-11,-47}$
12	$\frac{15}{3}$	21	21	$\mathrm{Gr}(12,21)$	$U(12)_{18,18} \oplus U\underbrace{(9)_{18,18} \times U(3)}_{0}{}_{-3,-3}$ $\oplus\ U\underbrace{(10)_{18,18} \times U(2)}_{0}{}_{-3,-3}$ $\oplus\ U\underbrace{(11)_{18,18} \times U(1)}_{0}{}_{-3}$
30	1	0	32	$\mathrm{Gr}(30,32)$	$U\underbrace{(15)_{17,17} \times U(15)}_{0}{}_{-15,-15}$ $\oplus\ U\underbrace{(16)_{17,17} \times U(14)}_{0}{}_{-15,-15}$ $\oplus\ U\underbrace{(17)_{-17,-17} \times U(13)}_{0}{}_{-15,-15}$

C.2 Minimally-chiral duality for $\xi > 0$

Here, we consider a few examples of the minimally-chiral duality with positive FI parameter:

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
3	$\frac{9}{2}$	-5	7	$\mathrm{Gr}(3,7)$ $\oplus\ U(3)_{8,-7}$	$\mathrm{Gr}(4,7) \times U\underbrace{(1)_0 \times U(1)}_{1}{}_{-4}$ $\oplus\ U\underbrace{(5)_{-8,-3} \times U(1)}_{1}{}_{-4}$

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
4	7	-2	8	$\mathrm{Gr}(4,8)$ $\oplus\ \mathrm{Gr}(3,8)\times U(1)_1$	$\mathrm{Gr}(4,8)\times \underbrace{U(3)_{-3,0}\times U(1)_{-1}}_{1}$ $\oplus\ \mathrm{Gr}(5,8)\times \underbrace{U(2)_{-3,-1}\times U(1)_{-1}}_{1}$
5	$-\frac{13}{2}$	5	9	$\mathrm{Gr}(5,9)$ $\oplus\ U(5)_{-11,14}$	$\mathrm{Gr}(4,9)\times \underbrace{U(2)_{2,0}\times U(1)_{4}}_{-1}$ $\oplus\ \underbrace{U(6)_{11,5}\times U(1)_{4}}_{-1}$
6	6	-5	8	$\mathrm{Gr}(6,8)$ $\oplus\ U(6)_{10,-20}$	$\mathrm{Gr}(2,8)\times \underbrace{U(2)_{-2,0}\times U(1)_{-4}}_{1}$ $\oplus\ \underbrace{U(4)_{-10,-6}\times U(1)_{-4}}_{1}$
11	12	0	12	$\mathrm{Gr}(11,12)$ $\oplus\ \mathrm{Gr}(10,12)\times U(1)_6$ $\oplus\ \mathrm{Gr}(9,12)\times U(2)_{6,6}$ $\oplus\ \mathrm{Gr}(8,12)\times U(3)_{6,6}$ $\oplus\ \mathrm{Gr}(7,12)\times U(4)_{6,6}$ $\oplus\ \mathrm{Gr}(6,12)\times U(5)_{6,6}$ $\oplus\ \mathrm{Gr}(5,12)\times U(6)_{6,6}$	$\mathbb{CP}^{11}\times \underbrace{U(6)_{-6,0}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(2,12)\times \underbrace{U(5)_{-6,-1}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(3,12)\times \underbrace{U(4)_{-6,-2}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(4,12)\times \underbrace{U(3)_{-6,-3}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(5,12)\times \underbrace{U(2)_{-6,-4}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(6,12)\times \underbrace{U(1)_{-5}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(7,12)\times U(1)_1$
12	$\frac{23}{2}$	-10	21	$\mathrm{Gr}(12,21)$ $\oplus\ U(12)_{22,-98}$	$\mathrm{Gr}(9,21)\times \underbrace{U(1)_{0}\times U(1)_{-9}}_{1}$ $\oplus\ \underbrace{U(10)_{-22,-12}\times U(1)_{-9}}_{1}$
20	$\frac{33}{2}$	0	27	$\mathrm{Gr}(20,27)$ $\oplus\ \mathrm{Gr}(19,27)\times U(1)_3$ $\oplus\ \mathrm{Gr}(18,27)\times U(2)_{3,3}$ $\oplus\ \mathrm{Gr}(17,27)\times U(3)_{3,3}$	$\mathrm{Gr}(7,27)\times \underbrace{U(3)_{-3,0}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(8,27)\times \underbrace{U(2)_{-3,-1}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(9,27)\times \underbrace{U(1)_{-2}\times U(1)_{1}}_{1}$ $\oplus\ \mathrm{Gr}(10,27)\times U(1)_1$

C.3 Maximally-chiral duality for $\xi > 0$

Here, we consider a few examples of the maximally-chiral duality with positive FI parameter:

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
3	$\frac{3}{2}$	-5	7	$\mathrm{Gr}(3,7) \oplus U(3)_{5,-10}$	$\mathrm{Gr}(4,7) \oplus U(\underbrace{2)_{-5,-15} \times U(2}_{-5})_{2,-8}$
4	3	-7	8	$\mathrm{Gr}(4,8) \oplus U(4)_{7,-21}$	$\mathrm{Gr}(4,8) \oplus U(\underbrace{3)_{-7,-28} \times U(1}_{-7})_{-6}$
5	-3	-5	10	$\mathrm{Gr}(5,10) \oplus U(\underbrace{2)_{2,-8} \times U(3}_{-5})_{-8,-23}$	$\mathrm{Gr}(5,10) \oplus U(5)_{8,-17}$
6	3	-12	10	$\mathrm{Gr}(6,10)$ $\oplus\, U(6)_{8,-64}$ $\oplus\, U(\underbrace{5)_{8,-52} \times U(1}_{-12})_{-14}$	$\mathrm{Gr}(4,10)$ $\oplus\, U(\underbrace{2)_{-8,-32} \times U(2}_{-12})_{2,-22}$ $\oplus\, U(\underbrace{3)_{-8,-44} \times U(1}_{-12})_{-10}$
14	-7	4	20	$\mathrm{Gr}(14,20)$ $\oplus\, U(14)_{-17,39}$ $\oplus\, U(\underbrace{1)_{7} \times U(13}_{4})_{-17,35}$	$\mathrm{Gr}(6,20)$ $\oplus\, U(\underbrace{3)_{-3,9} \times U(3}_{4})_{17,29}$ $\oplus\, U(\underbrace{2)_{-3,5} \times U(4}_{4})_{17,33}$
15	$\frac{11}{2}$	4	23	$\mathrm{Gr}(15,23) \oplus U(\underbrace{9)_{17,53} \times U(6}_{4})_{-6,18}$	$\mathrm{Gr}(8,23) \oplus U(8)_{-17,15}$
17	0	10	20	$\mathrm{Gr}(17,20)$ $\oplus\, U(\underbrace{7)_{10,80} \times U(10}_{10})_{-10,90}$	$\mathrm{Gr}(3,20)$ $\oplus\, U(3)_{-10,20}$
20	$\frac{13}{2}$	-4	25	$\mathrm{Gr}(20,25)$ $\oplus\, U(\underbrace{17)_{19,-49} \times U(3}_{-4})_{-6,-18}$ $\oplus\, U(\underbrace{18)_{19,-53} \times U(2}_{-4})_{-6,-14}$ $\oplus\, U(\underbrace{19)_{19,-57} \times U(1}_{-4})_{-10}$	$\mathrm{Gr}(5,25)$ $\oplus\, U(\underbrace{2)_{-19,-27} \times U(3}_{-4})_{6,-6}$ $\oplus\, U(\underbrace{1)_{-23} \times U(4}_{-4})_{6,-10}$ $\oplus\, U(5)_{6,-14}$

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
23	19	-23	42	$\mathrm{Gr}(23,42)$ $\oplus\, U(23)_{40,-489}$ $\oplus\, \underbrace{U(22)_{40,-466}\times U(1)_{-25}}_{-23}$	$\mathrm{Gr}(19,42)$ $\oplus\, \underbrace{U(17)_{-40,-431}\times U(2)_{2,-44}}_{-23}$ $\oplus\, \underbrace{U(18)_{-40,-454}\times U(1)_{-21}}_{-23}$
30	$\frac{11}{2}$	2	45	$\mathrm{Gr}(30,45)\oplus \underbrace{U(13)_{28,54}\times U(17)_{-17,17}}_{2}$	$\mathrm{Gr}(15,45)\oplus U(15)_{-28,2}$
40	9	-3	46	$\mathrm{Gr}(40,46)$ $\oplus\, \underbrace{U(32)_{32,-64}\times U(8)_{-14,-38}}_{-3}$	$\mathrm{Gr}(6,46)$ $\oplus\, U(6)_{14,-4}$

C.4 Minimally-chiral duality for $\xi < 0$

Next, we consider a few examples of the minimally-chiral duality with negative FI parameter:

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
3	$\frac{9}{2}$	-5	7	$\mathrm{Gr}(2,7)\times U(1)_{-4}$	$\mathrm{Gr}(5,7)\times U(1)_{-4}$
4	7	-2	8	$\mathrm{Gr}(2,8)\times U(2)_{3,-1}$ $\oplus\, \mathbb{CP}^7\times U(3)_{3,-3}$ $\oplus\, U(4)_{11,3}$	$\mathrm{Gr}(6,8)\times \underbrace{U(1)_{-2}\times U(1)_{-1}}_{1}$ $\oplus\, \mathrm{Gr}(7,8)\times U(1)_{-1}$ $\oplus\, \underbrace{U(7)_{-11,-4}\times U(1)_{-1}}_{1}$
5	$-\frac{13}{2}$	5	9	$\mathrm{Gr}(4,9)\times U(1)_3$ $\oplus\, \mathrm{Gr}(3,9)\times U(2)_{-2,8}$	$\oplus\, \mathrm{Gr}(5,9)\times \underbrace{U(1)_1\times U(1)_4}_{-1}$ $\mathrm{Gr}(6,9)\times U(1)_4$
6	6	-5	8	$\mathrm{Gr}(5,8)\times U(1)_{-3}$ $\oplus\, \mathrm{Gr}(4,8)\times U(2)_{2,-8}$	$\mathrm{Gr}(3,8)\times \underbrace{U(1)_{-1}\times U(1)_{-4}}_{1}$ $\oplus\, \mathrm{Gr}(4,8)\times U(1)_{-4}$

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
11	12	0	12	$U(11)_{18,18}$	$\underbrace{U(7)_{-18,-11}\times U(1)}_{1}{}_1$
12	$\frac{23}{2}$	-10	21	$\mathrm{Gr}(11,21)\times U(1)_{-9}$	$\mathrm{Gr}(10,21)\times U(1)_{-9}$
14	14	-7	24	$\mathrm{Gr}(13,24)\times U(1)_{-5}$ $\oplus\ \mathrm{Gr}(12,24)\times U(2)_{2,-12}$	$\mathrm{Gr}(11,24)\times \underbrace{U(1)_{-1}\times U(1)}_{1}{}_{-6}$ $\oplus\ \mathrm{Gr}(12,24)\times U(1)_{-6}$
23	$\frac{29}{2}$	-10	25	$\mathrm{Gr}(22,25)\times U(1)_{-8}$ $\oplus\ \mathrm{Gr}(21,25)\times U(2)_{2,-18}$	$\mathrm{Gr}(3,25)\times \underbrace{U(1)_{-1}\times U(1)}_{1}{}_{-9}$ $\oplus\ \mathrm{Gr}(4,25)\times U(1)_{-9}$
37	$\frac{53}{2}$	-5	45	$\mathrm{Gr}(36,45)\times U(1)_{-1}$ $\oplus\ \mathrm{Gr}(35,45)\times U(2)_{4,-6}$ $\oplus\ \mathrm{Gr}(34,45)\times U(3)_{4,-11}$ $\oplus\ \mathrm{Gr}(33,45)\times U(4)_{4,-16}$	$\mathrm{Gr}(9,45)\times \underbrace{U(3)_{-4,-1}\times U(1)}_{1}{}_{-4}$ $\oplus\ \mathrm{Gr}(10,45)\times \underbrace{U(2)_{-4,-2}\times U(1)}_{1}{}_{-4}$ $\oplus\ \mathrm{Gr}(11,45)\times \underbrace{U(1)_{-3}\times U(1)}_{1}{}_{-4}$ $\oplus\ \mathrm{Gr}(12,45)\times U(1)_{-4}$
40	-28	11	50	$\mathrm{Gr}(39,50)\times U(1)_{8}$ $\oplus\ \mathrm{Gr}(38,50)\times U(2)_{-3,19}$ $\oplus\ \mathrm{Gr}(37,50)\times U(3)_{-3,30}$	$\mathrm{Gr}(11,50)\times \underbrace{U(2)_{3,1}\times U(1)}_{-1}{}_{10}$ $\oplus\ \mathrm{Gr}(12,50)\times \underbrace{U(1)_{2}\times U(1)}_{-1}{}_{10}$ $\oplus\ \mathrm{Gr}(13,50)\times U(1)_{10}$
52	$\frac{77}{2}$	-3	73	$\mathrm{Gr}(51,73)\times U(1)_{-1}$ $\oplus\ \mathrm{Gr}(50,73)\times U(2)_{2,-4}$	$\mathrm{Gr}(22,73)\times \underbrace{U(1)_{-1}\times U(1)}_{1}{}_{-2}$ $\oplus\ \mathrm{Gr}(23,73)\times U(1)_{-2}$

C.5 Maximally-chiral duality for $\xi < 0$

Finally, here are a few examples of the maximally-chiral duality with negative FI parameter:

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
4	3	-7	8	$\underbrace{U(3)_{7,-14}\times U(1)}_{-7}{}_{-8}$	$U(4)_{-7,-35}$

N_c	k	l	n_f	Electric Vacua	Magnetic Vacua
3	$\frac{3}{2}$	-5	7	$\underbrace{U(1)_0 \times U(2)_{-2,-12}}_{-5}$ $\oplus\ \underbrace{U(2)_{5,-5} \times U(1)_{-7}}_{-5}$	$U(4)_{-5,-25}$ $\oplus\ \underbrace{U(3)_{-5,-20} \times U(1)_{-3}}_{-5}$
5	-3	-5	10	$U(5)_{-8,-33}$ $\oplus\ \underbrace{U(1)_{-3} \times U(4)_{-8,-28}}_{-5}$	$\underbrace{U(2)_{-2,-12} \times U(3)_{8,-7}}_{-5}$ $\oplus\ \underbrace{U(1)_{-7} \times U(4)_{8,-12}}_{-5}$
6	3	-12	10	$\underbrace{U(4)_{8,-40} \times U(2)_{-2,-26}}_{-12}$	$U(4)_{-8,-56}$
14	-7	4	20	$\underbrace{U(2)_{3,11} \times U(12)_{-17,31}}_{4}$ $\oplus\ \underbrace{U(3)_{3,15} \times U(11)_{-17,27}}_{4}$	$\underbrace{U(1)_1 \times U(5)_{17,37}}_{4}$ $\oplus\ U(6)_{17,41}$
15	$\frac{11}{2}$	4	23	$U(15)_{17,77}$ $\oplus\ \underbrace{U(10)_{17,57} \times U(5)_{-6,14}}_{4}$ $\oplus\ \underbrace{U(11)_{17,61} \times U(4)_{-6,10}}_{4}$ $\oplus\ \underbrace{U(12)_{17,65} \times U(3)_{-6,6}}_{4}$ $\oplus \underbrace{U(13)_{17,69} \times U(2)_{-6,2}}_{4}$ $\oplus\ \underbrace{U(14)_{17,73} \times U(1)_{-2}}_{4}$	$\underbrace{U(2)_{-17,-9} \times U(6)_{6,30}}_{4}$ $\oplus\ \underbrace{U(7)_{-17,11} \times U(1)_{10}}_{4}$ $\oplus\ \underbrace{U(6)_{-17,7} \times U(2)_{6,14}}_{4}$ $\oplus\ \underbrace{U(5)_{-17,3} \times U(3)_{6,18}}_{4}$ $\oplus\ \underbrace{U(4)_{-17,-1} \times U(4)_{6,22}}_{4}$ $\oplus\ \underbrace{U(3)_{-17,-5} \times U(5)_{6,26}}_{4}$
29	14	0	30	$U(29)_{29,29} \oplus U(28)_{29,29} \times U(1)_{-1}$	$U(1)_1 \oplus U(1)_{-29}$
35	20	-3	50	$\underbrace{U(30)_{45,-45} \times U(5)_{-5,-20}}_{-3}$ $\oplus\ \underbrace{U(31) \times U(4)_{-5,-17}}_{-3}$ $\oplus\ \underbrace{U(32)_{45,-51} \times U(3)_{-5,-14}}_{-3}$	$U(15)_{-45,-90}$ $\oplus\ \underbrace{U(14)_{-45,-87} \times U(1)_2}_{-3}$ $\oplus\ \underbrace{U(13)_{-45,-84} \times U(2)_{5,-1}}_{-3}$

APPENDIX D

THE JK RESIDUE FORMULA AND THE SUM OVER 3D VACUA

In this appendix, we further study the JK residue formula (5.1.12) for the genus-zero topologically twisted index of the 3d $\mathcal{N} = 2$ SQCD$[N_c, k, l, n_f, 0]$ theory. We show that the JK residues that contribute non-trivially to the index can be organised in terms of the 3d vacua that contribute to the Witten index that we analysed in section 3.2. In particular, for (k, l) in the geometric window as defined in section 5.1.2, only the so-called Higgs-branch singularities (5.1.26) contribute.

D.1 Singularity structure of the twisted index integrand

Let us first review the singularity structure of the twisted-index integrand (5.1.13). Consider the set of all codimension-one singularities of this integrand on:

$$\widetilde{\mathfrak{M}} \cong \left(\mathbb{P}^1\right)^{N_c} \ , \tag{D.1.1}$$

a natural compactification of the space spanned by the gauge variables x_a, $a = 1, \cdots, N_c$. Here, $\widetilde{\mathfrak{M}}$ is really a compactification of a covering space of the classical 3d Coulomb branch $\mathfrak{M} \cong (\mathbb{C}^*)^{N_c}/S_{N_c}$, and we are effectively dealing with an abelianised theory. Then, one can consider the 3d monopole operators $\mathfrak{T}_a^{\pm}$ for each factor $U(1)_a \subset U(N_c)$, whose charges are governed by the singularities of the integrand at $x_a = 0$ and $x_a = \infty$.

The integrand then has two types of codimension-one singularities, from either the matter fields or the monopole operators, in any given topological sector indexed by the abelianised magnetic flux $\mathfrak{m}$. To every hyperplane singularity H, we assign a gauge charge

$Q = (Q^a)$ under $\prod_{a=1}^{N_c} U(1)_a$, as follows [115]:

Matter field singularities. Depending on the value of the magnetic fluxes $\mathfrak{m} \in \mathbb{Z}^{N_c}$, there can be singularities that arise from the chiral multiplet contribution to $Z_{\mathfrak{m}}(x, y)$ in (5.1.13). They are characterised by the singular hyperplanes $H_{\rho_a,\alpha}$ defined by:

$$H_{\rho_a,\alpha} \equiv \{x \in \widetilde{\mathfrak{M}} : \quad x^\rho \, y_\alpha^{-1} = 1 \} \;, \qquad \alpha = 1, \cdots, n_f \;, \tag{D.1.2}$$

with ρ_a being the fundamental weights of $U(N_c)$. To such hyperplanes, we assign the gauge charges of the corresponding chiral multiplets:

$$Q_{a,\alpha} = \rho_a = (0, \cdots, 0, \underbrace{1}_{a-\text{th}}, 0, \cdots, 0) \;, \qquad \alpha = 1, \cdots, n_f \;, \qquad a = 1, \cdots, N_c \;. \tag{D.1.3}$$

Note that the charge vectors are the same for the n_f distinct fundamental chiral multiplets.

Monopole operators singularities. These singularities are associated to the monopole operators that we can define semi-classically in the asymptotic regions of the classical Coulomb branch $\mathfrak{M}$. Depending on the whether we consider the limit $x_a = \infty$ ($\sigma_a = -\infty$) or $x_a = 0$ ($\sigma_a = \infty$), we have the monopole operators $\mathfrak{T}_a^+$ or $\mathfrak{T}_a^-$, respectively. We define the corresponding hyperplanes:

$$H_{a,\pm} \equiv \{x \in \widetilde{\mathfrak{M}} : x_a = 0, \; \infty\} \;, \qquad a = 1, \cdots, N_c \;. \tag{D.1.4}$$

In these asymptotic regions, the JK form $\mathfrak{I}_{\mathfrak{m}}$ in (5.1.13) has the following behaviour:

$$\mathfrak{I}_{\mathfrak{m}} \sim x_a^{\pm\left(Q_a^{(\pm)}(\mathfrak{m}) - r_{a,\pm}\right)} \frac{dx_a}{x_a} \;, \tag{D.1.5}$$

with $Q_a^{(\pm)}(\mathfrak{m}) \equiv \sum_{b=1}^{N_c} Q_a^{(\pm)\,b} \mathfrak{m}_b$ being the 1-loop-exact gauge charges of the monopole operators under $\prod_{a=1}^{N_c} U(1)_a$. These are the gauge charges that we should assign to the hyperplanes (D.1.4). They take the explicit form [115]:

$$Q_a^{(\pm) b} = \delta_a^b \left(\pm k - \frac{n_f}{2} \right) \pm l \;, \qquad a, b = 1, \cdots, N_c \;. \tag{D.1.6}$$

Note also that $r_{a,\pm}$ are the R-charges of the monopole operators.

D.2 JK residue prescription and phases of SQCD

In order to define the JK residue, we need to consider the set of all codimension-N_c singularities in each flux sector $\mathfrak{m}$:

$$\widetilde{\mathfrak{M}}^{\mathfrak{m}}_{\text{sing}} \subset \widetilde{\mathfrak{M}} \ . \tag{D.2.1}$$

Such singularities arise from the intersection of $r_s \geq N_c$ hyperplanes – for generic y_i we always have $r_s = N_c$, which is the case we will focus on. The JK residue prescription [221] instructs us to pick 'the JK residue' at the singularity $x = x_* \in \widetilde{\mathfrak{M}}^{\mathfrak{m}}_{\text{sing}}$:

$$\underset{x=x_*}{\text{JK-Res}} \left[\mathbf{Q}(x_*), \eta_\xi\right] \mathfrak{I}_{\mathfrak{m}}[\mathscr{L}](x, y, q) \ . \tag{D.2.2}$$

This is defined in terms of the gauge charges Q assigned to the N_c hyperplanes, as follows. Let us first define the positive cone generated by the charges $\mathbf{Q}(x_*) = (Q_1, \cdots, Q_N)$ as:

$$\text{Cone}_+ \left(\mathbf{Q}(x_*)\right) \equiv \{c_1 Q_1 + \cdots + c_s Q_{N_c} : c_1, \cdots, c_{N_c} > 0\} \subset \mathbb{C}^{N_c} \ . \tag{D.2.3}$$

We note that, for the definition of this cone to make sense, we need these charges to be *projective* in the sense that they live in a half-space of $\mathbb{C}^{N_c}$. In the *non-projective* case, the JK residue is ill-defined. Then, we are instructed to pick an auxiliary parameter $\eta \in \mathbb{C}^{N_c}$, which could be arbitrary as long as it is not parallel to any of the JK charges. Here, we will choose to align $\eta \equiv \eta_\xi$ with the real 3d FI parameter ξ, as follows:

$$\eta_\xi \equiv \xi(1, \cdots, 1) \in \mathbb{C}^{N_c} \ . \tag{D.2.4}$$

We thus assume that $\xi \neq 0$ in the following. Then, the singular points $x_* \in \widetilde{\mathfrak{M}}^{\mathfrak{m}}_{\text{sing}}$ that contribute non-trivially to the JK residue are those such that $\eta \in \text{Cone}_+(\mathbf{Q}(x_*))$ – see *e.g.* [115] for further details.

Hence, to determine which singularities contribute to the correlation function (5.1.12) for SQCD$[N_c, k, l, n_f, 0]$, we need to determine all the possible positive cones that contain $\eta = \eta_\xi$. That is, we need to find all possible abelianised gauge charges $Q_1^{(p_1)}, \cdots, Q_{N_c}^{(p_{N_c})}$ such that:

$$\eta = \sum_{a=1}^{N_c} c_a \, Q_a^{(p_a)} \ , \qquad c_a > 0 \ , \ \forall a = 1, \ \cdots, N_c \ . \tag{D.2.5}$$

Here, we use the labels $p_a \in \{+, -, 1, \cdots, n_f\}$ depending on whether the associated singularity comes from the monopole operators $\mathfrak{T}^\pm$ or the fundamental chiral multiplets

Φ_α, $\alpha = 1, \cdots, n_f$, respectively. Let us first work out the case $N_c = 2$. We will then briefly discuss the general case.

D.2.1 For SQCD$[2, k, l, n_f, 0]$

Consider the $U(2)_{k,k+2l}$ gauge theory coupled with n_f matter multiplets in the fundamental representation. In this case, we have the following charges defining the singular hyperplanes:

$$Q_1^{(\alpha)} = (1,\ 0)\ , \qquad Q_2^{(\alpha)} = (0,\ 1) \qquad i = 1, \cdots, n_f\ , \tag{D.2.6}$$

for the matter singularities (D.1.3). And,

$$Q_1^{(\pm)} = \left(\pm k - \frac{n_f}{2} \pm l,\ \pm l\right)\ , \qquad Q_2^{(\pm)} = \left(\pm l,\ \pm k - \frac{n_f}{2} \pm l\right)\ , \tag{D.2.7}$$

for the monopole singularities (D.1.6). To find which singularities do contribute for different choices of CS levels k and l, and for a given n_f, we need to study the equations (D.2.5), namely:

$$\xi(1,\ 1) = c_1 Q_1^{(p_1)} + c_2 Q_2^{(p_2)}\ , \qquad p_1, p_2 \in \{+, -, 1, \cdots, n_f\}\ , \tag{D.2.8}$$

with the constraint that $c_1, c_2 > 0$. The solutions to these equations are closely related to the 3d vacua that contribute to the 3d Witten index, hence, we shall index the solutions as in 3.2. We have the following possibilities:

- ***Type I.*** In this case, all the contributing singularities come from the matter multiplets. In this case, equations (D.2.8) become:

  $$\xi(1,\ 1) = c_1(1,\ 0) + c_2(0,\ 1)\ , \tag{D.2.9}$$

  which has the unique solution $c_1 = c_2 = \xi$. Thus, this type of singularities does indeed contribute to the correlation function if $\xi > 0$.

  Let $\alpha_1, \alpha_2 \in \{1, \cdots, n_f\}$, $\alpha_1 \neq \alpha_2$, denote two possible distinct matter singularities. In this case, the JK residue (5.1.12) for $N_c = 2$ becomes:

  $$\langle \mathscr{L} \rangle_{\mathbb{P}^1 \times S^1} = \Theta(\xi) \sum_{\mathfrak{m} \in \mathbb{Z}^2} \sum_{1 \leq \alpha_1 < \alpha_2 \leq n_f} \operatorname*{Res}_{\substack{x_1 = y_{\alpha_1} \\ x_2 = y_{\alpha_2}}} \mathfrak{I}_{\mathfrak{m}}[\mathscr{L}](q, x, y)\ , \tag{D.2.10}$$

  where we used the residual gauge symmetry to cancel the 2! factor. Here, Θ is the

Heaviside step function, as defined around (3.1.7). We further note that, although the residue formula is formally given in terms of a sum over all topological sectors $\mathfrak{m} \in \mathbb{Z}^2$, only the topological sectors with $\mathfrak{m}_1, \mathfrak{m}_2 \geq 0$ actually contribute singularities. From the point of view of the semi-classical analysis of the vacua in section 3.2, we get a 3d $\mathrm{Gr}(2, n_f)$ Higgs-branch vacuum spanned by the matter multiplets (3.2.26).

- ***Type II***. In this case, we take one of the contributing singularities to be from the matter multiplets and the other from one of the monopoles $\mathfrak{T}^{(\pm)}_{1,2}$. The equations (D.2.8) become:

$$\xi(1,1) = c_1(1,\ 0) + c_2^{(\pm)}(\pm l,\ \pm k - \frac{n_f}{2} \pm l)\ , \tag{D.2.11}$$

which have the solution:

$$c_1 = \xi \frac{\left(\pm k - \frac{n_f}{2}\right)}{\pm k - \frac{n_f}{2} \pm l} > 0\ , \qquad c_2 = \frac{\xi}{\pm k - \frac{n_f}{2} \pm l} > 0\ . \tag{D.2.12}$$

The corresponding 3d supersymmetric vacuum is a hybrid topological-Higgs vacuum of the form: $\mathbb{P}^{n_f - 1} \times U(1)_{k + l \pm \frac{n_f}{2}}$ (3.2.29).

- ***Type III.*** In this case, all the contributing singularities come from the two monopole operators $\mathfrak{T}^{\pm}_{1,2}$ (D.1.6). We actually have three possible choices. The first two are where we take the singular hyperplanes $x^{(+)}_{1,2} = \infty$ or $x^{(-)}_{1,2} = 0$. The corresponding equations (D.2.8) take the following form:

$$\xi(1,\ 1) = c_1^{(\pm)} \left(\pm k - \frac{n_f}{2} \pm l,\ \pm l\right) + c_2^{(\pm)} \left(\pm l,\ \pm k - \frac{n_f}{2} \pm l\right)\ . \tag{D.2.13}$$

In either case, we have a unique solution:

$$c_1^{(\pm)} = c_2^{(\pm)} = \frac{\xi}{\pm k - \frac{n_f}{2} \pm 2l} > 0\ , \qquad \text{iff} \quad \xi \left(\pm k - \frac{n_f}{2} \pm 2l\right) > 0\ . \tag{D.2.14}$$

From the point of view of moduli space of vacua, in this case, we find topological vacua of the form 3d TQFT with gauge group $U(2)_{\pm k,\ \pm k - \frac{n_f}{2} \pm 2l}$ (3.2.37).

The third choice of intersecting hyperplanes is $x_1^{(+)} = \infty$ and $x_2^{(-)} = 0$. This leads to the equations:

$$\xi(1,\ 1) = c_1^{(+)} Q_1^{(+)} + c_2^{(-)} Q_2^{(-)}\ . \tag{D.2.15}$$

These two equations can be uniquely solved by:

$$\begin{aligned} c_1^{(+)} &= \xi \frac{k+\frac{n_f}{2}}{\left(k+\frac{n_f}{2}\right)\left(k-\frac{n_f}{2}\right)+2kl} > 0 \ , \\ c_2^{(-)} &= -\xi \frac{k-\frac{n_f}{2}}{\left(k+\frac{n_f}{2}\right)\left(k-\frac{n_f}{2}\right)+2kl} > 0 \ . \end{aligned} \tag{D.2.16}$$

We can simplify these constraints into the following:

$$|k| < \frac{n_f}{2} \ , \qquad \xi\left(k^2+2kl-\frac{n_f^2}{4}\right) > 0 \ . \tag{D.2.17}$$

In this case, the moduli space of vacua consists of topological ones of the form of a TQFT with a gauge group $\underbrace{U(1)_{k+l-\frac{n_f}{2}}}_{l} \times U(1)_{k+l+\frac{n_f}{2}}$ (3.2.32).

In the analysis above, we tacitly assumed that we are staying away from the *marginal case*, which is the case when $|k| = \frac{n_f}{2}$. In the marginal case, the monopole charges become parallel to each other and render the JK residue ill-defined – that is, the singularities become *non-projective* –, as we can see from (D.1.6). These non-projective singularities are conjecturally associated with strongly-coupled 3d vacua, which are not accounted for when solving the semi-classical 3d vacuum equations as we discussed in section 3.2. It may be interesting to apply the methods of [173] to better deal with the marginal case. We will not consider this issue further in this work.

D.2.2 For SQCD$[N_c, k, l, n_f, 0]$

It is straightforward to extend the analysis of the $N_c = 2$ case above to the general case. One finds a one-to-one correspondence between the possible mixtures of the singularities and types of supersymmetric vacua that we get from the semi-classical analysis of section 3.2.

APPENDIX E

OTHER GENERALISED QK(Gr(2, 4)) RINGS

In this appendix, we provide a few more examples for the non-equivariant rings $\mathcal{R}_{3\text{d}}[k, l]$ for the Gr(2, 4) 3d GLSM, for different values of the CS levels k, l in the geometric window, in addition to the ones discussed in subsection 5.3.3. Specifically, we write down the result for the cases $(k, l) = (0, 0), (1, 0)$ and $(2, -2)$ in the Schubert class basis. In each of these three cases, we give both the topological metric (5.3.25) and the 3d ring structure.

E.1 Case $(k, l) = (0, 0)$

In this case, we find that the 2-point function (5.3.25), computed up to degree $d = 4$, has the following form:

$$\langle \mathcal{O}_\mu \, \mathcal{O}_\nu \rangle^{(k,l)=(0,0)}_{\mathbb{P}^1 \times S^1_\beta} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{E.1.1}$$

Meanwhile, the generalised QK ring $\mathcal{R}_{3\text{d}}[0, 0]$ has the following structure:

$$\begin{aligned} \mathcal{O}_{\ydiagram{1}}^2 &= \mathcal{O}_{\ydiagram{1,1}} + \mathcal{O}_{\ydiagram{2}} - \mathcal{O}_{\ydiagram{2,1}} . \\ \mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2}} &= \mathcal{O}_{\ydiagram{2,1}} , \end{aligned} \tag{E.1.2}$$

$$
\begin{aligned}
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{1,1}} = \mathcal{O}_{\ydiagram{2,1}} \,, \\
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2,1}} = -q + 2q\mathcal{O}_{\ydiagram{1}} - q\mathcal{O}_{\ydiagram{1,1}} - q\mathcal{O}_{\ydiagram{2}} + q\mathcal{O}_{\ydiagram{2,1}} + \mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{1}} \star \mathcal{O}_{\ydiagram{2,2}} = -q\mathcal{O}_{\ydiagram{1}} + q\mathcal{O}_{\ydiagram{1,1}} + q\mathcal{O}_{\ydiagram{2}} - q\mathcal{O}_{\ydiagram{2,1}} \,, \\
&\mathcal{O}_{\ydiagram{1,1}}^2 = \mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2}} = -q + q\mathcal{O}_{\ydiagram{1}} \,, \\
&\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2,1}} = -q\mathcal{O}_{\ydiagram{1}} + q\mathcal{O}_{\ydiagram{1,1}} + q\mathcal{O}_{\ydiagram{2}} - q\mathcal{O}_{\ydiagram{2,1}} \,, \\
&\mathcal{O}_{\ydiagram{1,1}} \star \mathcal{O}_{\ydiagram{2,2}} = -q\mathcal{O}_{\ydiagram{2}} + q\mathcal{O}_{\ydiagram{2,1}} \,, \qquad \text{(E.1.3)} \\
&\mathcal{O}_{\ydiagram{2}}^2 = \mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2,1}} = -q\mathcal{O}_{\ydiagram{1}} + q\mathcal{O}_{\ydiagram{1,1}} + q\mathcal{O}_{\ydiagram{2}} - q\mathcal{O}_{\ydiagram{2,1}} \,, \\
&\mathcal{O}_{\ydiagram{2}} \star \mathcal{O}_{\ydiagram{2,2}} = -q\mathcal{O}_{\ydiagram{1,1}} + q\mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{2,1}}^2 = q^2 - 2q^2\mathcal{O}_{\ydiagram{1}} - \left(q - q^2\right)\mathcal{O}_{\ydiagram{1,1}} - \left(q - q^2\right)\mathcal{O}_{\ydiagram{2}} - \left(q^2 - 3q\right)\mathcal{O}_{\ydiagram{2,1}} - q\mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{2,1}} \star \mathcal{O}_{\ydiagram{2,2}} = -q^2 + 2q^2\mathcal{O}_{\ydiagram{1}} - q^2\mathcal{O}_{\ydiagram{1,1}} - q^2\mathcal{O}_{\ydiagram{2}} - \left(q - q^2\right)\mathcal{O}_{\ydiagram{2,1}} + q\mathcal{O}_{\ydiagram{2,2}} \,, \\
&\mathcal{O}_{\ydiagram{2,2}}^2 = q^2 - 2q^2\mathcal{O}_{\ydiagram{1}} + q^2\mathcal{O}_{\ydiagram{1,1}} + q^2\mathcal{O}_{\ydiagram{2}} - q^2\mathcal{O}_{\ydiagram{2,1}} \,.
\end{aligned}
$$

E.2 Case $(k, l) = (1, 0)$

In this case, we find that the 2-point function takes the form:

$$
\langle \mathcal{O}_\mu \, \mathcal{O}_\nu \rangle^{(k,l)=(1,0)}_{\mathbb{P}^1 \times S^1_\beta} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & -q \\ 1 & 1 & 0 & 0 & 0 & -q \\ 1 & 0 & 0 & -q & -q & 0 \end{pmatrix} , \qquad \text{(E.2.1)}
$$

and we have the following $\mathcal{R}_{\text{3d}}[1,0]$ ring structure:

$$
\begin{aligned}
&\mathcal{O}_{\yng(1)}^2 = \mathcal{O}_{\yng(1,1)} + \mathcal{O}_{\yng(2)} - \mathcal{O}_{\yng(2,1)} \, . \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(1,1)} = \mathcal{O}_{\yng(2,1)} \, , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(2)} = q - 2q\mathcal{O}_{\yng(1)} + q\mathcal{O}_{\yng(1,1)} + q\mathcal{O}_{\yng(2)} - (q-1)\mathcal{O}_{\yng(2,1)} \, , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(2,1)} = q - 2q\mathcal{O}_{\yng(1)} + q\mathcal{O}_{\yng(1,1)} + q\mathcal{O}_{\yng(2)} - q\mathcal{O}_{\yng(2,1)} + \mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(2,2)} = q\mathcal{O}_{\yng(1)} - 2q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + 2q\mathcal{O}_{\yng(2,1)} \, , \\
&\mathcal{O}_{\yng(1,1)}^2 = \mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2)} = q - q\mathcal{O}_{\yng(1)} - q\mathcal{O}_{\yng(1,1)} + q\mathcal{O}_{\yng(2,1)} \, , \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,1)} = \mathcal{O}_{\yng(1)} - 2q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + 2q\mathcal{O}_{\yng(2,1)} \, , \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,2)} = q\mathcal{O}_{\yng(2)} - 2q\mathcal{O}_{\yng(2,1)} + q\mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(2)}^2 = q^2 - \left(2q^2 - q\right)\mathcal{O}_{\yng(1)} - \left(q - q^2\right)\mathcal{O}_{\yng(1,1)} - \left(2q - q^2\right)\mathcal{O}_{\yng(2)} - \left(q^2 - 2q\right)\mathcal{O}_{\yng(2,1)} + \mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,1)} = q^2 - \left(2q^2 - q\right)\mathcal{O}_{\yng(1)} - \left(q - q^2\right)\mathcal{O}_{\yng(1,1)} - \left(q - q^2\right)\mathcal{O}_{\yng(2)} - q^2\mathcal{O}_{\yng(2,1)} + q\mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,2)} = q^2\mathcal{O}_{\yng(1)} - \left(2q^2 - q\right)\mathcal{O}_{\yng(1,1)} - q^2\mathcal{O}_{\yng(2)} - \left(q - 2q^2\right)\mathcal{O}_{\yng(2,1)} - q\mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(2,1)}^2 = q^2 - 2q^2\mathcal{O}_{\yng(1)} + \left(q^2 + q\right)\mathcal{O}_{\yng(1,1)} + \left(q^2 + q\right)\mathcal{O}_{\yng(2)} - \left(q^2 + 4q\right)\mathcal{O}_{\yng(2,1)} + 2q\mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(2,2)} = -q^2 + 3q^2\mathcal{O}_{\yng(1)} - 3q^2\mathcal{O}_{\yng(1,1)} - 2q^2\mathcal{O}_{\yng(2)} + \left(3q^2 + q\right)\mathcal{O}_{\yng(2,1)} - 2q\mathcal{O}_{\yng(2,2)} \, , \\
&\mathcal{O}_{\yng(2,2)}^2 = q^2 - 3q^2\mathcal{O}_{\yng(1)} + 3q^2\mathcal{O}_{\yng(1,1)} + 3q^2\mathcal{O}_{\yng(2)} - 5q^2\mathcal{O}_{\yng(2,1)} + q^2\mathcal{O}_{\yng(2,2)} \, .
\end{aligned}
\tag{E.2.2}
$$

E.3 Case $(k,l) = (2,-2)$

In this case, the 2-point function reads (here up to order q^4):

$$\langle \mathcal{O}_\mu \, \mathcal{O}_\nu \rangle^{(k,l)=(2,-2)}_{\mathbb{P}^1 \times S^1_\beta}$$

$$= \begin{pmatrix} q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 \\ q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2 \\ q^4+q^2+1 & q^4+q^2+1 & q^4+q^2+1 & q^4+q^2 & q^4+q^2 & q^4+q^2 \\ q^4+q^2+1 & q^4+q^2+1 & q^4+q^2 & q^4+q^2+1 & q^4+q^2 & q^4+q^2 \\ q^4+q^2+1 & q^4+q^2+1 & q^4+q^2 & q^4+q^2 & q^4+q^2 & q^4+q^2 \\ q^4+q^2+1 & q^4+q^2 & q^4+q^2 & q^4+q^2 & q^4+q^2 & q^4+q^2 \end{pmatrix} . \tag{E.3.1}$$

For the ring $\mathcal{R}_{\rm 3d}[2,-2]$, we find:

$$\begin{aligned}
&\mathcal{O}_{\yng(1)}^2 = \mathcal{O}_{\yng(1,1)} + \mathcal{O}_{\yng(2)} - \mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(1,1)} = \mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(2)} = -2q + q\mathcal{O}_{\yng(1)} + q\mathcal{O}_{\yng(1,1)} + \mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(2,1)} = -q + q\mathcal{O}_{\yng(1,1)} + \mathcal{O}_{\yng(2,2)} \ , \\
&\mathcal{O}_{\yng(1)} \star \mathcal{O}_{\yng(2,2)} = q^2 - q\mathcal{O}_{\yng(1)} + q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + q\mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(1,1)}^2 = \mathcal{O}_{\yng(2,2)} \ , \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2)} = q^2 - q - q\mathcal{O}_{\yng(1)} + 2q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + q\mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,1)} = q^2 - q\mathcal{O}_{\yng(1)} + q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + q\mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,2)} = q^2 - q\mathcal{O}_{\yng(2)} + q\mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(2)}^2 = -2q\mathcal{O}_{\yng(1)} - q\mathcal{O}_{\yng(1,1)} + q\mathcal{O}_{\yng(2)} + q\mathcal{O}_{\yng(2,1)} + (q+1)\mathcal{O}_{\yng(2,2)} \ , \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,1)} = q - q\mathcal{O}_{\yng(1)} - q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + 2q\mathcal{O}_{\yng(2,1)} \ , \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,2)} = q^2 - q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2,1)} + 2q\mathcal{O}_{\yng(2,2)} \ , \\
&\mathcal{O}_{\yng(2,1)}^2 = -q\mathcal{O}_{\yng(1,1)} - q\mathcal{O}_{\yng(2)} + q\mathcal{O}_{\yng(2,1)} + q\mathcal{O}_{\yng(2,2)} \ , \\
&\mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(2,2)} = q^2 - q\mathcal{O}_{\yng(2,1)} + q\mathcal{O}_{\yng(2,2)} \ , \\
&\mathcal{O}_{\yng(2,2)}^2 = q^2 \ .
\end{aligned} \tag{E.3.2}$$

APPENDIX F

ANOTHER EXAMPLE: THE 3D GLSM FOR Gr(3, 5)

In this appendix, we briefly display another example, namely the 3d GLSM onto Gr(3, 5) that gives us its ordinary QK ring. This is the $U(3)_{\frac{1}{2},-\frac{5}{2}}$ gauge theory with $n_f = 5$. We restrict ourselves to the non-equivariant limit for simplicity of exposition. The Grothendieck lines are shown explicitly in figures F.1 and F.2.

F.1 QK(Gr(3, 5))

$$\mathcal{O}_{(1)}^2 = \mathcal{O}_{(1,1)} + \mathcal{O}_{(2)} - \mathcal{O}_{(2,1)} \,, \qquad \mathcal{O}_{(1)} \star \mathcal{O}_{(1,1)} = \mathcal{O}_{(2,1)} + \mathcal{O}_{(1,1,1)} - \mathcal{O}_{(2,1,1)} \,,$$

$$\mathcal{O}_{(1)} \star \mathcal{O}_{(2)} = \mathcal{O}_{(2,1)} \,, \qquad \mathcal{O}_{(1)} \star \mathcal{O}_{(2,1)} = \mathcal{O}_{(2,2)} + \mathcal{O}_{(2,1,1)} - \mathcal{O}_{(2,2,1)} \,,$$

$$\mathcal{O}_{(1)} \star \mathcal{O}_{(1,1,1)} = \mathcal{O}_{(2,1,1)} \,, \qquad \mathcal{O}_{(1)} \star \mathcal{O}_{(2,2)} = \mathcal{O}_{(2,2,1)} \,,$$

$$\mathcal{O}_{(1)} \star \mathcal{O}_{(2,1,1)} = q - q\mathcal{O}_{(1)} + \mathcal{O}_{(2,2,1)} \,, \qquad \mathcal{O}_{(1)} \star \mathcal{O}_{(2,2,1)} = q\mathcal{O}_{(1)} - q\mathcal{O}_{(1,1)} + \mathcal{O}_{(2,2,2)} \,,$$

$$\mathcal{O}_{(1)} \star \mathcal{O}_{(2,2,2)} = q\mathcal{O}_{(1,1)} \,, \qquad \mathcal{O}_{(1,1)}^2 = \mathcal{O}_{(2,2)} + \mathcal{O}_{(2,1,1)} - \mathcal{O}_{(2,2,1)} \,,$$

$$\mathcal{O}_{(1,1)} \star \mathcal{O}_{(2)} = \mathcal{O}_{(2,1,1)} \,, \qquad \mathcal{O}_{(1,1)} \star \mathcal{O}_{(2,1)} = q - q\mathcal{O}_{(1)} + \mathcal{O}_{(2,2,1)} \,,$$

$$\mathcal{O}_{(1,1)} \star \mathcal{O}_{(1,1,1)} = \mathcal{O}_{(2,2,1)} \,, \qquad \mathcal{O}_{(1,1)} \star \mathcal{O}_{(2,2)} = q\mathcal{O}_{(1)} \,,$$

$$
\begin{aligned}
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,1,1)} = \mathcal{O}_{\yng(2,2,2)} + q\mathcal{O}_{\yng(1)} - q\mathcal{O}_{\yng(1,1)} \,, && \mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,2,1)} = q\mathcal{O}_{\yng(1,1)} + q\mathcal{O}_{\yng(2)} - q\mathcal{O}_{\yng(2,1)} \,, \\
&\mathcal{O}_{\yng(1,1)} \star \mathcal{O}_{\yng(2,2,2)} = q\mathcal{O}_{\yng(2,1)} \,, && \mathcal{O}^2_{\yng(2)} = \mathcal{O}_{\yng(2,2)} \,, \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,1)} = \mathcal{O}_{\yng(2,2,1)} \,, && \mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(1,1,1)} = q \,, \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,2)} = \mathcal{O}_{\yng(2,2,2)} \,, && \mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,1,1)} = q\mathcal{O}_{\yng(1)} \\
&\mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,2,1)} = q\mathcal{O}_{\yng(1,1)} \,, && \mathcal{O}_{\yng(2)} \star \mathcal{O}_{\yng(2,2,2)} = q\mathcal{O}_{\yng(1,1,1)} \,, \\
&\mathcal{O}^2_{\yng(2,1)} = q\mathcal{O}_{\yng(1)} - q\mathcal{O}_{\yng(1,1)} + \mathcal{O}_{\yng(2,2,2)} \,, && \mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(1,1,1)} = q\mathcal{O}_{\yng(1)} \,, \\
&\mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(2,2)} = q\mathcal{O}_{\yng(1,1)} \,, && \mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(2,1,1)} = q\mathcal{O}_{\yng(1,1)} + q\mathcal{O}_{\yng(2)} - q\mathcal{O}_{\yng(2,1)} \,, \\
&\mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(2,2,1)} = q\mathcal{O}_{\yng(1,1,1)} + q\mathcal{O}_{\yng(2,1)} - q\mathcal{O}_{\yng(2,1,1)} \,, && \mathcal{O}_{\yng(2,1)} \star \mathcal{O}_{\yng(2,2,2)} = q\mathcal{O}_{\yng(2,1,1)} \\
&\mathcal{O}^2_{\yng(1,1,1)} = \mathcal{O}_{\yng(2,2,2)} \,, && \mathcal{O}_{\yng(1,1,1)} \star \mathcal{O}_{\yng(2,1,1)} = q\mathcal{O}_{\yng(1,1)} \,, \\
&\mathcal{O}_{\yng(1,1,1)} \star \mathcal{O}_{\yng(2,2)} = q\mathcal{O}_{\yng(2)} \,, && \mathcal{O}_{\yng(1,1,1)} \star \mathcal{O}_{\yng(2,2,1)} = q\mathcal{O}_{\yng(2,1)} \,, \\
&\mathcal{O}_{\yng(1,1,1)} \star \mathcal{O}_{\yng(2,2,2)} = q\mathcal{O}_{\yng(2,2)} \,, && \mathcal{O}^2_{\yng(2,1,1)} = q\mathcal{O}_{\yng(2,1)} + q\mathcal{O}_{\yng(1,1,1)} - q\mathcal{O}_{\yng(2,1,1)} \,, \\
&\mathcal{O}_{\yng(2,1,1)} \star \mathcal{O}_{\yng(2,2)} = q\mathcal{O}_{\yng(2,1)} \,, && \mathcal{O}_{\yng(2,1,1)} \star \mathcal{O}_{\yng(2,2,1)} = q\mathcal{O}_{\yng(2,1,1)} + q\mathcal{O}_{\yng(2,2)} - q\mathcal{O}_{\yng(2,2,1)} \,, \\
&\mathcal{O}_{\yng(2,1,1)} \star \mathcal{O}_{\yng(2,2,2)} = q\mathcal{O}_{\yng(2,2,1)} \,, && \mathcal{O}^2_{\yng(2,2)} = q\mathcal{O}_{\yng(1,1,1)} \,, \\
&\mathcal{O}_{\yng(2,2)} \star \mathcal{O}_{\yng(2,2,1)} = q\mathcal{O}_{\yng(2,1,1)} \,, && \mathcal{O}_{\yng(2,2)} \star \mathcal{O}_{\yng(2,2,2)} = q^2 \,, \\
&\mathcal{O}^2_{\yng(2,2,1)} = q^2 - q^2\mathcal{O}_{\yng(1)} + q\mathcal{O}_{\yng(2,2,1)} \,, && \mathcal{O}_{\yng(2,2,1)} \star \mathcal{O}_{\yng(2,2,2)} = q^2\mathcal{O}_{\yng(1)} \,, \\
&\mathcal{O}^2_{\yng(2,2,2)} = q^2\mathcal{O}_{\yng(2)} \,.
\end{aligned}
\tag{F.1.1}
$$

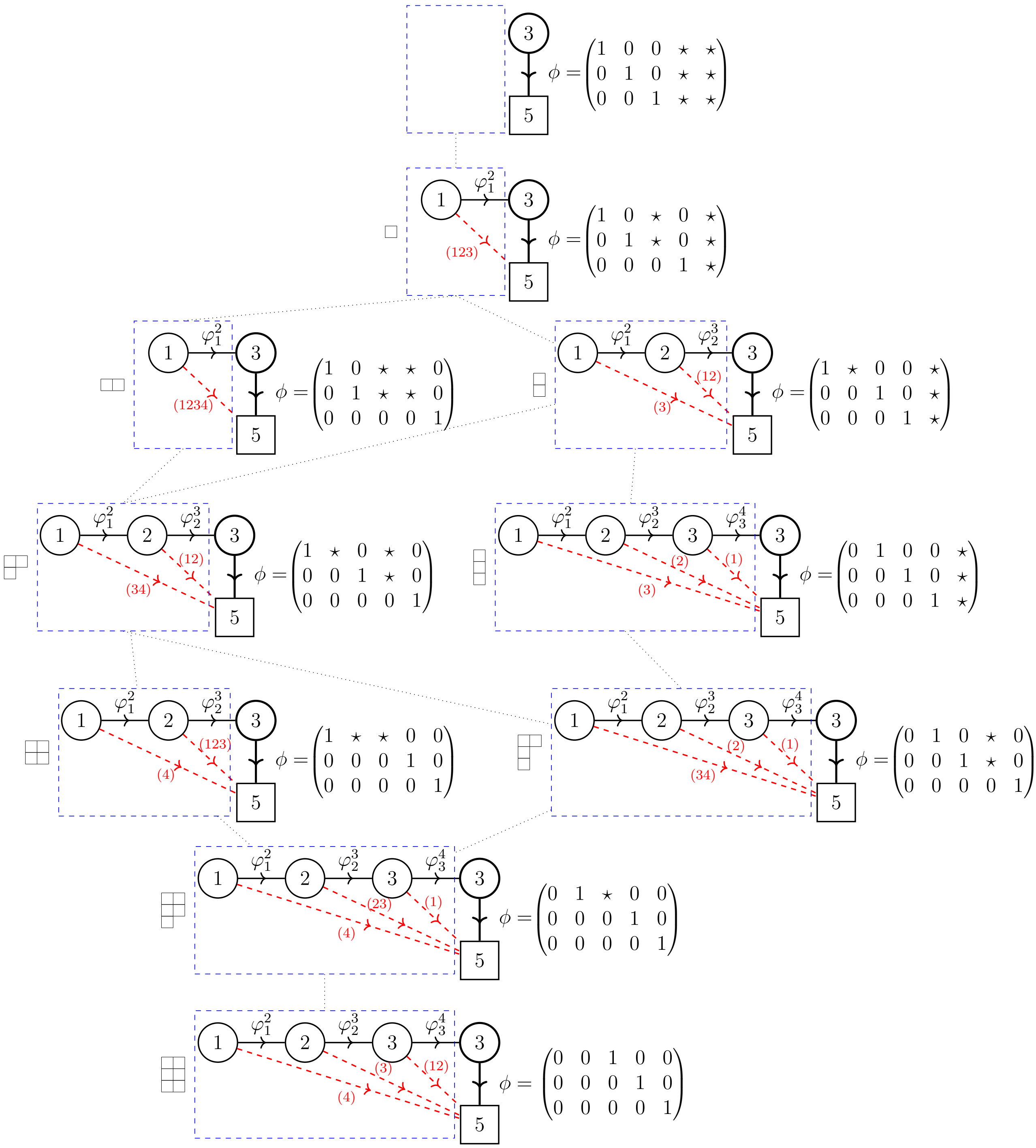


Figure F.1: The Hasse diagram associated with the Schubert subvarieties of $\mathrm{Gr}(3,5)$. The defining partitions are displayed at Young tableaux, and the 'generic' Grothendieck line defects are shown explicitly.

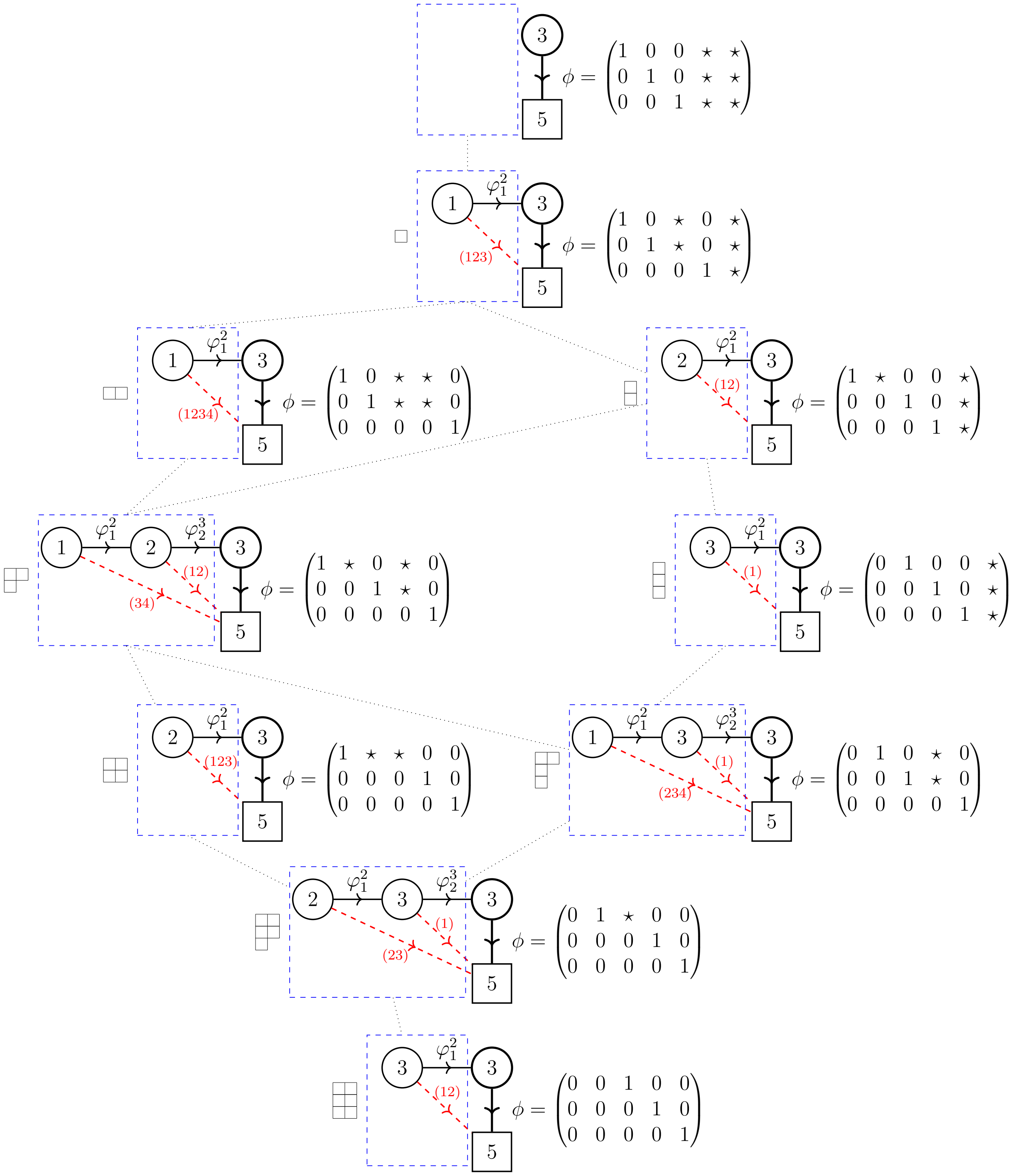


Figure F.2: The Hasse diagram associated with the Schubert subvarieties of Gr(3, 5), with the 1d quivers simplified using the duality moves (5.2.66).

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