CONVEX ANALYSIS AND OPTIMISATION ON HYPERBOLIC SPACE FORMS

by

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Abstract

This thesis is based on two published paper and one submitted paper in collaboration with co-authors Ferreira, O.P. and Németh, S.Z.. Some concepts of convex analysis on hyperbolic spaces are firstly studied. Next, we study the concept of convex sets and the intrinsic projection onto these sets. We also study the concept of convex functions and present first and second order characterizations of these functions. An extensive study of the hyperbolically convex quadratic functions is also presented. Unlike in the Euclidean space, the study of intrinsic convexity of non-homogeneous quadratic functions in the hyperbolic space is more elaborate than that of homogeneous quadratic functions. Then, we examine the gradient projection method as a solution approach for constrained optimization problems in κ-hyperbolic space forms. Moreover, we provide formulas for the intrinsic κ-projection into specific convex sets using the Euclidean orthogonal projection and the Lorentz projection. Regarding the convergence results of the gradient projection method, we establish two main findings. Firstly, we demonstrate that every accumulation point of the sequence generated by the method with backtracking step sizes is a stationary point for the given problem. Secondly, assuming the Lipschitz continuity of the gradient of the objective function, we show that each accumulation point of the sequence generated by the gradient projection method with a constant step size is also a stationary point. Finally, we explore the properties of the constrained Fermat-Weber problem, demonstrating that the sequence generated by the gradient projection method converges to its unique solution.

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LIST OF NOTATIONS

$\mathbb{R}^m$	m-dimensional Euclidean space.
$\mathbb{R}^m_+$	The nonnegative orthant of $\mathbb{R}^m$.
$\mathbb{R}^{m\times n}$	The set of all $m \times n$ matrices with real entries, $\mathbb{R}^m \equiv \mathbb{R}^{m\times 1}$.
$\|u\|_2$	$\sqrt{u^\top u}$, the Euclidean norm of $u \in \mathbb{R}^m$.
α^+	$\max(\alpha, 0)$, where $\alpha \in \mathbb{R}$.
α^-	$(-\alpha)^+$.
$M^\top$	The transpose of matrix M.
$\lambda_{\min}(M)$	The minimum eigenvalue of the matrix $M \in \mathbb{R}^{m\times m}$.
$\lambda_{\max}(M)$	The maximum eigenvalue of the matrix $M \in \mathbb{R}^{m\times m}$.
I	The identity matrix.
J	$\mathrm{diag}(1, \ldots, 1, -1) \in \mathbb{R}^{(n+1)\times(n+1)}$.
$\mathrm{diag}(u)$	A diagonal matrix with (i,i)-th entry equal to u_i, where $u \in \mathbb{R}^m$.
$\|A\|_2$	$\max\{\|Mu\|_2 : \|u\|_2 = 1, u \in \mathbb{R}^m\}$, where $M \in \mathbb{R}^{m\times m}$.
$\mathcal{L}$	$\left\{x \in \mathbb{R}^{n+1} : x^\top \mathrm{J} x \leq 0,\ x_{n+1} \geq 0\right\}$, the Lorentz cone.
$\partial\mathcal{L}$	$\left\{x \in \mathbb{R}^{n+1} : x^\top \mathrm{J} x = 0,\ x_{n+1} \geq 0\right\}$, the boundary of the Lorentz cone.

$\mathcal{K}^*$	The dual cone of a cone $\mathcal{K}$.
$\langle x, y\rangle$	$x^\top \mathrm{J} y, \forall x, y \in \mathbb{R}^{n+1}$, the Lorentzian inner product.
$\Vert x\Vert$	$\sqrt{\langle x, x\rangle}$, the Lorentzian norm.
$\mathbb{H}^n$	$\{p \in \mathbb{R}^{n+1} :\ \langle p, p\rangle = -1,\ p^{n+1} > 0\}$, the n-dimensional hyperbolic space.
$T_p\mathbb{H}^n$	$\{v \in \mathbb{R}^{n+1} \ :\ \langle p, v\rangle = 0\}$, the tangent hyperplane of the n-dimensional hyperbolic space at a point p.
$d(p, q)$	$\operatorname{arcosh}(-\langle p, q\rangle)$, the intrinsic distance on the hyperbolic space between two points $p, q \in \mathbb{H}^n$.
$\gamma_{pq}(t)$	The unique geodesic segment from p to q, where $t \in [0,\ d(p, q)$.
$\exp_p v$	$\gamma_v(1)$, the exponential mapping, where γ_v is the geodesic defined by its initial position p, with velocity v at p..
$\log_p q$	The inverse of the exponential mapping.
$\operatorname{grad} f(p)$	The gradient on the hyperbolic space of f, where $\Omega \ni p \mapsto \operatorname{grad} f(p) \in T_p\mathcal{M}$.
$Df(p)$	The usual gradient of f at $p \in \Omega$.
$X(p)$	A vector field on $\Omega \subseteq \mathbb{H}^n$.
$\nabla X(p)$	The covariant derivative of X at $p \in \Omega$.
$DX(p)$	The usual derivative of the vector field X at the point p.
$\operatorname{Hess} f(p)$	The Hessian on the hyperbolic space.
$D^2 f(p)$	The usual Hessian (Euclidean Hessian) of the function f at a point p.
P_{pq}	The parallel transport along the geodesic segment γ joining p to q.

$B_\delta(p)$	The open ball of radius $\delta > 0$ and centre $p \in \mathbb{H}^n$.
$\hat{B}_\delta(p)$	The closed ball of radius $\delta > 0$ and centre $p \in \mathbb{H}^n$.
$G_{\mathcal{L}}$	$\{Q \in \mathbb{R}^{(n+1)\times(n+1)} : \ Q^\top \mathrm{J} Q = \mathrm{J}\}$, the Lorentz group.
d_q	The intrinsic distance function on the hyperbolic space from the fixed point $q \in \mathbb{H}^n$.
$\operatorname{grad} d_q(p)$	The gradient of the distance from q at p.
$\operatorname{Hess} d_q(p)$	The Hessian of the distance from q at p.
$\mathcal{K}_{\mathcal{A}}$	The cone spanned by $\mathcal{A}$.
$\mathcal{P}_{\mathcal{C}}$	The projection mapping $\mathbb{H}^n \to \mathcal{C}$.
$\mathrm{epi} f$	$\{(p, \mu) : \ p \in \mathcal{C}, \ \mu \in \mathbb{R}, \ f(p) \leq \mu\}$, the epigraph.
$\operatorname{cone}\{\dots\}$	The simplicial cone.
$\Pi_{\mathcal{K}}$	The Euclidean orthogonal projection set-valued mapping into a convex cone $\mathcal{K}$.
$\mathcal{L}_\alpha$	α-circular cone, where $\alpha \geq 1$ is a real constant.
$\mathrm{int}(\mathcal{K})$	The interior of cone $\mathcal{K}$.
$\mathbb{H}^n_\kappa$	The n-dimensional κ-hyperbolic space form for a given $\kappa > 0$.
$T_p\mathbb{H}^n_\kappa$	The tangent hyperplane at a point p of the n-dimensional κ-hyperbolic space form.
Proj^κ_p	The Lorentzian projection into $T_p\mathbb{H}^n_\kappa$.
$\Lambda_{\mathcal{K}}(x)$	The set of L-projections of $x \in \mathbb{R}^{n+1}$ into $\mathcal{K}$.
$\overline{\mathcal{S}}$	The topological closure of a set $\mathcal{S} \subseteq \mathbb{R}^{n+1}$.

CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

The modelling of various classes of constrained optimization problems via Riemannian manifolds has gained increasing attention in the recent years in both the academic and business community as, in some sense, it outperforms the Euclidean counterpart. The development of numerical methods to address constrained optimization problems heavily relies on an understanding of the geometric structure of the constraint set. It is widely recognized that Riemannian manifolds are significant classes of sets with rich intrinsic geometric characteristics and a solid conceptual foundation (see, for instance, [12,43,48]). Many optimization problems formulated in Euclidean spaces have been reformulated as problems on Riemannian manifolds to leverage this knowledge and enhance the effectiveness of numerical optimization methods in tackling these problems (see [1,7]). Another point deserving to be highlighted is that endowing the set of constraints of the problem with a suitable Riemannian metric, it allows us to explore its intrinsics algebraic and geometric structures and then significantly reduce the cost of obtaining solutions of the problem in question. For example, a non-convex Euclidean problem can be seen as a convex Riemannian one, whose optimization methods for solving it have much less inherent computational complexity, see for instance [46, Example 13.4.1] and [17, 23]. It is worth noting that the concepts of convexity of sets and functions in the Riemannian optimization context is a topic that is of interest in itself, see for example [35, 45, 52, 60].

The hyperbolic space was discovered due to attempts to understand Euclid's axiomatic basis for geometry dating back to the 1800s. This space it is one of the most interesting models of non-Euclidean Riemannian manifold of negative constant sectional curvature, see for example [6, 8, 47]. Since the discovery of the hyperbolic space, several efforts have been made to understand its properties and several models of it have emerged over the years, including the hyperboloid model (also called Lorentz model), the Poincaré half-plane model, the Poincaré disc model and the Klein model, see [6]. The growing interest over the years in the hyperbolic space has resulted in a proven success story, helping to make impressive advances in many fields of science, one of the best known being general relativity, see [53]. It is also worth mentioning that in many practical applications the natural structure of the data is modelled in the hyperbolic space. Various topics of research use this type of modelling, including those related to machine learning, artificial intelligence, financial networks, Procrustes problem, and various practical applications (see references [9, 32–34, 39–42, 49–51, 54, 56, 57]).

All the aforementioned constrained problems can be modelled by treating the constraint set as a manifold, thus transforming them into unconstrained problems. However, when additional constraints are introduced within the manifold itself, the problems become intrinsically constrained. In such cases, optimization problems on manifolds with intrinsic constraints become significantly more challenging. One key difficulty arises in computing the intrinsic projection into convex sets of the manifold. This problem was initially addressed in the work of [55], where intriguing theoretical properties were discovered, but it remains a challenging task. While there are some papers that have tackled this problem on certain sets in Euclidean spaces (e.g., [4,18]), we note that the problem of intrinsically projecting into a convex subset of the sphere was also studied in [4, 15], where an explicit formula for the intrinsic projection was reduced to the Euclidean projection on a suitable cone. Additionally, the computational difficulties involved with computing intrinsic projections in convex sets limit the investigation of numerical methods for addressing problems with constraints in Riemannian manifolds. As a matter of fact, there are currently only a few studies that address the gradient projection method in the Riemannian context. However, earlier research has investigated the use of gradient projection for minimizing functions on Riemannian manifolds under convexity or coercivity assumptions, as well as the boundedness of the set of constraints. Notably, even for this method, which plays a fundamental role in constrained optimization, the conducted studies predominantly concentrate on theoretical findings regarding the computational complexity associated with solving problems with objective function convex or coercive, see [11, 14, 31, 38, 59].

The structure of this thesis is as follows. In Chapter 1, we recall some notations, basic results and properties about the geometry of the hyperbolic space used throughout the thesis. In Chapter 2, we present a characterization of convex sets in the hyperbolic space and the properties of the projection onto convex sets. In Chapter 3, we study the basic properties of convex functions on the hyperbolic space, the hyperbolically convex quadratic functions and some concepts of optimization related to hyperbolically convex functions. We conclude this thesis by providing an overview of the geometry of κ-hyperbolic space forms, including notations, definitions, and basic results as well as presenting some properties of the intrinsic κ-projection into κ-hyperbolically convex sets, with a focus on explicit computation of the intrinsic κ-projection for specific convex sets in Chapter 4. Moreover, we also address the constrained optimization problem and introduce two versions of the gradient projection method to solve it and finally explore some properties of the constrained version of the problem of computing the centre of mass in Chapter 4.

1.1 Notations and basic results

In this section we will introduce some basic notations and lemmas used in the thesis. For any real number α denote $\alpha^+ := \max(\alpha, 0)$ and $\alpha^- := (-\alpha)^+$. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. Denote by e^i is the i-th canonical unit vector in $\mathbb{R}^{n+1}$. The Euclidean norm of $u \in \mathbb{R}^m$ is denoted by $\|u\|_2 := \sqrt{u^\top u}$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{R}^m \equiv \mathbb{R}^{m\times 1}$. For $M \in \mathbb{R}^{m\times n}$ the matrix $M^\top \in \mathbb{R}^{n\times m}$ denotes the *transpose* of M. The operator norm associated to Euclidean norm of a matrix $M \in \mathbb{R}^{m\times m}$ is defined by $\|A\|_2 := \max\{\|Mu\|_2 : \ \|u\|_2 = 1, \ u \in \mathbb{R}^m\}$. The numbers $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ stand for the minimum and maximum eigenvalue of the matrix $M \in \mathbb{R}^{m\times m}$, respectively. If $u \in \mathbb{R}^m$, then $\mathrm{diag}(u) \in \mathbb{R}^{m\times m}$ denotes a diagonal matrix with (i,i)-th entry equal to u_i, $i = 1, \ldots, m$. The matrix I denotes the $m \times m$ identity matrix.

In the following we state a version of Finsler's lemma, see [25]. A proof of it can be found, for example, in [37, Theorem 2].

Lemma 1. Let $M, N \in \mathbb{R}^{n\times n}$ be two symmetric matrices with $N \neq 0$. If $x^\top N x =$

0 implies $x^\top Mx \geq 0$, then there exists $\lambda \in \mathbb{R}$ such that $M + \lambda N$ is positive semidefinite.

In order to state a special version of Lemma 1 in a convenient form, we take the diagonal matrix $\mathrm{J} \in \mathbb{R}^{(n+1)\times(n+1)}$ defined by

$$\mathrm{J} := \mathrm{diag}(1, \dots, 1, -1) \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{1.1}$$

By using the matrix (1.1) the Lorentz cone $\mathcal{L}$ and its *boundary* $\partial\mathcal{L}$ are defined, respectively, by

$$\mathcal{L} := \left\{x \in \mathbb{R}^{n+1} : x^\top \mathrm{J}x \leq 0,\ x_{n+1} \geq 0\right\},\ \partial\mathcal{L} := \left\{x \in \mathbb{R}^{n+1} : x^\top \mathrm{J}x = 0,\ x_{n+1} \geq 0\right\}. \tag{1.2}$$

A matrix M is called *$\partial\mathcal{L}$-copositive* if $z^\top Mz \geq 0$, for all $z \in \partial\mathcal{L}$. Then, combining Lemma 1 with the second equality in (1.2) we obtain the following special version of Lemma 1.

Corollary 1. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix. If M is $\partial\mathcal{L}$-copositive, then there exists $\lambda \in \mathbb{R}$ such that $M + \lambda\mathrm{J}$ is positive semidefinite.

The *dual cone* of a cone $\mathcal{K} \subset \mathbb{R}^m$ is a cone defined by $\mathcal{K}^* := \{x \in \mathbb{R}^m\ :\ x^\top y \geq 0,\ \forall y \in \mathcal{K}\}$. It is well known that $\mathcal{L} = \mathcal{L}^* = (\partial\mathcal{L})^*$.

The following characterizations of symmetric positive definite matrices can be found in [30, Theorem 7.2.5, pp. 404].

Lemma 2. Let $M \in \mathbb{R}^{n\times n}$ be symmetric. Then, M is positive definite if and only if $\det M_i > 0$, for $i = 1, \dots, n$, where M_i stands for the principal submatrix M determined by the first i rows and first i columns.

For a given $M \in \mathbb{R}^{(n+1)\times(n+1)}$, consider the following decomposition:

$$M := \begin{pmatrix} \bar{M} & b \\ b^\top & \theta \end{pmatrix}, \qquad \bar{M} \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^{n\times 1}, \quad \theta \in \mathbb{R}. \tag{1.3}$$

Using the decomposition (1.3) we have the following characterizations of positive definite matrices and positive semidefinite matrices, for the proof see [28, Propositions 16.1 and 16.2].

Lemma 3. Let $M = M^\top \in \mathbb{R}^{(n+1)\times(n+1)}$. Consider the decomposition of M in the form (1.3).

(i) M is positive definite if and only if $\theta > 0$ and $\bar{M} - \frac{1}{\theta}bb^\top$ is positive definite;

(ii) If $\theta > 0$, then M is positive semidefinite if and only if $\bar{M} - \frac{1}{\theta}bb^\top$ is positive semidefinite;

(iii) If $\bar{M}$ is positive definite, then M is positive semidefinite if and only if $\theta - b^\top \bar{M}^{-1} b \geq 0$.

Now, assume that $\bar{M}$ is invertible. In this case the determinant of M is given by the formula of the next lemma, see [30, Section 0.85, pp. 21].

Lemma 4. Let $M \in \mathbb{R}^{(n+1)\times(n+1)}$. Consider the decomposition of M in the form (1.3). Then, $\det(M) = (\theta - b^\top \bar{M}^{-1} b) \det \bar{M}$.

Next we present a version of the S-lemma which can be found for example in [44, Theorem 2.2].

Lemma 5. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. Assume that $\hat{x}^\top B \hat{x} < 0$ for some $\hat{x} \in \mathbb{R}^{n\times n}$. Then the following two statements are equivalent.

(i) If $x^\top Bx \le 0$, then $x^\top Ax \ge 0$;

(ii) There exists a $\beta \ge 0$ such that $A + \beta B$ is positive semidefinite.

1.2 Basic results about the hyperbolic space

In this section we recall some notations, definitions and basic properties about the geometry of the hyperbolic space used throughout the thesis. They can be found in many introductory books on Riemannian and Differential Geometry, for example in [6, 47], see also [7].

Let $\langle \cdot, \cdot \rangle$ be the *Lorentzian inner product* of $x := (x_1, \dots, x_n, x_{n+1})^\top$ and $y := (y_1, \dots, y_n, y_{n+1})^\top$ on $\mathbb{R}^{n+1}$ defined as follows

$$\langle x, y \rangle := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}. \tag{1.4}$$

For each $x \in \mathbb{R}^{n+1}$, the *Lorentzian norm (length)* of x is defined to be the complex number

$$\|x\| := \sqrt{\langle x, x \rangle}. \tag{1.5}$$

Here $\|x\|$ is either positive, zero, or positive imaginary.

By using (1.1), the Lorentz inner product (1.4) can be stated equivalently as follows

$$\langle x, y \rangle := x^\top \mathrm{J} y, \qquad \forall x, y \in \mathbb{R}^{n+1}. \tag{1.6}$$

Throughout the thesis the *n-dimensional hyperbolic space* and its *tangent hyper-*

plane at a point p are denoted by

$$\mathbb{H}^n := \left\{p \in \mathbb{R}^{n+1} : \ \langle p, p\rangle = -1, \ p^{n+1} > 0\right\}, \qquad T_p\mathbb{H}^n := \left\{v \in \mathbb{R}^{n+1} \ : \ \langle p, v\rangle = 0\right\}, \tag{1.7}$$

respectively. It is worth noting that the Lorentzian inner product defined in (1.4) is not positive definite in the entire space $\mathbb{R}^{n+1}$. However, one can show that its restriction to the tangent spaces of $\mathbb{H}^n$ is positive definite; see [7, Section 7.6]. Consequently, $\|v\| > 0$ for all $v \in T_p\mathbb{H}^n$ and all $p \in \mathbb{H}^n$ with $v \neq 0$. Therefore, $\langle \cdot, \cdot\rangle$ *and* $\|\cdot\|$ *are in fact a positive inner product and the associated norm in* $T_p\mathbb{H}^n$*, for all* $p \in \mathbb{H}^n$. Next we present a basic lemma used in the sequel.

Lemma 6. Let $p, q \in \mathbb{H}^n$. Then, $\langle p, q\rangle \leq -1$ and $\langle p, q\rangle = -1$ if and only if $p = q$.

Proof. Since $p, q \in \mathbb{H}^n$, we have $\langle p, p\rangle = -1$, $p_{n+1} > 0$, $\langle q, q\rangle = -1$ and $q_{n+1} > 0$. Thus, we have $p_{n+1} = \sqrt{u^\top u}$ and $q_{n+1} = \sqrt{v^\top v}$, where $u = (p_1, \ldots, p_n, 1)^\top$ and $v = (q_1, \ldots, q_n, 1)^\top$. Hence, it follows from (1.4) that $\langle p, q\rangle = p_1q_1 + \cdots + p_nq_n - \sqrt{u^\top u}\sqrt{v^\top v}$. On the other hand, by taking into account that Cauchy's inequality in the Euclidean space implies that $\sqrt{u^\top u}\sqrt{v^\top v} \geq u^\top v$ and the equality holds if and only if $u = v$, the result follows. □

Therefore, (1.4) actually defines a Riemannian metric on $\mathbb{H}^n$, see [8, pp. 67]. The *Lorentzian projection onto the tangent hyperplane* $T_p\mathbb{H}^n$ is the linear mapping defined by

$$\mathrm{I} + pp^\top \mathrm{J} : \mathbb{R}^{n+1} \to T_p\mathbb{H}^n, \qquad \forall p \in \mathbb{H}^n, \tag{1.8}$$

where $\mathrm{I} \in \mathbb{R}^{(n+1)\times(n+1)}$ is the identity matrix.

Remark 1. The Lorentzian projection (1.8) is self-adjoint with respect to the Lorentzian inner product (1.4). Indeed, $\langle (\mathrm{I} + pp^\top \mathrm{J})u, v\rangle = \langle u, (\mathrm{I} + pp^\top \mathrm{J})v\rangle$, for all

$u, v \in \mathbb{R}^{n+1}$ and all $p \in \mathbb{H}^n$. Moreover, we also have $(\mathrm{I}+pp^\top \mathrm{J})(\mathrm{I}+pp^\top \mathrm{J}) = \mathrm{I}+pp^\top \mathrm{J}$, for all $p \in \mathbb{H}^n$.

The *intrinsic distance on the hyperbolic space* between two points $p, q \in \mathbb{H}^n$ is defined by

$$d(p,q) := \operatorname{arcosh}(-\langle p, q\rangle). \tag{1.9}$$

It can be shown that $(\mathbb{H}^n, d)$ is a complete metric space, so that $d(p,q) \geq 0$ for all $p, q \in \mathbb{H}^n$, and $d(p,q) = 0$ if and only if $p = q$. Moreover, $(\mathbb{H}^n, d)$ has the same topology as $\mathbb{R}^n$. The intersection curve of a plane though the origin of $\mathbb{R}^{n+1}$ with $\mathbb{H}^n$ is called a *geodesic.* Moreover, each geodesic segment $\gamma : [a, b] \to \mathbb{H}^n$ is *minimal,* i.e., its arc length is equal to the intrinsic distance $\ell(\gamma) = \operatorname{arcosh}(-\langle \gamma(a), \gamma(b)\rangle)$ between its end points. We say that γ is a *normalized geodesic* if $\|\gamma'\| = 1$. If $p, q \in \mathbb{H}^n$ and $q \neq p$, then the unique *geodesic segment from p to q* is

$$\gamma_{pq}(t) = \left(\cosh t + \frac{\langle p, q\rangle \sinh t}{\sqrt{\langle p, q\rangle^2 - 1}}\right) p + \frac{\sinh t}{\sqrt{\langle p, q\rangle^2 - 1}}\, q, \qquad \forall t \in [0,\ d(p,q)].$$

The *exponential mapping* $\exp_p : T_p\mathbb{H}^n \to \mathbb{H}^n$ is defined by $\exp_p v = \gamma_v(1)$, where γ_v is the geodesic defined by its initial position p, with velocity v at p. Hence, $\exp_p v = p$ for $v = 0$, and

$$\exp_p v := \cosh(\|v\|)\, p + \sinh(\|v\|)\, \frac{v}{\|v\|}, \qquad \forall v \in T_p\mathbb{H}^n \setminus \{0\}.$$

It is easy to prove that $\gamma_{tv}(1) = \gamma_v(t)$ for any values of t. Therefore, for all $t \in \mathbb{R}$ we have

$$\exp_p tv := \cosh(t\|v\|)\, p + \sinh(t\|v\|)\, \frac{v}{\|v\|}, \qquad \forall v \in T_p\mathbb{H}^n \setminus \{0\}. \tag{1.10}$$

We will also use the expression above for denoting the geodesic starting at $p \in \mathbb{H}^n$ with velocity $v \in T_p\mathbb{H}^n$ at p. The *inverse of the exponential mapping* is given by $\log_p q = 0$, for $q = p$, and

$$\log_p q := \frac{\operatorname{arcosh}(-\langle p, q\rangle)}{\sqrt{\langle p, q\rangle^2 - 1}} \left[\mathrm{I} + pp^\top \mathrm{J}\right] q, \qquad q \neq p. \tag{1.11}$$

It follows from (1.9) and (1.11) that

$$d(p, q) = \| \log_p q\|, \qquad p, q \in \mathbb{H}^n.$$

Let $\Omega \subseteq \mathbb{H}^n$ be an open set and $f : \Omega \to \mathbb{R}$ be a differentiable function. The *gradient on the hyperbolic space* of f is the unique vector field $\Omega \ni p \mapsto \operatorname{grad} f(p) \in T_p\mathcal{M}$ such that $df(p)v = \langle \operatorname{grad} f(p), v\rangle$, see [7, Proposition 7-5, p.162]. Therefore, we have

$$\operatorname{grad} f(p) := \left[\mathrm{I} + pp^\top \mathrm{J}\right] \mathrm{J} \cdot Df(p) = \mathrm{J} \cdot Df(p) + \langle \mathrm{J} \cdot Df(p), p\rangle\, p, \tag{1.12}$$

where $Df(p) \in \mathbb{R}^{n+1}$ is the usual gradient of f at $p \in \Omega$. A *vector field* on $\Omega \subseteq \mathbb{H}^n$ is a smooth mapping $X : \Omega \to \mathbb{R}^{n+1}$ such that $X(p) \in T_p\mathbb{H}^n$. The *covariant derivative* of X at $p \in \Omega$ is map $\nabla X(p) : T_p\mathbb{H}^n \to T_p\mathbb{H}^n$ given by

$$\nabla X(p) := \left[\mathrm{I} + pp^\top \mathrm{J}\right] DX(p),$$

where $DX(p)$ denotes the usual derivative of the vector field X at the point p, see [7, Formula (7.62), p.162]. The *Hessian on the hyperbolic space* of a twice differentiable function $f : \Omega \to \mathbb{R}$ at a point $p \in \Omega$ is the mapping $\nabla \operatorname{grad} f(p) :=$

$\operatorname{Hess} f(p) : T_p\mathbb{H}^n \to T_p\mathbb{H}^n$ given by

$$\operatorname{Hess} f(p) := \left[\mathrm{I} + pp^{\top}\mathrm{J}\right]\left[\mathrm{J}\cdot D^2 f(p) + \langle \mathrm{J}\cdot Df(p), p\rangle \mathrm{I}\right], \tag{1.13}$$

where $D^2 f(p)$ is the usual Hessian (Euclidean Hessian) of the function f at a point p, see [7, Proposition 7.6, p.163]. Let $I \subseteq \mathbb{R}$ be an open interval, $\Omega \subseteq \mathbb{H}^n$ an open set and $\gamma : I \to \Omega$ a geodesic segment. Since $f : \mathcal{C} \to \mathbb{R}$ is a differentiable function and $\gamma'(t) \in T_{\gamma(t)}\mathbb{H}^n$ for all $t \in I$, equality (1.12) implies

$$\frac{d}{dt} f(\gamma(t)) = \langle \operatorname{grad} f(\gamma(t)), \gamma'(t)\rangle = \langle \mathrm{J}\cdot Df(\gamma(t)), \gamma'(t)\rangle, \qquad \forall\, t \in I. \tag{1.14}$$

Moreover, if the function f is twice differentiable then it holds that

$$\begin{aligned}\frac{d^2}{dt^2} f(\gamma(t)) &= \langle \operatorname{Hess} f(\gamma(t))\gamma'(t), \gamma'(t)\rangle \\ &= \left\langle \mathrm{J}\cdot D^2 f(\gamma(t))\gamma'(t), \gamma'(t)\right\rangle + \langle \mathrm{J}\cdot Df(\gamma(t)), \gamma(t)\rangle \langle \gamma'(t), \gamma'(t)\rangle, \ \forall\, t \in I.\end{aligned} \tag{1.15}$$

For each $p, q \in \mathbb{H}^n$ the covariant derivative induces the linear isometry relative to the Lorentzian inner product $\langle\cdot,\cdot\rangle$, $P_{pq}\colon T_p\mathbb{H}^n \to T_q\mathbb{H}^n$ defined by $P_{pq}v = V(t)$, where V is the unique vector field on the geodesic segment $\gamma : [0,1] \to \mathbb{H}^n$ from p to q, i.e., $\gamma(0) = p$ and $\gamma(1) = q$ such that $\nabla V(t)\gamma'(t) = \nabla_{\gamma'(t)}V(t) = 0$ and $V(0) = v$, the so-called *parallel transport* along the geodesic segment γ joining p to q. The explicit formula of P_{pq} is given by

$$P_{pq}(v) := v - \frac{\langle v, \log_q p\rangle}{\operatorname{arcosh}^2(-\langle p, q\rangle)}\left(\log_q p + \log_p q\right).$$

By using (1.11), after some algebraic manipulation, the last inequality becomes

$$P_{pq}(v) := \left[\mathrm{I} + \frac{1}{1-\langle p, q\rangle}(p+q)q^\top \mathrm{J}\right] v.$$

Note that for all geodesic segment $\gamma : [a, b] \to \mathbb{H}^n$ we have $\gamma'(t) = P_{pq}(\gamma'(a))$, for all $t \in [a, b]$ or equivalently that $\gamma''(t) = 0$, for all $t \in [a, b]$. Next we give two standard notations. We denote the *open* and *closed ball* of radius $\delta > 0$ and centre $p \in \mathbb{H}^n$ by $B_\delta(p) := \{q \in \mathbb{H}^n : d(p, q) < \delta\}$ and $\hat{B}_\delta(p) := \{q \in \mathbb{H}^n : d(p, q) \le \delta\}$, respectively.

Let us recall the *Lorentz group* $G_{\mathcal{L}}$, preserving the norm (1.5) and metric (1.6), defined by

$$G_{\mathcal{L}} := \left\{Q \in \mathbb{R}^{(n+1)\times(n+1)} :\ Q^\top \mathrm{J} Q = \mathrm{J}\right\}. \tag{1.16}$$

Note that $|\det Q| = 1$, for all $Q \in G_{\mathcal{L}}$. Moreover, $Q^{-1}, Q^\top \in G_{\mathcal{L}}$, for all $Q \in G_{\mathcal{L}}$.

Example 1. We exhibit two examples of matrices $Q \in G_{\mathcal{L}}$. For the first example, take $u \in \mathbb{R}^{n+1}$ such that $\|u\| > 0$. Thus, $Q = \mathrm{I} - (2/\|u\|^2)uu^\top \mathrm{J} \in G_{\mathcal{L}}$. For the second example, take $u, w \in \mathbb{R}^{n+1}$ such that $\|u\| = 1$ and $\|w\| = 1$. Therefore, $Q = \mathrm{I} + 2wu^\top \mathrm{J} - (1/(1 + u^\top \mathrm{J} w))(u + w)(u + w)^\top \mathrm{J} \in G_{\mathcal{L}}$.

We end this section by remarking that the Lorentz group (1.16) preserves geodesics of $\mathbb{H}^n$.

Remark 2. First note that for a given $Q \in G_{\mathcal{L}}$, we have $\|Qv\| = \|v\|$. Moreover, $p \in \mathbb{H}^n$ and $v \in T_p\mathbb{H}^n$ if and only if $Qp \in \mathbb{H}^n$ and $Qv \in T_p\mathbb{H}^n$, i.e., $v \in T_p\mathbb{H}^n \setminus \{0\}$ if and only if $Qv \in T_{Qp}\mathbb{H}^n \setminus \{0\}$. Consequently, it follows from (1.10) that

$$Q \exp_p tv = \exp_{Qp} tQv, \qquad \forall v \in T_p\mathbb{H}^n \setminus \{0\}.$$

Therefore, the Lorentz group preserves the geodesics of $\mathbb{H}^n$.

1.3 Properties of the intrinsic distance on the hyperbolic space

In this section we present some important properties of the intrinsic distance from a fixed point on the hyperbolic space. In particular, we present the spectral decomposition of the Hessian of the intrinsic distance. The *intrinsic distance function on the hyperbolic space from the fixed point* $q \in \mathbb{H}^n$ is the mapping $d_q : \mathbb{H}^n \to \mathbb{R}$ defined by

$$d_q(p) := \operatorname{arcosh}(-\langle p, q\rangle). \tag{1.17}$$

The intrinsic distance from q, denoted by d_q , is twice differentiable at $p \in \mathbb{H}^n \backslash \{q\}$. By combining (1.12) and (1.17), we can see that the *gradient of the distance from* q *at* p is given by

$$\operatorname{grad} d_q(p) := \frac{-1}{\sqrt{\langle p, q\rangle^2 - 1}} \left[\mathrm{I} + pp^\top \mathrm{J}\right] q, \qquad q \neq p. \tag{1.18}$$

Moreover, by using (1.13) and (1.17), we obtain that the *Hessian of the distance from* q at p is given by

$$\operatorname{Hess} d_q(p) := \frac{\langle p, q\rangle}{\sqrt{\langle p, q\rangle^2 - 1}} \left[\mathrm{I} + pp^\top \mathrm{J}\right] \left[\frac{1}{\langle p, q\rangle^2 - 1} qq^\top \mathrm{J} - \mathrm{I}\right], \qquad q \neq p. \tag{1.19}$$

Before presenting the spectral decomposition of the Hessian of intrinsic distance from a fixed point on the hyperbolic space, we need the following elementary result.

Lemma 7. Let $p, q \in \mathbb{H}^n$ with $q \neq p$. Then $\dim\left(T_p\mathbb{H}^n \cap T_q\mathbb{H}^n\right) = n - 1$ and $\langle q +$

$\langle p, q\rangle p, v\rangle = 0$, for all $v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$. As a consequence, taking an orthonormal basis of the subspace $T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$, say $\{v_1, \ldots, v_{n-1}\}$ and defining $v_n = (q + \langle p, q\rangle p)/\|q + \langle p, q\rangle p\|$, the set $\{v_1, \ldots, v_{n-1}, v_n\}$ is an orthonormal basis of $T_p\mathbb{H}^n$.

In the next lemma we present a spectral decomposition of the Hessian of the intrinsic distance from a fixed point on the hyperbolic space. The results in this lemma and the next one are closely related to [24], see also its counterparts on the sphere in [16, Lemma 2, Lemma 3].

Lemma 8. Take $q \in \mathbb{H}^n$ and let $\operatorname{Hess} d_q(p) : T_p\mathbb{H}^n \to T_p\mathbb{H}^n$ be the Hessian of the intrinsic distance from q at the point $p \in \mathbb{H}^n \backslash \{q\}$. Then,

$$\operatorname{Hess} d_q(p)\ (q + \langle p, q\rangle p) = 0, \quad \operatorname{Hess} d_q(p)\, v = \frac{-\langle p, q\rangle}{\sqrt{\langle p, q\rangle^2 - 1}}\, v, \quad \forall\, v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n. \tag{1.20}$$

Moreover, $\lambda_1 = 0$ and $\lambda_2 = -\langle p, q\rangle/\sqrt{\langle p, q\rangle^2 - 1} > 0$ are the unique eigenvalues of $\operatorname{Hess} d_q(p)$, with algebraic multiplicity 1 and $n-1$, respectively. Moreover, the Hessian $\operatorname{Hess} d_q(p)$ is positive semidefinite.

Proof. Since $p \neq q$, Lemma 6 implies that $\langle p, q\rangle \neq -1$ and from (1.19) the Hessian is well defined. As $q^\top \mathrm{J} q = -1$, simple calculations give

$$\left[\frac{1}{\langle p, q\rangle^2 - 1}\, q q^\top \mathrm{J} - \mathrm{I}\right] (q + \langle p, q\rangle p) = -\langle p, q\rangle p.$$

On the other hand, $[\mathrm{I} + p p^\top \mathrm{J}](-\langle p, q\rangle p) = 0$, which combined with the latter equality and (1.19), implies the first equality in (1.20), and we also have that $\lambda_1 = 0$ is an eigenvalue of the Hessian. For proving the second equality in (1.20), note that the definitions in (1.7) imply that

$$\langle p, v\rangle = 0, \qquad \langle q, v\rangle = 0, \qquad \forall\, v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n.$$

Thus, the second inequality in (1.20) follows from (1.19) and the last two equalities. In particular, the Hessian is a multiple of the identity in the subspace $T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$. Moreover, due to $\dim T_p\mathbb{H}^n = n$, we conclude, using Lemma 7, that the eigenvalues λ_1 and λ_2 have algebraic multiplicity 1 and $n-1$, respectively, proving the first statement. For proving the second statement, let $\{v_1, \ldots, v_{n-1}\}$ be an orthonormal basis of the subspace $T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$. Since $\langle p, q\rangle \neq 1$, we can define $v_n = (q + \langle p, q\rangle p)/\|q + \langle p, q\rangle p\|$. Hence, Lemma 7 implies that $\{v_1, \ldots, v_{n-1}, v_n\}$ is an orthonormal basis of $T_p\mathbb{H}^n$. Therefore, given $u \in T_p\mathbb{H}^n$, there exist $a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ such that $u = a_1v_1 + \cdots + a_{n-1}v_{n-1} + a_nv_n$, which, by using the first statement, entails $\langle \operatorname{Hess} d_q(p)u, u\rangle = \lambda_2(a_1^2 + \cdots + a_{n-1}^2)$, completing the proof of the second statement. □

Take $q \in \mathbb{H}^n$ and define $\rho_q : \mathbb{H}^n \to \mathbb{R}$ as

$$\rho_q(p) := \frac{1}{2}d_q^2(p). \tag{1.21}$$

By using the definition of ρ_q in (1.21) and (1.13), it is easy to conclude, after some algebra, that

$$\operatorname{Hess} \rho_q(p) = d_q(p) \operatorname{Hess} d_q(p) + [\mathrm{I} + pp^\top \mathrm{J}]\mathrm{J} \cdot Dd_q(p)Dd_q(p)^\top, \tag{1.22}$$

where $Dd_q(p)$ is the usual derivative of d_q at the point p.

Lemma 9. Take $q \in \mathbb{H}^n$ and define $\operatorname{Hess}\rho_q(p) : T_p\mathbb{H}^n \to T_p\mathbb{H}^n$ as the Hessian of ρ_q at the point $p \in \mathbb{H}^n \backslash \{q\}$. Then the following equalities hold:

$$\operatorname{Hess}\rho_q(p)\ (q + \langle p, q\rangle p) = q + \langle p, q\rangle p, \quad \operatorname{Hess}\rho_q(p)\, v = \frac{-\langle p, q\rangle \operatorname{arcosh}(-\langle p, q\rangle)}{\sqrt{\langle p, q\rangle^2 - 1}}\, v, \tag{1.23}$$

for all $v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$. As a consequence, $\mu_1 = 1$ and $\mu_2 = \frac{\langle p,q\rangle \operatorname{arcosh}(-\langle p,q\rangle)}{\sqrt{\langle p,q\rangle^2 - 1}}$ are the unique eigenvalues of $\operatorname{Hess}\rho_q(p)$, with algebraic multiplicity 1 and $n-1$, respectively. Moreover, the Hessian $\operatorname{Hess}\rho_q(p)$ is positive definite.

Proof. First note that taking into account that $Dd_q(p) = -(\mathrm{J}q)/\sqrt{\langle p, q\rangle^2 - 1}$, we have

$$\mathrm{J} \cdot Dd_q(p) Dd_q(p)^\top = \frac{1}{\langle p, q\rangle^2 - 1} q q^\top \mathrm{J}. \tag{1.24}$$

Due to $q^\top \mathrm{J} q = -1$, it follows from the last equality that $\mathrm{J} \cdot Dd_q(p) Dd_q(p)^\top (q + \langle p, q\rangle p) = q$. On the other hand, $[\mathrm{I} + pp^\top \mathrm{J}]q = q + \langle p, q\rangle p$. Hence, we obtain that

$$[\mathrm{I} + pp^\top \mathrm{J}]\mathrm{J} \cdot Dd_q(p) Dd_q(p)^\top (q + \langle p, q\rangle p) = q + \langle p, q\rangle p.$$

Therefore, combining the last equality, equation (1.22) and the first equality in (1.20), we get that

$$\operatorname{Hess}\rho_q(p)(q + \langle p, q\rangle p) = q + \langle p, q\rangle p,$$

which is the first equality in (1.23). For proving the second one, note first that the definition of $T_q\mathbb{H}^n$ implies that $q^\top \mathrm{J} v = 0$ for all $v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$. Then, by using

(1.24), we have

$$[\mathrm{I} + pp^\top \mathrm{J}]Dd_q(p)Dd_q(p)^\top v = 0, \qquad \forall\, v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n.$$

Hence, equation (1.22) implies that $\operatorname{Hess}\rho_q(p)v = d_q(p)\operatorname{Hess} d_q(p)v$ for all $v \in T_p\mathbb{H}^n \cap T_q\mathbb{H}^n$. Thus, by using the second equality in (1.20) and the definition of $d_q(p)$ in (1.17), the second equality in (1.23) follows. The remainder of our proof requires similar arguments to those in the proof of Lemma 8 (note that in the final part of the proof we must invoke the fact that $\operatorname{arcosh}(-\langle p, q\rangle) > 0$ and $\langle p, q\rangle < 0$, which holds by applying Lemma 6 together with $p \neq q$). $\square$

1.4 Conclusion

In this chapter, we reviewed the notations, definitions, and fundamental properties of hyperbolic space geometry, which are used throughout the thesis. In particular, we explored certain properties of the intrinsic distance from a fixed point. This review provided a solid foundation for the research presented in the following chapters.

CHAPTER 2

CONVEX SETS ON THE HYPERBOLIC SPACE

In this chapter[1] we first study some concepts related to the convexity of sets on the hyperbolic space. Although some of these concepts have already been studied in general Riemannian manifolds, we revisit them in this specific context in order to present some explicit formulas and new properties. For our study, we selected the hyperboloid model (also known as the Lorentz model) from among the existing hyperbolic geometry models (see [6,8]), as it appears to be better suited for studying Riemannian optimization problems and provides more numerical stability to solution methods (see [41, 42]). Some important properties of the intrinsic distance are studied, for instance, we present the spectral decomposition of its Hessian. We also study the concept of convex sets and the intrinsic projection onto these sets.

[1]The main result in this chapter has been published in the paper [20], which is joint work with Ferreira, O.P.and Németh, S.Z.

2.1 Preliminaries for hyperbolically convex

In this section we present some properties of the convex sets of the hyperbolic space. It is worth to remark that the convex sets on the hyperbolic space $\mathbb{H}^n$ are closely related to convex cones belonging to the interior of the Lorentz cone

$$\mathcal{L} := \left\{ x \in \mathbb{R}^{n+1} : x_{n+1} \geq \sqrt{x_1^2 + \cdots + x_n^2} \right\}. \tag{2.1}$$

Definition 1. The set $\mathcal{C} \subseteq \mathbb{H}^n$ is said to be *hyperbolically convex* if for any p, $q \in \mathcal{C}$ the geodesic segment joining p to q is contained in $\mathcal{C}$.

For each set $\mathcal{A} \subseteq \mathbb{H}^n$, let $\mathcal{K}_{\mathcal{A}}$ be the *cone spanned by* $\mathcal{A}$, namely,

$$\mathcal{K}_{\mathcal{A}} := \{ tp \, : \, p \in \mathcal{A}, \ t \in [0, +\infty) \}. \tag{2.2}$$

Clearly, $\mathcal{K}_{\mathcal{A}}$ is the smallest cone which contains $\mathcal{A}$ and belongs to the interior of the Lorentz cone $\mathcal{L}$. In the next result we relate a hyperbolically convex set with the cone spanned by it.

Proposition 1. The set $\mathcal{C} \subseteq \mathbb{H}^n$ is hyperbolically convex if and only if the cone $\mathcal{K}_{\mathcal{C}}$ is convex.

Proof. Assume that $\mathcal{C} \subseteq \mathbb{H}^n$ is a hyperbolically convex set. Let $x, y \in \mathcal{K}_{\mathcal{C}}$. For proving that $\mathcal{K}_{\mathcal{C}}$ is convex, it suffices to show that

$$z = x + y \in \mathcal{K}_{\mathcal{C}}. \tag{2.3}$$

The definition of $\mathcal{K}_{\mathcal{C}}$ implies that there exist $p, q \in \mathcal{C}$ and $s, t \in [0, +\infty)$ such that $x = sp$ and $y = tq$. Hence, due to $z = sp + tq$ with $p, q \in \mathcal{C} \subseteq \mathbb{H}^n$, (2.3) and $\langle p, q \rangle \leq$

-1 imply that $\langle z, z\rangle \leq -(t+s)^2$, which is equivalent to $0 < t+s \leq \sqrt{-\langle z, z\rangle}$. Now we take

$$\gamma_{pq}(t) = \Big(\cosh t + \frac{\langle p, q\rangle \sinh t}{\sqrt{\langle p, q\rangle^2 - 1}}\Big)p + \frac{\sinh t}{\sqrt{\langle p, q\rangle^2 - 1}}\, q, \qquad \forall t \in [0,\ d(p, q)]. \quad (2.4)$$

the normalized segment of geodesic from p to q. To proceed we first need to prove that

$$d\,(p, az) \leq d(p, q), \qquad \gamma_{pq}\,(d(p, az)) = az, \tag{2.5}$$

where $a := 1/\sqrt{-\langle z, z\rangle}$. Since the function $[0, +\infty] \ni \tau \mapsto \operatorname{arcosh}(\tau)$ is increasing, it follows from (1.9) that to prove the inequality in (2.5) it suffices to show that $-\langle p, az\rangle \leq -\langle p, q\rangle$ or the equivalent inequality

$$0 \leq \langle p, az\rangle - \langle p, q\rangle. \tag{2.6}$$

Due to $z = sp + tq$ and $\langle p, p\rangle = -1$, direct calculations yield $\langle p, az\rangle - \langle p, q\rangle = a(-s + t\langle p, q\rangle) - \langle p, q\rangle$. Thus, taking into account that $\langle p, q\rangle \leq -1$ and $s + t \leq \sqrt{-\langle z, z\rangle} = 1/a$, we conclude that

$$\langle p, az\rangle - \langle p, q\rangle \geq a(s\langle p, q\rangle + t\langle p, q\rangle) - \langle p, q\rangle = a(-\langle p, q\rangle)(1/a - s - t) \geq 0,$$

which implies that (2.6) holds and consequently the inequality in (2.5) also holds. Our next task is to prove the equality in (2.5). Thus, by using (2.4), we have to show that

$$\gamma_{pq}(d(p, az)) = \Big(\cosh d(p, az) + \frac{\langle p, q\rangle \sinh d(p, az)}{\sqrt{\langle p, q\rangle^2 - 1}}\Big)p + \frac{\sinh d(p, az)}{\sqrt{\langle p, q\rangle^2 - 1}}\, q = az. \tag{2.7}$$

It follows from (1.9) that $\cosh d(p, az) = -a\langle p, z\rangle$, which implies that $\sinh d(p, az) = \sqrt{a^2\langle p, z\rangle^2 - 1}$. Thus, considering that $a^2\langle z, z\rangle = -1$ and $z = sp + tq$ with $p, q \in \mathcal{C} \subseteq \mathbb{H}^n$ and $t \in [0, +\infty)$, we have

$$\sinh d(p, az) = \sqrt{a^2\langle p, z\rangle^2 - 1} = a\sqrt{\langle p, z\rangle^2 + \langle z, z\rangle} = at\sqrt{\langle p, q\rangle^2 - 1} \tag{2.8}$$

and $\cosh d(p, az) = -a\langle p, z\rangle = as - at\langle p, q\rangle$. Hence, substituting the last equality and (2.8) into (2.7) and taking into account that $z = sp + tq$, we obtain that

$$\begin{aligned}\gamma_{pq}(d(p, az)) &= \left(as - at\langle p, q\rangle + \frac{\langle p, q\rangle at\sqrt{\langle p, q\rangle^2 - 1}}{\sqrt{\langle p, q\rangle^2 - 1}}\right)p + \frac{at\sqrt{\langle p, q\rangle^2 - 1}}{\sqrt{\langle p, q\rangle^2 - 1}}q \\ &= asp + atq = az.\end{aligned}$$

which concludes the proof of the equality in (2.5). Since $\mathcal{C}$ is a hyperbolically convex set and $d(p, az) \le d(p, q)$ we obtain that $\gamma_{pq}(d(p, az)) \in \mathcal{C}$, which together with (2.2) and (2.5) implies that $z = (1/a)\gamma_{pq}(d(p, az)) \in \mathcal{K}_{\mathcal{C}}$. Thus, $\mathcal{K}_{\mathcal{C}}$ is convex.

Now, assume that the cone $\mathcal{K}_{\mathcal{C}}$ is convex. First note that $\mathcal{C} = \mathcal{K}_{\mathcal{C}} \cap \mathbb{H}^n$. Take $p, q \in \mathcal{C}$ with $q \ne p$. We must prove that the geodesic segment from p to q is contained in $\mathcal{C}$. As $p, q \in \mathcal{K}_{\mathcal{C}}$ and $\mathcal{K}_{\mathcal{C}} \subseteq \mathcal{L}$, we conclude that $q \ne -p$. Thus, $\langle p, q\rangle < -1$ and $d(p, q) > 0$. Let

$$[0,\ d(p, q)] \ni t \mapsto \gamma_{pq}(t) = \alpha(t)p + \beta(t)q$$

be the normalized geodesic segment from p to q, where

$$\alpha(t) := \cosh t + \frac{\langle p, q\rangle \sinh t}{\sqrt{\langle p, q\rangle^2 - 1}}, \qquad \beta(t) := \frac{\sinh t}{\sqrt{\langle p, q\rangle^2 - 1}}.$$

Since $\gamma_{pq}(t) \in \mathbb{H}^n$, $p, q \in \mathcal{K}_{\mathcal{C}}$ and $\mathcal{K}_{\mathcal{C}}$ is a convex cone, for proving that $\gamma_{pq}(t) \in \mathcal{C}$ for all $t \in [0,\ d(p,q)]$, it suffices to prove that $\alpha(t) \geq 0$ and $\beta(t) \geq 0$ for all $t \in [0,\ d(p,q)]$. Due to $\sinh t \geq 0$ for all $t \geq 0$ we conclude that $\beta(t) \geq 0$ for all $t \in [0,\ d(p,q)]$. We proceed to prove that $\alpha(t) \geq 0$ for all $t \in [0,\ d(p,q)]$. For that, we first note that due to hyperbolic tangent being an increasing function, $\cosh d(p,q) = -\langle p, q\rangle$ and $\sinh d(p,p) = \sqrt{\langle p, q\rangle^2 - 1}$, we have

$$\tanh t \leq \tanh d(p,q) = \frac{\sinh d(p,p)}{\cosh d(p,q)} = \frac{\sqrt{\langle p, q\rangle^2 - 1}}{-\langle p, q\rangle}, \qquad t \in [0,\ d(p,q)].$$

Hence, taking into account that $\cosh t \geq 0$ for all $t \in \mathbb{R}$ and $\langle p, q\rangle < -1$, we conclude that

$$\alpha(t) = \cosh t \left(1 + \frac{\langle p, q\rangle}{\sqrt{\langle p, q\rangle^2 - 1}} \tanh t\right) \geq 0, \qquad t \in [0,\ d(p,q)],$$

which completes the proof.

$\square$

Remark 3. The hyperbolically convex sets are intersections of the hyperboloid with convex cones which belong to the interior of $\mathcal{L}$. Indeed, it follows easily from Proposition 1, that if $\mathcal{K} \subseteq \operatorname{int}(\mathcal{L})$ is a convex cone, where $\mathcal{L}$ is the Lorentz cone, then $\mathcal{C} = \mathcal{K} \cap \mathbb{H}^n$ is a hyperbolically convex set and $\mathcal{K} = \mathcal{K}_{\mathcal{C}}$.

Remark 4. Let $\mathcal{C} \subseteq \mathbb{H}^n$ and $Q \in G_{\mathcal{L}}$. First note that $Q\mathcal{C} := \{Qp :\ p \in \mathbb{H}^n\}$. It follows from Remark 2 and Definition 1 that $\mathcal{C}$ is hyperbolically convex if and only if $Q\mathcal{C}$ is hyperbolically convex.

2.2 Intrinsic projection onto hyperbolically convex sets

In this section we present some properties of the intrinsic projection onto hyperbolically convex sets on hyperbolic spaces. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a closed hyperbolically convex set and $p \in \mathbb{H}^n$. Consider the following constrained optimization problem

$$\min_{q \in \mathcal{C}} d(p, q). \tag{2.9}$$

The minimal value of the function $\mathcal{C} \ni q \mapsto d(p, q)$ is called the *distance of p from* $\mathcal{C}$ and it is denoted by $d_{\mathcal{C}}(p)$, i.e., $d_{\mathcal{C}} : \mathbb{H}^n \to \mathbb{R}$ is defined by

$$d_{\mathcal{C}}(p) := \min_{q \in \mathcal{C}} d(p, q).$$

Since $(\mathbb{H}^n, d)$ is a complete metric space, we have the following results.

Proposition 2. Let $\mathcal{C} \in \mathbb{H}^n$ be a nonempty subset. Then, $|d_{\mathcal{C}}(p) - d_{\mathcal{C}}(q)| \leq d(p, q)$, for all $p, q \in \mathbb{H}^n$. In particular, the function $d_{\mathcal{C}}$ is continuous.

Note that due to $\mathcal{C}$ being a closed set and the distance function continuous, the problem (2.9) has a solution. The solution of the problem (2.9) is called *metric projection*, it was first studied in [55]. In the next proposition we explicitly give an important property of the metric projection.

Proposition 3. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a closed hyperbolically convex set and $p \in \mathbb{H}^n$. A point $y^p \in \mathcal{C}$ is a solution of the problem (2.9) if and only if

$$\left\langle \left(\mathrm{I} + y^p (y^p)^\top \mathrm{J}\right) p, \left(\mathrm{I} + y^p (y^p)^\top \mathrm{J}\right) q \right\rangle \leq 0, \qquad \forall q \in \mathcal{C}. \tag{2.10}$$

Furthermore, the solution of problem (2.9) is unique.

Proof. First we assume that y^p is a solution of (2.9). If $p \in \mathcal{C}$ i.e., $p = y^p$, then the inequality trivially holds. Assume that $p \notin \mathcal{C}$, i.e., $p \neq y^p$. Take $q \in \mathcal{C}$ such that $q \neq y^p$ and

$$[0,1] \ni t \mapsto \exp_{y^p}(t \, \log_{y^p} q) = \cosh(td(y^p, q))y^p + \frac{\sinh(td(y^p, q))}{d(y^p, q)} \, \log_{y^p} q \qquad (2.11)$$

be the geodesic segment from y^p to q. Thus, due to $\mathcal{C}$ being a hyperbolically convex set, it follows that $d(p, y^p) \leq d(p, \exp_{y^p}(t \, \log_{y^p} q))$, for all $t \in [0,1]$. Hence, using (1.9) and (2.11), we conclude that

$$\operatorname{arcosh}(-\langle p, y^p \rangle) \leq \operatorname{arcosh}\Big(- \Big\langle p, \cosh(td(y^p, q))y^p + \frac{\sinh(td(y^p, q))}{d(y^p, q)} \, \log_{y^p} q \Big\rangle \Big),$$

for all $t \in [0,1]$. Since $1 \leq -\langle p, y^p \rangle$, for all $p \in \mathcal{C}$, and the function $[1, +\infty] \ni s \mapsto \operatorname{arcosh}(s)$ is increasing, we obtain from (2.11) that

$$\Big\langle p \, , \, \cosh(td(y^p, q))y^p + \frac{\sinh(td(y^p, q))}{d(y^p, q)} \, \log_{y^p} q \Big\rangle \leq \langle p, y^p \rangle, \qquad \forall \, t \in [0,1].$$

After some algebra, we conclude from the previous inequality that

$$\frac{\sinh(t(y^p, q))}{td(y^p, q)} \, \big\langle p, \log_{y^p} q \big\rangle \leq \frac{1 - \cosh(td(y^p, q))}{td(y^p, q)} \, d(y^p, q) \, \langle p, y^p \rangle, \qquad \forall \, t \in [0,1].$$

Letting t go to zero in the last inequality we have $\langle p, \log_{y^p} q \rangle \leq 0$, which, in view of (1.11), yields

$$\frac{\operatorname{arcosh}(-\langle y^p, q \rangle)}{\sqrt{\langle y^p, q \rangle^2 - 1}} \, \big\langle p, \big(\mathrm{I} + y^p (y^p)^\top \mathrm{J}\big) \, q \big\rangle \leq 0.$$

Thus, due to $\operatorname{arcosh}(-\langle y^p, q\rangle) > 0$, we have $\left\langle p, \left(\mathrm{I} + y^p (y^p)^\top \mathrm{J}\right) q \right\rangle \leq 0$. Since

$$\left\langle y^p (y^p)^\top \mathrm{J} p, \left(\mathrm{I} + y^p (y^p)^\top \mathrm{J}\right) q \right\rangle = 0,$$

the desired inequality (2.10) follows. To establish the converse we assume that y^p satisfies (2.10). Direct computations show that (2.10) is equivalent to

$$\langle p, q\rangle + \langle p, y^p\rangle\langle y^p, q\rangle \leq 0, \qquad \forall\, q \in \mathcal{C}. \tag{2.12}$$

Since $\langle y^p, q\rangle \leq -1$ and $\langle p, y^p\rangle \leq -1$, we have $\langle p, y^p\rangle\langle y^p, q\rangle \geq -\langle p, y^p\rangle$. Thus, (2.12) implies that

$$1 \leq -\langle p, y^p\rangle \leq -\langle p, q\rangle, \qquad \forall\, q \in \mathcal{C}.$$

Due to the function $[0, +\infty] \ni t \mapsto \operatorname{arcosh}(t)$ being increasing, the last inequality implies that

$$\operatorname{arcosh}(-\langle p, y^p\rangle) \leq \operatorname{arcosh}(-\langle p, q\rangle), \qquad \forall q \in \mathcal{C},$$

or equivalently that $d(p, y^p) \leq d(p, q)$, for all $q \in \mathcal{C}$. Therefore, y^p is a solution of (2.9) and the converse is proved. For the uniqueness, let $p, \hat{p} \in \mathcal{P}_{\mathcal{C}}(p)$. Since $y^p, \hat{y}^p \in \mathcal{C}$ and $\langle y^p, \hat{y}^p\rangle \leq -1$ (see Lemma 6) , by the first statement, it follows from the equivalence of (2.10) and (2.12) that

$$\langle p, \hat{y}^p\rangle \leq -\langle p, \hat{y}^p\rangle\langle y^p, \hat{y}^p\rangle = \langle p, \hat{y}^p\rangle|\langle y^p, \hat{y}^p\rangle|,$$

which implies that $\langle p, \hat{y}^p\rangle \leq \langle p, \hat{y}^p\rangle\langle y^p, \hat{y}^p\rangle^2$. Due to $\langle p, \hat{y}^p\rangle \leq -1$, we obtain that $1 \geq \langle \hat{y}^p, \hat{y}^p\rangle^2$. Hence, taking into account that $\langle \hat{y}^p, \hat{y}^p\rangle \leq -1$, we conclude that

$\langle y^p, \hat{y}^p \rangle = -1$. Therefore, from Lemma 6 we conclude that $y^p = \hat{y}^p$ and the solution set of the problem (2.9) is a singleton set, which concludes the proof.

$\square$

It follows from Proposition 3 that the *projection mapping* $\mathcal{P}_{\mathcal{C}} : \mathbb{H}^n \to \mathcal{C}$ given by

$$\mathcal{P}_{\mathcal{C}}(p) := \arg\min_{q \in \mathcal{C}} d(p, q) \tag{2.13}$$

is well defined. Moreover, (2.13) is equivalent to the following inequality

$$\left\langle \left(\mathrm{I} + \mathcal{P}_{\mathcal{C}}(p)\mathcal{P}_{\mathcal{C}}(p)^{\top}\mathrm{J}\right) p, \left(\mathrm{I} + \mathcal{P}_{\mathcal{C}}(p)\mathcal{P}_{\mathcal{C}}(p)^{\top}\mathrm{J}\right) q \right\rangle \le 0, \qquad \forall q \in \mathcal{C},\ \forall p \in \mathbb{H}^n. \tag{2.14}$$

Considering that Lemma 6 implies that for all $p, q \in \mathbb{H}^n$ we have $\langle \mathcal{P}_{\mathcal{C}}(p), q \rangle \le -1$, we conclude from (1.11) that (2.14) can be equivalently stated as follows

$$\langle \log_{\mathcal{P}_{\mathcal{C}}(p)} p, \log_{\mathcal{P}_{\mathcal{C}}(p)} q \rangle \le 0, \qquad \forall q \in \mathcal{C},\ \forall p \in \mathbb{H}^n, \tag{2.15}$$

see [22, Corollary 3.1]. Furthermore, since the function $[0, +\infty] \ni \tau \mapsto \operatorname{arcosh}(\tau)$ is increasing, it follows from (1.9) that (2.13), (2.14) and (2.15) are also equivalent to

$$\mathcal{P}_{\mathcal{C}}(p) := \operatorname{argmin}_{q \in \mathcal{C}}(-\langle p, q \rangle). \tag{2.16}$$

An immediate consequence of (2.16) is the monotonicity of the projection mapping, stated as follows:

Proposition 4. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a nonempty closed hyperbolically convex set. Then

$$\langle \mathcal{P}_{\mathcal{C}}(p) - \mathcal{P}_{\mathcal{C}}(q), p - q\rangle \geq 0, \qquad \forall\, p, q \in \mathcal{C}.$$

Proof. Take $p, q \in \mathbb{H}^n$. Since $\mathcal{P}_{\mathcal{C}}(p)$, $\mathcal{P}_{\mathcal{C}}(q) \in \mathcal{C}$, it follows from (2.16) that $-\langle p, \mathcal{P}_{\mathcal{C}}(p)\rangle \leq -\langle p, \mathcal{P}_{\mathcal{C}}(q)\rangle$ and $-\langle q, \mathcal{P}_{\mathcal{C}}(q)\rangle \leq -\langle q, \mathcal{P}_{\mathcal{C}}(p)\rangle$. Hence, $\langle p, \mathcal{P}_{\mathcal{C}}(p) - \mathcal{P}_{\mathcal{C}}(q)\rangle \geq 0$ and $\langle -q, \mathcal{P}_{\mathcal{C}}(p) - \mathcal{P}_{\mathcal{C}}(q)\rangle \geq 0$. Therefore, summing the last two inequalities the desired inequality follows.

□

Proposition 5. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a nonempty closed hyperbolically convex set. Then $P_{\mathcal{C}}$ is continuous.

Proof. Let $\{p^k\} \subseteq \mathbb{H}^n$ be such that $\lim_{k\to+\infty} p^k = p$. Since Proposition 2 implies that $d_{\mathcal{C}}$ is continuous and taking into account that (2.13) implies

$$d\left(p^k, P_{\mathcal{C}}\left(p^k\right)\right) = d_{\mathcal{C}}\left(p^k\right), \tag{2.17}$$

we conclude that $\left(d\left(p^k, P_{\mathcal{C}}\left(p^k\right)\right)\right)_{k\in\mathbb{N}}$ is a bounded sequence. Consequently, considering that

$$d\left(p, P_{\mathcal{C}}\left(p^k\right)\right) \leq d\left(p, p^k\right) + d\left(p^k, P_{\mathcal{C}}\left(p^k\right)\right),$$

we also have that $\left(P_{\mathcal{C}}\left(p^k\right)\right)_{k\in\mathbb{N}}$ is also bounded. Let $q \in \mathcal{C}$ be a cluster point of $\left(P_{\mathcal{C}}\left(p^k\right)\right)_{k\in\mathbb{N}}$ and let $\left(p^{k_j}\right)_{j\in\mathbb{N}}$be such that $\lim_{j\to+\infty} P_{\mathcal{C}}\left(p^{k_j}\right) = q$. Hence, using (2.17) we have $d_{\mathcal{C}}\left(p^{k_j}\right) = d\left(p^{k_j}, P_{\mathcal{C}}\left(p^{k_j}\right)\right)$, for all $j \in \mathbb{N}$. Thus, letting j goes to $+\infty$ and using Proposition 2 we have $d_{\mathcal{C}}(p) = d(p, q)$, which is due to the second part of Proposition 3 implies that $q = P_{\mathcal{C}}(p)$. Consequently, $\left(P_{\mathcal{C}}\left(p^k\right)\right)_{k\in\mathbb{N}}$ has only one cluster point, namely, $P_{\mathcal{C}}(p)$. Thus, $\lim_{k\to+\infty} P_{\mathcal{C}}\left(p^k\right) = P_{\mathcal{C}}(p)$ and the proof is concluded.

$\square$

2.3 Conclusion

In this chapter, we provided a characterization of convex sets in the hyperbolic space and in Section 2.2 we presented properties of the projection onto convex sets. With the convexity of sets in hyperbolic space established, we can now proceed to the study of the convexity of quadratic functions in hyperbolic space.

CHAPTER 3

CONVEXITY OF QUADRATIC FUNCTIONS ON HYPERBOLIC SPACE

In this chapter[1,2] we present an extensive study of the hyperbolically convex quadratic functions. The convex quadratic functions are the most popular convex functions in the Euclidean space, occurring in many problems, such as eigenvalue optimisation, least square approximation and linear regression. We study the convexity of both homogeneous and non-homogeneous quadratic functions on the hyperbolic space in an intrinsic way. In particular, we present several characterizations that allow the construction of several examples. As it is well known, in Euclidean space there is no conceptual difference between the convexity of homogeneous and non-homogeneous quadratic functions. However, we will see that in the hyperboloid model of the hyperbolic space the conceptual intrinsic convexity of homogeneous and non-homogeneous quadratic functions are quite different, requiring much more effort than in the Euclidean scenario to understand it. The

[1]The main result in this chapter excluding the Section 3.3 has been published in the paper [20], which is joint work with Ferreira, O.P.and Németh, S.Z.

[2]The main result in the Section 3.3 has been published in the paper [21], which is joint work with Ferreira, O.P.and Németh, S.Z.

primary challenge lies in establishing the role of the linear term on the hyperbolic convexity of a non-homogeneous quadratic function. This is because introducing a linear term to a hyperbolically convex homogeneous function can result in the newly formed non-homogeneous quadratic function losing its hyperbolic convexity.

Definition 2. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a hyperbolically convex set and $I \subseteq \mathbb{R}$ an interval. A function $f : \mathcal{C} \to \mathbb{R}$ is said to be hyperbolically convex (respectively, strictly hyperbolically convex) if for any geodesic segment $\gamma : I \to \mathcal{C}$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense.

In the following remark we state some general properties of hyperbolically convex, which follow directly from Definition 2.

Remark 5. It follows from Definition 2 that $f : \mathcal{C} \to \mathbb{R}$ is a hyperbolically convex function if and only if the *epigraph* $\mathrm{epi}f := \{(p, \mu) : \ p \in \mathcal{C}, \ \mu \in \mathbb{R}, \ f(p) \leq \mu\}$, is convex in $\mathbb{H}^n \times \mathbb{R}$. Moreover, if $f : \mathcal{C} \to \mathbb{R}$ is a hyperbolically convex function, then the sub-level sets $\{p \in \mathcal{C} : \ f(p) \leq a\}$ are hyperbolically convex sets, for all $a \in \mathbb{R}$. Furthermore, if $f, f_1, \cdots f_n : \mathbb{H}^n \to \mathbb{R}$ are hyperbolically convex in $\mathcal{C}$, then ζf and $f_1 + \cdots + f_n$ are hyperbolically convex in $\mathcal{C}$, for all $\zeta \geq 0$.

The next proposition follows from Remark 2 and Remark 4, Definition 1 and Definition 2.

Proposition 6. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a hyperbolically convex set, $Q \in G_{\mathcal{L}}$ and $\mathcal{D} := \{Q^{-1}p : \ p \in \mathcal{C}\}$. The function $f : \mathcal{C} \to \mathbb{R}$ is hyperbolically convex if and only if $f \circ Q : \mathcal{D} \to \mathbb{R}$ defined by $f \circ Q(q) := f(Qq)$ is hyperbolically convex.

3.1 Hyperbolically convex functions

In this section we present first and second order characterization for hyperbolically convex functions on hyperbolic spaces.

Proposition 7. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be an open hyperbolically convex set and $f : \mathcal{C} \to \mathbb{R}$ be a differentiable function. The function f is hyperbolically convex if and only if $f(q) \geq f(p) + \langle \operatorname{grad} f(p), \log_p q \rangle$, for all $p, q \in \mathcal{C}$ and $q \neq p$, or equivalently,

$$f(q) \geq f(p) + \frac{\operatorname{arcosh}(-\langle p, q\rangle)}{\sqrt{1 - \langle p, q\rangle^2}} \left\langle [\mathrm{I} + pp^\top \mathrm{J}]\mathrm{J} \cdot Df(p)\,,\, q \right\rangle, \qquad \forall\, p, q \in \mathcal{C},\ q \neq p,$$

where Df is the usual gradient of f.

Proof. By using (1.14), the usual characterization of scalar convex functions implies that, for all minimal geodesic segment $\gamma : I \to \mathcal{C}$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex if and only if

$$f(\gamma(t_2)) \geq f(\gamma(t_1)) + \langle \mathrm{J} \cdot Df(\gamma(t_1)), \gamma'(t_1)\rangle\, (t_2 - t_1), \qquad \forall\; t_2, t_1 \in I.$$

Note that if $\gamma : [0, 1] \to \mathcal{C}$ is the geodesic segment from $p = \gamma(0)$ to $q = \gamma(1)$, then it may be represented as $\gamma(t) = \exp_p t \log_p q$. Moreover, $\gamma'(0) = \log_p q$ and $\gamma'(1) = -\log_q p$. Therefore, the first inequality of the proposition is an immediate consequence of the inequality above, Definition 2 and equation (1.11). For concluding the proof, note that equations (1.12) and (1.11) together with Remark 1 imply the equivalence between the two inequalities of the lemma.

□

Proposition 8. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be an open hyperbolically convex set and $f : \mathcal{C} \to \mathbb{R}$

a differentiable function. The function f is hyperbolically convex if and only if $\operatorname{grad} f$ satisfies the inequality $\langle \operatorname{grad} f(p), \log_p q\rangle + \langle \operatorname{grad} f(q), \log_q p\rangle \leq 0$, for all $p, q \in \mathcal{C}$ and $q \neq p$, or equivalently,

$$\langle \mathrm{J}\cdot Df(p) - \mathrm{J}\cdot Df(q), p - q\rangle - (\langle p, q\rangle + 1)\left[\langle \mathrm{J}\cdot Df(p), p\rangle + \langle \mathrm{J}\cdot Df(q), q\rangle\right] \geq 0,$$

$\forall\, p, q \in \mathcal{C},\ q \neq p$, where Df is the usual gradient of f.

Proof. Using (1.14), the usual first order characterization of convex functions implies that, for all minimal geodesic segments $\gamma : I \to \mathcal{C}$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex if and only if

$$\left[\langle \mathrm{J}\cdot Df(\gamma(t_2)), \gamma'(t_2)\rangle - \langle \mathrm{J}\cdot Df(\gamma(t_1)), \gamma'(t_1)\rangle\right](t_2 - t_1) \geq 0, \qquad \forall\, t_2, t_1 \in I.$$

Note that if $\gamma : [0,1] \to \mathcal{C}$ is the segment of geodesic from $p = \gamma(0)$ to $q = \gamma(1)$, then it may be represented as $\gamma(t) = \exp_p t \log_p q$. Moreover, $\gamma'(0) = \log_p q$ and $\gamma'(1) = -\log_q p$. Therefore, the first inequality of the proposition follows by combining the previous inequality with Definition 2 and (1.11). For concluding the proof, note that equations (1.12) and (1.11) imply the equivalence between the two inequalities of the lemma.

$\square$

Proposition 9. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be an open hyperbolically convex set and $f : \mathcal{C} \to \mathbb{R}$ be a twice differentiable function. The function f is hyperbolically convex if and only if the Hessian $\operatorname{Hess} f$ on the hyperbolic space satisfies the inequality

$\langle \text{Hess}\, f(p)v, v\rangle \geq 0$, for all $p \in \mathcal{C}$ and all $\; v \in T_p\mathbb{H}^n$, or equivalently,

$$\left\langle \mathrm{J} \cdot D^2 f(p)v, v\right\rangle + \langle \mathrm{J} \cdot Df(p), p\rangle \langle v, v\rangle \geq 0, \qquad \forall\, p \in \mathcal{C},\ \forall\, v \in T_p\mathbb{H}^n,$$

where $D^2 f(p)$ is the usual Hessian and $Df(p)$ is the usual gradient of f at a point $p \in \mathcal{C}$. If the above inequalities are strict then f is strictly hyperbolically convex.

Proof. By using (1.15), the usual second order characterization of hyperbolically convex functions implies that, for all minimal geodesic segment $\gamma : I \to \mathcal{C}$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex if and only if

$$\left\langle \mathrm{J} \cdot D^2 f(\gamma(t))\gamma'(t), \gamma'(t)\right\rangle + \langle \mathrm{J} \cdot Df(\gamma(t)), \gamma(t)\rangle \langle \gamma'(t), \gamma'(t)\rangle \geq 0, \qquad \forall\, t \in I.$$

If the last inequality is strict then $f \circ \gamma$ is strictly convex. Therefore, the result follows by combining the above inequality with Definition 2. For concluding the proof, note that equation (1.13) together with Remark 1 imply the equivalence between the two inequalities of the lemma.

$\square$

Example 2. Fix $q \in \mathbb{H}^n$. The function $d_q(\cdot) : \mathbb{H}^n \to \mathbb{R}$ is hyperbolically convex. In general, taking a hyperbolically convex set $\mathcal{C} \subseteq \mathbb{H}^n$, the function $d_q(\cdot) : \mathcal{C} \to \mathbb{R}$ is hyperbolically convex. Indeed, the hyperbolic convexity of $d_q(\cdot)$ follows by combining Lemma 8 with Proposition 9.

Example 3. Fix $q \in \mathbb{H}^n$. The function $\rho_q : \mathbb{H}^n :\to \mathbb{R}$ defined as $\rho_q(p) := \frac{1}{2}d_q^2(p)$ is strictly hyperbolically convex. In general, taking a hyperbolically convex set $\mathcal{C} \subseteq \mathbb{H}^n$, the function $\rho_q : \mathcal{C} \to \mathbb{R}$ is strictly hyperbolically convex. Indeed, the result follows by combining Lemma 9 with Proposition 9.

Example 4. Take $\tilde{p} = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ and the hyperbolically convex set $\mathcal{C} = \{p \in \mathbb{H}^n : \quad p_1 > 0, \ldots, p_n > 0\}$. The function $\psi : \mathcal{C} \to \mathbb{R}$ defined by $\psi(p) = -\ln(-1 - \langle \tilde{p}, p \rangle)$ is hyperbolically convex. Indeed, considering that $Df(p) = -(1 + \langle \tilde{p}, p \rangle)^{-1} \mathrm{J}\tilde{p}$ and $D^2 f(p) = (1 + \langle \tilde{p}, p \rangle)^{-2} \mathrm{J}\tilde{p}\tilde{p}^\top \mathrm{J}$, the hyperbolical convexity of ψ follows by combining Lemma 6 and Proposition 9.

3.2 Hyperbolically convex homogenous quadratic functions

In this section we study the hyperbolic convexity of the quadratic function $f(p) = p^\top Ap$, for $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$. We begin with a general characterization.

Corollary 2. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap$. The function f is hyperbolically convex if and only if

$$v^\top Av + p^\top Ap \geq 0, \qquad \forall p, v \in \mathbb{R}^{n+1} \quad \text{with} \quad p^\top Jp = -1, \quad v^\top Jv = 1, \quad p^\top Jv = 0.$$

Proof. Considering that $Df(p) = 2Ap$, $D^2 f(p) = 2A$ and $\mathrm{JJ} = \mathrm{I}$, we conclude that

$$\left\langle \mathrm{J} \cdot D^2 f(p)v, v \right\rangle + \langle \mathrm{J} \cdot Df(p), p \rangle \langle v, v \rangle = 2v^\top Av + 2p^\top Ap \langle v, v \rangle.$$

Thus, it follows from Proposition 9 that f is hyperbolically convex in $\mathbb{H}^n$ if and only if $v^\top Av + p^\top Ap \geq 0$, for all $p \in \mathbb{H}^n$, all $v \in T_p\mathbb{H}^n$ such that $v^\top Jv = 1$. Considering that $v \in T_p\mathbb{H}^n$ with $p \in \mathbb{H}^n$ if and only if $v^\top Jv = 1$ and $p^\top Jv = 0$ with $p \in \mathbb{H}^n$, the result follows.

□

Next, we use the Lorentz group (1.16) to present some examples of hyperbolically convex quadratic functions. Before that, we need the following result.

Corollary 3. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ and $Q \in G_{\mathcal{L}}$. Then, $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) = p^\top Ap$ is hyperbolically convex if and only if $g : \mathbb{H}^n \to \mathbb{R}$ defined by $g(q) = q^\top(Q^\top AQ)q$ is hyperbolically convex.

Proof. Since $g(q) = f(Qq)$, the equivalence follows from Proposition 6.

□

As an application of Corollaries 2 and 3 in the following we present an example of a hyperbolically convex quadratic function.

Example 5. Take a diagonal matrix $D \in \mathbb{R}^{(n+1)\times(n+1)}$, $D = \operatorname{diag}(d_1, \ldots, d_n, d_{n+1})$. Assume that $d_{min} + d_{n+1} \geq 0$, where $d_{min} := \min\{d_1, \ldots, d_n\}$. Then, for each $Q \in G_{\mathcal{L}}$, the function $g : \mathbb{H}^n \to \mathbb{R}$ defined by $g(p) = p^\top Q^\top DQp$ is hyperbolically convex. Indeed, take $q, u \in \mathbb{R}^{n+1}$ such that $q^\top Jq = -1$, $u^\top Ju = 1$ and $p^\top Jv = 0$. Thus, we have $q_{n+1}^2 = \sum_{i=1}^n q_i^2 + 1$ and $u_{n+1}^2 = \sum_{i=1}^n u_i^2 - 1$. Hence, since $d_{min} + d_{n+1} \geq 0$, we obtain that

$$u^\top Du + q^\top Dq = \sum_{i=1}^n (d_i + d_{n+1})u_i^2 + \sum_{i=1}^n (d_i + d_{n+1})q_i^2 \geq (d_{min} + d_{n+1})\Big(\sum_{i=1}^n u_i^2 + \sum_{i=1}^n q_i^2\Big) \geq 0.$$

Thus, Corollary 2 implies that $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) = p^\top Dp$ is hyperbolically convex. Therefore, applying Corollary 3 we conclude that g is hyperbolically convex.

To continue with our study of hyperbolic convexity of quadratic functions, we denote the boundary of the Lorentz cone (2.1) by

$$\partial\mathcal{L} := \left\{x \in \mathcal{L} : \; x^\top J x = 0\right\}.$$

In order to simplify notations, for a given $x \in \mathbb{R}^{n+1}$, we consider the following decomposition:

$$x = (\bar{x}^\top, x_{n+1}) \in \mathbb{R}^{n+1}, \qquad \bar{x} := (x_1, \ldots, x_n)^\top \in \mathbb{R}^n, \quad x_{n+1} \in \mathbb{R}. \tag{3.1}$$

Lemma 10. Let $x, y \in \partial\mathcal{L}$. The following three statements are equivalent:

(i) $x^\top J y \neq 0$;

(ii) $y \neq \alpha x$, for all $\alpha \in \mathbb{R}$;

(iii) $x^\top J y < 0$.

Proof. First, we prove (i) is equivalent to (ii). Assume (i) holds. If there exists $\alpha \in \mathbb{R}$ such that $y = \alpha x$, then $x^\top J y = x^\top J(\alpha x) = \alpha x^\top J x = 0$, which contradicts $x^\top J y \neq 0$. Hence, $y \neq \alpha x$, for all $\alpha \in \mathbb{R}$, and (ii) holds. For the converse, assume (ii) holds. By contradiction, assume that $x^\top J y = 0$. Since $y, z \in \partial\mathcal{L}$, by using the notation introduced in (3.1), we have $y^{n+1} = \sqrt{\bar{y}^\top \bar{y}}$ and $z_{n+1} = \sqrt{\bar{z}^\top \bar{z}}$. Thus, due to $y^\top z = 0$, we have $\bar{y}^\top \bar{z} = -y^{n+1} z_{n+1}$, or equivalently

$$\bar{y}^\top \bar{z} = -\sqrt{\bar{y}^\top \bar{y}}\sqrt{\bar{z}^\top \bar{z}}.$$

Hence, Cauchy's inequality implies that there exists a $\alpha \geq 0$ such that $\bar{y} = -\alpha \bar{z}$. Furthermore,

$$y^{n+1} = \sqrt{\bar{y}^\top \bar{y}} = \alpha\sqrt{\bar{z}^\top \bar{z}},$$

which gives $y^{n+1} = \alpha z_{n+1}$. Thus, we conclude that $y = -\alpha J z$, which implies $y = \alpha x$ and we have a contradiction. Therefore, $x^\top J y \neq 0$ and (i) holds.

Now, we prove (i) is equivalent to (iii). Assume (i) holds. Since $\mathcal{L}$ is a closed and convex cone, and $x, y \in \partial\mathcal{L} \subseteq \mathcal{L}$, we have $x + y \in \mathcal{L}$. Thus,

$$0 \geq (x+y)^\top J(x+y) = x^\top Jx + 2x^\top Jy + y^\top Jy = 2x^\top Jy.$$

Hence, $x^\top Jy \neq 0$ implies $x^\top Jy < 0$. Therefore, the item (iii) holds. Conversely, (iii) implies (i) is immediate, which concludes the proof.

□

If we make some transformations in Corollary 2 , we will obtain the following result.

Lemma 11. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$. The following three conditions are equivalent:

(i) The function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) = p^\top Ap$ is hyperbolically convex;

(ii) $x^\top Ax + y^\top Ay \geq 0$, for all $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial\mathcal{L}$ and $x^\top Jy = -1$;

(iii) $z^\top Az + w^\top Aw \geq 0$, for all $z, w \in \mathbb{R}^{n+1}$ with $z, w \in \partial\mathcal{L}$ and $z^\top Jw < 0$.

Proof. First we prove the equivalence between (i) and (ii). For that it is convenient to consider the following invertible transformations

$$p = \frac{1}{\sqrt{2}}(x+y), \quad v = \frac{1}{\sqrt{2}}(x-y) \quad \text{if and only if} \quad x = \frac{1}{\sqrt{2}}(p+v), \quad y = \frac{1}{\sqrt{2}}(p-v), \tag{3.2}$$

where $x, y, p, v \in \mathbb{R}^{n+1}$. By using the first two equalities in (3.2), after some calculations, we have

$$\begin{aligned} 2p^\top Jp &= x^\top Jx + 2x^\top Jy + y^\top Jy, \\ 2v^\top Jv &= x^\top Jx - 2x^\top Jy + y^\top Jy, \\ 2p^\top Jv &= x^\top Jx - y^\top Jy. \end{aligned} \tag{3.3}$$

On the other hand, by using the last two inequalities in (3.2) we obtain the following three equalities

$$\begin{aligned} 2x^\top Jx &= p^\top Jp + 2p^\top Jv + v^\top Jv, \\ 2y^\top Jy &= p^\top Jp - 2p^\top Jv + v^\top Jv, \\ 2x^\top Jy &= p^\top Jp - v^\top Jv. \end{aligned} \tag{3.4}$$

Moreover, the equalities in (3.2) also imply that

$$v^\top Av + p^\top Ap = x^\top Ax + y^\top Ay. \tag{3.5}$$

First we prove (i) implies (ii). Take $x, y \in \partial\mathcal{L}$ and $x^\top Jy = -1$, and consider the transformation (3.2). Thus, by using (3.3), we conclude that $p^\top Jp = -1$, $v^\top Jv = 1$ and $p^\top Jv = 0$. Hence, item (i) together with Corollary 2 implies that $v^\top Av + p^\top Ap \geq 0$. Therefore, by using (3.5), we conclude that $x^\top Ax + y^\top Ay \geq 0$ and item (ii) holds.

Next we prove that (ii) implies (i). Assume that the item (ii) holds, and take $p, v \in \mathbb{R}^{n+1}$ with $p^\top Jp = -1$, $v^\top Jv = 1$ and $p^\top Jv = 0$, and consider (3.2). Hence, by using (3.4) we have x (or $-\,x$) $\in \partial\mathcal{L}$, y (or $-\,y$) $\in \partial\mathcal{L}$ and $x^\top Jy = -1$, and item (ii) implies that $x^\top Ax + y^\top Ay \geq 0$. Thus, (3.5) implies that $v^\top Av + p^\top Ap \geq 0$, which implies that item (i) holds.

We proceed to prove the equivalence between (ii) and (iii). Assume that item (ii) holds and take $z, w \in \partial\mathcal{L}$ and $z^\top Jw < 0$. Since $z^\top Jw < 0$ we define

$$x = \frac{z}{\sqrt{-z^\top Jw}}, \qquad y = \frac{w}{\sqrt{-z^\top Jw}}. \tag{3.6}$$

Thus, considering that $z, w \in \partial\mathcal{L}$ and $z^\top Jw < 0$, some calculations show that $x, y \in \partial\mathcal{L}$ and $x^\top Jy = -1$. Therefore, using (3.6) together item (ii), we conclude that

$$z^\top Az + w^\top Aw = -z^\top Jw \left(x^\top Ax + y^\top Ay\right) \geq 0,$$

and item (iii) holds. Finally, (iii) implies (ii) is immediate, which concludes the proof.

□

In the following theorem we present a characterization for hyperbolically convex quadratic functions in term of the matrix defining it. In particular, we show that the study of hyperbolically convex quadratic functions reduces to the study of their behavior on the boundary of the Lorentz cone.

Theorem 1. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$. The following four conditions are equivalent:

(i) The quadratic function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) := p^\top Ap$ is hyperbolically convex;

(ii) The matrix A is $\partial\mathcal{L}$-copositive, i.e., $x^\top Ax \geq 0$ for all $x \in \partial\mathcal{L}$;

(iii) There exists $\alpha \in \mathbb{R}$ such that $A + \alpha \mathrm{J}$ is positive semidefinite;

(iv) The function f is bounded from below, i.e., there exists an $\alpha \in \mathbb{R}$ such that $f(p) \geq \alpha$, for all $p \in \mathbb{H}^n$.

Proof. We first prove that (i) implies (ii). Assume that f is hyperbolically convex. Let $x \in \partial\mathcal{L}$ such that $x \neq 0$. Take $y \in \partial\mathcal{L}$ not parallel to x, i.e., such that $y \neq \alpha x$, for all $\alpha \in \mathbb{R}$. Define the sequence $\left(y^k\right)_{k \in \mathbb{N}}$, where $y^k := (1/k)y$, for all $k \in \mathbb{N}$. Since $y \in \partial\mathcal{L}$ and $y \neq \alpha x$ for all $\alpha \in \mathbb{R}$, we also have $y^k \in \partial\mathcal{L}$ and $y^k \neq \alpha x$, for all $\alpha \in \mathbb{R}$ and all $k \in \mathbb{N}$. Thus, by using Lemma 10, we conclude that $x^\top J y^k < 0$, for all $k \in \mathbb{N}$. Hence, considering that $x, y^k \in \partial\mathcal{L}$ and $x^\top J y^k < 0$, for all $k \in \mathbb{N}$, and f is hyperbolically convex, by using Lemma 11, we obtain that $x^\top A x + \left(y^k\right)^\top A y^k \geq 0$, for all $k \in \mathbb{N}$. Therefore, owing to $y^k := (1/k)y$, for all $k \in \mathbb{N}$, we have

$$x^\top A x + \frac{1}{k^2} y^\top A y \geq 0, \qquad k \in \mathbb{N}.$$

Therefore, taking the limit in the latter inequality, we obtain that $x^\top A x \geq 0$. In conclusion, A is $\partial\mathcal{L}$-copositive.

Next we prove that (ii) implies (iii). Assume that A is $\partial\mathcal{L}$-copositive. Thus, for all $x \in \mathbb{R}^{n+1}$ with $x^\top J x = 0$ we have $x^\top A x \geq 0$. Consequently, for all $x \in \mathbb{R}^{n+1}$ with $x^\top J x = 0$ and $x \neq 0$ we have $x^\top (A + (1/k)\mathrm{I}) x > 0$, for all $k \in \mathbb{N}$. Hence, by Lemma 1, there exists $\alpha_k \in \mathbb{R}$ such that $A + (1/k)\mathrm{I} + \alpha_k \mathrm{J}$ is positive definite, for all $k \in \mathbb{N}$. We claim that the sequence $(\alpha_k)_{k \in \mathbb{N}}$ is bounded. Indeed, assume by absurd that $(\alpha_k)_{k \in \mathbb{N}}$ is unbounded. Since $A + (1/k)\mathrm{I} + \alpha_k \mathrm{J}$ is positive definite, for each $x \in \mathbb{R}^{n+1}$ with $x \neq 0$ we conclude that

$$\frac{1}{\alpha_k} x^\top A x + \frac{1}{k\alpha_k} x^\top x + x^\top \mathrm{J} x > 0,$$

for all $k \geq \bar{k}$ and some $\bar{k} \in \mathbb{N}$. Hence, by taking the limit in the last inequality as k goes to infinity, we conclude that $x^\top \mathrm{J} x \geq 0$ for all $x \in \mathbb{R}^{n+1}$, which is absurd. Therefore, the claim is proved. Since the sequence $(\alpha_k)_{k\in\mathbb{N}}$ is bounded, we can take a subsequence $(\alpha_{k_j})_{j\in\mathbb{N}}$ and $\alpha \in \mathbb{R}$ such that $\lim_{j\to\infty} \alpha_{k_j} = \alpha$. On the other hand, considering that $A + (1/k)\mathrm{I} + \alpha_k \mathrm{J}$ is positive definite for all $k \in \mathbb{N}$, we have

$$x^\top \left(A + \frac{1}{k_j}\mathrm{I} + \alpha_{k_j}\mathrm{J} \right) x > 0,$$

for all $x \in \mathbb{R}^{n+1}$ such that $x \neq 0$. Therefore, by taking the limit in the last inequality as k goes to infinity, we have $x^\top (A + \alpha \mathrm{J}) x \geq 0$, for all $x \in \mathbb{R}^{n+1}$, which implies that $A + \alpha \mathrm{J}$ is positive semidefinite.

Next we prove that (iii) implies (i). Assume that there is an $\alpha \in \mathbb{R}$ such that $A + \alpha \mathrm{J}$ is positive semidefinite. Due to $A + \alpha \mathrm{J}$ being positive semidefinite and $v^\top A v + p^\top A p = v^\top A v + \alpha + p^\top A p - \alpha$, some calculations show

$$v^\top A v + p^\top A p = v^\top (A + \alpha \mathrm{J}) v + p^\top (A + \alpha \mathrm{J}) p \geq 0, \tag{3.7}$$

for all $p, v \in \mathbb{R}^{n+1}$ with $p^\top J p = -1$, $v^\top J v = 1$ and $p^\top J v = 0$. Therefore, by using (3.7) and Corollary 2, we conclude that f is hyperbolically convex.

Next we prove that (iii) implies (iv). Assume that there exists $\alpha \in \mathbb{R}$ such that $A + \alpha \mathrm{J}$ is positive semidefinite. Hence, $p^\top A p + \alpha p^\top \mathrm{J} p = p^\top (A + \alpha \mathrm{J}) p \geq 0$, for all $p \in \mathbb{R}^n$. Since $p \in \mathbb{H}^n$ implies $p^\top \mathrm{J} p = -1$, we conclude that $p^\top A p \geq \alpha$, for all

$p \in \mathbb{H}^n$. Therefore, f is bounded from below on $\mathbb{H}^n$.

Finally, to conclude the proof, we prove that (iv) imply (iii). Assume that f is bounded from below on $\mathbb{H}^n$. Thus, there exists an $\alpha \in \mathbb{R}$ such that $f(p) \geq \alpha$, for all $p \in \mathbb{H}^n$, or equivalently,

$$p^\top (A + \alpha J)p \geq 0, \qquad \forall p \in \mathbb{H}^n. \tag{3.8}$$

In order to apply Lemma 5, we will first prove the statement: if $x^\top \mathrm{J} x \leq 0$, then $x^\top (A + \alpha J)x \geq 0$. Let $x \in \mathbb{R}^n$ such that $x^\top \mathrm{J} x \leq 0$. Take a sequence $(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^n$ such that $\lim_{k\to+\infty} x_k = x$ and ${x_k}^\top \mathrm{J} x_k < 0$. Define

$$y^k = \frac{x_k}{\sqrt{-{x_k}^\top J x_k}}, \qquad k \in \mathbb{N}. \tag{3.9}$$

Since ${x_k}^\top \mathrm{J} x_k < 0$, by using (3.9), we conclude that ${y^k}^\top \mathrm{J} y^k = -1$. Thus, $y^k \in \mathbb{H}^n$ and (3.8) implies that $\left(y^k\right)^\top (A + \alpha J) y^k \geq 0$. Using, again (3.9) we obtain that ${x_k}^\top (A + \alpha J) x_k \geq 0$, for all $k \in \mathbb{N}$. Hence, by letting $k \to \infty$, we conclude that $x^\top (A + \alpha J)x \geq 0$, which proves the statement. Therefore, after applying Lemma 5, we conclude that there exists a $\beta \in \mathbb{R}$ such that $A + (\alpha + \beta)\mathrm{J}$ is positive definite. Hence, by Theorem 1, the function f is hyperbolically convex.

$\square$

Example 6. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be a positive semidefinite matrix and $\alpha \in \mathbb{R}$. Since $A = (A + \alpha \mathrm{J}) - \alpha \mathrm{J}$ is a positive semidefinite matrix, by applying Theorem 1, we conclude that the function $f_\alpha : \mathbb{H}^n :\to \mathbb{R}$ defined by $f_\alpha(p) := p^\top (A + \alpha \mathrm{J})p$ is hyperbolically convex.

For simplifying the statement and proof of the next results it is convenient to introduce the following notation. For a given $A \in \mathbb{R}^{(n+1)\times(n+1)}$, consider the following decomposition:

$$A := \begin{pmatrix} \bar{A} & a \\ a^\top & \sigma \end{pmatrix}, \qquad \bar{A} \in \mathbb{R}^{n\times n}, \quad a \in \mathbb{R}^{n\times 1}, \quad \sigma \in \mathbb{R} \tag{3.10}$$

and denote by $\bar{I} \in \mathbb{R}^{n\times n}$ is the identity matrix. For any $\alpha \in \mathbb{R}$, the decomposition (3.10) yields

$$A + \alpha J = \begin{pmatrix} \bar{A} + \alpha\bar{I} & a \\ a^\top & \sigma - \alpha \end{pmatrix}. \tag{3.11}$$

Proposition 10. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be a symmetric matrix and $\alpha \in \mathbb{R}$. Consider the decomposition (3.10) and the following statements:

(i) The quadratic function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) := p^\top A p$ is hyperbolically convex;

(ii) The matrix $A + \alpha$J is positive definite;

(iii) The number $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ and $\det(A + \alpha J) > 0$;

(iv) The number $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ and $\sigma - \alpha - a^\top \left(\bar{A} + \alpha\bar{I}\right)^{-1} a > 0$.

Then (ii), (iii) and (iv) are equivalent and any of them implies item (i).

Proof. We first prove the equivalence between (ii) and (iii). Assume that (ii) holds. Hence, by applying Lemmas 2 and 3, and taking into account (3.11), we have $\det(A+\alpha J) > 0$, the matrix $\bar{A}+\sigma\bar{I}$ is positive definite and $\sigma - \alpha > 0$. Since $\bar{A} + \sigma\bar{I}$ is positive definite, we obtain that $\alpha > -\lambda_{\min}(\bar{A})$. Therefore, we conclude that $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ and $\det(A + \alpha J) > 0$. Hence, (iii) holds. Reciprocally, assume that (iii) holds. Thus, we have $\alpha > -\lambda_{\min}(\bar{A})$, $\sigma > \alpha$ and $\det(A+\alpha \mathrm{J}) > 0$. Hence, the matrix $\bar{A}+\sigma\bar{I}$ is positive definite, $\sigma - \alpha > 0$ and $\det(A+\alpha\mathrm{J}) > 0$. Therefore, by using Lemmas 2 and 3 and the decomposition (3.10), we conclude that $A + \alpha\mathrm{J}$ is positive definite and (ii) holds.

Next, we prove the equivalence between (iii) and (iv). First note that, for any $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$, we obtain that $\lambda_{\min}(\bar{A}) + \alpha > 0$. Thus, the matrix $\bar{A}+\alpha\bar{I}$ is positive definite, which implies that $\det\left(\bar{A}+\alpha\bar{I}\right) > 0$. In particular, the matrix $\bar{A}+\alpha\bar{I}$ is invertible. Thus, applying Lemma 4, we have

$$\det(A+\alpha J) = \left(\sigma - \alpha - a^\top \left(\bar{A}+\alpha\bar{I}\right)^{-1} a\right) \det\left(\bar{A}+\alpha\bar{I}\right). \tag{3.12}$$

Since under the assumption $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ we have $\det\left(\bar{A}+\alpha\bar{I}\right) > 0$, it follows from (3.12) that $\det(A+\alpha J) > 0$ is equivalent to $\sigma - \alpha - a^\top\left(\bar{A}+\alpha\bar{I}\right)^{-1} a > 0$. Therefore, (iii) is equivalent to (iv), and the proof of the first statement of the proposition is concluded.

By using the implication (iii) $\Longrightarrow$ (i) in Theorem 1, the last statement of the proposition follows from the first one. $\square$

Consider the decomposition (3.10) of a symmetric matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Then, it follows from Proposition 10 that we can decide if $f(p) := p^\top A p$ is hyperbolically convex by solving the following optimization problem:

$$\inf\left\{\sigma - \alpha - a^\top\left(\bar{A}+\alpha\bar{I}\right)^{-1} a : \ \alpha \in (-\lambda_{\min}(\bar{A}), \sigma)\right\}.$$

By using decompositions (3.1) and (3.10), Corollary 2 can be stated equivalently in the following form.

Corollary 4. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap$. The function f is hyperbolically convex if and only if

$$\bar{v}^\top \bar{A}\bar{v} + \bar{p}^\top \bar{A}\bar{p} + 2v_{n+1}\bar{v}^\top a + 2p^{n+1}\bar{p}^\top a + \sigma v_{n+1}^2 + \sigma \left(p^{n+1}\right)^2 \geq 0, \tag{3.13}$$

for all $p, v \in \mathbb{R}^{n+1}$ with

$$\bar{p}^\top \bar{p} - \left(p^{n+1}\right)^2 = -1 \quad \bar{v}^\top \bar{v} - v_{n+1}^2 = 1, \quad \bar{v}^\top \bar{p} - v_{n+1}p^{n+1} = 0. \tag{3.14}$$

Example 7. Let $A := \left(a_i^j\right) \in \mathbb{R}^{(n+1)\times(n+1)}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap$. If f is hyperbolically convex, then

$$\frac{1}{n}\sum_{i,j=1}^{n} a_i^j + \sigma \geq 0. \tag{3.15}$$

Indeed, take $\bar{v} = (1/\sqrt{n}, \ldots, 1/\sqrt{n}) \in \mathbb{R}^n$, $v_{n+1} = 0$ and $\bar{p} = 0 \in \mathbb{R}^n$, $p^{n+1} = 1$. Thus, (3.13) becomes

$$\bar{v}^\top \bar{A}\bar{v} + \bar{p}^\top \bar{A}\bar{p} + 2v_{n+1}\bar{v}^\top a + 2p^{n+1}\bar{p}^\top a + \sigma v_{n+1}^2 + \sigma \left(p^{n+1}\right)^2 = \frac{1}{n}\sum_{i,j=1}^{n} a_i^j + \sigma.$$

Since $p = (\bar{p}, p^{n+1}) \in \mathbb{R}^{n+1}$, $v = (\bar{v}, v_{n+1}) \in \mathbb{R}^{n+1}$ satisfy (3.14) and considering that f is hyperbolically convex, the inequality (3.15) follows from applying Corollary 4.

Theorem 2. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap$. Then, considering decompositions (3.1) and (3.10), the following statements hold:

(i) If f is hyperbolically convex, then $\lambda_{\min}(\bar{A}) \geq -\sigma$;

(ii) If $\sigma \geq -\lambda_{\min}(\bar{A})$ and $a = 0$, then f is hyperbolically convex;

(iii) If $\sigma + \lambda_{min}(\bar{A}) > 2\sqrt{a^\top a}$, then f is hyperbolically convex.

Proof. To prove (i), assume that f is hyperbolically convex and take any $p, v \in \mathbb{R}^{n+1}$ with $\bar{p} = 0$, $p^{n+1} = 1$, $v^{n+1} = 0$, $\bar{v}^\top \bar{v} = 1$. Then, conditions (3.14) in Corollary 4 are satisfied. Hence, it follows from (3.13) that $\bar{v}^\top \bar{A} \bar{v} \geq -\sigma$, for all $\bar{v} \in \mathbb{R}^n$ with $\bar{v}^\top \bar{v} = 1$. Therefore, the result follows.

We proceed to prove (ii). Assume that $\lambda_{\min}(\bar{A}) \geq -\sigma$ and $a = 0$. In this case, we have $\lambda_{\min}(\bar{A} + \sigma \bar{I}) \geq 0$, where $\bar{I} \in \mathbb{R}^{n \times n}$ is the identity matrix. Hence, both matrices $\bar{A} + \sigma \bar{I}$ and

$$A + \sigma J = \begin{pmatrix} \bar{A} + \sigma \bar{I} & 0 \\ 0 & 0 \end{pmatrix}$$

are positive semidefinite. Thus, by applying Proposition 10 with $\alpha = \sigma$, the proof of item (ii) follows.

To prove item (iii), first we introduce the auxiliary quadratic polynomial $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = (\sigma - t)\left(\lambda_{\min}(\bar{A}) + t\right) - a^\top a.$$

The roots of the quadratic polynomial g are given by

$$\mu = \frac{1}{2}\left(\sigma - \lambda_{\min}(\bar{A}) - \sqrt{\left[\sigma + \lambda_{\min}(\bar{A})\right]^2 - 4a^\top a}\right),$$

$$\eta = \frac{1}{2}\left(\sigma - \lambda_{\min}(\bar{A}) + \sqrt{\left[\sigma + \lambda_{\min}(\bar{A})\right]^2 - 4a^\top a}\right).$$

Since $\sigma + \lambda_{min}(\bar{A}) > 2\sqrt{a^\top a}$ we have $\mu < \eta$. Thus, take $\beta \in (\mu, \eta)$. Hence,

$g(\beta) > 0$. Since

$$\beta > \mu \geq \frac{1}{2}\left(\sigma - \lambda_{\min}(\bar{A}) + \sqrt{\left[\sigma + \lambda_{\min}(\bar{A})\right]^2}\right) = -\lambda_{\min}(\bar{A}),$$

we obtain $\lambda_{\min}(\bar{A}) + \beta > 0$. Thus, we conclude that $\bar{A} + \beta\bar{I}$ is positive definite. It follows that

$$\begin{aligned}\sigma - \beta - a^\top \left(\bar{A} + \beta\bar{I}\right)^{-1} a &\geq \sigma - \beta - \lambda_{max}\left(\bar{A} + \beta\bar{I}\right)^{-1} a^\top a \\ &= \sigma - \beta - \frac{1}{\lambda_{\min}(\bar{A}) + \beta} a^\top a \\ &= \frac{g(\beta)}{\lambda_{\min}(\bar{A}) + \beta} > 0.\end{aligned}$$

Therefore, by using Lemma 4, it follows from the positive definiteness of the matrix $\bar{A} + \beta\bar{I}$ that

$$\det(A + \beta J) = \left(\sigma - \beta - a^\top \left(\bar{A} + \beta\bar{I}\right)^{-1} a\right) \det\left(\bar{A} + \beta\bar{I}\right) > 0. \tag{3.16}$$

Since $\bar{A} + \beta\bar{I}$ is positive definite, combining Lemma 5 with (3.16), we conclude that $A + \beta J$ is positive definite. Therefore, Proposition 10 implies that f is hyperbolically convex, and the proof of item (iii) is concluded.

□

Corollary 5. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap$. Consider the decomposition (3.10) and assume that $a = 0$. Then, f is hyperbolically convex if and only if $\lambda_{\min}(\bar{A}) \geq -\sigma$.

Proof. The proof is an immediate consequence of items (i) and (ii) of Theorem 2.

□

In the next proposition we present a characterization for the case $a \neq 0$ in (3.10), which completes the result of Corollary 5.

Proposition 11. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be a symmetric matrix and the decomposition (3.10). If $a \neq 0$, then the following statements are equivalent:

(i) The quadratic function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) := p^\top A p$ is hyperbolically convex;

(ii) There exists $\alpha \in \mathbb{R}$ such that the matrix $A + \alpha \mathrm{J}$ is positive semidefinite;

(iii) There exists $\alpha \in \mathbb{R}$ such that $\sigma > \alpha$ and the matrix $\bar{A} + \alpha \bar{I} - \frac{1}{\sigma - \alpha} a a^\top$ is positive semidefinite;

Proof. First we prove that (ii) and (iii) are equivalent. Since the matrix $A + \alpha \mathrm{J}$ is positive semidefinite, by using the decompositions (3.11) and [30, Corollary 7.15, p. 398], we conclude that

$$(\bar{a}_{ii} + \alpha)(\sigma - \alpha) \geq a_i^2, \qquad i = 1, \ldots, n, \tag{3.17}$$

where $\bar{a}_{ii}$ is the ii-entry of the matrix $\bar{A}$ and a_i is the i-entry of the vector a. Considering that $a \neq 0$, and all elements in the diagonal of a positive semidefinite matrix are nonnegative, it follows from (3.17) that $\sigma > \alpha$. Thus, applying item (iii) of Lemma 3 we conclude that items (ii) and (iii) are equivalent. The equivalence of (i) and (ii) follows from the equivalence of (iii) and (i) in Theorem 1.

□

Proposition 12. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be a symmetric matrix and the decomposition (3.10). Consider the following statements:

(i) The quadratic function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) := p^\top Ap$ is hyperbolically convex;

(ii) There exists $\alpha \in \mathbb{R}$ such that $\bar{A}+\alpha\bar{I}$ is positive definite and $A+\alpha\mathrm{J}$ is positive semidefinite;

(iii) There exists $\alpha \in \mathbb{R}$ such that $\bar{A}+\alpha\bar{I}$ is positive definite and $\sigma - \alpha - a^\top(\bar{A}+\alpha\bar{I})^{-1}a \geq 0$.

Then, items (ii) and (iii) are equivalent and any of them implies (i).

Proof. It follows from item (iii) of Lemma 3 that items (ii) and (iii) are equivalent. The equivalence of (i) and (ii) follows from the equivalence of (iii) and (i) in Theorem 1.

□

Proposition 13. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be a symmetric matrix and the decomposition (3.10). Consider the following statements:

(i) The quadratic function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) := p^\top Ap$ is hyperbolically convex;

(ii) There exists $\alpha \in \mathbb{R}$ such that the matrix $A+\alpha\mathrm{J}$ is positive definite;

(iii) There exists $\alpha \in \mathbb{R}$ such that $\sigma > \alpha$ and the matrix $\bar{A}+\alpha\bar{I} - \frac{1}{\sigma-\alpha}aa^\top$ is positive definite.

Then, items (ii) and (iii) are equivalent and any of them implies item (i).

Proof. The equivalence between items (ii) and (iii) follows by direct application of item (i) of Lemma 3. By using the implication (iii) $\Longrightarrow$ (i) in Theorem 1, the last statement of the proposition follows from the first one.

$\square$

3.3 Hyperbolically convex non-homogenous quadratic functions

In this section we study the hyperbolic convexity of the non-homogeneous quadratic function $f : \mathbb{H}^n \to \mathbb{R}$ defined by

$$f(p) = p^\top Ap + b^\top p + c, \tag{3.18}$$

where $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$. For that we first recall some general characterizations for a non-homogeneous quadratic convex function. We begin with the general definition Definition 2 of a convex function on the hyperbolic space. In the following we recall the general second order characterization for hyperbolically convex functions on hyperbolic spaces Proposition 9 and we present a general characterization for convexity of the function (3.18), which is an immediate consequence of Proposition 9.

Corollary 6. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$. The function $f(p) = p^\top Ap + b^\top p + c$ is hyperbolically convex if and only if

$$2v^\top Av + 2p^\top Ap + b^\top p \geq 0, \ \forall p, v \in \mathbb{R}^{n+1} \quad \text{with} \quad p^\top \mathrm{J} p = -1, \ v^\top \mathrm{J} v = 1, \ p^\top \mathrm{J} v = 0.$$

Proof. Considering that $Df(p) = 2Ap + b$, $D^2 f(p) = 2A$ and $\mathrm{JJ} = \mathrm{I}$, we conclude that

$$\left\langle \mathrm{J}\cdot D^2f(p)v,v\right\rangle + \left\langle \mathrm{J}\cdot Df(p),p\right\rangle \left\langle v,v\right\rangle = 2v^\top Av + (2p^\top Ap + b^\top p)\left\langle v,v\right\rangle .$$

Thus, it follows from Proposition 9 that the function f is hyperbolically convex in $\mathbb{H}^n$ if and only if $2v^\top Av + 2p^\top Ap + b^\top p \geq 0$, for all $p \in \mathbb{H}^n$, all $v \in T_p\mathbb{H}^n$ with $v^\top \mathrm{J}v = 1$. Considering that $p \in \mathbb{H}^n$ and $v \in T_p\mathbb{H}^n$ with $v^\top \mathrm{J}v = 1$ if and only if $p^\top \mathrm{J}p = -1$, $v^\top \mathrm{J}v = 1$ and $p^\top \mathrm{J}v = 0$, the result follows. $\square$

Next we relate the hyperbolic convexity of f with the boundary of the Lorentz cone $\partial\mathcal{L}$.

Lemma 12. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$. The following three conditions are equivalent:

(i) The function $f : \mathbb{H}^n \to \mathbb{R}$ defined by $f(p) = p^\top Ap + b^\top p + c$ is hyperbolically convex;

(ii) $4x^\top Ax + 4y^\top Ay + \sqrt{2}b^\top(x+y) \geq 0$, for all $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial\mathcal{L}$ and $x^\top \mathrm{J}y = -1$;

(iii) $4z^\top Az + 4w^\top Aw + \sqrt{-2z^\top \mathrm{J}w}\, b^\top(z+w) \geq 0$, for all $z, w \in \mathbb{R}^{n+1}$ with $z, w \in \partial\mathcal{L}$ and $z^\top \mathrm{J}w < 0$.

Proof. First we prove the equivalence between (i) and (ii). For that it is convenient to first consider the following invertible transformations

$$p = \frac{1}{\sqrt{2}}(x+y), \quad v = \frac{1}{\sqrt{2}}(x-y) \quad \text{if and only if} \quad x = \frac{1}{\sqrt{2}}(p+v), \quad y = \frac{1}{\sqrt{2}}(p-v), \tag{3.19}$$

where $x, y, p, v \in \mathbb{R}^{n+1}$. By using the first two equalities in (3.19), after some calculations, we have

$$\begin{aligned} 2p^\top \mathrm{J}p &= x^\top \mathrm{J}x + 2x^\top \mathrm{J}y + y^\top \mathrm{J}y, \\ 2v^\top \mathrm{J}v &= x^\top \mathrm{J}x - 2x^\top \mathrm{J}y + y^\top \mathrm{J}y, \\ 2p^\top \mathrm{J}v &= x^\top \mathrm{J}x - y^\top \mathrm{J}y. \end{aligned} \tag{3.20}$$

On the other hand, by using the last two inequalities in (3.19) we obtain the following three equalities

$$\begin{aligned} 2x^\top \mathrm{J}x &= p^\top \mathrm{J}p + 2p^\top \mathrm{J}v + v^\top \mathrm{J}v, \\ 2y^\top \mathrm{J}y &= p^\top \mathrm{J}p - 2p^\top \mathrm{J}v + v^\top \mathrm{J}v, \\ 2x^\top \mathrm{J}y &= p^\top \mathrm{J}p - v^\top \mathrm{J}v. \end{aligned} \tag{3.21}$$

Moreover, the equalities in (3.19) also imply that

$$v^\top A v + p^\top A p = x^\top A x + y^\top A y. \tag{3.22}$$

First we prove (i) implies (ii). Take $x, y \in \partial\mathcal{L}$ and $x^\top \mathrm{J}y = -1$, and consider the transformation (3.19). Thus, by using (3.20), we conclude that $p^\top \mathrm{J}p = -1$, $v^\top \mathrm{J}v = 1$ and $p^\top \mathrm{J}v = 0$. Hence, item (i) together with Corollary 6 implies that $2v^\top Av + 2p^\top Ap + b^\top p \geq 0$. Therefore, by using (3.19) and (3.22), we conclude that $4x^\top Ax + 4y^\top Ay + \sqrt{2}b^\top(x + y) \geq 0$ and item (ii) holds. Next we prove that (ii) implies (i). Assume that the item (ii) holds, and take $p, v \in \mathbb{R}^{n+1}$ with $p^\top \mathrm{J}p = -1$, $v^\top \mathrm{J}v = 1$ and $p^\top \mathrm{J}v = 0$, and consider (3.19). Hence, by using (3.21) we have x (or $-x$) $\in \partial\mathcal{L}$, y (or $-y$) $\in \partial\mathcal{L}$ and $x^\top \mathrm{J} y = -1$, and item (ii) implies that $4x^\top Ax + 4y^\top Ay + \sqrt{2}b^\top(x+y) \geq 0$. Thus, (3.19) and (3.22) implies that $2v^\top Av + 2p^\top Ap + b^\top p \geq 0$, which implies that item (i) holds.

We proceed to prove the equivalence between (ii) and (iii). Assume that item (ii) holds and take $z, w \in \partial\mathcal{L}$ and $z^\top \mathrm{J}w < 0$. Since $z^\top \mathrm{J}w < 0$ we define

$$x = \frac{z}{\sqrt{-z^\top \mathrm{J}w}}, \qquad y = \frac{w}{\sqrt{-z^\top \mathrm{J}w}} \tag{3.23}$$

Thus, considering that $z, w \in \partial\mathcal{L}$ and $z^\top \mathrm{J}w < 0$, some calculations show that $x, y \in \partial\mathcal{L}$ and $x^\top \mathrm{J}y = -1$. Therefore, using (3.23) together item (ii), we conclude that

$$z^\top Az + w^\top Aw = -z^\top \mathrm{J}w \left(x^\top Ax + y^\top Ay\right) \geq 0,$$

and the item (iii) holds. Finally, (iii) implies (ii) is immediate, which concludes the proof. □

Proposition 14. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then there exists $\mu \in \mathbb{R}$ such that $A + \mu J$ is positive semidefinite.

Proof. Let $z, w \in \mathbb{R}^{n+1}$ with $z, w \in \partial\mathcal{L}$ and $z^\top \mathrm{J}w < 0$. Define the sequence $(w_k)_{k\in\mathbb{N}}$, where $w_k = (1/k)w$ with $k \neq 0$. Then, by (iii) of Lemma 12 we have $4z^\top Az + \frac{1}{k^2}w^\top Aw + \sqrt{-\frac{2}{k}z^\top \mathrm{J}w}\, b^\top \left(z + \frac{1}{k}w_k\right) = 4z^\top Az + 4w_k^\top Aw_k + \sqrt{-2z^\top \mathrm{J}w_k}\, b^\top(z + w_k) \geq 0$, for all $z, w \in \partial\mathcal{L}$ with $z^\top \mathrm{J}w < 0$ and all $k \in \mathbb{N}$ with $k \neq 0$. Hence, by tending with k to infinity, we obtain that $4z^\top Az \geq 0$, for all $z \in \partial\mathcal{L}$. Therefore, A is $\partial\mathcal{L}$-copositive and by using Corollary 1 the result follows. □

Corollary 7. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then A is $\partial\mathcal{L}$-copositive. As a consequence, $h : \mathbb{H}^n \to \mathbb{R}$ defined by $h(p) = p^\top Ap$ is hyperbolically convex.

Proof. Combining Proposition 14 with itens (i), (ii) and (iii) of [20, Theorem 5.1] the result follows. □

Since the condition $x^\top \mathrm{J}y \leq 0$, for any $x, y \in \partial\mathcal{L}$, is equivalent to the n-dimensional Cauchy inequality, it always holds. Then, the equivalence between items (i) and (iii) of Lemma 12 can be stated equivalently in the following form.

Proposition 15. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. Then, f is hyperbolically convex, if and only if

$$4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) \geq 0, \qquad \forall x, y \in \mathbb{R}^{n+1} \text{ with } x, y \in \partial\mathcal{L}.$$

In next corollary we present a characterization for a linear function to be hyperbolically convex.

Corollary 8. Let $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $g : \mathbb{H}^n \to \mathbb{R}$ be defined by $g(p) = b^\top p + c$. Then, g is hyperbolically convex, if and only if $b \in \mathcal{L}$.

Proof. Applying Proposition 15 with $A = 0$ and $g = f$ we obtain that g is hyperbolically convex if and only if

$$\sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) \geq 0, \qquad \forall x, y \in \mathbb{R}^{n+1} \text{ with } x, y \in \partial\mathcal{L}. \tag{3.24}$$

Suppose first that g is hyperbolically convex. Let $x \in \partial\mathcal{L} \setminus \{0\}$ arbitrary and

$y = -(1/k)\mathrm{J}x \in \partial\mathcal{L}$, where $k \in \mathbb{N}$ with $k \neq 0$. Thus, it follows from (3.24) that $b^\top(x - (1/k)\mathrm{J}x) \geq 0$, for any $x \in \partial\mathcal{L} \setminus \{0\}$ and $k \in \mathbb{N}$ with $k \neq 0$. By tending with k to infinity in the last inequality, we obtain $b^\top x \geq 0$ for any $x \in \partial\mathcal{L} \setminus \{0\}$. Hence, $b \in (\partial\mathcal{L} \setminus \{0\})^* = \mathcal{L}^* = \mathcal{L}$. Conversely, if $b \in \mathcal{L} = \mathcal{L}^*$, then (3.24) holds and hence g is hyperbolically convex. □

In the Euclidean context the linear term of a quadratic function has no influence on the convexity of the function. As we will see in the next corollary, this is not the case in the hyperbolic setting.

Corollary 9. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1} \setminus \mathcal{L}$ and $c \in \mathbb{R}$. Then, there exists a $\lambda > 0$ such that the function $f_\lambda : \mathbb{H}^n \to \mathbb{R}$ defined by $f_\lambda(p) = p^\top Ap + (\lambda b)^\top p + c$ is not hyperbolically convex.

Proof. Since $b \notin \mathcal{L} = \mathcal{L}^*$ and $\mathcal{L} = \partial\mathcal{L} + \partial\mathcal{L}$, it follows that there exists $x, y \in \partial\mathcal{L}$ such that $(\lambda b)^\top(x+y) < 0$, for all $\lambda > 0$. Hence, taking into account that $x^\top \mathrm{J}y < 0$, we conclude that

$$\lim_{\lambda \to +\infty} \left(4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top \mathrm{J}y}\,(\lambda b)^\top(x+y)\right) = -\infty.$$

It follows that if $\lambda > 0$ is sufficiently large, $4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top \mathrm{J}y}\,(\lambda b)^\top(x + y) < 0$. Therefore, by Proposition 15, the function f_λ is not hyperbolically convex.

□

Proposition 16. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b = (b_1, \ldots, b_{n+1}) \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then there exists $\mu \in \mathbb{R}$ such that

$$A + \mathrm{J}A\mathrm{J} + \frac{1}{2}b_{n+1}\mathrm{I} + \mu J$$

is a positive semidefinite matrix.

Proof. Applying Proposition 15 with $x := (x_1, \ldots, x_{n+1}) \in \partial\mathcal{L}$ and $y = -\mathrm{J}x \in \partial\mathcal{L}$, we obtain that

$$4x^\top(A + \mathrm{J}A\mathrm{J})x + \sqrt{2x^\top x}\, b^\top(\mathrm{I} - \mathrm{J})x \geq 0, \qquad \forall x \in \partial\mathcal{L}. \tag{3.25}$$

Since $2x_{n+1} = \sqrt{2x^\top x}$, for all $x \in \partial\mathcal{L}$, we obtain that $b^\top(\mathrm{I} - \mathrm{J})x = b_{n+1}(2x_{n+1}) = b_{n+1}\sqrt{2x^\top x}$. Thus, we conclude that

$$\sqrt{2x^\top x}\, b^\top(\mathrm{I} - \mathrm{J})x = 2b_{n+1}x^\top x, \qquad \forall x \in \partial\mathcal{L}.$$

Hence, combining the last equality with (3.25) we obtain, after some algebraic manipulations, that

$$4x^\top\left(A + \mathrm{J}A\mathrm{J} + \frac{1}{2}b_{n+1}\mathrm{I}\right)x = 4x^\top(A+\mathrm{J}A\mathrm{J})x+\sqrt{2x^\top x}\, b^\top(\mathrm{I}-\mathrm{J})x \geq 0, \qquad \forall x \in \partial\mathcal{L}.$$

Therefore, $A + \mathrm{J}A\mathrm{J} + \frac{1}{2}b_{n+1}\mathrm{I}$ is $\partial\mathcal{L}$-copositive and by using Corollary 1 the result follows. $\square$

Corollary 10. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b = (b_1, \ldots, b_{n+1}) \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then $g : \mathbb{H}^n \to \mathbb{R}$ defined by

$$g(p) = p^\top \left(A + \mathrm{J}A\mathrm{J} + \frac{1}{2} b_{n+1} \mathrm{I} \right) p$$

is hyperbolically convex.

Proof. The proof follows from Proposition 16 by using the items (i) and (iii) of [20, Theorem 5.1]. $\square$

Remark 6. If A is $\partial\mathcal{L}$-copositive and $b_{n+1} > 0$, then g in Corollary 10 is hyperbolically convex. Indeed, considering that for any $p \in \partial\mathcal{L}$ we have $q = -\mathrm{J}p \in \partial\mathcal{L}$, we conclude that

$$g(p) = p^\top \left(A + \mathrm{J}A\mathrm{J} + \frac{1}{2} b_{n+1} \mathrm{I} \right) p = p^\top A p + q^\top A q + b_{n+1} p_{n+1}^2 \geq 0,$$

which implies that g is bounded from below. Therefore, the result follows from items (i) and (iv) of [20, Theorem 5.1].

Proposition 17. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$, $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top A p + b^\top p + c$ and $h : \mathbb{H}^n \to \mathbb{R}$ be defined by $h(p) = p^\top A p$. Consider the following statements:

(i) The function f is hyperbolically convex.

(ii) The inequality $4x^\top A x + \sqrt{-2x^\top \mathrm{J} y}\, b^\top x \geq 0$ holds, for all $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial\mathcal{L}$.

(iii) The inequality $4y^\top A y + \sqrt{-2x^\top \mathrm{J} y}\, b^\top y \geq 0$ holds, for all $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial\mathcal{L}$.

(iv) The function h is hyperbolically convex and $b \in \mathcal{L}$.

(v) The matrix A is $\partial\mathcal{L}$-copositive and $b \in \mathcal{L}$.

(vi) The vector $b \in \mathcal{L}$ and there is a $\mu \in \mathbb{R}$ such that $A+\mu J$ is positive semidefinite

Then, (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (vi) $\Longrightarrow$ (i).

Proof. Items (ii)and (iii) are equivalent because they are obtained from each other by swapping x and y. Suppose that (ii) is true. Then (iii) is also true. By summing up the inequalities of (ii) and (iii) we obtain that

$$4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) \geq 0, \qquad \forall x, y \in \mathbb{R}^{n+1} \text{ with } x, y \in \partial\mathcal{L}.$$

Hence, (i) follows from Proposition 15. The equivalence of items (iv), (v), and (vi) follow from items (i), (ii) and (iii) of [20, Theorem 5.1]. To complete the proof we will show that (ii) $\Longleftrightarrow$ (v). Suppose that (v) holds. Then, the inequality of (ii) follows immediately from the $\partial\mathcal{L}$-copositivity of A and the self-duality of $\mathcal{L}$. Reciprocally, suppose that (ii) holds. Then, (i) also holds. Hence, it follows from Corollary 7 that h is a hyperbolically convex function. Thus, items (i) and (ii) of [20, Theorem 5.1] imply that the matrix A is $\partial\mathcal{L}$-copositive. On the other hand, since $x^k := (1/k)^2 x \in \partial\mathcal{L}$ for all $k \in \mathbb{N}$, it follows from the inequality of (ii) that

$$4(x^k)^\top Ax^k + \sqrt{-2(x^k)^\top \mathrm{J}y}\, b^\top x^k \geq 0,$$

for all $x, y \in \partial\mathcal{L}$ and $k \in N$. Substituting $x^k = (1/k)^2 x$ into the last inequity we conclude that

$$\frac{4}{k^4}x^\top Ax + \frac{1}{k^3}\sqrt{-2x^\top \mathrm{J}y}\, b^\top x \geq 0.$$

for all $x, y \in \partial\mathcal{L}$ and $k \in N$. Multiplying the latest inequality by k^3, then tending with k to infinity and finally dividing by $\sqrt{-2x^\top \mathrm{J}y}$, we obtain $b^\top x \geq 0$, for all $x \in \partial\mathcal{L}$. Hence, $b \in (\partial\mathcal{L})^* = \mathcal{L}$. Therefore, item (v) holds and the proof is completed. □

For simplifying the statement and proof of the next results it is convenient to introduce the following notations. For a given $A \in \mathbb{R}^{(n+1)\times(n+1)}$, consider the following decomposition:

$$A := \begin{pmatrix} \bar{A} & \bar{a} \\ \bar{a}^\top & \sigma \end{pmatrix}, \qquad \bar{A} \in \mathbb{R}^{n\times n}, \quad \bar{a} \in \mathbb{R}^{n\times 1}, \quad \sigma \in \mathbb{R}. \tag{3.26}$$

Denote $\bar{\mathrm{I}} \in \mathbb{R}^{n\times n}$ the identity matrix. In addition, for a given vector $z \in \mathbb{R}^{n+1}$, consider the following decomposition:

$$z := \begin{pmatrix} \bar{z} \\ z_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}, \qquad \bar{z} \in \mathbb{R}^n, \quad z_{n+1} \in \mathbb{R}. \tag{3.27}$$

By using the above decompositions we have the following result.

Proposition 18. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. Then, f is hyperbolically convex, if and only if

$$\begin{aligned} 4\bar{x}^\top(\bar{A} + \sigma\bar{I})\bar{x} + 4\bar{y}^\top(\bar{A} + \sigma\bar{I})\bar{y} + 8\bar{a}^\top \left(\|\bar{x}\|_2\bar{x} + \|\bar{y}\|_2\bar{y}\right) \\ + \sqrt{2\|\bar{x}\|_2\|\bar{y}\|_2 - 2\bar{x}^\top\bar{y}}\, \left(\bar{b}^\top(\bar{x}+\bar{y}) + b_{n+1}\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\right) \geq 0, \end{aligned}$$

for all $\bar{x}, \bar{y} \in \mathbb{R}^n$.

Proof. First note that, by using (1.2) and (3.27) we conclude that all $x, y \in \partial\mathcal{L}$ can be written

$$x := \begin{pmatrix} \bar{x} \\ \|\bar{x}\|_2 \end{pmatrix} \in \mathbb{R}^{n+1}, \qquad y := \begin{pmatrix} \bar{y} \\ \|\bar{y}\|_2 \end{pmatrix} \in \mathbb{R}^{n+1}. \tag{3.28}$$

Hence, using (3.26), (3.27) and (3.28) we obtain that

$$4x^\top Ax = 4\bar{x}^\top(\bar{A} + \sigma\bar{\mathrm{I}})\bar{x} + 8\|\bar{x}\|_2 \bar{a}^\top \bar{x} \tag{3.29}$$

$$4y^\top Ay = 4\bar{y}^\top(\bar{A} + \sigma\bar{\mathrm{I}})\bar{y} + 8\|\bar{y}\|_2 \bar{a}^\top \bar{y} \tag{3.30}$$

$$\sqrt{-2x^\top Jy} = \sqrt{2\|\bar{x}\|_2\|\bar{y}\|_2 - 2\bar{x}^\top \bar{y}} \tag{3.31}$$

$$b^\top(x+y) = \bar{b}^\top(\bar{x}+\bar{y}) + b_{n+1}(\|\bar{x}\|_2 + \|\bar{y}\|_2). \tag{3.32}$$

By using equations (3.29), (3.30), (3.31) and (3.32), we conclude that

$$4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) = 4\bar{x}^\top(\bar{A} + \sigma\bar{I})\bar{x} + 4\bar{y}^\top(\bar{A} + \sigma\bar{I})\bar{y} +$$
$$8\bar{a}^\top\left(\|\bar{x}\|_2\bar{x} + \|\bar{y}\|_2\bar{y}\right) + \sqrt{2\|\bar{x}\|_2\|\bar{y}\|_2 - 2\bar{x}^\top\bar{y}}\,\left(\bar{b}^\top(\bar{x}+\bar{y}) + b_{n+1}\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\right).$$

Therefore, by using the last equality together with Proposition 15 the desired result follows. $\square$

Remark 7. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then $4\bar{x}^\top(\bar{A} + \sigma\bar{I})\bar{x} + 8\bar{a}^\top\|\bar{x}\|_2\bar{x} \geq 0$, for all $\bar{x} \in \mathbb{R}^n$. To see that set $\bar{y} = 0$ in Proposition 18.

Corollary 11. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then

$$\bar{A} + \left(\sigma + \frac{1}{2}b_{n+1}\right)\bar{\mathrm{I}} \in \mathbb{R}^{n\times n}$$

is positive semidefinite.

Proof. First note that by letting $\bar{y} = -\bar{x}$ in Proposition 18 we obtain that if f is hyperbolically convex then $8\bar{x}^\top \left(\bar{A} + \sigma\bar{\mathrm{I}}\right)\bar{x} + 4b_{n+1}\|\bar{x}\|_2^2 \geq 0$, for all $\bar{x} \in \mathbb{R}^n$. Considering that

$$8\bar{x}^\top \left(\bar{A} + \left(\sigma + \frac{1}{2}b_{n+1}\right)\bar{\mathrm{I}}\right)\bar{x} = 8\bar{x}^\top \left(\bar{A} + \sigma\bar{\mathrm{I}}\right)\bar{x} + 4b_{n+1}\|\bar{x}\|_2^2, \qquad \forall \bar{x} \in \mathbb{R}^n.$$

and f is hyperbolically convex, the result follows. □

Proposition 19. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$. Then, there exists a $\lambda > 0$ such that the function $f_\lambda : \mathbb{H}^n \to \mathbb{R}$ defined by $f_\lambda(p) = p^\top Ap + \left(b - \lambda e^{n+1}\right)^\top p + c$ is not hyperbolically convex.

Proof. Consider the following matrix

$$\bar{A} + \left(\sigma + \frac{1}{2}(b_{n+1} - \lambda)\right)\bar{\mathrm{I}} \in \mathbb{R}^{n\times n} \tag{3.33}$$

The eigenvalues of the matrix (3.33) are of the form $\beta + \sigma + \frac{1}{2}(b_{n+1} - \lambda)$, where β is an eigenvalue of $\bar{A}$. Thus, the matrix (3.33) have negative eigenvalue by taking $\lambda > 0$ sufficiently large. Consequently, the matrix (3.33) will not be positive semidefinite for $\lambda > 0$ sufficiently large. Therefore, by applying Corollary 11 for $f = f_\lambda$ the result follows. □

Theorem 3. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ be a nonzero matrix, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$, $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$ and $g : \mathbb{H}^n \to \mathbb{R}$ be defined by $g(p) = p^\top p + b^\top p + c$. Then, the following statements hold:

(i) If f is hyperbolically convex, then the function $h : \mathbb{H}^n \to \mathbb{R}$ defined by

$$h(p) = p^\top p + \left(b^A\right)^\top p + c$$

is hyperbolically convex, where

$$b^A = \frac{1}{\|A\|_2} b. \tag{3.34}$$

(ii) If there exists a $\mu \in \mathbb{R}$ and a $\lambda \geq 0$ such that the matrix $A - \lambda I + \mu J$ is positive semidefinite and $b + 4\lambda e^{n+1} \in \mathcal{L}$, then f is hyperbolically convex.

(iii) If $b + 4e^{n+1} \in \mathcal{L}$, then g is hyperbolically convex.

Proof. Proof of item (i): Since f is hyperbolically convex, the Cauchy inequality and Proposition 15 imply that

$$4\|A\|_2\|x\|_2^2+4\|A\|_2\|y\|_2^2+\sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) \geq 4x^\top Ax+4y^\top Ay+\sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) \geq 0, \tag{3.35}$$

for any $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial\mathcal{L}$. Divide (3.35) by $\|A\|_2$ and use (3.34) to obtain

$$4x^\top x + 4y^\top y + \sqrt{-2x^\top \mathrm{J}y}\, \left(b^A\right)^\top (x + y) \geq 0,$$

for any $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial\mathcal{L}$. Thus, applying Proposition 15 with $f = h$, it follows that h is hyperbolically convex.

Proof of item (ii): First note that by using Cauchy and triangular inequalities we have

$$\frac{1}{\|\bar{x}\|_2 + \|\bar{y}\|_2}\bar{b}^\top(\bar{x}+\bar{y})+b_{n+1} \geq -\frac{\|\bar{b}\|_2\|\bar{x}+\bar{y}\|_2}{\|\bar{x}\|_2 + \|\bar{y}\|_2}+b_{n+1} \geq b_{n+1}-\|\bar{b}\|_2, \ \forall \bar{x},\bar{y} \in \mathbb{R}^n\backslash\{0\}.$$

Taking into account that $b + 4\lambda e^{n+1} \in \mathcal{L}$, we obtain that $b_{n+1} - \|\bar{b}\|_2 \geq -4\lambda$. Hence, we have

$$\frac{1}{\|\bar{x}\|_2 + \|\bar{y}\|_2}\bar{b}^\top(\bar{x}+\bar{y})+b_{n+1} \geq -4\lambda. \tag{3.36}$$

On the other hand, some calculations show that

$$-4 = \max_{\rho>0}\frac{-8-8\rho^2}{2\sqrt{\rho}(1+\rho)} \geq \frac{-8-8\left(\frac{\|\bar{y}\|_2}{\|\bar{x}\|_2}\right)^2}{2\sqrt{\frac{\|\bar{y}\|_2}{\|\bar{x}\|_2}}\left(1+\frac{\|\bar{y}\|_2}{\|\bar{x}\|_2}\right)} = \frac{-8\|\bar{x}\|_2^2 - 8\|\bar{y}\|_2^2}{\sqrt{4\|\bar{x}\|_2\|\bar{y}\|_2}(\|\bar{x}\|_2 + \|\bar{y}\|_2)}.$$

If $\bar{x}$ and $\bar{y}$ are not parallel, then Cauchy inequality implies that $4\|\bar{x}\|_2\|\bar{y}\|_2 \geq 2\|\bar{x}\|_2\|\bar{y}\|_2 - 2\bar{x}^\top\bar{y} > 0$. Thus, it follows from the last inequality that

$$-4 \geq \frac{-8\|\bar{x}\|_2^2 - 8\|\bar{y}\|_2^2}{\sqrt{2\|\bar{x}\|_2\|\bar{y}\|_2 - 2\bar{x}^\top\bar{y}}\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)}.$$

Combining the previous equality with (3.36) we obtain that

$$\frac{1}{\|\bar{x}\|_2 + \|\bar{y}\|_2}\bar{b}^\top(\bar{x}+\bar{y})+b_{n+1} \geq \lambda\frac{-8\|\bar{x}\|_2^2 - 8\|\bar{y}\|_2^2}{\sqrt{2\|\bar{x}\|_2\|\bar{y}\|_2 - 2\bar{x}^\top\bar{y}}\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)},$$

which, after some algebraic manipulations, can be rewritten equivalently as follows

$$8\lambda\|\bar{x}\|_2^2 + 8\lambda\|\bar{y}\|_2^2 + \sqrt{2\left(\|\bar{x}\|_2\|\bar{y}\|_2 - \bar{x}^\top\bar{y}\right)}\left[\bar{b}^\top(\bar{x}+\bar{y}) + b_{n+1}\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\right] \geq 0. \tag{3.37}$$

The last inequality holds for any $\bar{x}, \bar{y} \in \mathbb{R}^n$, since it is also true for $\bar{x}$ and $\bar{y}$ parallel or if any of x and y is zero. To proceed, note that by using (1.2) and (3.27) we conclude that all $x, y \in \partial\mathcal{L}$ can be written

$$x := \begin{pmatrix} \bar{x} \\ \|\bar{x}\|_2 \end{pmatrix} \in \mathbb{R}^{n+1}, \qquad y := \begin{pmatrix} \bar{y} \\ \|\bar{y}\|_2 \end{pmatrix} \in \mathbb{R}^{n+1}. \tag{3.38}$$

In addition, for any $x, y \in \partial\mathcal{L}$, we have

$$4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top \mathrm{J}y}\, b^\top(x+y) = 4x^\top(A+\mu\mathrm{J})x + 4y^\top(A+\mu\mathrm{J})y + \sqrt{-2x^\top\mathrm{J}y}\, b^\top(x+y).$$

On the other hand, since $A - \lambda I + \mu J$ is positive semidefinite we obtain

$$4x^\top(A+\mu\mathrm{J})x + 4y^\top(A+\mu\mathrm{J})y \geq 4\lambda x^\top x + 4\lambda y^\top y, \qquad \forall x, y \in \mathbb{R}^{n+1},$$

which combined with the last equality yields

$$4x^\top Ax + 4y^\top Ay + \sqrt{-2x^\top\mathrm{J}y}\, b^\top(x+y) \geq 4\lambda x^\top x + 4\lambda y^\top y + \sqrt{-2x^\top\mathrm{J}y}\, b^\top(x+y) \tag{3.39}$$

for any $x, y \in \mathbb{R}^{n+1}$. Now, by using (3.38), we obtain after some calculations that

$$
\begin{aligned}
4\lambda x^\top x + 4\lambda y^\top y + \sqrt{-2x^\top \mathrm{J} y}\, b^\top (x+y) &= 8\lambda \|\bar{x}\|_2^2 + 8\lambda \|\bar{y}\|_2^2 \\
&+ \sqrt{2\left(\|\bar{x}\|_2 \|\bar{y}\|_2 - \bar{x}^\top \bar{y}\right)}\, \left[\bar{b}^\top (\bar{x}+\bar{y}) + b_{n+1}\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\right].
\end{aligned}
$$

for any $x, y \in \partial\mathcal{L}$. Thus, combining (3.37) and (3.39) with the previous equality we obtain that

$$
4x^\top A x + 4y^\top A y + \sqrt{-2x^\top \mathrm{J} y}\, b^\top (x+y) \geq 0, \qquad \forall x, y \in \partial\mathcal{L}.
$$

Therefore, by applying Proposition 15, we conclude that f is hyperbolically convex.

Proof of item (iii): It is a particular case of (ii) with $A = I$, $\lambda = 0$ and $\mu = 1$. □

Corollary 12. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b := (\bar{b}^\top, b_{n+1})^\top \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top A p + b^\top p + c$. If there exists a $\mu \in \mathbb{R}$ such that

$$
A - \frac{1}{4}(\|\bar{b}\|_2 - b_{n+1})^+ \mathrm{I} + \mu \mathrm{J}
$$

is positive semidefinite, then f is hyperbolically convex.

Proof. If $b \in \mathcal{L}$, then $\frac{1}{4}(\|\bar{b}\|_2 - b_{n+1})^+ \mathrm{I} = 0$. Thus, in this case, we are under the following assumptions: $b \in \mathcal{L}$ and $A + \mu \mathrm{J}$ is positive semidefinite. Therefore, it follows from items (i) and (vi) of Proposition 17 that f is hyperbolically convex. Now, assume that $b \notin \mathcal{L}$. In this case, by setting

$$
\lambda = \frac{1}{4}(\|\bar{b}\|_2 - b_{n+1})^+ = \frac{1}{4}(\|\bar{b}\|_2 - b_{n+1}).
$$

we conclude that

$$b + 4\lambda e^{n+1} = \begin{pmatrix} \bar{b} \\ \left\|\bar{b}\right\|_2 \end{pmatrix} \in \mathcal{L}.$$

In this case, by applying item (ii) of Theorem 3, we also obtain that f is hyperbolically convex. $\square$

Example 8. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b := (\bar{b}^\top, b_{n+1})^\top \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. For all $b \notin \mathcal{L}$, we can construct many examples of hyperbolically convex quadratic functions f. Indeed, take any $b \notin \mathcal{L}$, any $\mu \in \mathbb{R}$, any positive semidefinite matrix P and $A = P + \frac{1}{4}(\left\|\bar{b}\right\|_2 - b_{n+1})^+\mathrm{I} + \mu\mathrm{J}$. Then, the hyperbolic convexity of f follows from Corollary 12.

Corollary 13. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then

$$b_{n+1} \geq -4\|A\|_2$$

Proof. From item (i) of Theorem 3 it follows that the function $h(p) = p^\top p + \left(b^A\right)^\top p + c$ is hyperbolically convex, where

$$b^A = \frac{1}{\|A\|_2} b. \tag{3.40}$$

Applying Corollary 11 with $f = h$, it follows that

$$\bar{\mathrm{I}} + \left(1 + \frac{1}{2} b^A_{n+1}\right) \bar{\mathrm{I}} \in \mathbb{R}^{n\times n}$$

is positive semidefinite, which implies that $2 + (1/2)b^A_{n+1} \geq 0$. Therefore, it follows from (3.40) that $2 + (1/2)(b_{n+1}/\|A\|_2) \geq 0$, which is equivalent to the required inequality. $\square$

Corollary 14. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. Then, the following statements hold:

(i) If f is hyperbolically convex, then there exists a $\mu \in \mathbb{R}$ such that $A + \mu J$ is positive semidefinite and $\lambda_{\max}(A + \mu \mathrm{J}) \geq \frac{1}{4} b^-_{n+1}$.

(ii) If there exists a $\mu \in \mathbb{R}$ such that $\lambda_{\min}(A + \mu \mathrm{J}) \geq \frac{1}{4}(\left\|\bar{b}\right\|_2 - b_{n+1})^+$, then f is hyperbolically convex. In particular, if $\bar{b} = 0$ and there exists a $\lambda \in \mathbb{R}$ such that $\lambda_{\min}(A + \mu \mathrm{J}) \geq \frac{1}{4} b^-_{n+1}$, then f is hyperbolically convex.

Proof. Proof of item (i): Suppose that f is hyperbolically convex. From Proposition 14 it follows that there exists a $\mu \in \mathbb{R}$ such that $A + \mu J$ is positive semidefinite. Since

$$f(p) = p^\top (A + \mu \mathrm{J})p + b^\top p + c + \mu$$

for any $p \in \mathbb{H}^n$, is hyperbolically convex, it follows from Corollary 13 that

$$b_{n+1} \geq -4\|A + \mu J\|_2 = -4\lambda_{\max}(A + \mu \mathrm{J}). \tag{3.41}$$

If $b_{n+1} \geq 0$, then $\lambda_{\max}(A + \mu \mathrm{J}) \geq 0 = \frac{1}{4} b^-_{n+1}$. On the other hand, if $b_{n+1} < 0$, then (3.41) implies that $\lambda_{\max}(A + \mu \mathrm{J}) \geq \frac{1}{4} b^-_{n+1}$, which concludes the proof of item (i).

Proof of item (ii): Assume that there exists a $\lambda \in \mathbb{R}$ such that $\lambda_{\min}(A + \lambda \mathrm{J}) \geq \frac{1}{4}(\left\|\bar{b}\right\|_2 - b_{n+1})^+$. Then, the following matrix

$$A - \frac{1}{4}(\left\|\bar{b}\right\|_2 - b_{n+1})^+ \mathrm{I} + \lambda \mathrm{J}$$

is positive semidefinite. Hence, by applying Corollary 12, we conclude that f is hyperbolically convex, which concludes the proof. $\square$

Proposition 20. Let $\rho, c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top p + \rho p_{n+1} + c$. Then, f is hyperbolically convex if and only if $\rho \geq -4$.

Proof. By applying Corollary 14 with $A = \mathrm{I}$, $\mu = 0$, $\bar{b} = 0$ and $b_{n+1} = \rho$, the result follows. $\square$

To state the next theorem it is convenient to introduce the following notation: Denote by $w \not\parallel z$ that neither of the vectors w and z is a nonnegative multiple of the other one.

Theorem 4. Let $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$ and $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top A p + b^\top p + c$. Define

$$\varphi\left(\bar{b}, A\right) = \inf_{\substack{\bar{x},\bar{y} \in \mathbb{R}^n \\ \bar{x} \not\parallel \bar{y}}} \left[\frac{4\left(\bar{x}^\top \bar{A} \bar{x} + \bar{y}^\top \bar{A} \bar{y}\right) + 8\bar{a}^\top \left(\|\bar{x}\|_2 \bar{x} + \|\bar{y}\|_2 \bar{y}\right) + 4\sigma\left(\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2\right)}{\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{2\left(\|\bar{x}\|_2 \|\bar{y}\|_2 - \bar{x}^\top \bar{y}\right)}} + \frac{\bar{b}^\top (\bar{x} + \bar{y})}{\|\bar{x}\|_2 + \|\bar{y}\|_2} \right].$$

Then, f is hyperbolically convex if and only if

$$b_{n+1} + \varphi\left(\bar{b}, A\right) \geq 0.$$

In particular, if $A = \mathrm{I}$, then f is hyperbolically convex if and only if

$$b_{n+1} + \varphi\left(\bar{b}\right) \geq 0,$$

where

$$\varphi\left(\bar{b}\right) = \inf_{\substack{\bar{x},\bar{y} \in \mathbb{R}^n \\ \bar{x} \not\parallel \bar{y}}} \left[\frac{8\left(\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2\right)}{\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{2\left(\|\bar{x}\|_2 \|\bar{y}\|_2 - \bar{x}^\top \bar{y}\right)}} + \frac{\bar{b}^\top (\bar{x} + \bar{y})}{\|\bar{x}\|_2 + \|\bar{y}\|_2} \right].$$

Proof. It follows from Proposition 18 that f is hyperbolically convex, if and only if

$$4\bar{x}^\top(\bar{A}+\sigma\bar{\mathrm{I}})\bar{x}+4\bar{y}^\top(\bar{A}+\sigma\bar{\mathrm{I}})\bar{y}+8\bar{a}^\top\left(\|\bar{x}\|_2\bar{x}+\|\bar{y}\|_2\bar{y}\right)$$
$$+\sqrt{2\|\bar{x}\|_2\|\bar{y}\|_2-2\bar{x}^\top\bar{y}}\left(\bar{b}^\top(\bar{x}+\bar{y})+b_{n+1}\left(\|\bar{x}\|_2+\|\bar{y}\|_2\right)\right)\geq 0,$$

for all $\bar{x},\bar{y}\in\mathbb{R}^n$. The first statement follows after dividing the last inequality by the quantity

$$\left(\|\bar{x}\|_2+\|\bar{y}\|_2\right)\sqrt{2\left(\|\bar{x}\|_2\|\bar{y}\|_2-\bar{x}^\top\bar{y}\right)}$$

and then using the definition of the infimum. The second statement is a particular instance of the first one. $\square$

Corollary 15. Let $A=A^\top\in\mathbb{R}^{(n+1)\times(n+1)}$ be a nonzero matrix, $b\in\mathbb{R}^{n+1}$, $c\in\mathbb{R}$, $f:\mathbb{H}^n\to\mathbb{R}$ be defined by $f(p)=p^\top Ap+b^\top p+c$ and $h:\mathbb{H}^n\to\mathbb{R}$ be defined by $h(p)=p^\top Ap$. Then, the following statements hold:

(i) If $\lambda_{\min}(\bar{A})+\sigma\geq 2\|\bar{a}\|_2$ and $b_{n+1}\geq\left\|\bar{b}\right\|_2+4\|\bar{a}\|_2-2\lambda_{\min}(\bar{A})-2\sigma$, then f is hyperbolically convex. In particular, if $A=\mathrm{I}$ and $b_{n+1}\geq\left\|\bar{b}\right\|_2-4$, then f is hyperbolically convex.

(ii) If $\lambda_{\min}(\bar{A})+\sigma\geq 2\|\bar{a}\|_2$, then h is hypebolically convex.

Proof. Proof of item (i): By using the notations of Theorem 2 and triangular and

Cauchy inequality we have

$$\varphi\left(\bar{b}, A\right) \geq \inf_{\substack{\bar{x},\bar{y}\in\mathbb{R}^n \\ \bar{x}\not\parallel\bar{y}}} \left[\frac{4\left(\bar{x}^\top \bar{A}\bar{x} + \bar{y}^\top \bar{A}\bar{y}\right) + 8\bar{a}^\top\left(\|\bar{x}\|_2\bar{x} + \|\bar{y}\|_2\bar{y}\right) + 4\sigma\left(\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2\right)}{2\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{\|\bar{x}\|_2\|\bar{y}\|_2}} - \left\|\bar{b}\right\|_2\right]$$

$$\geq -\left\|\bar{b}\right\|_2 + \inf_{\substack{\bar{x},\bar{y}\in\mathbb{R}^n \\ \bar{x}\not\parallel\bar{y}}} \frac{\left(2\lambda_{\min}(\bar{A}) - 4\|\bar{a}\|_2 + 2\sigma\right)\left(\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2\right)}{\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{\|\bar{x}\|_2\|\bar{y}\|_2}}$$

Considering that $\lambda_{\min}(\bar{A}) + \sigma \geq 2\|\bar{a}\|_2$ we obtain from the last inequality that

$$\varphi\left(\bar{b}, A\right) \geq -\left\|\bar{b}\right\|_2 + \left(2\lambda_{\min}(\bar{A}) - 4\|\bar{a}\|_2 + 2\sigma\right) \inf_{\substack{\bar{x},\bar{y}\in\mathbb{R}^n \\ \bar{x}\not\parallel\bar{y}}} \frac{\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2}{\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{\|\bar{x}\|_2\|\bar{y}\|_2}}. \tag{3.42}$$

Taking into account that

$$\inf_{\substack{\bar{x},\bar{y}\in\mathbb{R}^n \\ \bar{x}\not\parallel\bar{y}}} \frac{\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2}{\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{\|\bar{x}\|_2\|\bar{y}\|_2}} \geq 1,$$

it follows from (3.42) that $\varphi\left(\bar{b}, A\right) \geq -\left\|\bar{b}\right\|_2 + 2\lambda_{\min}(\bar{A}) - 4\|\bar{a}\|_2 + 2\sigma$. Therefore, considering that $b_{n+1} \geq \left\|\bar{b}\right\|_2 + 4\|\bar{a}\|_2 - 2\lambda_{\min}(\bar{A}) - 2\sigma$, we conclude that $\varphi\left(\bar{b}, A\right) \geq -b_{n+1}$ or equivalently that $b_{n+1} + \varphi\left(\bar{b}, A\right) \geq 0$. Hence, applying Theorem 4 we obtain that f is hyperbolically convex and the first statement of item (i) is proved. The second statement is an immediate consequence of the first one.

Proof of item (ii): Choose $b_{n+1} \geq \left\|\bar{b}\right\|_2 + 4\|\bar{a}\|_2 - 2\lambda_{\min}(\bar{A}) - 2\sigma$. It follows from item (i) that f is hyperbolically convex. Then, Corollary 7 implies that h is also hyperbolically convex. □

Remark 8. Item (ii) of Corollary 15 improves item (iii) of [20, Theorem 5.2], where the inequality is strict.

Example 9. For the particular case in item (i) of Corollary 15, by taking $A = \mathrm{I}$ any $\bar{b} \in \mathbb{R}^n$ and $b_{n+1} \in \left[\|\bar{b}\|_2 - 4, \|\bar{b}\|_2\right)$, another large class of hyperbolically convex quadratic functions f with $b \notin \mathcal{L}$ follows.

Corollary 16. Let $n \geq 3$ be an integer, $A = A^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ be a nonzero matrix, $b \in \mathbb{R}^{n+1}$, $c \in \mathbb{R}$, $f : \mathbb{H}^n \to \mathbb{R}$ be defined by $f(p) = p^\top Ap + b^\top p + c$. If f is hyperbolically convex, then

$$b_{n+1} \geq \frac{\|\bar{b}\|_2}{2} + 2\sqrt{2}\,\frac{\bar{a}^\top \bar{b}}{\|\bar{b}\|_2} - \sqrt{2}\,\frac{\bar{b}^\top \bar{A}\bar{b}}{\|\bar{b}\|_2^2} - \sqrt{2}\,\lambda_{\max}(\bar{A}) - 2\sqrt{2}\,\sigma. \tag{3.43}$$

In particular if f is hyperbolically convex and $A = \mathrm{I}$, then

$$b_{n+1} \geq \frac{\|\bar{b}\|_2}{2} - 4\sqrt{2}. \tag{3.44}$$

Proof. We will use the notation of Theorem 4. We have

$$\varphi(\bar{b}, A) = \inf_{\substack{\bar{x},\bar{y}\in\mathbb{R}^n \\ \bar{x}\not\parallel\bar{y}}} \psi(\bar{x}, \bar{y}, \bar{b}, A),$$

where

$$\psi(\bar{x}, \bar{y}, \bar{b}, A) = \frac{4\left(\bar{x}^\top \bar{A}\bar{x} + \bar{y}^\top \bar{A}\bar{y}\right) + 8\bar{a}^\top\left(\|\bar{x}\|_2\bar{x} + \|\bar{y}\|_2\bar{y}\right) + 4\sigma\left(\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2\right)}{\left(\|\bar{x}\|_2 + \|\bar{y}\|_2\right)\sqrt{2\left(\|\bar{x}\|_2\|\bar{y}\|_2 - \bar{x}^\top\bar{y}\right)}} + \frac{\bar{b}^\top(\bar{x}+\bar{y})}{\|\bar{x}\|_2 + \|\bar{y}\|_2}$$

Hence, $\psi(\bar{x}, \bar{y}, \bar{b}, A) \leq \zeta(\bar{x}, \bar{y}, \bar{b}, A)$, where

$$\zeta(\bar{x}, \bar{y}, \bar{b}, A) = \frac{4\big(\bar{x}^\top \bar{A}\bar{x} + \lambda_{\max}(\bar{A})\|\bar{y}\|_2^2\big) + 8\bar{a}^\top\big(\|\bar{x}\|_2\bar{x} + \|\bar{y}\|_2\bar{y}\big) + 4\sigma\left(\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2\right)}{\big(\|\bar{x}\|_2 + \|\bar{y}\|_2\big)\sqrt{2\left(\|\bar{x}\|_2\|\bar{y}\|_2 - \bar{x}^\top\bar{y}\right)}} + \frac{\bar{b}^\top(\bar{x} + \bar{y})}{\|\bar{x}\|_2 + \|\bar{y}\|_2}$$

and $\varphi(\bar{b}, A) \leq \inf_{\substack{\bar{x}, \bar{y} \in \mathbb{R}^n \\ \bar{x} \not\parallel \bar{y}}} \zeta(\bar{x}, \bar{y}, \bar{b}, A) \leq \zeta(\bar{x}, \bar{y}, \bar{b}, A)$ for any $\bar{x}, \bar{y} \in \mathbb{R}^n$ with $\bar{x} \not\parallel \bar{y}$. Thus, Theorem 4 implies

$$b_{n+1} + \zeta(\bar{x}, \bar{y}, \bar{b}, A) \geq b_{n+1} + \varphi(\bar{b}, A) \geq 0. \tag{3.45}$$

Let $\bar{x} := -\bar{b}$ and $\bar{y} \in \mathbb{R}^n$ such that $\bar{y}^\top\bar{a} = 0$, $\bar{y}^\top\bar{b} = 0$ and $\|y\|_2 = \|b\|_2$. Then, a simple calculation yields that the right hand side of equation (3.43) is $-\zeta(\bar{x}, \bar{y}, \bar{b}, A)$. Hence, (3.43) follows from (3.45) and (3.44) is a simple consequence of (3.43). $\square$

3.4 Optimization concepts on hyperbolic space

In this section we present some concepts of optimization related to hyperbolically convex function. In order to do that, consider a differentiable function $f : \mathbb{H}^n \to \mathbb{R}$ and the following unconstrained optimization problem

$$\text{Minimize}_{p \in \mathbb{H}^n} f(p). \tag{3.46}$$

It follows from (1.12) that a necessary optimality condition for the unconstrained problem (3.46) is:

$$\left[\mathrm{I} + pp^\top \mathrm{J}\right] \mathrm{J} \cdot Df(p) = 0. \tag{3.47}$$

Remark 9. If f is a hyperbolically convex function, then Proposition 7 implies that all points satisfying (3.47) are global solutions of problem (3.46), i.e., (3.47) is also a sufficient optimality condition.

Let $\mathcal{C} \subset \mathbb{H}^n$ be a hyperbolically convex set and consider the constrained optimization problem:

$$\text{Minimize}_{p \in \mathcal{C}} f(p). \tag{3.48}$$

In the following proposition we state the necessary optimality condition for the problem (3.48).

Proposition 21. If the point $\bar{p} \in \mathcal{C}$ is a solution of problem (3.48), then

$$\Big\langle [\mathrm{I} + \bar{p}\bar{p}^\top \mathrm{J}]\mathrm{J} \cdot Df(\bar{p}), p \Big\rangle \geq 0, \quad \forall\ p \in \mathcal{C},$$

where $Df(\bar{p}) \in \mathbb{R}^{n+1}$ is the usual gradient of f at $\bar{p}$.

Proof. Take $p \in \mathcal{C}$ and let $\bar{p} \in \mathcal{C}$ be a solution to (3.48). Let $[0,1] \ni t \mapsto \gamma_{\bar{p}p}(t) = \exp_{\bar{p}}(t \log_{\bar{p}} p)$, be the geodesic from $\bar{p}$ to p. Since $\mathcal{C}$ is hyperbolically convex and $p, \bar{p} \in \mathcal{C}$, we conclude that $\gamma_{\bar{p}p}(t) \in \mathcal{C}$ for all $t \in [0,1]$. Hence, as $\bar{p} \in \mathcal{C}$ is a solution to the problem in (3.48), we have $(f(\gamma_{\bar{p}p}(t)) - f(\bar{p}))/t \geq 0$, for all $t \in [0,1]$. Thus, taking the limit when t tends to zero, we obtain, by using (1.14) and $\gamma'_{\bar{p}p}(0) = \log_{\bar{p}} p$, that $\langle \operatorname{grad} f(\bar{p}), \log_{\bar{p}} p \rangle \geq 0$. Therefore, the result follows from (1.11) and (1.12), by taking into account that $\operatorname{arcosh}(-\langle \bar{p}, p \rangle) \geq 0$.

□

Proposition 22. Let f be a hyperbolically convex function in $\mathcal{C}$. The point $\bar{p} \in \mathcal{C}$

is a solution of the problem in (3.48) if and only if

$$\Big\langle [\mathrm{I} + \bar{p}\bar{p}^\top \mathrm{J}]\mathrm{J} \cdot Df(\bar{p}), p \Big\rangle \geq 0, \quad \forall\, p \in \mathcal{C},$$

where $Df(\bar{p}) \in \mathbb{R}^{n+1}$ is the usual gradient of f at $\bar{p}$.

Proof. If the point $\bar{p} \in \mathcal{C}$ is a solution of (3.48), then the inequality follows from Proposition 21. Conversely, take p, $\bar{p} \in \mathcal{C}$, $p \neq \bar{p}$ and assume that $\big\langle [\mathrm{I} + \bar{p}\bar{p}^\top \mathrm{J}]\mathrm{J} \cdot Df(\bar{p}), p \big\rangle \geq 0$. As f is hyperbolically convex in $\mathcal{C}$, we conclude from Proposition 7 that

$$f(p) \geq f(\bar{p}) + \frac{\operatorname{arcosh}(-\langle \bar{p}, p\rangle)}{\sqrt{1 - \langle \bar{p}, p\rangle^2}} \Big\langle [\mathrm{I} + \bar{p}\bar{p}^\top \mathrm{J}]\mathrm{J} \cdot Df(\bar{p}), p \Big\rangle, \qquad \forall\, p \in \mathcal{C},\ p \neq \bar{p}.$$

Since $\big\langle [\mathrm{I} + \bar{p}\bar{p}^\top \mathrm{J}]\mathrm{J} \cdot Df(\bar{p}), p \big\rangle \geq 0$ and $\operatorname{arcosh}(-\langle \bar{p}, p\rangle) \geq 0$, the latter inequality implies that $f(p) \geq f(\bar{p})$, for all $p \in \mathcal{C}$. Therefore, $\bar{p}$ is a global solution of the problem in (3.48).

□

Next we present an equivalent form for Proposition 3, whose proof follows by combining Proposition 9 with Lemma 9, Proposition 22 and (1.18),

Corollary 17. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a closed hyperbolically convex set and $\bar{p} \in \mathbb{H}^n$. Consider the function $\mathbb{H}^n \ni p \mapsto \rho_{\bar{p}}(p) := \frac{1}{2} d_{\bar{p}}^2(p)$ defined in (1.21). Then, $\mathcal{P}_{\mathcal{C}}(\bar{p}) = \arg\min_{p \in \mathcal{C}} \rho_{\bar{p}}(p)$ if and only if

$$\Big\langle \big(\mathrm{I} + \mathcal{P}_{\mathcal{C}}(\bar{p})\mathcal{P}_{\mathcal{C}}(\bar{p})^\top \mathrm{J}\big)\, \bar{p}, p \Big\rangle \leq 0, \qquad \forall p \in \mathcal{C}.$$

For $f : \mathbb{H}^n \to \mathbb{R}$ and $g_i : \mathbb{H}^n \to \mathbb{R}^m$, $i = 1, \ldots, m$ differentiable hyperbolically convex functions and

$$\mathcal{C} = \{p \in \mathbb{R}^n : g_i(p) \leq 0, i = 1, \ldots, m\}$$

a hyperbolically convex set, see Remark 5. Consider the following particular instance of the hyperbolically convex optimization problem (3.48):

$$\text{Minimize}_{x \in \mathcal{C}} f(x). \tag{3.49}$$

Proposition 23. Suppose that $\bar{p} \in \mathcal{C}$ and there exists $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m_+$ such that

$$\left[\mathrm{I} + \bar{p}\bar{p}^\top \mathrm{J}\right] \left[\mathrm{J} \cdot Df(\bar{p}) + \sum_{i=1}^{m} \mu_i \mathrm{J} \cdot Dg_i(\bar{p})\right] = 0, \qquad \sum_{i=1}^{m} \mu_i g_i(\bar{p}) = 0, \tag{3.50}$$

where $Df(\bar{p}) \in \mathbb{R}^{n+1}$ is the usual gradient of f at $\bar{p}$. Then $\bar{p}$ is a solution of the problem (3.49).

Proof. Since $f, g_i : \mathbb{H}^n \to \mathbb{R}$ are hyperbolically convex functions and $\mu_i \geq 0$, for $i = 1, \ldots, m$, it follows that $h : \mathbb{H}^n \to \mathbb{R}$ defined by

$$h(p) = f(p) + \sum_{i=1}^{m} \mu_i g_i(p)$$

is hyperbolically convex. Moreover, $f(p) \geq h(p)$, for all $p \in \mathcal{C}$. Now, from the first equality in (3.50) we obtain that $\left[\mathrm{I} + pp^\top \mathrm{J}\right] \mathrm{J} \cdot Dh(\bar{p}) = 0$. Thus, since h is hyperbolically convex, we can apply Proposition 22 with $f = h$ and $\mathcal{C} = \mathbb{R}^{n+1}$ to conclude that $\bar{p}$ is a minimizer of h in $\mathbb{R}^{n+1}$. Hence, from $f(p) \geq h(p)$, for all $p \in \mathcal{C}$ and the second equality in (3.50), we have $f(p) \geq h(p) \geq h(\bar{p}) = f(\bar{p})$, for all $p \in \mathcal{C}$. Since $\bar{p} \in \mathcal{C}$, the last inequality implies that it is a solution of (3.49).

$\square$

Remark 10. If $\bar{p}$ is a solution of the problem (3.49), then under mild conditions on the problem (3.49) the point $\bar{p}$ satisfies (3.50), for details see [52, Section 9]; see also [58, Theorem 4.4].

3.5 Conclusion

In this chapter, a comprehensive study of convex quadratic functions on hyperbolic spaces was conducted. Although some of the ideas are similar to those in the papers [16, 19], the convex functions in hyperbolic spaces turned out to have a completely different structure. For instance, there is no constant globally convex function on the whole sphere, but there are many such functions on the hyperbolic space, as demonstrated in Section 3.2. Section 3.3 focused on characterizing non-homogeneous quadratic functions on the hyperbolic space, complementing the findings presented in Section 3.2. Unlike in Euclidean space, studying non-homogeneous quadratic functions in hyperbolic space is more complex than studying homogeneous quadratic functions. This complexity arises because adding a linear term to a hyperbolically convex homogeneous function can cause the newly formed non-homogeneous quadratic function to lose its hyperbolic convexity. We concluded this chapter by presenting optimization concepts related to hyperbolically convex functions in Section 3.4, which will be further studied in Section 4.3.

CHAPTER 4

HYPERBOLIC SPACE FORMS

In this chapter[1] we rescale the Lorentzian metric to have constant negative curvature $-\kappa$ with $\kappa > 0$, which turns the hyperbolic space into the κ-hyperbolic space forms. This part, a continuation of Chapter 2, addresses the natural counterpart in hyperbolic space forms of the problem presented in [15], which involves computing a projection formula into a convex set of the sphere. However, as we will see, the problem in hyperbolic space forms is considerably more difficult. We will propose and analyse two intrinsic versions of the gradient projection method, one with constant step sizes and the other with backtracking step sizes, to solve problems constrained to convex sets and for potentially non-convex objective functions in κ-hyperbolic space forms with a constant sectional curvature $-\kappa < 0$. We begin by reviewing some fundamental properties of κ-hyperbolic space forms and then present novel properties of the intrinsic κ-projection into convex sets of κ-hyperbolic space forms. These properties are essential for analysing the gradient projection method and have intrinsic importance of their own. Specifically, we

[1]The main result in this chapter is based on a submitted paper, which is joint work with Ferreira, O.P.and Németh, S.Z.

introduce the concept of the Lorentz projection and establish the relationship between the intrinsic κ-projection, the Lorentz projection, and the Euclidean orthogonal projection. Additionally, we provide formulas for the intrinsic κ-projection into specific convex sets using the Euclidean orthogonal projection and the Lorentz projection. Concerning the convergence results obtained from the gradient projection method, we demonstrate that every accumulation point of the sequence generated by the method with backtracking step sizes is a stationary point for the problem. Under the assumption of Lipschitz continuity of the gradient of the objective function, we show that each accumulation point of the sequence generated by the gradient projection method with constant step size is a stationary point. Furthermore, we provide an iteration complexity bound that characterizes the number of iterations needed to achieve a suitable measure of stationarity for both step sizes. It is important to emphasize that the findings presented here do not rely on the convexity or coercivity of the objective function, nor on the compactness of the constraint set. We also explore the properties of the constrained version of the problem of computing the centre of mass, commonly referred to as the Fermat-Weber problem. Specifically, we demonstrate that the sequence generated by the gradient projection method with backtracking step sizes converges to the unique solution of this problem.

4.1 Basics results about the κ-hyperbolic space forms

In this section, we provide a review of the notations, definitions, and basic results related to the geometry of the κ-hyperbolic space forms that are utilized throughout the chapter. We present simplified proofs of a few of these results, as they are commonly presented in more general contexts in the existing literature. Hence, this section constitutes a review of the helpful κ-hyperbolic space forms results used in optimization. The introductory books on Riemannian and Differential Geometry that we refer to in this section are [6] and [47], and we also draw insights from [7].

Denote $\mathbb{R}^m_+ = \{x = (x^1, \dots, x^m) \in \mathbb{R}^m : x_1 \geq 0, \dots, x^m \geq 0\}$ the nonnegative orthant of $\mathbb{R}^m$. Let $\{v^1, v^2, \dots, v^m\}$ be a set of vectors of $\mathbb{R}^m$. Then, the cone defined by

$$\operatorname{cone}\left\{v^1, \dots, v^m\right\} := \left\{\lambda_1 v^1 + \cdots + \lambda_m v^m : \ \lambda_1 \geq 0, \dots, \lambda_m \geq 0\right\},$$

is called a *simplicial cone* or *finitely generated cone*. Let $\mathcal{K} \subseteq \mathbb{R}^{n+1}$ be a convex cone. The *Euclidean orthogonal projection set-valued mapping* $\Pi_{\mathcal{K}} : \mathbb{R}^{n+1} \multimap \mathcal{K}$ *into* $\mathcal{K}$ is defined by

$$\Pi_{\mathcal{K}}(x) := \left\{ y \in \mathbb{R}^{n+1} : \ y = \arg\min_{z \in \mathcal{K}} \sqrt{(x-z)^\top (x-z)} \right\}, \tag{4.1}$$

Since $\mathcal{K}$ is convex, $\Pi_{\mathcal{K}}(x)$ is either an empty set or contains a single point. Moreover, it is well known that, if $y \in \Pi_{\mathcal{K}}(x)$ then

$$(x-y)^\top z \leq 0, \ \forall\, z \in \mathcal{K} \qquad \text{and} \qquad (x-y)^\top y = 0, \tag{4.2}$$

see for example [29, Theorem 3.1.1]. Whenever the set $\mathcal{K}$ is closed and convex, the projection set is nonempty and has a unique point, which is characterised by

(4.2). In this case, we use the notation

$$\Pi_{\mathcal{K}}(x) = \arg\min_{z\in\mathcal{K}} \sqrt{(x-z)^\top (x-z)}.$$

Let $\alpha \geq 1$ be a real constant. Denote the *α-circular cone* $\mathcal{L}_\alpha$ by

$$\mathcal{L}_\alpha = \left\{ (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \; x_{n+1} \geq \alpha \sqrt{x_1^2 + \cdots + x_n^2} \right\}. \tag{4.3}$$

When $\alpha = 1$, the α-circular cone $\mathcal{L}_\alpha$ is known as the *Lorentz cone* and will simply be denoted by $\mathcal{L}$. Note that, for all $\alpha > 1$, we have $\mathcal{L}_\alpha \subseteq \operatorname{int}(\mathcal{L})$ and it is a closed and convex cone.

Remark 11. From (4.3) with $\alpha = 1$ and (1.4) we have $\mathcal{L} = \{x := (x_1, \ldots, x_{n+1})^\top \in \mathbb{R}^{n+1} : \; \langle x, x\rangle \leq 0, \; x_{n+1} \geq 0\}$. Furthermore, $\langle x, y\rangle < 0$, for all $x \in \mathcal{L}$ and all

$$y \in \operatorname{int} \mathcal{L} = \{x := (x_1, \ldots, x_{n+1})^\top \in \mathbb{R}^{n+1} : \; \langle x, x\rangle < 0, \; x_{n+1} > 0\},$$

because $\langle x, y\rangle = -y^\top(-\mathrm{J}x)$, $-\mathrm{J}x \in -\mathrm{J}\mathcal{L} = \mathcal{L}$ and $\mathcal{L}$ is self-dual.

Throughout the chapter, for a given $\kappa > 0$, the *n-dimensional κ-hyperbolic space form* and its *tangent hyperplane at p* are denoted by

$$\mathbb{H}^n_\kappa := \left\{ p \in \mathbb{R}^{n+1} : \; \langle p, p\rangle = -\frac{1}{\kappa}, \; p^{n+1} > 0 \right\}, \; T_p\mathbb{H}^n_\kappa := \left\{ v \in \mathbb{R}^{n+1} \; : \; \langle p, v\rangle = 0 \right\}, \tag{4.4}$$

respectively. It is worth noting that the Lorentzian inner product is not positive definite in the entire space $\mathbb{R}^{n+1}$. However, one can show that its restriction to the tangent spaces of $\mathbb{H}^n_\kappa$ is positive definite; see [7, Section 7.6]. Consequently, we have $\|v\| > 0$ for all $v \in T_p\mathbb{H}^n_\kappa$ and all $p \in \mathbb{H}^n_\kappa$ with $v \neq 0$. Hence, $\langle \cdot, \cdot \rangle$ *and* $\|\cdot\|$ *are*

in fact a positive inner product and the associated norm in $T_p\mathbb{H}^n_\kappa$*, for all* $p \in \mathbb{H}^n_\kappa$*.*
Therefore, (1.4) actually defines a Riemannian metric on $\mathbb{H}^n_\kappa$, see [8, pp. 67]. Moreover, $\langle p, q\rangle \leq -1/\kappa$, for all $p, q \in \mathbb{H}^n_\kappa$ and $\langle p, q\rangle = -1/\kappa$ if and only if $p = q$. The *Lorentzian projection into* $T_p\mathbb{H}^n_\kappa$ is the linear mapping $\operatorname{Proj}^\kappa_p : \mathbb{R}^{n+1} \to T_p\mathbb{H}^n_\kappa$ defined by

$$\operatorname{Proj}^\kappa_p x := x + \kappa\langle p, x\rangle p, \tag{4.5}$$

i.e., $\operatorname{Proj}^\kappa_p := \mathrm{I} + \kappa pp^\top \mathrm{J}$, where $\mathrm{I} \in \mathbb{R}^{(n+1)\times(n+1)}$ is the identity matrix.

Remark 12. The Lorentzian projection (4.5) is self-adjoint with respect to the Lorentzian inner product (1.4). Indeed, $\left\langle \operatorname{Proj}^\kappa_p u, v\right\rangle = \left\langle u, \operatorname{Proj}^\kappa_p v\right\rangle$, for all $u, v \in \mathbb{R}^{n+1}$ and all $p \in \mathbb{H}^n_\kappa$. Moreover, we also have $\operatorname{Proj}^\kappa_p \operatorname{Proj}^\kappa_p = \operatorname{Proj}^\kappa_p$, for all $p \in \mathbb{H}^n_\kappa$.

The *intrinsic* κ*-distance on the* κ*-hyperbolic space form* between two points $p, q \in \mathbb{H}^n_\kappa$ is given by

$$d_\kappa(p, q) := \frac{1}{\sqrt{\kappa}}\operatorname{arcosh}(-\kappa \langle p, q\rangle). \tag{4.6}$$

It can be shown that $(\mathbb{H}^n_\kappa, d_\kappa)$ is a complete metric space, so that $d_\kappa(p, q) \geq 0$ for all $p, q \in \mathbb{H}^n_\kappa$, and $d_\kappa(p, q) = 0$ if and only if $p = q$. Moreover, $(\mathbb{H}^n_\kappa, d)$ has the same topology as $\mathbb{R}^n$ and $(\mathbb{H}^n_\kappa, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold with constant sectional curvature $-\kappa < 0$. Next we give two standard notations. We denote the *open* and *closed ball* of radius $\delta > 0$ and center $p \in \mathbb{H}^n_\kappa$ by $B_\delta(p) := \{q \in \mathbb{H}^n_\kappa : d_\kappa(p, q) < \delta\}$ and $\bar{B}_\delta(p) := \{q \in \mathbb{H}^n_\kappa : d_\kappa(p, q) \leq \delta\}$, respectively. The intersection curve of a plane though the origin of $\mathbb{R}^{n+1}$ with $\mathbb{H}^n_\kappa$ is called a *geodesic*. Moreover, each geodesic segment $\gamma : [a, b] \to \mathbb{H}^n_\kappa$ is *minimal*, i.e., its arc length is equal to the intrinsic distance $\ell(\gamma) = \operatorname{arcosh}(-\langle\gamma(a), \gamma(b)\rangle_\kappa)$ between its end points. We say that γ is a *normalized geodesic* if $\|\gamma'\| = 1$. If $p, q \in \mathbb{H}^n_\kappa$ and $q \neq p$, then the unique *geodesic segment from p to q* is

$$\gamma_{pq}(t) = \left(\cosh(\sqrt{\kappa}t) + \frac{\kappa \langle p, q \rangle \sinh(\sqrt{\kappa}t)}{\sqrt{\kappa^2 \langle p, q \rangle^2 - 1}} \right) p + \frac{\sinh(\sqrt{\kappa}t)}{\sqrt{\kappa^2 \langle p, q \rangle^2 - 1}}\, q,\ \forall t \in [0,\ d_\kappa(p, q)].$$

The *κ-exponential mapping* $\exp^\kappa_p : T_p\mathbb{H}^n_\kappa \to \mathbb{H}^n_\kappa$ is defined by $\exp^\kappa_p v = \gamma_v(1)$, where γ_v is the geodesic defined by its initial position p, with velocity v at p. Hence, $\exp^\kappa_p v = p$ for $v = 0$, and

$$\exp^\kappa_p v := \cosh(\sqrt{\kappa}\|v\|)\, p + \sinh(\sqrt{\kappa}\|v\|)\, \frac{v}{\sqrt{\kappa}\|v\|}, \qquad \forall v \in T_p\mathbb{H}^n_\kappa \setminus \{0\}.$$

It is easy to prove that $\gamma_{tv}(1) = \gamma_v(t)$ for any values of t. Therefore, for all $t \in \mathbb{R}$ we have

$$\exp^\kappa_p tv := \cosh(t\sqrt{\kappa}\|v\|)\, p + \sinh(t\sqrt{\kappa}\|v\|)\, \frac{v}{\sqrt{\kappa}\|v\|}, \qquad \forall v \in T_p\mathbb{H}^n_\kappa \setminus \{0\}. \quad (4.7)$$

We will also use the expression above for denoting the geodesic starting at $p \in \mathbb{H}^n_\kappa$ with velocity $v \in T_p\mathbb{H}^n_\kappa$ at p. The *inverse of the κ-exponential mapping* is given by $\log^\kappa_p q = 0$, for $q = p$, and

$$\log^\kappa_p q := \frac{\sqrt{\kappa} d_\kappa(p, q)}{\sqrt{\kappa^2 \langle p, q \rangle^2 - 1}} \operatorname{Proj}^\kappa_p q = d_\kappa(p, q) \frac{\operatorname{Proj}^\kappa_p q}{\| \operatorname{Proj}^\kappa_p q\|}, \qquad q \neq p. \quad (4.8)$$

It follows from (4.8) that

$$d_\kappa(p, q) = \| \log^\kappa_p q\|, \qquad p, q \in \mathbb{H}^n_\kappa. \quad (4.9)$$

In the next lemma, we recall the well-known "cosine law" for triangles in the κ-hyperbolic space form. Although the proof is a straightforward application of (4.8), we include it here for the sake of completeness.

Lemma 13. Let $x, y, z \in \mathbb{H}_\kappa^n$. Let θ_x be the angle between the vectors $\log_x^\kappa y$ and $\log_x^\kappa z$. Then,

$$\begin{aligned}\cosh(\sqrt{\kappa}d_\kappa(y,z)) &= \cosh(\sqrt{\kappa}d_\kappa(x,y))\cosh(\sqrt{\kappa}d_\kappa(x,z)) \\ &\quad - \sinh(\sqrt{\kappa}d_\kappa(x,y))\sinh(\sqrt{\kappa}d_\kappa(x,z))\cos\theta_x.\end{aligned}$$

Proof. First we apply (4.9) to obtain that $\langle \log_x^\kappa y, \log_x^\kappa z\rangle = d_\kappa(x,y)d_\kappa(x,z)\cos\theta_x$. Now, by using (4.8) we obtain that

$$\langle \log_x^\kappa y, \log_x^\kappa z\rangle = \frac{\sqrt{\kappa}d_\kappa(x,y)}{\sqrt{\kappa^2\left\langle x,y\right\rangle^2-1}}\frac{\sqrt{\kappa}d_\kappa(x,z)}{\sqrt{\kappa^2\left\langle x,z\right\rangle^2-1}}\left\langle \left(\mathrm{I}+\kappa x x^\top \mathrm{J}\right)y, \left(\mathrm{I}+\kappa x x^\top \mathrm{J}\right)z\right\rangle.$$

By combining the two previous inequalities and performing some manipulations, we obtain that

$$\cos\theta_x = \frac{1}{\sqrt{\kappa^2\left\langle x,y\right\rangle^2-1}}\frac{1}{\sqrt{\kappa^2\left\langle x,z\right\rangle^2-1}}\left(\left\langle y,z\right\rangle+\kappa\left\langle x,z\right\rangle\left\langle y,x\right\rangle\right).$$

Since $\kappa\langle y,z\rangle = -\cosh(\sqrt{\kappa}d_\kappa(y,z))$, $\kappa\langle x,z\rangle = -\cosh(\sqrt{\kappa}d_\kappa(x,z))$ and $\kappa\langle y,x\rangle = -\cosh(\sqrt{\kappa}d_\kappa(y,x))$, the desired equality follows. $\square$

In the following, we will recall the well-known "comparison theorem" for triangles in the κ-hyperbolic space form, as stated in [48, Proposition 4.5]. However, we will provide a direct proof of the theorem using Lemma 13 as a reference.

Lemma 14. The following inequality holds:

$$d_\kappa^2(x,y) + d_\kappa^2(x,z) - 2\left\langle \log_x^\kappa y, \log_x^\kappa z\right\rangle \leq d_\kappa^2(y,z), \qquad \forall x,y,z \in \mathbb{H}_\kappa^n. \tag{4.10}$$

Proof. If we have two equal points, the inequality holds. Then, without loss of generality, assume that the three points are pairwise different. Let θ_x be the angle between the vectors $\log_x^\kappa y$ and $\log_x^\kappa z$. Thus, by using (4.9) we conclude that $\langle \log_x^\kappa y, \log_x^\kappa z\rangle = d_\kappa(x,y)d_\kappa(x,z)\cos\theta_x$. Hence, (4.10) is equivalent to

$$\sqrt{d_\kappa^2(x,y) + d_\kappa^2(x,z) - 2d_\kappa(x,y)d_\kappa(x,z)\cos\theta_x} \leq d_\kappa(y,z), \qquad \forall x,y,z \in \mathbb{H}_\kappa^n. \tag{4.11}$$

Since the function $[0,+\infty) \ni t \mapsto \cosh(t)$ is increasing the inequality (4.11) is equivalent to

$$\cosh\sqrt{\kappa\left(d_\kappa^2(x,y) + d_\kappa^2(x,z) - 2d_\kappa(x,y)d_\kappa(x,z)\cos\theta_x\right)} \leq \cosh\left(\sqrt{\kappa}d_\kappa(y,z)\right), \tag{4.12}$$

for all $x,y,z \in \mathbb{H}_\kappa^n$. On the other hand, it follows from Lemma 13 that

$$\begin{aligned}\cosh(\sqrt{\kappa}d_\kappa(y,z)) = {} & \cosh(\sqrt{\kappa}d_\kappa(x,y))\cosh(\sqrt{\kappa}d_\kappa(x,z)) \\ & - \sinh(\sqrt{\kappa}d_\kappa(x,y))\sinh(\sqrt{\kappa}d_\kappa(x,z))\cos\theta_x\end{aligned}$$

Hence, by combining (4.12) with the last equality, we conclude that (4.10) is also equivalent to

$$\cosh \sqrt{\kappa \left(d_\kappa^2(x,y) + d_\kappa^2(x,z) - 2d_\kappa(x,y)d_\kappa(x,z)\cos\theta_x\right)} \leq$$
$$\cosh(\sqrt{\kappa}d_\kappa(x,y))\cosh(\sqrt{\kappa}d_\kappa(x,z)) - \sinh(\sqrt{\kappa}d_\kappa(x,y))\sinh(\sqrt{\kappa}d_\kappa(x,z))\cos\theta_x, \tag{4.13}$$

for all $x, y, z \in \mathbb{H}_\kappa^n$. Therefore, our task is reduced to proving the inequality (4.13). For simplifying the notations, we set $a := \sqrt{\kappa}d_\kappa(x,y)$, $b := \sqrt{\kappa}d_\kappa(x,z)$ and $t := \cos\theta_x$. In view of (4.13), define the auxiliary function $\phi : [-1,1] \to \mathbb{R}$ as follows

$$\phi(t) := \cosh a \cosh b - t \sinh a \sinh b - \cosh \varphi(t),$$

where $\varphi(t) := \sqrt{a^2 + b^2 - 2abt}$. Hence, to prove (4.13) it is sufficient to show that $\phi(t) \geq 0$, for all $t \in [-1,1]$. To accomplish this, we first observe that $\phi(-1) = \cosh(a+b) - \cosh(a+b) = 0$ and $\phi(1) = \cosh(a-b) - \cosh|a-b| = 0$. Moreover, we have

$$\phi''(t) = (ab)^2 \frac{\sinh\varphi(t) - \varphi(t)\cosh\varphi(t)}{\varphi(t)^3}. \tag{4.14}$$

Since $a, b > 0$, we have $a^2 + b^2 - 2abt = (a-b)^2 + 2ab(1-t) > 0$, for all $t \in (-1,1)$. Thus, $\varphi(t) > 0$, for all $t \in (-1,1)$. By considering that $\sinh\tau < \tau\cosh\tau$, for all $\tau > 0$, it follows from (4.14) that $\phi''(t) < 0$, for all $t \in (-1,1)$. Therefore, ϕ is concave function in $t \in (-1,1)$ and taking into account that $\phi(-1) = \phi(1) = 0$ we conclude that $\phi(t) \geq 0$, for all $t \in [-1,1]$, which implies the desired inequality. □

Let $\Omega \subseteq \mathbb{H}_\kappa^n$ be an open set and $f : \Omega \to \mathbb{R}$ be a differentiable function. The *gradient on the κ-hyperbolic space form* of f is the unique vector field $\Omega \ni p \mapsto \operatorname{grad} f(p) \in T_p\mathcal{M}$ such that $df(p)v = \langle \operatorname{grad} f(p), v\rangle$, see [7, Proposition 7-5, p.162].

Therefore, we have

$$\operatorname{grad} f(p) := \operatorname{Proj}_p^{\kappa} \mathrm{J} f'(p) = \mathrm{J} f'(p) + \kappa \left\langle \mathrm{J} f'(p), p \right\rangle p, \tag{4.15}$$

where $f'(p) \in \mathbb{R}^{n+1}$ is the usual gradient of f at $p \in \Omega$. A *vector field* on $\Omega \subseteq \mathbb{H}_{\kappa}^{n}$ is a smooth mapping $X : \Omega \to \mathbb{R}^{n+1}$ such that $X(p) \in T_p\mathbb{H}_{\kappa}^{n}$. The *covariant derivative* of X at $p \in \Omega$ is map $\nabla X(p) : T_p\mathbb{H}_{\kappa}^{n} \to T_p\mathbb{H}_{\kappa}^{n}$ given by

$$\nabla X(p) := \operatorname{Proj}_p^{\kappa} X'(p),$$

where $X'(p)$ denotes the usual derivative of the vector field X at the point p, see [7, Formula (7.62), p.162]. The *Hessian on the κ-hyperbolic space form* of a twice differentiable function $f : \Omega \to \mathbb{R}$ at a point $p \in \Omega$ is the mapping $\nabla \operatorname{grad} f(p) := \operatorname{Hess} f(p) : T_p\mathbb{H}_{\kappa}^{n} \to T_p\mathbb{H}_{\kappa}^{n}$ given by

$$\operatorname{Hess} f(p) := \operatorname{Proj}_p^{\kappa} \left[\mathrm{J} f''(p) + \kappa \left\langle \mathrm{J} f'(p), p \right\rangle \mathrm{I} \right], \tag{4.16}$$

where $f''(p)$ is the usual Hessian (Euclidean Hessian) of f at p, see [7, Proposition 7.6, p.163]. For each $p, q \in \mathbb{H}_{\kappa}^{n}$ the covariant derivative induces the linear isometry relative to the Lorentzian inner product $\langle \cdot, \cdot \rangle$, namely, the *parallel transport* $P_{pq} \colon T_p\mathbb{H}_{\kappa}^{n} \to T_q\mathbb{H}_{\kappa}^{n}$ given by

$$P_{pq}(v) := v - \frac{\left\langle v, \log_q^{\kappa} p \right\rangle}{d_{\kappa}^2(p,q)} \left(\log_q^{\kappa} p + \log_p^{\kappa} q \right) = \left[\mathrm{I} - \frac{\kappa}{\kappa \left\langle p, q \right\rangle - 1} (p+q) q^{\top} \mathrm{J} \right] v.$$

Note that for any geodesic segment $\gamma : [a,b] \to \mathbb{H}_{\kappa}^{n}$ we have $\gamma'(t) = P_{\gamma(a)\gamma(t)}(\gamma'(a))$, for all $t \in [a,b]$, or equivalently $\gamma''(t) = 0$, for all $t \in [a,b]$.

Definition 3. Let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable function and $\mathcal{D} \subseteq \mathbb{H}^n_\kappa$. The gradient vector field $\operatorname{grad} f$ is said to be Lipschitz continuous on $\mathcal{D}$ with constant $L \geq 0$ if the following inequality holds $\|P_{pq} \operatorname{grad} f(p) - \operatorname{grad} f(q)\| \leq L d_\kappa(p, q)$, for all $p, q \in \mathcal{D}$.

The next lemma can be found, for example, in [7, Corollary 10.47].

Lemma 15. Let $f : \mathcal{D} \to \mathbb{R}$ be a twice continuously differentiable function and $\mathcal{D} \subseteq \mathbb{H}^n_\kappa$. The gradient vector field $\operatorname{grad} f$ is Lipschitz continuous on $\mathcal{D}$ with constant $L \geq 0$ if and only if there exists $L \geq 0$ such that $\|\operatorname{Hess} f(p)\| \leq L$, for all $p \in \mathcal{D}$.

Subject to a few minor adjustments, the next lemma closely mirrors [7, Proposition 10.54]

Lemma 16. Let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable function and $\mathcal{D} \subseteq \mathbb{H}^n_\kappa$. Assume that $\operatorname{grad} f$ is Lipschitz continuous on a convex set $\mathcal{C} \subseteq \mathcal{D}$ with constant $L \geq 0$. Then, it follows that

$$f(q) \leq f(p) + \left\langle \operatorname{grad} f(p), \log^\kappa_p q \right\rangle + \frac{L}{2} d^2_\kappa(p, q), \qquad \forall p, q \in \mathcal{C}.$$

It is worth noting that convex sets in κ-hyperbolic space forms $\mathbb{H}^n_\kappa$ have a close relationship with convex cones situated in the interior of the Lorentz cone.

Definition 4. The set $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ is called *κ-hyperbolically convex* if for any p, $q \in \mathcal{C}$ the geodesic segment joining p to q is contained in $\mathcal{C}$.

In the next proposition we relate a κ-hyperbolically convex set with the cone spanned by it, its proof can be found in [20, Proposition 1].

Proposition 24. The set $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ is κ-hyperbolically convex if and only if the cone $\mathcal{K}_\mathcal{C}$ is convex.

Remark 13. The κ-hyperbolically closed convex sets are intersections of the hyperboloid with convex cones which belong to the Lorentz cone $\mathcal{L}$. Indeed, it follows from Proposition 24, that if $\mathcal{K} \subseteq \mathcal{L}$ is a convex cone, where $\mathcal{L}$ is the Lorentz cone, then $\mathcal{C} = \mathcal{K} \cap \mathbb{H}^n_\kappa$ is a κ-hyperbolically convex set and $\mathcal{K} = \mathcal{K}_\mathcal{C}$.

4.2 Intrinsic κ-projection into κ-hyperbolically convex sets

In this section we present new properties of the intrinsic κ-projection into κ-hyperbolically convex sets on κ-hyperbolic space forms. Let $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ be a closed κ-hyperbolically convex set and $p \in \mathbb{H}^n_\kappa$. Consider the following constrained optimization problem

$$\min_{q \in \mathcal{C}} d_\kappa(p, q). \tag{4.17}$$

The minimal value of the function $\mathcal{C} \ni q \mapsto d_\kappa(p, q)$ is called the *distance of p from $\mathcal{C}$* and it is denoted by $d_\mathcal{C}(p)$, i.e., $d_{\kappa,\mathcal{C}} : \mathbb{H}^n_\kappa \to \mathbb{R}$ is defined by

$$d_{\kappa,\mathcal{C}}(p) := \min_{q \in \mathcal{C}} d_\kappa(p, q).$$

Since $(\mathbb{H}^n_\kappa, d_\kappa)$ is a complete metric space, [20, Proposition 2] implies that $|d_{\kappa,\mathcal{C}}(p) - d_{\kappa,\mathcal{C}}(q)| \leq d_\kappa(p, q)$, for all $p, q \in \mathbb{H}^n_\kappa$. In particular, the function $d_{\kappa,\mathcal{C}}$ is continuous. Note that the problem (4.17) has a solution because $\mathcal{C}$ is a closed set and the distance function is continuous. The solution of problem (4.17) is referred to as the "metric κ-projection", which was first studied in [55]. Similar to the proof in [20, Proposition 4.2], we can also demonstrate that the "intrinsic κ-projection mapping" $\mathcal{P}_{\mathcal{C}}^{\kappa} : \mathbb{H}_{\kappa}^{n} \to \mathcal{C}$ defined by

$$\mathcal{P}_{\mathcal{C}}^{\kappa}(p) := \arg\min_{q \in \mathcal{C}} d_{\kappa}(p, q). \tag{4.18}$$

is well-defined. In other words, for each $p \in \mathbb{H}^n$, the minimizer $\mathcal{P}_{\mathcal{C}}^{\kappa}(p)$ of problem (4.18) exists and is unique. The following proposition can also be proven in a similar manner to [20, Proposition 4.4].

Proposition 25. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a nonempty closed κ-hyperbolically convex set. Then $\mathcal{P}_{\mathcal{C}}^{\kappa}$ is continuous.

Note that the definition of the intrinsic κ-projection mapping (4.18) yields

$$d_{\kappa}(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(p)) \leq d_{\kappa}(p, q), \qquad \forall\, q \in \mathcal{C}. \tag{4.19}$$

Furthermore, we can prove that equations (4.18) and (4.19) are equivalent to the inequality:

$$\langle \mathrm{Proj}_{\mathcal{P}_{\mathcal{C}}^{\kappa}(p)}^{\kappa}\, p, \mathrm{Proj}_{\mathcal{P}_{\mathcal{C}}^{\kappa}(p)}^{\kappa}\, q \rangle \leq 0, \quad \forall q \in \mathcal{C}, \forall p \in \mathbb{H}_{\kappa}^{n}. \tag{4.20}$$

By considering that, for all $p, q \in \mathbb{H}_{\kappa}^{n}$, we have $\langle \mathcal{P}_{\mathcal{C}}^{\kappa}(p), q \rangle \leq -1/\kappa$, we can deduce from (4.8) that (4.20) can also be equivalently stated as follows:

$$\langle \log_{\mathcal{P}_{\mathcal{C}}^{\kappa}(p)}^{\kappa}\, p, \log_{\mathcal{P}_{\mathcal{C}}^{\kappa}(p)}^{\kappa}\, q \rangle \leq 0, \quad \forall q \in \mathcal{C}, \forall p \in \mathbb{H}_{\kappa}^{n}, \tag{4.21}$$

as shown in [22, Corollary 3.1]. Furthermore, since the function $[1, +\infty] \ni \tau \mapsto$

arcosh(τ) is increasing, it follows from (4.6) that (4.18), (4.20), and (4.21) are also equivalent to:

$$\mathcal{P}_{\mathcal{C}}^{\kappa}(p) := \arg\min_{q\in\mathcal{C}} (-\langle p, q\rangle). \tag{4.22}$$

Therefore, by combining (4.6) with (4.19), we conclude that (4.22) can be stated equivalently as follows:

$$\langle p, q\rangle \leq \langle p, \mathcal{P}_{\mathcal{C}}^{\kappa}(p)\rangle, \quad \forall q \in \mathcal{C}. \tag{4.23}$$

As an application of Lemma 14, we have the following property of the intrinsic κ projection.

Lemma 17. Let $\mathcal{C} \subseteq \mathbb{H}_{\kappa}^{n}$ be a closed κ-hyperbolically convex set, and let $q \in \mathbb{H}_{\kappa}^{n}$. Then, the following inequality holds:

$$d_{\kappa}^{2}(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(q)) \leq \langle \log_{p}^{\kappa} q, \log_{p}^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q)\rangle + \langle \log_{q}^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q), \log_{p}^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q)\rangle, \quad \forall p \in \mathcal{C}. \tag{4.24}$$

As a consequence, we have:

$$d_{\kappa}^{2}(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(q)) \leq \langle \log_{p}^{\kappa} q, \log_{p}^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q)\rangle, \quad \forall p \in \mathcal{C}. \tag{4.25}$$

Furthermore, we have $d_{\kappa}(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(q)) \leq d_{\kappa}(p, q)$ for all $p \in \mathcal{C}$.

Proof. Let $p \in \mathcal{C}$ and $q \in \mathbb{H}_{\kappa}^{n}$. By applying Lemma 14 with $x = p$, $y = \mathcal{P}_{\mathcal{C}}^{\kappa}(q)$ and $z = q$, we obtain:

$$d_{\kappa}^{2}(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(q)) + d_{\kappa}^{2}(p, q) - 2\langle \log_{p}^{\kappa} q, \log_{p}^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q)\rangle \leq d_{\kappa}^{2}(\mathcal{P}_{\mathcal{C}}^{\kappa}(q), q). \tag{4.26}$$

Similarly, by applying Lemma 14 with $x = \mathcal{P}_{\mathcal{C}}^{\kappa}(q)$, $y = p$ and $z = q$, we have:

$$d_{\kappa}^2(\mathcal{P}_{\mathcal{C}}^{\kappa}(q), p) + d_{\kappa}^2(\mathcal{P}_{\mathcal{C}}^{\kappa}(q), q) - 2\langle \log_q^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q), \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(q)\rangle \leq d_{\kappa}^2(p, q). \tag{4.27}$$

By combining (4.26) with (4.27), we conclude that (4.24) holds. Therefore, since $p \in \mathcal{C}$, by applying (4.21), the inequality (4.25) follows from (4.24). The last inequality of the lemma is a direct consequence of the second Cauchy inequality and (4.9). □

Based on Lemma 17, we can derive the following lemma, which highlights a crucial property of the intrinsic κ-projection. This property holds significant importance in the analysis of the gradient projection method.

Proposition 26. Let $p \in \mathcal{C}$, $0 \neq v \in T_p\mathbb{H}_{\kappa}^n$ and $\alpha > 0$. Then, we have

$$\left\langle v, \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}\left(\exp_p^{\kappa}(-\alpha v)\right)\right\rangle \leq -\frac{1}{\alpha} d_{\kappa}^2\left(p, \mathcal{P}_{\mathcal{C}}^{\kappa}\left(\exp_p^{\kappa}(-\alpha v)\right)\right). \tag{4.28}$$

Proof. For simplicity, let us define $\gamma(\alpha) := \exp_p(-\alpha v)$. This allows us to express the vector v as $v = -\frac{1}{\alpha} \log_p^{\kappa} \gamma(\alpha)$. After performing some algebraic manipulations, we can obtain the following expression:

$$\begin{aligned}\left\langle v, \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(\gamma(\alpha))\right\rangle &= -\frac{1}{\alpha} d_{\kappa}^2(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(\gamma(\alpha))) \\ &\quad + \frac{1}{\alpha}\left(d_{\kappa}^2(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(\gamma(\alpha))) - \left\langle \log_p^{\kappa} \gamma(\alpha), \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(\gamma(\alpha))\right\rangle\right). \end{aligned} \tag{4.29}$$

Since $p \in \mathcal{C}$, we can apply Lemma 17 with $q = \gamma(\alpha)$, which gives us

$$d_{\kappa}^2(p, \mathcal{P}_{\mathcal{C}}^{\kappa}(\gamma(\alpha))) - \left\langle \log_p^{\kappa} \gamma(\alpha), \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}^{\kappa}(\gamma(\alpha))\right\rangle \leq 0.$$

Combining this inequality with (4.29), we can conclude that:

$$\left\langle v, \log_p^\kappa \mathcal{P}_\mathcal{C}^\kappa(\gamma(\alpha))\right\rangle \leq -\frac{1}{\alpha} d_\kappa^2(p, \mathcal{P}_\mathcal{C}^\kappa(\gamma(\alpha))).$$

Since $\gamma(\alpha) := \exp_p(-\alpha v)$, the last inequality is equivalent to (4.28), and this concludes the proof. $\square$

4.2.1 Computing the intrinsic κ-projection into convex sets

In this section, we will demonstrate how to explicitly compute the intrinsic κ-projection for certain special convex sets. Specifically, for these special convex sets, we can simplify the computation of the intrinsic κ-projection on $\mathcal{C} \subseteq \mathbb{H}_\kappa^n$ by reducing it to the calculation of the orthogonal Euclidean projection on its associated cone $\mathcal{K}_\mathcal{C}$.

Proposition 27. Let $\mathcal{C} \subseteq \mathbb{H}_\kappa^n$ be a closed κ-hyperbolically convex set, $p := (p_1, \ldots, p_n, p_{n+1})^\top \in \mathbb{H}_\kappa^n$ and $u := (u_1, \ldots, u_n, u_{n+1})^\top \in \mathcal{K}_\mathcal{C} \setminus \{0\}$. Consider the following statements:

(i)

$$(p_{n+1} - u_{n+1}) \left(q^{n+1} - \frac{u^{n+1}}{\sqrt{-\kappa \langle u, u \rangle}}\right) \geq 0, \qquad \forall q := (q_1, \ldots, q_n, q_{n+1})^\top \in \mathcal{C}; \tag{4.30}$$

(ii)

$$\left(2p_{n+1}e^{n+1} - u\right)^\top \left(q - \frac{u}{\sqrt{-\kappa \langle u, u \rangle}}\right) \geq 0, \qquad \forall q \in \mathcal{C}; \tag{4.31}$$

(iii) If $u \in \Pi_{\mathcal{K}_{\mathcal{C}}}(p)$, then

$$\mathcal{P}^{\kappa}_{\mathcal{C}}(p) = \frac{u}{\sqrt{-\kappa \langle u, u \rangle}}. \tag{4.32}$$

Then, the following implications hold: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. To begin, for the sake of simplifying notation, let us consider a given $u \in \mathbb{H}^n_\kappa$. We will set

$$\eta := \sqrt{-\kappa\langle u, u\rangle}.$$

Proof of (i) $\Longrightarrow$ (ii): Let u and q be vectors in $\mathbb{H}^n_\kappa$. Since u/η and q are in $\mathbb{H}^n_\kappa$, we can conclude that

$$\left\langle \frac{u}{\eta}, \frac{u}{\eta} \right\rangle = -\frac{1}{\kappa}, \qquad \left\langle \frac{u}{\eta}, q \right\rangle \leq -\frac{1}{\kappa}. \tag{4.33}$$

By combining the equality and inequality in (4.33) and considering (1.6), we obtain

$$(-\mathrm{J}u)^\top \left(q - \frac{u}{\eta} \right) = -\eta \left\langle \frac{u}{\eta}, q - \frac{u}{\eta} \right\rangle \geq 0. \tag{4.34}$$

Since $-u + \mathrm{J}u = 2u_{n+1}e^{n+1}$, inequality (4.30) can be equivalently stated as

$$\frac{1}{2}(2p_{n+1}e^{n+1} - u + \mathrm{J}u)^\top \left(q - \frac{u}{\eta} \right) \geq 0, \tag{4.35}$$

where $e^{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Therefore, by combining (4.34) with (4.35), we obtain (4.31).

Proof of (ii) $\Longrightarrow$ (iii): Since $\mathcal{C} \subset \mathbb{H}^n_\kappa$ is a closed κ-hyperbolically convex set, Proposition 24 implies that $\mathcal{K}_{\mathcal{C}}$ is a convex cone. Take $q \in \mathcal{C}$ and $u \in \mathbb{R}^{n+1}$ such that $u \in \Pi_{\mathcal{K}_{\mathcal{C}}}(p)$. Due to $\eta q \in \mathcal{K}_{\mathcal{C}}$ and (4.2), we have $0 \geq (p-u)^\top(\eta q - u)$. Thus, we can conclude that

$$\left\langle p, \frac{u}{\eta} \right\rangle \geq \langle p, q\rangle - \langle p, q\rangle + \left\langle p, \frac{u}{\eta} \right\rangle + \frac{1}{\eta}(p-u)^\top(\eta q - u). \tag{4.36}$$

On the other hand, using (4.31), we have

$$\begin{aligned} -\langle p, q\rangle + \tfrac{1}{\eta}\langle p, u\rangle + \tfrac{1}{\eta}(p-u)^\top(\eta q - u) &= -(\mathrm{J}p)^\top q + \tfrac{1}{\eta}(\mathrm{J}p)^\top u + \tfrac{1}{\eta}(p-u)^\top(\eta q - u) \\ &= -(\mathrm{J}p)^\top\left(q - \tfrac{1}{\eta}u\right) + (p-u)^\top\left(q - \tfrac{1}{\eta}u\right) \\ &= (p - \mathrm{J}p - u)^\top\left(q - \tfrac{1}{\eta}u\right) \\ &= (2p_{n+1}e^{n+1} - u)^\top\left(q - \tfrac{1}{\eta}u\right) \geq 0, \end{aligned} \tag{4.37}$$

where $e^{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Hence, by combining (4.36) and (4.37), we have

$$\left\langle p, \frac{u}{\eta} \right\rangle \geq \langle p, q\rangle, \qquad \forall q \in \mathcal{C}.$$

Therefore, (4.23) implies (4.32). □

In the following examples we use Proposition 27 to present formulas to intrinsic κ-projection for certain special convex sets.

Example 10. Let $\mathcal{K} = \operatorname{cone}\{v^1, \ldots, v^n, v^{n+1}\}$ be a simplicial cone, where $(v^i)^\top e^{n+1} = 0$ with $v^i \neq 0$, for all $i = 1, \ldots, n$ and $v^{n+1} = e^{n+1}$. Consider a convex set $\mathcal{C}$ and corresponding cone spanned by the convex set $\mathcal{C}$ given, respectively, by

$$\mathcal{C} := \mathcal{K} \cap \mathbb{H}^n_\kappa, \qquad \mathcal{K}_{\mathcal{C}} := \{p : \ p \in \operatorname{Int}(\mathcal{L}) \cap \mathcal{K}\} \cup \{0\}.$$

We notice that $\mathcal{K}_{\mathcal{C}}$ is a convex cone. If $u \in \Pi_{\mathcal{K}_{\mathcal{C}}}(p)$, then the intrinsic κ-projection into the convex set $\mathcal{C}$ is given by

$$\mathcal{P}_{\mathcal{C}}^{\kappa}(p) = \frac{u}{\sqrt{-\kappa \langle u, u \rangle}}. \tag{4.38}$$

Indeed, take $p := (p_1, \ldots, p_n, p_{n+1}) \in \mathbb{H}_{\kappa}^n$ and $u := (u_1, \ldots, u_n, u_{n+1}) \in \mathcal{C}$. If $u \in \Pi_{\mathcal{K}_{\mathcal{C}}}(p)$, then $u_{n+1} = p_{n+1}$. In fact, suppose, on the contrary, that $u_{n+1} \neq p_{n+1}$. Since $u \in \Pi_{\mathcal{K}_{\mathcal{C}}}(p)$ and $u_{n+1} \neq p_{n+1}$, we infer that $p \notin \mathcal{K}_{\mathcal{C}}$ and u is on the boundary of $\mathcal{K}$. Hence, there exists $j \in \{1, \ldots, n, n+1\}$, such that

$$u \in \mathcal{F} := \operatorname{cone}\left\{\left\{v^1, \ldots, v^n, v^{n+1}\right\} \setminus \left\{v^j\right\}\right\}. \tag{4.39}$$

As $u \in \Pi_{\mathcal{K}_{\mathcal{C}}}(p)$, we infer that u belongs to the boundary of $\mathcal{K}$. Hence, u is on one of the $(n-1)$-dimensional faces of $\mathcal{K}$. If $j = n+1$, then $u \in \operatorname{cone}\{v^1, \ldots, v^n\}$. Thus, $u^\top e^{n+1} = 0$. Since, $e^{n+1} \in \mathcal{K}_{\mathcal{C}}$, it follows from (4.2) that $p_{n+1} = (p-u)^\top e^{n+1} \leq 0$, but this is absurd, because $p \in \mathbb{H}_{\kappa}^n$. Hence, it follows that $j \in \{1, \ldots, n\}$. By the definition of the projection and (4.39), it follows that $p - u$ must be perpendicular to the face $\mathcal{F}$. On the other hand, due to $j \in \{1, \ldots, n\}$, it follows that $e^{n+1} = v^{n+1} \in \mathcal{F}$. Hence, $p_{n+1} - u_{n+1} = (p-u)^\top e^{n+1} = 0$ and we have $u_{n+1} = p_{n+1}$. As consequence, item (i) of Proposition 27 holds, i.e., (4.30) holds. Therefore, by using item (iii) of Proposition 27, we obtain that the projection of p into $\mathcal{C}$ is in fact given by (4.38).

In the next example we use Example 10 to present an explicit formula to compute the intrinsic κ-projection into the following κ-hyperbolically convex set

$$\mathcal{C}_+ = \{p \in \mathbb{H}_{\kappa}^n : \ p \geq 0\}, \tag{4.40}$$

involving only nonnegative constraints, which are fundamental in many fields of science and engineering, see [10].

Example 11. Let us compute a formula for κ-projecting intrinsically a point into $\mathcal{C}_+$ given by (4.40). To achieve that, it is important to note that the cone spanned by the set $\mathcal{C}_+$ can be expressed as

$$\mathcal{K}_{\mathcal{C}_+} := (\mathbb{R}^{n+1}_+ \cap \operatorname{Int}(\mathcal{L})) \cup \{0\}, \tag{4.41}$$

which is a convex cone. It follows from $\mathcal{K}_{\mathcal{C}_+} \subseteq \mathbb{R}^n_+$ and (4.1) that

$$\inf_{z \in (\operatorname{Int}(\mathcal{L}) \cap \mathbb{R}^n_+) \cup \{0\}} \|p - z\| \geq \inf_{z \in \mathbb{R}^n_+} \|p - z\|. \tag{4.42}$$

On the other hand, we know that

$$\{p^+\} = \Pi_{\mathbb{R}^n_+}(p) = \arg\min_{z \in \mathbb{R}^n_+} \|p - z\|. \tag{4.43}$$

The combination of (4.3) for $\alpha = 1$ with (4.4) implies that $p = (p_1, \dots, p_n, p_{n+1}) \in \operatorname{Int}(\mathcal{L})$. Hence, considering the fact that $p_{n+1} \geq \frac{1}{\sqrt{\kappa}} > 0$, we obtain that

$$p^+_{n+1} = p_{n+1} > \sqrt{p_1^2 + \cdots + p_n^2} \geq \sqrt{(p_1^+)^2 + \cdots + (p_n^+)^2},$$

where $p^+ := (p_1^+, \dots, p_n^+, p_{n+1}^+)$ and $p_i^+ := \max\{p_i, 0\}$ for $i = 1, \dots, n, n+1$. Thus, we have $p^+ \in (\operatorname{Int}(\mathcal{L}) \cap \mathbb{R}^n_+) \cup \{0\}$, which combined with (4.42) and (4.43) yields

$$p^+ = \Pi_{\mathcal{K}_{\mathcal{C}_+}}(p).$$

Hence, as $\mathbb{R}^n_+ = \text{cone}\,\{e^1, \dots, e^n, e^{n+1}\}$ and $\mathcal{C}_+ := \mathbb{R}^n_+ \cap \mathbb{H}^n_\kappa$, it follows from Example 10 that the projection of $p \in \mathbb{H}^n_\kappa$ into the set $\mathcal{C}$ is given by

$$\mathcal{P}^\kappa_{\mathcal{C}_+}(p) = \frac{p^+}{\sqrt{-\kappa\,\langle p^+, p^+\rangle}}.$$

Example 12. Let $\mathcal{C} := \mathcal{L}_\alpha \cap \mathbb{H}^n_\kappa$, where $\mathcal{L}_\alpha$ is a circular cone and $\alpha \geq 1$ is a real constant. Since $\alpha \geq 1$, we have $\mathcal{K}_\mathcal{C} = \mathcal{L}_\alpha$ and it is a closed convex cone. If $p \in \mathbb{H}^n_\kappa$ and $u = \Pi_{\mathcal{K}_\mathcal{C}}(p)$, then $u \neq 0$ and (4.30) holds. Indeed, since $e^{n+1} \in \mathcal{K}_\mathcal{C}$ and $u = \Pi_{\mathcal{K}_\mathcal{C}}(p)$, it follows from (4.2) that

$$p_{n+1} - u_{n+1} = (p-u)^\top e^{n+1} \leq 0. \tag{4.44}$$

On the other hand, since $q \in \mathcal{C} = \mathcal{K}_\mathcal{C} \cap \mathbb{H}^n_\kappa$, we have $q^2_{n+1} \geq \alpha^2(q_1^2 + \cdots + q_n^2) = \alpha^2(q^2_{n+1} - \frac{1}{\kappa})$, which after some algebraic manipulations implies that

$$q_{n+1} \leq \frac{\alpha}{\sqrt{\kappa(\alpha^2-1)}}, \tag{4.45}$$

with equality holding if and only if q is on the boundary of $\mathcal{C}$. If $p \in \mathcal{C}$, then the projection formula is trivial. Suppose that $p \notin \mathcal{C}$. Thus, u is on the boundary of $\mathcal{K}_\mathcal{C}$, which implies that the point

$$\frac{u}{\sqrt{-\kappa\langle u, u\rangle}}$$

is on the boundary of the set $\mathcal{C}$. Hence, by using the same argument as the one used to show (4.45), we can show that

$$\frac{u_{n+1}}{\sqrt{-\kappa\langle u,u\rangle}} = \frac{\alpha}{\sqrt{\kappa(\alpha^2-1)}}.$$

Thus, by combining the last equality with (4.45), we obtain that

$$q_{n+1} - \frac{u_{n+1}}{\sqrt{-\kappa\langle u,u\rangle}} \leq 0. \tag{4.46}$$

It follows from (4.44) and (4.46) that (4.30) holds. Therefore, item (iii) of Proposition 27 implies that

$$\mathcal{P}^{\kappa}_{\mathcal{C}}(p) = \frac{u}{\sqrt{-\kappa\left\langle u,u\right\rangle}}. \tag{4.47}$$

4.2.2 The Lorentz projection into convex sets

In this section, we will introduce the Lorentz projection and explore its connections with the Euclidean orthogonal projection and the intrinsic κ-projection.

Definition 5. Let $\mathcal{K} \subseteq \mathbb{R}^{n+1}$ be a convex set, $x \in \mathbb{R}^{n+1}$ and $y \in \mathcal{K}$. The point y is called a *Lorentz projection* (or shortly *L-projection*) of x into $\mathcal{K}$ if $\langle x-y, z-y\rangle \leq 0$, for all $z \in \mathcal{K}$.

Denote by $\Lambda_{\mathcal{K}}(x)$ the set of *L-projections* of $x \in \mathbb{R}^{n+1}$ into $\mathcal{K}$, i.e.,

$$\Lambda_{\mathcal{K}}(x) := \{y \in \mathcal{K} : \ \langle x-y, z-y\rangle \leq 0, \ \forall z \in \mathcal{K}\}. \tag{4.48}$$

As the Lorentzian inner product is not positive definite, given $x \in \mathbb{R}^{n+1}$, the set $\Lambda_{\mathcal{K}}(x)$ might be empty or even unbounded. However, the general analysis of $\Lambda_{\mathcal{K}}(x)$

is out of the scope of this section. As we will see later, if $p \in \mathbb{H}^n_\kappa$ and $\mathcal{K} \subseteq \mathcal{L}$ is a cone such that $\mathcal{K} \cap \mathbb{H}^n_\kappa$ is a closed κ-hyperbolically convex set, then $\Lambda_{\mathcal{K}}(p)$ is nonempty and contains precisely one non-zero vector in addition to the zero vector. In the next remark we show that zero vector belongs to the L-projections.

Remark 14. If $\mathcal{K} \subseteq \mathcal{L}$ such that $\mathcal{K} \cap \mathbb{H}^n_\kappa$ is a κ-hyperbolically convex set and $p \in \mathbb{H}^n_\kappa$, then it follows from $p \in \mathbb{H}^n_\kappa \subseteq \operatorname{int}(\mathcal{L})$, (4.48) and Remark 11 that $0 \in \Lambda_{\mathcal{K}}(p)$, for all $p \in \mathbb{H}^n_\kappa$.

In the following lemma, we present equivalent conditions for a point to be an L-projection of a point into a cone. They are corresponding to relations (4.2) fulfilled by the Euclidean orthogonal projection.

Lemma 18. Let $\mathcal{K} \subseteq \mathbb{R}^{n+1}$ be a convex cone, $x \in \mathbb{R}^{n+1}$ and $y \in \mathcal{K}$. Then, $y \in \Lambda_{\mathcal{K}}(x)$ if and only if the following two relations hold:

$$\langle x - y, z \rangle \leq 0, \ \forall\, z \in \mathcal{K} \qquad \text{and} \qquad \langle x - y, y \rangle = 0. \tag{4.49}$$

Proof. We have that y is an L-projection of x into $\mathcal{K}$ if and only if

$$\langle x - y, z - y \rangle \leq 0, \tag{4.50}$$

for all $z \in \mathcal{K}$. Hence, we must prove that (4.50) is equivalent to (4.49). Suppose that (4.50) holds. Then, $\langle x - y, \lambda z - y \rangle \leq 0$, for all $z \in \mathcal{K}$ and $\lambda > 0$. By letting $\lambda = 2$ and $z = y$, we obtain $\langle x - y, y \rangle \leq 0$, while by letting $\lambda \to 0$ we obtain $\langle x - y, y \rangle \geq 0$. Hence, $\langle x - y, y \rangle = 0$, which inserted into (4.50) also implies $\langle x - y, z \rangle \leq 0$. Therefore, (4.50) holds. Next, suppose that (4.49) holds. Then, (4.50) follows from the bilinearity of the Lorentzian inner product. $\square$

In the following corollary we use Lemma 18 to compute a L-projection into a hyperplane.

Corollary 18. Let $a \in \mathbb{R}^n$ be such that $\langle a, a\rangle \neq 0$ and $\mathcal{V} := \{x \in \mathbb{R}^n : \; a^\top x = 0\}$ be a hyperplane. Then

$$\Lambda_{\mathcal{V}}(y) \setminus \{0\} \ni y - \frac{\langle y, \mathrm{J}a\rangle}{\langle a, a\rangle}\mathrm{J}a, \qquad \forall y \in \mathbb{R}^{n+1}.$$

Proof. Since $\mathcal{V}$ is a hyperplane, its follows from Lemma 18 that $u \in \Lambda_{\mathcal{V}}(y) \setminus \{0\}$ if and only if the following equality holds

$$(\mathrm{J}y - \mathrm{J}u)^\top x = \langle y - u, x\rangle \leq 0, \qquad \forall x \in \mathcal{V}. \tag{4.51}$$

Since a is the normal of the hyperplane $\mathcal{V}$, it follows from (4.51) that there exists $t \in \mathbb{R}$ such that $\mathrm{J}y - \mathrm{J}u = ta$, or equivalently,

$$u = y - t\mathrm{J}a \tag{4.52}$$

Using (4.52) we obtain that $\langle u, \mathrm{J}a\rangle = \langle y, \mathrm{J}a\rangle - t\langle \mathrm{J}a, \mathrm{J}a\rangle$. Since $u \in \mathcal{V}$, we have $\langle u, \mathrm{J}a\rangle = 0$, and taking into account that $\langle \mathrm{J}a, \mathrm{J}a\rangle = \langle a, a\rangle \neq 0$, we conclue that $t = \langle y, \mathrm{J}a\rangle / \langle a, a\rangle$. Substituting t given by the last equality into (4.52) we obtain the desired result. □

In the following, we will revisit the concept of a linear cone-complementarity problem and explore its potential connection to (4.49) in a particular case.

Definition 6. Let $\mathcal{K} \subseteq \mathbb{R}^{n+1}$ be a closed convex cone, $M \in \mathbb{R}^{(n+1)\times(n+1)}$ be a matrix and $q \in \mathbb{R}^{n+1}$ be a vector. Then, the *linear cone-complementarity problem*

LCP$(q, M, \mathcal{K})$ defined by q, M and $\mathcal{K}$ is the problem of finding an $y \in \mathcal{K}$ such that $z := q + My \in \mathcal{K}^*$ and $y^\top z = 0$.

In the upcoming corollary, we establish a connection between the L-projection and a particular linear cone-complementarity problem. To achieve that, denote by $\overline{\mathcal{S}}$ the *topological closure* of a set $\mathcal{S} \subseteq \mathbb{R}^{n+1}$.

Corollary 19. Let $\mathcal{K} \subseteq \mathbb{R}^{n+1}$ be a convex cone. Then, $y \in \Lambda_{\mathcal{K}}(x)$ if and only if $y \in \mathcal{K}$ and it is a solution of the linear cone-complementarity problem LCP$(-\mathrm{J}x, \mathrm{J}, \overline{\mathcal{K}})$.

Proof. From Lemma 18 we obtain that $y \in \Lambda_{\mathcal{K}}(x)$ if and only if $\langle x - y, y \rangle = 0$ and $\langle x - y, z \rangle \leq 0$, for any $z \in \mathcal{K}$. The latter two formulas are equivalent to $(\mathrm{J}y - \mathrm{J}x)^\top y = 0$ and $(\mathrm{J}y - \mathrm{J}x)^\top z \geq 0$, for any $z \in \mathcal{K}$, respectively. Hence, $y \in \Lambda_{\mathcal{K}}(x)$ if and only if $(\mathrm{J}y - \mathrm{J}x)^\top y = 0$ and $\mathrm{J}y - \mathrm{J}x \in \mathcal{K}^*$. Therefore, $y \in \Lambda_{\mathcal{K}}(x)$ if and only if it is a solution of the linear cone-complementarity problem LCP$(-\mathrm{J}(x), \mathrm{J}, \overline{\mathcal{K}})$. $\square$

In the subsequent theorem, we demonstrate that every nonzero L-projection gives rise to an intrinsic κ-projection, which can be obtained by solving the linear complementarity problem presented in the previous corollary.

Theorem 5. Let $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ be a closed κ-hyperbolically convex set, $p \in \mathbb{H}^n_\kappa$ and $u \in \Lambda_{\mathcal{K}_\mathcal{C}}(p) \setminus \{0\}$. Then,

$$\mathcal{P}^\kappa_\mathcal{C}(p) = \frac{u}{\sqrt{-\kappa \langle u, u \rangle}}. \tag{4.53}$$

Proof. Let $p \in \mathbb{H}^n_\kappa$ and $u \in \Lambda_{\mathcal{K}_\mathcal{C}}(p) \setminus \{0\} \subseteq \mathcal{L}$. Due to $u \in \Lambda_{\mathcal{K}_\mathcal{C}}(p) \setminus \{0\} \subseteq \operatorname{int} \mathcal{L}$, we have

$$\frac{u}{\sqrt{-\kappa \langle u, u \rangle}} \in \mathbb{H}^n_\kappa.$$

Hence, we obtain that $\langle u/\sqrt{-\kappa\langle u,u\rangle}, q\rangle \leq -1/\kappa$, for any $q \in \mathbb{H}^n_\kappa$. Thus, we conclude that

$$\langle u, q\rangle \leq -\frac{1}{\kappa}\sqrt{-\kappa\,\langle u,u\rangle} < 0, \qquad \forall q \in \mathbb{H}^n_\kappa. \tag{4.54}$$

On the other hand, the conditions (4.49) in Lemma 18 imply, respectively, that

$$\langle p, q\rangle \leq \langle u, q\rangle\,, \qquad \langle p, u\rangle = \langle u, u\rangle\,, \qquad \forall q \in \mathcal{C} \subseteq \mathcal{K}_{\mathcal{C}}. \tag{4.55}$$

By using the inequality in (4.55) together with the second inequality in (4.54), we conclude that $\langle p, q\rangle \leq \langle u, q\rangle < 0$, for all $q \in \mathcal{C}$. Since the second inequality in (4.54) yields $\langle u, q\rangle^2 \geq -(1/\kappa)\,\langle u, u\rangle$, we obtain from the last inequality and the equality in (4.55) that

$$\langle p, q\rangle^2 \geq \langle u, q\rangle^2 \geq -\frac{1}{\kappa}\,\langle u, u\rangle = -\frac{1}{\kappa}\,\langle p, u\rangle = \frac{\langle p, u\rangle^2}{-\kappa\,\langle u, u\rangle} = \left\langle p, \frac{u}{\sqrt{-\kappa\langle u,u\rangle}}\right\rangle^2, \ \forall q \in \mathcal{C}.$$

Since $p, q \in \mathbb{H}^n_\kappa$ implies $\langle p, q\rangle < 0$, it follows from the last inequality that

$$\langle p, q\rangle \leq \left\langle p, \frac{u}{\sqrt{-\kappa\langle u,u\rangle}}\right\rangle, \qquad \forall q \in \mathcal{C}.$$

Therefore, by using (4.23), we deduce the desired equality in (4.53). $\square$

The following theorem establishes an equivalence between the intrinsic κ-projection and the L-projection. In particular, it shows that the L-projection contains a nonzero vector.

Theorem 6. Let $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ be a closed κ-hyperbolically convex set, $p \in \mathbb{H}^n_\kappa$ and $v \in \mathcal{C}$. Then, the following statements hold:

(i) We have $v = \mathcal{P}^\kappa_{\mathcal{C}}(p)$, if and only if $-\kappa\,\langle p, v\rangle\, v \in \Lambda_{\mathcal{K}_{\mathcal{C}}}(p) \setminus \{0\}$.

(ii) The set $\Lambda_{\mathcal{K}_{\mathcal{C}}}(p) \setminus \{0\} \neq \varnothing$.

Proof. Item (i): Suppose $v = \mathcal{P}^{\kappa}_{\mathcal{C}}(p)$ and let $u := -\kappa \langle p, v \rangle v$. Since $p, v \in \mathbb{H}^n_\kappa$, it follows that $-\kappa \langle p, v \rangle > 0$ and $\langle p + \kappa \langle p, v \rangle v, v \rangle = 0$. Hence, we conclude that

$$u \in \mathcal{K}_{\mathcal{C}} \setminus \{0\}, \qquad \langle p - u, u \rangle = 0. \tag{4.56}$$

On the other hand, for any $z \in \mathcal{K}_{\mathcal{C}}$, we have

$$q := \frac{z}{\sqrt{-\kappa \langle z, z \rangle}} \in \mathcal{C}. \tag{4.57}$$

Thus, by combining (4.6) with Lemma 13 and denoting θ_v as the angle between the vectors $\log^{\kappa}_{v} p$ and $\log^{\kappa}_{v} q$, we can deduce, after performing some calculations, that

$$\kappa \langle p, q \rangle + \kappa^2 \langle v, p \rangle \langle v, q \rangle = \sqrt{1 - \kappa^2 \langle v, p \rangle^2} \sqrt{1 - \kappa^2 \langle v, q \rangle^2} \cos \theta_v. \tag{4.58}$$

Since $v = \mathcal{P}^{\kappa}_{\mathcal{C}}(p)$ and θ_v is the angle between the vectors $\log^{\kappa}_{v} p$ and $\log^{\kappa}_{v} q$, it follows from (4.21) that $\cos \theta_v \leq 0$. Thus, by using (4.58), we infer that

$$\kappa \langle p, q \rangle + \kappa^2 \langle v, p \rangle \langle v, q \rangle \leq 0.$$

Hence, (4.57), $u = -\kappa \langle p, v \rangle v$ and the last inequality imply that

$$\langle p - u, z \rangle = \frac{1}{\kappa} \sqrt{-\kappa \langle z, z \rangle} \left(\kappa \langle p, q \rangle + \kappa^2 \langle p, v \rangle \langle v, q \rangle \right) \leq 0, \qquad \forall z \in \mathcal{K}_{\mathcal{C}}.$$

Therefore, the latter inequality, (4.56) and Lemma 18 imply that $u \in \Lambda_{\mathcal{K}_{\mathcal{C}}}(p) \setminus \{0\}$, and the proof is concluded.

Conversely, suppose that $u := -\kappa \langle p, v \rangle v \in \Lambda_{\mathcal{K}_{\mathcal{C}}}(p) \setminus \{0\}$. Since $v \in \mathcal{C} \subseteq \mathbb{H}^n_\kappa$, it follows that $\langle v, v \rangle = -1/\kappa$. Hence, the results follow from Theorem 5 after performing some calculations.

Item (ii): Since $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ is closed κ-hyperbolically convex set, for each $p \in \mathbb{H}^n_\kappa$ there exists $v = \mathcal{P}^\kappa_{\mathcal{C}}(p)$. Therefore, the result is a direct consequence of item (i). □

In the following corollary, we assume that $\mathcal{K} \subseteq \mathcal{L}$ is a convex cone such that $\mathcal{K} \cap \mathbb{H}^n_\kappa$ is a closed κ-hyperbolically convex set, and 0 belongs to $\mathcal{K}$. We will prove various properties of the L-projection of a point $p \in \mathrm{int}(\mathcal{L})$ into $\mathcal{K}$.

Corollary 20. Let $p \in \mathrm{int}(\mathcal{L})$, $\mathcal{K} \subseteq \mathcal{L}$ be a convex cone, such that $\mathcal{C} := \mathcal{K} \cap \mathbb{H}^n_\kappa$ is a closed κ-hyperbolically convex set and $0 \in \mathcal{K}$. Then, the following statements hold:

(i) $0 \in \Lambda_{\mathcal{K}}(p)$ and $\Lambda_{\mathcal{K}}(p) \setminus \{0\}$ has exactly one element.

(ii) If $p \notin \mathcal{K}$, then $\Lambda_{\mathcal{K}}(p) \subseteq \partial \mathcal{K}$, where ∂ denotes boundary.

(iii) If $p \in \mathcal{K}$, then $\Lambda_{\mathcal{K}}(p) = \{0, p\}$.

(iv) $u = (u_1, \ldots, u_n, u_{n+1})^\top \in \Lambda_{\mathcal{K}}(p) \iff u = \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right) \iff u = \Pi_{\tau_u(\bar{\mathcal{K}})}(p)$, where

$$\tau_u(\bar{\mathcal{K}}) := \left\{\tau_u(x) : \; x = (x_1, \ldots, x_n, x_{n+1})^\top \in \bar{\mathcal{K}}\right\}$$

and $\tau_y : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined by $\tau_y(z) := z + 2(y_{n+1} - z_{n+1})e^{n+1}$ with $y = (y_1, \ldots, y_n, y_{n+1})^\top$ and $z = (z_1, \ldots, z_n, z_{n+1})^\top$.

(v) We have $\varnothing \neq \Lambda_{\mathcal{K}}(p) \setminus \{0\} = \Pi_{\mathcal{K}}(p)$, if and only if

$$(\Lambda_{\mathcal{K}}(p) \setminus \{0\}) \cap \{x \in \mathbb{R}^{n+1} : \ x_{n+1} = p_{n+1}\} \neq \varnothing.$$

(vi) Finding an $u = (u_1, \dots, u_n, u_{n+1})^\top \in \Lambda_{\mathcal{K}}(p)$ is equivalent to determining u_{n+1} from any of the following equations

$$u_{n+1} = \left(e^{n+1}\right)^\top \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right), u_{n+1} = \left(e^{n+1}\right)^\top \Pi_{\bar{\mathcal{K}}+2\left[u_{n+1}-(e^{n+1})^\top \bar{\mathcal{K}}\right]e^{n+1}}(p),$$

followed by using any of the following equations

$$u = \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right), \qquad u^{n+1} = \Pi_{\bar{\mathcal{K}}+2\left[u_{n+1}-(e^{n+1})^\top \bar{\mathcal{K}}\right]e^{n+1}}(p),$$

where $\bar{\mathcal{K}} + 2\left[u_{n+1} - (e^{n+1})^\top \bar{\mathcal{K}}\right] e^{n+1} := \left\{x + 2(u_{n+1} - (e^{n+1})^\top x)e^{n+1} : x \in \bar{\mathcal{K}}\right\}$.

(vii) Finding the $u = (u_1, \dots, u_n, u_{n+1})^\top \in \Lambda_{\mathcal{K}}(p) \setminus \{0\}$ is equivalent to determining $u_{n+1} \neq 0$ from the following equation

$$u_{n+1} = \left(e^{n+1}\right)^\top \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right),$$

followed by using the equation $u = \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right)$.

(viii) For any $\lambda > 0$ we have $\Lambda_{\mathcal{K}}(\lambda p) = \lambda\Lambda_{\mathcal{K}}(p)$.

(ix) If $\Lambda_{\mathcal{K}}(p) \setminus \{0\} = \lambda\Pi_{\mathcal{K}}(p)$, for some $\lambda > 0$, then

$$\Lambda_K(p) \setminus \{0\} = \frac{\langle p, \Pi_{\mathcal{K}}(p)\rangle}{\langle \Pi_{\mathcal{K}}(p), \Pi_{\mathcal{K}}(p)\rangle}\Pi_{\mathcal{K}}(p).$$

(x) Suppose that $\Pi_{\mathcal{K}}(p) \neq \varnothing$. We have $\Lambda_{\mathcal{K}}(p) \setminus \{0\} = \lambda \Pi_{\mathcal{K}}(p)$, for some $\lambda > 0$ if and only if

$$\Pi_{\mathcal{K}}(p) = \arg\min \left\{ \frac{\langle p, z \rangle}{\langle \Pi_{\mathcal{K}}(p), z \rangle} : z \in \mathcal{K} \setminus \{0\} \right\}. \tag{4.59}$$

(xi) Let $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ with $Q^\top \mathrm{J} Q = \mathrm{J}$. Then, the set $Q\mathcal{C}$ is κ-hyperbolically convex and

$$\Lambda_{\mathcal{K}_{Q\mathcal{C}}}(p) = \Lambda_{Q\mathcal{K}}(p) = \Lambda_{\mathcal{K}}\left(Q^\top \mathrm{J} p\right).$$

Proof. Item (i): We will use the following notations: For $\kappa > 0$ and $a \in \mathcal{L} \setminus \{0\}$ we denote

$$\eta(a) := \sqrt{-\kappa \langle a, a \rangle}, \qquad \mu(a) := \frac{a}{\eta(a)} \in \mathbb{H}^n_\kappa.$$

It follows from Remark 14 that $0 \in \Lambda_{\mathcal{K}}(p)$. Let $p \in \mathrm{int}(\mathcal{L})$ and $\mathbb{H}^n_\kappa$ be the κ-hyperbolic space form such that $\kappa := -1/\langle p, p \rangle$. We have $\mathcal{K} = \mathcal{K}_{\mathcal{C}}$. Since $\mathcal{C}$ is a closed κ-hyperbolically convex set, $\mathcal{P}^\kappa_{\mathcal{C}}(p)$ contains exactly one element. Let $\hat{u}, \tilde{u} \in \Lambda_{\mathcal{K}}(p) \setminus \{0\}$. Then, Theorem 5 implies that $\mu(\hat{u}) = \mu(\tilde{u}) = \mathcal{P}^\kappa_{\mathcal{C}}(p)$. Hence, $\tilde{u} = \lambda \hat{u}$, where $\lambda := \eta(\tilde{u})/\eta(\hat{u}) > 0$. On the other hand, Lemma 18 yields $\langle \hat{u} - p, \hat{u} \rangle = 0$ and

$$\langle p - \lambda \hat{u}, \hat{u} \rangle = \frac{1}{\lambda} \langle p - \tilde{u}, \tilde{u} \rangle = 0.$$

By summing up the last two inequalities, we get $(1 - \lambda) \langle \hat{u}, \hat{u} \rangle = 0$. Therefore, based on Remark 11, which implies that $\langle \hat{u}, \hat{u} \rangle < 0$, we conclude that $\lambda = 1$. Hence, $\hat{u} = \tilde{u}$. Hence, $\Lambda_{\mathcal{K}}(p) \setminus \{0\}$ contains at most one element. Therefore, item (ii) of Theorem 6 implies that $\Lambda_{\mathcal{K}}(p) \setminus \{0\}$ contains exactly one element.

Item (ii): Let $p \in \mathrm{int}(\mathcal{L})$ and $\mathbb{H}^n_\kappa$ be the κ-hyperbolic space form such that $\kappa :=$

$-1/\langle p,p\rangle$. We have $\mathcal{K}=\mathcal{K}_{\mathcal{C}}$, where $\mathcal{C}=\mathcal{K}\cap\mathbb{H}^n_\kappa$. Since $p\notin\mathcal{K}$ it follows that $p\notin\mathcal{C}$. This, together with $\mathcal{C}$ being a closed κ-hyperbolically convex set, implies $\mathcal{P}^\kappa_{\mathcal{C}}(p)\in\partial\mathcal{C}$. Hence, formula (4.53) of Theorem 5 implies that $\Lambda_{\mathcal{K}}(p)\subseteq\partial\mathcal{K}$.

Item (iii): From the definition of the L-projection and $p\in\mathcal{L}$ it follows that $p\in\Lambda_{\mathcal{K}}(p)$. Hence, the result is an immediate consequence of (i).

Item (iv): The first equivalence follows from Corollary 19, because the right hand side of the equivalence is the natural equation of $\operatorname{LCP}\big(-\mathrm{J}p,\mathrm{J},\overline{\mathcal{K}}\big)$, see [13, Proposition 1.5.8]. The second equivalence follows from the formula $\langle y-x,z-x\rangle=(y-x)^\top(\tau_y(z)-x)$.

Item (v): Suppose that $\varnothing\neq\Lambda_{\mathcal{K}}(p)\setminus\{0\}=\Pi_{\mathcal{K}}(p)$ and let $u\in\Lambda_{\mathcal{K}}(p)\setminus\{0\}=\Pi_{\mathcal{K}}(p)$. Then, Lemma 18 and equation (4.2) imply $0=\langle p-u,u\rangle-(p-u)^\top u=2\,(p_{n+1}-u_{n+1})\,u_{n+1}=0$. Hence, $u_{n+1}=0$ or $u_{n+1}=p_{n+1}$. If $u_{n+1}=0$, then item (iv) implies $u=\Pi_{\mathcal{K}}(\mathrm{J}p)=0$, where the latter equality follows from $\mathrm{J}p\in\mathrm{J}\mathcal{K}\subseteq\mathrm{J}\mathcal{L}=-\mathcal{L}=-\mathcal{L}^*$. Hence, $u=0$, which is a contradiction. Thus, $u_{n+1}\neq 0$ and therefore $u_{n+1}=p_{n+1}$. Conversely, suppose that $u\in\{x\in\mathbb{R}^{n+1}:x_{n+1}=p_{n+1}\}$. Then, $(p-u)^\top(u-z)=\langle p-u,u-z\rangle\leq 0$, for all $z\in\mathcal{K}$. Therefore, $u\in\Lambda_{\mathcal{K}}(p)\setminus\{0\}$ if and only if $u\in\Pi_{\mathcal{K}}(p)$.

Item (vi): It is an immediate consequence of (iv) and (i).

Item (vii): It is an immediate consequence of (vi) and $\Pi_{\mathcal{K}}(\mathrm{J}p)=0$, which follows from

$$\mathrm{J}p\in\mathrm{J}\mathcal{L}=-\mathcal{L}=-\mathcal{L}^*=-\left(\operatorname{int}(\mathcal{L})\cup\{0\}\right)^*\subseteq-\mathcal{K}^*.$$

Item (viii): It follows from Lemma 18.

Item (ix): Suppose that $\Lambda_{\mathcal{K}}(p)\setminus\{0\}=\lambda\Pi_{\mathcal{K}}(p)$, for some $\lambda>0$ and let $v\in\Pi_{\mathcal{K}}(p)$, which exists and it is unique because of our assumption and item (i). Then,

$\lambda v \in \Lambda_{\mathcal{K}}(p)$. Hence, it follows from Lemma 18 that $\langle p - \lambda v, v\rangle = 0$, which implies $\lambda = \langle p, v\rangle / \langle v, v, \rangle$. Therefore,

$$\Lambda_{\mathcal{K}}(p) \setminus \{0\} = \frac{\langle p, \Pi_{\mathcal{K}}(p)\rangle}{\langle \Pi_{\mathcal{K}}(p), \Pi_{\mathcal{K}}(p)\rangle} \Pi_{\mathcal{K}}(p).$$

Item (x): Suppose that $\Lambda_{\mathcal{K}}(p) \setminus \{0\} = \lambda \Pi_{\mathcal{K}}(p)$, for some $\lambda > 0$ and let $v \in \Pi_{\mathcal{K}}(p)$, which exists and it is unique because of our assumption and item (i). Then, item (ix) implies that $\lambda = \langle p, v\rangle / \langle v, v\rangle$. Then, by using Lemma 18, we have $\langle p - \lambda v, z\rangle \leq 0$, for any $z \in \mathcal{K} \setminus \{0\}$. Thus, by using Remark 11, it follows that

$$\frac{\langle p, z\rangle}{\langle v, z\rangle} \geq \lambda = \frac{\langle p, v\rangle}{\langle v, v\rangle}, \qquad \forall z \in \mathcal{K} \setminus \{0\},$$

which implies (4.59).Conversely, suppose that (4.59) holds: Let $v \in \Pi_{\mathcal{K}}(p) \neq \varnothing$, which is unique because $\mathcal{K}$ is convex. Denote

$$\lambda := \frac{\langle p, v\rangle}{\langle v, v\rangle}. \tag{4.60}$$

Then, $\lambda > 0$, because of Remark 11. Let any $z \in \mathcal{K}$. By using equation (4.60), we obtain that

$$\langle p - \lambda v, v\rangle = 0, \tag{4.61}$$

Due to equation (4.59), we also have $(\langle p, z\rangle / \langle v, z\rangle) \geq (\langle p, v\rangle / \langle v, v\rangle)$, which implies

$$\langle p - \lambda v, z\rangle \leq 0. \tag{4.62}$$

Hence, by using (4.61), (4.62), Lemma 18 and item (i), it follows that $\lambda v = \Lambda_{\mathcal{K}} \setminus \{0\}$. In conclusion, $\Lambda_{\mathcal{K}}(p) \setminus \{0\} = \lambda \Pi_{\mathcal{K}}(p)$, with $\lambda > 0$.

Item (xi): Since for any $x, y \in \mathbb{R}^{n+1}$ we have $\langle Qx, Qy\rangle = \langle x, y\rangle$ and $Q\mathbb{H}^n_\kappa = \mathbb{H}^n_\kappa$, it follows that Q is an isometry of $\mathbb{H}^n_\kappa$ and $Q^{-1} = \mathrm{J}Q^\top\mathrm{J}$. Hence, $Q\mathcal{C}$ is hyperbolically convex and $\mathcal{K}_{Q\mathcal{C}} = Q\mathcal{K}_\mathcal{C} = Q\mathcal{K}$. Let $\hat{p} := Q^\top\mathrm{J}p$ and $u \in \Lambda_\mathcal{K}(p)$. Since $\langle \hat{p} - u, \hat{p} - z\rangle = \langle Q\hat{p} - Qu, Q\hat{p} - Qz\rangle \le 0$, for any $z \in \mathcal{K}$, (4.48) implies $\Lambda_{\mathcal{K}_{QC}}(p) = \Lambda_{Q\mathcal{K}}(p) = \Lambda_{Q\mathcal{K}}(Q\hat{p}) = \Lambda_\mathcal{K}(\hat{p}) = \Lambda_K(Q^\top\mathrm{J}p)$. $\square$

In the following, we will show how to compute directly the intrinsic κ-projection for some specific sets using the L-projection. We begin by computing the projection on convex sets associated with half-spaces.

Example 13. Let $r \in \mathrm{int}(\mathcal{L})$ and $a \in \mathbb{R}^n$ such that $a \neq 0$ and $a^\top r = 0$. Consider the hyperplane

$$\mathcal{V} := \left\{x \in \mathbb{R}^n : a^\top x = 0\right\},$$

the corresponding half-spaces $\mathcal{V}_+ := \left\{x \in \mathbb{R}^n : a^\top x \ge 0\right\}$, $\mathcal{V}_- := \left\{x \in \mathbb{R}^n : a^\top x \le 0\right\}$ and the cone $\mathcal{K} = \mathcal{V}_+ \cap \mathrm{int}(\mathcal{L})$. Denote $\bar{\mathcal{K}} = \mathcal{V}_+ \cap \mathcal{L}$ the topological closure of $\mathcal{K}$. Let $p \in \mathcal{V}_- \cap \mathrm{int}(\mathcal{L})$. By item (i) of Corollary 20, $u \in \Lambda_\mathcal{K}(p) \setminus \{0\}$ exists and it is unique, and is given by

$$u = p - \frac{\langle a, \mathrm{J}p\rangle}{\langle a, a\rangle}\mathrm{J}a. \tag{4.63}$$

Let $\mathbb{H}^n_\kappa$ be the κ-hyperbolic space form such that $\kappa := -1/\langle p, p\rangle$. Then (4.63) also provides the κ-hyperbolic projection into $\mathcal{C} = \mathcal{V}_+ \cap \mathbb{H}^n_\kappa$ through (4.53) in Theorem 5, since $\mathcal{K}_\mathcal{C} = \mathcal{K}$, as follows:

$$\mathcal{P}^\kappa_\mathcal{C}(p) = \sqrt{\frac{\langle a, a\rangle}{\kappa\,\langle a, \mathrm{J}p\rangle^2 + \langle a, a\rangle}}\left(p - \frac{\langle a, \mathrm{J}p\rangle}{\langle a, a\rangle}\mathrm{J}a\right).$$

Indeed, $a \neq 0$ and $a^\top r = 0$ with $r \in \mathrm{int}(\mathcal{L})$ implies $\langle a, a\rangle \neq 0$. Otherwise, $a \in \mathcal{L} \cup -\mathcal{L}$. Hence, it follows from Remark 11 that $\langle a, a\rangle < 0$ or $\langle a, a\rangle > 0$, which contradicts $\langle a, a\rangle = 0$. On the other hand, $u \in \Lambda(\mathcal{K}) \setminus \{0\}$ if and only if it satisfies the equation

$$u = \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right) = \Pi_{\mathcal{V}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right), \tag{4.64}$$

where the first equality of (4.64) comes from item (iv) of Corollary 20. To justify the second equality of (4.64), note that the first equality of (4.64) implies $u_{n+1} \geq 0$. If $u = 0$, then $u \in \mathcal{V} \cap \bar{\mathcal{K}}$ and the second equality of (4.64) follows from its first equality because $u \in \mathcal{V} \cap \bar{\mathcal{K}} \subseteq \mathcal{V}$ and $u \in \mathcal{V} \cap \bar{\mathcal{K}} \subseteq \bar{\mathcal{K}}$ implies

$$u = \Pi_{\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right) = \Pi_{\mathcal{V}\cap\bar{\mathcal{K}}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right) = \Pi_{\mathcal{V}}\left(\mathrm{J}p + 2u_{n+1}e^{n+1}\right).$$

If $u \neq 0$, then $u_{n+1} > 0$ and we obtain the second equality of (4.64) in a similar way as above, because $u \in \mathcal{V} \cap \bar{\mathcal{K}} \subseteq \bar{\mathcal{K}}$. Indeed, if $u \notin \mathcal{V}$, then u is on $\partial\mathcal{L} \setminus \bar{\mathcal{K}}$ which contradicts item (ii) of Corollary 20. Equation (4.64) together with item (iv) of Corollary 20 implies that $u \in \Lambda_{\mathcal{V}}(p) \setminus \{0\}$. Hence, by using Corollary 18 and item (i) of Corollary 20, we obtain formula (4.63).

The next remark shows the linear independence of the Euclidean projection and L-projection.

Remark 15. Suppose that $n > 1$ and $a_{n+1} \neq 0$. Hence, a and $\mathrm{J}a$ are linearly independent. Take a point $p \in \mathrm{int}(\mathcal{L})$ that does not belong to $\mathcal{V}_+$, it is not in the span of $a, \mathrm{J}a$ and $\Pi_{\mathcal{V}}(p) = \Pi_{\mathcal{V}\cap\mathcal{L}}(p)$. In this case, we have

$$\Pi_{\bar{\mathcal{K}}}(p) = \Pi_{\mathcal{V}\cap\mathcal{L}}(p) = \Pi_{\mathcal{V}}(p) = p - \frac{\langle a, \mathrm{J}p\rangle}{\langle a, \mathrm{J}a\rangle}a.$$

Otherwise, $m = \Pi_{\bar{\mathcal{K}}}(p) \in \mathcal{V}_+ \cap \partial\mathcal{L}$. Connect m and p with a straight-line segment. Since $\mathcal{L}$ is convex this segment is in $\mathcal{L}$ and it intersects $\mathcal{V}$ at a point of $\bar{\mathcal{K}}$ closer to p. Bearing in mind Corollary 18 together with the formula of the Euclidean projection onto a hyperplane, and assuming that

$$p - \frac{\langle a, \mathrm{J}p\rangle}{\langle a, a\rangle}\mathrm{J}a = \lambda\left(p - \frac{\langle a, p\rangle}{\langle a, a\rangle}a\right),$$

for some $\lambda \in \mathbb{R}$, where a, $\mathrm{J}a$, and p are linearly independent, we find that $\langle a, \mathrm{J}p\rangle = 0$, which implies that p is perpendicular to a. However, this contradicts the initial choice of p.

Example 14. Let $\mathcal{C} := \mathcal{L}_\alpha \cap \mathbb{H}^n_\kappa$, where $\alpha > 1$ is a real constant and $\mathcal{L}_\alpha$ is an α-circular cone. Note that $\mathcal{C}$ is a closed and convex set, and $\mathcal{L}_\alpha$ is the cone spanned by $\mathcal{C}$. If $p \in \mathbb{H}^n_\kappa$ and $u = \Pi_{\mathcal{L}_\alpha}(p)$, then $u \neq 0$ and

$$v := \frac{\langle p, u\rangle}{\langle u, u\rangle}u \in \Lambda_{\mathcal{L}_\alpha}(p). \tag{4.65}$$

Indeed, the tangent plane in u of $\mathcal{L}_\alpha$ contains the ray starting at the origin through u and has normal $\alpha^2 u - (\alpha^2+1)u_{n+1}e^{n+1}$ which is in the 2-dimensional linear subspace $\mathcal{V}$ spanned by u and e^{n+1}. Since $p - u$ is parallel to the normal, it follows that $p - u \in \mathcal{V}$, $p \in \mathcal{V}$, $-\mathrm{J}u \in \mathcal{V}$ and $u = \Pi_{\mathcal{L}_\alpha \cap \mathcal{V}}(p)$, where we used the inequality characterizing the projection into a closed convex set. From Lemma 18 we have that $v \in \Lambda_{\mathcal{L}_\alpha\cap\mathcal{V}}(p)$ if and only if $v \in \mathcal{L}_\alpha \cap \mathcal{V}$, $\langle p - v, v\rangle = 0$ and $\langle p - v, z\rangle \leq 0$, for any $z \in \mathcal{L}_\alpha \cap \mathcal{V}$. Then, for v defined by (4.65), we have $\langle p - v, v\rangle = 0$. On the other hand, let any $z \in \mathcal{L}_\alpha \cap \mathcal{V}$. Since $\mathcal{L}_\alpha \cap \mathcal{V} = \operatorname{cone}\{u, -\mathrm{J}u\}$, it follows that

$z = \lambda u - \mu \mathrm{J} u$ for some $\lambda, \mu > 0$. Hence equation (4.2) implies that

$$\begin{aligned}\langle p - v, z\rangle &= \left\langle p - \frac{\langle p, u\rangle}{\langle u, u\rangle} u, \lambda u - \mu \mathrm{J} u\right\rangle = -\mu\left(p^\top u - \frac{\langle p, u\rangle}{\langle u, u\rangle} u^\top u\right)\\ &= -\mu u^\top u\left(1 - \frac{(p-u)^\top \mathrm{J} u + u^\top \mathrm{J} u}{u^\top \mathrm{J} u}\right) = \mu \frac{u^\top u}{u^\top(-\mathrm{J} u)}(p-u)^\top(-\mathrm{J} u) \le 0,\end{aligned}$$

because $u = \Pi_{\mathcal{L}_\alpha \cap \mathcal{V}}(p) \in \mathcal{L}_\alpha \cap \mathcal{V} \subseteq \mathcal{L}$, $-\mathrm{J} u \in \mathcal{L}_\alpha \cap \mathcal{V} \subseteq \mathcal{L}$ and $\mathcal{L}$ is self-dual. Since $\langle p - v, v\rangle = 0$ and $\langle p - v, z\rangle \le 0$, for any $z \in \mathcal{L}_\alpha \cap \mathcal{V}$, by using again Lemma 18, it follows that $v \in \Lambda_{\mathcal{L}_\alpha \cap \mathcal{V}}(p)$. Thus, item (iv) of Corollary 20 implies that

$$v = \Pi_{\mathcal{L}_\alpha \cap V}\left(\mathrm{J} p + 2 v_{n+1} e^{n+1}\right) = \Pi_{\mathcal{L}_\alpha}\left(\mathrm{J} p + 2 v_{n+1} e^{n+1}\right),$$

where we also used $\mathrm{J} p + 2 v^{n+1} e^{n+1} = p + 2\left(v^{n+1} - p^{n+1}\right) e^{n+1} \in \mathcal{V}$, $\Pi_{\mathcal{L}_\alpha \cap V}\left(\mathrm{J} p + 2 v_{n+1} e^{n+1}\right) \in \mathcal{L}_\alpha \cap \mathcal{V}$ and equation (4.1). Hence, by using again item (iv) of Corollary 20, we obtain $v \in \Lambda_{\mathcal{L}_\alpha}(p)$ and (4.65) holds. Note that (4.65) can be written as

$$0 \neq \frac{\langle p, \Pi_{\mathcal{L}_\alpha}(p)\rangle}{\langle \Pi_{\mathcal{L}_\alpha}(p), \Pi_{\mathcal{L}_\alpha}(p)\rangle} \Pi_{\mathcal{L}_\alpha(p)} \in \Lambda_{\mathcal{L}_\alpha}(p).$$

Therefore, Theorem 5 implies that

$$\mathcal{P}^\kappa_{\mathcal{C}}(p) = \frac{\Pi_{\mathcal{K}_\mathcal{C}}(p)}{\sqrt{-\kappa \langle \Pi_{\mathcal{K}_\mathcal{C}}(p), \Pi_{\mathcal{K}_\mathcal{C}}(p)\rangle}},$$

which has been already obtained in Equation (4.47) of Example 12. Note that this is an example when an Euclidean and an L-projection of p into $\mathcal{K}$ are nonzero, have the same direction, but are different (see also item (ix) of Corollary 20).

Example 15. Let $\mathcal{C}_+$ be as in (4.40) and $\mathcal{K} := \mathcal{K}_{\mathcal{C}_+}$ be the cone spanned by the set $\mathcal{C}_+$ as in (4.41). Then, $\Lambda_{\mathcal{K}}|_{\mathrm{int}(\mathcal{L})} \setminus \{0\} = \Pi_{\mathcal{K}}$, i.e.

$$\Lambda_{\mathcal{K}}(p) \setminus \{0\} = \Pi_{\mathcal{K}}(p), \qquad \forall p \in \mathrm{int}(\mathcal{L}). \tag{4.66}$$

Indeed, first note that in Example 11 we showed that $p^+ = \Pi_{\mathcal{K}_{\mathcal{C}_+}}(p)$. Let $p \in \mathrm{int}(\mathcal{L})$ and $\mathbb{H}^n_\kappa$ be the κ-hyperbolic space form such $\kappa := -1/\langle p, p\rangle$. By item (i) of Corollary 20, it is sufficient to prove that $\Pi_{\mathcal{K}}(p) = p^+ \in \Lambda_{\mathcal{K}}(p) \setminus \{0\}$. Since $p \in \mathrm{int}(\mathcal{L})$, we have $p_{n+1} > 0$, which implies $p^+_{n+1} = p_{n+1} > 0$. Thus, $p^+ \neq 0$. Moreover, considering that $p^+_{n+1} - p_{n+1} = 0$ and $p^+ = \Pi_{\mathcal{K}_{\mathcal{C}_+}}(p)$, we conclude that

$$\left\langle p - p^+, p^+ \right\rangle + 2\left(p_{n+1} - p^+_{n+1}\right) p^+_{n+1} = \left(p - p^+\right)^\top p^+ = 0,$$

$$\left\langle p - p^+, z \right\rangle + 2\left(p_{n+1} - p^+_{n+1}\right) z_{n+1} = \left(p - p^+\right)^\top z \leq 0,$$

for all $z \in \mathcal{K} \subseteq \mathbb{R}^n_+$. Hence, Lemma 18 implies $p^+ \in \Lambda_{\mathcal{K}}(p) \setminus \{0\}$. Therefore, (4.66) holds.

Example 16. Let $a \in \mathrm{int}(\mathcal{L}) \setminus \{e^{n+1}\}$ be a vector with $a^\top a = 1$,

$$\mathcal{C}_{\mathcal{L}(a)} = \left\{ q \in \mathbb{H}^n_\kappa : \; a^\top q \geq \sqrt{\frac{q^\top q}{2}} \right\}, \quad \mathcal{L}(a) := \left\{ x \in \mathbb{R}^{n+1} : a^\top x \geq \sqrt{\frac{x^\top x}{2}} \right\}.$$

Note that the cone spanned by the set $\mathcal{C}_{\mathcal{L}(a)}$ is $\mathcal{K}_{\mathcal{C}_{\mathcal{L}(a)}} = (\mathcal{L}(\alpha) \cap \mathrm{int}(\mathcal{L})) \cup \{0\}$, which is convex. Hence, $\mathcal{C}_{\mathcal{L}(a)}$ is κ-hyperbolically convex, and from its definition it follows that it is also closed. Next, we will show the steps of how to κ-project intrinsically onto $\mathcal{C}_{\mathcal{L}(a)}$. As the cone $\mathcal{L}(a)$ is a Lorentz cone with symmetry axis generated by $a \in \mathcal{L}(a)$, a simple adaptation of the projection formula in [27, Proposition 3.3], with e^{n+1} replaced by a, yields

$$\Pi_{\mathcal{L}(a)}(x) = \begin{cases} \frac{1}{2}\left[(\xi-\zeta)\frac{w}{\sqrt{w^\top w}} + (\zeta+\xi)a\right], & \text{if } w \neq 0 \\ z^+ a, & \text{if } w = 0 \end{cases}, \tag{4.67}$$

where $\zeta = \left(z - \sqrt{w^\top w}\right)^+$, $\xi = \left(z + \sqrt{w^\top w}\right)^+$, $w = x - za$, $\sqrt{w^\top w} = \sqrt{x^\top x - z^2}$ and $z = a^\top x$. Note that an alternative formula for $\mathcal{L}(a)$ is

$$\mathcal{L}(a) = \left\{w + za \in \mathbb{R}^{n+1} : z \geq \sqrt{w^\top w}\right\}.$$

If $p \in \mathbb{H}^n_\kappa \setminus \mathcal{C}_{\mathcal{L}(a)}$, then, by (4.67) and item (vii) of Corollary 20, the nonzero L-projection $u \in \Lambda_{\mathcal{K}_{\mathcal{C}_{\mathcal{L}(a)}}}$ of p into $\mathcal{K}_{\mathcal{C}_{\mathcal{L}(a)}}$ (which is unique according to item (i) of Corollary 20) can be obtained by first numerically solving the semi-smooth equation

$$2u_{n+1} = [\xi(u_{n+1}) - \zeta(u_{n+1})]\,\frac{w_{n+1}(u_{n+1})}{\sqrt{w(u_{n+1})^\top w(u_{n+1})}} + [\zeta(u_{n+1}) + \xi(u_{n+1})]\,a_{n+1},$$

in u_{n+1}, where
$\zeta(u_{n+1}) = \left[z(u_{n+1}) - \sqrt{w(u_{n+1})^\top w(u_{n+1})}\right]^+$,
$\xi(u_{n+1}) = \left[z(u_{n+1}) + \sqrt{w(u_{n+1})^\top w(u_{n+1})}\right]^+$, $w(u_{n+1}) = x(u_{n+1}) - z(u_{n+1})a$,
$\sqrt{w(u_{n+1})^\top w(u_{n+1})} = \sqrt{x(u_{n+1})^\top x(u_{n+1}) - z(u_{n+1})^2}$,
$z(u_{n+1}) = a^\top x(u_{n+1})$, $x(u_{n+1}) = \mathrm{J}p + 2u_{n+1}e^{n+1}$, and then inserting the obtained result into the equation

$$2u_{n+1} = [\xi(u_{n+1}) - \zeta(u_{n+1})]\,\frac{w(u_{n+1})}{\sqrt{w(u_{n+1})^\top w(u_{n+1})}} + [\zeta(u_{n+1}) + \xi(u_{n+1})]\,a.$$

After obtaining the value of u, we can calculate the κ-projection of p into $\mathcal{C}_{\mathcal{L}(a)}$ using the following equation, derived from equation (4.53) in Theorem 5:

$$\mathcal{P}^{\kappa}_{\mathcal{C}_{\mathcal{L}(a)}} = \frac{u}{\sqrt{-\kappa \langle u, u \rangle}}.$$

4.3 Gradient projection method on the κ-hyperbolic space forms

In this section, we focus on the constrained optimization problem on κ-hyperbolic space and present the gradient projection method as a solution approach. To do so, we consider a closed and κ-hyperbolically convex set $\mathcal{C} \subset \mathbb{H}^{\setminus}_{\kappa}$ and a differentiable function $f : \mathbb{H}^n_\kappa \to \mathbb{R}$. Recall that the specific form of the constrained optimization problem we are interested in is as follows:

$$\text{Minimize}_{p \in \mathcal{C}} f(p). \tag{4.68}$$

Let $\mathcal{C}^* \neq \varnothing$ be the *solution set* of the problem (4.68) and $-\infty < f^* := \inf_{p \in \mathcal{C}} f(p)$ be the *optimum value* of f. By using [20, Proposition 6.1], we conclude that if $\bar{p} \in \mathcal{C}$ is a solution of (4.68), then

$$\langle \operatorname{grad} f(\bar{p}), p \rangle = \left\langle \operatorname{grad} f(\bar{p}), \operatorname{Proj}_{\bar{p}} p \right\rangle \geq 0, \quad \forall\, p \in \mathcal{C}. \tag{4.69}$$

Any point $\bar{p}$ satisfying (4.69) is called a *stationary point* for Problem (4.68). In the following corollary we present two important properties of the projection, which are related to the stationary points of the problem (4.68).

Proposition 28. Let $\bar{p} \in \mathcal{C}$, such that $\operatorname{grad} f(\bar{p}) \neq 0$ and $\alpha > 0$. Then, there holds:

(i) The point $\bar{p}$ is stationary for problem (4.68) if and only if

$$\bar{p} = \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)$$

.

(ii) If p is a nonstationary point for problem (4.68) then

$$\left\langle \operatorname{grad} f(p), \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}\left(\exp_p\left(-\alpha \operatorname{grad} f(p)\right)\right)\right\rangle < 0. \tag{4.70}$$

Equivalently, if there exists $\bar{\alpha} > 0$ satisfying

$$\left\langle \operatorname{grad} f(\bar{p}), \log_p^{\kappa} \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\bar{\alpha} \operatorname{grad} f(\bar{p})\right)\right)\right\rangle \geq 0, \tag{4.71}$$

then p is stationary for problem (4.68).

Proof. To prove item (i), we first assume that $\bar{p} \in \mathcal{C}$ is a stationary point for Problem (4.68). It follows from (1.11) and (4.69) that

$$\left\langle \operatorname{grad} f(\bar{p}), \log_{\bar{p}}^{\kappa} \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)\right\rangle \geq 0.$$

Thus, by using Proposition 26, we conclude that $d(\bar{p}, \mathcal{P}_{\mathcal{C}}(\exp_{\bar{p}}(-\alpha \operatorname{grad} f(\bar{p})))) \leq 0$, which implies that $\bar{p} = \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)$. Conversely, assume that $\bar{p} = \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)$. Thus, (2.15) implies that

$$\left\langle \log_{\bar{p}}^{\kappa} \exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right), \log_{\bar{p}}^{\kappa} p\right\rangle \leq 0, \qquad \forall\, p \in \mathcal{C}.$$

or equivalently, $\langle \alpha \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} p \rangle \geq 0$, for all $p \in \mathcal{C}$. Thus, by using $\alpha > 0$ and (1.11), the last inequality implies that

$$\left\langle \operatorname{grad} f(\bar{p}), \operatorname{Proj}_{\bar{p}} p \right\rangle \geq 0, \quad \forall\, p \in \mathcal{C}.$$

Therefore, the point $\bar{p}$ is stationary for Problem (4.68) and (i) is proved. We proceed to prove item (ii). Take p a nonstationary point for Problem (4.68). Thus, item (i) implies that

$$\bar{p} \neq \mathcal{P}_{\mathcal{C}} \left(\exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right) \right).$$

Thus, by applying Proposition 26, we infer that

$$0 < \frac{1}{\alpha} d^2 \big(\bar{p}, \mathcal{P}_{\mathcal{C}} \big(\exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right) \big) \big) \leq - \big\langle \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} \mathcal{P}_{\mathcal{C}} \big(\exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right) \big) \big\rangle,$$

which implies (4.70) and the first statement of item (ii) is proved. Finally, note that the second statement of item (ii) is the contrapositive of the first statement. □

4.3.1 Gradient projection method with constant step size

In this section we analyse the gradient projection method with constant step size. For that, in this section, we need to assume that

(H1) The gradient vector field $\operatorname{grad} f$ is Lipschitz continuous on $\mathcal{C}$ with constant $L \geq 0$.

Next we present the conceptual version of the *gradient projection method* to solve the problem (4.68).

Algorithm 1 Gradient projection method on $\mathbb{H}^n_\kappa$

1: Take $p_0 \in \mathcal{C}$. Set $k = 0$;

2: If $\operatorname{grad} f(p_k) = 0$, then **stop** and returns p_k; otherwise, compute a step size $t_k > 0$ such that

$$p_{k+1} := \mathcal{P}^\kappa_{\mathcal{C}}(\exp^\kappa_{p_k}(-t_k \operatorname{grad} f(p_k))). \tag{4.72}$$

3: Update $k \leftarrow k+1$ and go to **2**

It follows from (4.69) that if $\operatorname{grad} f(p_k) = 0$, then p_k is a stationary point for Problem (4.68). Therefore, *from now on we assume that* $\operatorname{grad} f(p_k) \neq 0$*, for all* $k = 0, 1, \ldots$. To proceed, we assume that $(p_k)_{k\in\mathbb{N}}$ is generated by Algorithm 1 with the following constant stepsize:

$$t_k := \alpha, \qquad 0 < \alpha < \frac{1}{L}, \qquad k = 0, 1, \ldots. \tag{4.73}$$

To simply the notations of the next results, we define the following positive constant

$$\Gamma := (1 - \alpha L)/(2\alpha) > 0. \tag{4.74}$$

Lemma 19. There hold $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$ and

$$f(p_{k+1}) \leq f(p_k) - \Gamma d^2_\kappa(p_k, p_{k+1}), \qquad k = 0, 1, \ldots. \tag{4.75}$$

In particular, the sequence $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing and converges.

Proof. Since $p_0 \in \mathcal{C}$, it follows from (4.72) that $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$. By applying Lemma 16 with $p = p_k$ and $q = p_{k+1}$, we have

$$f(p_{k+1}) \leq f(p_k) + \left\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} p_{k+1} \right\rangle + \frac{L}{2} d^2_\kappa(p_k, p_{k+1}).$$

Therefore, by using Proposition 26 with $p = p_k$ and considering (4.72) along with $t_k = \alpha$, we can deduce that

$$f(p_{k+1}) \leq f(p_k) - \frac{1}{2\alpha} d_\kappa^2(p_k, p_{k+1}) + \frac{L}{2} d_\kappa^2(p_k, p_{k+1}) = f(p_k) - \left(\frac{1 - \alpha L}{2\alpha} \right) d_\kappa^2(p_k, p_{k+1}).$$

Therefore, we can establish the validity of (4.75) from (4.74). Furthermore, the formulas (4.73) and (4.75) indicate that the sequence $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing. Additionally, considering that $-\infty < f^*$ and $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing, we can deduce that the sequence converges. Thus, the proof is now complete. $\square$

In the following we prove that any cluster point of $(p_k)_{k\in\mathbb{N}}$ is a solution of the Problem (4.68).

Theorem 7. If $\bar{p} \in \mathcal{C}$ is a cluster point of the sequence $(p_k)_{k\in\mathbb{N}}$, then $\bar{p}$ is a stationary point for Problem (4.68).

Proof. If $\operatorname{grad} f(\bar{p})) = 0$, then (4.69) implies that $\bar{p} \in \mathcal{C}$ is a stationary point for Problem (4.68). Now, assume that $\operatorname{grad} f(\bar{p})) \neq 0$. By using (4.75), we obtain that

$$d_\kappa^2(p_k, p_{k+1}) \leq \frac{1}{\Gamma} \left(f(p_k) - f(p_{k+1}) \right), \qquad k = 0, 1, \ldots. \tag{4.76}$$

By applying Lemma 19, we can establish that the sequence $(f(p_k))_{k\in\mathbb{N}}$ converges. Taking the limit in (4.76), we can deduce that $\lim_{k\to+\infty} d_\kappa(p_k, p_{k+1}) = 0$. Let $\bar{p}$ be a cluster point of $(p_k)_{k\in\mathbb{N}}$ and $(p_{k_j})_{j\in\mathbb{N}}$ be a subsequence of $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty} p_{k_j} = \bar{p}$. As $\lim_{j\to+\infty} d_\kappa(p_{k_j+1}, p_{k_j}) = 0$, we have $\lim_{j\to+\infty} p_{k_j+1} = \bar{p}$. On the other hand,

$$p_{k_j+1} = P_C(\exp^\kappa_{p_{k_j}} \left(-\alpha \operatorname{grad} f(p_{k_j}) \right)), \qquad j = 0, 1, \ldots$$

and (2.14) imply

$$\left\langle \mathrm{Proj}^{\kappa}_{p_{k_j+1}} \exp^{\kappa}_{p_{k_j}}(-\alpha \,\mathrm{grad}\, f(p_{k_j})), \mathrm{Proj}^{\kappa}_{p_{k_j+1}} q \right\rangle \leq 0, \qquad \forall q \in \mathcal{C}.$$

Hence, by setting $v = -\,\mathrm{grad}\, f(\bar{p})$, and employing (1.10) while considering that $\bar{p} \in \mathcal{C} \subseteq \mathbb{H}^{\setminus}_{\kappa}$, we can conclude that

$$\left\langle \mathrm{Proj}^{\kappa}_{\bar{p}} \exp^{\kappa}_{\bar{p}}(-\alpha \,\mathrm{grad}\, f(\bar{p})), \mathrm{Proj}^{\kappa}_{\bar{p}} q \right\rangle \leq 0, \qquad \forall q \in \mathcal{C}.$$

which is equivalent to $\left\langle \exp^{\kappa}_{\bar{p}}(-\alpha \,\mathrm{grad}\, f(\bar{p})), \mathrm{Proj}^{\kappa}_{\bar{p}} q \right\rangle \leq 0$, for all $q \in \mathcal{C}$. Therefore, by setting $v = -\,\mathrm{grad}\, f(\bar{p})$, using (1.10) and considering that $\bar{p} \in \mathcal{C} \subseteq \mathbb{H}^{\setminus}_{\kappa}$, we can infer that

$$0 \geq \left\langle \cosh(\alpha\sqrt{\kappa}\|v\|)\bar{p} + \sinh(\alpha\sqrt{\kappa}\|v\|)\frac{v}{\sqrt{\kappa}\|v\|}, \mathrm{Proj}^{\kappa}_{\bar{p}} q \right\rangle = \frac{\sinh(\alpha\sqrt{\kappa}\|v\|)}{\sqrt{\kappa}\|v\|}\left\langle v, \mathrm{Proj}^{\kappa}_{\bar{p}} q \right\rangle,$$

for all $q \in \mathcal{C}$. Thus, as $\sinh(\alpha\sqrt{\kappa}\|v\|) > 0$ and $v = -\,\mathrm{grad}\, f(\bar{p})$, we have

$$\left\langle \mathrm{grad}\, f(\bar{p}), \mathrm{Proj}^{\kappa}_{\bar{p}} q \right\rangle \geq 0, \quad \forall\; q \in \mathcal{C},$$

which, by using (4.69), implies that $\bar{p} \in \mathcal{C}$ is a stationary point for problem (4.68).

□

The item (i) of Proposition 28 implies that if $p_k = \mathcal{P}_{\mathcal{C}}\left(\exp_{p_k}(-\alpha_k \,\mathrm{grad}\, f(p_k))\right)$, then p_k is stationary for Problem (4.68). Since $p_{k+1} = \mathcal{P}_{\mathcal{C}}\left(\exp_{p_k}(-\alpha_k \,\mathrm{grad}\, f(p_k))\right)$, the distance between the points p_k and p_{k+1}, i.e., $d(p_k, p_{k+1})$, can be seen as a measure of stationarity of p^k. The next theorem presents an iteration-complexity bound for this measure.

Theorem 8. For all $N \in \mathbb{N}$ there holds

$$\min\left\{d_\kappa(p_k, p_{k+1}) :\ k = 0, 1, \ldots, N\right\} \leq \sqrt{\frac{f(p_0) - f^*}{\Gamma(N+1)}},$$

Proof. By using (4.75), we have $d_\kappa^2(p_k, p_{k+1}) \leq \left(f(p_k) - f(p_{k+1})\right)/\Gamma$, for all $k = 0, 1, \ldots$. Since $f^* \leq f(p_k)$ for all k, the last inequality implies

$$\sum_{k=0}^{N} d_\kappa^2(p_{k+1}, p_k) \leq \frac{1}{\Gamma}\sum_{k=0}^{N}\left(f(p_k) - f(p_{k+1})\right) \leq \frac{1}{\Gamma}(f(p_0) - f^*).$$

Therefore, $(N+1)\min\left\{d_\kappa^2(p_k, p_{k+1}) :\ k = 0, 1, \ldots, N\right\} \leq (f(p_0) - f^*)/\Gamma$, which implies the desired inequality. □

4.3.2 Gradient projection method with backtracking step size

In Algorithm 1 we need the Lipschitz constant to compute the stepsize. However, this constant is not always known or computable. Therefore, in this section we also analyze a version of Algorithm 1 with a backtracking stepsize rule. The conceptual version of the gradient projection method on the κ-hyperbolic space with a backtracking stepsize rule to solve the problem (4.68) is as follows:

In the next proposition we prove that Algorithm 2 is well defined and two useful inequalities.

Proposition 29. The Algorithm 2 is well defined and generates a sequence $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$. Moreover, the following inequality hold:

$$\left\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} z_k \right\rangle \leq -\frac{1}{\alpha_k} d^2(p_k, z_k) < 0, \qquad \forall k \in \mathbb{N}. \tag{4.80}$$

Algorithm 2 Gradient projection method on $\mathbb{H}^n_\kappa$ with backtracking step size rule

1: Take constants $\rho, \beta, \theta_0 \in (0,1)$ and $\hat{\alpha}, \bar{\alpha} > 0$. Choose initial point $p_0 \in \mathcal{C}$. Set $k = 0$;

2: If $\operatorname{grad} f(p_k) = 0$, then **stop** and returns p_k; otherwise take

$$0 < \hat{\alpha} \leq \alpha_k \leq \bar{\alpha}, \tag{4.77}$$

3: Compute

$$y_k := \exp^\kappa_{p_k}\left(-\alpha_k \operatorname{grad} f(p_k)\right), \qquad z_k := \mathcal{P}^\kappa_{\mathcal{C}}\left(y_k\right), \tag{4.78}$$

$$\ell_k := \min\left\{\ell \in \mathbb{N} :\ f\big(q(\beta^\ell \theta_k)\big) \leq f(p_k) + \rho\big(\beta^\ell \theta_k\big)\big\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} z_k\big\rangle\right\}, \tag{4.79}$$

where $q(\tau) := \exp^\kappa_{p_k}\left(\tau \log^\kappa_{p_k} z_k\right)$ denotes the geodesic segment joining p_k to z_k;

4: Set $p_{k+1} := q(\beta^{\ell_k}\theta_k)$ and $\theta_{k+1} := \beta^{\ell_k - 1}\theta_k$;

5: Update $k \leftarrow k + 1$ and go to **2**.

Furthermore,

$$f(p_{k+1}) \leq f(p_k) + \rho\beta\theta_{k+1}\big\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} z_k\big\rangle, \qquad \forall k \in \mathbb{N}. \tag{4.81}$$

Proof. Bearing in mind that $p_0 \in \mathcal{C}$, without loss of generality we can assume that $p_k \in \mathcal{C}$. Since $\mathcal{C} \subset \mathbb{H}^n_\kappa$ is closed, $\mathcal{P}_{\mathcal{C}}\left(y_k\right)$ is a singleton. Hence, the point z_k in (4.78) is well defined and belongs to the set $\mathcal{C}$. Since y_k are given by (4.78), Proposition 26 with $v = \operatorname{grad} f(p_k)$, $p = p_k$ and $\alpha = \alpha_k$ implies that

$$\big\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} \mathcal{P}_{\mathcal{C}}\left(y_k\right)\big\rangle \leq -\frac{1}{\alpha_k} d^2_\kappa(p_k, \mathcal{P}_{\mathcal{C}}\left(y_k\right)).$$

Thus, due to $z_k = \mathcal{P}_{\mathcal{C}}\left(y_k\right)$, the inequality (4.80) follows from the previous one.

From the differentiability of f is and $0 < \beta < 1$, we conclude that

$$\lim_{\ell\to+\infty} \frac{f\left(\exp^{\kappa}_{p_k}\left(\beta^{\ell}\theta_k \log^{\kappa}_{p_k} z_k\right)\right) - f(p_k)}{\beta^{\ell}\theta_k} = \left\langle \operatorname{grad} f(p_k), \log^{\kappa}_{p_k} z_k \right\rangle. \tag{4.82}$$

Thus, taking into account that $0 < \rho < 1$ and using item (ii) of Proposition 28, the combination of (4.80) with (4.82) implies that

$$\lim_{\ell\to+\infty} \frac{f\left(\exp^{\kappa}_{p_k}\left(\beta^{\ell}\theta_k \log^{\kappa}_{p_k} z_k\right)\right) - f(p_k)}{\beta^{\ell}\theta_k} < \rho \left\langle \operatorname{grad} f(p_k), \log^{\kappa}_{p_k} z_k \right\rangle.$$

Therefore, there exists $\hat{\ell} \in \mathbb{N}$ such that

$$f\left(\exp^{\kappa}_{p_k}\left(\beta^{\ell}\theta_k \log^{\kappa}_{p_k} z_k\right)\right) < f(p_k) + \rho\big(\beta^{\ell}\theta_k\big) \left\langle \operatorname{grad} f(p_k), \log^{\kappa}_{p_k} z_k \right\rangle, \qquad \forall \ell \in \mathbb{N};\ \hat{\ell} \leq \ell,$$

which implies ℓ_k at **Step 3** is well defined, and then Algorithm 2 is also well defined. The definition of the next iterate p_{k+1} implies that it belongs to the geodesic through p_k and z_k. Since p_k and z_k belong to $\mathcal{C}$ and $\mathcal{C}$ is convex, we infer that p_{k+1} also belongs to $\mathcal{C}$. Therefore, the generated sequence $(p_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{C}$. The inequality (4.81) follows from the definition of ℓ_k at **Step 3** and definitions of p_{k+1} and θ_{k+1} at **Step 4**, which concludes the proof. □

Theorem 9. If $\bar{p} \in \mathcal{C}$ is a cluster point of the sequence $(p_k)_{k\in\mathbb{N}}$, then $\bar{p}$ is a stationary point for Problem (4.68).

Proof. Let $\bar{p}$ be a cluster point of the sequence $(p_k)_{k\in\mathbb{N}}$. If $\operatorname{grad} f(\bar{p}) = 0$, then (4.69) implies that $\bar{p} \in \mathcal{C}$ is a stationary point for Problem (4.68). Now, assume that $\operatorname{grad} f(\bar{p}) \neq 0$. Combining (4.80) with (4.81) we have

$$0 < -\rho\beta\theta_{k+1}\big\langle \operatorname{grad} f(p_k), \log^{\kappa}_{p_k} z_k \big\rangle \leq f(p_k) - f(p_{k+1}), \qquad \forall k \in \mathbb{N}. \tag{4.83}$$

It follows from (4.83) that $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing and considering that $-\infty < f^*$, we conclude that it converges. Since $(f(p_k))_{k\in\mathbb{N}}$ converges, we obtain from (4.83) that

$$\lim_{k\to+\infty} \theta_{k+1}\big\langle \operatorname{grad} f(p_k), \log^{\kappa}_{p_k} z_k\big\rangle = 0. \tag{4.84}$$

Since $\bar{p}$ is a cluster point of sequence $(p_k)_{k\in\mathbb{N}}$, we take $(p_{k_j})_{j\in\mathbb{N}}$ a subsequence of $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty} p_{k_j} = \bar{p}$. Considering that $(\alpha_k)_{k\in\mathbb{N}} \subseteq [\hat{\alpha}, \bar{\alpha}]$ and $(\theta_k)_{k\in\mathbb{N}} \subseteq (0, 1)$, we can assume without generality that

$$\lim_{j\to+\infty} \alpha_{k_j} =: \tilde{\alpha} \geq \hat{\alpha} > 0, \qquad \lim_{j\to+\infty} \theta_{k_j+1} =: \tilde{\theta} \in [0, 1]. \tag{4.85}$$

It follows from (4.78) that $z_{k_j} := \mathcal{P}^{\kappa}_{\mathcal{C}}\big(\exp^{\kappa}_{p_{k_j}}(-\alpha_{k_j}\operatorname{grad} f(p_{k_j}))\big)$. Thus, by using Proposition 25, taking into account that $\lim_{j\to+\infty} p_{k_j} = \bar{p}$, and (4.85) we infer that

$$\lim_{j\to+\infty} z_{k_j} := \mathcal{P}^{\kappa}_{\mathcal{C}}\big(\exp^{\kappa}_{\bar{p}}(-\tilde{\alpha}\operatorname{grad} f(\bar{p}))\big) =: \bar{z}. \tag{4.86}$$

From (4.84) we obtain that $\lim_{k\to+\infty} \theta_{k_j+1}\big\langle \operatorname{grad} f(p_{k_j}), \log^{\kappa}_{p_{k_j}} z_{k_j}\big\rangle = 0$. Hence, the combination of (4.85) and (4.86) with $\lim_{j\to+\infty} p_{k_j} = \bar{p}$ yields

$$\tilde{\theta}\big\langle \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} \bar{z}\big\rangle = 0. \tag{4.87}$$

Since $\tilde{\alpha} \in [0, 1]$, we have two possibilities that we must consider: $\tilde{\theta} > 0$ or $\tilde{\theta} = 0$.

First we assume that $\tilde{\theta} > 0$. In this case, by using (4.80) we have

$$d^2(p_{k_j}, z_{k_j}) \leq -\alpha_{k_j}\left\langle \operatorname{grad} f(p_{k_j}), \log^{\kappa}_{p_{k_j}} z_j\right\rangle, \qquad \forall j \in \mathbb{N}. \tag{4.88}$$

By taking limit in (4.88), and by using (4.85) and (4.86), together with the first equality in (4.87), we get $\lim_{k\to+\infty} d(p_{k_j}, z_{k_j}) = 0$. Thus, due to $\lim_{j\to+\infty} p_{k_j} = \bar{p}$ and $\lim_{j\to+\infty} z_{k_j} = \bar{z}$, we have $\bar{z} = \bar{p}$. Hence, using (4.86) we obtain that

$$\bar{p} = \mathcal{P}^{\kappa}_{\mathcal{C}}\big(\exp^{\kappa}_{\bar{p}}(-\tilde{\alpha}\operatorname{grad} f(\bar{p}))\big).$$

Therefore, item (i) of Proposition 28 implies that $\bar{p}$ is a stationary point for Problem (4.68).

Now, we assume that $\tilde{\theta} = 0$. In this case, we consider the following auxiliary sequence $(q_{k_j})_{j\in\mathbb{N}}$ defined by

$$q_{k_j} := \exp^{\kappa}_{p_{k_j}}\big(\theta_{k_j+1}\log^{\kappa}_{p_{k_j}} z_{k_j}\big), \qquad \forall k \in \mathbb{N}. \tag{4.89}$$

Since $\lim_{j\to+\infty} \theta_{k_j} = 0$, we have $\lim_{j\to+\infty} q_{k_j} = \bar{p}$. Moreover, the definition of ℓ_k is defined in (4.79) implies that $f(q_{k_j}) > f(p_{k_j}) + \rho\theta_{k_j+1}\big\langle \operatorname{grad} f(p_{k_j}), \log^{\kappa}_{p_{k_j}} z_{k_j}\big\rangle$. Via combining (4.89) with the last inequality, we obtain that

$$\frac{f\big(\exp^{\kappa}_{p_{k_j}}(\theta_{k_j+1}\log^{\kappa}_{p_{k_j}} z_{k_j})\big) - f(p_{k_j})}{\theta_{k_j+1}} > \rho\big\langle \operatorname{grad} f(p_{k_j}), \log^{\kappa}_{p_{k_j}} z_{k_j}\big\rangle.$$

Since $\lim_{j\to+\infty} \theta_{k_j+1} = 0$, $\lim_{j\to+\infty} p_{k_j} = \bar{p}$, $\lim_{j\to+\infty} z_{k_j} = \bar{z}$ and f is differentiable, By taking limit in the last inequality we arrive to $\big\langle \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} \bar{z}, \big\rangle \geq \rho\big\langle \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} \bar{z}, \big\rangle$. Due to $0 < \rho < 1$, it follows from the last inequality that $\big\langle \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} \bar{z}, \big\rangle \geq 0$. Thus, using (4.86) we obtain that

$$\big\langle \operatorname{grad} f(\bar{p}), \log^{\kappa}_{\bar{p}} \mathcal{P}^{\kappa}_{\mathcal{C}}\big(\exp^{\kappa}_{\bar{p}}(-\tilde{\alpha}\operatorname{grad} f(\bar{p}))\big)\big\rangle \geq 0.$$

Therefore, using (4.71) in item (ii) of Proposition 28 we deduce that $\bar{p}$ is a stationary point for Problem (4.68). Thus, the proof is now complete. $\square$

To proceed, we need the assumption (H1). For simplifying the notation, it is convenient to define the following positive constant

$$\bar{\theta} := \min\Big\{\theta_0, \frac{2}{\bar{\alpha}L}(1-\rho)\Big\}. \tag{4.90}$$

Lemma 20. The following inequality holds:

$$\theta_k \geq \bar{\theta}, \quad \forall k \in \mathbb{N}. \tag{4.91}$$

Proof. Since $p_0 \in \mathcal{C}$, it follows from Proposition 29 that $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$. The inequality (4.91) immediately holds for $k = 0$. Lemma 16 with $D = \mathbb{H}^n_\kappa$, $p = p_k$ and $q = \exp_{p_k}(\beta^{\ell_k-1}\theta_k \log^\kappa_{p_k} z_k)$ yield

$$f(\exp^\kappa_{p_k}(\beta^{\ell_k-1}\theta_k \log^\kappa_{p_k} z_k)) \leq f(p_k) + \beta^{\ell_k-1}\theta_k\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} z_k\rangle + \frac{L}{2}(\beta^{\ell_k-1}\theta_k)^2 d^2(p_k, z_k).$$

On the other hand, it follows from the definition of ℓ_k in (4.79) that

$$f(\exp^\kappa_{p_k}\big(\beta^{\ell_k-1}\theta_k \log^\kappa_{p_k} z_k\big)) > f(p_k) + \rho\beta^{\ell_k-1}\theta_k\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} z_k\rangle.$$

Hence, by combining the two previous inequalities and keeping in mind that $\theta_{k+1} = \beta^{\ell_k-1}\theta_k$, we get

$$-(1-\rho)\theta_{k+1}\langle \operatorname{grad} f(p_k), \log^\kappa_{p_k} z_k\rangle < \frac{L}{2}\theta^2_{k+1} d^2(p_k, z_k).$$

Having in mind that $0 < \rho < 1$, by applying Proposition 29, we infer from the last inequality that

$$(1-\rho)\theta_{k+1}\frac{1}{\alpha_k}d^2(p_k, z_k) < \frac{L}{2}\theta_{k+1}^2 d^2(p_k, z_k).$$

Thus, due to $0 < \alpha_k < \bar{\alpha}$, $\theta_{k+1} \neq 0$ and $d(p_k, z_k) \neq 0$, it follows from the previos inequality that

$$\frac{2}{\bar{\alpha}L}(1-\rho) < \theta_{k+1}.$$

Therefore, by using (4.90) we have $\bar{\theta} < \theta_{k+1}$. Because (4.91) holds for $k = 0$, the proof is complete. □

As the quantity $d(p_k, z_k)$ can be interpreted as a measure of the stationarity of p_k, the following theorem provides a bound on the iteration complexity for this measure.

Theorem 10. For all $N \in \mathbb{N}$ there holds

$$\min\{d(p_k, z_k) : \ k = 0, 1, \dots, N\} \leq \sqrt{\frac{\bar{\alpha}(f(p_0) - f^*)}{\rho\beta\bar{\theta}}}\frac{1}{\sqrt{(N+1)}}.$$

Proof. Inequalities (4.81), (4.77), (4.80) and (4.91) yield

$$\rho\beta\bar{\theta}\frac{1}{\bar{\alpha}}d^2(p_k, z_k) \leq -\rho\beta\theta_{k+1}\left\langle \operatorname{grad} f(p_k), \log_{p_k}^{\kappa} z_k \right\rangle \leq f(p_k) - f(p_{k+1}), \qquad \forall k \in \mathbb{N}.$$

Since $f^* \leq f(p_k)$ for all k, the last inequality implies

$$\sum_{k=0}^{N} d^2(p_k, z_k) \leq \frac{\bar{\alpha}}{\rho\beta\bar{\theta}}\sum_{k=0}^{N}(f(p_k) - f(p_{k+1})) \leq \frac{\bar{\alpha}}{\rho\beta\bar{\theta}}(f(p_0) - f^*).$$

Therefore, $(N+1)\min\{d^2(p_k, z_k) : \ k = 0, 1, \dots, N\} \leq \bar{\alpha}(f(p_0) - f^*)/(\rho\beta\bar{\theta})$, which implies the desired inequality. $\square$

4.4 The constrained center of mass in κ-hyperbolic space forms

Let us consider the problem of computing the center of mass, which is also known as the Fermat-Weber problem, the Fréchet mean, or the variance problem (see [3, 5]). This problem arises frequently in various practical applications without constraints, as demonstrated in [2, 26, 36]. However, in this section, we will also consider the constrained version of this problem, which may become important in the future.

Let $\{q_1, \ldots, q_m\} \subseteq \mathbb{H}^n_\kappa$ be a data set of distinct points, $q_i \neq q_j$ for $i \neq j$, and $\mu_1, \ldots \mu_m$ be positive weights with $\sum_{j=1}^m \mu_j = 1$ and $\sigma > 1$. Let $\mathcal{C} \subseteq \mathbb{H}^n_\kappa$ be a closed and κ-hyperbolically convex set. The constrained *center of mass* minimization problem is stated as follows:

$$\arg\min_{p\in\mathcal{C}} \sum_{i=1}^{m} \mu_i d^\sigma_\kappa(p, q_i), \tag{4.92}$$

where d_κ is the intrinsic distance on the κ-hyperbolic space form defined in (1.9). Assume that $\sigma \geq 2$. Then, the objective function of problem (4.92) is continuously differentiable. For simplifying the notations, let $\rho : \mathbb{H}^n_\kappa \to \mathbb{R}$ and $\rho_i : \mathbb{H}^n_\kappa \to \mathbb{R}$ be defined, respectively, by

$$\rho(p) := \sum_{i=1}^{m} \mu_i \rho_i(p), \qquad \rho_i(p) := d^\sigma_\kappa(p, q_i), \tag{4.93}$$

for all $p \in \mathbb{H}^n_\kappa$ and $i = 1, \dots, m$. The sub-level set of ρ associated to $c > 0$ is defined by

$$L_{\rho,\mathcal{C}}(c) := \{p \in \mathcal{C} : \ \rho(p) \leq c\}.$$

Proposition 30. For any $c > 0$, the sub-level $L_{\rho,\mathcal{C}}(c)$ is a compact set. As a consequence, the problem (4.92) has a global solution.

Proof. We can assume without loss of generality that the sub-level set $L_{\rho,\mathcal{C}}(c)$ is a nonempty set. Let $0 < \mu_0 := \min\{\mu_1, \dots, \mu_m\}$, $r := (c/\mu_0)^{1/\sigma}$ and $B(q_i, r) := \{p \in \mathbb{H}^n_\kappa : \ d_\kappa(p, q_i) \leq r\}$, for $i = 1, \dots, m$. Hence, we have $L_{\rho,\mathcal{C}}(c) \subseteq \cup_{i=1}^m B(q_i, r)$. Consequently, $L_{\rho,\mathcal{C}}(c)$ is bounded. Since ρ is continuous, we conclude that $L_{\rho,\mathcal{C}}(c)$ is also closed. Therefore, $L_{\rho,\mathcal{C}}(c)$ is compact, which proves the first part. For the second part, note that since the sub-level $L_{\rho,\mathcal{C}}(c)$ set is compact and nonempty, the constrained optimization problem $\arg\min_{p \in \mathcal{L}_{\rho,\mathcal{C}}(c)} \rho(p)$ has a global solution. Let us denote this solution by $\bar{p} \in L_{\rho,\mathcal{C}}(c)$. Thus, we have $\rho(\bar{p}) \leq \rho(p)$ if $p \in L_{\rho,\mathcal{C}}(c)$ and, for all $p \in \mathcal{C} \backslash L_{\rho,\mathcal{C}}(c)$ we have $\rho(p) \geq c \geq \rho(\bar{p})$. Therefore, $\rho(\bar{p}) \leq \rho(p)$ for all $p \in \mathcal{C}$, and the problem (4.92) has a global solution. $\square$

For proving the next proposition we will need the following definition.

Definition 7. Let $\mathcal{C} \subseteq \mathbb{H}^\backslash$ be a κ-hyperbolically convex set and $I \subseteq \mathbb{R}$ an interval. A function $f : \mathcal{C} \to \mathbb{R}$ is called κ-hyperbolically convex (respectively, strictly κ-hyperbolically convex) if for any geodesic segment $\gamma : I \to \mathcal{C}$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense.

Proposition 31. The problem (4.92) has global solution and the solution is unique.

Proof. Let $p_0 \in \mathcal{C}$. Thus, the sublevel set $\{p \in \mathbb{H}^n_\kappa : \ \rho(p) \leq \rho(p_0)\} \neq \varnothing$. Thus, Proposition 30 implies that the problem (4.92) has global solution. We proceed to prove the uniqueness. First note that, due to $\sigma \geq 2$, (1.13) and similar arguments to the ones used to prove [20, Lemma 2.4], we can prove that the Hessian $\operatorname{Hess} \rho_i(p)$ is positive definite. Thus [20, Proposition 5.4] implies that ρ_i is strictly κ-hyperbolically convex. Therefore, as $\mu_1, \ldots, \mu_m$ are positive, it can be inferred from (4.93) that ρ is strictly κ-hyperbolically convex. As a consequence, the global solution of problem 4.92 is unique. □

Now, let us analyze the convergence of Algorithm 2 when it is applied to the problem (4.92).

Theorem 11. Consider the sequence $(p_k)_{k\in\mathbb{N}}$ generated by Algorithm 2 to solve problem (4.92). Then, $(p_k)_{k\in\mathbb{N}}$ converges to the solution of problem (4.92).

Proof. Consider p_0 as the starting point of the sequence $(p_k)_{k\in\mathbb{N}}$. From Proposition 29, we can deduce that the sequence $(\rho(p_k))_{k\in\mathbb{N}}$ is non-increasing. Consequently, we have

$$(p_k)_{k\in\mathbb{N}} \subseteq \{p \in \mathbb{H}^n_\kappa : \ \rho(p) \leq \rho(p_0)\}.$$

By applying the first part of Proposition 30, we can establish that $(p_k)_{k\in\mathbb{N}}$ is bounded. Thus, $(p_k)_{k\in\mathbb{N}}$ has a cluster point $\bar{p} \in \mathcal{C}$. Utilizing Theorem 9, we conclude that $\bar{p}$ is a stationary point for the problem (4.92). Since ρ is convex and stationary points are global solutions, $\bar{p}$ is also a solution for the problem (4.92). Therefore, since Proposition 31 implies that problem (4.92) has only one solution, the sequence $(p_k)_{k\in\mathbb{N}}$ has only one cluster point and must converge. □

Let us proceed to provide a formula for the gradient of the objective function of problem (4.92). For $p \neq q_i$, (1.9) and (1.12) imply

$$\operatorname{grad} \rho_i(p) = -\sigma d_\kappa^{\sigma-2}(p, q_i) \log_p^\kappa q_i, \qquad i = 1, \ldots, m.$$

Thus, due to $\sigma \geq 2$, the objective function of problem (4.92) is continuously differentiable. Hence, it follows, from the last equality, (1.8), (1.9) and (1.11), that the gradient of the objective function in (4.92) is given by

$$\operatorname{grad} \rho(p) = -\sum_{i=1}^{m} \mu_i \frac{\sigma \kappa^{1-\frac{\sigma}{2}} (\operatorname{arcosh}(-\kappa \langle p, q_i \rangle))^{\sigma-1}}{\sqrt{\kappa^2 \langle p, q_i \rangle^2 - 1}} \left(\mathrm{I} + \kappa_p p^\top \mathrm{J} \right) q_i. \tag{4.94}$$

To implement Algorithm 2, it is necessary to utilize the formula to project intrinsically into convex set $\mathcal{C}$ in consideration, some of which were provided in Section 4.2, and (4.94).

4.5 Conclusion

In this chapter, we have examined the gradient projection method as a solution approach for constrained optimization problems in κ-hyperbolic space forms, particularly for potentially non-convex objective functions. We considered both constant and backtracking step sizes in our analysis. In our investigation, we presented several innovative properties of the intrinsic κ-projection into convex sets of κ-hyperbolic space forms. These properties are crucial for analysing the method and also hold independent significance. We discussed the relationship between the intrinsic κ-projection and the Euclidean orthogonal projection as well as the Lorentz projection. Furthermore, we derived formulas for the intrinsic κ-projection into specific convex sets using the Euclidean orthogonal projection and the Lorentz projection. Concerning the convergence results of the gradient projection method, we established two key findings. First, we showed that every accumulation point of the sequence generated by the method with backtracking step sizes is a stationary point of the problem. Second, assuming the Lipschitz continuity of the objective function's gradient, we showed that each accumulation point of the sequence produced by the gradient projection method with a constant step size is also a stationary point. Additionally, we provided an iteration complexity bound that characterizes the number of iterations needed to achieve a suitable measure of stationarity for both step sizes. Finally, the properties of the constrained Fermat-Weber problem were explored.

CHAPTER 5

CONCLUSION AND FUTURE WORK

This thesis demonstrates the characterization of convex sets on hyperbolic space, the characterization of hyperbolically convex quadratic functions, several innovative properties of the intrinsic κ-projection into convex sets of κ-hyperbolic space forms and the gradient projection method for minimizing potentially non-convex functions.

A related remark to Chapter 2 and Chapter 3 is that the class of convex functions on constant curvature manifolds is widening with the decrease of the sign of the curvature. We also expect that the class of convex functions on a proper convex subset of the hyperbolic space to be much more wider than the convex functions on the corresponding proper convex subset of the sphere. It is worth noting that when defining a quadratic function on the sphere and in the hyperbolic space, we rely on the fact that these manifolds are subsets of the Euclidean space. In fact, this kind of study could potentially be extended to manifolds that are subsets of matrix spaces. However, engaging in convex analysis within a general Riemannian manifold remains a challenging endeavour.

To further explore the topic in Chapter 4, an intriguing aspect that could be explored is conducting a comprehensive study on the asymptotic convergence of the method for optimization problem with convex objective functions. It is worth noting that the inclusion of the projection in the definition of the method poses a challenging problem, and we believe that additional properties of the projection would be necessary for a successful analysis. Given the significance of comprehending the computation of the projection in implementations of the gradient projection method, it would be worthwhile to pursue new research that offers a technique for computing the projection in additional convex sets, as well as obtaining new properties of it. This study can open up possibilities for designing other methods in hyperbolic spaces.

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