Baric Axial Algebras of Jordan Type

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Abstract

An axial algebra is a non-associative commutative algebra and generated by special idempotents (axes) satisfying a prescribed fusion law. An axial algebra is baric if there exists a surjective homomorphism from the algebra onto the ground field, which does not send axes to zero. In this text, we investigate baric algebras within the class of algebras of Jordan type η. We show that such algebras only exist when $\eta = 2$ or $\eta = \frac{1}{2}$. We completely classify the case $\eta = 2$ by showing that baric algebras of Jordan type 2 are exactly the Matsuo algebras of Moufang type and their factors. The case of $\eta = \frac{1}{2}$ is more complicated as it includes Jordan algebras. We demonstrate the existence of the universal k-generated baric algebra of Jordan type $\frac{1}{2}$ and in doing so we also establish a few interesting facts about more general axial algebras in terms of magma algebras. We also explicitly construct the 4-generated baric axial algebras of Jordan type $\frac{1}{2}$, which turns out to have dimension 54, which is quite below the known upper bound of 81. As a consequence of this calculation, we also deduce that the universal k-generated baric algebra of Jordan type $\frac{1}{2}$ is Jordan for all k.

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CONTENTS

CHAPTER 1

INTRODUCTION

Axial algebras are non-associative commutative algebras. Unlike other classes of algebras defined by identities, axial algebras are generated by a set of idempotents that follow a specific fusion law. In 1870 Peirce introduced a decomposition of an associative algebra as a sum of eigenspaces of an idempotent. This later became known as the Peirce decomposition. In 1947, during Albert's research of Jordan algebras, he generalised the Peirce decomposition to the case of Jordan algebras [1]. The Peirce decompositions are just instances of what we call fusion laws. After the construction of the Monster group and the Griess algebra, the concept of fusion law became widely recognised.

In 1982, Griess constructed the Monster group [15] whose existence was predicted independently around 1973 by Fischer in his unpublished work on 4-transposition groups and in 1975 by Griess at the Conference on Finite Groups at the University of Utah [14]. The Monster was constructed as the automorphism group of a $196,883$-dimensional algebra. Later Conway suggested a $196,884$-dimensional version of this algebra by adding the identity to the algebra [3]. This algebra V is now called the Griess algebra. It is a commutative non-associative algebra over the field of real numbers.

The research around the moonshine conjecture in the 1980s resulted in the construction by Frenkel, Lepowsky and Meurman [11] of the moonshine module $V^\natural$ and, in 1992, Borcherds [2] showed that the moonshine module is in fact a vertex operator algebra (VOA). Being a VOA, $V^\natural$ can be decomposed as $V^\natural = \oplus_{i\geq 0} V_i$, where V_i is called the weight-i component of $V^\natural$. The weight-2 component V_2 of $V^\natural$ is the Griess algebra. An OZ (one-zero) VOA is a VOA with a 1-dimensional V_0 and $V_1 = 0$. In 1996, Miyamoto [26] introduced Ising vectors, conformal vectors of central charge $\frac{1}{2}$ in V_2, and linked them to automorphisms of order 2 of OZ VOAs. Miyamoto started the classification of OZ VOAs generated by two Ising vectors and it was completed later by Sakuma [28].

In 2009, Ivanov [22] turned the properties of V used by Sakuma in his proof into axioms of a new class of algebras, called Majorana algebras. A Majorana algebra is a commutative non-associative algebra over the field of real numbers generated by special idempotents called Majorana axes that satisfy the fusion law of Monster type $\left(\frac{1}{4}, \frac{1}{32}\right)$. The adjoint

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 1.1: Fusion law $\mathcal{M}\left(\frac{1}{4}, \frac{1}{32}\right)$

endomorphism ad_a of the axis a is required to be semisimple with eigenvalues in the set $\left\{1, 0, \frac{1}{4}, \frac{1}{32}\right\}$. The fusion law also restricts the products of eigenvectors of ad_a. Apart from the fusion law, Majorana algebras also need to satisfy further axioms. To generalise Majorana algebras, Hall, Rehren and Shpectorov allowed an arbitrary ground field, an arbitrary fusion law and they removed all axioms not related to the fusion law. This led to the concept of axial algebras [17]. In the paper [16], Hall, Rehren and Shpectorov introduced and studied the class of algebras of Jordan type η. The fusion law $\mathcal{J}(\eta)$ for this class is as follows. This class of algebras includes Jordan algebras in the case where

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

Table 1.2: fusion law $\mathcal{J}(\eta)$

$\eta = \frac{1}{2}$ and Matsuo algebras where η can be arbitrary. Jordan algebras were introduced by a German physicist Ernst Pascual Jordan in 1933 [23] and the examples of Matsuo algebras were first described in 2003 by Matsuo [25] while Hall, Rehren and Shpectorov [16] defined this class of algebras in the general case. In order to define Matsuo algebras, we use 3-transposition groups, which are groups generated by 3-transpositions, a normal set of involutions, such that the product of any two elements has order at most 3. In 1971, Fischer [9] classified irreducible 3-transposition groups. Cuypers and Hall [5] then classified the general case.

Hall, Rehren and Shpectorov [16] (see also Hall, Segev and Shpectorov [18]) showed that the algebras of Jordan type $\eta \neq \frac{1}{2}$ are either Matsuo algebras or their quotients. In [16], they also determined 2-generated algebras of Jordan type, which will be very important in this text. In 2018, Hall, Segev and Shpectorov [19] also proved that every algebra of Jordan type admits a Frobenius form, that is, a non-zero bilinear form, that associates with the algebra product. Furthermore, the Frobenius form can be scaled so that every primitive axis has length 1.

More recently, the research on algebras of Jordan type has been focusing on the case $\eta = \frac{1}{2}$. In 2020, Gorshkov and Staroletov showed that every algebra of Jordan type $\frac{1}{2}$ generated by three primitive axes has a dimension at most nine [13]. Namely, they introduced a 4-parameter family of 9-dimensional algebras $A := A(\alpha, \beta, \gamma, \psi)$ and showed that every 3-generated algebra of Jordan type is a quotient of one of these. In 2024, De Medts, Rowen and Segev [7] proved that primitive 4-generated algebras of Jordan type η are at most 81-dimensional, for any η.

In recent years, baric axial algebras have become an interesting topic. A baric algebra is an algebra A over the field $\mathbb{F}$ endowed with a surjective weight homomorphism $w : A \to \mathbb{F}$. In 1939, Etherington [8] introduced the notion of a genetic algebra to model inheritance in genetics. He also introduced the class of baric algebras as one of the variations of genetic algebras. The formal definition of a genetic algebra was given by Schafer in 1949 [27]. This definition includes the requirement that every genetic algebra should be baric.

In general, baric algebras seem to appear as exceptional cases in most classification results on axial algebras. For example, in 2022, Franchi, Mainardis and Shpectorov [10] found an exceptional infinite-dimensional 2-generated axial algebra HW of Monster type $\left(2, \frac{1}{2}\right)$, which is baric. The classifications of 2-generated and 3-generated algebras of Jordan type, mentioned above, also include baric algebras where the algebra's identity element exceptionally becomes a nilpotent. In this text, we focus on the baric case within the general class of algebras of Jordan type $\frac{1}{2}$.

We start in Chapter 2 with the basic concepts of axial algebras, including fusion laws, axes and axial algebras, graded fusion laws, Miyamoto groups, closed sets of axes, radical and connectivity, and what the latter two look like in the presence of a Frobenius form. These results are based on the paper by Kharsaw, M$^{\text{c}}$Inroy and Shpectorov [24].

In Chapter 3, we discuss 3-transposition groups, related Fischer spaces and their connectivity. Towards the end of the chapter, we focus on the Moufang type of 3-transposition groups, which play a significant role in our text. A 3-transposition group is said to be of Moufang type if it contains no two distinct commuting 3-transpositions. This condition results in the group being an extension of a 3-group by the group of order 2. We use the results from [5] to add more details for groups of Moufang type. We also include a description of the Burnside group $B(d, 3)$ to give us a concrete idea about groups of Moufang type.

In Chapter 4, we discuss in detail the algebras of Jordan type and the available results. We also introduce Matsuo algebras and their properties, including a discussion of their radicals. We also present the classification of 2-generated algebras of Jordan type η by Hall, Rehren and Shpectorov [16], including a detailed investigation of ideals in these algebras. Next, we explain the classification of algebras of Jordan type $\eta \neq \frac{1}{2}$. In the final section, we also briefly discuss 3- and 4-generated algebras of Jordan type.

In Chapter 5, we focus on the baric property of axial algebras. We start with the definition, which needs to be slightly altered in the axial case. We provide some motivating examples, such as $1A$, $2B$, $3C(2)$ and $A\left(\frac{1}{2}, 1\right)$. This chapter contains our first result, a characterisation of baric axial algebras in terms of the fusion law.

Theorem 1.0.1. *A connected primitive axial algebra is baric if and only if it satisfies a fusion law $\mathcal{F}$ such that for all $\lambda, \mu \in \mathcal{F} \setminus \{1\}$, we have that $\lambda * \mu$ does not contain* 1.

This appears as Theorem 5.2.1 in Chapter 5. This result leads to the following simplified fusion law for baric algebras of Jordan type:

$*$	1	0	η
1	1		η
0		0	η
η	η	η	0

Then we use the information about 2-generated algebras of Jordan type from Chapter 4 to establish the following.

Theorem 1.0.2. *An algebra of Jordan type η is baric if and only if $(a, b) = 1$ for all primitive axes a and b. Consequently, η can only be 2 or $\frac{1}{2}$ for a baric algebra of Jordan type.*

This appears in Chapter 5 as Corollary 5.3.2.

Towards the end of the chapter, we complete the classification of baric algebras of Jordan type 2. Namely, we prove the following.

Theorem 1.0.3. *A connected Matsuo algebra $M = M_2(G, D)$ is baric if and only if the corresponding 3-transposition group (G, D) is of Moufang type.*

(This statement is the content of Theorem 5.4.1). Hence the baric algebras of Jordan type 2 are the Matsuo algebras of Moufang type and their quotients.

The remainder of the text focuses on the case $\eta = \frac{1}{2}$, which is much more difficult because it includes Jordan algebras. Our first goal is to establish the existence of the universal k-generated baric algebra of Jordan type $\frac{1}{2}$. In Chapter 6, we provide the necessary background on magmas, including the basic definitions, and examples, including the free commutative magma, and free commutative magma generated by idempotents (also known as free axial magma). We use graph theory language to describe the elements of various free magmas as binary trees with additional properties.

In Chapter 7, we use the free axial magma to construct the universal baric axial algebra of Jordan type $\frac{1}{2}$. The focus is on the relators that are needed to create such an algebra. These we split into two groups: (1) relators assuring the correct spectrum and primitivity; and (2) relators assuring the correct fusion law. We provide an explicit basis for the algebra where only the relators of type (1) are imposed and hence show that it is infinite-dimensional as long as the number of generators, k, is at least 2. Once both relator groups (1) and (2) are imposed, we conjecture that the algebra becomes finite-dimensional for all k, but this remains an open question.

In Chapter 8, we provide a simple example of a series of baric Jordan algebras of type $\frac{1}{2}$ with an increasing number of generators k. This demonstrates in particular that such algebras exist for all k and that the dimension increases with k.

In Chapter 9, we use GAP [12] and some ideas from [7] to construct the 4-generated universal baric algebra of Jordan type $\frac{1}{2}$ explicitly. The result (Theorem 9.5.4) is quite surprising: while the dimension of the general 4-generated algebra of Jordan type $\frac{1}{2}$ has the sharp upper bound of 81, in the baric case the dimension, 54, is significantly lower. We also obtain an important corollary (Corollary 9.5.5) that the baric algebras of Jordan type $\frac{1}{2}$ are necessarily Jordan.

The main body of text ends with a conclusion, an appendix containing the complete GAP code for our calculation in Chapter 9, and the bibliography.

CHAPTER 2

BASIC DEFINITIONS AND RESULTS

In this chapter, we review the basic notion of axial algebras. We use $\mathbb{F}$ to denote a field.

2.1 Axial algebras

Definition 2.1.1. Let $\mathcal{F} \subseteq \mathbb{F}$ be a finite set and a symmetric operation $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ (the power set of $\mathcal{F}$) be defined on $\mathcal{F}$. The pair $(\mathcal{F}, *)$ is said to be a *fusion law* over $\mathbb{F}$.

In the following text, we write $\mathcal{F}$ for the fusion law $(\mathcal{F}, *)$.

Let us see some examples of fusion laws. We normally use tables to represent fusion laws.

Let us explain these three tables. Each entry is a set. We remove the set brackets and leave the entry blank rather than write the empty set notation $\emptyset$. Notice that each entry $\lambda * \mu$ is a subset of the finite set $\mathcal{F}$. In Table 2.1 (a), the finite set is $\mathcal{A} = \{1, 0\}$ and this fusion law is satisfied by idempotents in commutative associative algebras over arbitrary fields. For non-associative cases, we need to introduce other elements which are shown

-

$*$	1	0
1	1	
0		0

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 2.1: Examples of fusion laws: (a) $\mathcal{A}$; (b) $\mathcal{J}(\eta)$; (c) $\mathcal{M}(\alpha, \beta)$

in Tables 2.1 (b) and (c). The second example is called the fusion law of *Jordan type* η, where $\mathcal{J}(\eta) = \{1, 0, \eta\}$ with $\eta \in \mathbb{F} \setminus \{1, 0\}$. It is a sub fusion of the third example, the fusion law of *Monster type* (α, β) (when we allow β to be η and remove α). The finite set in the third example is $\mathcal{M}(\alpha, \beta) = \{1, 0, \alpha, \beta\}$ where α and β are distinct and from $\mathbb{F} \setminus \{1, 0\}$.

Suppose that A is a commutative non-associative algebra. Recall that the adjoint endomorphism of an element a of A is $\mathrm{ad}_a : A \to A$ that is defined by $b \mapsto ab$ for all $b \in A$. The adjoint endomorphism ad_a is said to be *semisimple* if it is diagonalisable. Take $\lambda \in \mathbb{F}$. The λ-eigenspace of ad_a is $A_\lambda(a) = \{v \in A \mid av = \lambda v\}$. Note that $A_\lambda(a) = 0$ if λ is not an eigenvalue of ad_a. For $\Lambda \subseteq \mathbb{F}$, we write $A_\Lambda(a) = \bigoplus_{\lambda \in \Lambda} A_\lambda(a)$.

In the following text, we use $\mathcal{F} \subseteq \mathbb{F}$ to represent a fusion law.

Definition 2.1.2. A non-zero element $a \in A$ is an $\mathcal{F}$-*axis* if it satisfies:

(a) a is idempotent, i.e., $a^2 = a$;

(b) ad_a is semisimple and all eigenvalues of ad_a are in $\mathcal{F}$, i.e., ad_a has a basis of eigenvectors and $A = A_{\mathcal{F}}(a) = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$;

(c) $A_\lambda(a) \cdot A_\mu(a) \subseteq A_{\lambda * \mu}(a)$ for all $\lambda, \mu \in \mathcal{F}$, where $\lambda * \mu$ is a set.

Note that $\mathcal{F}$ is non-empty and always contains 1. We can show that by considering the adjoint endomorphism of an axis a from A. We have that $\mathrm{ad}_a(a) = a \cdot a = a^2 = a = 1 \cdot a$. This means that a is an eigenvector of ad_a and the corresponding eigenvalue is 1, i.e., $1 \in \mathcal{F}$ and $a \in A_1(a)$.

Definition 2.1.3. An $\mathcal{F}$-axis a is *primitive* if $A_1(a)$ is 1-dimensional.

Definition 2.1.4. Let X be a generating set of $\mathcal{F}$-axes from A. The pair (A, X) is said to be an *$\mathcal{F}$-axial algebra* and it is *primitive* if all axes in X are primitive.

Again, we use A for the pair (A, X).

Next, we state the definitions of two important classes of axial algebras.

Definition 2.1.5. We call A an *axial algebra of Jordan type* η if it is generated by a set of primitive axes satisfying the fusion law $\mathcal{J}(\eta)$ (see Table 2.1 (b)).

Definition 2.1.6. The algebra A is called an *axial algebra of Monster type* (α, β) if it is a primitive axial algebra whose generating axes satisfy the fusion law $\mathcal{M}(\alpha, \beta)$ (see Table 2.1 (c)).

Let us first consider an example of an axial algebra of Jordan type η.

Example 2.1.7. Suppose that $A := 3C(\eta) = \langle a, b, c \rangle$ where $\eta \in \mathbb{F} \setminus \{0, 1\}$ and $\mathrm{char}\, \mathbb{F} \neq 2$. The multiplication on this algebra is given in the following table.

$\cdot$	a	b	c
a	a	$\frac{\eta}{2}(a+b-c)$	$\frac{\eta}{2}(a+c-b)$
b	$\frac{\eta}{2}(b+a-c)$	b	$\frac{\eta}{2}(b+c-a)$
c	$\frac{\eta}{2}(c+a-b)$	$\frac{\eta}{2}(c+b-a)$	c

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 2.2: Monster fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$

Note that each element in $\{a, b, c\}$ is an idempotent and A is 3-dimensional. So, taking, for example, idempotent a, we have

$$\dim A_1(a) = \dim A_0(a) = \dim A_\eta(a) = 1.$$

We have that $A_1(a) = \langle a \rangle$ as a is an idempotent, that is, $a \cdot a = a = 1 \cdot a$. Also, $A_0(a) = \langle \eta a - b - c \rangle$ since $a(\eta a - b - c) = \eta a^2 - ab - ac = \eta a - \frac{\eta}{2}(a + b - c) - \frac{\eta}{2}(a + c - b) = 0$. Finally, since $a(b - c) = \frac{\eta}{2}(a + b - c) - \frac{\eta}{2}(a + c - b) = \eta(b - c)$, we have that $A_\eta(a) = \langle b - c \rangle$.

The Griess algebra is an axial algebra of Monster type and its axes satisfy the fusion law in Table 2.2.

2.2 Miyamoto Group

We first need to introduce the notion of gradings. Let us state some basic definitions. The exposition of this section is based on [6].

Definition 2.2.1. Suppose that $(X, *)$ and $(Y, *)$ are fusion laws. Then we call $\xi : X \to Y$ a *morphism* of fusion laws if it satisfies $\xi(x * y) \subseteq \xi(x) * \xi(y)$ for all $x, y \in X$.

Definition 2.2.2. Suppose that T is a group. The *group fusion law* $(T, *)$ is defined by $s * t = \{st\}$ for all $s, t \in T$.

Definition 2.2.3. Let $(X, *)$ be a fusion law and $(T, *)$ a group fusion law. A T-*grading* of $(X, *)$ is a morphism $\xi : (X, *) \to (T, *)$. The grading ξ is *Abelian* if T is an Abelian group and it is *adequate* if $\xi(X)$ generates T.

If $\mathcal{F}$ is T-graded and A is an $\mathcal{F}$-axial algebra with the generating set X, then for each $a \in X$ and $t \in T$, we set $A_t(a) = A_{\xi^{-1}(t)}(a)$ and note that $A = \bigoplus_{t \in T} A_t(a)$ is a T-grading of A.

Now consider the linear character group $T^* := \mathrm{Hom}(T, \mathbb{F}^\times)$. Let $\chi \in T^*$. Take a map $\tau_a(\chi) : A \to A$ defined via $u \mapsto \chi(t)u$ for all $u \in A_t(a)$ and extended linearly to A. Note that $\tau_a(\chi)$ is an automorphism of A. Indeed, if $u \in A_t(a)$ and $v \in A_s(a)$, then $uv \in A_{ts}(a)$. So $\tau_a(\chi)(uv) = \chi(ts)uv = \chi(t)\chi(s)uv = (\chi(t)u)(\chi(s)v) = \tau_a(\chi)(u)\tau_a(\chi)(v)$. Clearly, $\tau_a(\chi)$ is injective as $\chi(t) \neq 0$ always holds. Also, $\tau_a(\chi)$ is surjective because all the basis elements are in the image. Since $\tau_a(\chi)$ is both injective and surjective, $\tau_a(\chi)$ is bijective. Therefore, $\tau_a(\chi)$ is an automorphism of A. The map $T^* \to \mathrm{Aut}(A)$ given by $\chi \mapsto \tau_a(\chi)$ is a group homomorphism. Its image $\{\tau_a(\chi) \mid \chi \in T^*\}$ is the *axis subgroup* of $\mathrm{Aut}(A)$ corresponding to a.

From the above discussion, we have the following definition.

Definition 2.2.4. The *Miyamoto group* $\mathrm{Miy}\,(X)$ of A is the subgroup $\langle \tau_a(\chi) \mid a \in X, \chi \in T^* \rangle$ of $\mathrm{Aut}(A)$.

In other words, the Miyamoto group of A is the subgroup of $\mathrm{Aut}(A)$ generated by all the axis subgroups.

We now illustrate this by an example, where $T = C_2 = \{+, -\}$. Note that in this case,

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 2.3: C_2-grading of $\mathcal{M}(\alpha, \beta)$

there is only one non-trivial character χ (we assume in this section and the remainder of the thesis that $\operatorname{char} \mathbb{F} \neq 2$). So we can simplify the notation $\tau_a(\chi)$ to τ_a. The order of this automorphism τ_a divides 2. Hence, we call it the *Miyamoto involution*.

Example 2.2.5. The fusion law $\mathcal{F} = \mathcal{M}(\alpha, \beta)$ in Table 2.1 (c) is C_2-graded. We can take $\xi(1) = \xi(0) = \xi(\alpha) = +$ and $\xi(\beta) = -$ (this partition is shown in Table 2.3). In this case, τ_a acts as an identity on $A_+(a) = A_1(a) \oplus A_0(a) \oplus A_\alpha(a)$ and as a minus identity on $A_-(a) = A_\beta(a)$. Similarly, the fusion law $\mathcal{J}(\eta)$ in Table 2.1 (b) is also C_2-graded with $\xi(1) = \xi(0) = +$ and $\xi(\eta) = -$. So here τ_a acts as an identity on $A_+(a) = A_1(a) \oplus A_0(a)$ and as a minus identity on $A_-(a) = A_\eta(a)$.

2.3 Closed sets of axes

Let (A, X) be an axial algebra with a T-graded fusion law $\mathcal{F}$. We will first note the following fact.

Proposition 2.3.1. *If $a \in A$ is an $\mathcal{F}$-axis and $\sigma \in \operatorname{Aut} A$, then a^σ is an $\mathcal{F}$-axis.*

Proof. Let $b = a^\sigma$. Then $b^2 = (a^\sigma)^2 = (a^2)^\sigma = a^\sigma = b$. So b is an idempotent. Let us show that $A_\lambda(b) = A_\lambda(a)^\sigma$ for all $\lambda \in \mathbb{F}$. Take $u \in A_\lambda(a)^\sigma$. Then $u = v^\sigma$ for some

$v \in A_\lambda(a)$. We have that $bu = a^\sigma v^\sigma = (av)^\sigma = (\lambda v)^\sigma = \lambda u$. So $u \in A_\lambda(b)$. This means that $A_\lambda(a)^\sigma \subseteq A_\lambda(b)$. Conversely, we take $t \in A_\lambda(b)$. Set $s = t^{\sigma^{-1}}$ so that $t = s^\sigma$. Then $(as)^\sigma = a^\sigma s^\sigma = bt = \lambda t = \lambda s^\sigma$. Thus, $(as)^\sigma = \lambda s^\sigma$, and so $as = \lambda s$, meaning that $s \in A_\lambda(a)$. It follows that $t = s^\sigma \in A_\lambda(a)^\sigma$. Therefore, $A_\lambda(b) = A_\lambda(a)^\sigma$. This implies that $A_\Lambda(b) = A_\Lambda(a)^\sigma$ for all $\Lambda \subseteq \mathbb{F}$. Taking $\Lambda = \mathcal{F}$, we have that $A_\mathcal{F}(b) = (A_\mathcal{F}(a))^\sigma = A^\sigma = A$. Therefore, ad_b is semi-simple on A with all eigenvalues in $\mathcal{F}$. Also, $A_\lambda(b)A_\mu(b) = A_\lambda(a)^\sigma A_\mu(a)^\sigma = (A_\lambda(a)A_\mu(a))^\sigma \subseteq (A_{\lambda*\mu}(a))^\sigma = A_{\lambda*\mu}(b)$. So we have shown that b satisfies the fusion law $\mathcal{F}$. Since b is an idempotent, we have that b is an $\mathcal{F}$-axis. □

We can also state a similar result about Miyamoto involutions. Hence we additionally assume that $\mathcal{F}$ is graded by C_2 and so for each axis a we have a Miyamoto involution τ_a.

Proposition 2.3.2. *Under the above assumptions, if $a \in A$ is an axis and $\sigma \in \operatorname{Aut} A$ then $\tau_{a^\sigma} = (\tau_a)^\sigma$.*

Proof. By assumption, $\mathcal{F}$ is graded by $C_2 = \{+, -\}$. Let

$$\mathcal{F}_+ = \xi^{-1}(+) = \{\lambda \in \mathcal{F} \mid \xi(\lambda) = 1\}$$

and

$$\mathcal{F}_- = \xi^{-1}(-) = \{\lambda \in \mathcal{F} \mid \xi(\lambda) = -1\}.$$

Let $A_+(a) = A_{\mathcal{F}_+}(a)$ and $A_-(a) = A_{\mathcal{F}_-}(a)$. Then $A = A_+(a) \oplus A_-(a)$ and τ_a acts as identity on $A_+(a)$ and as minus identity on $A_-(a)$. Let $b = a^\sigma$. We similarly define $A_+(b)$ and $A_-(b)$. As above, $A = A_+(b) \oplus A_-(b)$ and τ_b acts on the summands as identity and minus identity, respectively. Furthermore, note that $A_+(b) = A_{\mathcal{F}_+}(b) = A_{\mathcal{F}_+}(a)^\sigma = A_+(a)^\sigma$ and, similarly, $A_-(b) = A_-(a)^\sigma$.

Take $u = u_+ + u_- \in A$, where $u_+ \in A_+(b)$ and $u_- \in A_-(b)$. By the above, $u^{\tau_b} = u_+^{\tau_b} + u_-^{\tau_b} = u_+ - u_-$. On the other hand, $u^{\tau_a^\sigma} = u_+^{\tau_a^\sigma} + u_-^{\tau_a^\sigma} = u_+^{\sigma^{-1}\tau_a\sigma} + u_-^{\sigma^{-1}\tau_a\sigma} =$

$((u_+^{\sigma^{-1}})^{\tau_a})^\sigma + ((u_-^{\sigma^{-1}})^{\tau_a})^\sigma = (u_+^{\sigma^{-1}})^\sigma + (-u_-^{\sigma^{-1}})^\sigma = u_+ - u_-$. Here we used that $u_+^{\sigma^{-1}} \in A_+(a)$ and $u_-^{\sigma^{-1}} \in A_-(a)$, and so $(u_+^{\sigma^{-1}})^{\tau_a} = u_+^{\sigma^{-1}}$ and $(u_-^{\sigma^{-1}})^{\tau_a} = -u_-^{\sigma^{-1}}$.

We have shown that $u^{\tau_b} = u_+ - u_- = u^{\tau_a^\sigma}$. This means that $\tau_b = \tau_a^\sigma$. □

Because of these observations, we can extend the generating set of axes by adding the conjugates under the action of the Miyamoto group.

Definition 2.3.3. Let X be a set of axes of an algebra A. Then the set $\bar{X} = \{x^\tau \mid x \in X, \tau \in \text{Miy}\,(X)\} = X^{\text{Miy}\,(X)}$ is called the *closure* of X.

The closure of the generating set of an axial algebra preserves many of its properties. For instance, if X generates the algebra A, then its closure, $\bar{X}$, also generates A. Similarly, if the axes in X are primitive, then so are the axes in $\bar{X}$. Also, the connectivity of the projection (di)graph (see Section 2.4 for the definition) is preserved under closure.

The following statement provides another property of axial algebras that is preserved under the operation of closure.

Lemma 2.3.4 ([24])**.** *Suppose that X is a set of axes. Then* $\text{Miy}\,(\bar{X}) = \text{Miy}\,(X)$.

Proof. Clearly, since $X \subseteq \bar{X}$, we have that $\text{Miy}\,(X) \leq \text{Miy}\,(\bar{X})$.

Conversely, take a generator $\tau_x(\chi) \in \text{Miy}\,(\bar{X})$. Then $x \in \bar{X}$, i.e., $x = y^g$ for some $y \in X$ and $g \in \text{Miy}\,(X)$. Note that $\tau_x(\chi) = \tau_{y^g}(\chi) = \tau_y(\chi)^g \in \text{Miy}\,(X)^g = \text{Miy}\,(X)$. So all the generators of $\text{Miy}\,(\bar{X})$ are contained in $\text{Miy}\,(X)$, i.e., $\text{Miy}\,(\bar{X}) \leq \text{Miy}\,(X)$. Therefore, $\text{Miy}\,(\bar{X}) = \text{Miy}\,(X)$. □

This statement tells us that the Miyamoto group does not change under the closure and it has the following corollary.

Corollary 2.3.5. *For a set of axes X, we have that the closure of $\bar{X}$ is $\bar{X}$.*

Proof. Note that the closure of $\bar{X}$ is $\bar{X}^{\text{Miy}(\bar{X})} = \bar{X}^{\text{Miy}(X)} = (X^{\text{Miy}(X)})^{\text{Miy}(X)} = X^{\text{Miy}(X)} = \bar{X}$. So the statement holds. □

Building on this corollary, we now give the following definition.

Definition 2.3.6. A set of axes X is *closed* if $X = \bar{X}$.

Using this terminology, Corollary 2.3.5 tells us that the closure of a set of axes is closed. From now on, we will typically consider closed sets of generators, $X = \bar{X}$. This has the advantage that the Miyamoto group $\text{Miy}(X)$ acts on X. Furthermore, this action is faithful.

2.4 The radical and Frobenius form

To investigate the properties of axial algebras, we need to introduce the notion of the radical and the Frobenius form. The discussion in this section is based on [24].

Definition 2.4.1. Let W be a subspace of an axial algebra A. Then W is said to be *invariant* under the adjoint map ad_a of an axis a if $\text{ad}_a(W) = aW \subseteq W$.

We note that all ideals of A are invariant under ad_a for all axes $a \in A$.

If a is an axis from an axial algebra A, then every $u \in A$ can be decomposed with respect to a as $u = \bigoplus_{\lambda \in \mathcal{F}} u_\lambda$ where each u_λ is the *component* of u contained in $A_\lambda(a)$.

Lemma 2.4.2. *Suppose that $a \in A$ is an axis. A subspace $W \subseteq A$ is invariant under ad_a if and only if, for every $w \in W$ and $\lambda \in \mathcal{F}$, the component w_λ of w is contained in W.*

Proof. Suppose that W is invariant under ad_a. Suppose by contradiction that the statement in the lemma is wrong. Then there exists $w = \sum_{\lambda \in \mathcal{F}} w_\lambda \in W$ such that $w_\lambda \notin W$

for at least one $\lambda \in \mathcal{F}$. Select such a w so that its number of non-zero components is minimal, say k. If $k = 1$, then $w = w_\lambda \notin W$. This is a contradiction. So $k \geq 2$.

Select $\mu \neq \lambda$ such that $w_\mu \neq 0$ and consider $w' = \mu w - aw$. We first note that $w' \in W$ since W is invariant under ad_a and so $aw \in W$. Also, $w' = \mu w - aw = \sum_{\nu\in\mathcal{F}} \mu w_\nu - \nu w_\nu = \sum_{\nu\in\mathcal{F}}(\mu - \nu)w_\nu$, which means that $w'_\nu = (\mu - \nu)w_\nu$ for each $\nu \in \mathcal{F}$. Take $\nu = \lambda$. We have that $w'_\lambda = (\mu - \lambda)w_\lambda$. Since $w_\lambda \notin W$, and $\mu - \lambda \neq 0$, we have that $w'_\lambda \notin W$. Next, we claim that the number of non-zero components of w' is less than k. If $w_\nu = 0$ for some $\nu \in \mathcal{F}$, then $w'_\nu = (\mu - \nu)w_\nu = 0$. Also, if $\nu = \mu$, then $w_\mu \neq 0$ but $w'_\mu = (\mu - \mu)w_\mu = 0$. Thus, w' has more zero components than w, and so it has fewer non-zero components. This is a contradiction.

Conversely, suppose that each component w_λ of w is contained in W. So $\mathrm{ad}_a(w) = \mathrm{ad}_a(\sum_{\lambda\in\mathcal{F}} w_\lambda) = \sum_{\lambda\in\mathcal{F}} \mathrm{ad}_a(w_\lambda) = \sum_{\lambda\in\mathcal{F}} aw_\lambda = \sum_{\lambda\in\mathcal{F}} \lambda w_\lambda \in W$. Therefore, W is invariant under ad_a. □

In particular, this means that every ad_a-invariant subspace J of A decomposes as $J = \bigoplus_{\lambda\in\mathcal{F}} J_\lambda(a)$, where $J_\lambda(a) = J \cap A_\lambda(a)$ for all $\lambda \in \mathcal{F}$.

When the axis a is primitive, the component u_1 will be called the *projection of u to a*. Indeed, in this case, $u_1 = \varphi a$ for some $\varphi \in \mathbb{F}$.

Corollary 2.4.3. *Suppose that $a \in X$ is a primitive axis and J is a subspace of A invariant under ad_a. Then $a \in J$ if and only if $J_1(a) = J \cap A_1(a)$ is not 0.*

Proof. Since a is a primitive axis, we have that $A_1(a) = \langle a\rangle$. If $a \in J$, then $J_1(a) = A_1(a) = \langle a\rangle = J \cap A_1(a) = J_1(a)$. Conversely, if $a \notin J$, then $J_1(a) = J \cap \langle a\rangle = 0$, as required. □

Our next goal is to introduce the concept of radical. We introduce a family of ideals

$\mathcal{J} = \{J \trianglelefteq A \mid J \cap X = \emptyset\}$. Note that the collection $\mathcal{J}$ is non-empty, since $0 \in \mathcal{J}$, and that each $J \in \mathcal{J}$ is ad_a-invariant for all $a \in X$. We claim that $\sum_{J \in \mathcal{J}} J = R \in \mathcal{J}$. Note that R is an ideal because any sum of ideals is an ideal. By Corollary 2.4.3, if $a \notin J$, then $J_1(a) = J \cap A_1(a) = 0$. So $J = \oplus_{\lambda \in \mathcal{F}} J_\lambda(a) = J_1(a) \oplus \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} J_\lambda(a) = 0 \oplus \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} J_\lambda(a) = \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} J_\lambda(a) \subseteq \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} A_\lambda(a)$. Let $W = \bigcap_{a \in X}(\bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} A_\lambda(a))$. Since $J \subseteq \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} A_\lambda(a)$ for each $a \in X$, we have that $J \subseteq W$. Note that W is a subspace of $\bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} A_\lambda(a)$ and $J \in \mathcal{J}$ are subspaces of W. Then $R \subseteq W$. Since $a \in X \notin J$, we have that $a \notin W \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} A_\lambda(a)$. Since $a \notin \bigoplus_{\lambda \in \mathcal{F}\setminus\{1\}} A_\lambda(a)$, the axis a in not contained in R.

This shows that R is in $\mathcal{J}$ and clearly, it is the unique largest ideal in $\mathcal{J}$. So we have the following definition.

Definition 2.4.4. Let A be an axial algebra with the generating set of primitive axes X. The *radical* $R(A, X)$ is the unique largest ideal of A containing no axes from X.

We usually write A for (A, X). In this case, we write $R(A)$ instead of $R(A, X)$.

We often see an axial algebra admit a non-zero bilinear form that associates with the algebra product. Let us state the definition of such a bilinear form.

Definition 2.4.5. A *Frobenius form* on an axial algebra A is a non-zero bilinear form $(\cdot,\cdot) : A \times A \to \mathbb{F}$ such that $(u, vw) = (uv, w)$ for all $u, v, w \in A$.

Note that the Frobenius forms on axial algebras are necessarily symmetric.

Now let us see some relevant statements. In the next two statements, we will see that A is an axial algebra admitting a Frobenius form.

Lemma 2.4.6. *Let a be an axis of an axial algebra A. Then $A = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$ is an orthogonal direct sum with respect to any Frobenius form $(\cdot,\cdot)$.*

Proof. Let $u \in A_\lambda(a)$ and $v \in A_\mu(a)$ with $\lambda \neq \mu$. Then $au = \lambda u$ and $av = \mu v$. We have that $\lambda(u, v) = (\lambda u, v) = (au, v) = (ua, v) = (u, av) = (u, \mu v) = \mu(u, v)$. Since λ and μ are distinct, $(u, v) = 0$. Therefore, eigenspaces are pairwise orthogonal with respect to the Frobenius form. This means that $A = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$ is an orthogonal direct sum. □

We now can investigate the link between the Frobenius form and the radical of A. We use $A^\perp := \{u \in A \mid (u, v) = 0 \text{ for all } v \in A\}$ to denote the radical of the form.

Lemma 2.4.7. *The radical $A^\perp$ is an ideal of an axial algebra A.*

Proof. Clearly, $A^\perp$ is non-empty. Let $x, y \in A^\perp$. Then $(x, v) = (y, v) = 0$ for all $v \in A$. We have that $(\alpha x + y, v) = \alpha(x, v) + (y, v) = 0$. So $\alpha x + y \in A^\perp$. This shows that $A^\perp$ is a subspace. Let $u \in A^\perp$ and $v, w \in A$. We have that $(uv, w) = (u, vw) = 0$. As this is true for every $w \in A$, we conclude that $uv \in A^\perp$. Since A is commutative, this means that $A^\perp$ is an ideal of A. □

Lemma 2.4.8. *Let $(\cdot, \cdot)$ be a Frobenius form on a primitive axial algebra A and $A^\perp$ be the radical of the form. Then a primitive axis a is contained in $A^\perp$ if and only if $(a, a) = 0$.*

Proof. Since a is primitive, we have that $A_1(a) = \langle a \rangle$. Take $x \in A$. Since

$$A = A_1(a) \oplus \bigoplus_{\lambda \in \mathcal{F} \setminus \{1\}} A_\lambda(a),$$

$x = \mu a + \sum_{\lambda \in \mathcal{F} \setminus \{1\}} x_\lambda$ for some $\mu \in \mathbb{F}$ and some $x_\lambda \in A_\lambda(a)$, $\lambda \in \mathcal{F} \setminus \{1\}$. By Lemma 2.4.6, we have that $A_1(a)$ is orthogonal to $\bigoplus_{\lambda \in \mathcal{F} \setminus \{1\}} A_\lambda(a)$, i.e., $(a, \sum_{\lambda \in \mathcal{F} \setminus \{1\}} x_\lambda) = 0$. So $(a, x) = \left(a, \mu a + \sum_{\lambda \in \mathcal{F} \setminus \{1\}} x_\lambda\right) = (a, \mu a) + \left(a, \sum_{\lambda \in \mathcal{F} \setminus \{1\}} x_\lambda\right) = \mu(a, a)$. If $(a, a) = 0$, then for any $x \in A$, we have that $(a, x) = \mu(a, a) = \mu 0 = 0$. This means that $a \in A^\perp$. Conversely, suppose that $a \in A^\perp$. Then a is perpendicular to itself, that is, $(a, a) = 0$. □

Theorem 2.4.9 ([24])**.** *Suppose that (A, X) is a primitive axial algebra admitting a*

Frobenius form $(\cdot,\cdot)$. Then the radical $A^\perp$ coincides with the radical $R(A,X)$ of A if and only if $(a,a)\neq 0$ for all $a\in X$.

Proof. Suppose that $A^\perp$ coincides with the radical $R(A,X)$ of A. Take $a\in X$. By definition of the radical, $a\notin R(A,X)=A^\perp$. So by Lemma 2.4.8, $(a,a)\neq 0$. Conversely, suppose that $(a,a)\neq 0$ for all $a\in X$. Then a is not perpendicular to itself. This means that $a\notin A^\perp$. That is, $A^\perp$ contains no elements from X. By Lemma 2.4.7, we have that $A^\perp$ is an ideal. Since it does not contain any axes, it must lie in the radical $R(A,X)$. It remains to show that $R(A,X)\subseteq A^\perp$. We first show that $R(A,X)$ is orthogonal to every $a\in X$. Indeed, suppose by contradiction that $(a,u)\neq 0$ for some $u\in R(A,X)$. Recall from the above that $(a,u)=\mu(a,a)$ for $\mu\in\mathbb{F}$ such that $u_1=\mu a$. So $\mu(a,a)\neq 0$. Since $\mu(a,a)\neq 0$, we have that $\mu\neq 0$ and so $u_1\neq 0$. It follows from Lemma 2.4.2 that $u_1\in R(A,X)$ as $u\in R(A,X)$. Therefore, $\mu a\in R(A,X)$. Since $\mu\neq 0$, we have that $a\in R(A,X)$. However, $R(A,X)$ contains no axes, which is a contradiction. Thus, $R(A,X)$ is orthogonal to every axis.

Now we prove that $R(A,X)$ is orthogonal to the entire algebra A. By linearity, it suffices to show that $R(A,X)$ is orthogonal to a spanning set of A. As this spanning set, we take the set P of all products of axes. We do induction on the length of the product. First, if $p\in P$ has length 1, then p is an axis, i.e., $p\in X$. By the above, $R(A,X)$ is orthogonal to all the axes. So the claim is true for the products of length 1. Suppose that the claim is true for the products of length at most $k\geq 1$. Consider a product $p\in P$ of length $k+1$. Then $p=p_1p_2$, where $p_1,p_2\in P$ and the lengths of p_1 and p_2 add up to $k+1$, i.e., they both have length at most k. By the inductive assumption, $R(A,X)$ is orthogonal to both p_1 and p_2. Take $u\in R(A,X)$. Then $(u,p)=(u,p_1p_2)=(up_1,p_2)$. Since $R(A,X)$ is an ideal, $up_1\in R(A,X)$. Also, since $R(A,X)$ is orthogonal to p_2, we have that $(up_1,p_2)=0$. Thus, $(u,p)=0$. This means that $R(A,X)$ is orthogonal to p.

Thus, by induction $R(A, X)$ is orthogonal to all products $p \in P$. Since P is a spanning set for A, we have shown that $R(A, X)$ is orthogonal to the entire A. That is, $R(A, X) \subseteq A^{\perp}$. Since $A^{\perp} \subseteq R(A, X)$ and $A^{\perp} \supseteq R(A, X)$, we have that $A^{\perp} = R(A, X)$. □

Note that the final part of the proof is more general. Namely, $R(A, X)$ is contained in $A^{\perp}$ for any primitive axial algebra admitting a Frobenius form.

Definition 2.4.10. Consider a primitive axial algebra (A, X). The *projection digraph* of the algebra is defined on the set X, where $x, y \in X$ form an arc $x \to y$ if the projection of x to y is non-zero. We call the algebra (A, X) *connected* if the projection digraph is strongly connected (i.e., we can reach any vertex from any other vertex via a sequence of arcs).

If the algebra admits a Frobenius form such that $(x, x) \neq 0$ for all $x \in X$, then the projection digraph is symmetric (i.e., it is a simple graph) and vertices x, y are adjacent whenever the condition $(x, y) \neq 0$ holds. So in this case, the connectivity of the algebra is equivalent to the connectivity of the projection graph.

The following proposition is a corollary of Lemma 2.4.2.

Proposition 2.4.11. *Let Γ be the projection graph of the primitive axial algebra (A, X). If $x \in X$ is contained in an ideal J, then so is every $y \in X$ such that there is an arc $x \to y$.*

Proof. Suppose that $x, y \in X$, $x \in J$ and there is an arc $x \to y$. By Lemma 2.4.2, we have that every component x_α of x with respect to y is contained in J. In particular, $x_1 \in J$. Since there is an arc $x \to y$, x_1 is non-zero. Also, the algebra is primitive, and so $x_1 = \mu y$ for some $\mu \in \mathbb{F}$. Since $x_1 \in J$, so is $y = \frac{1}{\mu} x_1$. □

The following corollary now is an immediate consequence.

Corollary 2.4.12 ([24, Lemma 4.14]). *If the primitive algebra is strongly connected, then every proper ideal is contained in the radical.*

Again, when the algebra admits a Frobenius form that is non-zero on the axes, things simplify. Instead of the projection digraph, we have the graph where adjacency is defined by non-orthogonality. Correspondingly, instead of strong connectivity, we can speak simply of the connectivity of this graph and the algebra.

CHAPTER 3

GROUPS OF 3-TRANSPOSITIONS

3.1 Groups of 3-transpositions

To define a Matsuo algebra we first need to have the definition of 3-transposition groups. The notion was introduced by Bernd Fischer [9].

Definition 3.1.1. Let D be a normal subset of a group G, that is, $D \subseteq G$ such that $D = D^g$ for all $g \in G$. A *group of* 3*-transpositions* (or a 3*-transposition group*) is a pair (G, D) if D satisfies the following conditions:

(i) D generates G;

(ii) $d^2 = 1 \neq d$ for all $d \in D$, i.e., each d is an *involution*;

(iii) the order of cd is at most 3 for all $c, d \in D$.

In general, the normal set D is a union of several conjugacy classes, $D_1, \ldots, D_k$. In the following section, we will explore the geometric meaning of this decomposition of D.

Let us consider an example of a 3-transposition group (G, D).

Example 3.1.2. Suppose that G is the symmetric group S_n and $D = (1,2)^G$ is the transposition (2-cycle) class of S_n. Note that the order of the product of any two elements from D is at most 3. Indeed, let $c, d \in D$. Then these two elements can be equal, i.e., $c = d$, and hence $o(cd) = 1$; or they share one element, and then $o(cd) = 3$; or they are disjoint, and so $o(cd) = 2$. Thus (G, D) is a 3-transposition group.

3.2 Fischer Spaces

A *point-line geometry* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set of points and $\mathcal{L} \subseteq 2^{\mathcal{P}}$ is a set of lines. A *partial linear space* is a geometry $(\mathcal{P}, \mathcal{L})$ where two points lie in at most one line. A *partial triple system* is a partial linear space $(\mathcal{P}, \mathcal{L})$ such that each line $\ell \in \mathcal{L}$ contains exactly three points.

We write $x \sim y$ if x and y are *collinear*, i.e., there is a line passing through x and y. If x and y are *non-collinear*, we write $x \not\sim y$.

A *subspace* of a partial linear space $(\mathcal{P}, \mathcal{L})$ is a subset $\mathcal{P}' \subseteq \mathcal{P}$ satisfying the condition that if a line from $\mathcal{L}$ contain at least two points from $\mathcal{P}'$, then this line is fully contained in $\mathcal{P}'$. The intersection of any collection of subspaces is a subspace. Take a subset $\mathcal{S}$ of $\mathcal{P}$. The subspace generated by $\mathcal{S}$ is the unique smallest subspace of $(\mathcal{P}, \mathcal{L})$ containing $\mathcal{S}$ and it can be constructed as the intersection of all subspaces of $\mathcal{P}$ containing $\mathcal{S}$.

Next, we introduce the notion of a Fischer space. The following definition relies on the geometries shown in Figure 3.1 and Figure 3.2.

Definition 3.2.1. A *Fischer space* is a partial triple system $(\mathcal{P}, \mathcal{L})$ in which two distinct, intersecting lines generate a subspace isomorphic to the dual affine plane of order 2 or the affine plane of order 3.

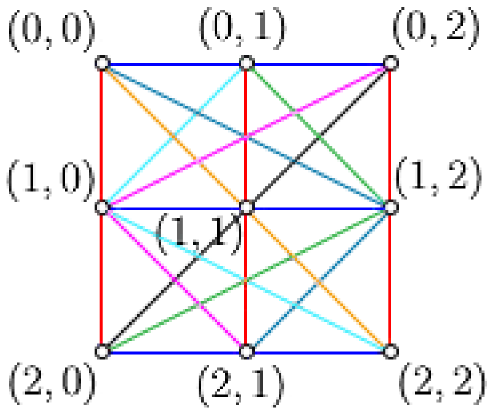


Figure 3.1: The affine plane of order 3

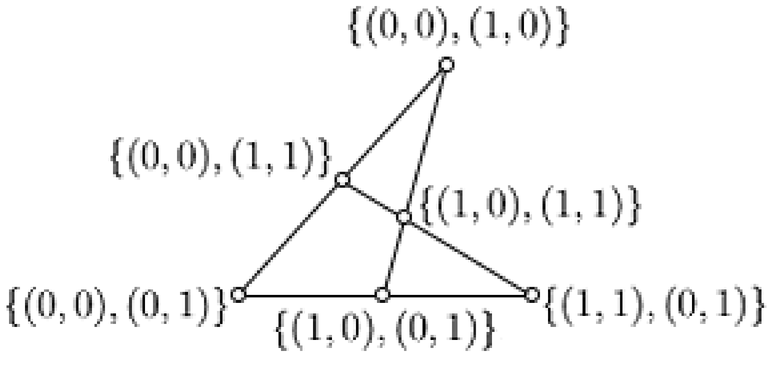


Figure 3.2: The dual affine plane of order 2

Given a 3-transposition group (G, D), we can construct a Fischer space $(\mathcal{P}, \mathcal{L})$ from it. Let $\mathcal{P}$ be D and $\mathcal{L}$ be the set of all triples of points $\{a, b, c\}$, where $a, b \in D$ such that $o(ab) = 3$ and $c = a^b = b^a$. Note that here $S := \langle a, b\rangle = \langle b, c\rangle = \langle a, c\rangle \cong S_3$ and $\{a, b, c\} = S \cap D$. This means that the lines $\{a, b, c\}$ from $\mathcal{L}$ are the intersections of D with S_3.

Conversely, every Fischer space $(\mathcal{P}, \mathcal{L})$ arises in this way from some 3-transposition group. For $a \in \mathcal{P}$, let σ_a be the permutation of $\mathcal{P}$ such that σ_a fixes a itself and the points in $\mathcal{P}$ that are not collinear with a. If the points a, b, c are on the same line, then σ_a switches b and c. This σ_a is an automorphism of $(\mathcal{P}, \mathcal{L})$ of order dividing 2. Define $D' = \{\sigma_a \mid a \in \mathcal{P}\}$ and $G' = \langle \sigma_a \mid a \in \mathcal{P}\rangle \leq \mathrm{Aut}\,(\mathcal{P}, \mathcal{L})$. Then (G', D') is a 3-transposition group. Furthermore, the Fischer space constructed from (G', D') is isomorphic to $(\mathcal{P}, \mathcal{L})$ via $a \mapsto \sigma_a$ and, similarly, if $(\mathcal{P}, \mathcal{L})$ is obtained from some (G, D) then $G' \cong \bar{G} := G/Z(G)$ and D' is corresponding to $\bar{D}$, the image of D in $\bar{G}$. Thus, (G, D) can be recovered from $(\mathcal{P}, \mathcal{L})$ up to the centre.

Now we introduce the collinearity graph Σ. The vertices of a graph are the elements of D. We draw an edge between distinct $a, b \in D$ if and only if a and b are collinear in the Fischer space of (G, D), i.e., if and only if $o(ab) = 3$.

Definition 3.2.2. A 3-transposition group is *connected* if the collinearity graph of the corresponding Fischer space is connected.

We will now investigate the connectivity of the collinearity graph of (G, D). Let $D = D_1 \cup D_2 \cup \ldots \cup D_k$ be the decomposition of the normal set D as a union of conjugacy classes of G. Let $G_i = \langle D_i\rangle$. Then we have the following statements, where we use $\circ$ to denote the central product.

Proposition 3.2.3. *The following hold:*

(i) $G = G_1 \circ G_2 \circ \ldots \circ G_k$*;*

(ii) D_i *is a conjugacy class of* G_i*;*

(iii) $D_1, \ldots, D_k$ *are the connected components of the collinearity graph of the Fischer space of* (G, D)*.*

Proof. Let $c \in D_i$ and $d \in D_j$, where $i \neq j$. Note that $o(cd) \in \{1, 2, 3\}$ because D is a set of 3-transpositions. If $o(cd) = 1$, then $cd = 1$ and so $c = d$. This is impossible because c and d are from different classes. If $o(cd) = 3$, then c and d generate a subgroup isomorphic to S_3. Then c and d are conjugate in this subgroup, but this is a contradiction because c and d lie in different conjugacy classes. So $o(cd)$ can only be 2, which means that c and d commute. We have shown that all elements of D_i commute with all elements of D_j. This means that all elements of $G_i = \langle D_i \rangle$ commute with all elements of $G_j = \langle D_j \rangle$. So $G_1 G_2 \ldots G_k = G_1 \circ G_2 \circ \ldots \circ G_k$ is a subgroup of G because G_i normalise each other. Since $D = D_1 \cup D_2 \cup \ldots \cup D_k \subseteq G_1 \circ G_2 \circ \ldots \circ G_k$ and $G = \langle D \rangle$, we have that $G = G_1 \circ G_2 \circ \ldots \circ G_k$. This proves (i).

Take $x_i \in D_i$. Then $D_i = x_i^G = x_i^{G_1 \circ G_2 \circ \ldots \circ G_k} = x_i^{G_i}$, since $x_i^{G_j} = \{x_i\}$ for $j \neq i$. So D_i is a single conjugacy class of G_i, proving (ii).

For (iii), it is sufficient to show that if $G = G_i$ and $D = D_i$, then D is connected. Take any $c, d \in D$. Then there exists $g \in G$ such that $c^g = d$. We have that $g = c_1 c_2 \ldots c_n$ for some $c_1, c_2, \ldots, c_n \in D$. We prove that there is a path in the collinearity graph from c to d by induction on the product length n. When $n = 0$, we have $g = 1$ and $c = d$. So the statement holds. Next, we consider the case where $n > 0$. Assume that the statement is true when the length is $n - 1$. Let $e = c^{c_1 \cdots c_{n-1}}$. By induction, c and e are connected by a path. It remains to show that e and d are connected. If $o(ec_n) = 1$ or 2, then $d = e^{c_n} = e$ and so the claim holds. On the other hand, when $o(ec_n) = 3$, $\{e, c_n, d = e^{c_n}\}$ is a line. So e and d are collinear, i.e., they are connected by a path of length 1. This means that c

and d are connected by a path. By induction, D is connected. □

Corollary 3.2.4. *Let (G, D) be a 3-transposition group. Then (G, D) is connected if and only if D is a single conjugacy class.*

For each $d \in D$, we write $A_d = \{x \in D \mid o(xd) = 3\}$ and $D_d = \{x \in D \mid o(xd) = 2\}$. We define two equivalence relations, τ and θ, on D as follows: for $c, d \in D$, $c\tau d$ if and only if $A_c = A_d$ and $c\theta d$ if and only if $D_c = D_d$. (Note that $D = \{d\} \cup A_d \cup D_d$; hence if $A_c = A_d$, then D_c and D_d differ in one element and, symmetrically, if $D_c = D_d$, then A_c and A_d differ in one element.) Let $\tau(G) = \langle de \mid d, e \in D, d\tau e\rangle$ and $\theta(G) = \langle de \mid d, e \in D, d\theta e\rangle$. Note that $\tau(G)$ is a normal 2-subgroup and $\theta(G)$ is a normal 3-subgroup.

Next, we summarise the results from [5], and we get the following theorem.

Theorem 3.2.5. *If $\tau(G) \neq 1$, then it is a non-central normal 2-subgroup. Similarly, if $\theta(G) \neq 1$, then $\theta(G)$ is a non-central normal 3-subgroup of G. Additionally, only one of the two subgroups, $\tau(G)$ and $\theta(G)$, can be non-trivial.*

So the cases where $\tau(G) \neq 1$ and $\theta(G) \neq 1$ are independent cases.

Definition 3.2.6. The 3-transposition group (G, D) is said to be *irreducible* if $\tau(G) = \theta(G) = 1$.

Assume that (G, D) is connected and reducible. Then we have two cases. If $\tau(G) \neq 1$, then $|D \cap d\tau(G)| = 2^n$ for all $d \in D$ and some fixed $n > 0$. We write $G = 2^{\bullet n} \cdot \bar{G}$, where $\bar{G} = G/\tau(G)$. Similarly, when $\theta(G) \neq 1$, then $|D \cap d\theta(G)| = 3^n$ for all $d \in D$ and a fixed $n > 0$. Here, we write $G = 3^{\bullet n} \cdot \bar{G}$, for $\bar{G} = G/\theta(G)$. Note that 2^n and 3^n are not the orders of $\tau(G)$ and $\theta(G)$, respectively. Rather, they are the ratios between the numbers of 3-transpositions in $2^{\bullet n} \cdot \bar{G}$ and $3^{\bullet n} \cdot \bar{G}$ and the number of 3-transpositions in $\bar{G}$.

Example 3.2.7. Let us consider the group $H := 3^n : S_n$, the semi-direct product of the group $Q := 3^n = \underbrace{C_3 \times C_3 \times \ldots \times C_3}_{n \text{ times}}$ and $G := S_n$, where S_n acts on 3^n by permuting a fixed set of generators of size n. An element of H can be written as $(u_1, u_2, \ldots, u_n : g)$, for some $u_i \in C_3 = \{1, \xi, \xi^{-1}\}$ and $g \in S_n$. The product in H is given by

$$(u_1, u_2, \ldots, u_n : g)(u'_1, u'_2, \ldots, u'_n : g') = (u_1(u'_1)^{g^{-1}}, u_2(u'_2)^{g^{-1}}, \ldots, u_n(u'_n)^{g^{-1}} : gg').$$

Clearly, G is a subgroup of H and its conjugacy class of transpositions D is a subset of a conjugacy class C of H. Take $\tau = (1, 2) \in D$. Then $C_G(\tau) \cong C_2 \times S_{n-2}$ and $[G : C_G(\tau)] = \frac{n!}{2(n-2)!} = \frac{n(n-1)}{2} = |D|$. Also, $C_Q(\tau) = \{(u_1, u_2, \ldots, u_n) \in Q \mid u_1 = u_2\}$. So $[Q : C_Q(\tau)] = \frac{3^n}{3^{n-1}} = 3$. Note that $C_H(\tau) \geq C_Q(\tau)C_G(\tau)$. So $|C| = \frac{|H|}{|C_H(\tau)|} \leq \frac{|Q||G|}{|C_Q(\tau)||C_G(\tau)|} = 3|D|$. Let us see that in fact $|C| = 3|D|$. For this, we will list all elements of C. These elements can be one of two types:

(i) $(i, j) = (1, 1, \ldots, 1 : (i, j)) \in D$;

(ii) $c_{ij}(i, j) = (1, \ldots, 1, \xi, 1, \ldots, 1, \xi^{-1}, 1, \ldots, 1 : (i, j)) \in C \setminus D$.

Here, $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Furthermore, in (ii), ξ is the i-th position and ξ^{-1} is the j-th position. For each $(i, j) \in D$, we have three elements from the above list involving (i, j). Namely, (i, j) under (i), and $c_{ij}(i, j)$ and $c_{ji}(j, i) = c_{ji}(i, j)$ under (ii). So the above list contains $3|D|$ elements. It remains to see that these elements are all conjugate. First, all $(i, j) \in D$ are conjugate. Let $c_i = (1, \ldots, 1, \xi, 1, \ldots, 1 : (\ ))$, where ξ is the i-th position. We have that $c_{ij} = c_i c_j^{-1}$ and $c_{ji} = c_i^{-1} c_j = c_{ij}^{-1}$. Also, $(i, j)^{c_i} = c_i^{-1}(i, j)c_i = c_i^{-1}(i, j)c_i(i, j)(i, j) = c_i^{-1}c_i^{(i,j)}(i, j) = c_i^{-1}c_j(i, j) = c_{ji}(i, j)$. Similarly, $(i, j)^{c_j} = c_{ij}(i, j)$. Thus, (i, j), $c_{ij}(i, j)$ and $c_{ji}(i, j)$ are in the same conjugacy class. We have determined the conjugacy class C and shown that the size of C is $3|D|$.

Next, we need to show that C is a class of 3-transpositions. First, it is clear that the elements of C are of order 2. We then take $x, y \in C$, $x \neq y$. Conjugating the pair if necessary, we can assume that $y = (i, j)$ for some $i, j \in \{1, \ldots, n\}$. Then for x, we have the following possibilities: (a) if $x = c_{ij}(i, j)$, then $xy = c_{ij}(i, j)(i, j) = c_{ij}$ has order 3; (b) if $x = (i, k)$, for $k \neq j$, then $xy = (i, k, j)$ is also of order 3; (c) if $x = c_{ik}(i, k)$ then we conjugate by c_k and reduce to case (b), that is, here we also have order 3; (d) if $x = (k, l)$, with $\{i, j\} \cap \{k, l\} = \emptyset$, then $xy = (k, l)(i, j)$ is of order 2; finally, (e) if $x = c_{kl}(k, l)$ then we conjugate by c_l and reduce to case (d), obtaining order 2. Therefore, C is indeed a class of 3-transpositions.

We note that C does not generate the entire group H. Let $H_0 := \langle C \rangle$. Since C contains (i, j) and $c_{ij}(i, j)$ for all i and j, this means that H_0 is also generated by c_{ij} and (i, j). Define $Q_0 = \{(u_1, u_2, \ldots, u_n : (\ )) \mid \prod_{i=1}^{n} u_i = 1\}$. Note that Q_0 contains all elements c_{ij} and is generated by them. Hence, $H_0 = Q_0 : G$, since all (i, j) generate $G = S_n$. We conclude that H_0 is a subgroup of index 3 in H.

If we use the bullet notation for the 3-transposition group (H_0, C), it will be $H_0 = 3^{\bullet 1} : S_n$.

3.3 Moufang type

In this section, we consider a special class of 3-transposition groups, which will be very useful later in the text.

Definition 3.3.1 ([5]). A 3-transposition group (G, D) is of *Moufang type* if no two distinct elements of D commute.

It follows from the definition that the collinearity graph of the Fischer space of (G, D) is complete. In particular, the group (G, D) is connected. We will now prove a characterisation of the class of 3-transposition groups of Moufang type.

Proposition 3.3.2. *A connected 3-transposition group (G, D) is of Moufang type if and only if G has no subgroup S_4 generated by 3-transpositions.*

Proof. Suppose that G contains two commuting elements $a, b \in D$, where $a \neq b$. Since the Fischer space of (G, D) is connected, we may consider the shortest path between a and b: $a = a_0 \sim a_1 \sim \ldots \sim a_n = b$. Note that n is at least 2 as $a \not\sim b$. Also, note that $a \not\sim a_2$ because the above is the shortest path. Let $H = \langle a, a_1, a_2 \rangle$. The diagram on the set $\{a, a_1, a_2\}$ is the Coxeter diagram A_3. So the group H is isomorphic to a quotient of the Coxeter group A_3 which is isomorphic to S_4. Note that the proper quotients of S_4 are isomorphic to C_2 and S_3 and they contain no commuting involutions. So $H \cong S_4$.

Conversely, if (G, D) contains a subgroup S_4 generated by 3-transpositions, then within the subgroup we have pairs of commuting 3-transpositions. □

The subgroups S_4 correspond to planes in the Fischer space of (G, D) that are isomorphic to a dual affine plane of order 2. Thus, we cannot have such planes for the Moufang type. So we have the following corollary.

Corollary 3.3.3. *A connected 3-transposition group (G, D) is of Moufang type if and only if every triple of points of its Fischer space generates an affine plane of order 3.*

Recall that Figure 3.1 contains the picture of the affine plane of order 3.

A 3-transposition group of Moufang type is always an extension $Q : 2$ of a normal 3-group Q by a group of order 2. Indeed, in a group (G, D) of Moufang type, we have that $x\theta y$ for all $x, y \in D$. This means that the entire D is the unique equivalence class for θ. Consequently, if we set $Q = \theta(G)$, then the group $\bar{G} = G/Q$ is generated by a single element of order 2, i.e., $\bar{G}$ is of order 2, which shows the claim.

In principle, this 3-group Q can be arbitrarily large depending on the number of generating 3-transpositions. Let us demonstrate this by presenting an example based on the Burnside group $B(d,3)$. The Burnside group $B(d,n)$ is defined as the quotient of the free group F_d on d generators by the normal subgroup generated by all the n-th powers of elements of F_d. In 1902, Burnside asked whether every finitely generated group of exponent n is finite. The group $B(d,n)$ is the unique largest d-generated group of exponent n, and so Burnside's question is equivalent to asking whether $B(d,n)$ is finite. It is now known that the Burnside problem has a negative solution, namely, $B(d,n)$ is infinite for all sufficiently large n and $d \geq 2$. However, when $n = 3$, the group $B(d,3)$ is finite for all d, and this is the case that we need.

Example 3.3.4. Let $Q := B(d,3)$ be the Burnside group of exponent 3 with d generators $q_1, q_2, \ldots, q_d$. Let σ be the automorphism of Q inverting each generator q_i. Then $G = Q\langle\sigma\rangle$ is generated by $k = d+1$ involutions $\sigma, q_1\sigma, \ldots, q_d\sigma$. Note that all these involutions are conjugate in G, because the latter has Sylow 2-subgroups of order 2, which means that it has a unique class of involutions, $D = \sigma^G$. If $z = q\sigma \in D$, where $q \in Q$, then $\sigma z = q$ is of order 3 (or 1 if $z = \sigma$, i.e., $q = 1$). Therefore, (G, D) is indeed a 3-transposition group of Moufang type.

Finiteness of the Burnside group $B(d,3)$ was first proven by William Burnside himself in [4]. Namely, he showed that $|B(d,3)| \leq 3^{2^d-1}$. Much later, Marshall Hall Jr. included the following exact theorem in his textbook [21, Theorem 18.2.1].

Theorem 3.3.5. *The Burnside group $B(d,3)$ is of order $3^{m(d)}$, where $m(d) = d + \binom{d}{2} + \binom{d}{3}$.*

This means that the order of the group Q in the above example does indeed rapidly increase with the number of generators.

CHAPTER 4

AXIAL ALGEBRAS OF JORDAN TYPE

4.1 Jordan algebras

In 1933, during his research in quantum mechanics, German physicist Pascual Jordan introduced a class of non-associative algebras. Jordan initially called them r-number systems, but later they were renamed 'Jordan algebras' in 1946 by Albert. This renaming marked a significant shift in how the mathematical community recognised and understood it. Let us first state the definition of a Jordan algebra.

Definition 4.1.1. A *Jordan algebra* is a commutative non-associative algebra A with the product satisfying the Jordan identity: $x^2(yx) = (x^2y)x$ for all $x, y \in A$.

This identity means that ad_x and ad_{x^2} commute.

There is a systematic way in which a Jordan algebra can be constructed from an associative algebra.

Proposition 4.1.2. *Let A be an associative algebra over a field $\mathbb{F}$ with $\mathrm{char}\,\mathbb{F} \neq 2$. Then A^+, the algebra A with the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, is a Jordan algebra.*

We will typically assume throughout the text that $\operatorname{char}\mathbb{F} \neq 2$.

Note that every subalgebra or quotient algebra of a Jordan algebra is again Jordan. We will particularly focus on the subalgebras of Jordan algebras. In the situation where we consider the Jordan algebra A^+ for an associative algebra A, every subalgebra of A is also a subalgebra of A^+. However, A^+ might have additional subalgebras because a subspace of A might not be closed under the original product but it may be closed under the Jordan product. We will now see an example of this.

Example 4.1.3. Recall that the tensor algebra over a vector space V is

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots,$$

where the product $\otimes$ on $T(V)$ is defined as following: for $u = u_1 \otimes u_2 \otimes \ldots \otimes u_m \in V^{\otimes m}$ and $v = v_1 \otimes v_2 \otimes \ldots \otimes v_n \in V^{\otimes n}$, we have $u \otimes v = u_1 \otimes u_2 \otimes \ldots \otimes u_m \otimes v_1 \otimes v_2 \otimes \ldots \otimes v_n$. This product is clearly associative, i.e., $T(V)$ is an associative algebra.

Next, we define the Clifford algebra. Suppose that q is a quadratic form on V. Consider the ideal $I_q = (v \otimes v - q(v) \mid v \in V)$ of the tensor algebra $T(V)$. Then the Clifford algebra corresponding to the quadratic form q is the algebra $C = Cl(V, q) = T(V)/I_q$. The dimension of C is $2^{\dim V}$. Furthermore, V isomorphically embeds into C in a natural way.

Since the Clifford algebra C is associative, we have that C^+ is a Jordan algebra. Consider the subspace $S = \mathbb{F} \oplus V$ of C^+. Note that S is not closed for the product in C but it is closed under the Jordan product, as for $\alpha, \beta \in \mathbb{F}$ and $u, v \in V$, we have $(\alpha+u)\circ(\beta+v) = \frac{1}{2}((\alpha+u)(\beta+v)+(\beta+v)(\alpha+u)) = \frac{1}{2}(\alpha\beta+\alpha v+u\beta+uv+\beta\alpha+\beta u+v\alpha+vu) = \alpha\beta + \frac{1}{2}(uv + vu) + \alpha v + \beta u \in S$. We call this subalgebra S the *spin factor* Jordan algebra.

We use the notation $S(V, q)$ for it.

Subalgebras of the Jordan algebras A^+ for associative algebras A are called *special* Jordan algebras. Not all Jordan algebras can be obtained in this way and then they are called *exceptional* Jordan algebras.

Let us see a further example. We will need the definition of a $*$-algebra.

Definition 4.1.4. A $*$-algebra is an algebra D over a field $\mathbb{F}$ along with a linear map $* : D \to D$ such that

(i) for all elements $a, b \in D$, $(ab)^* = b^* a^*$;

(ii) for all elements $a \in D$, $(a^*)^* = a$.

This map $*$ is called an involution of D.

Definition 4.1.5. Let D be a $*$-algebra. A square matrix $M = (m_{ij}) \in \mathcal{M}_{n\times n}(D)$ is called *Hermitian* if $m_{ij} = m_{ji}^*$ for all i, j. This means that M is equal to its own conjugate transpose: $M = \left(M^T\right)^*$.

We write the set of all Hermitian matrices as $\mathcal{H}_n(D)$. This set forms a subalgebra of $\mathcal{M}_{n\times n}(D)^+$. That is, although $\mathcal{H}_n(D)$ is not closed under the matrix product, it is closed under the Jordan product. Indeed, for $A, B \in \mathcal{H}_n(D)$, we have $\left((A \circ B)^T\right)^* = \left(\left(\frac{1}{2}(AB + BA)\right)^T\right)^* = (\frac{1}{2}(B^T A^T + A^T B^T))^* = \frac{1}{2}((B^T)^*)(A^T)^* + (A^T)^*(B^T)^*) = \frac{1}{2}(BA + AB) = \frac{1}{2}(AB + BA) = A \circ B$.

Note that if D is associative then $\mathcal{M}_{n\times n}(D)$ is associative. So in this case $\mathcal{H}_n(D)$ with the Jordan product is a special Jordan algebra.

When D is non-associative, there is no guarantee that $\mathcal{H}_n(D)$ is a Jordan algebra. However, there is an important example where D is non-associative, but $\mathcal{H}_n(D)$ is Jordan. If we take $D = \mathbb{O}$, the algebra of octonions over $\mathbb{R}$, with the conjugation map $*$, then $\mathcal{H}_3(\mathbb{O})$ is an exceptional 27-dimensional Jordan algebra, known as the Albert algebra.

Finally, we want to explain the relation between the Jordan algebras and the algebras of Jordan type η by using the following well-known result from Albert [1].

Theorem 4.1.6. *Let $J = J_1 \oplus J_0 \oplus J_{\frac{1}{2}}$ be the Peirce decomposition of the Jordan algebra J corresponding to the idempotent e. Then the following conditions hold:*

(a) $J_0(e) \cdot J_0(e) \subseteq J_0(e)$;

(b) $J_1(e) \cdot J_1(e) \subseteq J_1(e)$;

(c) $J_1(e) \cdot J_0(e) = 0$;

(d) $J_{\frac{1}{2}}(e) \cdot J_{\frac{1}{2}} \subseteq J_1(e) + J_0(e)$;

(e) $J_{\frac{1}{2}}(e)(J_0(e) + J_1(e)) \subseteq J_{\frac{1}{2}}(e)$.

These five statements correspond to the entries of the fusion law $\mathcal{J}(\frac{1}{2})$. Hence, we see that every idempotent of a Jordan algebra is an axis of Jordan type $\frac{1}{2}$. However, not every Jordan algebra contains non-zero idempotents. Thus, a Jordan algebra is an axial algebra of Jordan type half only when it is generated by primitive idempotents.

4.2 Matsuo algebras

Now we state the definition of a Matsuo algebra.

Definition 4.2.1. Suppose that (G, D) is a 3-transpostion group and $\operatorname{char} \mathbb{F} \neq 2$. For $\eta \in \mathbb{F} \setminus \{1, 0\}$, let $M = M_\eta(G, D)$ be the algebra with the basis D and the product on M defined as follows:

$$c \cdot d = \begin{cases} c, & \text{if } c = d, \\ 0, & \text{if } o(cd) = 2, \\ \frac{\eta}{2}(c + d - e), & \text{if } o(cd) = 3, \end{cases}$$

where $c, d \in D$ and $e = c^d$.

Note that here we distinguish the product cd in the group G and $c \cdot d$ in the algebra M. The basis elements $d \in D$ are primitive axes in the algebra M. These axes satisfy the fusion law $\mathcal{J}(\eta)$ (see Table 2.1 (b)). In other words, M is an axial algebra of Jordan type η.

Proposition 4.2.2 ([16])**.** *The Matsuo algebra $M_\eta(G, D)$ admits a Frobenius form. Namely, for $c, d \in D$, the value of the Frobenius form is given by*

$$(c, d) = \begin{cases} 1, & \text{if } c = d, \\ 0, & \text{if } o(cd) = 2, \\ \frac{\eta}{2}, & \text{if } o(cd) = 3. \end{cases}$$

Recall the definition of the projection graph we introduced in Chapter 2. The vertices of the projection graph of the Matsuo algebra $M = M_\eta(G, D)$ are the axes in D and there is an edge between distinct vertices $c, d \in D$ if and only if the value of the Frobenius form $(c, d) \neq 0$. This holds because $(c, c) \neq 0 \neq (d, d)$ (see the discussion after Definition 2.4.10). Note that $(c, d) \neq 0$ if and only if $o(cd) = 3$, i.e., when c, d are collinear in the Fischer space of (G, D). So the projection graph of M is the same as the collinearity graph of the Fischer space. This means that the projection graph is connected if and only if the group (G, D) is connected. By Theorem 3.2.4, this happens when D is a single conjugacy class. In summary, the Matsuo algebra $M = M_\eta(G, D)$ is connected if and only if (G, D) is connected if and only if D is a single conjugacy class in G.

Let us consider the disconnected case. Let $D = D_1 \cup D_2 \cup \ldots \cup D_k$ be the decomposition of D as a union of conjugacy classes of G. As in the previous chapter (see Proposition 3.2.3), we denote $G_i = \langle D_i \rangle$.

Proposition 4.2.3. *$M_\eta(G, D) = M_\eta(G_1, D_1) \oplus \ldots \oplus M_\eta(G_k, D_k)$.*

Proof. Recall that each D_i is a connected component of the Fischer space of (G, D) by Proposition 3.2.3 and it is a single conjugacy class in G_i. Let $M_i := \langle D_i \rangle \subseteq M$. Since D is a basis of M, it follows that M decomposes as a direct sum of subspaces M_i. Also, it follows that D_i is a basis of M_i. Take two elements from D_i, say a, b. If $a = b$, then $a \cdot b = a \in M_i$. When a and b commute, we have $a \cdot b = 0 \in M_i$. For the case where $o(ab) = 3$, there exists a line containing a and b in the Fischer space of (G_i, D_i). Since this line contains two points from D_i, this means that the third point, say e, is also in D_i, because D_i is a connected component and e is collinear to a and b. So the product $a \cdot b = \frac{\eta}{2}(a + b - e)$ is also in M_i. Thus, M_i is closed under the algebra multiplication, i.e., M_i is a subalgebra of M. Furthermore, comparing with Definition 4.2.1, we have that $M_i = M_\eta(G_i, D_i)$.

It remains to show that M is a direct sum of algebras. That is, we need to show that $M_i \cdot M_j = 0$ for all $i \neq j$. Let $c \in D_i$ and $d \in D_j$. Then $c \cdot d = 0$ because $o(cd) = 2$. Since $M_i = \langle D_i \rangle$ and $M_j = \langle D_j \rangle$, this means that the algebra product of M_i and M_j for $i \neq j$ is indeed zero. Therefore, $M = M_1 \oplus \ldots \oplus M_k$. $\square$

From the above discussion, we can see that most of the questions about Matsuo algebras can be reduced to the connected case. One of the questions we will look into is the radicals of Matsuo algebras.

4.3 Radicals of Matsuo algebras

In this section $M = M_\eta(G, D)$ is a Matsuo algebra. By Theorem 2.4.9, since $(a, a) = 1$ for all $a \in D$, we have that the radical $R(M)$ of M coincides with the radical $M^\perp$ of the Frobenius form on M.

We first consider the disconnected case. Let $D = D_1 \cup \ldots \cup D_k$ be a union of k conjugacy classes. We have shown in Proposition 4.2.3 that $M = M_1 \oplus \ldots \oplus M_k$, where $M_i = M_\eta(G_i, D_i)$ for $G_i = \langle D_i \rangle$. Note that for $c \in M_i$ and $d \in M_j$, $i \neq j$, we have $(c, d) = 0$ because $o(cd) = 2$. So M_i is perpendicular to M_j for all $i \neq j$. Thus, the decomposition of M is an orthogonal direct sum.

Theorem 4.3.1. *$R(M) = R(M_1) \oplus \ldots \oplus R(M_k)$.*

Proof. We first show that $R(M_i) \subseteq R(M)$. If $u \in R(M_i)$, then $u \perp M_i$. For $j \neq i$, we have $u \perp M_j$ because $M_i \perp M_j$. So $u \perp M_1 \oplus \ldots \oplus M_k$. This means that $u \in R(M)$. Thus, $R(M_i) \subseteq R(M)$. Since each $R(M_i)$ is contained in $R(M)$, we deduce that $R(M_1) \oplus \ldots \oplus R(M_k) \subseteq R(M)$.

Conversely, we take $u \in R(M)$. So $u \perp M$. Then $u = u_1 + \ldots + u_k$ for some $u_i \in M_i$ because $M = M_1 \oplus \ldots \oplus M_k$. Note that $u_j \perp M_i$ for $i \neq j$ because $M_i \perp M_j$. Since $u \perp M$ and $M_i \subseteq M$, we have that $u \perp M_i$. So $u_i = u - \sum_{j \neq i} u_j \perp M_i$. Thus, $u_i \in R(M_i)$. This means $R(M) \subseteq R(M_1) \oplus \ldots \oplus R(M_k)$. Therefore, $R(M) = R(M_1) \oplus \ldots \oplus R(M_k)$.□

Here the summands M_i are connected Matsuo algebras. Hence, the above theorem tells us how to find the radical of an arbitrary Matsuo algebra if we know how to find the radical in the connected case. Until the end of this section, we assume that M is connected, i.e., D is a single conjugacy class.

Note that $R(M) = M^\perp$ can be found as the 0-eigenspace of the Gram matrix of the

Frobenius form on M. So $R(M) \neq 0$ if and only if 0 is an eigenvalue of the Gram matrix. From Proposition 4.2.2, the Gram matrix is $R = I + \frac{\eta}{2}T$, where T is the collinearity matrix, that is, the adjacency matrix of the collinearity graph on D. In characteristic zero, since T is symmetric, we have that T is semisimple. Let $\theta_1, \ldots, \theta_k$ be the eigenvalues of T and $m_1, \ldots, m_k$ be their multiplicities. For connected (G, D), the eigenvalues θ_i and multiplicities m_i can be found in the tables in [20]. Once we know $\theta_1, \ldots, \theta_k$, we can find the eigenvalues $\rho_1, \ldots, \rho_k$ of R using the formula $\rho_i = 1 + \frac{\eta}{2}\theta_i$. In particular, for the radical $R(M)$ to be non-zero, we need to have some $\rho_i = 0$, which leads to $0 = 1 + \frac{\eta}{2}\theta_i$. Hence, $R(M) \neq 0$ (i.e., $R(M)$ is not simple) if and only if $\eta = -\frac{2}{\theta_i}$, for some eigenvalue $\theta_i \neq 0$ of T. Such values of η are called the *critical values* of the connected Matsuo algebra M.

We illustrate this with an example.

Example 4.3.2. Let $G = S_n$ and $D = (1,2)^G$ be the class of transpositions. According to [20], the eigenvalues of the collinearity matrix T in this case are $\theta_1 = 2(n-2)$, $\theta_2 = n-4$ and $\theta_3 = -2$ and the multiplicities are, correspondingly, $m_1 = 1$, $m_2 = n-1$ and $m_3 = \frac{n(n-3)}{2}$. Then the critical values of $M = M_\eta(G, D)$ are $\eta_1 = -\frac{2}{\theta_1} = -\frac{2}{2(n-2)} = -\frac{1}{n-2}$ (with $\dim R(M) = 1$), $\eta_2 = -\frac{2}{\theta_2} = -\frac{2}{n-4}$ (with $\dim R(M) = n-1$) and $\eta_3 = -\frac{2}{\theta_3} = -\frac{2}{-2} = 1$ (with $\dim R(M) = \frac{n(n-3)}{2}$). This calculation assumes that the field $\mathbb{F}$ is of characteristic zero. However, the above claims also remain true for the majority of positive characteristics p, as long as the eigenvalues θ_i (they are always integers) are pairwise distinct modulo p.

4.4 2-Generated algebras of Jordan type

Let us consider the 3-dimensional algebra $A := A(\eta, \varphi) = \langle a, b, \sigma \rangle$ over a field $\mathbb{F}$ and with the multiplication table

$\cdot$	a	b	σ
a	a	$\eta a + \eta b + \sigma$	πa
b	$\eta a + \eta b + \sigma$	b	πb
σ	πa	πb	$\pi \sigma$

where $\eta, \varphi \in \mathbb{F}$ and $\pi = \varphi - \varphi\eta - \eta$.

Lemma 4.4.1. *The form $(\cdot, \cdot)$ on the algebra $A = A(\eta, \varphi)$, whose Gram matrix with respect to the basis $\{a, b, \sigma\}$ is*

$$G := \begin{pmatrix} 1 & \varphi & \pi \\ \varphi & 1 & \pi \\ \pi & \pi & \pi(\varphi - 2\eta) \end{pmatrix},$$

is a Frobenius form on A.

Proof. Note that the condition $(u, vw) = (uv, w)$ is linear in u, v and w. Because of this, it suffices to check the identity only for $u, v, w \in \{a, b, \sigma\}$. Hence, we have $3^3 = 27$ cases to consider. All the cases where $u = w$ are clearly satisfied since the form is symmetric. Also, if we verified the identity for u, v, w, then by symmetry, it also holds for w, v, u. Additionally, there is a symmetry between a and b. These simple observations reduce the whole set of cases to just five, as follows:

$$\begin{aligned}
(a, ab) &= (a, \eta a + \eta b + \sigma) \\
&= (a, \sigma) + (a, \eta a) + (a, \eta b) \\
&= \pi + \eta + \eta\varphi \\
&= \varphi - \varphi\eta - \eta + \eta + \eta\varphi \\
&= \varphi \\
&= (a, b) \\
&= (aa, b),
\end{aligned}$$

$$(a, a\sigma) = (a, \pi a) = \pi(a, a) = \pi = (a, \sigma) = (aa, \sigma),$$

$$\begin{aligned}
(a, b\sigma) &= (a, \pi b) \\
&= \pi(a, b) \\
&= \pi\varphi = \eta\pi + \eta\pi + \pi(\varphi - 2\eta) \\
&= \eta(a, \sigma) + \eta(b, \sigma) + (\sigma, \sigma) \\
&= (\eta a, \sigma) + (\eta b, \sigma) + (\sigma, \sigma) \\
&= (\eta a + \eta b + \sigma, \sigma) \\
&= (ab, \sigma),
\end{aligned}$$

$$(a, \sigma b) = (a, \pi b) = \pi(a, b) = \pi\varphi = (\pi a, b) = (a\sigma, b)$$

and

$$(a, \sigma\sigma) = (a, \pi\sigma) = \pi(a, \sigma) = \pi^2 = (\pi a, \sigma) = (a\sigma, \sigma).$$

Since the identity is satisfied for all these five cases, we have that the form is a Frobenius form on A. $\square$

Proposition 4.4.2 ([16, (4.7)(a)])**.** *The eigenspaces of* ad_a *on* $A = A(\eta, \varphi)$ *are* $A_1(a) = \langle a \rangle$*,* $A_0(a) = \langle \pi a - \sigma \rangle$ *and* $A_\eta(a) = \langle (\eta - \varphi)a + \eta b + \sigma \rangle$*.*

Proof. We have that $a \cdot a = a = 1 \cdot a$. So $A_1(a) \supseteq \langle a \rangle$. Also, $a(\pi a - \sigma) = \pi a^2 - \sigma a = \pi a - \pi a = 0$. This means that $A_0(a) \supseteq \langle \pi a - \sigma \rangle$. Finally, we have that $a((\eta - \varphi)a + \eta b + \sigma) = (\eta - \varphi)a^2 + \eta ba + \sigma a = (\eta - \varphi)a + \eta(\eta a + \eta b + \sigma) + \pi a = \eta a - \varphi a + \eta^2 a + \eta^2 b + \eta\sigma + \pi a = (\eta - \varphi + \eta^2 + \pi)a + \eta^2 b + \eta\sigma = (\eta - \varphi + \eta^2 + \varphi - \varphi\eta - \eta)a + \eta^2 b + \eta\sigma = (\eta^2 - \varphi\eta)a + \eta^2 b + \eta\sigma = \eta((\eta - \varphi)a + \eta b + \sigma)$, and hence $A_\eta(a) \supseteq \langle (\eta - \varphi)a + \eta b + \sigma \rangle$.

Note that A is 3-dimensional and a, $\pi a - \sigma$ and $(\eta - \varphi)a + \eta b + \sigma$ are linearly independent. So each eigenspace is at most 1-dimensional. Thus, $A_1(a) = \langle a \rangle$, $A_0(a) = \langle \pi a - \sigma \rangle$ and $A_\eta(a) = \langle (\eta - \varphi)a + \eta b + \sigma \rangle$. $\square$

Now we state a proposition that is a consequence of [16], Proposition (4.8) but we will provide proof of it.

Proposition 4.4.3 ([16, (4.8)])**.** *Let* $\eta \in \mathbb{F} \setminus \{1, 0\}$ *and* $\varphi \in \mathbb{F}$*. Then* $A = A(\eta, \varphi)$ *is an axial algebra of Jordan type* η *with axes* a *and* b *if and only if one of the followings holds:*

(i) $\varphi = \frac{\eta}{2}$*; or*

(ii) $\eta = \frac{1}{2}$*.*

Proof. First of all, note that from Proposition 4.4.2, ad_a is semisimple with the eigenvalues 1, 0 and η. Next, we use the information about eigenvectors to verify the fusion law. From the fusion table, we know that there are six fusion rules that we need to consider because the algebra is commutative and the fusion table is symmetric. Clearly, the fusion rules $1 * 1 = \{1\}$, $1 * 0 = \emptyset$ and $1 * \eta = \{\eta\}$ hold. Then there are only three fusion rules left and they are $0 * 0 = \{0\}$, $0 * \eta = \{\eta\}$ and $\eta * \eta = \{1, 0\}$.

First, we multiply the 0-eigenvector with the 0-eigenvector. We have that $(\pi a - \sigma)(\pi a - \sigma) = \pi^2 a - 2\pi a\sigma + \pi\sigma = \pi^2 a - 2\pi^2 a + \pi\sigma = -\pi^2 a + \pi\sigma = -\pi(\pi a - \sigma)$. This means that $0 * 0 = \{0\}$ is satisfied.

We have that the 0-eigenvector times the η-eigenvector gives us $(\pi a - \sigma)((\eta - \varphi)a + \eta b + \sigma) = \pi(\eta - \varphi)a + \pi\eta(\eta a + \eta b + \sigma) + \pi^2 a - \pi(\eta - \varphi)a - \pi\eta b - \pi\sigma = \pi\eta^2 a + \pi\eta^2 b + \pi\eta\sigma + \pi^2 a - \pi\eta b - \pi\sigma = \pi((\eta^2 + \pi)a + \eta(\eta - 1)b + (\eta - 1)\sigma) = \pi((\eta^2 + \varphi - \varphi\eta - \eta)a + (\eta - 1)(\eta b + \sigma)) = \pi((\eta - \varphi)(\eta - 1)a + (\eta - 1)(\eta b + \sigma)) = \pi(\eta - 1)((\eta - \varphi)a + \eta b + \sigma)$. So $0 * \eta = \{\eta\}$ also holds.

Now we consider $\eta * \eta$. Since in the fusion law, $\eta * \eta = \{1, 0\}$, we need to have $A_\eta(a) \cdot A_\eta(a) \subseteq A_{\{1,0\}}(a) = A_1(a) \oplus A_0(a) = \langle a \rangle \oplus \langle \pi a - \sigma \rangle = \langle a, \pi a - \sigma \rangle = \langle a, \sigma \rangle$. This means that the coefficient of b should be zero in the product $((\eta - \phi)a + \eta b + \sigma)((\eta - \phi)a + \eta b + \sigma)$. This coefficient is $2\eta^3 - 2\varphi\eta^2 + \eta^2 + 2\eta\varphi - 2\varphi\eta^2 - 2\eta^2 = \eta(2\eta - 1)(\eta - 2\varphi)$. Since $\eta \neq 0$, we have $2\eta - 1 = 0$ or $\eta - 2\varphi = 0$. That is, $\eta = \frac{1}{2}$ or $\varphi = \frac{\eta}{2}$. ☐

Let us state one of the main results of [16] classifying 2-generated algebras of Jordan type η.

Theorem 4.4.4 ([16])**.** *Every 2-generated algebra of Jordan type η is isomorphic to $A(\eta, \varphi)$ or a quotient of it.*

Recall the definition of the algebra $3C(\eta)$ from Example 2.1.7.

Proposition 4.4.5 ([16, (1.1)(i)])**.** *For $\eta \in \mathbb{F} \setminus \{1, 0\}$, the algebra $A = A\left(\eta, \frac{\eta}{2}\right)$ is isomorphic to $3C(\eta)$.*

Proof. Take $c = -a - b - \frac{2}{\eta}\sigma$. Then $\langle a, b, c \rangle = A$, since $\sigma = -\frac{\eta}{2}(a + b + c)$. It follows that $\{a, b, c\}$ is a basis of A. Let us check the multiplication table with respect to this basis. Note that $\pi = \varphi - \varphi\eta - \eta = \frac{\eta}{2} - \frac{\eta}{2}\eta - \eta = -\frac{\eta}{2} - \frac{\eta^2}{2}$. We have $ab = \eta a + \eta b + \sigma =$

$\eta a + \eta b - \frac{\eta}{2}(a+b+c) = \frac{\eta}{2}(a+b-c)$. Also,

$$\begin{aligned}
ac &= a(-a-b-\tfrac{2}{\eta}\sigma) \\
&= -aa - ab - \tfrac{2}{\eta}a\sigma \\
&= -a - \tfrac{\eta}{2}(a+b-c) - \tfrac{2}{\eta}\pi a \\
&= -a - \tfrac{\eta}{2}a - \tfrac{\eta}{2}b + \tfrac{\eta}{2}c - \tfrac{2}{\eta}(-\tfrac{\eta}{2} - \tfrac{\eta^2}{2})a \\
&= -a - \tfrac{\eta}{2}a - \tfrac{\eta}{2}b + \tfrac{\eta}{2}c + a + \eta a \\
&= \tfrac{\eta}{2}(a-b+c).
\end{aligned}$$

Similarly, $bc = \frac{\eta}{2}(b-a+c)$. Finally, we have

$$\begin{aligned}
c^2 &= (-a-b-\tfrac{2}{\eta}\sigma)^2 \\
&= (a+b)^2 + 2(a+b)\tfrac{2}{\eta}\sigma + (\tfrac{2}{\eta}\sigma)^2 \\
&= a^2 + 2ab + b^2 + \tfrac{4}{\eta}a\sigma + \tfrac{4}{\eta}b\sigma + \tfrac{4}{\eta^2}\sigma^2 \\
&= a + \eta(a+b-c) + b + \tfrac{4}{\eta}\pi(a+b+\tfrac{1}{\eta}\sigma) \\
&= a + \eta a + \eta b - \eta c + b + \tfrac{4}{\eta}(-\tfrac{\eta}{2} - \tfrac{\eta^2}{2})(a+b+\tfrac{1}{\eta}\sigma) \\
&= a + \eta a + \eta b - \eta c + b + (-2a - 2b - \tfrac{2}{\eta}\sigma - 2\eta a - 2\eta b - 2\sigma) \\
&= a - \eta a - b - \eta b - \eta c - \tfrac{2}{\eta}\sigma - 2\sigma \\
&= -a - b - \tfrac{2}{\eta}\sigma + \eta(-a-b-\tfrac{2}{\eta}\sigma) - \eta c \\
&= c + \eta c - \eta c \\
&= c.
\end{aligned}$$

So the basis $\{a, b, c\}$ of A satisfies the multiplication table of $3C(\eta)$. Therefore, A is isomorphic to $3C(\eta)$. □

Corollary 4.4.6. *If A is a 2-generated algebra of Jordan type $\eta \neq \frac{1}{2}$, then A is isomorphic* *to* $2B = \mathbb{F} \oplus \mathbb{F}$ *or* $3C(\eta)$.

Proof. Let $A = \langle\langle a, b \rangle\rangle$. Note that if $\varphi = \frac{\eta}{2}$, then the algebra is isomorphic to $3C(\eta)$ by Proposition 4.4.5. Otherwise, $\eta = \frac{1}{2}$ by Proposition 4.4.3. However, we also assumed that $\eta \neq \frac{1}{2}$. This obvious contradiction means η cannot be an eigenvalue. So the fusion law for both a and b involves only 1 and 0, and so it is as shown in Table 2.1 (a). In particular, $A = A_1(a) \oplus A_0(a)$. We claim that $b \in A_0(a)$. Indeed, let $b = b_1 + b_0$, where $b_1 \in A_1(a)$ and $b_0 \in A_0(a)$. We have that $b^2 = (b_1 + b_0)^2 = b_1^2 + 2b_1b_0 + b_0^2 = b_1^2 + b_0^2$, because $b_1b_0 = 0$ since $1 * 0 = \emptyset$. Now recall that $b^2 = b$, i.e., $b_1^2 + b_0^2 = b_1 + b_0$. Since $1 * 1 = \{1\}$ and $0 * 0 = \{0\}$, we have that $b_1^2 \in A_1(a)$ and $b_0^2 \in A_0(a)$. So $b_1^2 = b_1$ and $b_0^2 = b_0$. That is, both b_1 and b_0 are idempotents. Suppose by contradiction that $b_1 \neq 0$. Since $A_1(a) = \langle a \rangle \cong \mathbb{F}$ has only two idempotents, 0 and a, if $b_1 \neq 0$, then $b_1 = a$. If so, then $ab = a(b_1 + b_0) = a(a + b_0) = a^2 + ab_0 = a + 0 = a$. This means that $a \in A_1(b)$. But also $b \in A_1(b)$, which contradicts the primitivity of the axis b. This means that $b_1 = 0$, i.e., $b = b_0 \in A_0(a)$. Now it is easy to see that $\langle a, b \rangle$ is closed under multiplication, forming a subalgebra that is isomorphic to $2B$. Since a and b generate A, we conclude that $A = \langle a, b \rangle \cong 2B$. □

This shows us what we can encounter when $\eta \neq \frac{1}{2}$. Now let us investigate the case $\eta = \frac{1}{2}$. Recall that in Example 4.1.3, we introduced the spin factor algebra $S(V, q)$. Now let V be a 2-dimensional vector space over $\mathbb{F}$ with the basis $\{v_1, v_2\}$ and the symmetric bilinear form b on V be given by $b(v_i, v_i) = 2$, $i \in \{1, 2\}$, $b(v_1, v_2) = \delta$ for some $\delta \in \mathbb{F}$. Take the quadratic form $q(v) = b(v, v)$ for all $v \in V$. We denote this spin factor algebra of V and q as $S_\delta = S(V, q)$.

Proposition 4.4.7 ([16]). *If* $\varphi \neq 1$ *then the algebra* $A = A(\frac{1}{2}, \varphi)$ *is isomorphic to the spin factor algebra* S_δ, *where* $\delta = 4\varphi - 2$.

When $\varphi = 1$, we have that $\delta = 4\varphi - 2 = 4 - 2 = 2$, but $A(\frac{1}{2}, 1)$ is not isomorphic to S_2. So this case is a true exception. In order to see why they are not isomorphic, we look at the element σ. Normally σ is a multiple of identity, but in this case, $\pi = \varphi - \varphi\eta - \eta = 1 - 2 \cdot \frac{1}{2} = 0$. This means that σ is in the nil-radical of the algebra. Now it is easy to see that the algebra $A(\frac{1}{2}, 1)$ has no identity element, whereas the spin factor always has identity.

4.5 Ideals in 2-generated algebras of Jordan type

Now we want to determine the values of η and φ, satisfying the conditions $\eta = \frac{1}{2}$ or $\varphi = \frac{\eta}{2}$, for which the algebra $A = A(\eta, \varphi)$ has proper non-trivial ideals.

Recall that there are two types of ideals: containing generating axes and not containing any generating axes. We first look at the case where the ideal contains generating axes. Recall the definition of the projection (di)graph, say, call it Γ, in Definition 2.4.10 and the discussion after it. The vertices of Γ are the generating axes a and b of A. Since A admits a Frobenius form such that $(a, a) = (b, b) = 1 \neq 0$, Γ is a simple graph and there is an edge between a and b if and only if $(a, b) = \varphi \neq 0$. Hence Γ is disconnected if and only if $\varphi = 0$. Clearly, this means that $\eta = \frac{1}{2}$.

Now we consider the case where the ideal does not contain any axes, i.e., the ideal is in the radical of A. Again, since A admits a Frobenius form that is non-zero on both a and b, the radical of A coincides with the radical of the Frobenius form. By Lemma 4.4.1, the Gram matrix of the Frobenius form is

$$G := \begin{pmatrix} 1 & \varphi & \pi \\ \varphi & 1 & \pi \\ \pi & \pi & \pi(\varphi - 2\eta) \end{pmatrix},$$

and its determinant is $-\pi\varphi(\varphi-1)^2$. The radical of the form is non-zero when the determinant is zero, i.e., $\pi = 0$, $\varphi = 0$ or $\varphi = 1$. The case $\varphi = 0$ has already been noted earlier. If $\varphi = 1$, we can have $\eta = 2\varphi = 2$ or $\eta = \frac{1}{2}$.

It remains to analyse the case $\pi = 0$. Equivalently, $\varphi - \eta\varphi - \eta = 0$. If $\varphi = \frac{\eta}{2}$, this simplifies to $\frac{\eta(\eta+1)}{2} = 0$. Clearly, $\eta \neq 0$, so the only possibility is $\eta = -1$ and $\varphi = -\frac{1}{2}$. If $\eta = \frac{1}{2}$ then the equality simplifies to $\frac{\varphi-1}{2} = 0$, i.e., $\varphi = 1$, which has already been considered.

Thus we can formulate the following.

Proposition 4.5.1. *Suppose that $\varphi = \frac{\eta}{2}$ or $\eta = \frac{1}{2}$. Then $A = A(\eta, \varphi)$ has a non-zero proper ideal if and only if $(\eta, \varphi) \in \{(2, 1), (\frac{1}{2}, 1), (\frac{1}{2}, 0), (-1, -\frac{1}{2})\}$.*

Now we look at these cases individually and determine all ideals. For this, we need the following fact from linear algebra.

Lemma 4.5.2. *Suppose that $\varphi : V \to V$ is a linear map and U is a φ-invariant subspace of V, i.e., $\varphi(U) \subseteq U$. Consider the restriction $\varphi|_U$. Then the minimal polynomial of the restriction $\varphi|_U$ divides the minimal polynomial of φ.*

Proof. For any polynomial f, we have $f(\varphi|_U) = f(\varphi)|_U$. If f is the minimal polynomial of φ, then $f(\varphi) = 0$ and so $f(\varphi)|_U = 0$. That is, $f(\varphi|_U) = 0$. So f is divisible by the minimal polynomial of $\varphi|_U$. □

Now let us apply this to the situation where $\varphi = \mathrm{ad}_a$ is the adjoint map of the axis a and $U = I$ is an ideal of A. Since ad_a is semisimple, its minimal polynomial f has no multiple roots. From the above lemma, the minimal polynomial g of $\mathrm{ad}_a|_I$ is a divisor of f. So g also has no multiple roots. This means that $\mathrm{ad}_a|_I$ is diagonalisable.

Let $I_\lambda(a) = \{u \in I : au = \lambda u\}$ be the λ-eigenspace of ad_a restricted to I.

Corollary 4.5.3. *If I is an ideal of $A = A(\eta, \varphi)$, then $I = \bigoplus_{\lambda \in \mathcal{F}} I_\lambda(a) = \bigoplus_{\lambda \in \mathcal{F}} (I \cap A_\lambda(a))$.*

Proof. By the above, ad_a acts on I in a semisimple way which means I is the direct sum of the eigenspaces $I_\lambda(a)$. Clearly, $I_\lambda(a) = I \cap A_\lambda(a)$ and so the claims follows. ☐

We will use these results to describe explicitly all ideals in 2-generated algebras based on four cases from Proposition 4.5.1.

Case 1: When $(\eta, \varphi) = (2, 1)$, $\pi = 1 - 2 - 2 = -3$. By Proposition 4.4.5 we know that $A(2, 1) \cong 3C(2)$. So we will work with the basis $\{a, b, c\}$. Note that $\sigma = ab - \eta a - \eta b = \frac{\eta}{2}(a + b - c) - \eta a - \eta b = -\frac{\eta}{2}(a + b + c) = -a - b - c$. From Proposition 4.4.2, we have that $A_1(a) = \langle a \rangle$, $A_0(a) = \langle \pi a - \sigma \rangle = \langle -3a + a + b + c \rangle = \langle -2a + b + c \rangle$ and $A_2(a) = \langle \varphi a + 2b + \sigma \rangle = \langle a + 2b - a - b - c \rangle = \langle b - c \rangle$. Let I be an ideal. Since all eigenspaces are 1-dimensional, $I \cap A_\lambda(a)$ is either $A_\lambda(a)$ or 0. Therefore, I is a direct sum of several eigenspaces, that is, we have that I is one of the following: 0, $\langle a \rangle$, $\langle b - c \rangle$, $\langle 2a - b - c \rangle$, $\langle a, b - c \rangle$, $\langle a, 2a - b - c \rangle$, $\langle b - c, 2a - b - c \rangle$ and $\langle a, b - c, 2a - b - c \rangle = A$. Since we want I to be non-zero and proper, we discard the first and the last possibilities. Also, since $\varphi \neq 0$, the projection graph is connected. So $a \notin I$. This leaves us only three possibilities: $\langle b - c \rangle$, $\langle 2a - b - c \rangle$ and $\langle b - c, 2a - b - c \rangle$.

Suppose that $b - c \in I$. Then I also contains

$$\begin{aligned}
(b - c)b &= b - (b + c - a) \\
&= -c + a \\
&= a - c \\
&= \left(a - \tfrac{1}{2}b - \tfrac{1}{2}c\right) + \left(\tfrac{1}{2}b - \tfrac{1}{2}c\right) \\
&= \tfrac{1}{2}(2a - b - c) + \tfrac{1}{2}(b - c).
\end{aligned}$$

Therefore, $2a - b - c \in I$, i.e.,

$$I \supseteq \langle b - c, 2a - b - c \rangle.$$

Since I is proper, we must have $I = \langle b - c, 2a - b - c \rangle = A_0(a) \oplus A_2(a)$. Now note that

$$I = \{\alpha a + \beta b + \gamma c \mid \alpha, \beta, \gamma \in \mathbb{F}, \alpha + \beta + \gamma = 0\}.$$

Indeed, the right hand side is a subspace of the algebra and it does not contain all linear combinations of the basis, the dimension should be less than 3. Also, the sum of coefficients of $2a - b - c$ and $b - c$ is 0. So the left hand side is contained in the right hand side. Thus, they are equal and the dimension is 2.

By the symmetry in a, b and c, we also have $I = A_0(b) \oplus A_2(b)$ and $I = A_0(c) \oplus A_2(c)$. In particular, I is invariant under multiplication with a, b and c. Since $\{a, b, c\}$ is a basis, $I = \{\alpha a + \beta b + \gamma c : \alpha, \beta, \gamma \in \mathbb{F}, \alpha + \beta + \gamma = 0\}$ is indeed an ideal.

Now suppose that I does not contain $b - c$. Then $I = \langle 2a - b - c \rangle$. However,

$$\begin{aligned}
(2a - b - c)b &= 2ab - b - cb \\
&= 2(a + b - c) - b - (c + b - a) \\
&= 2a + 2b - 2c - b - c - b + a \\
&= 3a - 3c
\end{aligned}$$

This is a contradiction unless $\operatorname{char} \mathbb{F} = 3$.

So we have the following statement.

Proposition 4.5.4. *If $A = A(2,1) \cong 3C(2)$, then*

$$\begin{aligned} I &= \langle b-c, 2a-b-c\rangle \\ &= \{\alpha a + \beta b + \gamma c \mid \alpha,\beta,\gamma \in \mathbb{F}, \alpha+\beta+\gamma = 0\} \end{aligned}$$

is the only non-zero proper ideal of A, unless $\operatorname{char}\mathbb{F} = 3$*, in which case, we have a second ideal $\langle 2a-b-c\rangle$.*

Case 2: When $(\eta,\varphi) = (\frac{1}{2},1)$, $\pi = 1-\frac{1}{2}-\frac{1}{2} = 0$. Then $A_1(a) = \langle a\rangle$, $A_0(a) = \langle -\sigma\rangle = \langle\sigma\rangle$ and $A_{\frac{1}{2}}(a) = \langle -\frac{1}{2}a + \frac{1}{2}b + \sigma\rangle$.

In this case, the projection graph is again connected. So a proper ideal I cannot contain a. This leaves us with three possibilities: $\langle\sigma\rangle$, $\langle -\frac{1}{2}a+\frac{1}{2}b+\sigma\rangle$ and $\langle \sigma, -\frac{1}{2}a+\frac{1}{2}b+\sigma\rangle$.

We have that $\sigma a = \sigma b = \sigma^2 = 0$. This means that $\langle\sigma\rangle$ is an ideal, in fact, the nil-ideal of A. Suppose that $w := -\frac{1}{2}a+\frac{1}{2}b+\sigma \in I$. Then $wb = -\frac{1}{2}ab+\frac{1}{2}b+\sigma b = -\frac{1}{2}(\frac{1}{2}a+\frac{1}{2}b+\sigma)+\frac{1}{2}b+0 = -\frac{1}{4}a+\frac{1}{4}b-\frac{1}{2}\sigma$ and hence $I \ni \frac{1}{2}w - wb = -\frac{1}{4}a+\frac{1}{4}b+\frac{1}{2}\sigma+\frac{1}{4}a-\frac{1}{4}b+\frac{1}{2}\sigma = \sigma$. Thus, $\langle w\rangle$ is not an ideal. On the other hand, since $a\sigma = b\sigma = \sigma^2 = 0$ and since also $aw = \frac{1}{2}w$, $bw = \frac{1}{2}w - \sigma$, and $w\sigma = 0$, we have that $\langle\sigma, -\frac{1}{2}a+\frac{1}{2}b+\sigma\rangle$ is an ideal.

Proposition 4.5.5. *For the algebra $A(\frac{1}{2},1)$, the proper non-zero ideals are $\langle\sigma\rangle$ and $\langle\sigma, -\frac{1}{2}a+\frac{1}{2}b+\sigma\rangle$.*

Case 3: When $(\eta,\varphi) = (\frac{1}{2},0)$, $\pi = 0-0-\frac{1}{2} = -\frac{1}{2}$. Then $A_1(a) = \langle a\rangle$, $A_0(a) = \langle -\frac{1}{2}a-\sigma\rangle$ and $A_{\frac{1}{2}}(a) = \langle \frac{1}{2}a+\frac{1}{2}b+\sigma\rangle$.

Suppose that $a \in I$. Then $I \ni ab = \frac{1}{2}a+\frac{1}{2}b+\sigma$, which is a $\frac{1}{2}$-eigenvector. We have that $(\frac{1}{2}a+\frac{1}{2}b+\sigma)b = \frac{1}{2}ab+\frac{1}{2}b+\sigma b = \frac{1}{2}(\frac{1}{2}a+\frac{1}{2}b+\sigma)+\frac{1}{2}b-\frac{1}{2}b = \frac{1}{2}(\frac{1}{2}a+\frac{1}{2}b+\sigma)$. So $\langle a, \frac{1}{2}a+\frac{1}{2}b+\sigma\rangle = \langle a, b+2\sigma\rangle$ is an ideal. Suppose that $-\frac{1}{2}a-\sigma \in I$. Then $(-\frac{1}{2}a-\sigma)b = -\frac{1}{2}ab-\sigma b = -\frac{1}{2}(\frac{1}{2}a+\frac{1}{2}b+\sigma)+\frac{1}{2}b = -\frac{1}{4}a-\frac{1}{4}b-\frac{1}{2}\sigma+\frac{1}{2}b = -\frac{1}{4}a+\frac{1}{4}b-\frac{1}{2}\sigma$.

So $I = \langle -\frac{1}{2}a - \sigma, b\rangle = \langle b, a + 2\sigma\rangle$. Suppose that $\frac{1}{2}a + \frac{1}{2}b + \sigma \in I$. Since it is in the $\frac{1}{2}$-eigenspace, we have that $(\frac{1}{2}a + \frac{1}{2}b + \sigma)a = \frac{1}{2}(\frac{1}{2}a + \frac{1}{2}b + \sigma)$. Symmetrically, also $(\frac{1}{2}a + \frac{1}{2}b + \sigma)b = \frac{1}{2}(\frac{1}{2}a + \frac{1}{2}b + \sigma)$. Additionally, $(\frac{1}{2}a + \frac{1}{2}b + \sigma)\sigma = -\frac{1}{2}(\frac{1}{2}a + \frac{1}{2}b + \sigma)$. So $\langle \frac{1}{2}a + \frac{1}{2}b + \sigma\rangle$ is invariant under a, b and σ. This means that it is also an ideal.

Proposition 4.5.6. *For the algebra $A(\frac{1}{2}, 0)$, the proper non-zero ideals are $\langle a, b + 2\sigma\rangle$, $\langle b, a + 2\sigma\rangle$ and $\langle \frac{1}{2}a + \frac{1}{2}b + \sigma\rangle$.*

Case 4: When $(\eta, \varphi) = (-1, -\frac{1}{2})$, $\pi = -\frac{1}{2} - \frac{1}{2} + 1 = 0$. Then $A_1(a) = \langle a\rangle$, $A_0(a) = \langle -\sigma\rangle = \langle \sigma\rangle$ and $A_{-1}(a) = \langle -\frac{1}{2}a - b + \sigma\rangle$. Since the projection graph is connected in this case, a cannot be contained in a proper ideal.

Suppose that $u := -\frac{1}{2}a - b + \sigma \in I$. Then $I \ni ub = (-\frac{1}{2}a - b + \sigma)b = -\frac{1}{2}ab - b + \sigma b = -\frac{1}{2}(-a - b + \sigma) - b = \frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}\sigma$. Since this decomposes as $\frac{3}{4}a + \frac{1}{2}(-\frac{1}{2}a - b + \sigma) - \sigma$, we conclude that both a and σ are contained in I, that is, $I = A$, which is a contradiction because we only consider proper ideals I. Thus, the only possible proper non-zero ideal in this case is $\langle \sigma\rangle$. And indeed, since $a\sigma = 0 = b\sigma = \sigma^2$, this is an ideal. Thus we can formulate the following.

Proposition 4.5.7. *For the algebra $A(-1, -\frac{1}{2})$, the only proper non-zero ideal is $\langle \sigma\rangle$.*

The case of $A(\frac{1}{2}, 1)$ will be especially important in later chapters.

4.6 Classification of algebras of Jordan type not half

In this section we will review the result from [16] that any axial algebra of Jordan type $\eta \neq \frac{1}{2}$ is either a Matsuo algebra or a quotient of Matsuo algebra. Recall the definition of the Miyamoto group we introduced in Chapter 2. If A is an axial algebra with the fusion law graded by the group T, then the Miyamoto group $\mathrm{Miy}(X)$ is the subgroup $\langle \tau_a(\chi) \mid a \in X, \chi \in T^* \rangle$ of $\mathrm{Aut}(A)$. In this case, since A is an algebra of Jordan type, the group T is of order 2.

Recall the definition of a closed set in Definition 2.3.6.

Lemma 4.6.1. *Let A be a primitive axial algebra of Jordan type η with a closed set X of generating axes. Then X spans A.*

Proof. Let $a, b \in X$. Then $b = b_1 + b_0 + b_\eta$, where $b_1 \in A_1(a)$, $b_0 \in A_0(a)$ and $b_\eta \in A_\eta(a)$. First of all, $b_1 = \varphi a$, for some $\varphi \in \mathbb{F}$, because A is primitive. Then $b^{\tau_a} = b_1 + b_0 - b_\eta$, because the Miyamoto involution τ_a acts as identity on $A_1(a) \oplus A_0(a)$ and as minus identity on $A_\eta(a)$. So $b - b^{\tau_a} = 2b_\eta$. Thus, $b_\eta = \frac{1}{2}(b - b^{\tau_a})$. Note that $ab = a(b_1 + b_0 + b_\eta) = 1b_1 + 0b_0 + \eta b_\eta = \varphi a + \eta(\frac{1}{2}(b - b^{\tau_a})) = \varphi a + \frac{\eta}{2} b - \frac{\eta}{2} b^{\tau_a}$. Since X is closed, we have that $b^{\tau_a} \in X$. So ab is a linear combination of three axes from X. This means that $\langle X \rangle$ is closed under the algebra product, i.e., it is a subalgebra of A. Since it also contains X and X generates A, we conclude that $A = \langle X \rangle$. □

Theorem 4.6.2 ([16]). *Let $\mathbb{F}$ be a field with $\mathrm{char}\,\mathbb{F} \neq 2$ and A be an axial algebra of Jordan type $\eta \neq \frac{1}{2}$ over $\mathbb{F}$. Then the Miyamoto involutions form a normal set D of 3-transpositions in the Miyamoto group $\mathrm{Miy}(X)$.*

Proof. We assume that $X = \bar{X}$ is closed. Let $G = \mathrm{Miy}(X)$ and $D = \{\tau_a \mid a \in X\}$. By the definition of the Miyamoto group, D generates G. Also, for $a \in X$ and $g \in G$, we have that $\tau_{a^g} = \tau_a^g \in D$. So D is a normal set of involutions.

Now let us investigate the order of $\tau_a \tau_b$ for $a, b \in X$. If $a = b$, then $\tau_a \tau_b = (\tau_a)^2 = 1$. That is, $o(\tau_a \tau_b) = 1$. When $a \neq b$, by Corollary 4.4.6, we have $\langle\langle a, b \rangle\rangle \cong 2B$ or $3C(\eta)$. If $\langle\langle a, b \rangle\rangle \cong 2B$, then $b^{\tau_a} = b$ because $b \in A_0(a)$. This implies that $\tau_b^{\tau_a} = \tau_{b^{\tau_a}} = \tau_b$. Thus, τ_a

and τ_b commute, i.e., $o(\tau_a\tau_b) = 2$. Now we consider this case: $\langle\langle a, b\rangle\rangle \cong 3C(\eta)$. In this case, we have $\langle\langle a, b\rangle\rangle \cap D = \{a, b, c\}$, where $c = a^{\tau_b} = b^{\tau_a}$. So $\tau_a^{\tau_b} = \tau_c = \tau_b^{\tau_a}$. This gives us $\tau_b\tau_a\tau_b = \tau_a\tau_b\tau_a$. Rearranging it, we get $(\tau_a\tau_b)^3 = 1$. Therefore, we conclude that D is a 3-transposition class. □

This theorem leads to the following conclusion.

Corollary 4.6.3. *An axial algebra of Jordan type $\eta \neq \frac{1}{2}$ over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ is either a Matsuo algebra or a quotient of a Matsuo algebra.*

4.7 Further information about algebras of Jordan type $\frac{1}{2}$

This section is based on the results from [13] and [7]. Because of our discussion in Section 4.6, from now on we will only look at the case $\eta = \frac{1}{2}$. We define the following algebra.

Definition 4.7.1. Let $A := A(\alpha, \beta, \gamma, \psi) = \langle\langle a, b, c, ab, ac, bc, a(bc), b(ac), c(ab)\rangle\rangle$ be the commutative algebra over $\mathbb{F}$ generated by idempotents a, b and c, where $\alpha = (a, b)$, $\beta = (b, c)$, $\gamma = (c, a)$ and $\psi = (a, bc) = (b, ca) = (c, ab)$. The algebra products on A are defined in Table 4.1, where $\{x, y, z\} = \{a, b, c\}$.

Table 4.1 skips the obvious products: $x^2 = x$, x times y is xy and x times yz is $x(yz)$. The algebra A is a Jordan algebra for all values of the four parameters. It admits a unique Frobenius form such that every generating axis has a length of 1. Furthermore, the parameters α, β, γ and ψ are indeed the values of this Frobenius form as shown in the definition.

Theorem 4.7.2 ([13])**.** *If we have an algebra A' of Jordan type $\frac{1}{2}$, defined over $\mathbb{F}$ and generated by primitive axes a', b' and c', then there is a homomorphism from $A(\alpha, \beta, \gamma, \psi)$*

$x(xy) = \frac{1}{2}((x,y)x + xy)$
$x(x(yz)) = \frac{1}{2}((x,yz)x + x(yz))$
$x(y(xz)) = \frac{1}{4}((x,yz)x + (x,z)xy + (x,y)xz + x(yz) + y(xz) - z(xy))$
$(xy)^2 = \frac{1}{4}(x,y)(x + y + 2xy)$
$(xy)(yz) = \frac{1}{4}((y,xz)y + (y,z)xy + (x,y)yz + x(yz) - y(xz) + z(xy))$
$(xy)(x(yz)) = \frac{1}{8}(((x,y)(x,z) + (x,yz))x + (x,yz)y$ $+2(x,yz)xy + (x,y)yz + 2(x,y)x(yz))$
$(xy)(z(xy)) = \frac{1}{8}((x,y)(y,z)x + (x,y)(x,z)y + 4(xy,z)xy + (x,y)yz + (x,y)xz$ $-2(x,y)x(yz) - 2(x,y)y(xz) + 4(x,y)z(xy))$
$(x(yz))^2 = \frac{1}{16}((y,z)((x,y) + (x,z) + 2(x,yz))x + (y,z)(x,z)y + (x,y)(y,z)z$ $+4(x,yz)yz + (2(y,z) + 8(x,yz))x(yz) - 2(y,z)y(xz) - 2(y,z)z(xy))$
$(x(yz))(y(xz)) = \frac{1}{16}((y,z)((x,z) + (x,yz))x + (x,z)((y,z) + (x,yz))y + (x,y)(x,yz)z$ $+2(x,yz)xy + 2((x,y)(x,z) + (x,yz))yz + 2((x,y)(y,z) + (x,yz))xz + (4(x,yz)$ $-(x,y) + (y,z) - (x,z))x(yz) + (4(x,yz) - (x,y) - (y,z) + (x,z))y(xz) + ((x,y)$ $-(y,z) - (x,z) - 4(x,yz))z(xy))$

Table 4.1: The algebra products on $A(\alpha,\beta,\gamma,\psi)$

onto A', sending a, b, c to a', b', c' respectively, provided that $\alpha = (a', b')$, $\beta = (b', c')$, $\gamma = (c', a')$ and $\psi = (a', b'c') = (b', c'a') = (c', a'b')$.

In particular, every 3-generated axial algebra of Jordan type $\frac{1}{2}$ has dimension at most 9. The dimension of the algebra is exactly 9 if $(\alpha + \beta + \gamma - 2\psi - 1)(\alpha\beta\gamma - \psi^2) \neq 0$.

In [7], De Medts, Rowen and Segev proved that primitive 4-generated axial algebras of Jordan type are at most 81-dimensional. Furthermore, they determined a specific spanning set of size 81 for such an algebra. However, the information they provided is not as detailed as in the case of 3-generated algebras of Jordan type. Thus, this case remains open to some extent.

CHAPTER 5

BARIC AXIAL ALGEBRAS OF JORDAN TYPE

5.1 Baric algebras

In this section, we provide the basic notions of baric axial algebras. We will also investigate their properties.

Definition 5.1.1. Let A be an algebra over $\mathbb{F}$. We say that A is *baric* if there exists a surjective algebra homomorphism $w : A \to \mathbb{F}$. The homomorphism w is known as the *weight* function and its kernel, $\ker w$, is known as the *baric radical.*

Since we want to deal with axial algebras, what can we say about idempotents in baric algebras?

Lemma 5.1.2. *Suppose that A is baric and $a \in A$ is an idempotent. Then $w(a) = 0$ or 1.*

Proof. Since a is idempotent, $a^2 = a$. Let $w(a) = \alpha$. Then $w(a) = w(a^2) = w(a)w(a)$. So $\alpha = \alpha^2$. Then $\alpha^2 - \alpha = 0$, i.e., $\alpha(\alpha - 1) = 0$. That is, $\alpha = 0$ or $\alpha = 1$, as required. □

Note that $w(a) = 0$ means that the idempotent a lies in the baric radical, which is an ideal of A. In view of this, we give the following adjusted definition.

Definition 5.1.3. Suppose that (A, X) is a primitive axial algebra. Then (A, X) is a *baric axial algebra* if A is baric and additionally, $w(a) = 1$ for each $a \in X$. This means that none of the generating axes is in the baric radical of A.

Proposition 5.1.4. *For a baric axial algebra (A, X), its radical coincides with the baric radical. In particular, the weight function on (A, X) is unique.*

Proof. Suppose that (A, X) is a baric axial algebra. Then the baric radical $I = \ker w$ is an ideal and it contains no axes from X by Definition 5.1.3. By Definition 2.4.4, this means that $I \subseteq R(A)$. Note that I is of codimension 1 in A since $A/I \cong \mathbb{F}$. So $R(A) = I$ because $R(A) \neq A$.

For the second claim, it is clear that the baric weight w must be equivalent to the homomorphism $A \to A/R(A) \cong \mathbb{F}$. More formally, the weight function w should take every generating axis $a \in X$ to $1 \in \mathbb{F}$. Since w is a homomorphism, w takes every product of axes to 1. On the other hand, X generates A, which means that A is spanned by products of axes from X. All those products are mapped by w to 1, and so by linearity, $w(u)$ can be computed uniquely for each $u \in A$. □

We can also formulate the following.

Corollary 5.1.5. *An axial algebra A is baric if and only if $R(A)$ is of codimension 1 in A.*

Note that in the axial context, when we talk about the baric property, we will always refer to Definition 5.1.1.

Let us see some examples.

Examples 5.1.6. *(i) $1A = \mathbb{F}$ (the only axis here is the identity 1) is baric trivially, because there is the identity homomorphism from $1A$ to $\mathbb{F}$. Since this homomorphism is injective, the baric radical of $1A$ is trivial.*

(ii) $2B = \mathbb{F} \oplus \mathbb{F}$ is baric as a general algebra but not baric as an axial algebra. It has two homomorphisms onto $\mathbb{F}$, namely, the projections to the first and second summands. However, the first projection maps the second generator to zero and, symmetrically, the second projection maps the first generator to zero. So neither of these two homomorphisms satisfies the definition. One may also notice that $R(2B) = 0$ has codimension 2.

(iii) Recall the definition of $3C(\eta)$ introduced in Example 2.1.7. If $\eta = 2$, then $A = 3C(2)$ is baric. Namely, the linear map $w : A \to \mathbb{F}$ sending the basis $\{a, b, c\}$ of A to 1 is an algebra homomorphism. Indeed, for $\{x, y, z\} = \{a, b, c\}$, we have that $w(xy) = w(\frac{\eta}{2}(x + y - z)) = w(x + y - z) = w(x) + w(y) - w(z) = 1 + 1 - 1 = 1 = w(x)w(y)$. Also, $w(x^2) = w(x) = 1 = w(x)^2$. The radical of A is 2-dimensional spanned by $a - b$ and $a - c$. Clearly, $w(a - b) = w(a) - w(b) = 1 - 1 = 0$ and similarly, $w(a - c) = 0$.

(iv) Take $\eta = \frac{1}{2}$ and $\varphi = 1$. Consider the algebra $A(\frac{1}{2}, 1)$. Recall the multiplication table introduced in Section 4.4. We compute that $\pi = \varphi - \varphi\eta - \eta = 1 - \frac{1}{2} - \frac{1}{2} = 0$. Thus, the multiplication table looks as follows

$\cdot$	a	b	σ
a	a	$\frac{1}{2}a + \frac{1}{2}b + \sigma$	0
b	$\frac{1}{2}a + \frac{1}{2}b + \sigma$	b	0
σ	0	0	0

.

From the table, we can see that $\langle \sigma \rangle$ is an ideal of $A(\frac{1}{2}, 1)$. Also, we claim that $J = \langle a - b, \sigma \rangle$ is an ideal. Note that $(a - b)a = aa - ba = a - (\frac{1}{2}a + \frac{1}{2}b + \sigma) =$

$\frac{1}{2}(a-b)-\sigma \in J$. Similarly, $(a-b)b = ab - bb = (\frac{1}{2}a+\frac{1}{2}b+\sigma)-b = \frac{1}{2}(a-b)+\sigma \in J$. Clearly, J is of codimension 1 and it does not contain the generating axes a and b. So J is the baric radical by Proposition 5.1.4. Thus, $A(\frac{1}{2}, 1)$ is a baric algebra.

An *axial subalgebra* of (A, X) is a subalgebra generated by a subset of X or $\bar{X}$ in the case of a graded fusion law. Note that we typically assume that X is closed, i.e., $\bar{X} = X$.

Proposition 5.1.7. *Suppose that (A, X) is a baric axial algebra. Then every non-zero axial subalgebra (B, Y) of (A, X) is baric.*

Proof. Since A is baric, there exists a surjective homomorphism $w : A \to \mathbb{F}$ such that $w(a) = 1$ for all $a \in X$. Let $0 \neq B \subseteq A$ be an axial subalgebra. That is, B is generated by Y where $Y \subseteq X$. Since $B \neq 0$, we have that the generating set Y of B contains at least one axis b. Take the restriction $w' = w|_B$ on B. Since w is a homomorphism, we have that w' is also a homomorphism. For every $\alpha \in \mathbb{F}$, $w'(\alpha b) = \alpha w'(b) = \alpha \cdot 1 = \alpha$. This means that w' is surjective on B. Therefore, B is a baric algebra. All the axes in the generating set Y of B satisfy $w'(b) = w(b) = 1$ since $Y \subseteq X$. Thus, B is a baric axial algebra. $\square$

Proposition 5.1.8. *Suppose that (A, X) is a baric axial algebra. Then A admits a Frobenius form given by the formula $(u, v) = w(u)w(v)$. For this Frobenius form, $(a, b) = 1$ for all $a, b \in X$.*

Proof. Let us check that this is a bilinear form. Take $\lambda \in \mathbb{F}$ and $t, u, v \in A$. Then

$$\begin{aligned}(\lambda u + v, t) &= w(\lambda u + v)w(t)\\ &= (\lambda w(u) + w(v))w(t)\\ &= \lambda w(u)w(t) + w(v)w(t)\\ &= \lambda(u, t) + (v, t).\end{aligned}$$

Since the form is clearly symmetric, we have that

$$(u, \lambda v + t) = \lambda(u, v) + (u, t).$$

Thus, $(\cdot, \cdot)$ is a bilinear form. Since $(u, vt) = w(u)w(vt) = w(u)w(v)w(t) = w(uv)w(t) = (uv, t)$, we have that $(\cdot, \cdot)$ is a Frobenius form. Also, for all $a, b \in X$, $(a, b) = w(a)w(b) = 1 \cdot 1 = 1$. □

A Frobenius form satisfying $(a, a) = 1$ for all generating axes a is called the *projection* form or *normalised* form. Note that the normalised Frobenius form on an algebra of Jordan type is unique by [19, Theorem 4.1], and so the above form is that unique normalised form on A.

5.2 Baric fusion law

As we have just seen, a baric axial algebra (A, X) always has a Frobenius form such that $(x, x) = 1$ for all $x \in X$. Furthermore, $(x, y) = 1$ for all $x, y \in X$. So a baric algebra is always connected.

Theorem 5.2.1. *A connected primitive axial algebra is baric if and only if it satisfies a fusion law $\mathcal{F}$ such that for all $\lambda, \mu \in \mathcal{F} \setminus \{1\}$, we have that $\lambda * \mu$ does not contain 1.*

Proof. Suppose that a primitive algebra A satisfies the fusion law $\mathcal{F}$ such that for all $\lambda, \mu \in \mathcal{F} \setminus \{1\}$, we have that $\lambda * \mu$ does not contain 1. Take a primitive axis $a \in A$. Define $J = \bigoplus_{\lambda \in \mathcal{F} \setminus \{1\}} A_\lambda(a)$. We claim that J is an ideal of A of codimension 1. The last claim is clear since $A = A_1(a) \oplus J = \langle a \rangle \oplus J$. We only need to check that J is an ideal. Select a basis $B := \{u_i \mid i \in I\}$ of J, where each u_i is an eigenvector of ad_a, say for eigenvalue

λ_i, and let $\hat{B} = \{a\} \cup B$ be the corresponding basis of A. By linearity, to show that J is an ideal, it suffices to check that $uv \in J$ for $u \in \hat{B}$ (i.e., $u = a$ or $u = u_i$ for some $i \in I$) and $v \in B$ (i.e., $v = u_j$ for some $j \in I$). If $u = a$, then $uv = au_j = \lambda_j u_j \in J$. If $u = u_i$, then $uv = u_i u_j$ is contained in $A_{\lambda_i}(a) A_{\lambda_j}(a) \subseteq A_{\lambda_i * \lambda_j}(a)$. Note that $\lambda_i \neq 1 \neq \lambda_j$. By the assumption, $\lambda_i * \lambda_j$ does not contain 1. So $A_{\lambda_i * \lambda_j}(a)$ is contained in J. We have shown that J is an ideal.

We now want to prove that none of the generating axes is in J. Suppose that some axis $x \in X$ is contained in J. By connectivity, using Corollary 2.4.12, we then see that all generating axes must be in J which would imply $J = A$. Since this is not the case, we conclude that none of the generating axes is in J. So J is in the radical of A. Since J has codimension 1 in A, J must be equal to the radical since clearly there cannot be a larger proper ideal. We have shown that the algebra is baric.

Now we prove the converse. Suppose that A is a baric axial algebra satisfying a fusion law $\mathcal{F}$. We claim that A also satisfies the fusion law $\mathcal{G}$ obtained from $\mathcal{F}$ by removing 1 from all the subsets $\lambda * \mu$, $\lambda, \mu \in \mathcal{F} \setminus \{1\}$.

Let J be the radical of A and a be a primitive axis. Then $a \notin J$. Therefore, $A = \langle a \rangle \oplus J$. Since J, being an ideal, is invariant under ad_a, we must have that $J = \bigoplus_{\lambda \in \mathcal{F}} J \cap A_\lambda(a)$. Clearly, this implies that $J = \bigoplus_{\lambda \in \mathcal{F} \setminus \{1\}} A_\lambda(a)$ since J is of codimension 1. So J contains all eigenspaces of ad_a except $A_1(a) = \langle a \rangle$.

If we multiply $A_\lambda(a)$ and $A_\mu(a)$, $\lambda, \mu \neq 1$, then both quotients are in J, and so the product $A_\lambda(a) A_\mu(a) \subseteq J$. This means that this product is contained in $A_{\lambda * \mu}(a) \cap J = A_{\lambda * \mu \setminus \{1\}}(a)$. So indeed, A obeys the fusion law $\mathcal{G}$. □

Applying the above operation to the fusion law $\mathcal{J}(\eta)$ (see Table 2.1), we obtain the following corollary.

Corollary 5.2.2. *A connected primitive axial algebra is a baric algebra of Jordan type* η *if and only if it satisfies the fusion law*

$*$	1	0	η
1	1		η
0		0	η
η	η	η	0

5.3 Relevant values of η

We first note the following fact.

Proposition 5.3.1. *A 2-generated algebra of Jordan type* η *is baric if and only if* $\varphi = 1$. *Consequently,* $\eta = 2$ *or* $\eta = \frac{1}{2}$.

Proof. Suppose that A is a 2-generated baric algebra of Jordan type, i.e., A is isomorphic to (a quotient of) $A(\eta, \varphi)$ for suitable η and φ. Let w be the weight function on A. Recall that the normalised Frobenius form on an algebra of Jordan type is unique. In particular, this Frobenius form must coincide with $(u, v) = w(u)w(v)$. Therefore, for the generating axes a and b, we have $\varphi = (a, b) = w(a)w(b) = 1 \cdot 1 = 1$. So $\varphi = 1$.

For the converse, assume that $\varphi = 1$. If $\eta \neq \frac{1}{2}$, every 2-generated algebra of Jordan type η is isomorphic either to $2B$, or to $A(\eta, \frac{\eta}{2}) \cong 3C(\eta)$, or to a quotient of $3C(\eta)$. However, $2B$ is not baric (as $\varphi = 0$ in $2B$). So the baric algebra has to be isomorphic to $3C(\eta)$ or a quotient of it. Recall that in $3C(\eta)$ (or quotient), $\varphi = \frac{\eta}{2}$. So if $\varphi = 1$, then $\eta = 2$. And indeed, in $3C(2)$, the Gram matrix with respect to the standard basis $\{a, b, c\}$ looks as follows:

$$\begin{pmatrix} 1 & \varphi & \varphi \\ \varphi & 1 & \varphi \\ \varphi & \varphi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Clearly, it has a rank of 1, and so this algebra is baric.

In the other case, $\eta = \frac{1}{2}$, the condition $\varphi = 1$ identifies the algebra as $A(\frac{1}{2}, 1)$ (or a quotient). By Lemma 4.4.1, the Gram matrix of the Frobenius form with respect to the basis $\{a, b, \sigma\}$ is

$$\begin{pmatrix} 1 & \varphi & \pi \\ \varphi & 1 & \pi \\ \pi & \pi & \pi(\varphi - 2\eta) \end{pmatrix} = \begin{pmatrix} 1 & \varphi & \frac{1}{2}(\varphi - 1) \\ \varphi & 1 & \frac{1}{2}(\varphi - 1) \\ \frac{1}{2}(\varphi - 1) & \frac{1}{2}(\varphi - 1) & \frac{1}{2}(\varphi - 1)^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, manifestly, this matrix has rank 1 and so the algebra is baric. □

Since in a baric axial algebra, every 2-generated subalgebra is also baric, we have the following corollary.

Corollary 5.3.2. *Suppose that A is a baric algebra of Jordan type η. Then $\eta = \frac{1}{2}$ or $\eta = 2$. Furthermore, $(a, b) = 1$ for any two axes a and b in A.*

5.4 Baric Matsuo algebras

In this section, we will discuss the case where $\eta = 2$. We have already seen in Corollary 4.6.3 that every axial algebra of Jordan type 2 is either a Matsuo algebra or a quotient of one. (This excludes the case where $\mathbb{F}$ has characteristic 3, because we must have that $2 \neq \frac{1}{2}$ to apply Corollary 4.6.3.) If a quotient $A = M/I$ of a connected Matsuo algebra

M is baric, then A admits a surjective weight homomorphism $w : A \to \mathbb{F}$ such that $w(a) = 1$ for every axis $a \in A$. Let $t : M \to M/I$ be the natural surjection. Then $w \circ t$ is a surjective algebra homomorphism $M \to \mathbb{F}$, hence a weight function on M. Furthermore, $\ker(w \circ t)$ is a proper ideal in M. Since M is connected, none of the axes from M can be in this ideal, and so M is a baric axial algebra. Vice versa, quotient algebras of baric axial algebras are clearly baric. Therefore, our task reduces to identifying which Matsuo algebras for $\eta = 2$ are baric.

Theorem 5.4.1. *A connected Matsuo algebra $M = M_2(G, D)$ is baric if and only if the corresponding 3-transposition group (G, D) is of Moufang type.*

Proof. Suppose that the Matsuo algebra M is baric. According to Proposition 5.1.8, we have that $\varphi = (a, b) = 1$ for all pairs of axes $a, b \in D$. Comparing with Proposition 4.2.2, we see that the order of ab in the group G is three, as long as $a \neq b$. Hence, we never have $o(ab) = 2$ and so (G, D) is of Moufang type.

Conversely, suppose that (G, D) is of Moufang type and $\eta = 2$. Then G contains no commuting 3-transpositions. Therefore, the order of ab is 3 for all $a, b \in D$, $a \neq b$. Hence, $(a, b) = 1$ for all axes $a \neq b$.

Consequently, the Gram matrix T of the Frobenius form with respect to the basis D of M is the all-one matrix and so the rank of this matrix is 1. Hence the dimension of the radical of M is $|D| - 1$, i.e., it has codimension 1. This means that M is baric. $\square$

This result gives us a complete description of baric algebras of Jordan type $\eta \neq \frac{1}{2}$.

In the remainder of the thesis, we turn to the exceptional case $\eta = \frac{1}{2}$. Here, in addition to Matsuo algebras, we also encounter Jordan algebras. In fact, the complete list of algebras of Jordan type $\frac{1}{2}$ is not known and so the answer in this exceptional case is quite different.

CHAPTER 6

PRELIMINARIES ON MAGMAS

We will need the concepts of magmas and universal magmas.

6.1 Magmas

Definition 6.1.1. A *magma* M is a set with a binary operation. If the binary operation is commutative, then we say that M is a *commutative* magma.

Examples of magmas are plentiful, for example, groups, monoids, semi-groups and quasi-groups are all magmas. If A is an algebra, then $(A, +)$ is a commutative magma, a group and even a vector space. Also, $(A, \cdot\,)$ is a magma, and we call it the *multiplicative magma* of A.

Definition 6.1.2. Suppose M is a magma. A subset S of M is a *submagma* if it is closed with respect to the magma operation.

In particular, the empty subset is trivially a submagma of every magma M.

Lemma 6.1.3. *Let M be a magma and M_i, $i \in I$, a collection of submagmas of M. Then the intersection $K := \bigcap_{i \in I} M_i$ is a submagma.*

Proof. If $a, b \in K$, then $a, b \in M_i$ for each i. Each M_i is a submagma, so it is closed under the magma product. So $ab \in M_i$ for each i. Thus, ab is also in K, proving that K is closed under the product. □

Corollary 6.1.4. *Given a subset X of a magma M, there exists a unique smallest submagma of M containing X.*

Proof. This unique smallest submagma is the intersection of all submagmas of M that contain X. Indeed, by the above lemma, this intersection is a submagma. □

Definition 6.1.5. For a magma M and a subset $X \subseteq M$, the unique smallest submagma containing X is called the submagma *generated* by X. This unique smallest submagma is denoted by $\langle X \rangle$.

In particular, the empty subset $\emptyset = X \subseteq M$ generates the empty submagma. That is, $\langle \emptyset \rangle = \emptyset$.

Example 6.1.6. *Suppose that (A, X) is an axial algebra. We can view A as a multiplicative magma by ignoring the addition operation. We will denote by $M(A, X)$ the submagma $\langle X \rangle$ generated by X within the magma A. In other words, $M(A, X)$ consists of all the products (of arbitrary length) that we can form using the elements of X. When A or X is clear, we can use $M(A)$ or $M(X)$ instead of the full notation $M(A, X)$. This example of a submagma will play an important role in the remainder of the text.*

Note that $M(A, X)$ spans A as a vector space, that is, $A = \langle M(A, X) \rangle$.

For a magma M and a subset $X \subseteq M$, let us inductively define the following increasing sequence of subsets:

(i) $\mathcal{M}_1 = X$;

(ii) for $i \geq 2$, $\mathcal{M}_i = \mathcal{M}_{i-1} \cup (\bigcup_{j=1}^{i-1} \mathcal{M}_j \mathcal{M}_{i-j})$.

It is clear that each $\mathcal{M}_i$ is contained in $\langle X \rangle$. On the other hand, $\bigcup_{i=1}^{\infty} \mathcal{M}_i$ is closed under the magma product, so we conclude that $\bigcup_{i=1}^{\infty} \mathcal{M}_i = \langle X \rangle$.

Definition 6.1.7. Suppose that M is a magma and $X \subseteq M$. Let the subsets $\mathcal{M}_i$ be as above. For $m \in \langle X \rangle$, we define the *length* $\ell(m)$ of m with respect to X as $i \in \mathbb{N}$ such that $m \in \mathcal{M}_i \setminus \mathcal{M}_{i-1}$.

Informally, the length $\ell(m)$ is equal to the number of quotients from X that are needed to obtain m as a product. We can also write $\ell(m) = \min \{\ell(n) + \ell(k) \mid n, k \in \langle X \rangle, m = nk\}$.

From the above discussion, we can now use the length of an element for inductive proofs.

Definition 6.1.8. Suppose that M, N are magmas and $\varphi : M \to N$ is a map. We say that φ is a *magma homomorphism* if $\varphi(mn) = \varphi(m)\varphi(n)$ for all $m, n \in M$.

Proposition 6.1.9. *Suppose that M is a magma with the generator set X. Then for every map $\varphi : X \to N$, where N is a magma, there exists at most one homomorphism $\psi : M \to N$ that extends φ. That is, $\psi|_X = \varphi$.*

Proof. By contradiction suppose that there are two extension homomorphisms ψ_1 and ψ_2. Let $m \in M$ be an element such that $\ell(m)$ is minimal subject to the condition that $\psi_1(m) \neq \psi_2(m)$. There are two cases. The first case is where $m \in X$, i.e., $\ell(m) = 1$. Clearly, in this case, $\psi_1(m) = \varphi(m) = \psi_2(m)$ since $\psi_1|_X = \varphi = \psi_2|_X$. Therefore, we have a contradiction in this case. The second case is where $m \notin X$. Then $m = nk$ where $\ell(m) = \ell(n) + \ell(k)$, which means that $\ell(n)$ and $\ell(k)$ are both smaller than $\ell(m)$. By the minimality of $\ell(m)$, $\psi_1(n) = \psi_2(n)$ and $\psi_1(k) = \psi_2(k)$. But $\psi_1(m) = \psi_1(nk) = \psi_1(n)\psi_1(k)$

and $\psi_2(m) = \psi_2(nk) = \psi_2(n)\psi_2(k)$. So $\psi_1(m) = \psi_2(m)$ which is a contradiction. Thus, there exists at most one homomorphism ψ extending φ. □

Definition 6.1.10. Suppose that a magma M is generated by its subset X. We say that M is a *free* magma with (free) generators X if, for any magma N, any map $\psi : X \to N$ extends to a homomorphism from M to N.

Our first goal is to show that the free magma exists. We use binary trees to represent its elements. We first recall the basic concepts from graph theory. A *tree* is a connected graph without cycles. We only consider finite trees. There is a unique shortest path between two vertices in a tree. A *rooted tree* is a tree with one vertex being designated as the *root*. In a rooted tree, the number of edges on the unique shortest path from the root to the particular vertex v is called the *depth* of v. So the depth of the root is 0 and the depth of the neighbours of the root is 1, etc. The maximum depth of a vertex in a tree is called the *height* of the tree.

Suppose that a vertex v is not the root. The *parent* of v is the unique neighbour u of v such that the depth of u is smaller than the depth of v. For any vertex u, the *children* of u are all vertices v such that u is the parent of v. We call a vertex without children a *leaf*.

A *binary tree* is a rooted tree such that every vertex has either zero or two children. An *ordered binary tree* is a binary tree where the children of every vertex in the tree are ordered. That is, there is the first child and the second child. In a picture, the children will be organised from left to right.

The above concepts are illustrated in Figure 6.1.

Let X be a set. We can think of X as an alphabet. We use the elements of X to mark the leaves of an ordered binary tree. In the remainder of this section, by a tree we will mean such an ordered binary tree with marked leaves. The elements of the free magma $M(X)$

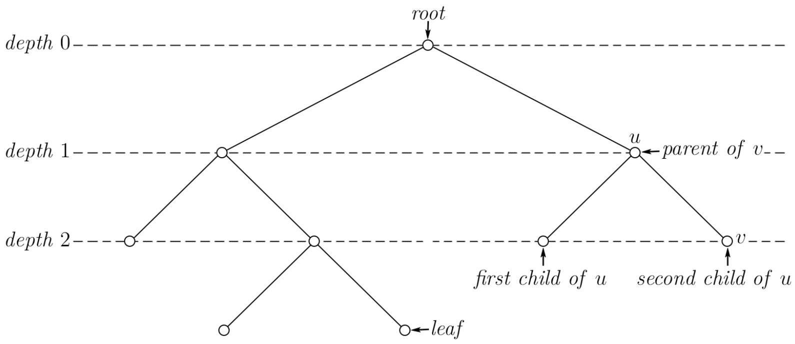
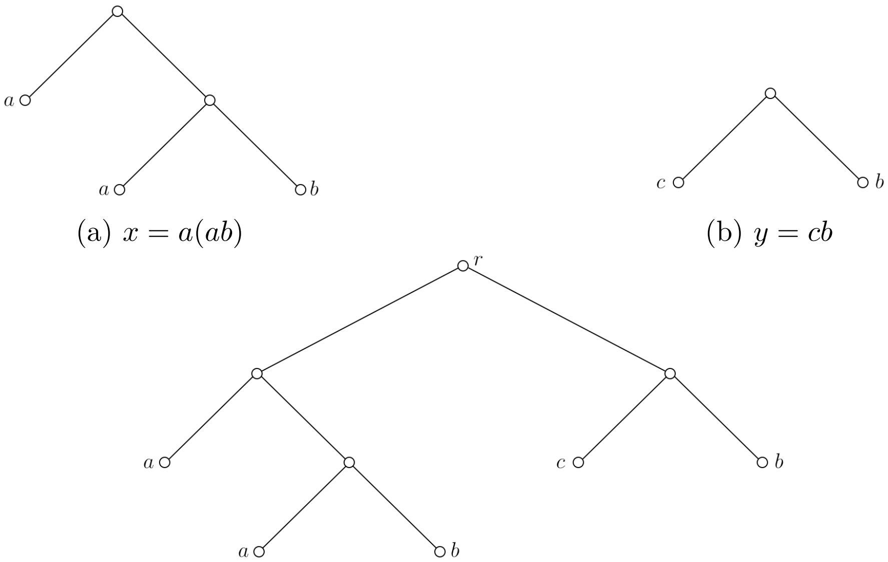


Figure 6.1: An ordered binary tree of height 3

we are building are all such trees. We define the product on $M(X)$ as follows. Suppose that $x, y \in M(X)$, i.e., x and y are trees. To construct the tree xy, we add a new root vertex r and the roots of x and y become the first and second child of r respectively. We call xy the *join* of x and y. We can see an example of a join in Figure 6.2.



(a) $x = a(ab)$

(b) $y = cb$

(c) $xy = (a(ab))(cb)$

Figure 6.2: A join of trees

We identify each element of X with a one-vertex tree labelled by that element (see Figure 6.3). So $X \subseteq M(X)$ and then $M(X)$ is generated by X.

Observe that every element w of $M(X)$ is either in X or it can be uniquely written as

$$\begin{array}{ccc} a & \longleftrightarrow & \circ\, a \\ b & \longleftrightarrow & \circ\, b \\ & \cdots & \end{array}$$

Figure 6.3: Graphs with one vertex

$w = xy$ for $x, y \in M(X)$. Let N be a magma and $\psi : X \to N$ be an arbitrary map. Let us construct a map $\varphi : M(X) \to N$ extending ψ. We do it inductively in terms of the height of the element. Let $w \in M(X)$. If the height of w is 0, then $w \in X$, and so we define $\varphi(w) = \psi(w)$. Otherwise, w can be written uniquely as $w = xy$. Note that the heights of x and y are lower than the height of w. So we can assume by induction that $\varphi(x)$ and $\varphi(y)$ have already been defined. Therefore, we can set $\varphi(w) = \varphi(x)\varphi(y)$.

It is now clear that φ is a homomorphism. Indeed, for $x, y \in M(X)$, if we set $w = xy$, then this is the only decomposition of w as a product, and so by the definition of φ, $\varphi(xy) = \varphi(w) = \varphi(x)\varphi(y)$. So we have $\varphi(xy) = \varphi(x)\varphi(y)$ and so φ is a homomorphism. It is clear that φ is unique for ψ because there was no choice at every step of its construction. Therefore, we have shown the following result.

Theorem 6.1.11. *The magma $M(X)$ is a free magma with respect to its subset X.*

The cardinality of the set X is called the *rank* of the free magma M.

Theorem 6.1.12. *The free magma of a fixed rank is unique up to isomorphism.*

Proof. Let M and N be free magmas with the generating sets X and Y respectively. Assume that M and N have the same rank, i.e., $|X| = |Y|$. There is a bijection ψ from X to $Y \subseteq N$. Since M is a free magma, this map ψ extends to a homomorphism φ from M and N. Now take the inverse map ψ^{-1} from Y to $X \subseteq M$. Since N is a free magma, this map ψ^{-1} extends to a homomorphism ξ from N to M. We claim that φ and ξ are inverses of each other. Indeed, the product $\xi\varphi$ is a homomorphism from M to itself extending the identity map $\psi^{-1}\psi$ from X to $X \subseteq M$. Since the extension is unique, the entire map $\xi\varphi$ is the identity map, because the identity map is a homomorphism and it extends $\psi^{-1}\psi$. Symmetrically, $\varphi\xi$ is also the identity map on N and so φ and ξ are inverses of each other. Thus, they are isomorphisms. □

6.2 Congruences

We have earlier defined the homomorphisms between magmas. In this section, we will define the congruences of magmas which is roughly equivalent to the concept of normal subgroups in group theory.

Definition 6.2.1. Let M be a magma. A *congruence* on M is an equivalence relation $\sim$ such that if $x \sim y$ and $z \sim t$ for $x, y, z, t \in M$, then $xz \sim yt$.

We have mentioned that this concept is similar to normal subgroups in group theory.

Example 6.2.2. Suppose that M is a group. Then a congruence $\sim$ on M is an equivalence relation corresponding to the partition of M as the set of cosets of a normal subgroup. Indeed, it is not difficult to verify that the set $N = \{x \in M \mid x \sim 1_M\}$ is a normal subgroup of M and every other equivalence class is a coset of N.

The equivalence classes of a congruence will be called *congruence classes*. The congruence class containing an element $x \in M$ will be denoted by $[x]$.

The set of congruence classes carries the structure of a magma where the product is defined as follows: $[x][y] := [xy]$ for all $x, y \in M$. Let us show that it is well-defined. Take $x' \in [x]$ and $y' \in [y]$. Since $\sim$ is a congruence on M and since $x' \sim x$ and $y' \sim y$, we conclude that $x'y' \sim xy$. That is, $[x'y'] = [xy]$. So the product of congruence classes is well-defined. The magma of congruence classes is called the *factor magma* and it is denoted by $M/\sim$.

We next have the following theorem that is similar to the First Isomorphism Theorem for groups.

Theorem 6.2.3. *Suppose that $\varphi : M \to N$ is a homomorphism of magmas. For $x, y \in M$, let $x \sim y$ if and only if $\varphi(x) = \varphi(y)$. Then this relation $\sim$ is a congruence on M and the submagma* $\operatorname{im}\varphi$ *of N is isomorphic to $M/\sim$.*

Proof. Let $x, y, z, t \in M$ with $x \sim y$ and $z \sim t$. Then $\varphi(x) = \varphi(y)$ and $\varphi(z) = \varphi(t)$. Since φ is a homomorphism, $\varphi(xz) = \varphi(x)\varphi(z) = \varphi(y)\varphi(t) = \varphi(yt)$. Therefore, $xz \sim yt$ and so $\sim$ is a congruence on M.

Let $p, q \in \operatorname{im}\varphi$. Then $\varphi(m) = p$ and $\varphi(n) = q$ for some $m, n \in M$. We have that $\varphi(mn) = \varphi(m)\varphi(n) = pq$. So $\operatorname{im}\varphi$ is closed with respect to the magma operation. Thus, $\operatorname{im}\varphi$ is a submagma of N.

Next, we show that $\operatorname{im}\varphi \cong M/\sim$. Consider the map $\varphi^{-1} : \operatorname{im}\varphi \to 2^M$ via $\varphi^{-1}(n) = \{m \in M \mid \varphi(m) = n\}$. We claim that $\varphi^{-1}(n)$ is an element of $M/\sim$, i.e., it is a congruence class. This follows from the definition of $\sim$ above. Thus, φ^{-1} can be viewed as a map from $\operatorname{im}\varphi$ to $M/\sim$. The inverse is induced by φ. This means that φ^{-1} is bijective. It remains to note that φ^{-1} is a homomorphism. Let $n, n' \in \operatorname{im}\varphi$. Say $n = \varphi(m)$ and $n' = \varphi(m')$. Then $\varphi^{-1}(nn') = [mm'] = [m][m'] = \varphi^{-1}(n)\varphi^{-1}(n')$. Thus, φ^{-1} is an isomorphism. □

Definition 6.2.4. Suppose that $\sim_i$ for $i \in I$ are equivalence relations on a set X. By the intersection $\bigcap_{i\in I} \sim_i$ we mean the relation $\sim$ such that $x \sim y$ if and only if $x \sim_i y$ for all i.

Note that the $\sim$ in this definition is again an equivalence relation. Indeed, since each $\sim_i$ is reflexive, $x \sim_i x$ for all i and $x \in M$. Thus, $x \sim x$ for all $x \in M$, i.e., $\sim$ is reflexive.

Let $x, y \in M$. Suppose that $x \sim y$. Then $x \sim_i y$ for each i. So $y \sim_i x$ for every i because $\sim_i$ is symmetric. Thus, $y \sim x$, i.e., $\sim$ is also symmetric. Let $x, y, z \in M$. Suppose that $x \sim y$ and $y \sim z$. Then for every i, $x \sim_i y$ and $y \sim_i z$. So $x \sim_i z$ for every i as $\sim_i$ is an equivalence relation. Thus, $x \sim y$ and $y \sim z$ imply that $x \sim z$, i.e., $\sim$ is transitive. Thus, $\sim$ is an equivalence relation.

Theorem 6.2.5. *If $\sim_i$, $i \in I$, are congruences on a magma M, then the intersection $\sim = \bigcap \sim_i$ is also a congruence of M.*

Proof. We have already seen that $\sim$ is an equivalence. So we just need to show that $\sim$ is a congruence. Let $x, y, z, t \in M$. Suppose that $x \sim y$ and $z \sim t$. Then $x \sim_i y$ and $z \sim_i t$. So $xy \sim_i zt$ as $\sim_i$ is a congruence. Since this is true for every i, we have $xz \sim yt$. Therefore, $\sim$ is a congruence on M. □

Corollary 6.2.6. *For a magma N and a set of pairs $(n_i, n_i') \in N \times N$, there is a unique finest congruence $\sim$ on N such that $n_i \sim n_i'$ for all $i \in I$.*

Proof. The set of pairs $R = \{(n_i, n_i') \mid i \in I\}$ is contained in at least one congruence, for example, $N \times N$ is such a congruence. Therefore, by Theorem 6.2.5 the intersection of all congruences containing R is the unique finest congruence containing R. □

We will denote the minimal congruence containing the set of pairs R by (R).

This result means that we can define magmas in terms of their generators and relations. For example, if a magma M is generated by m_i, for $i \in I$, then $M \cong M(X)/\sim$, where $X = \{x_i \mid i \in I\}$ and $\sim$ is a congruence on $M(X)$ coming from the homomorphism $\varphi : M(X) \to M$ sending each x_i to m_i. Then if we have a set of pairs $R = \{(n_i, n_i') \in M(X) \times M(X) \mid i \in I\}$ such that $\sim$ is the finest congruence containing all pairs from R, then we can view the pairs (n_i, n_i') as relators generating the congruence $\sim$. Note that we must have $\varphi(n_i) = \varphi(n_i')$, for all $i \in I$, for the pairs to be contained in $\sim$. In view of this, we will write our relators as $n_i = n_i'$ and we will also write $M \cong M(X)/(R)$.

Let us formulate the Third Isomorphism Theorem for magmas.

Theorem 6.2.7. *Let M be a magma and $\sim$ and $\approx$ be two congruences on M with $\sim$ contained in $\approx$. Then there exists a surjective homomorphism of magmas $\phi : M/\sim \to M/\approx$ sending $[m]_\sim \mapsto [m]_\approx$ for all $m \in M$. Furthermore, the congruence on $\overline{M} = M/\sim$ corresponding to ϕ is the relation $\approxeq$ defined as follows: the congruence classes $[m]_\sim$ and $[m']_\sim$, where $m, m' \in M$, are in the relation $\approxeq$ if and only if $m \approx m'$. Consequently, $\overline{M}/\approxeq \cong M/\approx$.*

Proof. Note that since $\sim$ is contained in $\approx$, every congruence class of $\sim$ is fully contained in some class of $\approx$. In particular, the map ϕ is well-defined. If $m, m' \in M$ then $\phi([m]_\sim[m']_\sim) = \phi([mm']_\sim) = [mm']_\approx = [m]_\approx[m']_\approx = \phi([m]_\sim)\phi([m']_\sim)$. This shows that ϕ is a homomorphism of magmas. Every class $[m]_\approx \in M/\approx$ is equal to $\phi([m]_\sim)$, which means that ϕ is surjective, as claimed.

Manifestly, $\approxeq$ is the congruence on $\overline{M} = M/\sim$ corresponding to ϕ and hence the final claim follows from Theorem 6.2.3. □

6.3 Example: magmas generated by idempotents

Consider the magma $M(X)$ for some set X, that is, the elements of $M(X)$ are (marked ordered binary) trees and let $\sim = (R)$ be the minimal congruence containing $R = \{(x^2, x) \mid x \in X\}$ (we could have written these relators as equalities $x^2 = x$; see Figure 6.4). Let $I(X) = M(X)/\sim$. This is the universal magma generated by a set X of idempotents. Indeed, if M is generated by a set Y of idempotents and $\varphi : X \to Y$ is a bijection,

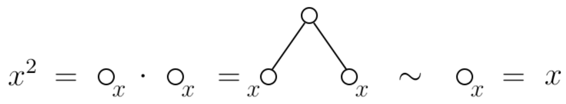


Figure 6.4: Generators of $\sim$

then, by Theorem 6.1.9, φ extends to a homomorphism $\psi : M(X) \to M$. Let $\approx$ be the congruence on $M(X)$ coming from the homomorphism ψ. Then $\psi(x^2) = \psi(x)^2 = \psi(x)$ since $\psi(x) = \varphi(x) \in Y$ (as ψ is an extension of φ) and the elements of Y are idempotents. This means that $x^2 \approx x$ for all $x \in X$, i.e., the congruence $\approx$ contains the relation R. Consequently, $\approx$ also contains the congruence $\sim = (R)$. Now our Theorems 6.2.3 and 6.2.7 are applicable. Namely, $M \cong (M(X)/\approx) \cong I(X)/\overline{\approx}$. Therefore, we have a homomorphism from $I(X)$ onto M sending every $[x]_\sim$ to $y = \varphi(x)$. Note that the classes $[x]_\sim$ remain distinct for distinct x.

Definition 6.3.1. A *twin pair* in a tree from $M(X)$ is a pair of leaves with the same parent and with the same mark $x \in X$.

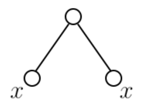


Figure 6.5: A twin pair

Proposition 6.3.2. *Consider the congruence $\sim = (x^2 = x \mid x \in X)$ on $M(X)$. Then every congruence class of $\sim$ contains the unique smallest tree which is the only tree in the congruence class that has no twin pairs.*

Proof. Let us define a process of reduction on trees from $M(X)$. If a tree T has a pair of twins, say with a mark $x \in X$, a single step of reduction removes these twins from T and labels the parent of these twins (which is now a leaf) with the mark x. Clearly, the resulting tree T' is again in $M(X)$ and it has fewer leaves by one. A full reduction consists of a number of such steps until the final tree has no twins at all.

Note that the pair (T, T') from the single step above belongs in the congruence $\sim$. Consequently, a full reduction of T produces a tree without twins in the congruence class $[T]_\sim$.

Clearly, when a tree contains many pairs of twins, there is more than one way to carry out a full reduction. The first step of the proof is to show that the result of full reduction is independent of the order of steps. We prove this by induction on the number of leaves. Namely, suppose by contradiction that the claim is not true and select a counterexample tree T for which the number of leaves is minimally possible. This means that T has two full reductions, say, R_1 and R_2, resulting in two different reduced trees T_1^* and T_2^*, respectively. (Note that T cannot be itself reduced, as it then would only admit the unique, zero-step full reduction, and so $T_1^* = T = T_2^*$, which is a contradiction.)

Suppose first that R_1 and R_2 start with the same first step, producing a tree T'. Since T' is smaller than T, it cannot be a counterexample, so it has a unique fully reduced form. Since R_1 and R_2 without the first step are full reductions of T', we conclude that $T_1^* = T_2^*$, which is a contradiction.

Now suppose that R_1 and R_2 have different first steps. Say the first step of R_1, denoted S_1, removes a pair of twins in T resulting in a tree T_1, and the first step of R_2, denoted S_2, removes a different pair of twins from T resulting in a tree T_2. The two pairs of twins are clearly disjoint in T. This means that the pair of twins used for S_2 is still present in T_1 and so S_2 can be applied to T_1 producing a tree T'. Symmetrically, S_1 can be applied to T_2 and the result of this step is the same tree T'. Let R' be any full reduction of T' with the resulting tree T^*. Note that both T_1^* and T^* are full reductions of T_1. Since

T_1 has fewer leaves than T, it cannot be a counterexamples, which means that $T_1^* = T^*$. Symmetrically, T_2^* and T^* are full reductions of T_2, and again this means that $T_2^* = T^*$. We conclude that $T_1^* = T_2^*$, which is the final contradiction showing that indeed every tree $T \in M(X)$ has a unique reduced form.

Now we are prepared to establish the claim of the lemma. Since $\sim$ is the congruence generated by all pairs (x^2, x), $x \in X$, any two trees in the same congruence class are connected via a path with a finite number of steps either removing a pair of twins, as above, or inserting a pair of twins (i.e., adding to a former leaf marked with $x \in X$ two children leaves marked x). In any case, any two consecutive trees on the path would have the same reduced form, which means that all trees in the congruence class have the same reduced form, proving the claim. □

Corollary 6.3.3. *The elements of $I(X)$, i.e., the congruence classes of $\sim$, are in a natural bijection with the trees not containing twin pairs. The multiplication on this set is the same as the multiplication from $M(X)$ with one exception: the product of the one-vertex tree T_x labelled by x with itself is again itself for every $x \in X$. That is, $T_x T_x = T_x$.*

Proof. By Proposition 6.3.2, every congruence class contains a unique tree without twin pairs. This gives us a bijective correspondence. Note that if T and Q are two such trees, then TQ also does not have twin pairs unless $T = Q$ consists of a single vertex. When TQ has no twin pairs, clearly we have that $[T]_\sim [Q]_\sim = [TQ]_\sim$, and so the multiplication of classes matches the multiplication of the trees T and Q. In the exceptional case, where $T = Q$ has only one vertex, $[T]_\sim [T]_\sim = [TT]_\sim = [T]_\sim$ since TT has a twin pair and its class is represented by T. □

6.4 Commutative magmas

In the section, we construct the free commutative magma. Let us first give the definition of this object.

Definition 6.4.1. Let a commutative magma M be generated by a subset $X \subseteq M$. Then M is said to be the *free commutative* magma with (free) generators X if for every commutative magma N, any map $\psi : X \to N$ extends to a homomorphism from M to N.

We will first show the existence of this object in terms of a presentation.

Proposition 6.4.2. *Let $C(X)$ be the factor magma $M(X)/\sim$ where $\sim = (TQ = QT \mid T, Q \in M(X))$. Then $C(X)$ is the free commutative magma generated by X.*

Proof. First of all, it is clear that $C(X)$ is commutative. Indeed, $[T]_\sim[Q]_\sim = [TQ]_\sim = [QT]_\sim = [Q]_\sim[T]_\sim$ for all $T, Q \in M(X)$, since $TQ \sim QT$. Consider now an arbitrary commutative magma N and an arbitrary map $\varphi : X \to N$. By Proposition 6.1.9, this map extends to a homomorphism $\psi : M(X) \to N$. Let $\approx$ be the congruence on $M(X)$ corresponding to ψ. Note that for T and Q in $M(X)$, we have that $\psi(TQ) = \psi(T)\psi(Q) = \psi(Q)\psi(T) = \psi(QT)$, since N is commutative. This means that $TQ \approx QT$ for all $T, Q \in M(X)$. Thus, $\approx$ contains the congruence $\sim = (TQ = QT \mid T, Q \in M(X))$. By Theorem 6.2.7, we now have a homomorphism $\xi : C(X) = M(X)/\sim \to N$ extending φ. So $C(X)$ is indeed the free commutative magma. □

Recall the uniqueness theorem for the free magma. Similarly, we have the uniqueness theorem for the free commutative magma.

Theorem 6.4.3. *The free commutative magma of a fixed rank is unique up to isomorphism.*

Proof. Suppose that M and N are free commutative magmas generated by the sets X and

Y respectively such that $|X| = |Y|$, i.e., the rank of M and the rank of N are equal. We can select a bijective map $\psi : X \to Y \subseteq N$ because $|X| = |Y|$. Since M is a free commutative magma and N is commutative, the map ψ extends to a homomorphism $\varphi : M \to N$. Consider the inverse map $\psi^{-1} : Y \to X \subseteq M$. Since N is a free commutative magma and M is commutative, the map ψ^{-1} extends to the homomorphism $\xi : N \to M$. Note that $\xi\varphi$ is a homomorphism from M to itself extending the identity map $\psi^{-1}\psi : X \to X \subseteq M$. Since the extension is unique, $\xi\varphi$ is also the identity map. Similarly, $\varphi\xi$ is the identity map from N to itself. Therefore, ξ and φ are isomorphisms. □

Let us now describe $C(X)$ explicitly. The description is similar to what we have in Section 6.1. The only difference is that here we consider the ordinary (unordered) binary trees instead of the ordered ones. Thus, let $C(X)$ be the set of all labelled binary trees. Recall that the free magma $M(X)$ is the set of all binary trees where two children of the same vertex are ordered by the first and the second children. The product on $M(X)$ is defined as follows: if $x, y \in M(X)$ are binary trees, then xy is obtained by adding a new root vertex and the roots of x and y become the first child and the second child of the new root vertex respectively. The product on $C(X)$ is the same but again we ignore the order of children. For example, in Figure 6.6, we show two representations of the same element $a(ab) = (ba)a$.

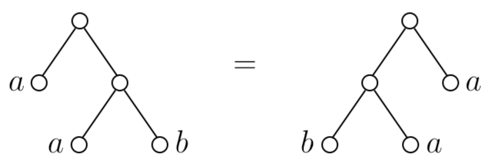


Figure 6.6: $a(ab) = (ba)a$

We need to show that this is a realisation of $C(X)$. Indeed, notice the following properties of binary trees: every unordered binary tree T has a unique set of factors F_1, F_2 such that $T = F_1F_2 = F_2F_1$. Using this, we can show by induction on the height of the tree that every map φ from X to a commutative magma N extends to a homomorphism ψ from this version of $C(X)$ to N. The map ψ is known for the one-vertex trees T (height 0), which are elements of X, and so $\psi(T) = \varphi(T)$. Let us suppose that the map is defined for all trees of height up to $k \geq 0$. If T is a tree of height $k+1$ and F_1, F_2 are its factors, then we define $\psi(T) = \psi(F_1)\psi(F_2)$ (note that $\psi(F_1)\psi(F_2) = \psi(F_2)\psi(F_1)$ as N is commutative). This gives us the definition of the map ψ and it is immediately from the definition that ψ is a homomorphism since every tree T with height $k \geq 1$ can be uniquely factorised as a product of two commuting factors. From now on, when we talk about $C(X)$, we mean its tree realisation.

6.5 Free commutative magma generated by idempotents

In this section, we summarise everything we have so far. We start with building an example of the free commutative magma generated by idempotents which will be used to construct the universal axial algebra.

Definition 6.5.1. Suppose that M is a magma with a set of generators X. We say that the pair (M, X) is an *axial magma* if M is commutative and all elements of X are idempotents.

Definition 6.5.2. An axial magma (M, X) is a *free axial magma* if for every axial magma (N, Y), every map $\psi : X \to Y$ extends to a homomorphism from M to N.

We now describe a particular axial magma $A(X)$ and further show that $A(X)$ is a free axial magma. The construction for a free axial magma via representation is very similar to the examples above. So we will skip this construction and instead describe $A(X)$ directly in terms of unordered binary trees. Since the generating elements are idempotents, we do not allow twin pairs in the tree. So the elements of $A(X)$ are unordered binary trees without twin pairs. The product on $A(X)$ is defined in the following way: if $x, y \in A(X)$, we add a new root vertex and the roots of x and y become the children of this new root. The only exception to this rule is when $x = y \in X$ (i.e., both factors are the one-vertex trees marked by x), then we set $xy = xx = x$.

Theorem 6.5.3. *The magma $A(X)$ is a free axial magma generated by X. In particular, $A(X) \cong M(X)/\sim$, where $\sim = (TQ = QT, x^2 = x \mid T, Q \in M(X), x \in X)$.*

Proof. Consider an axial magma (N, Y) and an arbitrary map $\varphi : X \to Y$. We need to show that φ extends to a homomorphism $\psi : A(X) \to N$. Clearly, for $x \in X$, $\psi(x) = \varphi(x)$. If T is a tree of height more than 0, then T decomposes uniquely as a product of two trees R and S of smaller height. Hence, inductively, we may assume that $\psi(R)$ and $\psi(S)$ are already defined and we can set $\psi(T) = \psi(RS) := \psi(R)\psi(S)$. (Recall that N is commutative. So $\psi(R)\psi(S) = \psi(S)\psi(R)$, making our construction well-defined.) The fact that this is a homomorphism follows immediately from the construction since every T of height at least one decomposes uniquely up to the order of the factors. $\square$

Now we state the corresponding uniqueness theorem.

Theorem 6.5.4. *The free axial magma $A(X)$ of a fixed rank $|X|$ is unique up to isomorphism.*

The proof of this theorem is very similar to the proof of Theorem 6.4.3. The only difference is that when we take (N, Y) to be the second free axial magma with $|X| = |Y|$.

CHAPTER 7

UNIVERSAL BARIC ALGEBRA OF JORDAN TYPE $\frac{1}{2}$

In this chapter, we establish the existence of the universal k-generated baric algebra of Jordan type $\frac{1}{2}$.

7.1 Magma algebra

Definition 7.1.1. Let $A(X)$ be a free axial magma. Then the *magma algebra* $\mathbb{F}[A(X)]$ over a field $\mathbb{F}$ has the basis $A(X)$ and the multiplication in $\mathbb{F}[A(X)]$ extends the multiplication in $A(X)$. That is,

$$\mathbb{F}[A(X)] = \langle A(X) \rangle = \{ \sum_{m \in A(X)} \alpha_m m \mid \alpha_m \in \mathbb{F} \}$$

and for $u = \sum_{m \in A(X)} \alpha_m m$ and $v = \sum_{n \in A(X)} \beta_n n$, we have $uv = \sum_{m,n \in A(X)} \alpha_m \beta_n mn$.

As usual, we only consider finite linear combinations, that is, for $u = \sum_{m \in A(X)} \alpha_m m \in \mathbb{F}[A(X)]$, all but finitely many coefficients α_m are zero, and similarly for v. This is why the product formula above makes sense as it is a finite sum.

Recall that the elements of $A(X)$ are special binary trees, hence every element of the magma algebra $\mathbb{F}[A(X)]$ can be represented by the linear combination of unordered binary trees without twin pairs.

Proposition 7.1.2. *Suppose that A is an axial algebra with a generating set of axes Y. Then every map from X to Y can be extended to a unique algebra homomorphism φ from $\mathbb{F}[A(X)]$ to A.*

Proof. Let us view A as a multiplicative magma (i.e., we ignore addition). Let M be the submagma of A generated by Y. Since $A(X)$ is a free axial magma, the map $\psi : X \to Y$ extends to a unique homomorphism $\varphi' : A(X) \to M$. Since each element of $\mathbb{F}[A(X)]$ can be expressed as a linear combination of elements of $A(X)$, the map $\psi : X \to Y$ extends the homomorphism $\varphi : \mathbb{F}[A(X)] \to A$ by linearity. □

In particular, this proposition tells us that as long as the number of the generating axes of the axial algebra A is no greater than the size of the set X, the algebra A is a factor of $\mathbb{F}[A(X)]$. So we can obtain it by factoring out some relations from $\mathbb{F}[A(X)]$. In the remainder of this chapter, we will analyse which relations hold for all baric algebras of Jordan type $\frac{1}{2}$, and in this way, we will construct the universal k-generated baric algebra of Jordan type half.

However, we first need to add a Frobenius form to $\mathbb{F}[A(X)]$ as it is known that every axial algebra of Jordan type admits a Frobenius form [19].

7.2 Frobenius form

We first need the following observation.

Lemma 7.2.1. *Suppose that A is a baric algebra of Jordan type $\frac{1}{2}$ with a generating set of axes X. Let $M = M(A)$ be the multiplicative submagma in A generated by the set of axes X. Let $m, m' \in M$. Then $(m, m') = 1$.*

Proof. Recall that $(u, v) = w(u)w(v)$, where w is the weight function. We claim that $w(m) = 1$ for all $m \in M$. Indeed, we do it by induction on the length $\ell(m)$ of m. If $\ell(m) = 1$, then $m \in X$ is an axis. So m is an idempotent and it is not in the radical. Hence, $w(m)$ is a non-zero idempotent in $\mathbb{F}$ and so $w(m) = 1$. Consider an arbitrary $m \in M$. It can be written as a product $m = m_1 m_2$ where $\ell(m_1)$, $\ell(m_2)$ are smaller than $\ell(m)$. By induction we know that $w(m_1) = 1 = w(m_2)$. Since w is a homomorphism, we have that $w(m) = w(m_1 m_2) = w(m_1)w(m_2) = 1 \cdot 1 = 1$. Therefore, indeed $w(m) = 1$ for all $m \in M$.

Now taking m and m' from the statement of the lemma, we have that $(m, m') = w(m)w(m') = 1$. $\square$

The above lemma allows us to introduce a suitable Frobenius form on the algebra $\mathbb{F}[A(X)]$. For an arbitrary element $u = \sum_{m \in A(X)} \alpha_m m \in \mathbb{F}[A(X)]$, we define $w(u) = \sum_{m \in A(X)} \alpha_m \in \mathbb{F}$.

Proposition 7.2.2. *The map $w : \mathbb{F}[A(X)] \to \mathbb{F}$ is an algebra homomorphism. The algebra $\mathbb{F}[A(X)]$ can be viewed as a baric algebra with the weight function w.*

Proof. Clearly, w is linear and surjective. So we just need to show that $w(uv) = w(u)w(v)$ for all $u, v \in \mathbb{F}[A(X)]$. Let $u = \sum_{m \in A(X)} \alpha_m m$ and $v = \sum_{n \in A(X)} \beta_n n$. Then $w(uv) =$

$w(\sum_{m,n\in A(X)} \alpha_m\beta_n mn) = \sum_{m,n\in A(X)} \alpha_m\beta_n = (\sum_{m\in A(X)} \alpha_m)(\sum_{n\in A(X)} \beta_n) = w(u)w(v)$. So w is an algebra homomorphism and a valid weight function.

Note that we treat the elements of $X \subset A(X) \subset \mathbb{F}[A(X)]$ as axes of $\mathbb{F}[A(X)]$ and $w(x) = 1$ for all $x \in X$ and this is exactly the additional condition we impose on baric axial algebras. $\square$

It is remarkable that the universal axial algebra (regardless of the fusion law) is baric. This underscores the importance of this property.

7.3 Spectrum relators

We now focus on the relators we need to add for the algebra to become an algebra of Jordan type half. Namely, we need to ensure that each element $x \in X$ is a primitive axis of Jordan type half. The first order of business is the spectrum of ad_x and primitivity.

Suppose that (A, X) is a baric algebra of Jordan type half and consider the multiplicative submagma $M := M(A, X)$ generated by X. Recall that $A = A_1(x) \oplus A_0(x) \oplus A_{\frac{1}{2}}(x)$ for every $x \in X$ and so for each $u \in A$, we have that $u = u_1 + u_0 + u_{\frac{1}{2}}$ for some $u_1 \in A_1(x)$, $u_0 \in A_0(x)$ and $u_{\frac{1}{2}} \in A_{\frac{1}{2}}(x)$. Consider the projection u_1 of u onto $A_1(x)$. Since x is primitive, $A_1(x) = \langle x\rangle$, i.e., u_1 is a multiple of x. More precisely, $u_1 = (x, u)x = w(u)x$, where w is the weight function on A. Thus, $u = w(u)x + u_0 + u_{\frac{1}{2}}$.

Recall that the weight function on a baric algebra has value 1 for all $m \in M$, i.e., $w(m) = 1$. That is, if $u = m \in M$, then $m = x + m_0 + m_{\frac{1}{2}}$.

In the remainder of this chapter, we will use the following notation for convenience: if u is an element of an axial algebra B over a field $\mathbb{F}$, x is an axis of B, and $f \in \mathbb{F}[t]$ is a polynomial, then by $u \cdot f(x)$ we mean $f(\mathrm{ad}_x)(u)$. For example, if $f(t) = t(t - \frac{1}{2}) = t^2 - \frac{1}{2}t$, then $u \cdot f(x) = u \cdot (x^2 - \frac{1}{2}x) = (ux)x - \frac{1}{2}ux$. Since ad_x is an element of the algebra of all linear maps from B to itself, which is associative, we have that $(u \cdot f(x)) \cdot g(x) = u \cdot (f(x)g(x))$.

Lemma 7.3.1. *Suppose that (A, X) is a baric axial algebra of Jordan type half. Let $M := M(A, X)$. Then for $x \in X$ and $m \in M$, we have that $m = x + m_0 + m_{\frac{1}{2}}$ for some $m_0 \in A_0(x)$ and $m_{\frac{1}{2}} \in A_{\frac{1}{2}}(x)$. Furthermore,*

(a) $m_0 = -2(m - x) \cdot (x - \frac{1}{2}) = -2mx + m + x$;

(b) $m_{\frac{1}{2}} = 2(m - x) \cdot x = 2mx - 2x$;

(c) $(m - x) \cdot (x^2 - \frac{1}{2}x) = (mx)x - \frac{1}{2}mx - \frac{1}{2}x = 0$ *for all* $m \in M$*; consequently,* $(u - w(u)x) \cdot (x^2 - \frac{1}{2}x) = (ux)x - \frac{1}{2}ux - \frac{1}{2}w(u)x = 0$ *for all* $u \in A$.

Proof. The claim that $m = x + m_0 + m_{\frac{1}{2}}$ has been shown before the lemma. We have that $(m-x)\cdot(x-\frac{1}{2}) = (m_0+m_{\frac{1}{2}})\cdot(x-\frac{1}{2}) = m_0x-\frac{1}{2}m_0+m_{\frac{1}{2}}x-\frac{1}{2}m_{\frac{1}{2}} = 0-\frac{1}{2}m_0+\frac{1}{2}m_{\frac{1}{2}}-\frac{1}{2}m_{\frac{1}{2}} = -\frac{1}{2}m_0$. Hence, (a) follows. Similarly, $(m-x)\cdot x = (m_0+m_{\frac{1}{2}})\cdot x = m_0x+m_{\frac{1}{2}}x = 0+\frac{1}{2}m_{\frac{1}{2}} = \frac{1}{2}m_{\frac{1}{2}}$. So (b) also holds. The final claim follows similarly from either (a) or (b). □

In view of part (c) of the above lemma, if we want to construct from $\mathbb{F}[A(X)]$ the universal baric axial algebra of Jordan type half, we need to impose at least the relations $(mx)x - \frac{1}{2}mx - \frac{1}{2}x = 0$ for all $x \in X$ and $m \in A(X)$.

Lemma 7.3.2. *The relator $(mx)x - \frac{1}{2}mx - \frac{1}{2}x$ is contained in the baric radical of $\mathbb{F}[A(X)]$ for all $x \in X$ and $m \in A(X)$.*

Proof. Since $w((mx)x - \frac{1}{2}mx - \frac{1}{2}x) = 1 - \frac{1}{2} - \frac{1}{2} = 0$, the claim follows, as the baric radical is the kernel of the weight function w. □

Let $I := ((mx)x - \frac{1}{2}mx - \frac{1}{2}x \mid x \in X, m \in A(X))$ be the ideal of $\mathbb{F}[A(X)]$ generated by the above relators. In view of Lemma 7.3.2, I is fully contained in the baric radical of $\mathbb{F}[A(X)]$. Consider the quotient algebra $U(X) := \mathbb{F}[A(X)]/I$ of $\mathbb{F}[A(X)]$. Clearly, $U(X)$ is a baric algebra with a weight function induced from $\mathbb{F}[A(X)]$. We will denote this new weight function with the same symbol w.

Theorem 7.3.3. *The algebra $U(X)$ is the universal k-generated axial algebra for the fusion law:*

$*$	1	0	η
1	1		η
0		$0, \eta$	$0, \eta$
η	η	$0, \eta$	$0, \eta$

Proof. We write $\bar{u}$ for the image of $u \in \mathbb{F}[A(X)]$ in $U = U(X) = \mathbb{F}[A(X)]/I$. Let $f(t) = t^2 - \frac{1}{2}t = t(t - \frac{1}{2})$.

Since we have relators $(m - x) \cdot (x^2 - \frac{1}{2}x) = (m - x) \cdot f(x) \in I$ for all $x \in X$ and $m \in A(X)$, we have that $(u - w(u)x) \cdot f(x) \in I$ for all $u \in \mathbb{F}[A(X)]$. Note that $\{u - w(u)x \mid u \in \mathbb{F}[A(X)]\} = R(\mathbb{F}[A(X)])$. So $f(\mathrm{ad}_x)R(\mathbb{F}[A(X)]) \subseteq I$, for each $x \in X$, or equivalently, $f(\mathrm{ad}_{\bar{x}}) = 0$ on $R(U)$. This means that the eigenvalues of $\mathrm{ad}_{\bar{x}}$ on $R(U)$ are restricted to the roots of f, i.e., $\{0, \frac{1}{2}\}$. Consequently, $R(U) \subseteq U_0(\bar{x}) \oplus U_{\frac{1}{2}}(\bar{x})$. Taking into account that $R(U)$ has codimension 1 in U, since U is baric, and that $U_1(\bar{x}) \neq 0$, since $\bar{x}$ is an idempotent, we conclude that $U_1(\bar{x})$ is 1-dimensional spanned by $\bar{x}$, $R(U) = U_0(\bar{x}) \oplus U_{\frac{1}{2}}(\bar{x})$, and $U = U_1(\bar{x}) \oplus U_0(\bar{x}) \oplus U_{\frac{1}{2}}(\bar{x})$. In particular, every $\bar{x}$ is primitive and the spectrum of $\mathrm{ad}_{\bar{x}}$ agrees with the fusion law above.

Primitivity of $\bar{x}$ assures the first column and first row of the fusion law. The remainder of it is due to the fact that $R(U) = U_0(\bar{x}) \oplus U_{\frac{1}{2}}(\bar{x})$ is an ideal and so

$$U_0(\bar{x})U_0(\bar{x}), U_0(\bar{x})U_{\frac{1}{2}}(\bar{x}), U_{\frac{1}{2}}(\bar{x})U_{\frac{1}{2}}(\bar{x}) \subseteq R(U) = U_0(\bar{x}) \oplus U_{\frac{1}{2}}(\bar{x}).$$

We have shown that U is a primitive axial algebra for the above fusion law.

If (B, Y) is any other primitive axial algebra with this fusion law, then any map X to Y extends to a magma homomorphism $A(X)$ to B since $A(X)$ is a free axial magma, and it also extends to an algebra homomorphism from $\mathbb{F}[A(X)]$ to B by Proposition 7.1.2. By Lemma 7.3.1, all relators of I are contained in the kernel of the algebra homomorphism from $\mathbb{F}[A(X)]$ to B. Since $U = \mathbb{F}[A(X)]/I$, we have that the algebra homomorphism can also be viewed as a homomorphism from U to B. This shows the universality of $U = U(X)$. □

It is an interesting task to understand the exact structure of $U(X)$ and find its dimension.

7.4 Basis

Recall that the elements of $A(X)$ are unordered binary trees without twin pairs. We will identify the trees from $A(X)$ with their images in $U(X)$.

Definition 7.4.1. Suppose that T is a tree in $A(X)$. We say that T has a *repetition* when it has a fragment shown in Figure 7.1 for some $x \in X$. In terms of products, T contains a sub-product $((mx)x)$, where $m \in A(X)$ is the sub-tree left of the bottom vertex x.

When we have a tree T with a repetition as above, we have that $(mx)x - \frac{1}{2}mx - \frac{1}{2}x \in I$

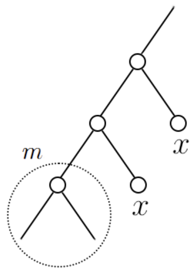


Figure 7.1: Repetition

and so $\bar{T}$, the image of T in $U(X)$, can be expressed via the elements of lower order. So we can exclude $\bar{T}$ from the basis of $U(X)$.

Of course, the pattern of multiplying something with x can persist more than two times. In this sense, the following lemma is useful.

Lemma 7.4.2. *In $U(X)$, we have that $((m\underbrace{x)x)\ldots x}_{k\ times} = \frac{1}{2^{k-1}}mx + \left(1 - \frac{1}{2^{k-1}}\right)x$, where $k \in \mathbb{N}$.*

Proof. We prove this statement by induction on k. When $k = 1$, both sides of the claimed equality are equal to mx. So it is true for $k = 1$. Assume by inductive hypothesis that the statement is true for a natural number k, i.e., $((m\underbrace{x)x)\ldots x}_{k \text{ times}} = \frac{1}{2^{k-1}}mx + \left(1 - \frac{1}{2^{k-1}}\right)x$. Then $((m\underbrace{x)x)\ldots x}_{k+1 \text{ times}} = \left(\frac{1}{2^{k-1}}mx + \left(1 - \frac{1}{2^{k-1}}\right)x\right)x = \left(\frac{1}{2^{k-1}}mx\right)x + \left(1 - \frac{1}{2^{k-1}}\right)x = \frac{1}{2^{k-1}}\left(\frac{1}{2}mx + \frac{1}{2}x\right) + \left(1 - \frac{1}{2^{k-1}}\right)x = \frac{1}{2^k}mx + \frac{1}{2^k}x + \left(1 - \frac{1}{2^{k-1}}\right)x = \frac{1}{2^k}mx + \left(1 - \frac{1}{2^{k-1}} + \frac{1}{2^k}\right)x = \frac{1}{2^k}mx + \left(1 - \frac{1}{2^k}\right)x$, as required. By the principle of mathematical induction, the statement is true for all natural numbers k. $\square$

We have shown that all elements of $A(X)$ having a repetition can be excluded from a basis of $U(X)$. We will show the following.

Theorem 7.4.3. *Unordered binary trees without twin pairs and repetitions form a basis of $U(X)$. In particular, $U(X)$ is infinite-dimensional.*

We will show it after some preparatory work.

Lexicographical order is commonly used to compare elements in various structures, including trees. Now we introduce lexicographical order on trees from $A(X)$ inductively. Recall that if a tree is not a one-vertex, it is a product of two factors. Furthermore, since the magma $A(X)$ is commutative, we will assume throughout that the first factor is greater than or equal to the second factor.

Let us select an arbitrary order on the alphabet X.

Definition 7.4.4. The lexicographical order on $A(X)$ is defined as follows. For $T_1, T_2 \in A(X)$, we have:

(i) if T_1 and T_2 are both one-vertex trees, say $T_1 = x_1$ and $T_2 = x_2$, then $T_1 < T_2$ when x_1 precedes x_2 in the order on X;

(ii) if only one of T_1 and T_2 is a one-vertex tree, then $T_1 < T_2$ when T_1 is the one-vertex tree;

(iii) if both trees are not one-vertex trees, say $T_1 = T_{11}T_{12}$ and $T_2 = T_{21}T_{22}$, then $T_1 < T_2$ when $T_{11} < T_{21}$ or $T_{11} = T_{21}$ and $T_{12} < T_{22}$. (Recall that we assume that $T_{11} \geq T_{12}$ and $T_{21} \geq T_{22}$.)

Recall the definition of the height of a tree we introduced after Definition 6.1.10. We will denote the height of a tree T by $h(T)$. The following lemma shows that the order we introduced on $A(X)$ is compatible with height.

Lemma 7.4.5. *Given two trees T_1 and T_2, if $h(T_1) < h(T_2)$ then $T_1 < T_2$.*

Proof. We prove this statement by induction on the height of T_1. If $h(T_1) = 0$, then T_1 is a one-vertex tree and T_2 is not a one-vertex tree. By the above definition, part (ii),

$T_1 < T_2$.

Assume that $h(T_1) = h > 0$ and that for smaller heights the claim is true. Let $T_1 = T_{11}T_{12}$ and $T_2 = T_{21}T_{22}$. We note that $h(T_1) = \max\{h(T_{11}) + 1, h(T_{12}) + 1\}$. Furthermore, since we assume that $T_{11} \geq T_{12}$, we must have, by the inductive assumption, that $h(T_{11}) \geq h(T_{12})$. Therefore, $h(T_1) = h(T_{11}) + 1$. Similarly, $h(T_2) = h(T_{21}) + 1$. Since $h(T_1) < h(T_2)$, we have that $h(T_{11}) = h(T_1) - 1 = h - 1 < h(T_2) - 1 = h(T_{21})$. Thus, $h(T_{11}) < h(T_{21})$, which gives us, again by the inductive assumption, that $T_{11} < T_{21}$. By the above definition, part (iii), we now obtain that $T_1 < T_2$. □

We now want to extend the order to the set of finite multisets (sets where elements can repeat) of trees. We will assume each multiset of trees is ordered in the descending order.

Definition 7.4.6. The order between multisets of trees is defined recursively as follows. Let S and S' be two different finite multisets of trees, say $S = \{T_1, T_2, \ldots, T_m\}$ and $S' = \{T'_1, T'_2, \ldots, T'_n\}$.

(i) If S is empty and S' is not empty, then $S < S'$.

(ii) If both S and S' are non-empty and $T_1 < T'_1$ then $S < S'$.

(iii) If $T_1 = T'_1$, then $S < S'$ when $S \setminus \{T_1\} < S' \setminus \{T'_1\}$.

Using the above definition, we can define a non-strict order on the set of all linear combinations $C = \sum_{i=1}^{n} \alpha_i T_i$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ are non-zero coefficients and $\{T_1, \ldots, T_n\}$ is a multiset of trees from $A(X)$. (That is, we allow repetitions among T_i.) Namely, we say that $\sum_{i=1}^{n} \alpha_i T_i \leq \sum_{i=1}^{m} \beta_i T'_i$ when $\{T_1, \ldots, T_n\} \leq \{T'_1, \ldots, T'_m\}$.

We will now describe two types of reduction that strictly decrease the order of a linear combination:

(a) **Combining summands.** If the linear combination $C = \sum_{i=1}^{n} \alpha_i T_i$ has repeated terms, i.e., $T_i = T_j$ for some $i < j$ then we can combine these two terms. Namely, if $\alpha_i + \alpha_j \neq 0$ then we substitute the summand $\alpha_i T_i$ with $(\alpha_i + \alpha_j)T_i$ and remove $\alpha_j T_j$ altogether. If $\alpha_i + \alpha_j = 0$ then we simply remove both summands $\alpha_i T_i$ and $\alpha_j T_j$. Clearly, this operation removes a term or two from the multiset involved in this linear combination, and so the order decreases.

(b) **Erasing a repetition**: If the linear combination $C = \sum_{i=1}^{n} \alpha_i T_i$ contains a tree T_i with a repetition, say, involving $x \in X$, we can utilise the relator $(mx)x - \frac{1}{2}mx - \frac{1}{2}x \in I$ and substitute the term $\alpha_i T_i$ with two terms $\frac{1}{2}\alpha_i T_i' + \frac{1}{2}\alpha_i T_i''$, where T_i' is obtained from T_i by shrinking the repetition and T_i'' is obtained by substituting the entire repetition subtree with a leaf labelled with x. (See Figure 7.2.) Note that the second tree, T_i'', may

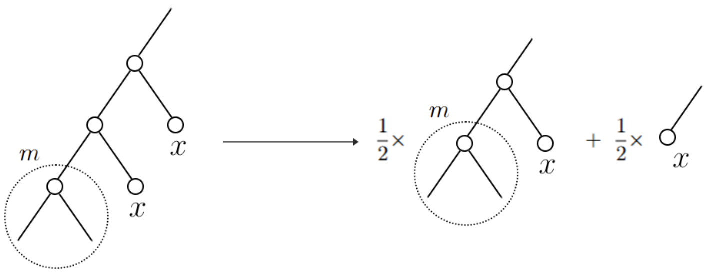


Figure 7.2: Erasing a repetition

end up having a twin pair (when our repetition overlaps with another, higher repetition with the same x), and so this twin pair also needs to be reduced, possibly repeatedly, making the tree even smaller. In any case, since we substitute a larger tree by two smaller ones, the order of the linear combination decreases.

We are going to do these reductions repeatedly until we arrive at a linear combination that cannot be reduced further. In particular, the multiset of terms for this resulting tree is a set (no repeated terms) and each tree involved in the linear combination contains no repetitions. We call such linear combinations *fully reduced*.

Clearly, it is possible that the reductions can be applied to a linear combination in different ways. However, this does not affect the result. Namely, we prove the following.

Proposition 7.4.7. *Any two fully reduced linear combinations obtained by reductions from the same linear combination $C = \sum_{i=1}^{n} \alpha_i T_i$ are equal.*

Proof. We prove it by induction on the non-strict order we introduced on the set of all linear combinations. Suppose by contradiction that the claim is not true and select the smallest counterexample, i.e., a linear combination $C = \sum_{i=1}^{n} \alpha_i T_i$, that can be reduced to two different fully reduced linear combinations, and also such that for every linear combination that is strictly smaller than C the full reduction is unique.

Now, we must have two different sequences of reductions $R_1, R_2, \ldots$ and $R'_1, R'_2, \ldots$ reducing C to two different fully reduced linear combinations. We have different cases depending on the pair (R_1, R'_1). Let, say, C_1 be the linear combination obtained from C after R_1 and C'_1 be the linear combination obtained from C after R'_1. Our strategy in each case will be to reduce C_1 and C'_1 further to the same smaller linear combination C'. If we do so, we will have that the full reductions of C_1 and C'_1 (which are unique since C_1 and C'_1 are smaller than C) must coincide with the full reduction of C' and hence they are the same, which is, clearly, a contradiction.

Before we look at the main cases, let us dispose of some easy possibilities. First of all, if $R_1 = R'_1$ then after this first step we have the same linear combination $C_1 = C' = C'_1$ and we have a contradiction right away. Hence $R_1 \neq R'_1$. Also, if R_1 and R_2 are independent, i.e., they involve distinct terms from C then we also get a contradiction. Indeed, in this case, the result C' of applying R'_1 to C_1 is the same as the result of applying R_1 to C'_1. So we have our common descendant C' and hence a contradiction. Thus, R_1 and R'_1 cannot be independent. Now we look at the main cases.

Case 1*: Both* R_1 *and* R_1' *involve combining terms.* Say, R_1 combines terms $\alpha_i T_i$ with $\alpha_j T_j$ and R_1' combines $\alpha_j T_j$ with $\alpha_s T_s$ (recall that R_1 and R_1' cannot be equal or independent). Then R_1 leaves us with the summands $(\alpha_i + \alpha_j)T_i$ and $\alpha_s T_s$, which can be combined further to form C' with the term $(\alpha_i + \alpha_j + \alpha_s)T_i$ (note that $T_i = T_j = T_s$). Similarly, R_1' leaves us with $\alpha_i T_i$ and $(\alpha_j + \alpha_s)T_j$, which can be combined further to obtain again the same term $(\alpha_i + \alpha_j + \alpha_s)T_i$ and the same C', which clearly leads to a contradiction as above. It may be that $\alpha_i + \alpha_j = 0$, or $\alpha_j + \alpha_s = 0$, or $\alpha_i + \alpha_j + \alpha_s = 0$, but in any case the result after two steps will be the same and yield a contradiction. So this case is impossible.

Case 2*:* R_1 *combines two terms and* R_1' *reduces a repetition in a term.* Let, say, R_1 combines terms $\alpha_i T_i$ with $\alpha_j T_j$ and R_1' reduces a repetition in $\alpha_j T_j$. Let us write T for T_i and T_j, since they are equal. Let R_1' substitute T with $\frac{1}{2}T' + \frac{1}{2}T''$. Then R_1 applied to C gives us the term $(\alpha_i + \alpha_j)T$ and we can apply R_1' to it, getting $\frac{1}{2}(\alpha_i + \alpha_j)T' + \frac{1}{2}(\alpha_i + \alpha_j)T''$. (The resulting linear combination will be C'.) If we, on the other hand, start with R_1' then we obtain the terms $\alpha_i T$ and $\frac{1}{2}\alpha_j T' + \frac{1}{2}\alpha_j T''$. We can apply R_1' to the former, getting $\frac{1}{2}\alpha_i T' + \frac{1}{2}\alpha_i T''$ and $\frac{1}{2}\alpha_j T' + \frac{1}{2}\alpha_j T''$. We can now combine the T' terms and T'' terms and this results in exactly the same result, the linear combination C' involving $\frac{1}{2}(\alpha_i + \alpha_j)T' + \frac{1}{2}(\alpha_i + \alpha_j)T''$. Again, this clearly leads to a contradiction. If $\alpha_i + \alpha_j = 0$ then we will have slightly different results but with the same outcome, the identical C' and a contradiction.

Case 3: Both R_1 *and* R_1' *reduce a repetition.* Say, R_1 reduces a repetition r in $\alpha T := \alpha_i T_i$ and then R_1' reduces a different repetition r' in the same term $\alpha_i T_i$ because R_1 and R_1' cannot be independent. We have several subcases here: (a) repetition r and r' can be overlapping (say, r' just below r); then they will involve the same $x \in X$; (b) r and r' are not overlapping but dependent: say, r' is below r (i.e., the subtree rooted at the top node of r contains r'); and (c) r and r' are independent: r' is not contained in the subtree rooted at the top node of r and, symmetrically, r is not contained in the subtree rooted at the top node of r'.

Let us do these subcases in turn. Assume we are in the situation (a). Then after R_1 and R_1', we have the same terms $\frac{1}{2}\alpha T' + \frac{1}{2}\alpha T''$, where T' is obtained by shrinking (any) one repetition and T'' is obtained by substituting both r and r' with the node x. (If we do R_1' then we get a twin pair and erasing it yields the result.) Thus, we obtain a common descendant C' and a contradiction (see Figure 7.3). Let us now suppose that we are in the situation (b). Say, R_1 involves $x \in X$ and R_1' involves $x' \in X$. Here we will encounter quite a few different trees, so let us arrange them in a systematic way. Let T' be the tree obtained by shrinking r and T'' by substituting r with the leaf x. Let T_* be the tree obtained by shrinking r' in T and T_*' by shrinking r' in T'. Also, let $T_\natural$ be the tree obtained from T by substituting r' with a leaf x' and, similarly, let $T_\natural'$ be the tree obtained from T' by substituting r' with an x'. Then, after R_1, we get $\frac{1}{2}\alpha T' + \frac{1}{2}\alpha T''$ and we follow this by R_1' applied to first summand. This gives us $\frac{1}{4}\alpha T_*' + \frac{1}{4}\alpha T_\natural' + \frac{1}{2}\alpha T''$ (see Figure 7.4). This will be C'. On the other hand, if we start with R_1' then we get $\frac{1}{2}\alpha T_* + \frac{1}{2}\alpha T_\natural$. We apply R_1 to the both terms to get $\frac{1}{4}\alpha T_*' + \frac{1}{4}\alpha T'' + \frac{1}{4}\alpha T_\natural' + \frac{1}{4}\alpha T''$. We now combine two equal terms to get exactly the same linear combination C' (see Figure 7.5). Thus this again leads to a contradiction. In the final subcase (c), we encounter even more (nine) different trees (see Figures 7.6 and 7.7). We skip the exact calculations, but the result is the same: we can reduce to the same linear combination C' after both R_1 and R_1', and so we have a contradiction, since we will then have the same fully reduced result. This completes the proof. □

This proposition tells us that for every linear combination C, there is a unique full reduction of C, which we will denote by $FR(C)$. In particular, this is true for all $T \in A(X)$ and,

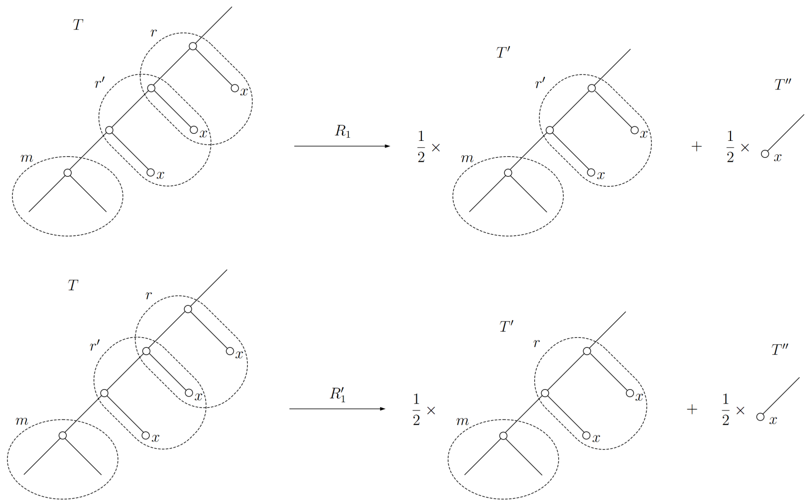


Figure 7.3: Case 3(a)

more generally, for all elements of $\mathbb{F}[A(X)]$.

Let J be the subspace of $\mathbb{F}[A(X)]$ spanned by all vectors $FR(T) - T$, $T \in A(X)$. It is quite clear that the operator FR is linear, so J contains $FR(u) - u$ for all $u \in \mathbb{F}[A(X)]$.

Proposition 7.4.8. *The subspace J is an ideal of* $\mathbb{F}[A(X)]$.

Proof. It suffices to show that $(FR(T) - T)T'$ is contained in J for all $T, T' \in A(X)$. Note that TT' can be reduced to $FR(T)T'$ if we do reductions only within the subtree T (or subtrees that were derived from T). This means that $C := FR(TT') = FR(FR(T)T')$. Now, $(FR(T) - T)T' = FR(T)T' - TT' = -(C - FR(T)T') + (C - TT') = -(FR(FR(T)T') - FR(T)T') + (FR(TT') - TT') \in J$. □

Finally, we can prove Theorem 7.4.3.

Proof. Clearly, all generating relators $(mx)x - \frac{1}{2}mx - \frac{1}{2}x$ are contained in J, as these are simply one-step reductions of trees. On the other hand, every reduction is either collecting terms (i.e., linear algebra) or adding a multiple of one of these relators. It follows that J is exactly the ideal spanned by all the relators, i.e., $U(X) = \mathbb{F}[A(X)]/J$.

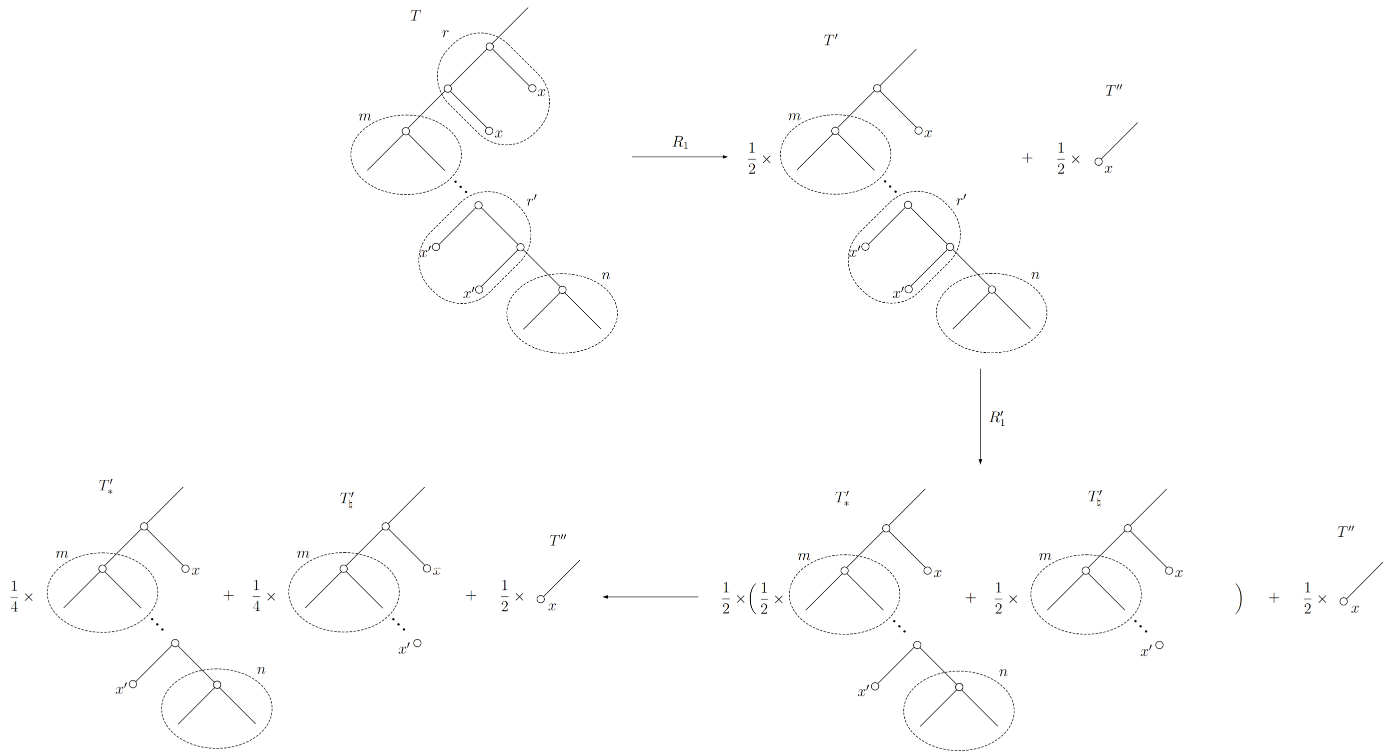


Figure 7.4: Case 3(b)(i)

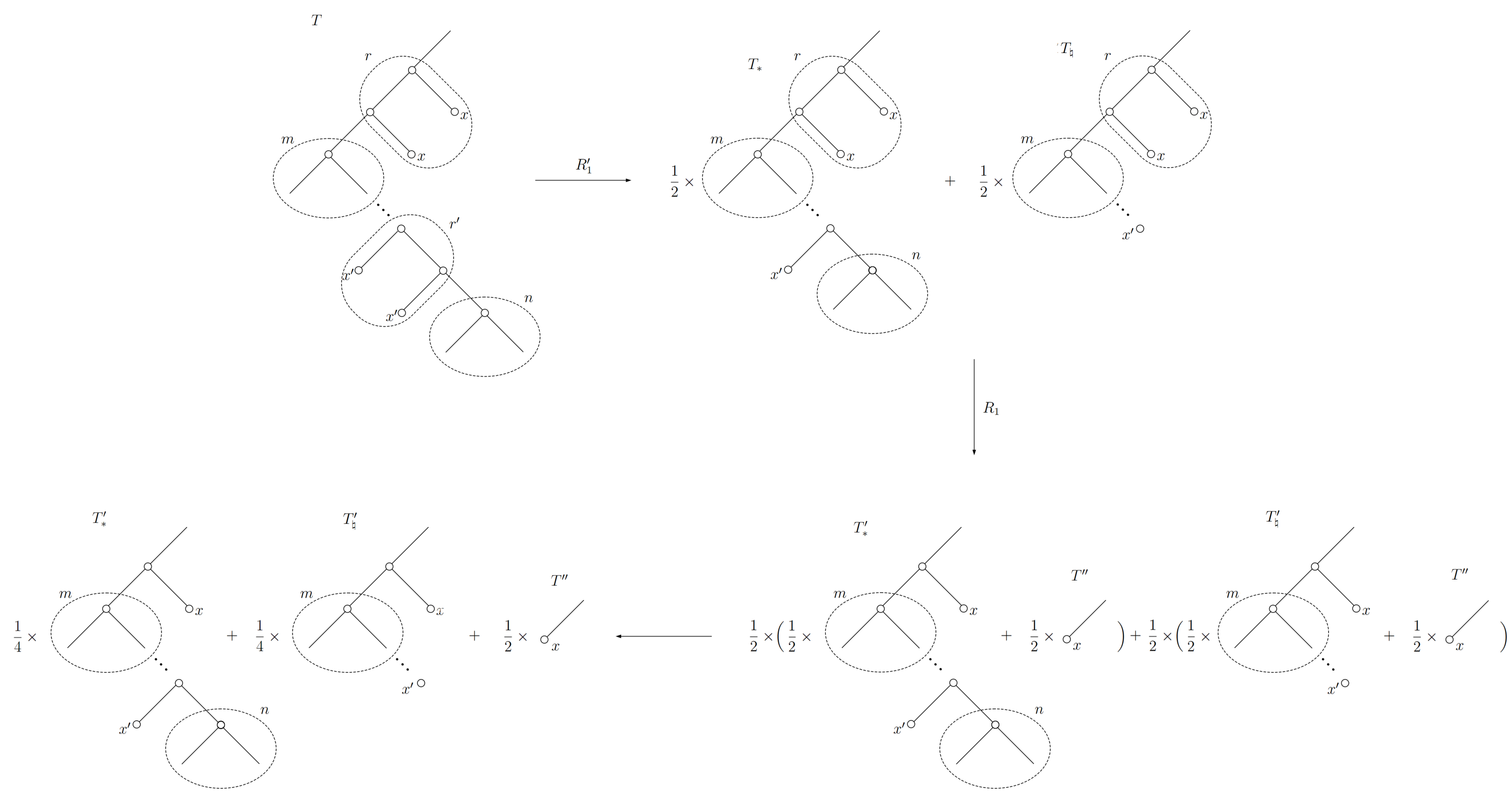


Figure 7.5: Case 3(b)(ii)

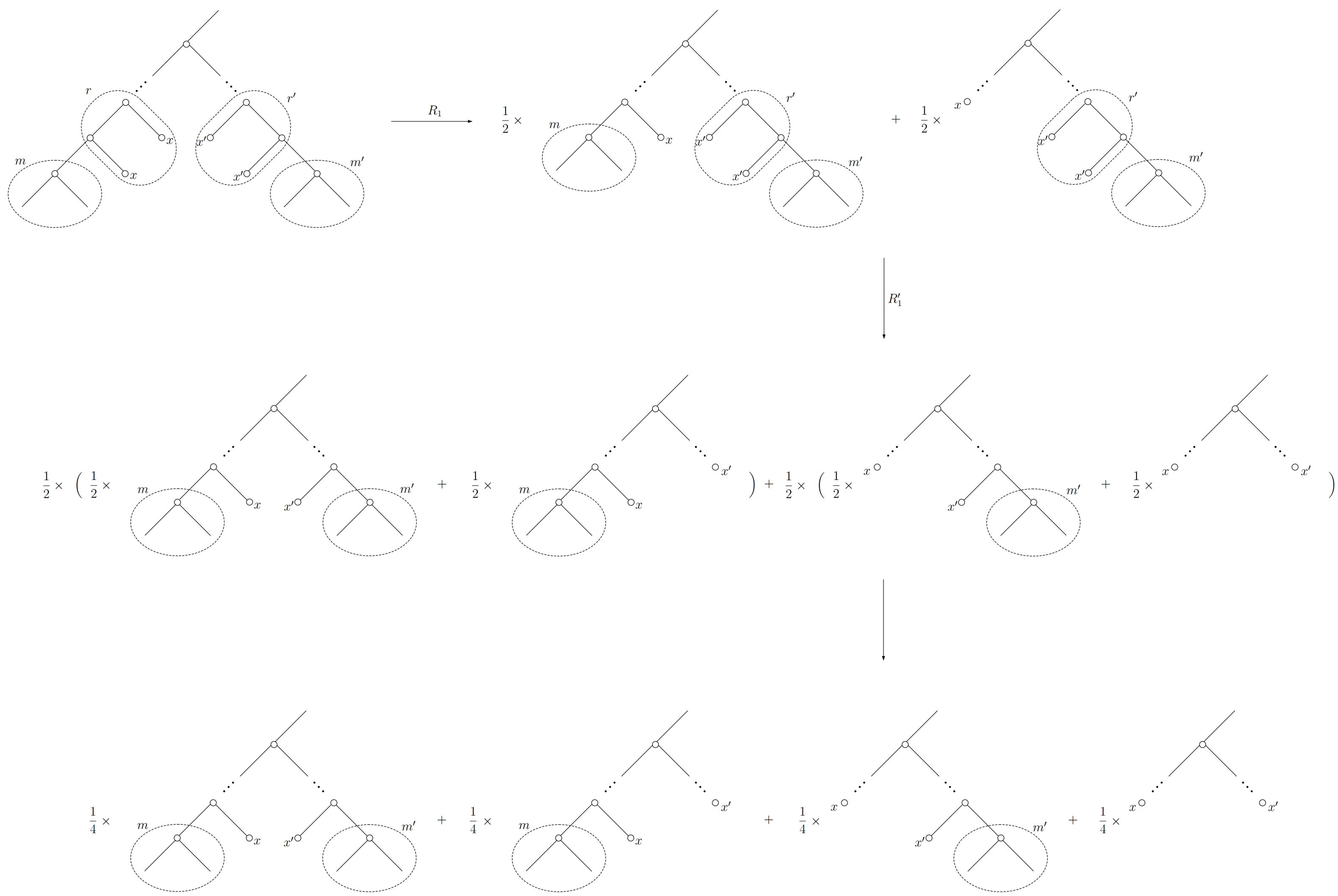


Figure 7.6: Case 3(c)(i)

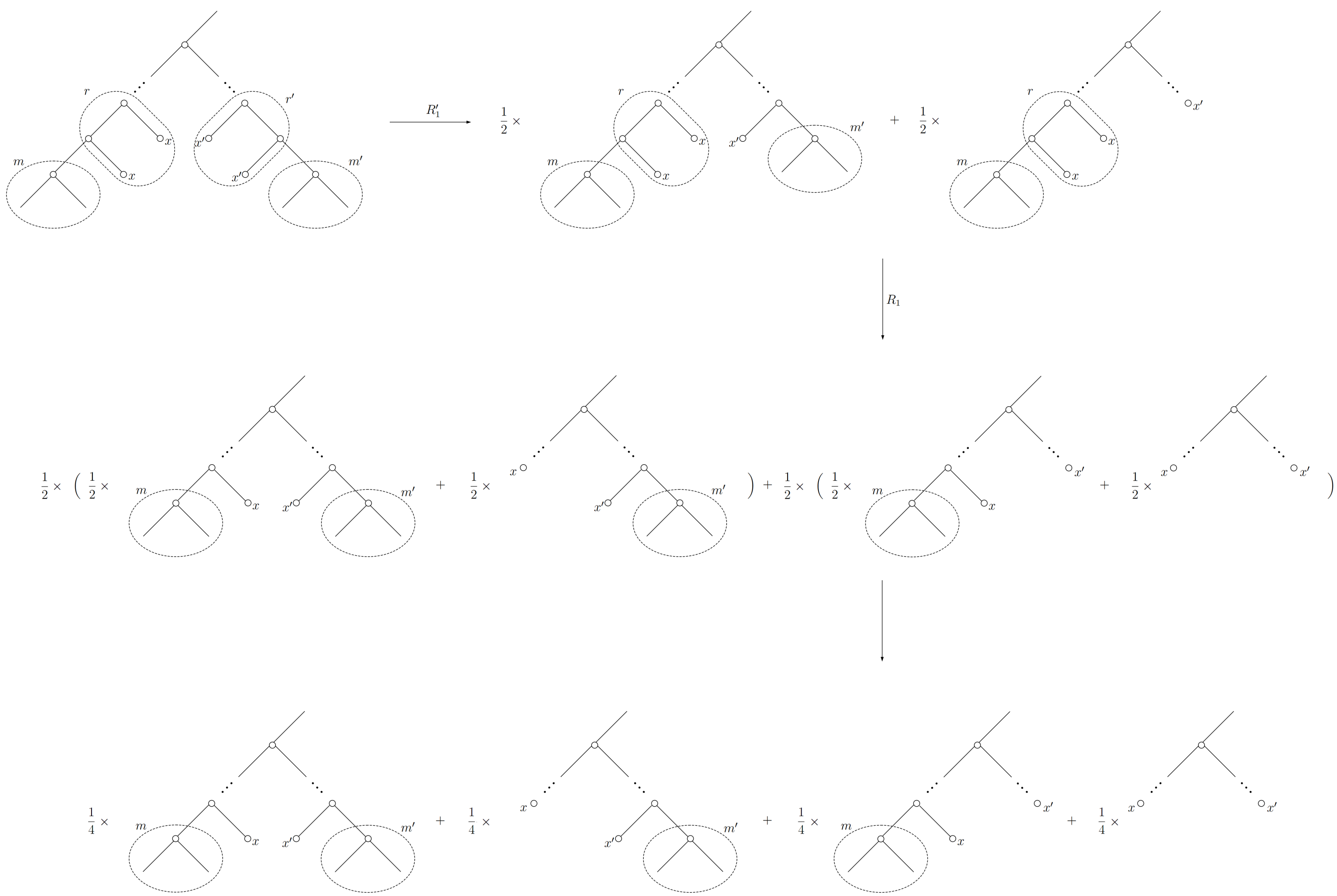


Figure 7.7: Case 3(c)(ii)

For the second claim, the elements $FR(T) - T$ of the spanning set for J are zero when T is fully reduced. The remaining non-zero terms are linearly independent, because they all contain different highest term T. Hence reducing modulo J simply eliminates all such non-fully-reduced T from the basis of the algebra. This means that the basis of $U(X)$ consists of the images of all fully reduced trees T, i.e., unordered binary trees without twins and without repetitions. □

7.5 Universal baric algebra of Jordan type $\frac{1}{2}$

We have seen above how the universal algebra $U(X)$ encodes the three target properties: primitivity of axes from X, spectrum $\left\{1, 0, \frac{1}{2}\right\}$, and the baric property. In this final section, we derive the additional relators that ensure that $x \in X$ satisfy the fusion law of Jordan type $\frac{1}{2}$. A way to do it was described in [17].

If we compare the fusion law of Jordan type $\frac{1}{2}$ in Table 2.1(b) with the fusion law satisfied by $U(X)$ (see the table in Theorem 7.3.3), we know that we only need to enforce the fusion rules $0 * 0 = \{0\}$, $0 * \frac{1}{2} = \left\{\frac{1}{2}\right\}$, and $\frac{1}{2} * \frac{1}{2} = \{0\}$.

Recall that in Lemma 7.3.1 we wrote expressions for the components $m_0 = -2mx + m + x$ and $m_{\frac{1}{2}} = 2mx - 2x$ of an arbitrary element $m \in A(X)$. Since the image of $A(X)$ spans $U(X)$, the components m_0 span $U_0(x)$ and the components $m_{\frac{1}{2}}$ span $U_{\frac{1}{2}}(x)$. The relators we record that $m_0 m'_0 \in U_0(x)$, $m_0 m'_{\frac{1}{2}} \in U_{\frac{1}{2}}(x)$, and $m_{\frac{1}{2}} m'_{\frac{1}{2}} \in U_0(x)$ for all $m, m' \in A(X)$. How can we recognise when a given element lies in $U_0(x)$ or $U_{\frac{1}{2}}(x)$? Clearly, we have the following.

Lemma 7.5.1. *For $u \in U(X)$, we have that*

(i) $u \in U_0(x)$ if and only if $ux = 0$;

(ii) $u \in U_{\frac{1}{2}}(x)$ *if and only if* $ux - \frac{1}{2}u = 0$.

Let K be the ideal in $U(X)$ generated by all relators of the form (a) $(m_0 m'_0)x$; (b) $(m_0 m'_{\frac{1}{2}})x - \frac{1}{2}m_0 m'_0$; and (c) $(m_{\frac{1}{2}} m'_{\frac{1}{2}})x$, for all $m, m' \in A(X)$. Let $B(X) = U(X)/K$.

Then we have the following.

Theorem 7.5.2. *The algebra* $B(X)$ *is the universal* k*-generated algebra of Jordan type* $\frac{1}{2}$.

The proof is straightforward and we omit it. Note that these additional relators are much more complicated than the ones we encountered before. While the above theorem gives us a realisation of the universal k-generated baric algebra of Jordan type $\frac{1}{2}$, it is not well suited for calculations. Also, the complexity of the problem is underscored by the expected finiteness of the dimension of $B(X)$ for each finite X.

In Chapter 9, we develop a different realisation of this universal algebra, inspired by some ideas from the paper by De Medts, Rowen and Segev [7]. This new realisation proves to be much nicer computationally, and in particular, it allows us to determine $B(X)$ in the case $k = |X| = 4$.

CHAPTER 8

AN EXAMPLE OF BARIC ALGEBRAS OF JORDAN TYPE $\frac{1}{2}$

In this short chapter, we present an example of a series of baric algebras of Jordan type $\frac{1}{2}$ with an increasing number of generators.

8.1 An example

Take an n-dimensional vector space V and define a commutative algebra $A = \langle a \rangle \oplus V$ with multiplication given by $a^2 = a$, $av = \frac{1}{2}v$ and $uv = 0$ for all $u, v \in V$. We write general elements of A as $\lambda a + u$ where $\lambda \in \mathbb{F}$ and $u \in V$.

What are the idempotents in the algebra A? Consider an arbitrary element $x = \lambda a + u \in A$.

Lemma 8.1.1. *We have that $x^2 = \lambda x$.*

Proof. Indeed, $x^2 = (\lambda a + u)^2 = \lambda^2 a^2 + 2\lambda au + u^2 = \lambda^2 a + 2\lambda(\frac{1}{2}u) + 0 = \lambda^2 a + \lambda u = \lambda(\lambda a + u) = \lambda x$. □

Corollary 8.1.2. *We have that $x = \lambda a + u \in A$ is an idempotent if and only if either $x = 0$ or $\lambda = 1$.*

So the non-zero idempotents of A are all the vectors $a + u$, $u \in V$.

Lemma 8.1.3. *Fix $u \in V$. Define the linear map $\varphi : A \to A$ via $\varphi(a) = a + u$ and $\varphi(v) = v$ for $v \in V$. Then φ is an automorphism of A.*

Proof. Indeed, there is such a linear map as it can be defined on the basis of A consisting of a and a basis of V. Let us show that φ is an automorphism of A. Take $x = \lambda a + v$ and $y = \mu a + w$ in A. We have

$$\begin{aligned}\varphi(xy) &= \varphi(\lambda\mu a + \lambda a w + \mu a v + wv)\\ &= \varphi(\lambda\mu a + \tfrac{\lambda}{2}w + \tfrac{\mu}{2}v)\\ &= \lambda\mu\varphi(a) + \tfrac{\lambda}{2}\varphi(w) + \tfrac{\mu}{2}\varphi(v)\\ &= \lambda\mu(a+u) + \tfrac{\lambda}{2}w + \tfrac{\mu}{2}v.\end{aligned}$$

Also,

$$\begin{aligned}\varphi(x)\varphi(y) &= \varphi(\lambda a + v)\varphi(\mu a + w)\\ &= (\lambda\varphi(a) + \varphi(v))(\mu\varphi(a) + \varphi(w))\\ &= (\lambda(a+u) + v)(\mu(a+u) + w)\\ &= \lambda\mu(a+u) + \lambda(a+u)w + \mu(a+u)v + vw\\ &= \lambda\mu(a+u) + \lambda a w + \mu a v\\ &= \lambda\mu(a+u) + \tfrac{\lambda}{2}w + \tfrac{\mu}{2}v.\end{aligned}$$

So $\varphi(xy) = \varphi(x)\varphi(y)$. Thus, φ is a homomorphism. Note that

$$\begin{aligned}
\ker\varphi &= \{x \in A \mid \varphi(x) = 0\} \\
&= \{\lambda a + v \in A \mid \varphi(\lambda a + v) = 0\} \\
&= \{\lambda a + v \in A \mid \lambda\varphi(a) + \varphi(v) = 0\} \\
&= \{\lambda a + v \in A \mid \lambda(a + u) + v = 0\} \\
&= \{\lambda a + v \in A \mid \lambda a + (\lambda u + v) = 0\} \\
&= \{\lambda a + v \in A \mid \lambda a = 0 \text{ and } \lambda u + v = 0\} \\
&= \{\lambda a + v \in A \mid \lambda = 0 \text{ and } v = 0\} \\
&= \{0\}.
\end{aligned}$$

So φ is injective. By the rank-nullity theorem, we have that $\dim \operatorname{im} \varphi = \dim A - \dim \ker \varphi = \dim A$. So $\operatorname{im} \varphi = A$. This means that φ is bijective. Therefore, $\varphi : A \to A$ is an automorphism. □

Since φ takes a to $a + u$ and u is arbitrary in V, the group of automorphisms of A acts transitively on the set of non-zero idempotents. Therefore, all non-zero idempotents satisfy the same fusion law and so we take all of them as the axes of our axial algebra A. Let us now determine this fusion law. We can focus on a particular axis. Namely, we choose the axis $a = a + 0$.

Lemma 8.1.4. *The eigenvalues of* ad_a *are* 1 *and* $\frac{1}{2}$*. Furthermore,* $A_1(a) = \langle a \rangle$ *and* $A_{\frac{1}{2}}(a) = V$*.*

Proof. Clearly, $A_1(a)$ contains $\langle a \rangle$, and so the dimension of $A_1(a)$ is at least one. Since $av = \frac{1}{2}v$ for every $v \in V$, we have that $V \subseteq A_{\frac{1}{2}}(a)$ which implies that the dimension of $A_{\frac{1}{2}}(a)$ is at least $\dim V$. Since the dimension of A is $1 + \dim V$ and different eigenspaces intersect trivially, we have that in fact $A_1(a) = \langle a \rangle$ and $A_{\frac{1}{2}}(a) = V$. Moreover, there are no further eigenvalues. $\square$

Next, we check the fusion law. We see that $A_1(a)A_1(a) = \langle a \rangle \langle a \rangle = \langle a^2 \rangle = \langle a \rangle$. Thus, $1 * 1 = \{1\}$. Also, $A_1(a)A_{\frac{1}{2}}(a) = \langle a \rangle V = V = A_{\frac{1}{2}}(a)$ which means that $1 * \frac{1}{2} = \{\frac{1}{2}\}$. Similarly, $\frac{1}{2} * 1 = \{\frac{1}{2}\}$. Finally, $A_{\frac{1}{2}}(a)A_{\frac{1}{2}}(a) = VV = 0$. So $\frac{1}{2} * \frac{1}{2} = \emptyset$.

We summarise this as follows.

Lemma 8.1.5. *The algebra A is an axial algebra for the fusion law in the following table.*

$*$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

This is the sub fusion law of the fusion law $\mathcal{J}(\frac{1}{2})$, where the eigenvalue 0 is removed. So our algebra A is in fact an algebra of Jordan type $\frac{1}{2}$. More precisely, we have the following lemma.

Lemma 8.1.6. *The algebra A is a Jordan algebra.*

Proof. Note that A is commutative by definition. Take $x = \lambda a + u$ and $y = \mu a + v$ be arbitrary elements of A. We have shown earlier that $x^2 = \lambda x$. Therefore,

$$(xy)x^2 = (xy)(\lambda x) = \lambda(xy)x = \lambda x(xy) = \lambda x(yx) = x(\lambda yx) = x(y(\lambda x)) = x(yx^2).$$

Since the Jordan identity is satisfied, A is a Jordan algebra. $\square$

Since $AV = (\langle a \rangle \oplus V)V = \langle a \rangle V + VV = V + 0 = V$, we see that V is an ideal. Furthermore, V contains no axes. So V is contained in the radical of A. Note that the codimension of V is 1. So V is the radical of A. This shows that A is a baric algebra of Jordan type $\frac{1}{2}$.

Let us see what the weight function is in this algebra.

Lemma 8.1.7. *The map $\psi : A \to \mathbb{F}$ sending each vector $\lambda a + u \in A$ to λ is an algebra homomorphism.*

Proof. Let us show that ψ is linear. Take $\lambda a + u, \mu a + v \in A$ and $\alpha \in \mathbb{F}$. We have that $\psi(\alpha(\lambda a+u)+(\mu a+v)) = \psi((\alpha\lambda a+\alpha u)+(\mu a+v)) = \psi((\alpha\lambda+\mu)a+\alpha u+v) = \alpha\lambda+\mu$ and $\alpha\psi(\lambda a + u) + \psi(\mu a + v) = \alpha\lambda + \mu$. So $\psi(\alpha(\lambda a + u) + (\mu a + v)) = \alpha\psi(\lambda a + u) + \psi(\mu a + v)$. Thus, ψ is linear.

Next, we show that ψ is a homomorphism. We have that $\psi((\lambda a + u)(\mu a + v)) = \psi(\lambda\mu a + \lambda av + \mu au + uv) = \psi(\lambda\mu a + \frac{\lambda}{2}v + \frac{\mu}{2}u) = \lambda\mu = \psi(\lambda a + u)\psi(\mu a + v)$. Thus, ψ is a homomorphism. $\square$

Clearly, $\ker\psi = V$ and so this ψ is the required weight function of the baric algebra A. What is the smallest number of generators for A? Clearly, by transitivity of the automorphism group of A on the set of axes, we can assume that a is one of our generating axes. It is clear from the definition of A that for every subspace $U \subseteq V$, we have that $B = \mathbb{F} \oplus U$ is a subalgebra of the algebra $A = \mathbb{F} \oplus V$. This suggests the following result.

Lemma 8.1.8. *Suppose that $X = \{a, a + u_1, a + u_2, \ldots, a + u_k\}$. Then $\langle\langle X\rangle\rangle = \langle a, u_1, u_2, \ldots, u_k\rangle = \mathbb{F} \oplus U$, where $U = \langle u_1, u_2, \ldots, u_k\rangle$.*

Proof. Clearly, $\langle a, u_1, u_2, \ldots, u_k\rangle \subseteq \langle\langle X\rangle\rangle$. On the other hand, the left side is closed under the algebra multiplication, i.e., it is a subalgebra. Also, it contains X, so we have equality. $\square$

In particular, this tells us that we get the smallest number of generators for A, when $\{u_1, u_2, \ldots, u_k\}$ is a basis for U. This gives us the following final result.

Corollary 8.1.9. *The algebra A is $(n + 1)$-generated as an axial algebra.*

So here we indeed see a sequence of baric algebras of Jordan type $\frac{1}{2}$ with an increasing number of generators.

We conclude this chapter with the following interesting conjecture.

Conjecture 8.1.10. *The algebra A above is the universal $(n+1)$-generated algebra for the fusion law in Lemma 8.1.5.*

CHAPTER 9

UNIVERSAL BARIC ALGEBRA OF JORDAN TYPE $\frac{1}{2}$

In this chapter, we develop an algorithm that allows us to construct the universal k-generated baric algebra of Jordan type $\frac{1}{2}$ and we apply it in the case $k = 4$. However, first we need to prove a lemma that is at the core of our new approach.

9.1 A lemma

In the following lemma, we show how to compute the product of two axes in a baric algebra A of Jordan type $\frac{1}{2}$.

Lemma 9.1.1. *Suppose that $y, z \in A$ are two primitive axes. Then $yz = y + \frac{1}{4}z - \frac{1}{4}z^{\tau_y}$.*

Proof. Write $z = y + z_0 + z_{\frac{1}{2}}$, where $z_0 \in A_0(y)$ and $z_{\frac{1}{2}} \in A_{\frac{1}{2}}(y)$. Note that $(y, z) = 1$, since A is baric, and so the projection of z to $A_1(y) = \langle y \rangle$ is indeed y. We have that $z^{\tau_u} = y + z_0 - z_{\frac{1}{2}}$, because τ_y acts as minus identity on $A_{\frac{1}{2}}(y)$. This allows us to express

$z_{\frac{1}{2}}$ as

$$z_{\frac{1}{2}} = \tfrac{1}{2}(z - z^{\tau_y}).$$

We now obtain $yz = y(y + z_0 + z_{\frac{1}{2}}) = y + 0 + \frac{1}{2} z_{\frac{1}{2}} = y + \frac{1}{2}(\frac{1}{2}(z - z^{\tau_y})) = y + \frac{1}{4} z - \frac{1}{4} z^{\tau_y}$, as claimed. □

This formula will play a key role below, but first we define a suitable candidate for the universal Miyamoto group.

9.2 The group

Let X be a finite alphabet of size $|X| = k$. Consider the group $G = G(X)$ given by the presentation

$$G(X) = \langle X \mid x^2 = 1, x \in X \rangle,$$

that is, G is isomorphic to the free product of k copies of the cyclic group of order 2. We write each element $g \in G$ uniquely in the shortest way as $g = x_1 x_2 \dots x_m$, where $(x_1, \dots, x_m) \in X^m$ is a non-repeating sequence, i.e., adjacent elements from this sequence are distinct.

Let us now discuss the set Y of involutions, i.e., elements of order 2, in G. Kurosh's Theorem implies that every such element is conjugate to a unique element of X. Consider an involution $y = x_0^g \in Y$, where $x_0 \in X$ and $g = x_1 x_2 \dots x_m$ is in the shortest form, i.e., $(x_1, x_2, \dots, x_m)$ is non-repeating. If $x_0 \neq x_1$ then $y = g^{-1} x_0 g = x_m x_{m-1} \dots x_1 x_0 x_1 x_2 \dots x_m$ is in the shortest form, as the sequence clearly has no repetitions. If $x_0 = x_1$ then $y = g^{-1} x_0 g = x_m x_{m-1} \dots x_1 x_0 x_1 x_2 \dots x_m = x_m x_{m-1} \dots x_2 x_0 x_2 \dots x_m$, as $x_1 x_0 x_1 = x_0^3 = x_0$. The final expression for y is again non-repeating and hence it is the unique shortest representation of y.

We will use the notation inspired by [7]: instead of x^g, we will write $x[g]$. Then, from the above, we have that every involution in G has a unique shortest representation as $x_0[x_1x_2\dots x_m]$ for a non-repeating sequence $(x_0, x_1, \dots, x_m)$. We will also use this notation more broadly, for arbitrary involutions x (not necessarily) in X.

In this respect, note the following rules for dealing with expressions $x[g]$.

Lemma 9.2.1. *If $y, z \in Y$ and $g, h \in G$, then*

(i) $(x[g])[h] = x[gh]$*; and*

(ii) $y[g][z[g]] = y[zg]$.

Proof. The first claim is clear, since $(x[g])[h] = (x^g)^h = x^{gh} = x[gh]$. Also, we have that $y[g][z[g]] = y[g][g^{-1}zg] = y[gg^{-1}zg] = y[zg]$. □

9.3 The algebra

Define a new algebra $\hat{A} = \hat{A}(X)$ over $\mathbb{F}$ with the basis Y and the product on $\hat{A}$ given by $y \cdot z = y + \frac{1}{4}z - \frac{1}{4}z[y]$. (This is inspired by the formula from Lemma 9.1.1.) This is an infinite-dimensional non-commutative algebra.

Proposition 9.3.1. *The natural action of $g \in G$ by conjugation on Y extends by linearity to an automorphism of $\hat{A}$.*

Proof. It suffices to check that this action preserves the product of basis vectors. Let $y, z \in Y$. Then $(y \cdot z)^g = (y + \frac{1}{4}z - \frac{1}{4}z[y])^g = y[g] + \frac{1}{4}z[g] - \frac{1}{4}z[y][g] = y[g] + \frac{1}{4}z[g] - \frac{1}{4}z[yg] = y[g] + \frac{1}{4}z[g] - \frac{1}{4}z[g][y[g]] = y[g] \cdot z[g] = y^g \cdot z^g$, where we used Lemma 9.2.1 with y and z swapped. □

Thus, G acts on $\hat{A}$ by conjugation.

Note that the elements $y \in Y$ are idempotents in $\hat{A}$, because $y \cdot y = y + \frac{1}{4}y - \frac{1}{4}y[y] = y + \frac{1}{4}y - \frac{1}{4}y = y$. Next, we investigate the fusion law satisfied by the left adjoint of y.

For $z \in Y$, let $z_0 = -y + \frac{1}{2}(z + z[y])$ and $z_{\frac{1}{2}} = \frac{1}{2}(z - z[y])$. For $\lambda \in \mathbb{F}$, we denote by $\hat{A}_\lambda(y)$ the λ-eigenspace of the left adjoint map of y; that is, $\hat{A}_\lambda(y) = \{u \in \hat{A} \mid y \cdot u = \lambda u\}$.

Lemma 9.3.2. *Let $y, z \in Y$. Then*

(i) $z = y + z_0 + z_{\frac{1}{2}}$*;*

(ii) $y \cdot z_0 = 0$*, i.e.,* $z_0 \in \hat{A}_0(y)$*;*

(iii) $y \cdot z_{\frac{1}{2}} = \frac{1}{2}z_{\frac{1}{2}}$*, i.e.* $z_{\frac{1}{2}} \in \hat{A}_{\frac{1}{2}}(y)$*.*

Proof. For (i), we have that $y + z_0 + z_{\frac{1}{2}} = y - y + \frac{1}{2}(z + z[y]) + \frac{1}{2}(z - z[y]) = \frac{1}{2}z + \frac{1}{2}z = z$, as required. For (ii), we have that $y \cdot z_0 = y \cdot (-y + \frac{1}{2}(z + z[y])) = -y + \frac{1}{2}y \cdot z + \frac{1}{2}y \cdot z[y] = -y + \frac{1}{2}(y + \frac{1}{4}z - \frac{1}{4}z[y]) + \frac{1}{2}(y + \frac{1}{4}z - \frac{1}{4}z[y])[y] = -y + \frac{1}{2}y + \frac{1}{8}z - \frac{1}{8}z[y] + \frac{1}{2}y + \frac{1}{8}z[y] - \frac{1}{8}z = 0$. So $z_0 \in \hat{A}_0(y)$. Finally, we have that $y \cdot z_{\frac{1}{2}} = y \cdot (\frac{1}{2}z - \frac{1}{2}z[y]) = \frac{1}{2}(y + \frac{1}{4}z - \frac{1}{4}z[y] - y[y] - \frac{1}{4}z[y] + \frac{1}{4}z[y][y]) = \frac{1}{2}(\frac{1}{2}(z - z[y])) = \frac{1}{2}z_{\frac{1}{2}}$. This means that $z_{\frac{1}{2}} \in \hat{A}_{\frac{1}{2}}(y)$. □

It follows from the above lemma, that since $\hat{A} = \langle Y \rangle$ and $Y \subseteq \langle y \rangle \oplus \hat{A}_0(y) \oplus \hat{A}_{\frac{1}{2}}(y)$, we have that $\hat{A} = \hat{A}_1(y) \oplus \hat{A}_0(y) \oplus \hat{A}_{\frac{1}{2}}(y)$, where $\hat{A}_1(y) = \langle y \rangle$. We also obtain from this lemma that $\hat{A}_0(y) = \langle z_0 \mid z \in Y \rangle$ and $\hat{A}_{\frac{1}{2}}(y) = \langle z_{\frac{1}{2}} \mid z \in Y \rangle$.

Now recall that y is an involution in G and so it also acts on $\hat{A}$ by conjugation.

Lemma 9.3.3. *The involution $y \in Y$ acts as identity on $\hat{A}_1(y) \oplus \hat{A}_0(y)$ and as minus identity on $\hat{A}_{\frac{1}{2}}(y)$.*

Proof. Since $\hat{A}_1(y) = \langle y \rangle$, clearly y acts as identity on $\hat{A}_1(y)$. Also, $\hat{A}_0(y) = \langle z_0 \mid z \in Y \rangle$

and $z_0[y] = (-y + \frac{1}{2}(z + z[y]))[y] = -y[y] + \frac{1}{2}(z[y] + z[y][y]) = -y + \frac{1}{2}(z[y] + z) = z_0$, which means that y also acts as identity on $\hat{A}_0(y)$. This shows the first claim. Similarly, we have that y acts as minus identity on $\hat{A}_{\frac{1}{2}}(y) = \langle z_{\frac{1}{2}} \mid z \in Y \rangle$, because $z_{\frac{1}{2}}[y] = \frac{1}{2}(z - z[y])[y] = \frac{1}{2}(z[y] - z[y][y]) = \frac{1}{2}(z[y] - z) = -z_{\frac{1}{2}}$. □

We next demonstrate the baric property of $\hat{A}$.

Lemma 9.3.4. *The function $w : \hat{A} \to \mathbb{F}$, given by $\sum_{z\in Y} \alpha_z z \mapsto \sum_{z\in Y} \alpha_z$, is an algebra homomorphism.*

Proof. Let $\hat{x}, \hat{z} \in \hat{A}$. Then $\hat{x} = \sum_{x\in Y} \alpha_x x$ and $\hat{z} = \sum_{z\in Y} \beta_z z$ for suitable coefficients $\alpha_x, \beta_z \in \mathbb{F}$. (As usual, in a linear combination, only a finite number of terms can be non-zero.) Clearly, w is linear. We have that

$$\begin{aligned} w(\hat{x} \cdot \hat{y}) &= w((\sum_{x\in Y} \alpha_x x) \cdot (\sum_{z\in Y} \beta_z z)) \\ &= \sum_{x\in Y} \alpha_x \sum_{z\in Y} \beta_z w(x \cdot z) \\ &= \sum_{x\in Y} \alpha_x \sum_{z\in Y} \beta_z w(x + \tfrac{1}{4}z - \tfrac{1}{4}z[x]) \\ &= (\sum_{x\in Y} \alpha_x)(\sum_{z\in Y} \beta_z) \\ &= w(\hat{x})w(\hat{y}), \end{aligned}$$

since $w(x + \frac{1}{4}z - \frac{1}{4}z[x]) = 1 + \frac{1}{4} - \frac{1}{4} = 1$. Therefore, w is an algebra homomorphism. □

Note that the axes of $\hat{A}$ are the elements $z \in Y$ and $w(z) = 1$. So w is indeed a baric weight on the algebra $\hat{A}$. Now we consider the baric radical of the algebra. Let $\hat{R} = \ker w$ be the baric radical of $\hat{A}$.

Recall that we fixed $y \in Y$.

Lemma 9.3.5. $\hat{R} = \hat{A}_0(y) \oplus \hat{A}_{\frac{1}{2}}(y)$.

Proof. First, note that for $z \in Z$, we have that $w(z_0) = w(-y + \frac{1}{2}(z + z[y])) = -w(y) + \frac{1}{2}(w(z)+w(z[y])) = -1+1 = 0$ and $w(z_{\frac{1}{2}}) = w(\frac{1}{2}(z-z[y])) = \frac{1}{2}(w(z)-w(z[y])) = \frac{1}{2}(1-1) = 0$. So $z_0, z_{\frac{1}{2}} \in \hat{R}$ for all $z \in Y$. Thus, $\hat{A}_0(y), \hat{A}_{\frac{1}{2}}(y) \subseteq \hat{R}$. Since $\hat{A} = \hat{A}_1(y) \oplus \hat{A}_0(y) \oplus \hat{A}_{\frac{1}{2}}(y)$ and $\hat{A}_1(y) = \langle y \rangle$, we have that $\hat{A}_0(y) \oplus \hat{A}_{\frac{1}{2}}(y)$ is of codimension 1 in $\hat{A}$. By the rank-nullity theorem, the codimension of $\hat{R} = \ker w$ is clearly 1. Hence, $\hat{R} = \hat{A}_0(y) \oplus A_{\frac{1}{2}}(y)$. $\square$

We are finally prepared to state the fusion law for the left adjoint action of y.

Proposition 9.3.6. *The left adjoint action of $y \in Y$ satisfies the following fusion law:*

$*$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0	0	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

Proof. Let $a' \in \hat{A}_1(y) = \langle y \rangle$. Then $y' = \alpha y$ for some $\alpha \in \mathbb{F}$. Let $\lambda \in \{1, 0, \frac{1}{2}\}$. Take $v \in \hat{A}_\lambda(y)$. Then $y'v = (\alpha y)v = \alpha(yv) = \alpha\lambda v$, which is either in $\hat{A}_\lambda(y)$ when $\lambda \neq 0$ or is zero when $\lambda = 0$. This means that the action of y satisfies the fusion rules $1 * 1 = \{1\}$, $1 * 0 = \emptyset$ and $1 * \frac{1}{2} = \{\frac{1}{2}\}$.

By Lemma 9.3.3, we know that the algebra $\hat{A}$ is C_2-graded, with $\hat{A}_+ = \hat{A}_1(y) \oplus \hat{A}_0(y)$ and $\hat{A}_- = \hat{A}_{\frac{1}{2}}(y)$. This means that $\hat{A}_+\hat{A}_+$ and $\hat{A}_-\hat{A}_-$ are contained in $\hat{A}_+$ and $\hat{A}_+\hat{A}_-$ and $\hat{A}_-\hat{A}_+$ are contained in $\hat{A}_-$. Combining this with the fact that $\hat{A}$ is baric, we deduce the remaining fusion rules $0 * 1 \subseteq \{0\}$, $0 * 0 \subseteq \{0\}$, $\frac{1}{2} * \frac{1}{2} \subseteq \{0\}$, $0 * \frac{1}{2} \subseteq \{\frac{1}{2}\}$, $\frac{1}{2} * 1 \subseteq \{\frac{1}{2}\}$, and $\frac{1}{2} * 0 \subseteq \{\frac{1}{2}\}$. $\square$

Note that the above fusion law is very much like the baric version of the fusion law $\mathcal{J}(\frac{1}{2})$, except $0 * 1 \neq \emptyset$. The following lemma makes some of the related products more specific.

Lemma 9.3.7. *For $z \in Y$, we have that $z_0 \cdot y = z_0 - \frac{1}{4}y[z]_0$ and $z_{\frac{1}{2}} \cdot y = z_{\frac{1}{2}} - \frac{1}{4}y[z]_{\frac{1}{2}}$.*

Proof. Note that $z \cdot y = z + \frac{1}{4}y - \frac{1}{4}y[z]$ and hence $(z \cdot y)_0 = z_0 - \frac{1}{4}y[z]_0$ and $(z \cdot y)_{\frac{1}{2}} = z_{\frac{1}{2}} - \frac{1}{4}y[z]_{\frac{1}{2}}$, since $y_0 = 0 = y_{\frac{1}{2}}$. Also, $z \cdot y = (y + z_0 + z_{\frac{1}{2}}) \cdot y = y + z_0 \cdot y + z_{\frac{1}{2}} \cdot y$. Since $0 * 1 = \{0\}$ and $\frac{1}{2} * 1 = \{\frac{1}{2}\}$, we have that $z_0 \cdot y = (z \cdot y)_0 = z_0 - \frac{1}{4}y[z]_0$ and $z_{\frac{1}{2}} \cdot y = (z \cdot y)_{\frac{1}{2}} = z_{\frac{1}{2}} - \frac{1}{4}y[z]_{\frac{1}{2}}$. □

We can now formulate the main result of this section. Recall the commutator notation $[y, z] = y \cdot z - z \cdot y$. Let $A(X) = \hat{A}(X)/([y, z] \mid y, z \in Y)$ be the largest commutative factor of $\hat{A}$.

Theorem 9.3.8. *The algebra $A(X)$ is the universal k-generated baric algebra of Jordan type $\frac{1}{2}$.*

Of course, as above, $k = |X|$. This theorem will be proved in the next section.

9.4 The commutator ideal

In this section, we develop a few properties of the commutator ideal $I = ([y, z] \mid y, z \in Y)$ that we need for the proof of Theorem 9.3.8.

Proposition 9.4.1. *For $y, z \in Y$, we have that $[y, z] = \frac{1}{4}(3y - 3z + y[z] - z[y])$.*

Proof. Indeed, $[y, z] = y \cdot z - z \cdot y = y + \frac{1}{4}z - \frac{1}{4}z[y] - (z + \frac{1}{4}y - \frac{1}{4}y[z]) = y - z + \frac{1}{4}z - \frac{1}{4}y - \frac{1}{4}z[y] + \frac{1}{4}y[z] = \frac{1}{4}(3y - 3z - z[y] + y[z])$. □

It will be convenient to consider, instead of $[y, z]$, its multiple $4[y, z] = 3y - 3z + y[z] - z[y]$. We can do it because $\mathbb{F}$ is not of characteristic 2.

Proposition 9.4.2. *The ideal I is spanned by the commutators $[y, z]$ (or $4[y, z]$) for $y, z \in Y$. That is, $I = \langle 3y - 3z + y[z] - z[y] \mid y, z \in Y \rangle$.*

Proof. It suffices to show that the span is an ideal. We first show that it is a left ideal. Take $y, z, t \in Y$. Then $t \cdot 4([y,z]) = t \cdot (3y - 3z + y[z] - z[y]) = 3t \cdot y - 3t \cdot z + t \cdot y[z] - t \cdot z[y] = 3(t + \frac{1}{4}y - \frac{1}{4}y[t]) - 3(t + \frac{1}{4}z - \frac{1}{4}z[t]) + (t + \frac{1}{4}y[z] - \frac{1}{4}y[zt]) - (t + \frac{1}{4}z[y] - \frac{1}{4}z[yt]) = 3t + \frac{3}{4}y - \frac{3}{4}y[t] - 3t - \frac{3}{4}z + \frac{3}{4}z[t] + t + \frac{1}{4}y[z] - \frac{1}{4}y[zt] - t - \frac{1}{4}z[y] + \frac{1}{4}z[yt] = \frac{1}{4}(3y - 3z + y[z] - z[y] - 3y[t] + 3z[t] - y[zt] + z[yt]) = [y,z] - [y[t], z[t]]$, which is in the span. So the commutator span is a left ideal.

Also, $4[y,z] \cdot t = (3y - 3z + y[z] - z[y]) \cdot t = t \cdot (3y - 3z + y[z] - z[y]) + 3[y,t] - 3[z,t] + [y[z], t] - [z[y], t]$ is contained in the span, since $t \cdot (3y - 3z + y[z] - z[y])$ is in it. So I is also a right ideal of $\hat{A}$. □

Now we are prepared for the proof of Theorem 9.3.8.

Proof. Note that $w(4[y,z]) = 3 - 3 + 1 - 1 = 0$ for all $y, z \in Y$. This means that $I \subseteq \hat{R}$, the radical of $\hat{A}$. It follows that $A = \hat{A}/I$ is baric. Since A is commutative, it clearly is a primitive axial algebra for the fusion law in Proposition 9.3.6. Furthermore, it satisfies the symmetrised version, where $0 * 1 = 1 * 0 = \emptyset$, and that is exactly the baric version of the fusion law of Jordan type $\frac{1}{2}$. Therefore, A is a baric algebra of Jordan type $\frac{1}{2}$. Clearly, for $y \in Y$, $\tau_y = y$ as an element of the group G acting on both $\hat{A}$ and $A = \hat{A}/I$. Hence the Miyamoto group of A is induced by the action of G. Since every element of Y is conjugate to an element of X by the action of G, it now follows that the images in A of the elements of X generate A. So A is k-generated.

For the universality property of A, consider an arbitrary algebra B of Jordan type $\frac{1}{2}$ generated by axes b_x, $x \in X$. Consider the homomorphism $\theta : G \to \mathrm{Miy}\,(B)$ sending each x to τ_{b_x}, which exists because G is the universal group generated by the involutions in X. For $y = x^g \in Y$, we have that $\theta(y) = \theta(x^g) = \theta(x)^{\theta(g)} = \tau_{b_x}^{\theta(g)} = \tau_{b_x^{\theta(g)}}$, i.e., $y = x^g$ is mapped by θ to the tau involution of the axis $b_x^{\theta(g)}$.

This gives us a map $\pi : Y \to B$ via $y = x^g \mapsto b_x^{\theta(g)}$. We will write $b_y = b_x^{\theta(g)}$ and note that $\tau_{b_y} = \theta(y)$.

Since Y is a basis of $\hat{A}$, the map π extends to a linear map from $\hat{A}$ to B, which we will again denote π. We claim that π is an algebra homomorphism. Indeed, for $y, z \in Y$, we have that

$$
\begin{aligned}
\pi(y \cdot z) &= \pi(y + \tfrac{1}{4}z - \tfrac{1}{4}z[y]) \\
&= \pi(y) + \tfrac{1}{4}\pi(z) - \tfrac{1}{4}\pi(z[y]) \\
&= b_y + \tfrac{1}{4}b_z - \tfrac{1}{4}\pi(z^y) \\
&= b_y + \tfrac{1}{4}b_z - \tfrac{1}{4}\pi(x^{gy}) \\
&= b_y + \tfrac{1}{4}b_z - \tfrac{1}{4}b_x^{\theta(gy)} \\
&= b_y + \tfrac{1}{4}b_z - \tfrac{1}{4}(b_x^{\theta(g)})^{\theta(y)} \\
&= b_y + \tfrac{1}{4}b_z - \tfrac{1}{4}b_z^{\theta(y)} \\
&= b_y + \tfrac{1}{4}b_z - \tfrac{1}{4}b_z^{\tau_{b_y}} \\
&= b_y b_z \\
&= \pi(y)\pi(z).
\end{aligned}
$$

Here we of course wrote $z = x^g$ for suitable $x \in X$ and $g \in G$. The last equality is due to the formula for the multiplication of two axes from Lemma 9.1.1. Note that π is surjective since the axes b_x, $x \in X$, generate B and these axes are in the image of π.

Thus $\pi : \hat{A} \to B$ is a surjective algebra homomorphism. It remains to note that, since B is commutative, all commutators $[y, z]$, $y, z \in Y$, are contained in the kernel $\ker(\pi)$ of π. Therefore, $I = ([y, z] \mid y, z \in Y) \subseteq \ker(\pi)$, and this means that π induces a surjective homomorphism from $A = \hat{A}/I$ onto B, proving universality. □

We now have a different realisation of the universal k-generated baric algebra of Jordan type $\frac{1}{2}$. This realisation is preferable to the one we obtained in Chapter 7 because Proposition 9.4.2 gives us an explicit spanning set for the ideal I. Thus, the computation of $A = A(X)$ is a matter of linear algebra, even though it is not a simple one, especially in view of the fact that Y is an infinite set.

9.5 Case $k = 4$

In this section, we describe a linear algebra algorithm constructing the universal k-generated baric algebra of Jordan type half. It is done for $k = 4$, which is the case of interest for us, but it easily generalises to an arbitrary k.

Let $X = \{a, b, c, d\}$, so $k = 4$. Recall from Section 9.2 that an element $y \in Y$ can be uniquely written as $y = x_0[x_1 \dots x_n]$ for some $n \geq 0$ and $x_0, x_1, \dots, x_n \in X$ such that $x_{i-1} \neq x_i$ for $1 \leq i \leq n$ (non-repeating sequence). This n is called the *length* of y. We represent y by the non-repeating array $[x_0, [x_1, \dots, x_n]]$ and order the set Y of all such arrays in the lexicographic order.

Let us state the following result about the number of axes of a given length.

Proposition 9.5.1. *For $n \geq 0$, the followings hold:*

(i) the number of $y \in Y$ of length n is $4 \cdot 3^n$;

(ii) the number of $y \in Y$ of length up to n is $2(3^{n+1} - 1)$.

Proof. First of all, there are four possibilities for x_0 because $|X| = 4$. Since $x_i \neq x_{i-1}$ for $i \geq 1$, we only have three options left for x_i. Therefore, the number of $y \in Y$ of length n is $4 \cdot 3^n$.

Now we use (i) to prove (ii). Indeed, the number of $y \in Y$ of length up to n is $\sum_{i=0}^{n} 4 \cdot 3^i = 4\sum_{i=0}^{n} 3^i = 4\left(\frac{3^{n+1}-1}{3-1}\right) = 2(3^{n+1}-1)$. □

Recall that Y is the basis of the algebra $\hat{A} = \hat{A}(X)$ while the kernel I that we need to factor out is spanned by all the relators $4[y, z] = 3y - 3z + y[z] - z[y]$, $y, z \in Y$. Our main operation is to express the highest terms of the relator $4[y, z] = 3y - 3z + y[z] - z[y]$ via the remaining terms and hence decrease the number of spanning elements for $A = \hat{A}/I$. Thus, we need to determine which term of $4[y, z]$ is the highest. Let $y = y_0[y_1 y_2 \dots y_n]$ and $z = z_0[z_1 z_2 \dots z_m]$ be in Y. Without loss of generality, we assume that $y < z$. So $n \leq m$. We might have that y and z have the same tail elements. Let $0 \leq r \leq n$ be maximal such that $y_{n-r+1} = z_{m-r+1}, y_{n-r+2} = z_{m-r+2}, \dots y_n = z_m$. Let $t = y_{n-r+1} y_{n-r+2} \dots y_n = z_{m-r+1} z_{m-r+2} \dots z_m$ be the *common tail* of y and z.

Lemma 9.5.2. *Let y and z be defined as above. Take $g = y_1 \dots y_{n-r}$ and $h = z_1 \dots z_{m-r}$. Here, when $r = n$, we set $g = 1$ and similarly, when $r = m$ (which can only be when $n = m$), we let $h = 1$. Then $y = y_0[gt]$, $z = z_0[ht]$ and $y[z] = y_0[gh^{-1}z_0 ht] = y'[z'][t]$ and, symmetrically, $z[y] = z_0[hg^{-1}y_0 gt] = z'[y'][t]$, where $y' = y_0[g]$ and $z' = z_0[h]$.*

Furthermore, if $r < n$ or $r = n$ and $y_0 \neq z_{m-n}$ then the above expressions are the shortest forms of $y[z]$ and $z[y]$.

Proof. Indeed, $y[z] = y_0[gt][z_0[ht]] = y_0[gt][t^{-1}h^{-1}z_0 ht] = y_0[gtt^{-1}h^{-1}z_0 ht] = y_0[gh^{-1}z_0 ht]$, which has no repetitions if the assumptions in the final claim are satisfied. Furthermore, $y_0[gh^{-1}z_0 ht] = y_0[g][z_0[h]][t] = y'[z'][t]$. Similarly, for $z[y]$. □

Now we can determine the highest term in each relator.

Proposition 9.5.3. *Let $y, z \in Y$ and $y < z$. The highest term of the relator $4[y, z] = 3y - 3z + y[z] - z[y]$ is $y[z]$ if z is longer than y, and it is $-z[y]$ if y and z have the same length.*

Proof. First suppose that $r < n$ or $r = n$ and $y_0 \neq z_{m-n}$. By Lemma 9.5.2, the length of $y[z]$ is $n - r + 2(m - r) + 1 + r = n + 2m - 2r + 1$ and, similarly, the length of $z[y]$ equal to $m + 2n - 2r + 1$. Clearly, both of these are greater than n and m, so $3y$ and $-3z$ cannot be the highest terms. If $n < m$ then $n + 2m - 2r + 1 > m + 2n - 2r + 1$ and so $y[z]$ is the highest term, as claimed. If $n = m$ then $y' \neq z'$ (or else $y = z$, a contradiction) and then $y' < z'$ since $y < z$. This in turn implies that $y[z] = y'[z'][t] < z'[y'][t]$, and so $-z[y]$ is the highest term in this case.

Now assume that $r = n$ (i.e., $g = 1$) and $y_0 = z_{m-n}$. If $m = n$, then we get that $y_0 = z_0$, $y = y_0[t] = z_0[t] = z$, a contradiction. Hence $m > n$. Then $y[z] = y_0[z_0[h]][t] = y_0[h^{-1}z_0ht]$ has length $n + 2m - 2r = 2m - n$, because it has only one repetition: $y_0 = z_{m-n}$. At the same time, $z[y] = z_0[hy_0t]$ has at least one repetition, so its length is less or equal to $m - r + r = m$. Clearly, $2m - n > n, m$, and so $y[z]$ is the highest term. □

Using this proposition, it is easy to create a function (we used the computer algebra system GAP [12]) that, for a given $u \in Y$ finds all pairs $y, z \in Y$ such that u (or $-u$) is the highest term of $4[y, z]$. Indeed, $u = y'[z'][t]$ or $z'[y'][t]$, and we just need to identify the possible t, y' and z' within u.

Now we are ready to outline our algorithm for finding $A[X]$. As already mentioned, it consists of several steps:

(1) Choose a depth d and enumerate all elements of Y of length up to d.

(2) For each u on this list, enumerate all relators with the highest term u. If such relators exist, use the first one to express u as a linear combination of lower terms. Reduce the remaining relators with the highest term u, going down the list. Each reduction ends either with 0 or with a new expression of a lower term in terms of even lower terms.

(3) After this is completed, we have two types of elements on the list: the ones that have expressions as linear combinations of lower terms; and the ones that do not have such expressions.

(4) Select a second depth $dd < d$. Select all u of length up to dd which have no expression as a linear combination of lower terms. Let B be the list of all such u. Furthermore, create a similar list B' for $dd' = dd + 1$. If $B = B'$, then we have success and the end of the main calculation. Indeed, this means that B provides a finite spanning set for $A = A(X)$. This is because the span A_0 of B (or rather its image in A) is invariant under the (left) adjoints of all $x \in X$. This is because, by Lemma 9.1.1, $x \cdot u = x + \frac{1}{4}u - \frac{1}{4}u[x]$ and none of the terms here has length more than $dd + 1$; hence the product can be expressed as a linear combination of B, i.e., it is in A_0.

In turn, the condition that A_0 is invariant under the left adjoints for all $x \in X$ implies that it is invariant under the action of G and hence it contains all of Y, i.e., $A_0 = A$. It is now easy to build the algebra and check its properties.

In the actual calculation we carried out for $k = 4$ over $\mathbb{F} = \mathbb{Q}$, the rationals, we achieved success for $d = 13$ and $dd = 6$. The spanning set B consisted of 55 elements. When we constructed the algebra of dimension 55, it turned out that this algebra is still non-commutative, with a 1-dimensional commutator ideal. The corresponding factor algebra of dimension 54 proved to be the actual realisation of the universal 4-generated algebra of Jordan type $\frac{1}{2}$. Thus, by our calculation, we established the following result.

Theorem 9.5.4. *The universal* 4*-generated baric* $\mathbb{Q}$*-algebra of Jordan type* $\frac{1}{2}$ *has dimension* 54*.*

This result is quite surprising because the known upper bound from [7] is 81. To clarify the relation between these numbers, we repeated our calculation for the same $d = 13$,

$dd = 6$, but over $\mathbb{F} = \mathbb{F}_3$, the field of order 3. This time the dimension of $A(X)$ was 81. We believe that the version of $A(X)$ over $\mathbb{Z}$ has $\mathbb{Z}$-rank 54 and a torsion ideal of exponent 3 to compensate for the difference between 81 and 54.

We can also state the following corollary of Theorem 9.5.4.

Corollary 9.5.5. *For all k, the universal k-generated baric $\mathbb{Q}$-algebra of Jordan type $\frac{1}{2}$ is a Jordan algebra.*

Proof. By calculation, for $k = 4$, the algebra is a Jordan algebra. On the other hand, the linearised version of the Jordan identity is 4-linear, and so every its instance, for any k, only depends on four axes. □

CONCLUSION

In this text, we investigated the class of baric algebras of Jordan type η. It turned out that the parameter η has to equal either 2 or $\frac{1}{2}$. In the first case, $\eta = 2$, we achieved a complete classification of all baric algebras of Jordan type 2. To complete the second case, $\eta = \frac{1}{2}$, we attempted two approaches. We first used the techniques involving universal magmas and magma algebras to show the existence of the universal k-generated baric algebra of Jordan type $\frac{1}{2}$. However, in this way, we did not obtain a realisation that is convenient for calculations. With the second approach, based on some ideas from [7], we get a much nicer realisation, where we can find the target algebra via a single (but rather big) linear algebra calculation. This has been realised in GAP for $k = 4$ and yielded a surprising result: the universal 4-generated baric algebra of Jordan type $\frac{1}{2}$ is much smaller than expected (dimension 54 against the expected 81). Furthermore, it turned out to be a Jordan algebra, at least in characteristic 0. This yielded an additional prize: all universal baric algebras of Jordan type $\frac{1}{2}$ in characteristic 0 are Jordan. This is expected to be true also in all positive characteristics other than 3.

There are still many interesting open questions and directions for research. In particular, it would be very interesting to generalise the results on the universal algebra $U(X)$ from Chapter 7 to arbitrary baric axial algebras or even arbitrary axial algebras. The related question is: is it true that there is an infinite-dimensional k-generated universal algebra assuring primitivity and the prescribed axis spectrum? We believe that this can be approached using ideas similar to what we did in this thesis.

APPENDIX A

ALGORITHM

```
#  Relations in the 4-dimensional baric algebra of Jordan type half
#
# (C) 2024 Yunxi Shi, Sergey Shpectorov

# key parameters:

d:=6;
dd:=13;

#
# the function computing the number of reduced axes, fast axis code
# function and axis for the given code
#

# number of axes of length up to k

NumberOfAxes:=k->2*(3^(k+1)-1);

# code of an axis

```

```
CodeOfX:=function(x)
  local k,c,i,z;
  k:=Length(x[2]);
  c:=NumberOfAxes(k-1);
  c:=c+(x[1]-1)*3^k;
  for i in [1..k] do
    z:=0;
    if (i=1 and x[1]<x[2][i]) or (i>1 and x[2][i-1]<x[2][i]) then
      z:=1;
    fi;
    c:=c+(x[2][i]-1-z)*3^(k-i);
  od;
  return c+1;
end;

# axis with the given code

AxisByCode:=function(c)
  local cc,k,r,i;
  cc:=c-1;
  for i in [0..100] do
    if NumberOfAxes(i)>cc then
      k:=i;
      break;
    fi;
  od;
  if k<>0 then
    cc:=cc-NumberOfAxes(k-1);
  fi;
  r:=[Int(cc/3^k)+1,[]];
  cc:=cc-(r[1]-1)*3^k;
  for i in [1..k] do
```

```
    r[2][i]:=Int(cc/3^(k-i))+1;
    cc:=cc-(r[2][i]-1)*3^(k-i);
    if (i=1 and r[2][i]>=r[1]) or (i>1 and r[2][i]>=r[2][i-1]) then
      r[2][i]:=r[2][i]+1;
    fi;
  od;
  return r;
end;

#
# axes may appear in a non-reduced form
#

ReduceAxis:=function(x)
  local xx,k,i;
  xx:=StructuralCopy(x);
  k:=Length(xx[2]);
  for i in [k,k-1..2] do
    if xx[2][i-1]=xx[2][i] then
      Unbind(xx[2][i-1]);
      Unbind(xx[2][i]);
      xx[2]:=Compacted(xx[2]);
      return ReduceAxis(xx);
    fi;
  od;
  if Length(xx[2])>0 and xx[1]=xx[2][1] then
    Unbind(xx[2][1]);
    xx[2]:=Compacted(xx[2]);
  fi;
  return xx;
end;

```

```
#
# action by conjugation on the set of axes
#

ActionConj:=function(x,y)
  return ReduceAxis([x[1],Concatenation(x[2],Reversed(y[2]),[y[1]],y
    [2])]);
end;

#
# ground field
#

F:=Rationals;

#
# commutator relations
# plus functions generating them
#

rels:=[];

# relator [x,y]=3x-3y+x^y-y^x

CommutatorRelation:=function(x,y)
  local a;
  if x=y then
    return [];
  fi;
  if 3*One(F)<>Zero(F) then
    a:=[[CodeOfX(x),3*One(F)],[CodeOfX(y),-3*One(F)],[CodeOfX(
    ActionConj(x,y)),One(F)],
```

```
        [CodeOfX(ActionConj(y,x)),-One(F)]];
  else
    a:=[[CodeOfX(ActionConj(x,y)),One(F)],[CodeOfX(ActionConj(y,x)),-
   One(F)]];
  fi;
  Sort(a);
  return a;
end;

# all relators with a given highest term

RelsForHighestTerm:=function(yz)
  local rels,y0,k,i,t,j,h,z0,s,good,g;
  rels:=[];
  y0:=yz[1];
  k:=Length(yz[2]);
  for i in [0..k-1] do
    t:=yz[2]{[k-i+1..k]};
    for j in [Int((k-i+1)/3)..Int((k-i)/2)] do
      good:=true;
      for s in [1..j-1] do
        if yz[2][k-i-j+s]<>yz[2][k-i-j-s] then
          good:=false;
          break;
        fi;
      od;
      if not good then
        continue;
      fi;
      h:=yz[2]{[k-i-j+1..k-i]};
      z0:=yz[2][k-i-j];
      if k=i+2*j then
```

```
      if h[j]<>y0 then
        continue;
      else
        g:=[];
        Add(rels,CommutatorRelation([y0,Concatenation(g,t)],[z0,
  Concatenation(h,t)]));
      fi;
    else
      if yz[2][k-i]<>yz[2][k-i-2*j] then
        continue;
      fi;
      if i>0 then
        if (k=i+2*j+1 and yz[2][k-i+1]=y0) or
           (k>i+2*j+1 and yz[2][k-i+1]=yz[2][k-i-2*j-1]) then
          continue;
        fi;
      fi;
      g:=yz[2]{[1..k-i-2*j-1]};
      if Length(g)=Length(h) then
        if CodeOfX([y0,g])<=CodeOfX([z0,h]) then
          continue;
        else
          Add(rels,CommutatorRelation([y0,Concatenation(g,t)],[z0,
  Concatenation(h,t)]));
        fi;
      else
        Add(rels,CommutatorRelation([y0,Concatenation(g,t)],[z0,
  Concatenation(h,t)]));
      fi;
    fi;
  od;
od;
```

```
  return rels;
end;

# filling the list of relators

FillAllRelators:=function(k)
  local i;
  for i in [1..NumberOfAxes(k)] do
    Print("            \r",i,"\c");
    rels[i]:=RelsForHighestTerm(AxisByCode(i));
  od;
  Print("\n");
end;

Print("\nFilling the list of relators\n");
FillAllRelators(dd);

#
# linear algebra in the relator space
#

AddRelations:=function(r,s)
  local a,i,j,c;
  a:=[];
  i:=1;
  j:=1;
  while i<=Length(r) and j <=Length(s) do
    if r[i][1]<s[j][1] then
      Add(a,ShallowCopy(r[i]));
      i:=i+1;
    elif r[i][1]>s[j][1] then
      Add(a,ShallowCopy(s[j]));
```

```
      j:=j+1;
    else
      c:=r[i][2]+s[j][2];
      if c<>Zero(F) then
        Add(a,[r[i][1],c]);
      fi;
      i:=i+1;
      j:=j+1;
    fi;
  od;
  if i<=Length(r) then
    Append(a,r{[i..Length(r)]});
  fi;
  if j<=Length(s) then
    Append(a,s{[j..Length(s)]});
  fi;
  return a;
end;


TimesRelation:=function(r,s)
  if s<>Zero(F) then
    return List(r,e->[e[1],s*e[2]]);
  else
    return [];
  fi;
end;

#
# reducing relations modulo the known ones
#
# (for each highest term, we view the first relation
```

```
# as known and all other as additional ones, to be
# reduced later)
#

# reduce a single relation

ReduceRelation:=function(r)
  local h,s,n,m,c;
  if r=[] then
    return;
  fi;
  h:=r[Length(r)][1];
  if rels[h]=[] then
    rels[h]:=[StructuralCopy(r)];
    return;
  else
    s:=rels[h][1];
    n:=r[Length(r)][2];
    m:=s[Length(s)][2];
    if F=Rationals then
      c:=GcdInt(n,m);
    else
      c:=One(F);
    fi;
    r:=AddRelations(TimesRelation(r,m/c),TimesRelation(s,-n/c));
    ReduceRelation(r);
    return;
  fi;
end;

# reduce all additional relations

```

```
ReduceAll:=function(k)
  local i,j;
  for i in [1..NumberOfAxes(k)] do
    Print("\r",i,"\c");
    if Length(rels[i])>1 then
      for j in [2..Length(rels[i])] do
        ReduceRelation(rels[i][j]);
        Unbind(rels[i][j]);
      od;
    fi;
  od;
  Print("\n");
end;

Print("\nReducing additional relations\n");
ReduceAll(dd);

#
# basis
#

bas:=[];

FindBasis:=function(k)
  bas:=Filtered([1..NumberOfAxes(k+1)],i->rels[i]=[]);
  if bas[Length(bas)]>NumberOfAxes(k) then
    Print("Not a basis!\n");
  fi;
end;

Print("\nSelecting basis\n");
FindBasis(d);
```

```
#
# algebra
#

Print("\nSetting the algebras\n");
N:=Length(bas);
A:=F^N;
a:=Basis(A);

#
# expressing axes in terms of a possible basis
#

Axes:=[];

ExpressAxis:=function(k)
  local r,n,c;
  if k in bas then
    Axes[k]:=ShallowCopy(a[Position(bas,k)]);
    return;
  fi;
  r:=rels[k][1];
  n:=Length(r);
  c:=-r[n][2];
  Axes[k]:=Sum(List(r{[1..n-1]},e->e[2]*Axes[e[1]]/c));
  return;
end;

ExpressAll:=function(k)
  local i;
  for i in [1..NumberOfAxes(k)] do
```

```
    Print("\r",i,"\c");
    ExpressAxis(i);
  od;
  Print("\n");
end;

ExpressAll(d+1);

#
# building the algebra
#

# actions of tau/Miyamoto involutions

# generating axes

BasicAction:=function(x)
  local mat,i;
  mat:=[];
  for i in bas do
    Add(mat,ShallowCopy(Axes[CodeOfX(ActionConj(AxisByCode(i),
   AxisByCode(x)))]));
  od;
  return mat;
end;

tau:=List([1..4],i->BasicAction(i));

# all axes in the basis

AxisAction:=function(r)
  local ans,i;
```

```
  ans:=StructuralCopy(tau[r[1]]);
  for i in [1..Length(r[2])] do
    ans:=ans^tau[r[2][i]];
  od;
  return ans;
end;

act:=List(bas,i->AxisAction(AxisByCode(i)));

# multiplication table

Mult:=List([1..N],i->[]);

for i in [1..N] do
  for j in [1..N] do
    Mult[i][j]:=a[i]+a[j]/4-act[i][j]/4;
  od;
od;

Times:=function(u,v)
  local ans,i,j;
  ans:=Zero(A);
  for i in [1..N] do
    for j in [1..N] do
      ans:=ans+u[i]*v[j]*Mult[i][j];
    od;
  od;
  return ans;
end;

#
# express arbitrary axis using the known action
```

```
#

ExpressArbitraryAxis:=function(r)
  local ans,i;
  ans:=ShallowCopy(Axes[r[1]]);
  for i in r[2] do
    ans:=ans*tau[i];
  od;
  return ans;
end;

# end
```

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