Extremal and probabilistic problems for discrete structures

by

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Abstract

In this thesis we present a collection of new results in extremal and probabilistic combinatorics. More precisely, we answer various questions from the following five topics.

The first topic is graph discrepancy. Balogh, Csaba, Jing and Pluhár determined the minimum degree threshold that ensures a 2-edge-coloured graph G contains a Hamilton cycle with *large discrepancy*, that is, a Hamilton cycle with significantly more than half of its edges in one colour. We prove an analogous result for r-colourings.

The second topic is the study of Dirac-type problems for (vertex) ordered graphs. For any fixed ordered graph H, we determine the asymptotic minimum degree threshold for forcing

(i) a perfect H-tiling in an ordered graph, under the assumption that $\chi_<(H) \geq 3$;

(ii) an H-tiling in an ordered graph G covering a fixed proportion of the vertices of G;

(iii) an H-cover in an ordered graph.

Our results combined with the work of Balogh, Li and Treglown determine completely the asymptotic minimum degree thresholds for forcing a perfect H-tiling and an almost perfect H-tiling in an ordered graph (for any fixed ordered graph H).

The third topic is graph deficiency. Given a property $\mathcal{P}$, the *deficiency* $\text{def}(G)$ of a graph G with respect to $\mathcal{P}$ is the smallest non-negative integer t such that $G * K_t$ has property $\mathcal{P}$, where $G * K_t$ is the graph obtained by adding t new vertices to G and all edges incident to such vertices. This notion was introduced by Nenadov, Sudakov and Wagner, who raised the question of determining how many edges an n-vertex graph G needs to ensure $G * K_t$ contains a K_r-factor (for any fixed $r \geq 3$). We present a full resolution of this problem.

The fourth topic is saturation for induced posets. For a fixed poset P, a family $\mathcal{F}$ of subsets of $[n] := \{1, 2, \dots, n\}$ is *induced P-saturated* if $\mathcal{F}$ does not contain an induced copy of P and, for every subset S of $[n]$ such that $S \notin \mathcal{F}$, the family $\mathcal{F} \cup \{S\}$ does contain

an induced copy of P. The size of the smallest such family $\mathcal{F}$ is denoted by $\text{sat}^*(n, P)$. Keszegh, Lemons, Martin, Pálvölgyi and Patkós proved that there is a dichotomy of behaviour for this parameter: for any poset P, either $\text{sat}^*(n, P) = O(1)$ or $\text{sat}^*(n, P) \geq \log_2 n$. Furthermore, they conjectured that the $\log_2 n$ term can be replaced with $n + 1$. We make progress towards this conjecture by showing that either $\text{sat}^*(n, P) = O(1)$ or $\text{sat}^*(n, P) \geq \min\{2\sqrt{n}, n/2 + 1\}$. Our proof makes use of a Turán-type result for digraphs. We also prove that the aforementioned conjecture holds for a certain class of posets P and determine the exact value of $\text{sat}^*(n, P)$ in some cases.

The fifth topic is the study of typical Ramsey properties of abelian groups. A classical result of Rado characterises all integer matrices A such that any finite colouring of $\mathbb{N}$ yields a monochromatic solution to the system of equations $Ax = 0$. Rödl and Ruciński, and Friedgut, Rödl and Schacht proved a random version of Rado's theorem where one considers a random subset of $[n]$ instead of $\mathbb{N}$. We investigate the analogous random Ramsey problem in the more general setting of abelian groups: given a sequence $(S_n)_{n \in \mathbb{N}}$ of finite subsets of abelian groups and a positive integer $r \geq 2$, we are interested in determining the probability threshold $\hat{p} := \hat{p}(n)$ such that

$$\lim_{n \to \infty} \mathbb{P} \left[\begin{array}{c} \text{Any } r\text{-colouring of } S_{n,p} \text{ yields a} \\ \text{monochromatic solution to } Ax = 0. \end{array} \right] = \begin{cases} 0 & \text{if } p = o(\hat{p}), \\ 1 & \text{if } p = \omega(\hat{p}). \end{cases}$$

where $S_{n,p}$ denotes the random subset of S_n obtained by including each element of S_n with probability p, independently. We develop a general black box, which we refer to as the *random Rado lemma*, to tackle problems of this type. Among various applications, we obtain a random version of the *van der Waerden theorem for the primes* (a consequence of the Green–Tao theorem) and an integer lattice generalisation of the random version of Rado's theorem (using a novel supersaturation result for $S_n := [n]^d$). Furthermore, we prove a 1-statement and a 0-statement for hypergraphs that imply several of the previously known 1- and 0-statements in various settings, as well as the random Rado lemma.

Declaration

This thesis comprises the following collaborative works.

The material in Chapter 2 is joint work with Joseph Hyde, Joanna Lada and Andrew Treglown, published in *SIAM Journal on Discrete Mathematics* [56].

The material in Chapter 3 is joint work with Andrew Treglown, published in *Forum of Mathematics, Sigma* [59].

The material in Chapter 4 is joint work with Joseph Hyde and Andrew Treglown, published in *Combinatorics, Probability and Computing* [57].

The material in Chapter 5 is joint work with Simón Piga, Maryam Sharifzadeh and Andrew Treglown, published in *Combinatorial Theory* [58].

The material in Chapter 6 and Appendices A and B is joint work with Robert Hancock and Andrew Treglown. The material of Chapter 6 and Appendix A appears on arXiv [55] and has been submitted for publication.

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Contents

Chapter 1

Introduction

Combinatorics is the area of mathematics that investigates problems about counting and arranging. This kind of question emerges frequently from practical situations. How many connections are needed to travel from one place to another? What is the optimal order of performing a certain set of tasks? On the other hand, combinatorial problems can stem from pure intellectual curiosity. What is the number of anagrams of a given word? How many knights can be placed on a chessboard so that no pair of knights are attacking each other?

The earliest recorded application of combinatorial techniques can be found in the Rhind papyrus [17], a mathematical text dating to around 1550 BC. Throughout history, the study of combinatorial problems attracted the attention of scholars from various geographical regions, but combinatorics itself did not constitute a formal subject until modern times. The *Traité du triangle arithmétique* by Pascal from 1654 is usually considered the first recognisable treaty in combinatorics [17]. Since the second half of the 20th Century, combinatorics has been experiencing rapid growth, and many connections with other fields have been established. Indeed, combinatorial techniques have been successfully employed in other mathematical areas, including number theory, analysis, geometry and probability, as well as to subjects outside the inner circle of pure mathematical disciplines such as computer science, statistical physics, biology and chemistry. Vice versa, methods and concepts from these fields have inspired many contributions in combinatorics.

One of the central areas of modern combinatorics is *graph theory*. The initiation of this topic of study is generally attributed to Euler, who introduced the notion of graphs to answer the famous problem of the *Seven Bridges of Königsberg* [47] in 1741. Formally, a *graph* G consists of two sets $V(G)$ and $E(G)$ where $E(G)$ is a set of unordered pairs of elements of $V(G)$. The elements of $V(G)$ and $E(G)$ are referred to as *vertices* and *edges* of G, respectively. A graph can be intuitively visualized as a drawing on the plane: vertices are represented as points and edges are represented as lines joining the corresponding points. In this thesis, we strictly consider *finite simple graphs*, that is, graphs with finitely many vertices, no edges from a vertex to itself (i.e., no loops) and with at most one edge between any given pair of vertices (i.e, no multiple edges). If two vertices x and y are connected by an edge, we may denote such edge as xy. We say a graph F is a *subgraph* of G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. Two graphs G_1 and G_2 are *isomorphic* if there is a bijection $\phi : V(G_1) \to V(G_2)$ which preserves edges, that is, if $xy \in E(G_1)$ then $\phi(x)\phi(y) \in E(G_2)$. Given two graphs G and H, a *copy* of H in G is a subgraph of G which is isomorphic to H.

In 1907 Mantel [100] proved that any triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges, where a graph is *triangle-free* if it does not contain three vertices x, y and z such that xy, yz and xz are edges of the graph. This is one of the earliest results in *extremal graph theory*, a sub-branch of graph theory whose central topic of investigation is how large, or small, the value of a given parameter can be for certain classes of graphs. For a fixed graph H, we say a graph is *H-free* if it does not contain a copy of H. Motivated by Mantel's theorem, it is natural to ask what is the maximum number of edges $ex(n, H)$ in an H-free graph on n vertices. Questions of this flavour are referred to as *Turán-type* problems, and $ex(n, H)$ is known as the *Turán number*. The name refers to a result of Turán [133] (1941) which provides the exact value of $ex(n, K_t)$ for any positive integer $t \geq 2$, where K_t denotes the *complete* graph on t vertices (a graph is complete if every pair of vertices forms an edge). Note that Mantel's theorem corresponds to the case $t = 3$ of Turán's theorem. The celebrated Erdős–Stone–Simonovits theorem [44, 46] determines $ex(n, H)$ for any fixed graph H, up to a $o(n^2)$ error term. To state this result, we first introduce a key parameter. The *chromatic number* $\chi(H)$ of a graph H is the smallest number of colours so that the vertices of H can be assigned colours in such a way that there is no edge between vertices of the same colour.

Theorem 1.0.1 (Erdős–Stone–Simonovits theorem [44, 46]). *Fix a graph H. Then*

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\frac{n^2}{2} + o(n^2).$$

Observe that Theorem 1.0.1 yields an asymptotic formula for $ex(n, H)$ when $\chi(H) \geq 3$. Conversely, the asymptotic behaviour of $ex(n, H)$ when H is bipartite (i.e., when $\chi(H) = 2$) is not well-understood in general, and its study is still an active area of research. A famous result in this field is the Kővári–Sós–Turán theorem [90] which states that $ex(n, K_{t,t}) \leq Cn^{2-1/t}$ for some constant $C = C(t)$. Here $K_{t,t}$ is the *complete bipartite graph* with parts of size t (a bipartite graph is complete if all edges between its two parts are present).

Besides Turán-type questions, many settings have been studied for parameters other than the number of edges and forbidding structures other than a fixed graph H. For example, the *degree* of a vertex is the number of edges connected to such vertex; the *minimum degree* of a graph G, denoted as $\delta(G)$, is the smallest degree among all vertices of G. A classic theorem of Dirac [38] from 1952 states that any graph G on $n \geq 3$ vertices which does not contain a *spanning cycle* (i.e., a cycle covering all vertices of G) has minimum degree $\delta(G) < n/2$. Spanning cycles are often referred to as *Hamilton cycles*. Dirac's theorem inspired a long line of research on how forbidding certain structures affects the minimum degree of a graph. These are known as *Dirac-type* problems.

Taking a step back, observe that Mantel's theorem can be rephrased in the following language. Suppose we were asked to exhibit a graph on n vertices and m edges that does not contain a triangle. Mantel's theorem states that if m is strictly greater than $\lfloor n^2/4 \rfloor$, then *regardless* of how we arrange the m edges, a triangle will inevitably appear. The theorems of Turán, Erdős–Stone–Simonovits, Kővári–Sós–Turán and Dirac can be rephrased similarly. This formulation naturally leads to a much broader combinatorial question:

Given a collection of elements to arrange, are there unavoidable patterns that emerge in any arrangement?

Numerous results in combinatorics can be seen as specific instances of this general problem. A prime example is Ramsey's theorem [111] (1930), a deeply influential result in graph theory: given positive integers r and t, there exists a positive integer n such that any r-colouring of the edges of K_n yields a monochromatic copy of K_t. That is, regardless of how the edges of K_n are coloured using the r colours, one can always find a copy of K_t whose edges have received the same colour. Determining the smallest n satisfying the above property (for t fixed) is a central problem in combinatorics which has inspired many ingenious techniques. A particularly important one is the *probabilistic method* pioneered by Erdős.

On this note, *probabilistic combinatorics* is the branch of combinatorics focusing on the study of random structures. In probabilistic graph theory, perhaps the most studied object is the Erdős–Rényi random graph $G(n,p)$. This is a graph on n vertices where every pair of vertices forms an edge with probability p, independently. Considerable effort has been focused on transferring classical results in graph theory to the probabilistic setting. For example, having Ramsey's theorem in mind, one may wonder what is the probability that the graph $G(n,p)$ has the property that any r-colouring of its edges yields a monochromatic copy of K_t. Indeed, a generalisation of this problem was settled by Rödl and Ruciński in a series of works [113, 114, 115]. Similarly, mirroring Dirac's theorem, one may ask what is the probability that the graph $G(n,p)$ contains a Hamilton cycle. A classic paper of Pósa [109] provides an insightful answer to this question, with more refined results obtained later by other authors. This type of statement for random graphs suggests a probabilistic analogue of the general question we previously stated:

Given a random collection of elements, what is the probability that a certain pattern emerges?

Beyond graph theory, results of similar flavour have been obtained in other settings. For example, Turán-type questions have been extensively studied for integer sets. A central result in *extremal set theory* is Sperner's theorem [126]: a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ such that no set contains another (i.e, $\mathcal{F}$ is an antichain) has size at most $\binom{n}{\lfloor n/2 \rfloor}$. Ramsey-type results have been studied for various discrete objects, particularly for arithmetic structures. For instance, a well-known Ramsey-type result by Schur [123] states that any finite colouring of the set of natural numbers $\mathbb{N}$ yields a monochromatic solution to the equation $x + y = z$. Probabilistic analogues of Turán-type and Ramsey-type questions have also attracted considerable attention, as discussed further in Chapter 6.

In this thesis, we present a collection of new results fitting the wide picture described so far. These results are independent of each other, and address problems of different flavours, from extremal to probabilistic questions on graphs, sets, and arithmetic structures. For the remainder of this chapter, we outline the results of this thesis and their specific backgrounds.

1.1 Hamilton cycles with large discrepancy

The study of colour-biased structures in graphs concerns the following problem. Given graphs H and G, what is the largest t such that in any r-colouring of the edges of G, there is always a copy of H in G that has at least t edges of the same colour? Note if H is a subgraph of G, one can trivially ensure a copy of H with at least $|E(H)|/r$ edges of the same colour; so one is interested in when one can achieve a colour-bias significantly above this. Colour-bias problems can be seen as a relaxation of classical Ramsey questions: instead of seeking a monochromatic copy of a certain structure, one is interested in finding a copy of the structure with large colour-bias.

The topic was first raised by Erdős in the 1960s (see [40, 45]). Erdős, Füredi, Loebl and Sós [41] proved the following: for some constant $c > 0$, given any 2-colouring of the edges of K_n and any fixed spanning tree T_n with maximum degree Δ, we have that K_n contains a copy of T_n such that at least $(n-1)/2 + c(n-1-\Delta)$ edges of this copy of T_n receive the same colour. In [6], Balogh, Csaba, Jing and Pluhár investigated the colour-bias problem in the case of spanning trees, paths and Hamilton cycles for various classes of graphs G. Note all their results concern 2-colourings and therefore were expressed in the equivalent language of *graph discrepancy*. The following result determines the minimum degree threshold for forcing a Hamilton cycle of significant colour bias in a 2-edge-coloured graph.

Theorem 1.1.1 (Balogh, Csaba, Jing and Pluhár [6])**.** *Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If G is an n-vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-colouring of $E(G)$ there is a Hamilton cycle in G with at least $(1/2 + c/64)n$ edges of the same colour. Moreover, if 4 divides n, there is an n-vertex graph G' with $\delta(G') = 3n/4$ and a 2-colouring of $E(G')$ for which every Hamilton cycle in G' has precisely $n/2$ edges in each colour.

In [66], Gishboliner, Krivelevich and Michaeli considered colour-bias Hamilton cycles in the random graph $G(n,p)$. Roughly speaking, their result states that if p is such that with high probability (w.h.p.) $G(n,p)$ has a Hamilton cycle, then in fact w.h.p., given any r-colouring of the edges of $G(n,p)$, one can guarantee a Hamilton cycle that is essentially as colour-bias as possible (see [66, Theorem 1.1] for the precise statement). A discrepancy (therefore colour-bias) version of the Hajnal–Szemerédi theorem was proven in [7].

We give a very short proof of the following multicolour generalisation of Theorem 1.1.1. We require the following definition to state it.

Definition 1.1.2. Let $t, r \in \mathbb{N}$ and H be a graph. We say that an r-colouring of the edges of H is *t-unbalanced* if at least $|E(H)|/r + t$ edges are coloured with the same colour.

Theorem 1.1.3. *Let $n, r, d \in \mathbb{N}$ with $r \geq 2$. Let G be an n-vertex graph with $\delta(G) \geq \left(\frac{1}{2} + \frac{1}{2r}\right) n + 6dr^2$. Then for every r-colouring of $E(G)$ there exists a d-unbalanced Hamilton cycle in G.*

Note that n, r and d may all be comparable in size. Hence Theorem 1.1.3 implies Theorem 1.1.1 (with a slightly better bound on the colour-bias). Theorem 1.1.3 is tight, in the sense that the $1/2 + 1/2r$ term cannot be replaced by a smaller number.

The proof of Theorem 1.1.3 is presented in Chapter 2, and is based on a coauthored paper [56] with Joseph Hyde, Joanna Lada and Andrew Treglown.

1.2 Dirac-type problems for vertex ordered graphs

In recent years there has been a significant effort to develop both Turán and Ramsey theories in the setting of *vertex ordered graphs*. A *(vertex) ordered graph* or *labelled graph* H on h vertices is a graph whose vertices have been labelled with $[h] := \{1, \dots, h\}$. An ordered graph G with vertex set $[n]$ *contains* an ordered graph H on $[h]$ if there is an injection $\phi : [h] \to [n]$ such that

(i) $\phi(i) < \phi(j)$ for all $1 \leq i < j \leq h$ and

(ii) $\phi(i)\phi(j)$ is an edge in G whenever ij is an edge in H.

In other words, the map ϕ is a graph isomorphism which preserves the order of the vertices.

The following parameter is the vertex ordered analogue of the chromatic number.

Definition 1.2.1 (Interval chromatic number)**.** Given $r \in \mathbb{N}$, an *interval r-colouring* of an ordered graph H is a partition of the vertex set $[h]$ of H into r (possibly empty) intervals so that no two vertices belonging to the same interval are adjacent in H. The *interval chromatic number* $\chi_<(H)$ of an ordered graph H is the smallest $r \in \mathbb{N}$ such that there exits an interval r-colouring of H.

As previously mentioned, Turán-type problems concern edge density conditions that force a fixed graph H as a subgraph in a host graph G. Whilst the Erdős–Stone–

Simonovits theorem [44, 46] determines, up to a quadratic error term, the number of edges in the densest H-free n-vertex graph, there is still active interest in the Turán problem for bipartite H. Similarly, a result of Pach and Tardos [107] determines asymptotically the number of edges an ordered graph requires to force a copy of a fixed ordered graph H with interval chromatic number $\chi_<(H)$ at least 3. Therefore, again there is significant interest in the 'bipartite' case of this problem (i.e., when $\chi_<(H) = 2$); see Tardos [132] for a recent survey on such results.

The study of Ramsey theory for ordered graphs has also gained significant traction. For example, results of Conlon, Fox, Lee and Sudakov [27], and of Balko, Cibulka, Král and Kynčl [5] demonstrate that there are ordered graphs H for which the behaviour of the Ramsey number is vastly different to their underlying unordered graph.

In a recent paper, Balogh, Li and Treglown [8] initiated the study of Dirac-type results for ordered graphs. Their main focus was on *perfect H-tilings*, though they also raised other Dirac-type problems (see [8, Section 8]). In both the ordered and unordered settings, an *H-tiling* in a graph G is a collection of vertex-disjoint copies of H contained in G. An H-tiling is *perfect* if it covers all the vertices of G. In the literature, as well as in this thesis, perfect H-tilings are also referred to as *H-factors*. H-tilings can be viewed as generalisations of both the notion of a matching (which corresponds to the case when H is a single edge) and the Turán problem (i.e., a copy of H in G is simply an H-tiling of size one).

Balogh, Li and Treglown [8] raised the following question.

Problem 1.2.2. *[8] Given any ordered graph H and any $n \in \mathbb{N}$ divisible by $|H|$, determine the smallest integer $\delta_<(H, n)$ such that every n-vertex ordered graph G with $\delta(G) \geq \delta_<(H, n)$ contains a perfect H-tiling.*

The analogous problem in the (unordered) graph setting had been studied since the 1960s (see, e.g., [3, 30, 74, 88, 93, 94]) and forty-five years later a complete solution, up to an additive constant term, was obtained via a theorem of Kühn and Osthus [94]. We will discuss this result further in Chapter 3, when comparing this problem with Problem 1.2.2.

In [8, Theorem 1.9], Balogh, Li and Treglown asymptotically resolved Problem 1.2.2 for H with $\chi_<(H) = 2$. Further, they developed approaches to the absorbing and regularity methods for ordered graphs, including providing general absorbing and almost perfect tiling lemmas. Building on their results, we asymptotically resolve Problem 1.2.2 in all remaining cases (i.e., for all H with interval chromatic number at least 3). In addition to this result, we also resolve the Dirac-type problems for *H-covers* in ordered graphs and H-tilings covering a fixed proportion x of the vertices of an ordered graph for $x \in (0, 1)$.

All the results mentioned in this section are based on a coauthored paper [59] with Andrew Treglown. The formal statements of these results and their proofs are presented in Chapter 3.

1.3 Deficiency problems for graphs

Steiner triple systems and Latin squares are two well-studied objects in combinatorics. A *Steiner triple system* of order n consists of a ground set X of n elements and a set T of triples of X such that every pair of elements in X occurs in exactly one triple in T. A *Latin square* of order n consists of an $n \times n$ grid and n symbols such that each symbol occurs precisely once in every column and every row. A natural question dating back to the 1970s asks for the order of the smallest Steiner triple system a fixed partial Steiner triple system can be embedded into (see e.g. [21, 98, 106]). Similarly, there has been interest in establishing the order of the smallest Latin square that a fixed partial Latin square can be embedded into (see e.g. [36, 48, 106]).

Motivated by these research directions, Nenadov, Sudakov and Wagner [106] introduced the notion of *graph deficiency*: for a graph G and integer $t \geq 0$, denote by $G * K_t$ the *join* of G and K_t, which is the graph obtained from G by adding t new vertices and adding all edges incident to at least one of the new vertices. Given a global spanning property $\mathcal{P}$ and a graph G, the *deficiency* $\mathrm{def}(G)$ of the graph G with respect to the property $\mathcal{P}$ is the smallest $t \geq 0$ such that the join $G * K_t$ has property $\mathcal{P}$.

Note that the following special type of deficiency problem has been previously studied: given a graph H and $n \in \mathbb{N}$, what is the minimum number of vertices needed to ensure any H-packing on n vertices (i.e., a collection of edge-disjoint copies of H that together form a graph on n vertices) can be extended to an H-design (i.e., an H-packing of a complete graph)? See e.g. [63, 64] for background and results on this problem.

One of the main results in [106] is a bound on $\mathrm{def}(G)$ with respect to the Hamiltonicity property for graphs G of a given density. More precisely, Nenadov, Sudakov and Wagner determined exactly the maximum number of edges in an n-vertex graph G such that $G * K_t$ does not contain a Hamilton cycle.

Theorem 1.3.1 (Nenadov, Sudakov and Wagner [106])**.** *Let n and t be integers and G an n-vertex graph so that $G * K_t$ does not contain a Hamilton cycle. Then we have the following sharp bounds on $e(G)$.*

$$\textit{If } n+t \textit{ is even:} \quad e(G) \leq \binom{n}{2} - \begin{cases} \left(t(n-1) - \binom{t}{2}\right) & \textit{if } t \leq (n+4)/5, \\ \left(\binom{\frac{n+t+2}{2}}{2} - 1\right) & \textit{if } t \geq (n+4)/5. \end{cases}$$

$$\textit{If } n+t \textit{ is odd:} \quad e(G) \leq \binom{n}{2} - \begin{cases} \left(t(n-1) - \binom{t}{2}\right) & \textit{if } t \leq (n+1)/5, \\ \binom{\frac{n+t+1}{2}}{2} & \textit{if } t \geq (n+1)/5. \end{cases}$$

Another line of inquiry in [106] concerns the deficiency problem for K_r-factors. The following seminal result of Hajnal and Szemerédi [74] determines the minimum degree threshold for forcing a K_r-factor in a graph G.

Theorem 1.3.2 (Hajnal and Szemerédi [74])**.** *Every graph G on n vertices with $r|n$ and whose minimum degree satisfies $\delta(G) \geq (1-1/r)n$ contains a K_r-factor. Moreover, there are n-vertex graphs G with $\delta(G) = (1-1/r)n - 1$ that do not contain a K_r-factor.*

As mentioned in the previous section, Kühn and Osthus [94] determined, up to an additive constant, the minimum degree threshold for forcing an H-factor, for any fixed graph H.

The following result of Nenadov, Sudakov and Wagner [106] determines how many edges an n-vertex graph G needs to guarantee that $G * K_t$ contains a K_3-factor (provided that t is not too big compared to n).

Theorem 1.3.3 (Nenadov, Sudakov and Wagner [106])**.** *There exists $n_0 \in \mathbb{N}$ such that the following holds. Let $n, t \in \mathbb{N}$ so that $n \geq n_0$ and $3|(n+t)$, and let G be an n-vertex graph such that $G * K_t$ does not contain a K_3-factor. If $t \leq n/1000$ then*

$$e(G) \leq \binom{n}{2} - \binom{k}{2} - \begin{cases} k(n-k) & \text{if } t \text{ is odd,} \\ k(n-k-1) & \text{if } t \text{ is even,} \end{cases}$$

where $k := \lceil (t+1)/2 \rceil$. This bound on $e(G)$ is sharp.

Nenadov, Sudakov and Wagner [106] state that *'the study of deficiency concept by itself leads to intriguing open problems'*. In particular, the first open problem [106, Section 7] they raise is to extend Theorem 1.3.3 to the full range of t, and moreover to resolve the analogous question for K_r-factors in general.

We provide a full resolution of this problem via the following theorem.

Theorem 1.3.4. *Let $n, t, r \in \mathbb{Z}$ with $n \geq 2$, $t \geq 0$ and $r \geq 3$ such that $t < (r-1)n$ and $r|(n+t)$. Further, let $k := \lceil \frac{t+1}{r-1} \rceil$ and q be the integer remainder when t is divided by $r-1$. Let G be a graph on n vertices such that $G * K_t$ does not contain a K_r-factor. Then*

$$e(G) \leq \max\left\{ \binom{n}{2} - \binom{\frac{n+t}{r}+1}{2}, \binom{n}{2} - \binom{k}{2} - k(n-k-(r-2-q)) \right\}.$$

The proof of Theorem 1.3.4 is covered in Chapter 4, and is based on a coauthored paper [57] with Joseph Hyde and Andrew Treglown.

1.4 Induced poset saturation

Saturation problems have been well studied in graph theory. A graph G is H-*saturated* if it does not contain a copy of the graph H, but adding any edge to G from its complement creates a copy of H. Turán's celebrated theorem [133] can be stated in the language of saturation: it determines the maximum number of edges in a K_t-saturated n-vertex graph. In contrast, Erdős, Hajnal and Moon [43] determined the minimum number of edges in a K_t-saturated n-vertex graph; see the survey [32] for further results in this direction.

In recent years there has been an emphasis on developing the theory of saturation for *partially ordered sets*, or *posets* for short. In this thesis, all posets considered are finite. Furthermore, we view posets as finite collections of finite subsets of $\mathbb{N}$ (see Section 1.6 for more details on the formal definition of posets). In particular, we say that P is a *poset on* $[p] = \{1, 2, \ldots, p\}$ if P consists of subsets of $[p]$. Given two posets P and Q, a *poset homomorphism* from P to Q is a function $\phi : P \to Q$ such that for every $A, B \in P$, if $A \subseteq B$ then $\phi(A) \subseteq \phi(B)$. We say that P is a *subposet* of Q if there is an injective poset homomorphism from P to Q; otherwise, Q is said to be P-*free*. Further we say P is an *induced subposet* of Q if there is an injective poset homomorphism ϕ from P to Q such that for every $A, B \in P$ we have $\phi(A) \subseteq \phi(B)$ if and only if $A \subseteq B$; otherwise, Q is said to be *induced* P-*free*.

Turán-type problems for posets have been extensively studied (see, e.g., the survey [72]). In Chapter 5 we consider *minimum* saturation questions à la Erdős–Hajnal–Moon. For a fixed poset P, we say that a family $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is P-*saturated* if $\mathcal{F}$ is P-free, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, then P is a subposet of $\mathcal{F} \cup \{S\}$. A family $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is *induced* P-*saturated* if $\mathcal{F}$ is induced P-free, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, then P is an induced subposet of $\mathcal{F} \cup \{S\}$.

The study of minimum saturated posets was initiated by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [65] in 2013. In their work the relevant parameter is the size $\mathrm{sat}(n, P)$ of the smallest P-saturated family of subsets of $[n]$. See, e.g., [65, 85, 103] for various results on $\mathrm{sat}(n, P)$.

The induced analogue of $\mathrm{sat}(n, P)$ – denoted by $\mathrm{sat}^*(n, P)$ – was first considered by Ferrara, Kay, Kramer, Martin, Reiniger, Smith and Sullivan [52]. Thus, $\mathrm{sat}^*(n, P)$ is defined to be the size of the smallest induced P-saturated family of subsets of $[n]$. The following result from [85] (and implicit in [52]) shows that the parameter $\mathrm{sat}^*(n, P)$ has a dichotomy of behaviour.

Theorem 1.4.1. *[52, 85] For any poset P, either there exists a constant K_P with $sat^*(n, P) \leq K_P$ or $sat^*(n, P) \geq \log_2 n$, for all $n \in \mathbb{N}$.*

Probably the most important open problem in the area is to obtain a tight version of Theorem 1.4.1; that is, to replace the $\log_2 n$ in Theorem 1.4.1 with a term that is as large as possible. In fact, Keszegh, Lemons, Martin, Pálvölgyi and Patkós [85] made the following conjecture in this direction.

Conjecture 1.4.2. *[85] For any poset P, either there exists a constant K_P with $sat^*(n, P) \leq K_P$ or $sat^*(n, P) \geq n + 1$, for all $n \in \mathbb{N}$.*

Note that the lower bound of $n + 1$ is rather natural here. For example, it is the size of the largest *chain* in $2^{[n]}$ as well as the smallest possible size of the union of two consecutive 'layers' in $2^{[n]}$, namely the layer containing $[n]$ and the layer containing all subsets of $[n]$ of size exactly $n - 1$. Furthermore, such structures form minimum induced saturated families for the so-called fork poset $\vee$, i.e., $\mathrm{sat}^*(n, \vee) = n + 1$ [52]; so the lower bound in Conjecture 1.4.2 cannot be increased. There are also no known examples of posets P for which $\mathrm{sat}^*(n, P) = \omega(n)$.

In contrast, Ivan [81, Section 3] presented evidence that led her to conjecture a rather different picture for the *diamond* poset $\Diamond$ (see Figure 1.1 for the Hasse diagram of $\Diamond$).

Conjecture 1.4.3. *[81] $sat^*(n, \Diamond) = \Theta(\sqrt{n})$.*

In Chapter 5 we present the following improvement of Theorem 1.4.1.

Theorem 1.4.4. *For any poset P, either there exists a constant K_P with $sat^*(n, P) \leq K_P$ or $sat^*(n, P) \geq \min\{2\sqrt{n}, n/2 + 1\}$ for all $n \in \mathbb{N}$.*

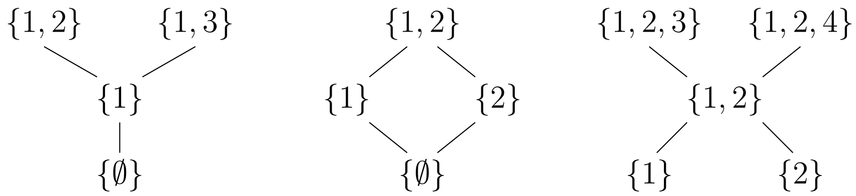

Figure 1.1: Hasse diagrams for the posets Y, $\Diamond$ and X. A segment indicates the set below is contained in the one above, and there is no intermediate set between them.

Note that if $n \geq 12$ then $n/2 + 1 \geq 2\sqrt{n}$. Thus, if Conjecture 1.4.3 is true, the lower bound in Theorem 1.4.4 would be tight up to a multiplicative constant.

On the other hand, we prove that Conjecture 1.4.2 holds for a certain class of posets (that does not include $\Diamond$), see Theorem 5.1.1.

Finally, another natural direction is to seek exact results on $\text{sat}^*(n, P)$. However, despite there already being several papers concerning $\text{sat}^*(n, P)$ [14, 34, 52, 80, 81, 85, 102], there are relatively few posets P for which $\text{sat}^*(n, P)$ is known precisely (see, e.g., Table 1 in [85] for a summary of most of the known results). We extends this limited pool of posets, determining $\text{sat}^*(n, X)$ and $\text{sat}^*(n, Y)$ (see Figure 1.1 for the Hasse diagrams of X and Y), see Theorem 5.1.2.

All the material mentioned in this section is covered in Chapter 5, and is based on a coauthored paper [58] with Simón Piga, Maryam Sharifzadeh and Andrew Treglown.

1.5 Typical Ramsey properties for abelian groups

Recall Ramsey's theorem asserts that, given any $r, t \in \mathbb{N}$, if $n \in \mathbb{N}$ is sufficiently large then however one partitions the edge set of the complete graph K_n into r colour classes, at least one of these colour classes must contain a copy of K_t. Analogous Ramsey-type behaviour is also exhibited in arithmetic settings. For example, Schur [123] (1916) proved that any finite colouring of $\mathbb{N}$ yields a monochromatic solution to the equation $x + y = z$, whereas van der Waerden [134] (1927) proved that any finite colouring of $\mathbb{N}$ yields arbitrarily long arithmetic progressions. In 1933, Rado [110] characterised all those homogeneous systems of linear equations $\mathcal{L}$ with integer coefficients for which every finite colouring of $\mathbb{N}$ yields a monochromatic solution to $\mathcal{L}$. In 1975, Deuber [37] resolved the analogous problem where now one works in the setting of abelian groups rather than $\mathbb{N}$. See, e.g., [16, 24, 119, 124, 128] for further Ramsey-type results for groups and other arithmetic structures.

Interestingly, certain arithmetic objects exhibit a *density-type* behaviour, that is, they already appear provided the underlying structure is not too sparse. For example, Szemerédi [129] (1975) famously proved that any subset S of $\mathbb{N}$ with positive upper density (i.e., such that $\frac{|S\cap[n]|}{n} \not\to 0$ as n goes to infinity) contains arbitrarily long arithmetic progressions. Generally, a density-type result immediately yields a Ramsey-type analogue. For instance, in any finite colouring of $\mathbb{N}$ one of the coloured sets must have positive upper density, and so Szemeredi's theorem implies the result of van der Waerden. Conversely, there are structures that exhibit a Ramsey-type behaviour but not a density-type one. For example, a density analogue of Schur's theorem does not hold since the set of odd numbers (which has positive upper density) does not contain a solution to the equation $x + y = z$. Other famous density-type results include Hilbert's theorem [78] (1892) and the Green-Tao theorem [71] (2008). The later appears in Chapter 6 (see Theorem 6.3.1).

Another natural Ramsey-type question is to investigate *how many* copies of a given structure one can guarantee within a partition of a mathematical object. This is the topic of a branch of Ramsey theory known as *Ramsey multiplicity*, in which researchers initially focused on graphs. For example, a seminal result of Goodman [68] from 1959 determines the minimum number of monochromatic triangles in a 2-edge-colouring of the complete graph K_n; see, e.g., the survey [22] for further results on the topic. More recently, there has been interest in studying Ramsey multiplicity in arithmetic structures, such as the integers (see, e.g., [31, 35, 69, 112, 122]) and abelian groups. In particular, after earlier work of Cameron, Cilleruelo and Serra [24], a 2017 paper of Saad and Wolf [119] initiated the systematic study of Ramsey multiplicity for systems of linear equations in abelian groups. A number of subsequent papers have made significant progress on the topic (see, e.g., [4, 39, 53, 83, 118, 136]).

In Chapter 6, we consider yet another Ramsey-type question, this time concerning the *typical Ramsey properties* of mathematical objects. Broadly speaking, once it is established that any finite partition of a set S yields a copy of a given structure, we are interested in whether we typically expect to witness the same Ramsey-type behaviour in subsets of S of a given size. As we will see, this problem is formalised by considering a random subset within S, and so we shall refer to it as the *random Ramsey problem*. This direction of research is part of the wider study of transferring (combinatorial) theorems into the random setting; see, e.g., the ICM survey of Conlon [26]. In particular, the random Ramsey problem for graphs and the integers has been well-studied, through the *random Ramsey theorem* [113, 114, 115] (see Theorem 6.5.1) and the *random Rado theorem [62, 116]* (see Theorems 6.5.3 and 6.5.4), respectively. The statements of both theorems, and of random Ramsey-type results in general, break down into two parts: the so-called 1-statement, asserting that random subsets of size above a certain threshold exhibit a Ramsey-type behaviour with high probability, and the 0-statement, asserting that random subsets of size below the threshold do not exhibit a Ramsey-type behaviour with high probability.

The main goal of Chapter 6 is to construct a general framework to study the random Ramsey problem for abelian groups. Our main result, which we call *the random Rado lemma* (Lemma 6.7.8), provides a machinery that, on input of a homogeneous system of linear equations $\mathcal{L}$ with integer coefficients, a sequence of 'well-behaved' subsets S_n of abelian groups and a Ramsey-type supersaturation result[1] for the number of monochromatic solutions of $\mathcal{L}$ in S_n, outputs an appropriate 1-statement and 0-statement. We give several highlight applications of the random Rado lemma. For example, we prove a random version of the *van der Waerden theorem for the primes* (Theorem 6.3.7) as well as a *random Rado theorem for integer lattices* (Theorem 6.6.1).[2]

Building towards the aforementioned theory for (subsets of) abelian groups, in Chap-

[1]A Ramsey-type supersaturation result states that in any r-colouring of a given mathematical object, a *constant proportion* of a certain class of substructures is monochromatic.

[2]In fact, we deduce Theorem 6.6.1 from a simplified version of the random Rado lemma (Lemma 6.7.33).

ter 6 we also survey the random Ramsey problem for graphs and the integers. As explained in more detail in Sections 6.5–6.8, there is a well-known heuristic for what the threshold in the 1- and 0-statements of a random Ramsey-type result should be. Indeed, this intuition turns out to be correct in the setting of graphs and the integers as well as for the various new results we prove. Therefore, it would be highly desirable to obtain a unifying theory which implies all previously known random Ramsey-type results. In this direction, we prove a general 1-statement and a general 0-statement for hypergraphs (Theorem 6.8.7) which imply several of the previously known 1- and 0-statements for discrete structures. Furthermore, the random Rado lemma arises as a direct corollary of this hypergraph result.

All the results mentioned in this section are covered in Chapter 6, and are based on a coauthored paper [55] with Robert Hancock and Andrew Treglown.

1.6 Notation

In this section we introduce various notation. Some notation which is relevant to single chapters may not appear here but rather at the beginning of the specific chapter.

We write $\mathbb{N}$ to denote the sets of positive integers. For any $n \in \mathbb{N}$, we write $[n]$ to denote the set of the first n positive integers $\{1, 2, \dots, n\}$. We write $2^{[n]}$ to denote the family consisting of all subsets of $[n]$. For $k \in \mathbb{N}$, we may refer to a set of size k as a *k-set* or a *k-tuple*.

Given two function $f, g : \mathbb{N} \to \mathbb{R} \setminus \{0\}$, we write

- $f = o(g)$ if $|f(n)/g(n)|$ tends to 0 as n goes to infinity.
- $f = \omega(g)$ if $|f(n)/g(n)|$ tends to infinity as n goes to infinity.
- $f = O(g)$ if there are $C, n_0 > 0$ such that $|f(n)/g(n)| \leq C$ for $n \geq n_0$.
- $f = \Omega(g)$ if there are $C, n_0 > 0$ such that $|f(n)/g(n)| \geq C$ for $n \geq n_0$.

We write $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. We write $f \sim g$ if $f(n)/g(n)$ tends to 1 as n goes to infinity.

The constants in the hierarchies used to state some of the results in this thesis are chosen from right to left. For example, if we claim that a result holds whenever $0 < a \ll b \ll c \leq 1$, then there are non-decreasing functions $f : (0, 1] \to (0, 1]$ and $g : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ with $b \leq f(c)$ and $a \leq g(b)$. Note that $a \ll b$ implies that we may assume in the proof that, e.g., $a < b$ or $a < b^2$.

Recall that the vertex and edge sets of a graph G are denoted as $V(G)$ and $E(G)$, respectively. We write $|G|$ and $e(G)$ to denote the cardinality of $V(G)$ and $E(G)$, respectively. Let $X \subseteq V(G)$. The graph *induced by X on G*, denoted as $G[X]$, has vertex set X and edge set $E(G[X]) := \{xy \in E(G) : x, y \in X\}$. For two disjoint subsets $A, B \subseteq V(G)$, the *induced bipartite subgraph* $G[A, B]$ is the subgraph of G whose vertex set is $A \cup B$ and whose edge set is $E(G[A, B]) := \{xy \in E(G) : x \in A\, ,\, y \in B\}$. We sometimes refer to $G[A, B]$ as the *pair* $(A, B)_G$, or simply (A, B) when the underlying graph is clear. For brevity, we write $G \setminus X := G[V(G) \setminus X]$ and $e(A, B) := e(G[A, B])$. For each $x \in V(G)$, we define the *neighbourhood of x in G* to be $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ and write $d_G(x) := |N_G(x)|$ for the degree of x in G. We define $d_G(x, X) := |N_G(x) \cap X|$. Furthermore, given a graph H, we say that $G[X]$ *spans a copy of H in G* if H is a spanning subgraph of $G[X]$.

A *digraph* D consists of a vertex set $V(D)$ and an edge set $E(D)$ where edges are ordered pairs of distinct vertices. Multiple copies of the same edge are not allowed. It is convenient to think of an edge $(v, u) \in E(D)$ as an edge which has been assigned an orientation from v to u. Note that in a digraph there are at most two edges between to vertices v and u, namely (v, u) and (u, v).

A *partially ordered set*, or *poset*, is a set X equipped with a binary relation $\leq$ between some elements of X such that: (i) $x \leq x$ for every $x \in X$, (ii) if $x \leq y$ and $y \leq x$ then $x = y$ and (iii) if $x \leq y$ and $y \leq z$ then $x \leq y$. It is well-known that if X is finite then there exists a collection $\mathcal{S}$ of finite subsets of $\mathbb{N}$ and a bijection $b : X \to \mathcal{S}$ such that $x \leq y$ if and only if $b(x) \subseteq b(y)$. In this thesis, all posets conisdered are finite and we view them as finite collections of finite subsets of $\mathbb{N}$.

Chapter 2

Hamilton cycles with large discrepancy

2.1 Introduction

In this chapter we present a proof of Theorem 1.1.3, which we now restate. Recall that a Hamilton cycle in a graph G is a cycle covering all vertices of G. For $t, r \in \mathbb{N}$, we say that an r-colouring of the edges of a graph H is t-unbalanced if at least $|E(H)|/r + t$ edges are coloured with the same colour.

Theorem 1.1.3. *Let $n, r, d \in \mathbb{N}$ with $r \geq 2$. Let G be an n-vertex graph with $\delta(G) \geq \left(\frac{1}{2} + \frac{1}{2r}\right) n + 6dr^2$. Then for every r-colouring of $E(G)$ there exists a d-unbalanced Hamilton cycle in G.*

In the following section we give constructions that show Theorem 1.1.3 is best possible; that is, there are n-vertex graphs G with minimum degree $\delta(G) = (1/2 + 1/2r)n$ such that for some r-colouring of $E(G)$, every Hamilton cycle in G uses precisely n/r edges of each colour.

2.2 The extremal constructions

Our first extremal example is a generalisation of a 2-colour construction from [6].

Extremal Example 1. *Let $r, n \in \mathbb{N}$ where $r \geq 2$ and such that $2r$ divides n. Then there exists a graph G on n vertices with $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$, and an r-colouring of $E(G)$, such that every Hamilton cycle uses precisely n/r edges of each colour.*

Proof. The vertex set of G is partitioned into r sets $V_1, \ldots, V_r$ such that $|V_i| = n/2r$ for every $i \in [r-1]$, and $|V_r| = (1/2 + 1/2r)n$; the edge set of G consists of all edges with at least one endpoint in V_r. Now colour the edges of G with colours $1, \ldots, r$ as follows:

- For each $i \in [r-1]$, colour every edge with one endpoint in V_i and one endpoint in V_r with colour i.
- Colour every edge with both endpoints in V_r with colour r (see Figure 2.1).

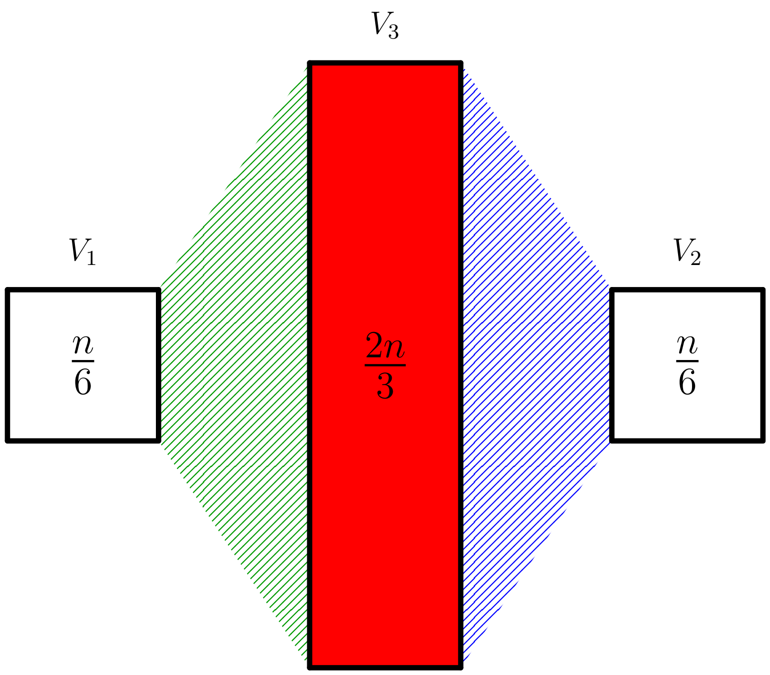

Figure 2.1: Extremal Example 1 for $r = 3$

Observe that $\delta(G) = \left(\frac{1}{2} + \frac{1}{2r}\right) n$, which is attained by every vertex in $V_1 \cup \ldots \cup V_{r-1}$. For each $i \in [r-1]$, every vertex in V_i is only adjacent to edges of colour i and $E(G[V_1 \cup$

$\ldots \cup V_{r-1}]) = \emptyset$. Hence every Hamilton cycle in G must contain precisely $2|V_i| = n/r$ edges of each colour $i \in [r-1]$. Since a Hamilton cycle has n edges, every Hamilton cycle in G must also contain n/r edges of colour r. Thus every Hamilton cycle in G uses precisely n/r edges of each colour. □

We also have an additional extremal example in the $r = 3$ case.

Extremal Example 2. *Let $n \in \mathbb{N}$ such that 3 divides n. Then there exists a graph G on n vertices with $\delta(G) = 2n/3$, and a 3-colouring of $E(G)$, such that every Hamilton cycle uses precisely $n/3$ edges of each colour and every vertex in G is incident to precisely two colours.*

Proof. Let G be the n-vertex 3-partite Turán graph. So G consists of three vertex sets V_1, V_2 and V_3, such that $|V_1| = |V_2| = |V_3| = n/3$, and all possible edges that go between distinct V_i and V_j. Colour all edges between V_1 and V_2 red; all edges between V_2 and V_3 blue; all edges between V_3 and V_1 green (see Figure 2.2).

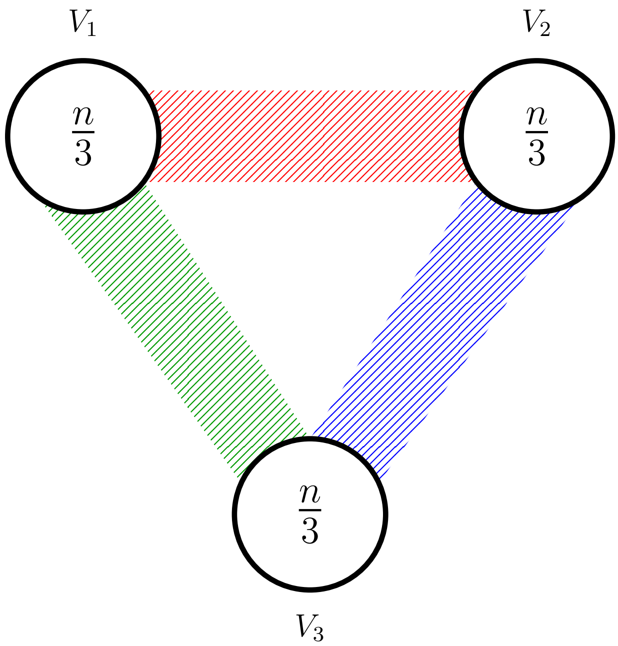

Figure 2.2: Extremal Example 2.

Clearly $\delta(G) = 2n/3$ and every vertex is incident to precisely two colours. Let H be a Hamilton cycle in G and let r, b and g be the number of red, blue and green edges in H, respectively. Since all red and green edges in H are incident to vertices in V_1, where V_1 is an independent set and $|V_1| = n/3$, we must have that $r + g = 2|V_1| = 2n/3$. Applying similar reasoning to V_2 and V_3, we have that $2n/3 = b + r$ and $2n/3 = g + b$. Hence $r = b = g = n/3$. Thus every Hamilton cycle in G uses precisely $n/3$ edges of each colour. □

2.3 Proof of Theorem 1.1.3

As in [6], we require the following generalisation of Dirac's theorem. Note that a *linear forest* is a disjoint union of paths.

Lemma 2.3.1 (Pósa [108]). *Let $1 \leq t \leq n/2$ and G be an n-vertex graph with $\delta(G) \geq \frac{n}{2}+t$. Let E' be a set of edges of a linear forest in G with $|E'| \leq 2t$. Then there is a Hamilton cycle in G containing E'.*

Proof of Theorem 1.1.3. Recall that G is a graph on n vertices with $\delta(G) \geq \left(\frac{1}{2} + \frac{1}{2r}\right) n + 6dr^2$ for some integers $r \geq 2$ and $d \geq 1$. Consider any r-colouring of $E(G)$. Given a colour c we define the function $L_c : E(G) \to \{0, 1\}$ as follows:

$$L_c(e) := \begin{cases} 1 & \text{if } e \text{ is coloured with } c, \\ 0 & \text{otherwise.} \end{cases}$$

Given a triangle xyz and a colour c, we define $\mathrm{Net}_c(xyz, xy)$ as follows:

$$\mathrm{Net}_c(xyz, xy) := L_c(xz) + L_c(yz) - L_c(xy).$$

This quantity comes from an operation we will perform later where we extend a cycle H, which contains the edge xy but not the vertex z, by deleting the edge xy from H and adding the edges xz and yz, to form a new cycle H' with vertex set $V(H') = V(H) \cup \{z\}$. One can see that $\mathrm{Net}_c(xyz, xy)$ is the change in the number of edges of colour c from H to H'.

Since $\delta(G) \geq \frac{1}{2}n$, by Dirac's theorem, G contains a Hamilton cycle C. If C is d-unbalanced we are done, so suppose it is not.

Claim 2.3.2. *Let $v \in V(G)$ and c be a colour. Let $Y \subseteq V(G)$ with $|Y| \leq 5dr^2$. There exists an edge xy which is vertex-disjoint from Y, is not coloured with c, and such that v and xy span a triangle in G.*

Proof. Let X be the set of neighbours of v, and X^+ the set of vertices whose 'predecessors' on C are neighbours of v, having arbitrarily chosen an orientation for C. We have

$$\begin{aligned} n \geq |X \cup X^+| = |X| + |X^+| - |X \cap X^+| &= 2d(v) - |X \cap X^+| \\ &\geq n + \frac{n}{r} + 12dr^2 - |X \cap X^+|. \end{aligned}$$

Hence $|X \cap X^+| \geq \frac{n}{r} + 12dr^2$. In particular, there are at least $\frac{n}{r} + 12dr^2$ edges e in C such that v and e span a triangle in G. Of these edges, at most $n/r + d$ are coloured with c (since C is not d-unbalanced) and at most $2|Y|$ are incident to Y (since each vertex in Y is incident to at most two edges in C). We have

$$\left(\frac{n}{r} + 12dr^2\right) - \left(\frac{n}{r} + d\right) - 2|Y| \geq \left(\frac{n}{r} + 12dr^2\right) - \left(\frac{n}{r} + dr^2\right) - 10dr^2 = dr^2 \geq 1.$$

Therefore, there exists an edge xy which forms a triangle with v, is vertex-disjoint from Y and is not coloured with c. □

Next, we repeatedly apply Claim 2.3.2 to obtain a collections of suitable "gadgets".

Claim 2.3.3. *There exist dr^2 vertex-disjoint sets of the form $\{v, x, y, w, z\} \subseteq V(G)$ so that the following holds for each set. Both vxy and vwz are triangles and there exists a colour c such that $Net_c(vxy, xy) \neq Net_c(vwz, wz)$.*

Proof. It suffices to show that, given ℓ vertex-disjoint sets $A_1 \ldots, A_\ell$ with the required properties for some $\ell < dr^2$, one can find an additional set.

Let Y be the set of all vertices which lie in $A_1 \cup \cdots \cup A_\ell$. We have $|Y| = 5\ell \leq 5dr^2 - 5$. Pick an arbitrary vertex $v \in V(G) \setminus Y$ (which exists, since $|Y| \leq 5dr^2 \leq \delta(G) < n$). By

Claim 2.3.2, there exists an edge xy which forms a triangle with v and is not incident to Y. Let c_1 be the colour of xy. By Claim 2.3.2 again, there exists an edge wz which forms a triangle with v, is not incident to $Y \cup \{x, y\}$, and receives a colour c_2 with $c_2 \neq c_1$.

By construction, we have $\{v, x, y, w, z\} \cap Y = \emptyset$. Therefore, if there exists a colour c such that $\mathrm{Net}_c(vxy, xy) \neq \mathrm{Net}_c(vwz, wz)$ then the set $\{v, x, y, w, z\}$ satisfies the required properties and we are done. If there is no such colour then we must have that $\mathrm{Net}_{c_1}(vxy, xy) = \mathrm{Net}_{c_1}(vwz, wz)$ and so

$$L_{c_1}(vx) + L_{c_1}(vy) - L_{c_1}(xy) = L_{c_1}(vw) + L_{c_1}(vz) - L_{c_1}(wz).$$

As xy and wz have colours c_1 and c_2 respectively, the above equality reduces to

$$L_{c_1}(vx) + L_{c_1}(vy) - 1 = L_{c_1}(vw) + L_{c_1}(vz) \geq 0.$$

Hence either $L_{c_1}(vx) = 1$ or $L_{c_1}(vy) = 1$. Without loss of generality, say $L_{c_1}(vx) = 1$, that is, vx is coloured with c_1. By following an analogous argument for $\mathrm{Net}_{c_2}(vxy, xy) = \mathrm{Net}_{c_2}(vwz, wz)$, we may assume that, without loss of generality, vw is coloured with c_2. Let c_3 be the colour of vy. Then $\mathrm{Net}_{c_3}(vxy, xy) = \mathrm{Net}_{c_3}(vwz, wz)$ and so

$$L_{c_3}(vx) + L_{c_3}(vy) - L_{c_3}(xy) = L_{c_3}(vw) + L_{c_3}(vz) - L_{c_3}(wz).$$

Note that $L_{c_3}(vx) = L_{c_3}(xy)$ since vx and xy are both coloured with c_1. Similarly, $L_{c_3}(vw) = L_{c_3}(wz)$ since vw and wz are both coloured with c_2. It follows that

$$L_{c_3}(vy) = L_{c_3}(vz).$$

Since vy is coloured with c_3, the above implies c_3 is also the colour of vz (see Figure 2.3). Since $c_1 \neq c_2$, either $c_1 \neq c_3$ or $c_2 \neq c_3$. We may assume, without loss of generality, that $c_1 \neq c_3$.

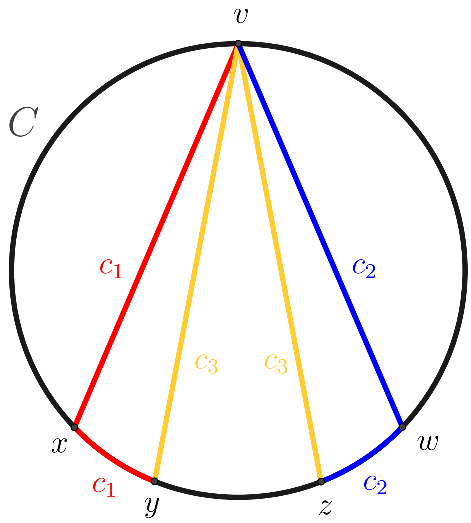

Figure 2.3: A Hamilton cycle C for G, with the structure we obtain when there is no colour c with $\text{Net}_c(vxy, xy) \neq \text{Net}_c(vzw, zw)$.

By Claim 2.3.2, there is an edge $w'z'$ which forms a triangle with x, is not incident to $Y \cup \{v, y\}$ and receives a colour c_4 with $c_4 \neq c_3$. Note that $\{x, v, y, w', z'\} \cap Y = \emptyset$.

Since xy and xv are coloured with c_1 and $c_1 \neq c_3$, we have

$$\text{Net}_{c_3}(xvy, vy) = L_{c_3}(xy) + L_{c_3}(xv) - L_{c_3}(vy) = -1.$$

Furthermore, since $w'z'$ is coloured with c_4 and $c_4 \neq c_3$, we have

$$\text{Net}_{c_3}(xw'z', w'z') = L_{c_3}(xw') + L_{c_3}(xz') - L_{c_3}(w'z') \geq 0.$$

In particular, $\text{Net}_{c_3}(xvy, vy) \neq \text{Net}_{c_3}(xw'z', w'z')$ and so the set $\{x, v, y, w', z'\}$ satisfies the required properties. $\square$

Let $\{\{v_i, x_i, y_i, w_i, z_i\} : i \in [dr^2]\}$ be a collection of sets as in Claim 2.3.3. In particular, for every $i \in [dr^2]$, there exists a colour c_i such that $\text{Net}_{c_i}(v_i x_i y_i, x_i y_i) \neq \text{Net}_{c_i}(v_i w_i z_i, w_i z_i)$.

By the pigeonhole principle, at least dr colours among $\{c_1, \ldots, c_{dr^2}\}$ must be the same. Without loss of generality we may assume $c^* = c_i$ for every $i \in [dr]$. By relabelling, we may further assume that for each $i \in [dr]$,

$$\mathrm{Net}_{c^*}(v_i x_i y_i, x_i y_i) > \mathrm{Net}_{c^*}(v_i w_i z_i, w_i z_i). \tag{2.1}$$

Consider the induced subgraph G' of G obtained from G by removing the vertices $v_1, \dots, v_{dr}$. Let $E' := \{x_i y_i : i \in [dr]\} \cup \{w_i z_i : i \in [dr]\}$. As $\delta(G') \geq \delta(G) - dr \geq n/2 + dr$, Lemma 2.3.1 implies that there exists a Hamilton cycle C' in G' which contains E'. Let C_1 be the Hamilton cycle of G obtained from C' by inserting each v_i between x_i and y_i; let C_2 be the Hamilton cycle of G obtained from C' by inserting each v_i between w_i and z_i. For $j \in \{1, 2\}$, write E_j for the number of edges in C_j of colour c^*. Note that (2.1) implies $E_1 - E_2 \geq dr$. If $E_1 \geq n/r + d$ then C_1 is d-unbalanced and we are done, so suppose $E_1 \leq n/r + d$. It follows that $(n/r + d) - E_2 \geq dr$ and so $E_2 \leq n/r - d(r-1)$. By the pigeonhole principle, there exists a colour c' with $c' \neq c^*$ such that there are $n/r + d$ edges coloured with c' in C_2, and so C_2 is d-unbalanced. This concludes the proof. □

2.4 Concluding remarks

As mentioned in [41, Section 7] there are many possible directions for future research. For example, it would be interesting to determine whether the term $6dr^2$ in Theorem 1.1.3 can be improved. It seems new ideas are required to obtain a term better than $\Theta(dr^2)$.

One natural extension of our work is to seek an analogue of Theorem 1.1.3 in the setting of digraphs.

Question 2.4.1. *Given any digraph G on n vertices with minimum in- and outdegree at least $(1/2 + 1/2r + o(1))n$, and any r-colouring of $E(G)$, can one always ensure a Hamilton cycle in G of significant colour-bias?*

Note that the natural digraph analogues of our extremal constructions for Theorem 1.1.3 show that one cannot lower the minimum degree condition in Question 2.4.1.

Chapter 3

Dirac-type problems for vertex ordered graphs

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3.1 Introduction

First, we recall some relevant definitions for this chapter. A *(vertex) ordered graph* H on h vertices is a graph with vertex set $[h]$. An ordered graph G with vertex set $[n]$ *contains* an ordered graph H on $[h]$ if (i) there is an injection $\phi : [h] \to [n]$ such that $\phi(i) < \phi(j)$ for all $1 \leq i < j \leq h$ and (ii) $\phi(i)\phi(j)$ is an edge in G whenever ij is an edge in H. The image of ϕ is referred to as a *copy* of H in G. In both the ordered and unordered settings, an H*-tiling* in a graph G is a collection of vertex-disjoint copies of H contained in G. An H-tiling is *perfect* if it covers all the vertices of G.

As mentioned in Section 1.2, Balogh, Li and Treglown raised the following question, which they asymptotically resolved for any ordered graph H such that $\chi_<(H) = 2$ [8, Theorem 1.9].

Problem 1.2.2. *[8] Given any ordered graph H and any $n \in \mathbb{N}$ divisible by $|H|$, determine the smallest integer $\delta_<(H, n)$ such that every n-vertex ordered graph G with $\delta(G) \geq \delta_<(H, n)$ contains a perfect H-tiling.*

In this chapter, we present an (asymptotic) resolution of Problem 1.2.2 in all remaining cases (i.e., for all H with interval chromatic number at least 3). Our main result, which is formally stated in the next section, shows that Problem 1.2.2 does exhibit a somewhat different behaviour when $\chi_<(H) \geq 3$ compared to when $\chi_<(H) = 2$; we discuss this further in Section 3.1.3.

In addition to this result, we also resolve the Dirac-type problems for *H-covers* in ordered graphs (see Section 3.1.2) and H-tilings covering a fixed proportion x of the vertices of an ordered graph for $x \in (0, 1)$ (see Section 3.1.4).

3.1.1 A Dirac-type theorem for perfect H-tilings

In this subsection we state Theorem 3.1.6, which asymptotically resolves Problem 1.2.2 when $\chi_<(H) \geq 3$. This result will depend on several definitions and parameters which we now introduce. Firstly, we recall the definitions of *interval colouring* and *interval chromatic number*.

Definition 1.2.1 (Interval chromatic number)**.** Given $r \in \mathbb{N}$, an *interval r-colouring* of an ordered graph H is a partition of the vertex set $[h]$ of H into r (possibly empty) intervals so that no two vertices belonging to the same interval are adjacent in H. The *interval chromatic number* $\chi_<(H)$ of an ordered graph H is the smallest $r \in \mathbb{N}$ such that there exits an interval r-colouring of H.

One can think of the interval chromatic number as the natural ordered analogue of the chromatic number of a graph. Moreover, whilst the Erdős–Stone–Simonovits theorem [44, 46] asserts that $1 - 1/(\chi(H) - 1) + o(1)$ is the edge density threshold for ensuring a copy of a graph H in G, the *Erdős–Stone–Simonovits theorem for ordered graphs* due to Pach and Tardos [107] asserts the corresponding threshold in the ordered graph setting is $1 - 1/(\chi_<(H) - 1) + o(1)$.

The next definition is the relevant parameter for studying *almost* perfect H-tilings in graphs.

Definition 3.1.1 (Critical chromatic number)**.** The *critical chromatic number* $\chi_{cr}(F)$ of an unordered graph F is defined as

$$\chi_{cr}(F) := (\chi(F) - 1)\frac{|F|}{|F| - \sigma(F)},$$

where $\sigma(F)$ denotes the size of the smallest possible colour class in any $\chi(F)$-colouring of F.

Note that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$ for all graphs H, and $\chi_{cr}(H) = \chi(H)$ precisely when every $\chi(H)$-colouring of H has colour classes of equal size.

We informally refer to an H-tiling in an n-vertex (ordered) graph G as an *almost perfect H-tiling* if it covers all but at most $o(n)$ vertices of G. Komlós [87] proved that $(1 - 1/\chi_{cr}(H))n$ is the minimum degree threshold for forcing an almost perfect H-tiling in an n-vertex graph G. In fact, it was later shown [125] that such graphs G contain H-tilings covering all but a constant number of the vertices in G. In the setting of ordered graphs, the relevant parameter for forcing an almost perfect H-tiling is the so-called *ordered critical chromatic number* $\chi^*_{cr}(H)$ (see Definition 3.1.3 below). To introduce this parameter we need the following definitions.

For two subsets X, Y of $[n]$, we write $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$. Let B be a complete k-partite unordered graph with parts $U_1, \dots, U_k$, and σ be a permutation of the set $[k]$. An *interval labelling* of B with respect to σ is a bijection $\phi : V(B) \to [|B|]$ such that $\phi(U_i) < \phi(U_j)$ if $\sigma(i) < \sigma(j)$; that is,

$$\phi(U_{\sigma^{-1}(1)}) < \cdots < \phi(U_{\sigma^{-1}(k)}).$$

For brevity, we will usually drop ϕ and just write $U_{\sigma^{-1}(1)} < \cdots < U_{\sigma^{-1}(k)}$. Given $t \in \mathbb{N}$, write $B(t)$ for the *blow-up of* B with vertex set $\bigcup_{x \in V(B)} V_x$, where the V_x's are sets of t independent vertices; so there are all possible edges between V_x and V_y in $B(t)$ if $xy \in E(B)$. Given an interval labelling ϕ of B, let $(B(t), \phi)$ be the ordered graph obtained from $B(t)$ by equipping $V(B(t))$ with a vertex ordering, satisfying $V_x < V_y$ for every $x, y \in V(B)$ with $\phi(x) < \phi(y)$. We refer to $(B(t), \phi)$ as an *ordered blow-up* of B.

Definition 3.1.2 (Bottlegraph)**.** For an ordered graph H, we say that a complete k-partite unordered graph B is a *bottlegraph of* H, if for every permutation σ of $[k]$ and every interval labelling ϕ of B with respect to σ, there exists a constant $t = t(B, H, \phi)$ such that the ordered blow-up $(B(t), \phi)$ contains a perfect H-tiling. We say that B is a *simple bottlegraph of* H if for any choice of σ and ϕ we can take $t = 1$.

As an example, consider the ordered graph H with $V(H) = [2h]$ for some $h \in \mathbb{N}$ and all edges between $\{1, \dots, h\}$ and $\{h+1, \dots, 2h\}$. Then any complete bipartite graph with parts of the same size is a bottlegraph of H. Conversely, any complete bipartite graph with parts of different size is not a bottlegraph of H.

We are now ready to give the formal definition of ordered critical chromatic number.

Definition 3.1.3 (Ordered critical chromatic number)**.** The *ordered critical chromatic number* $\chi^*_{cr}(F)$ of an ordered graph F is defined as

$$\chi^*_{cr}(F) := \inf\{\chi_{cr}(B) : B \text{ is a bottlegraph of } F\}.$$

We say a bottlegraph B of F is *optimal* if $\chi_{cr}(B) = \chi^*_{cr}(F)$.

As we will prove in Proposition 3.5.1, given any bottlegraph B of an ordered graph F, there is another bottlegraph B' of F such that $\chi_{cr}(B') = \chi_{cr}(B)$ and all parts of B' are of the same size, except for perhaps one smaller part. Therefore, in Definition 3.1.3, we could have equivalently defined $\chi^*_{cr}(F)$ by taking the infimum over all bottlegraphs of F whose parts are of the same size, except for perhaps one smaller part. This bottle-like structure is where the name bottlegraph is derived, and was first used by Komlós [87] in the setting of unordered graphs.

Notice that $\chi_<(H) - 1 \leq \chi^*_{cr}(H)$ for all ordered graphs H as each bottlegraph B of H must have chromatic number at least $\chi_<(H)$ and so $\chi_<(H) - 1 < \chi_{cr}(B)$. In fact, Proposition 3.2.6 in Section 3.2 yields a stronger lower bound on $\chi^*_{cr}(H)$. On the other hand, (in contrast to $\chi_{cr}(F)$ for unordered graphs F) we will also see examples of ordered graphs where $\chi^*_{cr}(H)$ is much larger than $\chi_<(H)$. Note though that $\chi^*_{cr}(H) \leq h$ for any ordered graph H on $[h]$ as K_h is a bottlegraph of H. In fact, this upper bound is attained when H is such that 1 and 2 are adjacent or $h-1$ and h are adjacent; this is an immediate consequence of Proposition 3.11.1. To aid the reader's intuition, in Section 3.3.3 we give examples of ordered graphs H where we compute $\chi^*_{cr}(H)$. Various bounds on $\chi^*_{cr}(H)$ are given in Section 3.11.

The next result, a simple corollary of [8, Theorem 4.3], shows that $\chi^*_{cr}(H)$ is a relevant parameter for forcing an almost perfect H-tiling in an ordered graph.[3]

Theorem 3.1.4 (Balogh, Li and Treglown [8]). *Let H be an ordered graph. Then for every $\eta > 0$, there exists an integer $n_0 = n_0(H, \eta)$ so that every ordered graph G on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n$$

contains an H-tiling covering all but at most ηn vertices.

From [8], it is not clear whether the minimum degree threshold in Theorem 3.1.4 is best possible. However, we prove in Theorem 3.12.1 (see Section 3.12) that Theorem 3.1.4 is optimal in the sense that one cannot replace the $1-1/\chi^*_{cr}(H)$ term in the minimum degree condition with any other fixed constant term $a < 1 - 1/\chi^*_{cr}(H)$. Thus, Theorem 3.12.1 and Theorem 3.1.4 provide an analogue of Komlós' theorem in the ordered setting.

Unusually, in the proof of Theorem 3.12.1, for most ordered graphs H we do not simply produce an explicit extremal example. Indeed, if one has not explicitly computed the value of $\chi^*_{cr}(H)$ and the 'reason' why it takes this value, then it seems difficult to produce such an explicit extremal example. Instead, the proof splits into a few cases and uses various tools and results that we introduce in the chapter.

[3]The reader may not immediately see why Theorem 3.1.4 is a corollary of Theorem 4.3 from [8] as there is an error term in the minimum degree condition in the latter theorem. However, a standard trick (see, e.g., the proof of Corollary 6.6 in [33]) always allows one to omit an error term in a minimum degree condition that forces an almost perfect H-tiling.

At this point the reader may wonder if the conclusion of Theorem 3.1.4 can be strengthened to ensure a *perfect* H-tiling. For some ordered graphs H this is possible. However, for other ordered graphs one will require a significantly higher minimum degree condition. The following definition is the critical concept for articulating this dichotomy for H with $\chi_<(H) \geq 3$.

Definition 3.1.5 (Local barrier)**.** Let H be an ordered graph on $[h]$ with $r := \chi_<(H) \geq 2$. We say that H has a *local barrier* if for some fixed $i \neq j \in [r+1]$ the following condition holds. Given any interval $(r+1)$-colouring of H with colour classes $V_1 < \cdots < V_{r+1}$ such that $V_i = \{v\}$ is a singleton class, there is at least one edge between v and V_j in H.

Note that in this definition we may have that a colour class V_k (where $k \neq i$) is empty. Next, we give a couple of examples.

If H is the ordered complete graph on r vertices then H does not have a local barrier; it is also easy to check that $\chi^*_{cr}(H) = \chi_<(H) = r$. Conversely, given $r \geq 2$, let H' be any complete r-partite (unordered) graph with at least 2 vertices in each colour class. Let H be any ordered graph obtained from H' by assigning an interval labelling to H'; so $\chi_<(H) = r$. Then one can check that H has a local barrier with parameters $i = 1$ and $j = r+1$ as in Definition 3.1.5. For instance, consider the case $r = 2$. Given any interval 3-colouring of H with colour classes $V_1 < V_2 < V_3$ and $|V_1| = 1$, if there is no edge between V_1 and V_3 then V_3 must be empty. It follows that $V_1 \cup V_2$ is a bipartition of H' with parts of size 1 and $|H| - 1$, which is impossible.

We are now able to state our main result which resolves Problem 1.2.2 for all ordered graphs H with $\chi_<(H) \geq 3$.

Theorem 3.1.6. *Let H be an ordered graph with $\chi_<(H) \geq 3$. Given any $\eta > 0$, there exists an integer $n_0 = n_0(H, \eta)$ so that if $n \geq n_0$ and $|H|$ divides n then*

(i) $\left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n \leq \delta_<(H, n) \leq \left(1 - \frac{1}{\chi^*_{cr}(H)} + \eta\right) n$ *if $\chi^*_{cr}(H) \geq \chi_<(H)$;*

(ii) $\left(1 - \frac{1}{\chi_<(H)}\right) n < \delta_<(H, n) \leq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n$ *if $\chi^*_{cr}(H) < \chi_<(H)$ and H has a local barrier;*

(iii) $\left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n \leq \delta_<(H, n) \leq \left(1 - \frac{1}{\chi^*_{cr}(H)} + \eta\right) n$ *if $\chi^*_{cr}(H) < \chi_<(H)$ and H has no local barrier.*

Therefore Problem 1.2.2 is now asymptotically resolved. The reader might find it hard to see why the value of $\delta_<(H, n)$ behaves as in Theorem 3.1.6, and indeed, it took quite some time to discover the correct behaviour of this problem. In Section 3.1.3 we give further intuition on this result. In Section 3.3 we give examples of H in each case (i)–(iii) of the theorem. In Section 3.2 we give the extremal constructions for Theorem 3.1.6. In particular, in cases (i) and (iii) the extremal examples are 'bottlegraphs' – complete multipartite ordered graphs where each part has the same size except at most one smaller part; in Section 3.11 we explicitly compute the value of $\chi^*_{cr}(H)$ for a range of H, and thus the minimum degree threshold in Theorem 3.1.6 also. The explanation for why these extremal 'bottlegraphs' do not have perfect H-tilings revolve around 'space barrier' constraints, (i.e., one runs out of space in some subset of vertices in the extremal bottlegraph).[4] However, the 'reason' for obtaining a space barrier can be somewhat involved (and unlike any other space barrier extremal example we have ever seen before); we discuss this in Section 3.3.1.

The proof of Theorem 3.1.6 applies an absorbing theorem from [8, Theorem 4.1] and Theorem 3.1.4 above. The main novelty is to prove an absorbing theorem for ordered graphs H as in Theorem 3.1.6(iii). See Section 3.9 for an overview of our absorbing strategy.

[4]See [84] for a discussion on space barriers.

3.1.2 A Dirac-type theorem for vertex covers

Given (ordered) graphs H and G, we say that G has an *H-cover* if every vertex in G lies in a copy of H. Note that the notion of an H-cover is an 'intermediate' between seeking a single copy of H and a perfect H-tiling; in particular, a perfect H-tiling in G is itself an H-cover. Given any $n \in \mathbb{N}$ and any (ordered) graph H, let $\delta_{\text{cov}}(H, n)$ denote the smallest integer k such that every n-vertex (ordered) graph G with $\delta(G) \geq k$ contains an H-cover. As noted in [95], an easy application of Szemerédi's regularity lemma asymptotically determines $\delta_{\text{cov}}(H, n)$ for all graphs H.

Proposition 3.1.7. *[95, Proposition 6] For every graph H and every $\eta > 0$, there exists an integer $n_0 = n_0(H, \eta)$ so that if $n \geq n_0$ then*

$$\left(1 - \frac{1}{\chi(H) - 1}\right) n - 1 \leq \delta_{cov}(H, n) \leq \left(1 - \frac{1}{\chi(H) - 1} + \eta\right) n.$$

Proposition 3.1.7 implies that, asymptotically, the minimum degree threshold for ensuring an H-cover in a graph G is the same as the minimum degree threshold for ensuring a *single copy* of H in G. In [95, Theorem 5], Kühn, Osthus and Treglown asymptotically determined the Ore-type degree condition that forces an H-cover for any fixed graph H. There has also been several recent papers concerning minimum ℓ-degree conditions that force H-covers in k-uniform hypergraphs; see, e.g., [50, 51, 75].

Falgas-Ravry [49] raised the question of determining $\delta_{\text{cov}}(H, n)$ for all ordered graphs H. Our next result asymptotically answers this question.

Theorem 3.1.8. *Let H be an ordered graph and $\eta > 0$. Then there exists an integer $n_0 = n_0(H, \eta)$ so that if $n \geq n_0$ then*

(i) $\left(1 - \frac{1}{\chi_<(H)-1}\right) n < \delta_{cov}(H, n) \leq \left(1 - \frac{1}{\chi_<(H)-1} + \eta\right) n$ *if H has no local barrier;*

(ii) $\left(1 - \frac{1}{\chi_<(H)}\right) n < \delta_{cov}(H, n) \leq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n$ *if H has a local barrier.*

Theorem 3.1.8 is a direct consequence of some of the auxiliary results we use in the proof of Theorem 3.1.6. Note that the behaviour of the threshold in Theorem 3.1.8

is perhaps unexpected. Indeed, unlike in the unordered setting, Theorem 3.1.8 and the Erdős–Stone–Simonovits theorem for ordered graphs imply that the asymptotic minimum degree thresholds for forcing a copy of H and an H-cover are different if H has a local barrier.

Furthermore, a key moral of the Erdős–Stone–Simonovits theorem (and Proposition 3.1.7) is that once a graph G is dense enough (or has large enough minimum degree) so as to ensure a copy of K_r (or a K_r-cover) then G must contain every fixed graph H (or an H-cover) for every H of chromatic number r. An intuition for this comes from Szemerédi's regularity lemma. However, the analogous moral is not true for H-covers in ordered graphs. Indeed, if H is an ordered complete graph on r vertices (so $\chi_<(H) = r$ and H has no local barrier) then Theorem 3.1.8 tells us the minimum degree threshold for forcing an H-cover in an n-vertex ordered graph is $(1 - \frac{1}{r-1} + o(1))n$ whilst the corresponding threshold for any 'blow-up' H' of H (so $\chi_<(H') = \chi_<(H) = r$ and H has a local barrier) is significantly higher, namely $(1 - \frac{1}{r} + o(1))n$. This should hint to the reader that the regularity method behaves differently in the ordered setting; in particular, if H has a local barrier this provides an obstruction when applying the regularity lemma. More discussion on the regularity method for ordered graphs can be found in [8, Section 3.1].

3.1.3 Intuition behind the threshold in Theorem 3.1.6

In this subsection we build up further intuition behind the threshold in Theorem 3.1.6. For this it will be useful to first take a step back and consider perfect H-tilings in unordered graphs. In this setting the Dirac-type threshold is governed by two factors:

$(C1)$ The minimum degree needs to be large enough to force an almost perfect H-tiling.

$(C2)$ The minimum degree must be large enough to prevent 'divisibility' barriers within the host graph that constrain us from turning an almost perfect H-tiling into a perfect H-tiling.

This is made precise by the following theorem of Kühn and Osthus [94]. (See [94, Section

1.2] for the definition of $\mathrm{hcf}(H) = 1$.)

Theorem 3.1.9 (Kühn and Osthus [94])**.** *Let $\delta(H, n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \geq k$ contains a perfect H-tiling. For every unordered graph H,*

$$\delta(H, n) = \left(1 - \frac{1}{\chi^*(H)}\right) n + O(1),$$

where $\chi^*(H) := \chi_{cr}(H)$ if $hcf(H) = 1$ and $\chi^*(H) := \chi(H)$ otherwise.

Recall that every graph H satisfies $\chi_{cr}(H) \leq \chi(H)$. So by Komlós' aforementioned almost perfect tiling theorem [87], the minimum degree condition in Theorem 3.1.9 is enough to ensure $(C1)$ holds. Meanwhile those graphs H with $\mathrm{hcf}(H) = 1$ are precisely the graphs for which, at the almost perfect tiling threshold, $(C2)$ is satisfied. Furthermore, for graphs H with $\mathrm{hcf}(H) \neq 1$, $(C2)$ is only guaranteed to be satisfied once the n-vertex host graph has minimum degree around $(1 - 1/\chi(H))n$.

We do not state the precise definition of $\mathrm{hcf}(H) = 1$ here, however, the following example is instructive. Let H be any connected bipartite graph and let $n \in \mathbb{N}$ be divisible by $|H|$. Consider the n-vertex graph G that consists of two disjoint cliques whose sizes are as equal as possible so that neither is divisible by $|H|$. Then whilst G contains an almost perfect H-tiling, the divisibility constraint on the clique sizes prevents a perfect H-tiling. Thus all such H are examples of graphs with $\mathrm{hcf}(H) \neq 1$. In particular, $\delta(G) = n/2 - O(1) = (1 - 1/\chi(H))n - O(1)$, so G is an extremal example for Theorem 3.1.9 in this case.

As mentioned earlier, another necessary condition for a Dirac-type threshold for perfect H-tilings is the following:

$(C3)$ The minimum degree needs to be large enough to force an H-cover.

Condition $(C3)$, however, does not factor into the statement of Theorem 3.1.9 as Proposition 3.1.7 shows that one can ensure an H-cover 'earlier' than an almost perfect H-tiling (recall that $\chi(H) - 1 < \chi_{cr}(H)$).

Interestingly, the opposite is true in the ordered graph setting when H has a local barrier and $\chi_<(H) \geq 3$ is such that $\chi_<(H) > \chi^*_{cr}(H)$. Indeed, in this case, Theorem 3.1.6(ii) essentially states that the H-cover condition is the 'last' of conditions $(C1)$–$(C3)$ to be satisfied. In particular, Extremal Example 1 in Section 3.2 shows that for every H satisfying Definition 3.1.5, there are n-vertex ordered graphs with $\delta(G) > (1 - 1/\chi_<(H))n - 1$ for which a certain vertex does not lie in a copy of H.

In all other cases when $\chi_<(H) \geq 3$, Theorem 3.1.6 essentially states that the almost perfect tiling condition is the 'last' of conditions $(C1)$–$(C3)$ to be satisfied. Therefore, surprisingly, divisibility barriers play no role in Problem 1.2.2 for H with $\chi_<(H) \geq 3$. In contrast, Theorem 1.9 in [8] shows that in the case when $\chi_<(H) = 2$, each of $(C1)$, $(C2)$ and $(C3)$ can be the condition that governs the Dirac-type threshold for perfect H-tiling, depending on the choice of H.

3.1.4 A Dirac-type theorem for H-tilings

In addition to determining the Dirac-type threshold for almost perfect tilings in unordered graphs, Komlós [87] provided a best-possible minimum degree condition for forcing an H-tiling covering a certain proportion of the vertices in a graph G.

Definition 3.1.10 ((x, H)-tilings)**.** Let G and H be (ordered) graphs and $x \in [0, 1]$. An (x, H)*-tiling in* G is an H-tiling covering at least $x|G|$ vertices. So a $(1, H)$-tiling is simply a perfect H-tiling.

Theorem 3.1.11 (Komlós [87])**.** *Let H be a graph and $x \in (0, 1)$. Define*

$$g(x, H) := \left(1 - \frac{1}{\chi(H) - 1}\right)(1 - x) + \left(1 - \frac{1}{\chi_{cr}(H)}\right)x.$$

Given any $\eta > 0$, there exists some $n_0 = n_0(x, H, \eta) \in \mathbb{N}$ such that if G is a graph on n vertices where $n \geq n_0$ and $\delta(G) \geq g(x, H) \cdot n$ then there exists an $(x - \eta, H)$-tiling in G.

Note that the minimum degree condition in Theorem 3.1.11 is best possible in the sense that given any fixed H and $x \in (0, 1)$, one cannot replace $g(x, H)$ with any fixed $g'(x, H) < g(x, H)$ (see [87, Theorem 7] for a proof of this).

The function $g(x, H)$ is quite well-behaved. Indeed, $g(x, H)$ grows linearly in x for fixed H. Note that $g(0, H) \cdot n$ and $g(1, H) \cdot n$ are the asymptotic minimum degree thresholds for ensuring an n-vertex graph contains a copy of H and an almost perfect H-tiling respectively. From this prospective, the function $g(x, H)$ can be viewed as a linear interpolation of these two thresholds.

The question of obtaining an ordered graph analogue of Theorem 3.1.11 was raised in [8, Question 8.2]. We provide an answer to this problem; for this we require the following definitions.

Definition 3.1.12 (x-bottlegraphs)**.** Let H be an ordered graph and $x \in (0, 1]$. An unordered graph B is an *x-bottlegraph of H* if it satisfies the following properties:

(i) B is a complete k-partite graph with parts $U_1, U_2, \dots, U_k$, for some $k \in \mathbb{N}$.

(ii) There exists some $m \in \mathbb{N}$ such that $|U_1| \leq m$ and $|U_i| = m$ for every $i > 1$.

(iii) Given any permutation σ of $[k]$ and any interval labelling of B with respect to σ, the resulting ordered graph contains an (x, H)-tiling.

Definition 3.1.13. Let H be an ordered graph and $x \in (0, 1]$. We define $\chi^*_{cr}(x, H)$ as

$$\chi^*_{cr}(x, H) := \inf\{\chi_{cr}(B) \,:\, B \,\text{is an}\, x\text{-bottlegraph of}\, H\}.$$

Given any ordered graph H, if B is an x-bottlegraph of H then $\chi(B) \geq \chi_<(H)$. This implies that $\chi_{cr}(B) > \chi_<(H) - 1$ and so

$$\chi^*_{cr}(x, H) \geq \chi_<(H) - 1. \tag{3.1}$$

An application of Theorem 3.1.11 together with a tool from [8, Lemma 6.2] yields the following minimum degree condition for the existence of (x, H)-tilings in ordered graphs.

Theorem 3.1.14. *Let H be an ordered graph, $x \in (0,1)$ and define*

$$f(x,H) := \left(1 - \frac{1}{\chi^*_{cr}(x,H)}\right).$$

Given any $\eta > 0$, there exists some $n_0 = n_0(x,H,\eta) \in \mathbb{N}$ such that if G is an ordered graph on n vertices with $n \geq n_0$ and $\delta(G) \geq (f(x,H)+\eta)n$ then G contains an (x,H)-tiling.

The minimum degree condition in Theorem 3.1.14 is best possible in the following sense. Let H and $x \in (0,1)$ be fixed. Given any $0 < a < 1 - 1/\chi^*_{cr}(x,H)$, and any sufficiently large $n \in \mathbb{N}$, consider any n-vertex graph B that satisfies (i) and (ii) of Definition 3.1.12 for some choice of $k, m \in \mathbb{N}$ and where

$$an < \delta(B) = n - m = \left(1 - \frac{1}{\chi_{cr}(B)}\right)n < f(x,H) \cdot n.$$

So $\chi_{cr}(B) < \chi^*_{cr}(x,H)$. (Note such a graph B exists for any choice of $0 < a < 1 - 1/\chi^*_{cr}(x,H)$.) Then by definition of $\chi^*_{cr}(x,H)$ there is a permutation σ of $[k]$ and an interval labelling ϕ of B with respect to σ, such the resulting ordered graph (B,ϕ) does not contain an (x,H)-tiling.

A draw-back of Theorem 3.1.14 is that it seems hard to compute $\chi^*_{cr}(x,H)$ in general. However, in Section 3.14 we describe the behaviour of the function $f(x,H)$ for some fixed ordered graphs H. In particular, akin to Theorem 3.1.11, if H has $\chi_<(H) = 2$ then $f(x,H)$ is linear in x. Perhaps surprisingly though, there are ordered graphs where $f(x,H)$ is only piecewise linear. We also compute $f(x,H)$ for every ordered graph H and every x that is not too big.

3.1.5 Organisation of the chapter

The rest of this chapter is organised as follows. In Section 3.2 we give the extremal constructions for Theorems 3.1.6 and 3.1.8. In Section 3.3 we give some examples of ordered graphs H that fall into each of the three cases of Theorem 3.1.6. In Section 3.4

we state a new absorbing theorem (Theorem 3.4.2) and an absorbing theorem from [8] and combine them with Theorem 3.1.4 to prove Theorem 3.1.6. The subsequent sections therefore build up tools for the proof of Theorem 3.4.2: in Section 3.5 we state a couple of useful properties of bottlegraphs; in Section 3.6 we introduce Szemerédi's regularity lemma and related useful results; some tools for absorbing are given in Section 3.7; Section 3.8 contains several results which give flexibility in how one can interval colour certain ordered graphs H.

In Section 3.9 we give a sketch of the proof of Theorem 3.4.2 before proving it and Theorem 3.1.8 in Section 3.10. In Section 3.11 we give general upper and lower bounds on $\chi^*_{cr}(H)$ and also compute $\chi^*_{cr}(H)$ for a few general classes of ordered graphs H.

In Section 3.12 we prove that the minimum degree condition in Theorem 3.1.4 is best possible. The proof of Theorem 3.1.14 is given in Section 3.13; in the subsequent section we describe the behaviour of the function $f(x, H)$ for some choices of H. We conclude the chapter with some open problems in Section 3.15.

3.1.6 Notation

A *nearly balanced interval partition* of $[n]$ is a partition of $[n]$ into intervals $W_1 < \cdots < W_t$ where $||W_i| - |W_j|| \leq 1$ for every $1 \leq i, j \leq t$. Similarly, a t-partite graph with vertex classes $V_1, \ldots, V_t$ is *nearly balanced* if $||V_i| - |V_j|| \leq 1$ for every $1 \leq i, j \leq t$.

Given an ordered graph G we say that *$V_1 < \cdots < V_r$ is an interval r-colouring of G* to mean that there is an interval r-colouring of G with colour classes $V_1 < \cdots < V_r$. We say that an ordered graph G is *complete r-partite* if there exists an interval r-colouring $V_1 < \cdots < V_r$ such that $xy \in E(G)$ for every $x \in V_i$ and $y \in V_j$ with $i \neq j$. We refer to the V_i's as the *parts* of G.

Given an unordered graph G and a positive integer t, let $G(t)$ be the graph obtained from G by replacing every vertex $x \in V(G)$ by a set V_x of t vertices spanning an independent set, and joining $u \in V_x$ to $v \in V_y$ precisely when xy is an edge in G; that is, we replace the edges of G by copies of $K_{t,t}$. We will refer to $G(t)$ as a *blown-up* copy of G. If U_i is a vertex class in G then we write $U_i(t)$ for the corresponding vertex class in $G(t)$. We use analogous notation when considering blown-up copies of complete k-partite ordered graphs. In particular, given a complete k-partite ordered graph B with parts $B_1 < \cdots < B_k$, the *ordered blow-up $B(t)$ of B* consists of parts $B_1(t) < \cdots < B_k(t)$ where $|B_i(t)| = t|B_i|$ for all $i \in [k]$.

Throughout the chapter, we omit all floor and ceiling signs whenever these are not crucial.

3.2 Extremal constructions

In this section we provide the extremal examples for Theorems 3.1.6 and 3.1.8. First, consider the case when H has a local barrier. We now construct an n-vertex ordered graph which does not contain an H-cover (and thus no perfect H-tiling), and whose minimum degree is more than $(1 - 1/\chi_<(H))n - 1$, thereby giving the lower bounds in Theorem 3.1.6(ii) and Theorem 3.1.8(ii).

Extremal Example 3. *Let $n, r \in \mathbb{N}$ and $i, j \in [r+1]$ with $i \neq j$. Let $F_1(n, r, i, j)$ be an n-vertex ordered graph consisting of vertex classes $U_1 < \cdots < U_{r+1}$ which satisfy the following conditions:*

- *$U_i = \{u\}$ is a singleton class while the remaining vertex classes are as equally sized as possible, and in particular, $|U_j| = \lfloor \frac{n-1}{r} \rfloor$;*
- *$F_1(n, r, i, j) \setminus \{u\}$ is a complete r-partite ordered graph with parts $U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{r+1}$;*
- *u is adjacent to all other vertices except those in U_j.*

Note that

$$\delta(F_1(n, r, i, j)) = n - 1 - |U_j| = n - 1 - \left\lfloor \frac{n-1}{r} \right\rfloor > \left(1 - \frac{1}{r}\right) n - 1. \qquad (3.2)$$

Furthermore, we now prove that $F_1(n, r, i, j)$ does not contain an H-cover (nor a perfect H-tiling) provided that $\chi_<(H) = r$ and H has a local barrier with respect to parameters $i, j \in [r + 1]$.

Lemma 3.2.1. *Let H be an ordered graph, let $r := \chi_<(H)$ and let $n \in \mathbb{N}$. If H has a local barrier then there exist $i, j \in \mathbb{N}$, with $i \neq j$, and a vertex $u \in F_1(n, r, i, j)$ such that there is no copy of H in $F_1(n, r, i, j)$ covering the vertex u. In particular, $F_1(n, r, i, j)$ does not contain an H-cover nor a perfect H-tiling.*

Proof. Suppose H has a local barrier with respect to $i \neq j \in [r + 1]$, as defined in Definition 3.1.5. Let u be the vertex in the singleton class U_i of $F_1(n, r, i, j)$. Suppose there is a copy of H in $F_1(n, r, i, j)$ covering the vertex u. Then the interval $(r + 1)$-colouring $U_1 < \cdots < U_{r+1}$ of $F_1(n, r, i, j)$ induces an interval $(r + 1)$-colouring $V_1 < \cdots < V_{r+1}$ of H such that $V_i = \{v\}$ is a singleton class and there is no edge between v and V_j. This contradicts the assumption that H has a local barrier with respect to i, j; thus, there is no copy of H in $F_1(n, r, i, j)$ covering the vertex u. □

We immediately obtain the following corollary of Lemma 3.2.1 and (3.2).

Corollary 3.2.2. *Let H be an ordered graph and let $n \in \mathbb{N}$. If H has a local barrier then*

$$\delta_{cov}(H, n) > \left(1 - \frac{1}{\chi_<(H)}\right) n$$

and, if $|H|$ divides n

$$\delta_<(H, n) > \left(1 - \frac{1}{\chi_<(H)}\right) n.$$

□

Next we prove a general lower bound on $\delta_<(H, n)$ which is asymptotically sharp if the ordered graph H does not have a local barrier. Similarly to before, we construct an n-vertex ordered graph which does not contain a perfect H-tiling and whose minimum degree is at least $(1 - 1/\chi^*_{cr}(H))n - 1$, thereby giving the lower bound in cases (i) and (iii) of Theorem 3.1.6.

Extremal Example 4. *Let H be an ordered graph and $n \in \mathbb{N}$. Set $\ell := \left\lfloor \frac{n}{\chi^*_{cr}(H)} + 1 \right\rfloor$. Define $F_2(H,n)$ to be the unordered complete $\lceil n/\ell \rceil$-partite graph on n vertices such that all classes have size ℓ except for one class of size at most ℓ.*

It is easy to check that the minimum degree of $F_2(H,n)$ is

$$\delta(F_2(H,n)) = n - \ell = n - \left\lfloor \frac{n}{\chi^*_{cr}(H)} + 1 \right\rfloor \geq \left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n - 1. \tag{3.3}$$

Additionally, there exists a certain ordering of the vertices of $F_2(H,n)$ such that the resulting ordered graph does not contain a perfect H-tiling:

Lemma 3.2.3. *Let H be an ordered graph and $n \in \mathbb{N}$ such that $|H|$ divides n. There exists an interval labelling ϕ of $F_2(H,n)$ such that the ordered graph $(F_2(H,n), \phi)$ does not contain a perfect H-tiling.*

Proof. Recall $F_2(H,n)$ is a complete $\lceil n/\ell \rceil$-partite graph with all parts of size ℓ, except one class of size at most ℓ, for some integer ℓ such that $\ell > n/\chi^*_{cr}(H)$. For brevity, let $F := F_2(H,n)$. It is easy to see that $\sigma(F)$ equals the size of the smallest part of F, thus $|F| - \sigma(F) = \ell(\chi(F) - 1)$. The critical chromatic number of F is

$$\chi_{cr}(F) = (\chi(F) - 1)\frac{|F|}{|F| - \sigma(F)} = \frac{n}{\ell} < \frac{n}{n/\chi^*_{cr}(H)} = \chi^*_{cr}(H).$$

By Definition 3.1.3, for every bottlegraph B of H we have $\chi_{cr}(B) \geq \chi^*_{cr}(H)$. It follows that $F_2(H,n)$ is not a bottlegraph of H. Hence, by definition, there exists a permutation σ of $[\lceil n/\ell \rceil]$ and an interval labelling ϕ of $F_2(H,n)$ with respect to σ such that $(F_2(H,n), \phi)$ does not contain a perfect H-tiling. $\square$

Lemma 3.2.3 and (3.3) immediately imply the following corollary.

Corollary 3.2.4. *Let H be an ordered graph. Then given any $n \in \mathbb{N}$ divisible by $|H|$,*

$$\delta_<(H,n) \geq \delta(F_2(H,n)) + 1 \geq \left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n.$$

Next we give a general lower bound for $\delta_{cov}(H,n)$ which is asymptotically sharp if the ordered graph H does not have a local barrier, thereby giving the lower bound in Theorem 3.1.8(i).

Extremal Example 5. *Let H be an ordered graph and $n \in \mathbb{N}$. Let $F_3(H,n)$ be the complete $(\chi_<(H)-1)$-partite ordered graph on n vertices with parts of size as equal as possible.*

It is easy to check that the minimum degree of $F_3(H,n)$ is

$$\delta(F_3(H,n)) = n - \left\lceil \frac{n}{\chi_<(H)-1} \right\rceil > \left(1 - \frac{1}{\chi_<(H)-1}\right) n - 1.$$

As $\chi_<(F_3(H,n)) < \chi_<(H)$, we have that $F_3(H,n)$ does not contain a copy of H and thus does not contain an H-cover. We therefore obtain the following result.

Lemma 3.2.5. *Let H be an ordered graph and $n \in \mathbb{N}$. Then*

$$\delta_{cov}(H,n) \geq \delta(F_3(H,n)) + 1 > \left(1 - \frac{1}{\chi_<(H)-1}\right) n.$$

□

For n divisible by $\chi_<(H)-1$, $F_3(H,n)$ also shows that the minimum degree threshold that ensures an almost perfect H-tiling in an n-vertex ordered graph is more than $(1 - 1/(\chi_<(H)-1))n$. Thus, combined with Theorem 3.1.4 this immediately implies that, for all ordered graphs H,

$$\chi_<(H) - 1 < \chi^*_{cr}(H). \tag{3.4}$$

Actually, we close the section by proving an even stronger lower bound on $\chi^*_{cr}(H)$.

Proposition 3.2.6 (A lower bound for $\chi^*_{cr}(H)$)**.** *Let H be an ordered graph on h vertices and $r := \chi_<(H)$. Then,*

$$\chi^*_{cr}(H) \geq (r-1) + \frac{r-1}{h-1}.$$

Proof. Let B be an arbitrary bottlegraph of H. It suffices to show that $\chi_{cr}(B) \geq (r - 1) + \frac{r-1}{h-1}$. If $\chi_{cr}(B) \geq r$ then we are done (since $h \geq r$), so for the rest of the proof we assume that $\chi_{cr}(B) < r$. In particular, as (3.4) implies that $\chi_{cr}(B) > r - 1$, this means that B has exactly r parts. Let B_1 denote the part of B of smallest size. Pick any interval labelling ϕ of B; then there exists some $t \in \mathbb{N}$ such that the ordered blow-up $(B(t), \phi)$ contains a perfect H-tiling $\mathcal{H}$. Since B has exactly r parts, it follows that every copy of H in $(B(t), \phi)$ intersects all parts of B. Hence,

$$|B_1(t)| \geq |\mathcal{H}| = \frac{|B(t)|}{|H|} = t|B|/h \implies |B_1| \geq |B|/h,$$

and so

$$\chi_{cr}(B) = (r-1)\frac{|B|}{|B| - |B_1|} \geq (r-1)\frac{|B|}{|B| - |B|/h} = (r-1) + \frac{r-1}{h-1}.$$

□

In Section 3.3.3 we give a family of ordered graphs H for which the lower bound on $\chi^*_{cr}(H)$ in Proposition 3.2.6 is tight.

3.3 Motivating examples

3.3.1 An example for Theorem 3.1.6(i)

Recall that Extremal Example 2 yields the lower bound in cases (i) and (iii) of Theorem 3.1.6. The argument in Lemma 3.2.3 is rather straightforward. This is because of the definition of $\chi^*_{cr}(H)$; if one takes a complete multipartite graph G with $\chi_{cr}(G) < \chi^*_{cr}(H)$, then by definition there is a vertex labelling of G so that the resulting ordered graph does not contain a perfect H-tiling.

Therefore, if one provides an argument that justifies why a bottlegraph of H is optimal, this equivalently can be translated into an argument which explains why an ordered graph is an extremal example for cases (i) and (iii) of Theorem 3.1.6. In this way, one can view $\chi^*_{cr}(H)$ as 'encoding' properties of the extremal example.

In Section 3.11 we will compute $\chi^*_{cr}(H)$ for various classes of ordered graphs H. Often these arguments will be somewhat involved; thus, in these cases the reason why the extremal example for Theorem 3.1.6 does not contain a perfect H-tiling is also 'involved'. That is, in general the reason why extremal examples do not contain perfect H-tilings is not as immediate as Lemma 3.2.3 might suggest. We illustrate this point through the following example.

Example 3.3.1 (An example for Theorem 3.1.6(i))**.** *Let $\ell \geq 2$ and let H be the complete 3-partite ordered graph with parts $H_1 < H_2 < H_3$ of size $\ell, 1, \ell$ respectively.*

For H as in Example 3.3.1, we have that $\chi^*_{cr}(H) = (4\ell^2 - 1)/\ell^2 > 3 = \chi_<(H)$ (this follows from Proposition 3.11.5, where we compute $\chi^*_{cr}(F)$ for all complete 3-partite ordered graphs F). Thus, for such H we are in case (i) of Theorem 3.1.6 and so

$$\delta_<(H, n) = \left(1 - \frac{\ell^2}{4\ell^2 - 1} + o(1)\right) n.$$

We now describe an extremal example for Theorem 3.1.6 for such H. Let $n \in \mathbb{N}$ such that $|H| = 2\ell + 1$ divides n and $n \geq 20$. Let G be the complete 4-partite ordered graph on n vertices with parts $G_1 < G_2 < G_3 < G_4$ where

$$|G_1| = |G_2| = |G_3| = \left\lfloor \frac{n\ell^2}{4\ell^2 - 1} \right\rfloor + 1 \quad \text{and} \quad |G_4| = n - 3\left\lfloor \frac{n\ell^2}{4\ell^2 - 1} \right\rfloor - 3.$$

Note that G_4 is the smallest part since $|G_i| \geq n/4$ for $i = 1, 2, 3$ and $|G_4| \leq n/4$. In particular,

$$\delta(G) = n - \left\lfloor \frac{n\ell^2}{4\ell^2 - 1} \right\rfloor - 1 \geq n - \frac{n\ell^2}{4\ell^2 - 1} - 1 = \left(1 - \frac{\ell^2}{4\ell^2 - 1}\right) n - 1.$$

Suppose for a contradiction that G contains a perfect H-tiling $\mathcal{H}$. Let $\mathcal{A} \subseteq \mathcal{H}$ be the set of copies of H in $\mathcal{H}$ which have exactly ℓ vertices in G_1 and set $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$. This immediately implies

$$|G_1| \geq \ell|\mathcal{A}|. \tag{3.5}$$

Note that if $H' \in \mathcal{A}$ then H' has at most $\ell+1$ vertices in $G_1 \cup G_2$ while if $H' \in \mathcal{B}$ then H' has at most ℓ vertices in $G_1 \cup G_2$. It follows that

$$|G_1| + |G_2| \leq (\ell+1)|\mathcal{A}| + \ell|\mathcal{B}|. \tag{3.6}$$

Combining (3.5) and (3.6) yields the following:

$$\begin{aligned} |\mathcal{A}| + |\mathcal{B}| &\overset{(3.6)}{\geq} \frac{|G_1| + |G_2| - |\mathcal{A}|}{\ell} \overset{(3.5)}{\geq} \frac{\ell|G_1| + \ell|G_2| - |G_1|}{\ell^2} \\ &= \frac{2\ell-1}{\ell^2}\left(\left\lfloor \frac{n\ell^2}{4\ell^2-1}\right\rfloor + 1\right) > \frac{2\ell-1}{\ell^2}\left(\frac{n\ell^2}{4\ell^2-1}\right) = \frac{n}{2\ell+1}. \end{aligned}$$

The above is a contradiction since

$$|\mathcal{A}| + |\mathcal{B}| = |\mathcal{H}| = \frac{|G|}{|H|} = \frac{n}{2\ell+1}.$$

Hence, G does not contain a perfect H-tiling.

Note that G is a 'space barrier' construction as our argument tells us that $G_1 \cup G_2$ is 'too big' to ensure a perfect H-tiling in G; moreover, the reason why $G_1 \cup G_2$ is 'too big', whilst not difficult, is not at first sight, obvious (i.e., we needed to consider how two types of copies of H intersect $G_1 \cup G_2$).

Space barrier constructions occur in many other settings too (e.g., the Kühn–Osthus perfect tiling theorem [93]). However, all previous graph space barrier constructions we are aware of have a different flavour to the above space barrier G. Indeed, previously known examples fail to contain the desired substructure due to some very immediate property that means *one* vertex class is 'too small' or 'too big'.

In Section 3.11 we compute $\chi^*_{cr}(H)$ precisely for several classes of ordered graphs.

In particular, we give other ordered graphs H that fall into case (i) of Theorem 3.1.6, namely all complete 3-partite ordered graphs and all complete r-partite ordered graphs whose smallest part is the first or last part (see Propositions 3.11.3 and 3.11.5).

3.3.2 An example for Theorem 3.1.6(ii)

The next example provides a family of ordered graphs that fall into case (ii) of Theorem 3.1.6.

Example 3.3.2 (An example for Theorem 3.1.6(ii))**.** *Let $r, k \geq 3$ and let H be the ordered graph with vertex set $V(H) = [(r-1)k+2]$ and edge set $E(H) = \{(1, (r-1)k+2)\} \cup \{((s-1)k+2, sk+2) : s \in [r-1]\}$ (see Figure 3.1). So $\chi_<(H) = r$.*

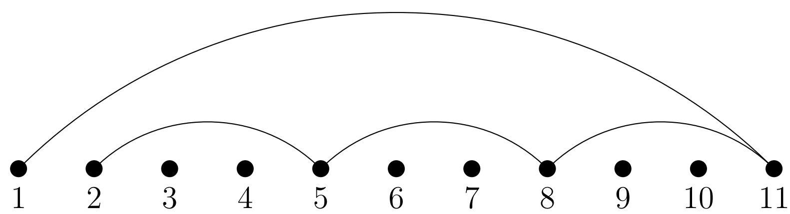

Figure 3.1: The ordered graph H as in Example 3.3.2 for $r = 4$ and $k = 3$.

Let B the complete r-partite graph with parts $B_1, \dots, B_r$ where $|B_i| = k$ for $i \in [r-1]$ and $|B_r| = 2$. Observe that $\chi_{cr}(B) = (r-1) + 2/k$ and $|B| = |H|$. Since all parts of B have size at most k and there is no edge in H whose endpoints differ by $k-1$ or less, we have that for any permutation σ of $[r]$ and any interval labelling ϕ of B with respect to σ, the ordered graph (B, ϕ) contains a spanning copy of H; hence B is a simple bottlegraph of H. It follows that

$$\chi^*_{cr}(H) \leq \chi_{cr}(B) = (r-1) + \frac{2}{k} < r = \chi_<(H).$$

Furthermore, H has a local barrier: for any interval $(r+1)$-colouring $\{1\} < V_1 < \cdots < V_r$ of H we have that $(r-1)k+2 \in V_r$ and thus there is one edge between $\{1\}$ and V_r.

3.3.3 An example for Theorem 3.1.6(iii)

Next we consider a family of ordered graphs which fall into case (iii) of Theorem 3.1.6.

Example 3.3.3 (An example for Theorem 3.1.6(iii)). *Let $r, k \geq 2$ and let H be the ordered graph with vertex set $V(H) = [(r-1)k+1]$ and edge set $E(H) = \{((s-1)k+1, sk+1) : s \in [r-1]\}$. So H is the path $(1)(k+1)(2k+1)\dots((r-1)k+1)$ and $\chi_<(H) = r$ (see Figure 3.2).*

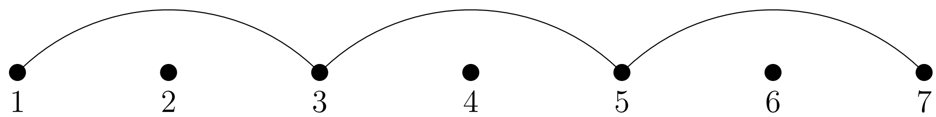

Figure 3.2: The ordered graph H as in Example 3.3.3 for $r = 4$ and $k = 2$.

We will explicitly compute $\chi^*_{cr}(H)$. In particular, we prove that $\chi^*_{cr}(H) < \chi_<(H)$ and that H does not have a local barrier. We first construct a bottlegraph of H. Let B denote the complete r-partite graph with parts $B_1, \dots, B_r$ where $|B_i| = k$ for $i \in [r-1]$ and $|B_r| = 1$. Observe that $\chi_{cr}(B) = (r-1) + 1/k$ and $|B| = |H|$. Since all parts of B have size at most k and there is no edge in H whose endpoints differ by $k-1$ or less, we have that for any permutation σ of $[r]$ and any interval labelling ϕ of B with respect to σ, the ordered graph (B, ϕ) contains a spanning copy of H. Thus, B is a simple bottlegraph of H and so $\chi^*_{cr}(H) \leq \chi_{cr}(B) = (r-1) + 1/k$. Moreover, Proposition 3.2.6 implies that in fact $\chi^*_{cr}(H) = (r-1) + 1/k$.

Finally, we show that H does not have a local barrier. Let $i \neq j \in [r+1]$. If $i \notin \{1, r+1\}$, there exists an interval $(r+1)$-colouring $V_1 < \dots < V_{r+1}$ of H such that $V_i = \{x\}$ with $x \neq (s-1)k+1$ for every $s \in [r]$. Then x is isolated in H and so clearly there is no edge between x and V_j. If $i = 1$, there exists an interval $(r+1)$-colouring $V_1 < \dots < V_{r+1}$ of H such that $V_i = \{1\}$ and $V_j = \emptyset$; so again, there is no edge between V_i and V_j. The case $i = r+1$ is analogous; we take $V_i = \{(r-1)k+1\}$ and $V_j = \emptyset$.

3.4 Proof of Theorem 3.1.6

In this section we present some intermediate results and explain how they imply Theorem 3.1.6. Crucial to our approach will be the use of the *absorbing method*, a technique that was pioneered by Erdős, Gyárfás and Pyber [42] (and by Krivelevich [92] in the context of random graphs) and later systematized by Rödl, Ruciński and Szemerédi [117]. Given ordered graphs G, H and a set $I \subseteq V(G)$, a set $S \subseteq V(G)$ is called an *H-absorbing set for I* if both $G[S]$ and $G[I \cup S]$ contain perfect H-tilings. In [8, Theorem 4.1], Balogh, Li and the Treglown provided a minimum degree condition that ensures an ordered graph G contains a set Abs that is an H-absorbing set for every not too large set $I \subseteq V(G) \setminus Abs$.[5]

Theorem 3.4.1 (Balogh, Li and Treglown [8])**.** *Let H be an ordered graph on h vertices and let $\eta > 0$. Then there exists an $n_0 \in \mathbb{N}$ and $0 < \nu \ll \eta$ so that the following holds. Suppose that G is an ordered graph on $n \geq n_0$ vertices and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n.$$

Then $V(G) = [n]$ contains a set Abs so that

- $|Abs| \leq \nu n$;
- *Abs is an H-absorbing set for every $I \subseteq V(G) \setminus Abs$ such that $|I| \in h\mathbb{N}$ and $|I| \leq \nu^3 n$.*

Theorems 3.1.4 and 3.4.1 can be combined to yield a minimum degree condition that forces a perfect H-tiling. Indeed, let G and H be ordered graphs and suppose that

$$\delta(G) \geq \left(1 - \frac{1}{\max\{\chi_<(H), \chi^*_{cr}(H)\}} + o(1)\right) n.$$

We first invoke Theorem 3.4.1 to find a set $Abs \subseteq V(G)$ which is an H-absorbing set for any not too large set $I \subseteq V(G) \setminus Abs$. Then we apply Theorem 3.1.4 to $G \setminus Abs$ to find an

[5] Note that this property of the set Abs does not hold for arbitrary H but rather for the specific H one is considering. A different H would require a different set Abs.

H-tiling $\mathcal{M}_1$ which covers all but a small proportion of vertices in $G \setminus Abs$. Let I denote the set of such vertices in $G \setminus Abs$. Since I is relatively small, Abs is an H-absorbing set for I, and thus $G[I \cup Abs]$ contains a perfect H-tiling $\mathcal{M}_2$. Finally, observe that $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect H-tiling in G.

Thus we have proven that

$$\delta_<(H,n) \leq \left(1 - \frac{1}{\max\{\chi_<(H), \chi^*_{cr}(H)\}} + o(1)\right) n.$$

In particular, this is asymptotically sharp if $\chi^*_{cr}(H) \geq \chi_<(H)$ (by Corollary 3.2.4) or if $\chi^*_{cr}(H) < \chi_<(H)$ and H has a local barrier (by Corollary 3.2.2), therefore proving cases (i) and (ii) of Theorem 3.1.6. However, if $\chi^*_{cr}(H) < \chi_<(H)$ and H does not have a local barrier then this minimum degree condition can be substantially lowered. To achieve this, we need the following new absorbing result, which is our main technical contribution.

Theorem 3.4.2 (Absorbing theorem for non-local barriers)**.** *Let H be an ordered graph on h vertices with $\chi_<(H) \geq 3$ and let $\eta > 0$. If H does not have a local barrier and $\chi^*_{cr}(H) < \chi_<(H)$, then there exists an $n_0 \in \mathbb{N}$ and $0 < \nu \ll \eta$ so that the following holds. Suppose that G is an ordered graph on $n \geq n_0$ vertices and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_<(H) - 1} + \eta\right) n.$$

Then $V(G) = [n]$ contains a set Abs so that

- *$|Abs| \leq \nu n$;*
- *Abs is an H-absorbing set for every $I \subseteq V(G) \setminus Abs$ such that $|I| \in h\mathbb{N}$ and $|I| \leq \nu^3 n$.*

Note that the statement of Theorem 3.4.2 is false if one allows $\chi_<(H) = 2$; indeed, the conclusion of the theorem fails for so-called divisibility barriers H.[6] However, one can adapt our proof and relax the hypothesis of Theorem 3.4.2 to $\chi_<(H) \geq 2$ if one

[6]More precisely, here *divisibility barrier* means an ordered graph H with $\chi_<(H) = 2$ that satisfies Property B as defined in [8, Page 3].

additionally assumes H is not a divisibility barrier. We will not do this in this chapter, however, as [8, Theorem 1.9] already resolves the perfect H-tiling problem for ordered graphs H with $\chi_<(H) = 2$.

We postpone the proof of Theorem 3.4.2 to Section 3.10. With Theorem 3.4.2 at hand, we can now give the proof of Theorem 3.1.6.

Proof of Theorem 3.1.6. First note that the lower bounds in parts (i)–(iii) of the theorem follow immediately from Corollary 3.2.4 (for (i) and (iii)) and Corollary 3.2.2 (for (ii)). Thus it remains to prove the upper bounds.

Let H be an ordered graph with $\chi_<(H) \geq 3$ and let $\eta > 0$. Let $n \in \mathbb{N}$ be sufficiently large and such that $|H|$ divides n. Let G be an ordered graph on n vertices with minimum degree so that

(i) $\delta(G) \geq \left(1 - \frac{1}{\chi^*_{cr}(H)} + \eta\right) n$ if $\chi^*_{cr}(H) \geq \chi_<(H)$;

(ii) $\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n$ if $\chi^*_{cr}(H) < \chi_<(H)$ and H has a local barrier;

(iii) $\delta(G) \geq \left(1 - \frac{1}{\chi^*_{cr}(H)} + \eta\right) n$ if $\chi^*_{cr}(H) < \chi_<(H)$ and H has no local barrier.

Recall that $\chi^*_{cr}(H) > \chi_<(H) - 1$. Thus, by Theorem 3.4.1 (for cases (i) and (ii)) and Theorem 3.4.2 (for case (iii)), there exists some $0 < \nu \ll \eta$ and a set $Abs \subseteq V(G)$ such that

- $|Abs| \leq \nu n$;
- Abs is an H-absorbing set for every $I \subseteq V(G) \setminus Abs$ such that $|I| \in |H|\mathbb{N}$ and $|I| \leq \nu^3 n$.

Let $G' := G \setminus Abs$. In all cases we have that $\delta(G') \geq (1 - 1/\chi^*_{cr}(H))|G'|$.

Since n was chosen to be sufficiently large, by Theorem 3.1.4 there exists an H-tiling $\mathcal{M}_1$ in G' covering all but at most $\nu^3 n$ vertices. Let $I \subseteq V(G')$ denote the set of vertices which are not covered by $\mathcal{M}_1$. Since $|H|$ divides n, $|V(\mathcal{M}_1)|$ and $|Abs|$, we have that $|H|$ divides $|I|$ too. Also, $|I| \leq \nu^3 n$, hence $G'[I \cup Abs]$ contains a perfect H-tiling $\mathcal{M}_2$. Finally, observe that $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect H-tiling of G, as desired. $\square$

3.5 Bottlegraphs

In the following proposition we show that it suffices to consider bottlegraphs where all parts are of the same size except for perhaps one smaller part.

Proposition 3.5.1. *Let H be an ordered graph and B be a bottlegraph of H. There exists a bottlegraph B' of H and an integer $m \in \mathbb{N}$ such that $\chi_{cr}(B') = \chi_{cr}(B)$ and all parts of B' have size m except for one part with size at most m.*

Proof. Let B be a bottlegraph of H; so B is a complete k-partite unordered graph with parts $B_1, \dots, B_k$ for some $k \in \mathbb{N}$. Without loss of generality, we may assume that $|B_1| \le |B_i|$ for every $i > 1$. Let B' be the complete k-partite unordered graph with parts $B_1', \dots, B_k'$ where

$$|B_i'| = \begin{cases} (k-1)|B_1| & \text{if } i = 1 \\ |B| - |B_1| & \text{otherwise.} \end{cases}$$

So $|B'| = (k-1)|B|$. Furthermore, we have

$$|B_1'| = (k-1)|B_1| = k|B_1| - |B_1| \le |B| - |B_1| = |B_i'|$$

for every $i > 1$. It follows that

$$\chi_{cr}(B') = \frac{(k-1)|B'|}{|B'| - |B_1'|} = \frac{(k-1)^2|B|}{(k-1)|B| - (k-1)|B_1|} = \frac{(k-1)|B|}{|B| - |B_1|} = \chi_{cr}(B).$$

It remains to show that B' is a bottlegraph of H. Observe that the vertices in B_1' can be partitioned into $(k-1)$ sets of size $|B_1|$ while the vertices in B_i' can be partitioned into $(k-1)$ sets of sizes $|B_2|, \dots, |B_k|$ respectively, for every $i > 1$. This implies that B' contains a perfect B-tiling consisting of $(k-1)$ copies of B.

Let $\{C_1, \dots, C_{k-1}\}$ be a perfect B-tiling in B' (i.e., each C_i is a copy of B in B'). Let σ be a permutation of $[k]$ and let ϕ be an interval labelling of B' with respect to σ. For every C_i, we have that ϕ induces an interval labelling ϕ_i of C_i with respect to some permutation σ_i of $[k]$. Since B is a bottlegraph, given any $i \in [k-1]$, there exists some $t_i \in \mathbb{N}$ such that the ordered blow-up $(C_i(t_i), \phi_i)$ contains a perfect H-tiling. Set $t := t_1 t_2 \dots t_{k-1}$. Then the ordered blow-up $(C_i(t), \phi_i)$ contains a perfect H-tiling $\mathcal{M}_i$, for each $i \in [k-1]$.

Finally, $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{k-1}$ is a perfect H-tiling of the ordered blow-up $(B'(t), \phi)$. Since σ, ϕ were arbitrary, B' is a bottlegraph of H. $\square$

Note that the notion of a bottlegraph of H (Definition 3.1.2) and 1-bottlegraph (Definition 3.1.12) are not quite the same. However, the next result implies that $\chi^*_{cr}(H) = \chi^*_{cr}(1, H)$.

Proposition 3.5.2. *Let H be an ordered graph. Then $\chi^*_{cr}(H) = \chi^*_{cr}(1, H)$.*

Proof. Let

$$\mathcal{X} := \{\chi_{cr}(B) : \ B \text{ is a bottlegraph of } H\}$$

and

$$\mathcal{X}_1 := \{\chi_{cr}(B) : \ B \text{ is a 1-bottlegraph of } H\}.$$

Thus, $\inf \mathcal{X} = \chi^*_{cr}(H)$ and $\inf \mathcal{X}_1 = \chi^*_{cr}(1, H)$. By definition of a bottlegraph and 1-bottlegraph we have that $\mathcal{X}_1 \subseteq \mathcal{X}$; so to prove the proposition it suffices to show that $\mathcal{X} \subseteq \mathcal{X}_1$.

Given any bottlegraph B of H, let B' be the bottlegraph of H obtained by applying Proposition 3.5.1. So B' satisfies conditions (i) and (ii) in the definition of a 1-bottlegraph of H and $\chi_{cr}(B') = \chi_{cr}(B)$. As B' is a bottlegraph of H, there is some $t \in \mathbb{N}$ so that $B'(t)$ satisfies condition (iii) of the definition of a 1-bottlegraph of H. Then $B'(t)$ is a 1-bottlegraph of H with $\chi_{cr}(B'(t)) = \chi_{cr}(B') = \chi_{cr}(B)$. Thus, $\mathcal{X} \subseteq \mathcal{X}_1$, as desired.

$\square$

3.6 The regularity lemma

In the proof of Theorem 3.4.2 we will make use of the regularity method. In this section we state a multipartite version of Szemerédi's regularity lemma and some other related tools. First, we introduce some basic notation.

The *density* of an (ordered) bipartite graph with vertex classes A and B is defined to be

$$d(A,B) := \frac{e(A,B)}{|A||B|}.$$

Given $\varepsilon > 0$, a graph G and two disjoint sets $A, B \subseteq V(G)$, we say that the pair $(A,B)_G$ is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $|d(A,B) - d(X,Y)| < \varepsilon$. Given $d \in [0,1]$, the pair $(A,B)_G$ is *(ε,d)-regular* if G is ε-regular, and $d(A,B) \geq d$.

We now state some well-known properties of ε-regular pairs. The first (see, e.g., [89, Fact 1.5]) implies that one can delete many vertices from an (ε,d)-regular pair and still retain such a regularity property.

Lemma 3.6.1 (Slicing lemma)**.** *Let $(A,B)_G$ be an ε-regular pair of density d, and for some $\alpha > \varepsilon$, let $A' \subseteq A$ with $|A'| \geq \alpha|A|$ and $B' \subseteq B$ with $|B'| \geq \alpha|B|$. Then $(A',B')_G$ is $(\varepsilon', d-\varepsilon)$-regular with $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$.*

The following theorem is a multipartite version of Szemerédi's regularity lemma [130] (presented, e.g., as Lemma 5.5 in [8]).

Theorem 3.6.2 (Multipartite regularity lemma)**.** *Given any integer $t \geq 2$, any $\varepsilon > 0$ and any $\ell_0 \in \mathbb{N}$ there exists $L_0 = L_0(\varepsilon, t, \ell_0) \in \mathbb{N}$ such that for every $d \in (0,1]$ and for every nearly balanced t-partite graph $G = (W_1, \dots, W_t)$ on $n \geq L_0$ vertices, there exists an $\ell \in \mathbb{N}$, a partition $W_i^0, W_i^1, \dots, W_i^\ell$ of W_i for each $i \in [t]$ and a spanning subgraph G' of G, such that the following conditions hold:*

1. $\ell_0 \leq \ell \leq L_0$;

2. $d_{G'}(x) \geq d_G(x) - (d+\varepsilon)n$ *for every* $x \in V(G)$;

3. $|W_i^0| \leq \varepsilon n/t$ *for every* $i \in [t]$;

4. $|W_i^j| = |W_{i'}^{j'}|$ *for every* $i, i' \in [t]$ *and* $j, j' \in [\ell]$;

5. *for every* $i, i' \in [t]$ *and* $j, j' \in [\ell]$ *either* $(W_i^j, W_{i'}^{j'})_{G'}$ *is an* (ε, d)*-regular pair or* $G'[W_i^j, W_{i'}^{j'}]$ *is empty.*

We call the W_i^j *clusters*, the W_i^0 the *exceptional sets* and the vertices in the W_i^0 *exceptional vertices*. We refer to G' as the *pure graph*. The *reduced graph R of G with parameters ε, d and ℓ_0* is the graph whose vertices are the W_i^j (where $i \in [t]$ and $j \in [\ell]$) and in which $W_i^j W_{i'}^{j'}$ is an edge precisely when $(W_i^j, W_{i'}^{j'})_{G'}$ is (ε, d)-regular. The following well-known corollary of the regularity lemma shows that the reduced graph almost inherits the minimum degree of the original graph.

Proposition 3.6.3. *Let* $0 < \varepsilon, d, k < 1$ *and let* G *be an* n*-vertex graph with* $\delta(G) \geq kn$. *If* R *is the reduced graph of* G *obtained by applying Theorem 3.6.2 with parameters* ε, d, ℓ_0, *then* $\delta(R) \geq (k - 2\varepsilon - d)|R|$.

A useful tool to embed subgraphs into G using the reduced graph R is the so-called *key lemma*.

Lemma 3.6.4 (Key lemma [89]). *Let* $0 < \varepsilon < d$ *and* $q, t \in \mathbb{N}$. *Let* R *be a graph with* $V(R) = \{v_1, \dots, v_k\}$. *We construct a graph* G *as follows: replace every vertex* $v_i \in V(R)$ *with a set* V_i *of* q *vertices and replace each edge of* R *with an* (ε, d)*-regular pair. For each* $v_i \in V(R)$, *let* U_i *denote the set of* t *vertices in* $R(t)$ *corresponding to* v_i. *Let* H *be a subgraph of* $R(t)$ *on* h *vertices with maximum degree* Δ. *Set* $\delta := d - \varepsilon$ *and* $\varepsilon_0 := \delta^\Delta/(2+\Delta)$. *If* $\varepsilon \leq \varepsilon_0$ *and* $t - 1 \leq \varepsilon_0 q$ *then there are at least* $(\varepsilon_0 q)^h$ *labelled copies of* H *in* G *so that if* $x \in V(H)$ *lies in* U_i *in* $R(t)$, *then* x *is embedded into* V_i *in* G.

As in [8], some of our applications of Lemma 3.6.4 will take the following form: suppose within an ordered graph G we have vertex classes $V_1 < \ldots < V_k$ so that each pair $(V_i, V_j)_G$ is (ε, d)-regular. Then Lemma 3.6.4 tells us G contains many copies of any fixed size ordered graph H with $\chi_<(H) = k$, where the ith vertex class of each such copy of H is embedded into V_i.

3.7 Absorbing tools

In this section we state a couple of results which are useful for proving the existence of H-absorbing sets. The first result is the following crucial lemma of Lo and Markström [99]; we present the ordered version of their result which appeared as Lemma 7.1 in [8].

Lemma 3.7.1 (Lo and Markström [99])**.** *Let $h, s \in \mathbb{N}$ and $\xi > 0$. Suppose that H is an ordered graph on h vertices. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is an ordered graph on $n \geq n_0$ vertices so that, for any $x, y \in V(G)$, there are at least ξn^{sh-1} many $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings. Then $V(G)$ contains a set M so that*

- *$|M| \leq (\xi/2)^h n/4$;*
- *M is an H-absorbing set for any $I \subseteq V(G) \setminus M$ such that $|I| \in h\mathbb{N}$ and $|I| \leq (\xi/2)^{2h} n/(32s^2h^3)$.*

Informally, we will sometimes refer to a set X satisfying the assumptions of Lemma 3.7.1 as a *chain* of size $|X|$ between vertices x and y. The next lemma states that it is, in some sense, possible to concatenate chains.

Lemma 3.7.2. *Let H be an (ordered) graph on h vertices, let $\alpha, \beta, \gamma > 0$ and $s_1, s_2, n \in \mathbb{N}$ where*

$$0 < 1/n \ll \gamma \ll \alpha, \beta, 1/h, 1/s_1, 1/s_2.$$

Let G be an (ordered) graph on n vertices, $x, y \in V(G)$ and $A \subseteq V(G) \setminus \{x, y\}$ where $|A| \geq \alpha n$. Suppose that for every $z \in A$ there exist at least $\beta n^{s_1 h-1}$ many $(s_1 h-1)$-sets X

in $V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{z\}]$ contain perfect H-tilings and similarly there exist at least $\beta n^{s_2 h-1}$ many $(s_2 h-1)$-sets $Y \subseteq V(G)$ such that both $G[Y \cup \{y\}]$ and $G[Y \cup \{z\}]$ contain perfect H-tilings. Then there exist at least $\gamma n^{(s_1+s_2)h-1}$ many $((s_1+s_2)h-1)$-sets $Z \subseteq V(G)$ such that both $G[Z \cup \{x\}]$ and $G[Z \cup \{y\}]$ contain perfect H-tilings.

Proof. Let $z \in A$ and let X, Y be an $(s_1 h-1)$-set and an $(s_2 h-1)$-set respectively which satisfy the above properties, with X, Y disjoint so that $y \notin X$ and $x \notin Y$. Then $X \cup \{z\} \cup Y$ is an $((s_1+s_2)h-1)$-set and both $G[(X \cup \{z\} \cup Y) \cup \{x\}]$ and $G[(X \cup \{z\} \cup Y) \cup \{y\}]$ contain perfect H-tilings. Thus, it is enough to lower bound the number of such triples (z, X, Y). There are at least αn choices for z. Given a fixed choice of z there are at least $\beta n^{s_1 h-1} - n^{s_1 h-2} \geq \beta n^{s_1 h-1}/2$ suitable choices for X. For a fixed X there are at least $\beta n^{s_2 h-1} - s_1 h n^{s_2 h-2} \geq \beta n^{s_2 h-1}/2$ choices for Y. Therefore, in total there are at least

$$\frac{(\alpha n) \cdot (\beta n^{s_1 h-1}/2) \cdot (\beta n^{s_2 h-1}/2)}{((s_1+s_2)h-1)!} \geq \gamma n^{(s_1+s_2)h-1}$$

choices for Z, where the denominator here is because different choices of the triple (z, X, Y) can yield the same set Z. □

3.8 Flexible colouring

In this section we prove several auxiliary results which will be particularly important for the proofs of Theorems 3.4.2 and 3.12.1. We start with the following definition.

Definition 3.8.1. Let H be an ordered graph and let $r := \chi_<(H)$. We say that H is *flexible* if, for every $i \in [r-1]$, there exists an interval $(r+1)$-colouring

$$V_1 < \cdots < V_i < \{x\} < V_{i+1} < \cdots < V_r$$

of H such that both $V_1 < \cdots < V_i \cup \{x\} < V_{i+1} < \cdots < V_r$ and $V_1 < \cdots < V_i < V_{i+1} \cup \{x\} < \cdots < V_r$ are interval r-colourings of H.

Our first goal is to show that if $\chi^*_{cr}(H) < \chi_<(H)$ then H is flexible. This property is crucial for the proof of Theorem 3.4.2 and it is one of the key results in this chapter. We start with the following lemma.

Lemma 3.8.2. *Let H be an ordered graph with vertex set $[h]$ and let $r := \chi_<(H)$. If H is not flexible, there exist some $i \in [r-1]$ such that the number of vertices lying in the first i intervals of any given interval r-colouring of H is fixed.*

Proof. Since H is not flexible, there exist some $i \in [r-1]$ such that there is no interval $(r+1)$-colouring $V_1 < \cdots < V_i < \{x\} < V_{i+1} < \cdots < V_r$ of H such that both $V_1 < \cdots < V_i \cup \{x\} < V_{i+1} < \cdots < V_r$ and $V_1 < \cdots < V_i < V_{i+1} \cup \{x\} < \cdots < V_r$ are interval r-colourings of H.

Let $U_1 < \cdots < U_r$ be an interval r-colouring of H. Let y be the largest vertex in U_i and let z be the smallest vertex in U_{i+1}; so $z = y+1$. Note that y must be adjacent to some vertex in U_{i+1}, as otherwise the interval $(r+1)$-colouring $U_1 < \cdots < U_i \setminus \{y\} < \{y\} < U_{i+1} < \cdots < U_r$ satisfies the flexibility property, a contradiction. Let y' be the smallest vertex in U_{i+1} adjacent to y. Similarly, let z' be the largest vertex in U_i adjacent to z.

Claim 3.8.3. *Given any r-colouring $Q_1 < \cdots < Q_r$ of H, we have $y \in Q_i$ and $z \in Q_{i+1}$.*

Fix an arbitrary interval r-colouring $Q_1 < \cdots < Q_r$ of H; say that z' lies in the kth interval Q_k. Note that $Q_1 < \cdots < Q_k \cap [z'] < (U_i \setminus [z']) < U_{i+1} < \cdots < U_r$ is an interval colouring of H. If (i) $k < i-1$ or (ii) $k = i-1$ and $z' = y$ then $\chi_<(H) < r$, a contradiction. If $k = i-1$ and $z' \neq y$ then the interval $(r+1)$-colouring $Q_1 < \cdots < Q_{i-1} \cap [z'] < (U_i \setminus [z']) < \{z\} < U_{i+1} \setminus \{z\} < \cdots < U_r$ satisfies the flexibility property, a contradiction to our initial assumption in the proof. Thus, $k \geq i$, which implies that z is contained in the $(i+1)$th interval Q_{i+1} or above since z and z' are adjacent. Similarly, we can show that y is contained in the ith interval Q_i or below. This implies that, since y, z are consecutive vertices in H, y lies in Q_i and z in Q_{i+1}. This completes the proof of the claim.

By the claim above, and as $z = y+1$, we conclude that y is the largest vertex in Q_i. Hence, the number of vertices in the first i intervals of any given interval r-colouring of H is exactly y, yielding the required result. $\square$

We are now ready to prove the aforementioned key property.

Corollary 3.8.4. *Let H be an ordered graph with vertex set $[h]$. If $\chi^*_{cr}(H) < \chi_<(H)$, then H is flexible.*

Proof. Let $r := \chi_<(H)$. Suppose for a contradiction that H is not flexible. By Lemma 3.8.2, there exist some $i \in [r-1]$ such that the number of vertices lying in the first i intervals of any given interval r-colouring of H is, say, y for some fixed $y \in \mathbb{N}$. Consider a bottle-graph B of H with $\chi_{cr}(B) < r$ (which exists by definition of $\chi^*_{cr}(H)$ and as $\chi^*_{cr}(H) < r$); note that B must consist of precisely r parts. By Proposition 3.5.1 we may assume that B has parts $B_1, \dots, B_r$ where $|B_1| < m$ and $|B_i| = m$ for some $m \in \mathbb{N}$. Let σ be a permutation of $[r]$ and ϕ be an interval labelling of B with respect to σ. Recall that the ordered graph (B, ϕ) has parts $B_{\sigma^{-1}(1)} < \cdots < B_{\sigma^{-1}(r)}$. Then, there exists some $t \in \mathbb{N}$ such that the ordered blow-up $(B(t), \phi)$ contains a perfect H-tiling. In particular, this perfect H-tiling consists of $t|B|/h$ copies of H. By assumption, each copy of H has exactly y vertices lying in the first i parts of $(B(t), \phi)$ which implies that the first i parts contain exactly $yt|B|/h$ vertices; that is, the total number of vertices in these parts is independent of the choice of σ. This is clearly a contradiction as if we pick σ such that $\sigma^{-1}(1) = 1$ then there are fewer than itm vertices in the first i parts of $(B(t), \phi)$, while if $\sigma^{-1}(r) = 1$ then there are precisely itm vertices in the first i parts of $(B(t), \phi)$. $\square$

Note that one does really require $\chi^*_{cr}(H) < \chi_<(H)$ in the statement of Corollary 3.8.4; consider, e.g., when H is a complete balanced r-partite ordered graph with parts of size at least 2.

We conclude this section with two slightly technical results which exploit the power of flexibility. In the next lemma we show that given a flexible ordered graph H, there exists a complete $\chi_<(H)$-partite ordered graph F such that F contains a perfect H-tiling and any complete r-partite ordered graph F', whose parts have approximately the same sizes as the corresponding parts in F, contains a perfect H-tiling too.

Lemma 3.8.5. *Let H be an ordered graph on h vertices and let $r := \chi_<(H)$. If H is flexible then the following holds. For every $i \in [r-1]$, let $V_1^i < \cdots < V_i^i < \{x_i\} < V_{i+1}^i < \cdots < V_r^i$ be an interval $(r+1)$-colouring of H as described in Definition 3.8.1. Let F be the complete r-partite ordered graph with parts $F_1 < \cdots < F_r$ such that*

$$|F_k| = \begin{cases} rh + 2rh \cdot \sum_{i=1}^{r-1} |V_k^i| & \text{for } k \in \{1, r\}; \\ 2rh + 2rh \cdot \sum_{i=1}^{r-1} |V_k^i| & \text{otherwise.} \end{cases}$$

Thus, $|F| = 2r(r-1)h^2$. Let $s_1, \ldots, s_r \in \mathbb{Z}$ such that $s_1 + \cdots + s_r = 0$ and $|s_i| \leq h$ for every $i \in [r]$. Let F' be the complete r-partite ordered graph with parts $F_1' < \cdots < F_r'$ such that $|F_k'| = |F_k| + s_k$ for every $k \in [r]$. Then both F and F' contain perfect H-tilings.

Proof. First, we explicitly construct a perfect H-tiling in F. For every $i \in [r-1]$, consider rh copies of the interval r-colouring $V_1^i < \cdots < V_i^i \cup \{x_i\} < V_{i+1}^i < \cdots < V_r^i$ of H and rh copies of the interval r-colouring $V_1^i < \cdots < V_i^i < V_{i+1}^i \cup \{x_i\} < \cdots < V_r^i$ of H. For every $k \in [r]$ we take the union of all the kth colour classes from the interval r-colourings above to obtain r new classes. It is easy to check that the size of the new kth class is exactly the size of F_k, so we have just constructed a perfect H-tiling in F. In particular, as there are $2r(r-1)h$ copies of H in this tiling, $|F| = 2r(r-1)h^2$.

Note that for every pair of consecutive classes (F_k, F_{k+1}) we could independently move rh vertices from F_k to F_{k+1} to yield a new complete r-partite ordered graph which still contains a perfect H-tiling; similarly, we could move rh vertices from F_{k+1} to F_k (these $2rh$ vertices fulfil the role of x_k in their respective interval r-colourings of H). We are going to use this observation to construct a perfect H-tiling in F'.

Set $t_k := s_1 + \cdots + s_k$ for $k \in [r]$ and $t_0 := 0$. For every pair of consecutive classes (F_k, F_{k+1}), if $t_k \geq 0$ move t_k vertices from F_{k+1} to F_k, otherwise move $-t_k$ vertices from F_k to F_{k+1}. This is possible since $|t_k| \leq |s_1| + \cdots + |s_r| \leq rh$ by assumption. Note that the size of the new kth class is $|F_k| + t_k - t_{k-1} = |F_k| + s_k = |F'_k|$, hence we just constructed a perfect H-tiling in F'. □

Observe that the ordered graphs F and F' in Lemma 3.8.5 have the same number of vertices. In the next corollary we ease this restriction and allow F' to have a few more vertices than F.

Corollary 3.8.6. *Let H be an ordered graph on h vertices and let $r := \chi_<(H)$. If H is flexible then the following holds. Let F be the complete r-partite ordered graph with parts $F_1 < \cdots < F_r$ as in Lemma 3.8.5 and let $t \in \mathbb{N}$. For any $s_1, \ldots, s_r, \ell \in \mathbb{N} \cup \{0\}$ such that $s_1 + \cdots + s_r = \ell h \leq th$ and any complete r-partite ordered graph F' with parts $F'_1 < \cdots < F'_r$ of size $|F'_k| = t|F_k| + s_k$ for every $k \in [r]$, both $F(t)$ and F' contain perfect H-tilings.*

Proof. By Lemma 3.8.5, F contains a perfect H-tiling and thus clearly the ordered blow-up $F(t)$ contains a perfect H-tiling too.

We prove that F' contains a perfect H-tiling by induction on ℓ. If $\ell = 0$ then $F' = F(t)$ and so F' contains a perfect H-tiling.

Assume $\ell > 0$. For every $k \in [r]$, let $s'_k \in \mathbb{N} \cup \{0\}$ such that $s'_k \leq s_k$ and $s'_1 + \cdots + s'_r = h$. Let $Q_1 < \cdots < Q_r$ be an interval r-colouring of H. Notice that $V(F')$ can be partitioned into four sets X, Y, W, Z such that

$$|X \cap F'_k| = (t-\ell)|F_k|, \quad |Y \cap F'_k| = (\ell-1)|F_k| + (s_k - s'_k), \quad |W \cap F'_k| = |Q_k|, \quad |Z| = |F_k| + s'_k - |Q_k|,$$

for every $k \in [r]$. Observe that

- If non-empty then $F'[X]$ is a copy of $F(t-\ell)$, thus $F'[X]$ contains a perfect H-tiling.
- If $\ell = 1$ then $s'_k = s_k$ for every $k \in [r]$ and so $F'[Y]$ is the empty graph. Otherwise, $F'[Y]$ contains a perfect H-tiling by the inductive hypothesis since $(s_1 - s'_1) + \cdots + (s_r - s'_r) = (\ell - 1)h$ and $s_k - s'_k \geq 0$ for every $k \in [r]$.
- $F'[W]$ spans a copy of H.

- $F'[Z]$ contains a perfect H-tiling by Lemma 3.8.5 since

$$\sum_{k=1}^{r}(s'_k - |Q_k|) = h - h = 0$$

  and $|s'_k - |Q_k|| \leq h$ for every $k \in [r]$.

Altogether this clearly implies F' contains a perfect H-tiling. □

3.9 Proof sketch of Theorem 3.4.2

In this section we briefly present the main ideas behind the proof of Theorem 3.4.2 before giving a rigorous argument in Section 3.10.1. Throughout this section we set $r := \chi_<(H) \geq 3$.

Let H be an ordered graph such that $\chi^*_{cr}(H) < \chi_<(H) = r$ and H has no local barrier. Let G be an n-vertex graph with n sufficiently large and minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + o(1)\right) n.$$

Our ultimate goal is to construct many chains between any given pair of vertices $x, y \in V(G)$; then Lemma 3.7.1 will conclude the proof.

To achieve this, we first divide $[n]$ into many nearly balanced intervals, remove all edges in G which lie completely in some interval and then apply the multipartite version of the regularity lemma to obtain a reduced graph R. This preconditioning process will make it convenient to work with cliques in the reduced graph: if two clusters W and W' are adjacent in R then by construction either $W < W'$ or $W > W'$.

We then show that given an arbitrary pair of clusters W and W' in R, for *almost* every pair of vertices $x \in W$ and $y \in W'$ we can find many chains between x and y. We achieve this gradually through various steps:

- Given a copy T of K_r in the reduced graph R and an arbitrary cluster W in T, we prove that for *almost* every pair of vertices $x, y \in W$ we can find many chains between x and y. This is quite straightforward and the only property of H we use is that $\chi_<(H) = r$.

- Given a copy T of K_r in the reduced graph R and two arbitrary clusters W, W' in T, we prove that for *almost* every pair of vertices $x \in W$ and $y \in W'$ we can find many chains between x and y. The "flexibility" guaranteed by the condition $\chi_<(H) < \chi^*_{cr}(H)$, formally stated in Corollary 3.8.4, will be the main ingredient here.

- Finally, we show that given any two arbitrary clusters W and W', for *almost* every pair of vertices $x \in W$ and $y \in W'$ we can find many chains between x and y. Since $r \geq 3$, this fairly straightforwardly follows from the minimum degree of R and the previous point.

Next, given an arbitrary vertex $x \in V(G)$, we use the minimum degree condition to find a particular structure $\mathcal{L}$ in G containing x and resembling an extremal construction for graphs which have a local barrier, namely Extremal Example 3; in particular, the vertex classes of Extremal Example 3 are replaced by some of the clusters in R. Assuming H does not have a local barrier, and using Corollary 3.8.6, we find many chains between x and *almost* every vertex lying in a certain cluster W in $\mathcal{L}$.

Analogously, given an arbitrary $y \in V(G) \setminus \{x\}$, we find many chains between y and *almost* every vertex lying in a certain cluster W'. By the previous steps, there exist many chains between almost every vertex in W and almost every vertex in W'. By concatenation, this ensures many chains between x and y and so the hypothesis of Lemma 3.7.1 is satisfied, thereby yielding Theorem 3.4.2.

3.10 Proof of Theorem 3.4.2 and Theorem 3.1.8

3.10.1 Proof of Theorem 3.4.2

Let H be an ordered graph on h vertices such that $\chi^*_{cr}(H) < \chi_<(H)$ and $\chi_<(H) \geq 3$. Assume further that H has no local barrier. Set $r := \chi_<(H)$.

Given any $\eta > 0$, define additional constants $d, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ and $\ell_0 \in \mathbb{N}$ so that

$$0 < 1/\ell_0 \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll d \ll \eta, 1/(rh). \tag{3.7}$$

Set $t := \lceil 4/\eta \rceil$ and let $L_0 = L_0(\varepsilon_1, t, \ell_0) \geq \ell_0$ be as in Theorem 3.6.2. Additionally, let $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathbb{R}$ and $n \in \mathbb{N}$ so that

$$0 < 1/n \ll \xi_5 \ll \xi_4 \ll \xi_3 \ll \xi_2 \ll \xi_1 \ll 1/L_0. \tag{3.8}$$

Define $\nu := (\xi_5/2)^h/4$ and $s := 2r(r-1)h + 1$.

Let G be an ordered graph on $n \geq L_0$ vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + \eta\right) n.$$

Let $W_1 < \cdots < W_t$ be a nearly balanced interval partition of $[n]$. Let G' be the ordered graph obtained by deleting all edges lying in each W_i from G. As $t = \lceil 4/\eta \rceil$ we have that $|W_i| \leq \lceil \eta n/4 \rceil$ for all $i \in [t]$ and so

$$\delta(G') \geq \left(1 - \frac{1}{r-1} + \frac{\eta}{2}\right) n. \tag{3.9}$$

Apply Theorem 3.6.2 to G' with parameters ε_1, t, ℓ_0 and d to obtain a pure graph G'' and a partition $W_i^0, \ldots, W_i^\ell$ for every W_i, where $\ell_0 \leq \ell \leq L_0$. All non-exceptional clusters W_i^j have size m where

$$m \geq \frac{1}{\ell}\left(\left\lfloor\frac{n}{t}\right\rfloor - \frac{\varepsilon_1 n}{t}\right) \geq \frac{\eta n}{5L_0}. \tag{3.10}$$

Let R be the corresponding reduced graph of G'. By (3.9), Proposition 3.6.3 implies that

$$\delta(R) \geq \left(1 - \frac{1}{r-1} + \frac{\eta}{3}\right)|R|. \tag{3.11}$$

Crucially, observe that if $W_i^j W_{i'}^{j'}$ is an edge in R then, by construction of G', we have $i \neq i'$ and so either $W_i^j < W_{i'}^{j'}$ or $W_i^j > W_{i'}^{j'}$.

First, we prove that given a copy T of K_r in R and an arbitrary cluster W in T, for *almost* every pair of vertices $x, y \in W$ there exist many chains between x and y.

Claim 3.10.1. *Let $T_1 < \cdots < T_r$ be r clusters which form a copy of K_r in R. Given any $i \in [r]$, there exists a set $A_i \subseteq T_i$ of size $|A_i| \geq (1 - \varepsilon_1 r)|T_i|$ such that the following holds: for every $x \in A_i$ there exists a set $C_x \subseteq T_i$ of size $|C_x| \geq (1 - 2\varepsilon_2 r)|T_i|$ such that for every $y \in C_x$ there are at least $\xi_1 n^{h-1}$ many $(h-1)$-sets $X \subseteq V(G)$ so that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain a copy of H.*

Proof. Fix $i \in [r]$. For every $k \neq i$, let

$$L_k := \{v \in T_i : d_{G''}(v, T_k) \leq (d - \varepsilon_1)|T_k|\}.$$

Observe that $d_{G''}(L_k, T_k) \leq d - \varepsilon_1$ for every $k \neq i$. Suppose $|L_{k_0}| \geq \varepsilon |T_i|$ for some $k_0 \neq i$. Since $(T_i, T_{k_0})_{G''}$ is (ε_1, d)-regular then

$$|d_{G''}(L_{k_0}, T_{k_0}) - d_{G''}(T_i, T_{k_0})| < \varepsilon_1$$

and so $d_{G''}(L_{k_0}, T_{k_0}) > d - \varepsilon_1$, a contradiction. Thus $|L_k| < \varepsilon_1 |T_i|$ for every $k \neq i$. In particular,

$$|L_1 \cup \cdots \cup L_r| < \varepsilon_1 (r-1)|T_i| < \varepsilon_1 r |T_i|.$$

Let $A_i := T_i \setminus (L_1 \cup \cdots \cup L_r)$. It follows from the previous inequality that

$$|A_i| \geq (1 - \varepsilon_1 r)|T_i|.$$

Let $x \in A_i$. Define $T'_k := T_k \cap N_{G''}(x)$ for every $k \neq i$ and $T'_i := T_i \setminus \{x\}$. By the slicing lemma (Lemma 3.6.1), the pair $(T'_a, T'_b)_{G''}$ is $(\varepsilon_2, d/2)$-regular for every $a \neq b \in [r]$. For every $k \neq i$, let

$$L'_k := \{v \in T'_i : d_{G''}(v, T'_k) \leq (d/2 - \varepsilon_2)|T'_k|\}.$$

By the same argument as before,

$$|L'_1 \cup \cdots \cup L'_r| < \varepsilon_2 r |T'_i|.$$

Let $C_x := T'_i \setminus (L'_1 \cup \cdots \cup L'_r)$. The above inequality implies that

$$|C_x| \geq (1 - \varepsilon_2 r)|T'_i| \geq (1 - 2\varepsilon_2 r)|T_i|.$$

Let $y \in C_x$. Define $T''_k := T'_k \cap N_{G''}(y) = T_k \cap N_{G''}(x) \cap N_{G''}(y)$ for every $k \neq i$ and $T''_i := T'_i \setminus \{y\} = T_i \setminus \{x, y\}$. By the slicing lemma, the pair $(T''_a, T''_b)_{G''}$ is $(\varepsilon_3, d/3)$-regular for every $a \neq b \in [r]$. Furthermore, by construction, x and y are adjacent to all vertices in T''_k for every $k \neq i$.

Let $V_1 < \cdots < V_r$ be an interval r-colouring of H. Pick any $v \in V_i$ and let H' be the complete r-partite ordered graph with parts $V_1 < \cdots < V_i \setminus \{v\} < \cdots < V_r$. Using (3.7), (3.8) and (3.10), the key lemma (Lemma 3.6.4) implies that there exist at least $\xi_1 n^{h-1}$ vertex sets X in G so that $G[X]$ spans a copy of H' where $V_k \subseteq T''_k$ for every $k \neq i$ and $V_i \setminus \{v\} \subseteq T''_i$. Because x and y are adjacent to all vertices in T''_k for every $k \neq i$, both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ are spanned by a copy of H. $\square$

We now prove that given a copy T of K_r in the reduced graph R and two arbitrary clusters W and W' in T, for *almost* every pair of vertices $x \in W$ and $y \in W'$ there exist many chains between x and y. Recall $s = 2r(r-1)h + 1$.

Claim 3.10.2. *Let $T_1 < \cdots < T_r$ be r clusters which form a copy of K_r in R. For every $i, j \in [r]$, there exist sets $A_i \subseteq T_i$ and $A_j \subseteq T_j$ of size $|A_i| \geq (1-\varepsilon_1 r)|T_i|$ and $|A_j| \geq (1-\varepsilon_1 r)|T_j|$ such that for every $x \in A_i$ and $y \in A_j$ there are at least $\xi_2 n^{sh-1}$ many $(sh-1)$-sets $X \subseteq V(G)$ for which both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.*

Proof. Fix $i, j \in [r]$. As in the proof of Claim 3.10.1, for every $k \neq i$, we define

$$L_k := \{v \in T_i : d_{G''}(v, T_k) \leq (d - \varepsilon_1)|T_k|\}$$

and set $A_i := T_i \setminus (L_1 \cup \cdots \cup L_r)$. In the proof of Claim 3.10.1 we saw that

$$|L_1 \cup \cdots \cup L_r| < \varepsilon_1(r-1)|T_i| < \varepsilon_1 r|T_i| \text{ and thus } |A_i| \geq (1-\varepsilon_1 r)|T_i|.$$

Let $x \in A_i$. As in the proof of Claim 3.10.1, we define $T'_k := T_k \cap N_{G''}(x)$ for every $k \neq i$ and $T'_i := T_i \setminus \{x\}$. The pair $(T'_a, T'_b)_{G''}$ is $(\varepsilon_2, d/2)$-regular for every $a \neq b \in [r]$.

For every $k \neq j$, let

$$L'_k := \{v \in T'_j : d_{G''}(v, T'_k) \leq (d/2 - \varepsilon_2)|T'_k|\}.$$

Note that L'_k is not quite the same as the corresponding set given in the proof of Claim 3.10.1 (it is defined it terms of j not i); however, as in the proof of Claim 3.10.1 we have that

$$|L'_1 \cup \cdots \cup L'_r| < \varepsilon_2 r|T'_j|.$$

Let $C := T'_j \setminus (L'_1 \cup \cdots \cup L'_r)$. The previous inequality implies that

$$|C| \geq (1-\varepsilon_2 r)|T'_j| \geq (1-\varepsilon_2 r)(d-\varepsilon_1)|T_j| \overset{(3.7)}{\geq} \frac{d}{2}|T_j|. \tag{3.12}$$

Let $z \in C$. Define $T''_k := T'_k \cap N_{G''}(z)$ for every $k \neq j$ and $T''_j := T'_j \setminus \{z\}$. By the slicing lemma and (3.7), the pair $(T''_a, T''_b)_{G''}$ is $(\varepsilon_3, d/3)$-regular for every pair $a \neq b \in [r]$. Furthermore, by construction, x is adjacent to all vertices in T''_k for every $k \neq i$ while z is adjacent to all vertices in T''_k for every $k \neq j$.

Since $\chi^*_{cr}(H) < \chi_<(H)$, by Corollary 3.8.4 H is flexible. Let F be the complete r-partite ordered graph with parts $F_1 < \cdots < F_r$ as defined in Lemma 3.8.5. Recall that $|F| = 2r(r-1)h^2 = (s-1)h$. Pick any $v \in F_i$ and let F^* be the ordered graph obtained by removing the vertex v from F. In particular, F^* is a complete r-partite ordered graph with parts $F_1 < \cdots < F_i \setminus \{v\} < \cdots < F_r$. By (3.7), (3.8) and (3.10), the key lemma (Lemma 3.6.4) implies that there exist at least $\xi_1 n^{|F|-1}$ vertex sets Y in G so that $G[Y]$ spans a copy of F^* where $F_k \subseteq T''_k$ for every $k \neq i$ and $F_i \setminus \{v\} \subseteq T''_i$.

Given any such set Y, since x is adjacent to all vertices in T''_k (for all $k \neq i$) then $G[Y \cup \{x\}]$ is spanned by a copy of F and thus it contains a perfect H-tiling. Similarly, since z is adjacent to all vertices in T''_k (for all $k \neq j$) then $G[Y \cup \{z\}]$ is spanned by a complete r-partite ordered graph F' on $|F|$ vertices where each part of F' differs in size by at most one compared to the corresponding part in F. Thus, Lemma 3.8.5 implies $G[Y \cup \{z\}]$ contains a perfect H-tiling.

Let $A_j \subseteq T_j$ be as in the statement of Claim 3.10.1; so $|A_j| \geq (1-\varepsilon_1 r)|T_j|$. Let $y \in A_j$. By Claim 3.10.1, there exists a set $C_y \subseteq T_j$ of size

$$|C_y| \geq (1 - 2\varepsilon_2 r)|T_j| \tag{3.13}$$

such that for any $z \in C_y$ there exist at least $\xi_1 n^{h-1}$ many $(h-1)$-sets $Z \subseteq V(G)$ so that both $G[Z \cup \{y\}]$ and $G[Z \cup \{z\}]$ contain spanning copies of H.

In summary, given any $x \in A_i$ and $y \in A_j$, we have shown that for every $z \in C \cap C_y$ there exist at least $\xi_1 n^{|F|-1}$ many $(|F|-1)$-sets Y such that both $G[Y \cup \{x\}]$ and $G[Y \cup \{z\}]$ contain perfect H-tilings and similarly there exist at least $\xi_1 n^{h-1}$ many $(h-1)$-sets Z such that both $G[Z \cup \{y\}]$ and $G[Z \cup \{z\}]$ contain perfect H-tilings. Furthermore, as

$C, C_y \subseteq T_j$ note that

$$|C \cap C_y| = |C| + |C_y| - |C \cup C_y| \geq |C| + |C_y| - |T_j| \overset{(3.7),(3.12),(3.13)}{\geq} \varepsilon_1 |T_j| = \varepsilon_1 m \overset{(3.10)}{\geq} \frac{\varepsilon_1 \eta n}{5L_0}.$$

Applying Lemma 3.7.2 with $C \cap C_y, \varepsilon_1\eta/(5L_0), \xi_1, \xi_2$ playing the roles of A, α, β, γ, we conclude that there exist at least $\xi_2 n^{|F|+h-1} = \xi_2 n^{sh-1}$ many $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings, as desired. $\square$

In the next claim we show that given any arbitrary pair of clusters W and W' in the reduced graph R, for *almost* every pair of vertices $x \in W$ and $y \in W'$ there exist many chains between x and y. The assumption $r \geq 3$ is crucial here.

Claim 3.10.3. *For every pair of clusters W and W' in R, there exist sets $A \subseteq W$ and $A' \subseteq W'$ of size $|A| \geq (1 - \varepsilon_1 r)|W|$ and $|A'| \geq (1 - \varepsilon_1 r)|W'|$ satisfying the following: for every pair of vertices $x \in A$ and $y \in A'$ there are at least $\xi_3 n^{2sh-1}$ many $(2sh-1)$-sets $X \subseteq V(G)$, such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.*

Proof. Recall the reduced graph R has minimum degree

$$\delta(R) \geq \left(1 - \frac{1}{r-1} + \frac{\eta}{3}\right)|R|.$$

Using the above minimum degree condition, given any two adjacent clusters in R, one can greedily construct a copy of K_r in R containing them both.

Since $r \geq 3$, we have $\delta(R) > |R|/2$ and so the clusters W and W' have a common neighbour U in R. Let K and K' be two copies of K_r in R containing W, U and W', U respectively.

Apply Claim 3.10.2 with K, W, U playing the roles of K_r, T_i and T_j to obtain sets $A \subseteq W$ and $D \subseteq U$ with $|A| \geq (1 - \varepsilon_1 r)|W|$ and $|D| \geq (1 - \varepsilon_1 r)|U|$. Similarly, apply Claim 3.10.2 with K', W', U playing the roles of K_r, T_i and T_j to obtain sets $A' \subseteq W'$ and $D' \subseteq U$ with $|A'| \geq (1 - \varepsilon_1 r)|W|$ and $|D'| \geq (1 - \varepsilon_1 r)|U|$.

By Claim 3.10.2, for any $x \in A$, $y \in A'$ and $z \in D \cap D'$, there exist at least $\xi_2 n^{sh-1}$

many $(sh-1)$-sets $Y \subseteq V(G)$ such that both $G[Y \cup \{x\}]$ and $G[Y \cup \{z\}]$ contain perfect H-tilings and $\xi_2 n^{sh-1}$ many $(sh-1)$-sets $Z \subseteq V(G)$ such that both $G[Z \cup \{y\}]$ and $G[Z \cup \{z\}]$ contain perfect H-tilings.

Furthermore, since $D, D' \subseteq U$ then

$$|D \cap D'| \geq |D| + |D'| - |U| \geq (1 - 2\varepsilon_1 r)|U| = (1 - 2\varepsilon_1 r)m \overset{(3.10)}{\geq} \frac{\eta n}{6L_0}.$$

Applying Lemma 3.7.2 with $D \cap D', \eta/(6L_0), \xi_2, \xi_3$ playing the roles of A, α, β, γ, we conclude that there exist at least $\xi_3 n^{2sh-1}$ many $(2sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings, as desired. □

In the next proposition we use the minimum degree condition of the reduced graph R to find a structure $\mathcal{L}$ containing an arbitrary vertex x and resembling the extremal construction for an ordered graph which has a local barrier. Furthermore, we prove that there exist many chains between x and *almost* every vertex in some cluster in $\mathcal{L}$.

Proposition 3.10.4. *Let $x \in V(G)$. There exists some cluster $W \in V(R)$ and a set $A \subseteq W$ of size $|A| \geq (1-\varepsilon_2 r)|W|$ satisfying the following: for every $y \in A$ there exist at least $\xi_1 n^{sh-1}$ many $(sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.*

Proof. Let $x \in V(G)$ and define

$$N^*(x) := \{W^i_j \in V(R) : d_{G'}(x, W^i_j) \geq \eta |W^i_j|/100\}.$$

Recall that every non-exceptional cluster W has size m. Hence, if $W \in N^*(x)$ then there are at most m neighbours of x in W, while if $W \notin N^*(x)$ then there are at most $\eta m/100$ neighbours of x in W. Finally, there are at most $\varepsilon_1 n$ vertices lying in exceptional clusters. Therefore,

$$\left(1 - \frac{1}{r-1} + \frac{\eta}{2}\right) n \overset{(3.9)}{\leq} d_{G'}(x) \leq m \cdot |N^*(x)| + \eta n/100 + \varepsilon_1 n.$$

and thus

$$\left(1 - \frac{1}{r-1} + \frac{\eta}{3}\right)|R| \leq |N^*(x)|. \tag{3.14}$$

Using (3.11) and (3.14), we can greedily find r clusters $T_1, \ldots, T_r$ in R such that

- the T_k's span a copy of K_r in R;
- $T_k \in N^*(x)$ for every $k \in [r-1]$;
- if $x \in W_{k_0}$ with $k_0 \in [t]$ then $T_k \not\subseteq W_{k_0}$ for all $k \in [r]$.

By the properties above, we may relabel indices so that there is an $i \in [r+1]$ and $j \in [r]$ for which

$$T_1 < \cdots < T_{i-1} < x < T_i < \cdots < T_r$$

and $T_k \in N^*(x)$ for every $k \neq j$.

Define $T'_k := T_k \cap N_{G'}(x)$ for $k \neq j$ and $T'_j := T_j$. By construction, $|T'_k| \geq \eta m/100$ for every $k \in [r]$. Thus, by the slicing lemma (Lemma 3.6.1), $(T'_a, T'_b)_{G''}$ is $(\varepsilon_2, d/2)$-regular for every $a \neq b \in [r]$.

We will show that the cluster $W := T_j$ is as desired for the claim. For every $k \neq j$, let

$$L_k := \{v \in T'_j : d_{G''}(v, T'_k) \leq (d/2 - \varepsilon_2)|T'_k|\}.$$

It is easy to show that $|L_k| < \varepsilon_2 |T'_j|$. Let $A := T'_j \setminus (L_1 \cup \cdots \cup L_r)$, then

$$|A| \geq (1 - \varepsilon_2 r)|T'_j| = (1 - \varepsilon_2 r)|T_j|.$$

Let $y \in A$. Define $T''_k := T'_k \cap N_{G''}(y)$ for every $k \neq j$ and $T''_j = T'_j \setminus \{y\}$. By the slicing lemma, $(T''_a, T''_b)_{G''}$ is $(\varepsilon_3, d/3)$-regular for every $a \neq b \in [r]$. Furthermore, x and y are adjacent to all vertices in T''_k for every $k \neq j$ (see Figure 3.3).

Since H has no local barrier, there exists an interval $(r+1)$-colouring

$$V_1 < \cdots < V_{i-1} < \{v\} < V_i < \cdots < V_r$$

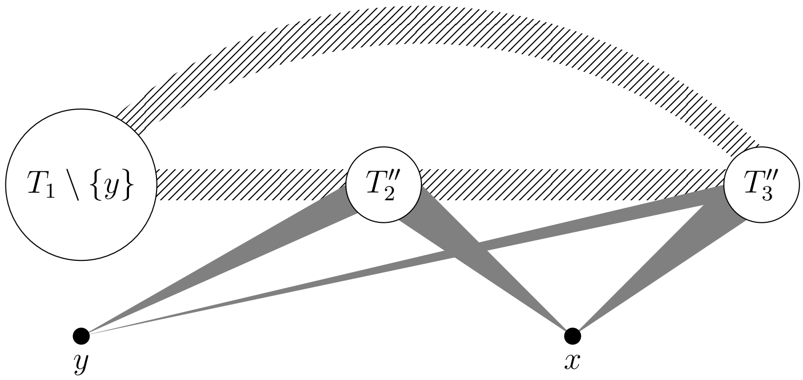

Figure 3.3: In this picture, we take $r = 3$, $T_1 < T_2 < x < T_3$ and $j = 1$. Note that $(T_1 \setminus \{y\}, T_2'')$, $(T_1 \setminus \{y\}, T_3'')$ and (T_2'', T_3') are regular pairs, while x and y are adjacent to all vertices in T_2'' and T_3''.

of H such that there is no edge between v and V_j. Furthermore, as $\chi^*_{cr}(H) < \chi_<(H)$, we have that H is flexible by Corollary 3.8.4. Let F be the complete r-partite ordered graph with parts $F_1 < \cdots < F_r$ as defined in Lemma 3.8.5 and let F' be the complete r-partite ordered graph with parts $F_1' < \cdots < F_r'$ where

$$|F_k'| = \begin{cases} |F_k| + |V_k| + 1 & \text{if } k = j, \\ |F_k| + |V_k| & \text{otherwise.} \end{cases}$$

Note that

$$1 + \sum_{k=1}^{r} |V_k| = 1 + (h-1) = h$$

thus, both F and F' contain perfect H-tilings by Corollary 3.8.6.

Pick any $u \in F_j'$. Note that $F' \setminus \{u\}$ is the complete r-partite ordered graph such that the kth part has size $|F_k| + |V_k|$ for every $k \in [r]$ and $|F' \setminus \{u\}| = |F| + h - 1 = sh - 1$. By the key lemma (Lemma 3.6.4), there exist at least $\xi_1 n^{sh-1}$ many $(sh-1)$-sets $X \subseteq V(G)$ such that $G[X]$ spans a copy of $F' \setminus \{u\}$ with $F_k' \subseteq T_k''$ for every $k \neq j$ and $F_j' \setminus \{u\} \subseteq T_j''$. It remains to show that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.

First, we consider $G[X \cup \{x\}]$. Since $G[X]$ spans a copy of $F' \setminus \{u\}$ with $F_k' \subseteq T_k''$ for every $k \neq j$ and $F'_j \setminus \{u\} \subseteq T''_j$, then X can be partitioned into two sets Y, Z such that

$$|Y \cap T''_k| = |V_k| \quad \text{and} \quad |Z \cap T''_k| = |F_k|$$

for every $k \in [r]$. Note that $G[Z]$ spans a copy of F, thus $G[Z]$ contains a perfect H-tiling. On the other hand, $G[Y]$ spans a copy of $H \setminus \{v\}$. Recall that x is adjacent to all vertices in T''_k for every $k \neq j$ and $T''_{i-1} < x < T''_i$. Thus, $G[Y \cup \{x\}]$ spans a copy of H in G where x plays the role of v. It follows that $G[X \cup \{x\}]$ contains a perfect H-tiling.

Next, we consider $G[X \cup \{y\}]$. Since $y \in T_j$ and y is adjacent to all vertices in T''_k for $k \neq j$, then $G[X \cup \{y\}]$ spans a copy of F' and so $G[X \cup \{y\}]$ contains a perfect H-tiling. □

Remark 3.10.5. *Note that Proposition 3.10.4 holds even if we relax the hypothesis of Theorem 3.4.2 to allow $\chi_<(H) \geq 2$; that is, we did not use the condition that $r \geq 3$ anywhere in the proof of this claim. This fact will be useful in the proof of Theorem 3.10.7 in the next subsection.*

Our final claim states that given arbitrary vertices $x, y \in V(G)$ there exist many chains of bounded size between x and y.

Claim 3.10.6. *Let $x, y \in V(G)$. There exist at least $\xi_5 n^{4sh-1}$ many $(4sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings.*

Proof. By Proposition 3.10.4 there exist clusters $W, W' \in V(R)$ a set $A \subseteq W$ of size $|A| \geq (1-\varepsilon_2 r)|W|$ and a set $A' \subseteq W'$ of size $|A'| \geq (1-\varepsilon_2 r)|W'|$ such that for every $z \in A$ and $z' \in A'$ there exist at least $\xi_1 n^{sh-1}$ many $(sh-1)$-sets $Y \subseteq V(G)$ such that both $G[Y \cup \{x\}]$ and $G[Y \cup \{z\}]$ contain perfect H-tilings and at least $\xi_1 n^{sh-1}$ many $(sh-1)$-sets $Y' \subseteq V(G)$ such that both $G[Y' \cup \{y\}]$ and $G[Y' \cup \{z'\}]$ contain perfect H-tilings.

Furthermore, by Claim 3.10.3 there exist sets $D \subseteq W$ and $D' \subseteq W'$ of size $|D| \geq (1-\varepsilon_1 r)|W|$ and $|D'| \geq (1-\varepsilon_1 r)|W'|$ respectively such that for every $z \in D$ and $z' \in D'$ there exist at least $\xi_3 n^{2sh-1}$ many $(2sh-1)$-sets $Z \subseteq V(G)$ such that both $G[Z \cup \{z\}]$

and $G[Z \cup \{z'\}]$ contain perfect H-tilings. Note that

$$|A \cap D| \geq (1 - 2\varepsilon_2 r)|W| = (1 - 2\varepsilon_2 r)m \overset{(3.10)}{\geq} \frac{\eta n}{6L_0},$$

$$|A' \cap D'| \geq (1 - 2\varepsilon_2 r)|W'| = (1 - 2\varepsilon_2 r)m \overset{(3.10)}{\geq} \frac{\eta n}{6L_0}.$$

Apply Lemma 3.7.2 to y and any $z \in A \cap D$ with $A' \cap D', \eta/(6L_0), \xi_3, \xi_4$ playing the roles of A, α, β, γ; we conclude that there are at least $\xi_4 n^{3sh-1}$ many $(3sh-1)$-sets $X' \subseteq V(G)$ such that both $G[X' \cup \{z\}]$ and $G[X' \cup \{y\}]$ contain perfect H-tilings.

Next apply Lemma 3.7.2 to x and y, with $A \cap D, \eta/(6L_0), \xi_4, \xi_5$ playing the roles of A, α, β, γ; this yields at least $\xi_5 n^{4sh-1}$ many $(4sh-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings, as desired. □

Observe that, by Claim 3.10.6, the hypothesis of Theorem 3.7.1 is satisfied with $4s$ and ξ_5 playing the roles of s and ξ. Thus, $V(G)$ contains a set Abs so that

- $|Abs| \leq (\xi_5/2)^h n/4$;
- Abs is an H-absorbing set for any $I \subseteq V(G) \setminus Abs$ such that $|I| \in h\mathbb{N}$ and $|I| \leq (\xi_5/2)^{2h} n/(512s^2h^3)$.

Recall that $\nu = (\xi_5/2)^h/4$. Therefore we have that $|Abs| \leq \nu n$ and by (3.7) and (3.8) we have that $\nu^3 < (\xi_5/2)^{2h}/(512s^2h^3)$; so Abs is as desired. □

3.10.2 Proof of Theorem 3.1.8

By arguing as in Proposition 3.10.4, we obtain the following result.

Theorem 3.10.7. *Let H be an ordered graph that does not have a local barrier and let $\eta > 0$. There exists an $n_0 \in \mathbb{N}$ so that the following holds. If G is an ordered graph on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_<(H) - 1} + \eta\right) n,$$

then for any vertex $x \in V(G)$ there exists a copy of H in G covering the vertex x.

Proof. Let $r := \chi_<(H)$. Given such an ordered graph G, define constants as in (3.7) and (3.8). As in the proof of Theorem 3.4.2, apply the regularity lemma, and then argue as in the proof of Proposition 3.10.4 to obtain the following: given any $x \in V(G)$, there are disjoint $T'_1, \dots, T'_r \subseteq V(G)$, where $|T'_k| \geq \eta^2 n/(500L_0)$, and integers $i \in [r+1]$ and $j \in [r]$ such that

$$T'_1 < \cdots < T'_{i-1} < x < T'_i < \cdots < T'_r,$$

(T'_a, T'_b) is $(\varepsilon_2, d/2)$-regular for every $a \neq b \in [r]$ and x is adjacent to all vertices in T'_k for any $k \neq j$.

As H does not have a local barrier, there exists an interval $(r+1)$-colouring

$$V_1 < \cdots < V_{i-1} < \{v\} < V_i < \cdots < V_r$$

of H such that there is no edge between v and V_j. By the key lemma, there exists some set $X \subseteq V(G)$ such that $G[X]$ spans a copy of $H \setminus \{v\}$ with $V_k \subseteq T'_k$ for every $k \in [r]$. Since x is adjacent to all vertices in T'_k for every $k \neq j$, we have that $G[X \cup \{x\}]$ spans a copy of H in G, as desired. □

We are now ready to prove Theorem 3.1.8 using Theorem 3.4.1 and Theorem 3.10.7.

Proof of Theorem 3.1.8. Note that the lower bounds stated in Theorem 3.1.8 follow immediately from Corollary 3.2.2 and Lemma 3.2.5. It remains to prove the upper bounds.

Let H be an ordered graph and $\eta > 0$. Let G be an ordered graph on n vertices with n sufficiently large and minimum degree

(i) $\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)} + \eta\right) n$ if H has a local barrier;

(ii) $\delta(G) \geq \left(1 - \frac{1}{\chi_<(H)-1} + \eta\right) n$ if H does not have a local barrier.

In case (ii) we obtain an H-cover by Theorem 3.10.7. In case (i), consider any $x \in V(G)$. Let $y \in V(G) \setminus \{x\}$. By Theorem 3.4.1 there exists a set $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect H-tilings. In particular, there exists a copy of H in G covering x. Thus, G has an H-cover, as desired. □

3.11 Properties of $\chi^*_{cr}(H)$

In this section, we provide various bounds on the parameter $\chi^*_{cr}(H)$. We start by showing some natural upper and lower bounds.

Proposition 3.11.1. *Let H be an ordered graph on h vertices and let $r := \chi_<(H)$. Let $\mathcal{C}$ denote the set of interval r-colourings of H. Define*

$$\ell_-(H) := \max_{(H_1<\cdots<H_r)\in\mathcal{C}} |H_1| \quad \textit{and} \quad \ell^*_-(H) := \max_{(H_1<\cdots<H_r)\in\mathcal{C}} |H_r|.$$

Then $\chi^*_{cr}(H) \geq \max\{h/\ell_-(H)\,,\, h/\ell^*_-(H)\}$.

Proof. We only prove that $\chi^*_{cr}(H) \geq h/\ell_-(H)$ as the proof that $\chi^*_{cr}(H) \geq h/\ell^*_-(H)$ is analogous. Let B be a bottlegraph of H with parts $B_1, \dots, B_k$ for some $k \in \mathbb{N}$. Our aim is to show that $\chi_{cr}(B) \geq h/\ell_-(H)$. By Proposition 3.5.1, we may assume there exists some $m \in \mathbb{N}$ such that

$$|B_k| \leq m \quad \text{and} \quad |B_i| = m$$

for every $i \in [k-1]$. Let ϕ be an interval labelling of B such that the ordered graph (B, ϕ) has parts $B_1 < \cdots < B_k$. Since B is a bottlegraph, there exists a $t \in \mathbb{N}$ so that $(B(t), \phi)$ contains a perfect H-tiling $\mathcal{H}$. Note $\mathcal{H}$ consists of $|B|t/h$ copies of H and every copy of H has at most $\ell_-(H)$ vertices in B_1, thus

$$mt = |B_1|t \leq \ell_-(H)|\mathcal{H}| = \ell_-(H)|B|t/h.$$

Since $\chi_{cr}(B) = |B|/m$, the above implies $\chi_{cr}(B) \geq h/\ell_-(H)$ and so $\chi^*_{cr}(H) \geq h/\ell_-(H)$. $\square$

Note Proposition 3.11.1 implies that if H is such that 1 and 2 are adjacent or $h-1$ and h are adjacent then $\chi^*_{cr}(H) = h$.

Proposition 3.11.2. *Let H be an ordered graph on h vertices and let $r := \chi_<(H)$. Let $\mathcal{C}$ denote the set of interval r-colourings of H. Define*

$$\ell_+(H) := \max_{(H_1<\cdots<H_r)\in\mathcal{C}} \left\{ \min_{i\in[r]} |H_i| \right\}.$$

Then $\chi^*_{cr}(H) \le h/\ell_+(H)$.

Proof. Let $H_1 < \cdots < H_r$ be an interval r-colouring of H such that

$$\ell_+(H) = \min_{i\in[r]} |H_i|.$$

Let $k := \lfloor h/\ell_+(H) \rfloor$ and let B be the complete $(k+1)$-partite unordered graph with parts $B_1, \dots, B_{k+1}$ such that

$$|B_{k+1}| = (h - k\ell_+(H)) \cdot (h!) \quad \text{and} \quad |B_s| = \ell_+(H) \cdot (h!)$$

for every $s \in [k]$. Note that B_{k+1} is the smallest part and may be empty. We have that $|B| = h \cdot h!$ and $\chi_{cr}(B) = h/\ell_+(H)$.

We will now prove that B is a simple bottlegraph of H. Let σ be a permutation of $[k+1]$ and let ϕ be an interval labelling of B with respect to σ. Note that $V((B,\phi)) = [h \cdot (h!)]$. Set $t_0 := 0$ and

$$t_i := (|H_1| + \cdots + |H_i|) \cdot (h!)$$

for every $i \in [r]$. For every $j \in [h!]$ and $i \in [r]$, let

$$T^i_j := [t_{i-1} + j|H_i|] \setminus [t_{i-1} + (j-1)|H_i|].$$

Observe that the set of intervals $\{T^i_j\}_{j\in[h!]}$ is a partition of $[t_i] \setminus [t_{i-1}]$ and in particular the set of all intervals T^i_j is a partition of $[h \cdot (h!)]$.

Relabel the parts of (B,ϕ) by $B'_1, \dots, B'_{k+1}$ so that $B'_1 < \cdots < B'_{k+1}$. Suppose that T^i_j intersects both B'_s and B'_{s+1} for some $i \in [r]$ and $j \in [h!]$ and $s \in [k]$. Since $|T^i_{j'}| = |H_i|$

for every $j' \in [h!]$, it follows that

$$|(B'_1 \cup \cdots \cup B'_s) \setminus [t_{i-1}]|$$

is not divisible by $|H_i|$, as otherwise no $T^i_{j'}$ would intersect both B'_s and B'_{s+1} for any $j' \in [h!]$, and in particular for $j' = j$, contradicting our assumption. However, $|B'_1 \cup \cdots \cup B'_s|$ and t_{i-1} are both divisible by $h!$ and thus $|(B'_1 \cup \cdots \cup B'_s) \setminus [t_{i-1}]|$ is divisible by $|H_i|$, reaching a contradiction. Therefore, every T^i_j is a subset of some B'_s.

Suppose T^i_j and T^{i+1}_j are both subsets of some B'_s, then

$$|B'_s| \geq (t_i + j|H_{i+1}|) - (t_{i-1} + (j-1)|H_i|) = (h! - j + 1)|H_i| + j|H_{i+1}| \geq (h! + 1) \cdot \ell_+(H),$$

again reaching a contradiction. It follows that the ordered graph T_j spanned by $T^1_j < \cdots < T^r_j$ is a complete r-partite ordered subgraph of (B, ϕ). Since $|T^i_j| = |H_i|$ then T_j spans a copy of H in (B, ϕ) for every $j \in [h!]$. The T_j's are disjoint and cover all the vertices of (B, ϕ), thus they yield a perfect H-tiling. Since σ, ϕ are arbitrary, B is a simple bottlegraph of H. In particular,

$$\chi^*_{cr}(H) \leq \chi_{cr}(B) = h/\ell_+(H).$$

□

The next result follows easily from the previous bounds.

Proposition 3.11.3. *Let H be a complete r-partite ordered graph on h vertices with parts $H_1 < \cdots < H_r$ such that (i) $|H_1| \leq |H_i|$ for every $i \in [r]$ or (ii) $|H_r| \leq |H_i|$ for every $i \in [r]$. Then $\chi^*_{cr}(H) = h/|H_1| \geq \chi_<(H)$ in case (i) and $\chi^*_{cr}(H) = h/|H_r| \geq \chi_<(H)$ in case (ii).*

Proof. For case (i), let $\ell_-(H)$ and $\ell_+(H)$ be as in Proposition 3.11.1 and Proposition 3.11.2 respectively. Note that $\ell_-(H) = \ell_+(H) = |H_1|$, hence Propositions 3.11.1 and 3.11.2 imply that $\chi^*_{cr}(H) = h/|H_1|$. Case (ii) follows similarly. $\square$

In [8], Balogh, Li and the Treglown (implicitly) computed $\chi^*_{cr}(H)$ for any H such that $\chi_<(H) = 2$. Their result can be easily recovered using Propositions 3.11.1 and 3.11.2.

Proposition 3.11.4 (Balogh, Li and Treglown [8])**.** *Let H be an ordered graph with vertex set $[h]$ such that $\chi_<(H) = 2$. Define $\alpha^+(H)$ to be the largest integer $t \in [h]$ so that $[1,t]$ is an independent set in H and similarly $\alpha^-(H)$ to be the largest integer $t \in [h]$ so that $[h-t+1,h]$ is an independent set in H. Let $\alpha(H) := \min\{\alpha^+(H), \alpha^-(H)\}$. Then*

$$\chi^*_{cr}(H) = \frac{h}{\alpha(H)}.$$

Proof. Let $\ell_-(H), \ell^*_-(H), \ell_+(H)$ be as in Propositions 3.11.1 and 3.11.2.

Observe that both $[1,\alpha^+(H)] < [\alpha^+(H)+1,h]$ and $[1,h-\alpha^-(H)] < [h-\alpha^-(H)+1,h]$ are interval 2-colourings of H. In particular,

$$\ell_-(H) = \alpha^+(H) \quad \text{and} \quad \ell^*_-(H) = \alpha^-(H),$$

and so $\chi^*_{cr}(H) \geq h/\alpha(H)$ by Proposition 3.11.1. It remains to show that $\chi^*_{cr}(H) \leq h/\alpha(H)$. If $\alpha(H) \leq h/2$ then $\ell_+(H) \geq \alpha(H)$[7] and so $\chi^*_{cr}(H) \leq h/\alpha(H)$ by Proposition 3.11.2. If $\alpha(H) > h/2$, then both $[1,\alpha(H)] < [\alpha(H)+1,h]$ and $[1,h-\alpha(H)] < [h-\alpha(H)+1,h]$ are interval 2-colourings of H. Therefore the complete bipartite graph B with parts U,V of size $|U| = \alpha(H)$ and $|V| = h-\alpha(H)$ is a simple bottlegraph of H. We have $\chi_{cr}(B) = h/\alpha(H)$ and thus $\chi^*_{cr}(H) \leq h/\alpha(H)$. $\square$

If $\chi_<(H) = 3$, finding a closed formula for $\chi^*_{cr}(H)$ already proves challenging. In the next result we determine $\chi^*_{cr}(H)$ for any complete 3-partite ordered graph H.

[7] In fact, if $\alpha(H) \leq h/2$ then $\ell_+(H) = \alpha(H)$.

Proposition 3.11.5. *Let H be a complete 3-partite ordered graph on h vertices with parts $H_1 < H_2 < H_3$ of size h_1, h_2, h_3 respectively and let*

$$g(H) := \left(2 - \frac{\min\{h_1, h_2, h_3\}}{\min\{h_1, h_3\}}\right) \cdot \frac{h}{\min\{h_1, h_3\}}.$$

Then $\chi^*_{cr}(H) = g(H)$.

Proof. We may assume that $h_1 \leq h_3$; the case $h_1 \geq h_3$ follows by the symmetry of the argument. If $h_1 \leq h_2$ then $\chi^*_{cr}(H) = h/h_1$ by Proposition 3.11.3 and the result follows. So for the rest of the proof we assume $h_2 < h_1 \leq h_3$. Under these assumptions, $g(H) = (2 - h_2/h_1)h/h_1 > 3$.

Claim 3.11.6. $\chi^*_{cr}(H) \geq g(H)$.

Let B be a bottlegraph of H with parts $B_1, \ldots, B_k$ for some $k \in \mathbb{N}$. Note that $k \geq 3$ since H is 3-partite. Our aim is to show that $\chi_{cr}(B) \geq g(H)$. By Proposition 3.5.1, we may assume that there exists some $m \in \mathbb{N}$ such that

$$|B_k| \leq m \quad \text{and} \quad |B_i| = m$$

for every $i \in [k-1]$. Pick an interval labelling ϕ of B such that the ordered graph (B, ϕ) has parts $B_1 < B_2 < \cdots < B_k$. By definition there exists some $t \in \mathbb{N}$ such that the ordered blow-up $(B(t), \phi)$ contains a perfect H-tiling $\mathcal{M}$. Recall that we denote the parts of $(B(t), \phi)$ by $B_1(t) < \cdots < B_k(t)$.

Let $\mathcal{M}_1$ be the set of copies of $H \in \mathcal{M}$ whose first part H_1 lies completely in $B_1(t)$ and let $\mathcal{M}_2 := \mathcal{M} \setminus \mathcal{M}_1$. Note that every copy of $H \in \mathcal{M}_1$ has exactly h_1 vertices in $B_1(t)$ and at most $h_1 + h_2$ vertices in $B_1(t) \cup B_2(t)$, while every copy of $H \in \mathcal{M}_2$ has at most h_1 vertices in $B_1(t) \cup B_2(t)$. Since $|B_1(t)| = |B_2(t)| = mt$, it follows that

$$mt \geq h_1 \cdot |\mathcal{M}_1| \quad \text{and} \quad 2mt \leq (h_1 + h_2) \cdot |\mathcal{M}_1| + h_1 \cdot |\mathcal{M}_2|.$$

The above implies that

$$|\mathcal{M}| = |\mathcal{M}_1| + |\mathcal{M}_2| \geq \left(2 - \frac{h_2}{h_1}\right)\frac{mt}{h_1}.$$

Since $\chi_{cr}(B) = |B|/m = |\mathcal{M}|h/(mt)$ then $\chi_{cr}(B) \geq g(H)$, proving the claim.

Claim 3.11.7. *If $h_1 = h_3$ then there exists a simple bottlegraph B of H with four parts such that three parts of B have size h_1^2, one part has size $h_1^2 - h_2^2$ and $\chi_{cr}(B) = g(H)$.*

Suppose $h_1 = h_3$. Then

$$g(H) = \left(2 - \frac{h_2}{h_1}\right)\frac{h}{h_1} = 4 - \left(\frac{h_2}{h_1}\right)^2$$

and so $3 < g(H) < 4$. Let B be the complete 4-partite graph with parts B_1, B_2, B_3, B_4 where

$$|B_1| = h_1^2 - h_2^2 \quad \text{and} \quad |B_i| = h_1^2$$

for every $i = 2, 3, 4$. Note that $\chi_{cr}(B) = |B|/h_1^2 = g(H)$. It remains to show that B is a simple bottlegraph of H.

Let σ be a permutation of $[4]$ and ϕ an interval labelling of B with respect to σ. Recall that the ordered graph (B, ϕ) has parts $B_{\sigma^{-1}(1)} < B_{\sigma^{-1}(2)} < B_{\sigma^{-1}(3)} < B_{\sigma^{-1}(4)}$. Either $\sigma^{-1}(1), \sigma^{-1}(2) \neq 1$ or $\sigma^{-1}(3), \sigma^{-1}(4) \neq 1$. We will now deal with the former case.

Observe that $V(B)$ can be partitioned into two sets X, Y of size $|X| = h_1(2h_1 + h_2)$ and $|Y| = (h_1 - h_2)(2h_1 + h_2)$ such that

$$|Y \cap B_{\sigma^{-1}(i)}| = \begin{cases} 0 & \text{if } i = 1, \\ h_1(h_1 - h_2) & \text{if } i = 2, \\ h_2(h_1 - h_2) & \text{if } i = 3, \\ h_1(h_1 - h_2) & \text{if } i = 4, \end{cases}$$

and

$$|X \cap B_{\sigma^{-1}(i)}| = \begin{cases} h_1^2 & \text{if } i = 1, \\ h_1 h_2 & \text{if } i = 2, \\ |B_{\sigma^{-1}(3)}| - h_1 h_2 + h_2^2 & \text{if } i = 3, \end{cases}$$

and all remaining vertices of X lie in $B_{\sigma^{-1}(4)}$.

In particular, (i) $|X \cap B_{\sigma^{-1}(1)}| = h_1^2$; $|X \cap B_{\sigma^{-1}(2)}| = h_1 h_2$; $|X \cap B_{\sigma^{-1}(3)}| = h_1^2 - h_1 h_2 + h_2^2$; $|X \cap B_{\sigma^{-1}(4)}| = h_1 h_2 - h_2^2$ or (ii) $|X \cap B_{\sigma^{-1}(1)}| = h_1^2$; $|X \cap B_{\sigma^{-1}(2)}| = h_1 h_2$; $|X \cap B_{\sigma^{-1}(3)}| = h_1^2 - h_1 h_2$; $|X \cap B_{\sigma^{-1}(4)}| = h_1 h_2$. Thus, we have that $(B, \phi)[X]$ contains a perfect H-tiling consisting of h_1 copies of H where the first part H_1 and the second part H_2 of every such copy of H lie in $B_{\sigma^{-1}(1)}$ and $B_{\sigma^{-1}(2)}$ respectively. Further, $(B, \phi)[Y]$ contains a perfect H-tiling consisting of $h_1 - h_2$ copies of H. The union of these two H-tilings yields a perfect H-tiling in (B, ϕ).

The only remaining case is when $\sigma^{-1}(3), \sigma^{-1}(4) \neq 1$. However, since $h_1 = h_3$, this case will follow by a symmetric argument to that of the previous case. Thus B is a simple bottlegraph of H.

Claim 3.11.8. *If $g(H)$ is an integer then there exists a simple bottlegraph B of H such that $\chi_{cr}(B) = g(H)$ (i.e., B has $g(H)$ parts) and all parts of B have size h_1^2.*

Set $k := g(H)$. Recall that $k > 3$ and so $k \geq 4$ since $k \in \mathbb{N}$. Let B be the complete k-partite graph with parts $B_1, \dots, B_k$ all of size h_1^2. Notice $\chi_{cr}(B) = k$, so it remains to show that B is a simple bottlegraph of H.

Let σ be a permutation of $[k]$ and ϕ an interval labelling of B with respect to σ. Recall that the ordered graph (B, ϕ) has parts $B_{\sigma^{-1}(1)} < \cdots < B_{\sigma^{-1}(k)}$. Let $X \subseteq V(B)$ be a set of size $4h_1^2 - h_2^2$ such that

$$|X \cap B_{\sigma^{-1}(i)}| = \begin{cases} h_1^2 & \text{if } i = 1, 2, 3, \\ h_1^2 - h_2^2 & \text{if } i = 4. \end{cases}$$

Let H' be the complete 3-partite ordered graph with parts $H'_1 < H'_2 < H'_3$ of size h_1, h_2, h_1 respectively. By Claim 3.11.7, $(B, \phi)[X]$ contains a perfect H'-tiling consisting of $2h_1 - h_2$ copies of H'. Observe that by definition of k,

$$|V(B) \setminus X| = kh_1^2 - (4h_1^2 - h_2^2) = (2h_1 - h_2)(h_3 - h_1).$$

Thus we can partition $V(B) \setminus X$ into $2h_1 - h_2$ sets of size $h_3 - h_1$ and assign each set to a copy of H'. Notice that every copy of H' together with its assigned set forms a copy of H; so we constructed a perfect H-tiling in (B, ϕ). Since σ and ϕ are arbitrary, B is a simple bottlegraph of H.

Claim 3.11.9. *There exists a simple bottlegraph B of H such that $\chi_{cr}(B) = g(H)$.*

Recall that $h_2 < h_1 \leq h_3$. We may assume that $g(H)$ is not an integer as otherwise we are done by Claim 3.11.8. Given $t \in \mathbb{N}$ and $\ell \in \mathbb{N} \cup \{0\}$, let $H(t, \ell)$ be the complete 3-partite ordered graph with parts $H'_1 < H'_2 < H'_3$ of size $th_1, th_2, th_3 - \ell$ respectively.

Set $k := \lfloor g(H) \rfloor$. We define t, ℓ, s and B as follows:

- If $g(H) \geq 4$, there exist $t, \ell \in \mathbb{N}$ such that

$$\frac{\ell}{t} = \frac{(g(H) - k)h_1}{(2 - h_2/h_1)},$$

since the right hand side of the above inequality is a positive rational number. Thus,

$$k = g(H) - \left(2 - \frac{h_2}{h_1}\right) \frac{\ell/t}{h_1} = \left(2 - \frac{h_2}{h_1}\right) \frac{h - \ell/t}{h_1} = \left(2 - \frac{th_2}{th_1}\right) \frac{th_1 + th_2 + th_3 - \ell}{th_1}. \tag{3.15}$$

Furthermore, we have

$$k \geq 4 > 4 - \left(\frac{h_2}{h_1}\right)^2 = \left(2 - \frac{th_2}{th_1}\right) \frac{th_1 + th_2 + th_1}{th_1}. \tag{3.16}$$

Equations (3.15) and (3.16) imply that $th_3 - \ell > th_1$. Hence $th_2 < th_1 < th_3 - \ell$

and so

$$g(H(t,\ell)) = \left(2 - \frac{th_2}{th_1}\right)\frac{th_1 + th_2 + th_3 - \ell}{th_1} = k.$$

Let B be the simple bottlegraph of $H(t,\ell)$ as in Claim 3.11.8 and set $s := 0$.

- If $3 < g(H) < 4$, let $t := 1$ and $\ell := h_3 - h_1$. Observe that the parts $H_1' < H_2' < H_3'$ of $H(t,\ell)$ have size h_1, h_2, h_1 respectively. Let B be the simple bottlegraph of $H(t,\ell)$ as in Claim 3.11.7. Set $s := h_1^2 - h_2^2$.

Note that in both cases, B is a complete $(k+1)$-partite graph where each part has size $(th_1)^2$ except one smaller part of size $t^2 s$ (which is empty if $g(H) \geq 4$).

Observe that $g(H) - k$ is a rational number and

$$\frac{s}{h_1^2} \leq g(H) - k < 1. \tag{3.17}$$

Indeed, the lower bound is trivial if $g(H) \geq 4$; if $3 < g(H) < 4$ then $s = h_1^2 - h_2^2$ and $k = 3$, so $g(H) - k \geq (2 - h_2/h_1)(2h_1 + h_2)/h_1 - k = 4 - (h_2/h_1)^2 - 3 = 1 - (h_2/h_1)^2 = s/h_1^2$.

Thus, by (3.17), there exist $a, b \in \mathbb{N} \cup \{0\}$ such that $(a,b) \neq (0,0)$ and

$$a(1 - g(H) + k) = b(g(H) - k - s/h_1^2) \implies a + b\frac{s}{h_1^2} = (a+b)(g(H) - k)$$

$$\implies a(th_1)^2 + bt^2 s = (a+b)(th_1)^2(g(H) - k). \tag{3.18}$$

Let B' be the complete $(k+1)$-partite graph with parts $B_1', \ldots, B_{k+1}'$ where

$$|B_1'| = (a+b)(th_1)^2(g(H) - k) \quad \text{and} \quad |B_i'| = (a+b)(th_1)^2$$

for every $i > 1$. Notice $\chi_{cr}(B') = g(H)$, so it remains to show that B' is a simple bottlegraph of H.

Let σ be a permutation of $[k+1]$ and let ϕ be an interval labelling of B' with respect to σ. Recall that the ordered graph (B', ϕ) has parts $B'_{\sigma^{-1}(1)} < \cdots < B'_{\sigma^{-1}(k+1)}$. Let X, Y be two disjoint sets in $V(B')$ of size $a|B|$ and $b|B|$ respectively such that

$$|X \cap B'_{\sigma^{-1}(i)}| = \begin{cases} a(th_1)^2 & \text{if } i < k+1, \\ a(t^2 s) & \text{if } i = k+1, \end{cases}$$

and

$$|Y \cap B'_{\sigma^{-1}(i)}| = \begin{cases} b(th_1)^2 & \text{if } \sigma^{-1}(i) \neq 1, \\ b(t^2 s) & \text{if } \sigma^{-1}(i) = 1. \end{cases}$$

Notice that by (3.18) and the choice of the sizes of the parts of B', all vertices in $B'_{\sigma^{-1}(i)}$ are in $X \cup Y$ for $i \leq k$; as $s \leq h_1^2$ it may be that some vertices in $B'_{\sigma^{-1}(k+1)}$ are not in $X \cup Y$.

Note that $(B', \phi)[X]$ is a copy of $(B(a), \phi')$ for some interval labelling ϕ'. So as B is a simple bottlegraph of $H(t, \ell)$, we have that $(B(a), \phi')$ contains a perfect $H(t, \ell)$-tiling $\mathcal{M}_1$ consisting of

$$\frac{a|B|}{|H(t,\ell)|} = \frac{a(th_1)^2 g(H(t,\ell))}{|H(t,\ell)|} = a(th_1)^2 \left(2 - \frac{h_2}{h_1}\right) \frac{h - \ell/t}{h_1} \frac{1}{th - \ell} = a(2h_1 - h_2)t \quad (3.19)$$

copies of $H(t, \ell)$. Similarly, $(B', \phi)[Y]$ is a copy of $(B(b), \phi'')$ for some interval labelling ϕ'' and it contains a perfect $H(t, \ell)$-tiling $\mathcal{M}_2$ consisting of

$$\frac{b|B|}{|H(t,\ell)|} = b(2h_1 - h_2)t \quad (3.20)$$

copies of $H(t, \ell)$. Note that the vertices in $V(B') \setminus (X \cup Y)$ lie in $B'_{\sigma^{-1}(k+1)}$. Moreover, as $g(H) = (2 - h_2/h_1)h/h_1$,

$$\begin{aligned} |V(B') \setminus (X \cup Y)| &= |B'| - (a+b)|B| = (a+b)(th_1)^2 g(H) - (a+b)|B| \\ &\overset{(3.19),(3.20)}{=} (a+b)\left((2h_1 - h_2)ht^2 - (2h_1 - h_2)(th - \ell)t\right). \end{aligned}$$

And so $|V(B') \setminus (X \cup Y)| = (a+b)(2h_1 - h_2)\ell t$. Thus, we can divide $V(B') \setminus (X \cup Y)$ into $(a+b)(2h_1 - h_2)t$ sets of size ℓ and assign each set to a copy of $H(t,\ell)$ in $\mathcal{M}_1 \cup \mathcal{M}_2$. Note that each copy of $H(t,\ell)$ together with its assigned set forms a copy of $H(t,0)$. Thus we constructed a perfect $H(t,0)$-tiling in (B',ϕ). Since $H(t,0) = H(t)$, this yields a perfect H-tiling in (B',ϕ), proving that B' is indeed a simple bottlegraph of H.

Claim 3.11.9 implies that $\chi^*_{cr}(H) \le g(H)$. Together with Claim 3.11.6, this concludes the proof. $\square$

Observe that Propositions 3.11.3 and 3.11.5 combined with Theorem 3.1.6 yield the following result which makes the threshold in Theorem 3.1.6 explicit for all complete 3-partite ordered graphs.

Theorem 3.11.10. *Let H be a complete r-partite ordered graph on h vertices with parts $H_1 < \cdots < H_r$ where $|H_i| = h_i$ for every $i \in [r]$.*

- *If $h_1 \le h_i$ for every $i > 1$, then $\delta_<(H,n) = \left(1 - \frac{h_1}{h} + o(1)\right) n$.*
- *If $h_r \le h_i$ for every $i \in [r-1]$, then $\delta_<(H,n) = \left(1 - \frac{h_r}{h} + o(1)\right) n$.*
- *If $r = 3$ and $h_2 \le h_1 \le h_3$ then $\delta_<(H,n) = \left(1 - \frac{h_1^2}{(2h_1 - h_2)h} + o(1)\right) n$.*
- *If $r = 3$ and $h_2 \le h_3 \le h_1$ then $\delta_<(H,n) = \left(1 - \frac{h_3^2}{(2h_3 - h_2)h} + o(1)\right) n$.*

3.12 The tightness of Theorem 3.1.4

In this section we show that the minimum degree condition in Theorem 3.1.4 is best possible. Given an ordered graph H, let $c_<(H)$ denote the smallest non-negative number which satisfies the following: for every $\eta > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that if G is an ordered graph on $n \ge n_0$ vertices and with minimum degree $\delta(G) \ge c_<(H)n$ then G contains an H-tiling covering all but at most ηn vertices. Observe Theorem 3.1.4

immediately implies that

$$c_<(H) \leq \left(1 - \frac{1}{\chi^*_{cr}(H)}\right). \tag{3.21}$$

In fact, we will show that equality holds.

Theorem 3.12.1. *Let H be an ordered graph on h vertices. Then*

$$c_<(H) = \left(1 - \frac{1}{\chi^*_{cr}(H)}\right).$$

Proof. By (3.21) it remains to prove that $c_<(H) \geq (1-1/\chi^*_{cr}(H))$. Throughout the proof, we let $c := c_<(H)$ and $r := \chi_<(H)$. Suppose for a contradiction that $c < (1-1/\chi^*_{cr}(H))$. We split the proof into three different cases.

Case 1: $\chi^*_{cr}(H) > r$.

Let $\eta > 0$ be sufficiently small and define c' such that

$$c' := \max\left\{\left(1 - \frac{1}{r}\right), c\right\} + \eta < 1 - \frac{1}{\chi^*_{cr}(H)}. \tag{3.22}$$

Let $0 < \nu \ll \eta$ and let $n \in \mathbb{N}$ be sufficiently large and divisible by h.

Let G be any ordered graph on n vertices with minimum degree

$$\delta(G) \geq c'n \overset{(3.22)}{\geq} \left(1 - \frac{1}{r} + \eta\right) n.$$

By Theorem 3.4.1, there exists a set $Abs \subseteq V(G)$ such that $|Abs| \leq \nu n$ and Abs is an H-absorbing set for every $W \subseteq V(G) \setminus Abs$ such that $|W| \in h\mathbb{N}$ and $|W| \leq \nu^3 n$. Let $G' := G \setminus Abs$. Note that

$$\delta(G') \geq (c' - \eta)\, n \overset{(3.22)}{\geq} cn.$$

By the definition of c, and as n is sufficiently large, there exists an H-tiling $\mathcal{M}_1$ covering all but at most $\nu^3 n$ vertices in G'. Let $W \subseteq V(G')$ be the set of vertices not covered by $\mathcal{M}_1$. Since $|W| \leq \nu^3 n$ and h divides $|W|$ (as h divides $n, |Abs|, |V(\mathcal{M}_1)|$), there exists a perfect H-tiling $\mathcal{M}_2$ in $G[Abs \cup W]$. Thus, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect H-tiling in G. Recall that G is an arbitrary ordered graph on n vertices with $\delta(G) \geq c'n$, hence

$$\delta_<(H,n) \leq c'n < \left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n.$$

This contradicts Corollary 3.2.4. Therefore, the initial assumption that $c < (1 - 1/\chi^*_{cr}(H))$ is false and the result follows in this case.

Case 2: $\chi^*_{cr}(H) \leq r$ and H is flexible.

As H is flexible, there exists a complete r-partite ordered graph F with parts $F_1 < \cdots < F_r$ as in Lemma 3.8.5. Pick $\eta > 0$ sufficiently small so that

$$c + \eta + \eta|F| \leq 1 - \frac{1}{\chi^*_{cr}(H)} \tag{3.23}$$

and

$$\frac{r-1}{1-\eta|F|} < \frac{\chi^*_{cr}(H)}{1+\eta\chi^*_{cr}(H)}. \tag{3.24}$$

Let $n \in \mathbb{N}$ be sufficiently large and divisible by h; we may choose η and n so that $\eta n \in \mathbb{N}$.

Recall that by (3.4), $r - 1 < \chi^*_{cr}(H) \leq r$. Thus, it is not difficult to show that there exist a complete r-partite unordered graph B on n vertices with parts $B_1, \dots, B_r$ such that all parts have the same size, except one smaller part, and

$$\frac{\chi^*_{cr}(H)}{1+\eta\chi^*_{cr}(H)} \leq \chi_{cr}(B) < \chi^*_{cr}(H). \tag{3.25}$$

Note that

$$\delta(B) = \left(1 - \frac{1}{\chi_{cr}(B)}\right) n \overset{(3.25)}{\geq} \left(1 - \frac{1}{\chi^*_{cr}(H)} - \eta\right) n.$$

Pick any permutation σ of $[r]$ and any interval labelling ϕ of B with respect to σ. Up to relabelling, we may assume that the complete r-partite ordered graph (B, ϕ) has parts $B_1 < \cdots < B_r$.

Notice that (3.24) and (3.25) imply $|B_k| \geq \eta|F|n$ for all $k \in [r]$. Let $X \subseteq V(B)$ such that $|X \cap B_k| = \eta|F_k|n$ for every $k \in [r]$. Clearly, $(B, \phi)[X]$ is a copy of the ordered blow-up $F(\eta n)$. Let B' be the ordered graph obtained from (B, ϕ) by deleting the vertices in X; so B' is a complete r-partite ordered graph with parts $B'_1 < \cdots < B'_r$ where $B'_k \subseteq B_k$ for every $k \in [r]$. Note that

$$\delta(B') \geq \delta(B) - \eta n|F| \geq \left(1 - \frac{1}{\chi^*_{cr}(H)} - \eta - \eta|F|\right) n \overset{(3.23)}{\geq} cn.$$

By the definition of c, there exists an H-tiling $\mathcal{M}_1$ in B' covering all but at most $(\eta h)n$ vertices. Let $W \subseteq V(B')$ be the set of vertices not covered by $\mathcal{M}_1$. Furthermore, let $s_k := |W \cap B'_k|$ for every $k \in [r]$. Note that $(B, \phi)[X \cup W]$ is a complete r-partite ordered graph with parts $L_1 < \cdots < L_r$ such that $|L_k| = \eta n|F_k| + s_k$. Since $s_1 + \cdots + s_r = |W| \leq (\eta n)h$ and h divides $|W|$, then by Corollary 3.8.6 there exists a perfect H-tiling $\mathcal{M}_2$ in $(B, \phi)[X \cup W]$. Finally, note that $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect H-tiling in (B, ϕ). Since σ, ϕ are arbitrary, this implies that B is a simple bottlegraph of H. This is a contradiction since $\chi_{cr}(B) < \chi^*_{cr}(H)$. Thus, the initial assumption that $c < (1 - 1/\chi^*_{cr}(H))$ is false and the result follows in this case.

Case 3: $\chi^*_{cr}(H) \leq r$ and H is not flexible.

Note that if $\chi^*_{cr}(H) < r$ then H is flexible by Corollary 3.8.4, a contradiction. Thus, $\chi^*_{cr}(H) = r$. In this case we actually produce an explicit extremal example to show that $c \geq 1 - 1/r$.

Since H is not flexible, by Lemma 3.8.2 there exist some $i \in [r-1]$ such that the number of vertices in the first i intervals of any given interval r-colouring of H is fixed. Clearly, this implies that the number of vertices in the last $(r-i)$ intervals of any given interval r-colouring of H is also fixed. Since H has h vertices, the number of vertices in the first i intervals is at least ih/r or the number of vertices in the last $(r-i)$ intervals is at least $(r-i)h/r$. Without loss of generality we may assume the former.

Fix $0 < \eta < 1$ and $n \in \mathbb{N}$ such that n/r and $\eta n/r$ are integers. Let G be the complete

r-partite ordered graph on n vertices with parts $G_1 < \cdots < G_r$ of size

$$|G_j| = \begin{cases} \frac{n}{r}(1-\eta) & \text{if } j < r, \\ \frac{n}{r}\left(1+(r-1)\eta\right) & \text{if } j = r. \end{cases}$$

Observe that

$$\delta(G) = n - \frac{n}{r}\left(1+(r-1)\eta\right) = \left(1 - \frac{1}{r} - \frac{(r-1)\eta}{r}\right) n.$$

By assumption, a copy of H in G has at least ih/r vertices in $G_1 \cup \cdots \cup G_i$. Suppose $\mathcal{M}$ is an H-tiling in G, then the number of copies of H in $\mathcal{M}$ is at most

$$\frac{|G_1 \cup \cdots \cup G_i|}{(ih/r)} = \frac{n(1-\eta)}{h}.$$

In particular, there are at least ηn vertices in G which are not covered by $\mathcal{M}$.

In summary what we have shown is that, given any $c' < 1 - 1/r$, we can choose $\eta > 0$ sufficiently small so that for an infinite number of choices of $n \in \mathbb{N}$, we can produce an n-vertex ordered graph G with $\delta(G) \geq c'n$ and so that G does not contain an H-tiling covering more than $(1-\eta)n$ vertices. So by definition $c \geq 1 - 1/r$, as desired. $\square$

3.13 Proof of Theorem 3.1.14

This section is devoted to the proof of Theorem 3.1.14. We will need the following result from [8, Lemma 6.2] which itself is a special case of a lemma of Bárány and Valtr [11].

Lemma 3.13.1 (Balogh, Li and Treglown [8]). *For $n \geq k \geq 2$, let $A_1, A_2, \ldots, A_k$ be non-empty disjoint subsets of $[n]$. Then there exist sets $S_1, S_2, \ldots, S_k$, where $S_i \subseteq A_i$, and a permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ of the set $[k]$, such that the following conditions hold for all $i, j \in [k]$: we have $|S_i| \geq \lfloor |A_i|/k \rfloor$ and $S_i < S_j$ if $\sigma(i) < \sigma(j)$.*

Proof of Theorem 3.1.14. Let H be an ordered graph on h vertices and let $x \in (0,1)$. Throughout the proof we set $r := \chi_<(H)$. If $r = 1$ the statement of the theorem holds trivially, so we may assume that $r \geq 2$. Observe that it suffices to prove that the theorem holds for any $\eta > 0$ sufficiently small. Fix constants $0 < \eta \ll 1/\chi^*_{cr}(x,H)$ and

$$0 < \varepsilon \ll (1-x)\eta/(xh). \tag{3.26}$$

By definition of $\chi^*_{cr}(x,H)$ there is an x-bottlegraph B of H with

$$\chi^*_{cr}(x,H) \leq \chi_{cr}(B) < \frac{\chi^*_{cr}(x,H)}{1-(\eta/12)\chi^*_{cr}(x,H)}. \tag{3.27}$$

Let $k := \chi(B) \geq r$ and let $B_1, \dots, B_k$ be the parts of B. Fix $t \in \mathbb{N}$ such that $k/t < \varepsilon$.

Let $n \in \mathbb{N}$ be sufficiently large and let G be an ordered graph on n vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*_{cr}(x,H)} + \eta\right) n.$$

Recall by (3.1), $\chi^*_{cr}(x,H) \geq r-1$; so $\delta(G) \geq (1 - 1/(r-1) + \eta)n$. By the *Erdős–Stone–Simonovits theorem for ordered graphs* [107] there exists a copy of H in G. We remove the vertices of H from G and repeat the same process until we obtain a set $W \subseteq V(G)$ such that $G[W]$ contains a perfect H-tiling $\mathcal{W}$ and $|W| = \left\lfloor \frac{\eta n}{2h} \right\rfloor h$. Let $G' := G \setminus W$. Observe that

$$\delta(G') \geq \left(1 - \frac{1}{\chi^*_{cr}(x,H)} + \frac{\eta}{3}\right)|G'| \overset{(3.27)}{\geq} \left(1 - \frac{1}{\chi_{cr}(B)} + \frac{\eta}{4}\right)|G'|.$$

Since $\chi_{cr}(B(t)) = \chi_{cr}(B)$ (and ignoring the ordering of $V(G')$), Theorem 3.1.11 implies there exists a $B(t)$-tiling $\mathcal{B}$ in G' which covers all but at most $\varepsilon|G'|$ vertices.

Consider a fixed copy of $B(t) \in \mathcal{B}$ whose parts are $A_1, \dots, A_k$ with $|A_i| = t|B_i|$ for every $i \in [k]$. By Lemma 3.13.1, there exist sets $S_i \subseteq A_i$ for every $i \in [k]$ such that $|S_i| \geq \lfloor |A_i|/k \rfloor \geq |B_i|$ and a permutation σ of $[k]$ such that $S_i < S_j$ if $\sigma(i) < \sigma(j)$. By discarding some vertices if necessary, we may assume that $|S_i| = |B_i|$ for every $i \in [k]$. Note that the sets $S_1, \dots, S_k$ span a copy of B and the ordering of $V(G')$ induces an interval labelling of B with respect to σ.

Crucially, $|A_i \setminus S_i| = (t-1)|B_i|$, so we can repeatedly apply Lemma 3.13.1 to find a B-tiling in G' covering all but at most

$$k|B| = (k/t)|B(t)|$$

vertices in our fixed $B(t) \in \mathcal{B}$. Furthermore, by Lemma 3.13.1, the ordering of $V(G')$ induces an interval labelling on each of these copies of B. Since B is an x-bottlegraph, each of these copies contains an (x, H)-tiling. Repeat this process for all $B(t) \in \mathcal{B}$ and denote the union of all these (x, H)-tilings as $\mathcal{M}$. The number of vertices in G covered by $\mathcal{M} \cup \mathcal{W}$ is at least

$$\begin{aligned} x(1-k/t)(1-\varepsilon)|G'| + |W| &\geq x(1-\varepsilon)^2(n - |W|) + |W| \\ &\geq xn - 2x\varepsilon n - x|W| + |W| \overset{(3.26)}{\geq} xn. \end{aligned}$$

Hence $\mathcal{M} \cup \mathcal{W}$ is an (x, H)-tiling in G, as desired. □

3.14 Computing the threshold in Theorem 3.1.14 for some choices of H

In this section we investigate the behaviour of the function $f(x, H)$ as defined in Theorem 3.1.14. Recall that for an unordered graph H, the analogous function $g(x, H)$ (defined in Theorem 3.1.11) is linear in x. On the other hand, for a vertex-ordered graph H, the function $f(x, H)$ can behave rather differently. However, we first give an instance where $f(x, H)$ does grow linearly in x.

Proposition 3.14.1. *Let H be an ordered graph with vertex set $[h]$ such that $\chi_<(H) = 2$. Define $\alpha^+(H)$ to be the largest integer $t \in [h]$ such that $[1, t]$ is an independent set in H and similarly $\alpha^-(H)$ to be the largest integer $t \in [h]$ such that $[h-t+1, h]$ is an* *independent set in H. Let $\alpha(H) := \min\{\alpha^+(H), \alpha^-(H)\}$. For any $x \in (0,1)$, we have*

$$f(x,H) = \frac{x(h-\alpha(H))}{h}.$$

Proof. Let H be an ordered graph with vertex set $[h]$ so that $\chi_<(H) = 2$, and let $x \in (0,1)$. Additionally, fix constants $0 < 1/N \ll \eta < 1$ where $N \in \mathbb{N}$. We first prove that $f(x,H) \leq x(h-\alpha(H))/h$.

By the definition of $\alpha^+(H)$ and $\alpha^-(H)$ and the fact that $\chi_<(H) = 2$, we have that $[1, \alpha^+(H)] < [\alpha^+(H)+1, h]$ and $[1, h-\alpha^-(H)] < [h-\alpha^-(H)+1, h]$ are interval 2-colourings of H. The maximality of $\alpha^+(H)$ implies $\alpha^+(H) \geq h-\alpha^-(H)$, i.e., $\alpha^+(H)+\alpha^-(H)-h \geq 0$. Furthermore, the previous observations imply that the vertices in $[h-\alpha^-(H)+1, \alpha^+(H)]$ are isolated in H and in particular $[1, h-\alpha^-(H)] < [h-\alpha^-(H)+1, \alpha^+(H)] < [\alpha^+(H)+1, h]$ is an interval 3-colouring of H. (Note that $[h-\alpha^-(H)+1, \alpha^+(H)]$ may be empty.)

Let H' be an ordered graph such that:

- $V(H') = [h']$ with $h' := Nh + \lfloor (1-x)Nh/x \rfloor$;
- $V(H')$ can be partitioned into four intervals $D_1 < D_2 < D_3 < D_4$ where

$$|D_1| = (h-\alpha^-(H))N, \quad |D_2| = (\alpha^+(H)+\alpha^-(H)-h)N,$$

$$|D_3| = \lfloor (1-x)Nh/x \rfloor, \quad |D_4| = (h-\alpha^+(H))N,$$

  such that $H'[D_1, D_4]$ is a complete 2-partite ordered graph and all vertices in $D_2 \cup D_3$ are isolated in H'.

As $[1, h-\alpha^-(H)] < [h-\alpha^-(H)+1, \alpha^+(H)] < [\alpha^+(H)+1, h]$ is an interval 3-colouring of H and $[h-\alpha^-(H)+1, \alpha^+(H)]$ is a set of isolated vertices, we have that $H'[D_1 \cup D_2 \cup D_4]$ contains N disjoint copies of H. These copies form an (x,H)-tiling in H' since $Nh \geq xh'$.

Notice $\alpha^+(H') = |D_1| + |D_2| + |D_3|$ and $\alpha^-(H') = |D_2| + |D_3| + |D_4|$, thus

$$\alpha(H') = h' - \max\{|D_1|, |D_4|\} = h' - N(h-\alpha(H)). \tag{3.28}$$

As $\chi_<(H') = 2$, Proposition 3.11.4 implies that

$$\chi^*_{cr}(H') = \frac{h'}{\alpha(H')}. \tag{3.29}$$

Since H' contains an (x, H)-tiling, it follows that every 1-bottlegraph of H' is an x-bottlegraph of H and so $\chi^*_{cr}(x, H) \leq \chi^*_{cr}(1, H')$. Proposition 3.5.2 implies that $\chi^*_{cr}(1, H') = \chi^*_{cr}(H')$, so

$$\begin{aligned} f(x,H) &= \left(1 - \frac{1}{\chi^*_{cr}(x,H)}\right) \leq \left(1 - \frac{1}{\chi^*_{cr}(H')}\right) \\ &\overset{(3.29)}{=} \left(1 - \frac{\alpha(H')}{h'}\right) \overset{(3.28)}{=} \left(1 - \frac{h' - N(h - \alpha(H))}{h'}\right) = \frac{x(h - \alpha(H))}{xh'/N}. \end{aligned}$$

Since N is arbitrarily large and $xh'/N \to h$ as $N \to \infty$, the above implies $f(x, H) \leq x(h - \alpha(H))/h$.

Next we prove that $f(x, H) \geq x(h - \alpha(H))/h$. Without loss of generality, we may assume that $\alpha(H) = \alpha^+(H)$. Let G be the ordered graph obtained by taking the complete 2-partite ordered graph with classes $U < V$ where

$$|U| = N\alpha^+(H) + \left\lceil \frac{1-x}{x} Nh \right\rceil + 1 \quad \text{and} \quad |V| = N(h - \alpha^+(H)) - 1,$$

and adding all possible edges to V. Suppose G contains an (x, H)-tiling $\mathcal{H}$. Since $|G| = Nh + \lceil \frac{1-x}{x} Nh \rceil$ then $\mathcal{H}$ covers at least $xNh + (1-x)Nh = Nh$ vertices of G. Let $\mathcal{H}' \subseteq \mathcal{H}$ where $\mathcal{H}'$ is an H-tiling consisting of exactly N copies of H in G. Every copy of H has at most $\alpha^+(H)$ vertices in U. Furthermore, there are precisely $\lceil \frac{1-x}{x} Nh \rceil$ vertices in G which are not covered by $\mathcal{H}'$, thus

$$|U| \leq N\alpha^+(H) + \left\lceil \frac{1-x}{x} Nh \right\rceil < |U|,$$

a contradiction. Therefore, the assumption that G contains an (x, H)-tiling is false. It follows from Theorem 3.1.14 that

$$\delta(G) < (f(x, H) + \eta)|G|.$$

Finally, we have

$$f(x, H) + \eta > \frac{\delta(G)}{|G|} = \frac{N(h - \alpha^+(H)) - 1}{Nh + \lceil \frac{1-x}{x} Nh \rceil} \geq \frac{N(h - \alpha^+(H)) - 1}{Nh + \frac{1-x}{x} Nh + 1} = \frac{x(h - \alpha^+(H)) - x/N}{h + x/N}.$$

As N can be chosen arbitrarily large and η arbitrarily small, it follows that $f(x, H) \geq x(h - \alpha(H))/h$. □

In general, for any ordered graph H, if x is not too big then the function $f(x, H)$ is linear in x.

Proposition 3.14.2. *Let H be an ordered graph on h vertices and $r := \chi_<(H)$. Let $\mathcal{C}$ be the set of interval r-colourings of H. Additionally, set*

$$T := \max_{1 \leq i \leq r} \left\{ \min_{(H_1 < \cdots < H_r) \in \mathcal{C}} |H_i| \right\}, \quad J := \max_{1 \leq i \leq r} \left\{ \max_{(H_1 < \cdots < H_r) \in \mathcal{C}} |H_i| \right\}$$

and $x_0 := \frac{h}{(r-1)J+T}$. Then for any $x \in (0, x_0]$ where $x < 1$,

$$f(x, H) = 1 - \frac{h - xT}{h(r-1)}.$$

Proof. Let H be an ordered graph on h vertices with $r := \chi_<(H)$ and $x \in (0, x_0]$ where $x < 1$. Fix constants $0 < 1/N \ll \eta < 1$ where $N \in \mathbb{N}$. We first show that $f(x, H) \leq 1 - (h - xT)/(h(r-1))$.

Let B be the complete r-partite unordered graph with parts $B_1, \dots, B_r$ where

$$|B_1| = NT \quad \text{and} \quad |B_i| = \left\lfloor \frac{N}{r-1} \left(\frac{h}{x} - T \right) \right\rfloor$$

for every $i \neq 1$. We now prove that B is an x-bottlegraph of H. Note that for every $i \neq 1$,

$$|B_i| \geq \left\lfloor \frac{N}{r-1}\left(\frac{h}{x_0} - T\right)\right\rfloor = NJ \geq NT = |B_1|. \tag{3.30}$$

Let σ be a permutation of $[r]$ and ϕ be an interval labelling of B with respect to σ. Recall that the ordered graph (B, ϕ) has parts $B_{\sigma^{-1}(1)} < \cdots < B_{\sigma^{-1}(r)}$. Let $H_1 < \cdots < H_r$ be an interval r-colouring of H which minimises $|H_{\sigma(1)}|$. By the definition of T we have that

$$|B_1| = NT \geq N|H_{\sigma(1)}|.$$

Furthermore, by the definition of J we have that for every $i \neq 1$,

$$|B_i| \overset{(3.30)}{\geq} NJ \geq N|H_{\sigma(i)}|.$$

Hence, the ordered graph (B, ϕ) contains an H-tiling $\mathcal{H}$ consisting of N disjoint copies of H. These copies cover exactly Nh vertices of (B, ϕ). In particular

$$x|B| \leq xNT + xN\left(\frac{h}{x} - T\right) = Nh,$$

therefore, $\mathcal{H}$ is an (x, H)-tiling in (B, ϕ). Since σ, ϕ are arbitrary, B is indeed an x-bottlegraph of H and thus $\chi^*_{cr}(x, H) \leq \chi_{cr}(B)$. Note that

$$\chi_{cr}(B) = \frac{|B|}{|B_2|} = \frac{NT + (r-1)\left\lfloor \frac{N}{r-1}\left(\frac{h}{x} - T\right)\right\rfloor}{\left\lfloor \frac{N}{r-1}\left(\frac{h}{x} - T\right)\right\rfloor} < \frac{NT + N\left(\frac{h}{x} - T\right)}{\frac{N}{r-1}\left(\frac{h}{x} - T\right) - 1} = \frac{h(r-1)}{(h - xT) - \frac{x(r-1)}{N}}$$

and so

$$f(x, H) = 1 - \frac{1}{\chi^*_{cr}(x, H)} \leq 1 - \frac{1}{\chi_{cr}(B)} < 1 - \frac{(h - xT) - \frac{x(r-1)}{N}}{h(r-1)}.$$

As N is arbitrarily large, the above implies $f(x, H) \leq 1 - (h - xT)/(h(r-1))$.

Next we show that $f(x, H) \geq 1-(h-xT)/(h(r-1))$. Suppose the value of T is achieved for some interval r-colouring $H^*_1 < \cdots < H^*_r$ of H and $i^* \in [r]$; that is, $T = |H^*_{i^*}|$. Let G

be the complete r-partite ordered graph with parts $G_1 < \cdots < G_r$ of where

$$|G_{i^*}| = NT \quad \text{and} \quad |G_i| = \left\lceil \frac{N}{r-1}\left(\frac{h}{x} - T\right)\right\rceil + 1$$

for every $i \neq i^*$. Notice that $|G_i| \geq |G_{i^*}|$ for every $i \in [r]$ since $x \leq x_0$. Suppose there exists an (x, H)-tiling $\mathcal{H}$ in G. By definition of T, every copy of $H \in \mathcal{H}$ has at least T vertices in G_{i^*}, thus

$$|\mathcal{H}| \leq \frac{|G_{i^*}|}{T} = N.$$

Hence, $\mathcal{H}$ covers at most Nh vertices of G. However,

$$|G| > NT + N\left(\frac{h}{x} - T\right) = \frac{Nh}{x} \implies x|G| > Nh,$$

contradicting the assumption that $\mathcal{H}$ is an (x, H)-tiling. In particular, G does not contain an (x, H)-tiling and so Theorem 3.1.14 implies

$$\delta(G) < (f(x, H) + \eta)|G|.$$

Let $i \neq i^*$. We have that

$$f(x,H) + \eta > \frac{\delta(G)}{|G|} = \frac{|G| - |G_i|}{|G|} = \left(1 - \frac{\left\lceil \frac{N}{r-1}\left(\frac{h}{x} - T\right)\right\rceil + 1}{NT + (r-1)\left\lceil \frac{N}{r-1}\left(\frac{h}{x} - T\right)\right\rceil + r - 1}\right)$$

$$\geq \left(1 - \frac{\frac{N}{r-1}\left(\frac{h}{x} - T\right) + 2}{NT + N\left(\frac{h}{x} - T\right)}\right) = 1 - \frac{(h - xT) + \frac{2x(r-1)}{N}}{h(r-1)}.$$

As N can be chosen arbitrarily large and η arbitrarily small, it follows that $f(x, H) \geq 1 - (h - xT)/(h(r-1))$. □

The following result states that $f(x, H)$ is piecewise linear for certain types of H. These different behaviours already suggest that computing $f(x, H)$ is likely to be a difficult task in general.

Proposition 3.14.3. *Let $\ell_1 \leq \cdots \leq \ell_r$ be positive integers and $x \in (0,1]$. Let H be a complete r-partite ordered graph on h vertices with parts $H_1 < \cdots < H_r$ where $|H_i| = \ell_i$ for every $i \in [r]$.*

- *If $t\ell_t - \sum_{i=1}^{t} \ell_i \leq \frac{1-x}{x}h < (t+1)\ell_{t+1} - \sum_{i=1}^{t+1} \ell_i$ for some $t \in [1, r-1]$ then*

$$f(x,H) = 1 - \frac{1}{ht}\left((1-x)h + x\sum_{i=1}^{t} \ell_i\right).$$

- *If $\frac{(1-x)h}{x} \geq r\ell_r - \sum_{i=1}^{r} \ell_i$ then*

$$f(x,H) = 1 - \frac{h - x\ell_r}{h(r-1)}.^{8}$$

In particular, for H fixed, $f(x,H)$ is continuous and piecewise linear in x with r' pieces where r' is the number of strict inequalities in $\ell_1 \leq \cdots \leq \ell_r$.

Proof. If $x = 1$, then the result follows from Propositions 3.5.2 and 3.11.3; so assume that $x < 1$. Let T, J, x_0 be as in Proposition 3.14.2. For our choice of H, we have $T = J = \ell_r$ and $x_0 = h/(r\ell_r)$. Note that if

$$\frac{(1-x)h}{x} \geq r\ell_r - \sum_{i=1}^{r} \ell_i,$$

then $x \leq x_0$. It follows from Proposition 3.14.2 that

$$f(x,H) = 1 - \frac{h - xT}{h(r-1)} = 1 - \frac{h - x\ell_r}{h(r-1)}.$$

Throughout the rest of the proof, we assume that

$$t\ell_t - \sum_{i=1}^{t} \ell_i \leq \frac{1-x}{x}h < (t+1)\ell_{t+1} - \sum_{i=1}^{t+1} \ell_i \tag{3.31}$$

[8] Observe that this value of $f(x,H)$ coincides with the value of $f(x,H)$ in the previous case when $t = r-1$.

for some $t \in [1, r-1]$. Fix constants $0 < 1/N \ll \eta \ll 1$ where $N \in \mathbb{N}$. We first prove that $f(x,H)$ is at most the claimed value. Let H' be the complete r-partite ordered graph with parts $H'_1 < \dots < H'_r$ such that the following holds:

- for every $i \le t$,
$$|H'_i| = \left\lfloor \frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right)\right\rfloor$$
and so $N\ell_t \le |H'_i| < N\ell_{t+1}$ by (3.31);

- for every $i \ge t+1$,
$$|H'_i| = N\ell_i.$$

As $N\ell_i \le |H'_i|$ for every $i \in [r]$, there exists an H-tiling $\mathcal{H}$ in H' consisting of exactly N copies of H. Note that $\mathcal{H}$ covers exactly Nh vertices in H' and

$$x|H'| \le xN\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right) + xN\sum_{i=t+1}^{r} \ell_i = xN\left(\frac{1-x}{x}h + h\right) = Nh.$$

Thus, $\mathcal{H}$ is an (x,H)-tiling in H'. Since H' contains an (x,H)-tiling, every 1-bottlegraph of H' is an x-bottlegraph of H, therefore $\chi^*_{cr}(x,H) \le \chi^*_{cr}(H')$.

Since $|H'_1| \le \dots \le |H'_r|$, Proposition 3.11.3 implies $\chi^*_{cr}(H') = |H'|/|H'_1|$. Therefore,

$$f(x,H) = \left(1 - \frac{1}{\chi^*_{cr}(x,H)}\right) \le \left(1 - \frac{1}{\chi^*_{cr}(H')}\right) = 1 - \frac{|H'_1|}{|H'|}$$

$$= 1 - \frac{\left\lfloor \frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right)\right\rfloor}{t\left\lfloor \frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right)\right\rfloor + N\sum_{i=t+1}^{r} \ell_i}$$

$$< 1 - \frac{\frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right) - 1}{N\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right) + N\sum_{i=t+1}^{r} \ell_i} = 1 - \frac{1}{ht}\left((1-x)h + x\sum_{i=1}^{t} \ell_i\right) + \frac{x}{Nh}.$$

Since N can be chosen arbitrarily large, the above implies

$$f(x,H) \leq 1 - \frac{1}{ht}\left((1-x)h + x\sum_{i=1}^{t} \ell_i\right).$$

Next we prove that $f(x,H)$ is at least the claimed value. Let G be the ordered graph obtained by taking the complete $(t+1)$-partite ordered graph with parts $G_1 < \cdots < G_{t+1}$ where

$$|G_{t+1}| = N\left(h - \sum_{i=1}^{t} \ell_i\right) - t \quad \text{and} \quad |G_i| = \left\lceil \frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right)\right\rceil + 1$$

for every $i \leq t$, and adding all missing edges to G_{t+1}. Suppose G contains an (x,H)-tiling $\mathcal{H}$. Observe that

$$|G| \geq N\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right) + t + N\left(h - \sum_{i=1}^{t} \ell_i\right) - t = \frac{Nh}{x} \implies x|G| \geq Nh.$$

Hence, $\mathcal{H}$ covers at least Nh vertices in G. Let $\mathcal{H}' \subseteq \mathcal{H}$ be an H-tiling consisting of exactly N copies of H. Each copy of H has at most $\sum_{i=1}^{t} \ell_i$ vertices in $G_1 \cup \cdots \cup G_t$. Also,

$$|G| < N\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right) + 2t + N\left(h - \sum_{i=1}^{t} \ell_i\right) - t = \frac{Nh}{x} + t,$$

and so there are fewer than $\frac{1-x}{x}Nh + t$ vertices in G which are not covered by $\mathcal{H}'$. This implies

$$|G_1 \cup \cdots \cup G_t| < N\sum_{i=1}^{t} \ell_i + \frac{(1-x)Nh}{x} + t = N\left(\frac{1-x}{x}h + \sum_{i=1}^{t} \ell_i\right) + t \leq |G_1 \cup \cdots \cup G_t|,$$

which is a contradiction; hence G does not contain an (x,H)-tiling. Therefore, Theorem 3.1.14 implies that $\delta(G) < (f(x,H)+\eta)|G|$ and so

$$f(x,H)+\eta > \frac{\delta(G)}{|G|} = \frac{|G|-|G_1|}{|G|} = 1 - \frac{\left\lceil \frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t}\ell_i\right)\right\rceil + 1}{t\left\lceil \frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t}\ell_i\right)\right\rceil + t + N\left(h - \sum_{i=1}^{t}\ell_i\right) - t}$$

$$\geq 1 - \frac{\frac{N}{t}\left(\frac{1-x}{x}h + \sum_{i=1}^{t}\ell_i\right) + 2}{N\left(\frac{1-x}{x}h + \sum_{i=1}^{t}\ell_i\right) + N\left(h - \sum_{i=1}^{t}\ell_i\right)} = 1 - \frac{1}{ht}\left((1-x)h + x\sum_{i=1}^{t}\ell_i\right) + \frac{2x}{Nh}.$$

Since η can be chosen arbitrarily small and N arbitrarily large, the above implies that

$$f(x,H) \geq 1 - \frac{1}{ht}\left((1-x)h + x\sum_{i=1}^{t}\ell_i\right).$$

□

We conclude this section with the following result.

Proposition 3.14.4. *Let H be an ordered graph, then*

$$\lim_{x\to 1}\chi^*_{cr}(x,H) = \chi^*_{cr}(H) \quad \textit{and} \quad \lim_{x\to 0}\chi^*_{cr}(x,H) = \chi_<(H) - 1.$$

Proof. Let T, J, x_0 be as in Proposition 3.14.2. If $x \in (0, x_0)$ then Proposition 3.14.2 implies

$$f(x,H) = 1 - \frac{h - xT}{h(\chi_<(H)-1)} \implies \lim_{x\to 0} f(x,H) = 1 - \frac{1}{\chi_<(H)-1}.$$

Since $f(x,H) = 1 - 1/\chi^*_{cr}(x,H)$, it follows from the above that

$$\lim_{x\to 0}\chi^*_{cr}(x,H) = \chi_<(H) - 1.$$

Let $y, z \in (0,1]$ with $y < z$. Let B be a z-bottlegraph of H. Clearly every z-bottlegraph of H is a y-bottlegraph of H, so $\chi^*_{cr}(y,H) \leq \chi^*_{cr}(z,H)$. It follows that, for $x \in (0,1]$, the function $\chi^*_{cr}(x, H)$ is non-decreasing. In particular,

$$\chi^*_{cr}(x, H) \leq \chi^*_{cr}(1, H) = \chi^*_{cr}(H)$$

for every $x \in (0, 1]$. The monotone convergence theorem implies that the limit $\ell := \lim_{x\to 1} \chi^*_{cr}(x, H)$ exists and $\ell \leq \chi^*_{cr}(H)$. Fix arbitrary constants $0 < \eta, \varepsilon < 1$. Pick $n \in \mathbb{N}$ sufficiently large and let G be an ordered graph on n vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{\ell} + \varepsilon\right) n \geq \left(1 - \frac{1}{\chi^*_{cr}(1-\eta, H)} + \varepsilon\right) n.$$

By Theorem 3.1.14, G contains a $(1 - \eta, H)$-tiling, i.e., an H-tiling covering all but at most ηn vertices. Recall the definition of $c_<(H)$ given at the beginning of Section 3.12: $c_<(H)$ denotes the smallest non-negative number such that for every $\eta > 0$, there exists an integer $n_0 \in \mathbb{N}$ so that if G is an ordered graph on $n \geq n_0$ vertices and with minimum degree $\delta(G) \geq c_<(H)n$ then G contains an H-tiling covering all but at most ηn vertices.

It follows that $c_<(H) \leq 1 - 1/\ell + \varepsilon$, and so $c_<(H) \leq 1 - 1/\ell$ since $\varepsilon > 0$ is arbitrary. By Theorem 3.12.1, $c_<(H) = 1 - 1/\chi^*_{cr}(H)$ and thus $\chi^*_{cr}(H) \leq \ell$. In particular, we have $\chi^*_{cr}(H) = \ell$. □

3.15 Further remarks and open problems

Theorem 3.1.6 together with [8, Theorem 1.9] asymptotically determine the minimum degree threshold for forcing a perfect H-tiling in an ordered graph, for any fixed ordered graph H. Depending on the structure of H, this threshold depends on one of three factors: $(C1)$ the existence of an almost perfect H-tiling; $(C2)$ the avoidance of divisibility barriers; $(C3)$ the existence of an H-cover. Analogous factors govern the threshold for other perfect H-tiling problems in a range of settings too.

Therefore, it would be extremely interesting to find a natural 'local' density condition (e.g., minimum degree, Ore-type, degree sequence) for an (ordered) graph, directed graph or hypergraph for which the corresponding perfect H-tiling threshold depends on another factor. We suspect no such problem exists. An alternative way to think about this question is as follows: are there barriers, other than local and divisibility barriers, that prevent absorbing for a perfect H-tiling problem?

Other than this general 'meta problem', it would be interesting to establish the Ore-type degree threshold that forces a perfect H-tiling in an ordered graph G (and compare this threshold to the corresponding Ore-type degree threshold for unordered graphs [95]).

In light of Theorem 3.1.4 it is natural to raise the following ordered graph analogue of the theorem of Shokoufandeh and Zhao [125].

Conjecture 3.15.1. *Let H be an ordered graph. Then there is a constant $C = C(H) \in \mathbb{N}$ so that the following holds. If G is an n-vertex ordered graph with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*_{cr}(H)}\right) n$$

then G contains an H-tiling covering all but at most C vertices.

Finally, whilst we have obtained some understanding of the function $f(x, H)$, it would be interesting to obtain a more complete understanding of how this function can behave in general. In particular, is it true that for any fixed ordered graph H, the function $f(x, H)$ is piecewise linear?

Chapter 4

Deficiency problems for graphs

4.1 Introduction

Recall that the *join* $G * K_t$ of G and K_t is the graph obtained from G by adding t new vertices and adding all edges incident to at least one of the new vertices. Nenadov, Sudakov and Wagner [106] raised the question of determining, for every $n, t, r \in \mathbb{Z}$, the maximum number of edges in an graph G on n vertices provided that the join $G * K_t$ does not contain a K_r-factor. Furthermore, they resolved the case $r = 3$ when t is not too large compared to n (see Theorem 1.3.3).

In this chapter we present the proof of Theorem 1.3.4, which completely resolves the above question. We state Theorem 1.3.4 once again.

Theorem 1.3.4. *Let $n, t, r \in \mathbb{Z}$ with $n \geq 2$, $t \geq 0$ and $r \geq 3$ such that $t < (r-1)n$ and $r|(n+t)$. Further, let $k := \lceil \frac{t+1}{r-1} \rceil$ and q be the integer remainder when t is divided by $r-1$. Let G be a graph on n vertices such that $G * K_t$ does not contain a K_r-factor. Then*

$$e(G) \leq \max\left\{ \binom{n}{2} - \binom{\frac{n+t}{r} + 1}{2}, \binom{n}{2} - \binom{k}{2} - k(n - k - (r - 2 - q)) \right\}.$$

When $(r-1)|(t+1)$, the first term is at most the second term precisely when $t \leq \frac{(r-1)n - r^2}{2r^2 - 2r + 1}$. Note that Theorem 1.3.4 considers all interesting values of n and t. Indeed, if $t \geq$

$(r-1)n$ and $r|(n+t)$, then $G*K_t$ trivially contains a K_r-factor, even if $e(G)=0$. Further, in Section 4.2 we provide extremal examples that demonstrate that the edge condition in Theorem 1.3.4 cannot be lowered. Perhaps surprisingly, the proof of Theorem 1.3.4 is short, making use of a couple of vertex-modification tricks (see the proofs of Lemmas 4.3.1 and 4.3.2) and Theorem 1.3.2.

Note that the $t=0$ case of Theorem 1.3.4 determines the edge density threshold for forcing a K_r-factor in a graph. In fact, this is an old result due to Akiyama and Frankl [2], so our result can be viewed as a deficiency generalisation of their theorem.

The chapter is organised as follows. In Section 4.2 we provide extremal examples for Theorem 1.3.4. In Section 4.3 we prove Theorem 1.3.4. Some concluding remarks are given in Section 4.4.

4.2 The extremal constructions for Theorem 1.3.4

In this section we will give the extremal constructions that match the upper bounds in Theorem 1.3.4.

Definition 4.2.1. Let $n, r, t \in \mathbb{Z}$ with $r \geq 3$, $t \geq 0$ and $n \geq 2$ such that $t < (r-1)n$ and $r|(n+t)$. Further, let $k := \lceil \frac{t+1}{r-1} \rceil$ and q be the integer remainder when t is divided by $r-1$. We define graphs $EX_1(n,t,r)$ and $EX_2(n,t,r)$ as follows:

- Let $K := K_n$ and $A \subseteq K$ such that $A = K_{\frac{n+t}{r}+1}$. Define $EX_1(n,t,r)$ to be the graph obtained by removing $E(A)$ from K.
- Consider a set of isolated vertices B where $|B| = k$ and K_{n-k}, and let $C \subseteq V(K_{n-k})$ where $|C| = r-2-q$. Define $EX_2(n,t,r)$ to be the graph obtained by taking the disjoint union of B and K_{n-k} and adding every edge incident to a vertex in C.

See Figure 4.1 for a picture of $EX_1(n,t,r)$ and $EX_2(n,t,r)$.

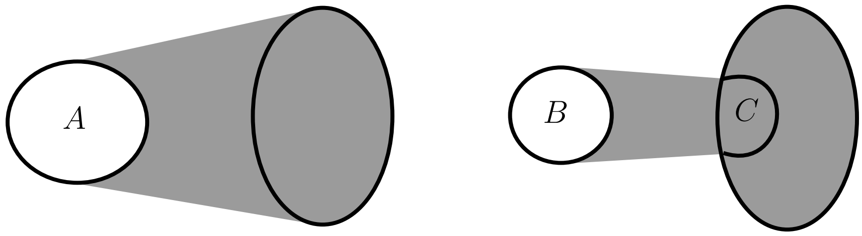

Figure 4.1: The graphs $EX_1(n,t,r)$ (left) and $EX_2(n,t,r)$ (right). The grey areas indicates complete sections of the graphs.

Observe that

$$e(EX_1(n,t,r)) = \binom{n}{2} - \binom{\frac{n+t}{r}+1}{2}$$

and

$$e(EX_2(n,t,r)) = \binom{n}{2} - \binom{k}{2} - k(n-k-(r-2-q)).$$

Hence

$$\max\left\{\binom{n}{2} - \binom{\frac{n+t}{r}+1}{2}, \binom{n}{2} - \binom{k}{2} - k(n-k-(r-2-q))\right\}$$

and

$$\max\{e(EX_1(n,t,r)), e(EX_2(n,t,r))\}$$

are equal. Next we show that $EX_1(n,t,r) * K_t$ and $EX_2(n,t,r) * K_t$ do not contain K_r-factors, that is, they are extremal graphs for Theorem 1.3.4.

Proposition 4.2.2. *$EX_1(n,t,r) * K_t$ and $EX_2(n,t,r) * K_t$ do not contain K_r-factors.*

Proof. Firstly, let us consider $EX_1(n,t,r) * K_t$ and, for a contradiction, assume that it contains a K_r-factor $\mathcal{T}$. Then each vertex in $V(A)$ belongs to a different copy of K_r in $\mathcal{T}$. This implies $\frac{n+t}{r} = |\mathcal{T}| \geq |V(A)| = \frac{n+t}{r} + 1$, a contradiction. Hence $EX_1(n,t,r) * K_t$ does not contain a K_r-factor.

Now let us consider $EX_2(n,t,r) * K_t$. For a contradiction, assume that $EX_2(n,t,r) * K_t$ contains a K_r-factor $\mathcal{T}$. Since B is an independent set, exactly $|B|$ copies of K_r in $\mathcal{T}$

intersect B. These copies cover $|B|(r-1)$ vertices in $V(K_t) \cup C$. However,

$$|B|(r-1) = \left\lceil \frac{t+1}{r-1} \right\rceil \cdot (r-1) = \left(\frac{t-q}{r-1} + 1 \right) \cdot (r-1) = t - q + r - 1 > t + |C|$$

a contradiction. Hence $EX_2(n,t,r) * K_t$ does not contain a K_r-factor. □

4.3 Proof of Theorem 1.3.4

The proof of Theorem 1.3.4 follows an inductive argument on the number of vertices n of G. Given a graph G such that $G * K_t$ does not contain a K_r-factor, we apply an appropriate vertex-modification procedure which, roughly speaking, allows us to assume G is locally isomorphic to one of the two extremal examples. This allows us to remove such local structure from G and apply induction.

The vertex-modification procedures are described by the following two structural lemmas regarding graphs G with the property that $G * K_t$ does not contain a K_r-factor. Lemma 4.3.1 allows us to assume that the degree of a vertex is either $n-1$ (which is the degree of all vertices in $V(EX_1(n,t,r)) \setminus A$ and $C \subseteq V(EX_2(n,t,r))$) or at most $n-1-\lceil \frac{t+1}{r-1} \rceil$ (which is the degree of all vertices in $V(EX_2(n,t,r)) \setminus (B \cup C)$). Lemma 4.3.2 allows us to assume that either each edge of G belongs to some r-clique or there is a vertex with degree $n-1$.

Lemma 4.3.1. *Let $t \geq 0$, $r \geq 3$ and G be a graph on n vertices such that $e(G)$ is maximal with respect to the property that $G * K_t$ does not contain a K_r-factor. Then for every vertex $v \in V(G)$ either $d_G(v) = n-1$ or $d_G(v) \leq n-1-\lceil \frac{t+1}{r-1} \rceil$.*

Proof. Suppose there exists a vertex $v \in V(G)$ such that

$$n - 1 - \left\lceil \frac{t+1}{r-1} \right\rceil < d_G(v) < n-1.$$

Let G' be the graph obtained from G by adding every possible edge incident to v, that is, $d_{G'}(v) = n-1$. Since $e(G)$ is maximal with respect to $G * K_t$ not containing a K_r-factor, we must have that $G' * K_t$ contains a K_r-factor $\mathcal{T}'$. Using $\mathcal{T}'$, we will now construct a K_r-factor $\mathcal{T}$ in $G * K_t$, giving us a contradiction. Let K^v be the copy of K_r in $\mathcal{T}'$ covering v.

Case 1: Suppose there exists a copy K' of K_r in $\mathcal{T}'$ that lies entirely in $\{v\} \cup V(K_t)$. If $K' = K^v$ then we can take $\mathcal{T} := \mathcal{T}'$, since $K' \subseteq G * K_t$. Hence suppose that $K' \neq K^v$, and so K' and K^v are vertex-disjoint. In particular, $v \notin V(K')$. Pick any $u \in V(K')$. As $K' \subseteq K_t$, the bipartite graph $(G*K_t)[V(K'), V(K^v)]$ is complete and so it is easy to see that both $A := (G*K_t)[\{u\} \cup (V(K^v) \setminus \{v\})]$ and $B := (G*K_t)[\{v\} \cup (V(K') \setminus \{u\})]$ are copies of K_r in $G * K_t$, see Figure 4.2. Thus, we can take $\mathcal{T} := (\mathcal{T}' \setminus \{K^v, K'\}) \cup \{A, B\}$.

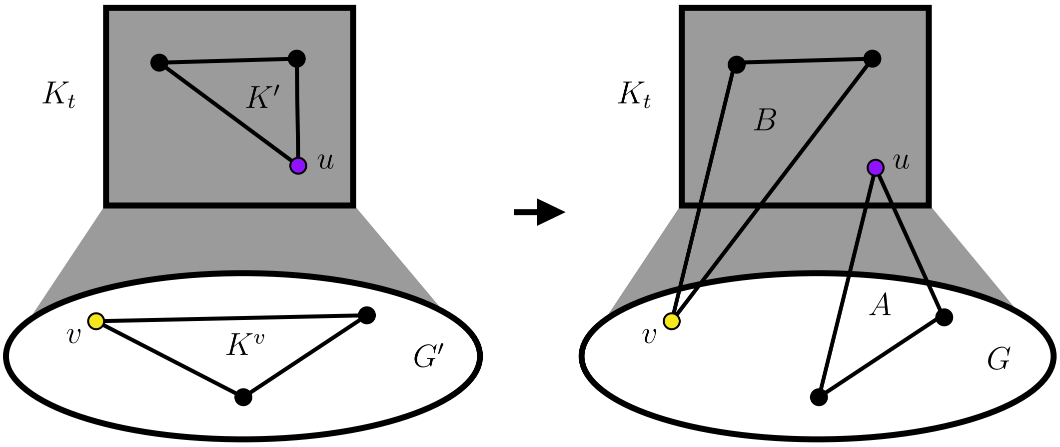

Figure 4.2: Case 1 when $r = 3$. The cliques K^v and K' in $G' * K_t$ (left) are replaced with the cliques A and B in $G * K_t$ (right). The grey areas indicate complete sections of the graphs.

Case 2: Suppose that no copy of K_r in $\mathcal{T}'$ lies entirely in $\{v\} \cup V(K_t)$. In particular, each copy of K_r in $\mathcal{T}'$ has at most $r-1$ vertices in $\{v\} \cup V(K_t)$. It follows that the number of copies of K_r in $\mathcal{T}'$ which cover some vertex of $\{v\} \cup V(K_t)$ in $G' * K_t$ is at least $\lceil \frac{t+1}{r-1} \rceil$. But $d_G(v) > n - 1 - \lceil \frac{t+1}{r-1} \rceil$, hence there exists a copy $\hat{K}$ of K_r in $\mathcal{T}'$ which intersects $\{v\} \cup V(K_t)$ and whose vertices are all neighbours of v in $G * K_t$ or v itself. If $K_v = \hat{K}$ then we can take $\mathcal{T} = \mathcal{T}'$, since $\hat{K} \subseteq G * K_t$. Hence assume $K^v \neq \hat{K}$ and so K^v and $\hat{K}$ are vertex-disjoint. In particular, $\hat{K}$ intersects $V(K_t)$ and so there exists $u \in V(\hat{K}) \cap V(K_t)$. As $u \in K_t$, it follows that $C := (G * K_t)[\{u\} \cup (V(K^v) \setminus \{v\})]$ is a clique. Furthermore, since $\hat{K}$ lies in the neighbourhood of v in G, we know $D := (G * K_t)[\{v\} \cup (V(\hat{K}) \setminus \{u\})]$

is also a clique (see Figure 4.3). Hence we can take $\mathcal{T} := (\mathcal{T}' \setminus \{K^v, \hat{K}\}) \cup \{C, D\}$. $\square$

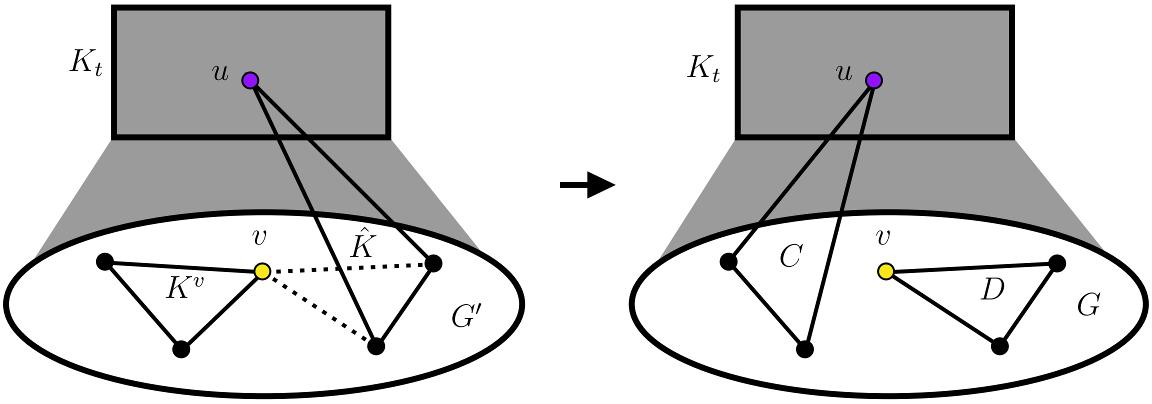

Figure 4.3: Case 2 when $r = 3$. The cliques K^v and $\hat{K}$ in $G' * K_t$ (left) are replaced with the cliques C and D in $G * K_t$ (right). Grey areas indicate complete sections.

Lemma 4.3.2. *Let $t \geq 0$, $r \geq 3$. Let G be a graph on n vertices such that $G * K_t$ does not contain a K_r-factor and suppose G contains an edge which is not contained in any copy of K_r in G. Then there exists a graph G' on n vertices such that $G' * K_t$ does not contain a K_r-factor, $e(G) \leq e(G')$ and G' has a vertex of degree $n - 1$.*

Proof. Let xy be an edge in G which is not contained in any copy of K_r. Let Q be a clique of maximal size containing xy and set $\ell := |V(Q)|$. Observe that every vertex in G has at most $\ell - 1$ neighbours in Q as otherwise xy would lie in an $(\ell + 1)$-clique. Thus

$$\sum_{v \in V(Q)} d_G(v) \leq n(\ell - 1).$$

Let G' be the graph obtained by deleting all edges between v and vertices in $V(G) \setminus V(Q)$ and, subsequently, adding any missing edge incident to any vertex in $V(Q) \setminus \{x\}$. Note that Q is still an ℓ-clique in G' and

$$\sum_{v \in V(Q)} d_{G'}(v) = n(\ell - 1).$$

Hence $e(G) \leq e(G')$. Also, G' is a graph on n vertices and $d_{G'}(y) = n - 1$. It remains to show that $G' * K_t$ does not contain a K_r-factor. Suppose, for a contradiction, that $G' * K_t$

does contain a K_r-factor $\mathcal{T}'$. We will now use $\mathcal{T}'$ to construct a new K_r-factor $\mathcal{T}$ in $G' * K_t$ which does not contain any edge vw where $v \in V(Q)$ and $w \in V(G') \setminus V(Q)$. Such a K_r-factor $\mathcal{T}$ is also a K_r-factor in $G * K_t$, giving us a contradiction.

Suppose that the copy K^x of K_r in $\mathcal{T}'$ covering x has exactly s vertices in K_t. Then K^x has exactly $r-s$ vertices in Q. Thus there are $\ell-r+s$ remaining vertices in $V(Q)\setminus V(K^x)$. Note that $\ell - r + s \leq s$, hence there is an injection $f : V(Q) \setminus V(K^x) \to V(K^x) \cap V(K_t)$. We construct our K_r-factor $\mathcal{T}$ as follows: for every copy of K_r in $\mathcal{T}'$ intersecting Q other than K^x, we substitute all vertices lying in its intersection with Q with their images under f. Finally, we take the copy of K_r formed by the ℓ-clique Q and the $s-(\ell-r+s) = r-\ell$ vertices in $V(K^x) \cap V(K_t)$ which do not appear in the image of f. Observe that $\mathcal{T}$ does not use any edge vw where $v \in V(Q)$ and $w \in V(G) \setminus V(Q)$, and we are done. $\square$

Before proceeding to the proof of Theorem 1.3.4, we prove the following technical lemma.

Lemma 4.3.3. *Let $n, t, r \in \mathbb{N}$ such that the following holds: $n, r \geq 3$; $r-1$ divides $t+1$; r divides $n+t$; $t+1 < (r-1)(n-1)$.*[9] *If*

$$e(EX_1(n-1, t+1, r)) < e(EX_2(n-1, t+1, r)) \tag{4.1}$$

then

$$e(EX_2(n-1, t+1, r)) + (n-1) \leq e(EX_2(n, t, r)). \tag{4.2}$$

Proof. Let $k := \frac{t+1}{r-1}$. Since $r-1$ divides $t+1$, we can compute $e(EX_2(n,t,r))$ and $e(EX_2(n-1,t+1,r))$ explicitly:

$$e(EX_2(n,t,r)) = \binom{n}{2} - \binom{k}{2} - k(n-k),$$

$$e(EX_2(n-1,t+1,r)) = \binom{n-1}{2} - \binom{k+2}{2} - (k+1)(n-k-r).$$

[9]Note that $EX_1(n-1, t+1, r)$, $EX_2(n-1, t+1, r)$ and $EX_2(n, t, r)$ are well-defined since the assumptions in Definition 4.2.1 are satisfied.

It follows that

$$e(EX_2(n,t,r)) - e(EX_2(n-1,t+1,r)) = (n-1) + k + (n-k-r) - kr.$$

Rearranging this, one obtains that (4.2) holds precisely when

$$t \le n - r - \frac{n}{r} =: g(n,r).$$

Further, one can calculate that $e(EX_2(n-1,t+1,r)) \le e(EX_1(n-1,t+1,r))$ precisely when

$$f_1(n,r) := \frac{n(r-1)}{(2r^2-2r+1)} - r \le t \le n(r-1) - r^2 =: f_2(n,r). \tag{4.3}$$

Since (4.1) holds, we have that (4.3) implies that $t < f_1(n,r)$ or $t > f_2(n,r)$. Observe that $f_1(n,r) \le g(n,r)$. Thus, if $t < f_1(n,r)$ then $t < g(n,r)$ and the claim holds.

Suppose $t > f_2(n,r)$. We will show that in this case the hypothesis of the lemma cannot actually hold. In particular, under the assumptions that $r-1$ divides $t+1$ and r divides $n+t$ (and $t > f_2(n,r)$), the graph $EX_2(n-1,t+1,r)$ is undefined. Indeed, for a contradiction let us assume that $EX_2(n-1,t+1,r)$ is well-defined in this case, thus the inequality $t+1 < (r-1)(n-1)$ must hold. By assumption, t satisfies the following modular equations:

$$t \equiv -1 \pmod{r-1} \quad \text{and} \quad t \equiv -n \pmod{r}. \tag{4.4}$$

For n and r fixed, the solution of (4.4) is unique modulo $r(r-1)$ by the Chinese Remainder Theorem, since r and $r-1$ are coprime. Note that $t' = (r-1)(n-1)-1$ is a solution of (4.4), hence $t = (r-1)(n-1) - 1 - kr(r-1)$ for some integer k. The constraint $t+1 < (r-1)(n-1)$ forces $k \ge 1$, hence

$$t \le (r-1)(n-1) - 1 - r(r-1) = n(r-1) - r^2 = f_2(n,r).$$

This contradicts the assumption that $t > f_2(n, r)$. $\square$

Proof of Theorem 1.3.4. We prove Theorem 1.3.4 by induction on n. If $n = 2$ then $e(G) \in \{0, 1\}$. Since $t < 2(r-1)$ and $r|(n+t)$, we must have $t = r-2$. If $e(G) = 1$ then $G * K_t$ is a copy of K_r, contradicting our choice of G. Thus $e(G) = 0$. Observe that $k = \lceil \frac{t+1}{r-1} \rceil = 1$ and $q = r-2$. Thus

$$\max\left\{\binom{n}{2} - \binom{\frac{n+t}{r}+1}{2}, \binom{n}{2} - \binom{k}{2} - k(n-k-(r-2-q))\right\} = \max\{0, 0\} = 0.$$

Thus Theorem 1.3.4 holds for $n = 2$.

For the inductive step, we may assume without loss of generality that G is a graph on n vertices such that $e(G)$ is maximal with respect to the property that $G * K_t$ does not contain a K_r-factor.

Case (i): G contains an isolated vertex v. If $t < r-2$, then one could add a single edge to v and G would still not contain a K_r-factor, contradicting our choice of G. Hence $t \geq r-2$. If $t = r-2$, then $k = \lceil \frac{t+1}{r-1} \rceil = 1$ and $q = r-2$. Hence

$$\begin{aligned} &\max\left\{\binom{n}{2} - \binom{\frac{n+t}{r}+1}{2}, \binom{n}{2} - \binom{k}{2} - k(n-k-(r-2-q))\right\} \\ = &\max\left\{\binom{n}{2} - \binom{\frac{n-2}{r}+2}{2}, \binom{n-1}{2}\right\}. \end{aligned}$$

Since G contains at least one isolated vertex,

$$e(G) \leq \binom{n-1}{2} \leq \max\left\{\binom{n}{2} - \binom{\frac{n-2}{r}+2}{2}, \binom{n-1}{2}\right\},$$

and we are done. If $t \geq r-1$, consider the graph G' obtained by deleting v from G. Since $G * K_t$ does not contain a K_r-factor, $G' * K_{t-r+1}$ does not contain a K_r-factor. Thus, by our inductive hypothesis

$$e(G') \leq \max\{e(EX_1(n-1, t-r+1, r)), e(EX_2(n-1, t-r+1, r))\}.$$

It follows from $e(G) = e(G')$ and $e(EX_i(n-1, t-r+1, r)) \leq e(EX_i(n, t, r))$[10] for $i = 1, 2$ that $e(G) \leq \max\{e(EX_1(n, t, r)), e(EX_2(n, t, r))\}$.

Case (ii): G contains a vertex of degree $n-1$. Consider the graph G' obtained by deleting such a vertex from G. Note that $G * K_t = G' * K_{t+1}$. If $t+1 \geq (r-1)(n-1)$, then trivially $G' * K_{t+1}$ contains a K_r-factor, a contradiction. So we must have that $t+1 < (r-1)(n-1)$ and hence by induction

$$e(G') \leq \max\{e(EX_1(n-1, t+1, r)), e(EX_2(n-1, t+1, r))\}.$$

Observe that $e(G) = e(G') + n - 1$. We aim to show that

$$e(G) \leq \max\{e(EX_1(n, t, r)), e(EX_2(n, t, r))\}. \tag{4.5}$$

If $e(G') \leq e(EX_1(n-1, t+1, r)$ then

$$e(G * K_t) = e(G' * K_{t+1}) \leq e(EX_1(n-1, t+1, r) * K_{t+1}) = e(EX_1(n, t, r) * K_t)$$

and thus

$$e(G) \leq e(EX_1(n, t, r)).$$

Similarly, if $e(G') \leq e(EX_2(n-1, t+1, r)$ and $r-1$ does not divide $t+1$ then

$$e(G * K_t) = e(G' * K_{t+1}) \leq e(EX_2(n-1, t+1, r) * K_{t+1}) = e(EX_2(n, t, r) * K_t)$$

and thus

$$e(G) \leq e(EX_2(n, t, r)).$$

[10] This holds since $EX_i(n-1, t-r+1, r)$ can be obtained by removing an appropriate vertex from $EX_i(n, t, r)$, for $i = 1, 2$.

It remains to check that (4.5) holds in the case when $(r-1)|(t+1)$ and

$$e(EX_1(n-1,t+1,r)) < e(EX_2(n-1,t+1,r)).$$

In this case Lemma 4.3.3 implies that

$$e(G) = e(G') + n - 1 \leq e(EX_2(n-1,t+1,r)) + (n-1) \leq e(EX_2(n,t,r)),$$

as desired.

Case (iii): G contains no vertex of degree 0 *or* $n-1$. If G contains an edge which is not contained in any copy of K_r, then by Lemma 4.3.2 there exists a graph G' on n vertices such that $G' * K_t$ does not contain a K_r-factor, $e(G) \leq e(G')$ and G' has a vertex of degree $n-1$. The argument from Case (ii) then implies that $e(G) \leq e(G') \leq \max\{e(EX_1(n,t,r)), e(EX_2(n,t,r))\}$.

We may therefore assume every edge of G is contained in some copy of K_r. Moreover, as no vertex in G has degree 0, every vertex in G is contained in a copy of K_r. Let w be a vertex of smallest degree in G. If $d_G(w) \geq n - \frac{n+t}{r}$ then

$$\delta(G * K_t) \geq n - \frac{n+t}{r} + t = \frac{r-1}{r}(n+t).$$

Hence by Theorem 1.3.2, we have that $G * K_t$ contains a K_r-factor, a contradiction.

Thus $d_G(w) < n - \frac{n+t}{r}$. Let K be a copy of K_r in G containing w. Consider the graph G' obtained by removing K and all its vertices from G. Then $G' * K_t$ does not contain a K_r-factor; this implies that $t < (r-1)(n-r)$. Hence, by our inductive hypothesis, we have

$$e(G') \leq \max\{e(EX_1(n-r,t,r)), e(EX_2(n-r,t,r))\}.$$

If $e(G') \leq e(EX_1(n-r,t,r))$ then $d_G(w) < n - \frac{n+t}{r}$ implies $e(G) \leq e(EX_1(n,t,r))$, as desired: this follows from the fact that $EX_1(n-r,t,r)$ can be obtained by removing a clique Q of size r from $EX_1(n,t,r)$ where Q has one vertex of degree $n - \frac{n+t}{r} - 1$. By

Lemma 4.3.1, we have that $\Delta(G) \leq n-1-\lceil \frac{t+1}{r-1} \rceil$ since we assumed G contains no vertices of degree $n-1$ and $e(G)$ is maximal with respect to the property that $G * K_t$ does not contain a K_r-factor. Thus, if $e(G') \leq e(EX_2(n-r,t,r))$ then applying this observation yields that $e(G) \leq e(EX_2(n,t,r))$, as desired: similarly as before, this follows from the fact that $EX_2(n-r,t,r)$ can be obtained by removing a clique Q of size r from $EX_2(n,t,r)$ where all vertices in Q have degree $n-1-\lceil \frac{t+1}{r-1} \rceil$. $\square$

4.4 Concluding remarks

In this chapter we resolved the deficiency problem for K_r-factors. For a general fixed graph H, it would be interesting to prove deficiency results regarding H-factors. As a starting point for this problem we pose the following question. Let $\alpha(H)$ denote the size of the largest independent set in H.

Question 4.4.1. *Let $K := K_n$ and $A \subseteq K$ such that $A = K_{\frac{\alpha(H)(n+t)}{|H|}+1}$. Define $EX_H(n,t)$ to be the graph obtained by removing $E(A)$ from K. Does there exist a constant $c := c(H) > 0$ such that if $t \geq cn$ and G is an n-vertex graph so that $G * K_t$ does not contain an H-factor then $e(G) \leq e(EX_H(n,t)) + o(n^2)$?*

On the other hand, at least for some H one cannot remove the $o(n^2)$ term in Question 4.4.1 completely. Indeed, let $H = K_{1,s}$ where $s \geq 2$ and consider the n-vertex graph $EX'_H(n,t)$ obtained from $EX_H(n,t)$ by adding a maximal matching in $V(A)$. It is easy to see that $EX'_H(n,t) * K_t$ does not contain a $K_{1,s}$-factor. This example suggests it might be rather challenging to resolve the H-factor deficiency problem completely for all graphs H.

Chapter 5

Induced poset saturation

5.1 Introduction

First, we recall some important definitions for this chapter. We say that P is a *poset on* $[p]$ if P consists of subsets of $[p]$. Let P, Q be posets. A *poset homomorphism* from P to Q is a function $\phi : P \to Q$ such that for every $A, B \in P$, if $A \subseteq B$ then $\phi(A) \subseteq \phi(B)$. We say P is an *induced subposet* of Q if there is an injective poset homomorphism ϕ from P to Q such that for every $A, B \in P$, we have $\phi(A) \subseteq \phi(B)$ if and only if $A \subseteq B$; otherwise, Q is said to be *induced P-free*. For a fixed poset P, we say that a family $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is *induced P-saturated* if $\mathcal{F}$ is induced P-free, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, then P is an induced subposet of $\mathcal{F} \cup \{S\}$. We write $\text{sat}^*(n, P)$ to denote the size of the smallest induced P-saturated family of subsets of $[n]$.

As described in Section 1.4, Keszegh, Lemons, Martin, Pálvölgyi and Patkós proposed the following conjecture.

Conjecture 1.4.2. *[85] For any poset P, either there exists a constant K_P with $sat^*(n, P) \leq K_P$ or $sat^*(n, P) \geq n + 1$, for all $n \in \mathbb{N}$.*

The main novel result discussed in this chapter is Theorem 1.4.4, which makes progress towards settling Conjecture 1.4.2. We state Theorem 1.4.4 once again.

Theorem 1.4.4. *For any poset P, either there exists a constant K_P with sat*$^*(n,P) \le K_P$ *or sat*$^*(n,P) \ge \min\{2\sqrt{n}, n/2+1\}$ *for all* $n \in \mathbb{N}$.

On the other hand, in the next theorem, we prove that Conjecture 1.4.2 does hold for a certain class of posets. Given $p \in \mathbb{N}$ and a poset P on $[p]$ we define the *dual* $\overline{P}$ of P as $\overline{P} := \{[p] \setminus F \colon F \in P\}$. We say a poset P *has legs* if there are distinct elements $L_1, L_2, H \in P$ such that L_1, L_2 are incomparable, $L_1, L_2 \subseteq H$ and for any other element $A \in P \setminus \{L_1, L_2, H\}$ we have $A \supseteq H$. The elements L_1 and L_2 are called *legs* and H is called a *hip*.

Theorem 5.1.1. *Let P be a poset with legs and $n \ge 3$. Then sat*$^*(n,P) \ge n+1$. *Moreover, if both P and $\overline{P}$ have legs, then sat*$^*(n,P) \ge 2n+2$.

Finally, we determine the exact values of $\text{sat}^*(n,X)$ and $\text{sat}^*(n,Y)$ (see Figure 5.1 for the Hasse diagrams of X and Y).

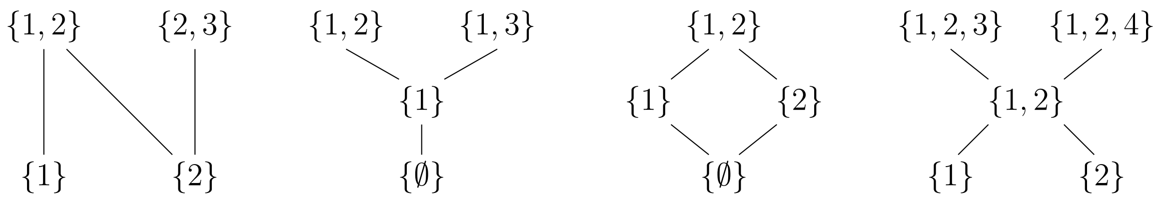

Figure 5.1: Hasse diagrams for the posets N, Y, $\Diamond$ and X. A segment indicates the set below is contained in the one above, and there is no intermediate set between them.

Theorem 5.1.2. *Given any $n \in \mathbb{N}$ with $n \ge 3$,*

(i) *sat*$^*(n,Y) = n+2$ *and*

(ii) *sat*$^*(n,X) = 2n+2$.

Note that $\overline{X} = X$, so Theorem 5.1.2(*ii*) easily follows via Theorem 5.1.1 and an extremal construction. An application of Theorem 5.1.1 to $\overline{Y}$ only yields that $\text{sat}^*(n,Y) = \text{sat}^*(n,\overline{Y}) \ge n+1$, so we require an extra idea to obtain Theorem 5.1.2(*i*).

In [102] a trick was introduced which can be used to prove lower bounds on $\text{sat}^*(n,P)$ for some posets P. Given an induced P-saturated family $\mathcal{F}$, the idea is to define an auxiliary digraph D whose vertex set is $\mathcal{F}$ and two vertices form an edge provided the corresponding elements of $\mathcal{F}$ satisfy a certain property. One then argues that there are many pairs of elements in $\mathcal{F}$ with such property, which provides a lower bound for the number of edges in the digraph D. In turn, this forces a lower bound on the size of the vertex set of D (i.e., lower bounds $|\mathcal{F}|$). This trick has been used to prove that $\mathrm{sat}^*(n, \Diamond) \geq \sqrt{n}$ [102, Theorem 6] and $\mathrm{sat}^*(n, N) \geq \sqrt{n}$ [80, Proposition 4] (see Figure 5.1 for the Hasse diagram of N and $\Diamond$).

Our proof of Theorem 1.4.4 utilises a variant of this digraph trick. In particular, by introducing an appropriate modification to the auxiliary digraph D used in [102], we are able to deduce certain Turán-type properties of D. Turán problems in digraphs are classical in extremal combinatorics and their study can be traced back to the work of Brown and Harary [20]. Here we prove a Turán-type result concerning *transitive cycles*.

Given $k \geq 3$, the *transitive cycle on k vertices* $\overrightarrow{TC}_k$ is a digraph with vertex set $[k]$ and every directed edge from i to $i+1$ for every $i \in [k-1]$, as well as the directed edge from 1 to k. We establish an upper bound on the number of edges of a digraph not containing any transitive cycle.

Theorem 5.1.3. *Let $n \in \mathbb{N}$ and let D be a digraph on n vertices. If D is $\overrightarrow{TC}_k$-free for all $k \geq 3$, then*

$$e(D) \leq \max\left\{2(n-1), \left\lfloor \frac{n^2}{4} \right\rfloor \right\}.$$

Note that the bound in Theorem 5.1.3 is best possible. Indeed, consider the n-vertex digraph D with vertex classes A, B of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively and all possible directed edges from A to B. So D has $\lfloor n^2/4 \rfloor$ edges and contains no transitive cycle. Similarly, pick an arbitrary n-vertex tree T and let D' be the digraph with vertex set $V(D') = V(T)$ and with an edge directed from i to j if and only if the unordered pair ij is an edge in T. So D' has $2(n-1)$ edges and contains no transitive cycle.

We would like to point out that the original proof of Theorem 5.1.3 caused a constant error term to appear in the statement of the theorem, and its current sharp form was achieved thanks to a suggestion by Louis DeBiasio.

In Section 5.3 we explain why exploiting the connection between the induced saturation problem for posets and the Turán problem for digraphs hits a natural barrier if we wish to further improve the lower bound in Theorem 1.4.4.

We conclude this introductory section by setting out notation that will be used in the rest of the chapter.

Notation

Given two elements A, B of a poset $\mathcal{F} \subseteq 2^{[n]}$ we say that A *dominates* B if $B \subseteq A$. We say A and B are *comparable* if one dominates the other; otherwise we say they are *incomparable*. An element F in $\mathcal{F}$ is *maximal* if there is no other element in $\mathcal{F}$ which dominates F.

For a digraph $D = (V(D), E(D))$, we write $\overrightarrow{xy}$ for the edge in D directed from the vertex x to the vertex y. We say that $\overrightarrow{xy}$ is an *out-edge* of x and an *in-edge* of y. For brevity, we write $e(D) := |E(D)|$. Given a set $U \subseteq V(D)$, we denote the induced subgraph of D with vertex set U as $D[U]$. The *underlying graph of* D is the graph with vertex set $V(D)$ whose edges are all (unordered) pairs $\{x, y\}$ such that $\overrightarrow{xy} \in E(D)$ or $\overrightarrow{yx} \in E(D)$.

Given $k \geq 2$, an *oriented path* $v_1 \dots v_k$ from v_1 to v_k consists of the edges $\overrightarrow{v_i v_{i+1}}$ for every $i \in [k-1]$. Similarly, an *oriented cycle* $v_1 \dots v_k$ consists of the edges $\overrightarrow{v_i v_{i+1}}$ for every $i \in [k-1]$ and $\overrightarrow{v_k v_1}$.

5.2 The proofs

5.2.1 Proof of Theorem 5.1.3

We proceed by induction on n. For the base case, it is easy to check that the statement of the theorem holds for $n \leq 3$. Next, we prove the inductive step.

Let D be a digraph on $n \geq 4$ vertices which is $\overrightarrow{TC}_k$-free for all $k \geq 3$.

Claim 5.2.1. *If D contains an induced oriented cycle then* $e(D) \leq \max\left\{2(n-1), \left\lfloor \frac{n^2}{4} \right\rfloor\right\}$.

Proof of the claim. Suppose D contains an induced oriented cycle C. For every $v \in V(D) \setminus V(C)$, it is straightforward to check that, since $D[V(C) \cup \{v\}]$ contains no transitive cycle, then

(i) there is at most one in-edge of v incident to $V(C)$ and

(ii) there is at most one out-edge of v incident to $V(C)$.

Let D' be the digraph obtained by contracting the cycle C into one vertex c. Namely, D' has vertex set $V(D') = (V(D) \cup \{c\}) \setminus V(C)$ and $E(D')$ is the union of the following sets:

- $E(D[V(D) \setminus V(C)])$,
- $\{\overrightarrow{xc} : \exists\, \overrightarrow{xy} \in E(D), x \notin V(C), y \in V(C)\}$ and
- $\{\overrightarrow{cx} : \exists\, \overrightarrow{yx} \in E(D), x \notin V(C), y \in V(C)\}$.

Note that properties (i) and (ii) imply that $e(D') = e(D) - e(C) = e(D) - |V(C)|$.

Suppose D' contains a transitive cycle $\overrightarrow{TC}_k$ on vertices $v_1, \ldots, v_k$ for some $k \geq 3$. Namely, $\overrightarrow{v_j v_{j+1}} \in E(D')$ for every $j \in [k-1]$ and $\overrightarrow{v_1 v_k} \in E(D')$. If $c \neq v_i$ for every $i \in [k]$ then $v_1, \ldots, v_k$ form a transitive cycle in D, a contradiction. Therefore, $c = v_i$ for some $i \in [k]$.

For brevity we only consider the case $i \neq 1, k$ (the cases $i = 1$ and $i = k$ can be handled with a similar argument). By the definition of D', there exist $c_1, c_2 \in V(C)$ such that $\overrightarrow{v_{i-1}c_1}, \overrightarrow{c_2 v_{i+1}} \in E(D)$. Furthermore, there exists an oriented path $u_1 \ldots u_\ell$ from u_1 to u_ℓ such that $u_1 = c_1$, $u_\ell = c_2$ and $u_j \in E(C)$ for all $j \in [\ell]$. Then $v_1 \ldots v_{i-1} u_1 \ldots u_\ell v_{i+1} \ldots v_k$ is an oriented path from v_1 to v_k in D. Together with $\overrightarrow{v_1 v_k} \in E(D)$, this forms a transitive cycle in D, a contradiction.

Therefore, D' is $\overrightarrow{TC}_k$-free for all $k \geq 3$; so by the induction hypothesis we have

$$e(D') \leq \max\left\{2(n - |V(C)|), \left\lfloor \frac{(n - |V(C)| + 1)^2}{4} \right\rfloor\right\}.$$

The inequalities $n \geq 4$ and $2 \leq |V(C)| \leq n$ imply

$$2(n - |V(C)|) + |V(C)| \leq 2(n-1) \quad \text{and} \quad \left\lfloor \frac{(n - |V(C)| + 1)^2}{4} \right\rfloor + |V(C)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In particular, $e(D) = e(D') + |V(C)| \leq \max\left\{2(n-1), \left\lfloor \frac{n^2}{4} \right\rfloor\right\}$. This concludes the proof of the claim.

■

Because of Claim 5.2.1 we may assume that D contains no double edges (which are induced oriented cycles on two vertices). Additionally, we may assume that the underlying graph of D contains no triangle, since such a triangle would either correspond to an induced oriented cycle or a transitive cycle $\overrightarrow{TC}_3$ in D. By Mantel's theorem, a triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges, hence $e(D) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. This concludes the inductive step. □

5.2.2 Proof of Theorem 1.4.4

We prove Theorem 1.4.4 using the following two lemmata.

Lemma 5.2.2. *Let $\mathcal{F} \subseteq 2^{[n]}$. If for every $i \in [n]$ there are elements $A, B \in \mathcal{F}$ such that $A \setminus B = \{i\}$ then $|\mathcal{F}| \geq \min\{2\sqrt{n}, n/2 + 1\}$.*

Notice that Lemma 5.2.2 is not specifically about induced saturated families. We will discuss this further in Section 5.3.

Lemma 5.2.3. *Given a poset P, let $\mathcal{F}_0 \subseteq 2^{[n_0]}$ be an induced P-saturated family. Suppose there is an $i \in [n_0]$ such that there are no elements $A, B \in \mathcal{F}_0$ satisfying $A \setminus B = \{i\}$. Then $sat^*(n, P) \leq |\mathcal{F}_0|$ for every $n \geq n_0$.*

Proof of Theorem 1.4.4. Let P be a poset. Suppose that for every $n \in \mathbb{N}$ and every induced P-saturated family $\mathcal{F} \subseteq 2^{[n]}$ we have that for every $i \in [n]$ there are elements

$A, B \in \mathcal{F}$ such that $A \setminus B = \{i\}$. Then Lemma 5.2.2 implies $|\mathcal{F}| \geq \min\{2\sqrt{n}, n/2+1\}$, and thus $\text{sat}^*(n, P) \geq \min\{2\sqrt{n}, n/2+1\}$.

Otherwise, there exists some $n_0 \in \mathbb{N}$ and $i \in [n_0]$ so that there is an induced P-saturated family $\mathcal{F}_0 \subseteq 2^{[n_0]}$ such that there are no elements $A, B \in \mathcal{F}_0$ with $A \setminus B = \{i\}$. Then Lemma 5.2.3 implies that $\text{sat}^*(n, P) \leq K_P$ for every $n \in \mathbb{N}$ where

$$K_P := \max\{|\mathcal{F}_0|, \text{sat}^*(m, P) : m < n_0\}.$$

$\square$

The rest of this subsection covers the proofs of Lemmata 5.2.2 and 5.2.3. For the proof of Lemma 5.2.2 we apply Theorem 5.1.3.

Proof of Lemma 5.2.2. Let D be a digraph with vertex set $\mathcal{F}$ and edge set $E(D)$ defined as follows: for every $i \in [n]$, choose precisely one pair $A, B \in \mathcal{F}$ with $A \setminus B = \{i\}$; add the edge $\overrightarrow{AB}$ to $E(D)$. Thus D has exactly n edges. Note that the hypothesis of the lemma ensures D is well-defined.

Claim 5.2.4. *For every $k \geq 3$, D is $\overrightarrow{TC}_k$-free.*

Proof of the claim. It suffices to show that given $\{A_1, \ldots, A_k\} \subseteq \mathcal{F}$ such that $\overrightarrow{A_j A_{j+1}} \in E(D)$ for every $j \in [k-1]$ then $\overrightarrow{A_1 A_k} \notin E(D)$. By definition of D, there are distinct $i_1, \ldots, i_{k-1} \in [n]$ such that

$$A_j \setminus A_{j+1} = \{i_j\} \qquad \text{for every } j \in [k-1]. \tag{5.1}$$

This implies $A_j \subseteq A_{j+1} \cup \{i_j\}$ for every $j \in [k-1]$, and thus $A_1 \subseteq A_k \cup \{i_1, \ldots, i_{k-1}\}$. Hence,

$$A_1 \setminus A_k \subseteq \{i_1, \ldots, i_{k-1}\}. \tag{5.2}$$

Note that if $\overrightarrow{A_1 A_k} \in E(D)$ then (5.2) implies that $A_1 \setminus A_k = \{i_j\}$ for some $j \in [k-1]$.

However, recall that for every $i \in [n]$ there is exactly one edge $\overrightarrow{AB}$ in D such that $A \setminus B = \{i\}$; so (5.1) implies that $\overrightarrow{A_1 A_k} \notin E(D)$, as desired.

■

By Claim 5.2.4, we can apply Theorem 5.1.3 to the digraph D. This yields

$$n = |E(D)| \leq \max\left\{2(|\mathcal{F}| - 1), \frac{|\mathcal{F}|^2}{4}\right\},$$

which implies $|\mathcal{F}| \geq \min\{2\sqrt{n}, n/2 + 1\}$. □

We now present the proof of Lemma 5.2.3.

Proof of Lemma 5.2.3. Observe that it is enough to prove that for every $n \geq n_0$ there exists a family $\mathcal{F} \subseteq 2^{[n]}$ such that

(*i*) $|\mathcal{F}| = |\mathcal{F}_0|$,

(*ii*) $\mathcal{F}$ is induced P-saturated and

(*iii*) there are no elements $A, B \in \mathcal{F}$ satisfying $A \setminus B = \{i\}$.

We proceed by induction on n and observe that the base case ($n = n_0$) follows directly from the assumption in the statement of the lemma.

Given an induced P-saturated family $\mathcal{F} \subseteq 2^{[n]}$ satisfying (*i*)-(*iii*) we consider the following function f from $2^{[n]}$ to $2^{[n+1]}$:

$$f(A) := \begin{cases} A & \text{if } i \notin A, \\ A \cup \{n+1\} & \text{if } i \in A. \end{cases}$$

We shall prove that the family $\mathcal{F}' := f(\mathcal{F}) \subseteq 2^{[n+1]}$ satisfies (*i*)-(*iii*).

First, note that (*i*) follows directly since f is injective. Second, (*iii*) also follows easily, since every element $f(A)$ either contains both i and $n+1$ or neither of them. Actually, because of this last property one might say that $n+1$ behaves as a 'copy' of i in $f(2^{[n]})$. We need to prove *(ii)*, i.e., that $\mathcal{F}'$ is induced P-saturated.

It is easy to check that f preserves the inclusion/incomparable relations between elements. More precisely, for every $A, B \in 2^{[n]}$ we have

$$A \subseteq B \iff f(A) \subseteq f(B). \tag{5.3}$$

Therefore, if P' forms an induced copy of P in $\mathcal{F}'$, then $f^{-1}(P') \subseteq \mathcal{F}$ forms an induced copy of P in $\mathcal{F}$. This means that, since $\mathcal{F}$ is induced P-free, $\mathcal{F}'$ must be induced P-free as well. It is left to prove that for any $S \in 2^{[n+1]} \setminus \mathcal{F}'$, there is an induced copy of P in $\mathcal{F}' \cup \{S\}$. There are four cases to consider depending on S. The first two cases are short and the fourth case is an easy consequence of the third. Let $S \in 2^{[n+1]} \setminus \mathcal{F}'$.

First case: $i, n+1 \notin S$. Note that $S \subseteq 2^{[n]}$ and $f(S) = S$, thus $S \notin \mathcal{F}$ (as otherwise we would have $S \in \mathcal{F}'$). As $\mathcal{F}$ is induced P-saturated, there exists an induced copy P' of P in $\mathcal{F} \cup \{S\}$. By (5.3) the set $\{f(F) : F \in P'\}$ is an induced copy of P in $\mathcal{F}' \cup \{S\}$.

Second case: $i, n+1 \in S$. Set $S^\star := S \setminus \{n+1\}$. Note that $S^\star \subseteq 2^{[n]}$ and $f(S^\star) = S$, thus $S^\star \notin \mathcal{F}$ (as otherwise we would have $S \in \mathcal{F}'$). As $\mathcal{F}$ is induced P-saturated, there exists an induced copy P' of P in $\mathcal{F} \cup \{S^\star\}$. By (5.3) the set $\{f(F) : F \in P'\}$ is an induced copy of P in $\mathcal{F}' \cup S$.

Third case: $i \in S$ and $n+1 \notin S$. For this case we use the assumption that there are no elements $A, B \in \mathcal{F}$ such that $A \setminus B = \{i\}$. Let $S^\star := S \setminus \{i\} \in 2^{[n]}$. We use the sets S and $S^\star$ to find elements $A, B \in \mathcal{F}$ that will contradict *(iii)* (for the family $\mathcal{F}$).

Claim 5.2.5. *Either $\mathcal{F}' \cup \{S\}$ contains an induced copy of P or there exists an element $A \in \mathcal{F}$ such that $A \subseteq S$ and $i \in A$.*

Proof of the claim. Assume that there is no $A \in \mathcal{F}$ satisfying the properties of the claim; notice that this implies $S \notin \mathcal{F}$. We need to prove that $\mathcal{F}' \cup \{S\}$ contains an induced copy of P.

Since $\mathcal{F}$ is induced P-saturated, there exists an induced copy P' of P in $\mathcal{F} \cup \{S\}$ that contains S. We shall prove that $\{f(F) : F \in P', F \neq S\} \cup \{S\}$ is an induced copy of P in $\mathcal{F}' \cup \{S\}$.

First, observe that (5.3) implies the set $\{f(F) : F \in P', F \neq S\}$ is an induced copy of the poset $P' \setminus \{S\}$. If for every $F \in P' \setminus \{S\}$ the relation (inclusion/incomparability) between F and S is the same as the one between $f(F)$ and S then we are done. Thus it is enough to prove for every $F \in P' \setminus \{S\}$ we have that

(*i*) if S and F are incomparable then S and $f(F)$ are incomparable,

(*ii*) if $S \subseteq F$ then $S \subseteq f(F)$ and

(*iii*) if $S \supseteq F$ then $S \supseteq f(F)$.

Notice that (*ii*) follows directly from $F \subseteq f(F)$. It is easy to check that (*i*) also holds by recalling that $i \in S$ and $n+1 \notin S$. So finally, for $F \in P' \setminus \{S\}$ as in (*iii*), observe that $i \notin F$ otherwise F would satisfy the properties of A in the statement of the claim. Then $f(F) = F$ and (*iii*) holds.

■

The proof of the following claim is very similar. We include it for completeness.

Claim 5.2.6. *Either $\mathcal{F}' \cup \{S\}$ contains an induced copy of P or there exists an element $B \in \mathcal{F}$ such that $S^\star \subseteq B$ and $i \notin B$.*

Proof of the claim. Assume that there is no $B \in \mathcal{F}$ satisfying the properties of the claim; notice that this implies $S^\star \notin \mathcal{F}$. It remains to show that $\mathcal{F}' \cup \{S\}$ contains an induced copy of P.

Since $\mathcal{F}$ is induced P-saturated, there exists an induced copy P' of P in $\mathcal{F} \cup \{S^\star\}$ that contains $S^\star$. We shall prove that $\{f(F) : F \in P', F \neq S^\star\} \cup \{S\}$ is an induced copy of P in $\mathcal{F}' \cup \{S\}$.

First, observe that (5.3) implies the set $\{f(F) : F \in P', F \neq S^\star\}$ is an induced copy of the poset $P' \setminus \{S^\star\}$. If for every $F \in P' \setminus \{S^\star\}$ the relation (inclusion/incomparability) between F and $S^\star$ is the same as the one between $f(F)$ and S then we are done. Thus, for every $F \in P' \setminus \{S^\star\}$ we must prove that

(i) if $S^\star$ and F are incomparable then S and $f(F)$ are incomparable,

(ii) if $S^\star \supseteq F$ then $S \supseteq f(F)$ and

(iii) if $S^\star \subseteq F$ then $S \subseteq f(F)$.

For (ii), notice that $S^\star \supseteq F$ implies $i \notin F$ and thus $S \supseteq S^\star \supseteq F = f(F)$. It is easy to check that (i) also holds by recalling that $i \in S$ and $n+1 \notin S$. Finally, for $F \in P' \setminus \{S^\star\}$ as in (iii), observe that $i \in F$ otherwise F would satisfy the properties of B in the statement of the claim. Then $f(F) = F \cup \{n+1\}$ and (iii) holds.

■

To finish the proof of this case, suppose that $\mathcal{F}' \cup \{S\}$ does not contain an induced copy of P.

By Claim 5.2.5 there is an element $A \in \mathcal{F}$ such that $A \subseteq S$ and $i \in A$, whereas by Claim 5.2.6 there is an element $B \in \mathcal{F}$ such that $S \setminus \{i\} \subseteq B$ and $i \notin B$. In particular, we have $A \setminus B = \{i\}$, a contradiction.

Fourth case: $i \notin S$ and $n+1 \in S$. Let $S' := (S \setminus \{n+1\}) \cup \{i\}$. Recall that every element of $\mathcal{F}'$ either contains both i and $n+1$ or neither of them. This property ensures that

(a) $S' \notin \mathcal{F}'$;

(b) for any $A \in \mathcal{F}'$, we have $A \subseteq S$ if and only if $A \subseteq S'$;

(c) for any $A \in \mathcal{F}'$, we have $S \subseteq A$ if and only if $S' \subseteq A$.

By the third case we have that $\mathcal{F}' \cup \{S'\}$ contains an induced copy of P. Clearly (b) and (c) then imply that $\mathcal{F}' \cup \{S\}$ contains an induced copy of P. □

5.2.3 Proof of Theorems 5.1.1 and 5.1.2

Given a poset P with legs and an induced P-saturated family $\mathcal{F} \subseteq 2^{[n]}$, the following lemma already implies that $\mathcal{F}$ has size at least n.

Lemma 5.2.7. *Let P be a poset with legs and $\mathcal{F} \subseteq 2^{[n]}$ be an induced P-saturated family. Then, there is an injective function $f\colon [n] \longrightarrow \mathcal{F} \setminus \{\emptyset\}$ such that*

$$f(i) = \begin{cases} \{i\} & \text{if } \{i\} \in \mathcal{F}, \\ H(i) & \text{if } \{i\} \notin \mathcal{F}, \text{ where } H(i) \text{ is the hip of an induced copy of } P \text{ in } \mathcal{F} \cup \{\{i\}\}. \end{cases}$$

Proof. Let $\mathcal{F}$ be an induced P-saturated family. For $\{i\} \notin \mathcal{F}$ we shall choose a suitable $f(i) = H(i)$ in such a way that f is injective. Given $i \in [n]$ with $\{i\} \notin \mathcal{F}$, let P' be an induced copy of P in $\mathcal{F} \cup \{\{i\}\}$ that contains $\{i\}$. Observe that $\{i\}$ can only be a leg of P'. Among all possible choices for P', we pick it under the following conditions.

(i) The leg L' of P' which is not $\{i\}$ is taken so that $|L'|$ is as large as possible;

(ii) under (i), the hip H' of P' is taken so that $|H'|$ is as small as possible.

Thus we define $f(i) := H'$. Note that since L' and $\{i\}$ are incomparable, $i \notin L'$.

Claim 5.2.8. $H' = L' \cup \{i\}$.

Proof of the claim. Suppose for a contradiction that $L' \cup \{i\} \subsetneq H'$. If $L' \cup \{i\} \in \mathcal{F}$ then the poset $(P' \cup \{L' \cup \{i\}\}) \setminus \{H'\}$ is an induced copy of P, which contradicts (ii). Therefore $L' \cup \{i\}$ is not in $\mathcal{F}$ and so $\mathcal{F} \cup \{L' \cup \{i\}\}$ contains an induced copy P'' of P which uses the set $L' \cup \{i\}$.

Denote the legs of P'' by L''_1, L''_2. If $L' \cup \{i\}$ is not a leg in P'' then $L''_1, L''_2 \subsetneq L' \cup \{i\}$. This implies that $(P' \cup \{L''_1, L''_2\}) \setminus \{\{i\}, L'\}$ is an induced copy of P in $\mathcal{F}$, a contradiction. Thus, $L' \cup \{i\}$ is a leg, and we may assume $L''_1 = L' \cup \{i\}$.

Note that if L' and L''_2 are incomparable, then $(P'' \cup \{L'\}) \setminus \{L''_1\}$ would be an induced copy of P in $\mathcal{F}$, a contradiction. Thus, L' is comparable with L''_2. In particular $L' \subsetneq L''_2$, otherwise we would have that $L''_2 \subseteq L' \cup \{i\} = L''_1$, which contradicts the fact that L''_1 and L''_2 are incomparable. Moreover, again because L''_1 and L''_2 are incomparable, we have that $i \notin L''_2$.

Finally, observe that $(P'' \cup \{\{i\}\}) \setminus \{L''_1\}$ is an induced copy of P in $\mathcal{F} \cup \{\{i\}\}$ with legs $\{i\}$ and $L''_2 \supset L'$, which contradicts (i).

■

Recall $f(i) := H' = H(i)$. We shall prove that f is injective. Suppose not, namely $f(i) = f(j)$ for some $i \neq j$. If either $\{i\} \in \mathcal{F}$ or $\{j\} \in \mathcal{F}$ then we get a contradiction. Thus, we have $\{i\}, \{j\} \notin \mathcal{F}$, and hence there exist induced copies $P' \subseteq \mathcal{F} \cup \{\{i\}\}$ and $P'' \subseteq \mathcal{F} \cup \{\{j\}\}$ of P such that H is the hip of both and $f(i) = f(j) = H$. Say $\{i\}$ and L' are the legs of P' while $\{j\}$ and L'' are the legs of P''. Because of Claim 5.2.8 we have that $H = L' \cup \{i\} = L'' \cup \{j\}$, and hence L' and L'' are incomparable. This implies that $(P' \cup \{L''\}) \setminus \{\{i\}\}$ is an induced copy of P in $\mathcal{F}$, a contradiction. □

Now we are ready to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Observe that for a poset P with legs, the inequality $\mathrm{sat}^*(n, P) \geq n + 1$ follows directly by applying Lemma 5.2.7 and by noticing that the empty set is in every induced P-saturated family.

For P such that P and $\overline{P}$ have legs, we proceed as follows. Let $\mathcal{F} \subseteq 2^{[n]}$ be an induced P-saturated family and observe that $\emptyset, [n] \in \mathcal{F}$. The poset $\overline{\mathcal{F}} \subseteq 2^{[n]}$ is induced $\overline{P}$-saturated and therefore we can apply Lemma 5.2.7 to both $\mathcal{F}$ and $\overline{\mathcal{F}}$ to obtain injective functions f and $\overline{f}$ respectively. Since both P and $\overline{P}$ have legs, it is easy to see that the hip of P is not a maximal element of P. It follows that $f(i) \neq [n]$ for every $i \in [n]$. That is, $f([n]) \subseteq \mathcal{F} \setminus \{\emptyset, [n]\}$; similarly $\overline{f}([n]) \subseteq \overline{\mathcal{F}} \setminus \{\emptyset, [n]\}$.

By taking $[n] \setminus \overline{f}(i)$ for every $i \in [n]$ we define a new injective function $\tilde{f}$ from $[n]$

to $\mathcal{F} \setminus \{\emptyset, [n]\}$:

$$\tilde{f}(i) := \begin{cases} [n] \setminus \{i\} & \text{if } \{i\} \in \overline{\mathcal{F}}, \\ H(i) & \text{if } \{i\} \notin \overline{\mathcal{F}}, \text{ where } [n] \setminus H(i) \text{ is the hip of an induced } \overline{P} \text{ in } \overline{\mathcal{F}} \cup \{\{i\}\}. \end{cases}$$

Now, since f and $\tilde{f}$ are injective, proving that $f([n]) \cap \tilde{f}([n]) = \emptyset$ yields that $|\mathcal{F}| \geq 2n + 2$ by recalling that the sets $\emptyset$ and $[n]$ belong to $\mathcal{F}$.

Suppose $f(i) = \tilde{f}(j)$ for $i, j \in [n]$. If $\{i\} \in \mathcal{F}$ or $[n] \setminus \{j\} \in \mathcal{F}$ then it is easy to check that such an identity is not possible. Hence, we have that $f(i) = \tilde{f}(j) = H$ where: H is the hip of an induced copy P_1 of P in $\mathcal{F} \cup \{\{i\}\}$ and $[n] \setminus H$ is the hip of an induced copy $\overline{P}_2$ of $\overline{P}$ in $\overline{\mathcal{F}} \cup \{\{j\}\}$. In particular, this means that there is an induced copy P_2 of P in $\mathcal{F} \cup \{[n] \setminus \{j\}\}$ where $[n] \setminus \{j\}$ plays the role in P_2 of one of the two maximal elements of P. Furthermore, H is in P_2 and dominates all elements of P_2 except for the two maximal elements.

Let L^1 be the leg of P_1 other than $\{i\}$. Let $L^2_1, L^2_2 \in \mathcal{F}$ be the legs of P_2; in particular, L^2_1 and L^2_2 are incomparable and $L^2_1, L^2_2 \subsetneq H$. This implies that for every $A \in P_1 \setminus \{L^1, \{i\}\}$, we have $L^2_1, L^2_2 \subsetneq A$. Thus, $(P_1 \cup \{L^2_1, L^2_2\}) \setminus \{L^1, \{i\}\}$ is an induced copy of P in $\mathcal{F}$, a contradiction. $\square$

Theorem 5.1.2 now follows by a simple application of Lemma 5.2.7 and Theorem 5.1.1, together with two upper bound constructions.

Proof of Theorem 5.1.2. Observe that $\text{sat}^*(n, Y) = \text{sat}^*(n, \curlywedge)$ where $\curlywedge := \overline{Y}$. Let $\mathcal{F} \subseteq 2^{[n]}$ be an induced $\curlywedge$-saturated family. Since $\curlywedge$ has legs, Lemma 5.2.7 yields an injective function f from $[n]$ to $\mathcal{F} \setminus \{\emptyset\}$ with $f(i) = \{i\}$ if $\{i\} \in \mathcal{F}$ and $f(i) = H$ otherwise, where H is the hip of an induced copy of $\curlywedge$ in $\mathcal{F} \cup \{\{i\}\}$. This already implies $|\mathcal{F}| \geq n+1$ as $\emptyset \in \mathcal{F}$.

Observe that $f(i) \neq [n]$ for every $i \in [n]$, therefore, if $[n] \in \mathcal{F}$ then $|\mathcal{F}| \geq n + 2$, as desired. Hence, assume that $[n] \notin \mathcal{F}$, which means that $\mathcal{F} \cup \{[n]\}$ contains an induced copy of $\curlywedge$ that uses $[n]$. Let L_1, L_2 be the legs of this copy of $\curlywedge$ and H be the hip. Assume there is an $i \in [n]$ such that $f(i) = H$; so H is the hip of an induced copy of $\curlywedge$

contained in $\mathcal{F} \cup \{\{i\}\}$. Let M be the maximal element of this copy of λ and observe that $\{L_1, L_2, H, M\}$ forms an induced copy of λ in $\mathcal{F}$, a contradiction. Hence, $f(i) \neq H$ for every $i \in [n]$, which means that H was not counted before and therefore $|\mathcal{F}| \geq n+2$. This argument implies that $\mathrm{sat}^*(n, Y) \geq n+2$.

For the poset X observe that X and $\overline{X} = X$ have legs, therefore Theorem 5.1.1 directly implies that $\mathrm{sat}^*(n, X) \geq 2n+2$. For the upper bounds, consider the posets

$$P := \{F \in 2^{[n]} \colon |F| \geq n-1 \text{ or } F = \emptyset\} \quad \text{and} \quad Q := \{F \in 2^{[n]} \colon |F| \geq n-1 \text{ or } |F| \leq 1\},$$

and notice that they are respectively induced Y-saturated and induced X-saturated. Furthermore, $|P| = n+2$ and $|Q| = 2n+2$. □

We conclude this subsection by exhibiting a class of posets for which the bound in Theorem 5.1.1 is sharp up to an additive constant.

For any $\ell \in \mathbb{N}$, let $\wedge_\ell$ denote the poset with the following properties:

- $\wedge_\ell$ has legs L_1, L_2 and hip H_1;
- $\wedge_\ell \setminus \{L_1, L_2\} = \{H_1, \dots, H_\ell\}$ where $H_j \subsetneq H_{j+1}$ for every $j \in [\ell-1]$.

Similarly, for any $\ell \in \mathbb{N}$, let X_ℓ denote the poset with the following properties:

- X_ℓ has legs L_1', L_2' and $X_\ell \setminus \{L_1', L_2'\} = \vee_\ell$ where $\vee_\ell := \overline{\wedge_\ell}$.

Clearly $X_1 = X$ and $\wedge_2 = \lambda$. Moreover, $\wedge_\ell, X_\ell$ and $\overline{X}_\ell$ have legs for every $\ell \in \mathbb{N}$, and therefore Theorem 5.1.1 implies that $\mathrm{sat}^*(n, \wedge_\ell) \geq n+1$ and $\mathrm{sat}^*(n, X_\ell) \geq 2n+2$. The next proposition states that these bounds are close to the exact values of $\mathrm{sat}^*(n, \wedge_\ell)$ and $\mathrm{sat}^*(n, X_\ell)$.

Proposition 5.2.9. *For all integers $n-1 > \ell \geq 2$ we have,*[11]

$$\begin{aligned} n+1 \leq \mathit{sat}^*(n, \wedge_{\ell+1}) &\leq n + 2^{\ell+1} - \ell - 1 \ \mathit{and} \\ 2n+2 \leq \mathit{sat}^*(n, X_\ell) &\leq 2n + 2^{\ell+1} - 2\ell. \end{aligned}$$

[11]Note that the case $\ell = 1$ is covered by Theorem 5.1.2 since $\wedge_2 = \lambda$ and $X_1 = X$.

Proof. First, we consider $\wedge_{\ell+1}$. Let $\mathcal{F} \subseteq 2^{[n]}$ be the family containing precisely the following sets:

- the empty set $\emptyset$;
- $\{i\}$ for every $i \in [n]$;
- all subsets of $\{1, 2, \ldots, \ell\}$;
- all proper supersets of $[n] \setminus \{1, 2, \ldots, \ell\}$.

It is straightforward to check that $\mathcal{F}$ has $n + 2^{\ell+1} - \ell - 1$ elements and is induced $\wedge_{\ell+1}$-saturated. Similarly, the family $\mathcal{F}' := \mathcal{F} \cup \overline{\mathcal{F}} \subseteq 2^{[n]}$ has $2n + 2^{\ell+1} - 2\ell$ elements and is induced X_ℓ-saturated. □

5.3 Further remarks and open problems

The following example shows that the bound in Lemma 5.2.2 is essentially tight.

Example 5.3.1. *Let $n \in \mathbb{N}$ be a perfect square, i.e., $\sqrt{n} \in \mathbb{N}$. Let $A_s, B_t \subseteq [n]$ be defined as follows.*

- $A_s := \{s\sqrt{n} + 1, s\sqrt{n} + 2, \ldots, s\sqrt{n} + \sqrt{n}\}$ *for every* $s \in [\sqrt{n} - 1] \cup \{0\}$*;*
- $B_t := [n] \setminus \{t, t + \sqrt{n}, t + 2\sqrt{n}, \ldots, t + (\sqrt{n} - 1)\sqrt{n}\}$ *for every* $t \in [\sqrt{n}]$*.*

Let $\mathcal{F}^\star := \{A_s : s \in [\sqrt{n} - 1] \cup \{0\}\} \cup \{B_t : t \in [\sqrt{n}]\} \subseteq 2^{[n]}$.

Observe that $\mathcal{F}^\star$ has $2\sqrt{n}$ elements. Furthermore, for every $i \in [n]$, there exists exactly one pair of elements $A, B \in \mathcal{F}^\star$ such that $A \setminus B = \{i\}$. Namely, if $i = s\sqrt{n} + t$ where $s \in [\sqrt{n} - 1] \cup \{0\}$ and $t \in [\sqrt{n}]$ then $A_s \setminus B_t = \{i\}$.

Note that the digraph D with $V(D) = \mathcal{F}^\star$ and edge set $E(D) = \{\overrightarrow{AB} : A \setminus B = \{i\}, i \in [n]\}$ is precisely the balanced oriented bipartite graph (i.e., an extremal example for Theorem 5.1.3).

This example shows that if one can improve the lower bound in Theorem 1.4.4 by using the auxiliary digraph approach, then one will really need to use the fact that the digraph is generated by an induced P-saturated family (recall this was not assumed in the statement of Lemma 5.2.2). On the other hand, if Theorem 1.4.4 is close to being best possible, then Example 5.3.1 points in the direction of potential extremal examples. That is, is there some poset P such that there is a minimum induced P-saturated family $\mathcal{F} \subsetneq 2^{[n]}$ that is 'close' to the family $\mathcal{F}^\star$ in Example 5.3.1?

Another natural question is to characterise those posets P for which $\mathrm{sat}^*(n, P)$ is bounded by a constant. Observe that Lemma 5.2.3 provides a method for determining such posets. That is, if one can exhibit an $n_0 \in \mathbb{N}$ and an induced P-saturated family $\mathcal{F} \subsetneq 2^{[n_0]}$ such that for some $i \in [n_0]$ there are no elements $A, B \in \mathcal{F}$ with $A \setminus B = \{i\}$, then $\mathrm{sat}^*(n, P) = O(1)$.

Along the lines of this research direction, Keszegh, Lemons, Martin, Pálvölgyi and Patkós [85] conjectured the following. Given a poset P on $[p]$, let $\dot{P}$ denote the poset on $[p+1]$ where $\dot{P} := P \cup \{[p+1]\}$ (i.e., $\dot{P}$ is obtained by adding an element to P which dominates all elements in P).

Conjecture 5.3.2. *[85] sat$^*(n, P) = O(1)$ if and only if sat$^*(n, \dot{P}) = O(1)$.*

Note that neither direction of Conjecture 5.3.2 has been verified except for some special cases (see [85, Theorem 3.6]). It would be interesting to investigate if Lemma 5.2.3 can help tackle Conjecture 5.3.2.

Another possible research direction is to consider induced saturation problems for families of posets. Given a family of posets $\mathcal{P}$, we say that $\mathcal{F} \subsetneq 2^{[n]}$ is *induced $\mathcal{P}$-saturated* if $\mathcal{F}$ contains no induced copy of any poset $P \in \mathcal{P}$ and for every $S \in 2^{[n]} \setminus \mathcal{F}$ there exists an induced copy of some poset $P \in \mathcal{P}$ in $\mathcal{F} \cup \{S\}$. We denote the size of the smallest such family by $\mathrm{sat}^*(n, \mathcal{P})$. By following the proof of Theorem 1.4.4 precisely, one obtains the following result.

Theorem 5.3.3. *For any family of posets $\mathcal{P}$, either there exists a constant $K_{\mathcal{P}}$ with $sat^*(n, \mathcal{P}) \leq K_{\mathcal{P}}$ or $sat^*(n, \mathcal{P}) \geq \min\{2\sqrt{n}, n/2 + 1\}$, for all $n \in \mathbb{N}$.*

In light of Theorem 5.3.3 it is natural to ask whether an analogue of Conjecture 1.4.2 is true in this more general setting, or whether (for example) the lower bound on $\mathrm{sat}^*(n, \mathcal{P})$ in Theorem 5.3.3 is best possible up to a multiplicative constant.

Finally, we remark that the problem of determining good upper bounds for $\mathrm{sat}^*(n, P)$ is also very natural. In this direction, Bastide, Groenland, Ivan and Johnston [13] proved the following polynomial upper bound for $\mathrm{sat}^*(n, P)$.

Theorem 5.3.4 ([13])**.** *Let P be a finite poset. Then $sat^*(n, P) = O(n^c)$, where $c \leq |P|^2/4 + 1$ is a constant depending on P only.*

It is an intriguing open problem whether this bound can be improved further. The authors proposed a conjecture (see [13, Conjecture 9]) which would imply $\mathrm{sat}^*(n, P) \leq O(n^{|P|-1})$ for every poset P. In [85] the question was raised whether $\mathrm{sat}^*(n, P) = O(n)$ for every poset P. Note that an affirmative answer to this question together with a proof of Conjecture 1.4.2 would imply that, for any fixed poset P , the function $\mathrm{sat}^*(n, P)$ is either bounded or linear in n.

Chapter 6

Typical Ramsey properties of abelian groups

6.1 Introduction

As anticipated in Section 1.5, the main goal of this chapter is to establish a framework to study random Ramsey-type questions for subsets of abelian groups. Our main result, which we refer to as the random Rado lemma (Lemma 6.7.8), is a black box that on input of:

- a homogeneous system of linear equations $\mathcal{L}$ with integer coefficients,
- a sequence of 'well-behaved' subsets S_n of abelian groups,
- a Ramsey-type supersaturation result for the number of monochromatic solutions of $\mathcal{L}$ in S_n,

outputs an appropriate 1-statement and 0-statement. In order to rigorously state the random Rado lemma, we need to introduce various terminology as well as recall some fundamental algebraic concepts. As such, we defer the statement of Lemma 6.7.8 to Section 6.7.

6.2 Organisation of the chapter

One of the most significant applications of our theory is to arithmetic progressions in the primes; we present our main contribution to this topic (Theorem 6.3.7) in Section 6.3. Sections 6.4 and 6.5 serve as a gentle introduction to some of the key results in the area. Namely, in Section 6.4 we briefly cover some of the classical themes in Ramsey theory, specifically partition regularity and supersaturation; in Section 6.5 we discuss the random Ramsey problem for graphs and the integers. In Section 6.6, we highlight the power of the random Rado lemma by presenting various novel results which we obtain from this black box. In Sections 6.7 and 6.8, we formally state the random Rado lemma and our general 1- and 0-statements for hypergraphs, respectively. Note that Figure 6.1 illustrates how these black box results are connected, and some of their implications.

The rest of the chapter is devoted to the technical proofs of our results. In Section 6.9, we prove the 1- and 0-statements for hypergraphs and in Section 6.10 we explain how to deduce the random Rado lemma from them. In Section 6.12, we give the proofs of all the applications of the random Rado lemma that appear in Sections 6.3 and 6.6. For these proofs, we apply a couple of new results that may be of independent interest: (i) a Ramsey-type supersaturation result for $[n]^d$ (Theorem 6.4.9) and (ii) an upper bound on the number of k-term arithmetic progressions in the primes that contain a given prime number (Lemma 6.3.3). The former generalises a well-known supersaturation result of Frankl, Graham and Rödl [54]. We prove these two results in Section 6.11.

In Section 6.13, we explain how, with the random Rado lemma at hand, one can attack further related problems. Indeed, we show further consequences of the method as well as discuss potential future research directions.

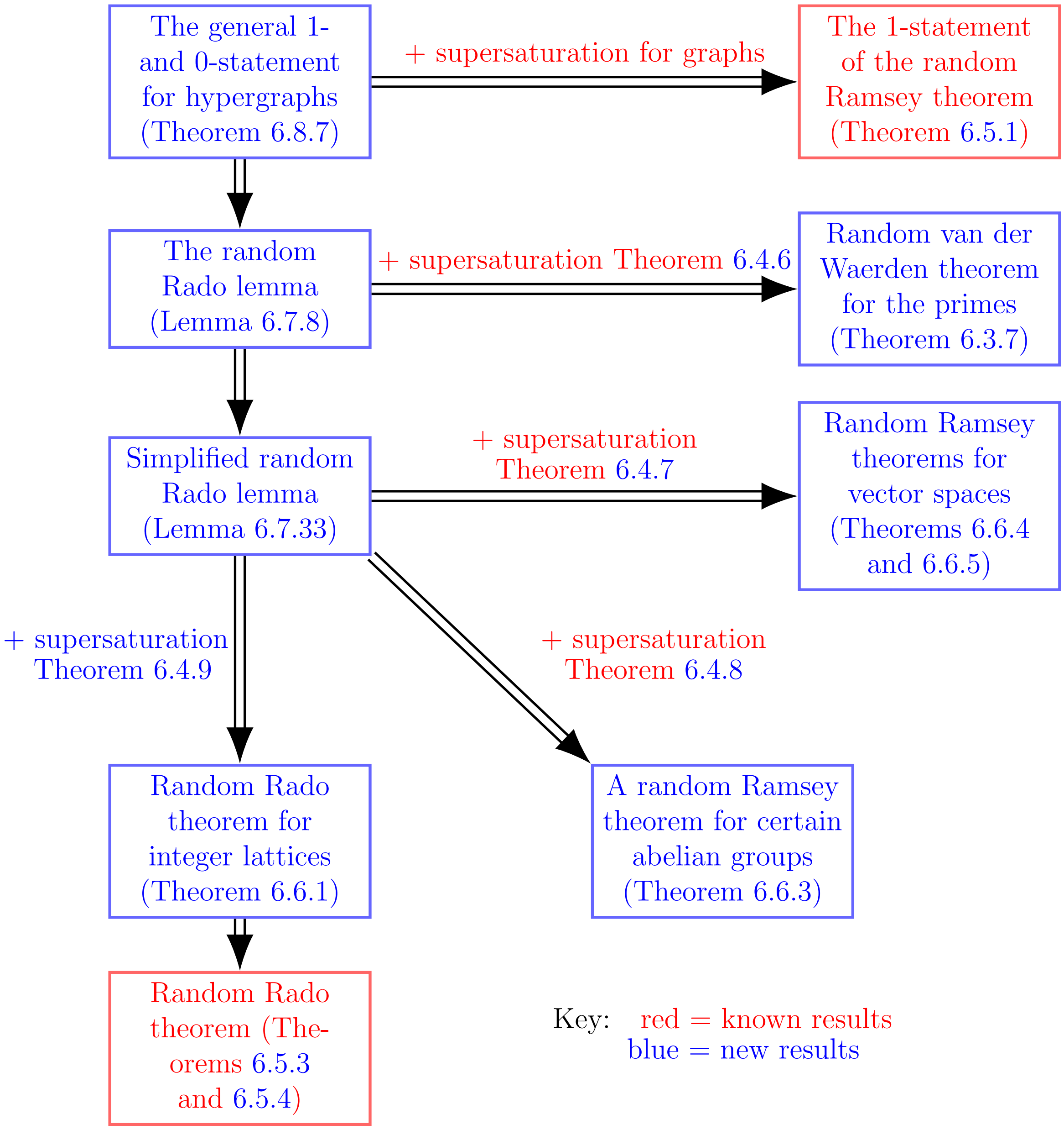

Figure 6.1: A visualisation of the main new (blue) and previously known (red) results covered in this chapter.

6.3 Arithmetic progressions in the primes

In this section, we highlight an application of the random Rado lemma to arithmetic progressions in the primes. We defer stating our other applications of the random Rado lemma until Section 6.6 as they require the introduction of a little more notation and a few definitions.

6.3.1 The Green–Tao theorem

One of the most celebrated results of the 21st Century is the following theorem of Green and Tao [71]. Here we write $\mathbf{P}$ for the set of prime numbers and, given $n \in \mathbb{N}$, we write $\mathbf{P}_n$ for the set of all prime numbers less than or equal to n. Recall the prime number theorem (see e.g., [67]) asserts that $|\mathbf{P}_n| \sim n/\log n$, where throughout this chapter we write log for the natural logarithm. We abbreviate k-term arithmetic progressions by k-APs.

Theorem 6.3.1 (Green and Tao [71])**.** *Let $k \geq 2$. The set of primes $\mathbf{P}$ contains arithmetic progressions of arbitrary length. Moreover, given any subset $S \subseteq \mathbf{P}_n$ of size $|S| = \Omega(|\mathbf{P}_n|)$, the number of k-APs in S is $\Theta(n^2/\log^k n)$.*[12]

The Green–Tao theorem therefore tells us that any linear size subset of $\mathbf{P}_n$ is rich in arithmetic progressions, for any large $n \in \mathbb{N}$. It is natural to ask about sparser subsets of $\mathbf{P}_n$: given some $p = p(n)$, does a typical subset S of $\mathbf{P}_n$ of size approximately $p|\mathbf{P}_n|$ contain k-APs? Let $\mathbf{P}_{n,p}$ denote the probability distribution of a random subset of $\mathbf{P}_n$ obtained by including each element independently with probability p. We may abuse notation and treat $\mathbf{P}_{n,p}$ as an actual subset of $\mathbf{P}_n$. The following new result determines the probability threshold for $\mathbf{P}_{n,p}$ containing a k-AP.

Theorem 6.3.2 (Random Green–Tao theorem)**.** *Let $k \geq 3$.*

$$\lim_{n\to\infty} \mathbb{P}[\mathbf{P}_{n,p} \text{ contains a } k\text{-AP}] = \begin{cases} 0 & \text{if } p = o(n^{-2/k}\log n), \\ 1 & \text{if } p = \omega(n^{-2/k}\log n). \end{cases}$$

Theorem 6.3.2 tells us that typical subsets of $\mathbf{P}_n$ whose size is significantly above $|\mathbf{P}_n|n^{-2/k}\log n \sim n^{(k-2)/k}$ contain k-APs; conversely, typical subsets of $\mathbf{P}_n$ of size significantly smaller than $n^{(k-2)/k}$ do not. Note that, since Theorem 6.3.2 is not a Ramsey-type result, we do not use the random Rado lemma to prove it. Instead, we make use of the following result.

[12]Note that the moreover part of Theorem 6.3.1 is implicit in [71]. An explicit proof of (a strengthening of) this part of Theorem 6.3.1 can be found in, e.g., [29, Theorem 1.2].

Lemma 6.3.3. *Let $k \in \mathbb{N}$ with $k \geq 2$ and let $\ell \in [k]$. For every prime $q \in \mathbf{P}_n$, the number of k-APs in $\mathbf{P}_n$ such that the ℓth term is equal to q is $O(n/\log^{k-1} n)$.*

We prove Lemma 6.3.3 in Section 6.11.1 via a sieve theory argument. With Lemma 6.3.3 and Theorem 6.3.1 at hand, the proof of Theorem 6.3.2 is just a simple application of the second moment method. For completeness, a full proof is given in Section A.2 of Appendix A.

Given $k \geq 3$ and $\delta > 0$, we say a finite set $X \subseteq \mathbb{N}$ is (δ, k)-*Szemerédi* if every subset of X of size at least $\delta|X|$ contains a k-AP. The following result tells us that not too sparse subsets of $\mathbf{P}_n$ are typically (δ, k)-Szemerédi.

Theorem 6.3.4 (Balogh, Liu and Sharifzadeh [9])**.** *For any $\delta, \gamma > 0$ and $k \geq 3$, if $p \geq n^{-\frac{1}{k-1}+\gamma}$ then*

$$\lim_{n\to\infty} \mathbb{P}[\,\mathbf{P}_{n,p} \textit{ is } (\delta, k)\textit{-Szemerédi}] = 1.$$

Note that the bound on p in Theorem 6.3.4 is significantly higher than the corresponding bound in the 1-statement of Theorem 6.3.2. Further, Theorem 6.3.4 is tight up to the γ term. Indeed, it is not hard to see that if $p = o(n^{-\frac{1}{k-1}} \log n)$ then

$$\lim_{n\to\infty} \mathbb{P}[\,\mathbf{P}_{n,p} \text{ is } (\delta, k)\text{-Szemerédi}] = 0.$$

This follows as for such p (suitably bounded away from 0), the expected number of k-APs is much less than the expected number of elements in $\mathbf{P}_{n,p}$. Hence, with high probability, greedily deleting a number from each k-AP produces a linear size subset of $\mathbf{P}_{n,p}$ containing no k-AP. In Section 6.13.2 we give a significant strengthening of Theorem 6.3.4; see Theorem 6.13.5.

6.3.2 Monochromatic arithmetic progressions in the primes

The following seminal result is one of the first proven in Ramsey theory.

Theorem 6.3.5 (van der Waerden [134])**.** *Any finite colouring of* $\mathbb{N}$ *yields a monochromatic arithmetic progression of arbitrary length.*

One beautiful consequence of the Green–Tao theorem is the following generalisation of van der Waerden's theorem.[13]

Theorem 6.3.6 (van der Waerden theorem for the primes [71])**.** *Any finite colouring of* $\mathbf{P}$ *yields a monochromatic arithmetic progression of arbitrary length.*

Given $k \geq 3$ and $r \geq 2$, we say a subset $X \subseteq \mathbb{N}$ is (r,k)*-van der Waerden* if whenever X is r-coloured there is a monochromatic k-AP. One of the main applications of our machinery is the following random analogue of Theorem 6.3.6.

Theorem 6.3.7 (Random van der Waerden theorem for the primes)**.** *Let* $k \geq 3$ *and* $r \geq 2$*. There are constants* $c, C > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathbf{P}_{n,p} \textit{ is } (r,k)\textit{-van der Waerden}] = \begin{cases} 0 & \textit{if } p \leq cn^{-\frac{1}{k-1}} \log n, \\ 1 & \textit{if } p \geq Cn^{-\frac{1}{k-1}} \log n. \end{cases}$$

Theorem 6.3.7 tells us that typical subsets of $\mathbf{P}_n$ whose size is significantly above $|\mathbf{P}_n| n^{-\frac{1}{k-1}} \log n \sim n^{\frac{k-2}{k-1}}$ are (r,k)-van der Waerden whilst typical subsets of $\mathbf{P}_n$ of size significantly smaller than $n^{\frac{k-2}{k-1}}$ are not.

Theorem 6.3.7 follows from the random Rado lemma combined with Theorem 6.3.1 and Lemma 6.3.3. The proof is postponed to Section 6.12.

[13] In fact, the Green–Tao theorem implies a supersaturation version of Theorem 6.3.6, see Theorem 6.4.6.

6.4 Partition regularity and supersaturation

In this section, we give a brief survey of Ramsey-type results for homogeneous systems of linear equations. Partition regularity is covered in the first subsection while supersaturation, including a new result for integer lattices, is covered in the second.

6.4.1 Partition regularity

An integer matrix A is called *partition regular* if every finite colouring of $\mathbb{N}$ yields a monochromatic solution to $Ax = 0$. A classical result of Rado [110] characterises all those matrices that are partition regular. In order to state Rado's theorem, and other results in this section, we require the following definition.

Definition 6.4.1 (The columns condition over R)**.** Let R be a commutative ring and A be a matrix with entries in R. We say A satisfies the *columns condition over R* if there exists a partition $C_0, C_1, ..., C_m$ of the columns of A such that the following holds. Let c_j denote the jth column of A and set

$$s_i := \sum_{c_j \in C_i} c_j$$

for every $0 \leq i \leq m$. Then

- $s_0 = 0$ and
- s_i can be written as a linear combination of the elements $\{c_j : c_j \in C_k,\ k < i\}$ over R, for every $1 \leq i \leq m$.

Theorem 6.4.2 (Rado [110])**.** *An integer matrix A is partition regular if and only if A satisfies the columns condition over $\mathbb{Q}$.*

There have been several works investigating the analogous problem for other algebraic structures. In general, given a ring R acting on an abelian group G, a subset $X \subseteq G$ and $r \in \mathbb{N}$, we say an $\ell \times k$ matrix A with entries in R is *r-partition regular in X* if every

r-colouring of X yields a monochromatic solution to $Ax = 0$, where 0 denotes the identity of G^ℓ. Furthermore, A is *partition regular in* X if it is r-partition regular in X for every $r \in \mathbb{N}$. Bergelson, Deuber, Hindman and Lefmann [16] gave some sufficient conditions for partition regularity in the case where $G = R$ and R is a commutative ring that acts on itself via ring multiplication. In a recent paper, Byszewski and Krawczyk [23] proved several interesting extensions of Rado's theorem for integral domains, noetherian rings and modules.

In this chapter, we are interested in monochromatic solutions within abelian groups. We mainly focus on matrices with integer entries since there is an obvious action from $\mathbb{Z}$ to abelian groups: given an arbitrary abelian group $(G, +)$, for $g \in G$ and $n \in \mathbb{N}$ we write ng to denote $g + \cdots + g$. If n is a negative integer, we write ng to denote the inverse of $|n|g$ in G. For the rest of the chapter, $\mathbb{Z}$ will act on abelian groups as we just described.

Given any abelian group G, Deuber [37] provided a necessary and sufficient condition for an integer matrix A to be partition regular in $G \setminus \{0\}$. For *finite* G though it is not interesting as it simply states that A is partition regular in $G \setminus \{0\}$ if and only if there exists $x \in G \setminus \{0\}$ such that $A(x, \ldots, x)^T = 0$, i.e., there is a *trivial* solution. Indeed, if the number of colours r is larger than $|G \setminus \{0\}|$ then colouring every element of $G \setminus \{0\}$ differently prevents any monochromatic solution other than trivial ones.

Therefore, in order to ensure a non-trivial monochromatic solution, one must insist that the size of the finite arithmetic structure is large compared to the number of colours. In this direction, Bergelson, Deuber and Hindman [15] proved the following analogue of Rado's theorem for finite vector spaces.

Theorem 6.4.3 (Bergelson, Deuber and Hindman [15]). *Let $\mathbb{F}$ be a finite field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. The following statements are equivalent:*

- *for all $r \in \mathbb{N}$, there exists $n_0 = n_0(r, |\mathbb{F}|, k) \in \mathbb{N}$ such that for all $n \geq n_0$ every r-colouring of $\mathbb{F}^n \setminus \{0\}$ yields a monochromatic solution to $Ax = 0$ in $\mathbb{F}^n \setminus \{0\}$;*
- *the matrix A is partition regular in $\mathbb{F}^{\mathbb{N}} \setminus \{0\}$;*

- *the matrix A satisfies the columns condition over $\mathbb{F}$.*

Note that Rado's theorem itself may be restated in a style similar to Theorem 6.4.3 by using the following well-known fact.

Fact 6.4.4. *For an integer matrix A, the following statements are equivalent:*

(A) A is partition regular;

(B) for every $r \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ every r-colouring of $[n]$ yields a monochromatic solution to $Ax = 0$ in $[n]$. □

Statement (B) trivially implies statement (A) while the converse can be proved using a standard compactness argument.

6.4.2 Supersaturation

As mentioned in the introduction, the topic of Ramsey multiplicity concerns how many copies of a given structure one can guarantee within a partition of a mathematical object. We are interested in a particular class of results in this area which we refer to as *supersaturation* results. In the context of this chapter, a supersaturation result essentially states that within any finite colouring of an arithmetic structure, a *constant proportion* of the total number of solutions of a system of linear equations are monochromatic. As a clarifying example, consider the following theorem of Frankl, Graham and Rödl [54].

Theorem 6.4.5 (Frankl, Graham and Rödl [54]). *Let $r \in \mathbb{N}$ and let A be a partition regular $\ell \times k$ integer matrix of rank ℓ. There exist $\delta, n_0 > 0$ such that the following holds: for all $n \geq n_0$, every r-colouring of $[n]$ yields at least $\delta n^{k-\ell}$ monochromatic solutions to $Ax = 0$ in $[n]$.*

It is easy to see that there are at most $n^{k-\ell}$ solutions to $Ax = 0$ in $[n]$ (see, e.g., case (ii) of Lemma 6.7.16). Thus, Theorem 6.4.5 states that in any r-colouring of $[n]$, a constant proportion of the total number of solutions are monochromatic.

Our motivation for considering supersaturation results is that they are a prerequisite of various well-known techniques used to prove random Ramsey-type statements. Specifically, supersaturation results can be used in combination with the hypergraph container method to give an upper bound on the probability that a Ramsey property is not satisfied, thereby yielding a proof for a 1-statement. See Section 6.9.1.1 for a more detailed sketch of how this calculation works. Note that supersaturation results were employed to prove 1-statements of Ramsey-type results even before the introduction of the hypergraph container method. In particular, in [62], Theorem 6.4.5 was used in the proof of the 1-statement of the random Rado theorem (Theorem 6.5.4 in the next section).

As we will see in Section 6.7, supersaturation is one of the key conditions to check in the random Rado lemma (Lemma 6.7.8). In some cases, supersaturation can be easily obtained from existing results. For example, Theorem 6.3.1 immediately implies the following supersaturation result for the primes.

Theorem 6.4.6. *Given any $r, k \in \mathbb{N}$ with $k \geq 3$, any r-colouring of $\mathbf{P}_n$ yields $\Theta(n^2/\log^k n)$ monochromatic k-APs in $\mathbf{P}_n$.* □

One can similarly obtain a supersaturation result for finite abelian groups and translation-invariant matrices. We say an $\ell \times k$ integer matrix A is *translation-invariant* with respect to a group G if $A(g, \dots, g)^T = 0$ for every $g \in G$.[14] Consider such a translation-invariant matrix A with respect to an abelian group G. As observed by Saad and Wolf [119, page 2], a generalised version of Szemerédi's theorem for finite abelian groups [70] yields that for any linear sized subset S of G, provided $|G|$ is sufficiently large, a constant proportion of all solutions to $Ax = 0$ in G are fully contained in S. This immediately implies that, given any $r \in \mathbb{N}$, any r-colouring of G is such that a constant proportion of all solutions to $Ax = 0$ in G are monochromatic, provided $|G|$ is sufficiently large.

We finish this section by highlighting three further supersaturation results that we require for our applications of the random Rado lemma appearing in Section 6.6. First,

[14] For example, any integer matrix A whose columns sum to 0 is translation-invariant with respect to any abelian group. This includes integer matrices for which the solutions to $Ax = 0$ in $\mathbb{Z}$ correspond to k-APs, for fixed $k \geq 3$.

Serra and Vena [124, Theorem 2.1] proved an analogue of Theorem 6.4.5 for finite vector spaces. This can be seen as a quantitative analogue of Theorem 6.4.3.

Theorem 6.4.7 (Serra and Vena [124])**.** *Let $r \in \mathbb{N}$, let $\mathbb{F}$ be a finite field and let A be an $\ell \times k$ matrix with entries in $\mathbb{F}$ and rank ℓ which satisfies the columns condition over $\mathbb{F}$. There exist $\delta, n_0 > 0$ such that, for all $n \geq n_0$, every r-colouring of $\mathbb{F}^n \setminus \{0\}$ yields at least $\delta|\mathbb{F}|^{n(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $\mathbb{F}^n \setminus \{0\}$.*

One can show that there are at most $|\mathbb{F}|^{n(k-\ell)}$ solutions to $Ax = 0$ in $\mathbb{F}^n$ and so the lower bound in Theorem 6.4.7 is linear in the total number of solutions.

The works of Serra and Vena [124, Theorem 1.3] and Vena [135] extend Theorem 6.4.7 to finite abelian groups. To state (a special case of) their result, we need a couple of definitions. We say an abelian group $(G, +)$ has *exponent* s if s is the smallest positive integer such that $sg = g + \cdots + g = 0$ for every $g \in G$. We say an integer matrix A satisfies the *s-columns condition* if A satisfies the columns condition over the ring $\mathbb{Z}_s$ (where the entries of A are treated as elements of $\mathbb{Z}_s$, i.e., residues modulo s). Finally, $S(A, G)$ denotes the set of solutions to $Ax = 0$ in G.

Theorem 6.4.8 (Serra and Vena [124], Vena [135])**.** *Let $r \in \mathbb{N}$, let G be a finite abelian group with exponent s and let A be an $\ell \times k$ integer matrix which satisfies the s-columns condition. There exist $\delta, n_0 > 0$ such that, for all $n \geq n_0$, every r-colouring of $G^n \setminus \{0\}$ yields at least $\delta|S(A, G^n)|$ monochromatic solutions to $Ax = 0$ in $G^n \setminus \{0\}$.*

Note that in [124], a slightly weaker quantitative bound was obtained for Theorem 6.4.8. However, the result used a previous weaker removal lemma of Král', Serra and Vena [91]. Vena [135] observed that by using his own removal lemma, one can obtain the stronger bound as stated in Theorem 6.4.8.

We conclude this section with a new supersaturation result for integer lattices.

Theorem 6.4.9. *Let $r, d \in \mathbb{N}$ and let A be a partition regular $\ell \times k$ integer matrix of rank ℓ. There exist $\delta, n_0 > 0$ such that the following holds: for all $n \geq n_0$, every r-colouring of $[n]^d$ yields at least $\delta n^{d(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[n]^d$.*

Again, the lower bound in Theorem 6.4.9 is a constant proportion of the total number of solutions. Observe that the case $d = 1$ of Theorem 6.4.9 is exactly Theorem 6.4.5. The proof of Theorem 6.4.9 is given in Section 6.11.

6.5 The random Ramsey problem for graphs and the integers

In this section we cover two settings where the random Ramsey problem has been extensively studied: edge-coloured graphs and the integers. Further background on these two problems can be found, e.g., in [26] and [76, Section 1]. We also discuss a few known random Ramsey-type results for $\mathbb{Z}_n$.

6.5.1 Ramsey properties of random graphs

The *random graph* model provides a framework for studying the typical Ramsey properties of graphs of a given density. The *random graph* $G_{n,p}$ has vertex set $[n]$ where each possible edge is present with probability p, independently of all other edges. An event occurs in $G_{n,p}$ *with high probability* (w.h.p.) if its probability tends to 1 as $n \to \infty$. For many properties $\mathcal{P}$ of $G_{n,p}$ (including Ramsey properties), the probability that $G_{n,p}$ has the property exhibits a phase transition, that is, there is a *threshold* for $\mathcal{P}$: a function $\hat{p} := \hat{p}(n)$ such that $G_{n,p}$ has $\mathcal{P}$ w.h.p. when $p = \omega(\hat{p})$ (the 1-*statement*), while $G_{n,p}$ does not have $\mathcal{P}$ w.h.p. when $p = o(\hat{p})$ (the 0-*statement*). Bollobás and Thomason [18] proved that every non-trivial *monotone* property $\mathcal{P}$ has a threshold.

Rödl and Ruciński [113, 114, 115] proved a general Ramsey-type result for the random graph. To state their result we need a few definitions. Recall that we write $V(H)$ and $E(H)$ for the vertex and edge sets of a graph H, respectively, and define $v(H) := |V(H)|$ and $e(H) := |E(H)|$. Given an $r \in \mathbb{N}$ and a graph G, an r-*edge-colouring* of G is a function $\sigma : E(G) \to [r]$. Given a graph H, we say that G is (H, r)-*Ramsey* if every r-edge-colouring of G yields a monochromatic copy of H in G. Given a graph H, set $d_2(H) := 0$ if $e(H) = 0$

and $d_2(H) := 1/2$ when H is precisely an edge. Define $d_2(H) := (e(H)-1)/(v(H)-2)$ otherwise. Then define $m_2(H) := \max_{H' \subseteq H} d_2(H')$ to be the 2-*density* of H. For example, if we consider the triangle K_3, then $m_2(K_3) = (e(K_3)-1)/(v(K_3)-2) = 2$. On the other hand, if we consider the disjoint union of K_3 and an edge, i.e., $H := K_3 \cup K_2$, then we also have that $m_2(H) = 2$, even though $d_2(H) = 1$, as the 'densest' part of H is the copy of K_3.

Theorem 6.5.1 (The random Ramsey theorem [113, 114, 115]). *Let $r \geq 2$ be a positive integer and let H be a graph that is not a forest consisting of stars and paths of length 3. There are positive constants c, C such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \text{ is } (H,r)\text{-Ramsey}] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(H)}, \\ 1 & \text{if } p \geq Cn^{-1/m_2(H)}. \end{cases}$$

Thus $n^{-1/m_2(H)}$ is the threshold for the (H,r)-Ramsey property. Roughly speaking, the random Ramsey theorem tells us that a typical n-vertex graph of edge density significantly greater than $n^{-1/m_2(H)}$ is (H,r)-Ramsey, whilst a typical n-vertex graph of edge density significantly less than $n^{-1/m_2(H)}$ is not (H,r)-Ramsey.

An intuition for the threshold in Theorem 6.5.1 is as follows: for small $c > 0$, if $p < cn^{-1/m_2(H)}$ then there is a subgraph H' of H for which the expected number of copies of H' in $G_{n,p}$ is much smaller than the expected number of edges in $G_{n,p}$. So intuitively this suggests the copies of H' (and therefore H) are 'spread out' in the random graph and so one may be able to colour the edges to avoid a monochromatic copy of H. On the other hand, for large $C > 0$, if $p > Cn^{-1/m_2(H)}$ then the expected number of copies of any subgraph H'' of H in $G_{n,p}$ is much larger than the expected number of edges; so in this sense, the copies of H in $G_{n,p}$ are more 'densely distributed' making it harder to avoid a monochromatic copy of H.

In more recent work, Nenadov and Steger [105] gave a short proof of Theorem 6.5.1 using the hypergraph container method. There has also been significant effort to generalise

Theorem 6.5.1 to the setting of *random hypergraphs* (see, e.g., [28, 62, 73]) and also to asymmetric Ramsey properties; that is, now the monochromatic graph one seeks can be different for each colour class (see, e.g., [19, 25, 76, 79, 86, 96, 97, 101, 104]).

6.5.2 Ramsey properties of random sets of integers

We start with some key definitions.

Definition 6.5.2 (k-distinct and (A,r)-Rado)**.** For an $\ell \times k$ integer matrix A and a set S of elements of an abelian group, if a vector $x = (x_1, \dots, x_k) \in S^k$ satisfies $Ax = 0$ and the x_is are pairwise distinct, we call x a *k-distinct solution* to $Ax = 0$ in S. We say that S is *(A,r)-Rado* if given any r-colouring of S, there is a monochromatic k-distinct solution $x = (x_1, \dots, x_k)$ to $Ax = 0$ in S.

In the study of random versions of Rado's theorem authors have (implicitly) considered the (A,r)-Rado property, rather than seeking a monochromatic solution that is not necessarily k-distinct (as in the original theorem of Rado); see [77, Section 4] for a discussion on why this has been the case.

An $\ell \times k$ integer matrix A is *irredundant* if there exists a k-distinct solution to $Ax = 0$ in $\mathbb{N}$. Otherwise, A is *redundant*. The study of random versions of Rado's theorem has focused on irredundant partition regular matrices. This is natural since for every redundant $\ell \times k$ matrix A for which $Ax = 0$ has solutions in $\mathbb{N}$, there exists an irredundant $\ell' \times k'$ matrix A' for some $\ell' \leq \ell$ and $k' \leq k$ with the same family of solutions (viewed as sets). See [116, Section 1] for a full explanation.

Index the columns of A by $[k]$. For a partition $W \dot\cup \overline{W} = [k]$ of the columns of A, we denote by $A_{\overline{W}}$ the matrix obtained from A by restricting to the columns indexed by $\overline{W}$. Let $\operatorname{rank}(A_{\overline{W}})$ be the rank of $A_{\overline{W}}$, where $\operatorname{rank}(A_{\overline{W}}) := 0$ for $\overline{W} = \emptyset$. We set

$$m(A) := \max_{\substack{W \dot\cup \overline{W} = [k] \\ |W| \geq 2}} \frac{|W| - 1}{|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \operatorname{rank}(A)}. \tag{6.1}$$

The definition of $m(A)$ was introduced in [116]. As noted there, the denominator of $m(A)$ is strictly positive provided that A is irredundant and partition regular, and so $m(A)$ is well-defined in this case. We say A is *strictly balanced* if the expression in (6.1) is maximised precisely when $W = [k]$. If A has precisely one row consisting of non-zero entries (i.e., it corresponds to a single linear equation) then A is strictly balanced and so

$$m(A) = \frac{k-1}{k-2}.$$

For example, if $Ax = 0$ corresponds to the equation $x + y = z$, then $m(A) = 2$.

We write $[n]_p$ for the random subset of $[n] = \{1, \dots, n\}$ where each element in $[n]$ is included with probability p independently of all other elements. Rödl and Ruciński [116] showed that $m(A)$ is an important parameter for determining whether $[n]_p$ is (A, r)-Rado or not. Indeed, one can view $m(A)$ as the arithmetic analogue of the 2-density $m_2(H)$ which we considered for the graph case in the previous subsection.

Theorem 6.5.3 (Rödl and Ruciński [116]). *For all irredundant partition regular full rank matrices A and all positive integers $r \geq 2$, there exists a constant $c > 0$ such that*

$$\lim_{n \to \infty} \mathbb{P}\left[[n]_p \textit{ is } (A, r)\textit{-Rado}\right] = 0 \quad \textit{if } p \leq cn^{-1/m(A)}.$$

Roughly speaking, Theorem 6.5.3 implies that a typical subset of $[n]$ with significantly fewer than $n^{1-1/m(A)}$ elements is not (A, r)-Rado for any irredundant partition regular matrix A. The following theorem of Friedgut, Rödl and Schacht [62] complements this result, implying that a typical subset of $[n]$ with significantly more than $n^{1-1/m(A)}$ elements is (A, r)-Rado.

Theorem 6.5.4 (Friedgut, Rödl and Schacht [62]). *For all irredundant partition regular full rank matrices A and all positive integers r, there exists a constant $C > 0$ such that*

$$\lim_{n \to \infty} \mathbb{P}\left[[n]_p \textit{ is } (A, r)\textit{-Rado}\right] = 1 \quad \textit{if } p \geq Cn^{-1/m(A)}.$$

So together, Theorems 6.5.3 and 6.5.4 form the *random Rado theorem* and show that the threshold for the property of being (A,r)-Rado is $p = n^{-1/m(A)}$. The intuition for this threshold is similar to that for the threshold in Theorem 6.5.1: in the case when A is strictly balanced, when $p = n^{-1/m(A)}$, in expectation there are roughly the same number of elements in $[n]_p$ as there are solutions to $Ax = 0$ in $[n]_p$. So if $p \leq cn^{-1/m(A)}$ for some small $c > 0$, then one would expect the solutions to $Ax = 0$ to be 'spread out' (i.e., one would expect most elements of $[n]_p$ lie in very few solutions to $Ax = 0$). Meanwhile, if $p \geq Cn^{-1/m(A)}$ for some large $C > 0$, then one would expect some clustering of solutions to $Ax = 0$, making it harder to avoid monochromatic solutions.

Note that Theorem 6.5.4 was confirmed earlier by Graham, Rödl and Ruciński [69] in the case where $r = 2$ and $Ax = 0$ corresponds to $x + y = z$, and then by Rödl and Ruciński [116] in the case when A is so-called density regular. We remark that both Theorem 6.5.3 and Theorem 6.5.4 are straightforward corollaries of the random Rado lemma (Lemma 6.7.8), which in turns follows from our general 1- and 0-statement for hypergraphs (Theorem 6.8.7). Furthermore, the 1-statement of the random Ramsey theorem (Theorem 6.5.1) also follows directly from the general 1-statement for hypergraphs (Theorem 6.8.7).

In [76, 127], Theorem 6.5.4 was generalised to allow one to consider matrices A that are not partition regular, but for which $\mathbb{N}$ is (A,r)-Rado for some choices of r. In particular, the condition that A is partition regular can be replaced with one of the following two equivalent definitions, subject to a suitable Ramsey-type supersaturation result holding. We say that an integer matrix A *satisfies property* $(*)$ if under Gaussian elimination A does not have any row which consists of precisely two non-zero rational entries. The definition of *abundant*, introduced in [121] and used in [127] is that, for an $\ell \times k$ matrix A, every $\ell \times (k-2)$ submatrix of A has the same rank as A. Clearly irredundant full rank matrices which satisfy $(*)$ are equivalent to irredundant full rank abundant matrices.

6.5.3 Ramsey properties of random sets of $\mathbb{Z}_n$

Recently there have been a few random Ramsey-type results proven for $\mathbb{Z}_n$. The first explicit result in this area is the following *sharp threshold* version of van der Waerden's theorem for random subsets of $\mathbb{Z}_n$ [60]. Recalling the definition of (r,k)-van der Waerden from earlier, note that, for example, $(r,3)$-van der Waerden and (A,r)-Rado are equivalent notions if $A=(1,1,-2)$. We write $\mathbb{Z}_{n,p}$ to denote the subset of $\mathbb{Z}_n$ obtained by including each element of $\mathbb{Z}_n$ independently and with probability p.

Theorem 6.5.5 (Friedgut, Hàn, Person and Schacht [60])**.** *For all $k \geq 3$, there exist constants $c_1, c_0 > 0$ and a function $c(n)$ satisfying $c_0 \leq c(n) \leq c_1$ such that for every $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{Z}_{n,p} \textit{ is } (2,k)\textit{-van der Waerden}] = \begin{cases} 0 & \textit{if } p \leq (1-\varepsilon)c(n)n^{-1/(k-1)}, \\ 1 & \textit{if } p \geq (1+\varepsilon)c(n)n^{-1/(k-1)}. \end{cases}$$

Note that the threshold in Theorem 6.5.5 is sharp compared to, e.g., the threshold given by Theorems 6.5.3 and 6.5.4. See, e.g., [26, Section 5.1] for further discussion on sharp thresholds.

Recently, Theorem 6.5.5 has been generalised to the (r,k)-van der Waerden property (i.e., r does not have to be 2 now) and an analogous sharp threshold version of the random Schur theorem for $\mathbb{Z}_n$ (where n is prime) has been obtained; see [61].

6.6 Further applications of the random Rado lemma

In addition to the random van der Waerden theorem for the primes (Theorem 6.3.7), we prove several other novel random Ramsey-type results via the random Rado lemma (Lemma 6.7.8). The proofs of all the results presented in this section are postponed to Section 6.12. Given a subset X of an abelian group, we write X_p to denote the subset of X obtained by including each element of X independently and with probability p.

Recall that Theorems 6.5.3 and 6.5.4 determine the threshold for the property that $[n]_p$ is (A, r)-Rado. The following result is a generalisation of this statement to $[n]_p^d$ for any fixed $d \in \mathbb{N}$.

Theorem 6.6.1 (Random Rado theorem for integer lattices)**.** *For all irredundant partition regular full rank matrices A and all positive integers $r \geq 2$ and $d \geq 1$, there exist constants $C, c > 0$ such that the following holds.*

$$\lim_{n\to\infty} \mathbb{P}[[n]_p^d \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq cn^{-d/m(A)}, \\ 1 & \text{if } p \geq Cn^{-d/m(A)}. \end{cases}$$

Observe that the $d = 1$ case corresponds precisely to Theorems 6.5.3 and 6.5.4. To deduce Theorem 6.6.1 from the random Rado lemma, we require a corresponding supersaturation result for $[n]^d$, namely Theorem 6.4.9.

Note that the probability threshold in Theorem 6.6.1 is independent of d in the following sense. As we are interested in how 'sparse' a subset can be while typically retaining its Ramsey properties, the probability threshold should always be compared to the size of the initial set we are considering. In this case, the size of $[n]^d$ is n^d and so we may rewrite the probability threshold $n^{-d/m(A)}$ as $|[n]^d|^{-1/m(A)}$. Indeed, this shows that the probability threshold depends exclusively on the matrix A. However, this phenomenon does not occur in general: for a fixed matrix A, the probability threshold may change depending on the underlying arithmetic structure. For instance, consider the equation $2x + 2y = 2z$ and the product group $(\mathbb{Z}_4)^n$.

Theorem 6.6.2. *Let $A = (2\ \ 2\ \ -2)$ and $r \geq 2$. There exist constants $c, C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[(\mathbb{Z}_4)_p^n \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c\,4^{-3n/4}, \\ 1 & \text{if } p \geq C\,4^{-3n/4}. \end{cases}$$

As the size of $(\mathbb{Z}_4)^n$ is 4^n, we may rewrite $4^{-3n/4}$ as $|(\mathbb{Z}_4)^n|^{-3/4}$. On the other hand,

$m(A) = 2$ and so the corresponding threshold when considering solutions to $Ax = 0$ in $[n]_p$ is $n^{-1/2}$. Thus, we can have different thresholds depending on the underlying abelian group.

Note that the probability threshold $p = 4^{-3n/4}$ is consistent with the heuristic that the expected size of $(\mathbb{Z}_4)^n_p$ should 'match' the expected number of solutions to $Ax = 0$ in $(\mathbb{Z}_4)^n_p$. Indeed, for $p = 4^{-3n/4}$, the expected size of $(\mathbb{Z}_4)^n_p$ is $4^n p = 4^{n/4}$. The total number of solutions of $2x + 2y = 2z$ in $(\mathbb{Z}_4)^n$ is $4^n \cdot 4^n \cdot 2^n = 4^{5n/2}$ and so the expected number of solutions in $(\mathbb{Z}_4)^n_p$ is $4^{5n/2}p^3 = 4^{n/4}$.

Observe that the total number of solutions to $2x+2y = 2z$ in $[n]$ is at most $n \cdot n \cdot 1 = n^2$ since for every choice of x and y there is at most one choice of z. The change of threshold in Theorem 6.6.2 (and more generally in Theorem 6.6.3 below) depends on the fact that we have many more choices for z in $(\mathbb{Z}_4)^n$.

Theorem 6.6.2 follows easily from the following application of the random Rado lemma to abelian groups. The definitions of $\operatorname{rank}_G(A)$ and $m_G(A)$ appearing in the statement require further motivation so we defer them to the next section; see Definitions 6.7.11 and 6.7.27. However, one may think of $m_G(A)$ as the natural generalisation of $m(A)$ which captures the intuition described for Theorem 6.6.2. Indeed, $m_{\mathbb{Z}_4}(A) = 4/3$ for $A = (2 \;\; 2 \;\; -2)$. For an $\ell \times k$ matrix A, we say that (A, G) is *abundant* if every $\ell \times (k-2)$ submatrix A' of A satisfies $\operatorname{rank}_G(A') = \operatorname{rank}_G(A)$. This naturally generalises the previous definition of abundant; see also Definition 6.7.22.

Theorem 6.6.3. *Let G be a finite abelian group with exponent $s \in \mathbb{N}$. Let A be an integer matrix which satisfies the s-columns condition, with $\operatorname{rank}_G(A) > 0$ and such that (A, G) is abundant. For all $r \geq 2$, there exist constants $C, c > 0$ such that the following holds.*

$$\lim_{n\to\infty} \mathbb{P}[G^n_p \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c|G|^{-n/m_G(A)}, \\ 1 & \text{if } p \geq C|G|^{-n/m_G(A)}. \end{cases}$$

Note that the condition $\operatorname{rank}_G(A) > 0$ in Theorem 6.6.3 is very mild: if A has dimension $\ell \times k$ then $\mathrm{rank}_G(A) > 0$ is equivalent to stating that not every ordered k-set over G is a solution to $Ax = 0$.

Given a finite subset S of an abelian group, we say that an $\ell \times k$ matrix A is *irredundant with respect to S* if there exists a k-distinct solution to $Ax = 0$ in S. So a matrix A is irredundant precisely if it is irredundant with respect to $\mathbb{N}$. Another application of the random Rado lemma is the following random version of the Ramsey-type result of Bergelson, Deuber and Hindman for vector spaces $\mathbb{F}^n$ (Theorem 6.4.3). For an $\ell \times k$ matrix A with entries from a field $\mathbb{F}$, we define $m_{\mathbb{F}}(A)$ analogously to (6.1), that is

$$m_{\mathbb{F}}(A) := \max_{\substack{W \dot\cup \overline{W} = [k] \\ |W| \geq 2}} \frac{|W| - 1}{|W| - 1 + \mathrm{rank}_{\mathbb{F}}(A_{\overline{W}}) - \mathrm{rank}_{\mathbb{F}}(A)}, \tag{6.2}$$

where we write $\mathrm{rank}_{\mathbb{F}}$ to denote that we calculate rank with respect to the field $\mathbb{F}$.

Theorem 6.6.4. *Let $\mathbb{F}$ be a non-trivial finite field. Consider any full rank matrix A with entries from $\mathbb{F}$ so that A satisfies the columns condition over $\mathbb{F}$ and A is also irredundant with respect to $\mathbb{F}^n$ for all $n \in \mathbb{N}$ sufficiently large. Given any positive integer $r \geq 2$, there exist constants $C, c > 0$ such that the following holds.*

$$\lim_{n \to \infty} \mathbb{P}[\mathbb{F}_p^n \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c|\mathbb{F}|^{-n/m_{\mathbb{F}}(A)}, \\ 1 & \text{if } p \geq C|\mathbb{F}|^{-n/m_{\mathbb{F}}(A)}. \end{cases}$$

We also obtain the analogue of Theorem 6.6.4 for matrices A which have integer entries. Here we must be slightly careful: as we are interested in solutions to $Ax = 0$ in $\mathbb{F}^n$, one must compute the rank of the integer matrix A with respect to the underlying field $\mathbb{F}$ (see Definition 6.7.14) and hence the threshold depends on $m_{\mathbb{F}}(A)$ rather than $m(A)$.[15] For example, the matrix $A = \begin{pmatrix} 3 & 3 & -3 \end{pmatrix}$ usually has $\mathrm{rank}(A) = 1$, however, in $\mathbb{Z}_3$, we have $3 \equiv 0 \pmod 3$ and so we have $\mathrm{rank}_{\mathbb{Z}_3}(A) = 0$.

Recall that an integer matrix A satisfies the q-columns condition if it satisfies the

[15] Note that the definition of $m_{\mathbb{F}}(A)$ for an integer matrix A is also consistent with putting $G = \mathbb{F}$ into the definition of $m_G(A)$ for groups (see Definition 6.7.27 and Fact 6.7.15).

columns condition over the ring $\mathbb{Z}_q$ (where the entries of A are treated as elements of $\mathbb{Z}_s$, i.e., residues modulo q).

Theorem 6.6.5. *Let $\mathbb{F}$ be a finite field of order q^k, for some prime q and $k \in \mathbb{N}$. Consider any full rank integer matrix A so that A satisfies the q-columns condition and A is also irredundant with respect to $\mathbb{F}^n$ for all $n \in \mathbb{N}$ sufficiently large. Given any positive integer $r \geq 2$, there exist constants $C, c > 0$ such that the following holds.*

$$\lim_{n \to \infty} \mathbb{P}[\mathbb{F}_p^n \textit{ is } (A,r)\textit{-Rado}] = \begin{cases} 0 & \textit{if } p \leq c|\mathbb{F}|^{-n/m_{\mathbb{F}}(A)}, \\ 1 & \textit{if } p \geq C|\mathbb{F}|^{-n/m_{\mathbb{F}}(A)}. \end{cases}$$

Theorem 6.6.5 is just a simple application of Theorem 6.6.4; its proof is given in Section 6.12.

6.7 The random Rado lemma

In this section, we rigorously state the random Rado lemma (Lemma 6.7.8). As previously mentioned, the random Rado lemma serves as a general black box that outputs a random Ramsey-type result for a given sequence of finite subsets of abelian groups provided that certain conditions, most notably supersaturation, are satisfied.

The content of this section is divided into four subsections. The first subsection contains the statement of Lemma 6.7.8 preceded by a list of all the required definitions. In the second subsection, we prove a general bound for the number of solutions to $Ax = 0$ in S when S is either a finite abelian group or a (power of a) finite subset of a field. This bound is used in the third subsection to prove that several technical requirements of the random Rado lemma trivially hold, or can be relaxed, in these specific settings. In light of these results, in the fourth subsection we formulate a simplified version of the random Rado lemma (Lemma 6.7.33) for finite abelian groups and (powers of) finite subsets of fields.

We stress once more that the random Rado lemma can be straightforwardly deduced from our general 1-statement and 0-statement for hypergraphs (Theorem 6.8.7). The proof of Lemma 6.7.8 is deferred to Section 6.10.

6.7.1 Definitions and main statement

Firstly, we recall the definitions of k-distinct and (A, r)-Rado.

Definition 6.5.2 (k-distinct and (A, r)-Rado)**.** For an $\ell \times k$ integer matrix A and a set S of elements of an abelian group, if a vector $x = (x_1, \dots, x_k) \in S^k$ satisfies $Ax = 0$ and the x_is are pairwise distinct, we call x a *k-distinct solution* to $Ax = 0$ in S. We say that S is *(A, r)-Rado* if given any r-colouring of S, there is a monochromatic k-distinct solution $x = (x_1, \dots, x_k)$ to $Ax = 0$ in S.

Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of abelian groups. We write $S_{n,p}$ to denote the subset of S_n obtained by including each element of S_n independently and with probability p. Given a function $p : \mathbb{N} \to [0, 1]$, we may abuse notation and write $S_{n,p}$ for $S_{n,p(n)}$. Given $p = p(n)$, we say an event occurs in $S_{n,p}$ *with high probability* (w.h.p.) if its probability tends to 1 as $n \to \infty$. Let A be an $\ell \times k$ integer matrix. We are interested in the probability threshold $\hat{p} := \hat{p}(n)$ such that, for some constants $0 < c < C$, we have that $S_{n,p}$ is (A, r)-Rado w.h.p. if $p \geq C\hat{p}$ and $S_{n,p}$ is not (A, r)-Rado w.h.p. if $p \leq c\hat{p}$.

Of course, as each subset S_n can lie in a different abelian group, it is easy to come up with a sequence $(S_n)_{n \in \mathbb{N}}$ for which no such probability threshold $\hat{p}$ exists. On the other hand, in Definition 6.7.2 we provide a candidate for $\hat{p}$ for 'well-behaved' sequences $(S_n)_{n \in \mathbb{N}}$. Given a k-set $(x_i)_{i \in [k]}$ and $W \subseteq [k]$, we write x_W for the $|W|$-set $(x_i)_{i \in W}$.

Definition 6.7.1 (Projected solutions)**.** Let S be a finite subset of an abelian group and A be an $\ell \times k$ integer matrix. Let $W \subseteq Y \subseteq [k]$ and $w_0 \in S^{|W|}$. The set of *projected solutions* $\mathrm{Sol}_S^A(w_0, W, Y)$ is the set of vectors $y \in S^{|Y|}$ such that there exists a solution $x \in S^k$ to $Ax = 0$ with $x_Y = y$ and $x_W = w_0$. For brevity, we write $\mathrm{Sol}_S^A(Y) := \mathrm{Sol}_S^A(\emptyset, \emptyset, Y)$, i.e., $\mathrm{Sol}_S^A(Y)$ is the set of vectors $y \in S^{|Y|}$ such that there exists a solution $x \in S^k$ to $Ax = 0$ with $x_Y = y$. In particular, $\mathrm{Sol}_S^A([k])$ is the set of all solutions to $Ax = 0$ in S.

Definition 6.7.2 (Probability threshold for abelian groups)**.** Let S be a finite subset of an abelian group and A be an $\ell \times k$ integer matrix. For every $W \subseteq [k]$, we let

$$p_W(A, S) := \left(\frac{|\operatorname{Sol}_S^A(W)|}{|S|} \right)^{-\frac{1}{|W|-1}}.$$

Additionally, we let

$$\hat{p}(A, S) := \max_{\substack{W \subseteq [k] \\ |W| \geq 2}} p_W(A, S).$$

The intuition for the choice of $\hat{p}(A, S)$ in Definition 6.7.2 aligns with the heuristic we discussed in Section 6.5, where we considered the random Ramsey problem for the integers. Indeed, for $p := p_W(A, S)$ the quantities $|\operatorname{Sol}_S^A(W)|p^{|W|}$, which is roughly[16] the expected number of projected solutions, and $|S|p$, which is the expected size of the ground set, are equal. The probability threshold $\hat{p}(A, S)$ is obtained by taking the largest of these $p_W(A, S)$.

Remark 6.7.3. *As the definition of (A, r)-Rado involves k-distinct solutions, the reader may wonder why, in Definition 6.7.1, we do not insist that the solutions to $Ax = 0$ must be k-distinct. This choice is designed to simplify later calculations. We will be able to restrict our attention to k-distinct solutions by imposing additional assumptions over the sequence $(S_n)_{n\in\mathbb{N}}$ (namely, conditions (A3) and (A6) of Lemma 6.7.8).*

Next, we introduce four properties that the sequence $(S_n)_{n\in\mathbb{N}}$ must satisfy in the hypothesis of the random Rado lemma. The first one is supersaturation, which we introduced and gave motivation for in Section 6.4.2. We will use the following formal definition.

Definition 6.7.4 ((A, r)-supersaturated)**.** Let $r \in \mathbb{N}$ and A be an $\ell \times k$ integer matrix. We say that a sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets of abelian groups is (A, r)-*supersaturated* if there exist an $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds: if $n \geq n_0$ then whenever S_n

[16]Note that the set $\operatorname{Sol}_S^A(W)$ includes projected solutions with repeated entries, and so the expected number of projected solutions may not be exactly equal to $|\operatorname{Sol}_S^A(W)|p^{|W|}$.

is r-coloured there are at least $\varepsilon|\operatorname{Sol}_{S_n}^A([k])|$ monochromatic solutions to $Ax=0$ in S_n.

The second property involves an upper bound on the number of projected solutions. Intuitively, this forces the solutions to be uniformly distributed over the ground set. This condition is rather natural since the heuristic argument for the probability threshold relies on the assumption that the solutions to $Ax=0$ are 'spread out'.

Definition 6.7.5 (B-extendable)**.** Let S be a finite subset of an abelian group, let A be an $\ell \times k$ integer matrix and let $B \geq 1$ be a real number. We say the pair (A,S) is *B-extendable* if for all non-empty $W \subseteq Y \subseteq [k]$ and any $w_0 \in S^{|W|}$ we have

$$|\operatorname{Sol}_S^A(w_0, W, Y)| \leq B \cdot \frac{|\operatorname{Sol}_S^A(Y)|}{|\operatorname{Sol}_S^A(W)|}.$$

The third property is more involved and perhaps the most artificial out of all requirements of the random Rado lemma. This property is tied to the calculations appearing in the proof of the general 0-statement for hypergraphs (Theorem 6.8.7) in Section 6.9.

Definition 6.7.6 (Compatibility)**.** Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of abelian groups and A be an $\ell \times k$ integer matrix with $k \geq 3$. We say $(S_n)_{n\in\mathbb{N}}$ is *compatible* with respect to A if there exists a sequence $(X_n)_{n\in\mathbb{N}}$ of subsets $X_n \subseteq [k]$ with $|X_n| \geq 3$ for all $n \in \mathbb{N}$ such that the following holds. We have $p_{X_n}(A, S_n) = \Omega(\hat{p}(A, S_n))$ and, for all sequences $(W_n)_{n\in\mathbb{N}}$ and $(W'_n)_{n\in\mathbb{N}}$ such that $W_n, W'_n \subsetneq X_n$ with $|W_n| = 2$ and $|W'_n| \geq 2$,

$$\frac{|S_n|^2}{|\operatorname{Sol}_{S_n}^A(W_n)|}\left(\frac{p_{W'_n}(A,S_n)}{p_{X_n}(A,S_n)}\right)^{|W'_n|-1} \to 0 \tag{6.3}$$

as $n \to \infty$.

The fourth property is a mild technical requirement that will be needed in the proof of Lemma 6.7.8 to allow us to restrict our attention to k-distinct solutions (see Lemma 6.7.9).

Definition 6.7.7 (Weak compatibility)**.** Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of abelian groups and A be an $\ell \times k$ integer matrix with $k \geq 2$. We say $(S_n)_{n\in\mathbb{N}}$ is *weakly* *compatible* with respect to A if for every $W \subseteq [k]$ with $|W| = 2$ we have

$$\frac{|S_n|}{|\operatorname{Sol}_{S_n}^A(W)|} \to 0$$

as $n \to \infty$.

We are now ready to formally state the random Rado lemma.

Lemma 6.7.8 (The random Rado lemma)**.** *Let $r \geq 2$, $k \geq 3$ and $\ell \geq 1$ be integers. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of abelian groups, A be an $\ell \times k$ integer matrix and set $\hat{p}_n := \hat{p}(A, S_n)$.*

Suppose that

(A1) $\hat{p}_n \to 0$ and $|S_n|\hat{p}_n \to \infty$ as $n \to \infty$;

(A2) $(S_n)_{n\in\mathbb{N}}$ is (A, r)-supersaturated;

(A3) there exists $B \geq 1$ such that the pair (A, S_n) is B-extendable for all sufficiently large $n \in \mathbb{N}$;

(A4) there exists $D > 0$ such that for every $W \subsetneq Y \subseteq [k]$ with $|W| = 1$ and $w_0 \in S_n$, we have $|\operatorname{Sol}_{S_n}^A(w_0, W, Y)| \leq D\frac{|\operatorname{Sol}_{S_n}^A(Y)|}{|S_n|}$ for all sufficiently large $n \in \mathbb{N}$;

(A5) $(S_n)_{n\in\mathbb{N}}$ is compatible with respect to A;

(A6) $(S_n)_{n\in\mathbb{N}}$ is weakly compatible with respect to A.

Then there exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c\hat{p}_n, \\ 1 & \text{if } p \geq C\hat{p}_n. \end{cases}$$

We find the generality of Lemma 6.7.8 very interesting - it confirms that for 'well-behaved' sequences $(S_n)_{n\in\mathbb{N}}$, the probability threshold essentially depends on the number of (projected) solutions to $Ax = 0$ in S_n.

Note that (A4) is a similar condition to the $|W| = 1$ case of the definition of B-extendable; we have written (A3) and (A4) as separate conditions as they correspond to conditions (P3) and (P4) in our general 1-statement and 0-statement for hypergraphs (Theorem 6.8.7). In fact, the random Rado lemma is a simple corollary of Theorem 6.8.7 and each of conditions (A1)–(A5) are adaptations of conditions (P1)–(P5) from Theorem 6.8.7. Condition (A6) is a mild requirement that we impose in order to restrict our attention to k-distinct solutions (see Lemma 6.7.9 below).

For most natural applications, conditions (A3)–(A6) can be further simplified. Thus, in practice, condition (A2) is the crucial assumption that one needs to verify. The goal of the next three subsections is to state a simplified version of the random Rado lemma (Lemma 6.7.33) which holds for sequences of finite abelian groups and sequences of (powers of) finite subsets of fields. Nonetheless, there are interesting examples where the generality of the random Rado lemma as stated in Lemma 6.7.8 is required. The most notable such example is the random van der Waerden theorem for the primes (Theorem 6.3.7), proved in Section 6.12.

It would be interesting to see if one can relax some of the assumptions in the random Rado lemma; if so, condition (A5) is the most likely candidate for where the hypothesis can be weakened. See Section 6.13.3 for a brief discussion related to this.

The next result states that conditions (A3) and (A6) imply a constant proportion of projected solutions are induced by k-distinct solutions. The proof is postponed to Section 6.10 but we state the result here in order to give the reader a better intuition why we did not impose in Definition 6.7.1 that the solutions are k-distinct.

Lemma 6.7.9. *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of abelian groups and let A be an $\ell \times k$ integer matrix with $k \geq 2$. Suppose that conditions (A3) and (A6) from Lemma 6.7.8 hold; so there is a $B \geq 1$ such that (A, S_n) is B-extendable for every $n \in \mathbb{N}$ sufficiently large.*

For each non-empty $Y \subseteq [k]$, let $k\text{-}\mathrm{Sol}^A_{S_n}(Y)$ be the set of $y \in S_n^{|Y|}$ such that there exists a k-distinct solution $x \in S^k$ to $Ax = 0$ with $x_Y = y$. For n sufficiently large, we *have*

$$\frac{1}{2B} \leq \frac{|k\text{-}\mathrm{Sol}^A_{S_n}(Y)|}{|\mathrm{Sol}^A_{S_n}(Y)|} \leq 1.$$

Furthermore, for $Y = [k]$ we have

$$\lim_{n\to\infty} \frac{|k\text{-}\mathrm{Sol}^A_{S_n}([k])|}{|\mathrm{Sol}^A_{S_n}([k])|} = 1.$$

Remark 6.7.10. *Recall that given a finite subset S of an abelian group, we say that an $\ell \times k$ matrix A is irredundant with respect to S if there exists a k-distinct solution to $Ax = 0$ in S. In many of the results mentioned previously in this chapter (e.g., Theorem 6.6.1) we require that the matrix considered is irredundant. However, we do not explicitly state that A is irredundant with respect to S_n in Lemma 6.7.8 as conditions (A3) and (A6) combined with Lemma 6.7.9 already imply that many solutions are k-distinct.*

6.7.2 Rank and counting solutions

The notion of extendability and (weak) compatibility introduced in the previous subsection are tied to the quantity $|\mathrm{Sol}^A_S(W)|$. In many natural cases we are able to approximate, or at least bound, this number using standard methods from group theory and linear algebra. For example, it is well known that the size of $\mathrm{Sol}^A_{[n]}([k])$, i.e., the number of solutions to $Ax = 0$ in $[n]$, is at most $n^{k-\mathrm{rank}(A)}$ (see, e.g., case (ii) of Lemma 6.7.16).

The notion of rank is useful here. For instance, the parameter $m(A)$ which governs the probability threshold for the random Rado theorem for the integers is expressed in terms of $\mathrm{rank}(A_W)$ for $W \subseteq [k]$. We now provide an appropriate generalisation of the notion of rank for the cases where S is either a finite abelian group or (a power of) a finite subset of a field.

Definition 6.7.11 (Rank for finite abelian groups)**.** Let S be a finite abelian group. Let A be an $\ell \times k$ integer matrix and let $f_A : S^k \to S^\ell$ be the function defined by $f_A : x \mapsto Ax$.

We define

$$\operatorname{rank}_S(A) := \begin{cases} \log_{|S|} |\operatorname{im}(f_A)| & \text{if } |S| > 1, \\ 0 & \text{if } |S| = 1. \end{cases}$$

By the first isomorphism theorem for groups, we have $|S|^k = |\operatorname{im}(f_A)| \cdot |\ker(f_A)|$. In particular, the number of solutions to $Ax = b$ is equal to

$$\begin{cases} |\ker(f_A)| = \frac{|S|^k}{|\operatorname{im}(f_A)|} = |S|^{k - \operatorname{rank}_S(A)} & \text{if } b \in \operatorname{im}(f_A), \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

A motivation for the choice of $\operatorname{rank}_S(A)$ in Definition 6.7.11 is that it will allow us to rewrite $p_W(A,S)$ from Definition 6.7.2 as a power of $|S|$. Indeed, in all applications of the random Rado lemma discussed in Sections 6.3 and 6.6, the probability threshold is expressed in such form. For example, it follows immediately from equation (6.4) that if A is an $\ell \times k$ integer matrix and S is a finite abelian group then $|\operatorname{Sol}_S^A([k])| = |S|^{k - \operatorname{rank}_S(A)}$ and so $p_{[k]}(A,S) = |S|^{-(k-1-\operatorname{rank}_S(A))/(k-1)}$.

Next, we consider the case where $S = L^d$ for some finite subset L of a field $\mathbb{F}$ and $d \in \mathbb{N}$. We are mostly interested in integer matrices, but it will be convenient to first consider matrices with entries in $\mathbb{F}$. We write $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ for the additive and multiplicative identities of $\mathbb{F}$, respectively.

Definition 6.7.12. Let $S = L^d$ where L is a finite subset of a field $\mathbb{F}$ and $d \in \mathbb{N}$. Let A be a matrix with entries from $\mathbb{F}$. If $L = \{0_{\mathbb{F}}\}$, let $\operatorname{rank}_S(A) := 0$. Otherwise, let $\operatorname{rank}_S(A)$ be the rank of A with respect to $\mathbb{F}$.

The following simple fact shows that $\operatorname{rank}_S(A)$ in Definition 6.7.12 is well-defined: the value of $\operatorname{rank}_S(A)$ is independent of the choice of $\mathbb{F}$.

Fact 6.7.13. *Let $\mathbb{F} \subseteq \mathbb{F}'$ be two fields. Let A be a matrix whose entries lie in $\mathbb{F}$. Then, the rank of A with respect to $\mathbb{F}$ is the same as the rank of A with respect to $\mathbb{F}'$.*

Proof. The rank of A is the size of the largest square submatrix of A with non-zero determinant. The determinant of a submatrix is computed by performing multiplications and additions using the entries of the submatrix. Let $\mathbb{F}_0$ be the field generated by the entries of A. Since the entries of A lie in $\mathbb{F}_0 \subseteq \mathbb{F} \subseteq \mathbb{F}'$ and $\mathbb{F}_0$ is closed under addition and multiplication, the determinant of any submatrix of A is the same whether we are working in $\mathbb{F}$ or $\mathbb{F}'$. $\square$

Definition 6.7.12 easily extends to matrices with integer entries by treating the entries of such matrices as elements of the field $\mathbb{F}$.

Definition 6.7.14 (Rank for powers of finite subsets of fields)**.** Let $S = L^d$ where L is a finite subset of a field $\mathbb{F}$ and $d \in \mathbb{N}$. Let $A = \{a_{ij}\}$ be an $\ell \times k$ matrix with integer entries. If $L = \{0_{\mathbb{F}}\}$, let $\operatorname{rank}_S(A) := 0$. Otherwise, let $\operatorname{rank}_S(A)$ be the rank of B with respect to $\mathbb{F}$, where $B := \{b_{ij}\}$ is the $\ell \times k$ matrix with $b_{ij} := a_{ij} \cdot 1_{\mathbb{F}} \in \mathbb{F}$.

Note that the entries of the matrix B in Definition 6.7.14 lie in the subfield $\langle 1_{\mathbb{F}} \rangle \subseteq \mathbb{F}$ generated by $1_{\mathbb{F}}$. Thus, by Fact 6.7.13, the rank of B is the same with respect to $\mathbb{F}$ and $\langle 1_{\mathbb{F}} \rangle$. If $\mathbb{F}$ and $\mathbb{F}'$ are non-trivial fields with $L \subseteq \mathbb{F}$ and $L \subseteq \mathbb{F}'$, then $1_{\mathbb{F}} = 1_{\mathbb{F}'}$ and in particular $\langle 1_{\mathbb{F}} \rangle = \langle 1_{\mathbb{F}'} \rangle$. Thus $\operatorname{rank}_S(A)$ from Definition 6.7.14 is independent from different choices of $\mathbb{F}$ and so it is well-defined.

The following fact states that Definition 6.7.14 is consistent with Definition 6.7.11.

Fact 6.7.15. *Let $A = \{a_{ij}\}$ be an $\ell \times k$ integer matrix and $S = L^d$ where L is a finite subset of a field $\mathbb{F}$ and $d \in \mathbb{N}$. If S is a finite abelian group, then* $\operatorname{rank}_S(A)$ *from Definition 6.7.14 is the same as* $\operatorname{rank}_S(A)$ *from Definition 6.7.11.*

Proof. As S is a finite abelian group, L is a finite abelian group too. If $L = \{0_{\mathbb{F}}\}$ then $\operatorname{rank}_S(A) = 0$ in both Definitions 6.7.11 and 6.7.14; so suppose $L \neq \{0_{\mathbb{F}}\}$.

It is well-known that every element in $(\mathbb{F}, +)$ other than $0_{\mathbb{F}}$ has (additive) order exactly p where either $p = \infty$ or p is a prime number. Since L is a finite abelian group, we must have $L \cong ((\mathbb{Z}_p)^t, +)$ for some prime p and $t \in \mathbb{N}$. Furthermore, we have $\langle 1_{\mathbb{F}} \rangle \cong (\mathbb{Z}_p, +, \times)$.

Let $B := \{b_{ij}\}$ where $b_{ij} := a_{ij} \cdot 1_{\mathbb{F}} \in \langle 1_{\mathbb{F}} \rangle \subseteq \mathbb{F}$ and let r be the rank of B with respect to $\mathbb{F}$. By Fact 6.7.13, r equals the rank of B with respect to $\langle 1_{\mathbb{F}} \rangle$. This in turn is equal to the rank of A with respect to $\mathbb{Z}_p$, where we treat the entries of A modulo p, by the definition of B and since $\langle 1_{\mathbb{F}} \rangle \cong (\mathbb{Z}_p, +, \times)$.

Let $f_A : S^k \to S^\ell$, $g_A : L^k \to L^\ell$, $h_A : (\mathbb{Z}_p)^k \to (\mathbb{Z}_p)^\ell$ be defined by $f_A, g_A, h_A : x \mapsto Ax$. Note that the rank of h_A is precisely r by the previous observations, and so $|\operatorname{im}(h_A)| = |\mathbb{Z}_p|^r = p^r$. It is easy to see that

$$|\operatorname{im}(f_A)| = |\operatorname{im}(g_A)|^d = |\operatorname{im}(h_A)|^{dt}.$$

In particular, $|\operatorname{im}(f_A)| = p^{rdt} = |S|^r$ and so $r = \log_{|S|} |\operatorname{im}(f_A)|$, as required. □

Next, we prove a general upper bound for the number of projected solutions for finite abelian groups and powers of finite subsets of fields. We will use this tool in Section 6.7.3 to simplify conditions (A3)–(A6) for these settings. Given a matrix A with k columns and $W \subseteq [k]$, recall A_W denotes the matrix formed by the columns of A indexed with W. We also write $\overline{W} := [k] \setminus W$, and so $A_{\overline{W}}$ denotes the matrix formed by the columns of A which are not indexed with W.

Lemma 6.7.16 (Key bounding result). *Let A be an $\ell \times k$ integer matrix. Suppose that either*

(i) S is a finite abelian group or

(ii) $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$.

Then for all $W \subseteq Y \subseteq [k]$ and $w_0 \in S^{|W|}$ we have

$$|\operatorname{Sol}_S^A(w_0, W, Y)| \leq |S|^{|Y| - |W| - \operatorname{rank}_S(A_{\overline{W}}) + \operatorname{rank}_S(A_{\overline{Y}})}. \tag{6.5}$$

Note that here we define $\operatorname{rank}_S(A_{\overline{W}}) := 0$ if $\overline{W} = \emptyset$.

Proof. Fix $W \subseteq Y \subseteq [k]$ and $w_0 \in S^{|W|}$. If $W = Y$ then $|\operatorname{Sol}_S^A(w_0, W, Y)| = 1$ and (6.5) holds. So we may assume that $W \subsetneq Y$.

Case (i): S is a finite abelian group. We double count the quantity $|\operatorname{Sol}_S^A(w_0, W, [k])|$. This is equal to the number of solutions x to $A_{\overline{W}}x = -A_W w_0$ which in turn is equal to either 0 or $|S|^{k-|W|-\operatorname{rank}_S(A_{\overline{W}})}$ by (6.4). In the former case, $|\operatorname{Sol}_S^A(w_0, W, [k])| = 0$ implies $|\operatorname{Sol}_S^A(w_0, W, Y)| = 0$ and so (6.5) holds. Hence, suppose that

$$|\operatorname{Sol}_S^A(w_0, W, [k])| = |S|^{k-|W|-\operatorname{rank}_S(A_{\overline{W}})}. \tag{6.6}$$

In particular, we have $|\operatorname{Sol}_S^A(w_0, W, Y)| > 0$. Let $y_0 \in \operatorname{Sol}_S^A(w_0, W, Y)$. By the same argument as above, we have $|\operatorname{Sol}_S^A(y_0, Y, [k])| = |S|^{k-|Y|-\operatorname{rank}_S(A_{\overline{Y}})}$. Thus,

$$|\operatorname{Sol}_S^A(w_0, W, [k])| = \sum_{y_0 \in \operatorname{Sol}_S^A(w_0, W, Y)} |\operatorname{Sol}_S^A(y_0, Y, [k])| = |\operatorname{Sol}_S^A(w_0, W, Y)| \cdot |S|^{k-|Y|-\operatorname{rank}_S(A_{\overline{Y}})}.$$

Combining the above with (6.6) yields

$$|\operatorname{Sol}_S^A(w_0, W, Y)| = |S|^{|Y|-|W|-\operatorname{rank}_S(A_{\overline{W}})+\operatorname{rank}_S(A_{\overline{Y}})},$$

as required.

Case (ii): $S = L^d$ for some finite subset L of a field $\mathbb{F}$ and $d \in \mathbb{N}$. If $L = \{0_{\mathbb{F}}\}$ then $|\operatorname{Sol}_S^A(w_0, W, Y)| \leq 1$ and so (6.5) holds. Hence, assume $L \neq \{0_{\mathbb{F}}\}$. For the rest of this case, we treat the entries of A as elements of $\mathbb{F}$ (namely, we view $n \in \mathbb{N}$ as $n \cdot 1_{\mathbb{F}}$). Let $I_0 \subseteq [k]$ be the indices of a maximal set of linearly independent columns in $A_{\overline{Y}}$ and I_1 be the indices of a maximal set of columns in $A_{Y \setminus W}$ such that the columns with indices $I_0 \cup I_1$ are linearly independent. Then recalling $W \subseteq Y$, we have $\operatorname{rank}_S(A_{\overline{W}}) = |I_0 \cup I_1|$.

Claim 6.7.17. *Let u, v be two solutions $Au = Av = 0$ in $\mathbb{F}$. If $u_{Y \setminus I_1} = v_{Y \setminus I_1}$ then $u_{I_1} = v_{I_1}$.*

Proof of the claim. Since $A(u-v)=0$ and $u_{Y\setminus I_1}=v_{Y\setminus I_1}$

$$A_{I_1}(u-v)_{I_1}+A_{\overline{Y}}(u-v)_{\overline{Y}}=0,$$

and thus $A_{I_1}(u-v)_{I_1}\in\langle A_{\overline{Y}}\rangle$, where $\langle A_{\overline{Y}}\rangle$ denotes the vector space spanned by the columns of $A_{\overline{Y}}$. By the maximality of I_0 we have $\langle A_{\overline{Y}}\rangle=\langle A_{I_0}\rangle$ and so $A_{I_1}(u-v)_{I_1}\in\langle A_{I_0}\rangle$. Since the columns with indices $I_0\cup I_1$ are linearly independent, this implies that $(u-v)_{I_1}=0$.

■

Now we bound $|\operatorname{Sol}_S^A(w_0,W,Y)|$ from above. Recall $w_0\in S^{|W|}$ and $S=L^d$. We can therefore write w_0 in the form $w_0=(t_1,\dots,t_d)$ where $t_i\in L^{|W|}$. Note that

$$|\operatorname{Sol}_S^A(w_0,W,Y)|=\prod_{i=1}^{d}|\operatorname{Sol}_L^A(t_i,W,Y)|,$$

so it suffices to prove that $|\operatorname{Sol}_L^A(t_i,W,Y)|\le|L|^{|Y|-|W|-\operatorname{rank}_S(A_{\overline{W}})+\operatorname{rank}_S(A_{\overline{Y}})}$ for every i. By Claim 6.7.17, there is at most one projected solution for every choice of the entries with indices $Y\setminus I_1$. Note that the entries with indices in W are already fixed and W and I_1 are disjoint, thus

$$|\operatorname{Sol}_L^A(t_i,W,Y)|\le|L|^{|Y|-|W|-|I_1|}=|L|^{|Y|-|W|-|I_0\cup I_1|+|I_0|}=|L|^{|Y|-|W|-\operatorname{rank}_S(A_{\overline{W}})+\operatorname{rank}_S(A_{\overline{Y}})}.$$

□

Applying inequality (6.5) from Lemma 6.7.16 with $W=\emptyset$ and $Y=[k]$ immediately implies that there are at most $|S|^{k-\operatorname{rank}_S(A)}$ solutions to $Ax=0$ in S. Informally, we say (A,S) is *rich* if this bound is close to tight.

Definition 6.7.18 (ε-rich)**.** Let A be an $\ell\times k$ integer matrix, let $\varepsilon>0$ and let S be either a finite abelian group or a power of a finite subset of a field. We say (A,S) is ε*-rich* if there are at least $\varepsilon|S|^{k-\operatorname{rank}_S(A)}$ solutions to $Ax=0$ in S.

Intuitively, richness is a measure of how strongly the elements of S are related within the arithmetic structure of S. For example, if S is a finite abelian group then (A, S) is 1-rich by (6.4) for any integer matrix A. If A is partition regular then Theorem 6.4.5 implies (in a very strong form) that $S = [n]$ is δ-rich for some $\delta > 0$ and $n \in \mathbb{N}$ sufficiently large. On the other hand, this is not the case for $S = \mathbf{P}_n$ (see, e.g., Theorem 6.3.1), and indeed, the primes have a very loose arithmetic structure.

The next corollary states that if (A, S) is rich then the bound given by Lemma 6.7.16 is close to tight for certain projected solutions.

Corollary 6.7.19. *Let A be an $\ell \times k$ integer matrix and let $0 < \varepsilon \leq 1$. Suppose that either*

(i) S is a finite abelian group or

(ii) $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$.

If (A, S) is ε-rich, then for every non-empty $Z \subseteq [k]$ we have

$$\varepsilon |S|^{|Z| - \operatorname{rank}_S(A) + \operatorname{rank}_S(A_{\overline{Z}})} \leq |\operatorname{Sol}_S^A(Z)| \leq |S|^{|Z| - \operatorname{rank}_S(A) + \operatorname{rank}_S(A_{\overline{Z}})}. \tag{6.7}$$

Proof. Fix $Z \subseteq [k]$. Applying (6.5) from Lemma 6.7.16 with $Y := Z$ and $W := \emptyset$ yields the upper bound

$$|\operatorname{Sol}_S^A(Z)| \leq |S|^{|Z| - \operatorname{rank}_S(A) + \operatorname{rank}_S(A_{\overline{Z}})}.$$

Fix $z_0 \in \operatorname{Sol}_S^A(Z)$. By applying (6.5) with $Y := [k]$ and $W := Z$, we obtain that

$$|\operatorname{Sol}_S^A(z_0, Z, [k])| \leq |S|^{k - |Z| - \operatorname{rank}_S(A_{\overline{Z}})}. \tag{6.8}$$

The total number of solutions to $Ax = 0$ in S is

$$|\operatorname{Sol}_S^A([k])| = \sum_{z_0 \in \operatorname{Sol}_S^A(Z)} |\operatorname{Sol}_S^A(z_0, Z, [k])| \overset{(6.8)}{\leq} |\operatorname{Sol}_S^A(Z)| \cdot |S|^{k - |Z| - \operatorname{rank}_S(A_{\overline{Z}})}.$$

Combining the above with $|\operatorname{Sol}_S^A([k])| \geq \varepsilon|S|^{k-\operatorname{rank}_S(A)}$ yields

$$|\operatorname{Sol}_S^A(Z)| \geq \varepsilon|S|^{k-\operatorname{rank}_S(A)} \cdot |S|^{-k+|Z|+\operatorname{rank}_S(A_{\overline{Z}})} = \varepsilon|S|^{|Z|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{Z}})}.$$

□

6.7.3 Conditions (A3)–(A6) for finite groups and powers of finite subsets of fields

In this subsection, we use the key bounding results (Lemma 6.7.16 and Corollary 6.7.19) from the previous subsection to investigate conditions (A3)–(A6) when the sets S_n in the sequence $(S_n)_{n\in\mathbb{N}}$ are either finite abelian groups or powers of finite subsets of fields. We start by showing that in this case, richness implies condition (A3).

Lemma 6.7.20. *Let A be an $\ell \times k$ integer matrix. Suppose that either*

(i) S is a finite abelian group or

(ii) $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$.

If there exists $B > 0$ such that (A, S) is $(1/B)$-rich, then (A, S) is B-extendable.

Proof. Let $W \subseteq Y \subseteq [k]$ and $w_0 \in S^{|W|}$. It follows from Lemma 6.7.16 and Corollary 6.7.19 that

$$\begin{aligned}\frac{B|\operatorname{Sol}_S^A(Y)|}{|\operatorname{Sol}_S^A(W)|} &\overset{(6.7)}{\geq} \frac{|S|^{|Y|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{Y}})}}{|S|^{|W|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{W}})}} \\ &= |S|^{|Y|-|W|-\operatorname{rank}_S(A_{\overline{W}})+\operatorname{rank}_S(A_{\overline{Y}})} \overset{(6.5)}{\geq} |\operatorname{Sol}_S^A(w_0, W, Y)|.\end{aligned}$$

Hence (A, S) is B-extendable. □

For finite abelian groups, we deduce the following corollary.

Corollary 6.7.21. *Let A be an $\ell \times k$ integer matrix and S be a finite abelian group. Then (A, S) is 1-extendable.*

Proof. By (6.4), (A, S) is 1-rich. Lemma 6.7.20 implies that (A, S) is 1-extendable. $\square$

Next, we consider conditions (A4), (A5) and (A6). First, we give the definition of *abundance*. This notion appeared in Section 6.5 when we considered the random Ramsey problem for the integers.

Definition 6.7.22 (Abundancy)**.** Let A be an $\ell \times k$ integer matrix and S be either a finite abelian group or a power of a finite subset of a field. We say (A, S) is *abundant* if $\mathrm{rank}_S(A) = \mathrm{rank}_S(A_{\overline{W}})$ for every $W \subseteq [k]$ with $|W| = 2$.

Before we show how abundancy can help simplify conditions (A4), (A5) and (A6), we show it is a mild requirement in the case of powers of finite subsets of a field $\mathbb{F}$. For brevity, we will write 0 and 1 for the additive and multiplicative identities of $\mathbb{F}$, respectively. The next lemma establishes that irredundancy and 3-partition regularity imply abundancy. We remark that the field $\mathbb{F}$ in the statement of the lemma may be finite or infinite.

Lemma 6.7.23. *Let A be an $\ell \times k$ integer matrix and $S = L^d$ for some finite subset L of a field $\mathbb{F}$ and $d \in \mathbb{N}$. If A is irredundant with respect to $\mathbb{F}^d$ and 3-partition regular in $S \setminus \{0\}^d$, then (A, S) is abundant.*

Proof. If $L = \{0\}$ then $\mathrm{rank}_S(A') = 0$ for any integer matrix A', and we are done. Thus, suppose that $L \neq \{0\}$. As before, we can view A as a matrix with entries in $\mathbb{F}$. Let $C_1, \ldots, C_k$ denote the columns of A. To prove that (A, S) is abundant it suffices to show that C_1 and C_2 are both spanned by $C_3, ..., C_k$ in $\mathbb{F}$. For the rest of the proof, given a solution x to $Ax = 0$, we write x_i for the ith entry of x.

If C_1 cannot be expressed as a linear combination of $C_2, \ldots, C_k$, then a solution x to $Ax = 0$ in $\mathbb{F}$ satisfies $x_1 = 0$. In turn, this implies that for every solution x to $Ax = 0$ in $\mathbb{F}^d$ the entry x_1 is the zero-vector. This contradicts the assumption that A is 3-partition regular in $S \setminus \{0\}^d$.

Hence, C_1 is spanned by $C_2, \ldots, C_k$. In particular, there exists $\alpha \in \mathbb{F}$ such that $C_1 + \alpha C_2$ is spanned by $C_3, \ldots, C_k$. If α is not unique then one can write C_2 as a linear combination of $C_3, \ldots, C_k$ and thus we are done.

Suppose therefore that α is unique. Thus, any solution x to $Ax = 0$ in $\mathbb{F}^d$ must satisfy $x_2 = \alpha \cdot x_1$. If $\alpha = 1$ then $x_1 = x_2$ and A is therefore not irredundant with respect to $\mathbb{F}^d$, a contradiction. If $\alpha = 0$ then x_2 is the zero-vector, and so A is not 3-partition regular in $S \setminus \{0\}^d$, a contradiction. Thus, $\alpha \neq 0, 1$.

Consider the graph G with vertex set $V(G) = S$ and edge set $E(G) = \{ss' : s' = \alpha \cdot s\}$ (here the edges are unordered pairs). Since $\alpha \neq 0$, each $s \in V(G)$ is incident to at most two edges, namely $\alpha \cdot s$ and $\alpha^{-1} \cdot s$. Thus $\Delta(G) \leq 2$. In particular, there exists a proper 3-vertex-colouring of G which induces a 3-colouring of S.

Recall that for any solution x of $Ax = 0$ in $\mathbb{F}^d$ we have $x_2 = \alpha \cdot x_1$. Since $\alpha \neq 1$, the entries x_1 and x_2 are distinct unless both of them are the zero-vector. In the former case, x_1 and x_2 are adjacent in G and so they are coloured differently. It follows that there is no monochromatic solution to $Ax = 0$ in $S \setminus \{0\}^d$. Hence A is not 3-partition regular in $S \setminus \{0\}^d$, a contradiction. $\square$

Note that the statement of Lemma 6.7.23 does not necessarily hold for finite abelian groups. Indeed, consider the example $A = \begin{pmatrix} 2 & 2 & 1 & 1 \end{pmatrix}$ and $S = \mathbb{Z}_6^n$. The columns of A sum to zero (mod 6), so A satisfies the 6-columns condition. By Theorem 6.4.8, for fixed r and n sufficiently large, every r-colouring of $\mathbb{Z}_6^n \setminus \{0\}^n$ yields (many) monochromatic solutions to $Ax = 0$. Additionally A is irredundant with respect to $\mathbb{Z}_6^n$. This follows easily by observing that $(2, 4, 1, 5)$ is a 4-distinct solution to $Ax = 0$ in $\mathbb{Z}_6$. However $(A, \mathbb{Z}_6^n)$ is not abundant, since the rank decreases after deleting the last two columns of A.

Next, we show how abundancy can help simplify conditions (A4), (A5) and (A6). First, if (A, S) is abundant and rich then condition (A4) trivially holds.

Lemma 6.7.24. *Let A be an $\ell \times k$ integer matrix. Suppose that either*

(i) S is a finite abelian group or

(ii) $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$.

Furthermore, assume that (A, S) is abundant and ε-rich for some $0 < \varepsilon \leq 1$. Then, for every $W \subseteq Y \subseteq [k]$ with $|W| = 1$ and $w_0 \in S$ we have $|\operatorname{Sol}_S^A(w_0, W, Y)| \leq |\operatorname{Sol}_S^A(Y)|/(\varepsilon|S|)$.

Proof. By Lemma 6.7.16 we have

$$|\operatorname{Sol}_S^A(w_0, W, Y)| \le |S|^{|Y|-1-\operatorname{rank}_S(A_{\overline{W}})+\operatorname{rank}_S(A_{\overline{Y}})},$$

while by Corollary 6.7.19 we have

$$|\operatorname{Sol}_S^A(Y)| \ge \varepsilon |S|^{|Y|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{Y}})}.$$

Combining the two inequalities above yields

$$|\operatorname{Sol}_S^A(w_0, W, Y)| \le \frac{1}{\varepsilon} \cdot \frac{|\operatorname{Sol}_S^A(Y)|}{|S|} \cdot |S|^{\operatorname{rank}_S(A)-\operatorname{rank}_S(A_{\overline{W}})}. \tag{6.9}$$

Let $W \subseteq W'$ with $|W'| = 2$ and note that $\operatorname{rank}_S(A_{\overline{W'}}) \le \operatorname{rank}_S(A_{\overline{W}}) \le \operatorname{rank}_S(A)$. We have $\operatorname{rank}_S(A_{\overline{W'}}) = \operatorname{rank}_S(A)$ since $|W'| = 2$ and (A, S) is abundant. Thus, $\operatorname{rank}_S(A_{\overline{W}}) = \operatorname{rank}_S(A)$ and so (6.9) implies that $|\operatorname{Sol}_S^A(w_0, W, Y)| \le |\operatorname{Sol}_S^A(Y)|/(\varepsilon|S|)$. □

Next, we consider conditions (A5) and (A6). If (A, S) is abundant and rich then the following bound holds.

Lemma 6.7.25. *Let A be an $\ell \times k$ integer matrix. Suppose that either*

(i) S is a finite abelian group or

(ii) $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$.

Furthermore, assume that (A, S) is abundant and ε-rich for some $0 < \varepsilon \le 1$. Then for any $W \subseteq [k]$ with $|W| = 2$, we have

$$\frac{|S|^2}{|\operatorname{Sol}_S^A(W)|} \le \frac{1}{\varepsilon}.$$

Proof. We have

$$\frac{|S|^2}{|\operatorname{Sol}_S^A(W)|} \overset{(6.7)}{\le} \frac{|S|^2}{\varepsilon|S|^{|W|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{W}})}} = \frac{|S|^2}{\varepsilon|S|^2} = \frac{1}{\varepsilon},$$

where the equality follows from the fact that $|W| = 2$ and $\mathrm{rank}_S(A) = \mathrm{rank}_S(A_{\overline{W}})$ since (A, S) is abundant. $\square$

Observe that if A and $(S_n)_{n\in\mathbb{N}}$ satisfy the hypothesis of Lemma 6.7.25 with the same $\varepsilon > 0$ for all $n \in \mathbb{N}$, and $|S_n| \to \infty$ as $n \to \infty$, then condition (A6) trivially holds. Condition (A5) is also close to being satisfied: it would suffice to additionally prove that there exists some sequence $(X_n)_{n\in\mathbb{N}}$ of subsets $X_n \subseteq [k]$ with $|X_n| \geq 3$ for all $n \in \mathbb{N}$ where $p_{X_n}(A, S_n) = \Omega(\hat{p}(A, S_n))$ and such that

$$\frac{p_{W'_n}(A, S_n)}{p_{X_n}(A, S_n)} \to 0$$

as $n \to \infty$ for any sequence $(W'_n)_{n\in\mathbb{N}}$ with $W'_n \subsetneq X_n$ and $|W'_n| \geq 2$. The next result states that this is the case for powers of subsets of fields as long as we have abundancy and richness.

Lemma 6.7.26. *Let $0 < \varepsilon \leq 1$ and let A be an $\ell \times k$ integer matrix and $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$. Suppose* $\mathrm{rank}_S(A) > 0$. *If (A, S) is ε-rich and abundant, then there exists $X \subseteq [k]$ with $|X| \geq 3$ satisfying the following. We have $p_X(A, S) \geq \varepsilon\hat{p}(A, S)$ and for every $W' \subsetneq X$ with $|W'| \geq 2$ we have*

$$\frac{p_{W'}(A, S)}{p_X(A, S)} \leq (1/\varepsilon)|S|^{-1/k^2}.$$

Proof. For every $Z \subseteq [k]$ with $|Z| \geq 2$ set

$$D(Z) := \frac{|Z| - 1 - \mathrm{rank}_S(A) + \mathrm{rank}_S(A_{\overline{Z}})}{|Z| - 1}.$$

As $0 < \varepsilon \leq 1$, by definition of $p_Z(A, S)$ and Corollary 6.7.19, we have that

$$|S|^{-D(Z)} \overset{(6.7)}{\leq} p_Z(A, S) \overset{(6.7)}{\leq} (1/\varepsilon)|S|^{-D(Z)}. \tag{6.10}$$

Pick $X \subseteq [k]$ with $|X| \geq 2$ so that the function $D(X)$ is minimised; additionally choose X so that $|X|$ is as small as possible under this assumption. We have

$$p_X(A,S) \overset{(6.10)}{\geq} |S|^{-D(X)} \geq |S|^{-D(Z)} \overset{(6.10)}{\geq} \varepsilon p_Z(A,S)$$

for every $Z \subseteq [k]$ with $|Z| \geq 2$. By maximising over Z, it follows that $p_X(A,S) \geq \varepsilon\hat{p}(A,S)$, as required. Moreover, $|X| \geq 3$. Indeed, for any $Z \subseteq [k]$ with $|Z| = 2$, we have $\mathrm{rank}_S(A) = \mathrm{rank}_S(A_{\overline{Z}})$ since (A,S) is abundant. In particular, we have $D(Z) = 1$. On the other hand, $D([k]) = (k-1-\mathrm{rank}_S(A))/(k-1) < 1$ since $\mathrm{rank}_S(A) > 0$. Thus, $|X| \neq 2$.

Let $W' \subsetneq X$ with $|W'| \geq 2$. We have

$$\frac{p_{W'}(A,S)}{p_X(A,S)} \overset{(6.10)}{\leq} (1/\varepsilon)|S|^{-D(W')+D(X)}.$$

By the minimality of X, we have $D(X) < D(W')$. Since $S = L^d$ for some finite subset L of a field, $\mathrm{rank}_S(\cdot)$ is always an integer and so

$$D(W') - D(X) \geq \frac{1}{(|X|-1)(|W'|-1)} \geq \frac{1}{k^2}.$$

It follows that

$$\frac{p_{W'}(A,S)}{p_X(A,S)} \leq (1/\varepsilon)|S|^{-1/k^2}.$$

□

6.7.4 Simplified random Rado lemma

In this subsection we state a simplified version of the random Rado lemma for finite abelian groups and powers of finite subsets of fields. This version of the random Rado lemma will be used to prove most of our applications (see Section 6.12). First, we introduce a generalisation of the parameter $m(A)$ to this setting.

Definition 6.7.27. Let S be either a finite abelian group or a power of a finite subset of a field. Let A be an $\ell \times k$ integer matrix. We set

$$m_S(A) := \max_{\substack{W \subseteq [k] \\ |W| \geq 2}} \frac{|W| - 1}{|W| - 1 + \operatorname{rank}_S(A_{\overline{W}}) - \operatorname{rank}_S(A)}. \tag{6.11}$$

Here we recall that $\operatorname{rank}_S(A_{\overline{W}}) := 0$ if $\overline{W} = \emptyset$. We call A *strictly balanced with respect to* S if the expression in (6.11) is maximised precisely when $W = [k]$.

Note that if $S := [n]$ then as S is a subset of the field $\mathbb{Q}$, we have that $m_S(A) = m(A)$. Indeed, in this setting the function rank_S is the same as the rank function appearing in the definition of $m(A)$.

Remark 6.7.28. *Observe that $m_S(A)$ is not always well-defined since the denominator in (6.11) might be zero. However, if (A, S) is abundant then for all $|W| \geq 2$ we have*

$$\operatorname{rank}_S(A_{\overline{W}}) \geq \operatorname{rank}_S(A) - |W| + 2, \tag{6.12}$$

since the rank of a matrix can decrease by at most one by deleting a column.[17] *If (6.12) is satisfied then the denominator in (6.11) is always strictly positive and so $m_S(A)$ is well-defined in this case.*

Before proceeding further, we provide several examples and facts about the parameter $m_S(A)$. Firstly, we show that $m_{\mathbb{Z}_4}(A) = 4/3$ for $A = (2 \ \ 2 \ \ -2)$, as stated in Section 6.6.

Example 6.7.29. *Let $A = (2 \ \ 2 \ \ -2)$. The number of solutions $x = (x_1, x_2, x_3)$ to $Ax = 0$ in $\mathbb{Z}_4$ is precisely $2 \cdot 4^2 = 4^{3-1/2}$: we have 4^2 choices for x_1, x_2 and, for each fixed choice, we have 2 choices for x_3. It follows from (6.4) that $\operatorname{rank}_{\mathbb{Z}_4}(A) = 1/2$. Now, let $W \subseteq [3]$. If $|W| = 3$ we trivially have $\operatorname{rank}_{\mathbb{Z}_4}(A_{\overline{W}}) = 0$. If $|W| = 2$ then $A_{\overline{W}}x = 0$ is the same as $2x = 0$, which has $2 = 4^{1-1/2}$ solutions in $\mathbb{Z}_4$. So we have $\operatorname{rank}_{\mathbb{Z}_4}(A_{\overline{W}}) = 1/2$ again.*

[17] This fact follows immediately from the definition of $\operatorname{rank}_S(A)$ when S is the power of a subset of a field F. When S is a finite abelian group, it can be proved using equation (6.4).

Thus,

$$m_{\mathbb{Z}_4}(A) = \max\left\{\frac{3-1}{3-1+0-1/2}, \frac{2-1}{2-1+1/2-1/2}\right\} = 4/3.$$

Next, we highlight certain similarities and differences between the parameters $m(A)$ and $m_S(A)$. Recall that if A is a $1 \times k$ integer matrix with non-zero entries then A is strictly balanced and $m(A) = \frac{k-1}{k-2}$. This is not always the case for $m_S(A)$, as shown in the next example.

Example 6.7.30. *Let $m \in \mathbb{N}$ and $A = (1 \quad m \quad m)$. The number of solutions $x = (x_1, x_2, x_3)$ to $Ax = 0$ in $\mathbb{Z}_{2m}$ is precisely $(2m)^2 = (2m)^{3-1}$. It follows that $\mathrm{rank}_{\mathbb{Z}_{2m}}(A) = 1$. If $W = [3]$ we trivially have $\mathrm{rank}_{\mathbb{Z}_4}(A_{\overline{W}}) = 0$. If $W = \{2,3\}$ then $A_{\overline{W}}x = 0$ is the same as $x = 0$, which has $1 = (2m)^{1-1}$ solution in $\mathbb{Z}_{2m}$, and so $\mathrm{rank}_{\mathbb{Z}_{2m}}(A_{\overline{W}}) = 1$. If $W = \{1,2\}$ or $W = \{1,3\}$ then $A_{\overline{W}}x = 0$ is the same as $mx = 0$, which has $m = (2m)^{1-\log_{2m} 2}$ solutions in $\mathbb{Z}_{2m}$, and so $\mathrm{rank}_{\mathbb{Z}_{2m}}(A_{\overline{W}}) = \log_{2m} 2$. Thus,*

$$\begin{aligned} m_{\mathbb{Z}_{2m}}(A) &= \max\left\{\frac{3-1}{3-1+0-1}, \frac{2-1}{2-1+1-1}, \frac{2-1}{2-1+\log_{2m} 2-1}\right\} \\ &= \max\{2, 1, (\log_{2m} 2)^{-1}\}. \end{aligned}$$

In particular, for $m \geq 2$ we have $m_{\mathbb{Z}_{2m}}(A) = (\log_{2m} 2)^{-1}$. Furthermore, for $m > 2$, the value of $m_{\mathbb{Z}_{2m}}(A)$ is achieved precisely by $W = \{1,2\}$ and $W = \{1,3\}$, and so A is not strictly balanced with respect to $\mathbb{Z}_{2m}$.

Despite the example above, by imposing some additional restrictions on the $1 \times k$ matrix A, one can infer that A is strictly balanced with respect to S and, furthermore, $m_S(A) = m(A) = \frac{k-1}{k-2}$.

Example 6.7.31. *Let A be a $1 \times k$ integer matrix with non-zero entries. Let S be a finite subset of an abelian group G. Suppose that $S \neq \{0_G\}$ and that either*

- *S is a finite abelian group or*
- *S is a power of a finite subset of a field $\mathbb{F}$.*

Furthermore, suppose that

(α) *for every entry a of A and $s \in S \setminus \{0_G\}$ we have $a \cdot s \neq 0$.*

Then A is strictly balanced with respect to S and $m_S(A) = m(A) = \frac{k-1}{k-2}$.

Indeed, let $W \subseteq [k]$ be non-empty. If S is a power of a finite subset of a field $\mathbb{F}$ then (α) implies A_W is not the $1 \times |W|$ zero vector in $\mathbb{F}$ and thus A_W has rank 1 with respect to $\mathbb{F}$. In particular, $\mathrm{rank}_S(A_W) = 1$. Similarly, if S is a finite abelian group then the number of solutions to $A_W x = 0$ in S is precisely $|S|^{|W|-1}$: we can pick the first $|W|-1$ entries of x arbitrarily and for each such choice we have 1 choice for the last entry (this follows from (α)). Thus, we have $\mathrm{rank}_S(A_W) = 1$. In both cases, we have

$$m_S(A) = \max_{2 \leq w \leq k-1} \left\{ \frac{k-1}{k-1+0-1}, \frac{w-1}{w-1+1-1} \right\} = \frac{k-1}{k-2}.$$

In particular, the value of $m_S(A)$ is achieved precisely when $W = [k]$ and so A is strictly balanced with respect to S.

Note that if a $1 \times k$ integer matrix A is strictly balanced then $m(A) = \frac{k-1}{k-2}$. Conversely, if A is strictly balanced with respect to S, it may not be the case that $m_S(A) = \frac{k-1}{k-2}$. For instance, in Example 6.7.29, we saw $A = (2 \;\; 2 \;\; -2)$ is strictly balanced with respect to $\mathbb{Z}_4$ but $m_{\mathbb{Z}_4}(A) = 4/3 < 2$.

Another observation is that, for A fixed, $m_S(A)$ may change depending on S. For example, consider $A = (1 \;\; 3 \;\; 3)$. From Example 6.7.30 we have that $m_{\mathbb{Z}_6}(A) = (\log_6 2)^{-1} \approx 2.58$, while from Example 6.7.31 we have $m_{\mathbb{Z}_2}(A) = \frac{3-1}{2-1} = 2$ (since A and $\mathbb{Z}_2$ satisfy property (α)).

There are cases where $m_S(A)$ is independent of S, as long as S is non-trivial. Indeed, let A be a $1 \times k$ matrix whose entries are all from $\{1, -1\}$. Then (α) from Example 6.7.31 holds for every S which is either a finite abelian group or a power of a finite subset of a field, provided that S includes a non-identity element. In particular, we have $m_S(A) = \frac{k-1}{k-2}$ for any such S.

The next result states that $\hat{p}(A, S)$ and $|S|^{-1/m_S(A)}$ are within a constant factor of each other, as long as we are in a rich setting.

Lemma 6.7.32. *Let A be an $\ell \times k$ integer matrix and $0 < \varepsilon \leq 1$. Suppose that either*

(i) S is a finite abelian group or

(ii) $S = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$.

If $m_S(A)$ is well-defined and (A, S) is ε-rich then

$$|S|^{-\frac{1}{m_S(A)}} \leq \hat{p}(A, S) \leq (1/\varepsilon^2)|S|^{-\frac{1}{m_S(A)}}.$$

Proof. For every $Z \subseteq [k]$ with $|Z| \geq 2$ let

$$D(Z) := \frac{|Z| - 1 - \mathrm{rank}_S(A) + \mathrm{rank}_S(A_{\overline{Z}})}{|Z| - 1}.$$

As $0 < \varepsilon \leq 1$, by definition of $p_Z(A, S)$ and Corollary 6.7.19, we have that

$$|S|^{-D(Z)} \overset{(6.7)}{\leq} p_Z(A, S) \overset{(6.7)}{\leq} (1/\varepsilon)|S|^{-D(Z)}. \tag{6.13}$$

Pick $X \subseteq [k]$ with $|X| \geq 2$ so that the function $D(X)$ is minimised; so $m_S(A) = 1/D(X)$. For any $Z \subseteq [k]$ with $|Z| \geq 2$, we have

$$p_X(A, S) \overset{(6.13)}{\geq} |S|^{-D(X)} \geq |S|^{-D(Z)} \overset{(6.13)}{\geq} \varepsilon p_Z(A, S).$$

As Z was arbitrary, it follows that

$$p_X(A, S) \leq \hat{p}(A, S) \leq (1/\varepsilon)p_X(A, S).$$

Combining the above with (6.13) yields

$$|S|^{-\frac{1}{m_S(A)}} = |S|^{-D(X)} \overset{(6.13)}{\leq} \hat{p}(A, S) \overset{(6.13)}{\leq} (1/\varepsilon^2)|S|^{-D(X)} = (1/\varepsilon^2)|S|^{-\frac{1}{m_S(A)}}.$$

$\square$

We can now state the simplified version of the random Rado lemma.

Lemma 6.7.33 (Simplified random Rado lemma)**.** *Let $r \geq 2$, $k \geq 3$ and $\ell \geq 1$ be integers. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence where S_n is either a finite abelian group or $S_n = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$, for every $n \in \mathbb{N}$. Let A be an $\ell \times k$ integer matrix and set $\hat{p}_n := \hat{p}(A, S_n)$. Suppose that*

(C1) $\hat{p}_n \to 0$ and $|S_n|\hat{p}_n \to \infty$ as $n \to \infty$;

(C2) $(S_n)_{n\in\mathbb{N}}$ is (A, r)-supersaturated;

(C3) there exists $0 < \varepsilon \leq 1$ such that (A, S_n) is ε-rich for every sufficiently large $n \in \mathbb{N}$;

(C4) (A, S_n) is abundant for every sufficiently large $n \in \mathbb{N}$;

(C5) given any $n \in \mathbb{N}$ there exists $X_n \subseteq [k]$ with $|X_n| \geq 3$ such that $p_{X_n}(A, S_n) = \Omega(\hat{p}(A, S_n))$ and for every sequence $(W'_n)_{n\in\mathbb{N}}$ with $W'_n \subsetneq X_n$ and $|W'_n| \geq 2$ we have

$$\frac{p_{W'_n}(A, S_n)}{p_{X_n}(A, S_n)} \to 0$$

as $n \to \infty$.

Then there exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c|S_n|^{-\frac{1}{m_{S_n}(A)}}, \\ 1 & \text{if } p \geq C|S_n|^{-\frac{1}{m_{S_n}(A)}}. \end{cases}$$

Remark 6.7.34. *Note that Lemma 6.7.33 can be further simplified if we consider finite abelian groups and powers of finite subsets of fields separately. Namely, if all sets in $(S_n)_{n\in\mathbb{N}}$ are finite abelian groups then condition (C3) holds since by (6.4) (A, S_n) is 1-rich. On the other hand, if all sets in $(S_n)_{n\in\mathbb{N}}$ are powers of finite subsets of fields and* $\mathrm{rank}_{S_n}(A) > 0$ *for all $n \in \mathbb{N}$, then condition (C5) follows from conditions (C1), (C3)* *and (C4) combined with Lemma 6.7.26. Furthermore, in this case, condition (C4) can be replaced by irredundancy and* 3*-partition regularity; see Lemma 6.7.23.*

If we set $S_n := [n]$ and let A be an irredundant partition regular matrix, then it is easy to check that (C1) and (C3)–(C5) hold. Theorem 6.4.5 gives (C2), and therefore Lemma 6.7.33 recovers the random Rado theorem (i.e., Theorems 6.5.3 and 6.5.4).[18]

Recall that Lemma 6.7.8 confirms the probability threshold essentially depends on the number of (projected) solutions to $Ax = 0$ in S_n. For sequences that satisfy the hypothesis of Lemma 6.7.33, we see that we can express this phenomenon cleanly in terms of the parameter $m_{S_n}(A)$. In particular, if $(S_n)_{n\in\mathbb{N}}$ is a sequence as in Lemma 6.7.33 then, for large $n \in \mathbb{N}$, a typical subset of S_n of size significantly greater than $|S_n|^{1-1/m_{S_n}(A)}$ will be (A, r)-Rado, whilst a typical subset of size significantly smaller than $|S_n|^{1-1/m_{S_n}(A)}$ will not be (A, r)-Rado.

The proof of Lemma 6.7.33 follows easily from our original random Rado lemma, Lemma 6.7.8, combined with the results from the previous subsection.

Proof of Lemma 6.7.33. Let $r \geq 2$, $k \geq 3$ and $\ell \geq 1$ be integers. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence where S_n is either a finite abelian group or $S_n = L^d$ for some finite subset L of a field and $d \in \mathbb{N}$, for every $n \in \mathbb{N}$. Suppose that conditions (C1)–(C5) hold. First we show that conditions (A1)–(A6) hold too.

Conditions (A1) and (A2) are identical to (C1) and (C2) respectively. Condition (A3) follows from condition (C3) and Lemma 6.7.20.

Pick arbitrary $W \subsetneq Y \subseteq [k]$ with $|W| = 1$. As (C3) and (C4) hold, we can apply Lemma 6.7.24 and deduce that $|\operatorname{Sol}^A_{S_n}(w_0, W, Y)| \leq |\operatorname{Sol}^A_{S_n}(Y)|/(\varepsilon|S_n|)$ for every sufficiently large $n \in \mathbb{N}$ and every $w_0 \in S_n$. This immediately implies condition (A4).

As (C3) and (C4) hold, we can apply Lemma 6.7.25 to A and S_n, for any sufficiently

[18] We formally verify this in the more general setting of integer lattices in the proof of Theorem 6.6.1 in Section 6.12.

large $n \in \mathbb{N}$, to obtain the following: for any $W \subseteq [k]$ with $|W| = 2$, we have

$$\frac{|S_n|^2}{|\operatorname{Sol}_{S_n}^A(W)|} \leq \frac{1}{\varepsilon}.$$

Combining the above with conditions (C1) and (C5) yields conditions (A6) and (A5) respectively.

As conditions (A1)–(A6) hold, by Lemma 6.7.8 there exist constants $c_0, C_0 > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c_0 \cdot \hat{p}(A, S_n), \\ 1 & \text{if } p \geq C_0 \cdot \hat{p}(A, S_n). \end{cases} \tag{6.14}$$

Condition (C4) implies $m_{S_n}(A)$ is well-defined for every sufficiently large $n \in \mathbb{N}$ (see, e.g., Remark 6.7.28). Condition (C3) combined with Lemma 6.7.32 implies that, for sufficiently large $n \in \mathbb{N}$,

$$|S_n|^{-\frac{1}{m_{S_n}(A)}} \leq \hat{p}(A, S_n) \leq (1/\varepsilon^2)|S_n|^{-\frac{1}{m_{S_n}(A)}}. \tag{6.15}$$

Let $C := (1/\varepsilon^2)C_0$ and $c := c_0$. Combining (6.14) and (6.15) yields

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c|S_n|^{-\frac{1}{m_{S_n}(A)}}, \\ 1 & \text{if } p \geq C|S_n|^{-\frac{1}{m_{S_n}(A)}}. \end{cases}$$

□

6.8 The general 1-statement and 0-statement for hypergraphs

In this section, we formulate our general random Ramsey 1-statement and 0-statement for hypergraphs. We combine both statements in a single result (Theorem 6.8.7), but first we provide some definitions and motivation.

6.8.1 Basic definitions for hypergraphs

Definition 6.8.1 (Hypergraphs)**.** Let $k \in \mathbb{N}$. A *k-uniform hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ is a family of k-sets $\{v_1, \dots, v_k\} \in V(\mathcal{H})^k$ with $v_i \neq v_j$ for $i \neq j$. A *k-uniform ordered hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ is a set of *ordered* k-sets $(v_1, \dots, v_k) \in V(\mathcal{H})^k$ with $v_i \neq v_j$ for $i \neq j$.

We refer to $V(\mathcal{H})$ and $E(\mathcal{H})$ as the *vertex set* and the *edge set* of $\mathcal{H}$, respectively. We write $v(\mathcal{H}) := |V(\mathcal{H})|$ and $e(\mathcal{H}) := |E(\mathcal{H})|$. Next, we define the analogues of the notions of *partition regularity* and *supersaturation* for hypergraphs and ordered hypergraphs.

Definition 6.8.2 (r-Ramsey, (r, ε)-supersaturated and r-supersaturated)**.** Let $\mathcal{H}$ be an (ordered) hypergraph, $r \in \mathbb{N}$ and $\varepsilon > 0$. We say $\mathcal{H}$ is *r-Ramsey* if every r-colouring of $V(\mathcal{H})$ yields a monochromatic edge in $E(\mathcal{H})$. Similarly, $\mathcal{H}$ is *(r, ε)-supersaturated* if every r-colouring of $V(\mathcal{H})$ yields at least $\varepsilon e(\mathcal{H})$ monochromatic edges. We say that a sequence $(\mathcal{H}_n)_{n \in \mathbb{N}}$ of (ordered) hypergraphs is *r-supersaturated* if there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the (ordered) hypergraph $\mathcal{H}_n$ is (r, ε)-supersaturated.

Before proceeding further, we give an informal explanation of how a Ramsey problem can be rephrased in terms of (ordered) hypergraphs. Say $\mathcal{F}$ is some discrete object (e.g., a graph, an abelian group etc.) over a ground set of elements. Given a colouring of the elements in $\mathcal{F}$, we are interested in finding a monochromatic copy of a certain structure F in $\mathcal{F}$. We consider the (ordered) hypergraph $\mathcal{H}$ whose vertex set is the set of elements of $\mathcal{F}$ and the edge set is the set of copies of F in $\mathcal{F}$. Clearly, the colouring of the elements of $\mathcal{F}$ induces a colouring of the vertices of $\mathcal{H}$. In particular, every monochromatic copy of F in $\mathcal{F}$ corresponds to a monochromatic edge in $\mathcal{H}$ and vice versa. Consider the following example, where $\mathcal{F} := [n]$ and F is a 3-distinct solution to the equation $x + y = z$.

Example 6.8.3. *Theorem 6.4.5 implies that there exists $c > 0$ so that, if n is sufficiently large, then any 2-colouring of $[n]$ yields at least cn^2 many 3-distinct monochromatic so-*

lutions to the equation $x + y = z$.[19] *This is equivalent to say that the* 3*-uniform ordered hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = [n]$ and $E(\mathcal{H}) = \{(x, y, z) : x+y = z\}$ is $(2, c')$-supersaturated for some constant $c' > 0$, provided $v(\mathcal{H})$ is sufficiently large. Similarly, saying that any* 2*-colouring of $[n]$ yields a monochromatic solution to the equation $x+y = z$ (provided n is sufficiently large) is equivalent to $\mathcal{H}$ being* 2*-Ramsey (provided $v(\mathcal{H})$ is sufficiently large).*

In practice, we can usually work in the setting of *unordered* hypergraphs.

Definition 6.8.4. Given an ordered hypergraph $\mathcal{H}$, the *unordered restriction of $\mathcal{H}$* is the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by ignoring the order of the edges in $E(\mathcal{H})$. Formally, $V(\mathcal{H}') := V(\mathcal{H})$ and

$$E(\mathcal{H}') := \{\{v_1, \dots, v_k\} : (v_1, \dots, v_k) \in E(\mathcal{H})\}.$$

Note that we do not allow multiple edges here; that is, if two ordered edges e_1, e_2 become the same unordered edge e, then we only keep one copy of e.

Remark 6.8.5. *By definition, a k-uniform ordered hypergraph $\mathcal{H}$ is r-Ramsey if and only if the unordered restriction $\mathcal{H}'$ of $\mathcal{H}$ is also r-Ramsey. Similarly, a sequence of k-uniform ordered hypergraphs $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is r-supersaturated if and only if the sequence $(\mathcal{H}'_n)_{n\in\mathbb{N}}$ is r-supersaturated, where $\mathcal{H}'_n$ is the unordered restriction of $\mathcal{H}_n$.*

Next, we define notions of maximum degree for (ordered) hypergraphs. Let $\mathcal{H}$ be a k-uniform ordered hypergraph and let $W \subseteq [k]$, say $W = \{i_1, i_2, \dots, i_{|W|}\}$ where $i_1 < i_2 < \cdots < i_{|W|}$. The *restriction* $\mathcal{H}_W$ of $\mathcal{H}$ is the $|W|$-uniform ordered hypergraph with $V(\mathcal{H}_W) := V(\mathcal{H})$ and

$$E(\mathcal{H}_W) := \{(x_{i_1}, \dots, x_{i_{|W|}}) : \exists\, (y_1, \dots, y_k) \in E(\mathcal{H}) \text{ with } y_{i_s} = x_{i_s} \text{ for each } s \in [|W|]\}.$$

Again, we do not allow multiple edges. Given sets $W \subseteq Y \subseteq [k]$, we define $\Delta_W(\mathcal{H}_Y)$ to

[19] While the statement of Theorem 6.4.5 does not insist that the solutions are 3-distinct, since there are at most $O(n)$ solutions to $x + y = z$ which are not 3-distinct, it is easy to obtain the result as stated here.

be the maximum number of edges in $\mathcal{H}_Y$ that restrict to the same edge in $\mathcal{H}_W$. More precisely, given any ordered tuple $e^* \in E(\mathcal{H}_W)$, let $E_{\mathcal{H}}(e^*, Y)$ be the set of ordered tuples $e \in E(\mathcal{H}_Y)$ so that e^* equals the ordered tuple obtained from e by deleting the entries of e in the positions corresponding to $Y \setminus W$. Then we define

$$\Delta_W(\mathcal{H}_Y) := \max_{e^* \in E(\mathcal{H}_W)} |E_{\mathcal{H}}(e^*, Y)|.$$

Let $\mathcal{H}'$ be a k-uniform hypergraph. For $i \in [k]$, we set

$$\Delta_i(\mathcal{H}') := \max_{e^* \subseteq V(\mathcal{H}'),\, |e^*|=i} |\{e \in E(\mathcal{H}') : e^* \subseteq e\}|.$$

Note that if $\mathcal{H}'$ is the unordered restriction of an ordered hypergraph $\mathcal{H}$ then

$$\frac{1}{k!} \cdot \max_{\substack{W \subseteq [k] \\ |W|=i}} \Delta_W(\mathcal{H}) \leq \frac{1}{(k-i)!} \cdot \max_{\substack{W \subseteq [k] \\ |W|=i}} \Delta_W(\mathcal{H}) \leq \Delta_i(\mathcal{H}') \tag{6.16}$$

$$\Delta_i(\mathcal{H}') \leq \binom{k}{i} \cdot i! \cdot \max_{\substack{W \subseteq [k] \\ |W|=i}} \Delta_W(\mathcal{H}) \leq k! \cdot \max_{\substack{W \subseteq [k] \\ |W|=i}} \Delta_W(\mathcal{H}). \tag{6.17}$$

Inequality (6.17) uses that, given any edge e in $\mathcal{H}'$, there are $(k-i)!$ ordered k-sets that have e as their underlying (unordered) set and have the entries in i of their positions fixed.

For an (ordered) hypergraph $\mathcal{H}$, we write $\mathcal{H}_p^{\mathrm{ind}}$ to denote the hypergraph obtained from $\mathcal{H}$ by including each *vertex* of $\mathcal{H}$ with probability p independently of all other vertices, while including each edge of $\mathcal{H}$ if and only if all of its vertices are kept. The superscript "ind" stand for "induced", as $\mathcal{H}_p^{\mathrm{ind}}$ can be seen as the subhypergraph of $\mathcal{H}$ induced by the vertices which are selected at random.

6.8.2 Heuristic for the probability threshold

The random Ramsey-type results we covered in Section 6.3 (for the primes), in Section 6.5 (for graphs and integers) and in Section 6.6 (for abelian groups) are formulated as follows: given a sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of objects that exhibit a Ramsey property in a robust way (i.e., supersaturation), there is probability threshold $\hat{p} = \hat{p}(n)$ and constants $0 < c < C$ such that, as n tends to infinity, the random binomial subset $\mathcal{F}_{n,p}$ retains the Ramsey property w.h.p. if $p \geq C\hat{p}$ and loses it w.h.p. if $p \leq c\hat{p}$. Our general 1-statement and 0-statement for hypergraphs are formulated in a similar fashion.

Definition 6.8.6 (Probability threshold for hypergraphs)**.** Given a sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$ of k-uniform ordered hypergraphs and $W \subseteq [k]$, let

$$f_{n,W} := \left(\frac{e(\mathcal{H}_{n,W})}{v(\mathcal{H}_n)}\right)^{-\frac{1}{|W|-1}} \quad \text{and} \quad \hat{p}(\mathcal{H}_n) := \max_{\substack{W\subseteq[k]\\|W|\geq 2}} f_{n,W}. \tag{6.18}$$

Note that in Definition 6.8.6 we slightly abuse notation, writing $\mathcal{H}_{n,W}$ for the restriction $\mathcal{H}_{n,W}$ of $\mathcal{H}_n$.

Our choice of $\hat{p}(\mathcal{H}_n)$ follows precisely the same heuristic as the probability threshold for the random Ramsey problem for graphs and the integers (see Section 6.5). Namely, consider the expected number of vertices and edges in the random hypergraph $\mathcal{H}^{\text{ind}}_{n,p}$. Assuming that the edges in $\mathcal{H}^{\text{ind}}_{n,p}$ are distributed uniformly in some way, if $\mathbb{E}(e(\mathcal{H}^{\text{ind}}_{n,p}))$ is much less than $\mathbb{E}(v(\mathcal{H}^{\text{ind}}_{n,p}))$ then intuitively the edges in $\mathcal{H}^{\text{ind}}_{n,p}$ should be fairly spread out. Therefore, one should be able to 2-colour the vertices of $\mathcal{H}^{\text{ind}}_{n,p}$ without creating monochromatic edges.

However, similarly to the random Ramsey problem for graphs and the integers, it may be the case that the above reasoning is applicable to a 'lower' level in the following sense. For any $W \subseteq [k]$, a 2-colouring that does not yield monochromatic edges in $\mathcal{H}^{\text{ind}}_{n,W,p}$ also does not yield monochromatic edges in $\mathcal{H}^{\text{ind}}_{n,p}$. By the previous reasoning, if $\mathbb{E}(e(\mathcal{H}^{\text{ind}}_{n,W,p}))$ is much less than $\mathbb{E}(v(\mathcal{H}^{\text{ind}}_{n,W,p}))$ then one should be able to 2-colour the vertices of $\mathcal{H}^{\text{ind}}_{n,W,p}$ while avoiding any monochromatic edge in $\mathcal{H}^{\text{ind}}_{n,W,p}$, and therefore in $\mathcal{H}^{\text{ind}}_{n,p}$. For any $W \subseteq [k]$ with $|W| \geq 2$, we have

$$\mathbb{E}(v(\mathcal{H}^{\text{ind}}_{n,W,p})) = pv(\mathcal{H}_n) \qquad \text{and} \qquad \mathbb{E}(e(\mathcal{H}^{\text{ind}}_{n,W,p})) = p^{|W|}e(\mathcal{H}_{n,W}).$$

In particular, $\mathbb{E}(e(\mathcal{H}^{\mathrm{ind}}_{n,W,p})) \leq \mathbb{E}(v(\mathcal{H}^{\mathrm{ind}}_{n,W,p}))$ if and only if

$$p \leq \left(\frac{e(\mathcal{H}_{n,W})}{v(\mathcal{H}_n)}\right)^{-\frac{1}{|W|-1}} = f_{n,W}.$$

In conclusion, according to the heuristic argument above, if $p \leq \hat{p}(\mathcal{H}_n)$ then one should be able to 2-colour the vertices of $\mathcal{H}^{\mathrm{ind}}_{n,p}$ while avoiding any monochromatic edge in $\mathcal{H}^{\mathrm{ind}}_{n,p}$.

6.8.3 General 1- and 0-statements for hypergraphs

We are now ready to state our general 1-statement and 0-statement for hypergraphs.

Theorem 6.8.7. *Let $r, k \geq 2$ be integers, and let $b > 0$. Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform ordered hypergraphs and set $\hat{p}_n := \hat{p}(\mathcal{H}_n)$ as given by (6.18). Let (P1)–(P5) be the following properties.*

(P1) Probabilities: $\hat{p}_n \to 0$ and $\hat{p}_n v(\mathcal{H}_n) \to \infty$ as $n \to \infty$.

(P2) Supersaturation: $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is r-supersaturated.

(P3) Bounded degree: for any $W \subseteq Y \subseteq [k]$, and any sufficiently large $n \in \mathbb{N}$ we have $\Delta_W(\mathcal{H}_{n,Y}) \leq b\frac{e(\mathcal{H}_{n,Y})}{e(\mathcal{H}_{n,W})}$.

(P4) Bounded 1-degree: for any $W \subsetneq Y \subseteq [k]$ with $|W| = 1$ and any sufficiently large $n \in \mathbb{N}$, we have $\Delta_W(\mathcal{H}_{n,Y}) \leq b\frac{e(\mathcal{H}_{n,Y})}{v(\mathcal{H}_n)}$.

(P5) Bounded 2-degree: there exists a sequence $(X_n)_{n\in\mathbb{N}}$ of subsets $X_n \subseteq [k]$ with $|X_n| \geq 3$ for all $n \in \mathbb{N}$ such that

$$f_{n,X_n} = \Omega(\hat{p}_n) \tag{6.19}$$

and

$$\Delta_{W_n}(\mathcal{H}_{n,X_n}) \cdot \Delta_{W'_n}(\mathcal{H}_{n,X_n}) \cdot e(\mathcal{H}_{n,X_n}) \cdot (\hat{p}_n)^{3|X_n|-2-|W'_n|} = o(1) \tag{6.20}$$

for any sequences $(W_n)_{n\in\mathbb{N}}$ and $(W'_n)_{n\in\mathbb{N}}$ such that $W_n, W'_n \subsetneq X_n$ with $|W_n| = 2$ and $|W'_n| \geq 2$.

If (P1)–(P4) hold, then there exists a constant $C > 0$ such that if $q_n \geq C\hat{p}_n$ then

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{H}^{ind}_{n,q_n} \textit{ is } r\textit{-Ramsey}] = 1.$$

If (P1) and (P3)–(P5) hold, then there exists a constant $c > 0$ such that if $q_n \leq c\hat{p}_n$ then

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{H}^{ind}_{n,q_n} \textit{ is } r\textit{-Ramsey}] = 0.$$

All of our new random Ramsey-type results are essentially corollaries of Theorem 6.8.7. Indeed, we use Theorem 6.8.7 to prove the random Rado lemma (Lemma 6.7.8) which we have in turn used to prove the simplified random Rado lemma (Lemma 6.7.33). Further, the proofs of our new results for the primes (Theorem 6.3.7), integer lattices (Theorem 6.6.1), groups (Theorem 6.6.3) and finite vector spaces (Theorems 6.6.4 and 6.6.5) all apply one of these two random Rado lemmas (see Section 6.12). This in turn implies that the random Rado theorem (Theorems 6.5.3 and 6.5.4) is a consequence of Theorem 6.8.7. As we describe in Section A.3 of Appendix A, the 1-statement of the random Ramsey theorem (Theorem 6.5.1) can also be deduced from Theorem 6.8.7.

The reader should therefore take away the following philosophy. Theorem 6.8.7 is the most general of our black box results, and its applications are not just restricted to systems of linear equations. On the other hand, the random Rado lemma is more readily applicable to the setting of Ramsey properties in arithmetic structures. The simplified random Rado lemma has the most restricted hypothesis of these three results, but has the advantage that it explicitly produces the corresponding probability threshold.

The proof of Theorem 6.8.7 is given in Section 6.9. The conditions required for the 1-statement to hold, namely (P1)–(P4), are natural. Condition (P1) is needed to ensure the expected value of $v(\mathcal{H}^{\text{ind}}_{n,p})$ tends to infinity. In Section 6.4.2, we already discussed the notion of supersaturation (condition (P2)) and how it is typically required in random Ramsey-type results. Properties (P3) and (P4) are recurring assumptions when applying the hypergraph container method, which we indeed use to prove Theorem 6.8.7. They are also natural conditions to make to ensure our heuristic described in Section 6.8.2 is valid. Indeed, (P3) and (P4) ensure that edges are fairly spread out in $\mathcal{H}_n$ and its associated restrictions $\mathcal{H}_{n,Y}$.

On the other hand, condition (P5) is more artificial: it arises naturally in the setting of abelian groups but fails to hold in general. Indeed, while the 0-statement of the random Rado lemma (Lemma 6.7.8) follows from Theorem 6.8.7, the 0-statement for random Ramsey theorem (Theorem 6.5.1) does not. In Section 6.9 we explain how the 0-statement of Theorem 6.8.7 follows from the combination of a *deterministic lemma*, stating that certain substructures must arise in the hypergraph assuming it is r-Ramsey, and a *probabilistic lemma*, stating that such structures do not appear w.h.p. The proof of the latter consists of bounding the expected number of the forbidden structures so that the expectation tends to 0 as n goes to infinity. Condition (P5) is precisely designed to achieve these bounds.

Returning to Theorem 6.5.1, its 0-statement is proved by using a much stronger deterministic lemma than what we use here. In particular, this statement says that graphs with density at most $m_2(H)$ can be 2-coloured so to avoid a monochromatic copy of H. Using this strong result, one can prove a suitable corresponding probabilistic lemma without needing (P5) to be satisfied.

Note that the use of ordered hypergraphs in Theorem 6.8.7 is only really necessary for the 0-statement. Indeed, we will deduce the 1-statement of Theorem 6.8.7 very quickly from a theorem for unordered hypergraphs, Theorem 6.9.1. The proof of Theorem 6.9.1 is obtained via the method of containers, which was first formalised in the setting of hypergraphs by Balogh, Morris and Samotij [10] and independently by Saxton and Thomason [120]. The use of containers is now standard in proofs of 1-statements for various Ramsey-type problems, see, e.g., [1, 10, 12, 60, 61, 73, 76, 104, 105, 127]. We emphasise that the proof of Theorem 6.9.1 is not particularly difficult. Indeed, a researcher familiar with the container method would be able to come up with such a proof, and in particular, our proof here is an easy adaption of the short proof of the 1-statement of Theorem 6.5.1 given by Nenadov and Steger [105]. The proof of the 0-statement of Theorem 6.8.7 adapts a recent argument of Hancock and Treglown [77].

There are several results in the literature of a similar flavour to Theorem 6.8.7. Indeed, Friedgut, Rödl and Schacht proved a similar result to our 1-statement; see [62, Theorem 2.5]. In their hypothesis though, they require the sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$ to be R-supersaturated for some R much larger than r. Furthermore, Aigner-Horev and Person [1, Theorem 2.8] proved an *asymmetric* version of the 1-statement of Theorem 6.8.7, where one considers a sequence of r-sets of hypergraphs $(\mathcal{H}_{n,1},\dots,\mathcal{H}_{n,r})_{n\in\mathbb{N}}$ on the same vertex set V_n and ask for the probability threshold such that whenever the vertices are r-coloured, there is some $i\in[r]$ such that there is an edge in $\mathcal{H}^{\mathrm{ind}}_{n,i,p}$ which is monochromatic in colour i. Note that their result requires some rather technical conditions on the $\mathcal{H}_{n,i}$ which are not necessary in the symmetric setting.

Finally, in their recent paper on sharp Ramsey thresholds, Friedgut, Kuperwasser, Samotij and Schacht [61, Theorem 2.1] gave general criteria for a sequence of hypergraphs $(\mathcal{H}_n)_{n\in\mathbb{N}}$ to have a sharp threshold for being r-Ramsey. The threshold function in their theorem is $\Theta(f_{n,[k]})$, which is the value of our $\hat{p}_n$ in the case where $W=[k]$ is the maximiser of the $f_{n,W}$'s: note that this is indeed the case when considering the Schur property or sets avoiding monochromatic k-APs in $\mathbb{Z}_n$; as mentioned at the end of Section 6.5, these are two of the specific settings for which sharp thresholds are obtained in [61].

6.9 Proof of Theorem 6.8.7

6.9.1 Proof of the 1-statement of Theorem 6.8.7

In this subsection we state a general 1-statement, Theorem 6.9.1 for *hypergraphs*. We then show that the 1-statement for ordered hypergraphs (Theorem 6.8.7) follows immediately from it. We prove Theorem 6.9.1 in Section 6.9.1.3.

6.9.1.1 The Ramsey property in supersaturated hypergraphs

Theorem 6.9.1. *Let $r, k \geq 2$ be integers and let $c > 0$. Suppose that $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of k-uniform hypergraphs. Let $(p_n)_{n\in\mathbb{N}}$ with $p_n \in (0,1)$ be such that $p_n \to 0$ and $p_n v(\mathcal{H}_n) \to \infty$ as $n \to \infty$, and for each sufficiently large $n \in \mathbb{N}$ and each $\ell \in [k]$, we have*

$$\Delta_\ell(\mathcal{H}_n) \leq c(p_n)^{\ell-1}\frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}.$$

Further, suppose that $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is r-supersaturated. Then there exists $C \geq 1$ such that for $q_n \geq Cp_n$,

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{H}^{ind}_{n,q_n} \text{ is } r\text{-Ramsey}] = 1.$$

The proof of Theorem 6.9.1 is in the same spirit as that of the proof of the 1-statement of Theorem 6.5.1 given by Nenadov and Steger in [105]. In particular, as mentioned in Section 6.8, it utilises the hypergraph container method.

For a set $\mathcal{A}$, let $\mathcal{P}(\mathcal{A})$ denote the powerset of $\mathcal{A}$. For a hypergraph $\mathcal{H}$, let $\mathcal{I}_r(\mathcal{H})$ be the collection of r-tuples $(I_1, \dots, I_r)$ such that each $I_i \in \mathcal{P}(V(\mathcal{H}))$ is an independent set in $\mathcal{H}$ and $I_i \cap I_j = \emptyset$ for every $i \neq j \in [r]$. For a sequence of hypergraphs $(\mathcal{H}_n)_{n\in\mathbb{N}}$, to show that w.h.p. $\mathcal{H}^{\mathrm{ind}}_{n,p}$ is r-Ramsey, it is equivalent to show that with probability tending to 0 the vertex set of $\mathcal{H}^{\mathrm{ind}}_{n,p}$ can be partitioned into r parts such that each part induces an independent set, i.e., it admits an r-partition $I \in \mathcal{I}_r(\mathcal{H}_n)$. Bounding this probability by using a union bound over all r-tuples in $\mathcal{I}_r(\mathcal{H}_n)$ does not work because $|\mathcal{I}_r(\mathcal{H}_n)|$ is too large. The power of a hypergraph container result is that one can instead use a union bound over all 'containers', since informally such a result says that every independent set lies in a container, and there are not too many containers overall.

6.9.1.2 Deducing the 1-statement of Theorem 6.8.7 from Theorem 6.9.1

Proof of the 1*-statement of Theorem 6.8.7.* Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform ordered hypergraphs satisfying (P1)–(P4). For all $W \subseteq [k]$ with $|W| \geq 2$ we have

$$\hat{p}_n = \max_{\substack{X\subseteq[n] \\ |X|\geq 2}} \left(\frac{e(\mathcal{H}_{n,X})}{v(\mathcal{H}_n)}\right)^{\frac{-1}{|X|-1}} \geq \left(\frac{e(\mathcal{H}_{n,W})}{v(\mathcal{H}_n)}\right)^{\frac{-1}{|W|-1}};$$

$$\implies e(\mathcal{H}_{n,W}) \geq (\hat{p}_n)^{-(|W|-1)} \cdot v(\mathcal{H}_n). \tag{6.21}$$

Therefore, for such $W \subseteq [k]$ with $|W| \geq 2$, and each $n \in \mathbb{N}$ sufficiently large, we have

$$\Delta_W(\mathcal{H}_n) \overset{\text{(P3)}}{\leq} b\frac{e(\mathcal{H}_n)}{e(\mathcal{H}_{n,W})} \overset{(6.21)}{\leq} b \cdot (\hat{p}_n)^{|W|-1} \cdot \frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}. \tag{6.22}$$

For any $W \subseteq [k]$ with $|W| = 1$, and each $n \in \mathbb{N}$ sufficiently large, we have

$$\Delta_W(\mathcal{H}_n) \overset{\text{(P4)}}{\leq} b\frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)} = b \cdot (\hat{p}_n)^{|W|-1} \cdot \frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}. \tag{6.23}$$

For each $n \in \mathbb{N}$, let $\mathcal{H}'_n$ be the unordered restriction of $\mathcal{H}_n$. We now check the hypothesis of Theorem 6.9.1 with respect to $(\mathcal{H}'_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ where $p_n := \hat{p}_n$. (P1) implies $p_n \to 0$ and $p_n v(\mathcal{H}'_n) \to \infty$ as $n \to \infty$. Remark 6.8.5 and (P2) imply $(\mathcal{H}'_n)_{n\in\mathbb{N}}$ is r-supersaturated. Finally, equations (6.22), (6.23), (6.16) and (6.17) yield the required upper bound on $\Delta_\ell(\mathcal{H}'_n)$ where $c := b \cdot k!$.

Now, by Remark 6.8.5, the conclusion of Theorem 6.9.1 implies the 1-statement of Theorem 6.8.7. □

6.9.1.3 Proof of Theorem 6.9.1

First, we introduce some notation. Let $\mathcal{H}$ be a hypergraph and let $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$. We say $\mathcal{F}$ is *increasing* if $A \in \mathcal{F}$ and $A \subseteq B \subseteq V(\mathcal{H})$ imply $B \in \mathcal{F}$. We write $\overline{\mathcal{F}}$ to denote the complement family of $\mathcal{F}$, i.e., the family of sets $A \in \mathcal{P}(V(\mathcal{F}))$ which are not in $\mathcal{F}$. Finally, for $\varepsilon \in (0, 1]$, we say that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense if $e(\mathcal{H}[A]) \geq \varepsilon e(\mathcal{H})$ for every $A \in \mathcal{F}$.

For this proof, when we consider r-tuples of sets, the r-tuples are *always* ordered. For two r-tuples of sets $I = (I_1, \dots, I_r)$ and $J = (J_1, \dots, J_r)$ we write $I \subseteq J$ if $I_i \subseteq J_i$ for all $i \in [r]$; we define $I \cup J := (I_1 \cup J_1, \dots, I_r \cup J_r)$. If $\mathcal{X}$ is a collection of sets, then we write $\mathcal{X}^r$ for the collection of r-tuples $(X_1, \dots, X_r)$ so that $X_i \in \mathcal{X}$ for all $i \in [r]$.

We now state the following container result from [76], from which Theorem 6.9.1 will follow. Note that Proposition 6.9.2 itself follows from the general hypergraph container theorem of Balogh, Morris and Samotij [10, Theorem 2.2].

Proposition 6.9.2. *[76, Proposition 3.2] For every $k, r \in \mathbb{N}$ and all $c, \varepsilon > 0$, there exists $D > 0$ such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and let $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ be an increasing family of sets such that $|A| \geq \varepsilon v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0,1)$ is such that, for every $\ell \in [k]$,*

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then there exists a family $\mathcal{S}_r \subseteq \mathcal{I}_r(\mathcal{H})$ and functions $f : \mathcal{S}_r \to (\overline{\mathcal{F}})^r$ and $g : \mathcal{I}_r(\mathcal{H}) \to \mathcal{S}_r$ such that the following conditions hold:

1. *If $(S_1, \dots, S_r) \in \mathcal{S}_r$ then $\sum_{i=1}^r |S_i| \leq Dp \cdot v(\mathcal{H})$;*

2. *for every $(I_1, \dots, I_r) \in \mathcal{I}_r(\mathcal{H})$, we have that $S \subseteq (I_1, \dots, I_r) \subseteq S \cup f(S)$ where $S := g(I_1, \dots, I_r)$.*

Proof of Theorem 6.9.1. Note that it suffices to prove the theorem under the additional assumption that $c \geq k$. Pick $\varepsilon > 0$ so that, for all sufficiently large $n \in \mathbb{N}$, the hypergraph $\mathcal{H}_n$ is (r, ε)-supersaturated. Define

$$\varepsilon' := \frac{\varepsilon}{3}, \quad \varepsilon'' := \frac{\varepsilon' k}{c} \quad \text{and} \quad \delta := \frac{\varepsilon}{3c}. \tag{6.24}$$

As $c \geq k$ we have that $\varepsilon'' \leq \varepsilon'$. Apply Proposition 6.9.2 with ε'' playing the role of ε to obtain $D > 0$. Since $p_n \to 0$ as $n \to \infty$, if n is sufficiently large then

$$cDp_n \leq \varepsilon'. \tag{6.25}$$

Next, we pick a sufficiently large constant $C > 0$ so that the last equality (†) in (6.26) below holds for all sufficiently large $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be sufficiently large and set $\mathcal{F}_n := \{F \subseteq V(\mathcal{H}_n) : e(\mathcal{H}_n[F]) \geq \varepsilon' e(\mathcal{H}_n)\}$. Clearly $\mathcal{F}_n$ is increasing and $\mathcal{H}_n$ is $(\mathcal{F}_n, \varepsilon')$-dense, and thus also $(\mathcal{F}_n, \varepsilon'')$-dense. Further, for all $F \in \mathcal{F}_n$, we have[20]

$$k\varepsilon' e(\mathcal{H}_n) \leq ke(\mathcal{H}_n[F]) = \sum_{y \in F} \deg_{\mathcal{H}_n[F]}(y) \leq |F|\Delta_1(\mathcal{H}_n) \leq \frac{|F|c \cdot e(\mathcal{H}_n)}{v(\mathcal{H}_n)},$$

where the last inequality uses the bound on $\Delta_1(\mathcal{H}_n)$ in the statement of Theorem 6.9.1. Rearranging, we obtain

$$|F| \geq \frac{\varepsilon' k}{c} v(\mathcal{H}_n) \stackrel{(6.24)}{=} \varepsilon'' v(\mathcal{H}_n),$$

for all $F \in \mathcal{F}_n$. Thus $\mathcal{H}_n$ and $\mathcal{F}_n$ satisfy the hypothesis of Proposition 6.9.2 with ε'' and p_n playing the roles of ε and p respectively. Apply Proposition 6.9.2 to obtain the family $\mathcal{S}_r \subseteq \mathcal{I}_r(\mathcal{H}_n)$ and functions $f : \mathcal{S}_r \to (\overline{\mathcal{F}}_n)^r$ and $g : \mathcal{I}_r(\mathcal{H}_n) \to \mathcal{S}_r$.

Now let $q_n \geq Cp_n$ and let B_{q_n} be the event that $\mathcal{H}^{\mathrm{ind}}_{n,q_n}$ is not r-Ramsey. It suffices to show that $\mathbb{P}[B_{q_n}] \to 0$ as $n \to \infty$. Since $q_n \leq q'_n \leq 1$ implies $\mathbb{P}[B_{q'_n}] \leq \mathbb{P}[B_{q_n}]$, we may assume that $q_n = Cp_n$.

Suppose that $\mathcal{H}^{\mathrm{ind}}_{n,q_n}$ is not r-Ramsey, i.e., there exists a partition $V(\mathcal{H}^{\mathrm{ind}}_{n,q_n}) = X_1 \cup \cdots \cup X_r$ with $(X_1, \ldots, X_r) \in \mathcal{I}_r(\mathcal{H}_n)$. By Proposition 6.9.2, there exists $S := (S_1, \ldots, S_r) := g(X_1, \ldots, X_r) \in \mathcal{S}_r$ such that $S \subseteq (X_1, \ldots, X_r) \subseteq S \cup f(S)$ and $|\cup_{i \in [r]} S_i| \leq Dp_n v(\mathcal{H}_n)$. For brevity, we write $f(S) =: (f(S_1), \ldots, f(S_r))$, $X(S) := \cup_{i\in[r]} S_i$, $Y(S) := \cup_{i\in[r]} f(S_i)$ and $Z(S) := X(S) \cup Y(S)$.[21]

[20]Here $\deg_{\mathcal{H}_n[F]}(y)$ denotes the number of edges y lies in within $\mathcal{H}_n[F]$.
[21]Note the slight abuse of the f notation here.

Claim 6.9.3. *We have $|Z(S)| \leq (1-\delta)v(\mathcal{H}_n)$.*

Proof of the claim. Since $\mathcal{H}_n$ is (r,ε)-supersaturated, in any partition $V(\mathcal{H}_n) = V_1 \cup \cdots \cup V_r$ there exists $i \in [r]$ such that $e(\mathcal{H}_n[V_i]) \geq \varepsilon e(\mathcal{H}_n)$. Any vertex in $\mathcal{H}_n$ is contained in at most $\Delta_1(\mathcal{H}_n)$ edges, and thus by deleting a set U of fewer than $\delta v(\mathcal{H}_n)$ vertices, we delete at most $|U|\Delta_1(\mathcal{H}_n) < \delta c e(\mathcal{H}_n)$ edges. Thus, given any induced subhypergraph of $\mathcal{H}_n$ with more than $(1-\delta)v(\mathcal{H}_n)$ vertices, for any partition $W_1 \cup \cdots \cup W_r$ of it, there exists $i \in [r]$ such that $e(\mathcal{H}_n[W_i]) > (\varepsilon - \delta c)e(\mathcal{H}_n) = 2\varepsilon' e(\mathcal{H}_n)$ edges (where the last equality follows from (6.24)).

Now we bound $e(\mathcal{H}_n[S_i \cup f(S_i)])$ for each $i \in [r]$. Since $f(S_i) \in \overline{\mathcal{F}}_n$ we have $e(\mathcal{H}_n[f(S_i)]) < \varepsilon' e(\mathcal{H}_n)$. The number of edges in $\mathcal{H}_n[S_i \cup f(S_i)]$ containing at least one vertex from S_i is at most

$$|S_i| \cdot \Delta_1(\mathcal{H}_n) \leq Dp_n v(\mathcal{H}_n) \cdot c\frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)} \overset{(6.25)}{\leq} \varepsilon' e(\mathcal{H}_n).$$

Thus, in total we have $e(\mathcal{H}_n[S_i \cup f(S_i)]) < 2\varepsilon' e(\mathcal{H}_n)$. Overall we see that the set $Z(S)$ can be partitioned into r sets so that each set induces fewer than $2\varepsilon' e(\mathcal{H}_n)$ edges in $\mathcal{H}_n$. Combining with the conclusion of the previous paragraph, this implies that $|Z(S)| \leq (1-\delta)v(\mathcal{H}_n)$ as required.

■

As $X_1 \cup \cdots \cup X_r$ is a partition of $V(\mathcal{H}^{\text{ind}}_{n,q_n})$, we must have that $X(S) \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n}) \subseteq Z(S)$. Crucially, this event depends uniquely on S and thus

$$\mathbb{P}[B_{q_n}] \leq \sum_{S \in \mathcal{S}_r} \mathbb{P}[X(S) \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n}) \subseteq Z(S)] \leq \sum_{S \in \mathcal{S}_r} q_n^{|X(S)|}(1-q_n)^{\delta v(\mathcal{H}_n)} \leq \sum_{S \in \mathcal{S}_r} q_n^{|X(S)|} e^{-\delta q_n v(\mathcal{H}_n)},$$

where the first inequality follows from the union bound, while the second inequality follows from Claim 6.9.3.

Recall that for every $S = (S_1, \ldots, S_r) \in \mathcal{S}_r$ we have $|\cup_{i \in [r]} S_i| \leq Dp_n v(\mathcal{H}_n)$. We may bound $|\mathcal{S}_r|$ from above as follows: choose a set of $i \leq Dp_n v(\mathcal{H}_n)$ vertices from $V(\mathcal{H}_n)$ and then take an (ordered) partition of these vertices into r classes. Therefore we have,

$$\begin{aligned}
\mathbb{P}[B_{q_n}] \leq \sum_{S\in\mathcal{S}_r} q_n^{|X(S)|} e^{-\delta q_n v(\mathcal{H}_n)} &\leq \sum_{i=0}^{Dp_n v(\mathcal{H}_n)} \binom{v(\mathcal{H}_n)}{i} r^i q_n^i e^{-\delta q_n v(\mathcal{H}_n)} \\
&\leq (Dp_n v(\mathcal{H}_n)+1)(rq_n)^{Dp_n v(\mathcal{H}_n)} \binom{v(\mathcal{H}_n)}{Dp_n v(\mathcal{H}_n)} e^{-\delta q_n v(\mathcal{H}_n)} \\
&\leq (Dp_n v(\mathcal{H}_n)+1) \left(\frac{rq_n e v(\mathcal{H}_n)}{Dp_n v(\mathcal{H}_n)}\right)^{Dp_n v(\mathcal{H}_n)} e^{-\delta q_n v(\mathcal{H}_n)} \\
&= (Dp_n v(\mathcal{H}_n)+1) \left(\frac{reC}{D}\right)^{\frac{D}{C} q_n v(\mathcal{H}_n)} e^{-\delta q_n v(\mathcal{H}_n)} \\
&\overset{(\dagger)}{\leq} e^{-\frac{\delta}{2} q_n v(\mathcal{H}_n)}. \qquad (6.26)
\end{aligned}$$

Note that the equality follows since $q_n = Cp_n$. Now using that $p_n v(\mathcal{H}_n)$ and hence $q_n v(\mathcal{H}_n)$ tends to infinity as $n \to \infty$, we have $\mathbb{P}[B_{q_n}] \to 0$ as $n \to \infty$, as required. □

6.9.1.4 A remark on container theorems for solutions to systems of linear equations in abelian groups

Note that we could have used Proposition 6.9.2 to obtain a container theorem for solutions to systems of linear equations in abelian groups, and then used this directly to prove the 1-statement of Lemma 6.7.8. Saxton and Thomason also gave a container theorem in this setting; see Theorem 4.2 in [121]. However, we do not use their result, as they give an alternative definition of $m_G(A)$ which we now state. (See Definitions 1.6 and 4.1 of their paper.) First define an $\ell \times k$ matrix A to have *full image* if given any $b \in G^\ell$, there exists $x \in G^k$ such that $Ax = b$. Now if G is a finite abelian group which is not a field, define $m'_G(A) := \frac{\ell+t-1}{t-1}$, where t is the maximum j for which $A_{\overline{W}}$ has full image for all $W \subseteq [k]$ with $|W| = j$.[22]

One can show that $m_G(A) \leq m'_G(A)$ for all finite abelian groups G and full image matrices A such that (A, G) is abundant. Further, there exist examples for which this

[22]Note that Saxton and Thomason only consider G for which $t \geq 2$, and so $m'_G(A)$ is well-defined in this case.

inequality is strict. For example, consider the matrix

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

together with any finite abelian group G which is not a field, for example $\mathbb{Z}_6$. We have that (A, G) is abundant and in particular $t \geq 2$. By deleting the first three columns we obtain a submatrix which does not have full image, and so $t = 2$ and $m'_G(A) = 3$. However, under our definition, it is easy to compute that $m_G(A) = 2$. (Indeed, we have that $\mathrm{rank}_G(A_{\overline{W}})$ is equal to the number of distinct columns in $\overline{W}$, except for when there are 3 distinct columns, in which case it equals 2.) Consequently, using Theorem 4.2 from [121] for such an example would lead to a suboptimal 1-statement.

6.9.2 Proof of the 0-statement of Theorem 6.8.7

The following proof adapts an argument of Hancock and Treglown [77] (which in turn builds on work from [116]). Suppose $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of k-uniform ordered hypergraphs that satisfies conditions (P1) and (P3)–(P5) of Theorem 6.8.7. In particular, (P5) yields a sequence $(X_n)_{n\in\mathbb{N}}$.

Let $c > 0$ be sufficiently small and pick $q_n \leq c\hat{p}(\mathcal{H}_n)$. Our aim is to show that w.h.p. there is a red-blue colouring of the vertices of $\mathcal{H}^{\mathrm{ind}}_{n,X_n,q_n}$ so that there are no monochromatic edges. By definition of $\mathcal{H}^{\mathrm{ind}}_{n,X_n,q_n}$, such a vertex colouring also does not produce a monochromatic edge in $\mathcal{H}^{\mathrm{ind}}_{n,q_n}$, so this would prove the 0-statement of Theorem 6.8.7. We may further assume that $q_n = c\hat{p}(\mathcal{H}_n)$.

We say an (ordered) hypergraph is *Rado* if it has the property that however its vertices are red-blue coloured, there is always a monochromatic edge.[23] So it suffices to prove that $\mathcal{H}^{\mathrm{ind}}_{n,X_n,q_n}$ is not Rado with high probability. We will do this by combining two lemmata;

- *A deterministic lemma*, which states that if $\mathcal{H}^{\mathrm{ind}}_{n,X_n,q_n}$ is Rado then it must contain

[23]So being Rado is equivalent to being 2-Ramsey.

one of a finite list of subhypergraphs.

- *A probabilistic lemma*, which states that w.h.p. $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ does not contain any of these subhypergraphs.

Before we state these two lemmata, we provide some motivation and state some required definitions.

A Rado (ordered) hypergraph is *Rado minimal* if it is no longer Rado under the deletion of any edge. The following claim is an analogue of Proposition 7.4 from [116] and Claim 1 from [77]. The proof of the claim follows in the same manner.

Claim 6.9.4. *Let H be a Rado minimal m-uniform (ordered) hypergraph for some $m \geq 2$. Then for every edge a of H and every vertex $v \in a$, there exists an edge b in H such that $a \cap b = v$.*

Proof. Let a be an edge, and let $v \in a$ be such that for all edges b such that $v \in b$, there exists another vertex $w \in a$ such that $w \in b$. Since H is Rado minimal, it is possible to red-blue colour the vertices of $H - a$ so that there are no red or blue edges. Thus once we add a back, it must be the case that a is (w.l.o.g.) red since H is Rado. But then change the colour of v to blue. If there is a monochromatic edge now, it must be a blue edge which contains v. However all edges containing v also contain another vertex from a which is red, thus we obtain a contradiction. $\square$

If $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ is Rado then, by repeatedly deleting edges, we can obtain a Rado minimal hypergraph H. Using the above claim, one can build up more and more structure in H. Eventually, one can obtain enough structure so that H, and thus $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$, contains one of the subhypergraphs listed in Lemma 6.9.5 below. We now define some hypergraph notation in order to state it. In what follows, t is a positive integer. Further, in these definitions we just view edges as sets of vertices (i.e., we are not concerned about the order of an edge).

- A *simple path of length* t consists of edges $e_1, \dots, e_t$ such that $|e_i \cap e_j| = 1$ if $j = i+1$, and $|e_i \cap e_j| = 0$ if $j > i+1$.

- A *fairly simple cycle of length* t ($t \geq 2$) consists of a simple path $e_1, \ldots, e_t$ and an edge e_0 such that $|e_0 \cap e_1| = 1$; $|e_0 \cap e_i| = 0$ for $2 \leq i \leq t-1$; $|e_0 \cap e_t| = s \geq 1$.

- A *simple cycle* is a fairly simple cycle with $s = 1$.

- A simple path $P = e_1, \ldots, e_t$ in H is called *spoiled* if there exists an edge $e^* \in E(H) \setminus E(P)$ such that $e^* \subseteq V(P)$ and $|e^* \cap e_1| = 1$ where the vertex $e^* \cap e_1$ does not lie in e_2.

- A subhypergraph H_0 of H is said to have a *handle* if there is an edge e^* in H such that $|e^*| > |e^* \cap V(H_0)| \geq 2$.

- A *faulty simple path of length* t ($t \geq 3$) is a simple path $e_1, \ldots, e_t$ together with two edges e_x and e_z such that e_1, e_2, e_x form a simple cycle with $|e_x \cap e_i| = 0$ for $i \geq 3$; e_{t-1}, e_t, e_z form a simple cycle with $|e_z \cap e_i| = 0$ for $i \leq t-2$; each edge has size 3; the edges e_x and e_z may or may not be disjoint.

- A *bad triple* is a set of three edges e_1, e_x, e_y with $e_1 \cap e_x = \{x\}$ and $e_1 \cap e_y = \{y\}$, where $x \neq y$, and $|e_x \cap e_y| \geq 2$.

- A *bad tight path* is a set of three edges e_1, e_2, e_3 each of size 3 such that $|e_1 \cap e_2| = 2$, $|e_1 \cap e_3| = 1$ and $|e_2 \cap e_3| = 2$.

We also require the following definition in the proof of Lemma 6.9.5.

- A *Pasch configuration* is a set of four edges e_1, e_2, e_3, e_4 of size 3 such that $v_{ij} = e_i \cap e_j$ is a distinct vertex for each pair $i < j$.

Write $N := n \cdot q_n = \mathbb{E}(v(\mathcal{H}^{\text{ind}}_{n,X_n,q_n}))$ for brevity.

Lemma 6.9.5 (Deterministic lemma)**.** *If $\mathcal{H}^{ind}_{n,X_n,q_n}$ is Rado and H is a Rado minimal subhypergraph of $\mathcal{H}^{ind}_{n,X_n,q_n}$, then H contains at least one of the following structures:*

(i) A simple path of length at least $\log N$.

(ii) A spoiled simple path of length at most $\log N$.

(iii) A fairly simple cycle of length at most $\log N$*, with a handle.*

(iv) A faulty simple path of length at most $\log N$*.*

(v) A bad triple.

(vi) A bad tight path.

Lemma 6.9.6 (Probabilistic lemma)**.** *If* $c > 0$ *is sufficiently small then w.h.p.* $\mathcal{H}^{ind}_{n,X_n,q_n}$ *does not contain any of the hypergraphs (i)–(vi) listed in Lemma 6.9.5.*

These two results immediately prove the 0-statement of Theorem 6.8.7.

6.9.2.1 Proof of Lemma 6.9.5

Suppose for a contradiction that H does not contain any of the structures defined in (i)–(vi). Note that H is an $|X_n|$-uniform ordered hypergraph, and $|X_n| \geq 3$ by (P5).

Let $P = e_1, \dots, e_t$ be the longest simple path in H. As we have no simple paths of length at least $\log N$, we have $t \leq \log N$. By Claim 6.9.4, $t \geq 2$. Let x, y be two vertices which belong only to e_1 in P, and let e_x and e_y be two edges of H such that $e_z \cap e_1 = \{z\}$ for $z = x, y$ (such e_x, e_y exist by Claim 6.9.4). By the maximality of P, we have $h_z := |V(P) \cap e_z| \geq 2$ for $z = x, y$.

If $h_z = |X_n|$ for some $z = x, y$, then P together with e_z is a spoiled simple path, a contradiction. Otherwise, let $i_z := \min\{i \geq 2 : e_z \cap e_i \neq \emptyset\}$ for $z = x, y$, and assume without loss of generality that $i_y \leq i_x$.

Claim 6.9.7. *We have* $|X_n| = 3$ *and* e_1, e_x, e_y, e_{i_x} *form a Pasch configuration.*

Proof of the claim. We know $e_1, \dots, e_{i_x}, e_x$ must not form a fairly simple cycle for which e_y is a handle. This implies $e_y \subseteq e_1 \cup \dots \cup e_{i_x} \cup e_x$. In particular, this means e_x must contain all those vertices in e_y which do not lie on P. If $|e_y \cap e_x| \geq 2$ then e_1, e_x and e_y form a bad triple, a contradiction. Thus, $e_y \cap e_x$ consists of precisely one vertex v_{xy} (and v_{xy} lies outside of P). Now consider $e_1, \dots, e_{i_y}, e_y$. This is a fairly simple cycle that e_x intersects in at least two vertices (namely x and v_{xy}). Thus, we obtain a fairly simple cycle with a handle unless all the vertices in e_x lie in $e_1, \ldots, e_{i_y}, e_y$. In particular, $e_x \subseteq (e_1 \cup e_{i_y} \cup e_y)$ as $i_y \leq i_x$. This in turn implies $e_{i_y} = e_{i_x}$. Indeed, otherwise e_x must contain one vertex from e_1 and $|X_n|-1 \geq 2$ vertices from e_y, a contradiction as we already observed that e_x only intersects e_y in one vertex.

In summary, we have that $i_x = i_y$ and e_x and e_y intersect in a single vertex v_{xy} outside of P. As mentioned in the last paragraph, we must have $e_x \subseteq (e_1 \cup e_{i_y} \cup e_y)$. Similarly, we have that $e_1, \ldots, e_{i_x}, e_x$ form a fairly simple cycle for which e_y is a handle (a contradiction) unless $e_y \subseteq (e_1 \cup e_{i_x} \cup e_x)$.

The two previous inequalities imply $e_x \setminus \{x, v_{xy}\} \subseteq e_{i_y}$ and $e_y \setminus \{y, v_{xy}\} \subseteq e_{i_x}$. Recall here that $e_{i_x} = e_{i_y}$. We must have that $|e_{i_x} \setminus (e_x \cup e_y)| \geq 1$; indeed, otherwise either e_x or e_y contains the vertex $e_{i_x} \cap e_{i_x - 1}$, which either contradicts the minimality of i_x or i_y. Note that $e_x \setminus \{x, v_{xy}\}$ and $e_y \setminus \{y, v_{xy}\}$ are disjoint, they have size $|X_n| - 2$ each, and their union is a strict subset of e_{i_x}. Thus $2(|X_n| - 2) \leq |e_{i_x}| - 1 = |X_n| - 1$. This implies $|X_n| \leq 3$, and so $|X_n| = 3$.

If $i_x \geq 3$ then e_1, e_x, e_y form a fairly simple cycle for which e_{i_x} is a handle (since $|e_{i_x} \setminus (e_x \cup e_y)| \geq 1$ and $e_1 \cap e_{i_x} = \emptyset$), a contradiction. Thus we have that $i_x = 2$, and so e_1, e_x, e_y, e_{i_x} form a Pasch configuration.

■

Now repeat the maximal path process which we did for e_1 to find e_x and e_y, except from the other end of the path. That is, there must exist edges e_z and e_w such that $e_z \cap e_t = \{z\}$ and $e_w \cap e_t = \{w\}$, where z, w are vertices in e_t that are not in e_{t-1}. By an analogous argument to that used in Claim 6.9.7, we arrive at the conclusion that e_{t-1}, e_t, e_z, e_w must also form a Pasch configuration where $e_z \cap e_w$ is a vertex v_{zw} outside of P. If $t \geq 3$, then $e_1, \ldots, e_t, e_x, e_z$ together form a faulty simple path. Hence we must have $t = 2$.

If the union of these two Pasch configurations contains 7 vertices (i.e., $v_{xy} \neq v_{zw}$), then e_1, e_2, e_x form a fairly simple cycle for which e_z is a handle. So we now suppose that the two Pasch configurations cover the same 6 vertices. If we do not have $\{e_x, e_y\} = \{e_z, e_w\}$ then e_x, e_z, e_y form a bad tight path. Hence we do have equality and the two Pasch configurations we found are identical.

Relabel the edges and vertices as in the definition of a Pasch configuration. We observe that H cannot be just these four edges: such a hypergraph is not Rado, e.g., colour v_{12}, v_{13}, v_{34} red, and the remaining vertices blue. Also, by definition of Rado minimal, this cannot be a component of H. That is, there is an edge e_5 in H, where $e_5 \neq e_i$ for $i \in [4]$, and e_5 contains s vertices from inside the Pasch configuration, where $s \geq 1$. If $s = 1$ then w.l.o.g. e_5 contains $v_{1,2}$; then e_5, e_1, e_3 is a simple path of length 3, a contradiction to the longest path in H of length 2 found earlier. If $s = 2$ then whichever two vertices of the Pasch configuration e_5 contains, taking any of the simple cycles of the Pasch configuration together with e_5 gives a (fairly) simple cycle with a handle. If $s = 3$, first suppose $e_5 = \{v_{1,2}, v_{1,3}, v_{2,3}\}$. Then e_1, e_5, e_2 is a bad tight path. If $e_5 = \{v_{1,2}, v_{1,3}, v_{2,4}\}$, then again e_1, e_5, e_2 is a bad tight path. For all other 3-sets of vertices e_5 could contain, a symmetrical argument shows that we find a bad tight path. Since all three values of s give a contradiction, this concludes the proof. $\square$

6.9.2.2 Proof of Lemma 6.9.6

Let $c > 0$ be sufficiently small and $n \in \mathbb{N}$ be sufficiently large. Recall that $q_n = c\hat{p}_n(\mathcal{H}_n)$. Given an $|X_n|$-uniform ordered hypergraph G, we will obtain an upper bound on the expected number of copies of G in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ by obtaining upper bounds on the expected number of possible assignments of vertices $v \in V(\mathcal{H}^{\text{ind}}_{n,X_n,q_n})$ to an edge in which some vertices are already fixed, and others are to be assigned. To make this precise, we need some notation. Given an edge order $e_1, \ldots, e_t$ of $E(G)$, we call a vertex v *new in* e_i if $v \in e_i$ but $v \notin e_j$ for all $j < i$. Otherwise we call $v \in e_i$ *old in* e_i. Clearly (provided G has no isolated vertices) each vertex in G is new in one edge, and old in any subsequent edge that it appears in. Now, for $0 \leq w \leq |X_n| - 1$, we will shortly provide an upper bound P_w on the expected number of edges in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ that contain a fixed set of w (old) vertices; crucially, P_w is independent of the exact set of w vertices considered. Then one can obtain an upper bound on the expected number of copies of G in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ directly as some function of the P_w's (which will depend on G). We first obtain bounds for these P_w's.

Given a fixed set of w vertices in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$, there are $\binom{|X_n|}{w}w! \leq |X_n|! \leq k!$ ways of selecting some $W \subseteq X_n$ with $|W| = w$ and picking an ordering $v_1, \ldots, v_w$ of these w vertices. Then, there are at most $\Delta_W(\mathcal{H}_{n,X_n})$ choices for an edge e in $\mathcal{H}_{n,X_n}$ such that $e_W = (v_1, \ldots, v_w)$. Each vertex in $\mathcal{H}_{n,X_n}$ is included with probability q_n in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$, thus we obtain the upper bound

$$P_w = (k!) \cdot \left(\max_{W \subseteq X_n,\, |W|=w} \Delta_W(\mathcal{H}_{n,X_n}) \right) \cdot (q_n)^{|X_n|-w}.$$

For $w = 0$, we will use the bound

$$P_0 \leq e(\mathcal{H}_{n,X_n}) \cdot (q_n)^{|X_n|} \leq n^{|X_n|} \cdot (q_n)^{|X_n|} \leq N^k.$$

We now use conditions (P3), (P4) and (P5) to bound P_w further for values of w greater than 0. First, condition (P4) implies that,

$$P_1 \leq (k!) \cdot b \cdot \frac{e(\mathcal{H}_{n,X_n})}{v(\mathcal{H}_n)} \cdot (q_n)^{|X_n|-1}.$$

By (6.19) and since $n \in \mathbb{N}$ is sufficiently large, we have that $f_{n,X_n} \geq \eta \hat{p}_n(\mathcal{H}_n)$ for some constant $\eta > 0$. It follows from the above that

$$\begin{aligned} P_1 &\leq (k!) \cdot b \cdot \frac{e(\mathcal{H}_{n,X_n})}{v(\mathcal{H}_n)} \cdot (c\hat{p}_n(\mathcal{H}_n))^{|X_n|-1} \\ &\overset{(6.19)}{\leq} (k!) \cdot b \cdot \frac{e(\mathcal{H}_{n,X_n})}{v(\mathcal{H}_n)} \cdot (f_{n,X_n})^{|X_n|-1} \cdot (c/\eta)^{|X_n|-1} \\ &\overset{(6.18)}{=} (k!) \cdot b \cdot (c/\eta)^{|X_n|-1} \leq \sqrt{c}, \end{aligned}$$

where the last inequality follows from $c > 0$ being sufficiently small.

Similar calculations hold for higher values of w. Recall we have $q_n = c\hat{p}_n \geq cf_{n,W}$ for every $W \subseteq [k]$ with $|W| \geq 2$. Let $2 \leq w \leq |X_n| - 1$ and pick $W \subsetneq X_n$ with $|W| = w$ maximising $\Delta_W(\mathcal{H}_{n,X_n})$. As $n \in \mathbb{N}$ is sufficiently large we have

$$\begin{aligned}
P_w &\overset{(P3)}{\leq} (k!) \cdot b \cdot \frac{e(\mathcal{H}_{n,X_n})}{e(\mathcal{H}_{n,W})} \cdot (q_n)^{|X_n|-w} = (k!) \cdot b \cdot \frac{e(\mathcal{H}_{n,X_n})}{v(\mathcal{H}_n)} \cdot (q_n)^{|X_n|-1} \cdot \frac{v(\mathcal{H}_n)}{e(\mathcal{H}_{n,W})} \cdot (q_n)^{-(w-1)} \\
&\leq \left((k!) \cdot b \cdot (c/\eta)^{|X_n|-1}\right) \cdot \frac{v(\mathcal{H}_n)}{e(\mathcal{H}_{n,W})} \cdot (cf_{n,W})^{-(w-1)} \\
&\overset{(6.18)}{=} \left((k!) \cdot b \cdot (c/\eta)^{|X_n|-1}\right) \cdot c^{-(w-1)} \leq (k!) \cdot b \cdot c \cdot \eta^{-(|X_n|-1)} \leq \sqrt{c},
\end{aligned}$$

where the last two inequalities follow from $|X_n| > w$ and $c > 0$ being sufficiently small.

Finally, for any $2 \leq w \leq |X_n| - 1$, we have that $P_0 \cdot P_2 \cdot P_w$ is at most

$$\begin{aligned}
&(k!)^2 \cdot e(\mathcal{H}_{n,X_n}) \cdot \left(\max_{W \subseteq X_n, |W|=2} \Delta_W(\mathcal{H}_{n,X_n})\right) \cdot \left(\max_{W \subseteq X_n, |W|=w} \Delta_W(\mathcal{H}_{n,X_n})\right) \cdot (q_n)^{3|X_n|-2-w} \\
&\overset{(6.20)}{=} o(1).
\end{aligned}$$

For brevity, let $P_{\geq w} := \max_{i \geq w} P_i$. To recap, we have shown that

$$P_0 \leq N^k, \quad P_{\geq 1} \leq \sqrt{c} \quad \text{and} \quad P_0 \cdot P_2 \cdot P_{\geq 2} = o(1). \tag{6.27}$$

To prove the lemma it suffices to show that, for each set of hypergraphs stated in Lemma 6.9.5, the expected number of such structures in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ is $o(1)$. Indeed, then by Markov's inequality w.h.p. $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ does not contain any of these structures. In each case, we will find an edge order and then use (6.27) to bound the expected number of these structures. Let G be an ordered hypergraph. Given an edge order $e_1, \ldots, e_t$ of $E(G)$, we call an edge e_i

- *initial* if all its vertices are new;
- *normal* if it has precisely one old vertex;
- *2-bad* if it has precisely two old vertices;

- *bad* if it has at least two and at most $|X_n|-1$ old vertices.

Now observe the bounds P_0, P_1, P_2 and $P_{\geq 2}$ correspond to upper bounds on the expected number of ways one can place an initial, normal, 2-bad and bad edge in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ respectively.

Case (i). Note that for this case it suffices to show that w.h.p. $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ contains no simple path of length exactly $s := \lceil \log N \rceil$; thus we only need to consider the expected number of simple paths of length s. Let $e_1, \dots, e_s$ be the simple path. Here, e_1 is initial, and all other edges are normal. Thus, we can bound the expected number of simple paths of length s in $\mathcal{H}^{\text{ind}}_{n,X_n,q_n}$ by

$$P_0 \cdot P_1^{s-1} \overset{(6.27)}{\leq} N^k \cdot (\sqrt{c})^{s-1} \leq (\sqrt{c})^{(\log N)-1} \cdot N^k \overset{(\text{P1})}{=} o(1),$$

which follows since $c > 0$ is sufficiently small.

Case (ii). Let $e_1, \dots, e_t$ be the simple path and let e^* be the spoiling edge. Choose i to be the smallest $i \geq 2$ such that $|e_i \cap e^*| \geq 1$. Consider the edge order

$$e^*, e_i, e_{i-1}, \dots, e_1, e_{i+1}, \dots, e_t.$$

Here e^* is initial. If $|e_i \cap e^*| \geq 2$ then e_i is bad, the e_j for $2 \leq j \leq i-1$ are normal, e_1 is 2-bad, and each e_j for $i+1 \leq j \leq t$ is normal or bad. If $|e_i \cap e^*| = 1$ then e_j for $2 \leq j \leq i$ are normal, e_1 is 2-bad, and each e_j for $i+1 \leq j \leq t$ is normal or bad. In particular, for at least one such j, we know e_j is bad, since e^* must intersect at least one of these edges (as $|X_n| \geq 3$). In either case, we have found an edge order with one initial edge, one 2-bad edge, at least one further bad edge, and all other edges normal or bad edges. We have $t \leq \log N$ and there are at most tk choices for the location on the simple path of each vertex of e^* intersecting it. Overall, the expected number of spoiled paths of length at most $\log N$ is at most

$$\sum_{t=2}^{\log N}(tk)^k \cdot P_0 \cdot P_{\geq 1}^{t-2} \cdot P_2 \cdot P_{\geq 2} \overset{(6.27)}{\leq} \sum_{t=2}^{\log N}(tk)^k \cdot (\sqrt{c})^{t-2} \cdot o(1) = o(1),$$

where the last inequality holds as $c > 0$ is sufficiently small.

Case (iii). Let $e_1, \dots, e_t, e_0$ be the fairly simple cycle and let e^* be the handle. Consider the edge order $e_0, e_t, \dots, e_1, e^*$. Here e_0 is initial, e_t is normal or bad, the e_j for $2 \leq j \leq t-1$ are normal, e_1 is 2-bad and e^* is bad. We have $t \leq \log N$ and there are at most tk choices for the location on the fairly simple cycle for the choice of where each of the u vertices of the handle e^* intersect it, where $2 \leq u \leq k-1$. Overall, the expected number of fairly simple cycles of length at most $\log N$ with a handle is at most

$$\sum_{t=2}^{\log N}\sum_{u=2}^{k-1}(tk)^u \cdot P_0 \cdot P_{\geq 1}^{t-1} \cdot P_2 \cdot P_{\geq 2} \overset{(6.27)}{\leq} \sum_{t=2}^{\log N}\sum_{u=2}^{k-1}(tk)^u \cdot (\sqrt{c})^{t-1} \cdot o(1) = o(1).$$

Case (iv). Let $e_1, \dots, e_t, e_x, e_z$ be the faulty simple path. Consider the edge order $e_1, e_x, e_2, \dots,\ e_{t-1}, e_z, e_t$. Here e_1 is initial, e_2 and e_t are 2-bad, e_z is normal or 2-bad (depending whether e_z intersects e_x), and all other edges are normal. We have $t \leq \log N$, and a choice of whether e_x and e_z intersect or not. Overall, we can bound the expected number of faulty simple paths of length at most $\log N$ by

$$\sum_{t=3}^{\log N} P_0 \cdot P_1^{t-1} \cdot P_2^2 + \sum_{t=3}^{\log N} P_0 \cdot P_1^{t-2} \cdot P_2^3 \overset{(6.27)}{\leq} 2\sum_{t=3}^{\log N}(\sqrt{c})^{t-1} \cdot o(1) = o(1).$$

Case (v). Let e_1, e_x, e_y be the bad triple. Consider the edge order e_x, e_y, e_1. Here, e_x is initial, e_y is bad and e_1 is 2-bad. We have a choice of the size of $t := |e_x \cap e_y|$. Overall, the expected number of bad triples is at most

$$\sum_{t=2}^{k-1} P_0 \cdot P_2 \cdot P_{\geq 2} \overset{(6.27)}{=} o(1).$$

Case (vi). Let e_1, e_2, e_3 be a bad tight path. In this edge order, e_1 is initial, and e_2 and e_3 are 2-bad. Overall, we can bound the expected number of bad tight paths by

$$P_0 \cdot P_2^2 \overset{(6.27)}{=} o(1).$$

□

6.10 Proof of the random Rado lemma

Before we deduce the random Rado lemma (Lemma 6.7.8) from Theorem 6.8.7, we next prove Lemma 6.7.9.

Proof of Lemma 6.7.9. We first prove the second statement of the lemma.

Claim 6.10.1.

$$\lim_{n\to\infty} \frac{|k\text{-}\mathrm{Sol}^A_{S_n}([k])|}{|\mathrm{Sol}^A_{S_n}([k])|} = 1.$$

Proof of the claim. We bound from above the number of non k-distinct solutions in $\mathrm{Sol}^A_{S_n}([k])$. For each such solution, we have $|S_n|$ choices for the repeated entry. Using conditions (A3) and (A6), for n sufficiently large, we obtain

$$\begin{aligned}\sum_{z\in S_n} \sum_{\substack{W\subseteq[k]\\|W|=2}} |\mathrm{Sol}^A_{S_n}((z,z),W,[k])| &\overset{(A3)}{\leq} B \sum_{z\in S_n} \sum_{\substack{W\subseteq[k]\\|W|=2}} \frac{|\mathrm{Sol}^A_{S_n}([k])|}{|\mathrm{Sol}^A_{S_n}(W)|} \\ &= B|S_n| \sum_{\substack{W\subseteq[k]\\|W|=2}} \frac{|\mathrm{Sol}^A_{S_n}([k])|}{|\mathrm{Sol}^A_{S_n}(W)|} \overset{(A6)}{=} o(|\mathrm{Sol}^A_{S_n}([k])|).\end{aligned}$$

In other words, $|\operatorname{Sol}^A_{S_n}([k])| - |k\text{-}\operatorname{Sol}^A_{S_n}([k])| = o(|\operatorname{Sol}^A_{S_n}([k])|)$ which immediately implies the claim.

■

Next, we prove the inequality stated in the lemma. Let $Y \subseteq [k]$. Note that $k\text{-}\operatorname{Sol}^A_{S_n}(Y) \subseteq \operatorname{Sol}^A_{S_n}(Y)$, so the upper bound follows trivially. By definition, if $y \in k\text{-}\operatorname{Sol}^A_{S_n}(Y)$ then there exists some $x \in k\text{-}\operatorname{Sol}^A_{S_n}([k])$ such that $x_Y = y$. Thus,

$$|k\text{-}\operatorname{Sol}^A_{S_n}(Y)| \cdot \max_{y_0 \in S^{|Y|}} |\operatorname{Sol}^A_{S_n}(y_0, Y, [k])| \geq |k\text{-}\operatorname{Sol}^A_{S_n}([k])|.$$

As (A, S_n) is B-extendable, for n large the above implies

$$|k\text{-}\operatorname{Sol}^A_{S_n}(Y)| \cdot \frac{B|\operatorname{Sol}^A_{S_n}([k])|}{|\operatorname{Sol}^A_{S_n}(Y)|} \geq |k\text{-}\operatorname{Sol}^A_{S_n}([k])|.$$

Rearranging gives

$$\frac{|k\text{-}\operatorname{Sol}^A_{S_n}(Y)|}{|\operatorname{Sol}^A_{S_n}(Y)|} \geq \frac{|k\text{-}\operatorname{Sol}^A_{S_n}([k])|}{B|\operatorname{Sol}^A_{S_n}([k])|} \geq \frac{1}{2B},$$

where the last inequality holds for n sufficiently large by Claim 6.10.1. □

Lemma 6.7.8 follows easily from Theorem 6.8.7. We use Lemma 6.7.9 to restrict our attention to k-distinct solutions.

Proof of Lemma 6.7.8. Let $r \geq 2$, $k \geq 3$ and $\ell \geq 1$ be integers. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of abelian groups and A be an $\ell \times k$ integer matrix. Suppose that conditions (A1)–(A6) from Lemma 6.7.8 hold. Let $\mathcal{H}_n$ be the k-uniform ordered hypergraph with vertex set S_n, and edge set consisting of all k-distinct solutions $x = (x_1, \ldots, x_k)$ to $Ax = 0$ in S_n.

Let $Y \subseteq [k]$. Observe that $e(\mathcal{H}_{n,Y}) = |k\text{-}\operatorname{Sol}^A_{S_n}(Y)|$. As (A3) and (A6) hold, Lemma 6.7.9 implies that

$$e(\mathcal{H}_{n,Y}) = \Theta(|\operatorname{Sol}^A_{S_n}(Y)|) \quad \text{and} \quad e(\mathcal{H}_n) \sim |\operatorname{Sol}^A_{S_n}([k])|. \tag{6.28}$$

It follows that, for $|Y| \geq 2$,

$$f_{n,Y} = \left(\frac{e(\mathcal{H}_{n,Y})}{v(\mathcal{H}_n)}\right)^{-\frac{1}{|Y|-1}} = \Theta\left(\frac{|\operatorname{Sol}^A_{S_n}(Y)|}{|S_n|}\right)^{-\frac{1}{|Y|-1}} = \Theta(p_Y(A, S_n)) \tag{6.29}$$

and thus $\hat{p}(\mathcal{H}_n) = \Theta(\hat{p}(A, S_n))$. We now prove that conditions (P1)–(P5) hold.

(P1): Since $\hat{p}_n(\mathcal{H}_n) = \Theta(\hat{p}_n(A, S_n))$ and $|S_n| = v(\mathcal{H}_n)$, (A1) implies that (P1) holds.

(P2): Since $e(\mathcal{H}_n) = |k\text{-}\operatorname{Sol}^A_{S_n}([k])| \sim |\operatorname{Sol}^A_{S_n}([k])|$ by (6.28), (A2) implies that (P2) holds.

(P3): For any $W \subseteq Y \subseteq [k]$ we have

$$\Delta_W(\mathcal{H}_{n,Y}) \leq \max_{w_0 \in S_n^{|W|}} \{|\operatorname{Sol}^A_{S_n}(w_0, W, Y)|\} \overset{\text{(A3)}}{\leq} \Theta\left(\frac{|\operatorname{Sol}^A_{S_n}(Y)|}{|\operatorname{Sol}^A_{S_n}(W)|}\right) \overset{(6.28)}{\leq} \Theta\left(\frac{|e(\mathcal{H}_{n,Y})|}{|e(\mathcal{H}_{n,W})|}\right),$$

and so (P3) holds.

(P4): Pick $W \subsetneq Y \subseteq [k]$ with $|W| = 1$. We have

$$\Delta_W(\mathcal{H}_{n,Y}) \leq \max_{w_0 \in S_n} |\operatorname{Sol}^A_{S_n}(w_0, W, Y)| \overset{\text{(A4)}}{=} O\left(\frac{|\operatorname{Sol}^A_{S_n}(Y)|}{|S_n|}\right) \overset{(6.28)}{=} O\left(\frac{e(\mathcal{H}_{n,Y})}{v(\mathcal{H}_n)}\right).$$

(P5): Pick a sequence $(X_n)_{n \in \mathbb{N}}$ of subsets $X_n \subseteq [k]$ with $|X_n| \geq 3$ for all $n \in \mathbb{N}$ as in Definition 6.7.6. Note that

$$f_{n,X_n} = \left(\frac{e(\mathcal{H}_{n,X_n})}{v(\mathcal{H}_n)}\right)^{-\frac{1}{|X_n|-1}} \overset{(6.29)}{=} \Theta\left(\frac{\operatorname{Sol}^A_{S_n}(X_n)}{|S_n|}\right)^{-\frac{1}{|X_n|-1}} = \Theta(\hat{p}(A, S_n)) = \Theta(\hat{p}(\mathcal{H}_n)),$$

and so $f_{n,X_n} = \Omega(\hat{p}(\mathcal{H}_n))$. It remains to verify equation (6.20). Pick sequences

$(W_n)_{n\in\mathbb{N}}$ and $(W'_n)_{n\in\mathbb{N}}$ such that $W_n, W'_n \subsetneq X_n$ where $|W_n| = 2$ and $|W'_n| \geq 2$. Then

$$\begin{aligned}
&\Delta_{W_n}(\mathcal{H}_{n,X_n}) \cdot \Delta_{W'_n}(\mathcal{H}_{n,X_n}) \cdot e(\mathcal{H}_{n,X_n}) \cdot \hat{p}(\mathcal{H}_n)^{3|X_n|-2-|W'_n|} \\
\overset{(P3),(6.28)}{\leq}\; &\Theta\left(\frac{|\operatorname{Sol}^A_{S_n}(X_n)|}{|\operatorname{Sol}^A_{S_n}(W_n)|} \cdot \frac{|\operatorname{Sol}^A_{S_n}(X_n)|}{|\operatorname{Sol}^A_{S_n}(W'_n)|} \cdot |\operatorname{Sol}^A_{S_n}(X_n)| \cdot \left(\frac{|\operatorname{Sol}^A_{S_n}(X_n)|}{|S_n|}\right)^{-\frac{3|X_n|-2-|W'_n|}{|X_n|-1}}\right) \\
=\; &\Theta\left(\frac{|S_n|^2}{|\operatorname{Sol}^A_{S_n}(W_n)|} \cdot \frac{|S_n|}{|\operatorname{Sol}^A_{S_n}(W'_n)|} \cdot \left(\frac{|\operatorname{Sol}^A_{S_n}(X_n)|}{|S_n|}\right)^{\frac{|W'_n|-1}{|X_n|-1}}\right) \\
=\; &\Theta\left(\frac{|S_n|^2}{|\operatorname{Sol}^A_{S_n}(W_n)|} \cdot \left(\frac{p_{W'_n}(A,S_n)}{p_{X_n}(A,S_n)}\right)^{|W'_n|-1}\right) \overset{(A5)}{=} o(1).
\end{aligned}$$

As conditions (P1)–(P5) are satisfied, by Theorem 6.8.7 there exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{H}^{\text{ind}}_{n,q_n} \text{ is } r\text{-Ramsey}] = \begin{cases} 0 & \text{if } q_n \leq c\hat{p}(\mathcal{H}_n), \\ 1 & \text{if } q_n \geq C\hat{p}(\mathcal{H}_n). \end{cases}$$

Note that $\mathcal{H}^{\text{ind}}_{n,q_n}$ is r-Ramsey if and only if S_{n,q_n} is (A,r)-Rado. Furthermore, we have $\hat{p}(\mathcal{H}_n) = \Theta(\hat{p}(A,S_n))$ and so there exist constants c_0, C_0 such that

$$\lim_{n\to\infty} \mathbb{P}[S_{n,q_n} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } q_n \leq c_0 \cdot \hat{p}(A,S_n), \\ 1 & \text{if } q_n \geq C_0 \cdot \hat{p}(A,S_n). \end{cases}$$

□

6.11 Proof of two auxiliary results

In this section, we prove two auxiliary results. Lemma 6.3.3 gives an upper bound on the number of k-APs in $\mathbf{P}_n$ containing a fixed prime number and it is used in the proofs of Theorem 6.3.2 and Theorem 6.3.7. Theorem 6.4.9 is a supersaturation result for integer lattices which is used to prove Theorem 6.6.1.

6.11.1 Proof of Lemma 6.3.3

We make use of the following sieve result.

Lemma 6.11.1 ([131]). *Let $k \in \mathbb{N}$ with $k \geq 2$ and let q be a prime number. For each prime $p \geq k$ with $p \neq q$, let E_p be the union of $k-1$ residue classes modulo p. Then $|E(n)| = O(n/\log^{k-1} n)$ where*

$$E(n) := [n] \setminus \bigcup_{\substack{p \neq q \text{ prime} \\ k \leq p \leq \sqrt{n}}} E_p.$$

Lemma 6.11.1 is a weakened version of Theorem 32 from [131]. A precise explanation of how to deduce Lemma 6.11.1 from the original statement in [131] can be found in Section A.1 of Appendix A.

Proof of the Lemma 6.3.3. Let k, ℓ, n and q be as in the statement of Lemma 6.3.3. Let $\mathcal{A}(n)$ be the set of k-APs in $\mathbb{Z}$ such that the ℓth term of the progression is q and the common difference is an integer between 1 and n inclusive. In particular, we have $|\mathcal{A}(n)| = n$. Furthermore, the collection of k-APs in $\mathbf{P}_n$ whose ℓth element is q is a subset of $\mathcal{A}(n)$. Thus, it suffices to show that the number of k-APs in $\mathcal{A}(n)$ consisting entirely of primes is $O(n/\log^{k-1} n)$.

For every prime number $p \neq q$, let $\mathcal{A}_p$ be the set of k-APs in $\mathcal{A}(n)$ with at least one term divisible by p. Consider the set

$$\mathcal{E}(n) := \mathcal{A}(n) \setminus \bigcup_{\substack{p \neq q \text{ prime} \\ k \leq p \leq \sqrt{n}}} \mathcal{A}_p.$$

In other words, $\mathcal{E}(n)$ is the set of all k-APs in $\mathcal{A}(n)$ such that no term of the progression is divisible by a prime between k and $\sqrt{n}$ (except q). In particular, $\mathcal{E}(n)$ contains all k-APs in $\mathcal{A}(n)$ whose terms (other than perhaps q) are all primes greater than $\sqrt{n}$.

Conversely, the number of k-APs in $\mathcal{A}(n)$ containing at least one prime $p' \leq \sqrt{n}$ where $p' \neq q$ is $o(n/\log^{k-1} n)$. This follows from (i) the fact that there are at most k arithmetic progressions of length k containing both q (in the ℓth position) and a fixed prime $p \neq q$ and as (ii) $\mathbf{P}_{\sqrt{n}} \sim \frac{2\sqrt{n}}{\log n}$ by the prime number theorem.

By the observations above, the number of k-APs in $\mathcal{A}(n)$ consisting entirely of primes is at most $o(n/\log^{k-1} n) + |\mathcal{E}(n)|$; so it suffices to prove that $|\mathcal{E}(n)| = O(n/\log^{k-1} n)$.

Now we reformulate the problem in terms of common differences. Since the ℓth term of all k-APs in $\mathcal{A}(n)$ is equal to q, each k-AP in $\mathcal{A}(n)$ is uniquely determined by its common difference. For every prime $p \neq q$, let E_p be the set of all common differences of k-APs in $\mathcal{A}_p$. It follows immediately that $|E_p| = |\mathcal{A}_p|$. Furthermore, the set of common differences of k-APs in $\mathcal{A}(n)$ is exactly $[n]$ by definition. Thus we have $|\mathcal{E}(n)| = |E(n)|$ where

$$E(n) := [n] \setminus \bigcup_{\substack{p \neq q \text{ prime} \\ k \leq p \leq \sqrt{n}}} E_p.$$

Applying Theorem 6.11.1 yields $|\mathcal{E}(n)| = |E(n)| = O(n/\log^{k-1} n)$, as required. The only assumption of Theorem 6.11.1 that needs to be checked is that, for every prime $p \geq k$ with $p \neq q$, the set E_p is the union of $k-1$ residue classes modulo p.

Let $a \in [n]$ and $b_i := q + a(i-\ell)$ for every $i \in [k]$. Then, $a \in E_p$ if and only if $(b_1, \ldots, b_k)$ is a k-AP in $\mathcal{A}_p$, i.e., p divides some b_i. Now, p divides $b_i = q + a(i-\ell)$ if and only if $i \neq \ell$ and $a \equiv -q(i-\ell)^{-1} \pmod{p}$. Note that $i - \ell \neq 0$ is always invertible modulo p since $|i-\ell| < k \leq p$. It follows that $E_p = \{a \in [n] : a \equiv -q(i-\ell)^{-1} \pmod{p},\ i \in [k], i \neq \ell\}$. Hence, E_p is the set of precisely $k-1$ residue classes modulo p, as required. □

6.11.2 Supersaturation in the integer lattice

To prove Theorem 6.4.9, we first need a few intermediate results. Throughout this subsection, we will repeatedly use the following fact.

Fact 6.11.2. *Let $d, \ell, k \in \mathbb{N}$ and A be an $\ell \times k$ integer matrix of rank ℓ. Let $b \in (\mathbb{Z}^d)^\ell$*

and $S \subseteq \mathbb{Z}^d$ be a finite subset. The number of solutions to $Ax = b$ in S is at most $|S|^{k-\ell}$.

Proof. As A has rank ℓ, there are ℓ linearly independent columns in A, w.l.o.g. the first ℓ columns. We bound the number of solutions $x = (x_1, \dots, x_k)$ to $Ax = 0$ in S as follows. There are at most $|S|^{k-\ell}$ choices for $x_{\ell+1}, \dots, x_k$. For each such choice, $x' := (x_1, \dots, x_\ell)$ is a solution to $A'x' = b'$ in $S \subseteq \mathbb{Z}^d$ where A' is the $\ell \times \ell$ matrix consisting of the first ℓ columns of A and b' is some vector in $(\mathbb{Z}^d)^\ell$. Since the columns of A' are linearly independent, there is at most one solution to $A'x' = b'$ in $\mathbb{Z}^d$. Thus, there are at most $|S|^{k-\ell}$ solutions to $Ax = 0$ in S, as required. $\square$

The following lemma essentially states that if a supersaturation result holds for $S \subseteq \mathbb{Z}^d$ then it also holds for $S' \subseteq S$, provided that S' contains almost all elements of S.

Lemma 6.11.3. *Let $r, d, \ell, k \in \mathbb{N}$ and $\gamma > 0$. Let A be a partition regular $\ell \times k$ integer matrix of rank ℓ and let S and S' be finite sets such that $S' \subseteq S \subseteq \mathbb{Z}^d$ and $|S'| \geq (1 - \gamma/(2k))|S|$. Fix an r-colouring of S (and thus S'). If there are at least $\gamma|S|^{k-\ell}$ monochromatic solutions to $Ax = 0$ in S, then there are at least $(\gamma/2)|S|^{k-\ell}$ monochromatic solutions to $Ax = 0$ in S'.*

Proof. We determine an upper bound on the number of monochromatic solutions $(\underline{\mathbf{z}}_1, \dots, \underline{\mathbf{z}}_k)$ which lie in S but not in S'. For one such solution, there must be some $\underline{\mathbf{z}}_i$ which lies in S but not in S'. We have k choices for i and $|S| - |S'| \leq \gamma|S|/(2k)$ choices for $\underline{\mathbf{z}}_i$. The remaining elements $(\underline{\mathbf{z}}_1, \dots, \underline{\mathbf{z}}_{i-1}, \underline{\mathbf{z}}_{i+1}, \dots, \underline{\mathbf{z}}_k)$ form a solution to $A'x = \underline{\mathbf{b}}$ in S where A' is obtained from A by removing the ith column and $\underline{\mathbf{b}}$ is some vector. Note that A' has rank ℓ. If not, then the ith column of A cannot be expressed as a linear combination of the columns of A'. In particular, the ith entry of any solution to $Ax = 0$ in $\mathbb{Z}$ must be 0, contradicting the assumption that A is partition regular. Since A' has rank ℓ, there are at most $|S|^{k-1-\ell}$ solutions to $A'x = \underline{\mathbf{b}}$ in S by Fact 6.11.2. Overall, there are at most $k \cdot \frac{\gamma|S|}{2k} \cdot |S|^{k-\ell-1} = \frac{\gamma}{2}|S|^{k-\ell}$ possible choices for $(\underline{\mathbf{z}}_1, \dots, \underline{\mathbf{z}}_k)$.

Therefore, the number of monochromatic solutions to $Ax = 0$ in S' is at least

$$\gamma|S|^{k-\ell} - \frac{\gamma}{2}|S|^{k-\ell} = \frac{\gamma}{2}|S|^{k-\ell}.$$

$\square$

The following proposition asserts that if the statement of Theorem 6.4.9 holds for $d \in \mathbb{N}$ then it also holds for $d-1$.

Proposition 6.11.4. *Let $r, d, n \in \mathbb{N}$ with $d \geq 2$ and $\delta > 0$. Let A be an $\ell \times k$ integer matrix of rank ℓ. If every r-colouring of $[n]^d$ yields at least $\delta n^{d(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[n]^d$, then every r-colouring of $[n]^{d-1}$ yields at least $\delta n^{(d-1)(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[n]^{d-1}$.*

Proof. Let A, r, d, n and δ be as in the statement of the proposition. Let $f : [n]^d \to [n]^{d-1}$ be such that for $\underline{\mathbf{x}} = (x_1, \dots, x_d) \in [n]^d$ we have $f(\underline{\mathbf{x}}) := (x_1, \dots, x_{d-1})$, i.e., $f(\underline{\mathbf{x}})$ is the vector consisting of the first $d-1$ entries of $\underline{\mathbf{x}}$. We first prove the following property of f.

Claim 6.11.5. *Given a solution $(\underline{\mathbf{y}}_1, \dots, \underline{\mathbf{y}}_k)$ to $Ax = 0$ in $[n]^{d-1}$, there are at most $n^{k-\ell}$ solutions $(\underline{\mathbf{x}}_1, \dots, \underline{\mathbf{x}}_k)$ to $Ax = 0$ in $[n]^d$ such that $f(\underline{\mathbf{x}}_i) = \underline{\mathbf{y}}_i$ for every $i \in [k]$.*

Proof of the claim. Let S be the set of solutions $(\underline{\mathbf{x}}_1, \dots, \underline{\mathbf{x}}_k)$ in $[n]^d$ such that $f(\underline{\mathbf{x}}_i) = \underline{\mathbf{y}}_i$ for every $i \in [k]$. Let $q : S \to [n]^k$ with $q((\underline{\mathbf{x}}_1, \dots, \underline{\mathbf{x}}_k)) := (x_1, \dots, x_k)$ where x_i is the dth entry of $\underline{\mathbf{x}}_i$. Since S is a set of solutions in $[n]^d$, it follows that the image $q(S)$ of q is a set of solutions to $Ax = 0$ in $[n]$. In particular, $|q(S)| \leq n^{k-\ell}$ by Fact 6.11.2. Note that the assumption $f(\underline{\mathbf{x}}_i) = \underline{\mathbf{y}}_i$ for every $i \in [k]$ implies q is injective, hence $|S| = |q(S)| \leq n^{k-\ell}$, as required.

$\blacksquare$

Pick an arbitrary r-colouring $\mathcal{C} : [n]^{d-1} \to [r]$ of $[n]^{d-1}$. Let $\mathcal{C}^* : [n]^d \to [r]$ be the r-colouring of $[n]^d$ where $\mathcal{C}^*(\underline{\mathbf{x}}) := \mathcal{C}(f(\underline{\mathbf{x}}))$. If $(\underline{\mathbf{x}}_1, \dots, \underline{\mathbf{x}}_k)$ is a monochromatic solution to $Ax = 0$ in $[n]^d$ (with respect to $\mathcal{C}^*$) then $(f(\underline{\mathbf{x}}_1), \dots, f(\underline{\mathbf{x}}_k))$ is a monochromatic solution to $Ax = 0$ in $[n]^{d-1}$ (with respect to $\mathcal{C}$). By assumption, there exist at least $\delta n^{d(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[n]^d$ with respect to $\mathcal{C}^*$. Hence, by Claim 6.11.5, the number of monochromatic solutions in $[n]^{d-1}$ with respect to $\mathcal{C}$ is at least

$$\frac{\delta n^{d(k-\ell)}}{n^{k-\ell}} = \delta n^{(d-1)(k-\ell)},$$

as required. $\square$

By Proposition 6.11.4, it is clear that if the statement of Theorem 6.4.9 holds for an infinite sequence $d_1 < d_2 < \dots$ of positive integers then it holds for any $d \in \mathbb{N}$. Therefore, it suffices to prove the following relaxation of Theorem 6.4.9.

Theorem 6.11.6. *Let $r \in \mathbb{N}$, $c \in \mathbb{N} \cup \{0\}$ and let A be a partition regular $\ell \times k$ integer matrix of rank ℓ. There exist $\delta, n_0 > 0$ such that for all $n \geq n_0$ the following holds: every r-colouring of $[n]^{2^c}$ yields at least $\delta n^{2^c(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[n]^{2^c}$.*

Proof of Theorem 6.4.9. Let $r, d \in \mathbb{N}$ and let A be a partition regular $\ell \times k$ integer matrix of rank ℓ. Choose $c \in \mathbb{N}$ such that $d \leq 2^c$. Let δ, n_0 be the constants obtained by applying Theorem 6.11.6 with parameters r, c and A. By Proposition 6.11.4, the statement of Theorem 6.4.9 holds for A, r, δ, n_0 and every $d' \in \mathbb{N}$ with $d' \leq 2^c$. In particular, it holds for A, r, d, δ, n_0. $\square$

We conclude this subsection with the proof of Theorem 6.11.6. Before proceeding with the proof, we give a rough intuition of the main idea by considering an easier case. For integers $a, b \in \mathbb{Z}$, we write $[a, b] := \{m \in \mathbb{Z} : a \leq m \leq b\}$. Let $A = (1 \quad 1 \quad -1)$. Given an arbitrary r-colouring $\mathcal{C}$ of $[-n, n]^2$, our aim is to find $\Theta(n^4)$ monochromatic solutions to $Ax = 0$.

Define a colouring $\mathcal{C}^*$ of $[(\varepsilon n)^2]$, where $\varepsilon > 0$ is a small constant, as follows. If $z \in [(\varepsilon n)^2]$, then $z = x + (\varepsilon n)y$ for a unique pair $(x, y) \in [\varepsilon n] \times [0, \varepsilon n - 1]$. Then colour z with $(c_{-2}, c_{-1}, c_0, c_1, c_2)$ where c_λ is the colour of $(x - \lambda(\varepsilon n), y + \lambda) \in [-n, n]^2$ with respect to $\mathcal{C}$. Note that if $z_1 + z_2 - z_3 = 0$ is a solution in $[(\varepsilon n)^2]$, then $(x_1 + x_2 - x_3) +$

$(\varepsilon n)(y_1 + y_2 - y_3) = 0$. Since the x_i's lie in $[\varepsilon n]$, we have $|y_1 + y_2 - y_3| \leq 2$. Thus, there exists $\lambda \in \mathbb{Z}$ with $|\lambda| \leq 2$ such that $(y_1 + \lambda) + (y_2 + \lambda) - (y_3 + \lambda) = 0$, which implies $(x_1 - (\varepsilon n)\lambda) + (x_2 - (\varepsilon n)\lambda) - (x_3 - (\varepsilon n)\lambda) = 0$. In particular, $\{(x_j - \lambda(\varepsilon n), y_j + \lambda)\}_{j=1,2,3}$ is a solution to $Ax = 0$ in $[-n, n]^2$. Furthermore, suppose $\{z_j\}_{j=1,2,3}$ is a monochromatic solution to $Ax = 0$ in $[(\varepsilon n)^2]$ with respect to $\mathcal{C}^*$; say the colour is $(c_{-2}, c_{-1}, c_0, c_1, c_2)$. Then there exists $\lambda \in \mathbb{Z}$ with $|\lambda| \leq 2$ such that $\{(x_j - \lambda(\varepsilon n), y_j + \lambda)\}_{j=1,2,3}$ is a monochromatic solution to $Ax = 0$ in $[-n, n]^2$ with respect to $\mathcal{C}$, the colour being c_λ.

It is not difficult to see that different monochromatic solutions in $[(\varepsilon n)^2]$ with respect to $\mathcal{C}^*$ give rise to different monochromatic solutions in $[-n, n]^2$ with respect to $\mathcal{C}$. That is, we have defined an injection from monochromatic solutions in $[(\varepsilon n)^2]$ with respect to $\mathcal{C}^*$ to monochromatic solutions in $[-n, n]^2$ with respect to $\mathcal{C}$. By Theorem 6.4.5 there are $\Theta(n^4)$ many of the former, yielding the required result.

The proof of Theorem 6.11.6 proceeds by induction on c and follows a similar idea, where instead of Theorem 6.4.5 we invoke the induction hypothesis. Essentially, we project a solution to $Ax = 0$ in $[(\varepsilon n)^2]^{2^c}$ to an 'almost' solution in $[\varepsilon n]^{2^{c+1}}$, and then apply a shifting procedure to obtain an exact solution in $[n]^{2^{c+1}}$. The colouring $\mathcal{C}^*$ we require is more involved than the one above due to two technicalities: (i) ensuring that the parameter $\underline{\lambda}$ (which is the analogue of λ in the general case) is a vector with integer entries, thereby implying that the shifting procedure is well-defined; (ii) ensuring that solutions generated by the shifting procedure lie in $[n]^{2^{c+1}}$ rather than $[-n, n]^{2^{c+1}}$.

We will use the following notation. Given two vectors $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$, we denote their concatenation as $\underline{\mathbf{x}} * \underline{\mathbf{y}}$. Furthermore, we let $abs[\underline{\mathbf{x}}]$ be the vector obtained by taking the absolute value of all entries of $\underline{\mathbf{x}}$. For example, if $\underline{\mathbf{x}} = (1, -2)$ and $\underline{\mathbf{y}} = (3, 4)$ then $\underline{\mathbf{x}} * \underline{\mathbf{y}} = (1, -2, 3, 4)$ and $abs[\underline{\mathbf{x}}] = (1, 2)$. We write $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$ for the scalar product of $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$. Let $\underline{\mathbf{1}}$ denote the all one vector.

Proof of Theorem 6.11.6. Let $r \in \mathbb{N}$ and let A be a partition regular $\ell \times k$ integer matrix of rank ℓ. We proceed by induction on c. The case $c = 0$ follows immediately from Theorem 6.4.5. For the induction hypothesis, suppose that the statement of Theorem 6.11.6

holds for some $c \in \mathbb{N} \cup \{0\}$.

Denote the ith row of A as $\underline{\mathbf{a}}_i$. Let $C := \max_{i\in[\ell]}\{|\underline{\mathbf{a}}_i \cdot \underline{\mathbf{1}}|, 1\}$. By relabelling the rows of A and changing their sign (these operations do not affect the set of solutions), we may assume that $C = \max\{\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{1}}, 1\} \geq 1$. Let $E \geq 0$ be the largest entry of A in absolute value. Fix constants ε, P such that

$$Ek \ll 1/\varepsilon \ll P,$$

where P is a prime number that does not divide C. Let $d := 2^c$ and $R := r^{(2Ek+1)^d} \cdot (CP)^d$. By assumption, we can apply Theorem 6.11.6 with input A, R and c to obtain $\gamma, n_0 > 0$. Finally, we take $n \in \mathbb{N}$ with $n \geq \max\{n_0/\varepsilon, 2^{d+2}Ek^2d/(\varepsilon\gamma)\}$. For simplicity, we will assume that εn is an integer.

Let $\mathcal{C}$ be an arbitrary r-colouring of $[0,n]^{2d}$. We define an R-colouring $\mathcal{C}^*$ of $[Ek(\varepsilon n)+1, (\varepsilon n)^2]^d$ as follows. For any $\underline{\mathbf{z}} \in [Ek(\varepsilon n)+1, (\varepsilon n)^2]^d$, there exists a unique pair of vectors $\underline{\mathbf{x}} \in [\varepsilon n]^d$ and $\underline{\mathbf{y}} \in [Ek, \varepsilon n]^d$ such that $\underline{\mathbf{z}} = \underline{\mathbf{x}} + \underline{\mathbf{y}}(\varepsilon n)$. Let $I := [-Ek, Ek]^d$. We colour $\underline{\mathbf{z}}$ with $(\{c_{\underline{\mathbf{s}}}\}_{\underline{\mathbf{s}}\in I}, \underline{\mathbf{t}})$ where $c_{\underline{\mathbf{s}}}$ is the colour of $abs[(\underline{\mathbf{y}} + \underline{\mathbf{s}}) * (\underline{\mathbf{x}} - \underline{\mathbf{s}}(\varepsilon n))] \in [0,n]^{2d}$ with respect to $\mathcal{C}$, for each $\underline{\mathbf{s}} \in I$, and $\underline{\mathbf{t}} \in [CP]^d$ is the unique vector such that $\underline{\mathbf{y}} \equiv \underline{\mathbf{t}} \pmod{CP}$. Note that all entries of $(\underline{\mathbf{y}} + \underline{\mathbf{s}}) * (\underline{\mathbf{x}} - \underline{\mathbf{s}}(\varepsilon n))$ have absolute value at most $(\varepsilon n)(Ek+1) \ll n$, so $abs[(\underline{\mathbf{y}} + \underline{\mathbf{s}}) * (\underline{\mathbf{x}} - \underline{\mathbf{s}}(\varepsilon n))]$ indeed lies in $[0,n]^{2d}$.

Observe that $\mathcal{C}^*$ is a well-defined R-colouring of $[Ek(\varepsilon n)+1, (\varepsilon n)^2]^d$: $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ and $\underline{\mathbf{t}}$ are unique, so the colour of $\underline{\mathbf{z}} \in [Ek(\varepsilon n)+1, (\varepsilon n)^2]^d$ in $\mathcal{C}^*$ is uniquely determined. There are r possible colours for each $c_{\underline{\mathbf{s}}}$ and $(CP)^d$ possible values for $\underline{\mathbf{t}}$; since $|I| = (2Ek+1)^d$, at most R colours have been used.

Next, we describe a shifting procedure which generates a monochromatic solution in $[0,n]^{2d}$ with respect to $\mathcal{C}$ given a monochromatic solution in $[Ek(\varepsilon n)+1, (\varepsilon n)^2]^d$ with respect to $\mathcal{C}^*$.

Suppose $(\underline{\mathbf{z}}_1, \dots, \underline{\mathbf{z}}_k)$ is a monochromatic solution to $Ax = 0$ in $[Ek(\varepsilon n)+1, (\varepsilon n)^2]^d$ with respect to $\mathcal{C}^*$. For every $1 \leq i \leq k$ we have $\underline{\mathbf{z}}_i = \underline{\mathbf{x}}_i + \underline{\mathbf{y}}_i(\varepsilon n)$ for some unique $\underline{\mathbf{x}}_i \in [\varepsilon n]^d$

and $\underline{\mathbf{y}}_i \in [Ek, \varepsilon n]^d$. Let $\underline{\mathbf{z}}_i := (z_i^1, \dots, z_i^d)$, $\underline{\mathbf{x}}_i := (x_i^1, \dots, x_i^d)$ and $\underline{\mathbf{y}}_i := (y_i^1, \dots, y_i^d)$. Pick an arbitrary $j \in [d]$ and let $\underline{\mathbf{x}} := (x_1^j, \dots, x_k^j)$, $\underline{\mathbf{y}} := (y_1^j, \dots, y_k^j)$ and $\underline{\mathbf{z}} := (z_1^j, \dots, z_k^j)$. Then we have $A\underline{\mathbf{z}} = 0$ and $\underline{\mathbf{z}} = \underline{\mathbf{x}} + (\varepsilon n)\underline{\mathbf{y}}$. In particular, for every $i \in [\ell]$ we have

$$\underline{\mathbf{a}}_i \cdot \underline{\mathbf{z}} = \underline{\mathbf{a}}_i \cdot \underline{\mathbf{x}} + (\varepsilon n)\underline{\mathbf{a}}_i \cdot \underline{\mathbf{y}} = 0. \tag{6.30}$$

Note that the absolute value of every entry of $\underline{\mathbf{a}}_i$ is at most E and every entry of $\underline{\mathbf{x}}$ lies in $[\varepsilon n]$, thus we obtain

$$|\underline{\mathbf{a}}_i \cdot \underline{\mathbf{x}}| \leq \varepsilon n E k. \tag{6.31}$$

Now (6.30) and (6.31) immediately imply

$$|\underline{\mathbf{a}}_i \cdot \underline{\mathbf{y}}| \leq Ek. \tag{6.32}$$

Next, we distinguish two cases.

Case 1. Suppose that $C = \underline{\mathbf{a}_1} \cdot \underline{\mathbf{1}} \geq 1$. Since $(\underline{\mathbf{z}}_1, \dots, \underline{\mathbf{z}}_k)$ is monochromatic with respect to $\mathcal{C}^*$, there exists some $t \in [CP]$ such that $\underline{\mathbf{y}} \equiv t\underline{\mathbf{1}} \pmod{CP}$. Hence, we have

$$\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{y}} \equiv \underline{\mathbf{a}}_1 \cdot (t\underline{\mathbf{1}}) \equiv t(\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{1}}) \equiv Ct \equiv 0 \pmod{C}.$$

Set $\lambda := -(\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{y}})/C$. The previous equality implies that λ is an integer, while by (6.32) we have $|\lambda| \leq Ek/C$. Observe that

$$\underline{\mathbf{a}}_1 \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}}) = \underline{\mathbf{a}}_1 \cdot \underline{\mathbf{y}} + \lambda C = 0. \tag{6.33}$$

As $\underline{\mathbf{y}} \equiv t\underline{\mathbf{1}} \pmod{CP}$, we have

$$0 \overset{(6.33)}{=} \underline{\mathbf{a}}_1 \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}}) \equiv (t+\lambda)\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{1}} \equiv C(t+\lambda) \pmod{P},$$

and so we must have that $t + \lambda \equiv 0 \pmod{P}$ since P is a prime number which does not divide C. It follows that

$$\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}}) \equiv (t+\lambda)(\underline{\mathbf{a}}_i \cdot \underline{\mathbf{1}}) \equiv 0 \pmod{P} \tag{6.34}$$

for every $i \in [\ell]$. Note that

$$|\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}})| \leq |\underline{\mathbf{a}}_i \cdot \underline{\mathbf{y}}| + |\lambda(\underline{\mathbf{a}}_i \cdot \underline{\mathbf{1}})| \overset{(6.32)}{\leq} Ek + |\lambda C| \leq 2Ek.$$

As we picked P so that $P \gg Ek$, the previous inequality and (6.34) imply that

$$\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}}) = 0$$

for every $i \in [\ell]$. Furthermore, this together with (6.30) implies that, for every $i \in [\ell]$ we have

$$\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{x}} - \lambda(\varepsilon n)\underline{\mathbf{1}}) = 0.$$

Case 2. Suppose $\underline{\mathbf{a}}_1 \cdot \underline{\mathbf{1}} = 0$ (and so $\underline{\mathbf{a}}_i \cdot \underline{\mathbf{1}} = 0$ for every $i \in [\ell]$). Then set $\lambda := 0$ and observe that

$$\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}}) = \underline{\mathbf{a}}_i \cdot \underline{\mathbf{y}} \equiv t(\underline{\mathbf{a}}_i \cdot \underline{\mathbf{1}}) \equiv 0 \pmod{P}.$$

By (6.32) and as $P \gg Ek$, we deduce that $\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{y}} + \lambda\underline{\mathbf{1}}) = 0$ and so $\underline{\mathbf{a}}_i \cdot (\underline{\mathbf{x}} - \lambda(\varepsilon n)\underline{\mathbf{1}}) = 0$, for every $i \in [\ell]$.

In both cases, we obtain two solutions $(y_i^j + \lambda)_{i\in[k]}$ and $(x_i^j - \lambda(\varepsilon n))_{i\in[k]}$ to $Ax = 0$ in $\mathbb{Z}$, where $|\lambda| \leq Ek$. In general, since j was arbitrarily chosen, for every $j \in [d]$ there exists $\lambda_j \in [-Ek, Ek]$ such that $(y_i^j + \lambda_j)_{i\in[k]}$ and $(x_i^j - \lambda_j(\varepsilon n))_{i\in[k]}$ are two solutions to $Ax = 0$ in $\mathbb{Z}$. By setting $\underline{\lambda} := (\lambda_1, \dots, \lambda_d)$, we obtain that $(\underline{\mathbf{y}}_i + \underline{\lambda})_{i\in[k]}$ and $(\underline{\mathbf{x}}_i - \underline{\lambda}(\varepsilon n))_{i\in[k]}$ are two solutions to $Ax = 0$ in $\mathbb{Z}^d$.

Note that $\underline{\mathbf{y}}_i + \underline{\lambda} \in [0, 2\varepsilon n]^d$ for every $i \in [k]$ since $\underline{\mathbf{y}}_i \in [Ek, \varepsilon n]^d$ and $\underline{\lambda} \in [-Ek, Ek]^d$. Furthermore, for every $m \in [d]$, the mth entries of the vectors $(\underline{\mathbf{x}}_i - \underline{\lambda}(\varepsilon n))_{i\in[k]}$ must have the same sign since $\underline{\mathbf{x}}_i \in [\varepsilon n]^d$ and $\underline{\lambda}$ has integer entries. Hence $(abs[(\underline{\mathbf{y}}_i + \underline{\lambda}) * (\underline{\mathbf{x}}_i -$

$\underline{\lambda}(\varepsilon n))])_{i\in[k]}$ is a solution to $Ax = 0$ in $[0, n]^{2d}$.

Crucially, all vectors $(abs[(\underline{\mathbf{y}}_i + \underline{\lambda}) * (\underline{\mathbf{x}}_i - \underline{\lambda}(\varepsilon n))])_{i\in[k]}$ have the same colour in $\mathcal{C}$ by the construction of $\mathcal{C}^*$ and the fact that $\underline{\lambda} \in I = [-Ek, Ek]^d$. Namely, if all $(\underline{\mathbf{z}}_1, \dots, \underline{\mathbf{z}}_k)$ are coloured with, say, $(\{c_{\underline{\mathbf{s}}}\}_{\underline{\mathbf{s}}\in I}, \underline{\mathbf{t}})$ in $\mathcal{C}^*$ then all $(abs[(\underline{\mathbf{y}}_i + \underline{\lambda}) * (\underline{\mathbf{x}}_i - \underline{\lambda}(\varepsilon n))])_{i\in[k]}$ are coloured with $c_{\underline{\lambda}}$ in $\mathcal{C}$. Hence, $(abs[(\underline{\mathbf{y}}_i + \underline{\lambda}) * (\underline{\mathbf{x}}_i - \underline{\lambda}(\varepsilon n))])_{i\in[k]}$ is a monochromatic solution to $Ax = 0$ in $[0, n]^{2d}$.

Arbitrarily extend the R-colouring $\mathcal{C}^*$ of $[Ek(\varepsilon n) + 1, (\varepsilon n)^2]^d$ to an R-colouring $\mathcal{C}^{**}$ of $[(\varepsilon n)^2]^d$. As $(\varepsilon n)^2 \geq n_0$, the induction hypothesis implies that there are at least $\gamma(\varepsilon n)^{2d(k-\ell)}$ monochromatic solutions in $[(\varepsilon n)^2]^d$ with respect to $\mathcal{C}^{**}$. Note that $[Ek(\varepsilon n) + 1, (\varepsilon n)^2]^d$ contains all but at most $dEk(\varepsilon n)^{2d-1} \leq \gamma(\varepsilon n)^{2d}/(2k)$ of the elements of $[(\varepsilon n)^2]^d$. Thus, by Lemma 6.11.3, there are at least $(\gamma/2)(\varepsilon n)^{2d(k-\ell)}$ monochromatic solutions in $[Ek(\varepsilon n) + 1, (\varepsilon n)^2]^d$ with respect to $\mathcal{C}^*$; moreover, using the procedure described above, we can produce a monochromatic solution in $[n]^{2d}$ with respect to $\mathcal{C}$ from each one of them. It remains to check how many times we could have counted each solution in $[n]^{2d}$.

Observe that a different monochromatic solution $(\underline{\mathbf{z}}_{\mathbf{1}}, \dots, \underline{\mathbf{z}}_k)$ in $[(Ek(\varepsilon n) + 1, (\varepsilon n)^2]^d$ would have generated different k-sets $(y_i^j + \lambda)_{i\in[k]}$ and $(x_i^j - \lambda(\varepsilon n))_{i\in[k]}$, since $\underline{\mathbf{x}}_i$ and $\underline{\mathbf{y}}_i$ are unique and by (6.30). On the other hand, the preimage of a singleton in $abs : [-n, n]^d \to [0, n]^d$ has size at most 2^d. Hence, a monochromatic solution in $[n]^{2d}$ with respect to $\mathcal{C}$ could be counted at most 2^d times.

In conclusion, there are at least $(\gamma/2^{d+1})(\varepsilon n)^{2d(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[0, n]^{2d}$ with respect to $\mathcal{C}$. Using Lemma 6.11.3, there are at least $(\gamma/2^{d+2})(\varepsilon n)^{2d(k-\ell)}$ monochromatic solutions to $Ax = 0$ in $[n]^{2d}$. Taking $\delta := (\gamma/2^{d+2})\varepsilon^{2d(k-\ell)}$, the statement of Theorem 6.11.6 holds for $c + 1$ since $2d = 2^{c+1}$. This concludes the inductive step and the proof. $\square$

6.12 Proof of the applications of the random Rado lemma

In this section, we prove the applications of the random Rado lemma presented in Sections 6.3 and 6.6, that is Theorems 6.3.7, 6.6.1, 6.6.2, 6.6.3, 6.6.4 and 6.6.5. The first uses the full version of the random Rado lemma, Lemma 6.7.8. The others all use the simplified version, Lemma 6.7.33.

Proof of Theorem 6.3.7. Let $r, k \in \mathbb{N}$ with $r \geq 2$ and $k \geq 3$. Let $A = (a_{ij})$ be the $(k-2) \times k$ matrix where

$$a_{ij} := \begin{cases} 1 & \text{if } i = j, \\ -2 & \text{if } i+1 = j, \\ 1 & \text{if } i+2 = j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $k\text{-}\mathrm{Sol}^A_{\mathbf{P}_n}([k])$ corresponds to the set of all k-APs in $\mathbf{P}_n$.

The prime number theorem yields

$$|\mathbf{P}_n| = \Theta\left(\frac{n}{\log n}\right). \tag{6.35}$$

Note that any solution to $Ax = 0$ that does not correspond to a k-AP must be a trivial solution (i.e., all entries in x are the same). Moreover, given fixed integers a, b and $1 \leq i < j \leq k$, there is at most one k-AP that has a in its ith position and b in its jth position. Thus, for any $Y \subseteq [k]$ with $2 \leq |Y| \leq k$, each element of $\mathrm{Sol}^A_{\mathbf{P}_n}(Y)$ corresponds to either a trivial solution to $Ax = 0$ or a single k-AP. Hence, by Theorem 6.3.1 we have that

$$|\mathrm{Sol}^A_{\mathbf{P}_n}(Y)| = \Theta\left(\frac{n^2}{\log^k n}\right) \quad \text{for each } Y \subseteq [k] \text{ with } 2 \leq |Y| \leq k. \tag{6.36}$$

Therefore, (6.35) and (6.36) imply that

$$p_Y(A, \mathbf{P}_n) = \Theta\left(\left(\frac{n}{\log^{k-1} n}\right)^{-\frac{1}{|Y|-1}}\right) \quad \text{for each } Y \subseteq [k] \text{ with } 2 \leq |Y| \leq k. \tag{6.37}$$

Note that $[k]$ maximises $p_Y(A, \mathbf{P}_n)$ over all $Y \subseteq [k]$ with $|Y| \geq 2$, and so

$$\hat{p}_n := \hat{p}(A, \mathbf{P}_n) = \Theta\left(n^{-\frac{1}{k-1}} \log n\right). \tag{6.38}$$

We now verify the conditions of Lemma 6.7.8.

(A1): By (6.35) and (6.38), we have $|\mathbf{P}_n|\hat{p}_n = \Theta(n^{\frac{k-2}{k-1}}) \to \infty$ and $\hat{p}_n \to 0$ as $n \to \infty$, since $k \geq 3$.

(A2): Theorem 6.4.6 immediately implies that $\mathbf{P}_n$ is (A, r)-supersaturated.

(A3): Let $W \subseteq Y \subseteq [k]$. First, consider the case $|W| = 1$. Fix a prime $q \in \mathbf{P}_n$. If $|Y| = 1$ then $|\operatorname{Sol}^A_{\mathbf{P}_n}(q, W, Y)| \leq 1$. If $|Y| \geq 2$, we have

$$|\operatorname{Sol}^A_{\mathbf{P}_n}(q, W, Y)| = O\left(\frac{n}{\log^{k-1} n}\right) \overset{(6.35),(6.36)}{=} O\left(\frac{|\operatorname{Sol}^A_{\mathbf{P}_n}(Y)|}{|\mathbf{P}_n|}\right) = O\left(\frac{|\operatorname{Sol}^A_{\mathbf{P}_n}(Y)|}{|\operatorname{Sol}^A_{\mathbf{P}_n}(W)|}\right), \tag{6.39}$$

where we used Lemma 6.3.3 for the first equality and $|\operatorname{Sol}^A_{\mathbf{P}_n}(W)| \leq |\mathbf{P}_n|$ for the third.

Now suppose $|W| \geq 2$. Since specifying two elements (and their position) of a k-AP uniquely determines the latter, for any $z_W \in S^{|W|}$, we have

$$|\operatorname{Sol}^A_{\mathbf{P}_n}(z_W, W, Y)| \leq 1 \overset{(6.36)}{=} O\left(\frac{|\operatorname{Sol}^A_{\mathbf{P}_n}(Y)|}{|\operatorname{Sol}^A_{\mathbf{P}_n}(W)|}\right).$$

Therefore, there exists $B \geq 1$ such that $(A, \mathbf{P}_n)$ is B-extendable for every n sufficiently large, as required.

(A4): Let $W \subsetneq Y \subseteq [k]$ with $|W| = 1$. Since $|Y| \geq 2$, equation (6.39) holds. The first two equalities of (6.39) immediately imply condition (A4).

(A5): We take $X_n = [k]$ for all $n \in \mathbb{N}$. We have $|X_n| = k \geq 3$ and $p_{X_n}(A, \mathbf{P}_n) = \hat{p}_n$. Let $W, W' \subsetneq [k]$ with $|W| = 2$ and $|W'| \geq 2$. As $|W'| \leq k-1$ we have

$$\frac{|\mathbf{P}_n|^2}{|\operatorname{Sol}^A_{\mathbf{P}_n}(W)|}\left(\frac{p_{W'}(A,\mathbf{P}_n)}{p_{[k]}(A,\mathbf{P}_n)}\right)^{|W'|-1} \overset{(6.35),(6.36),(6.37)}{=} O\left(\log^{k-2} n \left(\frac{n}{\log^{k-1} n}\right)^{\frac{|W'|-k}{k-1}}\right) \to 0,$$

as $n \to \infty$, as required.

(A6): Let $W \subseteq [k]$ with $|W| = 2$. Then (6.35) and (6.36) imply

$$\frac{|\mathbf{P}_n|}{|\operatorname{Sol}^A_{\mathbf{P}_n}(W)|} = O\left(\frac{\log^{k-1} n}{n}\right) \to 0,$$

as $n \to \infty$, as required.

Since conditions (A1)–(A6) hold, we can apply Lemma 6.7.8. Note that $\mathbf{P}_{n,p}$ is (A, r)-Rado if and only if $\mathbf{P}_{n,p}$ is (r, k)-van der Waerden. Thus Lemma 6.7.8 yields Theorem 6.3.7. □

Proof of Theorem 6.6.1. Let $r, d \in \mathbb{N}$ with $r \geq 2$. Let A satisfy the assumptions of Theorem 6.6.1. We will verify that the conditions of Lemma 6.7.33 hold, where we take $S_n := [n]^d$. Note that $[n]$ is a finite subset of the field $\mathbb{Q}$ and $m_{S_n}(A) = m(A)$.

(C2): Theorem 6.4.9 implies that $([n]^d)_{n\in\mathbb{N}}$ is (A, r)-supersaturated.

(C3): An immediate consequence of Theorem 6.4.9 is that there exists some $\delta > 0$ such that the number of solutions to $Ax = 0$ in $[n]^d$ is at least $\delta n^{d(k-\ell)}$, for all n sufficiently large. Since there are at most $n^{d(k-\ell)}$ solutions to $Ax = 0$ in $[n]^d$ (by e.g., Lemma 6.7.16), $(A, [n]^d)$ is δ-rich for n sufficiently large.

(C4): Since A is irredundant, A is irredundant with respect to $\mathbb{Q}^d$. Furthermore, by Theorem 6.4.9, A is 3-partition regular in $[n]^d$ for n sufficiently large. Thus, we can apply Lemma 6.7.23 with $\mathbb{F} := \mathbb{Q}$ and $S := [n]^d$ to obtain that $(A, [n]^d)$ is abundant for n sufficiently large.

(C1): By (C4), $m(A)$ is well-defined (see Remark 6.7.28). As A is full rank, considering $W = [k]$ yields $m(A) \geq \frac{k-1}{k-1-\ell} > 1$. Since (C3) holds, Lemma 6.7.32 yields $\hat{p}_n = \Theta(n^{-d/m(A)})$. Therefore, we have $|[n]^d|\hat{p}_n \to \infty$ and $\hat{p}_n \to 0$ as $n \to \infty$.

(C5): Note that $\mathrm{rank}_{S_n}(A) > 0$ for every n since $S_n \neq \{0\}$. As (C3) and (C4) are satisfied, we can apply Lemma 6.7.26. Lemma 6.7.26 and (C1) imply that (C5) holds.

As conditions (C1)–(C5) hold, we can apply Lemma 6.7.33 to obtain Theorem 6.6.1. □

Proof of Theorem 6.6.3. Let $r \in \mathbb{N}$ with $r \geq 2$ and let A and G be as in the statement of Theorem 6.6.3. Note that G^n is a finite abelian group. We will verify that the hypothesis of Lemma 6.7.33 holds, where we take $S_n := G^n$. We will repeatedly use the fact that $\mathrm{rank}_{G^n}(A_W) = \mathrm{rank}_G(A_W)$ for any $W \subseteq [k]$; this follows easily from Definition 6.7.11. Thus, $m_{G^n}(A) = m_G(A)$.

(C2): Since G is a finite abelian group with exponent s and A satisfies the s-columns condition, Theorem 6.4.8 implies $(G^n)_{n\in\mathbb{N}}$ is (A, r)-supersaturated.

(C3): Equation (6.4) implies (A, G^n) is 1-rich.

(C4): Let $W \subseteq [k]$ with $|W| = 2$. By assumption, (A, G) is abundant, and so $\mathrm{rank}_G(A_{\overline{W}}) = \mathrm{rank}_G(A)$. This implies $\mathrm{rank}_{G^n}(A_{\overline{W}}) = \mathrm{rank}_{G^n}(A)$. As W is arbitrary, (A, G^n) is abundant.

(C1): By (C4), $m_{G^n}(A)$ is well-defined and positive (see Remark 6.7.28). Since (C3) holds, Lemma 6.7.32 yields $\hat{p}_n := \hat{p}(A, G^n) = \Theta((|G|^n)^{-1/m_{G^n}(A)})$. As $\mathrm{rank}_{G^n}(A) = \mathrm{rank}_G(A) > 0$ and by using $W = [k]$, we have $m_{G^n}(A) \geq \frac{k-1}{k-1-\mathrm{rank}_{G^n}(A)} > 1$. Therefore we have $|G|^n\hat{p}_n \to \infty$ and $\hat{p}_n \to 0$ as $n \to \infty$.

(C5): Let $Z \subseteq [k]$ and

$$D(Z) := \frac{|Z| - 1 - \mathrm{rank}_G(A) + \mathrm{rank}_G(A_{\overline{Z}})}{|Z| - 1}.$$

As (A, G^n) is 1-rich, (6.7) implies that $p_Z(A, G^n) = |G|^{-n \cdot D(Z)}$ for every $n \in \mathbb{N}$. Pick $X \subseteq [k]$ with $|X| \geq 2$ so that the function $D(X)$ is minimised; additionally choose X so that $|X|$ is as small as possible under this assumption. It follows that $p_X(A, G^n) = \hat{p}(A, G^n)$. For every $|Z| = 2$ we have $D(Z) = 1$, by (C4), while $D([k]) < 1$ since $\mathrm{rank}_G(A) > 0$. Hence, $|X| \geq 3$.

Let $W' \subsetneq X$ with $|W'| \geq 2$. We have

$$\frac{p_{W'}(A, G^n)}{p_X(A, G^n)} = (|G|^n)^{-D(W') + D(X)}.$$

Since X is minimal and there are finitely many subsets $Z \subsetneq X$, there exists $\varepsilon > 0$ such that we have $D(Z) - D(X) \geq \varepsilon$ for every $Z \subsetneq X$. It follows that

$$\frac{p_{W'}(A, G^n)}{p_X(A, G^n)} \leq |G|^{-\varepsilon n} \to 0$$

as $n \to \infty$.

As conditions (C1)–(C5) hold, we can apply Lemma 6.7.33 to obtain Theorem 6.6.3.

$\square$

Proof of Theorem 6.6.2. Since $m_{\mathbb{Z}_4}(A) = 4/3$ when $A = (2 \;\; 2 \;\; -2)$ (see Example 6.7.29), to prove the theorem it suffices to verify that the hypothesis of Theorem 6.6.3 holds in our setting.

Firstly, $\mathbb{Z}_4$ is an abelian group with exponent 4. The matrix A satisfies the 4-columns condition: the last two entries of A sum to zero and they span the first entry.

Next, we check $(A, \mathbb{Z}_4)$ is abundant and $\mathrm{rank}_{\mathbb{Z}_4}(A) > 0$. The total number of solutions to $2x_1 + 2x_2 - 2x_3 = 0$ in $\mathbb{Z}_4$ is $4 \cdot 4 \cdot 2 = 4^{3-1/2}$; so $\mathrm{rank}_{\mathbb{Z}_4}(A) = 1/2 > 0$. For any

$W \subseteq [3]$ with $|W| = 2$, the equation $A_{\overline{W}} x = 0$ in $\mathbb{Z}_4$ is the same as $2x = 0$ in $\mathbb{Z}_4$. We have $2 = 4^{1-1/2}$ solutions to $2x = 0$ in $\mathbb{Z}_4$, thus $\mathrm{rank}_{\mathbb{Z}_4}(A_{\overline{W}}) = 1/2 = \mathrm{rank}_{\mathbb{Z}_4}(A)$. In particular, $(A, \mathbb{Z}_4)$ is abundant. $\square$

Proof of Theorem 6.6.4. Consider any full rank $\ell \times k$ matrix A with entries from $\mathbb{F}$ that satisfies the hypothesis of the theorem.

Observe that we cannot apply directly the simplified random Rado lemma (Lemma 6.7.33) as the matrix A does not have integer entries. However, all definitions, statements and proofs that led to Lemma 6.7.33 can be straightforwardly adapted to obtain an analogue of the simplified random Rado lemma for this setting, Lemma B.3.23. Similarly, the various results stated in Section 6.7, and their proofs, can be easily adapted to this setting almost word-by-word.

Lemma B.3.23 and the analogues of the results appearing in Section 6.7 are formally stated in Section B.3. For the rest of the proof, we abuse notation by utilizing the terms supersaturated, rich, abundant etc. although these have not been technically defined for matrices with entries from a field. Once again, the correct adaptation of these terms is immediate and it is rigorously explained in Section B.3.

Since $m_{\mathbb{F}^n}(A) = m_{\mathbb{F}}(A)$, to prove the theorem it suffices to show that conditions (C1)–(C5) of Lemma B.3.23 hold with respect to $S_n = \mathbb{F}^n$.

(C2): Theorem 6.4.7 implies that $(\mathbb{F}^n)_{n \in \mathbb{N}}$ is (A, r)-supersaturated.

(C3): Fact B.3.10 (the analogue of equation (6.4)) implies that $(A, \mathbb{F}^n)$ is 1-rich.

(C4): Theorem 6.4.3 implies that A is 3-partition regular in $\mathbb{F}^n \setminus \{0\}^n$ (for n sufficiently large). By assumption A is irredundant with respect to $\mathbb{F}^n$. By Lemma B.3.16 (analogue of Lemma 6.7.23), $(A, \mathbb{F}^n)$ is abundant for all sufficiently large n.

(C1): As $(A, \mathbb{F}^n)$ is abundant for sufficiently large n, we know $(A, \mathbb{F})$ is abundant. Remark B.3.21 (analogue of Remark 6.7.28) implies that $m_{\mathbb{F}}(A)$ is well-defined and positive. As A is full rank, considering $W = [k]$ yields $m_{\mathbb{F}}(A) \geq \frac{k-1}{k-1-\ell} > 1$. Since (C3) holds, Lemma B.3.22 (the analogue of Lemma 6.7.32) yields $\hat{p}_n = \Theta(|\mathbb{F}^n|^{-1/m_{\mathbb{F}}(A)})$. Therefore we have $|\mathbb{F}^n|\hat{p}_n \to \infty$ and $\hat{p}_n \to 0$ as $n \to \infty$.

(C5): Since (C3) and (C4) are satisfied, we can apply Lemma B.3.19 (the analogue of Lemma 6.7.26). This together with (C1) implies that (C5) holds.

□

Proof of Theorem 6.6.5. Let A be as in the theorem. Recall that as A is an integer matrix, we can view it as a matrix with entries from $\mathbb{F}$ (i.e., an entry a_{ij} corresponds to $a_{ij} \cdot 1_{\mathbb{F}} \in \mathbb{F}$). Thus, we show that the theorem follows directly from Theorem 6.6.4.

Note that the finite field $\mathbb{F}$ of order q^k has exponent q (in particular, $\langle 1_{\mathbb{F}} \rangle \cong \mathbb{Z}_q$) so the fact that A satisfies the q-columns condition immediately implies it also satisfies the columns condition over $\mathbb{F}$. We can therefore apply Theorem 6.6.4 to obtain the theorem. □

6.13 Further applications and future directions

We have seen a number of applications of the (simplified) random Rado lemma. Of course, there are many other applications; we give a couple more in the next subsection. In Section 6.13.2 we consider a resilience result. In the final subsection, we consider some possible future research directions. The proofs of the results from this section can be found in Appendix B.

6.13.1 Further applications of the random Rado lemma

The next result is an analogue of the random Rado theorem for $\mathbb{Z}_n$.

Theorem 6.13.1. *For all irredundant partition regular full rank integer matrices A, and* *all $r \geq 2$, there exist constants $C, c > 0$ such that the following holds.*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{Z}_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m(A)}, \\ 1 & \text{if } p \geq Cn^{-1/m(A)}. \end{cases}$$

The reader may wonder why the parameter $m(A)$ rather than $m_{\mathbb{Z}_n}(A)$ appears in the statement of Theorem 6.13.1. In particular, $m_{\mathbb{Z}_n}(A)$ may not equal $m(A)$ and it can be the case that $m_{\mathbb{Z}_n}(A)$ may not be the same for all values of n. For example, for $A = \begin{pmatrix} 2 & 2 & -2 \end{pmatrix}$ and even n we have $m_{\mathbb{Z}_n}(A) = 2/(1+\log_n 2)$. However, in this case we have $n^{-1/m_{\mathbb{Z}_n}(A)} = \frac{1}{\sqrt{2}} n^{-1/m(A)}$. More generally, we have that $n^{-1/m(A)} = \Theta(n^{-1/m_{\mathbb{Z}_n}(A)})$ and so we can use $m(A)$ in the statement of Theorem 6.13.1 instead of $m_{\mathbb{Z}_n}(A)$.

Theorem 6.13.1 follows from Lemma 6.7.8, though actually the 1-statement of Theorem 6.13.1 is also an immediate consequence of Theorem 6.5.4. Note that we cannot apply the simplified random Rado lemma (Lemma 6.7.33) to prove Theorem 6.13.1, as condition (C4) may not hold. Indeed, consider the following example.

Example 6.13.2. *Let $B := \begin{pmatrix} 2 & 2 & 2 & 1 & -1 \end{pmatrix}$ and $B_{\{1,2,3\}} = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$. Observe that B is an irredundant partition regular full rank integer matrix. However, for $n \in \mathbb{N}$ even, $(B, \mathbb{Z}_n)$ is not abundant since*

$$\operatorname{rank}_{\mathbb{Z}_n}(B_{\{1,2,3\}}) = 1 - \log_n 2 \neq 1 = \operatorname{rank}_{\mathbb{Z}_n}(B).$$

So condition (C4) does not hold for $(\mathbb{Z}_n)_{n\in\mathbb{N}}$ and B.

In general, our machinery ensures that, given any 'well behaved' sequence $(S_n)_{n\in\mathbb{N}}$ of finite abelian groups, one can obtain a corresponding random Ramsey-type theorem if one can prove a supersaturation result. It would therefore be interesting to prove other supersaturation results for abelian groups à la Theorem 6.4.8. Furthermore, it is likely that 'projection' type arguments such as in the proof of Theorem 6.4.9 can be employed to obtain new supersaturation results from existing results (such as Theorems 6.4.8 and 6.4.9).

As noted in Section 6.4.2, sometimes supersaturation results are easy to obtain, for example, for matrices A that are translation-invariant. In such cases there are fewer conditions to check in order to prove a random Ramsey-type result. By applying Lemma 6.7.33 we obtain the following.

Theorem 6.13.3. *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite abelian groups and A be an integer matrix that is translation-invariant with respect to S_n for all sufficiently large $n \in \mathbb{N}$. If (C1), (C4) and (C5) are satisfied then there exist constants $C, c > 0$ such that the following holds.*

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_{S_n}(A)}, \\ 1 & \text{if } p \geq Cn^{-1/m_{S_n}(A)}. \end{cases}$$

6.13.2 Resilience in the primes

The following is an immediate consequence of Theorem 6.3.1.

Lemma 6.13.4. *Let $r \geq 1$ and $k \geq 3$. Given any $\gamma > 0$ there exist $n_0, \varepsilon > 0$ such that for all $n \geq n_0$, the following holds. Suppose $X \subseteq \mathbf{P}_n$ with $|X| \geq \gamma|\mathbf{P}_n|$. Then for any r-colouring of X there are at least $\varepsilon n^2/\log^k n$ k-APs which are monochromatic in the same colour.*

Using Lemma 6.13.4 one can obtain the following resilience-type result.

Theorem 6.13.5. *Let $r \geq 1$, $k \geq 3$ and $\delta > 0$. There exists a constant $C > 0$ such that if $p \geq Cn^{-1/(k-1)} \log n$, then w.h.p. $\mathbf{P}_{n,p}$ has the property that for any subset $X \subseteq \mathbf{P}_{n,p}$ with $|X| \geq \delta|\mathbf{P}_{n,p}|$, whenever X is r-coloured, there is a monochromatic k-AP.*

By setting $r = 1$ in Theorem 6.13.5, we obtain a sharpening of Theorem 6.3.4, since the property considered is that of being (δ, k)-Szemerédi. For $r \geq 2$, Theorem 6.13.5 strengthens the 1-statement of Theorem 6.3.7 as it states that w.h.p. not only can we find a monochromatic k-AP in $\mathbf{P}_{n,p}$ whenever r-coloured, but we can afford to delete a

$(1-\delta)$-proportion of $\mathbf{P}_{n,p}$ and still guarantee such a monochromatic k-AP.

The proof of Theorem 6.13.5 is an easy adaption of the proof of the 1-statement of Theorem 6.8.7, using ideas from the proof of a resilience-type result in [76].

6.13.3 Further directions and examples

Another direction in which one can generalise the random Rado lemma is by removing the requirement of matrices having integer entries. Given a matrix A with entries in a ring R one can ask whether a finite colouring of an R-module yields a monochromatic solution to $Ax = 0$ or not. If one can in fact prove a suitable supersaturation result, then our machinery could yield a random Ramsey-type theorem for this setting. Theorem 6.6.4 already provides one such result in this direction.

Condition (P5) in the statement of Theorem 6.8.7 arises as an artifact of our proof. It would be interesting to investigate possible relaxations of this condition; this could in turn lead to a relaxation of condition (A5) from Lemma 6.7.8.

Condition (C4) in Theorem 6.7.33 states that (A, S_n) is abundant for every sufficiently large $n \in \mathbb{N}$. This property is used to verify, e.g., (A5) when deducing Lemma 6.7.33 from Lemma 6.7.8. However, (as briefly touched upon when discussing Theorem 6.13.1) just because a sequence $(S_n)_{n\in\mathbb{N}}$ of finite abelian groups does not satisfy (C4) does not necessarily mean one cannot apply Lemma 6.7.8 instead.

The following is another example of this. Let $A := \begin{pmatrix} 2 & -2 & 63 & 65 \end{pmatrix}$ and $S_n := \mathbb{Z}_{128}^n$ for every $n \in \mathbb{N}$. The pair (A, S_n) is not abundant since $\mathrm{rank}_{S_n}(A) = 1$, whereas $\mathrm{rank}_{S_n}(A_{\{1,2\}}) = 6/7$, so we cannot apply Lemma 6.7.33. However, conditions (A1)–(A6) do hold for $(S_n)_{n\in\mathbb{N}}$ and A, and so we can apply Lemma 6.7.8 to deduce the following theorem.

Theorem 6.13.6. *Let* $A = \begin{pmatrix} 2 & -2 & 63 & 65 \end{pmatrix}$ *and* $r \geq 2$. *There exist constants* $c, C > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}[(\mathbb{Z}_{128})_p^n \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c\,128^{-2n/3}, \\ 1 & \text{if } p \geq C\,128^{-2n/3}. \end{cases}$$

Conversely, consider $B := \begin{pmatrix} 4 & -4 & 63 & 65 \end{pmatrix}$. The sequence $(S_n)_{n\in\mathbb{N}}$ is (B,r)-supersaturated for every $r \in \mathbb{N}$: this follows from Theorem 6.4.8, as $\mathbb{Z}_{128}$ is an abelian group with exponent 128 and B satisfies the 128-columns condition as $4 - 4 + 63 + 65 \equiv 0 \pmod{128}$. However, we cannot apply Lemma 6.7.8, as condition (A5) does not hold. Indeed, it is easy to check that, for $W \subseteq [4]$,

$$|\operatorname{Sol}_{S_n}^B(W)| = \begin{cases} |S_n|^3 & \text{if } |W| = [4], \\ |S_n|^{|W|-2/7} & \text{if } \{3,4\} \subseteq W \neq [4], \\ |S_n|^{|W|} & \text{otherwise.} \end{cases}$$

It immediately follows that, for $|W| \geq 2$,

$$p_W(B, S_n) = \left(\frac{|\operatorname{Sol}_{S_n}^B(W)|}{|S_n|}\right)^{-\frac{1}{|W|-1}} = \begin{cases} |S_n|^{-2/3} & \text{if } |W| = [4], \\ |S_n|^{-(|W|-9/7)/(|W|-1)} & \text{if } \{3,4\} \subseteq W \neq [4], \\ |S_n|^{-1} & \text{otherwise.} \end{cases}$$

Note that if $(X_n)_{n\in\mathbb{N}}$ is a sequence such that $X_n \subseteq [4]$ and $p_{X_n}(B, S_n) = \Omega(\hat{p}(B, S_n))$, then $X_n = [4]$ for all sufficiently large $n \in \mathbb{N}$. In this case, for $W_n := \{3,4\} \subsetneq [4]$ we have

$$\frac{|S_n|^2}{|\operatorname{Sol}_{S_n}^B(W_n)|}\left(\frac{p_{W_n}(B, S_n)}{p_{[4]}(B, S_n)}\right) = |S_n|^{5/21} = (128)^{5n/21} \not\to 0$$

as $n \to \infty$. So $(S_n)_{n\in\mathbb{N}}$ is not compatible with respect to B.

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Appendix A

The content of this appendix is related to Chapter 6. Namely, this appendix includes the proofs of Lemma 6.11.1 and Theorem 6.3.2, as well as a formal explanation of how to deduce the 1-statement of the classic random Ramsey theorem (Theorem 6.5.1) from our Theorem 6.8.7.

A.1 Proof of Lemma 6.11.1

Recall Lemma 6.11.1 is the sieve result we used in the proof of Lemma 6.3.3 in Section 6.11. Lemma 6.11.1 is a corollary of the following result from [131].

Lemma A.1.1 ([131, Theorem 32])**.** *Let t and C be fixed natural numbers and let $n \in \mathbb{N}$ with $n \geq 2$. For each prime number $p \leq \sqrt{n}$, let E_p be the union of $\omega(p)$ residue classes modulo p, where $\omega(p) = t$ for all $p \geq C$, and $\omega(p) < p$ for all p. Then for any $\varepsilon > 0$, one has*

$$|[n] \setminus \bigcup_{\substack{p \text{ prime} \\ p \leq \sqrt{n}}} E_p| \leq (2^t t! + \varepsilon)\mathfrak{S}\frac{n}{\log^t n}$$

whenever n is sufficiently large depending on k, C, ε, and where $\mathfrak{S}$ is the singular series

$$\mathfrak{S} := \prod_{\text{prime } p} \left(1 - \frac{1}{p}\right)^{-t} \left(1 - \frac{\omega(p)}{p}\right).$$

We now explain how to deduce Lemma 6.11.1 from Lemma A.1.1.

Proof of Lemma 6.11.1. Let $k \in \mathbb{N}$ with $k \geq 2$ and let q be a prime number. For each prime $p \geq k$ with $p \neq q$, let E_p be the union of $k-1$ residue classes modulo p. We want to prove that $|E(n)| = O(n/\log^{k-1} n)$ where

$$E(n) := [n] \setminus \bigcup_{\substack{p \neq q \text{ prime} \\ k \leq p \leq \sqrt{n}}} E_p.$$

Fix $C := k$, $t := k-1$, $\varepsilon := 1$ and pick $n \in \mathbb{N}$ sufficiently large so that Lemma A.1.1 holds. For every prime $p < k$, let $\omega(p) := 0$ and $E_p := \emptyset$. We have

$$E(n) = [n] \setminus \bigcup_{\substack{p \neq q \text{ prime} \\ p \leq \sqrt{n}}} E_p.$$

Note that if $q < k$ then $E_q = \emptyset$ by definition. In this case set $E'(n) := E(n) \setminus E_q = E(n)$.

If $q \geq k$, let E_q be the union of $k-1$ residue classes modulo q that maximises $|E'(n)|$ where

$$E'(n) := E(n) \setminus E_q = [n] \setminus \bigcup_{\substack{p \text{ prime} \\ p \leq \sqrt{n}}} E_p.$$

Since the residue classes modulo q induce a partition of $E(n)$, by the maximality of E_q we obtain

$$|E'(n)| = |E(n) \setminus E_q| \geq \left(1 - \frac{k-1}{q}\right)|E(n)| \geq \frac{1}{k}|E(n)|.$$

Applying Lemma A.1.1 with respect to parameters t, C, ε and the E_p's, we obtain $|E'(n)| \leq Tn/\log^{k-1} n$ where T is a constant depending uniquely on k. Thus, $|E(n)| \leq kTn/\log^{k-1} n$ and so indeed $|E(n)| = O(n/\log^{k-1} n)$, as required. □

A.2 Proof of Theorem 6.3.2

The proof of Theorem 6.3.2 is an easy application of the second moment method and it is nearly identical to the analogous proof for arithmetic progressions in the integers (see [82, Example 3.2]). We include it here for completeness.

Proof of Theorem 6.3.2. Let X be the number of k-APs in $\mathbf{P}_{n,p}$. By Theorem 6.3.1 there are $\Theta(n^2/\log^k n)$ k-APs in $\mathbf{P}_n$ and so we have $\mathbb{E}(X) = \Theta(p^k n^2/\log^k n)$. If $p = o(n^{-2/k}\log n)$ then $\mathbb{E}(X) = o(1)$. By Markov's inequality, $\mathbb{P}(X \geq 1) \leq \mathbb{E}(X) = o(1)$ and in particular $\mathbb{P}(X = 0) = 1 - o(1)$, as required.

For the 1-statement, pick an arbitrary ordering of the k-APs in $\mathbf{P}_n$ and let X_i be the indicator variable for the ith k-AP to appear in $\mathbf{P}_{n,p}$. Formally,

$$X_i = \begin{cases} 1 & \text{if the } i\text{th } k\text{-AP lies in } \mathbf{P}_{n,p}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we bound the number of pairs of k-APs in $\mathbf{P}_n$ that share exactly one element: first fix a prime $q \in \mathbf{P}_n$ ($O(n/\log n)$ choices) and then pick two k-APs in $\mathbf{P}_n$ that contain q ($O(n^2/\log^{2k-2} n)$ choices by Lemma 6.3.3). Thus, there are $O(n^3/\log^{2k-1} n)$ such pairs. Furthermore, there are $O(n^2/\log^k n)$ pairs of k-APs in $\mathbf{P}_n$ that share at least two elements (since a k-AP is uniquely determined by two elements and their positions within the k-AP).

Note that $X = \sum_i X_i$. It follows that

$$\mathrm{Var}(X) = \sum_{i,j} \mathrm{Cov}(X_i, X_j) \leq \sum_{i,j} \mathbb{E}(X_i X_j) = O\left(\frac{n^3}{\log^{2k-1} n} p^{2k-1} + \sum_{t=2}^{k} \frac{n^2}{\log^k n} p^{2k-t}\right),$$

where here the sums are only for i, j (including $i = j$) such that the ith and jth k-APs in $\mathbf{P}_n$ intersect. Chebyshev's inequality yields

$$\mathbb{P}(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}(X)^2} = O\left(\frac{\log n}{np} + \sum_{t=2}^{k} \frac{\log^k n}{n^2 p^t}\right).$$

If $p = \omega(n^{-2/k}\log n)$ the RHS of the equality above is $o(1)$ and so $\mathbb{P}(X = 0) = o(1)$, as required. $\square$

A.3 Deducing the 1-statement of the random Ramsey theorem from Theorem 6.8.7

In this section we show that Theorem 6.8.7 implies the 1-statement of Theorem 6.5.1.

Let H be as in the statement of Theorem 6.5.1. Write $E(H) = \{h_1, \dots, h_{e(H)}\}$ and let $r \geq 2$. Without loss of generality we may assume that H has no isolated vertices. For $n \in \mathbb{N}$, let $\mathcal{H}_n$ be the $e(H)$-uniform ordered hypergraph with vertex set $V(\mathcal{H}_n) := E(K_n)$ where an ordered $e(H)$-set $(e_1, \dots, e_{e(H)}) \in (E(K_n))^{e(H)}$ is an edge of $\mathcal{H}_n$ if and only if there is an injective graph homomorphism $g : H \to K_n$ such that $g(h_i) = e_i$ for every $i \in [e(H)]$; in other words, there is a copy of H in K_n with edges $e_1, \dots, e_{e(H)}$ where e_i plays the role of h_i for every $i \in [e(H)]$.

For brevity, we write $\mathcal{H} := \mathcal{H}_n$. For $W \subseteq [e(H)]$, let H_W be the subgraph of H spanned by the set of edges $\{h_i : i \in W\} \subseteq E(H)$; so H_W contains no isolated vertices. Recall that the restriction $\mathcal{H}_W$ of $\mathcal{H}$ was defined just after Remark 6.8.5.

Claim A.3.1. *We have $e(\mathcal{H}_W) = \Theta(n^{v(H_W)})$ for every $W \subseteq [e(H)]$.*

Proof of the claim. Say $W = \{t_1, \dots, t_{|W|}\}$ where $t_1 < \cdots < t_{|W|}$. Then, provided $n \geq v(H)$, an ordered $|W|$-set $(e_1, \dots, e_{|W|}) \in (E(K_n))^{|W|}$ is an edge of $\mathcal{H}_W$ if and only if there is an injective graph homomorphism $g : H_W \to K_n$ such that $g(h_{t_i}) = e_i$ for every $i \in [|W|]$. There are $\Theta\left(n^{v(H_W)}\right)$ such homomorphisms. Conversely, at most $2^{|W|}$ injective homomorphisms $g : H_W \to K_n$ correspond to the same edge in $\mathcal{H}_W$.[24] Hence $e(\mathcal{H}_W) = \Theta\left(n^{v(H_W)}\right)$, as claimed.

■

We can now compute $\hat{p}(\mathcal{H})$. We have $v(\mathcal{H}) = e(K_n) = \binom{n}{2}$. This and Claim A.3.1

[24] More precisely, there are exactly 2^a injective homomorphisms $g : H_W \to K_n$ that corresponds to the same edge in $\mathcal{H}_W$, where a is the number of isolated edges in H which lie in H_W.

yield

$$f_{n,W} = \left(\frac{e(\mathcal{H}_W)}{v(\mathcal{H})}\right)^{-\frac{1}{|W|-1}} = \Theta\left(\left(\frac{n^{v(H_W)}}{n^2}\right)^{-\frac{1}{e(H_W)-1}}\right) = \Theta\left(n^{-\frac{v(H_W)-2}{e(H_W)-1}}\right). \tag{A.1}$$

Note that $\frac{e(H_W)-1}{v(H_W)-2} = d_2(H_W)$ when $|W| \geq 2$. Thus, we have

$$\max_{\substack{W\subseteq[k]\\|W|\geq 2}} \frac{e(H_W)-1}{v(H_W)-2} \leq \max_{H'\subseteq H} d_2(H') = m_2(H). \tag{A.2}$$

By assumption, H contains a connected component K with at least two edges. We have $d_2(K) = \frac{e(K)-1}{v(K)-2} \geq 1$ since K is connected. Suppose the value of $m_2(H)$ is achieved by $H' \subseteq H$. Then H' contains at least two edges, since $d_2(H') \geq d_2(K) \geq 1$. Furthermore, H' has no isolated vertices. Indeed, deleting an isolated vertex from H' would yield a subgraph H'' of H with $d_2(H'') > d_2(H')$, a contradiction. Set $W := \{i : h_i \in E(H')\}$. Since H' has no isolated vertices, it follows that $H_W = H'$. Furthermore, $|W| \geq 2$ since H' has at least two edges. We have $d_2(H_W) = d_2(H') = m_2(H)$, and so equality holds in (A.2). That is,

$$\max_{\substack{W\subseteq[k]\\|W|\geq 2}} \frac{e(H_W)-1}{v(H_W)-2} = m_2(H).$$

Using the equality above, taking the maximum over all $W \subseteq [e(H)]$ of size at least 2 in (A.1) yields

$$\hat{p}(\mathcal{H}) = \Theta\left(n^{-\frac{1}{m_2(H)}}\right).$$

Next, we verify conditions (P1)–(P4).

(P1): Recall we proved that $m_2(H) \geq 1$. This inequality implies $n^{-\frac{1}{m_2(H)}} \to 0$ and $n^{2-\frac{1}{m_2(H)}} \to \infty$ as $n \to \infty$. In particular, $\hat{p}(\mathcal{H}) \to 0$ and $\hat{p}(\mathcal{H})v(\mathcal{H}) \to \infty$ as $n \to \infty$.

(P2): By Ramsey's theorem, there exists an $m \in \mathbb{N}$ such that any r-edge-colouring of K_m yields a monochromatic copy of H. For $n \geq m$, pick an r-edge-colouring of K_n. Every subset of $V(K_n)$ of size m contains a monochromatic copy of H, and each copy of H belongs to $\binom{n-v(H)}{m-v(H)}$ subsets of $V(K_n)$ of size m. Thus, the number of monochromatic copies of H in K_n is at least

$$\frac{\binom{n}{m}}{\binom{n-v(H)}{m-v(H)}} \geq \left(\frac{n}{m}\right)^{v(H)}.$$

Equivalently, given any r-colouring of the vertices of $\mathcal{H}$, there are at least $(n/m)^{v(H)}$ monochromatic edges in $\mathcal{H}$. By Claim A.3.1 we have $e(\mathcal{H}) = e(\mathcal{H}_{[e(H)]}) = \Theta(n^{v(H)})$. Together, this implies that the sequence of ordered hypergraphs $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is r-supersaturated.

(P3): Let $W \subseteq Y \subseteq [e(H)]$. Given $(e_1, \ldots, e_{|W|}) \in \mathcal{H}_W$, there are at most $n^{v(H_Y)-v(H_W)}$ edges in $\mathcal{H}_Y$ that restrict to $(e_1, \ldots, e_{|W|})$. This is because each such edge corresponds to an injective graph homomorphism $g : H_Y \to K_n$ with $g(h_i) = e_i$ for every $i \in W$. For all such homomorphisms, the images of the edges in $E(H_W)$ are fixed, and we have at most $n^{v(H_Y)-v(H_W)}$ choices for the image of $V(H_Y) \setminus V(H_W)$. Thus,

$$\Delta_W(\mathcal{H}_Y) \leq n^{v(H_Y)-v(H_W)}.$$

By Claim A.3.1, we have $e(\mathcal{H}_W) = \Theta(n^{v(H_W)})$ and $e(\mathcal{H}_Y) = \Theta(n^{v(H_Y)})$. Combining this with the previous inequality yields

$$\Delta_W(\mathcal{H}_Y) \leq \Theta\left(\frac{e(\mathcal{H}_Y)}{e(\mathcal{H}_W)}\right),$$

and so condition (P3) holds.

(P4): Let $W \subsetneq Y \subseteq [e(H)]$ with $|W| = 1$. Note that $E(\mathcal{H}_W) = V(\mathcal{H})$. This combined with condition (P3) gives

$$\Delta_W(\mathcal{H}_Y) \leq \Theta\left(\frac{e(\mathcal{H}_Y)}{e(\mathcal{H}_W)}\right) = \Theta\left(\frac{e(\mathcal{H}_Y)}{v(\mathcal{H}_W)}\right),$$

and so condition (P4) holds.

Since conditions (P1)–(P4) hold, by Theorem 6.8.7 we obtain the following 1-statement. There exists some $C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left[\mathcal{H}^{\mathrm{ind}}_{n,q} \text{ is } r\text{-Ramsey}\right] = 1 \quad \text{ if } q \geq Cn^{-\frac{1}{m_2(H)}}.$$

This is equivalent to

$$\lim_{n\to\infty} \mathbb{P}\left[G_{n,p} \text{ is } (H,r)\text{-Ramsey}\right] = 1 \quad \text{ if } p \geq Cn^{-\frac{1}{m_2(H)}},$$

which is precisely the 1-statement of Theorem 6.5.1.

Appendix B

The content of this appendix is related to Chapter 6. Namely, this appendix includes rigorous proofs of the results stated in Section 6.13, as well as the version of the simplified random Rado lemma used in the proof of Theorem 6.6.4. Section B.1 covers the proofs of Theorems 6.13.1, 6.13.3, and 6.13.6; these proofs involve applying the random Rado lemma and its simplified version in the relevant settings. Section B.2 covers the proof of Theorem 6.13.5, the resilience-type result about primes presented in Section 6.13. Finally, in section B.3, we provide further details about an analogue of the simplified random Rado lemma which is applicable to matrices whose entries are elements from a field. This is the tool used in the proof of Theorem 6.6.4.

The proofs appearing in this appendix are the result of collaborative work with Robert Hancock and Andrew Treglown, although they are omitted from the arXiv version of the paper [55] covering the material of Chapter 6.

B.1 Proofs of Theorems 6.13.1, 6.13.3 and 6.13.6

B.1.1 Proof of Theorems 6.13.1

We will use the following fact.

Fact B.1.1. *Let A be an integer matrix. There exists some constant $C = C(A) > 0$ such that $|\operatorname{rank}_{\mathbb{Z}_n}(A) - \operatorname{rank}(A)| \leq \log_n C$ for all $n \in \mathbb{N}$ with $n \geq 2$.*

Proof. Let A be an $\ell \times k$ integer matrix and $n \in \mathbb{N}$ with $n \geq 2$.

First, we bound the number of solutions to $Ax = 0$ in $\mathbb{Z}_n$ from below. Let B be a matrix formed by a maximal set of linearly independent rows of A. In particular, B is a $\operatorname{rank}(A) \times k$ matrix and $\operatorname{rank}(B) = \operatorname{rank}(A)$. Pick $W \subseteq [k]$ maximal such that the columns of B_W are linearly independent. Hence, we have $|W| = \operatorname{rank}(A)$ and B_W is a $\operatorname{rank}(A) \times \operatorname{rank}(A)$ invertible matrix. Thus, there exists an inverse matrix B_W^{-1} of B_W whose entries are from $\mathbb{Q}$. Let $D \in \mathbb{N}$ be so that for every entry e of B_W^{-1} we have $e \cdot D \in \mathbb{Z}$.

Pick $y_0 \in [n]^{|\overline{W}|}$ such that every entry of y_0 is divisible by D. For $n \geq 2D$, there are at least

$$\left(\frac{n}{D} - 1\right)^{|\overline{W}|} \geq \left(\frac{n}{2D}\right)^{|\overline{W}|} = \left(\frac{n}{2D}\right)^{k-\operatorname{rank}(A)}$$

possible choices for y_0. For a fixed choice of y_0, let $y_1 := -B_W^{-1}(B_{\overline{W}} y_0)$. Note that $B_{\overline{W}} y_0$ is a vector whose entries are integers divisible by D. By definition of D, it follows that y_1 is a vector with integer entries. Let $y \in [n]^k$ with $y_W = y_1$ and $y_{\overline{W}} = y_0$. We have

$$By = B_W y_W + B_{\overline{W}} y_{\overline{W}} = B_W y_1 + B_{\overline{W}} y_0 = 0.$$

Furthermore, since the rows of B span the rows of A, we have $Ay = 0$. Treating the entries of y modulo n yields a solution to $Ax = 0$ in $\mathbb{Z}_n$. Note that different choices of y_0 produce different solutions in $\mathbb{Z}_n$. Hence, the number of solutions $Ax = 0$ in $\mathbb{Z}_n$ is at least $(n/(2D))^{k-\operatorname{rank}(A)}$ (for $n \geq 2D$). In particular,

$$\left(\frac{n}{2D}\right)^{k-\operatorname{rank}(A)} \leq n^{k-\operatorname{rank}_{\mathbb{Z}_n}(A)}$$

and so

$$\operatorname{rank}_{\mathbb{Z}_n}(A) \leq \operatorname{rank}(A) + \log_n (2D)^k. \tag{B.1}$$

Note that for $2 \leq n \leq 2D$ we have $\operatorname{rank}_{\mathbb{Z}_n}(A) \leq k = \log_{2D}(2D)^k \leq \log_n (2D)^k$. Thus, inequality (B.1) holds for all $n \in \mathbb{N}$ with $n \geq 2$.

Next, we bound the number of solutions to $Ax = 0$ in $\mathbb{Z}_n$ from above. Pick $U \subseteq [k]$

maximal such that the columns of A_U are linearly independent. In particular, $|U| = \operatorname{rank}(A)$. Let a be the largest entry of A in absolute value.

We have $n^{|\overline{U}|} = n^{k-\operatorname{rank}(A)}$ choices for $x_{\overline{U}}$. For $x_{\overline{U}}$ fixed, the number of choices for x_U is precisely the number of solutions y to $A_U y = -A_{\overline{U}} x_{\overline{U}}$ in $\mathbb{Z}_n$. In turn, this is at most the number of solutions $A_U y = b$ in $[n]$ where b is a vector with integer entries which lies in the residue class $b \equiv -A_{\overline{U}} x_{\overline{U}}$ modulo n. Note that $A_U y = b$ cannot have a solution in $[n]$ if one of the entries of b has absolute value larger than $a \cdot k \cdot n$. Hence, there are at most $2ak+1$ choices for each entry of b, and therefore at most $(2ak+1)^k$ choices for b, such that $A_U y = b$ has at least one solution in $[n]$. In fact, for b fixed, there is at most one solution to $A_U y = b$ in $[n]$ since the columns of A_U are linearly independent.

It follows that

$$n^{k-\operatorname{rank}_{\mathbb{Z}_n}(A)} \leq (2ak+1)^k n^{k-\operatorname{rank}(A)}$$

and so

$$\operatorname{rank}(A) \leq \operatorname{rank}_{\mathbb{Z}_n}(A) + \log_n (3ak)^k. \tag{B.2}$$

Inequalities (B.1) and (B.2) imply the required statement, where we take $C := \max\{(2D)^k, (3ak)^k\}$

$\square$

The following is a simple corollary of Fact B.1.1.

Corollary B.1.2. *Let A be an irredundant partition regular matrix. For n sufficiently large, $m(A)$ and $m_{\mathbb{Z}_n}(A)$ are well-defined and $n^{-1/m(A)} = \Theta(n^{1/m_{\mathbb{Z}_n}(A)})$.*

Proof. Say A is an $\ell \times k$ matrix. Firstly, by Fact B.1.1, there is a constant $C > 1$ such that $|\operatorname{rank}_{\mathbb{Z}_n}(A_W) - \operatorname{rank}(A_W)| \leq \log_n C$ for every $W \subseteq [k]$ and $n \in \mathbb{N}$ with $n \geq 2$.

As A is irredundant and partition regular, it is also abundant. Hence, for every $W \subseteq [k]$ with $|W| \geq 2$, we have $\operatorname{rank}(A_{\overline{W}}) \geq \operatorname{rank}(A) - |W| + 2$, as the rank decreases by at most one after deleting a column. This implies

$$|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \operatorname{rank}(A) \geq 1$$

and, for $n > C^2$,

$$|W| - 1 + \operatorname{rank}_{\mathbb{Z}_n}(A_{\overline{W}}) - \operatorname{rank}_{\mathbb{Z}_n}(A) \geq 1 - 2\log_n C > 0.$$

Therefore, $m(A)$ and $m_{\mathbb{Z}_n}(A)$ are well-defined.

Furthermore, we have

$$\left| \frac{|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \operatorname{rank}(A)}{|W| - 1} - \frac{|W| - 1 + \operatorname{rank}_{\mathbb{Z}_n}(A_{\overline{W}}) - \operatorname{rank}_{\mathbb{Z}_n}(A)}{|W| - 1} \right| \leq 2\log_n C.$$

The above inequality implies $|1/m(A) - 1/m_{\mathbb{Z}_n}(A)| \leq 2\log_n C$. It follows that, for n large, $n^{|1/m(A) - 1/m_{\mathbb{Z}_n}(A)|} \leq C^2$ and so $n^{-1/m(A)} = \Theta(n^{-1/m_{\mathbb{Z}_n}(A)})$. □

Proof of Theorem 6.13.1. We apply Lemma 6.7.8. As $\mathbb{Z}_n$ is an abelian group, $(A, \mathbb{Z}_n)$ is 1-rich. Furthermore, $m_{\mathbb{Z}_n}(A)$ is well-defined for n large. Hence, by Lemma 6.7.32, we have $\hat{p} := \hat{p}(A, \mathbb{Z}_n) = n^{-1/m_{\mathbb{Z}_n}(A)}$. By Corollary B.1.2, we have $\hat{p} = \Theta(n^{-1/m(A)})$. It remains to verify that conditions (A1)–(A6) hold. The statement of Theorem 6.13.1 then follows from the conclusion of Lemma 6.7.8.

Say A is an $\ell \times k$ matrix. By Fact B.1.1, there is a constant $C > 1$ such that $|\operatorname{rank}_{\mathbb{Z}_n}(A_W) - \operatorname{rank}(A_W)| \leq \log_n C$ for every $W \subseteq [k]$ and $n \in \mathbb{N}$ with $n \geq 2$. Note that, since A is irredundant and partition regular, it is also abundant.

(A1): We have $\hat{p} = \Theta(n^{-1/m(A)}) \to 0$ and $|\mathbb{Z}_n|\hat{p} = \Theta(n^{1-1/m(A)}) \to \infty$ as $n \to \infty$, since $m(A) \geq \frac{k-1}{k-\ell-1} > 1$. The first inequality follows since A is full rank.

(A2): Let $r \in \mathbb{N}$. Given an r-colouring $c : \mathbb{Z}_n \to [r]$, let $c^* : [n] \to [r]$ where $c^*(x) = c([x]_{\text{mod } n})$. By Theorem 6.4.5, there is a constant $\delta > 0$ such that, for n large enough, there are at least $\delta n^{k-\ell}$ monochromatic solutions (with respect to c^*) to

$Ax = 0$ in $[n]$. Each such solution induces a distinct monochromatic solution (with respect to c) to $Ax = 0$ in $\mathbb{Z}_n$. Hence, there are at least $\delta n^{k-\ell}$ monochromatic solutions to $Ax = 0$ in $\mathbb{Z}_n$. Since there are at most $\Theta(n^{k-\ell})$ solutions to $Ax = 0$ in $\mathbb{Z}_n$ (see, e.g., inequality (B.2) in the proof of Fact B.1.1), $(\mathbb{Z}_n)_{n\in\mathbb{N}}$ is (A, r)-supersaturated for every $r \in \mathbb{N}$.

(A3): By Corollary 6.7.21, $(A, \mathbb{Z}_n)$ is 1-extendable for every $n \in \mathbb{N}$.

(A4): Let $W \subsetneq Y \subseteq [k]$ with $W = 1$ and $w_0 \in \mathbb{Z}_n$. We have

$$|\operatorname{Sol}^A_{\mathbb{Z}_n}(W)| = n^{1-\operatorname{rank}_{\mathbb{Z}_n}(A)+\operatorname{rank}_{\mathbb{Z}_n}(A_{\overline{W}})} \geq (1/C^2)\cdot n^{1-\operatorname{rank}(A)+\operatorname{rank}(A_{\overline{W}})} = (1/C^2)\cdot n$$

where the first equality follows from Corollary 6.7.19 ($(A, \mathbb{Z}_n)$ is 1-rich), the second inequality follows from Fact B.1.1, and the last equality follows from $\operatorname{rank}(A) = \operatorname{rank}(A_{\overline{W}})$ (A is abundant).

Since $(A, \mathbb{Z}_n)$ is 1-extendable, we have

$$|\operatorname{Sol}^A_{\mathbb{Z}_n}(w_0, W, Y)| \leq \frac{|\operatorname{Sol}^A_{\mathbb{Z}_n}(Y)|}{|\operatorname{Sol}^A_{\mathbb{Z}_n}(W)|} \leq C^2 \cdot \frac{|\operatorname{Sol}^A_{\mathbb{Z}_n}(Y)|}{n}.$$

(A5): Let $W \subseteq [k]$ with $|W| = 2$. By Corollary 6.7.19 ($(A, \mathbb{Z}_n)$ is 1-rich), we have

$$\frac{|\mathbb{Z}_n|^2}{|\operatorname{Sol}^A_{\mathbb{Z}_n}(W)|} = \frac{|\mathbb{Z}_n|^2}{|\mathbb{Z}_n|^{|W|-\operatorname{rank}_{\mathbb{Z}_n}(A)+\operatorname{rank}_{\mathbb{Z}_n}(A_{\overline{W}})}} \leq C^2 \cdot \frac{|\mathbb{Z}_n|^2}{|\mathbb{Z}_n|^{|W|-\operatorname{rank}(A)+\operatorname{rank}(A_{\overline{W}})}}.$$

Since $\operatorname{rank}(A) = \operatorname{rank}(A_{\overline{W}})$ (A is abundant), the above implies

$$\frac{|\mathbb{Z}_n|^2}{|\operatorname{Sol}^A_{\mathbb{Z}_n}(W)|} \leq C^2. \tag{B.3}$$

Recall $(A, \mathbb{Z}_n)$ is 1-rich. Furthermore, $(A, [n])$ is δ-rich, where $\delta \in (0, 1)$ is the constant given by, e.g., Theorem 6.4.5. Let $Y \subseteq [k]$ with $|Y| \geq 2$. Applying

Corollary 6.7.19 to both $(A, \mathbb{Z}_n)$ and $(A, [n])$, and using Fact B.1.1, it is easy to show that $|\operatorname{Sol}^A_{\mathbb{Z}_n}(Y)| = \Theta(|\operatorname{Sol}^A_{[n]}(Y)|)$. In particular, $p_Y(A, \mathbb{Z}_n) = \Theta(p_Y(A, [n]))$.

Since $(A, [n])$ is γ-rich and abundant, and $\operatorname{rank}(A) > 0$ (as A is full rank), by Lemma 6.7.26 there exists $X \subseteq [k]$ with $|X| \geq 3$ such that $p_X(A, [n]) = \Theta(\hat{p}(A, [n]))$ and, for every $W' \subsetneq X$ with $|W'| \geq 2$,

$$\frac{p_{W'}(A, [n])}{p_X(A, [n])} \leq \Theta(n^{-1/k^2}).$$

It follows that

$$\frac{p_{W'}(A, \mathbb{Z}_n)}{p_X(A, \mathbb{Z}_n)} = \Theta\left(\frac{p_{W'}(A, [n])}{p_X(A, [n])}\right) \to 0 \tag{B.4}$$

as $n \to \infty$. Furthermore, $\hat{p}(A, [n]) = \Theta(n^{-1/m(A)}) = \Theta(\hat{p})$ by Lemma 6.7.32, and so $p_X(A, \mathbb{Z}_n) = \Theta(p_X(A, [n])) = \Theta(\hat{p})$.

Combining (B.3) and (B.4) yields that $(\mathbb{Z}_n)_{n \in \mathbb{N}}$ is compatible with respect to A.

(A6): Inequality (B.3) immediately implies $(\mathbb{Z}_n)_{n \in \mathbb{N}}$ is weakly compatible with respect to A.

□

B.1.2 Proof of Theorem 6.13.3

We apply Lemma 6.7.33. It suffices to verify conditions (C1)–(C5): the statement of the theorem follows immediately from the conclusion of the simplified random Rado lemma (Lemma 6.7.33). Note that conditions (C1), (C4) and (C5) hold by assumption.

(C2): Say A is an $\ell \times k$ matrix. By Green's generalised version of Szemerédi's theorem for finite abelian groups [70], as A is translation-invariant with respect to S_n for every n, there exists some $\varepsilon > 0$ such that the following holds. For every $n \in \mathbb{N}$, any

2-colouring of S_n yields at least $\varepsilon|\operatorname{Sol}_{S_n}^A([k])|$ monochromatic solutions to $Ax = 0$ in S_n. In particular, the sequence $(S_n)_{n\in\mathbb{N}}$ is $(A, 2)$-supersauturated.

(C3): As S_n is an abelian group, (A, S_n) is 1-rich for every $n \in \mathbb{N}$.

□

B.1.3 Proof of Theorem 6.13.6

Let $S_n := (\mathbb{Z}_{128})^n$ for every $n \in \mathbb{N}$. For $W \subseteq [4]$, it is easy to check that

$$|\operatorname{Sol}_{S_n}^A(W)| = \begin{cases} |S_n|^3 & \text{if } |W| = [4], \\ |S_n|^{|W|-1/7} & \text{if } \{3,4\} \subseteq W \neq [4], \\ |S_n|^{|W|} & \text{otherwise.} \end{cases} \tag{B.5}$$

It immediately follows that, for $|W| \geq 2$,

$$p_W(A, S_n) = \left(\frac{|\operatorname{Sol}_{S_n}^A(W)|}{|S_n|}\right)^{-\frac{1}{|W|-1}} = \begin{cases} |S_n|^{-2/3} & \text{if } |W| = [4], \\ |S_n|^{-(|W|-8/7)/(|W|-1)} & \text{if } \{3,4\} \subseteq W \neq [4], \\ |S_n|^{-1} & \text{otherwise.} \end{cases} \tag{B.6}$$

In particular, we have $\hat{p}(A, S_n) = p_{[4]}(A, S_n) = |S_n|^{-2/3} = (128)^{-2n/3}$.

It remains to verify that conditions (A1)–(A6) of Lemma 6.7.8 hold for $(S_n)_{n\in\mathbb{N}}$ and A: the statement of Theorem 6.13.6 follows from the conclusion of Lemma 6.7.8 and $\hat{p} := \hat{p}(A, S_n) = (128)^{-2n/3}$.

(A1): We have $\hat{p} = (128)^{-2n/3} \to 0$ and $\hat{p}|S_n| = (128)^{n/3} \to \infty$ as $n \to \infty$.

(A2): Observe that $\mathbb{Z}_{128}$ is an abelian group with exponent 128 and A satisfies the 128-columns condition as $2 + (-2) + 63 + 65 \equiv 0 \pmod{128}$. By Theorem 6.4.8, the sequence $((\mathbb{Z}_{128})^n)_{n\in\mathbb{N}}$ is (A, r)-supersaturated for every $r \in \mathbb{N}$.

(A3): As $(\mathbb{Z}_{128})^n$ is an abelian group, by Corollary 6.7.21 $(A, (\mathbb{Z}_{128})^n)$ is 1-extendable for every $n \in \mathbb{N}$.

(A4): Let $W \subsetneq Y \subseteq [4]$ with $|W| = 1$ and $w_0 \in S_n$. By (B.5), $|\operatorname{Sol}^A_{S_n}(W)| = |S_n|$. Since (A, S_n) is 1-extendable, we have

$$|\operatorname{Sol}^A_{S_n}(w_0, W, Y)| \leq \frac{|\operatorname{Sol}^A_{S_n}(Y)|}{|\operatorname{Sol}^A_{S_n}(W)|} = \frac{|\operatorname{Sol}^A_{S_n}(Y)|}{|S_n|}$$

and so condition (A4) holds.

(A5): Let $W, W' \subsetneq [4]$ with $|W| = 2$ and $|W'| \geq 2$. We have $|\operatorname{Sol}^A_{S_n}(W)| \geq |S_n|^{1+6/7}$ by (B.5) and $p_{W'}(A, S_n) \leq |S_n|^{-6/7}$ by (B.6). Thus

$$\begin{aligned}\frac{|S_n|^2}{|\operatorname{Sol}^A_{S_n}(W)|}\left(\frac{p_{W'}(A, S_n)}{p_{[4]}(A, S_n)}\right)^{|W'|-1} &\leq \frac{|S_n|^2}{|S_n|^{1+6/7}}\left(\frac{|S_n|^{-6/7}}{|S_n|^{-2/3}}\right)^{2-1} \\ &= |S_n|^{-1/21} = (128)^{-n/21} \to 0\end{aligned}$$

as $n \to \infty$. Since $|[4]| \geq 3$, it follows that $(S_n)_{n\in\mathbb{N}}$ is compatible with respect to A

(A6): For every $W \subseteq [4]$ with $|W| = 2$ we have $|\operatorname{Sol}^A_{S_n}(W)| \geq |S_n|^{1+6/7}$ and thus

$$\frac{|S_n|}{|\operatorname{Sol}^A_{S_n}(W)|} \leq |S_n|^{-6/7} = (128)^{-6n/7} \to 0$$

as $n \to \infty$. It follows that $(S_n)_{n\in\mathbb{N}}$ is weakly compatible with respect to A.

□

B.2 Proof of Theorem 6.13.5

We deduce Theorem 6.13.5 from the following statement.

Theorem B.2.1. *Let $r \geq 1$, $k \geq 2$ be integers and let $\delta, c > 0$. Suppose that $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of k-uniform hypergraphs. Let $(p_n)_{n\in\mathbb{N}}$ with $p_n \in (0,1)$ be such that $p_n \to 0$ and*

$p_n v(\mathcal{H}_n) \to \infty$ as $n \to \infty$, and for each sufficiently large $n \in \mathbb{N}$ and each $\ell \in [k]$, we have

$$\Delta_\ell(\mathcal{H}_n) \leq c(p_n)^{\ell-1} \frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}.$$

Moreover, suppose that for any $\gamma > 0$ there exist $\varepsilon, n_0 > 0$ such that for all $n \geq n_0$ the following holds. However any $X \subseteq V(\mathcal{H}_n)$ with $|X| \geq \gamma v(\mathcal{H}_n)$ is r-coloured, there are at least $\varepsilon e(\mathcal{H}_n)$ monochromatic edges.

Then there exists $C \geq 1$ such that, for $q_n \geq Cp_n$, w.h.p. $\mathcal{H}^{ind}_{n,q_n}$ has the property that for any subset $X \subseteq V(\mathcal{H}^{ind}_{n,q_n})$ with $|X| \geq \delta v(\mathcal{H}^{ind}_{n,q_n})$, any r-colouring of X yields a monochromatic edge.

B.2.1 Deducing Theorem 6.13.5 from Theorem B.2.1

Pick $r \geq 1$, $k \geq 3$ and $\delta > 0$. Let $c = c(r, k, \delta) > 0$ be sufficiently large. For every $n \in \mathbb{N}$, let $\mathcal{H}_n$ be the k-uniform hypergraph with $V(\mathcal{H}_n) = \mathbf{P}_n$ and $\{v_1, \ldots, v_k\}$ is a k-edge of $\mathcal{H}_n$ if and only if its elements form a k-AP in $\mathbf{P}_n$. In particular, we have $v(\mathcal{H}_n) = \Theta(n/\log n)$ by the prime number theorem and $e(\mathcal{H}_n) = \Theta(n^2/\log^k n)$ by Theorem 6.3.1.

Let $p_n = n^{-1/(k-1)} \log n$. We have $p_n \to 0$ and $v(\mathcal{H}_n)p_n \to \infty$ as $n \to \infty$, since $k \geq 3$. Note that there are at most $\binom{k}{2}$ k-APs in $\mathbb{Z}$ containing two given fixed integers. Thus, for $2 \leq \ell \leq k$, we have $\Delta_\ell(\mathcal{H}_n) \leq \binom{k}{2}$ and in particular

$$\Delta_\ell(\mathcal{H}_n) = \Theta(1) = \Theta\left(\frac{\log^{k-1} n}{n} \cdot \frac{n}{\log^{k-1} n}\right) \leq \Theta\left(\frac{\log^{\ell-1} n}{n^{\frac{\ell-1}{k-1}}} \cdot \frac{n}{\log^{k-1} n}\right) \leq c(p_n)^{\ell-1} \frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}$$

where the last inequality holds for $c > 0$ large enough.

By Lemma 6.3.3, we have

$$\Delta_1(\mathcal{H}_n) = O\left(\frac{n}{\log^{k-1} n}\right) \leq c \cdot \frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)}$$

where again the last inequality holds for $c > 0$ large enough.

Finally, observe that the moreover statement in Theorem B.2.1 holds by Lemma 6.13.4.

As the assumptions of Theorem B.2.1 hold, we conclude that there exists $C > 0$ such that if $q_n \geq Cp_n$ then w.h.p. for any subset $X \subseteq \mathbf{P}_{n,p}$ with $|X| \geq \delta|\mathbf{P}_{n,p}|$, whenever X is r-coloured there is a monochromatic edge in $\mathcal{H}_n$, that is, a monochromatic k-AP. $\square$

B.2.2 Proof of Theorem B.2.1 via containers

We deduce Theorem B.2.1 from Proposition 6.9.2. Note that we use the same notation introduced in Section 6.9.1.

Let $r \geq 1$, $k \geq 2$ be integers and $\delta, c > 0$. Note that it suffices to prove the theorem under the additional assumptions that $\delta \leq 1$ and $c \geq k$. Pick $\gamma := \delta/4$ and let $\varepsilon > 0$ be so that the moreover part of Theorem B.2.1 holds with respect to γ and ε. Define

$$\varepsilon' := \frac{\varepsilon}{2r}, \quad \text{and} \quad \varepsilon'' := \frac{\varepsilon' k}{c}. \tag{B.7}$$

As $c \geq k$, we have $\varepsilon'' \leq \varepsilon'$. Apply Proposition 6.9.2 with k, r, c and ε'' to obtain $D > 0$. Since $p_n \to 0$ as $n \to \infty$, if n is sufficiently large then

$$cDp_n \leq \varepsilon'. \tag{B.8}$$

Next we pick a constant $C \geq 1$ large enough so that the last inequality (†) in (B.10) below holds for all n sufficiently large and

$$\frac{\delta}{2} - \frac{D}{C} - \gamma > 0. \tag{B.9}$$

Let $\mathcal{F}_n := \{F \subseteq V(\mathcal{H}_n) : e(\mathcal{H}_n[F]) \geq \varepsilon' e(\mathcal{H}_n)\}$. Clearly $\mathcal{F}_n$ is increasing and $\mathcal{H}_n$ is $(\mathcal{F}_n, \varepsilon')$-dense. Since $\varepsilon'' \leq \varepsilon'$, we know $\mathcal{H}_n$ is also $(\mathcal{F}_n, \varepsilon'')$-dense. Further, for all $F \in \mathcal{F}_n$, we have

$$k\varepsilon' e(\mathcal{H}_n) \leq ke(\mathcal{H}_n[F]) = \sum_{y \in F} \deg_{\mathcal{H}_n[F]}(y) \leq |F|\Delta_1(\mathcal{H}_n) \leq \frac{|F|c \cdot e(\mathcal{H}_n)}{v(\mathcal{H}_n)},$$

where the last inequality uses the bound on $\Delta_1(\mathcal{H}_n)$. Rearranging gives

$$|F| \geq \frac{\varepsilon' k}{c} v(\mathcal{H}_n) \overset{(B.7)}{=} \varepsilon'' v(\mathcal{H}_n),$$

for all $F \in \mathcal{F}_n$, thus $\mathcal{H}_n$ and $\mathcal{F}_n$ satisfy the hypotheses of Proposition 6.9.2 with parameters k, r, c, ε'' and p_n, provided n is sufficiently large. Apply Proposition 6.9.2 to obtain the family $\mathcal{S}_r \subseteq \mathcal{I}_r(\mathcal{H}_n)$ and functions $f : \mathcal{S}_r \to (\overline{\mathcal{F}}_n)^r$ and $g : \mathcal{I}_r(\mathcal{H}_n) \to \mathcal{S}_r$.

Now let $q_n \geq Cp_n$ and let Q_{q_n} be the event that there exists $W \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})$ such that $|W| \geq (\delta/2) q_n v(\mathcal{H}_n)$ and $\mathcal{H}^{\text{ind}}_{n,q_n}[W]$ is not r-Ramsey.

Claim B.2.2. *It suffices to show that $\mathbb{P}(Q_{q_n}) \to 0$ as $n \to \infty$.*

Proof of the claim. Let B_{q_n} be the event that there exists $W \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})$ such that $|W| \geq \delta v(\mathcal{H}^{\text{ind}}_{n,q_n})$ and $\mathcal{H}^{\text{ind}}_{n,q_n}[W]$ is not r-Ramsey. Let T_{q_n} be the event that $v(\mathcal{H}^{\text{ind}}_{n,q_n}) \geq (1 - \delta/2) q_n v(\mathcal{H}_n)$.

If both events B_{q_n} and T_{q_n} occur then Q_{q_n} occurs too. This is because $\delta(1-\delta/2) \geq \delta/2$, as $\delta \in (0, 1]$. In particular, $\mathbb{P}(T_{q_n}) \cdot \mathbb{P}(B_{q_n} | T_{q_n}) \leq \mathbb{P}(Q_{q_n})$. Furthermore, as $v(\mathcal{H}^{\text{ind}}_{n,q_n})$ follows a binomial distribution and $\mathbb{E}(v(\mathcal{H}^{\text{ind}}_{n,q_n})) = q_n v(\mathcal{H}_n)$, then $\mathbb{P}(\overline{T}_{q_n}) \to 0$ as $n \to \infty$.

Suppose $\mathbb{P}(Q_{q_n}) \to 0$ as $n \to \infty$. We have

$$\begin{aligned}\mathbb{P}(B_{q_n}) &= \mathbb{P}(T_{q_n}) \cdot \mathbb{P}(B_{q_n} | T) + \mathbb{P}(\overline{T}_{q_n}) \cdot \mathbb{P}(B_{q_n} | \overline{T}_{q_n}) \\ &\leq \mathbb{P}(Q_{q_n}) + \mathbb{P}(\overline{T}_{q_n}) \cdot 1 \to 0\end{aligned}$$

as $n \to \infty$. In other words, w.h.p. if $X \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})$ and $|X| \geq \delta v(\mathcal{H}^{\text{ind}}_{n,q_n})$ then any r-colouring of X yields a monochromatic edge, which is precisely the conclusion of Theorem B.2.1.

■

Since $q_n \leq q'_n \leq 1$ implies $\mathbb{P}[Q_{q'_n}] \leq \mathbb{P}[Q_{q_n}]$, we may assume that $q_n = Cp_n$. Suppose that Q_{q_n} does occur, that is, there exists $W \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})$ such that $|W| \geq (\delta/2) q_n v(\mathcal{H}_n)$

and $\mathcal{H}^{\text{ind}}_{n,q_n}[W]$ is not r-Ramsey. Then there is a partition $(X_1,\dots,X_r) \in \mathcal{I}_r(\mathcal{H}_n)$ of W. By Proposition 6.9.2, there exists $S := (S_1,\dots,S_r) := g(X_1,\dots,X_r) \in \mathcal{S}_r$ such that $S \subseteq (X_1,\dots,X_r) \subseteq S \cup f(S)$ and $|\cup_{i\in[r]} S_i| \le Dp_n v(\mathcal{H}_n)$. For brevity, we write $f(S) =: (f(S_1),\dots,f(S_r))$, $X(S) := \cup_{i\in[r]}S_i$, $Y(S) := \cup_{i\in[r]}f(S_i)$ and $Z(S) := X(S)\cup Y(S)$.[25]

Claim B.2.3. *We have $|Z(S)| \le \gamma v(\mathcal{H}_n)$.*

Proof of the claim. We bound $e(\mathcal{H}_n[S_i \cup f(S_i)])$ for each $i \in [r]$. Since $f(S_i) \in \overline{\mathcal{F}_n}$ we have $e(\mathcal{H}_n[f(S_i)]) \le \varepsilon' e(\mathcal{H}_n)$. The number of edges in $\mathcal{H}_n[S_i \cup f(S_i)]$ using at least one vertex from S_i is at most

$$|S_i| \cdot \Delta_1(\mathcal{H}_n) \le Dp_n v(\mathcal{H}_n) \cdot c\frac{e(\mathcal{H}_n)}{v(\mathcal{H}_n)} \overset{(B.8)}{\le} \varepsilon' e(\mathcal{H}_n).$$

Thus, in total we have $e(\mathcal{H}_n[S_i \cup f(S_i)]) \le 2\varepsilon' e(\mathcal{H}_n)$. Overall we see that the set $Z(S)$ can be partitioned into r sets $\{S_i \cup f(S_i)\}_{i\in[r]}$ so that each set induces at most $2\varepsilon' e(\mathcal{H}_n)$ edges in $\mathcal{H}_n$.

Since $2\varepsilon' = \varepsilon/r$, it follows from the moreover part of Theorem B.2.1 that $|Z(S)| \le \gamma v(\mathcal{H}_n)$.

■

As $X_1 \cup \dots \cup X_r$ is a partition of W, we must have $X(S) \subseteq W \subseteq Z(S)$. Also, we have $|W| \ge (\delta/2)q_n v(\mathcal{H}_n)$. It follows that $X(S) \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})$ and $V(\mathcal{H}^{\text{ind}}_{n,q_n})$ contains at least $(\delta/2)q_n v(\mathcal{H}_n) - X(S)$ vertices in $Z(S) \setminus X(S)$.

From the above observations and the union bound, we have

$$\begin{aligned}\mathbb{P}(Q_{q_n}) &\le \sum_{S\in\mathcal{S}_r} \mathbb{P}(X(S) \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n}) \;\wedge\; |V(\mathcal{H}^{\text{ind}}_{n,q_n}) \cap (Z(S)\setminus X(S))| \ge (\delta/2)q_n v(\mathcal{H}_n) - |X(S)|) \\ &= \sum_{S\in\mathcal{S}_r} \mathbb{P}(X(S) \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})) \cdot \mathbb{P}(|V(\mathcal{H}^{\text{ind}}_{n,q_n}) \cap (Z(S)\setminus X(S))| \ge (\delta/2)q_n v(\mathcal{H}_n) - |X(S)|).\end{aligned}$$

We now bound $\mathbb{P}(Q_{q_n})$ further. Firstly, it is easy to see that $\mathbb{P}(X(S) \subseteq V(\mathcal{H}^{\text{ind}}_{n,q_n})) =$

[25]Note the slight abuse of the f notation here.

$q_n^{|X(S)|}$.

Since $|X(S)| \le Dp_n v(\mathcal{H}_n)$ and $q_n = Cp_n$, we have

$$(\delta/2)q_n v(\mathcal{H}_n) - |X(S)| \ge (\delta/2)q_n v(\mathcal{H}_n) - Dp_n v(\mathcal{H}_n) = (\delta/2 - D/C)q_n v(\mathcal{H}_n).$$

Thus,

$$\begin{aligned}\mathbb{P}(Q_{q_n}) &\le \sum_{S\in\mathcal{S}_r} q_n^{|X(S)|}\cdot \mathbb{P}(|V(\mathcal{H}^{\text{ind}}_{n,q_n})\cap (Z(S)\setminus X(S))| \ge (\delta/2)q_n v(\mathcal{H}_n) - |X(S)|)\\ &\le \sum_{S\in\mathcal{S}_r} q_n^{|X(S)|}\cdot \mathbb{P}(|V(\mathcal{H}^{\text{ind}}_{n,q_n})\cap Z(S)| \ge (\delta/2 - D/C)q_n v(\mathcal{H}_n)).\end{aligned}$$

Let $R := |V(\mathcal{H}^{\text{ind}}_{n,q_n})\cap Z(S)|$, $t := (\delta/2 - D/C - \gamma)/\gamma$ and $\eta := t^2\gamma/3$. Note that $t > 0$ by (B.9). By Chernoff's bound, for any $\beta \ge 0$ we have $\mathbb{P}(R \ge (1+\beta)\mathbb{E}[R]) \le 2\exp(-\beta^2\mathbb{E}(R)/3) \le 2\exp(-(\beta-1)\mathbb{E}(R)/3)$. As $\mathbb{E}(R) = q_n|Z(S)|$ and $|Z(S)| \le \gamma v(\mathcal{H}_n)$ (by Claim B.2.3), we have

$$\begin{aligned}&\mathbb{P}(|V(\mathcal{H}^{\text{ind}}_{n,q_n})\cap Z(S)| \ge (\delta/2 - D/C)q_n v(\mathcal{H}_n)) = \mathbb{P}(|V(\mathcal{H}^{\text{ind}}_{n,q_n})\cap Z(S)| \ge (t+1)q_n\gamma v(\mathcal{H}_n))\\ &= \mathbb{P}\left((|V(\mathcal{H}^{\text{ind}}_{n,q_n})\cap Z(S)| \ge \frac{(t+1)\gamma v(\mathcal{H}_n)}{Z(S)}\mathbb{E}[R]\right) \le 2\exp\left(-\left(\frac{(t+1)\gamma v(\mathcal{H}_n)}{Z(S)} - 1\right)\mathbb{E}(R)/3\right)\\ &\le 2\exp\left(-\left(\frac{t\gamma v(\mathcal{H}_n)}{Z(S)}\right)\mathbb{E}(R)/3\right) = 2\exp\left(-(t\gamma/3)q_n v(\mathcal{H}_n)\right) = 2\exp\left(-\eta q_n v(\mathcal{H}_n)\right)\end{aligned}$$

It follows that

$$\mathbb{P}(Q_{q_n}) \le 2\sum_{S\in\mathcal{S}_r} q_n^{|X(S)|}\exp(-\eta q_n v(\mathcal{H}_n)).$$

Recall that for every $S = (S_1,\ldots,S_r) \in \mathcal{S}_r$ we have $|\cup_{i\in[r]} S_i| \le Dp_n v(\mathcal{H}_n)$. We may bound $|\mathcal{S}_r|$ from above as follows: choose a set of $i \le Dp_n v(\mathcal{H}_n)$ vertices from $V(\mathcal{H}_n)$ and then take an (ordered) partition of these vertices into r classes. Therefore we have,

$$\begin{aligned}\mathbb{P}(Q_{q_n}) \leq 2\sum_{S\in\mathcal{S}_r} q_n^{|X(S)|}e^{-\eta q_n v(\mathcal{H}_n)} &\leq 2\sum_{i=0}^{Dp_nv(\mathcal{H}_n)}\binom{v(\mathcal{H}_n)}{i}r^i q_n^i e^{-\eta q_n v(\mathcal{H}_n)}\\ &\leq 2(Dp_nv(\mathcal{H}_n)+1)(rq_n)^{Dp_nv(\mathcal{H}_n)}\binom{v(\mathcal{H}_n)}{Dp_nv(\mathcal{H}_n)}e^{-\eta q_n v(\mathcal{H}_n)}\\ &\leq 2(Dp_nv(\mathcal{H}_n)+1)\left(\frac{rq_n e v(\mathcal{H}_n)}{Dp_nv(\mathcal{H}_n)}\right)^{Dp_nv(\mathcal{H}_n)}e^{-\eta q_n v(\mathcal{H}_n)}\\ &= 2(Dp_nv(\mathcal{H}_n)+1)\left(\frac{reC}{D}\right)^{\frac{D}{C}q_nv(\mathcal{H}_n)}e^{-\eta q_n v(\mathcal{H}_n)}\\ &\overset{(\dagger)}{\leq} 2e^{-\eta q_n v(\mathcal{H}_n)/2}. \end{aligned} \tag{B.10}$$

Note that the equality follows since $q_n = Cp_n$. Now using that $p_nv(\mathcal{H}_n)$ and hence $q_nv(\mathcal{H}_n)$ tends to infinity as $n\to\infty$, the above probability tends to zero as $n\to\infty$, as required. □

B.3 A simplified random Rado lemma for Theorem 6.6.4

In the proof of Theorem 6.6.4, we require an analogue of the simplified random Rado lemma (Lemma 6.7.33) which can be applied to the case where A is a matrix with entries from a field $\mathbb{F}$ and the sequence $\{S_n\}_{n\in\mathbb{N}}$ satisfies $S_n = \mathbb{F}^n$ for every $n\in\mathbb{N}$. For completeness, in this section we rigorously state such result (Lemma B.3.23) and provide the reader further details about its proof.

All the terminology and proofs are almost word-by-word identical to the case where the entries of the matrix are integers. Indeed, the following four subsections mirror the four subsection of Section 6.7.

B.3.1 Analogue of the random Rado lemma for fields

First, we prove an analogue (Lemma B.3.8) of the random Rado lemma (Lemma 6.7.8) for matrices with entries from a field. The following are the formal definitions of k-distinct, irredundant and (A,r)-Rado for this setting.

Definition B.3.1. Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let S be a finite subset of $\mathbb{F}^d$ for some $d \in \mathbb{N}$.

- We say a solution $x \in S^k$ to $Ax = 0$ is *k-distinct* if the entries of x are pairwise distinct.
- We say A is *irredundant with respect to S* if there exists a k-distinct solution to $Ax = 0$ in S.
- For $r \in \mathbb{N}$, we say S is *(A,r)-Rado* if any r-colouring of S yields a k-distinct monochromatic solution to $Ax = 0$.

The following definitions are identical to those from Section 6.7, except that A is a matrix whose entries are elements from a field $\mathbb{F}$ and S, S_n, etc. are subsets of powers of $\mathbb{F}$. Given a k-set $(x_i)_{i\in[k]}$ and $W \subseteq [k]$, we write x_W for the $|W|$-set $(x_i)_{i\in W}$.

Definition B.3.2 (Projected solutions)**.** Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let S be a finite subset of $\mathbb{F}^d$ for some $d \in \mathbb{N}$. Let $W \subseteq Y \subseteq [k]$ and $w_0 \in S^{|W|}$. The set of *projected solutions* $\operatorname{Sol}_S^A(w_0, W, Y)$ is the set of vectors $y \in S^{|Y|}$ such that there exists a solution $x \in S^k$ to $Ax = 0$ with $x_Y = y$ and $x_W = w_0$. For brevity, we write $\operatorname{Sol}_S^A(Y) := \operatorname{Sol}_S^A(\emptyset, \emptyset, Y)$, i.e., $\operatorname{Sol}_S^A(Y)$ is the set of vectors $y \in S^{|Y|}$ such that there exists a solution $x \in S^k$ to $Ax = 0$ with $x_Y = y$. In particular, $\operatorname{Sol}_S^A([k])$ is the set of all solutions to $Ax = 0$ in S.

Definition B.3.3 (Probability threshold for fields)**.** Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let S be a finite subset of $\mathbb{F}^d$ for some $d \in \mathbb{N}$. For every $W \subseteq [k]$, we let

$$p_W(A,S) := \left(\frac{|\operatorname{Sol}_S^A(W)|}{|S|}\right)^{-\frac{1}{|W|-1}}.$$

Additionally, we let

$$\hat{p}(A, S) := \max_{\substack{W \subseteq [k] \\ |W| \geq 2}} p_W(A, S).$$

Definition B.3.4 (Supersaturation)**.** Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of finite sets such that for every $n \in \mathbb{N}$ there is a $d \in \mathbb{N}$ such that $S_n \subseteq \mathbb{F}^d$. For $r \in \mathbb{N}$, we say that $(S_n)_{n \in \mathbb{N}}$ is (A, r)-*supersaturated* if there exist an $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds: if $n \geq n_0$ then whenever S_n is r-coloured there are at least $\varepsilon |\operatorname{Sol}^A_{S_n}([k])|$ monochromatic solutions to $Ax = 0$ in S_n.

Definition B.3.5 (Extendability)**.** Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let S be a finite subset of $\mathbb{F}^d$ for some $d \in \mathbb{N}$. Let $B \geq 1$ be a real number. We say the pair (A, S) is B-*extendable* if for all non-empty $W \subseteq Y \subseteq [k]$ and any $w_0 \in S^{|W|}$ we have

$$|\operatorname{Sol}^A_S(w_0, W, Y)| \leq B \cdot \frac{|\operatorname{Sol}^A_S(Y)|}{|\operatorname{Sol}^A_S(W)|}.$$

Definition B.3.6 (Compatibility)**.** Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries in $\mathbb{F}$ and $k \geq 3$. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of finite sets such that for every $n \in \mathbb{N}$ there is a $d \in \mathbb{N}$ such that $S_n \subseteq \mathbb{F}^d$. We say $(S_n)_{n \in \mathbb{N}}$ is *compatible* with respect to A if there exists a sequence $(X_n)_{n \in \mathbb{N}}$ of subsets $X_n \subseteq [k]$ with $|X_n| \geq 3$ for all $n \in \mathbb{N}$ such that the following holds. We have $p_{X_n}(A, S_n) = \Omega(\hat{p}(A, S_n))$ and, for all sequences $(W_n)_{n \in \mathbb{N}}$ and $(W'_n)_{n \in \mathbb{N}}$ with $W_n, W'_n \subsetneq X_n$ such that $|W_n| = 2$ and $|W'_n| \geq 2$,

$$\frac{|S_n|^2}{|\operatorname{Sol}^A_{S_n}(W_n)|} \left(\frac{p_{W'_n}(A, S_n)}{p_{X_n}(A, S_n)} \right)^{|W'_n|-1} \to 0$$

as $n \to \infty$.

Definition B.3.7 (Weak compatibility)**.** Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries in $\mathbb{F}$ and $k \geq 2$. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of finite sets such that for every $n \in \mathbb{N}$

there is a $d \in \mathbb{N}$ such that $S_n \subseteq \mathbb{F}^d$. We say $(S_n)_{n\in\mathbb{N}}$ is *weakly compatible* with respect to A if for every $W \subseteq [k]$ with $|W| = 2$ we have

$$\frac{|S_n|}{|\operatorname{Sol}^A_{S_n}(W)|} \to 0$$

as $n \to \infty$.

We can now state the analogue of the random Rado lemma for matrices with entries from a field.

Lemma B.3.8 (The random Rado lemma for fields)**.** *Let $r \geq 2$, $k \geq 3$ and $\ell \geq 1$ be integers. Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite sets such that for every $n \in \mathbb{N}$ there is a $d \in \mathbb{N}$ such that $S_n \subseteq \mathbb{F}^d$. Set $\hat{p}_n := \hat{p}(A, S_n)$.*

Suppose that

(A1) $\hat{p}_n \to 0$ and $|S_n|\hat{p}_n \to \infty$ as $n \to \infty$;

(A2) $(S_n)_{n\in\mathbb{N}}$ is (A, r)-supersaturated;

(A3) there exists $B \geq 1$ such that the pair (A, S_n) is B-extendable for all sufficiently large $n \in \mathbb{N}$;

(A4) there exists $D > 0$ such that for every $W \subsetneq Y \subseteq [k]$ with $|W| = 1$ and $w_0 \in S_n$, we have $|\operatorname{Sol}^A_{S_n}(w_0, W, Y)| \leq D\frac{|\operatorname{Sol}^A_{S_n}(Y)|}{|S_n|}$ for all sufficiently large $n \in \mathbb{N}$;

(A5) $(S_n)_{n\in\mathbb{N}}$ is compatible with respect to A;

(A6) $(S_n)_{n\in\mathbb{N}}$ is weakly compatible with respect to A.

Then there exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A, r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c\hat{p}_n, \\ 1 & \text{if } p \geq C\hat{p}_n. \end{cases}$$

To prove Lemma B.3.8, we require the following analogue of Lemma 6.7.9.

Lemma B.3.9. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$ and $k \geq 2$. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite sets such that for every $n \in \mathbb{N}$ there is a $d \in \mathbb{N}$ with $S_n \subseteq \mathbb{F}^d$. Suppose that conditions (A3) and (A6) from Lemma B.3.8 hold; so there is a $B \geq 1$ such that (A, S_n) is B-extendable for every $n \in \mathbb{N}$ sufficiently large.*

For each $Y \subseteq [k]$, let $k\text{-}\mathrm{Sol}^A_{S_n}(Y)$ be the set of $y \in S_n^{|Y|}$ such that there exists a k-distinct solution $x \in S^k$ to $Ax = 0$ with $x_Y = y$. For n sufficiently large, we have

$$\frac{1}{2B} \leq \frac{|k\text{-}\mathrm{Sol}^A_{S_n}(Y)|}{|\,\mathrm{Sol}^A_{S_n}(Y)|} \leq 1.$$

Furthermore, for $Y = [k]$ we have

$$\lim_{n\to\infty} \frac{|k\text{-}\mathrm{Sol}^A_{S_n}([k])|}{|\,\mathrm{Sol}^A_{S_n}([k])|} = 1.$$

Proof. The proof is word-by-word identical to the proof of of Lemma 6.7.9. □

Proof of Lemma B.3.8. The proof of Lemma B.3.8 is word-by-word identical to the proof of the random Rado lemma (Lemma 6.7.8), except for:

- We start by considering a matrix A with entries from a field $\mathbb{F}$ (instead of integer entries) and a sequence $(S_n)_{n\in\mathbb{N}}$ of finite sets such that for every $n \in \mathbb{N}$ there is $d \in \mathbb{N}$ such that $S_n \subseteq \mathbb{F}^d$ (instead of finite subsets of abelian groups).

- We use Lemma B.3.9 instead of Lemma 6.7.9.

□

B.3.2 Rank and relevant bounds

In this subsection we prove some relevant bounds on the number of (projected) solutions which will help simplifying conditions (A1)–(A6) of Lemma B.3.8.

Recall that Definition 6.7.12 provides a definition of $\mathrm{rank}_S(A)$ when S is a subset of a power of a field $\mathbb{F}$ and A is a matrix with entries in $\mathbb{F}$. When $S = \mathbb{F}^d$ for some $d \in \mathbb{N}$, the rank $\mathrm{rank}_S(A)$ can be equivalently defined in terms of the number of solutions to $Ax = 0$ in S, as shown by the following fact. This is, in a sense, the analogue of Fact 6.7.15.

Fact B.3.10. *Let $\mathbb{F}$ be a finite field and $S = \mathbb{F}^d$ for some $d \in \mathbb{N}$. Let A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. The number of solutions to $Ax = 0$ in S is $|S|^{k-\mathrm{rank}_S(A)}$.*

Proof. If $\mathbb{F} = \{0_{\mathbb{F}}\}$ then the number of solutions to $Ax = 0$ in S is precisely $|S|^k$. We have $\mathrm{rank}_S(A) = 0$ and so the required statement holds. Therefore, suppose $\mathbb{F} \neq \{0_{\mathbb{F}}\}$ and in particular $\mathrm{rank}_S(A) = \mathrm{rank}(A)$, where $\mathrm{rank}(A)$ denotes the rank of A with respect to $\mathbb{F}$.

Let $f_A : S^k \to S^\ell$ be defined as $f_A(x) = Ax$. Note that f_A is a group homomorphism. In particular, by the first isomorphism theorem we have $|S|^k = |\,\mathrm{im}(f_A)| \cdot |\ker(f_A)|$. Let $W \subseteq [k]$ so that the columns of A_W are linearly independent with respect to $\mathbb{F}$ and W is maximal with respect to this property. Thus, we have $|W| = \mathrm{rank}(A)$. Let $f_{A_W} : S^{\mathrm{rank}(A)} \to S^\ell$ be defined as $f_{A_W}(x) = A_W x$. Since the columns of A are spanned by the columns of A_W in $\mathbb{F}$ and $S = \mathbb{F}^d$, we have $\mathrm{im}(f_A) = \mathrm{im}(f_{A_W})$. Furthermore, $x \neq y \in S^{\mathrm{rank}(A)}$ implies $f_{A_W}(x) \neq f_{A_W}(y)$, and so $|\,\mathrm{im}(f_A)| = |\,\mathrm{im}(f_{A_W})| = |S|^{\mathrm{rank}(A)}$.

We conclude that $|\ker(f_A)| = |S|^k / |\,\mathrm{im}(f_A)| = |S|^{k-\mathrm{rank}(A)}$. In other words, there are precisely $|S|^{k-\mathrm{rank}(A)}$ solutions to $Ax = 0$ in S, as required. □

Remark B.3.11. *Note that the statement of Fact B.3.10 is not true if we replace the condition $S = \mathbb{F}^d$ with $S \subseteq \mathbb{F}^d$ being a finite abelian group of size at least 2. Indeed, consider the field $\mathbb{F} := \{0, 1, w, w+1\}$ on 4 elements ($0, 1$ are the additive and multiplicative identities, $1 + 1 = w + w = 0$ and $w^2 = w + 1$) and the matrix $A := \begin{pmatrix} 1 & w \end{pmatrix}$. Clearly $\mathrm{rank}_S(A) = \mathrm{rank}(A) = 1$ since A is a non-zero matrix. Note that $S := \{0, 1\} \subseteq \mathbb{F}$ is an abelian group, and the only solution to $Ax = 0$ in S is $x = (0, 0)$. Thus, it is not true that there are $|S|^{2-1} = 2$ solutions to $Ax = 0$ in S.*

The following statements and definitions are identical to those in Section 6.7.2, except that A is a matrix whose entries are elements from a field $\mathbb{F}$ and S, S_n, etc. are powers of subsets of $\mathbb{F}$.

Lemma B.3.12 (Key bounding result). *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. Then for all $W \subseteq Y \subseteq [k]$ and $w_0 \in S^{|W|}$ we have*

$$|\operatorname{Sol}_S^A(w_0, W, Y)| \leq |S|^{|Y|-|W|-\operatorname{rank}_S(A_{\overline{W}})+\operatorname{rank}_S(A_{\overline{Y}})}. \tag{B.11}$$

Proof. The proof is word-by-word identical to Case (ii) of the proof of Lemma 6.7.16. The only exception is that in the proof of Lemma 6.7.16 we first argue that an integer matrix can be seen as a matrix with entries from $\mathbb{F}$. We simply skip this step for the proof of Lemma B.3.12. □

Definition (Richness). *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $\varepsilon > 0$ and let $S = L^d$ where L is a finite subset of $\mathbb{F}$ and $d \in \mathbb{N}$. We say (A, S) is ε-rich if there are at least $\varepsilon|S|^{k-\operatorname{rank}_S(A)}$ solutions to $Ax = 0$ in S.*

Corollary B.3.13. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. Let $0 < \varepsilon \leq 1$. If (A, S) is ε-rich, then for every $Z \subseteq [k]$ we have*

$$\varepsilon|S|^{|Z|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{Z}})} \leq |\operatorname{Sol}_S^A(Z)| \leq |S|^{|Z|-\operatorname{rank}_S(A)+\operatorname{rank}_S(A_{\overline{Z}})}. \tag{B.12}$$

Proof. The proof is word-by word identical to the proof of Corollary 6.7.19, except we use Lemma B.3.12 and inequality (B.11) instead of Lemma 6.7.16 and inequality (6.5). □

B.3.3 Simplifying conditions (A1)–(A6)

We use Lemma B.3.12 and Corollary B.3.13 from the previous subsection to simplify conditions (A3)–(A6) in Lemma B.3.8. The various statements in this subsection are identical to the ones found in Subection 6.7.3, except that A is a matrix whose entries are elements from a field $\mathbb{F}$ and S, S_n, etc. are powers of subsets of $\mathbb{F}$. Unless otherwise stated, the proofs in this subsection are word-by-word identical to the ones from Subection 6.7.3, except we use Lemma B.3.12, Corollary B.3.13 and equations (B.11) and (B.12) in place of Lemma 6.7.16, Corollary 6.7.19 and equations (6.5) and (6.7).

Lemma B.3.14. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. If there exists $B > 0$ such that (A, S) is $(1/B)$-rich, then (A, S) is B-extendable.*

Proof. Word-by-word identical to to the proof of Lemma 6.7.20. □

Corollary B.3.15. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Then $(A, \mathbb{F}^d)$ is 1-extendable for every $d \in \mathbb{N}$.*

Proof. By Fact B.3.10, $(A, \mathbb{F}^d)$ is 1-rich. Lemma B.3.14 implies that $(A, \mathbb{F}^d)$ is 1-extendable. □

Definition (Abundancy)**.** *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. We say (A, S) is abundant if $\operatorname{rank}_S(A) = \operatorname{rank}_S(A_{\overline{W}})$ for every $W \subseteq [k]$ with $|W| = 2$.*

For brevity, we will write 0 and 1 for the additive and multiplicative identities of $\mathbb{F}$, respectively. The next lemma is an analogue to Lemma 6.7.23.

Lemma B.3.16. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. If A is irredundant with respect to $\mathbb{F}^d$ and 3-partition regular in $S \setminus \{0\}^d$, then (A, S) is abundant.*

Proof. The proof is word-by-word identical to the proof of Lemma 6.7.23. The only exception is that in the proof of Lemma 6.7.23 we first argue that an integer matrix can be seen as a matrix with entries from $\mathbb{F}$. We simply skip this step for the proof of Lemma B.3.16. □

Lemma B.3.17. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. Furthermore, assume that (A, S) is* *abundant and ε-rich for some $0 < \varepsilon \leq 1$. Then, for every $W \subseteq Y \subseteq [k]$ with $|W| = 1$ and $w_0 \in S$ we have $|\operatorname{Sol}_S^A(w_0, W, Y)| \leq |\operatorname{Sol}_S^A(Y)|/(\varepsilon|S|)$.*

Proof. Word-by-word identical to the proof of Lemma 6.7.24. □

Lemma B.3.18. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. Furthermore, assume that (A, S) is abundant and ε-rich for some $0 < \varepsilon \leq 1$. Then for any $W \subseteq [k]$ with $|W| = 2$, we have*

$$\frac{|S|^2}{|\operatorname{Sol}_S^A(W)|} \leq \frac{1}{\varepsilon}.$$

Proof. Word-by-word identical to the proof of Lemma 6.7.25. □

Lemma B.3.19. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. Let $0 < \varepsilon \leq 1$. Suppose $\operatorname{rank}_S(A) > 0$. If (A, S) is ε-rich and abundant, then there exists $X \subseteq [k]$ with $|X| \geq 3$ such that $p_X(A, S) \geq \varepsilon \hat{p}(A, S)$ and for every $W' \subsetneq X$ with $|W'| \geq 2$ we have*

$$\frac{p_{W'}(A, S)}{p_X(A, S)} \leq (1/\varepsilon)|S|^{-1/k^2}.$$

Proof. Word-by-word identical to the proof of Lemma 6.7.26. □

B.3.4 Analogue of the random Rado lemma for fields

We are now ready to formally state the analogue (Lemma B.3.23) of the simplified random Rado lemma that we use in the proof of Theorem 6.6.4. The following statements and definitions are identical to those in Section 6.7.4, except that A is a matrix whose entries are elements from a field $\mathbb{F}$ and S, S_n, etc. are powers of subsets of $\mathbb{F}$.

Definition B.3.20. Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let

$S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. We set

$$m_S(A) := \max_{\substack{W \subseteq [k] \\ |W| \geq 2}} \frac{|W| - 1}{|W| - 1 + \operatorname{rank}_S(A_{\overline{W}}) - \operatorname{rank}_S(A)}. \tag{B.13}$$

Here $\operatorname{rank}_S(A_{\overline{W}}) := 0$ if $\overline{W} = \emptyset$. We call A *strictly balanced with respect to S* if the expression in (B.13) is maximised precisely when $W = [k]$.

Remark B.3.21. *Observe that $m_S(A)$ is not always well-defined since the denominator in (B.13) might be zero. However, if (A, S) is abundant then for all $|W| \geq 2$ we have*

$$\operatorname{rank}_S(A_{\overline{W}}) \geq \operatorname{rank}_S(A) - |W| + 2, \tag{B.14}$$

since the rank of a matrix can decrease by at most one by deleting a column. If (B.14) is satisfied then the denominator in (B.13) is always strictly positive and so $m_S(A)$ is well-defined in this case.

Lemma B.3.22. *Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $S = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$. Let $0 < \varepsilon \leq 1$. If $m_S(A)$ is well-defined and (A, S) is ε-rich then*

$$|S|^{-\frac{1}{m_S(A)}} \leq \hat{p}(A, S) \leq (1/\varepsilon^2)|S|^{-\frac{1}{m_S(A)}}.$$

Proof. Word-by-word identical to the proof of Lemma 6.7.32, except we use Corollary B.3.13 and inequality (B.12) instead of Corollary 6.7.19 and inequality (6.7). □

We can now state the simplified version of Lemma B.3.8.

Lemma B.3.23 (Simplified random Rado lemma for fields)**.** *Let $r \geq 2$, $k \geq 3$ and $\ell \geq 1$ be integers. Let $\mathbb{F}$ be a field and A be an $\ell \times k$ matrix with entries from $\mathbb{F}$. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence where $S_n = L^d$ for some finite subset $L \subseteq \mathbb{F}$ and $d \in \mathbb{N}$, for every $n \in \mathbb{N}$. Set $\hat{p}_n := \hat{p}(A, S_n)$. Suppose that*

(C1) $\hat{p}_n \to 0$ and $|S_n|\hat{p}_n \to \infty$ as $n \to \infty$;

(C2) $(S_n)_{n\in\mathbb{N}}$ *is* (A, r)*-supersaturated;*

(C3) there exists $0 < \varepsilon \leq 1$ *such that* (A, S_n) *is* ε*-rich for every sufficiently large* $n \in \mathbb{N}$*;*

(C4) (A, S_n) *is abundant for every sufficiently large* $n \in \mathbb{N}$*;*

(C5) given any $n \in \mathbb{N}$ *there exists* $X_n \subseteq [k]$ *with* $|X_n| \geq 3$ *such that* $p_{X_n}(A, S_n) = \Omega(\hat{p}(A, S_n))$ *and for every sequence* $(W'_n)_{n\in\mathbb{N}}$ *with* $W'_n \subsetneq X_n$ *and* $|W'_n| \geq 2$ *we have*

$$\frac{p_{W'_n}(A, S_n)}{p_{X_n}(A, S_n)} \to 0$$

as $n \to \infty$.

Then there exist constants $c, C > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}[S_{n,p} \text{ is } (A,r)\text{-Rado}] = \begin{cases} 0 & \text{if } p \leq c|S_n|^{-\frac{1}{m_{S_n}(A)}}, \\ 1 & \text{if } p \geq C|S_n|^{-\frac{1}{m_{S_n}(A)}}. \end{cases}$$

Proof. Word-by-word identical to the proof of Lemma 6.7.33, except for:

- We start by considering a matrix A with entries from a field $\mathbb{F}$ (instead of integer entries) and a sequence $(S_n)_{n\in\mathbb{N}}$ of powers of finite subsets of $\mathbb{F}$.
- We apply Lemma B.3.14, Corollary B.3.15, Lemma B.3.16, Lemma B.3.17, Lemma B.3.18, Lemma B.3.19 and Lemma B.3.22 from the previous subsection instead of Lemma 6.7.20, Corollary 6.7.21, Lemma 6.7.23, Lemma 6.7.24, Lemma 6.7.25, Lemma 6.7.26 and Lemma 6.7.32 from Section 6.7.3.

□