ON RESOLVABLE DECOMPOSITION PROBLEMS

by

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Abstract

Given graphs F and G, a resolvable F-decomposition of G is a partition of its edges into F-factors. We study the existence of resolvable cycle and clique decompositions in graphs with high minimum degree and pseudo-random graphs.

We show a Dirac-type result for the uniform case of the Oberwolfach problem and make progress towards a conjecture by Glock, Joos, Kim, Kühn and Osthus. Specifically, we prove that for any $\alpha > 0$ there is an integer r_0 such that for any $r \geq r_0$, any sufficiently large graph G on n vertices, with $r \mid n$, even degree, and minimum degree $(1/2 + \alpha)n$, has a resolvable C_r-decomposition. The term $1/2$ in the minimum degree bound is best possible.

We show that any sufficiently large pseudo-random graph that satisfies the necessary divisibility conditions has a resolvable K_r-decomposition.

We also prove the analogue for multipartite graphs. That is, any sufficiently large r-partite pseudo-random graph that satisfies the necessary divisibility conditions has a resolvable K_r-decomposition.

Our methods are purely combinatorial and combine 'iterative absorption' with a new technique on finding a fractional decomposition in an 'extended graph'. Finally, we discuss the consequences of our results and some potential applications of the methods into other resolvable decomposition problems.

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DISCLAIMER

Chapters 2, 3 and 4 are joint work with Abhishek Methuku and Dong-yeap Kang.

CONTENTS

CHAPTER 1

INTRODUCTION

The study of decomposition problems in combinatorics dates back to the 18th century with Euler's study on the existence of mutually orthogonal Latin squares. Ever since, new decomposition problems have been formulated and extensively studied; some of them becoming notorious and of high interest among mathematicians. In essence, a decomposition problem asks for the partition of a large structure into smaller, and often organised, substructures. In particular a graph decomposition consists of partitioning the edge set of a given graph. Algebraic methods have often been used to obtain decomposition results in graphs with a highly symmetric structure (e.g. the complete graph K_n). However, during the 20th century the use of probabilistic techniques in graph theory was initiated and popularised by Paul Erdős and led to many graph decomposition results relying on combinatorial and probabilistic arguments. These methods allowed to obtain decomposition results on graphs which are far from being complete and satisfy instead certain global properties. In this thesis we will study decompositions of graphs with high minimum degree into cycles of fixed length and decompositions of pseudo-random non-partite and partite graphs into cliques of fixed size.

1.1 Graph decompositions

Given graphs F and G, an *F-decomposition of G* is a partition of its edge set $E(G)$ into copies of F. Let K_n denote the complete graph on n vertices. A K_3-decomposition of K_n is known as a Steiner triple system of order n. In 1847, Kirkman [50] showed that a Steiner triple system exists if and only if $n \equiv 1,3 \mod 6$. This became one of the first results in design theory. It is not difficult to see that $n \equiv 1,3 \mod 6$ is a necessary condition for the existence of a Steiner triple system. Indeed, if a graph G admits a K_r-decomposition then $e(G)$ must be divisible by $\binom{r}{2}$ and the degree of each vertex must be divisible by $r-1$. A graph satisfying such conditions is called *K_r-divisible*. These type of necessary conditions are known in general as *divisibility conditions* and arise in essentially every decomposition problem. In particular, given graphs F and G we say that G is *F-divisible* if $e(F) \mid e(G)$ and $\gcd(F) \mid d_G(v)$ for every $v \in V(G)$ where $\gcd(F) := \gcd\{d_F(v) : v \in V(F)\}$. Note that a graph G must be F-divisible in order to have an F-decomposition. A *Hamilton cycle* (or *path*) is a cycle (or path) that contains all vertices of a graph. Also in the 19th century, Walecki [62] showed that K_n can be decomposed into edge-disjoint Hamilton cycles if n is odd and into edge-disjoint Hamilton paths if n is even. Years later, Wilson [79, 80, 81, 82] generalised Kirkman's theorem by proving that, given a graph F, every large enough F-divisible complete graph K_n has an F-decomposition. When considering the host graph G to be non-complete, it is not true that every large enough graph satisfying the required divisibility conditions has an F-decomposition. In fact, determining whether a given graph G has an F-decomposition is NP-complete [20]. For this reason, we are interested in adding additional conditions to the graph G

which are simple and sufficient to guarantee the existence of a decomposition. Two of the most studied examples are minimum degree conditions, also known as Dirac-type conditions, and pseudo-random conditions.

1.1.1 Dirac-type results

Dirac-type results refer to problems on graphs that satisfy a minimum degree condition. In this framework, extending Walecki's theorem, Csaba, Kühn, Lo, Osthus and Treglown [15] solved the so-called Hamilton decomposition conjecture: there exists $n_0 \in \mathbb{N}$ such that every r-regular graph G on $n \geq n_0$ vertices with $r \geq \lfloor n/2 \rfloor$ has a decomposition into edge-disjoint Hamilton cycles and at most one perfect matching.

There is a large literature on Dirac-type results about graph decompositions. One of the most notorious problems is the following conjecture posed by Nash-Williams in 1970 which extends Kirkman's theorem to graphs with high minimum degree.

Conjecture 1.1.1 (Nash-Williams [69])**.** *There exists $n_0 \in \mathbb{N}$ such that every K_3-divisible graph G on $n \geq n_0$ vertices satisfying $\delta(G) \geq 3n/4$ has a K_3-decomposition.*

It is not hard to check that the bound on the minimum degree is best possible. Indeed, consider the following extremal example. Let G_1 and G_2 be vertex-disjoint $(6k+2)$-regular graphs on $12k+6$ vertices. Let $G := G_1 \cup G_2 \cup G_3$ where G_3 is the complete bipartite graph with parts $V(G_1)$ and $V(G_2)$. Note that the degree of each vertex of G is $18k+8 \equiv 0 \mod 2$ and that $|V(G)| = 24k+12 \equiv 0 \mod 3$. Thus, G is K_3-divisible and satisfies $\delta(G) \geq 3|V(G)|/4$. However, since every triangle of G contains at least one edge of $G_1 \cup G_2$ and $2|E(G_1 \cup G_2)| < |E(G_3)|$ then G cannot have a K_3-decomposition.

When considering K_r-decompositions for $r \geq 3$ the following natural extension of Conjecture 1.1.1 was proposed by Gustavsson.

Conjecture 1.1.2 (Gustavsson [32]). *Given $r \geq 3$ there exists $n_0 \in \mathbb{N}$ such that every K_r-divisible graph G on $n \geq n_0$ vertices satisfying $\delta(G) \geq (1 - 1/(r+1))n$ has a K_r-decomposition.*

Given a graph F let δ_F be defined as the minimum $\delta > 0$ such that for any $\alpha > 0$ there exists n_0 such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \alpha)n$ has an F-decomposition. Thus, Conjecture 1.1.2 states that $\delta_{K_r} \leq 1 - 1/(r+1)$ and in particular, the Nash-Williams conjecture would imply $\delta_{K_3} \leq 3/4$. We refer to δ_F as the *decomposition threshold of F*. Similarly, let δ_F^{0+} be defined as the minimum $\delta > 0$ such that for any $\alpha, \gamma > 0$ there exists n_0 such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \alpha)n$ contains an F-decomposable subgraph $H \subseteq G$ that covers all but at most γn^2 edges of G. We refer to δ_F^{0+} as the *approximate decomposition threshold of F*. A result by Glock, Kühn, Lo, Montgomery and Osthus [28] bounds the decomposition threshold of F by δ_F^{0+} and the chromatic number $\chi(F)$.

Theorem 1.1.3 ([28]). *For any graph F, $\delta_F \leq \max\{\delta_F^{0+}, 1 - 1/(\chi(F)+1)\}$.*

In particular Theorem 1.1.3 implies $\delta_{K_r} \leq \max\{\delta_{K_r}^{0+}, 1 - 1/(r+1)\}$ for every $r \geq 3$. The result for the case $F = K_3$ was originally proved by Barber, Kühn, Lo and Osthus [9] and a shorter proof was given later by Barber, Glock, Kühn, Lo, Montgomery and Osthus [7]. Hence, the problem of finding bounds on δ_F gets reduced to finding bounds on δ_F^{0+}. As we will discuss in Section 1.1.2 distinct bounds on δ_F^{0+} have been obtained via the so-called *fractional decomposition threshold*.

In the case of cycle decompositions, Barber, Kühn, Lo and Osthus [9] gave explicit values of the decomposition threshold of a cycle.

Theorem 1.1.4 ([9]). *Let C_r denote the cycle of length r. Then*

$$\delta_{C_r} = \begin{cases} 2/3 & \text{if } r = 4, \\ 1/2 & \text{if } r \geq 6 \text{ is even}, \\ \delta_{C_r}^{0+} & \text{if } r \geq 3 \text{ is odd}. \end{cases}$$

In addition to Theorem 1.1.3 the authors in [28] explicitly determined the decomposition threshold of any bipartite graph. It is always either 0, 1/2 or 2/3 depending on the graph.

1.1.2 Fractional decompositions

Given graphs F and G, a *fractional F-decomposition of G* is a function ω from the set of copies of F in G to $[0, 1]$ such that for each edge $e \in E(G)$

$$\omega(e) := \sum_{F' \subseteq G \,:\, e \in E(F')} \omega(F') = 1$$

where the sum is taken over all copies F' of F in G that contain e.

Clearly, finding a fractional F-decomposition is easier than finding an F-decomposition. For instance, the complete graph K_n contains the trivial fractional F-decomposition ω obtained by assigning, to each copy of F in G, a uniform weight equal to the inverse of the number of copies F' of F that contain a given edge $e \in E(G)$.

Let δ^*_F be the minimum $\delta > 0$ such that for all $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ such that every F-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \alpha)n$ has a fractional F-decomposition. We refer to δ^*_F as the *fractional decomposition threshold of F*. Using a known result by Haxell and Rödl [37] that transforms a fractional F-decomposition into an F-decomposition that covers all but $o(n^2)$ edges of G one can show that $\delta^{0+}_F \leq \delta^*_F$ (see [31, Corollary 11.4]). Hence, results such as Theorem 1.1.3 and 1.1.4 can be used to transfer any upper bound of δ^*_F into an upper bound of δ_F. Currently, the best known bound on the fractional decomposition threshold for an arbitrary graph F is due to Montgomery [65] who showed that $\delta^*_F \leq 1 - 1/(100\chi(F))$. In particular, $\delta^*_{K_r} \leq 1 - 1/(100r)$ for $r \geq 4$. In the triangle case, Dross [21] gave a simple and elegant proof of $\delta^*_{K_3} \leq 9/10$. This result was improved by Delcourt and Postle [18] who showed that $\delta^*_{K_3} \leq (7+\sqrt{21})/14 \approx 0.827$. For the case of cycles observe that Theorem 1.1.4 determines the decomposition threshold of cycles of even length. In the case of odd cycles, a recent result by Joos and Kühn [40] shows that given $\varepsilon > 0$ if r is large enough then $\delta^*_{C_r} \leq 1/2 + \varepsilon$.

1.1.3 Pseudo-random graphs and typicality

Other conditions that have been widely studied in order to guarantee the existence of graph decompositions are pseudo-random conditions. These are properties that a dense random graph would likely satisfy. The one we are interested in is the so-called *typicality*.

Definition 1.1.5. Given $\xi, p > 0$ and $s \in \mathbb{N}$ we say that a graph G is (ξ, p, s)-*typical* if for every set of vertices $S \subseteq V(G)$ of size at most s we have

$$d_G(S) = (1 \pm \xi)p^{|S|}|V(G)|$$

where $d_G(S) := |\bigcap_{v \in S} N_G(v)|$.

Note that, given $\xi, p > 0$ and $s \in \mathbb{N}$, for large enough n a random graph on n vertices where each edge is chosen independently with probability p is likely to be (ξ, p, s)-typical.

The existence of clique decompositions in typical graphs has been proved by Keevash [42]. Glock, Kühn, Lo and Osthus [30] generalised the result to F-decompositions for an arbitrary graph F by using an innovative method known as iterative absorption (see Section 1.5.1). It is worth mentioning that the results in [30, 42] cover also hypergraph decompositions.

1.2 Graph factorization

1.2.1 Graph factors

An F-*factor* of G is a collection of vertex-disjoint copies of F that spans $V(G)$. A classical theorem of Hajnal and Szemerédi [35] gives a minimum degree condition on a graph G to ensure the existence of a K_r-factor.

Theorem 1.2.1 (Hajnal and Szemerédi [35])**.** *Given $r \geq 2$, every graph G on n vertices such that $r \mid n$ and $\delta(G) \geq (1 - 1/r)n$ has a K_r-factor.*

The case $r = 2$ is equivalent to finding perfect matchings and can be deduced using Tutte's or Dirac's theorem. The case $r = 3$ was first solved by Corrádi and Hajnal [14]. For arbitrary graphs F, Alon and Yuster [2] showed that given $\alpha > 0$ any large enough graph G where $|V(F)|$ divides $|V(G)|$ and $\delta(G) \geq (1 - 1/\chi(F) + \alpha)n$ has an F-factor. Komlós, Sárközy and Szemerédi [54] improved the result by replacing the term αn in the minimum degree condition by a constant depending only on $|V(F)|$. Finally, Kühn and Osthus [57] showed that the minimum degree threshold for the existence of an F-factor is either $1 - 1/\chi(F)$ or $1 - 1/\chi_{cr}(F)$, where $\chi_{cr}(F)$ denotes the *critical chromatic number*, depending on the structure of F.

The study of K_r-factors in pseudo-random graphs was initiated by Krivelevich, Sudakov and Szabó [56] who showed the existence of a triangle factor in a graph with bounded second eigenvalue. A recent result by Morris [68] gives the existence of K_r-factors in pseudo-random bijumbled graphs improving previous results by Nenadov [70] and Han, Kohayakawa, Morris and Person [36].

1.2.2 Resolvable decompositions

Given graphs F and G a *resolvable F-decomposition of G* is a decomposition of G into edge-disjoint F-factors. Such a decomposition is also known as an *F-factorization of G*. We say that an F-decomposition of G is *resolvable* if the collection of copies of F in the decomposition can be partitioned into F-factors. The search for resolvable decompositions dates back to the 18th century with the publication of a playing cards puzzle by Ozanam in 1725 and the thirty-six officers problem, also known as Euler's officers problem, which inspired Euler to develop a particular interest in the study of mutually orthogonal Latin squares (see Section 1.4.4). They asked for a K_4-decomposition of the complete graphs $K_{4,4,4,4}$ and $K_{6,6,6,6}$ respectively. It is easy to see that the problems are equivalent to finding resolvable K_3-decompositions of $K_{4,4,4}$ and $K_{6,6,6}$. Years later, in 1850, Kirkman formulated the well-known Kirkman's schoolgirl problem:

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

The problem asks for a resolvable K_3-decomposition of the complete graph K_{15}. This triggered an interest on resolvable decompositions of the complete graph. The question was solved by Ray-Chaudhuri and Wilson [73] who determined for which n the complete graph K_n has a resolvable K_r-decomposition.

In 1967, Ringel posed his famous Oberwolfach problem which generalises Kirkman's and Walecki's problems by asking for a decomposition of the complete graph K_n into copies of a given 2-factor.

Problem 1.2.2 (Oberwolfach problem)**.** *Let F be a 2-regular graph on n vertices. For which odd n does the complete graph K_n decompose into copies of F?*

Glock, Joos, Kim, Kühn and Osthus [27] answered the problem positively for large enough n. Their result also allowed for the decomposition into a given family $F_1, \ldots, F_m$ of 2-factors instead of a fixed factor F provided that one of the factors in the family appeared a linear amount of times. Shortly afterwards, Keevash and Staden [47] generalised the result by replacing the complete graph in the statement by any large enough typical graph as well as digraph, and by proving a generalised version of the Oberwolfach problem in which the host graph can be decomposed into any given family $F_1, \ldots, F_m$ of 2-factors. The results in [27, 47] solves another famous problem, the Hamilton–Waterloo problem, which asks for a decomposition of the complete graph K_n into copies of two given 2-factors F_1, F_2. The case where a 2-factor consists of cycles of the same length is known as *uniform*. Our first result consists of a Dirac-type result for the uniform case of the Oberwolfach problem and makes progress towards a conjecture by Glock, Joos, Kim, Kühn and Osthus [27, Conjecture 1.5].

Theorem 1.2.3. *For all $\alpha > 0$ there exists $r_0 \in \mathbb{N}$ with the property that for any $r \geq r_0$ there is $n_0 \in \mathbb{N}$ such that any C_r-divisible regular graph G on $n \geq n_0$ vertices such that $r \mid n$ and $\delta(G) \geq (1/2 + \alpha)n$ has a resolvable C_r-decomposition.*

Note that C_r-divisibility, regularity and $r \mid |V(G)|$ are necessary conditions for the existence of a resolvable C_r-decomposition. Moreover, the bound of $1/2$ is best possible:

Proposition 1.2.4 ([9])**.** *Let $r \in \mathbb{N}$ be odd, $r \geq 3$ and let $\delta := \frac{r}{2(r-1)}$. There are infinitely many C_r-divisible regular graphs G with $r \mid |V(G)|$ and $\delta(G) \geq \delta|V(G)| - 1$ which do not admit a C_r-decomposition.*

Proof. Let $n \in \mathbb{N}$ be odd and satisfy $r \mid n$. Let G be the graph obtained by 'blowing up' each vertex of the complete bipartite graph $K_{r-1,r-1}$ into a clique of size n. In other words, G is obtained from $K_{r-1,r-1}$ by replacing each vertex v by a clique K_v of size n and replacing each edge vu by a complete bipartite graph between K_v and K_u. Note that G is $(nr-1)$-regular. Since $|V(G)| = 2n(r-1)$ then $r \mid |V(G)|$ and $\delta(G) = nr-1 = \delta|V(G)|-1$. Note also that $e(G) = (r-1)n(nr-1)$ which implies that $r \mid e(G)$ so G is C_r-divisible. Finally observe that since r is odd any cycle of G of length r must contain at least one edge of some clique

K_v, $v \in V(K_{r-1,r-1})$. Since the number of edges of the union of such cliques is $(r-1)n(n-1) < e(G)/r$ then G can not have a C_r-decomposition. $\square$

Thus, Theorem 1.2.3 can be viewed as the minimum degree threshold for the existence of a resolvable C_r-decomposition. As we shall discuss with more detail in Section 1.5, our proof is based on purely combinatorial and probabilistic arguments. The proof is simple and could be useful for finding resolvable F-decompositions in graphs with high minimum degree and pseudo-random graphs. In fact, we apply a similar approach to show the existence of resolvable K_r-decompositions in typical graphs.

Theorem 1.2.5. *For every $r \geq 3$ and every $p > 0$ there exists $\xi > 0$ and $n_0 \in \mathbb{N}$ such that any K_r-divisible $(\xi, p, 3r)$-typical regular graph G on $n \geq n_0$ vertices such that $r \mid n$ has a resolvable K_r-decomposition.*

The existence of resolvable F-decompositions for an arbitrary graph F was proved by Keevash [43] in large enough pseudo-random graphs and hypergraphs. In particular, this covered the case $F = K_r$. However, their result required the typicality parameter s to be significantly larger than r whereas ours only requires $s \geq 3r$. In fact, our proof only requires a linear size common neighbourhood of sets of size at most $3r$. This could be useful to deduce Dirac-type results. For instance, a graph G on n vertices with minimum degree at least $(1 - 1/3r + \alpha)n$ guarantees a size of at least αn in all common neighbourhoods of at most $3r$ vertices. We will discuss further the differences of our method when applied to graphs with high minimum degree and when applied to pseudo-random graphs in Chapter 6.

1.3 Related decomposition problems

1.3.1 Decompositions into trees

A big part of the literature on graph decompositions concerns decompositions into trees. Since the term 'tree decomposition' is widely used to refer a mapping of a graph into a tree these type of decomposition are known as 'tree packings'. Although our results are not directly related to trees we will present some of the most famous results on this topic.

Besides the Oberwolfach problem, Ringel also posed the following notorious conjecture in 1963.

Conjecture 1.3.1 (Ringel [74]). *For any tree T with n edges the complete graph K_{2n+1} has a T-decomposition.*

The conjecture has been solved for large n independently by Montgomery, Pokrovskiy and Sudakov [67] and Keevash and Staden [48]. Keevash and Staden also extended the result to typical graphs.

Another famous problem is the so-called 'tree packing conjecture' by Gyárfás and Lehel [33].

Conjecture 1.3.2 (Tree packing conjecture [33]). *For every n, the complete graph K_n can be decomposed into a sequence of trees $T_1, \ldots, T_n$ where T_i has i vertices.*

Joos, Kim, Kühn and Osthus [39] proved the conjecture for bounded degree trees and large enough n. Recently, Allen, Böttcher, Clemens, Hladký, Piguet and Taraz [1] improved the result to trees with maximum degree $cn/\log n$ for some $c > 0$. Both the results in [1, 39] allowed to replace the complete graph by a dense pseudo-random graph.

1.3.2 High-girth decompositions

Given a set of r-cliques $\mathcal{K}$, the *girth of* $\mathcal{K}$ is the smallest $g \geq 2$ such that there is a set of g cliques of $\mathcal{K}$ whose union contains at most $(r-2)g$ vertices.

Another extension of Kirkman's theorem on triangle decompositions of the complete graph was proposed by Erdős in 1973.

Conjecture 1.3.3 (Erdős [23]). *For any $g \geq 2$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ where $n \equiv 1, 3 \mod 6$ the complete graph K_n has a K_3-decomposition with girth at least g.*

The conjecture was proven recently by Kwan, Sah, Sawhney and Simkin [59]. Extending Erdős's conjecture, Glock, Kühn, Lo and Osthus [29], and Keevash and Long [44], conjectured that given $g \geq 2$ any large enough K_r-divisible complete graph K_n has a K_r-decomposition with girth at least g. This conjecture has been confirmed very recently by Delcourt and Postle [19].

1.4 Partite graphs and Latin squares

1.4.1 Decompositions in partite graphs

A graph G is *r-partite* if there exists a partition $V_1, \ldots, V_r$ of $V(G)$ such that $G[V_i]$ has no edges for every $i \in [r]$. We say that an r-partite graph G with parts $V_1, \ldots, V_r$ is *balanced* if $|V_1| = \cdots = |V_r|$. We say that an r-partite graph G is *K_r-divisible* if for every $i, j \in [r]$ and $v \in V(G) \setminus (V_i \cup V_j)$

$$d_G(v; V_i) = d_G(v; V_j)$$

where $d_G(v; V_i) := |N_G(v) \cap V_i|$. Given $r \geq 3$ observe that an r-partite graph G must be K_r-divisible in order to admit a K_r-decomposition. Moreover, this definition of divisibility automatically implies $r \mid d_G(v)$ for every $v \in V(G)$, $e(V_{i_1}, V_{i_2}) = e(V_{j_1}, V_{j_2})$ for every $i_1, i_2, j_1, j_2 \in [r]$ with $\{i_1, i_2\} \neq \{j_1, j_2\}$, and $\binom{r}{2} \mid e(G)$. Let

$$\hat{\delta}(G) := \min_{i \in [r], v \in V(G) \setminus V_i} d_G(v; V_i).$$

It makes sense for r-partite graphs to consider $\hat{\delta}(G)$ instead of $\delta(G)$ for the minimum degree threshold of a K_r-decomposition due to the fact that every r-clique in G contains exactly one edge between each pair of parts (V_i, V_j).

In this setting, a natural analogue of Conjecture 1.1.2 is the following:

Conjecture 1.4.1 (Gustavsson [32]). *Given $r \geq 3$ there exists $n_0 \in \mathbb{N}$ such that every K_r-divisible balanced r-partite graph G with parts of size n satisfying $\hat{\delta}(G) \geq (1 - 1/(r+1))n$ has a K_r-decomposition.*

Significant progress towards the conjecture was made by Barber, Kühn, Lo, Osthus and Taylor [10] who proved an analogue to Theorem 1.1.3 restricted to $F = K_r$ for r-partite graphs. To clarify, they showed that the minimum degree threshold for the existence of a K_r-decomposition in a large enough K_r-divisible r-partite graph was the minimum between $1 - 1/(r+1)$ and the fractional decomposition threshold. For the case $r = 3$, Bowditch and Dukes [11] showed that any K_3-divisible tripartite graph satisfying $\hat{\delta}(G) \geq 0.96n$ has a fractional K_3-decomposition. For the case $r \geq 4$, the current best bound to guarantee a fractional K_r-decomposition in an r-partite graph is due to Montgomery [64]. Combining their results the following theorem is obtained.

Theorem 1.4.2 ([10, 11, 64]). *For every $r \geq 3$ and every $\alpha > 0$ there exists*

$n_0 \in \mathbb{N}$ such that every K_r-divisible balanced r-partite graph G with parts of size $n \geq n_0$ satisfying

$$\hat{\delta}(G) \geq \begin{cases} (0.96 + \alpha)n & \text{if } r = 3, \\ (1 - \frac{1}{10^6 r^3} + \alpha)n & \text{if } r \geq 4, \end{cases}$$

has a K_r-decomposition.

1.4.2 Factors in partite graphs

A multipartite version of the Hajnal-Szemerédi theorem (Theorem 1.2.1) was conjectured by Fischer [25] and solved asymptotically by Keevash and Mycroft [45] and independently by Lo and Markström [60]. The exact result was solved by Keevash and Mycroft [46].

Theorem 1.4.3 ([46])**.** *For every $r \geq k$ there exists $n_0 \in \mathbb{N}$ such that every r-partite graph G with parts of size n such that $k \mid rn$ and $\hat{\delta}(G) \geq (1 - 1/k)n$ has a K_k-factor unless rn/k is odd, $k \mid n$ and $G \simeq \Gamma_{n,r,k}$.*

The family of graphs $\Gamma_{n,r,k}$ is an extremal counterexample to Fischer's original conjecture and were constructed by Catlin [13].

In the pseudo-random setting, the celebrated Blow-up Lemma by Komlós, Sárközy Szemerédi [53] can be used to find spanning subgraphs of bounded degree (e.g. F-factors for arbitrary graphs F) in typical graphs. A hypergraph version of the theorem was given by Keevash [41].

1.4.3 Resolvable decompositions in partite graphs

The existence of resolvable F-decompositions for arbitrary graphs F in multipartite pseudo-random graphs and hypergraphs has been settled by Keevash [43]. We combine ideas from [43] ('extended graph') and [28] ('iterative absorption') with a new result on fractional K_r-decompositions in pseudo-random graphs to show the following result (see Section 1.5 for further details).

Definition 1.4.4. Given $\xi, p > 0$ and $r, s \in \mathbb{N}$ we say that an r-partite graph G with parts $V_1, \dots, V_r$ is (ξ, p, s)-typical if for every $i \in [r]$ and every set of vertices $S \subseteq V(G) \setminus V_i$ of size at most s we have

$$d_G(S; V_i) = (1 \pm \xi) p^{|S|} |V_i|$$

where $d_G(S; V_i) := |\bigcap_{v \in S} N_G(v) \cap V_i|$.

Theorem 1.4.5. *For every $r \geq 3$ and $p > 0$ there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that any balanced K_r-divisible $(\xi, p, 3(r-1))$-typical r-partite graph G with parts of size $n \geq n_0$ has a resolvable K_r-decomposition.*

Theorem 1.4.5 can be viewed as an analogue to Theorem 1.2.5 in the multipartite setting. Again, our result only requires $s \geq 3(r-1)$ in the typicality condition whereas the results in [43] require s to be much larger than r. In order to prove Theorem 1.4.5 we will need a fractional K_r-decomposition in a pseudo-random r-partite graph (see Lemma 4.3.1). We accomplish this by adapting Montgomery's approach [64] to the pseudo-random setting. In addition, our fractional decomposition will be 'balanced' in the sense that the final weight on each clique will be close to uniform.

1.4.4 Completion of Latin squares

A *Latin square of order* n is an $n \times n$ array filled with the symbols from $[n]$ so that each symbol appears exactly once in each row and each column. Let $K_{n,n,n}$ denote the complete tripartite graph with vertex parts of size n. There is a natural bijection between Latin squares of order n and K_3-decompositions of $K_{n,n,n}$. To see this, label the three vertex parts of $K_{n,n,n}$ by 'row', 'column' and 'symbol' and label the vertices in each part with the numbers $[n]$. Any triangle rcs of $K_{n,n,n}$ contains one vertex from each part (r from 'row', c from 'column' and s from 'symbol') and can be viewed as placing the symbol s into the position given by r and c. Thus, a K_3-decomposition can be viewed as a filled $n \times n$ array where in each position rc we place the symbol s given by the triangle rcs in the decomposition. The fact that the triangles in a K_3-decomposition are edge-disjoint and cover all the edges of $K_{n,n,n}$ imply the properties that define a Latin square. For this reason, the study of Latin squares is closely related to the study of graph decompositions. For instance, a tripartite graph G which is obtained from $K_{n,n,n}$ by removing triangles can be viewed as a partially filled Latin square, or *partial Latin square* (each of the removed triangles would correspond to a filled entry). Whether such a partial Latin square can be completed to a proper Latin square is a question that has been extensively studied. Regarding this matter, Daykin and Häggkvist [17] made the following conjecture:

Conjecture 1.4.6 (Daykin and Häggkvist [17])**.** *Any partial Latin square of order n in which each row, column and symbol has been used at most $n/4$ times can be completed to a Latin square of order n.*

The conjecture is best possible as constructions matching the bound are given by Wanless [78]. Note that the conjecture can be viewed as a natural analogue to the Nash-Williams conjecture.

Consider an $n \times n$ array where each cell contains a list of 'permitted' symbols from $[n]$. Can we pick a symbol from each list to obtain a Latin square? Or equivalently, consider an $n \times n$ array where each cell contains a list of 'forbidden' symbols from $[n]$. Is there a Latin square that avoids using symbols from the corresponding lists? These questions were first posed by Häggkvist [34] who conjectured that any $n \times n$ array with lists of size at most $n/3$ such that every symbol appears in at most $n/3$ lists is avoidable by some Latin square. A construction by Pebody shows that the conjecture is best possible (see e.g. [16]). Formally, a *list Latin square* $\mathcal{L}$ of order n is an $n \times n$ array in which each entry contains a subset of $[n]$. We say that $\mathcal{L}$ *induces* a Latin square L if the symbol on each entry (i, j) of L is contained in the subset of symbols given by the entry (i, j) of $\mathcal{L}$. Confirming a conjecture by Häggkvist, it was proved by Andrén [5] for even n, and by Andrén, Casselgren and Öhman [4] for odd n, that there exists $\beta > 0$ such that any list Latin square of order n such that the subsets on each entry have size at least $(1-\beta)n$ and each symbol appears in at least $(1-\beta)n$ of the lists from each row and column, induces a Latin square of order n. The concepts of completing partial Latin squares and list Latin squares can be combined. The result is a *partial list Latin square of order* n which consists of an $n \times n$ array in which each entry contains either a number from $[n]$ ('filled entries') or a subset of $[n]$ ('lists on non-filled entries'). Andrén, Casselgren and Markström [6] explored this notion and showed that there are constants $\alpha, \beta > 0$ such that any partial list Latin square of order n in which each row, column and symbol has been used at most αn times, and each empty cell containing a list of size at least $(1-\beta)n$, can be completed into a proper Latin

A	B	C
B	C	A
C	A	B

,

α	β	γ
γ	α	β
β	γ	α

$\rightarrow$

Aα	Bβ	Cγ
Bγ	Cα	Aβ
Cβ	Aγ	Bα

Figure 1.1: Two orthogonal Latin squares of order 3.

square using only symbols from the corresponding lists. On the other hand, for $\alpha + \beta > 1/3$, they constructed examples of such partially filled Latin squares that could not be completed [6, Section 5]. We give explicit bounds on α, β for large enough n.

Theorem 1.4.7. *For any $\alpha < 1/(27 \cdot 10^6)$ and $\beta < 1/320$ there exists $n_0 \in \mathbb{N}$ such that any partial list Latin square of order $n \geq n_0$ in which each row, symbol and column has been used at most αn times, the lists for each empty entry have size at least $(1-\beta)n$, and each symbol appears in at least $(1-\beta)n$ of the lists from each row and each column, can be completed into a Latin square of order n.*

Two Latin squares of order n are said to be *orthogonal* if when superposed the ordered pairs of entries are all distinct, see e.g. Figure 1.1. A set of *mutually orthogonal Latin squares (MOLS)* is a set of Latin squares which are pairwise orthogonal. The study of MOLS dates back to Euler's officer problem, asking for the existence of a pair of MOLS of order six. As with Latin squares, MOLS are closely related to decompositions. More generally, the problem of finding a K_r-decomposition of an r-partite graph G which is obtained from the complete r-partite graph with parts of size n by removing r-cliques is equivalent to the completion of $(r-2)$ partial Latin squares into a set of MOLS of order n. This question is studied in [10] provided that G satisfies a minimum degree condition. On the other hand, Theorem 1.4.5 asserts that this can be done as long as G

satisfies a pseudo-random condition. In fact, Theorem 1.4.5 will be deduced from a more general result in which the K_r-decomposition can only use a certain set of 'permitted' cliques (see Theorem 4.2.1). This relates to the completion of a set of random-like partial list Latin squares into a set of MOLS.

1.5 Methods

1.5.1 Iterative absorption

The proofs of Theorems 1.2.3, 1.2.5 and 1.4.5 rely on 'iterative absorption', a method that was introduced by Kühn and Osthus [58] and has proven to be very successful for finding spanning structures and decompositions in large enough graphs and hypergraphs (see e.g. [9, 10, 12, 28, 30, 39, 52, 59, 61]). There are known methods to find 'almost-spanning' structures in a graph such as the Rödl Nibble, introduced by Rödl [75], or the random greedy algorithm proposed by Spencer [77]. The idea of the absorption method is, before applying any such algorithms, to set aside a suitable subgraph, called 'absorber', that will be used to transform the leftover from an almost-spanning structure so that it completes it into a proper one. Such an approach was introduced systematically by Rödl, Ruciński and Szemerédi [76] in order to find spanning structures in hypergraphs, but similar methods had already been used before (see e.g. [24, 55]). Roughly speaking, we don't have information about how the leftover might look like, so in order to be able to find an appropriate absorber, we need to modify the algorithm that finds the almost-spanning structure so that some information about the leftover is known beforehand. One way to achieve this is to somehow control the leftover at each step of the algorithm. Indeed, suppose we want to find an edge-decomposition of a graph G. The 'iterative absorption' method splits up the procedure of finding an approximate decomposition into many steps, and at each step applies a 'partial absorbing' procedure to absorb part of the leftover so the remaining one is under control. At the end of the overall iteration, the leftover is controlled and can be covered via the absorbers that were set aside at the beginning.

In our case, we will find a sequence of vertex sets $V(G) = U_0 \supseteq \cdots \supseteq U_\ell$, which we call the 'vortex' sequence, where $|U_i| = \varepsilon|U_{i-1}|$ and $|U_\ell| \leq m$ for some fixed $\varepsilon > 0$ and $m \in \mathbb{N}$. At the very beginning, we will set aside absorbers for all possible leftover configurations contained in $G[U_\ell]$. Then, iteratively, at the i-th step we will find an approximate decomposition of $G[U_i]$ and apply the so called 'Cover Down Lemma' to cover all edges that lie outside of $G[U_{i+1}]$. These two independent results: the existence of a fractional decomposition (which is transformed into an approximate decomposition) and the 'Cover Down Lemma', form the main core of the method. With this procedure, we will end up with the final leftover contained in $G[U_\ell]$ which can be 'absorbed'. This approach was first introduced by Barber, Lo, Kühn and Osthus [9] and further developed in [28, 30]. See e.g. [7] for a simple illustration of the method.

1.5.2 The extended graph

The problem of finding a resolvable decomposition in a graph G can be reduced to finding a decomposition in an auxiliary graph G', which we refer to as the *extended graph*. This method was first used by Keevash [43] in order to solve the existence of resolvable designs. Indeed, suppose we want to find a resolvable K_r-decomposition in a graph G of size n where $r \mid n$. Let $G' = G \cup H$ where H is the complete bipartite graph between $V(G)$ and a new set of vertices U of size $2e(G)/((r-1)n)$. The size of U is actually given by total number of edges of G divided by the number of edges in a K_r-factor of G. Let $\mathcal{K}$ be the set of $(r+1)$-cliques in G' that contain exactly one vertex in u and suppose that we are able to find a K_{r+1}-decomposition $\mathcal{D}$ of G' that uses only cliques in $\mathcal{K}$. For each $u \in U$ let $\mathcal{K}_u$ be the set of $(r+1)$-cliques in the decomposition $\mathcal{D}$ that contain u. Since each vertex $u \in U$ is adjacent (via H) to all vertices in $V(G)$ then $\mathcal{K}_u$ must cover all vertices G. Let $F_u := \bigcup_{K \in \mathcal{K}_u} K \cap V(G)$ be the edge-disjoint union of the r-cliques obtained when restricting $\mathcal{K}_u$ to $V(G)$. Since the union is edge-disjoint and cover all vertices in $V(G)$ then F_u must be a K_r-factor of G. Moreover, $\bigcup_{u \in U} F_u = G$ and all the K_r-factors $\{F_u\}_{u \in U}$ are edge-disjoint. Hence, G has a resolvable K_r-decomposition.

Theorems 1.2.3, 1.2.5 and 1.4.5 will actually be deduced from the more general results on the decomposition of their corresponding extended graphs (Theorems 2.2.3, 3.2.1 and 4.2.1 respectively).

1.5.3 Extended fractional decomposition

As mentioned in Section 1.5.1, the core of the 'iterative absorption' method that we use is made of two independent results, the existence of a fractional decomposition and the 'Cover Down Lemma'. If we want to find a resolvable decomposition of a graph G we will then have to prove such results in the corresponding extended graph G'. In particular, we are interested in finding a fractional decomposition in each of the corresponding extended graphs. However, the structure of the extended graphs will be distinct in each case and there are no previous results on the existence of such fractional decompositions. In this work, we develop a new technique which takes a given fractional decomposition of a graph G and transforms it into a fractional decomposition of the extended graph G'. We will achieve this with the use of customised gadgets for the case of cycles (see Lemma 2.3.3) and the case of cliques (Lemma 3.3.2). In the partite case we will construct it directly in the extended graph G' by making use of the gadgets introduced by Montgomery [64] (see Lemma 4.3.1).

1.5.4 Hypergraph matchings

A known result by Haxell and Rödl [37] transforms a fractional F-decomposition of a graph G on n vertices into an F-decomposition that covers all but $o(n^2)$ edges. Instead, we are interested in finding an F-decomposable subgraph $H \subseteq G$ such that $\Delta(G - H) = o(n)$. For this we will view the problem of finding an approximate edge-decomposition of G as finding an almost-perfect matching in an auxiliary hypergraph $\mathcal{H}$. Indeed, let $\mathcal{H}$ be the hypergraph whose vertices are the edges of G and whose hyperedges are the copies of F in G. Observe that a perfect matching of $\mathcal{H}$ corresponds to an F-decomposition of G. Starting with the introduction of the 'nibble' technique by Rödl [75] renowned results on hypergraph matchings were obtained by Frankl and Rödl [26] and Pippenger and Spencer [71]. Building on the results in [71], Alon and Yuster [3] showed the existence of 'well-distributed' matchings in almost-regular hypergraphs.

Definition 1.5.1. Given a hypergraph $\mathcal{H}$ and a collection $\mathcal{F}$ of subsets of $V(\mathcal{H})$, we say that a matching $\mathcal{M}$ in $\mathcal{H}$ is $(\gamma, \mathcal{F})$-*perfect* if for each $F \in \mathcal{F}$, at most $\gamma \cdot \max\{|F|, |V(\mathcal{H})|^{2/5}\}$ vertices of F are left uncovered by $\mathcal{M}$.

Theorem 1.5.2 ([3, Theorem 1.2]). *Suppose $1/n \ll \varepsilon \ll \gamma, 1/r$. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices such that for some $D \in \mathbb{N}$, we have $d_{\mathcal{H}}(x) = (1 \pm \varepsilon)D$ for all $x \in V(\mathcal{H})$, and $\Delta_2(\mathcal{H}) \leq D/\log^{9r} n$. Suppose that $\mathcal{F}$ is a collection of subsets of $V(\mathcal{H})$ such that $|\mathcal{F}| \leq n^{\log n}$. Then there exists a $(\gamma, \mathcal{F})$-perfect matching in $\mathcal{H}$.*

We will make use of Alon and Yuster's result in order to find an F-decomposable subgraph $H \subseteq G$ such that $\Delta(G - H) = o(n)$ as follows. Suppose that G has a fractional F-decomposition ω. It is then easy to obtain an 'almost-regular' collection $\mathcal{F}$ of copies of F where each edge $e \in E(G)$ is contained in roughly the same number of copies of F in $\mathcal{F}$. Indeed, this can be achieved by randomly sampling each copy of F with probability proportional to its weight $\omega(F)$. Since $\omega(e) = 1$ for each edge $e \in E(G)$ then the expected number of copies of F containing each edge will be the same. Thus, using probabilistic arguments one can show that with positive probability $\mathcal{F}$ will be 'almost-regular' in the sense mentioned above. Define the hypergraph $\mathcal{H}$ whose vertices are the edges of G and whose hyperedges are the sets in $\mathcal{F}$. For each $v \in V(G)$ let E_v be the set of edges containing v. We will then apply Theorem 1.5.2 to $\mathcal{H}$ with the sets $\{E_v\}_{v \in V(G)}$ playing the role of $\mathcal{S}$ to obtain a $(\gamma, \{E_v\}_{v \in V(G)})$-perfect matching in $\mathcal{H}$, which corresponds to a collection H of edge-disjoint copies of F where for each vertex $v \in V(G)$ only $o(n)$ edges in E_v are not covered by H.

1.6 Organisation

The thesis is organised as follows. In Chapter 2, Chapter 3 and Chapter 4 we will prove Theorem 1.2.3, Theorem 1.2.5 and Theorem 1.4.5 respectively. Each of these chapters starts with a short introduction to its corresponding theorem as well as a preliminaries section in which we introduce the terminology that will be used during the chapter. In particular, in Section 2.1 we describe the standard notation that shall be used throughout the thesis. Chapters 2, 3 and 4 have a similar structure that follows the proof approach showcased in the previous section. A more detailed description of the organisation of each chapter can be found in its corresponding introduction. In Chapter 5 we will outline the proof of Theorem 1.4.7. Finally, in Chapter 6 we will discuss the consequences of our results and methods as well as some potential applications.

CHAPTER 2

DIRAC-TYPE RESULT ON RESOLVABLE CYCLE DECOMPOSITIONS

The main aim of this chapter is to prove Theorem 1.2.3. Given a graph G recall that a *C_r-factor of G* is a collection of vertex-disjoint r-cycles that spans $V(G)$. Recall that a *resolvable C_r-decomposition of G* is a collection of edge-disjoint C_r-factors that cover all edges in $E(G)$. Finally, recall that G is said to be *C_r-divisible* if $d_G(v)$ is even for each vertex $v \in V(G)$ and $r \mid e(G)$. We now restate Theorem 1.2.3.

Theorem 1.2.3. For all $\alpha > 0$ there exists $r_0 \in \mathbb{N}$ with the property that for any $r \geq r_0$ there is $n_0 \in \mathbb{N}$ such that any C_r-divisible regular graph G on $n \geq n_0$ vertices such that $r \mid n$ and $\delta(G) \geq (1/2 + \alpha)n$ has a resolvable C_r-decomposition.

As discussed in Section 1.5.2, Theorem 1.2.3 will be deduced from a more general result, Theorem 2.2.3, on a 'wheel'-decomposition in the corresponding extended graph. Before stating Theorem 2.2.3 we shall introduce the standard notation used throughout the thesis as well as some useful probabilistic results. This is done in Section 2.1. We then state Theorem 2.2.3 and deduce Theorem 1.2.3 in Section 2.2. The rest of the chapter is focused on proving Theorem 2.2.3 and is organised as follows. In Section 2.3 we prove the fractional decomposition result in the extended graph (Lemma 2.3.3). Recall that this is, together with the 'Cover Down Lemma', one of the two key results used in the main proof. We transform the fractional decomposition into an approximate decomposition in Section 2.4. In Section 2.5 we find the 'vortex' sequence. In order to prove the 'Cover Down Lemma' we will need to find edge-disjoint factors in a family of sets with small intersection. These results are discussed in Section 2.6. We prove the 'Cover Down Lemma' (Lemma 2.7.1) in Section 2.7. In Section 2.8 we introduce the final absorbers that will be used in the main proof. Finally, the proof of Theorem 2.2.3 is presented in Section 2.9.

2.1 Preliminaries

2.1.1 Notation

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is a set of elements called *vertices* and $E(G)$ is a set of 2-sets of $V(G)$ called *edges*. We write $e(G) := |E(G)|$. An *r-clique* K of G is a subgraph of G isomorphic to the complete graph on r vertices. An *r-cycle* $C = (v_1, \dots, v_r)$ of G is a subgraph of G with vertex set $\{v_1, \dots, v_r\}$ and edge set $\{v_i v_{i+1}\}_{i \in [r] \cup \{0\}}$ where $v_0 = v_r$. We often identify a clique K or a cycle C with its vertex set $V(K)$ or $V(C)$ respectively. We denote by $\mathcal{K}_r(G)$ the set of r-cliques of G and by $\mathcal{C}_r(G)$ the set of r-cycles of G. Given graphs $G = (V, E)$ and $G' = (V', E')$ we say that G' *is a subgraph of* G and write $G' \subseteq G$ if $V' \subseteq V$ and $E' \subseteq E$. Given $U \subseteq V$ we write $G[U]$ for the graph with vertex set U and edge set $\{e \in E : e \subseteq U\}$. We write $G \setminus G'$ for the graph $G[V \setminus (V \cap V')]$ and $G - G'$

for the graph with vertex set $V(G)$ and edge set $E(G) \setminus (E(G) \cap E(G'))$. Given subsets $U_1, U_2 \subseteq V$, we write $G[U_1, U_2]$ for the graph with vertex set $U_1 \cup U_2$ and edge set $\{v_1v_2 \in E(G) \,:\, v_1 \in U_1, v_2 \in U_2\}$. Given a set of edges $E' \subseteq E(G)$, we write $G[E']$ for the graph with vertex set $V(G)$ and edge set E'.

Given a graph $G = (V, E)$, a set of vertices $S \subseteq V$ and a subset $V' \subseteq V$ we define $N_G(S; V') := \{v \in V' \,:\, xv \in E \text{ for all } x \in S\}$ to be the *common neighbourhood of* S *in* V'. We write $d_G(S; V') := |N_G(S; V')|$. We use the standard notation $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degree of G respectively.

Given a graph G and a subset $U \subseteq V(G)$ we say that the *degeneracy of* G *rooted at* U *is* r if there is an ordering $v_1, \ldots, v_m$ of the vertices in $V(G) \setminus U$ such that for all $i \in [m]$, $d_G(v_i; \{v_1, \ldots, v_{i-1}\} \cup U) \leq r$. If G has degeneracy r rooted at $\emptyset$ we say that G is r-*degenerate*.

We write $a = b \pm c$ if $a \in [b - c, b + c]$. We write $a = b \pm c = d \pm e$ if $a \in [b - c, b + c] \subseteq [d - e, d + e]$. The notation extends naturally to more equations. Equations containing $\pm$ are always meant to be read from left to right and might not commute.

We use the symbol '$\ll$' to define a hierarchy of constants. We will write $x \ll y$ to mean that for every $y > 0$ there exists $x_0 > 0$ such that for any $x \leq x_0$ the subsequent statement holds. In simple words, once y is fixed we can consider x to be as small as necessary for our computations. If a constant appears in a hierarchy with the form of $1/x$ it is assumed that $x \in \mathbb{N}$.

We will often omit the use of floors and ceilings whenever it does not affect the argument.

2.1.2 Tools

In this section we state some basic properties on error terms and some probabilistic results that we will use during the thesis. The following proposition presents a list of properties on the behaviour of error terms with a hierarchy.

Proposition 2.1.1. *Let* $\xi \ll 1/m, 1/c, 1/c_1, \ldots, 1/c_m$ *and* $A, B, B_1, \ldots, B_m > 0$.

(p1) $(1 \pm \xi)^m = 1 \pm 2^m \xi$.

(p2) If $\xi \ll c$ *then* $\frac{1}{1 \pm c\xi} = 1 \pm (c+1)\xi$.

(p3) If $B < \xi A$ *then* $(1 \pm c\xi)A \pm B = (1 \pm (c+1)\xi)A$.

(p4) If $A > (c + c_1 + 1)B_1 + \cdots + (c + c_m + 1)B_m$ *then*

$$(1 \pm c\xi)A - (1 \pm c_1\xi)B_1 - \cdots - (1 \pm c_m\xi)B_m = (1 \pm (c+1)\xi)(A - B_1 - \cdots - B_m).$$

(p5) If $A = (1 \pm c_1\xi)B$ *then* $(1 \pm c_2\xi)B = (1 \pm (c_1 + c_2 + 1))A$.

We will often use properties (p1)-(p5) implicitly during some of our calculations. It is important, therefore, to bear them in mind for the rest of this work.

Let $m, n, N \in \mathbb{N}$ with $\max\{m, n\} < N$. A random variable X has *hypergeometric distribution with parameters* N, n, m if $X := |S \cap [m]|$, where S is a random subset of $[N]$ of size n. We write $X \sim Bin(n, p)$ if X has binomial distribution with parameters n, p. We will often use the following Chernoff-type bound.

Lemma 2.1.2 (see [38, Corollary 2.3, Remark 2.5, Theorem 2.8 and Theorem 2.10])**.** *Let* X *be the sum of* n *independent Bernoulli random variables (with possibly different probability) or let* X *have a hypergeometric distribution with parameters* N, n, m*. Then the following hold.*

(i) *For all* $t \geq 0$, $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2\mathrm{e}^{-2t^2/n}$.

(ii) *For all* $0 \leq \varepsilon \leq 3/2$, $\mathbb{P}[X \neq (1 \pm \varepsilon)\mathbb{E}[X]] \leq 2\mathrm{e}^{-\varepsilon^2 \mathbb{E}[X]/3}$.

We will also need the following simple result.

Fact 2.1.3 (see e.g. [72, Lemma 8])**.** Let $X_1, \ldots, X_n$ be Bernoulli random variables such that for all $i \in [n]$, we have $\mathbb{P}[X_i = 1 \mid X_1, \ldots, X_{i-1}] \leq p$. Let $B \sim Bin(n, p)$ and $X := \sum_{i=1}^{n} X_i$. Then $\mathbb{P}[X \geq a] \leq \mathbb{P}[B \geq a]$ for all $a \geq 0$.

Finally, the following result is a straightforward consequence of Lemma 2.1.2.

Proposition 2.1.4. *Let* $1/n \ll \varepsilon \ll \rho \ll 1/s$. *Let* $m \in \mathbb{N}$ *where* $m < n^s$. *Let* $X_1, \ldots, X_m$ *be random variables where* $X_i \sim Bin(n_i, \rho)$ *and* $n_i > \rho n$ *for each* $i \in [m]$. *Then with positive probability*

$$X_i = (1 \pm \varepsilon)\mathbb{E}[X_i]$$

for every $i \in [m]$.

Proof. For each $i \in [m]$ we apply Lemma 2.1.2 to see that

$$\mathbb{P}\left[X_i \neq (1 \pm n_i^{-1/3})\mathbb{E}[X_i]\right] \leq 2\mathrm{e}^{-n^{-2/3}\mathbb{E}[X_i]/3} < 2\mathrm{e}^{-\rho^2 n^{1/3}/3} < \mathrm{e}^{-n^{1/4}}.$$

Let A_i be the event $X_i \neq (1 \pm n_i^{-1/3})\mathbb{E}[X_i]$ for each $i \in [m]$. We then apply a union bound to the events $\{A_i\}_{i \in [m]}$ to obtain

$$\mathbb{P}\left[\bigcup_{i \in [m]} A_i\right] \leq \sum_{i \in [m]} \mathbb{P}[A_i] \leq \sum_{i \in [m]} \mathrm{e}^{-n^{1/4}} \leq \frac{n^s}{\mathrm{e}^{n^{1/4}}} < 1$$

where we have used in both equations that n is large enough. Thus, with positive probability $X_i = (1 \pm \varepsilon)\mathbb{E}[X_i]$ for every $i \in [m]$ as desired. □

2.2 Main theorem

Theorem 1.2.3 will be deduced from a particular case of Theorem 2.2.3, a more general result on the existence of a 'wheel'-decomposition in an auxiliary graph. Before stating our main theorem we shall introduce some new concepts.

Definition 2.2.1. An *extended graph* $\mathcal{G} = (V, U, G, H)$ is a graph $\mathcal{G} = G \cup H$ where V and U are disjoint sets of vertices, G is a graph on V and H is a bipartite graph on $[V, U]$.

Let $\mathcal{G} = (V, U, G, H)$ and $\mathcal{G}' = (V', U', G', H')$ be two extended graphs. We say that $\mathcal{G}'$ *is an extended subgraph of* $\mathcal{G}$ and write $\mathcal{G}' \subseteq \mathcal{G}$ if $V' \subseteq V$, $U' \subseteq U$, $G' \subseteq G$ and $H' \subseteq H$. We say that $\mathcal{G}'$ is *an extended graph contained in* $[V, U]$ if $V' \subseteq V$ and $U' \subseteq U$.

Let $\varepsilon > 0$. We say that a graph G is (ε, d)-*regular* if $d_G(v) = (1 \pm \varepsilon)d$ for every $v \in V(G)$. We say that a bipartite graph H on $[V, U]$ is ε-*complete* if $d_H(v) \geq (1 - \varepsilon)|U|$ for every $v \in V$ and $d_H(u) \geq (1 - \varepsilon)|V|$ for every $u \in U$.

An $(r + 1)$-*wheel* W is a graph obtained from an r-cycle C by adding an extra vertex v and all the edges between v and C. We refer to the vertex v as the *hub of* W. Given an extended graph $\mathcal{G} = (V, U, G, H)$ we denote by $\mathcal{W}_{r+1}(\mathcal{G})$ the set of $(r + 1)$-wheels of $\mathcal{G}$ that contain exactly one vertex, the hub, in U.

Definition 2.2.2. A W_{r+1}-*decomposition of* $\mathcal{G}$ is a collection of edge-disjoint wheels in $\mathcal{W}_{r+1}(\mathcal{G})$ that cover all edges of $\mathcal{G}$.

Given $r \in \mathbb{N}$ we say that $\mathcal{G}$ is W_{r+1}-*divisible* if $d_G(v) = 2d_H(v)$ for every $v \in V$ and $r \mid d_H(u)$ for every $u \in U$. Note that W_{r+1}-divisibility is a necessary condition for the existence of a W_{r+1}-decomposition.

We state now the main theorem of this chapter.

Theorem 2.2.3. *Let $1/n \ll 1/r \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended W_{r+1}-divisible graph where $|V| = n$ and H is a complete bipartite graph. If G is d-regular with $d \geq (1/2 + \alpha)n$ then $\mathcal{G}$ has a W_{r+1}-decomposition.*

It is now straightforward to prove Theorem 1.2.3.

Proof of Theorem 1.2.3. Let $1/n \ll 1/r \ll \alpha$ and let G be a C_r-divisible regular graph on n vertices where $r \mid n$ and $\delta(G) \geq (1/2 + \alpha)n$. Note that G is regular so let $d := d_G(v)$ for any $v \in V(G)$. Since G is C_r-divisible then $r \mid e(G)$ and $2 \mid d$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $V = V(G)$, $|U| = e(G)/n$ and H is the complete bipartite graph on $[V, U]$. Note that $d_H(v) = |U| = e(G)/n = d/2$ for every $v \in V$ and $r \mid n = d_H(u)$ for every $u \in U$. Thus, $\mathcal{G}$ is W_{r+1}-divisible. Then by Theorem 2.2.3 we know that there exists a W_{r+1}-decomposition $\mathcal{D}$ of $\mathcal{G}$. For each $W \in \mathcal{D}$ let C_W be the r-cycle obtained from W by removing the hub (recall that the hub is the only vertex in U). For each $u \in U$ let $\mathcal{W}_u$ be the set of $(r+1)$-wheels in $\mathcal{D}$ that contain u and let $\mathcal{C}_u = \{C_W\}_{W \in \mathcal{W}_u}$. Since u is adjacent to all of V then $\mathcal{C}_u$ must be a C_r-factor. Finally, observe that since $\mathcal{D}$ covered all edges of $\mathcal{G}$ then the set $\{\mathcal{C}_u\}_{u \in U}$ is a set of edge-disjoint C_r-factors that cover all edges in G. Hence, G has a resolvable C_r-decomposition. $\square$

2.3 Extended fractional decomposition

In this section we will extend a given fractional C_r-decomposition in G to an 'almost' fractional wheel-decomposition of the extended graph $\mathcal{G}$ (Lemma 2.3.3). This will be used during the Cover down lemma (Lemma 2.7.1). Specifically, given an extended graph $\mathcal{G} = (V, U, G, H)$ where G is almost d-regular, $d \geq (1/2 + \alpha)n$ and H is almost complete, we will construct a weighting $\omega : \mathcal{W}_{r+1}(G) \to [0, 1]$ such that $\omega(e) := \sum_{W \in \mathcal{W}_{r+1} : e \subseteq W} \omega(W) = 1 \pm \eta$ for every edge $e \in E(\mathcal{G})$ for some small $\eta > 0$. To achieve this we will need a recent result by Joos and Kühn [40] which finds a fractional C_r-decomposition in a graph $\mathcal{G}$ with minimum degree $\delta(G) \geq (1/2 + \alpha)n$ for any $\alpha > 0$ provided that r and n are large enough. Moreover, if the graph G is almost regular then the fractional decomposition is 'balanced' in the sense that the final weight on all cycles is almost uniform. In fact, the result in [40] is more general and applies to tight cycles in highly connected hypergraphs. For simplicity, we state their result restricted to the case of graphs with high minimum degree.

Theorem 2.3.1 ([40, Theorem 1.4])**.** *Let $1/n \ll 1/r \ll \xi, \alpha$. Let G be a graph on n vertices such that $\delta(G) \geq (1/2+\alpha)n$. Then there is a fractional C_r-decomposition ω_G of G such that for every $C \in \mathcal{C}_r(G)$*

$$(1 - \xi)\frac{2e(G)}{\Delta(G)^r} \leq \omega_G(C) \leq (1 + \xi)\frac{2e(G)}{\delta(G)^r}. \tag{2.1}$$

We will now introduce the gadgets that we shall use to transform the fractional C_r-decomposition ω_G of G into an 'almost' fractional wheel-decomposition of the extended graph $\mathcal{G}$. Let $v \in V$ and $u_1, u_2 \in U$ such that $vu_1, vu_2 \in E(\mathcal{G})$ and let

$x \in \mathbb{R}$. A (vu_1, vu_2, x)-shifter is a function $\varphi : \mathcal{W}_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that

$$\varphi(e) := \sum_{W \in \mathcal{W}_{r+1}(\mathcal{G}) \,:\, e \subseteq W} \varphi(W) = \begin{cases} x & \text{if } e = vu_1, \\ -x & \text{if } e = vu_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

A (v, u_1, u_2)-*gadget* J is a subgraph of $\mathcal{G}$ consisting of a vertex $v \in V$, two vertices $u_1, u_2 \in U$ and r cycles $C_1, \ldots, C_r \in \mathcal{C}_r(G)$ with three distinguished and consecutive (in C_i) vertices $x_i, y_i, z_i \in V(C_i)$ for each $i \in [r]$ such that v is adjacent to the vertices x_i, z_i for each $i \in [r]$, $C_0 := (y_1, \ldots, y_r)$ is an r-cycle in J and $H[V(J)]$ is a complete bipartite graph. Note that J has $r^2 + 3$ vertices. For each $i \in [r]$ let C_i' be the r-cycle obtained from C_i by replacing y_i by v. We denote by $\mathcal{J}(v, u_1, u_2)$ the set of all (v, u_1, u_2)-gadgets in $\mathcal{G}$. Given $J \in \mathcal{J}(v, u_1, u_2)$ let $\varphi_J : \mathcal{W}_{r+1}(\mathcal{G}) \to \mathbb{R}$ be the $(vu_1, vu_2, 1)$-shifter defined as

$$\varphi_J(W) = \begin{cases} 1/r & \text{if } u_1 \text{ is the hub of } W \text{ and } C_0 \subseteq W \text{ or } C_i' \subseteq W \text{ for some } i \in [r], \\ -1/r & \text{if } u_1 \text{ is the hub of } W \text{ and } C_i \subseteq W \text{ for some } i \in [r], \\ -1/r & \text{if } u_2 \text{ is the hub of } W \text{ and } C_0 \subseteq W \text{ or } C_i' \subseteq W \text{ for some } i \in [r], \\ 1/r & \text{if } u_2 \text{ is the hub of } W \text{ and } C_i \subseteq W \text{ for some } i \in [r], \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

It is important to notice that φ_J is indeed an $(vu_1, vu_2, 1)$-shifter.

The following proposition gives a lower bound on the number of (v, u_1, u_2)-gadgets for any $v \in V$ and $u_1, u_2 \in U$.

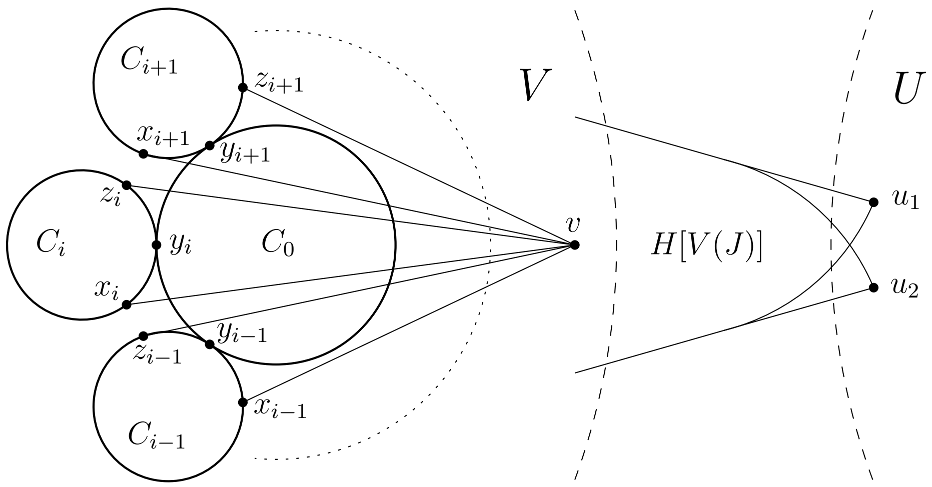

Figure 2.1: A (v, u_1, u_2)-gadget.

Proposition 2.3.2. *Let $1/n \ll \varepsilon \ll 1/r \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $|V| = n$, $\delta(G) \geq (1/2 + \alpha)n$ and H is ε-complete. Let $v \in V$ and $u_1, u_2 \in U$ such that $vu_1, vu_2 \in E(\mathcal{G})$. Then $|\mathcal{J}(v, u_1, u_2)| > \alpha^{3r+1}(1/2)^{r^2-3r}n^{r^2} > (\alpha n)^{r^2}$.*

Proof. We will first choose the cycle $C_0 = (y_1, \ldots, y_r)$. Since $\delta(G) \geq (1/2 + \alpha)n$ then $|\mathcal{C}_r(G)| \geq \alpha(1/2)^{r-2}n^r$ and $|\{C \in \mathcal{C}_r(G) : v \in C\}| < n^{r-1}$. Hence there are at least $\alpha(1/2)^{r-1}n^r$ different r-cycles that don't contain v. Since H is ε-complete at least $\alpha(1/2)^r n^r$ of such cycles are adjacent to both u_1 and u_2. Choose one of these cycles as C_0. For each $i \in [r]$ there are at least $\alpha^2 n^2$ pairs of vertices x_i, z_i that are adjacent to y_i, v, u_1, u_2 and distinct from previously chosen vertices. Finally, for each $i \in [r]$ there are at least $\alpha(1/2)^{r-4}n^{r-3}$ $(r-3)$-paths from x_i to z_i that are adjacent to u_1 and u_2 and avoid previous chosen vertices. $\square$

We are now ready to state and prove the main result of this section.

Lemma 2.3.3. *Let $1/n \ll \eta \ll \varepsilon \ll 1/r \ll \xi \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $|V| = n$, G is (ε, d)-regular with $d \geq (1/2 + \alpha)n$, H is*

ε-complete and $d_G(v) = (1 \pm \eta)2d_H(v)$ for every $v \in V$. Then there is a weighting $\omega : \mathcal{W}_{r+1}(\mathcal{G}) \to [0,1]$ such that for every edge $e \in E(\mathcal{G})$

$$\omega(e) = 1 \pm \eta \tag{2.4}$$

and for every wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$

$$\omega(W) = (1 \pm 4\xi)\frac{2e(G)}{d^r|U|} \tag{2.5}$$

Proof. First of all observe that for every vertex $v \in V$ we have $d_G(v) = (1 \pm \varepsilon)d$ and $d_G(v) = (1 \pm \eta)2d_H(v) = (1 \pm \eta)2(1 \pm \varepsilon)|U|$ and thus $|U| = (1 \pm 3\varepsilon)d/2$. Observe also that $2e(G) = \sum_{v \in V} d_G(v) = (1 \pm \varepsilon)dn$.

We start by Theorem 2.3.1 to G to find a weighting $\omega_G : \mathcal{C}_r(G) \to [0,1]$ such that

$$\omega_G(e) = \sum_{C \in \mathcal{C}_r(G)\,:\,e \subseteq C} \omega_G(C) = 1$$

for every $e \in E(G)$ and

$$\omega_G(C) = (1 \pm \xi)\frac{2e(G)}{(1 \pm \varepsilon)^r d^r} = (1 \pm 2\xi)\frac{2e(G)}{d^r}$$

for every $C \in \mathcal{C}_r(G)$.

Observe that for every vertex $v \in V$

$$\omega_G(v) := \sum_{e \in E(G)\,:\,v \in e} \omega_G(e) = d_G(v) = (1 \pm \varepsilon)d. \tag{2.6}$$

Given an r-cycle $C \in \mathcal{C}_r(G)$ let $N^*_{\mathcal{G}}(C) := \{W \in \mathcal{W}_{r+1}(\mathcal{G}) \;:\; C \subset W\}$ and

$d^*_{\mathcal{G}}(C) := |N^*_{\mathcal{G}}(C)|$. We now define $\omega' : \mathcal{W}_{r+1}(\mathcal{G}) \to [0,1]$ as

$$\omega'(W) = \frac{\omega_G(C_W)}{d^*_{\mathcal{G}}(C_W)}$$

where C_W is the r-cycle of G contained in W. Then, for every $C \in \mathcal{C}_r(G)$ we have that

$$\sum_{W \in N^*_{\mathcal{G}}(C)} \omega'(W) = \sum_{W \in N^*_{\mathcal{G}}(C)} \frac{\omega_G(C)}{d^*_{\mathcal{G}}(C)} = \omega_G(C). \tag{2.7}$$

Note that for every edge $e \in E(G)$

$$\begin{aligned}\omega'(e) &= \sum_{W \in \mathcal{W}_{r+1}(\mathcal{G}) \,:\, e \subseteq W} \omega'(W) = \sum_{C \in \mathcal{C}_r(G) \,:\, e \subseteq C} \sum_{W \in N^*_{\mathcal{G}}(C)} \omega'(W) \\ &\overset{(2.7)}{=} \sum_{C \in \mathcal{C}_r G \,:\, e \subseteq C} \omega_G(C) = \omega_G(e) = 1.\end{aligned} \tag{2.8}$$

Given $v \in V$, let E_v be the set of edges in $E(H)$ incident to v. Then, for every $v \in V$

$$\begin{aligned}\omega_H(v) &:= \sum_{e \in E_v} \omega'(e) = \sum_{e \in E_v} \sum_{W \in \mathcal{W}_{r+1}(\mathcal{G}) \,:\, e \subseteq W} \omega'(W) = \sum_{C \in \mathcal{C}_r(G) \,:\, v \in C} \sum_{W \in N^*_{\mathcal{G}}(C)} \omega'(W) \\ &\overset{(2.7)}{=} \sum_{C \in \mathcal{C}_r(G) \,:\, v \in C} \omega_G(C) = \frac{1}{2} \sum_{e \in E(G) \,:\, v \in e} \omega_G(e) \overset{(2.6)}{=} \frac{d_G(v)}{2}.\end{aligned}$$

Suppose that for each $v \in V$

$$\omega'(e_1) = \omega'(e_2) \text{ for every pair of edges } e_1, e_2 \in E_v. \tag{2.9}$$

Then, for any $e \in E_v$

$$\omega_H(v) = \sum_{e' \in E_v} \omega'(e') = d_H(v)\omega'(e)$$

and thus,

$$\omega'(e) = \frac{\omega_H(v)}{d_H(v)} = \frac{d_G(v)}{2d_H(v)} = 1 \pm \eta \tag{2.10}$$

which together with (2.8) implies (2.4).

Our goal then is to ensure that (2.9) holds for every vertex $v \in V$. To do this, we will use several shifters which will modify our current weighting ω' into the desired one.

Fix $v \in V$. For each $e \in E_v$, let $x_e := \omega_H(v)/d_H(v) - \omega'(e)$ be the "missing weight" of e.

Claim 1: For every $e \in E(H)$, $|x_e| \leq \varepsilon^{1/2}$.

Proof of claim: Let $e = vu \in E(H)$ with $v \in V$ and $u \in U$. On one hand

$$\begin{aligned}
\omega'(e) &= \sum_{W \in \mathcal{W}_{r+1}(\mathcal{G})\,:\,e \subseteq W} \omega'(W) = \sum_{\substack{C \in \mathcal{C}_r(G)\,:\,v \in C \\ C \subseteq N_H(u)}} \frac{\omega_G(C)}{d^*_{\mathcal{G}}(C)} \\
&= \sum_{C \in \mathcal{C}_r(G)\,:\,v \in C} \frac{\omega_G(C)}{d^*_{\mathcal{G}}(C)} - \sum_{\substack{C \in \mathcal{C}_r(G)\,:\,v \in C \\ C \nsubseteq N_H(u)}} \frac{\omega_G(C)}{d^*_{\mathcal{G}}(C)}.
\end{aligned}$$

And on the other hand,

$$\begin{aligned}
0 \leq \sum_{\substack{C \in \mathcal{C}_r(G)\,:\,v \in C \\ C \nsubseteq N_H(u)}} \frac{\omega_G(C)}{d^*_{\mathcal{G}}(C)} &\leq (1+2\xi)\frac{2e(G)}{d^r} \cdot \varepsilon n^{r-1} \cdot \frac{1}{(1-r\varepsilon)|U|} \\
&\leq (1+4\xi)2^{r+1}\varepsilon \leq \varepsilon^{1/2}
\end{aligned}$$

Hence, for every $v \in V$ and every pair of edges $e_1, e_2 \in E_v$ we have $|\omega'(e_1) - \omega'(e_2)| \le \varepsilon^{1/2}$ And so $|x_e| \le \varepsilon^{1/2}$ for every edge $e \in E(H)$. $-$

Note that $\sum_{e \in E_v} x_e = 0$.

The following claim finds triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{i \in I}$ where $e_{i_1}, e_{i_2} \in E_v$ so that by finding an (e_{i_1}, e_{i_2}, x_i)-shifter for each $i \in I$ we will compensate the missing weight of each $e \in E_v$ and ensure that (2.9) holds.

Claim 2: There exists a finite set A_v of ordered triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{i \in I}$ where $e_{i_1}, e_{i_2} \in E_v$ and $x_i \in \mathbb{R}$ satisfying that

(S1) for each $e \in E_v$ let $I_e \subseteq I$ be the set of indices i such that $e \in \{e_{i_1}, e_{i_2}\}$, then

$$\sum_{i \in I_e} |x_i| = |x_e|,$$

(S2) suppose that φ_i is an (e_{i_1}, e_{i_2}, x_i)-shifter for every $i \in I$, then

$$\omega'(e) + \sum_{i \in I} \varphi_i(e) = \omega_H(v)/d_H(v)$$

for every $e \in E_v$.

Proof of claim: The proof consists of the following algorithm. Suppose that at the j-th step we have already found triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{1 \le i < j}$. For each $e \in E_v$, let $I_j^+(e)$ be the set of indices i such that $e = e_{i_1}$ and let $I_j^-(e)$ be the set of indices i such that $e = e_{i_2}$. If $x_e \ge 0$ let $z_j(e) := \sum_{i \in I_j^+(e)} x_i$ and if $x_e < 0$ let $z_j(e) := \sum_{i \in I_j^-(e)} -x_i$. Let $E_j := \{e \in E_v \,:\, z_j(e) = x_e\}$. Suppose that the triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{1 \le i < j}$ satisfy that for each $e \in E_v$,

(s1) if $x_e \ge 0$ then $|I_j^-(e)| = 0$ and if $x_e < 0$ then $|I_j^+(e)| = 0$, (s2) $|z_j(e)| \leq |x_e|$,

and suppose that

(s3) $|E_j| \geq j$.

The algorithm stops if $E_j = E_v$. In this case we have that $z_j(e) = x_e$ for each $e \in E_v$ which implies (S1). Suppose that φ_i is an (e_{i_1}, e_{i_2}, x_i)-shifter for every $1 \leq i < j$. Then for every edge $e \in E_v$

$$\omega'(e) + \sum_{1 \leq i < j} \varphi_i(e) = \omega'(e) + z_j(e) = \omega'(e) + x_e = \omega_H(v)/d_H(v)$$

which implies (S2).

Else, if $|E_j| < |E_v|$, let e_j be an edge in $E_v \setminus E_j$ that minimises $|x_e - z_j(e)|$. Observe that

$$\sum_{e \in E_v} (x_e - z_j(e)) = \sum_{e \in E_v} x_e - \sum_{e \in E_v} z_j(e) = 0 - \sum_{e \in E_v \,:\, x_e \geq 0} \sum_{i \in I_j^+(e)} x_i + \sum_{e \in E_v \,:\, x_e < 0} \sum_{i \in I_j^-(e)} x_i = 0.$$

Then there must exist some edge $e'_j \in E_v \setminus E_j$ such that $\operatorname{sgn}(x_{e_j} - z_j(e_j)) \neq \operatorname{sgn}(x_{e'_j} - z_j(e'_j))$. We choose the j-th triple to be $(e_j, e'_j, (x_{e_j} - z_j(e_j)))$ if $(x_{e_j} - z_j(e_j)) > 0$ and $(e'_j, e_j, -(x_{e_j} - z_j(e_j)))$ if $(x_{e_j} - z_j(e_j)) < 0$. Observe that $z_{j+1}(e_j) = x_{e_j}$, $|z_{j+1}(e'_j)| \leq |x_{e'_j}|$, and $z_{j+1}(e) = z_j(e)$ for all edges except e_j and e'_j. It follows that the new set of triples satisfy (s1)-(s3) when replacing j by $j+1$. Note that $|E_{j+1}| > |E_j|$ guarantees that the algorithm will stop at most at the $d_H(v)$-th step. $-$

Next, for each $(e_1, e_2, x) \in A_v$ let $\varphi_{(e_1,e_2,x)} : \mathcal{W}_{r+1}(\mathcal{G}) \to \mathbb{R}$ be defined as $\varphi_{(e_1,e_2,x)}(W) = \frac{x}{|\mathcal{J}(v,u_1,u_2)|} \sum_{J \in \mathcal{J}(v,u_1,u_2)} \varphi_J(W)$ where $e_1 = vu_1$ and $e_2 = vu_2$. Recall

(2.3). It follows that $\varphi_{(e_1,e_2,x)}$ is an (e_1, e_2, x)-shifter. Moreover, since J has r^2 vertices in addition of v, u_1, u_2, given $W \in \mathcal{W}_{r+1}(\mathcal{G})$ the amount of gadgets $J \in \mathcal{J}(v, u_1, u_2)$ that contain W is at most n^{r^2+1-r} if e_1 or $e_2 \subseteq W$, n^{r^2-r} if u_1 or $u_2 \in W$ but $v \notin W$, and zero otherwise. Thus, using Proposition 2.3.2 we obtain

$$|\varphi_{(e_1,e_2,x)}(W)| \leq \begin{cases} |x|\alpha^{-r^2}n^{-(r-1)} & \text{if } e_1 \text{ or } e_2 \subseteq W, \\ |x|\alpha^{-r^2}n^{-r} & \text{if } u_1 \text{ or } u_2 \in W \text{ and } v \notin W, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Let $\varphi_v := \sum_{a \in A_v} \varphi_a$. Then (S2) implies that

$$\omega'(e) + \varphi_v(e) = \omega'(e) + \sum_{a \in A_v} \varphi_a(e) = \omega_H(v)/d_H(v) \tag{2.12}$$

for every $e \in E_v$. On the other hand, consider a wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$ with $\{u\} = W \cap U$. If W contains v and $vu \in E(H)$,

$$|\varphi_v(W)| \leq \sum_{a \in A_v} |\varphi_a(W)| = \sum_{a \in A_v \,:\, vu \in a} |\varphi_a(W)| \overset{(S1),(2.11)}{\leq} |x_{vu}|\alpha^{-r^2}n^{-(r-1)}. \tag{2.13}$$

If W does not contain v but $vu \in E(H)$,

$$|\varphi_v(W)| \leq \sum_{a \in A_v} |\varphi_a(W)| = \sum_{a \in A_v \,:\, vu \in a} |\varphi_a(W)| \overset{(S1),(2.11)}{\leq} |x_{vu}|\alpha^{-r^2}n^{-r}. \tag{2.14}$$

Else, if $vu \notin E(H)$, $\varphi_v(W) = 0$.

Let $\omega = \omega' + \sum_{v \in V} \varphi_v$. First observe that $\omega(e) = \omega'(e)$ for every edge $e \in E(G)$.

Second, for every $v \in V$ and every $e \in E_v$

$$\omega(e) = \omega'(e) + \sum_{v' \in V} \varphi_{v'}(e) = \omega'(e) + \varphi_v(e) = \omega_H(v)/d_H(v).$$

Thus, equations (2.9) and (2.10) holds when replacing ω' with ω, so (2.4) holds as desired.

It only remains to check equation (2.5). Fix $W \in \mathcal{W}_{r+1}(\mathcal{G})$ and let $W = \{v_1, \ldots, v_r, u\}$ with $u \in U$. Using Claim 1 and equations (2.13) and (2.14) we obtain

$$\begin{aligned} |\sum_{v \in V} \varphi_v(W)| &\leq \sum_{v \in V \,:\, v \in W} |\varphi_v(W)| + \sum_{v \in V \,:\, v \notin W} |\varphi_v(W)| \\ &\leq r\varepsilon^{1/2}\alpha^{-r^2} n^{-(r-1)} + n\varepsilon^{1/2}\alpha^{-r^2} n^{-r} \leq \varepsilon^{1/4} n^{-(r-1)} \end{aligned}$$

and

$$\omega'(W) = \frac{\omega_G(C_W)}{d^*_{\mathcal{G}}(C_W)} = (1 \pm 2\xi)\frac{2e(G)}{d^r} \cdot \frac{1}{(1 \pm r\varepsilon)|U|} = (1 \pm 3\xi)\frac{2e(G)}{d^r|U|}.$$

Hence,

$$\omega(W) = (1 \pm 3\xi)\frac{2e(G)}{d^r|U|} \pm \varepsilon^{1/4} n^{-(r-1)} = (1 \pm 4\xi)\frac{2e(G)}{d^r|U|}.$$

□

2.4 Approximate decomposition

In this section we will transform the fractional decomposition found by Lemma 2.3.3 into an approximate decomposition that covers, for each vertex, all but $o(n)$ of its incident edges. The proof is sketched in Section 1.5.4 and will make use of Theorem 1.5.2.

Lemma 2.4.1. *Let $1/n \ll \eta \ll \gamma \ll \varepsilon \ll 1/r \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $|V| = n$, G is (ε, d)-regular with $d \geq (1/2 + \alpha)n$, H is ε-complete and $d_G(v) = (1 \pm \eta)2d_H(v)$ for every $v \in V$. Then there is a W_{r+1}-decomposable subgraph $\mathcal{F} \subseteq \mathcal{G}$ such that $\Delta(\mathcal{G} - \mathcal{F}) \leq \gamma n$.*

Proof. Let $\rho, \xi, \eta' > 0$ such that

$$1/n \ll \eta \ll \eta' \ll \gamma \ll \varepsilon \ll 1/r \ll \xi \ll \rho \ll \alpha$$

We start applying Lemma 2.3.3 to obtain a weighting $\omega : \mathcal{W}_{r+1}(\mathcal{G}) \to [0, 1]$ such that

$$\omega(e) = 1 \pm \eta$$

for each edge $e \in E(\mathcal{G})$, and

$$\omega(W) = (1 \pm \xi)\frac{2e(G)}{d^r|U|}$$

for each wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$.

Since G is (ε, d)-regular, H is ε-complete and $d_G(v) = (1 \pm \eta)2d_H(v)$ for each vertex $v \in V(G)$ we have that $d_G(v) = (1 \pm \varepsilon)d$ and $d_G(v) = (1 \pm \eta)2d_H(v) = (1 \pm \eta)(1 \pm \varepsilon)2|U|$ so $|U| = (1 \pm 5\varepsilon)d$. Then

$$\frac{2e(G)}{d^r|U|} = \frac{(1 \pm \varepsilon)dn}{(1 \pm 5\varepsilon)d^{r+1}} = \frac{(1 \pm 8\varepsilon)n}{d^r}.$$

For each wheel $w \in \mathcal{W}_{r+1}(\mathcal{G})$, let $p(W) := \omega(W)\rho d^r/n$ and note that $0 <$

$p(W) < 1$.

Let $\mathcal{W}$ be a set of wheels obtained by choosing each wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$ with probability $p(W)$. Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H}) = E(\mathcal{G})$ and hyperedge set $E(\mathcal{H}) = \mathcal{W}$. We have that for each edge $e \in E(G)$

$$\mathbb{E}\left[d_{\mathcal{H}}(e)\right] = \sum_{W \in \mathcal{W}_{r+1}(\mathcal{G})\,:\,e \subset W} p(W) = \sum_{W \in \mathcal{W}_{r+1}(\mathcal{G})\,:\,e \subset W} \omega(W)\rho d^r/n = (1 \pm \eta)\rho d^r/n.$$

Note that since $d \geq n/2$ then using Lemma 2.1.2 and a union bound is it straightforward to see that with positive probability

$$d_{\mathcal{H}}(e) = (1 \pm \eta')\rho d^r/n \text{ for all } e \in E(G). \tag{2.15}$$

Fix $\mathcal{H}$ so that (2.15) holds and let $D := \rho d^r/n$.

Observe that $\Delta_2(\mathcal{H}) \leq n^{r-2}$ as every pair of edges determine at least three vertices so there are at most n^{r-2} $(r+1)$-wheels containing them. On the other hand, since $D = \rho d^r/n \geq \rho n^{r-1}/2^r$ then $n^{r-2} \leq \rho n^{r-1}/(2^r \log^{9r} n) \leq D/(\log^{9r} n)$. For each $v \in V(\mathcal{G})$ let $S_v = \{e \in E(\mathcal{G}) \,:\, v \in e\}$. Let $\mathcal{S} = \{S_v\}_{v \in V(\mathcal{G})}$ and note that $|\mathcal{S}| = n + |U| \leq n + 2d \leq 3n$. We apply Theorem 1.5.2 to the hypergraph $\mathcal{H}$ (with η' playing the role of ε) to obtain a $(\gamma, \mathcal{S})$-perfect matching $\mathcal{M}$. Let E_L be the set of edges in $E(\mathcal{G})$ which are left uncovered by $\mathcal{M}$. Observe that a perfect matching in $\mathcal{H}$ corresponds to a W_{r+1}-decomposition of $\mathcal{G}$. Hence, $\mathcal{F} := \mathcal{G} - \mathcal{G}[E_L]$ has a W_{r+1}-decomposition. Finally, $\Delta(\mathcal{G} - \mathcal{F}) = \Delta(\mathcal{G}[E_L]) \leq \gamma n$ since for each vertex $v \in V(\mathcal{G})$ there are at most γn edges e containing v which are left uncovered by $\mathcal{M}$ in $\mathcal{H}$. $\square$

2.5 Vortex sequence

As sketched in Section 1.5 our main proof is based on iterative absorption. The first step of the proof is to find a sequence of vertex sets $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ that end in a set V_ℓ of constant size. We call such sequence a 'vortex'. By choosing the sets at random we will be able to maintain the properties from the host graph in each vertex set V_i. The vortex sequence was introduced by Glock, Kühn, Lo, Montgomery and Osthus [28].

Definition 2.5.1 (Vortex)**.** Given $r, m \in \mathbb{N}$ and $\eta, \varepsilon, \alpha > 0$, an $(\alpha, \eta, \varepsilon, m)$-*vortex* of an extended graph $\mathcal{G} = (V, U, G, H)$ is a pair of sequences $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ such that $|V_\ell| = m$ and for all $j \in [\ell]$

(V1) $|V_j| = \varepsilon|V_{j-1}|$ and $|U_j| = \varepsilon|U_{j-1}|$,

(V2) $d_G(v_1) = (1 \pm \eta)d_G(v_2)$ for every $v_1, v_2 \in V_j$,

(V3) $d_G(v; V_j) \geq (1/2 + \alpha)|V_j|$ for all $v \in V_{j-1}$,

(V4) $H[V_j, U_j]$ is complete,

(V5) $d_G(v, V_j) = (1 \pm \eta)2d_H(v, U_j)$ for all $v \in V_{j-1}$.

The next result finds a vortex sequence in an extended graph $\mathcal{G}$ that satisfies the same conditions as the statement of Theorem 2.2.3. Applying Lemma 2.5.2 will be the first step in the proof of the main theorem.

Lemma 2.5.2. *Let $1/n \ll 1/m' \ll \eta \ll \varepsilon \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $|V| = n$, H is a complete bipartite graph, G is d-regular with $d > (1/2 + 2\alpha)n$ and $d_G(v) = 2d_H(v)$ for every $v \in V$. Then $\mathcal{G}$ contains an $(\alpha, \eta, \varepsilon, m)$-vortex where $\varepsilon m' \leq m \leq m'$.*

Proof. Let $n_0 := n$ and define $n_i := \varepsilon n_{i-1}$ recursively for every $i \in [\ell]$. Let $\ell := 1 + \max\{i \geq 0 : n_i \geq m'\}$ and let $m := n_\ell$. Note that by definition $\varepsilon m' \leq m \leq m' \leq n_{\ell-1}$. Let $\alpha_0 := 2\alpha$ and $\eta_0 := 0$ and define recursively for every $i \in [\ell]$

$$\alpha_i := \alpha_{i-1} - 2n_{i-1}^{-1/3}$$

and

$$\eta_i := \eta_{i-1} + 4n_{i-1}^{-1/3}.$$

Suppose that for some $i \in [\ell]$ we have already found a sequence of sets $V_0 \supseteq \cdots \supseteq V_{i-1}$ and $U_0 \supseteq \cdots \supseteq U_{i-1}$ that form an $(\alpha_{i-1}, \eta_{i-1}, \varepsilon, n_{i-1})$-vortex of $\mathcal{G}$ (so properties (V1)-(V5) holds for all $j = [i]$). Note that this is trivially true for $i = 0$.

Let V_i be a random subset of V_{i-1} of size $\varepsilon|V_{i-1}|$ and let U_i be an arbitrary subset of U_{i-1} of size $\varepsilon|U_{i-1}|$. Note that $|V_i| = n_i$.

For each $v \in V_{i-1}$ note that $d_G(v; V_i)$ is a random variable having an hypergeometric distribution with parameters n_{i-1}, $d_G(v; V_{i-1})$, n_i satisfying $\mathbb{E}[d_G(v; V_i)] = \varepsilon d_G(v; V_{i-1})$. Hence Lemma 2.1.2 implies that

$$\mathbb{P}\left[d_G(v; V_i) \neq (1 \pm n_{i-1}^{-1/3})\varepsilon d_G(v; V_{i-1})\right] \leq 2\mathrm{e}^{-n_{i-1}^{-2/3}\varepsilon d_G(v;V_{i-1})/3} \leq 2\mathrm{e}^{-\varepsilon n_{i-1}^{1/3}/6}$$

where we have used $d_G(v; V_{i-1}) \geq n_{i-1}/2$ by (V3). Let A_v be the event $d_G(v; V_i) \neq (1 \pm n_{i-1}^{-1/3})\varepsilon d_G(v; V_{i-1})$. Then, using a union bound we can see that

$$\mathbb{P}\left[\bigcup_{v \in V_{i-1}} A_v\right] \leq \sum_{v \in V_{i-1}} \mathbb{P}[A_v] \leq n_{i-1} 2\mathrm{e}^{-\varepsilon n_{i-1}^{1/3}/6} < 1.$$

Thus, we have shown that with positive probability for any vertex $v \in V_{i-1}$

$$d_G(v; V_i) = (1 \pm n_{i-1}^{-1/3})\varepsilon d_G(v; V_{i-1}).$$

Fix such choice of V_i. Then, for every $v_1, v_2 \in V_i$ we have

$$\begin{aligned} d_G(v_1; V_i) &= (1 \pm n_{i-1}^{-1/3})\varepsilon d_G(v_1; V_{i-1}) = (1 \pm n_{i-1}^{-1/3})(1 \pm \eta_{i-1})\varepsilon d_G(v_2; V_{i-1}) \\ &= (1 \pm n_{i-1}^{-1/3})(1 \pm \eta_{i-1})(1 \pm 2n_{i-1}^{-1/3})d_G(v_2; V_i) \\ &= (1 \pm \eta_i)d_G(v_2; V_i). \end{aligned}$$

For every $v \in V_{i-1}$ we have

$$d_G(v; V_i) \geq (1 - n_{i-1}^{-1/3})\varepsilon d_G(v; V_{i-1}) \geq (1 - n_{i-1}^{-1/3})(1/2 + \alpha_{i-1})\varepsilon n_{i-1} \geq (1/2 + \alpha_i)n_i,$$

and

$$\begin{aligned} d_G(v; V_i) &= (1 \pm n_{i-1}^{-1/3})\varepsilon d_G(v; V_{i-1}) = (1 \pm n_{i-1}^{-1/3})\varepsilon(1 \pm \eta_{i-1})d_H(v; U_{i-1}) \\ &= (1 \pm \eta_i)\varepsilon|U_{i-1}| = (1 \pm \eta_i)|U_i| = (1 \pm \eta_i)d_H(v; U_i). \end{aligned}$$

Hence, $V_0 \supseteq \cdots \supseteq V_i$ and $U_0 \supseteq \cdots \supseteq U_i$ form an $(\alpha_i, \eta_i, \varepsilon, n_i)$-vortex of $\mathcal{G}$. Repeating iteratively the above arguments we end up with an $(\alpha_\ell, \eta_\ell, \varepsilon, n_\ell)$-vortex $V_0 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq \cdots \supseteq U_\ell$ of $\mathcal{G}$. Note that

$$\sum_{i=0}^{\ell-1} n_i^{-1/3} = \sum_{i=0}^{\ell-1}(\varepsilon^i n)^{-1/3} = n^{-1/3}\frac{\varepsilon^{-\ell/3} - 1}{\varepsilon^{-1/3} - 1} \leq \frac{(\varepsilon^\ell n)^{-1/3}}{\varepsilon^{-1/3} - 1} \leq \frac{(\varepsilon^{\ell-1} n)^{-1/3}}{1 - \varepsilon^{1/3}} \leq (m')^{-1/3}.$$

Thus, we have

$$\alpha_\ell = \alpha_0 - \sum_{i=0}^{\ell-1} 2n_i^{-1/3} \geq \alpha_0 - 2(m')^{-1/3} \geq \alpha$$

and

$$\eta_\ell = \eta_0 + \sum_{i=0}^{\ell-1} 4n_i^{-1/3} \leq 4(m')^{-1/3} \leq \eta$$

so $V_1 \supseteq \cdots \supseteq V_\ell$ and $U_1 \supseteq \cdots \supseteq U_\ell$ is an $(\alpha, \eta, \varepsilon, m)$-vortex of $\mathcal{G}$ as desired. □

2.6 Edge-disjoint factors

Let G be a graph and let $V_1, \ldots, V_m \subseteq V(G)$ be a sequence of vertex subsets. Can we find, for each $i \in [m]$, a factor F_i of $G[V_i]$ so that all the factors $\{F_i\}_{i\in[m]}$ are edge-disjoint? During the proof of the 'Cover Down Lemma' we will encounter the situation where we need to find such factors. Following a method introduced by Barber, Kühn, Lo and Osthus [9] we prove two distinct results, Lemmas 2.6.2 and 2.6.3, that allow us to do such thing. Of course, each of the graphs $G[V_i]$ will require some condition to guarantee the existence of a factor. Moreover, in order to assure that the factors are edge-disjoint we will need the sets to be large enough and to have small intersection. Our first result, Lemma 2.6.2, will find edge-disjoint C_r-factors. We will need the following result due to Kühn and Osthus [57] that shows the existence of a C_r-factor in a graph satisfying Dirac's condition.

Theorem 2.6.1 ([57]). *Let $1/n \ll 1/r \ll \alpha$ with $r \mid n$. Let G be a graph on n vertices satisfying that $\delta(G) \geq (1/2 + \alpha)n$. Then G has a C_r-factor.*

We will use Theorem 2.6.1 successively in each of the graphs $G[V_i]$ to find many edge-disjoint C_r-factors and pick one at random. A probabilistic argument will then show that with positive probability we can randomly choose a C_r-factor F_i for each $G[V_i]$ so that all the factors $\{F_i\}_{i\in[m]}$ end up being edge-disjoint. This method was first used by Barber, Kühn, Lo and Osthus to find edge-disjoint K_r-factors in a graph with high minimum degree, see [9, Lemma 10.7].

Lemma 2.6.2. *Let $1/n \ll \rho \ll 1/r \ll \alpha$ and let $N \in \mathbb{N}$. Let G be a graph on n vertices and let $V_1, \dots, V_N \subseteq V(G)$ such that*

(F1) $r \mid |V_i|$ and $|V_i| \geq \rho^{4/3} n$ for all $i \in [N]$,

(F2) $\delta(G[V_i]) \geq (1/2 + \alpha)|V_i|$ for all $i \in [N]$,

(F3) $|V_i \cap V_j| \leq \rho^2 n$ for all $1 \leq i < j \leq N$,

(F4) every vertex $v \in V$ is contained in at most ρn of the sets $\{V_i\}_{i \in [N]}$.

Then for each $i \in [N]$ there exists a C_r-factor F_i of $G[V_i]$ such that all the factors $\{F_i\}_{i \in [N]}$ are edge-disjoint.

Proof. Let $m := 2\rho^{3/2} n$. The proof consists of a random algorithm where for each $i = 1, \dots, N$ we find a C_r-factor of $G[V_i]$ at random and show that with positive probability at the end of the algorithm the chosen factors are edge-disjoint. Suppose that we have already found C_r-factors $F_1, \dots, F_{i-1}$ of $G[V_1], \dots, G[V_{i-1}]$ respectively for some $i \in [m]$. Let $H_i := \bigcup_{j \in [i-1]} F_j$ and let $G_i := G - H_i$. We now consider two cases depending on the maximum degree of H_i.

Case 1: $\Delta(H_i) \leq \rho^{3/2} n$.

Observe that

$$\delta(G_i[V_i]) \geq \delta(G[V_i]) - \Delta(H_i) \geq (1/2 + \alpha)n - \rho^{3/2} n \geq (1/2 + \alpha/2)n$$

since $\rho \ll \alpha$. We can then apply Theorem 2.6.1 successively to find m edge-disjoint C_r-factors $A_1, \dots, A_m$ of $G_i[V_i]$.

Case 2: $\Delta(H_i) > \rho^{3/2}n$.

If this is the case, let $A_1, \dots, A_m$ be defined as empty graphs on the set V_i and continue with the next iteration.

Observe that in any case the graphs $A_1, \dots, A_m$ are pairwise edge-disjoint and edge-disjoint from all the previous graphs $F_1, \dots, F_{i-1}$. Let F_i be chosen uniformly at random from the set of candidates $A_1, \dots, A_m$. Note that at the end of the algorithm the graphs $F_1, \dots, F_m$ will be actually edge-disjoint C_r-factors of $G[V_1], \dots, G[V_m]$ if and only if

$$\Delta(H_i) \leq \rho^{3/2}n \tag{2.16}$$

for each $i = 1, \dots, m$. We will check that with positive probability this is the case.

For each $i \in [N]$ and $v \in V_i$ let $J^{i,v} := \{j \in [i-1] \; : \; v \in V_j\}$ and for each $j \in J^{i,v}$ let $Y_j^{i,v}$ be the indicator variable of the event that $vv' \in E(F_j)$ for some $v' \in V_i$. In other words, $Y_j^{i,v}$ indicates the event that some edge e of $G[V_i]$ incident to v already appears in the graph F_j, and therefore could not be used when choosing F_i. Observe that

$$d_{H_i[V_i]}(v) = \sum_{j \in J^{i,v}} Y_j^{i,v}. \tag{2.17}$$

Let $i \in [N]$ and $x \in V_i$ be fixed. Since $|V_i \cap V_j| \leq \rho^2 n$ for each $j \in J^{i,v}$ by (F3) it follows that at most $\rho^2 n$ of the edge-disjoint graphs $A_1, \dots, A_m$ that were candidates for F_i share an edge incident to v with G_i. Let $j_1, \dots, j_{|J^{i,v}|}$ be the elements of $J^{i,v}$ arranged in ascending order. Then, for all $k \in [|J^{i,v}|]$ we have that

$$\mathbb{P}\left[Y^{i,v}_{j_k} = 1 \mid Y^{i,v}_{j_1}, \ldots, Y^{i,v}_{j_{k-1}}\right] \leq \frac{\rho^2 n}{m} \leq \frac{\rho^{1/2}}{2}.$$

Let $B \sim Bin(|J^{i,v}|, \rho^{1/2}/2)$ and note that $|J^{i,v}| \leq \rho n$ by (F4). Thus,

$$\mathbb{E}[B] = |J^{i,v}|\frac{\rho^{1/2}}{2} \leq \frac{\rho^{3/2} n}{2}.$$

All in all we have

$$\mathbb{P}\left[\sum_{j \in J^{i,v}} Y^{i,v}_j > \rho^{3/2} n\right] \overset{\text{Fact 2.1.3}}{\leq} \mathbb{P}\left[B > \rho^{3/2} n\right] \leq \mathbb{P}\left[B > \mathbb{E}[B] + \rho^{3/2} n/2\right]$$
$$\overset{\text{Lemma 2.1.2}}{\leq} 2e^{-\rho^3 n/2}.$$

Finally, since there are at most ρn^2 pairs (i, v) with $v \in V_i$ (because of (F4)), a union bound implies that with positive probability $\sum_{j \in J^{i,v}} Y^{i,v}_j \leq \rho^{3/2} n$ for all $i \in [N]$ and $v \in V_i$. This together with (2.17) implies that equation (2.16) holds with positive probability for every $i = 1, \ldots, N$ concluding thus the proof. $\square$

We refer as a 2-*path* to a path of length 2, i.e, a graph consisting of three vertices and two edges. Our next result finds edge-disjoint 2-path-factors in a bipartite graph satisfying Dirac's condition.

Lemma 2.6.3. *Let $1/n \ll \rho \ll \alpha$ and $N \in \mathbb{N}$. Let H be a bipartite graph on $[V, U]$ where $|V| = n$ and let $V_1, \ldots, V_N \subseteq V$ and $U_1, \ldots, U_N \subseteq U$ such that*

(P1) $|V_i| = 2|U_i|$ for all $i \in [N]$,

(P2) $|V_i| \geq \rho^{4/3} n$ for all $i \in [N]$,

(P3) $d_H(v; U_i) \geq (1/2+\alpha)|U_i|$ *and* $d_H(u; V_i) \geq (1/2+\alpha)|V_i|$ *for all* $i \in [N]$, $v \in V_i$ *and* $u \in U_i$.

(P4) $|V_i \cap V_j| \leq \rho^2 n$ *and* $|U_i \cap U_j| \leq \rho^2 n$ *for all* $1 \leq i < j \leq [N]$,

(P5) every vertex $v \in V$ *and* $u \in U$ *is contained in at most* ρn *of the sets* $\{V_i\}_{i \in N}$ *and* $\{U_i\}_{i \in [N]}$ *respectively.*

Then for each $i \in [N]$ *there is a 2-path factor* P_i *of* $H[V_i, U_i]$ *so that all the factors* $\{P_i\}_{i \in [N]}$ *are edge-disjoint.*

To avoid reiteration we will omit the proof of Lemma 2.6.3. The lemma could be proven using a similar approach to that of Lemma 2.6.2 by replacing the use of Theorem 2.6.1 with the following consequence of Hall's theorem:

Proposition 2.6.4. *Let* H *be a bipartite graph with parts of size* n *such that* $d_H(v) \geq n/2$ *for each vertex* $v \in V(H)$. *Then* H *has a perfect matching.*

It is straightforward now to obtain a 2-path-factor in a bipartite graph H on $[V, U]$ where $|V| = 2|U|$, $d_H(v) \geq |U|/2$ for each $v \in V$ and $d_H(u) \geq |V|/2$ for each $u \in U$. Indeed, by duplicating each vertex $u \in U$ one obtain a bipartite graph H' satisfying the conditions of Proposition 2.6.4. Finally, a perfect matching in H' corresponds, by identifying the duplicated vertices, to a 2-path-factor of H.

2.7 Cover Down Lemma

In this section we prove the 'Cover Down Lemma', Lemma 2.7.1, which is the key machinery behind the iterative absorption method used in the proof of Theorem 2.2.3. Recall that in the Cover Down Lemma we are given a graph G and a subset $U \subseteq V(G)$ and wish to find a decomposable subgraph $F \subseteq G$ that covers all edges in $G - G[U]$ and only a sparse set of edges in $G[U]$. The proof is divided into three main steps. In the first step we will set aside a sparse set of edges R that will act later as a partial absorber. In the second step we remove the reservoir edges and find an approximate decomposition, in our case by using Lemma 2.4.1. Specifically, we find a decomposable subgraph in $G - R$ that covers all but $o(n^2)$ leftover edges. Finally, in the third step, we use the reservoir edges and possibly some edges within $G[U]$ to cover the leftover edges. It is during the third step where we will need to find edge-disjoint factors using Lemmas 2.6.2 and 2.6.3.

Lemma 2.7.1 (Cover down lemma)**.** *Let $1/n \ll \eta \ll \gamma' \ll \varepsilon \ll 1/r \ll \alpha$. Let $\mathcal{G} = (V_0, U_0, G, H)$ be an extended graph where G is (ε, d)-regular with $d \geq (1/2 + \alpha)n$, H is $2\varepsilon^2$-complete and $d_G(v) = (1 \pm \eta)2d_H(v)$ for every $v \in V$. Let $V_1 \subseteq V_0$ and $U_1 \subseteq U_0$ be subsets satisfying*

(C1) $|V_1| = \varepsilon|V_0| = \varepsilon n$ and $|U_1| = \varepsilon|U_0|$,

(C2) $d_G(v; V_1) \geq (1/2 + 2\alpha)|V_1|$ for all $v \in V_0$,

(C3) for every $v \in V_0$, $d_G(v; V_1) = (1 \pm \eta)2d_H(v; U_1)$,

(C4) for every $v \in V_0 \setminus V_1$, $d_G(v) = 2d_H(v)$,

(C5) for every $u \in U_0 \setminus U_1$, $r \mid d_H(u)$.

Then there is a W_{r+1}-decomposable subgraph $\mathcal{F} \subseteq \mathcal{G}$ such that $E(\mathcal{G} - \mathcal{F}) \subseteq E(\mathcal{G}[V_1 \cup U_1])$ and $\Delta(\mathcal{F}[V_1 \cup U_1]) \leq \gamma' n$.

Proof. Let $\gamma, \rho > 0$ such that

$$1/n \ll \eta \ll \gamma \ll \rho \ll \gamma' \ll \varepsilon \ll 1/r \ll \alpha. \tag{2.18}$$

Let $V' := V_0 \setminus V_1$ and $U' := U_0 \setminus U_1$.

Using (C2) and the fact that H is $2\varepsilon^2$-complete we have

$$d_G(\{v_1, v_2\}; V_1) \geq 4\alpha|V_1| \tag{2.19}$$

for every $v_1, v_2 \in V_0$ and

$$d_{\mathcal{G}}(\{v, u\}; V_1) \geq (1/2 + \alpha)|V_1| \tag{2.20}$$

for every $v \in V_0$ and $u \in U_0$.

Since $d_G(v) = (1 \pm \eta)2d_H(v)$ for every $v \in V$ and using (C3) we obtain

$$\begin{aligned} d_G(v; V') &= d_G(v) - d_G(v; V_1) = (1 \pm \eta)2d_H(v) - (1 \pm 2\eta)2d_H(v; U_1) \\ &= (1 \pm 2\eta)2d_H(v; U'). \end{aligned} \tag{2.21}$$

Step 1: Choosing the reservoir edges.

Let $R_1 \subseteq G[V', V_1]$, $R_2 \subseteq H[V', U_1]$ and $R_3 \subseteq H[V_1, U']$ be subgraphs obtained by choosing each edge independently at random with probability ρ. Observe that for each $v \in V'$, $d_{R_1}(v; V_1)$ can be seen as a random variable X_v where $X_v \sim Bin(d_G(v; V_1), \rho)$. And this is also true for the common degree of distinct vertices through R_1, R_2 and R_3. In other words, each degree through R_1, R_2 or R_3 can be seen as a binomial random variable where the expected size is a ρ proportion of the degree that the vertices originally had in $\mathcal{G}$. We shall use this observation to make use of Proposition 2.1.4 (with η playing the role of ε) together with the properties given by (C2)-(C4), (2.19)-(2.21), to show that the following list of properties is satisfied with positive probability.

–*Expected degree sizes:*

(R1) $d_{R_1}(v; V_1) = (1 \pm \eta)\rho d_G(v; V_1)$ for every $v \in V'$,

(R2) $d_{R_1}(v; V') = (1 \pm \eta)\rho d_G(v; V')$ for every $v \in V_1$,

(R3) $d_{R_2}(v; U_1) = (1 \pm \eta)\rho d_H(v; U_1)$ for every $v \in V'$,

(R4) $d_{R_2}(u; V') = (1 \pm \eta)\rho d_H(u; V')$ for every $u \in U_1$,

(R5) $d_{R_3}(u; V_1) = (1 \pm \eta)\rho d_H(u; V_1)$ for every $u \in U'$,

(R6) $d_{R_3}(v; U') = (1 \pm \eta)\rho d_H(v; U')$ for every $v \in V_1$.

–*Expected divisibility relations:*

(R7) $d_{R_1}(v; V_1) = (1 \pm 2\eta)2d_{R_2}(v; U_1)$ for every $v \in V'$,

(R8) $d_{R_1}(v; V') = (1 \pm 3\eta)2d_{R_3}(v; U')$ for every $v \in V_1$.

–*Expected common neighbourhood into V_1:*

(R9) $(1 + \eta)\rho^2|V_1| \geq |N_{R_1}(v_1) \cap N_{R_1}(v_2)| \geq 3\rho^2\alpha|V_1|$ for every $v_1, v_2 \in V'$,

(R10) $|N_{R_1}(v_1) \cap N_G(v_2)| \geq 3\rho\alpha|V_1|$ for every $v_1 \in V'$ and $v_2 \in V_1$,

(R11) $|N_{R_1}(v) \cap N_{R_3}(u)| \geq (1/2 + \alpha)\rho^2|V_1|$ for every $v \in V'$ and $u \in U'$,

(R12) $|N_{R_1}(v) \cap N_H(u)| \geq (1/2 + \alpha)\rho|V_1|$ for every $v \in V'$ and $u \in U_1$,

(R13) $|N_{R_3}(u) \cap N_G(v)| \geq (1/2 + \alpha)\rho|V_1|$ for every $u \in U'$ and $v \in V_1$,

(R14) $|N_{R_3}(u_1) \cap N_{R_3}(u_2)| \leq (1 + \eta)\rho^2|V_1|$ for every $u_1, u_2 \in U'$.

–*Expected common neighbourhood into U_1:*

(R15) $(1+\eta)\rho^2|U_1| \geq |N_{R_2}(v_1) \cap N_{R_2}(v_2)| \geq (1-5\varepsilon)\rho^2|U_1|$ for every $v_1, v_2 \in V'$,

(R16) $|N_{R_2}(v_1) \cap N_H(v_2)| \geq (1-5\varepsilon)\rho|U_1|$ for every $v_1 \in V'$ and $v_2 \in V_1$.

Fix such choice of R_1, R_2 and R_3 and let $\widetilde{\mathcal{G}} = (V_0, U_0, \widetilde{G}, \widetilde{H})$ where $\widetilde{G} = G - R_1 - G[V_1]$ and $\widetilde{H} = H - R_2 - R_3 - H[V_1, U_1]$.

Step 2: Approximate decomposition.

Observe that since $\Delta(R_1) \leq \rho n$ and $\Delta(G[V_1]) \leq \varepsilon n$ then $\widetilde{G}$ is $(2\varepsilon, d)$-regular and since $\Delta(R_2), \Delta(R_3) \leq \rho n$ and $\Delta(H[V_1, U_1]) \leq \varepsilon n$ then $\widetilde{H}$ is 2ε-complete. On the other hand, for every vertex $v \in V'$

$$d_{\widetilde{G}}(v) = d_G(v) - d_{R_1}(v) = 2d_H(v) - (1 \pm 2\eta)2d_{R_2}(v) = (1 \pm 2\eta)2d_{\widetilde{H}}(v) \tag{2.22}$$

where we have used (R7), and for every $v \in V_1$

$$\begin{aligned} d_{\widetilde{G}}(v) &= d_G(v) - d_{R_1}(v) - d_G(v; V_1) \\ &= (1 \pm \eta)2d_H(v) - (1 \pm 3\eta)2d_{R_3}(v) - (1 \pm \eta)2d_H(v; U_1) \\ &= (1 \pm 2\eta)2d_{\widetilde{H}}(v) \end{aligned} \tag{2.23}$$

where we have used (R8) and (C3). Thus, we can apply Lemma 2.4.1 to $\widetilde{\mathcal{G}}$ to obtain a W_{r+1}-decomposable subgraph $\mathcal{F}_1 \subseteq \widetilde{G}$ such that $\Delta(\mathcal{G} - \mathcal{F}_1) \leq \gamma n$. Let $L := (\mathcal{G} - \mathcal{F}_1)[V' \cup U']$. In particular,

$$\Delta(L) \leq \gamma n. \tag{2.24}$$

Step 3: Covering down the leftover edges.

Let $\mathcal{G}_1 := \mathcal{G} - \mathcal{F}_1$. Note that $\mathcal{G}_1[V_1 \cup U_1] = \mathcal{G}[V_1 \cup U_1]$.

Our goal now is to cover all the edges in $\mathcal{G}_1 - \mathcal{G}[V_1 \cup U_1]$, which includes the edges in R_1, R_2, R_3 and L, by using only a sparse set of edges within $\mathcal{G}[V_1 \cup U_1]$. We shall achieve this in three steps represented by Claims 1, 2 and 3. Specifically, in Claim 1 we will cover one by one each edge $e \in E(L)$ by finding an $(r+1)$-wheel in $\mathcal{G}_1$ that contains e. After this, all edges in $\mathcal{G}_1[V' \cup U']$ will have been covered. Next, in Claim 2, we will cover, for each $u' \in U'$, all the edges incident to u' that remain uncovered. These will be edges between U' and V_1. Here we shall make use of Lemma 2.6.2. Finally, in Claim 3, we will cover, for each $v' \in V'$, all the edges incident to v' that remain uncovered. For this we will need Lemma 2.6.3. Note that after applying the three claims we will have covered all edges in $\mathcal{G}_1 - \mathcal{G}[V_1 \cup U_1]$ as desired. We will make sure during the whole process that the edges in $\mathcal{G}[V_1 \cup U_1]$ covered after applying the three claims is sparse.

Claim 1*: For each $e \in E(L)$ there is a wheel $W_e \in \mathcal{W}_{r+1}(\mathcal{G} - \mathcal{F}_1)$ that contains e such that all the wheels $\{W_e\}_{e \in E(L)}$ are edge-disjoint and if $\mathcal{F}_2 := \bigcup_{e \in E(L)} W_e$ then $\Delta(\mathcal{F}_2) \leq \gamma^{1/3} n$.*

Proof of claim: Suppose that for some set $E' \subseteq E(L)$ we have already found edge-disjoint wheels $\{W_e\}_{e \in E'}$ such that $\mathcal{F}' := \bigcup_{e \in E'} W_e$ satisfies

$$\Delta(\mathcal{F}') \leq \gamma^{1/3} n. \tag{2.25}$$

We say that a vertex $x \in V_1 \cup U_1$ is *good* if there are less than $\gamma^{1/2} n$ wheels in $\mathcal{F}'$ containing x. Otherwise we say that it is *bad*. Let X be the set of bad vertices and suppose that $|X| > \gamma^{1/3} n$. By definition, for each bad vertex $x \in X$ there are at least $\gamma^{1/2} n$ edge-disjoint wheels in $\mathcal{F}'$ containing x, which implies $d_{\mathcal{F}'}(x) > 2\gamma^{1/2} n$.

It follows that

$$|E(\mathcal{F}')| \geq \frac{1}{2}\sum_{x\in X} d_{\mathcal{F}'}(x) > \gamma^{5/6}n^2.$$

On the other hand, we have that

$$|E(\mathcal{F}')| \leq 2r|E(L)| \overset{(2.24)}{\leq} 2r\gamma n^2$$

which is a contradiction. Hence,

$$|X| \leq \gamma^{1/3}n. \tag{2.26}$$

Let $G' := (G[V_1] - \mathcal{F}'[V_1]) \setminus X$ and observe that $\delta(G') \geq (1/2+\alpha)|V_1|$ using (C2), (2.25) and (2.26). Let $e \in E(L) \setminus E'$ be an edge. Suppose first that $e \in v_1'v_2'$ with $v_1', v_2' \in V'$. Using (R15), (R12), (2.25) and (2.26) we can greedily choose good vertices $u \in N_{R_2-\mathcal{F}'}(e)$, $v_1 \in N_{R_1-\mathcal{F}'}(v_1') \cap N_{H-\mathcal{F}'}(u)$ and $v_2 \in N_{R_1-\mathcal{F}'}(v_2') \cap N_{H-\mathcal{F}'}(u)$. Then, since the graph $G'[N_H(u;V_1)]$ satisfies Dirac's minimum degree condition we can find a path P_e with $r-3$ edges from v_1 to v_2. If $e = v'u$ with $v' \in V'$ and $u \in U'$, using (R11) we choose instead two good vertices $v_1, v_2 \in N_{R_1-\mathcal{F}'}(v') \cap N_{R_3-\mathcal{F}'}(u')$ and then find a path P_e with $r-2$ edges from v_1 to v_2 in $G'[N_H(u';V_1)]$. In any case we have that $W_e := e \cup P_e \cup \{u\}$ is a wheel of $\mathcal{W}_{r+1}(\mathcal{G})$, edge-disjoint of $\mathcal{F}'$ and such that all the vertices of W_e are good. Hence, $\Delta(\mathcal{F}' \cup W_e) \leq \gamma^{1/3}n$ so we can apply the same arguments by replacing E' by $E' \cup e$ and $\mathcal{F}'$ by $\mathcal{F}' \cup W_e$. –

Using Claim 1 we can find a W_{r+1}-decomposable subgraph $\mathcal{F}_2 \subseteq \mathcal{G} - \mathcal{F}_1$ such that $E(L) \subseteq E(\mathcal{F}_2)$ and

$$\Delta(\mathcal{F}_2) \leq \gamma^{1/3}n. \tag{2.27}$$

Let $\mathcal{G}_2 := \mathcal{G} - \mathcal{F}_1 - \mathcal{F}_2$, $G_2 := \mathcal{G}_2[V]$ and $H_2 := \mathcal{G}_2[V, U]$. Since $\mathcal{G}_2[V' \cup U']$ has no edges then for every $v' \in V'$ and $u' \in U'$ we have $N_{G_2}(v') = N_{G_2}(v'; V_1)$, $N_{H_2}(v') = N_{H_2}(v'; U_1)$ and $N_{H_2}(u') = N_{H_2}(u'; V_1)$. Moreover, since $\mathcal{G}_2$ is obtained from $\mathcal{G}$ by removing W_{r+1}-decomposable graphs we know by (C4) and (C5) that $d_{G_2}(v') = 2d_{H_2}(v')$ and $r \mid d_{H_2}(u')$ for every $v' \in V'$ and $u' \in U'$.

Claim 2: For each $u' \in U'$ there is a C_r-factor $F_{u'}$ of $N_{H_2}(u')$ in G_2 such that all the factors $\{F_{u'}\}_{u' \in U'}$ are edge-disjoint.

Proof of claim: We just need to check that the sets $\{N_{H_2}(u')\}_{u' \in U'}$ satisfy the conditions of the sets $\{V_i\}_{i \in N}$ in Lemma 2.6.2. As mentioned above we have $r \mid d_{H_2}(u')$ for each $u' \in U'$. Since $R_3 \subseteq H_2$ then $d_{H_2}(u') \geq d_{R_3}(u') \geq (1-\eta)\rho(1-2\varepsilon)\varepsilon n \geq (\rho\varepsilon^{-1})^{4/3}(\varepsilon n)$ where we have used (R5) and $d_H(u'; V_1) \geq |V_1| - 2\varepsilon^2 n$. Using (R14) we have $d_{R_3}(u'_1, u'_2) \leq (1+\eta)\rho^2|V_1| \leq (1+\eta)\rho^2\varepsilon n \leq (\rho\varepsilon^{-1})^2(\varepsilon n)$. For each $v_1 \in V_1$ we can see that $d_{H_2}(v_1) \leq d_{R_3}(v_1) + d_{\mathcal{F}_2}(v) \leq (1+\eta)\rho|U'| + \gamma^{1/3}n \leq \rho n = (\rho\varepsilon^{-1})(\varepsilon n)$ by using (R6). Finally, observe that for each $u' \in U'$ and each $v \in N_{H_2}(u')$ we have

$$\begin{aligned}
d_{G_2}(v; N_{H_2}(u')) &= |N_{G_2}(v; V_1) \cap N_{H_2}(u'; V_1)| \geq |N_G(v; V_1) \cap N_{R_3}(u')| - 2\Delta(\mathcal{F}_2) \\
&\geq (1/2 + \alpha)\rho|V_1| - 2\gamma^{1/3}n = (1/2 + \alpha/2)\rho\varepsilon n + (\alpha/2)\rho\varepsilon n - 2\gamma^{1/3}n \\
&\geq (1/2 + \alpha/2)\rho\varepsilon n + \rho(1/2 + \alpha/2)\rho\varepsilon n \\
&\geq (1/2 + \alpha/2)|N_{H_2}(u')|
\end{aligned}$$

where we have used that $G[V_1] \subseteq G_2 \cup \mathcal{F}_2$, $R_3 \subseteq H_2 \cup \mathcal{F}_2$, (R13), (2.27) and $|N_{H_2}(u')| \leq d_{R_3}(u') + d_{\mathcal{F}_2}(u') \leq (1+\rho)\rho\varepsilon n$ (because of (R5) and (2.27)). We can then apply Lemma 2.6.2 with G_2, $\alpha/2$, εn and $\rho\varepsilon^{-1}$ playing the role of G, α, n and ρ respectively. $\dashv$

Using Claim 2 we find edge-disjoint C_r-factors $F_{u'}$ of $N_{H_2}(u')$ in G_2 for each $u' \in U'$. For each $u' \in U'$ and each r-cycle $C \in F_{u'}$ let $W_C \in \mathcal{W}_{r+1}(\mathcal{G})$ be the wheel consisting of the cycle C and the hub u'. Let $\mathcal{F}_3 := \bigcup_{C\in F_{u'}, u'\in U'} W_C$. As $\mathcal{F}_3$ is obtained from the edge-disjoint union of wheels it is straightforward to see that $\mathcal{F}_3$ is W_{r+1}-decomposable. Moreover, $\mathcal{F}_3$ is edge-disjoint of $\mathcal{F}_1 \cup \mathcal{F}_2$ since $\mathcal{F}_3$ consists only of edges in H_2 and G_2. On the other hand, since $d_{H_2}(u') \leq (1+\rho)\rho\varepsilon n$ for each $u' \in U'$ and $d_{H_2}(v_1) \leq \rho n$ for each $v_1 \in V_1$ then

$$\Delta(\mathcal{F}_3) \leq 3\rho n. \tag{2.28}$$

Let $\mathcal{G}_3 := \mathcal{G} - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3$, $G_3 := \mathcal{G}_3[V]$ and $H_3 := \mathcal{G}_3[V, U]$. Note that $V' \cap V(\mathcal{F}_3) = \emptyset$ and thus $N_{\mathcal{G}_3}(v') = N_{\mathcal{G}_2}(v')$ for each $v' \in V'$. Hence, for each $v' \in V'$, $N_{G_3}(v') = N_{G_3}(v'; V_1)$, $N_{H_3}(v') = N_{H_3}(v'; U_1)$ and $d_{G_3}(v') = 2d_{H_3}(v')$.

Claim 3: For each $v' \in V'$ there is a collection $F_{v'}$ of $d_{H_3}(v')$ wheels that contain v' such that all the wheels $\{W : W \in F_{v'}, v' \in V'\}$ are edge-disjoint and if $\mathcal{F}_4 := \bigcup_{W\in F_{v'}, v'\in V'} W$ then $\Delta(\mathcal{F}_4) \leq (\gamma'/2)n$.

Proof of claim: For each $v' \in V'$ let $V_{v'} := N_{G_3}(v')$ and $U_{v'} := N_{H_3}(v')$. We will check that the sets $\{V_{v'}\}_{v'\in V'}$ and $\{U_{v'}\}_{v'\in V'}$ satisfy the properties of the sets $\{V_i\}_{i\in\mathbb{N}}$ and $\{U_i\}_{i\in\mathbb{N}}$ from Lemma 2.6.3 respectively. First of all note that $|V_{v'}| = 2|U_{v'}|$. Using (R1) and (2.27) we can see that

$$\begin{aligned} d_{G_3}(v') &= d_{R_1}(v') \pm d_{\mathcal{F}_2}(v') \\ &= (1 \pm \eta)\rho d_G(v'; V_1) \pm \gamma^{1/3} n = (1 \pm \rho)\rho d_G(v'; V_1) \\ &\geq (1 \pm \rho)\rho(1/2 + 2\alpha)|V_1| \geq (1/2 + \alpha)\rho\varepsilon n \end{aligned} \tag{2.29}$$

so $|V_{v'}| \geq \rho\varepsilon n/2$ and $|U_{v'}| \geq \rho\varepsilon n/4$ for all $v' \in V'$.

For each $v_1', v_2' \in V'$, $v_1' \neq v_2'$ we have

$$\begin{aligned} d_{H_3}(\{v_1', v_2'\}) &\leq d_{R_2}(\{v_1', v_2'\}) + d_{\mathcal{G}-\mathcal{F}_1}(\{v_1', v_2'\}) \\ &\overset{(R15)}{\leq} (1+\eta)\rho^2|U_1| + \gamma n \leq (1+2\varepsilon)\rho^2\varepsilon n \end{aligned}$$

and

$$\begin{aligned} d_{G_3}(\{v_1', v_2'\}) &\leq d_{R_1}(\{v_1', v_2'\}) + d_{\mathcal{G}-\mathcal{F}_1}(\{v_1', v_2'\}) \\ &\overset{(R9)}{\leq} (1+\eta)\rho^2|V_1| + \gamma n \leq (1+2\varepsilon)\rho^2\varepsilon n \end{aligned}$$

where we have used $|U_1|, |V_1| \leq \varepsilon n$.

On the other hand

$$d_{G_3}(v_1; V') \leq d_{R_1}(v_1; V') + d_{\mathcal{G}-\mathcal{F}_1}(v_1) \overset{(R2)}{\leq} (1+\eta)(1-\varepsilon)\rho n + \gamma n \leq \rho n$$

for every $v_1 \in V_1$ and

$$d_{H_3}(u_1; V') \leq d_{R_2}(u_1; V') + d_{\mathcal{G}-\mathcal{F}_1}(u_1) \overset{(R4)}{\leq} \rho n$$

for every $u_1 \in U_1$. Finally, we can see that for each $v' \in V'$ and $v_1 \in N_{G_3}(v')$

$$\begin{aligned} d_{G_3}(v_1; N_{H_3}(v')) &\geq |N_{R_2}(v') \cap N_H(v_1)| - d_{\mathcal{F}_2}(v') - d_{\mathcal{F}_2}(v_1) \\ &\overset{(R16)}{\geq} (1-5\varepsilon)\rho|U_1| - 2\gamma^{1/3}n \geq (1-\alpha)|N_{H_3}(v')| \end{aligned}$$

where we have used that $|N_{H_3}(v')| \leq (1+\eta)\rho|U_1|$ by (R3) and similarly, for each

$v' \in V'$ and $u_1 \in N_{H_3}(v')$

$$d_{H_3}(u_1; N_{G_3}(v')) \geq |N_{R_1}(v') \cap N_H(u_1)| - d_{\mathcal{F}_2}(v') - d_{\mathcal{F}_2}(u_1)$$
$$\overset{(R12)}{\geq} (1/2+\alpha)\rho|V_1| - 2\gamma^{1/3}n \geq (1/2+\alpha/2)|N_{G_3}(v')|$$

where we have used that $|N_{G_3}(v')| \leq (1+\eta)\rho|V_1|$ by (R1).

Hence we can apply Lemma 2.6.3 with $H_3[V_1, U_1]$, $\{V_{v'}\}_{v'\in V'}$, $\{U_{v'}\}_{v'\in V'}$, εn and $\alpha/2$ playing the role of $H[V,U]$, $\{V_i\}_{i\in[N]}$, $\{U_i\}_{i\in[N]}$, n and α respectively to find, for each $v' \in V'$, a 2-path factor $\mathcal{P}_{v'}$ of $H_3[V_{v'}, U_{v'}]$ so that all the factors $\{\mathcal{P}_{v'}\}_{v'\in V'}$ are edge-disjoint. We will now greedily extend the collection of paths $\{P : P \in \mathcal{P}_{v'}, v' \in V'\}$ into the desired collection of wheels $\{W : W \in F_{v'}, v' \in V'\}$.

Suppose that for some subset $S \subseteq V'$ we have already found for each $v' \in S$ a collection $F_{v'}$ of $d_{H_3}(v')$ wheels so that all the wheels $\{W \in F_{v'} : v' \in S\}$ are edge-disjoint and $\mathcal{F}_S := \bigcup_{W\in F_{v'}, v'\in S} W$ satisfies

$$\Delta(\mathcal{F}_S) \leq (\gamma'/2)n. \tag{2.30}$$

Similar to the proof of Claim 1 we say that a vertex $x \in V_1$ is *good* if there are less than $(\gamma'/8)n$ wheels in $\mathcal{F}_S$ containing x. Otherwise we say that it is *bad*. Let X be the set of bad vertices and suppose that $|X| > \gamma' n$. Then for each $x \in X$ there are at least $(\gamma'/8)n$ wheels in $\mathcal{F}_S$ containing x. It follows that

$$|E(\mathcal{F}_S)| \geq \frac{1}{2}\sum_{x\in X} d_{\mathcal{F}_S}(x) > \frac{1}{16}(\gamma' n)^2.$$

Let $D = \max_{v' \in V'} d_{H_3}(v') \leq 2\rho\varepsilon n$. On the other hand

$$|E(\mathcal{F}_S)| \leq 2r|V'|D \leq 4r\rho\varepsilon n^2$$

which is a contradiction since $\rho \ll \gamma'$. Hence

$$|X| \leq \gamma' n. \tag{2.31}$$

Now let $v' \in V' \setminus S$. We have

$$d_{H_3}(v') \leq d_{R_2}(v') + d_{\mathcal{G}-\mathcal{F}_1}(v') \overset{(R3)}{\leq} (1+\eta)\rho d_H(v'; U_1) + \gamma n \leq (1+2\varepsilon)\rho\varepsilon n$$

which implies $|\mathcal{P}_{v'}| = d_{H_3}(v') \leq 2\rho\varepsilon n$. Iteratively, for each path $v_1uv_2 \in \mathcal{P}_{v'}$ with $v_1, v_2 \in N_{G_3}(v')$ and $u \in N_{H_3}(v')$ we are going to find a path from v_1 to v_2 of length $r-2$ in $G_3[N_{H_3}(u; V_1)]$ made of good vertices and edge-disjoint from $\mathcal{F}_S$ and from the previously found paths. Let $P = vuv' \in \mathcal{P}_{v'}$ and let $\mathcal{P}$ be the set of paths of length $(r-2)$ that we have already found. Note that

$$|V(\mathcal{P})| \leq r|\mathcal{P}_{v'}| \leq 2r\rho\varepsilon n. \tag{2.32}$$

Let $Y := (V_1 \setminus N_{H_3}(u; V_1)) \cup N_{\mathcal{F}_S}(u; V_1)$. Since $d_{H_3}(u; V_1) \geq d_H(u; V_1) - \Delta(\mathcal{F}_2) \geq (1-2\varepsilon)|V_1| - \gamma^{1/3}n$ (where we have used that H is $2\varepsilon^2$-complete and equation (2.27)) then

$$|Y| \leq 2\varepsilon|V_1| + \gamma^{1/3}n + (\gamma'/2)n. \tag{2.33}$$

Let $G_P := (G_3[V_1] - \mathcal{F}_S) \setminus (Y \cup X \cup V(\mathcal{P}))$ and observe that $\delta(G_P) \geq \delta(G[V_1]) - \Delta(\mathcal{F}_2[V_1]) - \Delta(\mathcal{F}_3[V_1]) - \Delta(\mathcal{F}_S) - |Y| - |X| - |V(\mathcal{P})| \geq (1/2 + 2\alpha)|V_1| - 2\gamma^{1/3}n -$

$3\rho n - 2\gamma' n - 2\varepsilon|V_1| - 2r\rho\varepsilon n \geq (1/2+\alpha)|V_1|$ where we have used (C2), (2.27)-(2.28) and (2.30)-(2.33). Then we can greedily find the desired path from v_1 to v_2.

For each $P = v_1 u v_2 \in \mathcal{P}_{v'}$ let Q_P be the path of length $r-2$ from v_1 to v_2 found above. Note that by adding the edges $v_1 v'$ and $v' v_2$ to Q_P we obtain an r-cycle C_P. Since the vertices of Q_P were chosen disjoint from Y then all vertices in Q_P are adjacent to u via $H_3 - \mathcal{F}_S$. Let W_P be the wheel with cycle C_P and hub u.

Observe that the wheels $\{W_P\}_{P\in\mathcal{P}_{v'}}$ are pairwise edge-disjoint since the paths $\{Q_P\}_{P\in\mathcal{P}_{v'}}$ are chosen to be vertex-disjoint from the previously chosen paths. Moreover, all the paths $\{Q_P\}_{P\in\mathcal{P}_{v'}}$ are chosen to be edge-disjoint from $\mathcal{F}_S$ thus implying that all the wheels $\{W_P\}_{P\in\mathcal{P}_{v'}}$ are edge-disjoint from $\mathcal{F}_S$. Finally, let $F_{v'} := \{W_P\}_{P\in\mathcal{P}_{v'}}$, $S' := S \cup \{v'\}$ and $\mathcal{F}_{S'} := \mathcal{F}_S \cup \left(\bigcup_{W\in F_{v'}} W\right)$. We would now apply again the above arguments with S' playing the role of S.

At the end of the procedure we would have found, for each $v' \in V'$, a set of $d_{H_3}(v')$ wheels $F_{v'}$ containing v' such that all the wheels $\{W : W \in F_{v'}, v' \in V'\}$ are edge-disjoint. Let $\mathcal{F}_4 := \bigcup_{W\in F_{v'}, v'\in V'} W$. Each vertex $v' \in V'$ would be contained in exactly $d_{H_3}(v')$ wheels implying that $d_{\mathcal{F}_4}(v') \leq r d_{H_3}(v') \leq 2r\rho\varepsilon n \leq (\gamma'/2)n$. On the other hand, each vertex $u_1 \in U_1$ would be contained in exactly $d_{H_3}(u_1; V')$ wheels implying that $d_{\mathcal{F}_4}(u) \leq r d_{H_3}(u_1; V') \leq r\rho n \leq (\gamma'/2)n$. Finally, each vertex $v_1 \in V_1$ would be contained in exactly $d_{G_3}(v_1; V') + g(v_1)$ wheels where $g(v_1)$ denotes the number of times the vertex v_1 is chosen when finding the paths $\{Q_P : P \in \mathcal{P}_{v'}, v' \in V'\}$. Since the paths $\{Q_P : P \in \mathcal{P}_{v'}, v' \in V'\}$ were all found using only good vertices then $g(v_1) \leq (\gamma'/8)n$. On the other hand we have $d_{G_3}(v_1; V') \leq \rho n$ so $d_{\mathcal{F}_4}(v_1) \leq 3\rho n + 3(\gamma'/8)n \leq (\gamma'/2)n$. All in all we would have that $\Delta(\mathcal{F}_4) \leq (\gamma'/2)n$. Thus, (2.30) would actually hold for each $S \subseteq V'$ which ensures that we can apply the whole procedure until finding the desired collection of wheels $\mathcal{F}_4$.

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Let $\mathcal{F}_4$ be the graph obtained by applying Claim 3 and let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. Note that $E(\mathcal{G} - \mathcal{F}) \subseteq E(\mathcal{G}[V_1 \cup U_1])$. Since $\mathcal{F}_1[V_1 \cup U_1]$ has no edges, $\Delta(\mathcal{F}_2) + \Delta(\mathcal{F}_3) \leq (\gamma'/2)n$ by Claims 1 and 2 and $\Delta(\mathcal{F}_4) \leq (\gamma'/2)n$ by Claim 3, we have $\Delta(\mathcal{F}[V_1 \cup U_1]) \leq \gamma' n$ which concludes the proof. □

2.8 Final absorbers

In this section we construct the absorbers that will be used to cover the final remainder during the main proof. The construction is based on a sequence of 'transformers', a type of gadgets that allow to transform a given leftover into a 'new' leftover. The idea is to combine several transformers to end up with a leftover which is decomposable by itself. This method was introduced by Barber, Kühn, Lo and Osthus [9] and further developed in [10].

Given an extended graph $\mathcal{G} = (V, U, G, H)$, $r \in \mathbb{N}$ and a subgraph $L \subseteq \mathcal{G}$, an *absorber for* L is a graph $A_L \subseteq \mathcal{G}$ such that $V(L) \subseteq V(A_L)$, $A_L[V(L)]$ is independent (has no edges), and both A_L and $A_L \cup L$ have a W_{r+1}-decomposition in $\mathcal{G}$. The following result allows us to find an absorber A_L in $\mathcal{G}$ for any given W_{r+1}-divisible extended subgraph $L \subseteq \mathcal{G}$ of bounded size.

Lemma 2.8.1. *Let $1/n \ll 1/M \ll 1/m \ll \varepsilon \ll 1/r \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $|V| = n$, $\delta(G) \geq (1/2 + \alpha)n$ and H is ε-complete. Let $L \subseteq \mathcal{G}$ be an extended W_{r+1}-divisible graph where $L = (V_L, U_L, G_L, H_L)$ and $|V(L)| \leq m$. Then there is an absorber $A_L \subseteq \mathcal{G}$ for L such that $|V(A_L)| \leq M$.*

Before proving Lemma 2.8.1 we will first describe an explicit construction of an absorber A_L for $L \subseteq \mathcal{G}$. To make things easier, we will describe the construction of A_L inside a non-arbitrary extended graph $\mathcal{G}$, i.e., we may assume that any edge or subgraph required in the construction exists in $\mathcal{G}$. The construction of A_L will consist of the union of several graphs, which we call *transformers*, which will act as 'partial absorbers'. The construction of a single transformer is completed with equation (2.34) while the construction of A_L ends with equation (2.35). Once we have finished the construction, the proof of Lemma 2.8.1 will only consist of showing that a copy of A_L can be found in any given extended graph satisfying the conditions of the statement. We will make use of Lemma 2.8.5 which finds a copy of a single transformer.

Let $\mathcal{G} = (V, U, G, H)$ be a fixed extended graph. Given two vertex-disjoint extended subgraphs $L, L' \subseteq \mathcal{G}$ we say that *L' is obtained from L by identifying vertices* if there is a sequence of extended subgraphs $L = L_0, \ldots, L_\ell$ of $\mathcal{G}$ such that there is an isomorphism $f : L_\ell \to L'$ where $f(v) \in V$ if and only if $v \in V$ and for each $0 \leq i < \ell$ there are vertices $x_i, y_i \in V(L_i)$ satisfying that

(i) $x_i, y_i \in V$ or $x_i, y_i \in U$,

(ii) $N_{L_i}(x_i) \cap N_{L_i}(y_i) = \emptyset$,

(iii) L_{i+1} is the graph obtained from L_i by deleting the vertex y_i and adding the edges $\{x_i z \, : \, y_i z \in E(L_i)\}$.

Note that if L' is obtained from L by identifying vertices then there exists a graph homomorphism $\phi : L \to L'$ which is edge-bijective and satisfies that $\phi(v) \in V$ if and only if $v \in V$. An (L, L')-*transformer* is an extended subgraph

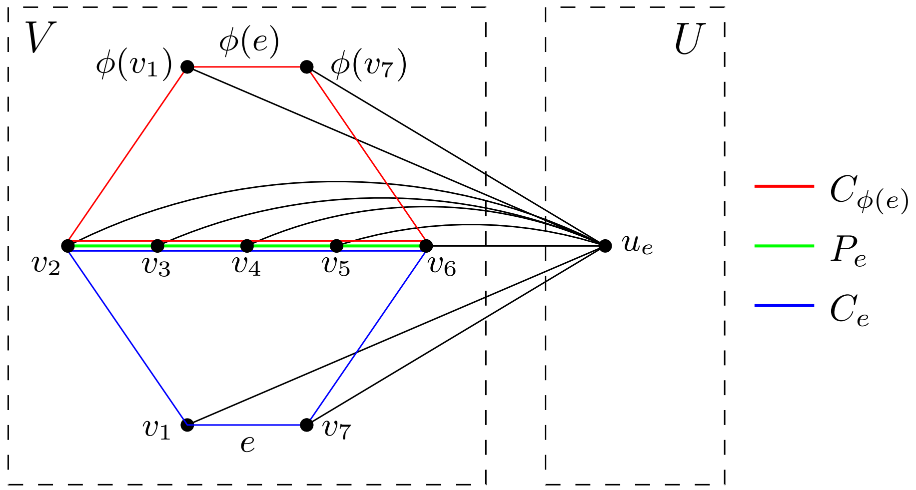

Figure 2.2: Illustration of $P_e, C_e, C_{\phi(e)}$ for the case $e \in E(G_L)$ and $r = 7$.

$T \subseteq \mathcal{G}$ edge-disjoint from L and L' such that both $L \cup T$ and $L' \cup T$ have a W_{r+1}-decomposition. Our absorber A_L will be made of the union of several transformers. Roughly speaking, each transformer acts as a 'partial' absorber that allows us to transform the leftover L into a new leftover L'. Our aim will be to reach a final leftover J that will be W_{r+1}-decomposable.

Let $\mathcal{G} = (V, U, G, H)$ be a fixed extended graph, let $L = (V_L, U_L, G_L, H_L)$ be a W_{r+1}-divisible extended subgraph of $\mathcal{G}$ and let $L' \subseteq \mathcal{G}$ be obtained from L by identifying vertices. Let $\phi : L \to L'$ be an edge-bijective homomorphism such that $\phi(v) \in V$ if and only if $v \in V$. For each $e = v_1 v_r \in E(G_L)$ let $P_e = (v_2, \ldots, v_{r-1})$ be an $(r-2)$-path of G where v_2 is adjacent to v_1 and $\phi(v_1)$, and v_{r-1} is adjacent to v_r and $\phi(v_r)$, and let u_e be a vertex in U adjacent to all of $v_1, \ldots, v_r, \phi(v_1), \phi(v_r)$. Note that $C_e := (v_1, \ldots, v_r)$ and $C_{\phi(e)} := (\phi(v_1), v_2, \ldots v_{r-1}, \phi(v_r))$ are r-cycles of G and let W_e and $W_{\phi(e)}$ be the $(r+1)$-wheels of $\mathcal{G}$ with hub u_e and cycles C_e and $C_{\phi(e)}$ respectively. For each $e = v_1 u \in E(H_L)$ with $v_1 \in V$ and $u \in U$ let $P_e = (v_2, \ldots, v_r)$ be an $(r-1)$-path of G where v_2 and v_r are adjacent to v_1

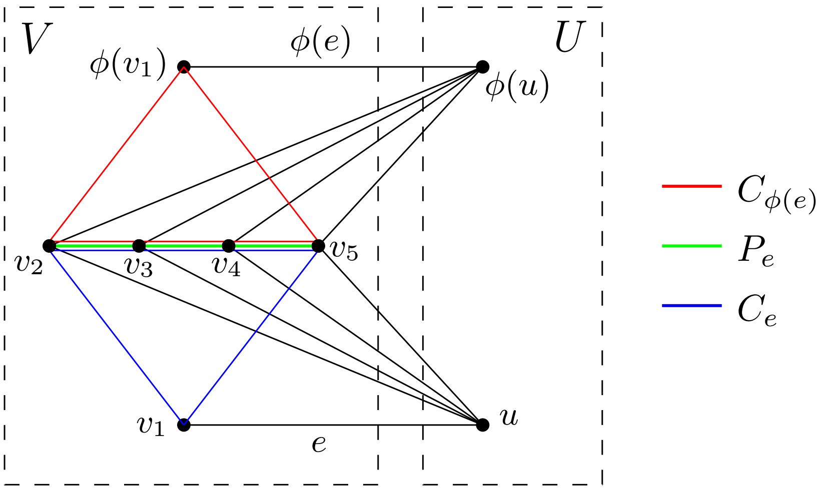

Figure 2.3: Illustration of $P_e, C_e, C_{\phi(e)}$ for the case $e \in E(H_L)$ and $r = 5$.

and $\phi(v_1)$, and u is adjacent to all of $v_2, \ldots, v_r$. Note that $C_e := (v_1, \ldots, v_r)$ and $C_{\phi(e)} := (\phi(v_1), v_2, \ldots v_r)$ are r-cycles of G and let W_e and $W_{\phi(e)}$ be the $(r+1)$-wheels of $\mathcal{G}$ with hub u and cycles C_e and $C_{\phi(e)}$ respectively. Let all the paths $\{P_e\}_{e\in E(L)}$ and all the vertices $\{u_e\}_{e\in E(L)\cap E(G)}$ be vertex-disjoint. Let $\mathcal{W}_L := \bigcup_{e\in E(L)} W_e$ and $\mathcal{W}_{L'} := \bigcup_{e\in E(L)} W_{\phi(e)}$ and note that both unions are edge-disjoint so $\mathcal{W}_L$ and $\mathcal{W}_{L'}$ are trivially W_{r+1}-decomposable.

Let $v \in V_L$ be a fixed vertex. Let X_v be the set of vertices $\{x \in P_e : vx \in C_e, v \in e \in E(L)\}$ and let Y_v be the set of vertices $\{u_e : v \in e \in E(G_L)\}$. Note that $|X_v| = d_{G_L}(v) + 2d_{H_L(v)}$ and $|Y_v| = d_{G_L}(v)$. Since L is W_{r+1}-divisible we have that $d_{G_L(v)} = 2d_{H_L(v)}$, thus $|X_v| = 2|Y_v|$. Let $x_1, \ldots, x_{2\ell}$ and $y_1, \ldots, y_\ell$ be an enumeration of the vertices in X_v and Y_v satisfying that for each $i \in [\ell]$ neither $x_i y_i$ nor $x_{\ell+i} y_i$ are edges of $\mathcal{W}_L$. For each $i \in [\ell]$ let P_i be an $(r-1)$-path starting at x_i and ending at $x_{\ell+i}$ so that all the paths $\{P_i\}_{i\in[\ell]}$ are vertex-disjoint. Let C_i be the r-cycle obtained by adding the edges $vx_i, vx_{\ell+i}$ to P_i and let W_i be the $(r+1)$-wheel with cycle C_i and hub y_i. Let $\mathcal{W}_v := \bigcup_{i\in m} W_i$. Analogously, let C_i'

be the r-cycle obtained by adding the edges $\phi(v)x_i, \phi(v)x_{\ell+i}$ to P_i, let W_i' be the $(r+1)$-wheel with cycle C_i' and hub y_i and let $\mathcal{W}_{\phi(v)} := \bigcup_{i \in \ell} W_i'$. Note that $\mathcal{W}_v$ and $\mathcal{W}_{\phi(v)}$ are both trivially W_{r+1}-decomposable.

Let $\{\mathcal{W}_v\}_{v \in V_L}$ be all edge-disjoint extended subgraphs. Then, by construction, $\{\mathcal{W}_{\phi(v)}\}_{v \in V_L}$ are also edge-disjoint. Observe that $\mathcal{W}_L$ is edge-disjoint from all of $\{\mathcal{W}_{\phi(v)}\}_{v \in V_L}$, and similarly, $\mathcal{W}_{L'}$ is edge-disjoint from all of $\{\mathcal{W}_v\}_{v \in V_L}$.

Let $s \in \mathbb{N}$ with $r \mid s$.

Now let $u \in U_L$ be fixed. Let X_u be the set of vertices $\{x \in P_e : u \in e \in E(H_L)\}$. Let $S_u \subseteq V$ be a set of vertices of size s and note that since L is W_{r+1}-divisible we have $r \mid d_{H_L}(u)$ so $r \mid (r-1)d_{H_L}(u) = |X_u|$. Let $\mathcal{C}_1$ be an C_r-factor of $G[S_u \cup X_u]$ such that X_u is independent in $\mathcal{C}_1$ and let $\mathcal{C}_2$ be an C_r-factor of $G[S_u]$ edge-disjoint from $\mathcal{C}_1$. For each r-cycle C in $\mathcal{C}_1$ or $\mathcal{C}_2$ let W_C and W_C' be the $(r+1)$-wheels with cycle C and hub u and $\phi(u)$ respectively. Let $\mathcal{W}_u^1 := \bigcup_{C \in \mathcal{C}_1} W_C$, $\mathcal{W}_u^2 := \bigcup_{C \in \mathcal{C}_2} W_C$, $\mathcal{W}_{\phi(u)}^1 := \bigcup_{C \in \mathcal{C}_1} W_C'$ and $\mathcal{W}_{\phi(u)}^2 := \bigcup_{C \in \mathcal{C}_2} W_C'$. Note that $\mathcal{W}_u^1, \mathcal{W}_u^2, \mathcal{W}_{\phi(u)}^1$ and $\mathcal{W}_{\phi(u)}^2$ are trivially W_{r+1}-decomposable.

Let $\{S_u\}_{u \in U_L}$ be pairwise disjoint sets. Finally let

$$T = \left(\mathcal{W}_L \cup \mathcal{W}_{L'} \cup \left(\bigcup_{v \in V_L} \mathcal{W}_v \cup \mathcal{W}_{\phi(v)} \right) \cup \left(\bigcup_{u \in U_L} \mathcal{W}_u^1 \cup \mathcal{W}_u^2 \cup \mathcal{W}_{\phi(u)}^1 \cup \mathcal{W}_{\phi(u)}^2 \right) \right) - L - L'. \tag{2.34}$$

Proposition 2.8.2. *T is an (L, L')-transformer.*

Proof. Note that $E(L) \subseteq E(\mathcal{W}_L)$ and $E(L') \subseteq E(\mathcal{W}_{L'})$. By construction we have that

$$T \cup L = \mathcal{W}_L \cup \left(\bigcup_{v \in V_L} \mathcal{W}_{\phi(v)} \right) \cup \left(\bigcup_{u \in U_L} \mathcal{W}_{\phi(u)}^1 \right) \cup \left(\bigcup_{u \in U_L} \mathcal{W}_u^2 \right)$$

where all the unions are edge-disjoint. Hence, $T \cup L$ is W_{r+1}-decomposable. Similarly we have that

$$T \cup L' = \mathcal{W}_{L'} \cup \left(\bigcup_{v \in V_L} \mathcal{W}_v \right) \cup \left(\bigcup_{u \in U_L} \mathcal{W}_u^1 \right) \cup \left(\bigcup_{u \in U_L} \mathcal{W}_{\phi(u)}^2 \right)$$

where the unions are edge-disjoint and thus $T \cup L'$ is W_{r+1}-decomposable. □

Proposition 2.8.3. $|V(T)| \leq 2(s+1)|V(L)| + 5r|V(L)|^2$.

Proof. First of all observe that $V(\mathcal{W}_L) \setminus V(L) = V(\mathcal{W}_{L'} \setminus V(L'))$, $\bigcup_{v \in V_L} V(\mathcal{W}_v) \setminus V(L) = \bigcup_{v \in V_L} V(\mathcal{W}_{\phi(v)}) \setminus V(L')$, $\bigcup_{u \in U_L} V(\mathcal{W}_u^1) \setminus V(L) = \bigcup_{u \in U_L} V(\mathcal{W}_{\phi(u)}^1) \setminus V(L')$, and $V(\mathcal{W}_u^2) \subseteq V(\mathcal{W}_u^1)$ and $V(\mathcal{W}_{\phi(u)}^2) \subseteq V(\mathcal{W}_{\phi(u)}^1)$ for every $u \in U_L$. By construction we have

$$|V(\mathcal{W}_L)| \leq (r+1)|E(L)| \leq 2r|V(L)|^2,$$

$$\left| \bigcup_{v \in V_L} V(\mathcal{W}_v) \right| \leq |V(L)|(r+1)d_{G_L}(v) \leq 2r|V(L)|^2$$

and

$$\left| \bigcup_{u \in U_L} V(\mathcal{W}_u^1) \right| \leq |V(L)|(r+1)\frac{s + (r-1)d_{H_L}(u)}{r} \leq 2s|V(L)| + r|V(L)|^2.$$

Hence,

$$\begin{aligned} |V(T)| &\leq |V(L)| + |V(L')| + |V(\mathcal{W}_L)| + \left| \bigcup_{v \in V_L} V(\mathcal{W}_v) \right| + \left| \bigcup_{u \in U_L} V(\mathcal{W}_u^1) \right| \\ &\leq 2(s+1)|V(L)| + 5r|V(L)|^2. \end{aligned}$$

□

An (L, L')-transformer allows us to transform a leftover L into a new leftover L' if L' is obtained from L by identifying vertices or vice-versa. We will now describe a sequence of graphs starting at L and ending in a W_{r+1}-decomposable graph J such that each pair of consecutive graphs will admit a transformation. By combining all such graphs and transformers we will obtain the desired absorber A_L for L.

Let $\mathcal{G}' = (V', U', G', H')$ be an extended subgraph of $\mathcal{G}$. Given an edge $v_1v_2 \in E(G')$, an *expansion of* v_1v_2 is the graph obtained by the union of G' and a vertex-disjoint wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$ by deleting an edge $v_1'v_2' \in E(W)$ with $v_1', v_2' \in V$ and the edge v_1v_2, and adding the edges v_1v_2' and v_2v_1'. Given an edge $v_1u_2 \in E(H')$ with $v_1 \in V'$ and $u_2 \in U'$, *an expansion of* v_1u_2 is the graph obtained from the union of G' and a vertex-disjoint wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$ by deleting an edge $v_1'u_2' \in E(W)$ with $v_1' \in V$ and $u_2' \in U$ and the edge v_1u_2, and adding the edges v_1u_2' and u_2v_1'. Observe that in both cases identifying the vertices v_1 and v_1' results in a graph obtained from G by attaching a wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$ into v_1. Let $\mathcal{G}'_{\text{exp}}$ be the graph obtained from $\mathcal{G}'$ by expanding every one of its edges. Let $\mathcal{G}'_{\text{att}}$ be the graph obtained from $\mathcal{G}'$ by, for each $e \in E(\mathcal{G}')$, choosing a vertex $v \in e \cap V$ and attaching an edge-disjoint wheel $W \in \mathcal{W}_{r+1}(\mathcal{G})$ to v. Note that $\mathcal{G}'_{\text{att}}$ is obtained from $\mathcal{G}'_{\text{exp}}$ by identifying vertices. Note also that $|V(\mathcal{G}'_{\text{att}})| = |V(\mathcal{G}')| + r|E(\mathcal{G}')|$ and $|V(\mathcal{G}'_{\text{exp}})| = |V(\mathcal{G}')| + (r+1)|E(\mathcal{G}')|$. Finally, given $v_0 \in V$ and $u_0 \in U$ let $\Gamma_{\mathcal{G}'}$ be the graph obtained by, consecutively, adding and expanding the loop v_0v_0 $|E(G')|$ times and adding and expanding the edge v_0u_0 $|E(H')|$ times. It follows that $\Gamma_{\mathcal{G}'}$ is obtained from $\mathcal{G}'_{\text{exp}}$ by identifying all the vertices in V' into v_0 and all the vertices in U' into u_0.

We have now all the ingredients to describe our absorber A_L. Let $L \subseteq \mathcal{G}$ be

W_{r+1}-divisible. Note that since L is W_{r+1}-divisible then $|E(L) \cap E(G)| = |E(L) \cap E(H)| =: m_L$ and $r \mid m_L$. Let J be the edge-disjoint union of m_L/r wheels of $\mathcal{W}_{r+1}(\mathcal{G})$. Then $|E(J) \cap E(G)| = |E(J) \cap E(H)| = m_L$. Let $L_{\text{att}}, L'_{\text{exp}}, \Gamma_L, J'_{\text{exp}}, J_{\text{att}} \subseteq \mathcal{G}$ be all pairwise edge-disjoint where L'_{exp} and J'_{exp} are copies of L_{exp} and J_{exp} respectively. Then we have the following relation

$$L_{\text{att}} \longleftarrow L'_{\text{exp}} \longrightarrow \Gamma_L \longleftarrow J'_{\text{exp}} \longrightarrow J_{\text{att}}$$

where $A \to B$ indicates that B is obtained from A by identifying vertices. Suppose that we have found edge-disjoint transformers T_1, T_2, T_3, T_4 satisfying

$$L_{\text{att}} \xleftarrow{T_1} L'_{\text{exp}} \xrightarrow{T_2} \Gamma_L \xleftarrow{T_3} J'_{\text{exp}} \xrightarrow{T_4} J_{\text{att}}$$

where $A \xrightarrow{T} B$ denotes that T is an (A, B)-transformer. Let

$$A_L = (L_{\text{att}} - L) \cup T_1 \cup L'_{\text{exp}} \cup T_2 \cup \Gamma_L \cup T_3 \cup J'_{\text{exp}} \cup T_4 \cup J_{\text{att}}. \tag{2.35}$$

Proposition 2.8.4. *A_L is an absorber for L.*

Proof. First note that that $V(L)$ is an independent set in $L_{\text{att}} - L$ and that $L_{\text{att}} - L$ is trivially W_{r+1}-decomposable as it consist of the edge-disjoint union of $|E(L)|$ wheels. Recall that if T is an (A, B)-transformer then both $A \cup T$ and $T \cup B$ have a W_{r+1}-decomposition.

Secondly, observe that A_L is the union of $(L_{\text{att}} - L)$, $T_1 \cup L'_{\text{exp}}$, $T_2 \cup \Gamma_L$, $T_3 \cup J'_{\text{exp}}$ and $T_4 \cup J_{\text{att}}$ which are all pairwise edge-disjoint W_{r+1}-decomposable extended graphs. Hence A_L is W_{r+1}-decomposable.

Finally, note that J_{att} is W_{r+1}-decomposable by construction. Then $A_L \cup L$ is the edge-disjoint union of $L_{\text{att}} \cup T_1$, $L'_{\text{exp}} \cup T_2$, $\Gamma_L \cup T_3$, $J'_{\text{exp}} \cup T_4$ and J_{att} which are all W_{r+1}-decomposable extended graphs. Thus, $A_L \cup L$ is also W_{r+1}-decomposable so A_L is an absorber for L. □

The following lemma shows the existence of a single transformer in an extended graph satisfying the conditions of Lemma 2.8.1.

Lemma 2.8.5. *Let $1/n \ll 1/s \ll 1/m \ll \varepsilon \ll 1/r \ll \alpha$ with $r \mid s$. Let $\mathcal{G} = (V, U, G, H)$ be an extended graph where $|V| = n$, $\delta(G) \geq (1/2 + \alpha)n$ and H is ε-complete. Let $L \subseteq \mathcal{G}$ be an extended W_{r+1}-divisible subgraph with $|V(L)| \leq m$ and let $L' \subseteq \mathcal{G}$ be obtained from L by identifying vertices. Then there is an (L, L')-transformer $T \subseteq \mathcal{G}$ with $|V(T)| \leq 2s^2$.*

Proof. Let $L = (V_L, U_L, G_L, H_L)$ and let $\phi : L \to L'$ be an edge-bijective homomorphism satisfying $\phi(v) \in V$ if and only if $v \in V$. We will find a copy of the (L, L')-transformer T described above. We will find it vertex by vertex and will refer as *new vertices* to the vertices that are not in $V(L \cup L')$ and that have not previously been chosen as vertices of T. Note that at any time at most s^2 vertices are not considered new vertices since $|V(T)| \leq s^2$ by Proposition 2.8.3 and the hierarchy of constants. In addition, observe that since $\delta(G) \geq (1/2 + \alpha)n$ the common neighbourhood of any two vertices in G is at least $2\alpha n$.

We will first find $\mathcal{W}_L$, $\mathcal{W}_{L'}$, $\{\mathcal{W}_v\}_{v \in V_L}$ and $\{\mathcal{W}_{\phi(v)}\}_{v \in V_L}$. For each $e = v_1 v_r \in E(G_L)$, we pick new vertices $v_2 \in N_G(\{v_1, \phi(v_1)\})$ and $v_{r-1} \in N_G(\{v_r, \phi(v_r)\})$. Next we find, using only new vertices, the $(r-2)$-path $P_e = (v_2, \dots, v_{r-1})$ in G. This can be done greedily since $2\alpha n - s^2 > 0$. For each $e = v_1 u \in E(H_L)$ with $v_1 \in V$ and $u \in U$ we find the r-cycle $C_e = (v_1, \dots, v_r)$ in G where $v_2, \dots, v_r$ are new vertices adjacent to u. We can find the cycles $\{C_e\}_{e \in E(H_L)}$ greedily since

$2\alpha n - s^2 - \varepsilon n > 0$. For each $v \in V_L$ let $d_v := d_{G_L}(v)$ and let $x_1^v, \dots, x_{d_v}^v$ be an enumeration of the vertices in $\{x \in P_e : vx \in C_e, v \in e \in E(G_L)\}$ and let $x_{d_v+1}^v, \dots, x_{2d_v}^v$ be an enumeration of the vertices in $\{x \in P_e : vx \in C_e, v \in e \in E(H_L)\}$. Note that $X_v = \{x_1^v, \dots, x_{2d_v}^v\}$. Now, for each $e = v_1v_2 \in E(G_L)$ let i_1 and i_2 be the indices such that $x_{i_1}^{v_1}, x_{i_2}^{v_2} \in V(C_e)$ and let $j_1 = i_1 \pmod{d_{v_1}} + 1$ and $j_2 = i_2 \pmod{d_{v_2}} + 1$. We pick a new vertex $u_e \in U$ that is adjacent to all of $v_1, v_2, \phi(v_1), \phi(v_2), V(P_e), x_{j_1}^{v_1}, x_{j_1+d_{v_1}}^{v_1}, x_{j_2}^{v_2}$ and $x_{j_2+d_{v_2}}^{v_2}$. This can be done greedily since $|U| - s^2 - (r+6)\varepsilon n > 0$. With this procedure we guarantee two things. First that for each $e \in E(L)$, $C_e \cup \{u_e\}$ and $C_{\phi(e)} \cup \{u_e\}$ are $(r+1)$-wheels which completes the embedding of $\mathcal{W}_L$ and $\mathcal{W}_{L'}$. And second that for each $v \in V_L$ there exists an enumeration $y_1, \dots, y_{d_v}$ of the vertices in Y_v such that for each $i \in [d_v]$ y_i is adjacent to x_i^v and $x_{i+d_v}^v$ and neither of vx_i^v nor $vx_{i+d_v}^v$ are edges of $\mathcal{W}_L$. Thus, for each $v \in V_L$ and $i \in d_v$, we will continue by finding an $(r-1)$-path P_i^v from x_i^v to $x_{i+d_v}^v$ in G made of new vertices (except from x_i^v and $x_{i+d_v}^v$) whose vertices are all adjacent to y_i. This can again be done greedily since $2\alpha n - s^2 - \varepsilon n > 0$. Finding these paths completes the embedding of $\{\mathcal{W}_v\}_{v \in V_L}$ and $\{\mathcal{W}_{\phi(v)}\}_{v \in V_L}$.

It only remains to find, for each $u \in U_L$, copies of $\mathcal{W}_u^1$, $\mathcal{W}_u^2$, $\mathcal{W}_{\phi(u)}^1$ and $\mathcal{W}_{\phi(u)}^2$. Fix $u \in U_L$ and let $N_u := N_H(u) \setminus X_u$ be the set of new vertices in V that are adjacent to u. Recall that $|X_u| = (r-1)d_{H_L}(u) \leq rm$. Let $S_u \subseteq N_u$ be a set of new vertices of size s chosen at random with uniform probability. Observe that $|N_u| \geq n - rm - \varepsilon n$ so for any $v \in V$ we have $d_G(v; N_u) \geq (1/2 + \alpha)n - rm - \varepsilon n \geq (1/2 + \alpha/2)n$. Then for every $v \in V$ we have

$$\mathbb{E}\left[d_G(v; S_u)\right] \geq (1/2 + \alpha/2)s.$$

For each $v \in V$ let A_v be the event that $d_G(v; S_u) \leq (1-\varepsilon)\mathbb{E}[d_G(v; S_u)]$. Lemma 2.1.2 implies that for any $v \in V$

$$\mathbb{P}[A_v] \leq 2\mathrm{e}^{-\varepsilon^2 \mathbb{E}[d_G(v;S_u)]/3} \leq 2\mathrm{e}^{-\varepsilon^2 s/6} \leq \mathrm{e}^{-s^{1/2}}.$$

Let A be the event that $d_G(v; S_u) \leq (1-\varepsilon)\mathbb{E}[d_G(v; S_u)]$ for some $v \in S_u \cup X_u$. Hence

$$\begin{aligned}\mathbb{P}[A] &\leq \sum_{v \in N_u} \mathbb{P}[v \in S_u \text{ and } A_v] + \sum_{v \in X_u} \mathbb{P}[A_v] \leq |N_u| \frac{s}{|N_u|} \mathrm{e}^{-s^{1/2}} + |X_u| \mathrm{e}^{-s^{1/2}} \\ &\leq (s + rm)\mathrm{e}^{-s^{1/2}} < 1\end{aligned}$$

which implies that A does not hold with positive probability. Thus, choose $S_u \subseteq N_u$ to be a set of new vertices satisfying that $d_G(v; S_u) \geq (1-\varepsilon)(1/2 + \alpha/2)s \geq (1/2 + \alpha/4)s$ for every $v \in S_u \cup X_u$. Since $\delta(G[S_u]) \geq (1/2 + \alpha/4)|S_u|$ we know by Theorem 2.6.1 that there exists a C_r-factor $\mathcal{C}_2$ of $G[S_u]$. This gives us the embeddings of $\mathcal{W}_u^2$ and $\mathcal{W}_{\phi(u)}^2$. Now let $G_u := G[S_u \cup X_u] - \mathcal{C}_2 - G[X_u]$. Observe that for any $v \in S_u \cup X_u$ we have

$$\begin{aligned}d_{G_u}(v; S_u \cup X_u) \geq d_{G_u}(v; S_u) \geq d_G(v; S_u) - \Delta(\mathcal{C}_2) &\geq (1/2 + \alpha/4)s - 2 \\ &\geq (1/2 + \alpha/8)s + (\alpha/8)s - 2 - (1/2 + \alpha/8)rm + (1/2 + \alpha/8)rm \\ &\geq (1/2 + \alpha/8)(s + rm) \geq (1/2 + \alpha/8)|S_u \cup X_u|\end{aligned}$$

where we have used that $(\alpha/8)s - 2 - (1/2 + \alpha/8)rm > 0$. We can then apply again Theorem 2.6.1 to find a C_r-factor $\mathcal{C}_1$ of G_u. It follows that $\mathcal{C}_1$ is a C_r-factor of $G[S_u \cup X_u]$ edge-disjoint from $\mathcal{C}_2$ and in which X_u form an independent set, which gives us copies of $\mathcal{W}_u^1$ and $\mathcal{W}_{\phi(u)}^1$.

This completes the embedding of T. Propositions 2.8.3 and 2.8.2 finish the proof. □

We are finally ready to prove Lemma 2.8.1.

Proof of Lemma 2.8.1. Let $s, s' \in \mathbb{N}$ such that $1/M \ll 1/s' \ll 1/s \ll 1/m$.

The proof consists of finding an embedding in $\mathcal{G}$ of the extended graph A_L described above which by Proposition 2.8.4 is an absorber for L. We will first find the extended graphs L_{att}, L'_{exp}, Γ_L, J'_{exp} and J_{att} in $\mathcal{G}$ and later find the transformers T_1, T_2, T_3 and T_4 by applying Lemma 2.8.5.

Recall that since $\delta(G) \geq (1/2 + \alpha)n$ and H is ε-complete then the common neighbourhood into $V \setminus V'$ of any two vertices $v_1, v_2 \in V$ and any vertex $u \in U$ is at least $2\alpha n - \varepsilon n - |V'| > \alpha n$ if $|V'| \leq s$. This allows us to find an $(r+1)$-wheel containing any given vertex in V and to expand any given edge $e \in E(\mathcal{G})$ even if we are not allowed to use a given set of vertices V' of size at most s. We will use this argument to find, vertex by vertex, the extended subgraphs L_{att}, L'_{exp}, Γ_L, J'_{exp} and J_{att} in $\mathcal{G}$. We shall then refer as *new vertices* to the vertices in $V \cup U$ that have not previously been used in the embedding of the absorber A_L. Since the size of the graphs L_{att}, L'_{exp}, Γ_L, J'_{exp} and J_{att} are all bounded polynomially by $V(L)$ and $E(L)$, and $|V(L)| \leq m$ and $|E(L)| \leq 2^m$, we can assume that at any point of the argument the number of used vertices is always bounded by s, allowing us to greedily find wheels or expand edges using only new vertices.

We start by finding an embedding of J in $\mathcal{G}$ using only new vertices. This can be trivially done since J consists of the union of $|E(L)|/(2r)$ edge-disjoint $(r+1)$-wheels. We can then, for each edge $e \in E(L) \cup E(J)$, choose a vertex $v \in e \cap V$ and find an $(r+1)$-wheel that contains v and uses only new vertices. This allows us to extend L and J into L_{att} and J_{att}. Let L' and J' be copies of L and J on the set of new vertices. L' and J' are not necessarily extended subgraphs of $\mathcal{G}$. However, we can greedily expand every edge of L' and J' to find vertex-disjoint copies of L'_{exp} and J'_{exp} in $\mathcal{G}$ using only new vertices. To find Γ_L we just need to pick new vertices $v_0 \in V$ and $u_0 \in U$ and iteratively add and expand the loop v_0v_0 and the edge v_0u_0 m_L times using only new vertices. We have then found vertex-disjoint embeddings of L_{att}, L'_{exp}, Γ_L, J'_{exp} and J_{att} in $\mathcal{G}$.

It only remains to find the transformers $T_1, \dots, T_4$. Let $i \in [4]$ and suppose that we have already found edge-disjoint copies of size at most s' of the transformers $T_1, \dots, T_{i-1}$. Let E' be the set of edges in $\mathcal{G}$ that have already been used in the embedding of A_L. We have $|E'| \leq 2^{10s'}$ with plenty of room to spare. Let $\mathcal{G}' = \mathcal{G} - \mathcal{G}[E']$ with $\mathcal{G}' = (V, U, G', H')$ and observe that $\delta(G') \geq (1/2 + \alpha/2)n$ and H' is $(\varepsilon/2)$-complete. We can then apply Lemma 2.8.5 (with $\mathcal{G}'$, $\alpha/2$, $\varepsilon/2$ and s' playing the role of $\mathcal{G}$, α, ε and s^2 respectively) to find a transformer $T_i \subseteq \mathcal{G}'$ with $|V(T_i)| \leq s'$. Thus, by induction, we can find edge-disjoint transformers $T_1, \dots, T_4$ of size at most s' which completes the embedding of A_L. Finally, since $s, s' \ll M$ then $|V(A_L)| \leq M$ as desired.

2.9 Proof of Theorem 2.2.3

We have now all the ingredients to prove the main theorem of this chapter which we restate below. Recall that the proof consist of three main steps. In the first step we find a vortex sequence $V_0 \supseteq \cdots \supseteq V_\ell$ where $V_0 = V(G)$ and V_ℓ has constant size and find and set aside a final absorber for all possible configurations of a leftover in $G[V_\ell]$. In the second step we iteratively apply the Cover Down Lemma to cover, at the i-th iteration, all edges in $G - G[V_i]$ to end up with a final leftover in $G[V_\ell]$. The final step consists of absorbing the final leftover using the absorbers that were set aside at the beginning. We now restate the main theorem.

Theorem 2.2.3. Let $1/n \ll 1/r \ll \alpha$. Let $\mathcal{G} = (V, U, G, H)$ be an extended W_{r+1}-divisible graph where $|V| = n$ and H is a complete bipartite graph. If G is d-regular with $d \geq (1/2 + \alpha)n$ then $\mathcal{G}$ has a W_{r+1}-decomposition.

Proof. Let $\gamma, \eta, \varepsilon > 0$ and $M, m' \in \mathbb{N}$ such that

$$1/n \ll 1/M \ll 1/m' \ll \eta \ll \gamma \ll \varepsilon \ll 1/r \ll \alpha. \tag{2.36}$$

We will write the proof with 10α playing the role of α. Hence we have by assumption $d \geq (1/2 + 10\alpha)n$.

Step 1: Find the vortex sequence and the final absorber.

We start by applying Lemma 2.5.2 to find a $(5\alpha, \eta, \varepsilon, m)$-vortex $V_0 \supseteq \cdots \supseteq V_\ell$, $U_0 \supseteq \cdots \supseteq U_\ell$ of $\mathcal{G}$ where $\varepsilon m' \leq m \leq m'$. Let $n_i := |V_i|$ for every $i \in [\ell] \cup \{0\}$. For every $i \in [\ell]$, since $|V_i| = n_i$, $d_G(v; V_i) = (1 \pm \eta) 2 d_H(v; U_i)$ for every $v \in V_i$ and $H[V_i, U_i]$ is complete we have that $|U_i| \leq n_i$. Let $\mathcal{L}$ be the set of all spanning W_{r+1}-divisible extended subgraphs of $\mathcal{G}[V_\ell \cup U_\ell]$. Suppose that for some $\mathcal{L}' \subseteq \mathcal{L}$ we have found for each $L \in \mathcal{L}'$ an absorber $A_L \subseteq \mathcal{G} - \mathcal{G}[V_1 \cup U_1]$ with $|V(A_L)| \leq M$ so that all the absorbers $\{A_L\}_{L \in \mathcal{L}'}$ are edge-disjoint. Let $\mathcal{A}' := \bigcup_{L \in \mathcal{L}'} A_L$ and note that $|V(\mathcal{A}')| \leq |\mathcal{L}| M \leq 2^{2m} M$. Let $L \in \mathcal{L} \setminus \mathcal{L}'$ and let $\mathcal{G}_L = (\mathcal{G} - \mathcal{G}[V_1 \cup U_1] - \mathcal{A}') \cup L$. Since $|V(\mathcal{A}')| \leq \varepsilon n$, $|V_1| = \varepsilon n$ and $|U_1| \leq \varepsilon n$ we have that $\delta(\mathcal{G}_L[V]) \geq (1/2 + \alpha)n$ and $\mathcal{G}_L[V, U]$ is 2ε-complete. We can then use Lemma 2.8.1 (with $\mathcal{G}_L$, L, $2m$ and 2ε playing the role of $\mathcal{G}$, L, m and ε) to find an absorber A_L for L in $\mathcal{G}_L$ such that $|V(A_L)| \leq M$. Repeating iteratively the above arguments we end up finding for each $L \in \mathcal{L}$ an absorber $A_L \subseteq \mathcal{G} - \mathcal{G}[V_1 \cup U_1]$ with $|V(A_L)| \leq M$ so that all the absorbers $\{A_L\}_{L\in\mathcal{L}}$ are edge-disjoint. Let $\mathcal{A} := \bigcup_{L\in\mathcal{L}} A_L$ and note that $\mathcal{A}$ satisfies that for any W_{r+1}-divisible extended subgraph $L \subseteq \mathcal{G}[V_\ell \cup U_\ell]$ $\mathcal{A} \cup L$ has a W_{r+1}-decomposition. Since $|V(\mathcal{A})| \leq 2^{2m}M$ and using (2.36) we can see that for every $v \in V$ $d_{\mathcal{G}-\mathcal{A}}(v; V) \geq (1/2 + 7\alpha)n$ and $d_{\mathcal{G}-\mathcal{A}}(v; V_1) \geq (1/2 + 3\alpha)n_1$ so $V_0 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq \cdots \supseteq U_\ell$ form a $(4\alpha, \eta, \varepsilon, m)$-vortex of $\mathcal{G} - \mathcal{A}$. Moreover, since $\mathcal{A}$ is W_{r+1}-decomposable then $\mathcal{G} - \mathcal{A}$ must be W_{r+1}-divisible.

Step 2: Iteratively apply the Cover Down Lemma.

Let $i \in [\ell] \cup \{0\}$ and suppose that we have found an extended subgraph $\mathcal{G}_i = (V_i, U_i, G_i, H_i)$ of $\mathcal{G}$ such that $\mathcal{G} - \mathcal{A} - \mathcal{G}_i$ is W_{r+1}-decomposable and the following properties hold:

(a) $\mathcal{G}_i$ is W_{r+1}-divisible,

(b) G_i is (ε^2, d_i)-regular for some $d_i \geq (1/2 + 2\alpha)n_i$,

(c) $d_{G_i}(v; V_{i+1}) \geq (1/2 + 3\alpha)n_{i+1}$ for every $v \in V_i$,

(d) H_i is γ-complete,

(e) $\mathcal{G}_i[V_{i+1} \cup U_{i+1}] = \mathcal{G}[V_{i+1} \cup U_{i+1}]$.

This is true for $i = 0$ since $\mathcal{G}_0 := \mathcal{G} - \mathcal{A}$ satisfies properties (a)-(e). Let $\mathcal{G}'_i := \mathcal{G}_i - \mathcal{G}_i[V_{i+2} \cup U_{i+2}]$ and let $\mathcal{G}'_i = (V_i, U_i, G'_i, H'_i)$. Using that $|V_{i+2}| = \varepsilon|V_{i+1}| = \varepsilon^2|V_i|$ and $|U_{i+2}| = \varepsilon^2|U_i|$ we can see that G'_i is (ε, d'_i)-regular for some $d'_i \geq (1/2 + \alpha)n_i$, H'_i is $2\varepsilon^2$-complete and $d_G(v; V_{i+1}) \geq (1/2 + 2\alpha)n_{i+1}$ for every $v \in V_i$. On the other hand by property (V5) of the vortex sequence and using (e) we know that $d_{G_i}(v; V_{i+2}) = (1 \pm \eta)2d_{H_i}(v; U_{i+2})$ for every $v \in V_{i+2}$. Thus, since $\mathcal{G}_i$ is W_{r+1}-divisible by (a) then $\mathcal{G}'_i$ satisfies that $d_{G'_i}(v) = (1 \pm \eta)2d_{H'_i}(v)$ for every $v \in V_i$,

$d_{G'_i}(v) = 2d_{H'_i}(v)$ for every $v \in V_i \setminus V_{i+2}$ and $r \mid d_{H'_i}(u)$ for every $u \in U_i \setminus U_{i+2}$. We can now apply Lemma 2.7.1 (the Cover Down Lemma) with $\mathcal{G}'_i = (V_i, U_i, G'_i, H'_i)$, V_{i+1} and U_{i+1} playing the role of $\mathcal{G} = (V, U, G, H)$, V' and U' respectively to find a W_{r+1}-decomposable subgraph $\mathcal{F} \subseteq \mathcal{G}'_i$ such that $E(\mathcal{G}'_i - \mathcal{F}) \subseteq E(\mathcal{G}'_i[V_{i+1} \cup U_{i+1}])$ and

$$\Delta(\mathcal{F}[V_{i+1} \cup U_{i+1}]) \leq \gamma |V_i|. \tag{2.37}$$

In other words, $\mathcal{F}$ covers all the edges in $E(\mathcal{G}_i - \mathcal{G}_i[V_{i+1} \cup U_{i+1}])$ and only a sparse set of edges in $\mathcal{G}_i[V_{i+1} \cup U_{i+1}]$. Let $\mathcal{G}_{i+1} := (\mathcal{G}_i - \mathcal{F})[V_{i+1} \cup U_{i+1}]$ and note that $\mathcal{G} - \mathcal{A} - \mathcal{G}_{i+1} = (\mathcal{G} - \mathcal{A} - \mathcal{G}_i) \cup \mathcal{F}$ is W_{r+1}-decomposable by assumption. We will now check that $\mathcal{G}_{i+1} = (V_{i+1}, U_{i+1}, G_{i+1}, H_{i+1})$ satisfies properties (a)-(e). First observe that $\mathcal{G}_i - \mathcal{F}$ is W_{r+1}-divisible since both $\mathcal{G}_i$ and $\mathcal{F}$ are W_{r+1}-divisible. Note that $\mathcal{G}_i - \mathcal{F}$ and $\mathcal{G}_{r+1}$ have the same set of edges since $\mathcal{F}$ covered all edges outside $\mathcal{G}_i[V_{i+1} \cup U_{i+1}]$). Thus $\mathcal{G}_{r+1}$ is also W_{r+1}-divisible. Using (e) we know that $\mathcal{G}_i[V_{i+1} \cup U_{i+1}]$ satisfies properties (V1)-(V5) (with 4α instead α). Hence from (V2) and (2.37) we can deduce that $d_{G_{i+1}}(v_1) = (1 \pm \varepsilon^2) d_{G_{i+1}}(v_2)$ for every $v_1, v_2 \in V_{i+1}$, (c) and (2.37) imply that $d_{G_{i+1}}(v) \geq (1/2 + 2\alpha) n_{i+1}$ for every $v \in V_{i+1}$, from (V3) and (2.37) we can see that $d_{G_{i+1}}(v; V_{i+2}) \geq (1/2 + 3\alpha) n_{i+2}$ for every $v \in V_{i+1}$, and (V4) and (2.37) imply that H_{i+1} is γ-complete. Finally since $\mathcal{F}[V_{i+2} \cup U_{i+2}]$ has no edges we have $\mathcal{G}_{i+1}[V_{i+2} \cup U_{i+2}] = \mathcal{G}[V_{i+2} \cup U_{i+2}]$.

It follows by induction that there exists a W_{r+1}-divisible extended subgraph $\mathcal{G}_\ell = (V_\ell, U_\ell, G_\ell, H_\ell)$ of $\mathcal{G}$ such that $\mathcal{G} - \mathcal{A} - \mathcal{G}_\ell$ is W_{r+1}-decomposable.

Step 3: Absorb the final leftover.

Since $\mathcal{G}_\ell$ is a W_{r+1}-divisible extended subgraph of $\mathcal{G}[V_\ell, U_\ell]$ we have $\mathcal{G}_\ell \in \mathcal{L}$ so

$\mathcal{A}$ contains an absorber $A_{\mathcal{G}_\ell}$ for $\mathcal{G}_\ell$. All in all we have

$$\mathcal{G} = (\mathcal{G} - \mathcal{A} - \mathcal{G}_\ell) \sqcup (\mathcal{A} - A_{\mathcal{G}_\ell}) \sqcup (\mathcal{G}_\ell \cup A_{\mathcal{G}_\ell})$$

where the unions are edge-disjoint and $(\mathcal{G} - \mathcal{A} - \mathcal{G}_\ell)$, $(\mathcal{A} - A_{\mathcal{G}_\ell})$ and $(\mathcal{G}_\ell \cup A_{\mathcal{G}_\ell})$ are all W_{r+1}-decomposable, implying that $\mathcal{G}$ has a W_{r+1}-decomposition. □

CHAPTER 3

RESOLVABLE CLIQUE DECOMPOSITIONS OF PSEUDO-RANDOM GRAPHS

In this chapter we will prove Theorem 1.2.5. Recall that a graph G is said to be (ξ, p, s)-*typical* if for every set of vertices $S \subseteq V(G)$ of size at most s we have that $d_G(S) = (1 \pm \xi)p^{|S|}n$. Note that this is a type of pseudo-random condition that the random graph $G(n, p)$ will satisfy with high probability if ξ, p, s are fixed an n is large enough. Recall that a graph G is K_r-*divisible* if $\binom{r}{2} \mid e(G)$ and $r - 1 \mid d_G(v)$ for every $v \in V(G)$. Finally recall that a *resolvable* K_r-*decomposition* is a decomposition of G into K_r-factors where a K_r-*factor* is a collection of vertex-disjoint r-cliques that span $V(G)$. We restate Theorem 1.2.5.

Theorem 1.2.5. For every $r \geq 3$ and every $p > 0$ there exists $\xi > 0$ and $n_0 \in \mathbb{N}$ such that any K_r-divisible $(\xi, p, 3r)$-typical regular graph G on $n \geq n_0$ vertices such that $r \mid n$ has a resolvable K_r-decomposition.

In order to prove Theorem 1.2.5 we will prove a more general result, Theorem 3.2.1, consisting of a K_{r+1}-decomposition of the corresponding 'extended graph', see Section 1.5.2. Furthermore, we will consider decompositions in 'complexes', which are a more general object than a graph. We will introduce all the terminology around complexes in Section 3.1 before stating Theorem 3.2.1 in Section 3.2. The rest of the chapter will be organised as follows. In Section 3.3 we will prove the existence of a fractional K_{r+1}-decomposition in the extended graph (Lemma 3.3.2) and transform it into an approximate decomposition in Section 3.4. In Section 3.5 we will find a vortex sequence in the extended graph. In Section 3.6 we prove the existence of a clique-factor in a multipartite pseudo-random graph (Theorems 3.6.1 and 3.6.3). We will use these results to find many edge-disjoint factors in Section 3.7. In Section 3.8 we prove the 'Cover Down Lemma' (Lemma 3.8.3) and in Section 3.9 we construct the final absorbers for the main proof. It is in this section where we will require $s \geq 3$ (see Lemma 3.9.1). Finally, the proof of the main theorem of this chapter, Theorem 3.2.1, is presented in Section 3.10.

3.1 Preliminaries

We will inherit the standard notation and the basic probabilistic results used in the previous chapter. For reference see Section 2.1.

3.1.1 Complexes

The concept of *complex* generalises that of a graph and consists of a family of sets of vertices which is closed under taking subsets. The union of the sets can be viewed as a ground vertex set V and the collection of 2-sets can be viewed as the edge set E. While graphs only capture pairwise interactions (edges), complexes can model interactions between multiple vertices simultaneously (e.g., r-sets for $r \geq 3$), offering a richer and more general framework, and connecting combinatorics to other fields in mathematics such as topology, where complexes are a fundamental object. This broader perspective has inspired mathematicians to generalise many classical results from graph theory to complexes. For the purpose of this thesis we will focus in complexes containing only 1-sets (vertices), 2-sets (edges) and r-sets (faces) for some fixed $r \geq 3$. We define formally a complex as follows.

Definition 3.1.1. An *r-complex* G is a triple $(V(G), E(G), F_r(G))$ where $V(G)$ is a set of *vertices*, $E(G)$ is a set of 2-sets of vertices called *edges*, and $F_r(G)$ is a set of r-sets of vertices called *faces*, such that for every $f \in F_r(G)$ and every 2-set $e \subset f$, $e \in E(G)$.

We extend all the standard notation of simple graphs to r-complexes by identifying an r-complex G with its underlying graph $(V(G), E(G))$.

Given two r-complexes G and H we write $H \subseteq G$ and say that H *is a subcomplex of* G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and $F_r(H) \subseteq F_r(G)$. Given a set of edges $E' \subseteq E(G)$ the *induced complex* $G[E']$ is the r-complex $G[E'] = (V(G), E', F')$ where $F' := \{f \in F_r(G) : e \in E', \forall e \subseteq f\}$. If G is fixed we might refer to $G[E']$ as the *induced complex by* E'. We write $G - H$ for the r-complex $G[E(G) \setminus E(H)]$. Given a set $T \subseteq \mathcal{K}_{r-1}(G)$ of $(r-1)$-cliques of G, we denote $N^*_G(T) := \{v \in V(G) : \{v\} \cup K \in F_r(G), \forall K \in T\}$ and $d^*_G(T) := |N^*_G(T)|$. In other words, $N^*_G(T)$ is the set of vertices which form a face with every $(r-1)$-clique in T. For convenience we define $N^*_G(\emptyset) := V(G)$. If T consists only of one $(r-1)$-clique K we will often write $N^*_G(K)$ and $d^*_G(K)$ instead of $N^*_G(\{K\})$ and $d^*_G(\{K\})$. Given $W \subseteq V(G)$, a set of vertices $S \subseteq V(G)$ and a set of

$(r-1)$-cliques $T \subseteq \mathcal{K}_{r-1}(G)$ we write $N^*_G(S,T;W) := N_G(S) \cap N^*_G(T) \cap W$ and $d^*_G(S,T;W) = |N^*_G(S,T;W)|$. We refer to W as the *target set*. If no target set is specified we assume it is $V(G)$. The *link complex of an k-clique $K \in \mathcal{K}_k(G)$ with $k < r$ into $W \subseteq V(G)$* is the $(r-k)$-complex (V', E', F') where $V' = N_G(K;W)$, $E' = E(G[V'])$ and $F' = \{K' \in \mathcal{K}_{r-k}(G[V]) : K' \cup K \in F_r(G)\}$. We denote it by $\mathrm{Lk}_G(K;W)$.

Definition 3.1.2. Let G be an r-complex and let $V \subseteq V(G)$. Given $\xi, p, q > 0$ and $s, t \in \mathbb{N}$, we say that G is *(ξ, p, q, s, t)-typical into V* if for every set $S \subseteq V(G)$ with $|S| \le s$ and $T \subseteq \mathcal{K}_{r-1}(G[S])$ with $|T| \le t$,

$$d^*_G(S,T;V) = (1 \pm \xi)p^{|S|}q^{|T|}|V|.$$

Note that a random r-complex on n vertices where each edge is chosen with probability p and each r-face is chosen with probability q will be (ξ, p, q, s, t)-typical with high probability if ξ, p, s, t are fixed and n is large enough. Hence this notion of typicality can be viewed as a pseudo-random condition and an extension to r-complexes of the typicality notion for graphs (Definition 1.1.5).

Given an r-complex G, an *F_r-decomposition of G* is a collection of edge-disjoint faces of $F_r(G)$ that cover all edges in $E(G)$. If such collection exists we say that G is *F_r-decomposable*. An *F_r-factor of G* is a collection of vertex-disjoint faces of $F_r(G)$ that cover all vertices in $V(G)$. Finally, a *resolvable F_r-decomposition of G* is a decomposition of G into F_r-factors.

3.1.2 Extended complex

As in Chapter 2, the main theorem of this chapter, Theorem 3.2.1, consists of finding a decomposition in an 'extended complex'. Such a result will then be used to deduce Theorem 1.2.5. To state the main theorem we need to further define some additional concepts.

Definition 3.1.3. An extended $(r+1)$-complex $\mathcal{G} = (V, U, G, H)$ is an $(r+1)$-complex with $V(\mathcal{G}) = V \cup U$ where V and U are disjoint sets of vertices, $E(\mathcal{G}) = E(G) \cup E(H)$ where G is an r-complex on V and H is a bipartite graph on $[V, U]$, and $F_{r+1}(\mathcal{G}) = \{f \cup \{u\} \in \mathcal{K}_{r+1}(G \cup H) \,:\, f \in F_r(G), u \in U\}$.

Since an extended $(r+1)$-complex is, in particular, an $(r+1)$-complex, we shall use the notation introduced in the previous section. Note that by definition, all the $(r+1)$-faces of an extended $(r+1)$-complex $\mathcal{G} = (V, U, G, H)$ contain exactly one vertex in U. One can view an extended $(r+1)$-complex $\mathcal{G} = (V, U, G, H)$ as an $(r+1)$-complex that has an inner structure defined by the r-complex G and the bipartite graph H.

Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-complex and suppose that an F_{r+1}-decomposition of $\mathcal{G}$ exists. For each $v \in V$ and $u \in U$ let c_v and c_u be the number of faces in the decomposition containing v and u respectively. Then

(a) $d_G(v) = (r-1)c_v$,

(b) $d_H(v) = c_v$,

(c) $d_H(u) = rc_u$,

which implies that $r - 1 \mid d_G(v)$, and $r \mid d_{\mathcal{G}}(v)$, and $r \mid d_{\mathcal{G}}(u)$.

Let C be the total number of faces in the decomposition. Then

(d) $\sum_{v\in V} c_v = rC$,

(e) $\sum_{u\in U} c_u = C$,

which implies that $\binom{r}{2} \mid e(G)$, and $r \mid e(H)$, and $\binom{r+1}{2} \mid e(\mathcal{G})$.

We say that $\mathcal{G}$ is F_{r+1}-*divisible* if there are constants $\{c_v\}_{v\in V}$, $\{c_u\}_{u\in U}$ and C satisfying properties (a)-(e). Observe that F_{r+1}-divisibility is a necessary condition for $\mathcal{G}$ to admit an F_{r+1}-decomposition. Given $\eta > 0$ we say that $\mathcal{G}$ is η-*divisible* if $d_G(v) = (1 \pm \eta)(r-1)d_H(v)$ for every $v \in V$.

A bipartite graph H on $[V, U]$ is (ξ, p, s)-*typical* if for every set of vertices $S_V \subseteq V$ with $|S_V| \leq s$,

$$d_H(S_V) = (1 \pm \xi)p^{|S_V|}|U|,$$

and for every set of vertices $S_U \subseteq U$ with $|S_U| \leq s$,

$$d_H(S_U) = (1 \pm \xi)p^{|S_U|}|V|.$$

Definition 3.1.4. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-complex and let $V' \subseteq V$ and $U' \subseteq U$. Given $\xi, p_G, p_H, q > 0$ and $s, t \in \mathbb{N}$ we say that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-*typical into* $V' \cup U'$ if for every set $S_V \subseteq V$ and $S_U \subseteq U$ with $|S_V| + |S_U| \leq s$ and for every set $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \leq t$,

$$d^*_{\mathcal{G}}(S_V \cup S_U, T; V') = (1 \pm \xi)p_G^{|S_V|}p_H^{|S_U|}q^{|T|}|V'|,$$

and for every set $S_V \subseteq V$ with $|S_V| \leq s$,

$$d_H(S_V; U') = (1 \pm \xi)p_H^{|S_V|}|U'|.$$

In the case where $V' = V$ and $U' = U$ we just say that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-*typical*. Observe that if $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical then, in particular, G is (ξ, p_G, q, s, t)-typical and H is (ξ, p_H, s)-typical.

The following result relates the size of U to the typicality and divisibility parameters.

Proposition 3.1.5. *Let $1/n \ll \eta \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical η-divisible extended $(r+1)$-complex where $|V| = n$. Then $|U| = (1 \pm 3\xi)(p_G/p_H)n/(r-1)$.*

Proof. Since $\mathcal{G}$ is η-divisible we have $d_G(v) = (1 \pm \eta)(r-1)d_H(v)$ for every vertex $v \in V$. On the other hand, typicality implies that for every $v \in V$, $d_G(v) = (1 \pm \xi)p_G n$ and $d_H(v) = (1 \pm \xi)p_H|U|$. Combining both observations we obtain

$$|U| = \frac{(1 \pm \xi)}{(1 \pm \xi)(1 \pm \eta)} \frac{p_G n}{(r-1)p_H} = (1 \pm 3\xi)(p_G/p_H)n/(r-1)$$

as desired. □

3.1.3 Partite complexes

While this chapter primarily concerns non-partite complexes, in Section 3.6 we will show the existence of face-factors in the partite setting. For this reason it is necessary to introduce some terminology related to partite complexes before delving into the main results of this chapter. It is worth noting that results on partite complexes will be the main focus of Chapter 4 and Chapter 5.

Definition 3.1.6. An r-complex G is r-*partite* if there exists a partition $V^1, \ldots, V^r$ of $V(G)$ such that $G[V^i]$ has no edges for every $i \in [r]$.

Just as in graphs, we say that G is *balanced* if $|V^1| = \cdots = |V^r|$.

Definition 3.1.7. Let G be an r-partite complex with parts $V^1, \ldots, V^r$. Given $\xi, p, q > 0$ and $s, t \in \mathbb{N}$, we say that an r-partite complex G is (ξ, p, q, s, t)-typical if for every $i \in [r]$, every set $S \subseteq V(G) \setminus V^i$ with $|S| \leq s$ and every $T \subseteq \mathcal{K}_{r-1}(G[S])$ with $|T| \leq t$,

$$d^*_G(S, T; V^i) = (1 \pm \xi)p^{|S|}q^{|T|}|V^i|.$$

Note that although an r-partite complex is, in particular, an r-complex, their corresponding definitions of typicality are different. In fact, an r-partite complex G could never be (ξ, p, q, s, t)-typical as in Definition 3.1.2 because each of the parts of G is an independent set which contradicts the conditions of typicality. Specifically, any vertex in S and the target set in the definition of typicality cannot belong to the same part. This is the reason why it is necessary to make a distinction in the definition of typicality between the two cases.

The following definitions are an analogue to Definition 3.1.3 and 3.1.4 respectively in the case of partite complexes.

Definition 3.1.8. An *extended* $(r+1)$*-partite complex* $\mathcal{G} = (V, U, G, H)$ *with parts* $V^1, \ldots, V^r$ is an $(r+1)$-complex with $V(\mathcal{G}) = V \cup U$, where V and U are disjoint sets of vertices with $V = V^1 \sqcup \cdots \sqcup V^r$; $E(\mathcal{G}) = E(G) \cup E(H)$, where G is an r-partite complex with parts $V^1, \ldots, V^r$ and H is a bipartite graph on $[V, U]$; and $F_{r+1}(\mathcal{G}) = \{f \cup \{u\} \in \mathcal{K}_{r+1}(G \cup H) \,:\, f \in F_r(G), u \in U\}$.

Definition 3.1.9. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-partite complex with parts $V^1, \ldots, V^r$. Given $\xi, p_G, p_H, q > 0$ and $s, t \in \mathbb{N}$ we say that $\mathcal{G}$ is (ξ, p_G, q_H, q, s, t)*-typical* if for every $i \in [r]$, every $S_V \subseteq V \setminus V^i$ and $S_U \subseteq U$ with

$|S_V| + |S_U| \leq s$ and every $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \leq t$ we have

$$d^*_{\mathcal{G}}(S_V \cup S_U, T; V^i) = (1 \pm \xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} |V^i|,$$

and for every $S_V \subseteq V$ with $|S_V| \leq s$

$$d_H(S_V; U) = (1 \pm \xi) p_H^{|S_V|} |U|.$$

3.2 Main results

We have now gathered all the necessary concepts to state the main theorem of this chapter.

Theorem 3.2.1. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq r^2+r$. Let $\mathcal{G} = (V, U, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$. Then $\mathcal{G}$ has an F_{r+1}-decomposition.*

We will now deduce Theorem 1.2.5 from a particular case of the following result.

Corollary 3.2.2. *Let $1/n \ll \xi \ll p, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq r^2 + r$. Let G be a K_r-divisible regular (ξ, p, q, s, t)-typical r-complex on n vertices where $r \mid n$. Then G has a resolvable F_r-decomposition.*

Proof. Since G is regular, let $d := d_G(v)$ for any $v \in V(G)$. Recall that since G is K_r-divisible then $\binom{r}{2} \mid E(G)$ and $(r-1) \mid d$. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-complex where $V = V(G)$, U is a set of vertices of size $2e(G)/((r-1)n)$ and H is the complete bipartite graph on $[V, U]$. For every $v \in V$ let $c_v := d/(r-1)$ and note that $d_H(v) = |U| = 2e(G)/((r-1)n) = d/(r-1) = c_v$. For each $u \in U$

let $c_u := n/r$ and note that $d_H(u) = |V| = rc_u$. Finally, let $C := e(G)/\binom{r}{2}$. Observe that the constants $\{c_v\}_{v\in V}$, $\{c_u\}_{u\in U}$ and C satisfy properties (a)-(e) so $\mathcal{G}$ is F_{r+1}-divisible. Moreover, since G is (ξ, p, q, s, t)-typical and H is complete then $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical with $p_G = p$ and $p_H = 1$. We can then apply Theorem 3.2.1 to find an F_{r+1}-decomposition $\mathcal{F}$ of $\mathcal{G}$. Recall that by definition of the extended complex each $(r+1)$-face $f \in F_{r+1}(\mathcal{G})$ contains exactly one vertex in U and satisfies that $f \cap V \in F_r(G)$. For each $u \in U$ let $F_u := \{f \cap V : f \in \mathcal{F}, u \in f\}$ be the set of r-faces of G that together with u form a face of the decomposition $\mathcal{F}$. Note that since u is adjacent via H to all vertices in V then F_u is an F_r-factor of G. Thus, $\{F_u\}_{u\in U}$ is a collection of edge-disjoint F_r-factors of G that cover all edges in $E(G)$. Hence G has a resolvable F_r-decomposition. □

Theorem 1.2.5 corresponds to the case $F_r(G) = \mathcal{K}_r(G)$, $q = 1$ and $t \geq r^2 + r$ of Corollary 3.2.2.

3.3 Extended fractional decomposition

In this section we prove the existence of a fractional face-decomposition of an extended complex (Lemma 3.3.2). This is one of the two key results to prove the main theorem of this chapter, Theorem 3.2.1, the other one being the Cover Down Lemma (Lemma 3.8.3). Similar to the proof of Lemma 2.3.3 in the previous chapter, the proof of Lemma 3.3.2 will consist of extending a fractional decomposition from the host complex G into a fractional decomposition of its corresponding extended complex. We will first find a fractional decomposition of G via the following result by Glock, Kühn, Lo and Osthus.

Lemma 3.3.1 ([30, Lemma 6.3])**.** *Let $1/n \ll \eta \ll \xi \ll \xi' \ll p, q, 1/s, 1/t, 1/r$ with $s \geq r+1$ and $t \geq r^2+r$. Let G be a (ξ, p, q, s, t)-typical complex on n vertices. Then there is a weighting $\omega : F_r(G) \to [0,1]$ such that for every edge $e \in E(G)$*

$$\omega(e) := \sum_{f \in F_r(G)\,:\, e \subseteq f} \omega(f) = 1$$

and for every $f \in F_r(G)$

$$\omega(f) = (1 \pm \xi') \frac{(r-2)!}{p^{\binom{r}{2}-1} q n^{r-2}}.$$

The result in [30] is actually more general than the stated above. One could prove Lemma 3.3.2 directly by using the 'edge-gadgets' introduced by Barber et al. [8] which allow to change the weight of a given edge without changing the weight of any other edge. For completeness we will outline this approach:

(i) start with the uniform weighting defined as $\omega'(f) := (r-2)!/(p^{\binom{r}{2}-1} q n^{r-2})$ for each $f \in F_r(G)$;

(ii) using the typicality of G check that $\omega'(e) = (1 \pm \Theta(\xi))$ for each $e \in E(G)$ and define its 'missing weight' as $x_e := 1 - \omega'(e)$;

(iii) using the typicality of G, check that each edge e is contained in $\Theta(n^r)$ distinct 'edge-gadgets' where each r-clique is an r-face;

(iv) for each $e \in E(G)$, add a weight of x_e to e using all the gadgets that contain e uniformly to achieve $\omega(e) = 1$;

(v) check that after the use of all the gadgets the final weight on each face is $\omega(f) = (1 \pm \Theta(\xi))\omega'(f)$.

See [7, Lemma 4.2] for a detailed proof of the triangle case.

We now describe the gadgets that will be used to 'extend' the fractional decomposition of G into the extended complex $\mathcal{G}$ and state Lemma 3.3.2. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-complex. Let $v \in V$ and $u_1, u_2 \in U$ such that $vu_1, vu_2 \in E(\mathcal{G})$ and let $x \in \mathbb{R}$. A (vu_1, vu_2, x)-shifter is a function $\varphi : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that

$$\varphi(e) := \sum_{f \in F_{r+1}(\mathcal{G})\,:\, e \subseteq f} \varphi(f) = \begin{cases} x & \text{if } e = vu_1, \\ -x & \text{if } e = vu_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

A (v, u_1, u_2)-gadget J is a set of $r+3$ vertices in $V(\mathcal{G})$ such that $J_U := J \cap U = \{u_1, u_2\}$, $J_V := J \cap V$ contains v and satisfies that $G[J_V]$ is an $(r+1)$-clique with the property that every r-clique in J_V is an r-face of G, and $H[J_V, J_U]$ is a complete bipartite graph. Note that a (v, u_1, u_2)-gadget contains $2(r+1)$ faces in $F_{r+1}(\mathcal{G})$. We denote by $\mathcal{J}(v, u_1, u_2)$ the set of all (v, u_1, u_2)-gadgets in $\mathcal{G}$.

We are now ready to state and prove the main result of this section.

Lemma 3.3.2. *Let $1/n \ll \eta \ll \eta' \ll \rho \ll \xi \ll \xi' \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq r+2$ and $t \geq r^2 + r$. Let $\mathcal{G} = (V, U, G, H)$ be an η-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$. Then there is a weighting $\omega : F_{r+1}(\mathcal{G}) \to [0, 1]$ such that for every edge $e \in E(\mathcal{G})$*

$$\omega(e) = (1 \pm \eta') \tag{3.2}$$

and for every $f \in F_{r+1}(\mathcal{G})$

$$\omega(f) = (1 \pm \xi') \frac{(r-1)!}{p_G^{\binom{r}{2}} p_H^{r-1} q n^{r-1}}. \tag{3.3}$$

Proof. Let $\xi'' > 0$ such that $\xi \ll \xi'' \ll \xi'$. We start by applying Lemma 3.3.1 to G to find a weighting $\omega_G : F_r(G) \to [0,1]$ such that

$$\omega_G(e) = \sum_{f' \in F_r(G) \,:\, e \subseteq f'} \omega_G(f') = 1$$

for every $e \in E(G)$ and

$$\omega_G(f) = (1 \pm \xi'') \frac{(r-2)!}{p_G^{\binom{r}{2}-1} q n^{r-2}}$$

for every $f \in F_r(G)$. Observe that for every vertex $v \in V$

$$\omega_G(v) := \sum_{e \in E(G) \,:\, v \in e} \omega_G(e) = d_G(v) = (1 \pm \xi) p_G n. \tag{3.4}$$

We now define $\omega' : F_{r+1}(\mathcal{G}) \to [0,1]$ as

$$\omega'(f) = \frac{\omega_G(f \cap V)}{d^*_{\mathcal{G}}(f \cap V)}$$

Then, for every $f' \in F_r(G)$ we have that

$$\sum_{f \in N^*_{\mathcal{G}}(f')} \omega'(f) = \sum_{f \in N^*_{\mathcal{G}}(f')} \frac{\omega_G(f')}{d^*_{\mathcal{G}}(f')} = \omega_G(f'). \tag{3.5}$$

Note that for every edge $e \in E(G)$

$$\begin{aligned}\omega'(e) &= \sum_{f \in F_{r+1}(\mathcal{G})\,:\,e \subseteq f} \omega'(f) = \sum_{f' \in F_r(G)\,:\,e \subseteq f'} \sum_{f \in N^*_{\mathcal{G}}(f')} \omega'(f) \\ &\overset{(3.5)}{=} \sum_{f' \in F_r G\,:\,e \subseteq f'} \omega_G(f') = \omega_G(e) = 1.\end{aligned} \tag{3.6}$$

Given $v \in V$, let E_v be the set of edges in $E(H)$ incident to v. Then, for every $v \in V$

$$\begin{aligned}\omega_H(v) &:= \sum_{e \in E_v} \omega'(e) = \sum_{e \in E_v} \sum_{f \in F_{r+1}(\mathcal{G})\,:\,e \subseteq f} \omega'(f) = \sum_{f' \in F_r(G)\,:\,v \in f'} \sum_{f \in N^*_{\mathcal{G}}(f')} \omega'(f) \\ &\overset{(3.5)}{=} \sum_{f' \in F_r(G)\,:\,v \in f'} \omega_G(f') = \frac{1}{(r-1)} \sum_{e \in E(G)\,:\,v \in e} \omega_G(e) \overset{(3.4)}{=} \frac{d_G(v)}{r-1}.\end{aligned}$$

Suppose that for each $v \in V$

$$\omega'(e_1) = \omega'(e_2) \text{ for every pair of edges } e_1, e_2 \in E_v. \tag{3.7}$$

Then, for any $e \in E_v$

$$\omega_H(v) = \sum_{e' \in E_v} \omega'(e') = d_H(v)\omega'(e)$$

and thus,

$$\omega'(e) = \frac{\omega_H(v)}{d_H(v)} = \frac{d_G(v)}{(r-1)d_H(v)} = (1 \pm 2\eta) \tag{3.8}$$

which together with (3.6) implies (3.2).

Our goal then is to ensure that (3.7) holds for every vertex $v \in V$. To do this, we will use several (vu_1, vu_2, x)-shifters which will modify our current weighting

ω' into the desired one.

Fix $v \in V$. For each $e \in E_v$, let $x_e := \omega_H(v)/d_H(v) - \omega'(e)$ be the "missing weight" of e.

Claim 1: For every $e \in E(H)$, $|x_e| \leq 3\xi''$.

Proof of claim: We know that for every $f \in F_r(G)$

$$\omega_G(f) = (1 \pm \xi'')\frac{(r-2)!}{p_G^{\binom{r}{2}-1} q n^{r-2}}.$$

Then, for every edge $e \in E(H)$

$$\begin{aligned}
\omega'(e) &= \sum_{f \in F_{r+1}(\mathcal{G}) \,:\, e \subseteq f} \omega'(f) = \sum_{f \in F_{r+1}(\mathcal{G}) \,:\, e \subseteq f} \frac{\omega_G(f \cap V)}{d^*_{\mathcal{G}}(f \cap V)} \\
&= (1 \pm \xi'')\frac{(r-2)!}{p_G^{\binom{r}{2}-1} q n^{r-2}} \frac{|\{f \in F_{r+1}(\mathcal{G}) \,:\, e \subseteq f\}|}{d^*_{\mathcal{G}}(f \cap V)} \\
&= (1 \pm \xi'')\frac{(r-2)!}{p_G^{\binom{r}{2}-1} q n^{r-2}} \frac{(1 \pm \xi)^r p_G^{\binom{r}{2}} p_H^{r-1} q n^{r-1}/(r-1)!}{(1 \pm \xi) p_H^r |U|} \\
&\overset{\text{Prop. 3.1.5}}{=} 1 \pm 2\xi''.
\end{aligned}$$

On the other hand $\omega_H(v)/d_H(v) = (1 \pm 2\eta)$ by (3.8). Hence,

$$|x_e| = |\omega_H(v)/d_H(v) - \omega'(e)| = |(1 \pm 2\eta) - (1 \pm 2\xi'')| \leq 3\xi''.$$

–

Note that $\sum_{e \in E_v} x_e = 0$.

The following claim finds triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{i \in I}$ where $e_{i_1}, e_{i_2} \in E_v$ so that by finding an (e_{i_1}, e_{i_2}, x_i)-shifter for each $i \in I$ we will compensate the missing weight of each $e \in E_v$ and ensure that (3.7) holds.

Claim 2: there exists a finite set A_v *of ordered triples* $\{(e_{i_1}, e_{i_2}, x_i)\}_{i \in I}$ *where* $e_{i_1}, e_{i_2} \in E_v$ *and* $x_i \in \mathbb{R}$ *satisfying that*

(S1) for each $e \in E_v$ *let* $I_e \subseteq I$ *be the set of indices* i *such that* $e \in \{e_{i_1}, e_{i_2}\}$*, then*

$$\sum_{i \in I_e} |x_i| = |x_e|,$$

(S2) suppose that φ_i *is an* (e_{i_1}, e_{i_2}, x_i)*-shifter for every* $i \in I$*, then*

$$\omega'(e) + \sum_{i \in I} \varphi_i(e) = \omega_H(v)/d_H(v)$$

for every $e \in E_v$*.*

Proof of claim: The proof consists of the following algorithm. Suppose that at the j-th step we have already found triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{1 \le i < j}$. For each $e \in E_v$, let $I_j^+(e)$ be the set of indices i such that $e = e_{i_1}$ and let $I_j^-(e)$ be the set of indices i such that $e = e_{i_2}$. If $x_e \ge 0$ let $z_j(e) := \sum_{i \in I_j^+(e)} x_i$ and if $x_e < 0$ let $z_j(e) := \sum_{i \in I_j^-(e)} -x_i$. Let $E_j := \{e \in E_v : z_j(e) = x_e\}$. Suppose that the triples $\{(e_{i_1}, e_{i_2}, x_i)\}_{1 \le i < j}$ satisfy that for each $e \in E_v$,

(s1) if $x_e \ge 0$ then $|I_j^-(e)| = 0$ and if $x_e < 0$ then $|I_j^+(e)| = 0$,

(s2) $|z_j(e)| \le |x_e|$,

and suppose that

(s3) $|E_j| \ge j$.

The algorithm stops if $E_j = E_v$. In this case we have that $z_j(e) = x_e$ for each $e \in E_v$ which implies (S1). Suppose that φ_i is an (e_{i_1}, e_{i_2}, x_i)-shifter for every $1 \leq i < j$. Then for every edge $e \in E_v$

$$\omega'(e) + \sum_{1 \leq i < j} \varphi_i(e) = \omega'(e) + z_j(e) = \omega'(e) + x_e = \omega_H(v)/d_H(v)$$

which implies (S2).

Else, if $|E_j| < |E_v|$, let e_j be an edge in $E_v \setminus E_j$ that minimizes $|x_e - z_j(e)|$. Observe that

$$\sum_{e \in E_v} (x_e - z_j(e)) = \sum_{e \in E_v} x_e - \sum_{e \in E_v} z_j(e) = 0 - \sum_{e \in E_v : x_e \geq 0} \sum_{i \in I_j^+(e)} x_i + \sum_{e \in E_v : x_e < 0} \sum_{i \in I_j^-(e)} x_i = 0.$$

Then there must exist some edge $e'_j \in E_v \setminus E_j$ such that $\text{sgn}(x_{e_j} - z_j(e_j)) \neq \text{sgn}(x_{e'_j} - z_j(e'_j))$. We choose the j-th triple to be $(e_j, e'_j, (x_{e_j} - z_j(e_j)))$ if $(x_{e_j} - z_j(e_j)) > 0$ and $(e'_j, e_j, -(x_{e_j} - z_j(e_j)))$ if $(x_{e_j} - z_j(e_j)) < 0$. Observe that $z_{j+1}(e_j) = x_{e_j}$, $|z_{j+1}(e'_j)| \leq |x_{e'_j}|$, and $z_{j+1}(e) = z_j(e)$ for all edges except e_j and e'_j. It follows that the new set of triples satisfy (s1)-(s3) when replacing j by $j+1$. Note that $|E_{j+1}| > |E_j|$ guarantees that the algorithm will stop at most at the $d_H(v)$-th step. $-$

Now, for each $(e_1, e_2, x) \in A_v$ we will construct an (e_1, e_2, x)-shifter as follows. Let u_1 and u_2 be the vertex in U contained in e_1 and e_2 respectively. Recall that a (v, u_1, u_2)-gadget J is a set of $r + 3$ vertices in $V(\mathcal{G})$ such that $J_U := J \cap U = \{u_1, u_2\}$, $J_V := J \cap V$ contains v and satisfies that $G[J_V]$ is an $(r+1)$-clique with the property that every r-clique in J_V is an r-face of G, and $H[J_V, J_U]$ is a complete bipartite graph. Recall also that $\mathcal{J}(v, u_1, u_2)$ denotes the set of all

(v, u_1, u_2)-gadgets in $\mathcal{G}$. For each gadget $J \in \mathcal{J}(v, u_1, u_2)$, let $\varphi_J : F_{r+1}(\mathcal{G}[J]) \to \mathbb{R}$ be defined as

$$\varphi_J(f) = \begin{cases} 1/r & \text{if } e_1 \subseteq f, \\ -1/r & \text{if } e_2 \subseteq f, \\ -(r-1)/r & \text{if } u_1 \in f \text{ and } v \notin f, \\ (r-1)/r & \text{if } u_2 \in f \text{ and } v \notin f. \end{cases} \tag{3.9}$$

Let $\varphi_{(e_1,e_2,x)} : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ be defined as

$$\varphi_{(e_1,e_2,x)}(f) = \frac{x}{|\mathcal{J}(v, u_1, u_2)|} \sum_{J \in \mathcal{J}(v,u_1,u_2)} \varphi_J(f)$$

where $\varphi_J(f) := 0$ if $f \notin F_{r+1}(\mathcal{G}[J])$. It is not difficult to check that $\varphi_{(e_1,e_2,x)}$ is indeed an (e_1, e_2, x)-shifter. Moreover, given $f \in F_{r+1}(\mathcal{G})$, the amount of gadgets $J \in \mathcal{J}(v, u_1, u_2)$ that contain f is linear in n if e_1 or $e_2 \subseteq f$, one if u_1 or $u_2 \in f$ but $v \notin f$, and zero otherwise. Thus,

$$|\varphi_{(e_1,e_2,x)}(f)| \leq \begin{cases} |x|\Theta(n^{-(r-1)}) & \text{if } e_1 \text{ or } e_2 \subseteq f, \\ |x|\Theta(n^{-r}) & \text{if } u_1 \text{ or } u_2 \in f \text{ and } v \notin f, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Let $\varphi_v := \sum_{a \in A_v} \varphi_a$. Then (S2) implies that

$$\omega'(e) + \varphi_v(e) = \omega'(e) + \sum_{a \in A_v} \varphi_a(e) = \omega_H(v)/d_H(v) \tag{3.11}$$

for every $e \in E_v$. On the other hand, consider a face $f \in F_{r+1}(\mathcal{G})$ with $\{u\} = f \cap U$.

If f contains v and $vu \in E(H)$,

$$|\varphi_v(f)| \leq \sum_{a \in A_v} |\varphi_a(f)| = \sum_{a \in A_v \,:\, vu \in a} |\varphi_a(f)| \overset{(S1),(3.10)}{\leq} |x_{vu}|\Theta(n^{-(r-1)}). \tag{3.12}$$

If f does not contain v but $vu \in E(H)$,

$$|\varphi_v(f)| \leq \sum_{a \in A_v} |\varphi_a(f)| = \sum_{a \in A_v \,:\, vu \in a} |\varphi_a(f)| \overset{(S1),(3.10)}{\leq} |x_{vu}|\Theta(n^{-r}). \tag{3.13}$$

Else, if $vu \notin E(H)$, $\varphi_v(f) = 0$.

Let $\omega = \omega' + \sum_{v \in V} \varphi_v$. First observe that $\omega(e) = \omega'(e)$ for every edge $e \in E(G)$. Second, for every $v \in V$ and every $e \in E_v$

$$\omega(e) = \omega'(e) + \sum_{v' \in V} \varphi_{v'}(e) = \omega'(e) + \varphi_v(e) = \omega_H(v)/d_H(v).$$

Thus, equations (3.7) and (3.8) holds when replacing ω' with ω, so (3.2) holds as desired.

It only remains to check equation (3.3). Fix $f \in F_{r+1}(\mathcal{G})$ and let $f = \{v_1, \dots, v_r, u\}$ with $u \in U$. Let $x_{\max} := \max_{e \in E(H)} |x_e|$. Then

$$\begin{aligned}
|\sum_{v \in V} \varphi_v(f)| &\leq \sum_{v \in V \,:\, v \notin f} |\varphi_v(f)| + \sum_{v \in f} |\varphi_v(f)| \leq \Theta(n)|x_{\max}|\Theta(n^{-r}) + |x_{\max}|\Theta(n^{-(r-1)}) \\
&\leq 3\xi''\Theta(n^{-(r-1)})
\end{aligned}$$

and

$$\begin{aligned}
\omega'(f) &= \frac{\omega_G(f \cap V)}{d^*_{\mathcal{G}}(f \cap V)} = (1 \pm \xi'') \frac{(r-2)!}{p_G^{\binom{r}{2}-1} q n^{r-2}} \cdot \frac{1}{(1 \pm \xi)^r p_H^r |U|} \\
&\overset{\text{Prop. } 3.1.5}{=} (1 \pm 2\xi'') \frac{(r-1)!}{p_G^{\binom{r}{2}} p_H^{r-1} q n^{r-1}}.
\end{aligned}$$

Thus,

$$\omega(f) = \omega'(f) + \sum_{v \in V} \varphi_v(f) = (1 \pm \xi') \frac{(r-1)!}{p_G^{\binom{r}{2}} p_H^{r-1} q n^{r-1}}$$

for each $f \in F_{r+1}(\mathcal{G})$. □

3.4 Approximate decomposition

In this section we prove Lemma 3.4.1 which finds an approximate face decomposition in an extended graph. The proof uses the existence of a fractional face decomposition given by Lemma 3.3.2 and the existence of an almost-perfect matching in an auxiliary hypergraph given by Theorem 1.5.2.

Lemma 3.4.1. *Let $1/n \ll \eta \ll \gamma \ll \xi \ll p_G, p_H$, $q, 1/s, 1/t, 1/r$ with $s \geq r+2$ and $t \geq r^2 + r$. Let $\mathcal{G} = (V, U, G, H)$ be an η-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$. If G is K_r-divisible then there exists an F_{r+1}-decomposable complex $\mathcal{F} \subseteq \mathcal{G}$ such that $\Delta(\mathcal{G} - \mathcal{F}) \leq \gamma n$.*

Proof. Let $\rho, \eta' > 0$ such that

$$1/n \ll \eta \ll \eta' \ll \gamma \ll \xi \ll \rho \ll p_G, p_H, q, 1/s, 1/t, 1/r$$

We start by applying Lemma 3.3.2 to $\mathcal{G}$ to obtain a weighting $\omega : F_{r+1}(\mathcal{G}) \to [0, 1]$ that satisfies (3.2) and (3.3).

For each $f \in F_{r+1}(\mathcal{G})$, let $p(f) := \omega(f)\rho n^{r-1}$. Note that $0 < p(f) < 1$. Let $E(\mathcal{H})$ be a set of faces obtained from $F_{r+1}(\mathcal{G})$ by including each $f \in F_{r+1}(\mathcal{G})$ with probability $p(f)$. Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be the $(r+1)$-uniform hypergraph with vertex set $V(\mathcal{H}) = E(\mathcal{G})$ and hyperedge set $E(\mathcal{H})$. Observe that for each $e \in E(\mathcal{G})$,

$$\mathbb{E}\left[d_{\mathcal{H}}(e)\right] = \sum_{f \in F_{r+1}(\mathcal{G})\,:\,e \subset f} p(f) = \sum_{f \in F_{r+1}(\mathcal{G})\,:\,e \subset f} \omega(f) \rho n^{r-1} = (1 \pm \eta'/2)\rho n^{r-1}.$$

Lemma 2.1.2 and a union bound implies that with positive probability

$$d_{\mathcal{H}}(e) = (1 \pm \eta')\rho n^{r-1} \text{ for all } e \in E(G). \tag{3.14}$$

Choose $\mathcal{H}$ so that (3.14) holds.

Observe that $\Delta_2(\mathcal{H}) \leq n^{r-2} \leq \rho n^{r-1}/\log^{9r} n$ as every pair of edges determine at least three vertices so there are at most n^{r-2} $(r+1)$-faces containing them. For each $v \in V(\mathcal{G})$ let $S_v = \{e \in E(\mathcal{G}) \,:\, v \in e\}$. Let $\mathcal{S} = \{S_v\}_{v \in V(\mathcal{G})}$ and note that $|\mathcal{S}| = n + pn/(r-1)$. We apply Theorem 1.5.2 to the hypergraph $\mathcal{H}$ (with η' playing the role of ε) to obtain a $(\gamma, \mathcal{S})$-perfect matching $\mathcal{M}$. Let E_L be the set of edges in $E(\mathcal{G})$ which are left uncovered by $\mathcal{M}$. Observe that a perfect matching in $\mathcal{H}$ corresponds to an F_{r+1}-decomposition of $\mathcal{G}$. Hence, $\mathcal{F} := \mathcal{G} - \mathcal{G}[E_L]$ has an F_{r+1}-decomposition. Finally, $\Delta(\mathcal{G} - \mathcal{F}) = \Delta(\mathcal{G}[E_L]) \leq \gamma n$ since for each vertex $v \in V(\mathcal{G})$ there are at most γn edges e containing v which are left uncovered by $\mathcal{M}$ in $\mathcal{H}$. □

3.5 Vortex sequence

In this section we find a vortex sequence in an extended graph. This will be the first step in the proof of Theorem 3.2.1. We will make sure that properties such as typicality and divisibility are passed down through the sequence by capturing them in following definition.

Let $n, r, s, t \in \mathbb{N}$ and let $\xi, p_G, p_H, q > 0$. Let $\mathcal{G} = (V, U, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$.

Definition 3.5.1 (Vortex)**.** Given $m \in \mathbb{N}$ and $\xi', \varepsilon, \eta > 0$, a $(\xi', \varepsilon, \eta, m)$-*vortex* of $\mathcal{G}$ is a pair of sequences $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ such that

(V1) $V_0 = V$ and $U_0 = U$,

(V2) For all $i \in [\ell]$, $|V_i| = \varepsilon |V_{i-1}|$ and $|U_i| = (p_G/p_H)|V_i|/(r-1)$,

(V3) $|V_\ell| = m$,

(V4) For all $i \in [\ell]$, $\mathcal{G}[V_{i-1} \cup U_{i-1}]$ is $(\xi', p_G, p_H, q, s, t)$-typical into $V_i \cup U_i$,

(V5) For all $i \in [\ell]$ and $v \in V_{i-1}$,

$$d_G(v, V_i) = (1 \pm \eta)(r-1) d_H(v, U_i).$$

Note that (V4) and (V5) imply that $\mathcal{G}[V_i \cup U_i]$ is $(\xi', p_G, p_H, q, s, t)$-typical and η-divisible for every $i \in [\ell]$.

Lemma 3.5.2. *Let* $1/n \ll 1/m' \ll \eta \ll \varepsilon \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. *Then* $\mathcal{G}$ *has a* $(2\xi, \varepsilon, \eta, m)$*-vortex where* $\varepsilon m' \leq m \leq m'$.

Proof. Let $n_0 := n$ and recursively define $n_i := \varepsilon n_{i-1}$. Let $\ell := 1 + \max\{i \geq 0 : n_i \geq m'\}$ and let $m := n_\ell$. Observe that $\varepsilon^i n \geq n_i \geq \varepsilon^i n - 1/(1-\varepsilon)$ and $\varepsilon m' \leq m \leq m'$.

Let $\xi_0 := \xi$ and $\eta_0 := 0$ and define for $i = 1, \dots, \ell$,

$$\xi_i := \xi_{i-1} + 2n_{i-1}^{-1/3} \tag{3.15}$$

and

$$\eta_i := \eta_{i-1} + 4n_{i-1}^{-1/3}. \tag{3.16}$$

Suppose that for some $i \in [\ell]$ we have already found sequences $V_0, \dots, V_{i-1}$ and $U_0, \dots, U_{i-1}$ which form a $(\xi_{i-1}, \varepsilon, \eta_{i-1}, n_{i-1})$-vortex of $\mathcal{G}$. Note that this is trivially true for $i = 0$ since $\mathcal{G}$ is F_{r+1}-divisible and (ξ, p_G, p_H, q, s, t)-typical.

Let V_i and U_i be random subsets of V_{i-1} and U_{i-1} of sizes n_i and $(p_G/p_H)n_i/(r-1)$ respectively. Lemma 2.1.2 and a union bound implies that with positive probability for any $S_V \subseteq V_{i-1}$ and $S_U \subseteq U_{i-1}$ with $|S_V| + |S_U| \leq s$ and $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \leq t$, we have

$$d^*_{\mathcal{G}}(S_V \cup S_U, T; V_i) = (1 \pm n_{i-1}^{-1/3}) d^*_{\mathcal{G}}(S_V \cup S_U, T; V_{i-1}) \frac{n_i}{n_{i-1}}, \tag{3.17}$$

and for any set $S_V \subseteq V_{i-1}$ with $|S_V| \leq s$

$$d_H(S_V; U_i) = (1 \pm n_{i-1}^{-1/3}) d_H(S_V; U_{i-1}) \frac{|U_i|}{|U_{i-1}|}. \tag{3.18}$$

Fix such choice of V_i and U_i.

Recall that, by assumption, $\mathcal{G}[V_{i-1} \cup U_{i-1}]$ is $(\xi_{i-1}, p_G, p_H, q, s, t)$-typical by (V4) and $d_G(v; V_{i-1}) = (1 \pm \eta_{i-1})(r-1) d_H(v; U_{i-1})$ for all $v \in V_{i-1}$ by (V5). Thus, using (3.17) we can deduce that for any $S_V \subseteq V_{i-1}$ and $S_U \subseteq U_{i-1}$ with $|S_V| + |S_U| \leq s$

and $\mathcal{T} \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|\mathcal{T}| \le t$

$$\begin{aligned} d^*_G(S_V \cup S_U, \mathcal{T}; V_i) &= (1 \pm n_{i-1}^{-1/3}) d^*_G(S_V \cup S_U, \mathcal{T}; V_{i-1}) \frac{n_i}{n_{i-1}} \\ &= (1 \pm n_{i-1}^{-1/3})(1 \pm \xi_{i-1}) p_G^{|S_V|} p_H^{|S_U|} q^{|\mathcal{T}|} n_{i-1} \frac{n_i}{n_{i-1}} \\ &= (1 \pm \xi_i) p_G^{|S_V|} p_H^{|S_U|} q^{|\mathcal{T}|} n_i. \end{aligned}$$

Similarly, using (3.18) we can deduce that for all $S_V \subseteq V_{i-1}$ with $|S_V| \le s$

$$\begin{aligned} d_H(S_V; U_i) &= (1 \pm n_{i-1}^{-1/3}) d_H(S_V; U_{i-1}) \frac{|U_i|}{|U_{i-1}|} \\ &= (1 \pm n_{i-1}^{-1/3})(1 \pm \xi_{i-1}) p_H^{|S_V|} |U_{i-1}| \frac{|U_i|}{|U_{i-1}|} \\ &= (1 \pm \xi_i) p_H^{|S_V|} |U_i|. \end{aligned} \tag{3.19}$$

On the other hand, for any $v \in V_{i-1}$

$$\begin{aligned} d_G(v; V_i) &\overset{(3.17)}{=} (1 \pm n_{i-1}^{-1/3}) d_G(v; V_{i-1}) \frac{n_i}{n_{i-1}} \\ &= (1 \pm n_{i-1}^{-1/3})(1 \pm \eta_{i-1})(r-1) d_H(v; U_{i-1}) \frac{n_i}{n_{i-1}} \\ &\overset{(3.18)}{=} (1 \pm 2n_{i-1}^{-1/3})(1 \pm \eta_{i-1})(r-1) d_H(v; U_i) \frac{|U_{i-1}|}{|U_i|} \frac{n_i}{n_{i-1}} \\ &= (1 \pm 3n_{i-1}^{-1/3})(1 \pm \eta_{i-1})(r-1) d_H(v; U_i) \\ &= (1 \pm \eta_i)(r-1) d_H(v; U_i). \end{aligned}$$

Thus, $V_0, \dots, V_i$ and $U_0, \dots, U_i$ form a $(\xi_i, \varepsilon, \eta_i, n_i)$-vortex of $\mathcal{G}$. It follows by induction that there exist sequences $V_0, \dots, V_\ell$ and $U_0, \dots, U_\ell$ which form a $(\xi_\ell, \varepsilon, \eta_\ell, n_\ell)$-vortex of $\mathcal{G}$.

Finally, observe that

$$\eta_\ell = \eta_0 + \sum_{j=0}^{\ell-1} 4n_j^{-1/3} \leq 4\frac{m'^{-1/3}}{1-\varepsilon^{1/3}} < \eta,$$

and likewise,

$$\xi_\ell = \xi_0 + \sum_{j=0}^{\ell-1} 2n_j^{-1/3} \leq \xi + 2\frac{m'^{-1/3}}{1-\varepsilon^{1/3}} < 2\xi.$$

$\square$

3.6 Face-factor in partite complexes

In this section we will prove several results on the existence of face-factors in partite and multipartite complexes. Though the existence of such factors could be proven using the hypergraph blow-up lemma by Keevash [41], a generalisation to hypergraphs of the celebrated Blow-up Lemma by Komlós, Sárközy and Szemerédi, [53], for the sake of completeness we will present a shorter and simpler proof following the methods used by Han, Kohayakawa, Morris and Person [36] to find a K_r-factor in a pseudorandom graph.

Given an r-complex G, recall that an *F_r-factor of G* is a collection of vertex-disjoint faces of $F_r(G)$ that cover all vertices in $V(G)$. It is straightforward to see that $r \mid V(G)$ is a necessary condition for the existence of an F_r-factor of G. The aim of this section is to prove the following results.

Theorem 3.6.1. *Let $1/n \ll \xi \ll p, q, 1/s, 1/t, 1/r$ with $s \geq 2(r-1)$ and $t \geq 2$. Let G be a balanced (ξ, p, q, s, t)-typical r-partite complex with parts $V^1, \dots, V^r$ of size n. Then G contains an F_r-factor.*

Corollary 3.6.2. *Let $1/n \ll \xi \ll p, q, 1/s, 1/t, 1/r$ with $r \mid n$, $s \geq 2(r-1)$ and $t \geq 2$. Let G be a (ξ, p, q, s, t)-typical r-complex on n vertices. Then G contains an F_r-factor.*

Theorem 3.6.3. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq 2$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ where $|V^i| = n$ for every $i \in [r+1]$. Then $\mathcal{G}$ contains an F_{r+1}-factor.*

Corollary 3.6.4. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq 2$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = r|U| = rn$. Then $\mathcal{G}$ contains an F_{r+1}-factor.*

Theorems 3.6.1 and 3.6.3 will be obtained by applying the slightly more general result stated below as Theorem 3.6.8. Corollaries 3.6.2 and 3.6.4 are deduced from Theorems 3.6.1 and 3.6.3 by randomly splitting the vertex set $V(G)$ into r parts $V^1, \dots, V^r$ of equal size and using a Chernoff bound and the typicality of G and $\mathcal{G}$ to see that with positive probability the parts $V^1, \dots, V^r$ induce a $(2\xi, p, q, s, t)$-typical r-partite complex G' and a $(2\xi, p_G, p_H, q, s, t)$-typical extended $(r+1)$-partite complex $\mathcal{G}'$ so we can apply Theorem 3.6.1 and 3.6.3 to G' and $\mathcal{G}'$ respectively.

Let G be an r-partite graph with parts $V^1, \dots, V^r$, let $\xi > 0$ and let $p_{ij} > 0$ for every $i \neq j$. Consider the following properties.

(P1) $d_G(v; V^j) > (1-\xi)p_{ij}|V^j|$ for every $v \in V^i$ and $i \neq j$.

(P2) $d_G(\{v_1, v_2\}; V^j) < (1+\xi)p_{ij}^2|V^j|$ for every $v_1, v_2 \in V^i$ and $i \neq j$.

Let G be a graph and let $A, B \subseteq V(G)$ be disjoint non-empty sets of vertices. The *density* of the pair (A, B) is defined as

$$d(A, B) := \frac{e(A, B)}{|A||B|},$$

where $e(A, B)$ denotes the number of edges in G between A and B. The pair (A, B) is δ-*regular* if for any two sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \delta|A|$ and $|Y| \geq \delta|B|$ we have

$$|d(X, Y) - d(A, B)| < \delta.$$

We shall use the following result on 'typicality' implies 'regularity'.

Proposition 3.6.5 ([22, Proposition 2.5])**.** *Let $1/n \ll \xi \ll \delta \ll \{p_{ij}\}_{i \neq j}, 1/r$. Let G be an r-partite graph with parts $V^1, \ldots, V^r$ of size greater than n. If G satisfies (P1)-(P2) then (V^i, V^j) is δ-regular for every $i \neq j$.*

Proposition 3.6.6. *Let $1/n \ll \xi \ll \delta \ll \{p_{ij}\}_{i \neq j}, 1/r$. Let G be an r-partite graph with parts $V^1, \ldots, V^r$ of size greater than n. Let $j \in [r]$ and $U \subseteq V^j$ such that $|U| \geq \delta n$. If G satisfies (P1)-(P2) then for every $i \in [r] \setminus \{j\}$ there are at most δn vertices w in V^i such that $d_G(w; U) < p_{ij}|U|/2$.*

Proof. Let $i \in [r] \setminus \{j\}$. We know by Proposition 3.6.5 that the pair (V^i, V^j) is δ-regular. Let $W \subseteq V^i$ be the set of vertices $w \in V^i$ such that $d_G(w; U) < p_{ij}|U|/2$. Suppose that $|W| \geq \delta n$, thus, it follows by δ-regularity that

$$|d(W, U) - d(V^i, V^j)| < \delta.$$

Using that $d(W,U) = \frac{e(W,U)}{|W||U|}$ and $d(V^i, V^j) \geq (1-\xi)p_{ij}$ we obtain

$$\frac{e(W,U)}{|W||U|} > (1-\xi)p_{ij} - \delta.$$

On the other hand, we know that $e(W,U) < |W|p_{ij}|U|/2$ by definition of W. Hence,

$$p_{ij}/2 > (1-\xi)p_{ij} - \delta$$

which is a contradiction since $\xi \ll \delta \ll p_{ij}$. Thus, we must have $|W| < \delta n$. □

Let G be an r-partite complex with parts $V^1, \dots, V^r$ of size greater than n and let $\varepsilon > 0$. Given a subset $U \subseteq V(G)$ we denote $U^i := U \cap V^i$ for all $i \in [r]$. Consider the following property.

(P3) For every $i \in [r]$, $v \in V^i$ and $U \subseteq N_G(v)$ with $|U^j| \geq \varepsilon n$ for each $j \in [r] \setminus \{i\}$, $G[U]$ contains an $(r-1)$-clique K such that $K \cup \{v\} \in F_r(G)$.

Proposition 3.6.7. *Let $1/n \ll \xi \ll \varepsilon \ll \{p_{ij}\}_{i \neq j}, 1/r$. Let G be an r-partite complex with parts $V^1, \dots, V^r$ of size greater than n. If G satisfies (P1)-(P3) then for any set of vertices $U \subseteq V(G)$ satisfying $|U^i| \geq \varepsilon n$ for every $i \in [r]$, $G[U]$ contains a face.*

Proof. Let $\delta > 0$ such that $\xi \ll \delta \ll \varepsilon$. Fix $i \in [r]$. For each $j \in [r] \setminus \{i\}$ we know by Proposition 3.6.6 that there are at most δn vertices w in V^i such that $d_G(w; U^j) < p_{ij}|U^j|/2$. Since $|U^i| \geq \varepsilon n > (r-1)\delta n$ we can greedily pick a vertex $v \in V^i$ such that $d_G(v; U^j) \geq p_{ij}|U^j|/2 > \delta n$ for every $j \in [r] \setminus \{i\}$. Then (P3) (with $N_G(v; U)$ and δ playing the role of U and ε) implies that $N_G(v; U)$ contains an $(r-1)$-clique K such that $K \cup \{v\}$ is a face of $G[U]$. □

Let $m \in \mathbb{N}$ and let Z be a set of $2m$ vertices. An *r-vertex-absorber for Z with flexibility m* is a graph A_Z containing Z such that for every subset $Z' \subseteq Z$ of size m the graph $A_Z \setminus Z'$ obtained by removing the vertices of Z' from A_Z has a K_r-factor. This absorbing technique is known as *distributive absorption* and was introduced by Montgomery [66].

Let G be a balanced r-partite complex with parts $V^1, \dots, V^r$ of size n, let $\varepsilon > 0$ and let $p_{ij} > 0$ for every $i \neq j$. Consider the following property.

(P4) For each $j \in [r]$ there is a set $Z^j \subseteq V^j$ and an r-vertex-absorber $A_{Z^j} \subseteq G$ for Z^j with flexibility εn such that for each $i \in [r] \setminus \{j\}$ and $v \in V^i$, $d_G(v; Z^j) \geq p_{ij}|Z^j|/2$. Moreover, $A := \bigcup_{j \in [r]} A_{Z^j}$ is vertex-disjoint with $|V(A)^1| = \dots = |V(A)^r| \leq 126r\varepsilon n$ and $\mathcal{K}_r(A) \subseteq F_r(G)$.

Theorem 3.6.8. *Let $1/n \ll \xi \ll \delta \ll \varepsilon \ll \{p_{ij}\}_{i \neq j}, 1/r$. Let G be a balanced r-partite complex with parts $V^1, \dots, V^r$ of size n. If G satisfies properties (P1)-(P4) then G contains an F_r-factor.*

Proof. Let $\xi', \varepsilon' > 0$, such that $\xi \ll \xi' \ll \varepsilon' \ll \varepsilon$. Let $(A_{Z^i})_{i \in [r]}$ be the r-vertex-absorbers for $(Z^i)_{i \in [r]}$ with flexibility εn given by property (P4) and let $Z = \bigcup_{i \in [r]} Z^i$ and $A = \bigcup_{i \in [r]} A_{Z^i}$. Let $\widetilde{V} = V(G) \setminus V(A)$ and $\widetilde{G} = G[\widetilde{V}]$. Observe that $|\widetilde{V}^1| = \dots = |\widetilde{V}^r| \geq (1 - 126r\varepsilon)n$. We iteratively apply Proposition 3.6.7 in $\widetilde{G}$ to greedily find a set $\mathcal{F}_1$ of vertex-disjoint faces of $F_r(\widetilde{G})$ until at most $\varepsilon' n$ vertices remain uncovered in each part of $\widetilde{G}$. We denote the set of all uncovered vertices in $\widetilde{G}$ by U. Thus, $|U^1| = \dots = |U^r| \leq \varepsilon' n$.

Next, for each $i \in [r]$ and $u \in U^i$ we will find an $(r-1)$-clique $K_u \in \mathcal{K}_{r-1}(Z \setminus Z^i)$ such that $K_u \cup \{u\} \in F_r(G)$ and all the cliques $\{K_u\}_{u \in U}$ are vertex-disjoint. Indeed, suppose we have already found cliques $\{K_{u'}\}_{u' \in U'}$ for some set of vertices $U' \subseteq U$

and let $u \in U \setminus U'$ and $i \in [r]$ such that $u \in U^i$. By property (P4) we know that $d_G(u; Z^j) \geq p_{ij}|Z^j|/2 = p_{ij}\varepsilon n$ for each $j \in [r]\setminus\{i\}$. Moreover, at most $r\varepsilon' n$ vertices in Z^j have been already used by some clique $K_{u'}$ with $u' \in U'$. Then, using $\varepsilon' \ll \varepsilon$ and (P3) (with u and $N_G(u; Z) \setminus \left(\bigcup_{u' \in U'} V(K_{u'})\right)$ playing the role of v and U) we can find an $(r-1)$-clique $K_u \in \mathcal{K}_{r-1}(Z \setminus Z^i)$ such that $K_u \cup \{u\} \in F_r(G)$ and K_u is vertex-disjoint from all $\{K_{u'}\}_{u' \in U'}$.

Let $\mathcal{F}_2 = \{K_u \cup \{u\} \ : \ u \in U\}$ and note that $\mathcal{F}_2$ is a set of vertex-disjoint faces covering all vertices in U and exactly $(r-1)|U|/r$ vertices in each Z^i, $i \in [r]$. Let $\widetilde{Z}$ be the set of uncovered vertices in Z. We have that $|\widetilde{Z}^1| = \cdots = |\widetilde{Z}^r| = 2\varepsilon n - (r-1)|U|/r \geq 2\varepsilon n - r\varepsilon' n \geq \varepsilon n$, so we can apply Proposition 3.6.7 (with $\widetilde{Z}$ playing the role of U) to greedily find a set $\mathcal{F}_3$ of vertex-disjoint faces until exactly εn vertices remain uncovered in each part of $\widetilde{Z}$. Let X be the set of uncovered vertices. Finally, for each $i \in [r]$ since A_{Z^i} is an r-vertex-absorber for Z^i with flexibility εn, $X^i \subseteq Z$ and $|X^i| = \varepsilon n$, we know that there is an F_r-factor $\mathcal{F}'_i$ of $A_{Z_i} \setminus (Z^i \setminus X^i)$ which covers the εn vertices in X^i together with the vertices in $V(A_{Z^i}) \setminus Z^i$ (that were set aside at the beginning). The union $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \left(\bigcup_{i \in [r]} \mathcal{F}'_i\right)$ is an F_r-factor of G. □

We are now ready to prove Theorems 3.6.1 and 3.6.3. Both proofs will follow the same approach and will be mostly analogous. We will only present the proof of Theorem 3.6.3 and will discuss how it can be adapted to deduce Theorem 3.6.1.

Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-partite complex as given in the statement of Theorem 3.6.3. We will first check that $\mathcal{G}$ satisfies properties (P1)-(P3) using the typicality of $\mathcal{G}$. On the other hand, checking property (P4) will be more elaborate because of the need of finding vertex-absorbers.

We now describe an explicit construction of sparse vertex-absorbers depicted for the non-partite case in [36]. To that end, we will make use of the following auxiliary graph. A *template* $T = (I, J_1, J_2)$ *with flexibility* m is a bipartite graph on $7m$ vertices with parts I and $J_1 \sqcup J_2$ of size $|I| = 3m$ and $|J_1| = |J_2| = 2m$ such that for every subset $J' \subseteq J_1$ of size m, the induced graph $T[V(T) \setminus J']$ has a perfect matching. Such templates were introduced by Montgomery [66] and were proven to exist with maximum degree bounded by 40 if m is large enough (see [63, Lemma 2.8]). Given a template $T = (I, J_1, J_2)$ of flexibility m, an *absorbing structure* $A = (T, \mathcal{K}, X, \mathcal{S}, Z_1, Z_2)$ is a graph obtained by taking the disjoint union of the vertex sets $X = \{x_{ij}\}_{ij \in E(T)}$, $Z_1 = \{z_j\}_{j \in J_1}$ and $Z_2 = \{z_j\}_{j \in J_2}$, and the sets of r-cliques $\mathcal{K} = \{K_i\}_{i \in I}$ and $\mathcal{S} = \{S_{ij}\}_{ij \in E(T)}$, satisfying that for all $i \in I$ and $j \in J$ with $ij \in E(T)$ the following holds.

- $\{x_{ij}\} \cup K_i$ is an $(r+1)$-clique;
- $\{x_{ij}\} \cup S_{ij}$ is an $(r+1)$-clique;
- $\{z_j\} \cup S_{ij}$ is an $(r+1)$-clique.

Observe that $|Z_1| = |Z_2| = 2m$, $|V(\mathcal{K})| = 3mr$, $|X| = |E(T)|$, $|V(\mathcal{S})| = r|E(T)|$ and $|E(T)| \leq 3m\Delta(T)$.

Fact 3.6.9 ([36, Fact 2.6])**.** An absorbing structure $A = (T, \mathcal{K}, X, \mathcal{S}, Z_1, Z_2)$, where T is a template of flexibility m, is an $(r+1)$-vertex-absorber for Z_1 with flexibility m.

Proof of Theorem 3.6.3. Let $\delta, \varepsilon > 0$ such that

$$1/n \ll \xi \ll \delta \ll \varepsilon \ll p_G, p_H, q, 1/r, 1/s, 1/t. \tag{3.20}$$

Our goal is to apply Theorem 3.6.8 to $\mathcal{G}$. For this reason we will check that $\mathcal{G}$ satisfies properties (P1)-(P4) (with $r+1$ playing the role of r). To make things simpler let us write $i \neq j$ instead of $i, j \in [r+1]$, $i \neq j$.

STEP 1: Checking properties (P1)-(P2).

Observe that for every $i \neq j$ and $v \in V^i$

$$d_{\mathcal{G}}(v; V^j) = \begin{cases} (1 \pm \xi) p_G |V^j| & \text{if } i, j \in [r], \\ (1 \pm \xi) p_H |V^j| & \text{if } i = r+1 \text{ or } j = r+1. \end{cases}$$

Similarly, for every $i \neq j$ and $v_1, v_2 \in V^i$

$$d_{\mathcal{G}}(\{v_1, v_2\}; V^j) = \begin{cases} (1 \pm \xi) p_G^2 |V^j| & \text{if } i, j \in [r], \\ (1 \pm \xi) p_H^2 |V^j| & \text{if } i = r+1 \text{ or } j = r+1. \end{cases}$$

Thus, $\mathcal{G}$ satisfies properties (P1)-(P2) with $p_{ij} = p_G$ if $i, j \in [r]$ and $p_{ij} = p_H$ if $i = r+1$ or $j = r+1$.

STEP 2: Checking property (P3).

Let $i \in [r+1]$, $v \in V^i$ and $U \subseteq N_{\mathcal{G}}(v)$ with $|U^j| \geq \varepsilon n$ for each $j \in [r+1] \setminus i$. We want to show that $\mathcal{G}[U]$ contains an r-clique K such that $K \cup \{v\} \subseteq F_{r+1}(\mathcal{G})$. We shall split the proof into two cases.

Case 1: $i = r+1$.

Consider the following algorithm for $k = 0, \ldots, r-1$. Suppose that at step k we have already found a k-clique $X_k = \{x_1, \ldots, x_k\}$ such that $x_j \in U^j$ for each $j \in [k]$ and $d_G(X_k; U^j) \geq p_G^k \varepsilon n / 2^k$ for all $j \in [r] \setminus [k]$.

If $k < r-2$, let $U_k := N_G(X_k; U)$ and note that $|U_k^j| \geq p_G^k \varepsilon n / 2^k$ for every

$j \in [r] \setminus [k]$ by assumption. Let $W := N_{\mathcal{G}}(X_k \cup \{v\})$ and observe that $|W^j| = (1 \pm \xi)p_G^k p_H n$ for any $j \in [r] \setminus [k]$. For any $j_1, j_2 \in [r] \setminus [k]$, $j_1 \neq j_2$ and $w, w_1, w_2 \in W^{j_1}$ we have

$$d_G(w; W^{j_2}) = d_{\mathcal{G}}(X_k \cup \{w, v\}; V^{j_2}) = (1 \pm \xi)p_G^{k+1} p_H n \stackrel{(p5)}{=} (1 \pm 3\xi)p_G |W^{j_2}|$$

and

$$d_G(\{w_1, w_2\}; W^{j_2}) = d_{\mathcal{G}}(X_k \cup \{w_1, w_2, v\}; V^{j_2}) = (1 \pm \xi)p_G^{k+2} p_H n \stackrel{(p5)}{=} (1 \pm 3\xi)p_G^2 |W^{j_2}|.$$

Thus, $G[W]$ satisfies (P1)-(P2) so by Proposition 3.6.6 (with U_k playing the role of U) for each $j \in [r] \setminus [k+1]$ there are at most δn vertices $w \in W^{k+1}$ such that $d_G(w; U_k^j) < p_G |U_k^j|/2$. Since $|U_k^{k+1}| \geq p_G^k \varepsilon n/2$ and using (3.20) we can greedily pick a vertex $x_{k+1} \in U_k^{k+1}$ such that $d_G(w; U_k^j) \geq p_G |U_k^j|/2 \geq p_G^{k+1} \varepsilon n/2^{k+1}$ for each $j \in [r] \setminus [k+1]$. Hence, $X = \{x_1, \ldots, x_{k+1}\}$ is a $(k+1)$-clique such that $d_G(X_{k+1}; U^j) \geq p_G^{k+1} \varepsilon n/2^{k+1}$ for each $j \in [r] \setminus [k+1]$ so we can proceed with the algorithm.

If $k = r-2$, let $U_{r-2} := N_G(X_{r-2}; U)$ and observe that $|U_{r-2}^{r-1}|, |U_{r-2}^r| \geq p_G^{r-2} \varepsilon n/2^{r-2}$ by assumption. Let $W^{r-1} := N_G(X_{r-2} \cup \{v\}; V^{r-1})$ and $W^r := N_G(X_{r-2} \cup \{v\}; V^r)$ and note that $|W^{r-1}|, |W^r| = (1 \pm \xi)p_G^{r-2} p_H n$. Let B be the bipartite graph on $[W^{r-1}, W^r]$ with edge set $\{e \in E(G[W^{r-1}, W^r]) : X_{r-2} \cup e \in F_r(G)\}$. Note that for each $w \in W^{r-1}$

$$d_B(w; W^r) = d_{\mathcal{G}}^*(K, \{K\}; V^r) = (1 \pm \xi)p_G^{r-1} p_H q n \stackrel{(p5)}{=} (1 \pm 3\xi)p_G q |W^r|$$

where $K = X_{r-2} \cup \{w, v\}$, and for each $w_1, w_2 \in W^{r-1}$

$$d_B(\{w_1, w_2\}; W^r) = d^*_{\mathcal{G}}(K_1 \cup K_2, \{K_1, K_2\}; V^r) = (1 \pm \xi) p_G^r p_H q^2 n$$
$$\stackrel{(p5)}{=} (1 \pm 3\xi)(p_G q)^2 |W^r|$$

where $K_1 = X_{r-2} \cup \{w_1, v\}$ and $K_2 = X_{r-2} \cup \{w_2, v\}$. And the same is true when swapping $r-1$ and r. Then B satisfies properties (P1)-(P2) so by Proposition 3.6.6 (with U^r_{r-2} playing the role of U) there are at most δn vertices w in W^{r-1} such that $d_B(w; U^r_{r-2}) < p_G q |U^r_{r-2}|/2$. Since $|U^{r-1}_{r-2}|, |U^r_{r-2}| \geq p_G^{r-2} \varepsilon n / 2^{r-2}$ by assumption and using (3.20) we can greedily pick a vertex $x_{r-1} \in U^{r-1}_{r-2}$ such that $d_B(x_{r-1}; U^r_{r-2}) \geq p_G^{r-1} q \varepsilon n / 2^{r-1}$. Let $x_r \in N_B(x_{r-1}; U^r_{r-2})$ be any vertex and observe that $K := \{x_1, \dots, x_r\}$ is an r-clique of $\mathcal{G}[U_0]$ such that $K \cup \{v\} \in F_{r+1}(\mathcal{G})$ as desired.

Case 2: $i \in [r]$.

The proof is analogous to the previous case so we will highlight the main differences. Suppose without loss of generality that $i = r$. We run the algorithm for $k = 0, \dots, r-2$ to find an $(r-1)$-clique $X_{r-1} = \{x_1, \dots, x_{r-1}\}$ of $\mathcal{G}[U_0]$ such that $X_{r-1} \cup \{v\} \in F_r(G)$. During the algorithm we replace the condition $d_G(X_k; U^r) \geq p_G^k \varepsilon n / 2^k$ by $d_G(X_k; U^{r+1}) \geq p_H^k \varepsilon n / 2^k$ for all $k = 1, \dots, r-2$. We then pick the vertex x_{r-1} so that $d_G(X_{r-1}; U^{r+1}) \geq p_H^{r-1} \varepsilon n / 2^{r-1}$. We finish by picking any vertex $x_{r+1} \in N_G(X_r; U^{r+1})$ and letting $K := X_{r-1} \cup x_{r+1}$. Since $X_{r-1} \cup \{v\} \in F_r(G)$ then $X_{r-1} \cup \{v, x_{r+1}\} \in F_{r+1}(\mathcal{G})$ as $\mathcal{G} = (V, V^{r+1}, G, H)$ is an extended $(r+1)$-complex. Then K is an r-clique such that $K \cup \{v\} \in F_{r+1}(\mathcal{G})$ which concludes property (P3).

Step 3: Checking property (P4).

It only remains to check that $\mathcal{G}$ satisfies (P4). Let $m = \varepsilon n$ and let $T =$

$(I, J_1 \sqcup J_2)$ be a template with flexibility m and $\Delta(T) \leq 40$. Suppose we have already found disjoint vertex absorbers $\{A_i\}_{i\in[k-1]}$ for sets $\{Z_1^i\}_{i\in[k-1]}$ with $|V(A_i)| \leq 126(r+1)m$ for some $k \in [r+1]$ and let $B = \bigcup_{i\in[k-1]} V(A_i)$. Then $|B| \leq 126(r+1)^2 m$. Let Z_1^k be a subset of size $2m$ chosen uniformly at random from $V^k \setminus B^k$. Lemma 2.1.2 and a union bound implies that with positive probability for each $v \in V(\mathcal{G}) \setminus V^k$

$$d_{\mathcal{G}}(v; Z_1^k) = (1 \pm n^{-1/3})(2m/(n - |B^k|)) d_{\mathcal{G}}(v; V^k \setminus B^k).$$

Fix such choice of Z_1^k. Let $j \in [r+1] \setminus \{k\}$ and note that

$$\begin{aligned} d_{\mathcal{G}}(v; Z_1^k) &\geq (1 - n^{-1/3}) 2\varepsilon d_{\mathcal{G}}(v; V^k \setminus B^k) \geq (1 - n^{-1/3}) 2\varepsilon \left(d_{\mathcal{G}}(v; V^k) - d_{\mathcal{G}}(v; B^k) \right) \\ &\geq (1 - n^{-1/3}) 2\varepsilon \left((1 - \xi) p_G n - 126(r+1)^2 \varepsilon n \right) \geq (1 - n^{-1/3}) 2\varepsilon (1 - \varepsilon^{1/2}) p_G n \\ &\geq 2\varepsilon p_G n/2 = p_G |Z_1^k|/2 \end{aligned}$$

if $j, k \in [r]$, and similarly

$$d_{\mathcal{G}}(v; V^k \setminus B^k) \geq p_H |Z_1^k|/2$$

if $j = r+1$ or $k = r+1$.

Let Z_2 be a set of $2m$ vertices in $V^k \setminus (Z_1^k \cup B^k)$ and let $\mathcal{K} = \{K_i\}_{i\in I}$ be a set of $3m$ vertex-disjoint r-cliques in $\mathcal{K}_r(\mathcal{G} \setminus \mathcal{G}[V^k \cup B])$ if $k \in [r]$ and in $F_r(G \setminus G[V^k \cup B])$ if $k = r+1$. We can find such cliques greedily because of (3.20), $|B| \leq 126(r+1)^2 \varepsilon n$, $3m = 3\varepsilon n$ and $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical. Let $\{z_j\}_{j\in J_1}$ and $\{z_j\}_{j\in J_2}$ be a labelling of the vertices in Z_1^k and Z_2 using the indices in J_1 and J_2 respectively. Next, we will find a set $\mathcal{S} = \{S_{ij}\}_{ij\in E(T)}$ of r-cliques such that $S_{ij} \cup \{z_j\} \in F_{r+1}(\mathcal{G})$ for each $i \in I$, $j \in J$. Suppose we have already found a set of r-cliques $\mathcal{S}' = \{S_{ij}\}_{ij \in E'}$ for some subset $E' \subseteq E(T)$. Let $ij \in E(T) \setminus E'$. We can then use property (P3) (with $N_{\mathcal{G}}(z_j) \setminus (B \cup V(\mathcal{S}'))$ playing the role of U) to find an r-clique S_{ij} vertex-disjoint of $V(\mathcal{S}')$ and B such that $S_{ij} \cup \{z_j\} \in F_{r+1}(\mathcal{G})$. Finally, for each $ij \in E(T)$ we greedily pick a distinct vertex x_{ij} from $N^*_{\mathcal{G}}(S_{ij} \cup K_i, \{S_{ij}, K_i\}; V^k \setminus (Z_1 \cup Z_2 \cup B))$. We can greedily do this because $d^*_{\mathcal{G}}(S_{ij} \cup K_i, \{S_{ij}, K_i\}; V^k) - |Z_1^k \cup Z_2| - |B| - |E(T)| > 0$ since $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical, $|Z_1^k|, |Z_2|, |B|, |E(T)| < \varepsilon^{1/2} n$ and $\varepsilon \ll p_G, p_H, q$. Hence, we can find a set of vertices $X = \{x_{ij}\}_{ij \in E(T)}$ in $V^k \setminus (Z_1^k \cup Z_2 \cup B^k)$ such that $\{x_{ij}\} \cup K_i$ and $\{x_{ij}\} \cup S_{ij} \in F_{r+1}(\mathcal{G})$ for all $i \in I, j \in J$. All in all, we have found a subcomplex $A_k \subseteq \mathcal{G}$ vertex-disjoint from B such that $A_k = (T, \mathcal{K}, X, \mathcal{S}, Z_1^k, Z_2)$ is an absorbing structure satisfying $\mathcal{K}_{r+1}(A_k) \subseteq F_{r+1}(\mathcal{G})$. By Fact 3.6.9, A_T is an $(r+1)$-vertex-absorber for Z_1^k with flexibility m. Moreover, note that $|V(A_k)^k| = |Z_1^k| + |Z_2| + |X| = 4m + |E(T)|$ and $|V(A_k)^j| = 3m + |E(T)|$ for each $j \in [r+1] \setminus \{k\}$.

At the end of the procedure we will have found a set $Z_1 = \bigcup_{j \in [r+1]} Z_1^j$ and a subcomplex $A = \bigcup_{j \in [r+1]} A_j$ satisfying that for each $j \in [r]$ A_j is an $(r+1)$-vertex-absorber for Z_1^j with flexibility $m = \varepsilon n$ such that $d_{\mathcal{G}}(v, Z_1^j) \geq p_{ij} |Z_1^j|/2$ for each $v \in V(\mathcal{G}) \setminus V^j$; $A_1, \ldots, A_{r+1}$ are vertex-disjoint; $|V(A)^1| = \cdots = |V(A^{r+1})| = 4m + |E(T)| + r(3m + |E(T)|) \leq 124(r+1)\varepsilon n$; and $\mathcal{K}_{r+1}(A) \subseteq F_{r+1}(\mathcal{G})$. This concludes property (P4). □

The proof of Theorem 3.6.1 follows exactly the same approach as the one above, which is checking properties (P1)-(P4) and then applying Theorem 3.6.8. However, unlike the extended $(r+1)$-partite complex $\mathcal{G}$ which didn't have the typicality property symmetric among all parts (because we might have $p_G \neq p_H$), the r-partite complex G given in the statement of Theorem 3.6.1 has a symmetric typicality property among all its parts (i.e., $p_{ij} = p$ for all $i \neq j$). This results in a simpler proof when checking properties (P1)-(P4) since there is no need to consider different cases depending on whether a vertex belongs to V^{r+1} or not.

3.7 Edge-disjoint factors

In this section we use Corollaries 3.6.2 and 3.6.4 together with a randomised algorithm to find many edge-disjoint face factors in an extended complex. This result will be used during the proof of the Cover Down Lemma. The proof follows a method introduced by Barber, Kühn, Lo and Osthus [9, Lemma 10.7] to find edge-disjoint K_r-factors in a graph with high minimum degree.

Lemma 3.7.1. *Let $1/n \ll \gamma \ll \rho \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2(r-1)$ and $t \geq 2$ and let $N, M \in \mathbb{N}$. Let V and U be sets of vertices such that $|V| = n$ and $|U| = (1 \pm \xi)(p_G/p_H)n/(r-1)$. For each $i \in [N]$ let $\mathcal{G}_i = (V_i, U_i, G_i, H_i)$ be a (ξ, p_G, p_H, q, s, t)-typical extended r-complex and for each $N < i \leq N + M$ let $\mathcal{G}_i$ be a (ξ, p_G, q, s, t)-typical r-complex on a set of vertices V_i. Suppose that*

(F1) $V_i \subseteq V$ for all $i \in [N+M]$ and $U_i \subseteq U$ for all $i \in [N]$,

(F2) $|V_i| = (r-1)|U_i|$ for all $i \in [N]$ and $r \mid |V_i|$ for all $N < i \leq N + M$,

(F3) $|V_i| \geq \rho^{4/3} n$ for all $i \in [N+M]$,

(F4) $|V_i \cap V_j| \leq \rho^2 n$ for all $1 \leq i < j \leq N + M$ and $|U_i \cap U_j| \leq \rho^2 n$ for all $1 \leq i < j \leq N$,

(F5) every vertex $v \in V$, $u \in U$ is contained in at most ρn of the sets $\{V_i\}_{i \in [N+M]}$, $\{U_i\}_{i \in [N]}$ respectively.

Then for each $i \in [N+M]$ there exists an F_r-factor $\mathcal{F}_i$ of $\mathcal{G}_i$ such that all the factors $\{\mathcal{F}_i\}_{i \in [N+M]}$ are edge-disjoint.

Proof. Let $m := 4\rho^{3/2}n$. The proof consists of a random algorithm with steps $i = 1, \dots, N+M$ that works as follows. Suppose that we have already found edge-disjoint F_r-factors $\mathcal{F}_1, \dots, \mathcal{F}_{i-1}$ of $\mathcal{G}_1, \dots, \mathcal{G}_{i-1}$ respectively for some $i \in [N+M]$. We will find an F_r-factor $\mathcal{F}_i$ of $\mathcal{G}_i$, edge-disjoint from all $\mathcal{F}_1, \dots, \mathcal{F}_{i-1}$, as follows.

Let $\mathcal{H}_i := \bigcup_{j=1}^{i-1} \mathcal{F}_j$ and let $\mathcal{G}'_i := \mathcal{G}_i - \mathcal{H}_i$. We now consider two cases depending on the maximum degree of $\mathcal{H}_i$.

Case 1: $\Delta(\mathcal{H}_i) \leq \rho^{3/2}n$.

If $i \in [N]$, $\mathcal{G}'_i$ is an extended r-complex (V_i, U_i, G'_i, H'_i). Observe that for every set $S_V \subseteq V_i$ and $S_U \subseteq U_i$ with $|S_V| + |S_U| \leq s$ and for every set $T \subseteq \mathcal{K}_{r-2}(G'_i[S_V])$ with $|T| \leq t$,

$$d^*_{\mathcal{G}_i}(S_V \cup S_U, T; V_i) \geq d^*_{\mathcal{G}'_i}(S_V \cup S_U, T; V_i) \geq d^*_{\mathcal{G}_i}(S_V \cup S_U, T; V_i) - s\Delta(\mathcal{H}_i)$$

so

$$(1+\xi')p_G^{|S_V|}p_H^{|S_U|}q^{|T|}|V_i| \geq d^*_{\mathcal{G}'_i}(S_V \cup S_U, T; V_i) \geq (1-\xi')p_G^{|S_V|}p_H^{|S_U|}q^{|T|}|V_i| + rsm,$$

and for every set $S_V \subseteq V_i$ with $|S_V| \leq s$,

$$d_{H_i}(S_V) \geq d_{H'_i}(S_V) \geq d_{H_i}(S_V) - s\Delta(\mathcal{H}_i)$$

so

$$(1+\xi')p_H^{|S_V|}|U_i| \geq d_{H_i'}(S_V) \geq (1-\xi')p_H^{|S_V|}|U_i| + rsm.$$

In particular, $\mathcal{G}_i'$ is $(\xi', p_G, p_H, q, s, t)$-typical. Moreover, we can successively apply Corollary 3.6.4 with $\mathcal{G}_i'$, ξ' and r playing the role of $\mathcal{G}$, ξ and $r+1$ to find m edge-disjoint F_r-factors $\mathcal{A}_1, \dots, \mathcal{A}_m$ of $\mathcal{G}_i'$ which will be candidates for $\mathcal{F}_i$.

If $N < i \leq N+M$, $\mathcal{G}_i'$ is an r-complex on V_i. Observe that for every set $S_V \subseteq V_i$ with $|S_V| \leq s$ and for every set $T \subseteq \mathcal{K}_{r-1}(G_i'[S_V])$ with $|T| \leq t$,

$$d^*_{\mathcal{G}_i}(S_V, T; V_i) \geq d^*_{\mathcal{G}_i'}(S_V, T; V_i) \geq d^*_{\mathcal{G}_i}(S_V, T; V_i) - s\Delta(\mathcal{H}_i)$$

so

$$(1+\xi)p_G^{|S_V|}q^{|T|}|V_i| \geq d^*_{\mathcal{G}_i'}(S_V, T; V_i) \geq (1-\xi')p_G^{|S_V|}q^{|T|}|V_i| + rsm.$$

Thus, $\mathcal{G}_i'$ is (ξ', p_G, q, s, t)-typical so we can successively apply Corollary 3.6.2 with $\mathcal{G}_i'$, ξ' playing the role of $\mathcal{G}$, ξ to find m edge-disjoint F_r-factors $\mathcal{A}_1, \dots, \mathcal{A}_m$ of $\mathcal{G}_i'$ which will be candidates for $\mathcal{F}_i$.

Case 2: $\Delta(\mathcal{H}_i) \geq \rho^{3/2}n$.

If $i \in [N]$ let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be defined as r-complexes on $V_i \cup U_i$ with no edges and if $N < i \leq N+M$ let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be defined as r-complexes on V_i with no edges.

In both cases $\mathcal{A}_1, \dots, \mathcal{A}_m$ are edge-disjoint subcomplexes of $\mathcal{G}_i'$. Next we choose $j \in [m]$ uniformly at random, set $\mathcal{F}_i := \mathcal{A}_i$ and continue with the next iteration. Observe that at the end of the algorithm all of $\mathcal{F}_1, \dots, \mathcal{F}_{N+M}$ will be edge-disjoint

F_r-factors of $\mathcal{G}_1, \ldots, \mathcal{G}_{N+M}$ if and only if

$$\Delta(\mathcal{H}_i) \leq \rho^{3/2} n \tag{3.21}$$

for all $i \in [N+M]$. Hence, the lemma follows if equation (3.21) holds for all $i \in [N+M]$ with positive probability.

For each $i \in [N+M]$ and $x \in V(\mathcal{G}_i)$ let $J^{i,x} := \{j \in [i-1] \ : \ x \in V(\mathcal{G}_j)\}$ and for each $j \in J^{i,x}$ let $Y_j^{i,x}$ be the indicator variable of the event $xy \in E(\mathcal{F}_j)$ for some $y \in V(\mathcal{G}_i)$. Observe that

$$d_{\mathcal{H}_i[V(\mathcal{G}_i) \cap V(\mathcal{H}_i)]}(x) = \sum_{j \in J^{i,x}} Y_j^{i,x}. \tag{3.22}$$

Fix $i \in [N+M]$ and $x \in V(\mathcal{G}_i)$. Note that (F4) implies that for each $j \in J^{i,x}$ at most $2\rho^2 n$ of the edge-disjoint complexes $\mathcal{A}_1, \ldots, \mathcal{A}_m$ that were candidates for $\mathcal{F}_j$ share an edge incident to x with $\mathcal{G}_i$. Let $j_1, \ldots, j_{|J^{i,x}|}$ be the elements of $J^{i,x}$ listed in increasing order. Then, for all $\ell \in [|J^{i,x}|]$ we have

$$\mathbb{P}\left[Y_{j_\ell}^{i,x} = 1 \mid Y_{j_1}^{i,x}, \ldots, Y_{j_{\ell-1}}^{i,x}\right] \leq \frac{2\rho^2 n}{m} \leq \frac{\rho^{1/2}}{2}.$$

Let $B \sim Bin(|J^{i,x}|, \rho^{1/2}/2)$ and observe that

$$\mathbb{E}[B] = |J^{i,x}| \frac{\rho^{1/2}}{2} \overset{(F5)}{\leq} \frac{\rho^{3/2} n}{2}.$$

Thus, by applying Fact 2.1.3 and Lemma 2.1.2 we obtain that

$$\mathbb{P}\left[\sum_{j\in J^{i,x}} Y_j^{i,x} > \rho^{3/2}n\right] \le \mathbb{P}\left[B > \rho^{3/2}n\right] \le \mathbb{P}\left[B > \mathbb{E}\left[B\right] + \rho^{3/2}n/2\right] \le 2e^{-\rho^3 n/2}.$$

Hence, since there are at most $\rho n|V(\mathcal{G})|$ pairs (i,x) with $x \in V(\mathcal{G}_i)$ (for each $x \in V(\mathcal{G})$ we use (F5)), and $|V(\mathcal{G})| \le |V| + |U| \le (1 + p_G/p_H)n$, a union bound implies that with positive probability $\sum_{j\in J^{i,x}} Y_j^{i,x} \le \rho^{3/2}n$ for all $i \in [N+M]$ and $x \in V(\mathcal{G}_i)$. Thus, by (3.22), it follows that equation (3.21) holds with positive probability for every $i \in [N+M]$, which concludes the proof. □

3.8 Cover Down Lemma

In this section we will prove the Cover Down Lemma (Lemma 3.8.3) which is, together with Lemma 3.3.2, one of the key results to prove the main theorem of this chapter. In order to do so, we will need the following two results. Recall that, given an r-complex G, a k-clique $K \in \mathcal{K}_k(G)$ with $k < r$, and a set $W \subseteq V(G)$, $\mathrm{Lk}_G(K;W)$ denotes the $(r-k)$-complex (V,E,F) where $V = N_G(K;W)$, $E = E(G[V])$ and $F = \{K' \in \mathcal{K}_{r-k}(G[V]) : K' \cup K \in F_r(G)\}$.

Lemma 3.8.1. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. Let $\mathcal{G} = (V,U,G,H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$. Let $V_1 \subseteq V$ and suppose that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into V_1. Let $K \cup \{u\}$ be a $(k+1)$-clique of $G \cup H$ where $0 \le k \le r$, $K \subseteq \mathcal{K}_k(G)$ and $u \in U$. Then $Lk_{\mathcal{G}}(K \cup \{u\}; V_1)$ is a $(3\xi, p_G, q, s-k-1, t)$-typical $(r-k)$-complex.*

Proof. Let $L = \mathrm{Lk}_{\mathcal{G}}(K \cup \{u\}; V_1)$. Let $S \subseteq V(L)$ with $|S| \le s-k-1$ and

$T' \subseteq \mathcal{K}_{r-k-1}(L[S])$ with $|T'| \leq t$. Let $T = \{K \cup K' : K' \in T'\}$ and note that $T \subseteq \mathcal{K}_{r-1}(G[S \cup K])$ and $|T| = |T'|$. Observe that

$$|V(L)| = d_{\mathcal{G}}(K \cup \{u\}; V_1) = (1 \pm \xi) p_G^k p_H |V_1|. \tag{3.23}$$

Then

$$\begin{aligned} d_L^*(S, T') &= d_{\mathcal{G}}^*(S \cup K \cup \{u\}, T; V_1) = (1 \pm \xi) p_G^{|S|+k} p_H q^{|T|} |V_1| \\ &\overset{(3.23),(p5)}{=} (1 \pm 3\xi) p_G^{|S|} q^{|T'|} V(L) \end{aligned}$$

which implies that L is $(3\xi, p_G, q, s-k-1, t)$-typical. □

Lemma 3.8.2. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$. Let $V_1 \subseteq V$ and $U_1 \subseteq U$ and suppose that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V_1 \cup U_1$. Let K be a k-clique of G where $0 \leq k \leq r-1$. Then $Lk_{\mathcal{G}}(K; V_1 \cup U_1)$ is a $(3\xi, p_G, p_H, q, s-k, t)$-typical extended $(r+1-k)$-complex (V', U', G', H') where $V' = N_{\mathcal{G}}(K; V_1)$, $U' = N_{\mathcal{G}}(K; U_1)$, $G' = Lk_G(K; V_1)$ and $H' = H[V', U']$.*

Proof. Let $L = \mathrm{Lk}_{\mathcal{G}}(K; V_1 \cup U_1)$, $V' = N_{\mathcal{G}}(K; V_1)$, $U' = N_{\mathcal{G}}(K; U_1)$, $G' = \mathrm{Lk}_G(K; V_1)$ and $H' = H[V', U']$. First we check that $L = (V', U', G', H')$ is an extended $(r+1-k)$-complex. Clearly, $V(L) = N_{\mathcal{G}}(K; V_1 \cup U_1) = V' \sqcup U'$. On the other hand, $E(L) = E(\mathcal{G}[V' \cup U']) = E(G[V']) \sqcup E(H[V', U']) = E(G') \sqcup E(H')$.

Finally we have

$$\begin{aligned} F_{r+1-k}(L) &= \{K' \in \mathcal{K}_{r+1-k}(\mathcal{G}[V' \cup U']) \,:\, K' \cup K \in F_{r+1}(\mathcal{G})\} \\ &= \{K' \cup \{u\} \in \mathcal{K}_{r+1-k}(\mathcal{G}[V' \cup U']) \,:\, K' \cup K \in F_r(G), u \in U\} \\ &= \{K' \cup \{u\} \in \mathcal{K}_{r+1-k}(G[V'] \cup H[V', U']) \,:\, K' \cup K \in F_r(G), u \in U\} \\ &= \{K' \cup \{u\} \in \mathcal{K}_{r+1-k}(G' \cup H') \,:\, K' \in F_r(G'), u \in U\}. \end{aligned}$$

We will now check that L is $(2\xi, p_G, p_H, q, s-k, t)$-typical. Observe that

$$|V'| = d_{\mathcal{G}}(K; V_1) = (1 \pm \xi) p_G^k |V_1| \tag{3.24}$$

and

$$|U'| = d_{\mathcal{G}}(K; U_1) = (1 \pm \xi) p_H^k |U_1|. \tag{3.25}$$

Let $S_V \subseteq V'$ and $S_U \subseteq U'$ with $|S_V| + |S_U| \leq s-k$ and $T' \subseteq \mathcal{K}_{r-k-1}(G'[S_V])$ with $|T'| \leq t$. Let $T = \{K \cup K' \,:\, K' \in T'\}$ and note that $T \subseteq \mathcal{K}_{r-1}(G[S_V \cup K])$ and $|T| = |T'|$. Then

$$\begin{aligned} d^*_L(S_V \cup S_U, T'; V') = d^*_{\mathcal{G}}(S_V \cup K \cup S_U, T; V_1) &= (1 \pm \xi) p_G^{|S_V|+k} p_H^{|S_U|} q^{|T|} |V_1| \\ &\overset{(3.24),(p5)}{=} (1 \pm 3\xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T'|} |V'|. \end{aligned} \tag{3.26}$$

Let $S_V \subseteq V'$ with $|S_V| \leq s-k$. Then

$$d_{H'}(S_V) = d_H(S_V \cup K; U_1) = (1 \pm \xi) p_H^{|S_V|+k} |U_1| \overset{(3.25),(p5)}{=} (1 \pm 3\xi) p_H^{|S_V|} |U'|. \tag{3.27}$$

Equations (3.26) and (3.27) imply that L is $(3\xi, p_G, p_H, q, s-k, t)$-typical. □

We have now all the ingredients to prove the Cover Down Lemma:

Lemma 3.8.3 (Cover Down Lemma). *Let $1/n \ll \eta \ll \gamma' \ll \varepsilon \ll \xi \ll p_G, p_H, q,$ $1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq r^2 + r$. Let $\mathcal{G} = (V, U, G, H)$ be an η-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex. Let $V_1 \subseteq V$ and $U_1 \subseteq U$ be subsets satisfying*

(C1) $|V_1| = \varepsilon|V| = \varepsilon n$ and $|U_1| = \varepsilon|U|$,

(C2) $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V_1 \cup U_1$,

(C3) for every $v \in V$, $d_G(v; V_1) = (1 \pm \eta)(r-1)d_H(v; U_1)$,

(C4) for every $v \in V \setminus V_1$, $d_G(v) = (r-1)d_H(v)$,

(C5) for every $u \in U \setminus U_1$, $r \mid d_H(u)$.

Then there is an F_{r+1}-decomposable subcomplex $\mathcal{H} \subseteq \mathcal{G}$ such that $E(\mathcal{G} - \mathcal{G}[V_1 \cup U_1]) \subseteq E(\mathcal{H})$ and $\Delta(\mathcal{H}[V_1 \cup U_1]) \leq \gamma'|V_1 \cup U_1|$.

Proof. Let $\gamma, \rho, \xi' > 0$ such that

$$1/n \ll \eta \ll \gamma \ll \rho \ll \gamma' \ll \varepsilon \ll \xi \ll \xi' \ll p_G, p_H, q, 1/s, 1/t, 1/r. \tag{3.28}$$

Let $V' := V \setminus V_1$ and $U' := U \setminus U_1$. Using (C2) we have that for every $S_V \subseteq V$ and $S_U \subseteq U$ with $|S_V| + |S_U| \leq s$ and for every $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \leq t$

$$\begin{aligned} d^*_G(S_V \cup S_U, T; V') &= d^*_G(S_V \cup S_U, T; V) - d^*_G(S_V \cup S_U, T; V_1) \\ &= (1 \pm 2\xi)p_G^{|S_V|}p_H^{|S_U|}q^{|T|}|V'| \end{aligned}$$

and for every $S_V \subseteq V$ with $|S_V| \leq s$

$$d_H(S_V; U') = d_H(S_V; U) - d_H(S_V; U_1) = (1 \pm 2\xi)p_H|U'|.$$

So $\mathcal{G}$ is $(2\xi, p_G, p_H, q, s, t)$-typical into $V' \cup U'$.

Using (C3) we obtain that for every $v \in V_1$

$$
\begin{aligned}
d_G(v; V') &= d_G(v) - d_G(v; V_1) = (1 \pm \eta)(r-1)d_H(v) - (1 \pm \eta)(r-1)d_H(v; U_1) \\
&= (1 \pm 2\eta)(r-1)d_H(v; U').
\end{aligned}
$$

To summarise, we have

(P1) $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V_1 \cup U_1$,

(P2) $\mathcal{G}$ is $(2\xi, p_G, p_H, q, s, t)$-typical into $V' \cup U'$,

(P3) for every $v \in V'$, $d_G(v; V_1) = (1 \pm \eta)(r-1)d_G(v; U_1)$,

(P4) for every $v \in V_1$, $d_G(v; V') = (1 \pm 2\eta)(r-1)d_G(v; U')$,

(P5) by Lemma 3.8.1, for every k-clique K of $G \cup H$ with $1 \leq k \leq r$ and $|K \cap U| = 1$, $\mathrm{Lk}_{\mathcal{G}}(K; V_1)$ is a $(3\xi, p_G, p_H, q, s-k, t)$-typical $(r+1-k)$-complex,

(P6) by Lemma 3.8.2, for every k-clique K of G with $1 \leq k \leq r-1$, $\mathrm{Lk}_{\mathcal{G}}(K, V_1 \cup U_1)$ is a $(3\xi, p_G, p_H, q, s-k, t)$-typical extended $(r+1-k)$-complex (V_K, U_K, G_K, H_K) where $V_K = N_G(K; V_1)$, $U_K = N_H(K; U_1)$, $G_K = \mathrm{Lk}_G(K; V_1)$ and $H_K = H[V_K, U_K]$.

Step 1: Choosing the reservoir edges.

Let $\mathcal{R} = (V, U, R_G, R_H) \subseteq \mathcal{G}$ be a subcomplex induced by a set of edges obtained by choosing each edge independently at random with probability ρ. Let $R_1 = R_G[V', V_1]$, $R_2 = R_H[V', U_1]$ and $R_3 = R_H[V_1, U']$. Let $\widetilde{\mathcal{G}} = (V, U, \widetilde{G}, \widetilde{H})$ where $\widetilde{G} = G - R_1 - G[V_1]$ and $\widetilde{H} = H - R_2 - R_3 - H[V_1, U_1]$. Note that properties (P1)-(P6) consist all of degree sizes and that each property describes the neighbourhood size of at most $\binom{s}{r}^t(n+|U|)^s < n^{3s}$ combinations of vertices and cliques. We can then apply Proposition 2.1.4 to the random variables given by each degree size described in (P1)-(P6) via $\mathcal{R}$ to see that with positive probability the following properties hold:

(R1) $\mathcal{R}$ is $(2\xi, \rho p_G, \rho p_H, q, s, t)$-typical into $V_1 \cup U_1$,

(R2) $\mathcal{R}$ is $(3\xi, \rho p_G, \rho p_H, q, s, t)$-typical into $V' \cup U'$,

(R3) for every $v \in V'$, $d_{R_1}(v) = (1 \pm 2\eta)(r-1)d_{R_2}(v)$,

(R4) for every $v \in V_1$, $d_{R_1}(v) = (1 \pm 3\eta)(r-1)d_{R_3}(v)$,

(R5) for every k-clique K of $G \cup H$ with $1 \leq k \leq r$ and $|K \cap U| = 1$, $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[K])}(K; V_1)$ is a $(6\xi, p_G, p_H, q, s-k, t)$-typical $(r+1-k)$-complex.

(R6) for every k-clique K of G with $1 \leq k \leq r-1$, $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[K])}(K, V_1 \cup U_1)$ is a $(6\xi, p_G, p_H, q, s-k, t)$-typical extended $(r+1-k)$-complex (V_K, U_K, G_K, H_K) where $V_K = N_{G-(\widetilde{G}-\widetilde{G}[K])}(K; V_1)$, $U_K = N_{H-(\widetilde{H}-\widetilde{H}[K])}(K; U_1)$, $G_K = \mathrm{Lk}_{G-(\widetilde{G}-\widetilde{G}[K])}(K; V_1)$ and $H_K = (H - (\widetilde{H} - \widetilde{H}[K]))[V_K, U_K]$.

Fix such a choice of $\mathcal{R}$.

Step 2: Approximate decomposition.

Observe that since $\eta, \rho, \varepsilon \ll \xi$, $\Delta(R_i) \leq 2\rho(p_G + p_H)n$ for each $i \in [3]$, $\Delta(G[V_1]) \leq 2\varepsilon p_G n$ and $\Delta(H[V_1, U_1]) \leq 2\varepsilon p_H n$, then $\widetilde{\mathcal{G}}$ is $(2\xi, p_G, p_H, q, s, t)$-typical. On the other hand, for every vertex $v \in V'$

$$\begin{aligned} d_{\widetilde{G}}(v) &= d_G(v) - d_{R_1}(v) \\ &= (1 \pm \eta)(r-1)d_H(v) - (1 \pm 2\eta)(r-1)d_{R_2}(v) \\ &= (1 \pm 3\eta)(r-1)d_{\widetilde{H}}(v) \end{aligned} \tag{3.29}$$

where we have used (R3), and for every $v \in V_1$

$$\begin{aligned} d_{\widetilde{G}}(v) &= d_G(v) - d_{R_1}(v) - d_G(v; V_1) \\ &= (1 \pm \eta)(r-1)d_H(v) - (1 \pm 3\eta)(r-1)d_{R_3}(v) \\ &\qquad - (1 \pm \eta)(r-1)d_H(v; U_1) \\ &= (1 \pm 3\eta)(r-1)d_{\widetilde{H}}(v) \end{aligned} \tag{3.30}$$

where we have used (R4) and (C3). Thus, $\widetilde{\mathcal{G}}$ is 3η-divisible. We can then apply Lemma 3.4.1 to $\widetilde{\mathcal{G}}$ to find an F_{r+1}-decomposable complex $\mathcal{F}_1$ such that $\Delta(\widetilde{\mathcal{G}} - \mathcal{F}_1) \leq \gamma^2 n$. Let L be the graph induced by the edges of $\widetilde{\mathcal{G}} - \mathcal{F}_1$ and note that

$$\Delta(L) \leq \gamma n. \tag{3.31}$$

Observe that

$$E(\mathcal{G} - \mathcal{F}_1) = E(L) \cup E(R_1) \cup E(R_2) \cup E(R_3) \cup E(\mathcal{G}[V_1 \cup U_1]) \tag{3.32}$$

and all the unions are disjoint.

Step 3: Cover down edges in L using edges from the reservoir.

Let $\mathcal{G}_1 = \mathcal{G} - \mathcal{F}_1$. We will now find an F_{r+1}-decomposable complex $\mathcal{F}_2 \subseteq \mathcal{G}_1$ with small maximum degree and such that $E(L) \subseteq E(\mathcal{F}_2)$.

Claim 1: For each $e \in E(L)$ there is a face $f_e \in F_{r+1}(\mathcal{G}_1)$ with $e \subseteq f_e$ such that all the faces $\{f_e\}_{e \in E(L)}$ are edge-disjoint. Moreover, if $\mathcal{F}_2$ is the complex induced by the faces in $\{f_e\}_{e \in E(L)}$ then

$$\Delta(\mathcal{F}_2) \leq r\gamma^{1/3} n. \tag{3.33}$$

Proof of claim: Suppose we have already chosen edge-disjoint faces $\{f_{e'}\}_{e' \in E(L')}$ for some subgraph $L' \subseteq L$. Let $\mathcal{F}' = \bigcup_{e' \in E(L')} f_{e'}$ and suppose that

$$\Delta(\mathcal{F}') \leq r\gamma^{1/3} n. \tag{3.34}$$

We say that a vertex $x \in V_1 \cup U_1$ is *good* if there are less than $\gamma^{1/3}n - 1$ faces in $\mathcal{F}'$ containing x. Otherwise we say that it is *bad*. Let X be the set of bad vertices and suppose that $|X| > \gamma^{1/3}n$. By definition, for each bad vertex $x \in X$ there are at least $\gamma^{1/3}n - 1$ edge-disjoint faces in $\mathcal{F}'$ containing x, which yields $d_{\mathcal{F}'}(x) \geq r\gamma^{1/3}n - r$. It follows that

$$|E(\mathcal{F}')| \geq \frac{1}{2} \sum_{x \in X} d_{\mathcal{F}'}(x) \geq \frac{1}{2}(\gamma^{1/3}n)(r\gamma^{1/3}n - r) \geq \frac{r}{4}\gamma^{2/3}n^2.$$

On the other hand, we have that

$$|E(\mathcal{F}')| \leq \binom{r+1}{2}|E(L)| \overset{(3.31)}{\leq} \binom{r+1}{2}\gamma n^2$$

which is a contradiction.

Hence we have

$$|X| \leq \gamma^{1/3} n. \tag{3.35}$$

Let $e \in E(L - L')$.

Suppose first that $e = vu$ with $v \in V$ and $u \in U$. We know by (R5) that $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[e])}(e; V_1)$ is a $(6\xi, p_G, p_H, q, s-2, t)$-typical $(r-1)$-complex. Let $\mathcal{Y}_e$ be the complex obtained from $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[e])}(e; V_1)$ by removing all edges that are in $E(\mathcal{F}')$ and all bad vertices. Using (3.34)-(3.35) and $\gamma \ll \rho, \xi$ we deduce that $\mathcal{Y}_e$ is a $(7\xi, p_G, p_H, q, s-2, t)$-typical $(r-1)$-complex. This allows us to greedily pick an $(r-1)$-face f of $\mathcal{Y}_e$. Let $f_e = f \cup e$ and observe that f_e is a face of $\mathcal{G} - \mathcal{F}_1$ which is edge-disjoint from $E(\mathcal{F}')$ and vertex-disjoint from X.

Suppose now that $e = v_1v_2$ with $v_1, v_2 \in V$. Using (R6) we know that $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[e])}(e; V_1 \cup U_1)$ is a $(6\xi, p_G, p_H, q, s-2, t)$-typical extended $(r-1)$-complex (V_e, U_e, G_e, H_e) where $V_e = N_{G-(\widetilde{G}-\widetilde{G}[e])}(e; V_1)$ and $U_e = N_{H-(\widetilde{H}-\widetilde{H}[e])}(e; U_1)$. Let $\mathcal{Y}_e$ be the extended complex obtained from $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[e])}(e; V_1 \cup U_1)$ by removing all edges that are in $E(\mathcal{F}')$ and all bad vertices. Using (3.34)-(3.35) and $\gamma \ll \rho, \xi$ we deduce that $\mathcal{Y}_e$ is a $(7\xi, p_G, p_H, q, s-2, t)$-typical extended $(r-1)$-complex. We can then greedily pick an $(r-1)$-face f of $\mathcal{Y}_e$ such that $|f \cap U| = 1$. Let $f_e = f \cup e$ and observe that f_e is a face of $\mathcal{G} - \mathcal{F}_1$ which is edge-disjoint from $E(\mathcal{F}')$ and vertex-disjoint from X.

In any case, given $e \in E(L - L')$ we are able to find a face f_e of $\mathcal{G} - \mathcal{F}_1$ containing e, edge-disjoint from $E(\mathcal{F}')$ and vertex-disjoint from X. Because all vertices of f_e are good and using (3.34), we have that $\Delta(\mathcal{F}' \cup f_e) \leq r\gamma^{1/3}n$.

By iteratively applying the previous arguments we end up with the desired collection $\mathcal{F}_2$ of faces. $-$

Step 4: Cover down remaining edges in R_1, R_2 and R_3.

Let $\mathcal{G}_2 = \mathcal{G} - \mathcal{F}_1 - \mathcal{F}_2$. In this step we will find an F_{r+1}-decomposable complex $\mathcal{F}_3 \subseteq \mathcal{G}_2$ such that $E(R_1 \cup R_2 \cup R_3) \subseteq E(\mathcal{F}_2) \cup E(\mathcal{F}_3)$. Let $G_2 \subseteq G$ and $H_2 \subseteq H$ such that $\mathcal{G}_2 = (V, U, G_2, H_2)$. Observe that for each $v \in V'$ we have $d_{G_2}(v) = (r-1)d_{H_2}(v)$ and for each $u \in U'$ we have $r \mid d_{H_2}(u)$ since $\mathcal{G}_2 = \mathcal{G} - \mathcal{F}_1 - \mathcal{F}_2$, $\mathcal{G}$ satisfies (C4) and (C5), and both $\mathcal{F}_1$ and $\mathcal{F}_2$ are F_{r+1}-divisible.

For each $v \in V'$, let $\mathcal{Y}_v = \mathrm{Lk}_{\mathcal{G}_2}(v; V_1 \cup U_1)$. Note that $\mathcal{G}_2 = \mathcal{G} - \widetilde{\mathcal{G}} - \mathcal{F}_2$.

Thus, using (R6), (3.33) and $\gamma \ll \rho, \xi$ we can deduce that, for each $v \in V'$, $\mathcal{Y}_v$ is a $(7\xi, p_G, p_H, q, s-1, t)$-typical extended r-complex (V_v, U_v, G_v, H_v) where $V_v = N_{G_2}(v; V_1)$, $U_v = N_{H_2}(v; U_1)$, $G_v = \mathrm{Lk}_{G_2}(v; V_1)$ and $H_v = H_2[V_v, U_v]$. Moreover, using (R1) we obtain that for each $v \in V'$

$$|V_v| = d_{G_2}(v; V_1) = d_{R_1}(v) \pm \Delta(\mathcal{F}_2) = (1 \pm 3\xi)\rho p_G |V_1| \geq (\rho\varepsilon^{-1})^{4/3}(\varepsilon n) \qquad (3.36)$$

and

$$|U_v| = d_{H_2}(v; U_1) = d_{R_2}(v) \pm \Delta(\mathcal{F}_2) = (1 \pm 3\xi)\rho p_H |U_1| \geq (\rho\varepsilon^{-1})^{4/3}(\varepsilon n), \qquad (3.37)$$

for each pair of distinct vertices $v_1, v_2 \in V'$

$$|V_{v_1} \cap V_{v_2}| = d_{R_1}(\{v_1, v_2\}) \pm 2\Delta(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_G)^2 |V_1| \leq (\rho\varepsilon^{-1})^2(\varepsilon n)$$

and

$$|U_{v_1} \cap U_{v_2}| = d_{R_2}(\{v_1, v_2\}) \pm 2\Delta(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_H)^2 |U_1| \leq (\rho\varepsilon^{-1})^2(\varepsilon n),$$

and using (R2) we obtain that each $v \in V_1$ is contained in

$$d_{G_2}(v; V') = d_{R_1}(v) \pm \Delta(\mathcal{F}_2) = (1 \pm 4\xi)(\rho p_G)|V'| \leq \rho\varepsilon^{-1}(\varepsilon n) \qquad (3.38)$$

of the sets $\{V_v\}_{v \in V'}$, and each $u \in U_1$ is contained in

$$d_{H_2}(u; V') = d_{R_2}(u) \pm \Delta(\mathcal{F}_2) = (1 \pm 4\xi)(\rho p_H)|V'| \leq \rho\varepsilon^{-1}(\varepsilon n) \qquad (3.39)$$

of the sets $\{U_v\}_{v\in V'}$.

On the other hand, for each $u \in U'$, let $\mathcal{Y}_u = \mathrm{Lk}_{\mathcal{G}_2}(u; V_1)$ and note that (R5), (3.33) and $\gamma \ll \rho, \xi$ imply that $\mathcal{Y}_u$ is a $(7\xi, p_G, p_H, q, s-1, t)$-typical r-complex with $V_u := V(\mathcal{Y}_u) = N_{H_2}(u; V_1)$. Again, using (R1) we obtain that for each $u \in U'$

$$|V_u| = d_{H_2}(u; V_1) = d_{R_3}(u) \pm \Delta(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_H)|V_1| \geq (\rho\varepsilon^{-1})^{4/3}(\varepsilon n), \quad (3.40)$$

and for each pair of distinct vertices $u_1, u_2 \in U'$

$$|V_{u_1} \cap V_{u_2}| = d_{R_3}(\{u_1, u_2\}) \pm 2\Delta(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_H)^2|V_1| \leq (\rho\varepsilon^{-1})^2(\varepsilon n),$$

and using (R2) we obtain that each $v \in V_1$ is contained in

$$d_{H_2}(v; U') = d_{R_3}(v) \pm \Delta(\mathcal{F}_2) = (1 \pm 4\xi)(\rho p_H)|U'| \leq \rho\varepsilon^{-1}(\varepsilon n) \quad (3.41)$$

of the sets $\{V_u\}_{u\in U'}$.

Finally, for each $v \in V'$ and $u \in U'$

$$|V_v \cap V_u| = d_{R_1\cup R_3}(\{v, u\}) \pm 2\Delta(\mathcal{F}_2) = (1 \pm 3\xi)\rho^2 p_G p_H |V_1| \leq (\rho\varepsilon^{-1})^2(\varepsilon n).$$

We can then apply Lemma 3.7.1 with V_1, U_1, $\rho\varepsilon^{-1}$, 7ξ, $|V'|$, $|U'|$, $\{\mathcal{Y}_v\}_{v\in V'}$, $\{\mathcal{Y}_u\}_{u\in U'}$ playing the role of V, U, ρ, ξ, N, M, $\{\mathcal{G}_i\}_{i\in[N]}$, $\{\mathcal{G}_i\}_{N<i\leq N+M}$ respectively to find for each $v \in V' \cup U'$ an F_r-factor $\mathcal{F}_v$ of $\mathcal{Y}_v$ such that all the factors $\{\mathcal{F}_v\}_{v\in V'\cup U'}$ are edge-disjoint. Observe that for each $v \in V' \cup U'$ and $f \in \mathcal{F}_v$, $f \cup \{v\}$ is an $(r+1)$-face of $\mathcal{G}_2$. Let $\mathcal{F}_3 := \bigcup_{v\in V'\cup U', f\in\mathcal{F}_v} f \cup \{v\}$ and note that $\mathcal{F}_3$ is an F_{r+1}-decomposable subcomplex of $\mathcal{G}_2$ since it consist of the union of pairwise

edge-disjoint $(r+1)$-faces of $\mathcal{G}_2$. Moreover note that each edge in $E(R_1 \cup R_2 \cup R_3)$ is covered by $\mathcal{F}_3$. Indeed, let $e \in E(R_1 \cup R_2 \cup R_3)$ then e contains a vertex v' in $V' \cup U'$ and a vertex v_1 in $V_1 \cup U_1$. Then $v_1 \subseteq V(\mathcal{Y}_{v'})$ so $\mathcal{F}_{v'}$ contains some face f that covers v_1 since $\mathcal{F}_{v'}$ is an F_r-factor of $\mathcal{Y}_{v'}$. Thus, by definition $f \cup \{v'\}$ is a face of $\mathcal{F}_3$ that contains both v_1 and v' and therefore covers e.

Finally, equations (3.36)-(3.41) imply that

$$\Delta(\mathcal{F}_3) \leq (\gamma'/2)n. \tag{3.42}$$

Step 5: Concluding the proof.

Let $\mathcal{H} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. First observe that since all of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are edge-disjoint F_{r+1}-decomposable subcomplexes of $\mathcal{G}$, $\mathcal{H}$ is an F_{r+1}-decomposable subcomplex of $\mathcal{G}$. Recall that $\mathcal{F}_1$ contains no edges in $E(\mathcal{G}[V_1 \cup U_1])$. This together with equations (3.33) and (3.42) imply that $\Delta(\mathcal{H}[V_1 \cup U_1]) \leq \gamma' n$. Note that $\mathcal{F}_1$ covers all edges in $E(\mathcal{G} - \mathcal{L} - \mathcal{R} - \mathcal{G}[V_1 \cup U_1])$, $\mathcal{F}_2$ covers all edges in $E(\mathcal{L})$ and $\mathcal{F}_2 \cup \mathcal{F}_3$ covers all edges in $E(\mathcal{R} - \mathcal{R}[V_1 \cup U_1])$. Thus, $\mathcal{H}$ covers all edges in $E(\mathcal{G} - \mathcal{G}[V_1 \cup U_1])$ as desired. □

3.9 Final absorbers

In this section we describe the 'final' absorber that will be used in the last step of the iterative absorption method. Let $\mathcal{G}$ be an extended $(r+1)$-complex and let L be a subgraph of $\mathcal{G}$. An *edge-absorber for* L is a graph $A_L \subseteq \mathcal{G}$ such that $V(L) \subseteq V(A_L)$ is independent in A_L and both A_L and $A_L \cup L$ have an F_{r+1}-decomposition in $\mathcal{G}$. Intuitively, we think of L as the leftover of an 'almost complete' F_{r+1}-

decomposition of $\mathcal{G}$. An edge-absorber A_L will allow us to 'absorb' L and complete the F_{r+1}-decomposition.

Lemma 3.9.1. *Let $1/n \ll 1/M \ll 1/m \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq 6$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex. Let $L \subseteq \mathcal{G}$ be an F_{r+1}-divisible subcomplex on m vertices. Then there is an edge-absorber $A_L \subseteq \mathcal{G}$ for L such that $|V(A_L)| \leq M$.*

Before proving Lemma 3.9.1 we shall describe an explicit construction of an edge-absorber. Our construction combines the ideas of [9, Section 8] and [10, Section 6]. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-complex. Given a subgraph $L \subseteq \mathcal{G}$ we write $V_L := V(L) \cap V$ and $U_L := V(L) \cap U$. Given vertex-disjoint subgraphs $L, L' \subseteq \mathcal{G}$ we say that L' *is obtained from L by identifying vertices* if there is a sequence of graphs $L = L_0, \dots, L_\ell$ on $V(L)$ and vertices $x_i, y_i \in V(L)$ for each $0 \leq i < \ell$ such that

(i) $x_i, y_i \in V_L$ or $x_i, y_i \in U_L$ for each $0 \leq i < \ell$,

(ii) $N_{L_i}(x_i) \cap N_{L_i}(y_i) = \emptyset$,

(iii) L_{i+1} is obtained from L_i by removing all edges incident to y_i and adding the edges $\{x_i z : y_i z \in E(L_i)\}$,

(iv) there is an isomorphism $f : L'_\ell \to L'$ such that $f(x) \in V$ if and only if $x \in V$ where L'_ℓ is the graph obtained from L_ℓ by removing all isolated vertices.

Note that if L' is obtained from L by identifying vertices then there exists a graph homomorphism $\phi : L \to L'$ which is edge-bijective and satisfies $\phi(x) \in V$ if and only if $x \in V$ for every $x \in V(L)$. An (L, L')-*transformer* is a graph

$T \subseteq \mathcal{G}$ edge-disjoint from L and L' such that both $L \cup T$ and $T \cup L'$ have an F_{r+1}-decomposition in $\mathcal{G}$. A transformer acts as a 'partial' absorber allowing us to transform the leftover L into a new leftover L'. By combining several transformers we desire to end with a final leftover that is F_{r+1}-decomposable in $\mathcal{G}$. We will first describe the construction of a single transformer and then explain how we can concatenate several of them to end up with an F_{r+1}-decomposable leftover.

Let L be an F_{r+1}-divisible subgraph of $\mathcal{G}$. Let $L' \subseteq \mathcal{G}$ be a graph obtained from L by identifying vertices and let $\phi : L \to L'$ be an edge-bijective homomorphism such that $\phi(v) \in V$ if and only if $v \in V$ for every $v \in V(L)$. Suppose that for each $e \in E(L)$ there is an $(r-1)$-clique $K_e \in \mathcal{K}_{r-1}(\mathcal{G})$ such that $e \cup K_e$ and $\phi(e) \cup K_e$ are faces of $\mathcal{G}$ and suppose that L, L' and all the cliques in $\{K_e\}_{e \in E(L)}$ are pairwise vertex-disjoint. Let $\mathcal{K} := \{K_e\}_{e \in E(L)}$, $V(\mathcal{K}) := \bigcup_{e \in E(L)} K_e$ and $E(\mathcal{K}) := \bigcup_{e \in E(L)} E(K_e)$. For each $v \in V(L)$ let $X_v := \{x \in K_e : v \in e \in E(L)\}$ and let $E_v := \{vx : x \in X_v\}$ and $E_{\phi(v)} := \{\phi(v)x : x \in X_v\}$. Observe that for each $v \in V_L$ there are $(r-2)d_L(v; V_L) + (r-1)d_L(v; U_L)$ vertices in $X_v \cap V$ and $d_L(v; V_L)$ vertices in $X_v \cap U$. Since L is F_{r+1}-divisible we have that $d_L(v; V_L) = (r-1)d_L(v; U_L)$ and thus $|X_v \cap V| = (r-1)|X_v \cap U|$. Suppose that there exists a K_r-factor $\mathcal{F}_v$ of $\mathcal{G}[X_v]$ such that for each r-clique $K \in \mathcal{F}_v$, both $K \cup \{v\}$ and $K \cup \{\phi(v)\}$ are faces of $\mathcal{G}$. On the other hand, for each $u \in U_L$ there are $(r-1)d_L(u; V_L)$ vertices in X_v and $X_v \subseteq V$. Since L is F_{r+1}-divisible we have that $r \mid d_L(u; V_L)$ and so $r \mid |X_v|$. Suppose that there exists an F_r-factor $\mathcal{F}_u$ of $G[X_u]$. Moreover, suppose that all of $E(L)$, $E(L')$, $E(\mathcal{K})$, $\{E_v\}_{v \in V(L)}$, $\{E_{v'}\}_{v' \in L'}$ and $\{E(\mathcal{F}_v)\}_{v \in V(L)}$ are edge-disjoint. Let T be the graph with vertex set

$$V(T) = V(L) \cup V(\mathcal{K}) \cup V(L')$$

and edge set

$$E(T) = E(\mathcal{K}) \cup \left(\bigcup_{v \in V(L)} E_v \cup E_{\phi(v)} \cup E(\mathcal{F}_v) \right).$$

Let $\mathcal{F}_T$ be the set of faces of $\mathcal{G}$ of the form

- $e \cup K_e$ or $\phi(e) \cup K_e$ with $e \in E(L)$,
- $\{v\} \cup f$ or $\{\phi(v)\} \cup f$ with $v \in V(L)$ and $f \in \mathcal{F}_v$.

Fact 3.9.2. $|V(T)| \leq r|V(L)|^2$.

Proof. $|V(T)| \leq |V(L)| + |V(L')| + (r-1)|E(L)| \leq 2|V(L)| + (r-1)|V(L)|^2 \leq r|V(L)|^2$. □

Fact 3.9.3. Every vertex in $V(\mathcal{K})$ is adjacent (in T) to $3r$ vertices in $V(T)$ and is contained in six faces of $\mathcal{F}_T$.

Proof. Let $v \in V(\mathcal{K})$. Then $v \in K_e$ for a unique $e \in E(L)$ so v is adjacent to $r-2$ vertices in K_e and is contained in the faces $e \cup K_e$ and $\phi(e) \cup K_e$. Let $e = v_1v_2$, then v must be adjacent to v_1, v_2, $\phi(v_1)$ and $\phi(v_2)$. Then v is covered by the K_r-factors $\mathcal{F}_{v_1}$ and $\mathcal{F}_{v_2}$ so it is adjacent to $r-1$ vertices in each factor and is contained in the faces $\{v_1\} \cup f_1$, $\{\phi(v_1)\} \cup f_1$, $\{v_2\} \cup f_2$ and $\{\phi(v_2)\} \cup f_2$, where $f_1 \in \mathcal{F}_{v_1}$ and $f_2 \in \mathcal{F}_{v_2}$. □

Fact 3.9.4. T is an (L, L')-transformer.

Proof. Observe that $L \cup T$ can be decomposed into the set of faces

$$\{e \cup K_e \, : \, e \in E(L)\} \; \cup \; \{\{\phi(v)\} \cup f \, : \, v \in V(L), f \in \mathcal{F}_v\},$$

and $T \cup L'$ can be decomposed into the set of faces

$$\{\phi(e) \cup K_e \,:\, e \in E(L)\} \;\cup\; \{\{v\} \cup f \,:\, v \in V(L), f \in \mathcal{F}_v\}.$$

□

Lemma 3.9.5. *Let $1/n \ll 1/m \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq 6$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex. Let L be an F_{r+1}-divisible subgraph of $\mathcal{G}$ on m vertices and let L' be a graph obtained from L by identifying vertices. Then $\mathcal{G}$ contains an (L, L')-transformer with at most rm^2 vertices.*

Proof. Since $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical, $s \geq 3r$ and $t \geq 6$, using Facts 3.9.2 and 3.9.3 we shall greedily find a copy of the graph T described above. Indeed, suppose we have already found a set V' of vertices of T in $\mathcal{G}$ and we want to find a new vertex v. By Fact 3.9.3 we know that v needs to be adjacent to at most $3r$ vertices and must form a face of $\mathcal{G}$ with at most six r-cliques of $\mathcal{G}$, and so, v must form a face of G with at most six $(r-1)$-cliques of G. Typicality implies that there are $\Theta(n)$ such vertices and Fact 3.9.2 implies that $|V' \cup V(L) \cup V(L')| \leq rm^2 = o(n)$ so we can greedily find such a new vertex v. Thus, we can find a copy of T in $\mathcal{G}$. By Fact 3.9.2 and 3.9.4 then T is an (L, L')-transformer with $|V(T)| \leq rm^2$. □

We say that a clique K of $\mathcal{G}$ is *proper* if $K \in \mathcal{K}_{r+1}(\mathcal{G})$ and $|K \cap U| = 1$. Note that if a clique is a face of $\mathcal{G}$ then it must be a proper clique. Given an edge $v_1v_2 \in E(L)$ with $v_1, v_2 \in V_L$, an *expansion of* v_1v_2 is the graph obtained from L and a vertex-disjoint proper clique K by deleting the edge v_1v_2, deleting an edge $v_1'v_2' \subseteq K$ with $v_1', v_2' \in V$, and adding the edges v_1v_2' and v_2v_1'. Similarly, given an edge $v_1u_2 \in E(L)$ with $v_1 \in V_L$ and $u_2 \in U_L$, an *expansion of* v_1u_2 is the graph obtained from L and a vertex-disjoint proper clique K by deleting the edge v_1u_2, deleting an edge $v_1'u_2' \subseteq K$ with $v_1' \in V$ and $u_2' \in U$, and adding the edges v_1u_2' and u_1v_2'. In both cases, identifying the vertices v_1 and v_1' results in a graph obtained from L by attaching a proper clique to the vertex v_1. Let L_{exp} be a graph obtained from L by expanding every edge in $E(L)$. Let L_{att} be a graph obtained from L by, for each edge $e \in E(L)$, choosing an endpoint $v \in e \cap V_L$ and attaching a proper clique to v. Observe that L_{att} is obtained from L_{exp} by identifying vertices. Note that $|V(L_{\text{exp}})| = |V(L)| + (r+1)|E(L)|$ and $|V(L_{\text{att}})| = |V(L)| + r|E(L)|$. Finally, let $m_1 = |E(L[V_L])|$ and $m_2 = |E(L[V_L, U_L])|$. Given $v_0 \in V$ and $u_0 \in U$ let $\Gamma(m_1, m_2)$ be the graph obtained by expanding the loop v_0v_0 m_1 times and by expanding the edge v_0u_0 m_2 times. Observe that $\Gamma(m_1, m_2)$ is the graph obtained from L_{exp} by identifying all its vertices in V_L into a single vertex $v_0 \in V$ and all its vertices in U_L into a single vertex $u_0 \in U$.

Proof of Lemma 3.9.1. Let $V_L := V(L) \cap V$ and $U_L := V(L) \cap U$. Let $m_1 = |E(L[V_L])|$ and $m_2 = |E(L[V_L, U_L])|$. Since L is F_{r+1}-divisible we have that $\binom{r+1}{2} \mid e(L)$, $\binom{r}{2} \mid m_1$ and $r \mid m_2$. For each edge $e \in E(L)$ we choose an endpoint $v \in e \cap V_L$ and find a face f_e of $\mathcal{G}$ containing v and no more vertices from $V(L)$ so that all the faces $\{f_e\}_{e \in E(L)}$ are vertex-disjoint. Note that by doing so we are finding a copy of L_{att} in $\mathcal{G}$ and observe that $L_{\text{att}} - L$ is F_{r+1}-decomposable since it consist of the vertex-disjoint union of $e(L)$ faces. Next, choose $|V_L|$ vertices V_L' in $V \setminus V(L_{\text{att}})$ and $|U_L|$ vertices U_L' in $U \setminus V(L_{\text{att}})$ and let L' be a copy of L on the vertices $V_L' \cup U_L'$. We find now a copy of L'_{exp} in $\mathcal{G}$. Choose vertices $v_0 \in V \setminus V(L_{\text{att}} \cup L'_{\text{exp}})$ and $u_0 \in U \setminus V(L_{\text{att}} \cup L'_{\text{exp}})$ and find a copy Γ of $\Gamma(m_1, m_2)$

in $\mathcal{G}$. Observe that we can actually find L_{att}, L'_{exp} and Γ so that they are all vertex-disjoint since in all cases whenever we want to find a new vertex it needs to be adjacent to at most r given vertices, $\mathcal{G}$ is typical with $s \geq r$, and $|V(L_{\text{att}} \cup L'_{\text{exp}} \cup \Gamma| \leq M = o(n)$. Note that L_{att} and Γ are obtained from L'_{exp} by identifying vertices. Let J be the union of $e(L)/\binom{r+1}{2}$ vertex-disjoint faces of $\mathcal{G} \setminus (L_{\text{att}} \cup L'_{\text{exp}} \cup \Gamma)$. Note that J is trivially F_{r+1}-decomposable and $|E(J[V])| = m_1$ and $|E(J[V,U])| = m_2$. We can then apply the same arguments we did with L to find graphs J_{att}, J'_{exp} so that all the graphs L_{att}, L'_{exp}, Γ, J'_{exp} and J_{att} are vertex-disjoint. Moreover note that J_{att} and Γ are obtained from J'_{exp} by identifying vertices. We can consecutively apply Lemma 3.9.5 to find edge-disjoint transformers T_1, T_2, T_3, T_4 such that

$$L_{\text{att}} \overset{T_1}{\longleftarrow} L'_{\text{exp}} \overset{T_2}{\longrightarrow} \Gamma \overset{T_3}{\longleftarrow} J'_{\text{exp}} \overset{T_4}{\longrightarrow} J_{\text{att}}$$

where $L_1 \overset{T}{\to} L_2$ denotes that T is an (L_1, L_2)-transformer.

Finally, let $A_L := (L_{\text{att}} \cup T_1 \cup L'_{\text{exp}} \cup T_2 \cup \Gamma \cup T_3 \cup J'_{\text{exp}} \cup T_4 \cup J_{\text{att}}) - L$. Note that $A_L \cup L$ is F_{r+1}-decomposable since $L_{\text{att}} \cup T_1$, $L'_{\text{exp}} \cup T_2$, $\Gamma \cup T_3$, $J'_{\text{exp}} \cup T_4$ and J_{att} are all edge-disjoint and F_{r+1}-decomposable in $\mathcal{G}$. On the other hand, A_L is also F_{r+1}-decomposable since $L_{\text{att}} - L$, $T_1 \cup L'_{\text{exp}}$, $T_2 \cup \Gamma$, $T_3 \cup J'_{\text{exp}}$ and $T_4 \cup J_{\text{att}}$ are all edge-disjoint and F_{r+1}-decomposable in $\mathcal{G}$. Hence, A_L is an edge-absorber for L. □

3.10 Proof of Theorem 3.2.1

We are finally ready to prove the main theorem of this chapter:

Theorem 3.2.1. Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq r^2+r$. Let $\mathcal{G} = (V, U, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-complex where $|V| = n$. Then $\mathcal{G}$ has an F_{r+1}-decomposition.

Proof. Let $\gamma, \eta, \varepsilon > 0$ and $M, m' \in \mathbb{N}$ such that

$$1/n \ll 1/M \ll 1/m' \ll \eta \ll \gamma \ll \varepsilon \ll \xi \ll p, q, 1/s, 1/t, 1/r. \tag{3.43}$$

Step 1: Find the vortex sequence and the final absorber.

We start by applying Lemma 3.5.2 to find a $(2\xi, \varepsilon, \eta, m)$-vortex $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ in $\mathcal{G}$ for some $m \leq m'$. Let $n_i := |V_i|$ for every $i \in [\ell] \cup \{0\}$.

Let $\mathcal{G}' := \mathcal{G} - \mathcal{G}[V_1 \cup U_1]$. Using (V2)-(V4) and $\varepsilon \ll \xi$ it can be checked that $\mathcal{G}'$ is $(2\xi, p_G, p_H, q, s, t)$-typical.

Let $\mathcal{L}$ be a collection of all the spanning F_{r+1}-divisible subcomplexes of $\mathcal{G}[V_\ell \cup U_\ell]$. Note that $|\mathcal{L}| \leq 2^{\binom{|V_\ell \cup U_\ell|}{2}} \leq M$. For each $L \in \mathcal{L}$ we find an edge-absorber A_L for L in $\mathcal{G} - \mathcal{G}[V_1 \cup U_1]$ so that $|V(A_L)| \leq M$ and $\mathcal{K}_{r+1}(A_L) \subseteq F_{r+1}(\mathcal{G})$ and all of $\{A_L\}_{L \in \mathcal{L}}$ are edge-disjoint. Indeed, suppose that we have already found absorbers $\{A_L\}_{L \in \mathcal{L}'}$ for some subset $\mathcal{L}' \subseteq \mathcal{L}$ with $|V(A_L)| \leq M$ and $\mathcal{K}_{r+1}(A_L) \subseteq F_{r+1}(\mathcal{G})$ for each $L \in \mathcal{L}'$. Let $L \in \mathcal{L} \setminus \mathcal{L}'$ and let $\mathcal{G}_L := \mathcal{G} - \mathcal{G}[V_1 \cup U_1] - (\bigcup_{L \in \mathcal{L}'} A_L)$. Note that $|\bigcup_{L \in \mathcal{L}'} V(A_L)| \leq |\mathcal{L}|M \leq M^2$, $|V_1| \leq \varepsilon|V|$ and $|U_1| \leq \varepsilon|U|$. Hence, $\mathcal{G}_L$ is $(2\xi, p_G, p_H, q, s, t)$-typical so we can apply Lemma 3.9.1 to find an edge-absorber A_L for L in $\mathcal{G}_L$ such that $|V(A_L)| \leq M$ and $\mathcal{K}_{r+1}(A_L) \subseteq F_{r+1}(\mathcal{G})$.

Let $\widetilde{A} := \bigcup_{L \in \mathcal{L}} A_L$ and note that $\widetilde{A}$ satisfies that for any F_{r+1}-divisible subcomplex $L \subseteq \mathcal{G}[V_\ell \cup U_\ell]$, $\widetilde{A} \cup L$ has an F_{r+1}-decomposition. Let $\widetilde{\mathcal{G}} := \mathcal{G} - \widetilde{A}$. Since all the graphs A_L have an F_{r+1}-decomposition $\widetilde{\mathcal{G}}$ must be F_{r+1}-divisible. Moreover, since $|V(\widetilde{A})| \leq |\mathcal{L}|M \leq M^2$ it is easy to check that $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ form a $(3\xi, \varepsilon, 2\eta, m)$-vortex in $\widetilde{\mathcal{G}}$. Note that $\widetilde{\mathcal{G}}[V_1 \cup U_1] = \mathcal{G}[V_1 \cup U_1]$.

Step 2: Iteratively apply the Cover Down Lemma.

We will now apply iteratively Lemma 3.8.3 to cover all edges of $\widetilde{\mathcal{G}}$ except possibly some edges of $\mathcal{G}[V_\ell \cup U_\ell]$. Let $i \in [\ell] \cup \{0\}$ and suppose there exists an extended $(r+1)$-complex $\mathcal{G}_i = (V_i, U_i, G_i, H_i)$ such that $\widetilde{\mathcal{G}} - \mathcal{G}_i$ has an F_{r+1}-decomposition and the following conditions hold:

(a) $\mathcal{G}_i$ is F_{r+1}-divisible,

(b) $\mathcal{G}_i$ is $(5\xi, p_G, p_H, q, s, t)$-typical,

(c) $\mathcal{G}_i$ is $(4\xi, p_G, p_H, q, s, t)$-typical into $V_{i+1} \cup U_{i+1}$,

(d) $\mathcal{G}_i[V_{i+1} \cup U_{i+1}] = \mathcal{G}[V_{i+1} \cup U_{i+1}]$.

Let $\mathcal{G}_0 := \widetilde{\mathcal{G}}$ and note that (a)-(d) hold for $i = 0$. Define $\widetilde{\mathcal{G}}_i = \mathcal{G}_i - \mathcal{G}_i[V_{i+2} \cup U_{i+2}] \overset{(d)}{=} \mathcal{G}_i - \mathcal{G}[V_{i+2} \cup U_{i+2}]$. We know that $\widetilde{\mathcal{G}}_i$ must be 2η-divisible since $\mathcal{G}[V_{i+2} \cup U_{i+2}]$ is 2η-divisible by (V5). Using that $|V_{i+2}| \leq \varepsilon^2 |V_i|$ it is not hard to check that $\widetilde{\mathcal{G}}_i$ is $(6\xi, p_G, p_H, q, s, t)$-typical and $(5\xi, p_G, p_H, q, s, t)$-typical into $V_{i+1} \cup U_{i+1}$. Thus, we can apply Lemma 3.8.3 to $\widetilde{\mathcal{G}}_i$ to find an F_{r+1}-decomposable subcomplex $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_i$ such that $E(\mathcal{G}_i - \mathcal{G}[V_{i+1} \cup U_{i+1}]) \subseteq E(\mathcal{H})$ and

$$\Delta(\mathcal{H}[V_{i+1} \cup U_{i+1}]) \leq \gamma |V_{i+1} \cup U_{i+1}|. \tag{3.44}$$

Roughly speaking, $\mathcal{H}$ covers all edges of $\widetilde{\mathcal{G}}_i$ that are not in $\mathcal{G}[V_{i+1} \cup U_{i+1}]$ and a sparse collection of edges inside $\mathcal{G}[V_{i+1} \cup U_{i+1}]$. Let $\mathcal{G}_{i+1} = (\mathcal{G}_i - \mathcal{H})[V_{i+1} \cup U_{i+1}]$. Observe that $\widetilde{\mathcal{G}} - \mathcal{G}_{i+1} = (\widetilde{\mathcal{G}} - \mathcal{G}_i) \cup (\mathcal{G}_i - \mathcal{G}_{i+1}) = (\widetilde{\mathcal{G}} - \mathcal{G}_i) \cup \mathcal{H}$ is F_{r+1}-decomposable.

We will now check that $\mathcal{G}_{i+1}$ satisfies conditions (a)-(d). Clearly, $\mathcal{G}_i$ must be F_{r+1}-divisible since both $\mathcal{G}_i$ and $\mathcal{H}$ are F_{r+1}-divisible. Using (c), (3.44) and $\gamma \ll \xi$ we obtain that $\mathcal{G}_{i+1}$ is $(5\xi, p_G, p_H, q, s, t)$-typical. Recall that $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ form a $(3\xi, \varepsilon, 2\eta, m)$-vortex in $\widetilde{\mathcal{G}}$. Thus, using (d), (3.44) and (V4), we obtain that $\mathcal{G}_{i+1}$ is $(4\xi, p_G, p_H, q, s, t)$-typical into $V_{i+2} \cup U_{i+2}$. Finally, $\mathcal{G}_{i+1}[V_{i+1} \cup U_{i+1}] = \mathcal{G}[V_{i+1} \cup U_{i+1}]$ since $\mathcal{H}[V_{i+2} \cup U_{i+2}]$ has no edges.

STEP 3: Absorb the final leftover.

Hence, conditions (a)-(d) are satisfied when replacing i by $i+1$. It follows by induction that there exists an F_{r+1}-divisible extended $(r+1)$-complex $\mathcal{G}_\ell = (V_\ell, U_\ell, G_\ell, H_\ell)$ such that $\widetilde{\mathcal{G}} - \mathcal{G}_\ell$ has an F_{r+1}-decomposition. Since $\mathcal{G}_\ell \subseteq \mathcal{G}[V_\ell \cup U_\ell]$ then $\mathcal{G}_\ell \in \mathcal{L}$, and so, $\widetilde{A} \cup \mathcal{G}_\ell$ is F_{r+1}-decomposable. All in all we have that $\mathcal{G} = (\widetilde{\mathcal{G}} - \mathcal{G}_\ell) \cup (\widetilde{A} \cup \mathcal{G}_\ell)$ has an F_{r+1}-decomposition. □

CHAPTER 4

RESOLVABLE CLIQUE DECOMPOSITIONS OF PSEUDO-RANDOM MULTIPARTITE GRAPHS

The main goal of this chapter is to prove Theorem 1.4.5. Recall that a graph G is *r-partite* if there is a partition $V^1, \dots, V^r$ of $V(G)$ such that $G[V^i]$ has no edges for every $i \in [r]$. We say that G is *balanced* if all its parts have the same size and *K_r-divisible* if for every $i, j \in [r]$ and $v \in V(G) \setminus (V^i \cup V^j)$, $d_G(v; V^i) = d_G(v; V^j)$. An r-partite graph is (ξ, p, s)-typical if for every $i \in [r]$ and for every set $S \subseteq V(G) \setminus V^i$ of size at most s, $d_G(S; V^i) = (1 \pm \xi)p^{|S|}|V^i|$. Recall that a *$K_r$-factor of G* is a collection of vertex-disjoint r-cliques that span $V(G)$ and a *K_r-decomposition of G* is a collection of edge-disjoint r-cliques that span $E(G)$. Finally recall that a *resolvable K_r-decomposition* is a K_r-decomposition that can be partitioned into K_r-factors. We now restate Theorem 1.4.5.

Theorem 1.4.5. For every $r \geq 3$ and $p > 0$ there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that any balanced K_r-divisible $(\xi, p, 3(r-1))$-typical r-partite graph G with parts of size $n \geq n_0$ has a resolvable K_r-decomposition.

Similar to Chapter 3, we will deduce Theorem 1.4.5 from a more general result, Theorem 4.2.1, consisting of the existence of a K_{r+1}-decomposition in the corresponding extended graph. Moreover, we will again extend the notion of a graph and prove all our results for 'complexes' (see Section 3.1). The chapter will be organised as follows. We will start by establishing all the necessary notions around 'partite complexes' in Section 4.1. Next, in Section 4.2, we will state Theorem 4.2.1, the main result of this chapter, and use it to deduce Theorem 1.4.5. As outlined in Section 1.5, the proof of the main theorem will consist of two key results, the existence of a fractional decomposition and the 'Cover Down Lemma'. The first is formalised by Lemma 4.3.1 and proven in Section 4.3. In Section 4.4 we find the vortex sequence which will be the first step in the proof of the main theorem. In Section 4.5 we find many edge-disjoint face-factors. This result will be used during the proof of the 'Cover Down Lemma'. In Section 4.6 we show how to make an r-partite graph K_r-divisible by removing a sparse set of edges. This will be used during the 'Cover Down Lemma' in order to apply Lemma 4.3.1. The 'Cover Down Lemma', Lemma 4.7.3, is proven in Section 4.7. Finally, the proof of the main theorem, Theorem 4.2.1, is presented in Section 4.8.

4.1 Preliminaries

We will inherit the standard notation and the basic probabilistic results used in the previous chapters. For reference see Section 2.1.

Let G be a fixed r-partite graph with parts $V^1, \dots, V^r$. For any $U \subseteq V(G)$, we write U^i instead of $U \cap V^i$ for all $1 \leq i \leq r$. Given subsets $W_1, W_2 \subseteq V(G)$, we sometimes write $E(W_1, W_2)$ instead of $E(G[W_1, W_2])$ and $e(W_1, W_2)$ instead of

$e(G[W_1, W_2])$. We denote $\hat{\delta}(G) := \min\{d(v; V^i) : i \in [r], v \in V(G) \setminus V^i\}$ and $\hat{\Delta}(G) := \max\{d_G(v; V^i) : i \in [r], v \in V(G) \setminus V^i\}$.

As in Chapter 3, we will prove our results in terms of 'complexes' instead of graphs. Recall that an *r-complex* G is a triple $(V(G), E(G), F_r(G))$ where $V(G)$ is a set of vertices, $E(G)$ is a set of edges, and $F_r(G)$ is a set of r-sets of vertices called *faces*, such that for every $f \in F_r(G)$ and every 2-set $e \subset f$, $e \in E(G)$. We shall inherit the notation around complexes used in the previous chapter. See Section 3.1. The main difference in this chapter is the consideration of partite complexes. Formally, an r-complex G is *r-partite* if there exists a partition $V^1, \dots, V^r$ of $V(G)$ such that $G[V_i]$ has no edges for every $i \in [r]$. Just as in graphs, we say that G is *balanced* if $|V^1| = \dots = |V^r|$. If G is clear from the context, given $v \in V(G)$ and $e \in E(G)$ we define $F_r(v) := \{f \in F_r(G) : v \in f\}$ and $F_r(e) := \{f \in F_r(G) : e \subset f\}$.

Similar to Theorem 3.2.1 from the previous chapter, the main theorem of this chapter, Theorem 4.2.1, guarantees a face-decomposition in a pseudo-random 'extended complex'. Let us recall the following definitions:

Definition 3.1.8. An *extended $(r+1)$-partite complex $\mathcal{G} = (V, U, G, H)$ with parts* $V^1, \dots, V^r$ is an $(r+1)$-complex with $V(\mathcal{G}) = V \cup U$, where V and U are disjoint sets of vertices with $V = V^1 \sqcup \dots \sqcup V^r$; $E(\mathcal{G}) = E(G) \cup E(H)$, where G is an r-partite complex with parts $V^1, \dots, V^r$ and H is a bipartite graph on $[V, U]$; and $F_{r+1}(\mathcal{G}) = \{f \cup \{u\} \in \mathcal{K}_{r+1}(G \cup H) : f \in F_r(G), u \in U\}$.

Definition 3.1.2. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ and let $V_1 \subseteq V$ and $U_1 \subseteq U$. Given $\xi, p_G, p_H, q > 0$ and $s, t \in \mathbb{N}$ we say that $\mathcal{G}$ is *(ξ, p_G, q_H, q, s, t)-typical into $V_1 \cup U_1$* if for every $i \in [r]$, every set $S_V \subseteq V \setminus V^i$ and $S_U \subseteq U$ with $|S_V| + |S_U| \leq s$ and every set $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \leq t$ we have

$$d^*_{\mathcal{G}}(S_V \cup S_U, T; V_1^i) = (1 \pm \xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} |V_1^i|,$$

and for every $S_V \subseteq V$ with $|S_V| \leq s$

$$d_H(S_V; U_1) = (1 \pm \xi) p_H^{|S_V|} |U_1|.$$

In the case where $V_1 = V$ and $U_1 = U$ we say just that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-*typical*.

Recall that an r-partite graph is K_r-divisible if for every $i, j \in [r]$ and $v \in V(G) \setminus (V^i \cup V^j)$, $d_G(v; V^i) = d_G(v; V^j)$. The same condition is required in the case of an r-partite complex to admit an F_r-decomposition. Hence, we say that an r-partite complex G is F_r-*divisible* if its underlying graph is K_r-divisible. Similarly, an extended $(r+1)$-partite graph $\mathcal{G} = (V, U, G, H)$ with parts $V^1, \ldots, V^r$ is F_{r+1}-*divisible* if the underlying $(r+1)$-partite graph with parts $V^1, \ldots, V^r, U$ is K_{r+1}-divisible. Most of the times however we will be working with complexes that are 'almost' F_{r+1}-divisible. We define this formally as follows.

Definition 4.1.1. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-partite complex with parts $V^1, \ldots, V^r$. Given $\eta > 0$, $V_1 \subseteq V$ and $U_1 \subseteq U$ we say that $\mathcal{G}$ is η-*divisible into* $V_1 \cup U_1$ if

$$d_G(v; V_1^{i_1}) = (1 \pm \eta) d_G(v; V_1^{i_2})$$

for every $i_1, i_2 \in [r]$ and $v \in V \setminus (V^{i_1} \cup V^{i_2})$,

$$d_G(v; V_1^i) = (1 \pm \eta) d_H(v; U_1)$$

for every $i \in [r]$ and $v \in V \setminus V^i$, and

$$d_H(u; V_1^{i_1}) = (1 \pm \eta) d_H(u; V_1^{i_2})$$

for every $i_1, i_2 \in [r]$ and $u \in U$.

Similar to Proposition 3.1.5, the following result relates the size of U to the typicality and divisibility parameters.

Proposition 4.1.2. *Let $1/n \ll \eta \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical η-divisible extended $(r+1)$-partite graph with parts $V^1, \ldots, V^r$ of size n. Then*

$$|U| = (1 \pm 3\xi) \frac{p_G}{p_H} n.$$

Proof. Since $\mathcal{G}$ is η-divisible we have that for any $i \in [r]$ and $v \in V^r$

$$d_G(v; V^i) = (1 \pm \eta) d_H(v; U).$$

On the other hand, since $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical we have

$$d_G(v; V^i) = (1 \pm \xi) p_G n$$

and

$$d_H(v; U) = (1 \pm \xi) p_H |U|.$$

Thus, combining all equations we obtain

$$|U| = \frac{(1 \pm \xi)p_G n}{(1 \pm \eta)(1 \pm \xi)p_H} = (1 \pm 3\xi)\frac{p_G}{p_H}n.$$

□

4.2 Main results

We are now ready to state the main theorem of this chapter:

Theorem 4.2.1. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3(r-1)$ and $t \geq 2r-1$. Let $\mathcal{G} = (V, U, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Then $\mathcal{G}$ has an F_{r+1}-decomposition.*

We will prove Theorem 4.2.1 in Section 4.8. On the other hand, Theorem 1.4.5 is deduced from a particular case of the following consequence of Theorem 4.2.1.

Corollary 4.2.2. *Let $1/n \ll \xi \ll p, q, 1/s, 1/t, 1/r$ with $s \geq 3(r-1)$ and $t \geq 2r-1$. Let G be a K_r-divisible regular (ξ, p, q, s, t)-typical r-partite complex with parts $V^1, \dots, V^r$ of size n. Then G has a resolvable F_r-decomposition.*

Proof. Let $d := d_G(v; V^i)$ for any $i \in [r]$ and $v \in V(G) \setminus V^i$. Note that d is a well-defined constant since G is regular and K_r-divisible. Let $\mathcal{G} = (V, U, G, H)$ be an extended $(r+1)$-partite complex where $V = V(G)$ has parts $V^1, \dots, V^r$, U is a set of vertices of size d and H is the complete bipartite graph on $[V, U]$. Observe that for every $i \in [r]$ and every $v \in V \setminus V^i$ we have $d_G(v; V^i) = d =$

$|U| = d_H(v;U)$, and for every $i,j \in [r]$ and $u \in U$ we have $d_H(u;V^i) = |V^i| = |V^j| = d_H(u;V^j)$. These together with the fact that G is K_r-divisible implies that $\mathcal{G}$ is F_{r+1}-divisible. Moreover, since G is (ξ, p, q, s, t)-typical and H is complete then $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical with $p_G = p$ and $p_H = 1$. Hence we can apply Theorem 4.2.1 to find an F_{r+1}-decomposition $\mathcal{F}$ of $\mathcal{G}$. For each $u \in U$ let $F_u := \{f \cap V : f \in \mathcal{F}, u \in f\}$ be the set of r-faces of G that together with u form a face of the decomposition $\mathcal{F}$. Since u is adjacent via H to all vertices in V then F_u is an F_r-factor of G. All in all, $\{F_u\}_{u \in U}$ is a collection of edge-disjoint F_r-factors of G that cover all edges in $E(G)$. Thus, G has a resolvable F_r-decomposition. □

It is now straightforward to deduce Theorem 1.4.5. Indeed, it follows directly from Corollary 4.2.2 by defining $F_r(G) = \mathcal{K}_r(G)$, $q = 1$ and $t = 2r - 1$.

4.3 Extended fractional decomposition

The goal of this section is to prove the existence of a fractional face-decomposition in a typical extended partite complex (Lemma 4.3.1). Recall that this is a fundamental result in order to apply the 'iterative absorption' method depicted in Section 1.5. We will start by stating Lemma 4.3.1 and then show how to use it to obtain an approximate face decomposition (Lemma 4.3.2). Next, in Section 4.3.1, we sketch the proof of Lemma 4.3.1 and describe the gadgets that will be used during its proof to modify the weight of the edges. In Section 4.3.2 we analyse how the gadgets can be used and how they affect the weight of each edge and each face. Finally, in Section 4.3.3 we prove Lemma 4.3.1. For convenience, during the rest of this section, we will denote the set U in an extended $(r+1)$-partite complex as V^{r+1} and save the letter 'U' to denote a different set.

Lemma 4.3.1. *Let $1/n \ll \xi \ll \xi'' \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq 2r-1$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Then there exists a weighting $\omega : F_{r+1}(\mathcal{G}) \to [0,1]$ such that for every edge $e \in E(\mathcal{G})$*

$$\omega(e) := \sum_{\substack{f \in F_{r+1}(\mathcal{G}) \\ e \subseteq f}} \omega(f) = 1,$$

and for every $f \in F_{r+1}(\mathcal{G})$

$$\omega(f) = \frac{(1 \pm \xi'')}{p^{\binom{r}{2}} p_H^{r-1} q n^{r-1}}.$$

The proof of Lemma 4.3.1 closely follows the approach used by Montgomery [64] to find a clique fractional decomposition in a partite graph with high minimum degree. Before delving into the proof, though, we will show how to obtain an approximate face decomposition in an extended partite complex using Lemma 4.3.1 and Theorem 1.5.2.

Lemma 4.3.2. *Let $1/n \ll \gamma \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq 2r-1$. Let $\mathcal{G} = (V, U, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Then there exists an F_{r+1}-decomposable complex $\mathcal{F} \subseteq \mathcal{G}$ such that $\hat{\Delta}(\mathcal{G} - \mathcal{F}) \leq \gamma n$.*

Proof. The proof is completely analogous to the proof of Lemma 3.4.1. The only differences are that we use Lemma 4.3.1 (instead of Lemma 3.3.2) to find the weighting ω, and that we define $\mathcal{S} := \{S_{v,i}\}_{v \in V(\mathcal{G}), i \in [r+1]}$ where $S_{v,i} := \{vv' \in$

$E(\mathcal{G})\ :\ v' \in V^i\}$ and $V^{r+1} := U$ to achieve, after applying Theorem 1.5.2, that $\hat{\Delta}(\mathcal{G} - \mathcal{F}) \leq \gamma n$ instead of $\Delta(\mathcal{G} - \mathcal{F}) \leq \gamma n$. □

The following useful result counts the number of faces containing a given edge in a typical extended partite complex.

Proposition 4.3.3. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq r$ and $t \geq 1$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n and $|V^{r+1}| = (1 \pm \xi)(p_G/p_H)n$. Then for every $e \in E(\mathcal{G})$*

$$|F_{r+1}(e)| = (1 \pm \xi)^r p_G^{\binom{r}{2}} p_H^{r-1} q n^{r-1},$$

and

$$|F_{r+1}(\mathcal{G})| = (1 \pm \xi)^{r+1} p_G^{\binom{r}{2}+1} p_H^{r-1} q n^{r+1}.$$

Proof. Let $e \in E(G)$ and suppose without loss of generality that $e \in E(G[V^1, V^2])$. We want to count how many ways we can extend the edge e into a face. Let M_i denote the number of i-cliques $X \subseteq V^1 \cup \dots \cup V^i$ in G that contain e. Recall that by typicality for every $i \in [r-1]$, each i-clique $X \subseteq V^1 \cup \dots \cup V^i$ satisfies $d_G(X; V^{i+1}) = (1 \pm \xi)p^i n$. Then, for each $i = 3, \dots, r-1$,

$$M_{i+1} = (1 \pm \xi) p_G^i n M_i$$

where $M_2 := 1$. Hence,

$$M_{r-1} = (1 \pm \xi)^{r-3} \left(\prod_{i=2}^{r-2} p_G^i \right) n^{r-3}.$$

Note that M_{r-1} counts the number of $(r-1)$-cliques in G containing e. Then number of faces of G that contain an $(r-1)$-clique K is

$$d^*_G(K; V^r) = (1 \pm \xi) p_G^{r-1} qn.$$

Finally, the number of vertices in V^{r+1} which are adjacent to a face $f \in F_r(G)$ is

$$d_H(f; V^{r+1}) = (1 \pm \xi) p_H^r |V^{r+1}| = (1 \pm \xi)^2 p_G p_H^{r-1} n.$$

Hence,

$$|F_{r+1}(e)| = (1 \pm \xi)^r \left(\prod_{i=1}^{r-1} p_G^i \right) p_H^{r-1} qn^{r-1} = (1 \pm \xi)^r p_G^{\binom{r}{2}} p_H^{r-1} qn^{r-1}.$$

Let $e \in E(H)$ and suppose without loss of generality that $e = vu$ with $v \in V^1$ and $u \in V^{r+1}$. Let M_i denote the number of i-cliques $X \subseteq V^1 \cup \cdots \cup V^i$ that contain v and lie in the neighbourhood of u. Then

$$M_{i+1} = (1 \pm \xi) p_G^i p_H n M_i$$

where $M_1 := 1$. Thus,

$$M_{r-1} = (1 \pm \xi)^{r-2} \left(\prod_{i=1}^{r-2} p_G^i p_H \right) n^{r-2}.$$

It remains to count the number of faces $f \in F_r(G)$ that contain an $(r-1)$-clique $K \in \mathcal{K}_{r-1}(G)$ and are adjacent to u. And that is

$$d^*_{\mathcal{G}}(K \cup \{u\}, K; V^r) = (1 \pm \xi) p_G^{r-1} p_H qn.$$

Hence,

$$|F_{r+1}(e)| = (1 \pm \xi)^{r-1} \left(\prod_{i=1}^{r-1} p_G^i p_H \right) qn^{r-1} = (1 \pm \xi)^r p_G^{\binom{r}{2}} p_H^{r-1} qn^{r-1}.$$

Finally, since each face contains exactly one edge between V^1 and V^2

$$\begin{aligned} |F_{r+1}(\mathcal{G})| &= \sum_{v_1 \in V^1} \sum_{v_2 \in N_G(v_1; V^2)} |F_{r+1}(v_1 v_2)| = |V^1|(1 \pm \xi) p_G n (1 \pm \xi)^r p_G^{\binom{r}{2}} p_H^{r-1} qn^{r-1} \\ &= (1 \pm \xi)^{r+1} p_G^{\binom{r}{2}+1} p_H^{r-1} qn^{r+1}. \end{aligned}$$

□

4.3.1 The gadgets

In order to prove Lemma 4.3.1 we will make use of two different gadgets that allow us to move weight between edges.

Definition 4.3.4. A function $\phi : \mathcal{A} \to \mathbb{R}$ is a *zero-sum function* if $\sum_{A \in \mathcal{A}} \phi(A) = 0$.

Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be an extended $(r+1)$-complex. Given a set of vertices $X \subseteq V(\mathcal{G})$ such that $|X \cap V^j| \leq 1$ for every $j \in [r+1]$, we denote $\{x_j\} = X \cap V^j$ and $X_{-j} := X \setminus \{x_j\}$.

Definition 4.3.5. Let $z \in \mathbb{R}$, $i \in [r+1]$, $v_1, v_2 \in V^i$ and $X = \{x_j\}_{j \in [r+1] \setminus \{i\}}$ with every $x_j \in N_{\mathcal{G}}(\{v_1, v_2\}; V^j)$. A (v_1, v_2, X, z)-*shifter* is a zero-sum function

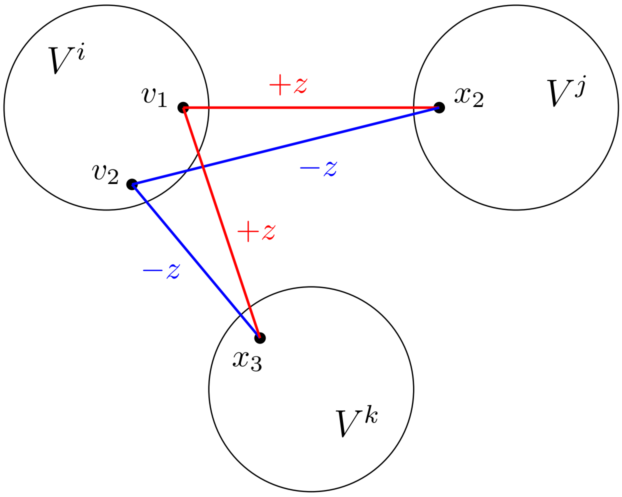

Figure 4.1: Changes on weights due to a (v_1, v_2, X, z)-shifter with $X = \{x_2, x_3\}$.

$\phi : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that for every $e \in E(\mathcal{G})$

$$\phi(e) = \begin{cases} z & \text{if } e = v_1x_j \text{ with } j \in [r+1] \setminus \{i\}, \\ -z & \text{if } e = v_2x_j \text{ with } j \in [r+1] \setminus \{i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Definition 4.3.6 (Gadget 1)**.** Let $i \in [r+1]$, $v_1, v_2 \in V^i$ and $X = \{x_j\}_{j\in[r+1]\setminus\{i\}}$. A (v_1, v_2, X)-*gadget* is a subgraph J of $\mathcal{G}$ such that $V(J) = \{v_1, v_2\} \cup X \cup Y^J$ where Y^J is a set of vertices $\{y_j\}_{j\in[r+1]\setminus\{i\}}$ with $y_j \in V^j$, and such that all of $\{v_1\} \cup Y^J$, $\{v_2\} \cup Y^J$, $\{\{v_1, x_j\} \cup Y^J_{-j}\}_{j\in[r+1]\setminus\{i\}}$ and $\{\{v_2, x_j\} \cup Y^J_{-j}\}_{j\in[r+1]\setminus\{i\}}$ are faces of $\mathcal{G}$. We denote by $\mathcal{J}(v_1, v_2, X)$ the set of all (v_1, v_2, X)-gadgets in $\mathcal{G}$. Given

$J \in \mathcal{J}(v_1, v_2, X)$ let $\phi_J : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ be the $(v_1, v_2, X, 1)$-shifter defined as

$$\phi_J(f) = \begin{cases} 1 & \text{if } f = \{v_1, x_j\} \cup Y^J_{-j} \text{ for some } j \in [r+1] \setminus \{i\}, \\ -1 & \text{if } f = \{v_2, x_j\} \cup Y^J_{-j} \text{ for some } j \in [r+1] \setminus \{i\}, \\ -(r-1) & \text{if } f = \{v_1\} \cup Y^J, \\ (r-1) & \text{if } f = \{v_2\} \cup Y^J, \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition finds a (v_1, v_2, X, z)-shifter in a typical extended $(r+1)$-partite graph and bounds the weight given to each face.

Proposition 4.3.7. *Let $1/n \ll \xi \ll 1/c \ll \alpha, p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq 2(r-1)$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n and $|V^{r+1}| = \alpha n$. Let $z \in \mathbb{R}$, $i \in [r+1]$, $v_1, v_2 \in V^i$ and $X = \{x_j\}_{j \in [r+1] \setminus \{i\}}$ with $x_j \in N_{\mathcal{G}}(\{v_1, v_2\}; V^j)$. Then there exists a (v_1, v_2, X, z)-shifter ϕ_1 such that for every $f \in F_{r+1}(\mathcal{G})$*

$$|\phi_1(f)| \leq \begin{cases} |z|cn^{-(r-1)} & \text{if } f = \{v_1, x_j\} \cup Y^J_{-j} \text{ or } \{v_2, x_j\} \cup Y^J_{-j} \\ & \text{for some } j \in [r+1] \setminus \{i\} \text{ and } J \in \mathcal{J}(v_1, v_2, X), \\ |z|cn^{-r} & \text{if } f = \{v_1\} \cup Y^J \text{ or } \{v_2\} \cup Y^J \\ & \text{for some } J \in \mathcal{J}(v_1, v_2, X), \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Proof. We start by counting the number of gadgets in $\mathcal{J}(v_1, v_2, X)$. We do so by iteratively, for each $j \in [r+1] \setminus \{i\}$, counting the number of choices for the vertex $y_j \in Y^J$. We also distinguish between two cases, whether $i \in [r]$ or $i \in [r+1]$. For convenience let $\mathcal{J} := \mathcal{J}(v_1, v_2, X)$.

Case 1: $i \in [r]$.

Without loss of generality, assume that $v_1, v_2 \in V^r$. Suppose that we have already found vertices $Y_{<j} := \{y_k\}_{k\in[j-1]}$. If $j \le r-3$ then y_j must be adjacent to all of v_1, v_2, X_{-j} and $Y_{<j}$. That is, $r+j-2$ vertices in $V \setminus V^{r+1}$ and one vertex, x_{r+1}, in V^{r+1}. Thus, the number of choices for y_j, for each $j \in [r-3]$, is

$$d_{\mathcal{G}}(\{v_1, v_2\} \cup X_{-j} \cup Y_{<j}; V^j) = (1 \pm \xi) p_G^{r+j-1} p_H n.$$

If $j = r-2$ then y_j must be again adjacent to all of v_1, v_2, X_{-j} and $Y_{<j}$ but, in addition, $Y_{<j} \cup \{y_j, x_{r-1}, v_1\}$ and $Y_{<j} \cup \{y_j, x_{r-1}, v_2\}$ must be r-faces of G. That is, y_j must be adjacent to $r+j-2$ vertices and form a face with two $(r-1)$-cliques of G. Let T denote the set of such two $(r-1)$-cliques. Since $j = r-2$ then the number of choices for y_{r-2} is

$$d^*_{\mathcal{G}}(\{v_1, v_2\} \cup X_{-j} \cup Y_{<j}, T; V^j) = (1 \pm \xi) p_G^{2r-3} p_H q^2 n.$$

If $j = r-1$ then all of $Y_{<j} \cup \{y_j, v_1\}$, $Y_{<j} \cup \{y_j, v_2\}$, $\{(Y_{<j} \setminus \{y_k\}) \cup \{x_k, y_j, v_1\}\}_{k\in[j-1]}$ and $\{(Y_{<j} \setminus \{y_k\}) \cup \{x_k, y_j, v_2\}\}_{k\in[j-1]}$ must be r-faces of G. Hence y_j must be adjacent to $r+j-2$ vertices and form a face with $2j$ $(r-1)$-cliques of G. Let T denote the set of such cliques. Using $j = r-1$ we obtain that the number of choices for y_{r-1} is

$$d^*_{\mathcal{G}}(\{v_1, v_2\} \cup X_{-j} \cup Y_{<j}, T; V^j) = (1 \pm \xi) p_G^{2(r-1)} p_H q^{2(r-1)} n.$$

Finally, if $j = r+1$, y_{r+1} must simply be adjacent to all of v_1, v_2, X_{-j} and $Y_{<j}$,

which adds up to $2r$ vertices. Then

$$d_H(\{v_1, v_2\} \cup X_{-j} \cup Y_{<j}; V^{r+1}) = (1 \pm \xi) p_H^{2r} |V^{r+1}|.$$

All in all we have

$$\begin{aligned} |\mathcal{J}| &= \left(\prod_{j=1}^{j-3} (1 \pm \xi) p_G^{r+j-1} p_H n \right) \left((1 \pm \xi) p_G^{2r-3} p_H q^2 n \right) \\ &\qquad \cdot \left((1 \pm \xi) p_G^{2(r-1)} p_H q^{2(r-1)} n \right) \left((1 \pm \xi) p_H^{2r} |V^{r+1}| \right) \\ &= (1 \pm \xi)^r p_G^{P(r)} p_H^{3r-1} q^{2r} \alpha n^r \end{aligned} \tag{4.3}$$

where $P(r) = \frac{3r^2-5r+2}{2}$.

Case 2: $i = r + 1$.

Using a similar analysis to the previous case one obtains that

$$\begin{aligned} |\mathcal{J}| &= \left(\prod_{m=1}^{r-2} (1 \pm \xi) p_G^{r-2+m} p_H^2 n \right) \left((1 \pm \xi) p_G^{2r-3} p_H^2 q^2 n \right) \\ &\qquad \cdot \left((1 \pm \xi) p_G^{2(r-1)} p_H^2 q^{2(r-1)} n \right) \\ &= (1 \pm \xi)^r p_G^{P(r)} p_H^{2r} q^{2r} n^r \end{aligned} \tag{4.4}$$

where $P(r) = \frac{3r^2-3r}{2}$.

Now, let $\phi_1 := \frac{z}{|\mathcal{J}|} \sum_{J \in \mathcal{J}} \phi_J$. It follows by the fact that ϕ_J is a zero-sum function for each $J \in \mathcal{J}$, that ϕ_1 is also a zero-sum function. Observe also that, since ϕ_J is a $(v_1, v_2, X, 1)$-shifter for each $J \in \mathcal{J}$, then for each $e \in E(G)$

$$\phi_1(e) = \begin{cases} z & \text{if } e = v_1x_j \text{ with } j \in [r+1] \setminus \{i\}, \\ -z & \text{if } e = v_2x_j \text{ with } j \in [r+1] \setminus \{i\}, \\ 0 & \text{otherwise} \end{cases}$$

as desired. It only remains to check that ϕ_1 satisfies (4.2).

Let $f \in F_{r+1}(\mathcal{G})$ be a face of the form $f = \{v_1, x_j\} \cup Y_{-j}$ where $j \in [r+1] \setminus \{i\}$ and Y_{-j} is a set of vertices $\{y_k\}_{k \in [r+1] \setminus \{i,j\}}$ with $y_k \in V^k$. Given $J \in \mathcal{J}$, then $\phi_J(f)$ is equal to 1 if $Y_{-j} = Y^J_{-j}$, or equal to zero otherwise. How many gadgets $J \in \mathcal{J}$ satisfy that $Y_{-j} = Y^J_{-j}$? The question is equivalent to determining how many vertices $y_j \in V^j \setminus \{x_j\}$ satisfy that $y_j \cup Y_{-j} = Y^J$ for some $J \in \mathcal{J}$. Suppose first that $i = r+1$. Let T be the set of $(r-1)$-cliques K of G of the form $K = \{x_k\} \cup (Y_{-j} \setminus \{y_k\})$ with $k \in [r] \setminus \{j\}$, or $K = Y_{-j}$. We have $|T| = r$. Note that $y_j \cup Y_{-j} = Y^J$ for some $J \in \mathcal{J}$ if and only if y_j is adjacent to all vertices in $\{v_1, v_2\} \cup X_j \cup Y_{-j}$ and forms a face with all the $(r-1)$-cliques in T. Then, the number of gadgets $J \in \mathcal{J}$ such that $Y_{-1} = Y^J$ is

$$d^*_{\mathcal{G}}(\{v_1, v_2\} \cup X_j \cup Y_{-j}, T; V^j \setminus \{x_j\}) = (1 \pm \xi) p_G^{2(r-1)} p_H^2 q^r n$$

so

$$|\phi_1(f)| \leq |z| \frac{(1+\xi) p_G^{2(r-1)} p_H^2 q^r n}{|\mathcal{J}|} \overset{(4.4)}{\leq} |z| c n^{-(r-1)}.$$

Suppose now that $i \neq r+1$ and $j \neq r+1$. Let T be the set of $(r-1)$-cliques K of G of the form $K = \{v_1, x_k\} \cup (Y_{-j} \setminus \{y_k, y_{r+1}\})$ or $\{v_2, x_k\} \cup (Y_{-j} \setminus \{y_k, y_{r+1}\})$ with $k \in [r] \setminus \{i, j\}$, or $K = \{v_1\} \cup (Y_{-j} \setminus \{y_{r+1}\})$ or $\{v_2\} \cup (Y_{-j} \setminus \{y_{r+1}\})$. Note that

$|T| = 2(r-1)$. In this case, the number of gadgets $J \in \mathcal{J}$ such that $Y_{-1} = Y^J$ is

$$d^*_{\mathcal{G}}(\{v_1, v_2\} \cup X_j \cup Y_{-j}, T; V^j \setminus \{x_j\}) = (1 \pm \xi) p_G^{2(r-1)} p_H^2 q^{2(r-1)} n$$

so

$$|\phi_1(f)| \leq |z| \frac{(1+\xi) p_G^{2(r-1)} p_H^2 q^{2(r-1)} n}{|\mathcal{J}|} \overset{(4.3)}{\leq} |z| c n^{-(r-1)}.$$

Finally suppose that $i \neq r+1$ and $j = r+1$. In this case $y_j \cup Y_{-j} = Y^J$ for some $J \in \mathcal{J}$ if and only if $y_j \in V^{r+1} \setminus x_j$ and y_j is adjacent to all vertices in $\{v_1, v_2\} \cup X_j \cup Y_{-j}$. Then

$$d_H(\{v_1, v_2\} \cup X_j \cup Y_{-j}; V^{r+1} \setminus \{x_j\}) = (1 \pm \xi) p_H^{2r} |V^{r+1}| = (1 \pm \xi) p_H^{2r} \alpha n$$

so

$$|\phi_1(f)| \leq |z| \frac{(1+\xi) p_H^{2r} \alpha n}{|\mathcal{J}|} \overset{(4.3)}{\leq} |z| c n^{-(r-1)}.$$

Analogously, we obtain that $|\phi_1(f)| \leq |z| c n^{-(r-1)}$ for any face $f \in F_{r+1}(\mathcal{G})$ of the form $f = \{v_2, x_j\} \cup Y_{-j}$ with $j \in [r+1] \setminus \{i\}$ and $Y_{-j} = \{y_k\}_{k \in [r+1] \setminus \{i,j\}}$ with $y_k \in V^k$.

Let $f \in F_{r+1}(\mathcal{G})$ be a face of the form $f = \{v_1\} \cup Y$ or $\{v_2\} \cup Y$ where Y is a set of vertices $\{y_j\}_{j \in [r+1] \setminus \{i\}}$. In this case we have that $\phi_J(f) = (r-1)$ if $Y = Y^J$ for some $J \in \mathcal{J}$ and $\phi_J(f) = 0$ otherwise. Since there can only exist one gadget $J \in \mathcal{J}$ with $Y = Y^J$ then

$$|\phi_1(f)| = |z|(r-1)/|\mathcal{Y}| \overset{(4.3),(4.4)}{\leq} |z| c n^{-r}.$$

Lastly, observe that if a face $f \in F_{r+1}(\mathcal{G})$ satisfies that for all $J \in \mathcal{J}$, $f \neq \{v_m\} \cup Y^J$

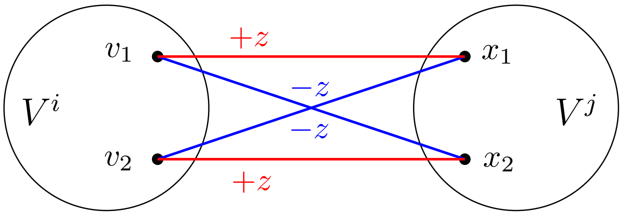

Figure 4.2: Changes on weights due to a (v_1, v_2, x_1, x_2)-shifter.

for all $m = 1, 2$ and $f \neq \{v_m, x_j\} \cup Y^J_{-j}$ for all $m = 1, 2$ and $j \in [r+1] \setminus \{i\}$, then $\phi_J(f) = 0$ for all $J \in \mathcal{J}$ so $\phi_1(f) = 0$. □

Definition 4.3.8. Let $z \in \mathbb{R}$, $i, j \in [r+1]$ with $i \neq j$, $v_1, v_2 \in V^i$ and $x_1, x_2 \in N_{\mathcal{G}}(\{v_1, v_2\}; V^j)$. A (v_1, v_2, x_1, x_2, z)-*shifter* is a zero-sum function $\phi : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that for every edge $e \in E(\mathcal{G})$

$$\phi(e) = \begin{cases} z & \text{if } e = v_1x_1 \text{ or } v_2x_2, \\ -z & \text{if } e = v_1x_2 \text{ or } v_2x_1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Definition 4.3.9 (Gadget 2)**.** Let $i, j \in [r+1]$ with $i \neq j$, $v_1, v_2 \in V^i$ and $x_1, x_2 \in V^j$. A (v_1, v_2, x_1, x_2)-*gadget* is a subgraph J of $\mathcal{G}$ such that $V(J) = \{v_1, v_2, x_1, x_2\} \cup Y^J$ where Y^J is a set of vertices $\{y_k\}_{k \in [r+1] \setminus \{i,j\}}$ with $y_k \in V^k$, and such that $\{v_{m_1}, x_{m_2}\} \cup Y^J$ is a face of $\mathcal{G}$ for all $m_1, m_2 = 1, 2$. Let $\mathcal{J}(v_1, v_2, x_1, x_2)$ denote the set of all (v_1, v_2, x_1, x_2)-gadgets in $\mathcal{G}$. Given $J \in \mathcal{J}(v_1, v_2, x_1, x_2)$, let

$\phi_J : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ be the $(v_1, v_2, x_1, x_2, 1)$-shifter defined as

$$\phi_J(f) = \begin{cases} 1 & \text{if } f = \{v_1, x_1\} \cup Y^J \text{ or } \{v_2, x_2\} \cup Y^J, \\ -1 & \text{if } f = \{v_1, x_2\} \cup Y^J \text{ or } \{v_2, x_1\} \cup Y^J, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition finds a (v_1, v_2, x_1, x_2, z)-shifter in a typical extended $(r+1)$-partite graph and bounds the weight on each face.

Proposition 4.3.10 (Gadget 2). *Let $1/n \ll \xi \ll 1/c \ll \alpha, p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq r+2$ and $t \geq 4$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n and $|V^{r+1}| = \alpha n$. Let $z \in \mathbb{R}$, $i, j \in [r+1]$ with $i \neq j$, $v_1, v_2 \in V^i$ and $x_1, x_2 \in N_{\mathcal{G}}(\{v_1, v_2\}; V^j)$. Then there exists a (v_1, v_2, x_1, x_2, z)-shifter ϕ_2 such that for every $f \in F_{r+1}(\mathcal{G})$*

$$|\phi_2(f)| \leq \begin{cases} |z|cn^{-(r-1)} & \text{if } f = \{v_{m_1}, x_{m_2}\} \cup Y^J \\ & \text{for some } m_1, m_2 \in [2] \text{ and } J \in \mathcal{J}(v_1, v_2, x_1, x_2) \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Proof. The proof is very similar to the proof of Proposition 4.3.7. Let $\mathcal{J} := \mathcal{J}(v_1, v_2, x_1, x_2)$. The first step is to count the number of gadgets in $\mathcal{J}$. If $i, j \neq r+1$ then one obtains that

$$\begin{aligned} |\mathcal{J}| = & \left(\prod_{m=1}^{r-3} (1 \pm \xi) p_G^{3+m} n \right) \left((1 \pm \xi) p_G^{r+1} q^4 n \right) \left((1 \pm \xi) p_H^{r+2} |V^{r+1}| \right) \\ = & (1 \pm \xi)^{r-1} p_G^{P(r)} p_H^{r+2} q^4 \alpha n^{r-1} \end{aligned} \tag{4.7}$$

where $P(r) = \frac{r^2+3r-10}{2}$.

Otherwise, if $i = r+1$ or $j = r+1$ we have

$$\begin{aligned} |\mathcal{J}| &= \left(\prod_{m=1}^{r-2} (1 \pm \xi) p_G^{1+m} p_H^2 n \right) \left((1 \pm \xi) p_G^r p_G^2 p_H^2 q^4 n \right) \\ &= (1 \pm \xi)^{r-1} p_G^{P(r)} p_H^{2(r-1)} q^4 n^{r-1} \end{aligned} \tag{4.8}$$

where $P(r) = \frac{r^2+r-2}{2}$.

Let $\phi_2 := \frac{z}{|\mathcal{Y}|} \sum_{J \in \mathcal{J}} \phi_J$. Again, since ϕ_J is a zero-sum function for all $J \in \mathcal{J}$ then ϕ_2 is also zero-sum function. Note that ϕ_2 satisfies

$$\phi_2(e) = \begin{cases} z & \text{if } e = v_1x_1 \text{ or } v_2x_2, \\ -z & \text{if } e = v_1x_2 \text{ or } v_2x_1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, observe that for each $f \in F_{r+1}(\mathcal{G})$, $\phi_2(f) \neq 0$ if and only if $f = \{v_{m_1}, x_{m_2}\} \cup Y^J$ for some $m_1, m_2 \in [2]$ and $J \in \mathcal{J}(v_1, v_2, x_1, x_2)$, and in such case, $\phi_2(f) = \phi_J(f) = 1$. Thus,

$$|\phi_2(f)| \leq \begin{cases} \frac{|z|}{|\mathcal{J}|} \overset{(4.7),(4.8)}{\leq} |z|cn^{-(r-1)} & \text{if } f = \{v_{m_1}, x_{m_2}\} \cup Y^J \\ & \text{for some } m_1, m_2 \in [2] \text{ and } J \in \mathcal{J}(v_1, v_2, x_1, x_2) \\ 0 & \text{otherwise.} \end{cases}$$

□

4.3.2 Moving weight through edges

The following two lemmas are the key ingredients to the proof of Lemma 4.3.1 and contain implicitly the use of the two gadgets to move weight around the edges. To get some intuition on the statements, we will view the numbers $\{z_e\}_{e\in E(G)}$ as the 'excess/missing' weight of each edge that we want to compensate.

In the following lemma we are given a fixed vertex $v \in V(\mathcal{G})$, a set of excess/missing weights $\{z_e\}_{e\in E_v}$ for each edge $e \in E(\mathcal{G})$ incident to v, and a set of vertices $U \subseteq V(\mathcal{G})$, and we find a zero-sum function $\varphi_v : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ that compensates all the weights $\{z_e\}_{e\in E_v}$ while only modifying the weight of edges incident to U. We also give bounds on the added or subtracted weight on those edges and on the weight of each face $f \in F_{r+1}(\mathcal{G})$ in terms of a constant C_U that depends on the set U. The proof consists of iteratively using the shifters given by Propositions 4.3.7 and 4.3.10 while carrying an analysis of the added/subtracted weight on each edge and face.

Lemma 4.3.11. *Let $1/n \ll \xi \ll 1/c \ll \alpha, p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq 2(r-1)$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \ldots, V^r$ of size n and $|V^{r+1}| = \alpha n$. Let $C_U \in \mathbb{N}$ and $U \subseteq V(\mathcal{G})$ be a set such that for every $i \in [r+1]$ and every set of r vertices $X \subseteq V(\mathcal{G}) \setminus V^i$ we have $d_{\mathcal{G}}(X; U^i) \geq C_U$. Let $i \in [r+1]$ and $v \in V^i \setminus U^i$ be fixed and let z_v and $\{z_e\}_{e\ni v}$ be real numbers satisfying*

$$z_v = \sum_{x_j \in N_{\mathcal{G}}(v; V^j)} z_{vx_j} \tag{4.9}$$

for all $j \in [r+1] \setminus \{i\}$.

Then there is a zero-sum function $\varphi_v : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ *such that* $\varphi_v(e) = z_e$ *for all edges incident to* v, $|\varphi_v(e)| \leq \frac{|z_{vx}|}{C_U}$ *for all edges* $e = ux$ *with* $u \in U^i$ *and* $x \in N_{\mathcal{G}}(v)$, *and* $\varphi_v(e) = 0$ *for all other edges. Moreover, for every* $f \in F_{r+1}(\mathcal{G})$ *with* $f = \{x_j\}_{j \in [r+1]}$ *and* $x_j \in V^j$,

$$|\varphi_v(f)| \leq \begin{cases} C_v(f) & \text{if } x_i = v, \\ C_v(f)/C_U & \text{if } x_i \in U^i, \\ 0 & \text{otherwise}, \end{cases} \tag{4.10}$$

where $C_v(f) := \left(\sum_{j \in [r+1] \setminus \{i\}} |z_{vx_j}|\right) cn^{-(r-1)} + |z_v| cn^{-r}$.

Proof. The proof consists of a backward induction argument on the number of edges adjacent to v which have a nonzero excess/missing weight. More formally, at each step of the induction process we will find mappings φ_l and z_l and define the set $N_l := \{x \in N_{\mathcal{G}}(v) \ : \ z_l(vx) \neq 0\}$. The sets will satisfy $0 < |N_{l+1}| < |N_l|$ implying the existence of φ_m and z_m such that $|N_m| = 0$.

We start defining $\varphi_0 : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ as $\varphi_0(f) = 0$ and $z_0 : E_v \to \mathbb{R}$ as $z_0(e) = z_e - \varphi_0(e)$ where $E_v := \{e \in E(\mathcal{G}) \ : \ e \ni v\}$. Let $N_0 := \{x \in N_{\mathcal{G}}(v) \ : \ z_0(vx) \neq 0\}$ and note that $|N_0| \leq d_{\mathcal{G}}(v)$.

Suppose that, for some $l \in \mathbb{N}$, we have a zero-sum function $\varphi_l : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ and a mapping $z_l : E_v \to \mathbb{R}$ defined as $z_l(e) := z_e - \varphi_l(e)$ such that for all $j, k \in [r+1] \setminus \{i\}$

$$\sum_{x \in N_{\mathcal{G}}(v; V^j)} z_l(vx) = \sum_{x \in N_{\mathcal{G}}(v; V^k)} z_l(vx) \tag{4.11}$$

and for all $e \in E_v$

$$\operatorname{sgn}(\varphi_l(e)) = \operatorname{sgn}(z_e) = \operatorname{sgn}(z_l(e)). \tag{4.12}$$

Let $N_l := \{x \in N_{\mathcal{G}}(v) \,:\, z_l(vx) \neq 0\}$.

Choose $j \in [r+1] \setminus \{i\}$ and $x_j \in N_{\mathcal{G}}(v; V^j)$ so that $\varphi_l(vx_j) \neq z_{vx_j}$ and $|z_l(vx_j)|$ is minimized. If there is no such vertex x_j then we set $m = l$. Assume without loss of generality that $z_l(vx_j) > 0$ (the other case is symmetric by inverting all signs in the argument).

First case. Suppose there is some other vertex $x^- \in N_{\mathcal{G}}(v; V^j)$ such that $z_l(vx^-) < 0$. Let $x^+ = x_j$. Now, for each vertex $u \in N_{\mathcal{G}}(\{x^+, x^-\}; U^i)$, we use Proposition 4.3.10 (with v, u, x^+, x^- and $z_l(vx_j)$ playing the role of v_1, v_2, x^+, x^- and z respectively) to find a $(v, u, x^+, x^-, z_l(vx_j))$-shifter $\phi_2^u : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ that satisfies (4.6). Thus, ϕ_2^u adds a weight of $z_l(vx_j)$ to the edges vx^+ and ux^+ while subtracting it from the edges vx^- and ux^- without modifying the weight of any other edge. We then define

$$\varphi_{l+1} := \varphi_l + \frac{1}{d_{\mathcal{G}}(\{x^+, x^-\}; U^i)} \sum_{u \in N_{\mathcal{G}}(\{x^+, x^-\}; U^i)} \phi_2^u$$

Observe that, by (4.5),

$$\varphi_{l+1}(vx) = \begin{cases} z_{vx_j} & \text{if } x = x^+ = x_j, \\ \varphi_l(vx) - z_l(vx_j) & \text{if } x = x^-, \\ \varphi_l(vx) & \text{otherwise.} \end{cases} \tag{4.13}$$

In particular, (4.5) and (4.13) implies

$$|\varphi_{l+1}(e)| \leq \begin{cases} |z_e| & \text{if } e \in E_v, \\ |\varphi_l(e)| + \frac{|z_l(vx_j)|}{C_U} & \text{if } e = ux^+ \text{ or } e = ux^- \text{ with } u \in U^i, \\ 0 & \text{otherwise,} \end{cases} \tag{4.14}$$

where we have used $d_{\mathcal{G}}(\{x^+, x^-\}; U^i) \geq C_U$ in the second inequality.

Let $z_{l+1} : E_v \to \mathbb{R}$ be defined as $z_{l+1}(e) = z_e - \varphi_{l+1}(e)$. Let $N_{l+1} := \{x \in N_{\mathcal{G}}(v) : z_{l+1}(vx) \neq 0\}$ and observe that $N_{l+1} \subseteq N_l \setminus \{x_j\}$ and $|N_{l+1}| \leq |N_l| - 1$. Furthermore, (4.13) implies that equations (4.11) and (4.12) holds when replacing l by $l+1$. Finally, equation (4.6) implies that for each $f \in F_{r+1}(\mathcal{G})$

$$|\varphi_{l+1}(f)| \leq |\varphi_l(f)| + \begin{cases} |z_l(vx_j)|cn^{-(r-1)} & \text{if } f \supset \{v, x^+\} \text{ or } f \supset \{v, x^-\}, \\ \frac{|z_l(xv_j)|}{C_U}cn^{-(r-1)} & \text{if } f \supset \{u, x^+\} \text{ or } f \supset \{u, x^-\} \\ & \text{ with } u \in U^i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Second case. Else, all vertices $x \in N_{\mathcal{G}}(v; V^j)$ satisfy $z_l(vx) \geq 0$. Then, (4.11) implies that for each $k \in [r+1] \setminus \{i, j\}$ there is some vertex $x_k \in N_{\mathcal{G}}(v; V^k)$ such that $z_l(vx_k) > 0$. Let $X = \{x_k\}_{k \in [r+1] \setminus \{i\}}$. Now, for each $u \in N_{\mathcal{G}}(X; U^i)$, we can apply Proposition 4.3.7 (with v, u, X and $z_l(vx_j)$ playing the role of v_1, v_2, X and z) to get a $(v, u, X, z_l(vx_j))$-shifter $\phi_1^u : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ that satisfies (4.2). In particular, ϕ_1^u adds a weight of $z_l(vx_j)$ to the edges $\{vx_k\}_{k \in [r+1] \setminus \{i\}}$ while subtracting it from the edges $\{ux_k\}_{k \in [r+1] \setminus \{i\}}$ without modifying the weight of any other edge. We then define

$$\varphi_{l+1} := \varphi_l + \frac{1}{d_{\mathcal{G}}(X;U^i)} \sum_{u \in N_{\mathcal{G}}(X;U^i)} \phi_1^u$$

Similarly to the previous case, (4.1) implies that

$$\varphi_{l+1}(vx) = \begin{cases} z_{vx_j} & \text{if } x = x_j, \\ \varphi_l(vx) + z_l(vx_j) & \text{if } x = x_k \text{ with } k \in [r+1] \setminus \{i,j\}, \\ \varphi_l(vx) & \text{otherwise,} \end{cases} \tag{4.16}$$

which together with (4.1) yields

$$|\varphi_{l+1}(e)| \leq \begin{cases} |z_e| & \text{if } e \in E_v, \\ |\varphi_l(e)| + \frac{|z_l(vx_j)|}{C_U} & \text{if } e = ux_k \text{ with } u \in U^i \\ & \text{ and } k \in [r+1] \setminus \{i\}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.17}$$

where we have used $d_{\mathcal{G}}(X;U^i) \geq C_U$ in the second inequality.

Let $z_{l+1} : E_v \to \mathbb{R}$ be defined as $z_{l+1}(e) = z_e - \varphi_{l+1}(e)$. Let $N_{l+1} := \{x \in N_{\mathcal{G}}(v) \ : \ z_{l+1}(vx) \neq 0\}$ and note that $N_{l+1} \subseteq N_l \setminus \{x_j\}$ and $|N_{l+1}| \leq |N_l| - 1$. Moreover, (4.16) and (4.17) implies that equations (4.11) and (4.12) hold when replacing l by $l+1$. Finally, equation (4.2) implies that for each $f \in F_{r+1}(\mathcal{G})$

$$|\varphi_{l+1}(f)| \leq |\varphi_l(f)| + \begin{cases} |z_l(vx_j)|cn^{-(r-1)} & \text{if } f \ni v \text{ and } |f \cap X| = 1, \\ |z_l(vx_j)|cn^{-r} & \text{if } f \ni v \text{ and } |f \cap X| = 0, \\ \frac{|z_l(xv_j)|}{C_U}cn^{-(r-1)} & \text{if } f \ni u \text{ and } |f \cap X| = 1, \\ \frac{|z_l(xv_j)|}{C_U}cn^{-r} & \text{if } f \ni u \text{ and } |f \cap X| = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Hence, we can iteratively repeat the above argument until finding $m \in \mathbb{N}$ such that $|N_m| = 0$. Then φ_m must satisfy $\varphi_m(e) = z_e$ for every $e \in E_v$ as desired.

Observe that

$$\sum_{l=1}^{m} |\varphi_l(e) - \varphi_{l-1}(e)| \overset{(4.12)}{=} \sum_{l=1}^{m} |\varphi_l(e)| - |\varphi_{l-1}(e)| = |\varphi_m(e)| - |\varphi_0(e)| = |z_e|. \tag{4.19}$$

Let $e = ux$ be an edge with $u \in U^i$ and $x \in N_{\mathcal{G}}(v)$. Note that for every $l \in [m]$, $|\varphi_l(e) - \varphi_{l-1}(e)| > 0$ if and only if we were adding/subtracting weight to the edge vx via some gadgets at the l-th step. Thus,

$$|\varphi_m(e)| = \sum_{l=1}^{m} |\varphi_l(e)| - |\varphi_{l-1}(e)| \overset{(4.14),(4.17)}{\leq} \sum_{l=1}^{m} \frac{|\varphi_l(vx) - \varphi_{l-1}(vx)|}{C_U} \overset{(4.19)}{=} \frac{|z_{vx}|}{C_U}.$$

And for all other edges, i.e. edges not in $E(\mathcal{G}[\{v\} \cup U^i, N_{\mathcal{G}}(v)])$, we have $\varphi_m(e) = 0$.

It only remains to check that φ_m satisfies (4.10).

Let $f \in F_{r+1}(\mathcal{G})$ with $f = \{v\} \cup Y$ where $Y = \{y_j\}_{j \in [r+1] \setminus \{i\}}$ and $y_j \in V^j$. Let $L_1 \subseteq [m]$ consist of all $l \in [m]$ such that, at the l-th step, we were adding/subtracting weight to at least one of the edges vy with $y \in Y$. This can happen during the first case if $y = x^+$ or $y = x^-$, or during the second case if $|f \cap X| = 1$. And let $L_2 \subseteq [m] \setminus L_1$ consist of all $l \in [m]$ such that at the l-th step we were adding/subtracting weight to a set of edges $\{vx_k\}_{k\in[r+1]\setminus\{i\}}$ with $x_k \in N_{\mathcal{G}}(v; V^k) \setminus \{y_k\}$. This can only happen during the second case when $|f \cap X| = 0$. Observe that $|\varphi_l(f) - \varphi_{l-1}(f)| > 0$ if and only if $l \in L_1$ or $l \in L_2$. Then

$$|\varphi_m(f)| = \sum_{l=1}^{m} |\varphi_l(f)| - |\varphi_{l-1}(f)| = \sum_{l\in L_1} |\varphi_l(f)| - |\varphi_{l-1}(f)| + \sum_{l\in L_2} |\varphi_l(f)| - |\varphi_{l-1}(f)|.$$

On one hand,

$$\begin{aligned}\sum_{l\in L_1} |\varphi_l(f)| - |\varphi_{l-1}(f)| &\overset{(4.15),(4.18)}{\leq} \sum_{j\in[r+1]\setminus\{i\}} \sum_{l\in L_1} |\varphi_l(vy_j) - \varphi_{l-1}(vy_j)| cn^{-(r-1)} \\ &= cn^{-(r-1)} \sum_{j\in[r+1]\setminus\{i\}} |z_{vy_j}|.\end{aligned}$$

And on the other hand,

$$\begin{aligned}\sum_{l\in L_2} |\varphi_l(f)| - |\varphi_{l-1}(f)| &\overset{(4.18)}{\leq} \sum_{x_k\in N_{\mathcal{G}}(v;V^k)} \sum_{l\in L_2} |\varphi_l(vx_k) - \varphi_{l-1}(vx_k)| cn^{-r} \\ &\leq cn^{-r} \sum_{x_k\in N_{\mathcal{G}}(v;V^k)} |z_{vx_k}| = |z_v| cn^{-r}\end{aligned}$$

where $k \in [r+1] \setminus \{i\}$ is chosen arbitrarily (since we added the same weight to all edges vx_k).

All in all, we have

$$|\varphi_m(f)| \leq \left(\sum_{j\in[r+1]\setminus\{i\}} |z_{vy_j}| \right) cn^{-(r-1)} + |z_v| cn^{-r}.$$

Let $f \in F_{r+1}(\mathcal{G})$ with $f = \{u\} \cup Y$ where $u \in U^i$ and $Y = \{y_j\}_{j \in [r+1] \setminus \{i\}}$ with $y_j \in V^j$. Following the same arguments as before we get

$$|\varphi_m(f)| \leq \frac{1}{C_U} \left(\sum_{j \in [r+1] \setminus \{i\}} |z_{vy_j}| \right) cn^{-(r-1)} + \frac{1}{C_U} |z_v| cn^{-r}.$$

Finally, the rest of the faces f satisfy $\varphi_m(f) = 0$. □

In the next lemma we are given excess/missing weights $\{z_e\}_{e \in E(\mathcal{G})}$ for all edges, and a set $U \subseteq V(\mathcal{G})$, and we find a zero-sum function $\omega : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ that compensates the excess/missing weight of all edges not contained in U. We also bound the weight added/subtracted from each edge in $\mathcal{G}[U]$ and the weight given to each face of $\mathcal{G}$ in terms of a constant C_U depending on U and a constant ξ_E defined as the maximum of $\{|z_e|\}_{e \in E(\mathcal{G})}$. The proof will consist of applying Lemma 4.3.11 twice: a first time to compensate the excess/missing weight of all edges not incident to U, and a second time to compensate the excess/missing weight of all edges not contained within U. The difficult part of the proof lies in the analysis of the resulting changes on the weights of the edges and faces each time we apply Lemma 4.3.11.

Lemma 4.3.12. *Let $1/n \ll \xi \ll 1/c \ll \alpha, p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq 2(r-1)$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \ldots, V^r$ of size n and $|V^{r+1}| = \alpha n$. Let $C_U \subseteq \mathbb{N}$ and $U \subseteq V(\mathcal{G})$ be a set such that for every $i \in [r+1]$ and every set of r vertices $X \subseteq V(\mathcal{G}) \setminus V^i$ we have $d_{\mathcal{G}}(X; U^i) \geq C_U$. Let $\xi_E > 0$. Let $\{z_v\}_{v \in V(\mathcal{G})}$ and*

$\{z_e\}_{e\in E(\mathcal{G})}$ *be real numbers satisfying* $|z_e| \leq \xi_E$ *for every* $e \in E(\mathcal{G})$ *and*

$$z_v = \sum_{x_j \in N_{\mathcal{G}}(v;V^j)} z_{vx_j}$$

for all $i \in [r+1]$, $v \in V^i$ *and* $j \in [r+1] \setminus \{i\}$.

Then there is a zero-sum function $\omega : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ *such that* $\omega(e) = z_e$ *for all edges not contained in* U *and*

$$|\omega(e)| \leq \frac{\xi_E n}{C_U} + \frac{2\xi_E n^2}{C_U^2} \tag{4.20}$$

for all edges $e \in E(\mathcal{G}[U])$. *Moreover, for every* $0 \leq k \leq r+1$ *and* $f \in F_{r+1}(\mathcal{G})$ *with* $|f \cap U| = k$

$$|\omega(f)| \leq \frac{\xi_E c^2}{n^{r-1}} + \left(k + \frac{|U|}{n}\right)\frac{\xi_E c^2}{C_U n^{r-2}} + \frac{k}{C_U}\left(k - 1 + \frac{|U|}{n}\right)\frac{\xi_E c^2}{C_U n^{r-3}}. \tag{4.21}$$

Proof. We start applying Lemma 4.3.11 to obtain, for each $i \in [r+1]$ and $v \in V^i \setminus U^i$, a zero-sum function $\varphi_v : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that $\varphi_v(e) = z_e$ for each edge e incident to v, $|\varphi_v(e)| \leq |z_{vx}|/C_U$ for all edges $e = ux$ with $u \in U^i$ and $x \in N_{\mathcal{G}}(v)$, and $\varphi_v(e) = 0$ for the rest of the edges.

Let $\varphi := \frac{1}{2}\sum_{v \in V(\mathcal{G})\setminus U} \varphi_v$. Then, for each edge v_1v_2 not incident to U we have

$$\varphi(v_1v_2) = \frac{1}{2}\sum_{v \in V(\mathcal{G})\setminus U} \varphi_v(v_1v_2) = \frac{1}{2}\Big(\varphi_{v_1}(v_1v_2) + \varphi_{v_2}(v_1v_2)\Big) = z_{v_1v_2}.$$

Moreover, for each edge ux with $u \in U^i$, $i \in [r+1]$ and $x \in V(\mathcal{G}) \setminus U$ we have

$$\begin{aligned}\varphi(ux) &= \frac{1}{2} \sum_{v \in V(\mathcal{G}) \setminus U} \varphi_v(ux) = \frac{1}{2}\left(\varphi_x(ux) + \sum_{v \in N_{\mathcal{G}}(x; V^i \setminus U^i)} \varphi_v(ux)\right) \\ &= \frac{z_{ux}}{2} + \frac{1}{2}\left(\sum_{v \in N_{\mathcal{G}}(x; V^i \setminus U^i)} \varphi_v(ux)\right)\end{aligned}$$

Finally, for each edge $u_i u_j$ with $u_i \in U^i$, $u_j \in U^j$ and $i, j \in [r+1]$ we have

$$\varphi(u_i u_j) = \frac{1}{2} \sum_{v \in V(\mathcal{G}) \setminus U} \varphi_v(u_i u_j) = \frac{1}{2}\left(\sum_{v \in N_{\mathcal{G}}(u_i; V^j \setminus U^j)} \varphi_v(u_i u_j) + \sum_{v \in N_{\mathcal{G}}(u_j; V^i \setminus U^i)} \varphi_v(u_1 u_j)\right).$$

Note that for each $i \in [r+1]$, $u \in U^i$ and $x \in N_{\mathcal{G}}(u)$ we have

$$\sum_{v \in N_{\mathcal{G}}(x; V^i \setminus U^i)} |\varphi_v(ux)| \leq \sum_{v \in N_{\mathcal{G}}(x; V^i)} \frac{\xi_E}{C_U} \leq \frac{\xi_E n}{C_U} \tag{4.22}$$

where we have used $d_{\mathcal{G}}(x; V^i) \leq n$.

For each edge $e \in E(\mathcal{G})$, let $z'_e := z_e - \varphi(e)$. Hence,

$$|z'_e| \leq \begin{cases} 0 & \text{if } e \text{ is not incident to } U, \\ \frac{1}{2}\left(\xi_E + \frac{\xi_E n}{C_U}\right) \leq \frac{\xi_E n}{C_U} & \text{if } e \in E(\mathcal{G}[U, V(\mathcal{G}) \setminus U]) \\ \xi_E + \frac{\xi_E n}{C_U} \leq \frac{2\xi_E n}{C_U} & \text{if } e \in E(\mathcal{G}[U]). \end{cases} \tag{4.23}$$

We have bounded the 'new' excess/missing weights $\{z'_e\}_{e \in E(\mathcal{G})}$.

Observe that for each $i \in [r+1]$, $v \in V^i$ and $j \in [r+1] \setminus \{i\}$

$$
\begin{aligned}
z'_v &:= \sum_{x \in N_{\mathcal{G}}(v;V^j)} z'_{vx} = \sum_{x \in N_{\mathcal{G}}(v;V^j)} z_{vx} - \varphi(vx) \\
&= z_v - \sum_{x \in N_{\mathcal{G}}(v;V^j)} \sum_{f \in F_{r+1}(vx)} \varphi(f) = z_v - \sum_{f \in F_{r+1}(v)} \varphi(f)
\end{aligned}
$$

which does not depend on j.

Thus, we can apply again Lemma 4.3.11 (with $\{z'_v\}_{v \in V(\mathcal{G})}$ and $\{z'_e\}_{e \in E(\mathcal{G})}$ playing the role of $\{z_v\}_{v \in V(\mathcal{G})}$ and $\{z_e\}_{e \in E(\mathcal{G})}$) to obtain, for each $i \in [r+1]$ and $v \in V^i \setminus U^i$, a zero-sum function $\varphi'_v : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that $\varphi'_v(e) = z'_e$ for each edge e incident to v, $|\varphi'_v(e)| \leq |z'_{vx}|/C_U$ for all edges $e = ux$ with $u \in U^i$ and $x \in N_{\mathcal{G}}(v)$, and $\varphi'_v(e) = 0$ for the rest of edges. Is important to note that this time $|\varphi'_v(e)| \leq |z'_{vx}|/C_U = 0$ if $e = ux$ with $u \in U$ and $x \in V(\mathcal{G}) \setminus U$.

Let $\varphi' := \sum_{v \in V(\mathcal{G}) \setminus U} \varphi'_v$. For each edge $e \in E(\mathcal{G})$, let $z''_e := z'_e - \varphi'(e)$. We now do the same analysis we did before. First, for each edge v_1v_2 not incident to U we have

$$
\varphi'(v_1v_2) = \sum_{v \in V(\mathcal{G}) \setminus U} \varphi'_v(v_1v_2) = \Big(\varphi'_{v_1}(v_1v_2) + \varphi'_{v_2}(v_1v_2)\Big) = 2z'_{v_1v_2} = 0.
$$

Second, for each edge ux with $u \in U^i$, $i \in [r+1]$ and $x \in V(\mathcal{G}) \setminus U$ we have

$$
\varphi'(ux) = \sum_{v \in V(G) \setminus U} \varphi'_v(ux) = \left(\varphi'_x(ux) + \sum_{v \in N_{\mathcal{G}}(x;V^i \setminus U^i)} \varphi'_v(ux)\right) = \varphi'_x(ux) + 0 = z'_{ux}.
$$

And lastly, for each edge u_iu_j with $u_i \in U^i$, $u_j \in U^j$ and $i, j \in [r+1]$ we have

$$\varphi'(u_iu_j) = \sum_{v \in V(\mathcal{G}) \setminus U} \varphi'_v(u_iu_j) = \sum_{v \in N_{\mathcal{G}}(u_i; V^j \setminus U^j)} \varphi'_v(u_iu_j) + \sum_{v \in N_{\mathcal{G}}(u_j; V^i \setminus U^i)} \varphi'_v(u_1u_j),$$

and

$$\sum_{v \in N_{\mathcal{G}}(u_i; V^j \setminus U^j)} |\varphi'_v(u_iu_j)| \le \sum_{v \in N_{\mathcal{G}}(u_i; V^j)} \frac{|z'_{vu_i}|}{C_U} \le \sum_{v \in N_{\mathcal{G}}(u_i; V^j)} \frac{\xi_E n}{C_U^2} \le \frac{\xi_E n^2}{C_U^2}$$

where again we have used $d_{\mathcal{G}}(u_i; V^j) \le n$.

Hence,

$$|z''_e| \le \begin{cases} 0 & \text{if } e \text{ is not contained in } U, \\ \frac{2\xi_E n}{C_U} + \frac{2\xi_E n^2}{C_U^2} = \frac{2\xi_E n}{C_U}\left(1 + \frac{n}{C_U}\right) & \text{if } e \in E(\mathcal{G}[U]). \end{cases} \tag{4.24}$$

Let $\omega := \varphi + \varphi'$ and observe that $z''_e = z_e - \omega(e)$. Then (4.24) implies that $\omega(e) = z_e$ for every edge not contained in U and $|\omega(e)| \le |\varphi(e)| + |\varphi'(e)| \le \frac{\xi_E n}{C_U} + \frac{2\xi_E n^2}{C_U^2}$ for every edge $e \in E(\mathcal{G}[U])$.

It only remains to bound the added/subtracted weight of the faces. First of all, we have that for every $i \in [r+1]$ and $v \in V^i \setminus U^i$, and an arbitrary $j \in [r+1] \setminus \{i\}$,

$$|z_v| \le \sum_{x_j \in N_{\mathcal{G}}(v; V^j)} |z_{vx_j}| \le \xi_E n \tag{4.25}$$

and

$$|z'_v| \le \sum_{x_j \in N_{\mathcal{G}}(v; V^j)} |z'_{vx_j}| \overset{(4.23)}{\le} \sum_{u_j \in N_{\mathcal{G}}(v; U^j)} |z'_{vu_j}| \overset{(4.23)}{\le} |U^j| \frac{\xi_E n}{C_U} \le |U| \frac{\xi_E n}{C_U}. \tag{4.26}$$

Let $f \in F_{r+1}(\mathcal{G})$ with $f = \{x_i\}_{i\in[r+1]}$ and $x_i \in V^i$ for each $i \in [r+1]$, and let $I_U = \{i \in [r+1] : x_i \in U^i\}$ and $I_V = \{i \in [r+1] : x_i \in V^i \setminus U^i\}$. Then by (4.10) we have that

$$\begin{aligned} |\varphi(f)| &\leq \frac{1}{2}\sum_{v\in V(\mathcal{G})\setminus U} |\varphi_v(f)| = \frac{1}{2}\sum_{i\in I_U}\sum_{v\in V^i\setminus U^i} |\varphi_v(f)| + \frac{1}{2}\sum_{i\in I_V}\sum_{v\in V^i\setminus U^i} |\varphi_v(f)| \\ &\leq \frac{1}{2}\sum_{i\in I_U}\sum_{v\in V^i\setminus U^i} \frac{C_v(f)}{C_U} + \frac{1}{2}\sum_{i\in I_V} C_{x_i}(f) \end{aligned}$$

where

$$\begin{aligned} C_v(f) &= \left(\sum_{j\in[r+1]\setminus\{i\}} |z_{vx_j}|\right) cn^{-(r-1)} + |z_v|cn^{-r} \overset{(4.25)}{\leq} r\xi_E cn^{-(r-1)} + \xi_E ncn^{-r} \\ &\leq 2r\xi_E cn^{-(r-1)} \end{aligned}$$

and thus,

$$|\varphi(f)| \leq \left(|I_U|\frac{|V(\mathcal{G})\setminus U|}{C_U} + |I_V|\right)\frac{r\xi_E c}{n^{r-1}} \leq \left(\frac{|I_U|n}{C_U} + (r+1)\right)\frac{r\xi_E c}{n^{r-1}}.$$

On the other hand,

$$\begin{aligned} |\varphi'(f)| &\leq \sum_{v\in V(\mathcal{G})\setminus U} |\varphi'_v(f)| = \sum_{i\in I_U}\sum_{v\in V^i\setminus U^i} |\varphi'_v(f)| + \sum_{i\in I_V}\sum_{v\in V^i\setminus U^i} |\varphi'_v(f)| \\ &\leq \sum_{i\in I_U}\sum_{v\in V^i\setminus U^i} \frac{C'_v(f)}{C_U} + \sum_{i\in I_V} C'_{x_i}(f) \end{aligned}$$

where

$$\begin{aligned} C'_v(f) &= \left(\sum_{j\in[r+1]\setminus\{i\}} |z'_{vx_j}|\right) cn^{-(r-1)} + |z'_v|cn^{-r} \\ &\overset{(4.23),(4.26)}{\leq} |(f\setminus\{x_i\})\cap U|\,\frac{\xi_E n}{C_U}cn^{-(r-1)} + |U|\frac{\xi_E n}{C_U}cn^{-r}. \end{aligned}$$

Hence,

$$
\begin{aligned}
|\varphi'(f)| &\le |I_U|\frac{|V(G)\setminus U|}{C_U}\left(|I_U|-1+\frac{|U|}{n}\right)\frac{\xi_E c}{C_U n^{r-2}} + |I_V|\left(|I_U|+\frac{|U|}{n}\right)\frac{\xi_E c}{C_U n^{r-2}} \\
&\le \left(\frac{|I_U|n}{C_U}\left(|I_U|-1+\frac{|U|}{n}\right)+(r+1)\left(|I_U|+\frac{|U|}{n}\right)\right)\frac{\xi_E c}{C_U n^{r-2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
|\omega(f)| &\le |\varphi(f)| + |\varphi'(f)| \\
&\le \left(\frac{|I_U|n}{C_U}+(r+1)\right)\frac{r\xi_E c}{n^{r-1}} \\
&\qquad + \left(\frac{|I_U|n}{C_U}\left(|I_U|-1+\frac{|U|}{n}\right)+(r+1)\left(|I_U|+\frac{|U|}{n}\right)\right)\frac{\xi_E c}{C_U n^{r-2}} \\
&\le \frac{\xi_E c^2}{n^{r-1}} + \left(|I_U|+\frac{|U|}{n}\right)\frac{\xi_E c^2}{C_U n^{r-2}} + \frac{|I_U|}{C_U}\left(|I_U|-1+\frac{|U|}{n}\right)\frac{\xi_E c^2}{C_U n^{r-3}}.
\end{aligned}
$$

□

4.3.3 Proof of Lemma 4.3.1

We are finally ready to prove the main result of this section restated below. The proof goes as follows. We will start with a constant weighting that will be already close to a fractional face-decomposition of $\mathcal{G}$. This will define small excess/missing weights $\{z_e\}_{e\in E(\mathcal{G})}$ with the property that their sum is equal to zero. For a fixed face $f \in F_{r+1}(\mathcal{G})$ we will find a set of vertices U of linear size in its neighbourhood. We will then use Lemma 4.3.12 to compensate the weight of all edges not contained in U and we will update the excess/missing weights of each edge of $\mathcal{G}[U]$ to some new values $\{z'_e\}_{e\in E(\mathcal{G}[U])}$. Next, we will apply Lemma 4.3.12 again, this time with $\mathcal{G}[U]$, f and $\{z'_e\}_{e\in E(\mathcal{G}[U])}$ playing the role of $\mathcal{G}$, U and $\{z_e\}_{e\in E(\mathcal{G})}$. This will compensate the weight of all edges of $\mathcal{G}$ except those contained in the face f. However, since the functions given by Lemma 4.3.12 are zero-sum functions and the sum of the values $\{z_e\}_{e\in E(\mathcal{G})}$ is zero, the edges of f will end up also compensated. With this approach we will be able to compensate the weight of all edges of $\mathcal{G}$. However, those faces that intersect f might end with a weight outside of $[0,1]$ (remember that a fractional face decomposition is a function from $F_{r+1}(\mathcal{G})$ to $[0,1]$). To solve this, we use the above approach for each face $f \in F_{r+1}(\mathcal{G})$ as the starting fixed face and we then average by $|F_{r+1}(\mathcal{G})|$.

Lemma 4.3.1. Let $1/n \ll \xi \ll \xi'' \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3r$ and $t \geq 2r-1$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Then there exists a weighting $\omega : F_{r+1}(\mathcal{G}) \to [0,1]$ such that for every edge $e \in E(\mathcal{G})$

$$\omega(e) := \sum_{\substack{f\in F_{r+1}(\mathcal{G})\\ e\subseteq f}} \omega(f) = 1, \tag{4.27}$$

and for every $f \in F_{r+1}(\mathcal{G})$

$$\omega(f) = \frac{(1\pm\xi'')}{p^{\binom{r}{2}} p_H^{r-1} q n^{r-1}}. \tag{4.28}$$

Proof. Let $\xi' > 0$ and $c \in \mathbb{N}$ satisfying

$$1/n \ll \xi \ll \xi' \ll 1/c \ll \xi'' \ll p_G, p_H, q, 1/s, 1/t, 1/r. \tag{4.29}$$

We start considering the constant weighting $\omega' : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ defined as

$$\omega'(f) = \frac{|E(\mathcal{G})|}{\binom{r+1}{2}|F_{r+1}(\mathcal{G})|}$$

for every face $f \in F_{r+1}(\mathcal{G})$. Next, for each edge $e \in E(\mathcal{G})$, let $z_0(e) := 1 - \omega'(e)$ be its 'excess/missing' weight.

Observe that from divisibility and typicality for every $i, j \in [r]$, $i \neq j$ we have

$$|E(G[V^1, V^2])| = \sum_{v \in V^i} d_G(v; V^j) = (1 \pm \xi) p_G n^2$$

and for every $i \in [r]$ we have

$$|E(H[V^i, V^{r+1}])| = \sum_{v \in V^i} d_G(v; V^{r+1}) = (1 \pm \xi) p_H |V^{r+1}| n \overset{\text{Prop. 4.1.2}}{=} (1 \pm 5\xi) p_G n^2.$$

Thus,

$$|E(\mathcal{G})| = \binom{r}{2}(1 \pm \xi) p_G n^2 + r(1 \pm 5\xi) p_G n^2 = \binom{r+1}{2}(1 \pm 5\xi) p_G n^2. \tag{4.30}$$

It follows by (4.30) and Proposition 4.3.3 that for every $f \in F_{r+1}(\mathcal{G})$

$$|\omega_0(f)| = \frac{|E(\mathcal{G})|}{\binom{r+1}{2}|F_{r+1}(\mathcal{G})|} = \frac{(1 \pm 5\xi) p_G n^2}{(1 \pm \xi)^r p^{\binom{r}{2}+1} p_H^{r-1} q n^{r+1}} \overset{(4.29)}{=} \frac{(1 \pm \xi')}{p^{\binom{r}{2}} p_H^{r-1} q n^{r-1}} \tag{4.31}$$

and for every edge $e \in E(\mathcal{G})$

$$\begin{aligned} |z_0(e)| = |1 - \omega'(e)| &= \left|1 - \sum_{f \in F_{r+1}(e)} \omega'(f)\right| = \left|1 - \frac{|F_{r+1}(e)||E(\mathcal{G})|}{\binom{r+1}{2}|F_{r+1}(\mathcal{G})|}\right| \\ &= \left|1 - \frac{(1 \pm \xi)^{r-1}(1 \pm 5\xi)}{(1 \pm \xi)^r}\right| = |1 - (1 \pm \xi')| \leq \xi'. \end{aligned} \tag{4.32}$$

Observe that, for any $i \in [r+1]$, $v \in V^i$ and $j \in [r+1] \setminus \{i\}$,

$$
\begin{aligned}
z_0(v) := \sum_{x \in N_{\mathcal{G}}(v;V^j)} z_0(vx) &= \sum_{x \in N_{\mathcal{G}}(v;V^j)} 1 - \omega'(vx) \\
&= d_{\mathcal{G}}(v;V^j) - \sum_{x \in N_{\mathcal{G}}(v;V^j)} \sum_{f \in F_{r+1}(vx)} \omega'(f) \\
&= d_{\mathcal{G}}(v,V^j) - \sum_{f \in F_{r+1}(v)} \omega'(f)
\end{aligned}
$$

does not depend on j since $\mathcal{G}$ is F_{r+1}-divisible.

Fix a face $f' = \{x'_1, \ldots, x'_{r+1}\} \in F_{r+1}(\mathcal{G})$ with $x'_i \in V^i$ for each $i \in [r+1]$ and recall that we write $f'_i := f' \setminus \{x'_i\}$. Let $U_{f'} = \bigcup_{i \in [r+1]} U^i_{f'}$ where $U^i_{f'} = N^*_{\mathcal{G}}(f'_i; V^i)$ and note that for every $i \in [r]$

$$
|U^i_{f'}| = d^*_{\mathcal{G}}(f'_i; V^i) = (1 \pm \xi) p_G^{r-1} p_H qn \tag{4.33}
$$

and

$$
\begin{aligned}
|U^{r+1}_{f'}| &= d^*_{\mathcal{G}}(f'_i; V^{r+1}) = d_H(f'_i; V^{r+1}) = (1 \pm \xi) p_H^r |V^{r+1}| \\
&\overset{\text{Prop. 4.1.2}}{=} (1 \pm 5\xi) p_G p_H^{r-1} n.
\end{aligned} \tag{4.34}
$$

Let $U \subseteq U_{f'}$ be a subset such that $|U^i| = (1-\xi) p_G^{r-1} p_H qn$ for every $i \in [r]$ and $U^{r+1} = U^{r+1}_{f'}$. In other words, U is obtained from $U_{f'}$ by removing at most a 2ξ proportion of vertices from each part so that $|U^1| = \cdots = |U^r|$.

Observe that by typicality for every $i \in [r]$ and every set $X \subseteq V(G) \setminus V^i$ and $X' \subseteq V^{r+1}$ with $|X| + |X'| \leq r$ we have

$$
\begin{aligned}
d_{\mathcal{G}}(X \cup X'; U^i) = d_{\mathcal{G}}(X \cup X'; U^i_{f'}) \pm 2\xi |U^i_{f'}| &= d^*_{\mathcal{G}}(X \cup X' \cup f'_i, f'_i; V^i) \pm 2\xi |U^i_{f'}| \\
&\geq (1-\xi) p_G^{|X|+r-1} p_H^{|X'|+1} qn - 2\xi p_G^{r-1} p_H qn
\end{aligned}
$$

and for every set $X \subseteq V(G)$ of at most r vertices we have

$$d_{\mathcal{G}}(X; U^{r+1}) = d_H(X \cup f'_{r+1}; V^{r+1}) \geq (1-5\xi)p_G p_H^{|X|+r-1} n.$$

Let $C_U := p_G^{2r-1} p_H^{2r-1} qn$ and $\alpha := |U^{r+1}|/n$. Then, by Lemma 4.3.12 (with ξ', $\{z_0(v)\}_{v \in V(G)}$ and $\{z_0(e)\}_{e \in E(G)}$ playing the role of ξ_E, $\{z_v\}_{v \in V(G)}$ and $\{z_e\}_{e \in E(G)}$), we know that there exists a zero-sum function $\omega_1 : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ such that $\omega_1(e) = z_0(e)$ for all edges not contained in U,

$$|\omega_1(e)| \leq c\xi' \tag{4.35}$$

for all edges $e \in E(\mathcal{G}[U])$, and

$$|\omega_1(f)| \leq \frac{\xi' c^3}{n^{r-1}}. \tag{4.36}$$

To obtain equations (4.35)-(4.36) note that C_U and $|U|$ are both linear in n if we treat the right side of the hierarchy (4.29) as constants. We can then replace C_U and $|U|$ in equations (4.20)-(4.21) and bound all the remaining constant terms by c.

Let $z_1(e) := z_0(e) - \omega_1(e)$. Then

$$|z_1(e)| \leq \begin{cases} 0 & \text{if } e \notin E(\mathcal{G}[U]), \\ |z_0(e)| + |\omega_1(e)| \leq 2c\xi' & \text{if } e \in E(\mathcal{G}[U]). \end{cases} \tag{4.37}$$

We will now look to the graph $\mathcal{G}[U]$ with excess/missing weights $z_1(e)$.

Since $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical we have that for every $i \in [r]$, every set $S \subseteq$

$U\setminus(U^i\cup U^{r+1})$ and $S'\subseteq U^{r+1}$ with $|S|+|S'|\leq s-r$ and every set $T\subseteq \mathcal{K}_{r-1}(G[S])$ with $|T|\leq t-1$

$$\begin{aligned}
d^*_{\mathcal{G}}(S\cup S',T;U^i) &= d^*_{\mathcal{G}}(S\cup S'\cup f'_i,T\cup f'_i;V^i)\pm 2\xi|U^i_{f'}| \\
&= (1\pm\xi)p_G^{|S|+|S'|+r-1}p_H^{|S'|+1}q^{|T|+1}n\pm 2\xi|U^i_{f'}| \\
&\overset{(4.33)}{=} (1\pm\xi')p_G^{|S|}p_H^{|S'|}q^{|T|}|U^i|
\end{aligned}$$

and for every set $S\subseteq U\setminus U^{r+1}$ with $|S|\leq s-r$

$$\begin{aligned}
d_{\mathcal{G}}(S;U^{r+1}) &= d_{\mathcal{G}}(S\cup f'_{r+1};V^{r+1}) = (1\pm 5\xi)p_G p_H^{|S|+r-1}n \\
&\overset{(4.34)}{=} (1\pm\xi')p_H^{|S|}|U^{r+1}|.
\end{aligned}$$

Hence, $\mathcal{G}[U]$ is $(\xi',p_G,p_H,q,s-r,t-1)$-typical. Recall that $s\geq 3r$ and $t\geq 2r-1$.

Note that for every $i\in[r+1]$, $u\in U^i$ and $j\in[r+1]\setminus\{i\}$,

$$\begin{aligned}
z_1(u) &:= \sum_{x\in N_{\mathcal{G}}(u;U^j)} z_1(ux) \overset{(4.37)}{=} \sum_{x\in N_{\mathcal{G}}(u;V^j)} z_1(ux) = \sum_{x\in N_{\mathcal{G}}(u;V^j)} (z_0(ux)-\omega_1(ux)) \\
&= z_0(u) - \sum_{x\in N_{\mathcal{G}}(u;V^j)}\ \sum_{f\in F_{r+1}(ux)} \omega_1(f) = z_0(u) - \sum_{f\in F_{r+1}(u)} \omega_1(f)
\end{aligned}$$

does not depend on j.

Thus, we can apply again Lemma 4.3.12 to $\mathcal{G}[U]$ (with $\mathcal{G}[U]$, $(1-\xi)p_G^{r-1}p_Hqn$, ξ', $s-r$, $t-1$, f', 1, $2c\xi'$, $\{z_1(u)\}_{u\in U}$ and $\{z_1(e)\}_{e\in E(G[U])}$ playing the role of G, n, ξ, s, t, U, C_U, ξ_E, $\{z_v\}_{v\in V(G)}$ and $\{z_e\}_{e\in E(G)}$ respectively) to obtain a zero-sum function $\widetilde{\omega}_2 : F_{r+1}(\mathcal{G}[U])\to\mathbb{R}$ such that $\widetilde{\omega}_2(e)=z_1(e)$ for all edges $e\in E(\mathcal{G}[U])$ not contained in f', and

$$|\widetilde{\omega}_2(f)|\leq \frac{\xi'c^3}{n^{r-1}}+|f\cap f'|\frac{\xi'c^3}{n^{r-2}}+(|f\cap f'|-1)\frac{\xi'c^3}{n^{r-3}} \tag{4.38}$$

for all $f \in F_{r+1}(\mathcal{G}[U])$. We now define $\omega_2 : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ as $\omega_2(f) = \widetilde{\omega}_2(f)$ if $f \in F_{r+1}(\mathcal{G}[U])$, and $\omega_2(f) = 0$ otherwise.

Let $z_2(e) := z_1(e) - \omega_2(e)$. Then $z_2(e) = 0$ for all $e \not\subset f'$. Hence, for every $i, j \in [r+1]$ with $i \neq j$ we have

$$\begin{aligned} z_2(x'_i x'_j) &= \sum_{e \in E(\mathcal{G}[V^i, V^j])} z_2(e) = \sum_{e \in E(\mathcal{G}[V^i, V^j])} (1 - \omega'(e) - \omega_1(e) - \omega_2(e)) \\ &= |E(\mathcal{G}[V^i, V^j])| - \sum_{e \in E(\mathcal{G}[V^i, V^j])} \sum_{f \in F_{r+1}(e)} (\omega'(f) + \omega_1(f) + \omega_2(f)) \\ &= |E(\mathcal{G}[V^i, V^j])| - \sum_{f \in F_{r+1}(\mathcal{G})} \omega'(f) = |E(\mathcal{G}[V^i, V^j])| - \frac{E(\mathcal{G})}{\binom{r+1}{2}} = 0 \end{aligned}$$

where we have used the fact that ω_1 and ω_2 are zero-sum functions, and for every $i, j \in [r+1]$, $i \neq j$, $|E(\mathcal{G}[V^i, V^j])| = E(\mathcal{G})/\binom{r+1}{2}$ since $\mathcal{G}$ is F_{r+1}-divisible.

Let $\omega_{f'} := \omega' + \omega_1 + \omega_2$. Observe that for every edge $e \in E(\mathcal{G})$

$$\begin{aligned} \omega_{f'}(e) &= \omega'(e) + \omega_1(e) + \omega_2(e) = (1 - z_0(e)) + (z_0(e) - z_1(e)) + (z_1(e) - z_2(e)) \\ &= 1 - z_2(e) = 1. \end{aligned}$$

Thus, $\omega_{f'}$ satisfies (4.27). However, equation (4.38) implies that the faces that intersect f' will not satisfy (4.28). Indeed, it follows by (4.36) and (4.38) that

$$|\omega_{f'}(f) - \omega'(f)| \leq |\omega_1(f)| + |\omega_2(f)| \overset{(4.29)}{\leq} \begin{cases} \xi' c^4 n^{-(r-1)} & \text{if } |f \cap f'| = 0, \\ \xi' c^4 n^{-(r-2)} & \text{if } |f \cap f'| = 1, \\ \xi' c^4 n^{-(r-3)} & \text{if } |f \cap f'| \geq 2. \end{cases} \tag{4.39}$$

Note that f' was chosen arbitrarily, which means that for every $f' \in F_{r+1}(\mathcal{G})$ there exists such a weighting $\omega_{f'}$. Thus, we can solve this issue by averaging among all weights $\{\omega_{f'}\}_{f'\in F_{r+1}(\mathcal{G})}$. Let $\omega : F_{r+1}(\mathcal{G}) \to \mathbb{R}$ be defined as $\omega = \frac{1}{|F_{r+1}(\mathcal{G})|}\sum_{f'\in F_{r+1}(\mathcal{G})}\omega_{f'}$. It is clear that ω also satisfies equation (4.27) since each $\omega_{f'}$ satisfies it. Given $f \in F_{r+1}(\mathcal{G})$, note that the number of cliques $f' \in F_{r+1}(\mathcal{G})$ such that $|f \cap f'| \geq 2$ is bounded by

$$\sum_{e\subset f} |F_{r+1}(e)| \overset{\text{Prop. 4.3.3}}{\leq} \binom{r+1}{2}(1\pm\xi)^r p^{\binom{r}{2}} p_H^{r-1} q n^{r-1} \leq 2\binom{r+1}{2} n^{r-1}.$$

Similarly, the number of cliques $f' \in F_{r+1}(\mathcal{G})$ such that $|f \cap f'| = 1$ is bounded by

$$\sum_{v\in f} |F_{r+1}(v)| \leq \sum_{v\in f}\sum_{e\ni v} |F_{r+1}(e)| \overset{\text{Prop. 4.3.3}}{\leq} (r+1)\hat{\Delta}(\mathcal{G})(1\pm\xi)^r p^{\binom{r}{2}} p_H^{r-1} q n^{r-1}$$
$$\leq (r+1)(1+5\xi)p_G n n^{r-1} \leq 2(r+1)n^r.$$

And the number of cliques $f' \in F_{r+1}(\mathcal{G})$ such that $|f \cap f'| = 0$ is bounded trivially by

$$|F_{r+1}(\mathcal{G})| \leq n^{r+1}.$$

Hence, for every $f \in F_{r+1}(\mathcal{G})$

$$\begin{aligned}
\omega(f) &= \frac{1}{|F_{r+1}(\mathcal{G})|}\sum_{f'\in F_{r+1}(\mathcal{G})} \omega_{f'}(f) = \frac{1}{|F_{r+1}(\mathcal{G})|}\sum_{f'\in F_{r+1}(\mathcal{G})} (\omega_0(f) \pm |\omega_{f'}(f) - \omega'(f')|) \\
&\overset{(4.39)}{=} \omega'(f) \pm \frac{1}{|F_{r+1}(\mathcal{G})|}\left(\sum_{\substack{f'\in F_{r+1}(\mathcal{G})\\ |f'\cap f|=0}} \frac{\xi' c^4}{n^{r-1}} + \sum_{\substack{f'\in F_{r+1}(\mathcal{G})\\ |f'\cap f|=1}} \frac{\xi' c^4}{n^{r-2}} + \sum_{\substack{f'\in F_{r+1}(\mathcal{G})\\ |f'\cap f|\geq 2}} \frac{\xi' c^4}{n^{r-3}}\right) \\
&= \omega'(f) \pm \frac{1}{|F_{r+1}(\mathcal{G})|}\left(\xi' c^4 n^2 + 2(r+1)\xi' c^4 n^2 + 2\binom{r+1}{2}\xi' c^4 n^2\right) \\
&\overset{\text{Prop. 4.3.3}}{=} \omega'(f) \pm \frac{\xi' c^5}{n^{r-1}} \overset{(4.31),(4.29)}{=} \frac{(1\pm\xi'')}{p^{\binom{r}{2}} p_H^{r-1} q n^{r-1}}.
\end{aligned}$$

Thus, ω satisfies (4.28) which concludes the proof. □

4.4 Vortex sequence

The following definition defines a vortex sequence in an extended partite complex so that the properties of divisibility and typicality get passed down through the sequence.

Definition 4.4.1 (Vortex)**.** Given $m \in \mathbb{N}$ and $\xi, \varepsilon, \eta > 0$, a $(\xi, \varepsilon, \eta, m)$-*vortex* of an extended $(r+1)$-partite complex $\mathcal{G} = (V, U, G, H)$ with parts $V^1, \dots, V^r$ is a pair of sequences $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ where $|V_\ell^k| = m$ for each $k \in [r]$ such that for all $i \in [\ell]$

(V1) $|V_i^k| = \varepsilon |V_{i-1}^k|$ for each $k \in [r]$ and $|U_i| = \varepsilon |U_{i-1}|$,

(V2) $\mathcal{G}[V_{i-1} \cup U_{i-1}]$ is (ξ, p_G, p_H, q, s, t)-typical into $V_i \cup U_i$,

(V3) $\mathcal{G}[V_{i-1} \cup U_{i-1}]$ is η-divisible into $V_i \cup U_i$.

The following result allows us to find a vortex sequence in an extended partite complex.

Lemma 4.4.2. *Let* $1/n \ll 1/m' \ll \eta \ll \varepsilon \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. *Let* $\mathcal{G} = (V, U, G, H)$ *be an* F_{r+1}*-divisible* (ξ, p_G, p_H, q, s, t)*-typical extended* $(r+1)$*-partite complex with parts* $V^1, \dots, V^r$ *of size* n*. Then* $\mathcal{G}$ *has a* $(2\xi, \varepsilon, \eta, m)$*-vortex where* $\varepsilon m' \leq m \leq m'$*.*

Proof. Let $n_0 := n$ and recursively define $n_i := \varepsilon n_{i-1}$. Let $\ell := 1 + \max\{i \geq 0 : n_i \geq m'\}$ and let $m := n_\ell$. Observe that $\varepsilon^i n \geq n_i \geq \varepsilon^i n - 1/(1-\varepsilon)$ and $\varepsilon m' \leq m \leq m'$.

Let $\xi_0 := \xi$ and $\eta_0 := 0$ and define for $i = 1, \ldots, \ell$,

$$\xi_i := \xi_{i-1} + 2n_{i-1}^{-1/3} \tag{4.40}$$

and

$$\eta_i := \eta_{i-1} + 4n_{i-1}^{-1/3}. \tag{4.41}$$

Suppose that for some $i \in [\ell]$ we have already found sequences $V_0, \ldots, V_{i-1}$ and $U_0, \ldots, U_{i-1}$ which form a $(\xi_{i-1}, \varepsilon, \eta_{i-1}, n_{i-1})$-vortex of $\mathcal{G}$. This is true for $i = 0$ since $\mathcal{G}$ is F_{r+1}-divisible and (ξ, p_G, p_H, q, s, t)-typical by assumption.

For each $k \in [r]$ let V_i^k be a random subset of V_{i-1}^k of size n_i and let U_i be a random subset of U_{i-1} of size $\varepsilon|U_{i-1}|$. Lemma 2.1.2 and a union bound implies that with positive probability for any $k \in [r]$, $S_V \subseteq V_{i-1} \setminus V_{i-1}^k$ and $S_U \subseteq U_{i-1}$ with $|S_V| + |S_U| \le s$, and $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \le t$, we have

$$d^*_{\mathcal{G}}(S_V \cup S_U, T; V_i^k) = (1 \pm n_{i-1}^{-1/3}) d^*_{\mathcal{G}}(S_V \cup S_U, T; V_{i-1}^k) \frac{n_i}{n_{i-1}}, \tag{4.42}$$

and for any set $S_V \subseteq V_{i-1}$ with $|S_V| \le s$

$$d_H(S_V; U_i) = (1 \pm n_{i-1}^{-1/3}) d_H(S_V; U_{i-1}) \frac{|U_i|}{|U_{i-1}|}. \tag{4.43}$$

Fix such a choice of V_i and U_i.

Recall that, by assumption, $\mathcal{G}[V_{i-1} \cup U_{i-1}]$ is $(\xi_{i-1}, p_G, p_H, q, s, t)$-typical by (V2) and η_{i-1}-divisible by (V3). Thus, using (4.42) we can deduce that for any $k \in [r]$, $S_V \subseteq V_{i-1} \setminus V_{i-1}^k$ and $S_U \subseteq U_{i-1}$ with $|S_V| + |S_U| \le s$, and $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with

$|T| \leq t$

$$
\begin{aligned}
d^*_{\mathcal{G}}(S_V \cup S_U, T; V_i^k) &= (1 \pm n_{i-1}^{-1/3}) d^*_{\mathcal{G}}(S_V \cup S_U, T; V_{i-1}^k) \frac{n_i}{n_{i-1}} \\
&\overset{(V2)}{=} (1 \pm n_{i-1}^{-1/3})(1 \pm \xi_{i-1}) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} n_{i-1} \frac{n_i}{n_{i-1}} \\
&= (1 \pm \xi_i) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} n_i.
\end{aligned}
$$

Similarly, using (4.43) we can deduce that for all $S_V \subseteq V_{i-1}$ with $|S_V| \leq s$

$$
\begin{aligned}
d_H(S_V; U_i) &= (1 \pm n_{i-1}^{-1/3}) d_H(S_V; U_{i-1}) \frac{|U_i|}{|U_{i-1}|} \\
&\overset{(V2)}{=} (1 \pm n_{i-1}^{-1/3})(1 \pm \xi_{i-1}) p_H^{|S_V|} |U_{i-1}| \frac{|U_i|}{|U_{i-1}|}, \qquad (4.44) \\
&= (1 \pm \xi_i) p_H^{|S_V|} |U_i|.
\end{aligned}
$$

Thus, $\mathcal{G}$ is $(\xi_i, p_G, p_H, q, s, t)$-typical into $V_i \cup U_i$.

On the other hand, for any $k_1, k_2 \in [r]$ and $v \in V_{i-1} \setminus (V_{i-1}^{k_1} \cup V_{i-1}^{k_2})$

$$
\begin{aligned}
d_G(v; V_i^{k_1}) &\overset{(4.42)}{=} (1 \pm n_{i-1}^{-1/3}) d_G(v; V_{i-1}^{k_1}) \frac{n_i}{n_{i-1}} \overset{(V3)}{=} (1 \pm n_{i-1}^{-1/3})(1 \pm \eta_{i-1}) d_G(v; V_{i-1}^{k_2}) \frac{n_i}{n_{i-1}} \\
&\overset{(4.42),(p5)}{=} (1 \pm 3n_{i-1}^{-1/3})(1 \pm \eta_{i-1}) d_G(v; V_i^{k_2}) = (1 \pm \eta_i) d_G(v; V_i^{k_2}).
\end{aligned}
$$

Similarly, for any $k \in [r]$ and $v \in V_{i-1} \setminus V_{i-1}^k$

$$
\begin{aligned}
d_G(v; V_i^k) &\overset{(4.42)}{=} (1 \pm n_{i-1}^{-1/3}) d_G(v; V_{i-1}^k) \frac{n_i}{n_{i-1}} \overset{(V3)}{=} (1 \pm n_{i-1}^{-1/3})(1 \pm \eta_{i-1}) d_H(v; U_{i-1}) \frac{n_i}{n_{i-1}} \\
&\overset{(4.43),(p5)}{=} (1 \pm 3n_{i-1}^{-1/3})(1 \pm \eta_{i-1}) d_H(v; U_i) = (1 \pm \eta_i) d_H(v; U_i),
\end{aligned}
$$

and for any $k_1, k_2 \in [r]$ and $u \in U_{i-1}$

$$d_H(u; V_i^{k_1}) \overset{(4.42)}{=} (1 \pm n_{i-1}^{-1/3}) d_H(u; V_{i-1}^{k_1}) \frac{n_i}{n_{i-1}} \overset{(V3)}{=} (1 \pm n_{i-1}^{-1/3})(1 \pm \eta_{i-1}) d_H(u; V_{i-1}^{k_2}) \frac{n_i}{n_{i-1}}$$
$$\overset{(4.42),(p5)}{=} (1 \pm 3n_{i-1}^{-1/3})(1 \pm \eta_{i-1}) d_H(u; V_i^{k_2}) = (1 \pm \eta_i) d_H(u; V_i^{k_2}),$$

so $\mathcal{G}$ is η_i-divisible into $V_i \cup U_i$. It follows that $V_0, \ldots, V_i$ and $U_0, \ldots, U_i$ form a $(\xi_i, \varepsilon, \eta_i, n_i)$-vortex of $\mathcal{G}$. Then, by induction, there exist sequences $V_0, \ldots, V_\ell$ and $U_0, \ldots, U_\ell$ which form a $(\xi_\ell, \varepsilon, \eta_\ell, n_\ell)$-vortex of $\mathcal{G}$.

Finally, observe that

$$\eta_\ell = \eta_0 + \sum_{j=0}^{\ell-1} 4n_j^{-1/3} \leq 4\frac{m'^{-1/3}}{1 - \varepsilon^{1/3}} < \eta,$$

and likewise,

$$\xi_\ell = \xi_0 + \sum_{j=0}^{\ell-1} 2n_j^{-1/3} \leq \xi + 2\frac{m'^{-1/3}}{1 - \varepsilon^{1/3}} < 2\xi.$$

□

4.5 Edge-disjoint factors

During the proof of the 'Cover Down Lemma', Lemma 4.7.3, we will need to find many edge-disjoint face-factors in an extended partite complex. This is achieved by the subsequent result.

Lemma 4.5.1. *Let $1/n \ll \gamma \ll \rho \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 2r$ and $t \geq 2$ and let $N, M \in \mathbb{N}$. Let $V^1, \ldots, V^r$ and U be sets of vertices such that $|V^i| = n$*

for every $i \in [r]$ and $|U| = (1 \pm \xi)(p_G/p_H)n$. For each $i \in [N]$ let $k_i \in [r]$ and let $\mathcal{G}_i = (V_i, U_i, G_i, H_i)$ be a balanced (ξ, p_G, p_H, q, s, t)-typical extended r-partite complex with parts $V_i^1, \ldots, V_i^{k_i-1}, V_i^{k_i+1}, \ldots, V_i^r$ and for each $N < i \leq N + M$ let $k_i := 0$ and let $\mathcal{G}_i$ be a balanced (ξ, p_G, q, s, t)-typical r-partite complex with parts $V_i^1, \ldots V_i^r$. Suppose that

(F1) $V_i^k \subseteq V^k$ for all $i \in [N + M]$ and $k \in [r]$ and $U_i \subseteq U$ for all $i \in [N]$,

(F2) $|V_i^k| = |U_i|$ for all $i \in [N]$ and $k \in [r] \setminus \{k_i\}$,

(F3) $|V_i^k| \geq \rho^{4/3}n$ for all $i \in [N + M]$ and $k \in [r] \setminus \{k_i\}$,

(F4) $|V_i^k \cap V_j^k| \leq \rho^2 n$ for all $1 \leq i < j \leq N + M$ and $k \in [r] \setminus \{k_i, k_j\}$ and $|U_i \cap U_j| \leq \rho^2 n$ for all $1 \leq i < j \leq N$,

(F5) every vertex $v \in V$, $u \in U$ is contained in at most $r\rho n$ of the sets $\{V_i\}_{i \in [N+M]}$, $\{U_i\}_{i \in [N]}$ respectively.

Then for each $i \in [N + M]$ there exists an F_r-factor $\mathcal{F}_i$ of $\mathcal{G}_i$ such that all the factors $\{\mathcal{F}_i\}_{i \in [N+M]}$ are edge-disjoint.

Proof. The proof is analogous to the proof of Lemma 3.7.1 but we present it here for completion. Let $m := 2r^2\rho^{3/2}n$. Suppose that we have found edge-disjoint F_r-factors $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}$ of $\mathcal{G}_1, \ldots, \mathcal{G}_{i-1}$ respectively for some $i \in [N + M]$. We will find an F_r-factor $\mathcal{F}_i$ of $\mathcal{G}_i$, edge-disjoint from all $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}$ as follows.

Let $\mathcal{H}_i := \bigcup_{j=1}^{i-1} \mathcal{F}_j$ and let $\mathcal{G}'_i := \mathcal{G}_i - \mathcal{H}_i$. We consider two cases depending on the maximum degree of $\mathcal{H}_i$.

Case 1: $\Delta(\mathcal{H}_i) \leq \rho^{3/2}n$.

If $i \in [N]$, $\mathcal{G}'_i$ is an extended r-partite complex (V_i, U_i, G'_i, H'_i). We have that for every $k \in [r]\setminus\{k_i\}$, every pair of sets $S_V \subseteq V_i \setminus V^k$ and $S_U \subseteq U_i$ with $|S_V| + |S_U| \leq s$, and every set $T \subseteq \mathcal{K}_{r-2}(G'_i[S_V])$ with $|T| \leq t$,

$$d^*_{\mathcal{G}_i}(S_V \cup S_U, T; V_i^k) \geq d^*_{\mathcal{G}'_i}(S_V \cup S_U, T; V_i^k) \geq d^*_{\mathcal{G}_i}(S_V \cup S_U, T; V_i^k) - s\Delta(\mathcal{H}_i)$$

so

$$(1+\xi')p_G^{|S_V|}p_H^{|S_U|}q^{|T|}|V_i^k| \geq d^*_{\mathcal{G}'_i}(S_V \cup S_U, T; V_i^k) \geq (1-\xi')p_G^{|S_V|}p_H^{|S_U|}q^{|T|}|V_i^k| + rsm,$$

and for every set $S_V \subseteq V_i$ with $|S_V| \leq s$,

$$d_{H_i}(S_V) \geq d_{H'_i}(S_V) \geq d_{H_i}(S_V) - s\Delta(\mathcal{H}_i)$$

so

$$(1+\xi')p_H^{|S_V|}|U_i| \geq d_{H'_i}(S_V) \geq (1-\xi')p_H^{|S_V|}|U_i| + rsm.$$

So $\mathcal{G}'_i$ is $(\xi', p_G, p_H, q, s, t)$-typical. Thus, we can successively apply Theorem 3.6.3 with $\mathcal{G}'_i$, ξ' and r playing the role of $\mathcal{G}$, ξ and $r+1$ to find m edge-disjoint F_r-factors $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of $\mathcal{G}'_i$ which will be candidates for $\mathcal{F}_i$.

On the other hand, if $N < i \leq N + M$ then $\mathcal{G}'_i$ is an r-partite complex on V_i. Observe that for every $k \in [r]$, every set $S_V \subseteq V_i \setminus V_i^k$ with $|S_V| \leq s$ and every set $T \subseteq \mathcal{K}_{r-1}(G'_i[S_V])$ with $|T| \leq t$,

$$d^*_{\mathcal{G}_i}(S_V, T; V_i^k) \geq d^*_{\mathcal{G}'_i}(S_V, T; V_i^k) \geq d^*_{\mathcal{G}_i}(S_V, T; V_i^k) - s\Delta(\mathcal{H}_i)$$

so

$$(1+\xi)p_G^{|S_V|}q^{|T|}|V_i^k| \geq d^*_{\mathcal{G}'_i}(S_V, T; V_i^k) \geq (1-\xi')p_G^{|S_V|}q^{|T|}|V_i^k| + rsm.$$

Thus, $\mathcal{G}_i'$ is (ξ', p_G, q, s, t)-typical so we can successively apply Theorem 3.6.1 with $\mathcal{G}_i'$, ξ' playing the role of $\mathcal{G}$, ξ to find m edge-disjoint F_r-factors $\mathcal{A}_1, \dots, \mathcal{A}_m$ of $\mathcal{G}_i'$ which will be candidates for $\mathcal{F}_i$.

Case 2: $\Delta(\mathcal{H}_i) \geq \rho^{3/2}n$.

If $i \in [N]$ let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be defined as r-complexes on $V_i \cup U_i$ with no edges and if $N < i \leq N + M$ let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be defined as r-complexes on V_i with no edges.

Note that in both cases $\mathcal{A}_1, \dots, \mathcal{A}_m$ are edge-disjoint subcomplexes of $\mathcal{G}_i'$. We now choose $j \in [m]$ uniformly at random, set $\mathcal{F}_i := \mathcal{A}_i$ and continue with the next iteration. Observe that at the end of the algorithm all of $\mathcal{F}_1, \dots, \mathcal{F}_{N+M}$ will be edge-disjoint F_{r+1}-factors of $\mathcal{G}_1, \dots, \mathcal{G}_{N+M}$ if and only if

$$\Delta(\mathcal{H}_i) \leq \rho^{3/2}n \tag{4.45}$$

for all $i \in [N + M]$. Hence, the lemma follows if equation (4.45) holds for all $i \in [N + M]$ with positive probability.

For each $i \in [N + M]$ and $x \in V(\mathcal{G}_i)$ let $J^{i,x} := \{j \in [i-1] \ : \ x \in V(\mathcal{G}_j)\}$ and for each $j \in J^{i,x}$ let $Y_j^{i,x}$ be the indicator variable of the event $xy \in E(\mathcal{F}_j)$ for some $y \in V(\mathcal{G}_i)$. Observe that

$$d_{\mathcal{H}_i[V(\mathcal{G}_i) \cap V(\mathcal{H}_i)]}(x) = \sum_{j \in J^{i,x}} Y_j^{i,x}. \tag{4.46}$$

Fix $i \in [N + M]$ and $x \in V(\mathcal{G}_i)$. Note that (F4) implies that for each $j \in J^{i,x}$ at most $r\rho^2 n$ of the edge-disjoint complexes $\mathcal{A}_1, \dots, \mathcal{A}_m$ that were candidates for $\mathcal{F}_j$ share an edge incident to x with $\mathcal{G}_i$. Let $j_1, \dots, j_{|J^{i,x}|}$ be the elements of $J^{i,x}$ listed in increasing order. Then, for all $\ell \in [|J^{i,x}|]$ we have

$$\mathbb{P}\left[Y^{i,x}_{j_\ell} = 1 \mid Y^{i,x}_{j_1}, \dots, Y^{i,x}_{j_{\ell-1}}\right] \leq \frac{r\rho^2 n}{m} \leq \frac{\rho^{1/2}}{2r}.$$

Let $B \sim Bin(|J^{i,x}|, \rho^{1/2}/(2r))$ and observe that

$$\mathbb{E}\left[B\right] = |J^{i,x}|\frac{\rho^{1/2}}{2r} \overset{(F5)}{\leq} \frac{\rho^{3/2}n}{2}.$$

Thus, by applying Fact 2.1.3 and Lemma 2.1.2 we obtain that

$$\mathbb{P}\left[\sum_{j\in J^{i,x}} Y^{i,x}_j > \rho^{3/2}n\right] \leq \mathbb{P}\left[B > \rho^{3/2}n\right] \leq \mathbb{P}\left[B > \mathbb{E}\left[B\right] + \rho^{3/2}n/2\right] \leq 2e^{-\rho^3 n/2}.$$

Observe that there are at most $r\rho n|V(\mathcal{G})|$ pairs (i,x) with $x \in V(\mathcal{G}_i)$ since for each $x \in V(\mathcal{G})$ at most $r\rho n$ of the graphs $\{\mathcal{G}_i\}_{i\in[N+M]}$ contain x by (F5). On the other hand we have $|V(\mathcal{G})| \leq |V| + |U| \leq rn + (2p_G/p_H)n$. Hence, a union bound implies that with positive probability $\sum_{j\in J^{i,x}} Y^{i,x}_j \leq \rho^{3/2}n$ for all $i \in [N+M]$ and $x \in V(\mathcal{G}_i)$. Thus, by (4.46), it follows that equation (4.45) holds with positive probability for every $i \in [N+M]$ which concludes the proof. □

4.6 Divisibility

In this section we show how to transform 'almost-divisibility' into 'divisibility' in an extended partite complex by removing only a sparse set of edges. Our approach is inspired by the work of Barber, Kühn, Lo, Osthus and Taylor [10, Lemma 10.1] although the statements are different.

Lemma 4.6.1. *Let $1/n \ll \eta \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq r$. Let $\mathcal{G} = (V, V^{r+1}, G, H)$ be an η-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \ldots, V^r$ of size n. Then there exists a subgraph $B \subseteq \mathcal{G}$ satisfying $\hat{\Delta}(B) \leq \eta^{1/10}n$ such that $\mathcal{G} - B$ is F_{r+1}-divisible.*

Proof. The proof is divided in two main steps. On the first step, we will find a subgraph $B_{\text{edges}} \subseteq \mathcal{G}$ with $\Delta(B_{\text{edges}}) \leq \eta^{1/3}n$ such that the graph $\mathcal{G} - B_{\text{edges}}$ will satisfy the following property

(a) $e(V^{i_1}, V^{j_1}) = e(V^{i_2}, V^{j_2})$ for every $i_1, i_2, j_1, j_2 \in [r+1]$.

In other words, we will use the graph B_{edges} to balance the total number of edges among every pair of parts. Then, the second step will consist of finding a graph B_{degree} with $\Delta(B_{\text{edges}} \cup B_{\text{degree}}) \leq \eta^{1/3}n$ such that the graph $\mathcal{G} - B_{\text{edges}} - B_{\text{degree}}$ satisfies

(b) $d(v; V^i) = d(v; V^j)$ for every $i, j \in [r+1]$ and $v \in V(\mathcal{G}) \setminus (V^i \cup V^j)$.

The graph B_{degree} will be used to balance the degree of each vertex to all parts so that $\mathcal{G} - B_{\text{edges}} - B_{\text{degree}}$ is F_{r+1}-divisible.

Step 1: Balancing the number of edges among parts.

First of all, observe that since $\mathcal{G}$ is η-divisible we have

$$|d_{\mathcal{G}}(v; V^i) - d_{\mathcal{G}}(v; V^j)| = |d_{\mathcal{G}}(v, V^i) - d_{\mathcal{G}}(v, V^i) \pm \eta d_{\mathcal{G}}(v, V^i)| \leq \eta n \qquad (4.47)$$

for every $i, j \in [r+1]$ and $v \in V(\mathcal{G}) \setminus (V^i \cup V^j)$, and

$$|e(V^i, V^j) - e(V^i, V^k)| = \left| \sum_{v \in V^i} d(v, V^j) - \sum_{v \in V^i} d(v, V^k) \right| \\ \leq \sum_{v \in V^i} \left| d(v, V^j) - d(v, V^k) \right| \overset{(4.47)}{\leq} \eta n^2 \tag{4.48}$$

for every $i, j, k \in [r+1]$, $i \neq j, k$. Then

$$|e(V^{i_1}, V^{j_1}) - e(V^{i_2}, V^{j_2})| \leq |e(V^{i_1}, V^{j_1}) - e(V^{i_1}, V^{j_2})| \\ + |e(V^{i_1}, V^{j_2}) - e(V^{i_2}, V^{j_2})| \\ \leq 2\eta n^2 \tag{4.49}$$

for every distinct $i_1, i_2, j_1, j_2 \in [r+1]$.

Let $e_{\min} := \min\{e(V^i, V^j) \,:\, 1 \leq i < j \leq r+1\}$. Equations (4.48) and (4.49) imply that

$$e(V^i, V^j) - e_{\min} \leq 2\eta n^2 \tag{4.50}$$

for every $1 \leq i < j \leq r+1$.

Now, for each $1 \leq i < j \leq r+1$, we will greedily choose $e(V^i, V^j) - e_{\min}$ edges from $E(V^i, V^j)$ that will be part of the subgraph B_{edges}. It follows from (4.50) that the total number of edges we want to choose is at most $\binom{r+1}{2} 2\eta n^2 \leq (r+1)^2 \eta n^2$.

Suppose we have already chosen some set of edges $E' \subseteq E(\mathcal{G})$. Let $B' := \mathcal{G}[E']$ and suppose that $\Delta(B') \leq \eta^{1/3} n$. Let X be the set of vertices $v \in V(\mathcal{G})$ such that $d_{B'}(v) > \eta^{1/2} n$. Observe that

$$|X| \eta^{1/2} n \leq 2|E'| \leq 2(r+1)^2 \eta n^2.$$

Then,

$$|X| \leq 2(r+1)^2\eta^{1/2}n \leq \eta^{1/3}n. \tag{4.51}$$

Suppose we want now to choose an edge in $E(V^i, V^j)$ for some $1 \leq i < j \leq r+1$. First, we will fix some vertex $v \in V^i \setminus X$. Since $v \notin X$ we know that $d_{B'}(v; V^j) \leq \eta^{1/2}n$. On the other hand, $d_{\mathcal{G}}(v; V^j) \geq (1-5\xi)p_G n$ since $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical and η-divisible (so we can apply Proposition 4.1.2). Thus,

$$d_{\mathcal{G}}(v; V^j) - d_{B'}(v; V^j) - |X| \geq (1-5\xi)p_G n - \eta^{1/2}n - \eta^{1/3}n > 0$$

so we can greedily pick a vertex $u \in V^j \setminus X$ such that $vu \in E(V^i, V^j) \setminus E(B')$. Consider the subgraph $\mathcal{G}[E' \cup \{vu\}]$. Since $v \notin X$ we have $d_{\mathcal{G}[E' \cup \{vu\}]}(v) \leq \eta^{1/2}n + 1 < \eta^{1/3}n$, and the same is true for the vertex u. Hence, $\Delta(\mathcal{G}[E' \cup \{vu\}]) \leq \eta^{1/3}n$.

By iteratively applying the above argument, we can find a set of edges $E_{\text{edges}} \subseteq E(\mathcal{G})$ such that

$$E_{\text{edges}} \cap E(V^i, V^j) = e(V^i, V^j) - e_{\min}$$

for every $1 \leq i < j \leq r+1$, and the subgraph $B_{\text{edges}} := \mathcal{G}[E_{\text{edges}}]$ satisfies $\Delta(B_{\text{edges}}) \leq \eta^{1/3}n$. It follows that the graph $\mathcal{G} - B_{\text{edges}}$ satisfies property (a).

Step 2: Balancing the degree of each vertex to each part.

Let $\mathcal{G}' = \mathcal{G} - B_{\text{edges}}$. Observe that

$$\begin{aligned} |d_{\mathcal{G}'}(v; V^i) - d_{\mathcal{G}'}(v; V^j)| &\leq |d_{\mathcal{G}}(v; V^i) - d_{\mathcal{G}}(v; V^i)| + \Delta(B_{\text{edges}}) \\ &\leq \eta n + \eta^{1/3}n \leq 2\eta^{1/3}n \end{aligned} \tag{4.52}$$

for every $i, j \in [r+1]$ and $v \in V(\mathcal{G}) \setminus (V^i \cup V^j)$.

Let $i, j, k \in [r+1]$ with $i \neq j, k$ be fixed. Now, for each vertex $v \in V^i$, let $d_v := d_{\mathcal{G}'}(v; V^j) - d_{\mathcal{G}'}(v; V^k)$. equation (4.52) implies that

$$|d_v| \leq 2\eta^{1/3} n. \tag{4.53}$$

Let V^+ be the set of vertices $v \in V^i$ such that $d_v > 0$ and let V^- be the set of vertices $v \in V^i$ such that $d_v < 0$. Let U^+ be a multiset where each $v \in V^+$ appears exactly d_v times and let U^- be a multiset where each $v \in V^-$ appears exactly $-d_v$ times. Observe that

$$\begin{aligned} |U^+| - |U^-| &= \sum_{v \in V^+} d_v + \sum_{v \in V^-} d_v \\ &= \sum_{v \in V^+} d_{\mathcal{G}'}(v; V^j) - d_{\mathcal{G}'}(v; V^k) + \sum_{v \in V^-} d_{\mathcal{G}'}(v; V^j) - d_{\mathcal{G}'}(v; V^k) \\ &= \sum_{v \in V^i} d_{\mathcal{G}'}(v; V^j) - d_{\mathcal{G}'}(v; V^k) = |E(\mathcal{G}'[V^i, V^j])| - |E(\mathcal{G}'[V^i, V^j])| \overset{(a)}{=} 0. \end{aligned}$$

Hence, $|U^+| = |U^-|$. Note also that

$$\begin{aligned} |U^+| &= |\sum_{v \in V^+} d_{\mathcal{G}'}(v; V^j) - d_{\mathcal{G}'}(v; V^k)| \\ &\leq \sum_{v \in V^+} |d_{\mathcal{G}'}(v; V^j) - d_{\mathcal{G}'}(v; V^k)| \overset{(4.52)}{\leq} 2\eta^{1/3} n^2. \end{aligned} \tag{4.54}$$

Given vertices $x, y \in V^i$, let $D_{x \to y}$ be a graph consisting of an r-clique Z on $V(\mathcal{G}) \setminus V^i$ plus the edges $\{yz \ : \ z \in Z \setminus Z^j\}$ and the edge xz with $z \in Z^j$.

Recall the definition of *degeneracy* from the beginning of Section 2.1 and note that $D_{x \to y}$ satisfies the following properties.

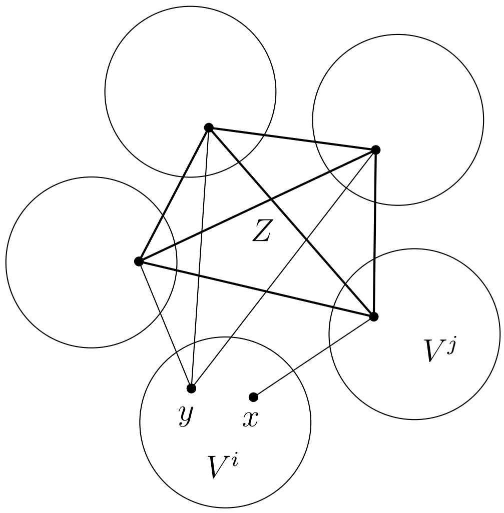

Figure 4.3: The edges of a gadget $D_{x\to y}$ when $r+1=5$.

(A1) $|V(D_{x\to y})| = r+2$ and $|E(D_{x\to y})| = \binom{r+1}{2}$,

(A2) $D_{x\to y}$ has degeneracy r rooted at $\{x,y\}$,

(A3) for every $j_1, j_2 \in [r+1]$, $j_2 \neq j$ and any $v \in V(\mathcal{G}) \setminus (V^{j_1} \cup V^{j_2})$

$$d_{D_{x\to y}}(v; V^{j_1}) - d_{D_{x\to y}}(v; V^{j_2}) = \begin{cases} 1 & \text{if } v = x \text{ and } j_1 = j, \\ -1 & \text{if } v = y \text{ and } j_1 = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|U^+| = |U^-|$ then there exists a bijection $\phi : U^+ \to U^-$. Suppose that we can find edge-disjoint graphs $\{D_{x\to\phi(x)}\}_{x\in U^+}$ in $\mathcal{G}'$. Let $D_i^{j\to k} := \bigcup_{x\in U^+} D_{x\to\phi(x)}$. Then it follows by (A3) that $\mathcal{G}' - D_i^{j\to k}$ satisfies that

(B1) for every $j_1 \in [r+1]$, $j_1 \neq i, k$ and $v \in V^i$

$$d_{\mathcal{G}'-D_i^{j\to k}}(v; V^{j_1}) - d_{\mathcal{G}'-D_i^{j\to k}}(v; V^k) = \begin{cases} 0 & \text{if } j_1 = j, \\ d_{\mathcal{G}'}(v; V^{j_1}) - d_{\mathcal{G}'}(v; V^k) & \text{if } j_1 \neq j, \end{cases}$$

(B2) for every $j_1, j_2 \in [r+1]$ and any $v \in V(\mathcal{G}) \setminus (V^i \cup V^{j_1} \cup V^{j_2})$

$$d_{\mathcal{G}'-D_i^{j\to k}}(v; V^{j_1}) - d_{\mathcal{G}'-D_i^{j\to k}}(v; V^{j_2}) = d_{\mathcal{G}'}(v; V^{j_1}) - d_{\mathcal{G}'}(v, V^{j_2}).$$

Now, suppose that we can find edge-disjoint graphs $\{D_i^{j'\to k}\}_{j'\in[r+1]\setminus\{i,k\}}$ in $\mathcal{G}'$. Let $D_i := \bigcup_{j'\in[r+1]\setminus\{i,k\}} D_i^{j'\to k}$. Then we have by (B1) and (B2) that the graph $\mathcal{G}' - D_i$ satisfies that

(C1) for every $j_1 \in [r+1]$, $j_1 \neq i, k$ and $v \in V^i$

$$d_{\mathcal{G}'-D_i}(v; V^{j_1}) = d_{\mathcal{G}'-D_i}(v; V^k),$$

(C2) for every $j_1, j_2 \in [r+1]$ and any $v \in V(\mathcal{G}) \setminus (V^i \cup V^{j_1} \cup V^{j_2})$

$$d_{\mathcal{G}'-D_i}(v; V^{j_1}) - d_{\mathcal{G}'-D_i}(v; V^{j_2}) = d_{\mathcal{G}'}(v; V^{j_1}) - d_{\mathcal{G}'}(v; V^{j_2}).$$

In particular, (C1) implies that in the graph $\mathcal{G}' - D_i$ each vertex in V^i has been balanced in the sense that it has the same degree to each of the parts, whereas (C2) implies that, for the rest of vertices, the degree difference among each part remains the same as the one they had in $\mathcal{G}'$.

Finally, suppose we are able to find edge-disjoint graphs $\{D_i\}_{i\in[r+1]}$ in $\mathcal{G}'$. Let

$D := \bigcup_{i\in[r+1]} D_i$. Then, (C1) and (C2) implies that the graph $\mathcal{G}' - D$ satisfies that for every $j_1, j_2 \in [r+1]$ and $v \in V(\mathcal{G}) \setminus (V^{j_1} \cup V^{j_2})$

$$d_{\mathcal{G}'-D}(v; V^{j_1}) = d_{\mathcal{G}'-D}(v; V^{j_2}).$$

In other words, $\mathcal{G}' - D$ is F_{r+1}-divisible.

STEP 3: Finding the gadgets.

It remains to show that we can indeed find such a graph D as a subgraph of $\mathcal{G}'$. In order to do so, first observe that

$$D = \bigcup_{i\in[r+1]} D_i = \bigcup_{i\in[r+1]} \Big(\bigcup_{j\in[r+1]\setminus\{i,k\}} D_i^{j\to k} \Big) = \bigcup_{i\in[r+1]} \Big(\bigcup_{j\in[r+1]\setminus\{i,k\}} \Big(\bigcup_{x\in U^+} D_{x\to\phi(x)} \Big)\Big) \tag{4.55}$$

where all the unions are edge-disjoint unions, and the set U^+ depends on i, j, k. Recall that when defining U^+, i, j, k were chosen arbitrarily. Hence, equation (4.54) holds for any choice of distinct $i, j, k \in [r+1]$. Thus, D consists of the edge-disjoint union of at most $(r+1)(r-1)2\eta^{1/3}n^2$ gadgets $D_{x\to y}$. So

$$D = \bigcup_{i\in[\ell]} D_{x_i\to y_i}$$

where

$$\ell \le 2r^2\eta^{1/3}n^2. \tag{4.56}$$

On the other hand, each vertex $v \in V(\mathcal{G})$ appears playing the role of x_i or y_i in some gadget $D_{x_i\to y_i} \subseteq D$ at most $|d_v| \le 2\eta^{1/3}n$ times (by equation (4.52)).

Suppose that for some $j \in [\ell]$ we have already found edge-disjoint gadgets $\{D_{x_i\to y_i}\}_{i\in[j]}$ and let $D' := \bigcup_{i\in[j]} D_{x_i\to y_i}$. Suppose that $\Delta(D') \le \eta^{1/8}n$. We want now to find a gadget $D_{x_{j+1}\to y_{j+1}}$ in $\mathcal{G}'$ edge-disjoint from D' and such that $\Delta(D' \cup D_{x_{j+1}\to y_{j+1}}) \le \eta^{1/8}n$. Let X be the set of vertices $v \in V(\mathcal{G})$ such that $d_{D'}(v) \ge \eta^{1/6}n$ and observe that

$$|X|\eta^{1/6}n \le 2|E(D')| \le 2\ell|E(D_{x\to y})| \overset{(A1),(4.56)}{\le} 4r^2\binom{r+1}{2}\eta^{1/3}n^2$$

which implies

$$|X| \le 4r^2\binom{r+1}{2}\eta^{1/3}n \le \eta^{1/4}n. \tag{4.57}$$

Using $\Delta(B_{\text{edges}}) \le \eta^{1/3}n$ and the fact that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical it is straightforward to check that $\mathcal{G}'$ must be $(2\xi, p_G, p_H, q, s, t)$-typical. Then, for any $i \in [r]$, $S_V \subseteq V \setminus V^i$ and $S_U \subseteq V^{r+1}$ with $|S_V| + |S_U| \le s$ we have

$$d_{\mathcal{G}'}(S_V \cup S_U; V^i) = (1 \pm 2\xi)p_G^{|S_V|}p_H^{|S_U|}n \ge (p_G p_H)^s n.$$

On the other hand, for every set $S_V \subseteq V$ with $|S_V| \le s$ we have

$$d_{\mathcal{G}'}(S_V; V^{r+1}) = (1 \pm 2\xi)p_H^{|S_V|}|V^{r+1}| \overset{\text{Prop. 4.1.2}}{=} (1 \pm 6\xi)p_G p_H^{|S_V|-1}n \ge (p_G p_H)^s n.$$

So in any case, for every $i \in [r+1]$ and every set of vertices $S \subseteq V(\mathcal{G}) \setminus V^i$ with $|S| \le s$ we have $d_{\mathcal{G}'}(S; V^i) \ge (p_G p_H)^s n$. Recall that $D_{x_{j+1}\to y_{j+1}}$ has degeneracy r rooted at $\{x_{j+1}, y_{j+1}\}$ by (A2) and note that $s \ge r$. Then, we can find the vertices of $V(D_{x_{j+1}\to y_{j+1}}) \setminus \{x_{j+1} \cup y_{j+1}\}$ one by one, avoiding any edge from D' and any vertex from X since for every $i \in [r+1]$ and every set of vertices $S \subseteq V(\mathcal{G}) \setminus V^i$

with $|S| \leq s$

$$d_{\mathcal{G}'}(S; V^i) - |X| - \Delta(D') \geq (p_G p_H)^s n - \eta^{1/4} n - \eta^{1/6} n > 0.$$

Because each chosen vertex was not in X then

$$d_{D' \cup D_{x_{j+1} \to y_{j+1}}}(v) \leq \eta^{1/6} n + |V(D_{x_{j+1} \to y_{j+1}})| \overset{(A1)}{\leq} \eta^{1/6} n + (r+2) \leq 2\eta^{1/6} n$$

for every $v \in V(D_{x_{j+1} \to y_{j+1}}) \setminus \{x_{j+1}, y_{j+1}\}$. On the other hand, the degree of x_{j+1} and y_{j+1} in $D' \cup D_{x_{j+1} \to y_{j+1}}$ is bounded by $2\eta^{1/6} n + 2(r+2)\eta^{1/3} n \leq \eta^{1/8} n$ since, as seen in the previous equation, the last time they were chosen for a gadget they had a maximum degree of $2\eta^{1/6} n$, and after that they might have appeared at most $2\eta^{1/3} n$ times playing the role of x_i or y_i in some gadget $D_{x_i \to y_i}$ with $i \in [j]$ and $|V(D_{x_i \to y_i})| \leq r + 2$. Hence, $\Delta(D' \cup D_{x_{j+1} \to y_{j+1}}) \leq \eta^{1/8} n$.

Altogether, we can find edge-disjoint gadgets $\{D_{x_i \to y_i}\}_{i \in [\ell]}$ in $\mathcal{G}'$ such that $B_{\text{degree}} := \bigcup_{i \in [\ell]} D_{x_i \to y_i}$ satisfies $\Delta(B_{\text{degree}}) \leq \eta^{1/8} n$. Therefore, we have

$$\Delta(B_{\text{edges}} \cup B_{\text{degree}}) \leq \Delta(B_{\text{edges}}) + \Delta(B_{\text{degree}}) \leq \eta^{1/3} n + \eta^{1/8} n \leq \eta^{1/10} n$$

and $\mathcal{G} - B_{\text{edges}} - B_{\text{degree}}$ is F_{r+1}-divisible. □

4.7 Cover Down Lemma

In this section we prove the 'Cover Down Lemma' (Lemma 4.3.1) which forms the core of the 'iterative absorption' method used in the proof of the main theorem.

Before doing so, we need the following two results.

Lemma 4.7.1. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Let $V_1 \subseteq V$ and suppose that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into V_1. Let $K \cup \{u\}$ be a $(k+1)$-clique of $G \cup H$ where $0 \leq k \leq r$, $K \subseteq \mathcal{K}_k(G)$ and $u \in U$. Then $Lk_{\mathcal{G}}(K \cup \{u\}; V_1)$ is a $(3\xi, p_G, q, s-k-1, t)$-typical $(r-k)$-partite complex.*

Lemma 4.7.2. *Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$. Let $\mathcal{G} = (V, U, G, H)$ be a (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Let $V_1 \subseteq V$ and $U_1 \subseteq U$ and suppose that $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V_1 \cup U_1$. Let K be a k-clique of G where $0 \leq k \leq r-1$. Then $Lk_{\mathcal{G}}(K; V_1 \cup U_1)$ is a $(3\xi, p_G, p_H, q, s-k, t)$-typical extended $(r+1-k)$-partite complex (V', U', G', H') where $V' = N_{\mathcal{G}}(K; V_1)$, $U' = N_{\mathcal{G}}(K; U_1)$, $G' = Lk_G(K; V_1)$ and $H' = H[V', U']$.*

We omit the proofs of Lemmas 4.7.1 and 4.7.2 due to their similarities with the proofs of Lemmas 3.8.1 and 3.8.2 respectively. We now state and prove the 'Cover Down Lemma'.

Lemma 4.7.3 (Cover Down Lemma)**.** *Let $1/n \ll \eta \ll \gamma' \ll \varepsilon \ll \xi \ll p, q, 1/s, 1/t, 1/r$ with $s \geq 3(r-1)$ and $t \geq 2r-1$. Let $\mathcal{G} = (V, U, G, H)$ be an η-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Let $V_1 \subseteq V$ and $U_1 \subseteq U$ be subsets satisfying*

(C1) $|V_1^i| = \varepsilon n$ for every $i \in [r]$ and $|U_1| = \varepsilon |U|$,

(C2) $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V_1 \cup U_1$,

(C3) $\mathcal{G}$ is η-divisible into $V_1 \cup U_1$,

(C4) for every $i \in [r]$ and $v \in V \setminus (V_1 \cup V^i)$, $d_G(v; V^i) = d_H(v; U)$,

(C5) for every $i, j \in [r]$ and $u \in U \setminus U_1$, $d_H(u; V^i) = d_H(u; V^j)$.

Then there is an F_{r+1}-decomposable subcomplex $\mathcal{H} \subseteq \mathcal{G}$ such that $E(\mathcal{G} - \mathcal{G}[V_1 \cup U_1]) \subseteq E(\mathcal{H})$ and $\hat{\Delta}(\mathcal{H}[V_1 \cup U_1]) \leq \gamma' n$.

Proof. Let $\gamma, \rho, \xi' > 0$ such that

$$1/n \ll \eta \ll \gamma \ll \rho \ll \gamma' \ll \varepsilon \ll \xi \ll \xi' \ll p_G, p_H, q, 1/s, 1/t, 1/r. \tag{4.58}$$

We will follow the same steps of the proof of Lemma 3.8.3.

Let $V_0 := V \setminus V_1$ and $U_0 := U \setminus U_1$. First observe that since $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V \cup U$ and into $V_1 \cup U_1$ then $\mathcal{G}$ must be $(2\xi, p_G, p_H, q, s, t)$-typical into $V_0 \cup U_0$. Indeed, for every $i \in [r]$, $S_V \subseteq V \setminus V^i$ and $S_U \subseteq U \setminus U^i$ with $|S_V| + |S_U| \leq s$, and $T \subseteq \mathcal{K}_{r-1}(G[S_V])$ with $|T| \leq t$ we have

$$\begin{aligned} d^*_{\mathcal{G}}(S_V \cup S_U, T; V_0^i) &= d^*_{\mathcal{G}}(S_V \cup S_U, T; V^i) - d^*_{\mathcal{G}}(S_V \cup S_U, T; V_1^i) \\ &= (1 \pm 2\xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} |V_0| \end{aligned}$$

and for every $S_V \subseteq V$ with $|S_V| \leq s$

$$d_H(S_V; U_0) = d_H(S_V; U) - d_H(S_V; U_1) = (1 \pm 2\xi) p_H^{|S_V|} |U_0|.$$

Similarly, since $\mathcal{G}$ is η-divisible into $V \cup U$ and into $V_1 \cup U_1$ we will check that

$\mathcal{G}$ is 2η-divisible into $V_0 \cup U_0$. For every $i_1, i_2 \in [r]$ and $v \in V \setminus V^{i_1} \cup V^{i_2}$ we have

$$d_G(v; V_0^{i_1}) = d_G(v; V^{i_1}) - d_G(v; V_1^{i_1}) = (1 \pm \eta) d_G(v; V^{i_2}) - (1 \pm \eta) d_G(v; V_1^{i_2})$$
$$\overset{(p4)}{=} (1 \pm 2\eta)(d_G(v; V^{i_2}) - d_G(v; V_1^{i_2})) = (1 \pm 2\eta) d_G(v; V_0^{i_2}).$$

Analogously, for every $i_1, i_2 \in [r]$ and $u \in U$

$$d_H(u; V_0^{i_1}) = (1 \pm 2\eta) d_H(u; V_0^{i_2}).$$

and for $i \in [r]$ and $v \in V \setminus V^i$

$$d_G(v; V_0^i) = (1 \pm 2\eta) d_H(v; U_0).$$

Hence, $\mathcal{G}$ satisfies the following properties:

(P1) $\mathcal{G}$ is (ξ, p_G, p_H, q, s, t)-typical into $V \cup U$ and $V_1 \cup U_1$,

(P2) $\mathcal{G}$ is $(2\xi, p_G, p_H, q, s, t)$-typical into $V_0 \cup U_0$,

(P3) $\mathcal{G}$ is η-divisible into $V \cup U$ and $V_1 \cup U_1$,

(P4) $\mathcal{G}$ is 2η-divisible into $V_0 \cup U_0$,

(P5) by Lemma 4.7.1, for every k-clique K of $G \cup H$ with $1 \leq k \leq r$ and $|K \cap U| = 1$, $\mathrm{Lk}_{\mathcal{G}}(K; V_1)$ is a $(3\xi, p_G, q, s-k, t)$-typical $(r+1-k)$-partite complex.

(P6) by Lemma 4.7.2, for every k-clique K of G with $1 \leq k \leq r-1$, $\mathrm{Lk}_{\mathcal{G}}(K, V_1 \cup U_1)$ is a $(3\xi, p_G, p_H, q, s-k, t)$-typical extended $(r+1-k)$-partite complex (V_K, U_K, G_K, H_K) where $V_K = N_G(K; V_1)$, $U_K = N_H(K; U_1)$, $G_K = \mathrm{Lk}_G(K; V_1)$ and $H_K = H[V_K, U_K]$.

Step 1: Choosing the reservoir edges.

Let $\mathcal{R}$ be a random subcomplex of $\mathcal{G}$ where each edge $e \in E(\mathcal{G})$ is included in $E(\mathcal{R})$ independently at random with probability ρ. Let $R := \mathcal{R}[V_0 \cup U_0, V_1 \cup U_1]$ and note that R is a $2(r+1)$-partite graph with parts $V_1^1, \dots, V_1^r, U_1$ and $V_0^1, \dots, V_0^r, U_0$. Let $\widetilde{\mathcal{G}} = (V, U, \widetilde{G}, \widetilde{H})$ be the $(r+1)$-extended complex where $\widetilde{G} := G - R[V] - G[V_1]$ and $\widetilde{H} := H - R[V, U] - H[V_1, U_1]$.

Note that each property (P1)-(P6) describes the neighbourhood size of at most $\binom{s}{r}^t n^s < n^{2s}$ combinations of vertices and cliques. We can then apply Proposition 2.1.4 to each degree size described in (P1)-(P6) when restricted to $\mathcal{R}$ to see that with positive probability:

(R1) $\mathcal{R}$ is $(2\xi, \rho p_G, \rho p_H, q, s, t)$-typical into $V \cup U$ and $V_1 \cup U_1$,

(R2) $\mathcal{R}$ is $(3\xi, \rho p_G, \rho p_H, q, s, t)$-typical into $V_0 \cup U_0$,

(R3) $\mathcal{R}$ is 2η-divisible into $V \cup U$ and $V_1 \cup U_1$,

(R4) $\mathcal{R}$ is 3η-divisible into $V_0 \cup U_0$,

(R5) for every k-clique K of $G \cup H$ with $1 \leq k \leq r$ and $|K \cap U| = 1$, $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[K])}(K; V_1)$ is a $(6\xi, p_G, q, s-k, t)$-typical $(r+1-k)$-partite complex.

(R6) for every k-clique K of G with $1 \leq k \leq r-1$, $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[K])}(K, V_1 \cup U_1)$ is a $(6\xi, p_G, p_H, q, s-k, t)$-typical extended $(r+1-k)$-partite complex (V_K, U_K, G_K, H_K) where $V_K = N_{G-(\widetilde{G}-\widetilde{G}[K])}(K; V_1)$, $U_K = N_{H-(\widetilde{H}-\widetilde{H}[K])}(K; U_1)$, $G_K = \mathrm{Lk}_{G-(\widetilde{G}-\widetilde{G}[K])}(K; V_1)$ and $H_K = (H - (\widetilde{H} - \widetilde{H}[K]))[V_K, U_K]$.

Let $\mathcal{R}$ satisfy properties (R1)-(R6).

Step 2: Finding an approximate decomposition.

We will now cover all edges of $\widetilde{\mathcal{G}}$, i.e., all edges contained in $V_0 \cup U_0$ and all edges that go from $V_0 \cup U_0$ to $V_1 \cup U_1$ except those from the reservoir $\mathcal{R}$. For this we will apply Lemma 4.3.2 to $\widetilde{\mathcal{G}}$ but first we need to make $\widetilde{\mathcal{G}}$ F_{r+1}-divisible. We can achieve this by using Lemma 4.6.1. Note that for every $i \in [r]$, $S_V \subseteq V \setminus V^i$ and $S_U \subseteq U \setminus U^i$ with $|S_V \cup S_U| \leq s$, and $T \subseteq \mathcal{K}_{r-1}(\widetilde{G}[S_V])$ with $|T| \leq q$ we have

$$\begin{aligned} d^*_{\widetilde{\mathcal{G}}}(S_V \cup S_U, T; V^i) &= d^*_{\mathcal{G}}(S_V \cup S_U, T; V^i) \pm d^*_{\mathcal{R}}(S_V \cup S_U, T; V^i) \pm |V_1^i| \\ &\overset{(R1),(C1)}{=} (1 \pm \xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} n \pm (1 \pm 2\xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} \rho^{|S_V \cup S_U|} n \pm \varepsilon n \\ &\overset{(4.58)}{=} (1 \pm 2\xi) p_G^{|S_V|} p_H^{|S_U|} q^{|T|} n. \end{aligned}$$

Similarly, for every $S_V \subseteq V$ with $|S_V| \leq s$ we obtain

$$d_{\widetilde{H}}(S_V; U) = (1 \pm 2\xi) p_H^{|S_V|} |U|.$$

Thus, $\widetilde{\mathcal{G}}$ is $(2\xi, p_G, p_H, q, s, t)$-typical. On the other hand, for every $i_1, i_2 \in [r]$ and $v \in V \setminus V^{i_1} \cup V^{i_2}$ we have

$$\begin{aligned} d_{\widetilde{G}}(v; V^{i_1}) &= d_G(v; V^{i_1}) - d_R(v; V^{i_1}) - d_G(v; V_1^{i_1}) \\ &\overset{(P3),(R3),(R4)}{=} (1 \pm \eta) d_G(v; V^{i_2}) - (1 \pm 3\eta) d_R(v; V^{i_2}) - (1 \pm \eta) d_G(v; V_1^{i_2}) \\ &\overset{(p4)}{=} (1 \pm 4\eta)(d_G(v; V^{i_2}) - d_R(v; V^{i_2}) - d_G(v; V_1^{i_2})) = (1 \pm 4\eta) d_{\widetilde{G}}(v; V^{i_2}). \end{aligned}$$

Analogously, it is easy to check that for every $i_1, i_2 \in [r]$ and $u \in U$

$$d_{\widetilde{H}}(u; V^{i_1}) = (1 \pm 4\eta) d_{\widetilde{H}}(u; V^{i_2})$$

and for every $i \in [r]$ and $v \in V \setminus V^i$

$$d_{\widetilde{G}}(v; V^i) = (1 \pm 4\eta) d_{\widetilde{H}}(v; V^i).$$

Hence, $\widetilde{\mathcal{G}}$ is 4η-divisible. We can then apply Lemma 4.6.1 to $\widetilde{\mathcal{G}}$ to find a graph $B \subseteq \widetilde{\mathcal{G}}$ such that $\hat{\Delta}(B) \leq (4\eta)^{1/10} n$ and $\widetilde{\mathcal{G}} - B$ is F_{r+1}-divisible. Since $\hat{\Delta}(B) \leq (4\eta)^{1/10} n$ and using (4.58) it is straightforward to check that $\widetilde{\mathcal{G}} - B$ is $(3\xi, p_G, p_H, q, s, t)$-typical. We can now apply Lemma 4.3.2 to $\widetilde{\mathcal{G}} - B$ to find an F_{r+1}-decomposable complex $\mathcal{F}_1 \subseteq \widetilde{\mathcal{G}} - B$ such that $\hat{\Delta}(\mathcal{G} - \mathcal{F}_1) \leq \gamma^2 n$. Let L be the graph induced by the edges of $\widetilde{\mathcal{G}} - \mathcal{F}_1$ and note that

$$\hat{\Delta}(L) \leq \gamma n. \tag{4.59}$$

Step 3: Covering the leftover edges using the reservoir.

Let $\mathcal{G}_1 := \mathcal{G} - \mathcal{F}_1$ and note that $\mathcal{F}_1$ does not have common edges with R nor with $\mathcal{G}[V_1 \cup U_1]$. Our aim now is to cover all edges from L using a sparse collection of faces. We achieve this via the following claim:

Claim 1: For each $e \in E(L)$ there is a face $f_e \in F_{r+1}(\mathcal{G}_1)$ with $e \subseteq f_e$ such that all the faces $\{f_e\}_{e \in E(L)}$ are edge-disjoint. Moreover, if $\mathcal{F}_2$ is the complex induced by the faces in $\{f_e\}_{e \in E(L)}$ then

$$\hat{\Delta}(\mathcal{F}_2) \leq \gamma^{1/3} n. \tag{4.60}$$

Proof of claim: Suppose that we have already found edge-disjoint faces $\{f_e\}_{e \in E(L')}$

for some subgraph $L' \subseteq L$. Let $\mathcal{F}' := \bigcup_{e \in E(L')} f_e$ and assume that

$$\hat{\Delta}(\mathcal{F}') \leq \gamma^{1/3} n. \tag{4.61}$$

We want to show that for any $e \in E(L - L')$ we can choose a face $f_e \in F_{r+1}(\mathcal{G})$ containing e which is edge-disjoint from $\mathcal{F}'$ and such that $\hat{\Delta}(\mathcal{F}' \cup f_e) \leq \gamma^{1/3} n$. If this was the case, we could iteratively find faces for every edge $e \in E(L)$ until ending with the desired collection of faces $\mathcal{F}_2$.

We say that a vertex $x \in V_1 \cup U_1$ is *good* if there are less than $\gamma^{1/3} n - 1$ faces in $\mathcal{F}'$ containing x. Else, we say it is *bad*. Let X be the set of bad vertices and suppose that $|X| > \gamma^{1/3} n$. Then, for every $x \in X$ there are at least $\gamma^{1/3} n - 1$ faces containing x. Thus,

$$|E(\mathcal{F}')| \geq \frac{1}{2} \sum_{x \in X} d_{\mathcal{F}'}(x) \geq \frac{1}{2} |X| r (\gamma^{1/3} n - 1) > \frac{r}{2} (\gamma^{2/3} n^2 - \gamma^{1/3} n) \geq \frac{r}{4} \gamma^{2/3} n^2.$$

On the other hand,

$$\begin{aligned} |E(\mathcal{F}')| &\leq \binom{r+1}{2} |E(L)| \leq (r+1)^2 |V(\mathcal{G})| \hat{\Delta}(L) \\ &\overset{\text{Prop. 4.1.2,(4.59)}}{\leq} (r+1)^2 (r + 2p_G/p_H) \gamma n^2 < \frac{r}{4} \gamma^{2/3} n^2 \end{aligned}$$

which is a contradiction. We must then have

$$|X| \leq \gamma^{1/3} n. \tag{4.62}$$

Pick an arbitrary edge $e \in E(L - L')$ and suppose first that $e = v_1 v_2$ with $v_1, v_2 \in V$. Let $\mathcal{Y}_e$ be the extended $(r-1)$-partite complex obtained from $\mathrm{Lk}_{\mathcal{G} - (\widetilde{\mathcal{G}} - \widetilde{\mathcal{G}}[e])}(e; V_1 \cup$

U_1) by removing all its edges that are in $E(\mathcal{F}')$ and all its bad vertices. Using (R6), (4.61), (4.62) and $\gamma \ll \xi$ it is not hard to check that $\mathcal{Y}_e$ is $(7\xi, p_G, p_H, q, s-2, t)$-typical. We can then greedily pick any face $f \in F_{r-1}(\mathcal{Y}_e)$ and let $f_e := f \cup e$. Observe that f_e is a face of $\mathcal{G}$ that contains only edges from R and $\mathcal{G}[V_1 \cup U_1]$, with the exception of e. Moreover, f_e is edge-disjoint from $\mathcal{F}'$ and doesn't contain any bad vertex.

Suppose now that $e = vu$ with $v \in V$ and $u \in U$. Analogously, let $\mathcal{Y}_e$ be the $(r-1)$-partite complex obtained from $\mathrm{Lk}_{\mathcal{G}-(\widetilde{\mathcal{G}}-\widetilde{\mathcal{G}}[e])}(e; V_1)$ by removing all its edges that are in $E(\mathcal{F}')$ and all its bad vertices. Using (R5), (4.61), (4.62) and $\gamma \ll \xi$ we can see that $\mathcal{Y}_e$ is $(7\xi, p_G, q, s-2, t)$-typical. We then pick an arbitrary face $f \in F_{r-1}(\mathcal{Y}_e)$ and let $f_e := f \cup e$. Again, we have that f_e is an $(r+1)$-face of $\mathcal{G}$ that contains only edges from R and $\mathcal{G}[V_1 \cup U_1]$ with the exception of e, and that is edge-disjoint from $\mathcal{F}'$ and vertex-disjoint from X.

In any case, we are able to find a face $f_e \in F_{r+1}(\mathcal{G}_1)$ that contains e, that is edge-disjoint from $\mathcal{F}'$ and which contains no bad vertices. Since all vertices in f_e are good then $\hat{d}_{\mathcal{F}' \cup f_e}(v) \leq \hat{d}_{\mathcal{F}'}(v) + 1 \leq \gamma^{1/3} n$ for any vertex $v \in f_e$. Hence, we have $\hat{\Delta}(\mathcal{F}' \cup f_e) \leq \gamma^{1/3} n$ which concludes the proof as discussed above. –

STEP 4: Cover down remaining reservoir edges.

Using Claim 1 we are able to find, for each $e \in E(L)$, a face $f_e \in F_{r+1}(\mathcal{G}_1)$ containing e such that all the faces $\{f_e\}_{e \in E(L)}$ are edge-disjoint and $\mathcal{F}_2 := \bigcup_{e \in E(L)} f_e$ satisfies $\hat{\Delta}(\mathcal{F}_2) \leq \gamma^{1/3} n$. Let $\mathcal{G}_2 := \mathcal{G} - \mathcal{F}_1 - \mathcal{F}_2$. Note that all edges of $\mathcal{G}_2$ are either contained in $V_1 \cup U_1$ or belong to the reservoir R. For this reason if we are able to find an F_{r+1}-decomposable complex $\mathcal{F}_3 \subseteq \mathcal{G}_2$ with bounded maximum degree that covers all the reservoir edges from $\mathcal{G}_2$ we will be done.

Let $G_2 \subseteq G$ and $H_2 \subseteq H$ such that $\mathcal{G}_2 = (V, U, G_2, H_2)$. For each $v \in V_0$ let $\mathcal{Y}_v := \mathrm{Lk}_{\mathcal{G}_2}(v; V_1 \cup U_1)$ be an extended r-partite complex $\mathcal{Y}_v = (V_v, U_v, G_v, H_v)$ where $V_v = N_{G_2}(v; V_1)$, $U_v = N_{H_2}(v; U_1)$, $G_v = \mathrm{Lk}_{G_2}(v; V_1)$ and $H_v = H_2[V_v, U_v]$. Since $\mathcal{G}_2$ is obtained from $\mathcal{G}$ by removing F_{r+1}-decomposable graphs it follows by (C4) that for every $i \in [r]$ and $v \in V_0 \setminus V_0^i$ we have $|V_v^i| = |U_v|$. In particular, $\mathcal{Y}_v$ is balanced for every $v \in V_0$. For each $u \in U_0$ let $\mathcal{Y}_u := \mathrm{Lk}_{\mathcal{G}_2}(u; V_1)$ be an r-partite complex on the vertex set $V_u := N_{H_2}(u; V_1)$. Similarly, it follows by (C5) that $\mathcal{Y}_u$ is balanced for every $u \in U_0$. Observe that $\mathcal{G}_2 = \mathcal{G} - \widetilde{\mathcal{G}} - \mathcal{F}_2$. Then, using $\hat{\Delta}(\mathcal{F}_2) \leq \gamma^{1/3} n$ and the hierarchy $\gamma \ll \rho, \xi$, from (R5) and (R6) we can deduce that $\mathcal{Y}_v$ is $(7\xi, p_G, p_H, q, s-1, t)$-typical for every $v \in V_0$ and $\mathcal{Y}_u$ is $(7\xi, p_G, q, s-1, t)$-typical for every $u \in U_0$. Note that $\mathcal{G}_2[V_0 \cup U_0, V_1 \cup U_1] = \mathcal{R}[V_0 \cup U_0, V_1 \cup U_1] - \mathcal{F}_2[V_0 \cup U_0, V_1 \cup U_1]$. Then, for every $i \in [r]$ and $v \in V_0 \setminus V_0^i$ we have

$$\begin{aligned} |V_v^i| = d_{G_2}(v; V_1^i) = d_{\mathcal{R}}(v; V_1^i) \pm \hat{\Delta}(\mathcal{F}_2) &\overset{(R1),(4.60)}{=} (1 \pm 3\xi)\rho p_G \varepsilon n \\ &\geq (\rho/\varepsilon)^{4/3}(\varepsilon n) \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} |U_v| = d_{H_2}(v; U_1) = d_{\mathcal{R}}(v; U_1) \pm \hat{\Delta}(\mathcal{F}_2) = (1 \pm 3\xi)\rho p_H |U_1| \\ \overset{\text{Prop. 4.1.2}}{=} (1 \pm 7\xi)\rho p_G \varepsilon n \geq (\rho/\varepsilon)^{4/3}(\varepsilon n), \end{aligned} \tag{4.64}$$

for every $i \in [r]$ and each pair of distinct vertices $v_1, v_2 \in V_0 \setminus V_0^i$ we have

$$|V_{v_1}^i \cap V_{v_2}^i| = d_{\mathcal{R}}(\{v_1, v_2\}; V_1^i) \pm 2\hat{\Delta}(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_G)^2 \varepsilon n \leq (\rho/\varepsilon)^2(\varepsilon n)$$

and

$$|U_{v_1} \cap U_{v_2}| = d_{\mathcal{R}}(\{v_1, v_2\}; U_1) \pm 2\hat{\Delta}(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_H)^2 |U_1|$$
$$= (1 \pm 7\xi)\rho^2 p_G p_H \varepsilon n \leq (\rho/\varepsilon)^2(\varepsilon n).$$

Also, given $i, j \in [r]$, $i \neq j$, each vertex $v \in V_1^i$ is contained in

$$d_{G_2}(v; V_0^j) = d_{\mathcal{R}}(v; V_0^j) \pm \hat{\Delta}(\mathcal{F}_2) \overset{(R2),(4.60)}{=} (1 \pm 4\xi)(\rho p_G)|V_0^j| \leq (\rho/\varepsilon)(\varepsilon n) \quad (4.65)$$

of the sets $\{V_x^i\}_{x \in V_0^j}$, and each $u \in U_1$ is contained in

$$d_{H_2}(u; V_0^j) = d_{\mathcal{R}}(u; V_0^j) \pm \hat{\Delta}(\mathcal{F}_2) = (1 \pm 4\xi)(\rho p_H)|V_0^j| \leq (\rho/\varepsilon)(\varepsilon n) \quad (4.66)$$

of the sets $\{U_x\}_{x \in V_0^j}$.

On the other hand, for each $u \in U_0$ and $i \in [r]$ we have

$$|V_u^i| = d_{H_2}(u; V_1^i) = d_{\mathcal{R}}(u; V_1^i) \pm \hat{\Delta}(\mathcal{F}_2) \overset{(R1),(4.60)}{=} (1 \pm 3\xi)(\rho p_H)\varepsilon n$$
$$\geq (\rho/\varepsilon)^{4/3}(\varepsilon n), \quad (4.67)$$

for each $i \in [r]$ and each pair of distinct vertices $u_1, u_2 \in U_0$ we have

$$|V_{u_1}^i \cap V_{u_2}^i| = d_{\mathcal{R}}(\{u_1, u_2\}; V_1^i) \pm 2\hat{\Delta}(\mathcal{F}_2) = (1 \pm 3\xi)(\rho p_H)^2 \varepsilon n \leq (\rho/\varepsilon)^2(\varepsilon n),$$

and finally, for each $i \in [r]$, each vertex $v \in V_1^i$ is contained in

$$d_{H_2}(v; U_0) = d_{\mathcal{R}}(v; U_0) \pm \hat{\Delta}(\mathcal{F}_2) \overset{(R2),(4.60)}{=} (1 \pm 4\xi)(\rho p_H)|U_0|$$
$$\overset{\text{Prop. 4.1.2}}{=} (1 \pm 8\xi)(\rho p_G)|V_0| \leq (\rho/\varepsilon)(\varepsilon n) \quad (4.68)$$

of the sets $\{V_x^i\}_{x \in U_0}$.

We apply now Lemma 4.5.1 with V_1, U_1, ρ/ε, 8ξ, $|V_0|$, $|U_0|$, $\{\mathcal{Y}_v\}_{v\in V_0}$, $\{\mathcal{Y}_u\}_{u\in U_0}$ playing the role of V, U, ρ, ξ, N, M, $\{\mathcal{G}_i\}_{i\in[N]}$, $\{\mathcal{G}_i\}_{N<i\leq N+M}$ respectively to find for each vertex $x \in V_0\cup U_0$ an F_r-factor $\mathcal{F}_x$ of $\mathcal{Y}_x$ such that all the factors $\{\mathcal{F}_x\}_{x\in V_0\cup U_0}$ are edge-disjoint. Given $x \in V_0 \cup U_0$ recall that $\mathcal{Y}_0 = \mathrm{Lk}_{\mathcal{G}_2}(x; V_1 \cup U_1)$ which by definition implies that any r-face $f \in F_r(\mathcal{Y}_0)$ satisfies that $f \cup \{x\} \in F_{r+1}(\mathcal{G}_2)$. Hence, for each $f \in \mathcal{F}_x$ we have that $f \cup \{x\}$ is an $(r + 1)$-face of $\mathcal{G}_2$. Moreover, $\mathcal{F}_x$ is an F_r-factor of $\mathcal{Y}_x$ which implies that $\{f \cup \{v\}\}_{f\in\mathcal{F}_x}$ is a set of edge-disjoint $(r + 1)$-faces of $\mathcal{G}_2$ that cover all the edges from $\{v\}$ to $V_1 \cup U_1$ contained in $\mathcal{G}_2$. Since all the factors $\{\mathcal{F}_x\}_{x\in V_0\cup U_0}$ are edge-disjoint then

$$\mathcal{F}_3 := \bigcup_{\substack{x\in V_0\cup U_0\\ f\in\mathcal{F}_x}} f \cup \{x\}$$

is an F_{r+1}-decomposable subcomplex of $\mathcal{G}_2$ that covers all edges in $E(\mathcal{G}_2[V_0\cup U_0, V_1\cup U_1])$.

Finally, equations (4.63)-(4.68) imply that

$$\Delta(\mathcal{F}_3) \leq (\gamma'/2)n. \tag{4.69}$$

Step 5: Conclusion of the proof.

Let $\mathcal{H} := \mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3$. It is straightforward to see that $\mathcal{H}$ is F_{r+1}-decomposable since it consists of the edge-disjoint union of F_{r+1}-decomposable complexes. Recall that we started by finding $\mathcal{F}_1$ and defining $\mathcal{G}_1 = \mathcal{G} - \mathcal{F}_1$, then $\mathcal{F}_2$ was covering the edges of $\mathcal{G}_1[V_0\cup U_0]$, we then defined $\mathcal{G}_2 = \mathcal{G} - \mathcal{F}_1 - \mathcal{F}_2$ and finally $\mathcal{F}_3$ covered all the edges of $\mathcal{G}_2$ between $V_0\cup U_0$ and $V_1\cup U_1$. Thus, $\mathcal{H}$ covers all edges of $\mathcal{G} - \mathcal{G}[V_1\cup U_1]$.

Lastly, since $\mathcal{F}_1$ contains no edges within $V_1 \cup U_1$ then

$$\hat{\Delta}(\mathcal{H}[V_1 \cup U_1]) \leq \hat{\Delta}(\mathcal{F}_2) + \hat{\Delta}(\mathcal{F}_3) \overset{(4.60),(4.69)}{\leq} \gamma' n.$$

□

4.8 Proof of Theorem 4.2.1

Before the main proof, we introduce the absorbers that will be used to absorb the final leftover.

Definition 4.8.1. Let L be an $(r+1)$-partite graph. An *edge-absorber for L* is an $(r+1)$-partite graph A_L such that $V(L) \subseteq V(A_L)$ is independent in A_L, the parts of $V(L)$ are correspondingly contained in the parts of $V(A_L)$, and both A_L and $A_L \cup L$ have a K_{r+1}-decomposition.

The following lemma arises from a construction of edge-absorbers due to Barber, Kühn, Lo, Osthus and Taylor [10, Section 6] which develops ideas from [9]. In fact, the existence of edge-absorbers for K_r-divisible r-partite graphs follows directly from [10, Lemma 6.5]. The additional properties of such absorbers listed below is proved implicitly in their construction.

Lemma 4.8.2 ([10, Section 6])**.** *Let $1/M \ll 1/m, 1/r$. Let L be a K_{r+1}-divisible $(r+1)$-partite graph such that $|V(L)| \leq m$ and let U_L be a vertex part of L. Then there exists an $(r+1)$-partite edge-absorber A_L for L such that*

(i) $|V(A_L)| \leq M$,

(ii) A_L *has degeneracy* $2r+1$ *rooted at* $V(L)$*,*

(iii) A_L *and* $A_L \cup L$ *have* K_{r+1}*-decompositions* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *respectively,*

(iv) every vertex $v \in V(A_L) \setminus V(L)$ *is contained in at most six* $(r+1)$*-cliques from* $\mathcal{K}_1$ *and* $\mathcal{K}_2$*,*

(v) every $(r+1)$*-clique from* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *contains at most one edge of* L*.*

We have now all the ingredients to prove our main result.

Theorem 4.2.1. Let $1/n \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r$ with $s \geq 3(r-1)$ and $t \geq 2r-1$. Let $\mathcal{G} = (V, U, G, H)$ be an F_{r+1}-divisible (ξ, p_G, p_H, q, s, t)-typical extended $(r+1)$-partite complex with parts $V^1, \dots, V^r$ of size n. Then $\mathcal{G}$ has an F_{r+1}-decomposition.

Proof. Let $\gamma, \eta, \varepsilon > 0$ and $M, m' \in \mathbb{N}$ such that

$$1/n \ll 1/M \ll 1/m' \ll \eta \ll \gamma \ll \varepsilon \ll \xi \ll p_G, p_H, q, 1/s, 1/t, 1/r. \tag{4.70}$$

Step 1: Find the vortex sequences.

We start by finding a $(2\xi, \varepsilon, \eta, m)$-vortex $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ and $U = U_0 \supseteq, \dots, \supseteq U_\ell$ for some $m \leq m'$ using Lemma 4.4.2 in $\mathcal{G}$. Let $n_i := |V_i^1|$ for every $i \in [\ell] \cup \{0\}$ and recall that for every $i \in [\ell]$, since $\mathcal{G}[V_i \cup U_i]$ is $(2\xi, p_G, p_H, q, s, t)$-typical by (V2) Proposition 4.1.2 implies that $|U_i| = (1 \pm 6\xi)(p_G/p_H)n_i$. Thus,

$$|V_i \cup U_i| \leq |V_i| + |U_i| \leq rn_i + (1+6\xi)(p_G/p_H)n_i \leq (r + 2p_G/p_H)n_i$$

for every $i \in [\ell] \cup \{0\}$.

Step 2: Find the final absorber.

Let $\mathcal{G}' := \mathcal{G} - \mathcal{G}[V_1 \cup U_1]$. Since $|V_1 \cup U_1| \leq (r + 2p_G/p_H)\varepsilon n$ and $\varepsilon \ll \xi$ it follows that $\mathcal{G}'$ must be $(2\xi, p_G, p_H, q, s, t)$-typical. Let $\mathcal{L}$ be a collection of all the spanning K_{r+1}-divisible subgraphs of $\mathcal{G}[V_\ell \cup U_\ell]$. Note that

$$|\mathcal{L}| \leq 2^{\binom{|V_\ell \cup U_\ell|}{2}} \leq 2^{\binom{(r+2p_G/p_H)m}{2}} \overset{(4.70)}{<} n^{1/3}.$$

We will now find edge-disjoint edge-absorbers $\{A_L\}_{L\in\mathcal{L}}$ in $\mathcal{G}'$ satisfying that $|V(A_L)| \leq M$ for each $L \in \mathcal{L}$ and both A_L and $A_L \cup L$ are F_{r+1}-decomposable. Suppose that for some subset $\mathcal{L}' \subseteq \mathcal{L}$ we have already found edge-disjoint absorbers $\{A_L\}_{L\in\mathcal{L}'}$ and let $\mathcal{A}' = \bigcup_{L\in\mathcal{L}'} A_L$. Since $|\mathcal{L}| < n^{1/3}$ and $|V(A_L)| \leq M$ for every $L \in \mathcal{L}'$ by (i) then $|V(\mathcal{A}')| \leq |\mathcal{L}|M < n^{2/3}$. Thus, $\mathcal{G}' - \mathcal{A}'$ is $(3\xi, p_G, p_H, q, s, t)$-typical. Let $L \in \mathcal{L} \setminus \mathcal{L}'$ be fixed. We know by Lemma 4.8.2 that there exists an edge-absorber A'_L for L satisfying (i)-(v). Then, we can greedily find a copy A_L of A'_L in $\mathcal{G}'$ so that every $(r+1)$-clique used in the K_{r+1}-decomposition of A_L or $A_L \cup L$ is an $(r+1)$-face of $\mathcal{G}$. Indeed, we can embed the vertices of $V(A_L) \setminus V(L)$ one by one following the order given by the degeneracy property (ii). Next, it follows by (iv) that every vertex that we want to embed must form an $(r+1)$-face with at most six r-cliques of $\mathcal{G}$, and since each $(r+1)$-face of $\mathcal{G}$ contains exactly one r-face of G, then it must form an r-face with at most six $(r-1)$-cliques of G. Lastly, (iii) implies that each vertex that we want to embed must be adjacent to at most $2r+1$ of the preceding vertices. Hence, since $\mathcal{G}' - \mathcal{A}'$ is $(3\xi, p_G, p_H, q, s, t)$-typical with $s \geq 2r+1$ and $t \geq 6$ we can always embed the next vertex. Thus, we can find A_L for every $L \in \mathcal{L}$.

Let $\mathcal{A} := \bigcup_{L\in\mathcal{L}} A_L$. Observe that $\mathcal{A} \cup L$ has an F_{r+1}-decomposition for any

$L \in \mathcal{L}$ since both $A_L \cup L$ and $\mathcal{A} - A_L$ are F_{r+1}-decomposable. Let $\widetilde{\mathcal{G}} := \mathcal{G} - \mathcal{A}$. Since $|V(\mathcal{A})| \leq |\mathcal{L}|M < n^{2/3}$ and $E(\mathcal{A}[V_1 \cup U_1]) = \emptyset$ it is straightforward to check that $\widetilde{\mathcal{G}}$ is $(2\xi, p_G, p_H, q, s, t)$-typical and $V_0 \supseteq \cdots \supseteq V_\ell$ and $U_0 \supseteq \cdots \supseteq U_\ell$ form a $(3\xi, \varepsilon, 2\eta, m)$-vortex of $\widetilde{\mathcal{G}}$. The fact that $\mathcal{G}$ is F_{r+1}-divisible and $\mathcal{A}$ is F_{r+1}-decomposable implies that $\widetilde{\mathcal{G}}$ must also be F_{r+1}-divisible.

Step 3: Iteratively apply the Cover down lemma.

Let $i \in [\ell] \cup \{0\}$ and suppose there is an extended $(r+1)$-partite complex $\mathcal{G}_i = (V_i, U_i, G_i, H_i)$ with $G_i \subseteq G[V_i]$ and $H_i \subseteq H[V_i \cup U_i]$ such that $\widetilde{\mathcal{G}} - \mathcal{G}_i$ is F_{r+1}-decomposable and the following holds:

(a) $\mathcal{G}_i$ is F_{r+1}-divisible,

(b) $\mathcal{G}_i$ is $(5\xi, p_G, p_H, q, s, t)$-typical,

(c) $\mathcal{G}_i$ is $(4\xi, p_G, p_H, q, s, t)$-typical into $V_{i+1} \cup U_{i+1}$,

(d) $\mathcal{G}_i[V_{i+1} \cup U_{i+1}] = \mathcal{G}[V_{i+1} \cup U_{i+1}]$.

Note that this is clearly true for $i = 0$ and $\mathcal{G}_0 := \widetilde{\mathcal{G}}$. Let $\widetilde{\mathcal{G}}_i := \mathcal{G}_i - \widetilde{\mathcal{G}}_i[V_{i+2} \cup U_{i+2}] \overset{(d)}{=} \mathcal{G}_i - \mathcal{G}[V_{i+2} \cup U_{i+2}]$. Since

$$|V_{i+2} \cup U_{i+2}| \leq (r + 2p_G/p_H)n_{i+2} = (r + 2p_G/p_H)\varepsilon^2 n_i \overset{(4.70)}{<} \varepsilon^{3/2} n_i$$

using (b) and (c) it is not hard to check that $\widetilde{\mathcal{G}}_i$ is $(6\xi, p_G, p_H, q, s, t)$-typical into $V_i \cup U_i$ and $(5\xi, p_G, p_H, q, s, t)$-typical into $V_{i+1} \cup U_{i+1}$. Moreover, since $\mathcal{G}[V_{i+2} \cup U_{i+2}]$ is 2η-divisible by the vortex property (V3), using (a) we obtain that $\widetilde{\mathcal{G}}_i$ must also be 2η-divisible. We can then apply Lemma 4.7.3 to find an F_{r+1}-decomposable subcomplex $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_i$ that covers all edges of $\widetilde{\mathcal{G}}_i - \widetilde{\mathcal{G}}_i[V_{i+1} \cup U_{i+1}]$ and such that

$$\hat{\Delta}(\mathcal{H}[V_{i+1} \cup U_{i+1}]) \leq \gamma|V_{i+1} \cup U_{i+1}|. \tag{4.71}$$

Define $\mathcal{G}_{i+1} = (\mathcal{G}_i - \mathcal{H})[V_{i+1} \cup U_{i+1}]$. Note that $\mathcal{G} - \mathcal{G}_{i+1}$ is F_{r+1}-decomposable since both $\mathcal{G} - \mathcal{G}_i$ and $\mathcal{G}_i - \mathcal{G}_{i+1} = \mathcal{H}$ are. It only remains to check that $\mathcal{G}_{i+1}$ satisfies properties (a)-(d). First, note that $\mathcal{G}_i - \mathcal{H}$ has no edges outside of $V_{i+1} \cup U_{i+1}$. Hence, since both $\mathcal{G}_i$ and $\mathcal{H}$ are F_{r+1}-divisible, then $\mathcal{G}_{i+1}$ must be as well. Using (c) and equation (4.71) it is easy to see, since $\gamma \ll \xi$, that $\mathcal{G}_{i+1}$ must be $(5\xi, p_G, p_H, q, s, t)$-typical. For the third property, observe that by (d) and the vortex property (V2) we have that $\mathcal{G}_i[V_{i+1} \cup U_{i+1}]$ is $(3\xi, p_G, p_H, q, s, t)$-typical into $V_{i+2} \cup U_{i+2}$. Thus, using again (4.71) and $\gamma \ll \xi$ we obtain that $\mathcal{G}_{i+1}$ must be $(4\xi, p_G, p_H, q, s, t)$-typical into $V_{i+2} \cup U_{i+2}$. Finally, since $\widetilde{\mathcal{G}}_i[V_{i+2} \cup U_{i+2}]$ had no edges and $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_i$ we have that $\mathcal{G}_{i+1}[V_{i+2} \cup U_{i+2}] = \mathcal{G}_i[V_{i+2} \cup U_{i+2}] \overset{(d)}{=} \mathcal{G}[V_{i+2} \cup U_{i+2}]$.

Step 4: Absorb the final leftover

It follows by induction that there exists an extended $(r+1)$-partite complex $\mathcal{G}_\ell = (V_\ell, U_\ell, G_\ell, H_\ell) \subseteq \mathcal{G}$ such that $\widetilde{\mathcal{G}} - \mathcal{G}_\ell$ is F_{r+1}-decomposable and $\mathcal{G}_\ell$ is F_{r+1}-divisible by (a). In particular, $\mathcal{G}_\ell$ is a K_{r+1}-divisible graph on $V_\ell \cup U_\ell$ so $\mathcal{G}_\ell \in \mathcal{L}$. Thus, $\mathcal{A} \cup \mathcal{G}_\ell$ is also F_{r+1}-decomposable. Altogether, $\mathcal{G} = (\widetilde{\mathcal{G}} - \mathcal{G}_\ell) \cup (\mathcal{G}_\ell \cup \mathcal{A})$ has an F_{r+1}-decomposition. □

CHAPTER 5

COMPLETION OF PARTIAL LIST LATIN SQUARES

In this chapter we will discuss our results on the completion of Latin squares with lists of permitted entries. Recall that a *Latin square of order* n is an $n \times n$ array filled with the symbols $1, \dots, n$ so that each symbol appears exactly once in each row and each column. A *partial Latin square of order* n is an $n \times n$ array where some entries are filled with a symbol from $[n]$ and the rest of entries are empty such that each symbol appears at most once in each row and each column. In this setting the question of whether a given partial Latin square can be completed into a proper Latin square by filling the empty entries arises. On the other hand, recall that a *list Latin square of order* n is an $n \times n$ array in which each entry contains a subset of $[n]$. Here, a natural question consists of picking one symbol from each list so that it produces a proper Latin square. Recall that a *partial list Latin square of order* n is an $n \times n$ array where each entry contains either a symbol from $[n]$ ('filled entries') or a subset of $[n]$ ('empty entries with a given list') such that each symbol appears at most once in each row and each column, and each list does not contain symbols that appear in the same row or column. A combination of both the previous problems consists of the completion of a partial list Latin square, i.e., given a partial list Latin square whether we can pick a symbol from each list so that, together with the filled entries, it produces a proper Latin square. Theorem 1.4.7, stated below, gives sufficient conditions for the completion of partial list Latin squares.

Theorem 1.4.7. For any $\alpha < 1/(27{\cdot}10^6)$ and $\beta < 1/320$ there exists $n_0 \in \mathbb{N}$ such that any partial list Latin square of order $n \geq n_0$ in which each row, symbol and column has been used at most αn times, the lists for each empty entry have size at least $(1-\beta)n$, and each symbol appears in at least $(1-\beta)n$ of the lists from each row and each column, can be completed into a Latin square of order n.

5.1 Equivalence to tripartite graphs

As seen in Section 1.4.4, there is a natural bijection between Latin squares of order n and K_3-decompositions of the complete tripartite graph $K_{n,n,n}$ by identifying each part of $K_{n,n,n}$ with 'rows', 'columns' and 'symbols', respectively, and identifying each triangle (r, c, s) in the decomposition (where $r \in$ 'rows', $c \in$ 'columns' and $s \in$ 'symbols') with the entry (r, c) filled with the symbol s. Similarly, recall that a tripartite graph G which is obtained from $K_{n,n,n}$ by removing triangles can be viewed as partial Latin square of order n (each one of the removed triangles corresponds to a filled entry in the $n \times n$ array). In the case of partial list Latin squares, one can find an analogue in terms of balanced tripartite complexes. Recall from Section 3.1 that a *3-complex* G is a triple $(V(G), E(G), F_3(G))$ where $V(G)$ is a set of vertices, $E(G)$ is a set of edges and $F_3(G)$ is a set of 3-sets of vertices called *faces* such that for each $f \in F_3(G)$ and each 2-set e of f, $e \in E(G)$. A *balanced tripartite complex* G is a 3-complex such that $V(G) = V^1 \cup V^2 \cup V^3$ where V^1, V^2, V^3 are disjoint sets of vertices of the same size and $G[V^i]$ has no edges for each $i \in [3]$. Observe that there is a one to one correspondence between partial list Latin squares of order n and balanced tripartite complexes G with parts of size n whose underlying graph is obtained from $K_{n,n,n}$ by removing triangles. Indeed, let L be a partial list Latin square of order n. Let V^1, V^2, V^3 be three sets of vertices of size n (each one labelled from 1 to n) identified with 'rows', 'columns' and 'symbols' respectively. Let G' be the graph obtained from $K_{n,n,n}$ by removing, for each filled entry (r, c) with the symbol s in L, the edges rc, rs and cs where $r \in V^1$, $c \in V^2$ and $s \in V^3$. Note that for each edge $v_1v_2 \in E(G[V^1, V^2])$ the entry (v_1, v_2) in L is empty and thus contains a list $S_{v_1v_2} \subseteq [n]$. For each $e \in E(G[V^1, V^2])$ let $T_e := \{\{v\} \cup e \ : \ v \in S_e\}$ be the set of triangles corresponding to S_e. Let G be the balanced tripartite complex with parts V^1, V^2, V^3 where $E(G) = E(G')$ and $F_3(G) := \bigcup_{e \in E(G[V^1,V^2])} T_e$. Observe that an F_3-decomposition of G corresponds to a completion of L. We now state Theorem 1.4.7 in terms of tripartite complexes.

Theorem 5.1.1. *For any $\alpha < 1/(27{\cdot}10^6)$ and $\beta < 1/320$ there exists $n_0 \in \mathbb{N}$ such that any K_3-divisible balanced tripartite complex G with parts of size $n \geq n_0$ such that $d^*_G(e) \geq (1-\beta)n$ for each $e \in E(G)$ and $\hat{\delta}(G) \geq (1-\alpha)n$ has an F_3-decomposition.*

Actually, Theorem 5.1.1 is slightly more general than Theorem 1.4.7 since it considers any K_3-divisible graph and not just graphs obtained from $K_{n,n,n}$ by removing triangles.

The main goal of this chapter is then to prove Theorem 5.1.1. We will first prove the existence of a 'balanced' fractional F_3-decomposition in Section 5.2 and then discuss how to adapt the 'iterative absorption' method described in Section 1.5.1 to prove Theorem 5.1.1 in Section 5.3. We shall use the standard notation and probabilistic results described in Section 2.1 as well as the notation for complexes described in Section 3.1.

5.2 Fractional triangle decomposition

The aim of this section is to prove the following lemma.

Lemma 5.2.1. *For any $\alpha < 1/(27{\cdot}10^6)$ and $\beta < 1/320$ there exists $n_0 \in \mathbb{N}$ such that any K_3-divisible balanced tripartite complex G with parts of size $n \geq n_0$ such that $d^*_G(e) \geq (1-\beta)n$ for each $e \in E(G)$ and $\hat{\delta}(G) \geq (1-\alpha)n$ has a fractional F_3-decomposition $\omega : F_3(G) \to [0,1]$ such that for every $f \in F_3(G)$*

$$0 < \omega(f) < \frac{11}{(1-2\beta)n}.$$

The proof of Lemma 5.2.1 goes as follows. We will start by finding a fractional K_3-decomposition of the underlying graph $(V(G), E(G))$ by using a result of Montgomery [64]. Since $d^*_G(e) \geq (1-\beta)n$ for every edge $e \in E(G)$ we will check that the restriction of the fractional decomposition to the set of faces $F_3(G)$ gives a weighting ω' that is already 'close' to a fractional F_3-decomposition. We will then make use of certain gadgets to compensate the weight of some edges and transform ω' into the desired weighting ω.

In [64], Montgomery shows the existence of fractional K_r-decomposition in an r-partite K_r-divisible graph G with high minimum degree. Their proof starts with an initial constant weighting $\omega_0 : \mathcal{K}_r(G) \to [0,1]$ defined as $\omega_0(K) = e(G)/(\binom{r}{2}|\mathcal{K}_r(G)|)$ for each $K \in \mathcal{K}_r(G)$ and then they find a weighting $\varphi : \mathcal{K}_r(G) \to \mathbb{R}$ such that $\omega_0+\varphi$ is a fractional K_r-decomposition. At the very end of their proof they show that for each $K \in \mathcal{K}_r(G)$

$$|\varphi(K)| \leq \frac{n^2/10^2 + 2n^2/5 + r^3n/20}{|\mathcal{K}_r(G)|} \leq \frac{4n^2/5 - 9n^2/10^6r^3}{|\mathcal{K}_r(G)|}.$$

Since

$$\left(1 - \frac{1}{10^6r^3}\right)n^2 \leq \frac{e(G)}{\binom{r}{2}} \leq n^2$$

then it follows that

$$\max_{K\in\mathcal{K}_3(G)}(\omega_0 + \varphi)(K) \leq 9 \min_{K\in\mathcal{K}_3(G)}(\omega_0 + \varphi)(K).$$

Theorem 5.2.2 ([64]). *Let G be a K_r-divisible balanced r-partite graph with parts of size n such that $\hat{\delta}(G) \geq (1 - 1/10^6r^3)n$. Then G has a fractional K_r-decomposition $\omega' : \mathcal{K}_r(G) \to [0,1]$ such that*

$$\max_{K\in\mathcal{K}_r(G)} \omega'(K) \leq 9 \min_{K\in\mathcal{K}_r(G)} \omega'(K). \tag{5.1}$$

Let G be a balanced tripartite complex with parts of size n. Note that $\delta^*_G(e) \geq (1-\beta)n$ for every $e \in E(G)$ trivially implies $\hat{\delta}(G) \geq (1-\beta)n$ since $F_3(G) \subseteq \mathcal{K}_3(G)$. Let $G' = (V(G), E(G))$ be its underlying graph and let ω' be a fractional K_3-decomposition of G' that satisfies (5.1). Observe that for each edge $e \in E(G)$ we have

$$1 = \omega'(e) := \sum_{T \in \mathcal{K}_3(G)} \omega'(T) \geq d_G(e) \min_{T \in \mathcal{K}_3(G)} \omega'(T) \geq \frac{d_G(e)}{9} \max_{T \in \mathcal{K}_3(G)} \omega'(T)$$
$$\geq \frac{(1-\beta)n}{9} \max_{T \in \mathcal{K}_3(G)} \omega'(T)$$

so

$$\max_{T \in \mathcal{K}_3(G)} \omega'(T) \leq \frac{9}{(1-\beta)n}. \tag{5.2}$$

On the other hand, since $F_3(G) \subseteq \mathcal{K}_3(G)$ we have that for any $e \in E(G)$,

$$\omega'(e) = \sum_{T \in \mathcal{K}_3(G)\,:\,e \subset T} \omega'(T) = \sum_{T \in F_3(G)\,:\,e \subset T} \omega'(T) + \sum_{T \in \mathcal{K}_3(G) \setminus F_3(G)\,:\,e \subset T} \omega'(T) = 1$$

Thus, the restriction of ω' on $F_3(G)$ induces a weighting $\omega_0 : F_3(G) \to [0,1]$ of G such that for all $e \in E(G)$

$$\omega_0(e) = \sum_{T \in F_3(G)\,:\,e \subset T} \omega'(T) = 1 - \sum_{T \in \mathcal{K}_3(G) \setminus F_3(G)\,:\,e \subset T} \omega'(T).$$

Let $\mathcal{T}(G) := \mathcal{K}_3(G) \setminus F_3(G)$ denote the set of *forbidden triangles*, i.e., the set of triangles of G that are not 3-faces and thus can not be used in an F_3-decomposition. Rewriting the previous equation we have

$$\omega_0(e) = 1 - \sum_{T \in \mathcal{T}(G)\,:\,e \subset T} \omega'(T) \text{ for each } e \in E(G).$$

We would like to compensate the weight loss produced by the presence of each forbidden triangle. To do that, we start with the weighting $\omega_0 : F_3(G) \to [0,1]$ induced by ω'. Next, for each forbidden triangle $T_f \in \mathcal{T}(G)$, we shall find a gadget

S, consisting of a set of faces together with a weighting ω_S of S, which allows us to add a weight of $\omega'(T_f)$ to each edge in T_f without modifying the weight of any other edge in G. Hence, we obtain the same result as if replacing the forbidden triangle T_f by a face and considering the weighting ω' for such triangle. By finding a gadget for each forbidden triangle and adding the corresponding weights, we end up with a new weighting $\omega : F_3(G) \to [0,1]$ which compensates the weight loss on every edge $e \in E(G)$ so that $\omega(e) = \omega'(e) = 1$.

Lemma 5.2.3. *The number of forbidden triangles of G is at most βn^3.*

Proof. Let m be the number of edges between any two parts of G. For each of these edges, the number of faces containing them is at least $d^*_G(e) \geq (1-\beta)n$. Therefore G has at least $(1-\beta)nm$ different faces. On the other hand, $|\mathcal{K}_3(G)| \leq mn$. Thus, the number of forbidden triangles is at most $mn - (1-\beta)nm = \beta mn \leq \beta n^3$. □

Given a forbidden triangle T_f of G, a gadget for T_f is defined as a set of vertices $S = T_f \cup T_S$ where $T_S \in F_3(G)$ and $T_f \cap T_S = \emptyset$, such that all partite 3-sets in S, except for T_f, are faces of G. We shall classify the triangles of S depending on their intersection with T_S. A triangle T of S is *of type* 1,2 or 3 if $|T \cap T_S| = 1, 2$ or 3 respectively. Observe that T_S is the only triangle of type 3 and that there are three triangles of each type 1 and 2. See Figure 5.1. Now, we define ω_S as follows

$$\omega_S(T) = \begin{cases} \omega'(T_f)/10 & \text{if } T \text{ is of type 1 or 3,} \\ -\omega'(T_f)/10 & \text{if } T \text{ is of type 2,} \\ 0 & \text{otherwise.} \end{cases}$$

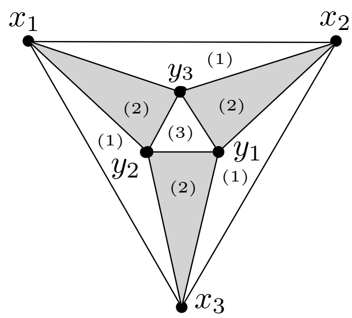

Figure 5.1: A gadget S for the triangle $T_f = \{x_1, x_2, x_3\}$ with $T_S = \{y_1, y_2, y_3\}$. The number between brackets indicates the type of each triangle.

Observe that $\omega_S(e) = \omega'(T_f)/10$ for each edge in T_f and $\omega_S(e) = 0$ for the rest of edges in S. Thus, adding the weights given by ω_S to the triangles of S increases by $\omega'(T_f)/10$ the weight of the edges in T_f and does not modify the weight of the rest of edges in G. Hence, by finding 10 gadgets for T_f, we are able to add a weight of $\omega'(T_f)$ to the edges of T_f without modifying the weight of the rest of edges. Moreover, equation (5.1) will guarantee that the weight of the triangles used in the gadget remain positive. We say that two gadgets S_1 and S_2 are *type-disjoint* if they share no triangles of the same type. In other words, if for any $k \in [3]$ there is no triangle T of type k such that T is a triangle of type k for both S_1 and S_2.

Lemma 5.2.4. *Let G and ω' be as in the statement of Theorem 5.2.2 and let $\omega_0 : F_3(G) \to [0,1]$ be the restriction of ω' to $F_3(G)$. Suppose that we can find 10 gadgets for each forbidden triangle $T_f \in \mathcal{T}(G)$ so that all the gadgets $\{S_i\}_{i\in[N]}$ with $N = 10|\mathcal{T}(G)|$ are pairwise type-disjoint. Then the weighting $\omega : F_3(G) \to [0,1]$ defined as $\omega(T) := \omega_0(T) + \sum_{i\in[N]} \omega_{S_i}(T)$ satisfies $\omega(e) = 1$ for each edge $e \in E(G)$ and $0 < \omega(T) < 11/(1-\beta)n$ for each $T \in F_3(G)$.*

Proof. It is clear that $\omega(e) = 1$ for each $e \in E(G)$ so it is enough to check that

$0 < \omega(T) < 11/(1-\beta)n$ for each $T \in F_3(G)$. Let $T \in F_3(G)$. Note that since all the gadgets are pairwise type-disjoint then T is in at most one of the gadgets in $\{S_i\}_{i\in[N]}$ as a triangle of type 2. Then

$$\omega(T) = \omega_0(T) + \sum_{i\in[N]} \omega_{S_i}(T) \geq \min_{T\in\mathcal{K}_3(G)} \omega'(T) - \frac{\max_{T\in\mathcal{K}_3(G)} \omega'(T)}{10}$$
$$\overset{(5.1)}{\geq} \frac{\max_{T\in\mathcal{K}_3(G)} \omega'(T)}{90} > 0.$$

Similarly, T appears in a gadget at most once as a triangle of type 1 and at most once as a triangle of type 3. Hence

$$\omega(T) = \omega_0(T) + \sum_{i\in[N]} \omega_{S_i}(T) \leq \max_{T\in\mathcal{K}_3(G)} \omega'(T) + 2\frac{\max_{T\in\mathcal{K}_3(G)} \omega'(T)}{10}$$
$$\leq \frac{6}{5} \max_{T\in\mathcal{K}_3(G)} \omega'(T) \overset{(5.2)}{\leq} \frac{54}{5(1-\beta)n} < \frac{11}{(1-\beta)n}.$$

□

Thus, the proof of Lemma 5.2.1 will consist of finding 10 gadgets for each forbidden triangle so that all the gadgets are type-disjoint. The following result counts the number of available gadgets for a given forbidden triangle.

Lemma 5.2.5. *Let T_f be a forbidden triangle and let $\mathcal{S}$ be the set of gadgets for T_f. Then $|\mathcal{S}| \geq (1-\beta)(1-2\beta)(1-4\beta)n^3$.*

Proof. First, note that $1-4\beta > 0$ since $\beta < 1/320$. Let $T_f = \{x_1, x_2, x_3\}$. We want to find a face $T_S = \{y_1, y_2, y_3\}$ such that all partite 3-sets in $T_f \cup T_S$, except for T_f, are faces of G. The number of vertices $y_1 \in X$ such that $\{y_1, x_2, x_3\}$ is a face of G is equal to $d^*_G(x_2x_3) \geq (1-\beta)n$. For fixed such y_1, the number of vertices $y_2 \in Y$ such that both $\{x_1, y_2, x_3\}$ and $\{y_1, y_2, x_3\}$ are faces is equal to

$|N_G^*(x_1x_3) \cap N_G^*(y_1x_3)| \geq (1-2\beta)n$. Finally, given y_1 and y_2, the number of vertices $y_3 \in Z$ such that all $\{x_1, x_2, y_3\}$, $\{y_1, x_2, y_3, \}$, $\{x_1, y_2, y_3\}$ and $\{y_1, y_2, y_3\}$ are faces of G is equal to $|N_G^*(x_1x_2) \cap N_G^*(y_1x_2) \cap N_G^*(x_1y_2) \cap N_G^*(y_1y_2)| \geq (1-4\beta)n$. □

The following proposition bounds the number of gadgets that contain a given face and a given forbidden triangle.

Proposition 5.2.6. *Let T_f be a forbidden triangle and let $T \in F_3(G)$. For each $i \in [3]$*

(i) a gadget S for T_f contains T as a triangle of type i if only if $T \subset S$ and $|T \cap T_f| = 3 - i$.

(ii) if $|T_f \cap T| = 3-i$, the number of gadgets S for T_f that contain T as a triangle of type i is at most n^{3-i}.

Proof. Let $S = T_f \sqcup T_S$. Then it is straightforward to see that $|T \cap T_S| = i$ if and only if $|T \cap T_f| = 3 - i$.

Suppose that $|T_f \cap T| = 3 - i$ and let S be a gadget for T_f that contains T as a triangle of type i. Then $T_f \cup T \subset S$ and $|T_f \cup T| = 3 + i$. Since S is formed of six vertices, and $3 + i$ of them are fixed, then there are at most n^{3-i} choices for the remaining vertices. □

Using Proposition 5.2.6 it is not difficult to check the following results.

Proposition 5.2.7. *Let T_1 and T_2 be forbidden triangles such that $|T_1 \cap T_2| = 0$ and let S_1 be a gadget for T_1. Then*

(i) S_1 shares no triangles of type 1 with any gadgets for T_2.

(ii) if $|T_{S_1} \cap T_2| \geq 1$ then S_1 shares a triangle of type 2 with at most $2n$ gadgets for T_2, else it shares no triangles of type 2 with any gadget for T_2.

Proposition 5.2.8. *Let T_1 and T_2 be forbidden triangles such that $|T_1 \cap T_2| = 1$ and let S_1 be a gadget for T_1. Then*

(i) if $|T_{S_1} \cap T_2| = 1$ then S_1 shares a triangle of type 1 with at most n^2 gadgets for T_2, else it shares no triangles of type 1 with any gadget for T_2,

(ii) S_1 shares a triangle of type 2 with at most n gadgets for T_2.

Proposition 5.2.9. *Let T_1 and T_2 be forbidden triangles such that $|T_1 \cap T_2| = 2$ and let S_1 be a gadget for T_1. Then*

(i) S_1 shares a triangle of type 1 with at most n^2 gadgets for T_2,

(ii) S_1 shares a triangle of type 2 with at most $2n$ gadgets for T_2.

We have now the main ingredients to prove Lemma 5.2.1.

Proof of Lemma 5.2.1. Let $N := 10|\mathcal{T}(G)|$. As previously discussed, it is enough to find 10 distinct gadgets for each forbidden triangle T_f of G so that all the gadgets $\{S_i\}_{i \in N}$ are pairwise type-disjoint.

Let $T_1, \ldots, T_N$ be an enumeration of the forbidden triangles of G, each one appearing 10 times. Let $\mathcal{S}_i$ be the set of gadgets for T_i in G. By Lemma 5.2.3 and 5.2.5 we have $N \leq 10\beta n^3$ and $|\mathcal{S}_i| \geq \gamma n^3$ for every $i \in [N]$, where $\gamma := (1-\beta)(1-2\beta)(1-4\beta)$. Note that $\gamma > (1-7\beta) > 313/320$ since $\beta < 1/320$.

Our aim is to find a gadget $S_i \in \mathcal{S}_i$ for T_i for every $i \in [N]$, such that the gadgets $\{S_i\}_{i \in [N]}$ are pairwise type-disjoint. For this, we shall use a randomised algorithm.

Suppose we have already found gadgets $S_1, \ldots, S_{i-1}$ for $T_1, \ldots, T_{i-1}$ for some $i \in [N]$. Let X_i be the set consisting of all gadgets in $\mathcal{S}_i$ that share a triangle of the same type with a gadget S_j for some $j \in [i-1]$. If $|X_i| < n^3/2$, let $W_i = \mathcal{S}_i \backslash X_i$. Otherwise, let $W_i = \mathcal{S}_i$. In any case, we have $|W_i| \geq (\gamma - 1/2)n^3$ and we choose S_i uniformly at random from W_i. Observe that if $|X_i| < n^3/2$ we ensure that S_i is type-disjoint from all gadgets S_j, $j \in [i-1]$.

Hence, the theorem follows if

$$|X_i| < n^3/2 \text{ for every } i \in [N]$$

with positive probability.

Let $J_0^i, J_1^i, J_2^i, J_3^i$ be the set of indices $j \in [i-1]$ such that $|T_j \cap T_i| = 0, 1, 2, 3$, respectively. Note that $J_0^i \sqcup J_1^i \sqcup J_2^i \sqcup J_3^i = [i-1]$. Let $X_0^i, X_1^i, X_2^i, X_3^i$ be the set of gadgets in $\mathcal{S}_i$ that share a triangle of the same type with a gadget S_j for some $j \in J_0^i, J_1^i, J_2^i, J_3^i$, respectively. Observe that $X_i = X_0^i \cup X_1^i \cup X_2^i \cup X_3^i$.

Claim 1*:* $|X_3^i| < 28n^2$*.*

Proof of claim: Note that $|J_3^i| \leq 9$ since each forbidden triangle is considered 10 times in the enumeration. Suppose $j \in J_3^i$ and consider the gadget S_j, which is also a gadget for T_i. It follows by Proposition 5.2.6 that $|X_3^i| \leq (3n^2+3n+1)|J_3^i| < 28n^2$ since S_j contains exactly three triangles of type 1, three triangles of type 2 and one triangle of type 3. —

Claim 2*:* $|X_2^i| < 4\beta n^3$*.*

Proof of claim: First, let us see that $|J_2^i| < 3\beta n$. The number of triangles $T \in \mathcal{K}_3(G)$ such that $|T \cap T_i| = 2$ is at most $3(n-1)$. On the other hand, the number of faces $f \in F_3(G)$ such that $f \cap T_i = 2$ is at least $3(1-\beta)(n-1)$. Thus, the number of forbidden triangles that intersect T_i in exactly two vertices is at most $3\beta(n-1)$. In particular, $|J_2^i| < 3\beta n$.

Thus, by Propositions 5.2.6 and 5.2.9 we have that $|X_2^i| \leq (n^2+2n+1)|J_2^i| < 4\beta n^3$. $-$

Suppose now that $X_0^i < 138\beta n^2$ and $X_1^i < 18\beta n^3$. Then

$$|X_i| \leq |X_0^i| + |X_1^i| + |X_2^i| + |X_3^i| < 138\beta n^3 + 18\beta n^3 + 4\beta n^3 + 28n^2 < n^3/2$$

since $\beta < 1/320$ and $n \geq n_0$ is large enough. Thus, it is enough to prove that with positive probability

$$|X_0^i| < 138\beta n^3 \text{ and } |X_1^i| < 18\beta n^3 \text{ for every } i \in [N]. \tag{5.3}$$

For each $j \in J_0^i$, let Y_j^i be the indicator variable for the event $|T_{S_j} \cap T_i| \geq 1$. By Propositions 5.2.6 and 5.2.7 we have that

$$|X_0^i| \leq 2n \left(\sum_{j \in [J_0^i]} Y_j^i \right) + |J_0^i|.$$

Let $j_1, \ldots, j_{|J_0^i|}$ be an enumeration of J_0^i. Then

$$\mathbb{P}\left[Y_{j_l}^i = 1 \mid Y_{j_1}^i, \ldots, Y_{j_{l-1}}^i \right] \leq \frac{3n^2}{(\gamma - 1/2)n^3} = \frac{3}{(\gamma - 1/2)n}$$

since $|W_j| \geq (\gamma - 1/2)n^3$ for all $j \in [i-1]$, and there are at most $3n^2$ triangles T_{S_j} such that $|T_{S_j} \cap T_i| \geq 1$.

Let $B_0 \sim Bin(|J_0^i|, \frac{3}{(\gamma-1/2)n})$. Since $N \leq 10\beta n^3$, in particular we have $|J_0^i| \leq$

$10\beta n^3$, then

$$\mathbb{E}[B_0] = \frac{3|J_0^i|}{(\gamma - 1/2)n} \leq \frac{30\beta}{(\gamma - 1/2)} n^2.$$

Note that $t_0 := 63\beta - \frac{30\beta}{(\gamma - 1/2)} > 0$ since $(\gamma - 1/2) > 30/63$. Thus, it follows by Lemma 2.1.2 and Fact 2.1.3 that

$$\begin{aligned} \mathbb{P}\left[\sum_{j \in [J_0^i]} Y_j^i \geq 63\beta n^2\right] &\leq \mathbb{P}\left[B_0 \geq 63\beta n^2\right] = \mathbb{P}\left[B_0 \geq \mathbb{E}\left[B_0\right] + t_0 n^2\right] \\ &\leq 2e^{-2t_0^2 n^3}. \end{aligned} \tag{5.4}$$

For each $j \in J_1^i$, let Z_j^i be the indicator variable for the event $|T_{S_j} \cap T_i| = 1$. Recall that for all $j \in J_1^i$ we have $|T_j \cap T_i| = 1$. It follows by Propositions 5.2.6 and 5.2.8 that

$$|X_1^i| \leq n^2 \left(\sum_{j \in [J_1^i]} Z_j^i\right) + (n+1)|J_1^i|.$$

Let $j_1, \ldots, j_{|J_1^i|}$ be an enumeration of J_1^i. Then

$$\mathbb{P}\left[Z_{j_l}^i = 1 \mid Z_{j_1}^i, \ldots, Z_{j_{l-1}}^i\right] \leq \frac{2n^2}{(\gamma - 1/2)n^3} = \frac{2}{(\gamma - 1/2)n}$$

since $|W_j| \geq (\gamma - 1/2)n^3$ for all $j \in [i-1]$, and there are at most $2n^2$ triangles T_{S_j} such that $|T_{S_j} \cap T_i| \geq 1$ and $|T_{S_j} \cap T_j| = 0$.

Now, let us check that $|J_1^i| < 3\beta n^2$. The number of triangles $T \in \mathcal{K}_3(G)$ such that $|T \cap T_i| = 1$ is at most $3(n-1)^2$, which is the total number of partial 3-sets intersecting T_i on precisely one vertex. On the other hand, the number of faces $f \in F_3(G)$ such that $|f \cap T_i| = 1$ is at least $3(1-\beta)(n-1)^2$. Thus, the number of forbidden triangles that intersect T_i in exactly two vertices is at most $3\beta(n-1)^2$.

In particular, $|J_1^i| < 3\beta n^2$.

Let $B_1 \sim Bin(|J_1^i|, \frac{2}{(\gamma-1/2)n})$. Then

$$\mathbb{E}[B_1] = \frac{2|J_1^i|}{(\gamma-1/2)n} \leq \frac{6\beta}{(\gamma-1/2)}n.$$

Note that $t_1 := 13\beta - \frac{6\beta}{(\gamma-1/2)} > 0$ since $(\gamma - 1/2) > 6/13$. Thus, it follows by Lemma 2.1.2 and Fact 2.1.3 that

$$\mathbb{P}\left[\sum_{j\in[J_1^i]} Z_j^i \geq 13\beta n\right] \leq \mathbb{P}\left[B_1 \geq 13\beta n\right] = \mathbb{P}\left[B_1 \geq \mathbb{E}\left[B_1\right] + t_1 n\right] \leq 2e^{-2t_1^2 n}. \quad (5.5)$$

Finally, applying (5.4) and (5.5), a union bound implies that with positive probability $\sum_{j\in[J_0^i]} Y_j^i < 63\beta n^2$ and $\sum_{j\in[J_1^i]} Z_j^i < 13\beta n$ for every $i \in [N]$. Hence,

$$|X_0^i| \leq 2n\left(\sum_{j\in[J_0^i]} Y_j^i\right) + |J_0^i| < 126\beta n^3 + 10\beta n^3 < 138\beta n^3$$

and

$$|X_1^i| \leq n^2\left(\sum_{j\in[J_1^i]} Z_j^i\right) + (n+1)|J_1^i| < 13\beta n^3 + 3\beta n^3 + 3\beta n^2 < 18\beta n^3$$

holds for all $i \in [N]$ with positive probability as required by (5.3). □

5.3 Proof of Theorem 5.1.1

In this section we will discuss the proof of Theorem 5.1.1. The proof closely follows the approach used by Barber, Glock, Kühn, Lo, Montgomery and Osthus [7] to find a triangle decomposition in a non-partite graph with high minimum degree. Recall Section 1.5.1 for an overview of the approach. Instead of presenting the full proof we will sketch the main steps and highlight any substantial difference from the method.

Let G be a K_3-divisible balanced tripartite complex with parts V^1, V^2, V^3 of size n such that $d^*_G(e) \geq (1-\beta)n$ for each $e \in E(G)$ for some $c < 1/320$. Let $\varepsilon > 0$ and assume the hierarchy of constants $1/n \ll \varepsilon \ll c$. The first step is to find a vortex sequence in G that will pass down the minimum degree property. Formally, we define a vortex sequence as follows.

Definition 5.3.1. A (δ, ε, m)-*vortex* in G is a sequence $V(G) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_\ell$ where $|V^k_\ell| = m$ for each $k \in [3]$ such that for all $i \in [\ell]$

(V1) $|V^k_i| = \varepsilon|V^k_{i-1}|$ for each $k \in [3]$,

(V2) $d^*_G(e; V^k_i) \geq \delta|V^k_i|$ for each $k \in [3]$ and $e \in E(G[V_{i-1} \setminus V^k_{i-1}])$.

By iteratively choosing each set V^k_i as a random subset of V^k_{i-1} of size $\varepsilon|V^k_{i-1}|$ for each $k \in [3]$, it is not hard to check that G contains a $((1-\beta-\varepsilon), \varepsilon, m)$-vortex where $1/n \ll 1/m \ll \varepsilon \ll \beta$. Note that using $\beta < 1/320$ and the hierarchy we can guarantee that $\beta + \varepsilon < 1/320$.

The next step is to set aside the final absorbers for all the possible leftover configurations in $G[V_\ell]$. Using Lemma 4.8.2 (with $r = 2$) and the fact that $d^*_G(e) \geq (1-\beta)n$ for all $e \in E(G)$ it is possible to greedily find such absorbers.

We want now to iteratively apply a 'Cover Down Lemma' until the final leftover is contained within $G[V_\ell]$ and thus can be covered using the final absorbers that were set aside. In our case, the Cover Down Lemma would look as follows.

Lemma 5.3.2. *Let $1/n \ll \gamma \ll \varepsilon \ll 1/\beta, 1/\alpha$ with $\alpha < 1/(27{\cdot}10^6)$ and $\beta < 1/320$. Let G be a balanced tripartite complex with parts V^1, V^2, V^3 of size n such that $d^*_G(e) \geq (1-\beta+\varepsilon)n$ for each $e \in E(G)$ and $\hat{\delta}(G) \geq (1-\alpha)n$. Let $U \subseteq V(G)$ with $|U^k| = \varepsilon n$ for each $k \in [3]$. Suppose that $d^*_G(e; U^k) \geq (1-\beta+\varepsilon)|U^k|$ for all $k \in [3]$ and $e \in E(G[V \setminus V^k])$ and $d_G(v; U^k) \geq (1-\alpha)|U^k|$ for all $k \in [3]$ and $v \in V \setminus V^k$. Suppose also that $d_G(v; V^i) = d_G(v; V^j)$ for all $i, j \in [3]$ and $v \in V(G) \setminus (V^i \cup V^j \cup U)$. Then there exists an F_3-decomposable subcomplex $H \subseteq G$ such that $G - G[U] \subseteq H$ and $\Delta(H[U]) \leq \gamma n$.*

Proof sketch. In the tripartite case the proof of the Cover Down Lemma is much simpler than, say, the proof of Lemma 4.7.3. After removing the 'reservoir' graph R and finding an approximate F_3-decomposition in $G - R - G[U]$ (by using Lemma 5.2.1), it is now easier to cover down the leftover edges that lie outside $G[U]$. Indeed, let L be the graph induced by the leftover edges. For each edge $e \in L[V \setminus U]$ we can greedily pick a vertex v inside $N^*_R(e)$ so that $e \cup \{v\}$ is a face of G. Once all edges in $L[V \setminus U]$ are covered, we find for each $v \in V \setminus U$ a matching in the link graph $\mathrm{Lk}_{G'}(v; U)$ where G' is the complex obtained from G by removing all the covered edges so far. The existence of edge-disjoint matchings for each $v \in V \setminus U$ can be shown using the minimum degree of G' and the randomised algorithm used in the proof of Lemma 2.6.2. With this all the leftover edges in L are covered and the proof would be completed. □

By iteratively applying Lemma 5.3.2 to each $G[V_{i-1}]$ (with V_i playing the role of U) we can cover down all the edges of G until a final leftover is left in $G[V_\ell]$. These edges are finally covered using the absorbers that were set aside.

CHAPTER 6

FINAL REMARKS

In this section we will summarise our results and discuss some further consequences and some potential applications. In a nutshell, the methods we use to find a resolvable decomposition consists of, first, transforming the problem into finding a graph decomposition in an auxiliary graph, which we call the 'extended graph', and then applying 'iterative absorption' to the extended graph in order to find the desired decomposition. In our case, in order to prove the existence of a resolvable decomposition in Theorems 1.2.3, 1.2.5 and 1.4.5 we prove the existence of a decomposition in their corresponding extended graphs in Theorems 2.2.3, 3.2.1 and 4.2.1, respectively. Recall that the use of the extended graph to find resolvable decompositions was introduced by Keevash [43]. Recall also that the iterative absorption approach that we use relies fundamentally on three key results: the existence of a fractional decomposition, the 'Cover Down Lemma', and the existence of final absorbers. While the last two have already been developed and successfully used together to solve several decomposition problems, e.g. [9, 28, 30, 59, 61], the existence of a fractional decomposition is often treated as an independent problem. For this reason, in addition to proving a 'Cover Down Lemma' and constructing final absorbers for the extended graphs, we have had to develop a new technique that allows us to find a fractional decomposition in each extended graph (see Section 2.3 and Section 3.3). More precisely, given a 'balanced' fractional decomposition of the host graph –where by 'balanced' we refer to fractional decomposition which is close to a uniform weighting– we are able, through the use of customised gadgets, to 'extend' the fractional decomposition of the host graph into a fractional decomposition of the extended graph. This new technique together with the implementation of the 'iterative absorption' method allows us to reduce the problem of finding a resolvable decomposition in a graph into finding a balanced fractional decomposition. We believe this is a powerful tool to solve problems on the existence of resolvable decompositions.

6.1 Applications

The existence of resolvable designs in typical graphs was settled by Keevash [43]. However, their proof involved algebraic techniques and required $\xi, 1/s \ll p$ in the assumption of (ξ, p, s)-typicality. As shown by Theorem 1.2.5 and Theorem 1.4.5 our methods can be used to show the existence of resolvable decompositions in (ξ, p, s)-typical pseudo-random graphs where $\xi \ll p, 1/s$. This difference in the typicality assumption is due to the fact that our methods only rely on probabilistic arguments and the existence of constant size gadgets. In fact, in order to apply our arguments we only require a linear size common neighbourhood of sets of constant size. This could be very useful to deduce analogous results in the minimum degree setting. For instance, to prove Theorem 1.2.5 we require a linear size common neighbourhood of sets of size at most $3r$ which is guaranteed by the typicality condition. A graph G on n vertices with minimum degree at least $(1 - 1/3r + \alpha)n$ guarantees a size of at least αn in all common neighbourhoods of at most $3r$ vertices. Assuming the existence of a balanced fractional decomposition we could apply the same approach as Theorem 1.2.5 to G and show the existence of a resolvable K_r-decomposition. The big constraint when deducing analogous results in the minimum degree setting is then the existence of the balanced fractional decomposition. In the pseudo-random setting it is not hard to find a balanced fractional decomposition since the graph is 'highly regular' due to the typicality condition. But this is not the case in the minimum degree setting. Thanks to the recent result by Joos and Kühn [40] (see Theorem 2.3.1) we are able to apply our methods and prove Theorem 1.2.3. The authors in [27] made the following conjecture which can be viewed as a Dirac-type version of the Oberwolfach problem.

Conjecture 6.1.1 ([27, Conjecture 1.5])**.** *For every $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ such that for any 2-regular graph F on $n \geq n_0$ vertices, any regular graph G on n vertices with even degree and such that $\delta(G) \geq (3/4+\alpha)n$ has an F-decomposition.*

The authors in [31] conjectured that if the girth of F is large compared to $1/\alpha$ then the term $3/4$ could be replaced by $1/2$. Theorem 1.2.3 answers this affirmatively if F consists of cycles of the same length.

It is also possible to deduce results in the pseudo-random setting from the approach used in the minimum degree setting. In fact, following the proof of Theorem 1.2.3 one can prove the subsequent result.

Theorem 6.1.2. *For every $r \geq 3$ and $p > 0$ there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that any C_r-divisible $(\xi, p, 4)$-typical regular graph G on $n \geq n_0$ vertices such that*

$r \mid n$ has a resolvable C_r-decomposition.

Of course the proof of Theorem 1.2.3 would have to be adapted. For example, the use of Theorem 2.3.1 could be replaced by the use of Lemma 4.2 in [7], and property (V3) in Definition 2.5.1 would have to be replaced by $d_G(S; V_i) = (1 \pm \xi)p|V_i|$ for every set $S \subseteq V_{i-1}$ with $|S| \leq 4$. On the other hand the construction of the 'shifters' in Section 2.3 and the final absorbers in Section 2.8 would not need any change.

On the other hand, a direct application of Theorem 1.4.5 leads to the following 'partite' version of Theorem 6.1.2. Given an r-partite graph G, a *resolvable partite C_r-decomposition of G* is a resolvable C_r-decomposition where each cycle contains exactly one vertex from each part of G.

Corollary 6.1.3. *For every $r \geq 3$ and $p > 0$ there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let $V_0 := V_r$ and $V_{r+1} := V_1$. Let G be a balanced regular r-partite graph with parts $V_1, \dots, V_r$ of size $n \geq n_0$ such that $E(G) = \bigcup_{i \in [r]} E(G[V_{i-1}, V_i])$ and, for each $i \in [r]$, $G[V_{i-1}, V_i]$ is a $(\xi, p, 3(r-1))$-typical bipartite graph and $d_G(v; V_{i-1}) = d_G(v; V_{i+1})$ for every $v \in V_i$. Then G has a resolvable partite C_r-decomposition.*

To prove Corollary 6.1.3 we shall use a recent result from Klimošová, Reiher, Ruciński and Šileikis [51] that confirms the Kim-Vu Sandwich Conjecture [49] in regular dense bipartite graphs. Let $G(n, n, p)$ denote the random bipartite graph obtained from $K_{n,n}$ by choosing each edge independently at random with probability p, and let $R(n, n, p)$ be a random bipartite graph chosen uniformly at random from the set of pn-regular bipartite graphs with parts of size n.

Theorem 6.1.4 ([51]). *Let $1/n \ll \xi \ll p$. There exists a joint distribution of the bipartite random graphs $G_1 \sim G(n, n, (1-\xi)p)$, $R \sim R(n, n, p)$ and $G_2 \sim G(n, n, (1+\xi)p)$ such that $G_1 \subset R \subset G_2$ with probability $1 - o(1)$.*

Proof of Corollary 6.1.3. Let $p' := d_G(v; V_i)/n$ for some $i \in [r]$ and $v \in V_{i-1}$. Note that p' is well defined since G is regular and $d_G(v; V_{i-1}) = d_G(v; V_{i+1})$ for every $v \in V_i$. Observe also that $p' = (1 \pm \xi)p$ because of typicality. For each $i, j \in [r]$, $i \neq j-1, j, j+1$, let $H[V_i, V_j]$ be a random $(p'n)$-regular bipartite graph on $[V_i, V_j]$ and let

$$H = \bigcup_{\substack{i,j \in [r] \\ i \neq j-1, j, j+1}} H[V_i, V_j].$$

Let $G' := G \cup H$. For each $i, j \in [r]$, $i \neq j-1, j, j+1$ we use Theorem 6.1.4 to find a joint distribution of $G_1[V_i, V_j] \sim G(n, n, (1-\xi)p')$, $H[V_i, V_j]$ and $G_2[V_i, V_j] \sim G(n, n, (1+\xi)p')$ such that $G_1[V_i, V_j] \subset H[V_i, V_j] \subset G_2[V_i, V_j]$ with probability almost 1. Thus, using Lemma 2.1.2 and a union bound we can see that with positive probability G' is a $(\xi', p, 3(r-1))$-typical r-partite graph for some $\xi' > 0$ where $1/n \ll \xi' \ll \xi$. Hence, we can choose H so that G' is a K_r-divisible $(\xi', p, 3(r-1))$-typical r-partite graph with parts $V_1, \ldots, V_r$. We can now apply Theorem 1.4.5 to find a resolvable K_r-decomposition of G'. Finally, observe that the restriction of the K_r-decomposition to G induces a resolvable partite C_r-decomposition. □

In their proof of the Oberwolfach problem, Glock, Joos, Kim, Kühn and Osthus [27] make use of a result by Keevash [43] on the existence of resolvable designs. In particular, they use it to prove two specific results [27, Corollary 3.7 and Corollary 3.9]. Theorem 6.1.2 and Corollary 6.1.3 imply those results respectively thus allowing for a purely combinatorial alternative proof.

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