ON THE RELATION BETWEEN ASYMPTOTIC BEHAVIOUR OF NEUMANN EIGENVALUE COUNTING FUNCTIONS AND THE INNER MINKOWSKI CONTENT OF FRACTAL DRUMS

by

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Abstract

We define a class of well-foliated domains for which explicit lower bounds for the first non-trivial eigenvalue of the Neumann Laplacian are obtained. We show that p-Rohde snowflakes for $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$ fall into this class. We establish conditions on snowflake-like domains and quasidisks that ensure well-foliatedness via bi-Lipschitz invariance. We prove that domains for which any sufficiently small inner ϵ-parallel neighbourhood can be covered (with uniformly bounded multiplicity of the cover) by at most $C\epsilon^{-\delta}$ well-foliated domains allow an upper bound of the remainder term of the spectral counting function of the Neumann Laplacian at t asymptotic to $t^{\delta/2}$. It is discussed how the cardinality of this cover relates to the upper inner Minkowski content and in particular we show that p-Rohde snowflakes satisfy this property whenever their upper inner Minkowski content is positive and finite. Applying the above results we obtain explicit bounds in the cases of homogeneous p-Rohde snowflakes for $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$ and of the classical Koch snowflake. Finally we construct a family of fractal sprays whose gaps have fractal boundary and find non-trivial terms in the asymptotic expansion of the spectral counting function and the volume of the inner ϵ-parallel neighbourhood with comparable scaling behaviour.

Contents

Chapter 1

Introduction

The celebrated 1966 article [52] by Kac asks which geometric information can be inferred from the Laplace spectrum of a domain coining the famous question: *Can one hear the shape of a drum?* Parallel to this well-known problem which naturally involves Dirichlet boundary conditions one may ask the same question for Neumann boundary conditions. These inverse problems, i. e. computing the "drum" (that is the domain of definition of the Laplace eigenfunctions) given its spectrum, were shown to be ultimately unsolvable in 1992 by Gordon-Webb-Wolpert [37] for domains with piecewise smooth boundary and subsequently for domains with fractal boundary by Chen-Sleemann in [18] by showing that different domains with identical spectrum exist. Nevertheless it is well known that for example the volume of a bounded domain can be deduced from the spectrum of the Laplace operator under Dirichlet boundary conditions. Other geometric quantities however are often far less attainable in particular if the boundary is less regular or even fractal.

The present work is focused on the particular situation of Neumann boundary conditions on a domain (that is an open, bounded and connected subset of Euclidean space) which has a fractal boundary, meaning that its Minkowski dimension might be non-integral, emphasising the complementary forward problem, namely to study the influence of a domain's geometry on its Laplace spectrum. This forward problem has attracted an immense amount of attention throughout the years (see for example [5, 48] for a summary). Since the Laplace eigenvalue equation can be physically interpreted as the wave equation of a function that is periodic in time, an eigenvalue λ can be understood as acoustic frequency of a wave mode. Heuristically, higher harmonics correspond to higher order eigenfunctions displaying similar geometric aspects of their domain. As a consequence, it is particularly interesting how the geometry of a domain dictates the asymptotic behaviour of eigenvalues of the Laplace operator. The idea goes back to 1911 when Weyl established a remarkable law for the eigenvalue counting function $N_D(\Omega, t)$ (i. e. the number of eigenvalues less or equal to t for the Dirichlet Laplace problem counted with multiplicity) for open and bounded $\Omega \subset \mathbb{R}^2$ in [112, 113], which had a tremendous impact in mathematics [23].

The very nature of fractals often renders analytical tools inapplicable. It should be pointed out that for this reason several different approaches to spectral analysis of fractals have been introduced over time and drastically depend on the nature of the fractal in question and in particular whether the fractal has non-empty interior. On one side, Laplacians on gasket-like fractals — the Sierpiński gasket being a leading example — have been studied extensively based on ideas of Goldbach, Kigami and Strichartz in [36, 54, 103] by introducing Dirichlet forms on corresponding prefractal structures and thus studying the limiting behaviour of graph-Laplacians. Recently Post-Zimmer complemented the spectral analysis on fractals with an analytical approach in [87] based on smooth Laplacians defined on a shrinking ϵ-parallel neighbourhood around prefractal structures. On the other side one may define a Laplacian *off* the domain (often called "spectral analysis off fractals"), meaning to define the Laplace operator on the interior of the union of bounded connected components of the complement; again the Sierpiński gasket is the leading example. Renewal theoretical spectral analysis was successfully applied in such cases (see also the works of Kigami-Lapidus and Lalley in [55, 64]). Whenever the fractal object is a domain, a direct analytical definition of differential operators *on* the domain is possible and this approach is followed here for the most part. The leading example shall be the Koch snowflake introduced by Koch in [57]. In Ch. 5 however we take the point of view of spectral analysis off fractals, too.

More precisely we are interested in the asymptotic behaviour of the spectral counting functions of the Laplace operator subject to Neumann boundary conditions on snowflake-like domains with fractal boundary and we aim at obtaining explicit asymptotic upper bounds of the remainder term (that is the second order asymptotic term). While Dirichlet boundary conditions in the fractal setting have been explored to a good extent, such a treatment is yet to be completed for other boundary conditions. One may regard the Neumann boundary condition as the most irregular amongst all mixed boundary conditions since it has minimal eigenvalues and therefore leads to fastest growth of the spectral counting function. Moreover, Dirichlet boundary conditions avoid complications with non-empty essential spectra and allow for simpler estimates of spectral counting functions based on domain monotonicity (i.e. $\lambda_k^D(\Omega') \geq \lambda_k^D(\Omega)$ if $\Omega' \subset \Omega$, where $\lambda_k^D(X)$ is the k^{th} eigenvalue of a domain X under Dirichlet boundary conditions). Neither of these properties are generally available under non-Dirichlet boundary conditions. We therefore restrict ourselves to extension domains for which the essential spectrum is known to be empty. We then obtain an explicit Poincaré-Wirtinger inequality (sometimes also called Poincaré-Sobolev inequality) for $W^{1,2}(\Omega)$ (the usual Sobolev space) where Ω has fractal boundary that is not necessarily the graph of a function and neither needs a cone condition nor an explicit extension property with the following result:

Theorem (Cor. 3.9). *Let D be a well-foliated domain (cf. Def. 3.3) with parameters $(r, L, \mathcal{I}_\beta, \beta_{\inf}, E)$ and let $\Lambda_2^N(X)$ be the infimum over all positive elements of the spectrum of the Laplacian on $X \in \{D, E\}$ subject to Neumann boundary conditions. Suppose $\Lambda_2^N(E)\epsilon^2$, r/ϵ, L/ϵ, $\mathcal{I}_\beta/\epsilon$ and $\beta_{\inf} \asymp 1$ are bounded away from 0 and ∞. Then there is an explicitly known constant $C \in \mathbb{R}$ depending only on D and $(r, L, \mathcal{I}_\beta, \beta_{\inf}, E)$ such that*

$$\Lambda_2^N(D) \geq C\epsilon^{-2}.$$

In a secondary step this then allows us to use a modified Whitney cover to obtain estimates on the spectral counting function $N_N(\Omega, t)$ of Neumann eigenvalues of snowflake-like domains Ω:

Theorem (cf. Thm. 4.15). *Let $\Omega \subset \mathbb{R}^n$ be uniformly well-covered (cf. Def. 4.1, 4.6) by $\{D_i^\epsilon\}_{i\in I_\epsilon}$ with $\#I_\epsilon \leq C\epsilon^{-\delta}$ for fixed C, δ. Then there is a constant C_M (given explicitly in*

Sec. 4.4) depending only on Ω *such that*

$$N_N(\Omega, t) \leq (2\pi)^{-n} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \operatorname{vol}_n(\Omega) t^{n/2} + C_M t^{\delta/2}.$$

In many cases this δ is given by the upper inner Minkowski dimension of the boundary.

Recent relatable results include a treatment of mixed boundary conditions by Dekkers, Hinz, Rozanova-Pierrat and Teplyaev in [24, 43] where, in particular for our purposes, Poincaré-Wirtinger inequalities are obtained for extension domains $\Omega \subset \mathbb{R}^n$. However the corresponding results are far from explicit as they rely on Rellich-Kondrachov embeddings $W^{1,p}(\Omega) \subset\subset L^{1,p}(\mathbb{R}^n)$. Therefore such results, while very general, cannot be used for our purpose and instead we employ a modification of a method introduced by Netrusov-Safarov in [80]. This is originally formulated for bounded variation domains with the crucial property that their boundary can locally be described as a graph of a sufficiently regular function. In this context we introduce a new framework of well-foliated domains (Def. 3.3) allowing us to generalise results of Netrusov and Safarov in particular to domains that cover an inner ϵ-parallel neighbourhood of snowflake-like domains and thus partially resolving a question about possible generalisations raised in the aforementioned work. This generalisation is achieved in Sec. 3.2 by introducing a new method of constructing paths from the interior of a domain heading to its boundary, foliating this region but leaving the Jacobian of the corresponding coordinate transformation sufficiently regular but potentially unbounded. This reflects the fact that domains with fractal boundary often have infinite circumference. The framework of well-foliated domains is designed to embed well into the concept of quasidisks (i.e. the quasiconformal image of a disk) based on a result by Rohde in [91]. There it is shown that any quasidisk is bi-Lipschitz equivalent to a specific kind of snowflake-like domain called Rohde snowflake. We show (Prop. 3.14) that the aforementioned method of constructing paths to the boundary applies to many Rohde snowflakes and thus making them well-foliated. Since any bi-Lipschitz image of a well-foliated domain is again well-foliated (Prop. 3.7), this allows a relatively general class of planar domains for which explicit results are obtainable. By a famous result of Jones in [49], any planar topological disk for which the Neumann Laplacian has empty essential spectrum is a quasidisk. In higher dimensions however, analogous statements are more involved.

Our approach on the asymptotic behaviour of the spectral counting function is based on a classical application of Dirichlet-Neumann bracketing using a modified Whitney cover (see the proof of Thm. 4.15). Directly finding an explicit lower bound for the first non-trivial eigenvalue of covering domains under Neumann boundary conditions allows us to avoid using non-constructive methods based on the compactness of embeddings of Sobolev spaces. It also allows a simple proof that the essential spectra of the Laplace operator subject to Neumann boundary conditions on well-covered domains (cf. Def. 4.1 and Cor. 4.18) is empty. Such explicit lower bounds for the first non-trivial Neumann eigenvalues of covering domains then put us in the position to formulate explicit upper bounds on remainder terms of spectral counting functions for snowflake-like domains. We include an explicit application to Koch snowflake-like domains in Sec. 4.3.1 and find that

$$N_N(K,t) \leq \frac{\mathrm{vol}_2(K)}{4\pi}\, t + 104326\, t^{\log_3 2},$$

where $N_N(K,t)$ denotes the number (with multiplicity) of Neumann eigenvalues $\leq t$ of the Koch snowflake K. To the author's knowledge this is the first explicit upper bound for the remainder term of the Neumann counting function of the classical Koch snowflake. This complements existing results on such asymptotic behaviour as found by Lapidus in [65]:

Theorem (Thm. 2.1 in [65])**.** *Let Ω be a domain and $N_D(\Omega,t)$ and $N_N(\Omega,t)$ be the counting function of Dirichlet, resp. Neumann eigenvalues on Ω. Suppose the upper Minkowski dimension of $\partial\Omega$ is given by $\delta \in (n-1,n)$. If Ω has upper inner Minkowski content, the remainder term of $N_D(\Omega,t)$ is $\mathcal{O}(t^{\delta/2})$. If the upper "full" Minkowski content exists then the remainder term of $N_N(\Omega,t)$ is $\mathcal{O}(t^{\delta/2})$.*

Besides making the above result explicit and allowing $\delta = n-1$, we are able to refine Lapidus' result by weakening the conditions under which such asymptotic upper bounds exist by showing that the existence of an upper inner Minkowski content is in fact sufficient versus the stronger condition of existence of the upper "full" Minkowski content.

Theorem (cf. Prop. 4.4, Thm. 4.7)**.** *Thm. 4.15 applies to uniformly well-covered domains with finite and positive upper inner Minkowski content.*

By an observation of Netrusov-Safarov, the result on asymptotic remainder terms of spectral counting functions is order-sharp in the sense that there are domains which have the growth behaviour obtained as upper bound (in fact such domains can even be Lipschitz domains). Finally it is worth mentioning an independent result by Gol'dshtein-Pchelintsev-Ukhlov in [34] on estimates of the first non-trivial Neumann eigenvalue in quasidisks. This approach directly uses a quasiconformal map which we avoid in our approach which does also not explicitly rely on quasiconformality.

Since remainder term estimates for Neumann counting functions hold true for Dirichlet counting functions, there are numerous possibilities to further analyse spectra of fractal objects. One option is to apply renewal theoretic ideas based on Lalley's work in [64] to so-called fractal sprays, i.e. a disjoint union of infinitely many images of a common domain (often called *gap*). This concept was introduced by Lapidus (see for example [70]) and analysed extensively therein. However the analysis was limited to fractal sprays where the gap itself had regular boundary. Based on results of Kombrink in [58, 59] it is possible to construct fractal sprays with several non-trivial asymptotic terms that correspond to geometrical information of the fractal spray as discussed in [61] by Kombrink and the author.

Structure. This work is structured as follows. In Ch. 2 we summarise several background results and concepts used throughout. In Sec. 2.1-2.2 we cover necessary functional analytic background focusing on the effect of Dirichlet and Neumann boundary conditions on the spectrum of the Laplacian on a domain with possibly fractal boundary. In Sec. 2.3 we then treat the necessary elements of fractal geometry, namely Hausdorff measure and dimension as well as Minkowski content and dimension. Three classical variants of construction of domains with fractal boundary are mentioned in Sec. 2.4 and in particular we define p-Rohde snowflakes. We finish the chapter constructing Whitney covers in Sec. 2.5 and finding bounds for the number of Whitney cubes necessary to cover a domain up to an inner ϵ-parallel neighbourhood and summarising known results on asymptotic behaviour of spectral counting functions of Laplacians in Sec. 2.6. In Ch. 3 we prove an explicit spectral gap (Sec. 3.1) and present the framework of well-foliated domains (Sec. 3.2). We show how p-Rohde snowflakes are included in this framework and discuss limiting cases and counter examples in Sec. 3.3. Ch. 4 aims at using the results from the previous chapter to obtain upper bounds for remainder terms of spectral counting functions. In Sec. 4.1 we then introduce the concept of well-covered domains as domains that allow a well-controlled cover by well-foliated domains and show that often p-Rohde snowflakes and similar domains fall into this class. We recover existing results by Netrusov and Safarov in Sec. 4.2 and finally show how well-covered domains allow upper bounds for remainder terms in Sec. 4.3. Ch. 5 is devoted to fractal sprays and in particular to domains consisting of infinitely many disjoint images of a common domain with fractal boundary. We focus on a class of examples constructed in Sec. 5.2 and combine renewal theoretic results with remainder term estimates for the Koch snowflake to obtain asymptotic expansions with several terms for such fractal sprays. We then compare the asymptotic behaviour of spectral counting functions with the asymptotic behaviour of the area of inner ϵ-parallel neighbourhoods. Finally we collect different possible directions of further research in Ch. 6.

Notation. We use the following definitions throughout this document.

- $B_r(x) := \{y : \|y - x\| < r\}$ denotes the open ball of radius r centered at x in a metric space.
- $0 \notin \mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $A \subset B$ includes equality.
- The cardinality of a set X is denoted by $\#X$.
- $\mathbb{R}^n$ is understood as Euclidean space.
- $\overline{X}$ is the closure of $X \subset \mathbb{R}^n$, $\mathrm{int}(X)$ is its interior.

- If the meaning is not affected, we sometimes omit brackets (for example $\operatorname{int} X = \operatorname{int}(X)$).
- The n-dimensional Lebesgue measure of a measurable set X is denoted by $\mathrm{vol}_n(X)$. For intervals I the Lebesgue measure is sometimes denoted by $|I|$.
- For any set of real valued functions from an $(n-1)$-dimensional cube Q_{n-1}, denoted by $X \subset \mathrm{Map}(Q_{n-1}, \mathbb{R})$ we define X-*domains* as the set of domains Ω for which every point $x \in \partial\Omega$ admits an open neighbourhood $U_x \subset \partial\Omega$ and an orthogonal transformation $O_x \in O(n)$ such that $O_x(U_x)$ is a graph of some $f \in X$. The topological dimension of these sets will always be clear from context. We then write $\Omega \in X(Q_{n-1})$ or $\Omega \in X(\mathbb{R}^{n-1})$ if a cube Q_{n-1} is not specified.
- $X \sqcup Y$ denotes the union of two disjoint sets X, Y.
- To simplify notation, we sometimes use $\ll$, $\gg$ and $\asymp$ with the following meaning: For two real-valued functions f, g with common domain of definition D we write $f \ll g$ if there exists a constant $c \in \mathbb{R}$ such that $f(x) \leq cg(x)$ holds for all $x \in D$. Moreover, we write $f \asymp g$ if $f \ll g$ and $f \gg g$.

Chapter 2

Background

We collect relevant results and concepts from functional and fractal analysis used throughout. We start by discussing standard results on the theory of Laplace operators and their spectra including Poincaré inequalities, Courant's Min-Max-Principle, and differences between the Laplace spectrum subject to Dirichlet and Neumann boundary conditions. It turns out that the spectrum of Laplace operators subject to Dirichlet boundary conditions is significantly simpler than under Neumann boundary conditions since the essential spectrum is always empty in the first case. We cover criteria that ensure emptiness of the essential spectrum under Neumann boundary conditions and discuss domain monotonicity. Moreover we define spectral counting functions as well as fractal dimensions and Whitney covers. The relevant functional analytic results stated here may be found for example in [22, 26, 51, 76, 89, 98, 111] whenever not explicitly referenced. References to the necessary fractal geometric background can be found in Falconer's works [27, 28].

2.1 The Laplace operator and its spectrum

Let $\Omega \subset \mathbb{R}^n$ be a domain, i. e. a bounded open connected subset of Euclidean $\mathbb{R}^n$ and denote its boundary by $\partial\Omega$. Let $H^1(\Omega) = W^{1,2}(\Omega)$ denote the usual Sobolev space, i. e. the set of all weakly differentiable $u \in L^2(\Omega)$ with a weak derivative $\nabla u \in L^2(\Omega)$. We equip $H^1(\Omega)$ with the usual inner product $(u, v)_{H^1(\Omega)} := (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$ with which $H^1(\Omega)$ becomes a Hilbert space. Further, $H_0^1(\Omega) \subset H^1(\Omega)$ denotes the closure of the space of compactly supported smooth functions in $H^1(\Omega)$. We consider the positive Laplacian $-\Delta := -\sum_{i=1}^n \partial_i^2$ on Ω and the corresponding eigenvalue equation of $-\Delta$ subject to Neumann (2.1) or Dirichlet (2.2) boundary conditions

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \qquad (2.1) \qquad\qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (2.2)$$

where $\mathbf{n}$ denotes the exterior normal to $\partial\Omega$ if $\partial\Omega$ is differentiable. Regardless of the regularity of $\partial\Omega$, the variational formulation, implied by Green's identity, of the problem (2.1) is stated as follows: Find $u \in H^1(\Omega)$ such that $(\nabla u, \nabla v)_{L^2(\Omega)} = \lambda(u, v)_{L^2(\Omega)}$ for all $v \in H^1(\Omega)$. Note that the space in which this problem is studied dictates the boundary condition; the variational formulation of (2.2) replaces $H^1(\Omega)$ with $H_0^1(\Omega)$: Find $u \in H_0^1(\Omega)$ such that $(\nabla u, \nabla v)_{L^2(\Omega)} = \lambda(u, v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$. Analogously, replacing $H^1(\Omega)$ with any other closed subspace V satisfying $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ gives rise to variational problems with more general boundary conditions. Whenever needed, we distinguish between different boundary conditions by an index D for Dirichlet boundary condition and N for Neumann boundary condition. In particular we write Δ_D and Δ_N for the Laplace operator with Dirichlet, resp. Neumann boundary condition.

Next, we cover necessary notation for the definition of a spectral counting function stressing the implications and differences between Dirichlet and Neumann boundary conditions. In this work, we focus on the Neumann case and to understand the difficulties that arise from fractal boundaries, it makes sense to discuss where the spectral differences to the Dirichlet case lie. Recall that an eigenvalue of a bounded self-adjoint operator $T : X \to X$ on a Hilbert space X is a $\lambda \in \mathbb{C}$ for which $T - \lambda \operatorname{id}_X$ is not injective.

Definition 2.1 (Spectrum of a bounded operator)**.** Let $T : X \to X$ be a bounded self-adjoint operator on a Hilbert space. Then we define the *spectrum of* T as $\sigma(T) := \{\lambda \in \mathbb{C} : (T - \lambda \operatorname{id}_X)$ does not have a bounded inverse$\}$. We decompose $\sigma(T) = \sigma_d(T) \sqcup \sigma_{\text{ess}}(T)$ into the *discrete spectrum* and the *essential spectrum*: The *essential spectrum* $\sigma_{\text{ess}}(T)$ is defined as the set of all $\lambda \in \sigma(T)$ for which $\ker(T - \lambda \operatorname{id}_X)$ or $\operatorname{coker}(T - \lambda \operatorname{id}_X)$ is infinite dimensional. It turns out (see for example the spectral characterisation by Reed-Simon in [89, VII.3]) that indeed $\sigma_d(T)$ is discrete and only contains eigenvalues of finite multiplicity.

A classical result of early functional analysis is the characterisation of the spectrum of $-\Delta$ under Dirichlet boundary conditions. The bilinear Dirichlet form $\mathcal{E} : H^1(\Omega) \times H^1(\Omega), (f, g) \mapsto \int_\Omega \nabla f \nabla g dx =: (\nabla f, \nabla g)_{L^2(\Omega)}$ is bounded, i.e. $|\mathcal{E}(f, g)| \leq \|f\|\|g\|$, by Cauchy-Schwartz. Poincaré inequalities (see below) then show that $\mathcal{E}$ is also coercive, i.e. $\exists m$ (namely $m = \frac{1}{1+C^2}$) such that $\mathcal{E}(f, f) \geq m\|f\|^2$.

Theorem 2.2 (Poincaré inequality for $H_0^1(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be any domain and $1 \leq p < \infty$. Then there is C (called* Poincaré constant*) such that for any $f \in H_0^1(\Omega)$ one has $\|f\|_{L^p(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)}$.*

The following is essentially a corollary of Riesz' representation theorem.

Theorem 2.3 (Lax-Milgram)**.** *Let B be a bounded and coercive bilinear form on a Hilbert space H. Then for any $g \in H'$ there is a unique f with $B(f, \varphi) = g(\varphi)$ for all $\varphi \in H$. This map $H' \to H, g \mapsto f$ is bounded has a bounded inverse.*

Now for any $g \in L^2(\Omega)$, the operator $(\varphi \mapsto (g, \varphi)_{L^2(\Omega)})$ is $\in H_0^1(\Omega)'$ so that there is a unique $f \in H_0^1(\Omega)$ with $\mathcal{E}(f, \varphi) = (g, \varphi)_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$. We denote this map as $-\Delta^{-1} : L^2(\Omega) \to H_0^1(\Omega), g \mapsto f$. By Green's identity this map is formally self-adjoint. In many cases, one can deduce the discreteness of the spectrum of Δ as described above by the spectral theorem. In the Dirichlet case this is achieved by Sobolev embedding theorems implying that $\iota : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact embedding. In particular this shows $\iota\Delta^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a compact self-adjoint operator.

Theorem 2.4 (Spectral theorem for compact operators)**.** *Let $K : H \to H$ be a compact self-adjoint operator on a Hilbert space H. Then, up to eigenvalue 0, the operator K has bounded discrete spectrum $\sigma(K) = \sigma_d(K)$ consisting of a null sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with finite multiplicity and only accumulation point 0.*

With the definition of the Rayleigh quotient one finds a formal expression for the n^{th} Laplace eigenvalue as follows:

Theorem 2.5 (Max-Min and Min-Max-Principle for Dirichlet and Neumann Laplacian). *Let Ω be an arbitrary fixed domain. Define the Rayleigh quotient $\rho(f) := \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2}$ and write $\Lambda_n^B(\Omega) := \min\{\inf \sigma_{\text{ess}}(-\Delta_B(\Omega)), \lambda_n^B(\Omega)\}$ for a boundary condition $B \in \{N, D\}$ (i.e. Neumann or Dirichlet) where $\lambda_n^B(\Omega)$ denotes the n^{th} entry (counted with multiplicity) in the discrete spectrum of $-\Delta_B$ defined on Ω. Then*

$$\begin{aligned}
\Lambda_n^D(\Omega) &= \sup_{\psi_1,\cdots,\psi_{n-1}} \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \{\rho(\psi) : \psi \perp \psi_1, \cdots, \psi_{n-1}\} \\
&= \inf_{\substack{W \in H_0^1(\Omega): \\ \dim(W)=n}} \sup_{w \in W} \rho(w). \\
\Lambda_n^N(\Omega) &= \sup_{\psi_1,\cdots,\psi_{n-1}} \inf_{\psi \in H^1(\Omega)\setminus\{0\}} \{\rho(\psi) : \psi \perp \psi_1, \cdots, \psi_{n-1}\} \\
&= \inf_{\substack{W \in H^1(\Omega): \\ \dim(W)=n}} \sup_{w \in W} \rho(w).
\end{aligned}$$

In particular, since the constant function trivially minimises the Rayleigh quotient in the Neumann case,

$$\Lambda_2^N(\Omega) = \inf_{\substack{v \in H^1(\Omega)\setminus\{0\}: \\ \int_\Omega v dx = 0}} \frac{\|\nabla v\|_{L^2(\Omega)}^2}{\|v\|_{L^2(\Omega)}^2}. \tag{2.3}$$

The following corollary then explains why Poincaré inequalities (and Poincaré-Wirtinger inequalities as presented in Thm. 2.8) are sometimes referred to as spectral gap theorems:

Corollary 2.6 (Spectral gap). *Let Ω be an arbitrary domain. Writing $1^\perp$ for the set of functions on Ω with vanishing integral,*

$$\Lambda_2^N(\Omega) \geq L^{-1} \Leftrightarrow \left(\|u\|_{L^2(\Omega)}^2 \leq L \|\nabla u\|_{L^2(\Omega)}^2 \quad \forall u \in H^1(\Omega) \cap 1^\perp \right).$$

Under both, Dirichlet and Neumann boundary conditions, the spectrum of the corresponding Laplacian Δ_D, resp. Δ_N is well-known to be non-negative.

The Dirichlet case. Since any eigenfunction is necessarily non-zero, $0 < \lambda_1^D(\Omega)$ on any domain Ω. The Poincaré inequality for $H_0^1(\Omega)$ for an arbitrary domain Ω implies that the spectrum of the Dirichlet Laplacian is a discrete set $\lambda_1^D(\Omega) \leq \lambda_2^D(\Omega) \cdots$ with finite multiplicity which has a single accumulation point at ∞. In other words, $\sigma_{\text{ess}}(\Delta_D) = \emptyset$.

The Neumann case. In sharp contrast to the Dirichlet case, for any domain Ω and therefore independent of geometrical properties, one has $\lambda_1^N(\Omega) = 0$. Secondly the essential spectrum of the Neumann Laplacian on a domain Ω is harder to control. As sketched above, the existence of a Poincaré inequality paired with a suitable Sobolev embedding theorem implies the discreteness of the spectrum of the Laplace operator on a given domain. Focusing on explicit estimates of eigenvalues subject to Neumann boundary conditions on domains with irregular boundary, this leaves us with two problems: Find a suitable formulation of a Poincaré inequality for $H^1(\Omega)$ and explain how a spectral theorem can be applied to an inverse Neumann Laplacian $\Delta_N^{-1} : L^2(\Omega) \to H^1(\Omega)$. To the first problem, variants of the Poincaré inequality exist that remain valid on $H^1(\Omega)$ for certain classes of domains. Often formulated for Lipschitz domains $\Omega \in \text{Lip}(\mathbb{R}^n)$ or C^1-domains, more general versions of Thm. 2.8 are also known for continuous domains, i.e. $\Omega \in C^0(\mathbb{R}^n)$ as presented for example by Burenkov in [14, 4.3-4.4]. We will discuss more general versions after defining extension domains in Def. 2.13. From Ch. 3 onwards we focus on domains with fractal boundary which are not necessarily locally graphs. For the second problem we will define classes of domains (and in particular quasidisks) that allow a compact embedding analogous to the case of Dirichlet boundary conditions.

Definition 2.7. A domain $X \subset \mathbb{R}^n$ satisfies the *cone condition* if there is a height $h > 0$ and opening angle $\alpha \in (0, \pi/2)$ such that for any point $x \in X$ there is a cone $\text{Cone}_{h,\alpha}(x) \subset X$ of height h and opening angle α with cusp x.

Theorem 2.8 (Poincaré-Wirtinger inequality for Lipschitz domains or domains with cone condition, cf. [15, 24], Thm. 5.4 in [2])**.** *Let $1 \leq p \leq \infty$ and $\Omega \in \mathbb{R}^n$ be a $\text{Lip}(\mathbb{R}^n)$-domain or a domain with cone property. Then there is a C depending only on Ω such that for any*

$u \in H^1(\Omega)$

$$\left\| u - \frac{1}{\operatorname{vol}_n \Omega} \int_\Omega u dx \right\|_{L^p(\Omega)} \leq C \left\| \nabla u \right\|_{L^p(\Omega)}.$$

The constant C *will be called* Poincaré-Sobolev constant.

Remark 2.9. Under certain variants of definitions of Lipschitz domains and cone conditions, cone condition domains and Lipschitz domains coincide. See for example Def. 2.4.1; 2.4.5 and Thm. 2.4.7 in [42] by Henrot-Pierre. In our definition an open slit unit square $(0,1)^2 \setminus (\{1/2\} \times (0,1/2))$ has a cone property but is not Lipschitz while every Lipschitz domain also satisfies a cone condition as shown for example by Burenkov in [14, 4.3].

Results such as Thm. 2.8 are restricted by the regularity conditions on $\partial\Omega$. This restriction is fundamentally unavoidable as can be shown by the following construction based on an example of Courant-Hilbert in [21, Ch. VI, §2.6], cf. also the variants by Cardone-Khrabustovskyi and Hempel-Seco-Simon in [16, 41].

Example 2.10. There are domains in which Thm. 2.8 is false. In other words, there are domains D with $\inf \sigma_{\mathrm{ess}}(-\Delta_N(D)) = 0$. To see this, consider the following example of a "room with infinitely many narrow passages" as shown in Fig. 2.1. Let $1 > \epsilon > 0$ be sufficiently small and consider the domain D given by a unit square Q and a family of squares Q_k with parallel sides and side lengths $(\epsilon^k)_{k\in\mathbb{N}}$ attached to Q by narrow passages of width $\eta_k = \epsilon^{4k}$ and length ϵ^k. Now for every $m \in \mathbb{N}$ define ψ_m on D piecewise as follows: $\psi_m|_{Q_m} = \epsilon^{-k}$, $\psi_m|_Q = c_m$ and ψ_m changes linearly in the passage between Q and Q_m with c_m fixed such that $\int_D \psi_m dx = 0$.[1] Then $\psi_m \in H^1(D)$ and $\psi_m \perp 1$ for all m. But since $\int_{Q_m} |\psi_m|^2 dx = 1$ for all m, $c_m \to 0$ as $\epsilon^m \to 0$ and $\lim_{m\to\infty} \|\nabla\psi\|^2_{L^2(D)} \epsilon^{3m}/\eta_m = 1$. Therefore $\lim_{m\to\infty} \|\nabla\psi_m\|^2_{L^2} = 0$ and $\lim_{m\to\infty} \|\psi_m\|^2_{L^2} = 1$.

In particular this shows that in difference to the Dirichlet spectrum, the Neumann spectrum of a domain Ω does not allow for estimates under general transformations of the domain. Indeed the spectrum of Neumann Laplacians is hardly controlled in the general

[1] Of course one can drop the condition of $\int_D \psi_m dx = 0$ and instead consider $\psi_m - \frac{1}{\operatorname{vol}_n D} \int_D \psi_m dx$ as done in Thm. 2.8.

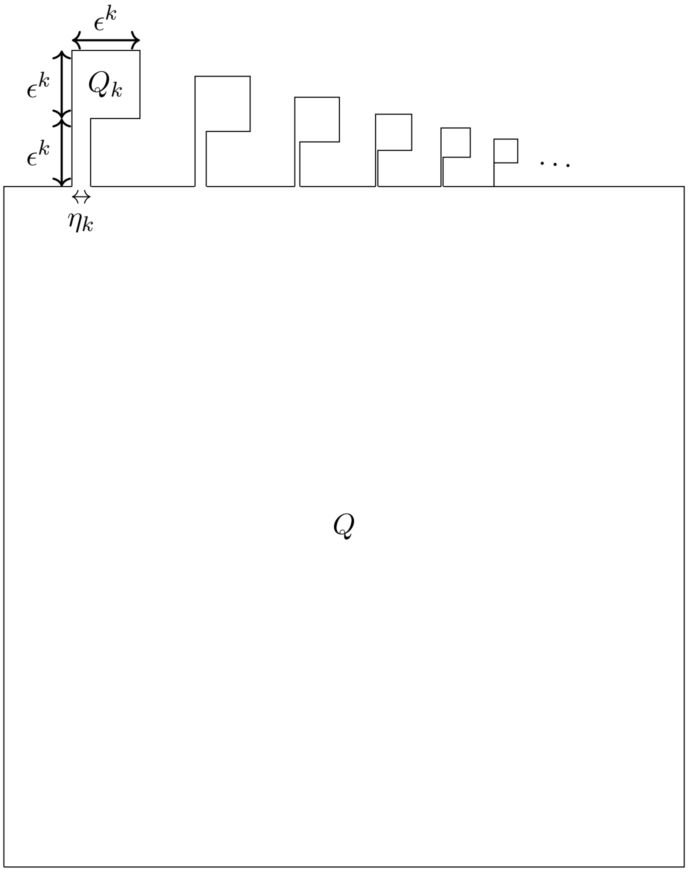


Figure 2.1: The domain D as constructed in Ex. 2.10 whose essential spectrum of the Neumann Laplacian reaches 0, $\inf \sigma_{\text{ess}}(-\Delta_N) = 0$. For better visualisation, the width η_k is not to scale. Since the size of each attached small "room with narrow passage" is scaled down by a factor of ϵ and $\eta_k = \epsilon^{4k}$, the total measure necessary to accommodate $\{Q_k\}_{k\in\mathbb{N}}$ is finite.

case as was shown in the following result by de Verdière and Hempel-Seco-Simon:

Proposition 2.11 ([19, 41])**.** *Any closed subset of $\mathbb{R}_{\geq 0}$ is the essential spectrum of the Neumann Laplacian on a planar domain. Any set of N positive values coincides with the first N non-trivial Neumann eigenvalues on a planar domain.*

Several criteria have been found which ensure the emptiness of the essential spectrum. As motivated above, one has the following criterion based on compact embeddings originally formulated by Rellich.

Theorem 2.12 ([90], see also Prop. 10.6 in [98])**.** *The Neumann spectrum is discrete iff* *the inclusion* $\iota : H^1(\Omega) \hookrightarrow L^2(\Omega)$ *is compact.*[2]

In terms of capacities, Maz'ya showed in [75] that ι is compact iff

$$\lim_{M\to 0} \inf_{t\in(0,M)} t^{-1} \inf_{\substack{\mathrm{vol}_n(F_1)\geq t\\ \mathrm{vol}_n(F_2)\leq M}} c(F_1,F_2) = 0.$$

Here, $c(F_1,F_2) := \inf_{f\in V(F_1,F_2)} \int_\Omega |\nabla f|^2 dx$ is called *relative capacity* where $V(F_1,F_2) := \{f \in C^\infty(\Omega) : f|_{F_1} = 1, f|_{F_2} = 0\}$ for any two disjoint $F_1, F_2 \subset \Omega$ that are closed in Ω giving possible descriptions of such domains.

In contrast, Dirichlet boundary conditions allow for a simple but crucial observation: Any function $f \in H^1_0(\Omega)$ can be extended by 0 to any superset of Ω without altering its H^1-norm. We finish this section by collecting some results relating the geometry of the boundary to vanishing essential spectra.

Definition 2.13. A domain $\Omega \subset \mathbb{R}^n$ is called *extension domain* if there is a bounded linear extension operator $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$.[3]

For any such extension domain the embedding $H^1(\Omega) \overset{E}{\to} H^1(\mathbb{R}^n) \to H^1(B_r(0)) \to L^2(B_r(0)) \to L^2(\Omega)$ is compact with $r = \operatorname{diam}\Omega$. In recent years results on compact embeddings where formulated in a vast variety of settings. In particular in [35] Gol'dshtein-Ramm proved a compact embedding if the domain can be approximated from above and below by Lipschitz functions. While it is known that a Poincaré-Wirtinger inequality exists for extension domains, as shown for example by Arfi-Rozanova-Pierrat and Dekkers-Rozanova-Pierrat-Teplyaev in [6, 24], the existence of such Poincaré-Sobolev constants is commonly shown in a non-constructive way by contradiction using the compactness of embedding operators.

An early breakthrough discovery regarding extension domains was the following observation by Jones and Vodop'yanov-Gol'dshtein-Latfullin: Planar extension domains are characterised as images of disks under quasiconformal maps.

[2] The original German publication uses the concept of completely continuous operators („vollstetige Operatoren“). Whenever the domain is a Hilbert space, this notion coincides with the usual notion of a compact operator.

[3] Sometimes one instead requests existence of an extension for all $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

Theorem 2.14 ([49, 109]). *Any planar connected and simply connected extension domain $\Omega \subset \mathbb{R}^2$ is a* quasidisk*, i.e. the image of the unit disk under a quasiconformal map. In higher dimensions, any domain $D \subset \mathbb{R}^n$ is an extension domain if it is an* (ϵ, ∞)-domain*, i.e. if for any two points $x, y \in \Omega$ there is a rectifiable curve c in Ω connecting x and y of length $\ell(c) \leq \frac{|x-y|}{\epsilon}$ and for all $z \in c$ one has $\operatorname{dist}(z, \partial\Omega) \geq \epsilon \frac{|x-z||y-z|}{|x-y|}$. In particular, quasidisks are always extension domains.*

Here, quasiconformal maps are defined as homeomorphisms with bounded distortion. Precisely, at any point x in a domain $X \subset \mathbb{R}^n$ we define the distortion of a homeomorphism $f : X \to \mathbb{R}^n$ as

$$H(x, f) := \limsup_{r \searrow 0} \frac{\max_{|h|=r} |f(x+h) - f(x)|}{\min_{|h|=r} |f(x+h) - f(x)|}.$$

Notice that $|f(x+h) - f(x)|$ is defined for all $|h| < r$ with $0 < r < \operatorname{dist}(x, \partial X)$ so that $H(x, f)$ exists for all $x \in X$.

Definition 2.15 (Quasiconformal map, [33]). Let $X, Y \subset \mathbb{R}^n$ be two domains. Then a map $f : X \to Y$ is called *quasiconformal* if it is homeomorphic and its distortion is bounded from above everywhere in X, i.e. if $\sup_{x \in X} H(x, f) < \infty$.

Jones correctly suspected that in higher dimensions quasidisks would no longer exhaust the set of extension domains, cf. Rem. 2.32. The extensive literature on quasiconformal maps in turn reveals simple criteria for quasidisks. We mention two well-known results by Ahlfors (Prop. 2.16) and Jones (Prop. 2.18).

Proposition 2.16 (Ahlfors criterion, [3, 4, 76]). *A Jordan curve γ is quasicircular (i.e. the image of S^1 under a quasiconformal map) iff the following is satisfied: for any two points $a, b \in \gamma$ one has* $\operatorname{diam} \Gamma \leq c|a - b|$*, where $\Gamma \subset \gamma \backslash \{a, b\}$ is a connected component with smallest diameter.*

As an application the Koch snowflake can be shown to be a quasidisk and therefore to have vanishing essential Neumann spectrum.

Example 2.17. Let K be the Koch snowflake (cf. Sec. 2.4). Then ∂K is a quasicircle based on the Ahlfors' criterion. To see this let $x, y \in \partial K$ be arbitrary. Stressing the self-similarity of the Koch snowflake, we may assume without loss of generality that x lies in the leftmost or second leftmost segment (red and green in Fig. 2.2) and y lies in one of the three other segments. If instead both x and y lie in the same segment, there is a largest scaled copy of the Koch curve within that segment containing only one of the two so that the next larger copy of the Koch curve within the segment contains both x, y in different sub-cylinders; if no such segment existed, both x, y would coincide, which is trivial. We now discuss these five cases individually.

(i). If x lies in the red segment and y lies in the blue segment, we find the smallest circle covering all of the red and all of the blue segment. Let the radius of this circle be $r_{\text{blue}} := r_b$. Let $\text{dist}(\text{red}, \text{blue}) := d_{rb}$. We define $c_{rb} := \frac{2r_b}{d_{rb}}$.

(ii). If x lies in the red segment and y lies in the orange segment, we find the smallest circle covering all of the red and all of the orange segment. Let the radius of this circle be $r_{\text{orange}} := r_o$. Let $\text{dist}(\text{red}, \text{orange}) := d_{ro}$. We define $c_{ro} := \frac{2r_o}{d_{ro}}$.

(iii). If x lies in the red segment and y lies in the green segment, we may regard the red and green segments as boundary of another Koch snowflake (indicated in gray in Fig. 2.2). This reduces the case again to one of the three other cases.

(iv). If x lies in the green segment and y lies in the blue segment, we differ between two cases.

(a) If at least one of the two points lies outside the pale-red part of the union of green and blue, both points are separated and we again find the smallest circle covering all of the green and all of the blue segment. Let its radius be r_{gb+}. Let $\text{dist}(\text{green but not pale-red}, \text{blue}) = d_{rb+}$. We define $c_{rb+} := \frac{2r_{gb+}}{d_{gb+}}$.

(b) If both points lie in the pale-red segment, one may start over with the above classification.

It remains to explain what happens if the above classification does not terminate. This can only occur the case (iv)b is called infinitely often. But then $x = y$ which is trivial. Therefore $c := \max\{c_{rb}, c_{ro}, c_{gb+}\}$ satisfies the Ahlfors criterion.

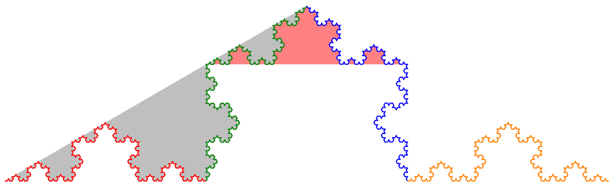


Figure 2.2: Koch snowflake dissected into disjoint segments to show that the Koch snowflake is a quasicircle by the Ahlfors criterion as described in Ex. 2.17. This argument is based on the idea that at sufficiently small scale any two points will eventually lie in distinct rescaled segments as shown above or the two points coincide.

Proposition 2.18 ([49])**.** *Let $\gamma \subset \mathbb{R}^n$ be a quasicircle. Then $\mathrm{vol}_n(\gamma) = 0$.*

Finally, Rohde gave a complete classification of quasicircles up to bi-Lipschitz maps. One goal of Ch. 3 is to provide explicit Poincaré-Sobolev constants for domains covering ϵ-parallel neighbourhoods of quasicircles up to bi-Lipschitz maps. To formulate the classification it is necessary to define *p-Rohde snowflakes* which is done in Def. 2.33. In contrast to more general mapping theorems that guarantee the existence of a, say bi-holomorphic, map $\Omega \to \Omega'$ for sufficiently regular domains, we will see in Prop. 2.30 that bi-Lipschitz maps preserve the Minkowski dimension of the boundary.

Theorem 2.19 ([91])**.** *Let Ω be a quasidisk. Then there is a global bi-Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps Ω to a p-Rohde snowflake with $p \in \left[\frac{1}{4}, \frac{1}{2}\right)$.*

Independent of this work and using a different techniques Gol'dshtein-Pchelintsev-Ukhlov recently found estimates for Poincaré-Sobolev constants for p-Rohde snowflakes in [34] based on quasiconformal geometrical information. In 2021 an approach to domains with locally self-similar boundary was developed by Banjai-Boulton in [8].

2.2 Dirichlet-Neumann bracketing

We now present the technique known as *Dirichlet-Neumann bracketing* which allows treating a domain as a union of much simpler covering domains. To this end with the spectral counting function we now define spectral counting functions as crucial object in this work.

Definition 2.20 (Spectral counting function)**.** For a domain Ω and boundary condition $B \in \{D, N\}$ with $\sigma_{\text{ess}}(-\Delta_B(\Omega)) = \emptyset$, we define the *spectral counting function* (also called eigenvalue counting function) $t \mapsto N_B(\Omega, t)$ as the number of eigenvalues less or equal to $t \in \mathbb{R}$ (with multiplicity) of $-\Delta_B$ on Ω. $N_N(\Omega, t)$ will often be called *Neumann counting function* and analogously $N_D(\Omega, t)$ will often be called *Dirichlet counting function*.

Thm. 2.5 implies a number of regularity results of Dirichlet and Neumann eigenvalues of the Laplace operator. In particular it provides an estimate of eigenvalues that can be applied to open covers:

(i). Assuming $\sigma(-\Delta_N(\Omega)) = \emptyset$ one has $\lambda_k^N \leq \lambda_k^D$ $\forall k \in \mathbb{N}$ because $H_0^1(\Omega)$ is isometrically embedded in $H^1(\Omega)$ by extension with 0. In terms of the spectral counting functions this may be written as $N_D(\Omega, t) \leq N_N(\Omega, t)$ $\forall t$. In fact more is true: By the Min-Max principle in Thm. 2.5, λ_k^N is minimal amongst all k^{th} eigenvalues for mixed boundary conditions corresponding to some closed V with $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ as long as the essential spectrum is empty.

(ii). (Domain monotonicity). Let $\Omega' \subset \Omega$ be two domains. Then $\lambda_n^D(\Omega') \geq \lambda_n^D(\Omega)$, since $H_0^1(\Omega')$ is isometrically embedded in $H_0^1(\Omega)$ be extending with 0. Contrary to common belief and occasional statements like [50, Remark to Cor. 8.5.2][4], there is no analogous domain monotonicity for Neumann eigenvalues. See Rem. 2.21 for an example.

(iii). Let $\Omega_1, \Omega_2 \subset \Omega$ be two disjoint subdomains on which the Neumann Laplacian has empty essential spectra such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$. Then $\lambda_n^D(\Omega) \leq \min(\lambda_n^D(\Omega_1), \lambda_n^D(\Omega_2))$ and $\lambda_n^N(\Omega) \geq \max(\lambda_n^N(\Omega_1), \lambda_n^N(\Omega_2))$. This follows from the maps $H_0^1(\Omega_1) \oplus H_0^1(\Omega_2) \to$

[4]This was corrected in the 2nd and 3rd edition.

$H_0^1(\Omega)$ via $(f_1, f_2) \mapsto \left(\Omega \ni x \mapsto \begin{cases} f_1 & \text{if } x \in \Omega_1 \\ f_2 & \text{if } x \in \Omega_2 \end{cases} \right)$ and $H^1(\Omega) \to H^1(\Omega_1) \oplus H^1(\Omega_2)$ via $f \mapsto (f|_{\Omega_1}, f|_{\Omega_2})$.

(iv). Writing $\alpha\Omega := \{x \in \mathbb{R}^n : \alpha^{-1}x \in \Omega\}$ for $\alpha > 0$, one has $N_B(\alpha\Omega, t) = N_B(\Omega, \alpha^2 t)$ for $B \in \{D, N\}$. This is because of an isomorphism $(\alpha^{-1})^* : H^1(\Omega) \to H^1(\alpha\Omega), u \mapsto u \circ \alpha^{-1}$ and equally $(\alpha^{-1})^* : H_0^1(\Omega) \to H_0^1(\alpha\Omega)$.

Remark 2.21. As shown by Ni-Wang in [82], even for convex planar domains there is no analogous statement about domain monotonicity as in (ii) above for the Neumann case. Consider for example the domains $\Omega', \Omega'' \subset \Omega$ as shown in Fig. 2.3. Then the first non-trivial Neumann eigenvalues is given by $\frac{\pi^2}{L}$ where L is the longer side of each rectangle. Therefore $d'' < d < d'$ implies

$$\lambda_2^N(\Omega') \leq \lambda_2^N(\Omega) \leq \lambda_2^N(\Omega'').$$

A second example for smooth domains is given by nested annuli. For $r < R$ let $A_{r,R}$ be the open annulus with inner radius r and outer radius R. Then

$$\begin{aligned} \lambda_2^N(A_{a,1}) &> \lambda_2^N(A_{b,1}) \quad \text{while} \quad A_{a,1} \supset A_{b,1} \quad \text{if } a < b < 1, \\ \lambda_2^N(A_{1,a}) &> \lambda_2^N(A_{1,b}) \quad \text{while} \quad A_{1,a} \subset A_{1,b} \quad \text{if } 1 < a < b. \end{aligned}$$

In spite of the observation from Rem. 2.21, Funano very recently found a variant of domain monotonicity for convex domains in [31]: (Piecewise) smooth convex domains satisfy domain monotonicity for Neumann eigenvalues up to a constant only depending on the ambient dimension. More precisely:

Theorem 2.22 ([31])**.** *Let $\Omega' \supset \Omega$ be convex domains in $\mathbb{R}^n$ with piecewise smooth boundary. Then $\lambda_k^N(\Omega') \leq (92n)^2\, \lambda_k^N(\Omega)$ for all $k \in \mathbb{N}$.*

Definition 2.23 (Volume cover and multiplicity of a cover)**.** For any domain $\Omega \subset \mathbb{R}^n$ a *volume cover* $\{\Omega_i\}_{i \in I}$ of Ω consists of at most countably many open sets $\Omega_i \subset \Omega$ with

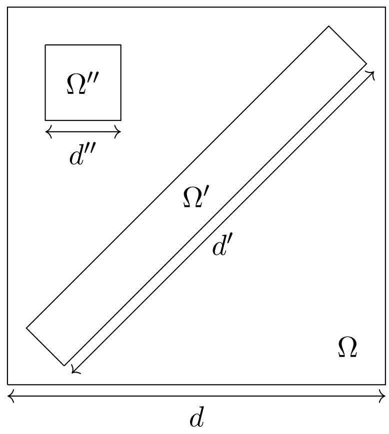


Figure 2.3: Example of the violation of domain monotonicity of Neumann eigenvalues.

$\mathrm{vol}_n(\Omega) = \mathrm{vol}_n\left(\bigcup_{i\in I}\Omega_i\right)$. For any domain Ω and any cover or volume cover $\{\Omega_i\}_{i\in I}$ we define its *multiplicity* by $\mu(\{\Omega_i\}_{i\in I}) := \sup_{x\in\Omega}\#(j\in I : x\in\Omega_j)$. It is not to be confused with the multiplicity of eigenvalues.

Next and in addition to the classical Dirichlet-Neumann bracketing linking the spectral counting functions of Dirichlet and Neumann eigenvalues, we mention a version that allows for non-disjoint covers as long as the elements of the cover do not intersect too often. More precisely one has the following result which also follows from the Min-Max-Principle.

Proposition 2.24 (Dirichlet-Neumann bracketing with multiplicity, [80])**.** *Let $\{\Omega_i\}_{i\in I}$ be a volume cover of a domain Ω. If the elements of the cover are pairwise disjoint,*

$$\sum_{i\in I} N_D(\Omega_i,t) \le N_D(\Omega,t) \le N_N(\Omega,t) \le \sum_{i\in I} N_N(\Omega_i,t)$$

If the volume cover has finite multiplicity μ, then

$$N_N(\Omega,t) \le \sum_{i\in I} N_N(\Omega_i,\mu t).$$

An immediate corollary relates the difference of Neumann and Dirichlet counting functions with the difference of Dirichlet counting functions and Dirichlet counting functions on a cover.

Corollary 2.25. *Let $\{\Omega_n\}$ be a finite volume cover of a domain Ω. We define $Q(\Omega', t) := N_N(\Omega', t) - N_D(\Omega', t)$ for any bounded open set Ω' and $R_D(t) := N_D(\Omega, t) - \sum_{n\in\mathbb{N}} N_D(\Omega_n, t)$. Then*

$$R_D(t) \leq -Q(\Omega, t) + \sum_{n\in\mathbb{N}} Q(\Omega_n, t).$$

Proof. Prop. 2.24 shows that $Q(\Omega', t) \geq 0$ and $N_N(\Omega, t) - \sum_n N_N(\Omega_n, t) \leq 0$. Moreover,

$$\begin{aligned} R_D(t) &= N_D(\Omega, t) - \sum_n N_D(\Omega_n, t) = N_N(\Omega, t) - \sum_n N_D(\Omega_n, t) - Q(\Omega, t) \\ &\leq \sum_n N_N(\Omega_n, t) - \sum_n N_D(\Omega_n, t) - Q(\Omega, t) = \sum_n Q(\Omega_n, t) - Q(\Omega, t). \end{aligned}$$ □

Therefore any estimate of $N_N(\Omega, t)$ and $N_D(\Omega, t)$ with the same asymptotic remainder term yields a asymptotic estimate of $R_D(t)$ by the same asymptotic behaviour as the remainder of the spectral counting functions; for certain bounded variation domains this was done by Safarov-Filonov in [29]. This result shows that the error committed by erroneously assuming a cover of a domain to be disjoint will not change the asymptotic behaviour as long as Neumann and Dirichlet counting functions have a common upper bound.

2.3 Dimensions and contents

We cover the relevant notions of dimensions briefly. The algebraic notion of a dimension for example as dimension of the tangent space to, say a manifold, is of very limited use in the context of fractal geometry. Instead scaling properties of volumes of ϵ-parallel neighbourhoods are used. Let X be any non-empty open bounded subset of $\mathbb{R}^n$ with boundary ∂X. We write the *(inner) ϵ-parallel neighbourhood* as

$$X_\epsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, X) < \epsilon\} \text{ and}$$
$$X_{-\epsilon} := \{x \in X : \operatorname{dist}(x, \partial X) < \epsilon\} = (\partial X)_\epsilon \cap X.$$

We define Hausdorff measure and dimension. Based on this, in a second step we define a upper and lower Minkowski dimension and content.

Definition 2.26. Let $X \subset \mathbb{R}^n$ be bounded. Then we define

$$\mathcal{H}_s^\delta(X) := \inf_{\{U_i\}_{i\in I}} \sum_{i\in I} \operatorname{diam}(U_i)^s,$$

where the infimum is taken over all finite open covers $\{U_i\}_{i\in I}$ of X with $\operatorname{diam} U_i \leq \delta \; \forall i \in I$. As any cover with smaller diameters will also be a cover of larger diameters, the limit $\delta \to 0$ exists. We define the *s-Hausdorff measure* as

$$\mathcal{H}_s(X) := \lim_{\delta\to 0} \mathcal{H}_s^\delta(X).$$

Since for $d' \leq d$ and sufficiently small δ, $\sum_{i\in I} \operatorname{diam}(U_i)^d \leq \sum_{i\in I} \operatorname{diam}(U_i)^{d'} \delta^{d-d'}$. This shows that in the limit $\delta \to 0$, $\mathcal{H}_{d'}(X) < \infty \Rightarrow \mathcal{H}_d(X) = 0$ and $\mathcal{H}_d(X) > 0 \Rightarrow \mathcal{H}_{d'}(X) = \infty$. Therefore there is a unique jump from in the values of $H_s(X)$ from ∞ to 0 as s increases.

Definition 2.27. The *Hausdorff dimension* of a bounded set $X \subset \mathbb{R}^n$ is defined as $\dim_H(X) := \inf\{\delta : \mathcal{H}_\delta(X) = 0\}$.

Notice that the Hausdorff dimension coincides with the usual concept of a dimension for smooth submanifolds. Moreover one may define a generalised version of the "manifold dimension" that can be applied to non-differentiable objects.

Definition 2.28 (Topological dimension)**.** Let $X \subset \mathbb{R}^n$. Then we define the *topological dimension* as $\dim_{\mathrm{Top}}(X) := \inf\{\dim_H X' : X' \text{ is homeomorphic to } X\}$.

With the definition of the Hausdorff dimension in mind, we now define the Minkowski content and corresponding dimension similarly. Often the Minkowski dimension is defined differently as scaling power of the remainder term in the expansion of $\operatorname{vol}_n(X_\epsilon)$. For later results (cf. Prop. 4.4) the following approach appears better suited.

Definition 2.29. Let $X, A \subset \mathbb{R}^n$ be bounded. Then we define the *ϵ-content at scaling dimension s relative to A, $M_s^\epsilon(X, A)$* as $M_s^\epsilon(X, A) := \operatorname{vol}_n(X_\epsilon \cap A)\epsilon^{-(n-s)}$ and define the upper/lower Minkowski content at scaling dimension s relative to A as

$$\overline{M_s}(X,A) := \limsup_{\epsilon \searrow 0} M_s^\epsilon(X,A) \text{ and } \underline{M_s}(X,A) := \liminf_{\epsilon \searrow 0} M_s^\epsilon(X,A)$$

For a domain $\Omega \subset \mathbb{R}^n$, we then define the *upper inner Minkowski content of* Ω *at dimension* s as $M_s(\partial\Omega,\Omega) := \overline{M_s}(\partial\Omega,\Omega)$ whenever this is finite. Likewise one may define a lower inner Minkowski content as scaling dimension s. Whenever $M_s(\partial\Omega,\Omega) \in (0,\infty)$, we say that Ω is *upper inner Minkowski measurable at dimension* s. We say that the domain is *Minkowski measurable* if it is upper and lower Minkowski measurable and both contents relative to the ambient space $A = \mathbb{R}^n$ coincide.

Analogously to a scaling property of the s-Hausdorff content we notice that for fixed X, A as above and $s < t$ one has

$$M_s^\epsilon(X,A) := \mathrm{vol}_n(X_\epsilon \cap A)\epsilon^{-(n-s)} = \mathrm{vol}_n(X_\epsilon \cap A)\epsilon^{-(n-t)}\epsilon^{-(t-s)} = M_t^\epsilon(X,A)\epsilon^{-(t-s)}.$$

This implies that $M_s(X,A) < \infty \Rightarrow M_t(X,A) = 0$ and $M_t(X,A) > 0 \Rightarrow M_s(X,A) = \infty$. We therefore define the *upper Minkowski dimension relative to* A as $\overline{\dim_M}(X,A) := \inf\{s : \overline{M}_s(X.A) = 0\}$. The *upper inner Minkowski dimension* of a set X is the upper Minkowski dimension relative to itself. For later applications it will be crucial that this dimension is bi-Lipschitz invariant.

Proposition 2.30. *Let* $F : X \to Y$ *be a bi-Lipschitz map between two bounded sets* $X, Y \in \mathbb{R}^n$*. Then* $\dim_M(X,A) = \dim_M(F(X),A)$ *whenever any of the two expressions exist.*

We finish this section by citing a recent result by Balka-Keleti giving some additional context about the position of the Minkowski dimension in the spectrum of dimensions. We refer the reader to Keleti's paper for details.

Proposition 2.31 ([7])**.** *Let* D *be a dimension function on the set of compact subsets of* $\mathbb{R}^n$*. Here, a dimension function is a function that satisfies the following three conditions.*

(i). If $f : X \to \mathbb{R}^n$ is a Lipschitz map, $D(f(X)) \le D(x)$

(ii). D is monotone with respect to inclusion, $X' \subset X \Rightarrow D(X') \le D(X)$

(iii). If X is homogeneous self-similar and satisfies the strong open set condition (meaning the open set condition as in Thm. 2.36 with $X \cap O \neq \emptyset$), then $D(X) = \dim_H(X)$

then $\dim_H(X) \le D(X) \le \overline{\dim_M}(X)$.

Remark 2.32. Referring to Prop. 2.18 we now have a stronger result formulated in a recent paper by García-Bravo-Rajala-Takanen in [32]: The Hausdorff dimension of a quasisphere (or any (ϵ, δ)-domain is never full. However there are extension domains whose boundary has full Hausdorff dimension. Indeed there exist extension domains in $\mathbb{R}^3$ that are homeomorphic to S^2 but have full Hausdorff dimension.

2.4 Snowflakes, limit sets of IFS and Moran sets

We present three different constructions of domains with fractal boundary. Besides the already mentioned p-Rohde snowflakes we include the classical construction of Hutchinson as well as Moran sets.

Definition 2.33 (p-Rohde curves and snowflakes)**.** Let $p \in \left[\frac{1}{4}, \frac{1}{2}\right)$ be fixed. Define four maps $(\phi_i[p] : \mathbb{R}^2 \to \mathbb{R}^2)_{i=1,\cdots,4}$ as

$$\phi_1[p] : x \mapsto px, \qquad \phi_2[p] : x \mapsto \begin{pmatrix} p \\ 0 \end{pmatrix} + pR_\alpha(x),$$
$$\phi_3[p] : x \mapsto \begin{pmatrix} 1/2 \\ \frac{\sqrt{4p-1}}{2} \end{pmatrix} + pR_{-\alpha}(x), \qquad \phi_4[p] : x \mapsto px + \begin{pmatrix} 1-p \\ 0 \end{pmatrix},$$

where R_α is the usual planar rotation by α about the origin and $\alpha = \arccos\left(\frac{1}{2p} - 1\right)$. A *p-Rohde curve* is now built by the following iterative procedure: Starting with the unit interval $I_0 := [0,1] \times \{0\} \subset \mathbb{R}^2$ one acts with $\Phi[q] := \bigcup_{i=1}^4 \phi_i[q]$ with $q \in \{\frac{1}{4}, p\}$ on I_0. The outcome of this is then read as union of disjoint smaller intervals and one repeats the procedure on each interval (see Fig. 2.4). A p-Rohde snowflake is built out of 4 p-Rohde curves arranged on the boundary of a unit square. If at every level of iteration the q is the same on all intervals, the curve and snowflake thus constructed is called *homogeneous.* If $q = p$ at every level and every interval, the curve and snowflake is called *fully homogeneous.*

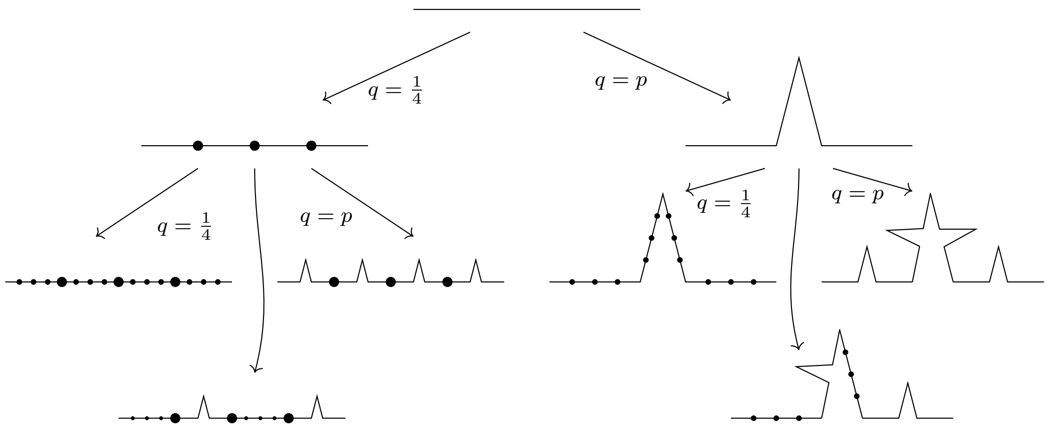


Figure 2.4: Illustration of the two possible iterations at the first two steps of the construction of a p-Rohde curve as described in Def. 2.33 starting with a unit interval $I_0 = [0,1] \times \{0\}$ at the top. The individual images of $\phi_i[\frac{1}{4}]$ are distinguished by dots in $\Phi[\frac{1}{4}]$. The two iterations on the bottom line represent examples of non-homogeneous iterations.

Example 2.34. Any $\frac{1}{4}$-Rohde curve is the unit interval. The Koch curve is the fully homogeneous $\frac{1}{3}$-Rohde curve. The Koch snowflake is then defined as the domain whose boundary is given by the union of three equilateral $\frac{1}{3}$-Rohde curves. The Koch curve may also be defined as the limit set of the IFS consisting of $\phi_1[\frac{1}{3}], \cdots, \phi_4[\frac{1}{3}]$ as defined below.

It is common terminology to denominate a (finite) set of contractions defined on $\mathbb{R}^n$ as *iterated function set (IFS).* In other words, an IFS is a (finite) set of contractions $\phi_i : \mathbb{R}^n \to \mathbb{R}^n$. We call $\sup_{x\in\mathbb{R}^n} |\phi_i'(x)|$ the *contraction ratio* of ϕ_i.

Consider the set of non-empty compact subsets $\mathrm{Comp}(\mathbb{R}^n)$ of $\mathbb{R}^n$ or more generally of any complete metric space. Then $(\mathrm{Comp}(\mathbb{R}^n), d_H)$ is again a complete metric space with the *Hausdorff metric* given by $d_H(A, B) := \max\{\sup_{x\in A} \mathrm{dist}(x, B), \sup_{y\in B} \mathrm{dist}(A, b)\}$. If a sequence $(X_n)_{n\in\mathbb{N}}$ of non-empty compact subsets converges to a limit X in Hausdorff metric we write $X_n \overset{\mathrm{H}}{\to} X$.

Any IFS gives rise to a map $\Phi := \bigcup_{i\in\Sigma} \phi_i$. If the IFS is finite, Φ is a contraction on $\mathrm{Comp}(\mathbb{R}^n)$ so that Banach's Fixed-point theorem gives the following celebrated result showing the existence of a *limit set* (also called *attractor*), i.e. a non-empty compact set A with $A = \bigcup_{i\in\Sigma} \phi_i A$.

Theorem 2.35 (Hutchinson, [45]). *Let $\{\phi_i\}_{i\in\Sigma}$ be a finite IFS on a complete metric space X. Then there is a unique non-empty compact set A with $A = \bigcup_{i\in\Sigma} \phi_i A$. Moreover $\left(\bigcap_{k\leq n} \Phi^k X\right)_n \xrightarrow{\mathrm{H}} A$ for any non-empty compact X.*

Under certain conditions of regularity, simple expressions for dimensions exists for such attractors. We recall the following classical result for the sake of completeness.

Theorem 2.36 (Moran-Hutchinson formula). *Let $\{\phi_i : \mathbb{R}^n \to \mathbb{R}^n\}_{i\in\Sigma}$ be a finite IFS. Suppose all ϕ_i are* similarity maps*, i.e. if there are $M_i \in O(n), v_i \in \mathbb{R}^n, r_i \in (0,1)$ with $\phi_i(x) = r_i M_i(x) + v_i$. Moreover suppose the IFS satisfies the* open set condition*, i.e. there is an open set O with $\phi_i O \cap \phi_j O = \emptyset\ \forall i \neq j$ and $\phi_i O \subset O\ \forall i \in \Sigma$. Then $\dim_H A = \overline{\dim_M}(A, \mathbb{R}^n) = s$ where s is the unique solution of $1 = \sum_{i\in\Sigma} {r_i}^s$.*

We finally turn our attention to Moran sets which will play a role in Thm. 4.7. We first recap some common notation for later use (see also Sec. 5.1). For any finite index set Σ (called an *alphabet*), we define the set of finite words over Σ as $\Sigma^{\mathrm{fin}} := \bigcup_{k\in\mathbb{N}_0} \Sigma^k$, where $\Sigma^0 := \{\emptyset\}$. The set of infinite words is defined as the set of sequences in Σ, $\Sigma^\infty := \Sigma^{\mathbb{N}}$. Finally we set $\Sigma^* := \Sigma^{\mathrm{fin}} \cup \Sigma^\infty$. On Σ^{fin} we define the *right-shift* $\overline{\sigma}$ as $\overline{\sigma} : w = (w_1 w_2 \cdots w_m) \mapsto (w_1 w_2 \cdots w_{m-1})$ and $\overline{\sigma}(\emptyset) := \emptyset$. If $\{\phi_i\}_{i\in\Sigma}$ is a family of maps, any finite word (also called *code*) $\Sigma^{\mathrm{fin}} \ni w = (w_1 w_2 \cdots w_m)$ defines a map by composition, $\phi_w := \phi_{w_1} \circ \cdots \circ \phi_{w_m}$. We define $\phi_\emptyset :=$ id. Lastly for a finite word $w \in \Sigma^k$, $|w| := k$ is defined as the length of the word.

Definition 2.37 ([38, 78]). Let Σ^{fin} be as above and suppose $J \in \mathrm{Comp}(\mathbb{R}^n)$ has non-empty interior. A collection of sets $\{J_u : u \in \Sigma^{\mathrm{fin}}\}$ is called *Moran structure* if

(i). To each $u \in \Sigma^{\mathrm{fin}}$ there is a similarity map ϕ_u so that $\phi_u J = J_u$.

(ii). For any $u \in \Sigma^{m-1}$, the sets $\{J_{ui} : i \in \Sigma\}$ all lie in J_u, have disjoint interior and $\frac{\operatorname{diam} J_{ui}}{\operatorname{diam} J_u} = c_{m,i}$ with $\sum_{k=1}^{\#\Sigma} c_{m,k}^n \leq 1$.

Then we define the *Moran set* by $\bigcap_{k=1}^{\infty} \bigcup_{u\in\Sigma^k} J_u$.

While the above version of the Moran-Hutchinson formula is often used, the following variant of the Moran-Hutchinson formula includes the situation of Moran sets.

Proposition 2.38 (Cf. [38] and the references therein)**.** *Let M be a Moran set with ratios $c_{m,i}$ as above with $\inf_{m,i}\{c_{m,i}\} > 0$. Let s_q be the unique solution of*

$$s \mapsto \prod_{m=1}^{q} \sum_{i=1}^{\#\Sigma} c_{m,i}^s = 1.$$

Then $\dim_H M = \liminf_{q\to\infty} s_q$ *and* $\overline{\dim_M} M := \limsup_{q\to\infty} s_q$.

Remark 2.39 (p-Rohde snowflakes as Moran sets). Let $p \in [1/4, 1/2)$ as before and J be the equilateral triangle of base length $I_0 = [0, 1]$ and legs of length $\sqrt{p}$. Setting $\phi_i := \phi_i[q]$ for $q \in \{1/4, p\}$ defines an index set $\Sigma := \{1, \cdots, 4\}$ and for any $\Sigma^{\text{fin}} \ni (u_1 u_2 \cdots u_k) = u$ we define ϕ_u by concatenation of maps as usual. By construction of p-Rohde snowflakes for any $u \in \Sigma^{\text{fin}}$, maps $\phi_{ui} = \phi_u \circ \phi_i[q]$ have the same q for all $i \in \Sigma$. In this setup $\phi_{ui} J \subset J_u$ with $\operatorname{int} \phi_{ui} J \cap \operatorname{int} \phi_{uj} J = \emptyset \, \forall i \neq j$ as desired and the corresponding $c_{m,i}$ coincide with the contraction ratios of the maps ϕ_i.

Example 2.40. A simple application of the above Moran-Hutchinson formula yields: The boundary of a fully homogeneous p-Rohde snowflake has Minkowski dimension is $-\log_p 4$. In particular the Minkowski dimension of the boundary of a Koch snowflake is $\log_3 4$.

2.5 Regularisation of $\Omega \setminus \Omega_{-\epsilon}$ by Whitney covers

Whitney covers are a common construction used to prove estimates on spectral counting functions by approximating a domain by a discrete union of cubes. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. A Whitney cover of Ω is a volume cover of Ω by cubes of different sizes such that a cube containing $x \in \Omega$ has a diameter that is uniformly comparable to $\operatorname{dist}(x, \partial\Omega)$. The construction of such a cover is well known and sometimes attributed to Whitney, who used it to study extensions of functions for example in [114]. However, a similar idea had in fact already been published by Courant-Hilbert in [20, Ch. VI. §4.4] in 1924 when Whitney was only 17. For the sake of completeness we include the construction from [102, Ch. VI. §1], as the notation employed here will be used in later results: Consider the nested family of lattices $\left\{2^{-k}\mathbb{Z}^n\right\}_{k\in\mathbb{Z}}$. To each point $p = (p_1, \dots, p_n)$ in the lattice $2^{-k}\mathbb{Z}^n$ we take the open cube $\prod_{i=1}^n \left(p_i, p_i + 2^{-k}\right)$ with diagonal of length $2^{-k}\sqrt{n}$ in the positive quadrant of p. The set of these cubes is denoted by $\mathrm{Cubes}(2^{-k}\mathbb{Z}^n)$. We then slice up Ω into sectors and cover them individually:

$$\begin{aligned} \mathcal{W}_k &:= \left\{Q \in \mathrm{Cubes}(2^{-k}\mathbb{Z}^n) : Q \cap \left(\overline{\Omega_{-2^{-k+2}\sqrt{n}}} \backslash \Omega_{-2^{-k+1}\sqrt{n}}\right) \neq \emptyset\right\} \\ \mathcal{W}' &:= \bigcup_{k\in\mathbb{Z}} \mathcal{W}_k. \end{aligned} \tag{2.4}$$

Since $\Omega = \bigcup_{k\in\mathbb{Z}} \left(\overline{\Omega_{-2^{-k+2}\sqrt{n}}} \backslash \Omega_{-2^{-k+1}\sqrt{n}}\right)$, it follows that $\mathrm{vol}_n\, \Omega = \mathrm{vol}_n \left(\bigcup_{Q\in\mathcal{W}'} Q\right)$. For the following, let $Q \in \mathcal{W}'$ and let $k \in \mathbb{Z}$ be so that $Q \in \mathcal{W}_k$. Then by definition, $\mathrm{diam}(Q) = 2^{-k}\sqrt{n}$ and there exists $x \in Q \cap \left(\overline{\Omega_{-2^{-k+2}\sqrt{n}}} \backslash \Omega_{-2^{-k+1}\sqrt{n}}\right)$. For this x we have $2^{-k+1}\sqrt{n} \leq \mathrm{dist}(x, \partial\Omega) \leq 2^{-k+2}\sqrt{n}$. Furthermore, $\mathrm{dist}(Q, \partial\Omega) \leq 2^{-k+2}\sqrt{n} = 4\,\mathrm{diam}(Q)$. Moreover, we have $\mathrm{diam}(Q) = 2^{-k+1}\sqrt{n} - 2^{-k}\sqrt{n} \leq \mathrm{dist}(x, \partial\Omega) - \mathrm{diam}(Q) \leq \mathrm{dist}(Q, \partial\Omega)$. However, $\mathcal{W}'$ might contain overlapping cubes. To rule out such cubes, observe that any two intersecting cubes are nested. Suppose two cubes Q, Q' intersect with $Q \subset Q'$. Then by the above

$$\mathrm{diam}(Q') \leq \mathrm{dist}(Q', \partial\Omega) \leq \mathrm{dist}(Q, \partial\Omega) \leq 4\,\mathrm{diam}(Q).$$

This shows that to any nested sequence of intersecting cubes in $\mathcal{W}'$ there is a unique maximal cube containing all others. We define $\mathcal{W} \subset \mathcal{W}'$ as the set of all those maximal cubes, which then are pairwise disjoint and $\mathcal{W}$ inherits the other properties from $\mathcal{W}'$. Such a volume cover will be called a *Whitney cover*. The cubes used in this cover shall be called *Whitney cubes*. The above consideration proves:

Lemma 2.41 (Existence of Whitney covers, Ch. VI. §1 in [102])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

Then there is a volume cover of Ω *consisting of pairwise disjoint cubes* $\{Q\}_{Q\in\mathcal{W}}$ *and*

$$\operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial\Omega) \leq 4\operatorname{diam}(Q) \qquad \forall Q \in \mathcal{W}. \tag{2.5}$$

Based on this, we have an immediate estimate on the number of cubes of a certain size in a Whitney cover. This result will play a crucial role in the proof of Thm. 4.15 since it allows us to estimate the number of cubes that are necessary to cover $\Omega \setminus \Omega_{-\epsilon}$.

Proposition 2.42 (Cardinality of slices of Whitney covers)**.** *Let* $\mathcal{W}$ *be a Whitney cover of a domain* $\Omega \subset \mathbb{R}^n$*, as constructed above. Suppose that there exists* $\delta \in [n-1, n)$ *such that* $M_\delta(\partial\Omega, \Omega) \in (0, \infty)$*. Then there is an* $\mathfrak{M}_\Omega \in \mathbb{R}$ *such that* $\#\mathcal{W}_k \leq \mathfrak{M}_\Omega 2^{k\delta}$*.*

A similar result with the upper inner Minkowski content replaced with the δ-Hausdorff measure $\mathcal{H}_\delta$ was obtained by Käenmäki-Lehrbäck-Vuorinen in [63].

Proof. Let $d_k := 2^{-k}\sqrt{n}$. By construction of $\mathcal{W}_k$ as in (2.4),

$$\#\mathcal{W}_k \leq \frac{\operatorname{vol}_n\left(\Omega_{-(4d_k+d_k)}\right) - \operatorname{vol}_n\left(\Omega_{-(2d_k-d_k)}\right)}{2^{-kn}} < \frac{\operatorname{vol}_n(\Omega_{-5\cdot 2^{-k}\sqrt{n}})}{2^{-kn}}.$$

Defining $\epsilon'(\epsilon) := \epsilon^{\delta-n}\operatorname{vol}_n(\Omega_{-\epsilon})/\overline{\mathcal{M}}_\delta(\partial\Omega, \Omega) - 1$, we have

$$\#\mathcal{W}_k \leq \overline{\mathcal{M}}_\delta(\partial\Omega, \Omega)\left(1 + \epsilon'\left(5\sqrt{n}2^{-k}\right)\right)\left(5\sqrt{n}\right)^{n-\delta} 2^{k\delta}.$$

Then for $\mathfrak{M}_\Omega := \sup_k \overline{\mathcal{M}}_\delta(\partial\Omega, \Omega)\left(1 + \epsilon'\left(5\sqrt{n}2^{-k}\right)\right)\left(5\sqrt{n}\right)^{n-\delta}$ and the assertion follows from $\limsup_{\epsilon\to 0}\epsilon'(\epsilon) = 0$. ☐

Proposition 2.43. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded with* $\delta := \overline{\dim}_M(\partial\Omega, \Omega) < \infty$ *and upper inner Minkowski content* $\overline{\mathcal{M}}_\delta(\partial\Omega, \Omega) \in (0, \infty)$*. Let* $\mathcal{W}$ *be a Whitney cover of* Ω *and for any* $\epsilon > 0$*, let*

$$\mathcal{W}_\epsilon := \{Q \in \mathcal{W} : Q \cap (\Omega \setminus \Omega_{-\epsilon}) \neq \emptyset\},$$

in other words, $\mathcal{W}_\epsilon$ *is the smallest collection of Whitney cubes in* $\mathcal{W}$ *that covers* $\{x \in \Omega :$

$\operatorname{dist}(x, \partial\Omega) \geq \epsilon\}$. *Then there is an* $A_\Omega \in \mathbb{R}$ *such that*

$$\operatorname{vol}_{n-1} \partial \left(\overline{\bigcup_{Q \in \mathcal{W}_\epsilon} Q} \right) \leq A_\Omega \cdot \epsilon^{(n-1)-\delta}.$$

Proof. Since the closure of the outermost cubes in $\mathcal{W}_\epsilon$ intersect $\partial\Omega_{-\epsilon}$, the $(n-1)$-volume of the boundary is bounded by the $(n-1)$-volumes of the boundaries intersecting $\partial\Omega_{-\epsilon}$: $\operatorname{vol}_{n-1} \partial \left(\overline{\bigcup_{Q \in \mathcal{W}_\epsilon} Q} \right) \leq \sum_{Q \in \mathcal{W}_\epsilon : \overline{Q} \cap \partial\Omega_{-\epsilon} \neq \emptyset} \operatorname{vol}_{n-1} \partial Q$. Suppose that $\overline{Q}$ has non-empty intersection with $\partial\Omega_{-\epsilon}$. Then there exists $x \in \overline{Q}$ such that $\operatorname{dist}(x, \partial\Omega) = \epsilon$. Now with Lem. 2.41 and $\operatorname{dist}(Q, \partial\Omega) + \operatorname{diam}(Q) \geq \sup_{x \in Q} \operatorname{dist}(x, \partial\Omega) \geq \epsilon \geq \operatorname{dist}(Q, \partial\Omega)$, we observe that

$$5 \operatorname{diam}(Q) \geq \epsilon \geq \operatorname{diam}(Q) \geq \frac{\epsilon}{5}.$$

With $k_{\min} := \min\{k \in \mathbb{Z} : 2^{-k}\sqrt{n} \leq \epsilon\}$, the diameter of such a cube is therefore restricted to one of at most three possible sizes, $\operatorname{diam}(Q) \in \{2^{-k_{\min}}\sqrt{n}, 2^{-(k_{\min}+1)}\sqrt{n}, 2^{-(k_{\min}+2)}\sqrt{n}\}$. Hence, an upper bound of the circumference of $\overline{\bigcup_{Q \in \mathcal{W}_\epsilon} Q}$ is

$$\begin{aligned} \operatorname{vol}_{n-1} \partial \left(\overline{\bigcup_{Q \in \mathcal{W}_\epsilon} Q} \right) &\leq \sum_{k=k_{\min}}^{k_{\min}+2} \sum_{Q \in \mathcal{W}_k} \operatorname{vol}_{n-1} \partial Q = 2n \sum_{k=k_{\min}}^{k_{\min}+2} \#\mathcal{W}_k \cdot 2^{-k(n-1)} \\ &\leq 2n \mathfrak{M}_\Omega \sum_{k=0}^{2} 2^{-k((n-1)-\delta)} \cdot 2^{-k_{\min}((n-1)-\delta)} \leq A_\Omega \epsilon^{(n-1)-\delta}, \end{aligned}$$

where $\mathfrak{M}_\Omega$ is defined in Prop. 2.42, $A_\Omega := 2n(2\sqrt{n})^{\delta-(n-1)} \mathfrak{M}_\Omega \sum_{k=0}^{2} 2^{-k((n-1)-\delta)}$ since one has $2 \cdot 2^{-k_{\min}}\sqrt{n} \geq \epsilon$ and $n - 1 \leq \delta$. □

In particular in the regular case (i.e. when $\delta = n - 1$), Prop. 2.43 shows that the circumference of the closure of the union of all large enough cubes of a Whitney cover is bounded.

Proposition 2.44. *In the setting of Prop. 2.43 almost all Billard trajectories in* $\overline{\bigcup_{Q \in \mathcal{W}_\epsilon} Q}$ *are non-periodic.*

Proof. We may assume that $\mathcal{W}_\epsilon$ consists only of cubes that have at least one face shared with another cube in $\mathcal{W}_\epsilon$, as the statement follows from known results for isolated cubes. Then there is $k_{\max} \in \mathbb{Z}$ such that $\mathcal{W}_\epsilon \subset \bigcup_{k \le k_{\max}} \mathcal{W}_k$ and all cubes in $\mathcal{W}_\epsilon$ have vertices in the $2^{-k_{\max}}\mathbb{Z}^n$-lattice. Let $x \in \partial\overline{\bigcup_{Q\in\mathcal{W}_\epsilon} Q}$ and v be any direction of a Billard. Then, up to lattice symmetry, the set of reflection points of such a trajectory in $\overline{\bigcup_{Q\in\mathcal{W}_\epsilon} Q}$ is contained in $\{x + tv : t \in \mathbb{R}\backslash\{0\}\}$. Up to lattice symmetry, this set contains the origin x iff v is rational up to normalisation. □

2.6 Weyl asymptotics of spectral counting functions

Suppose $\Omega \subset \mathbb{R}^n$ is a domain with $\sigma_{\text{ess}}(-\Delta_N) = \emptyset$ as discussed in Sec. 2.1. Then one may define the spectral counting function as in Def. 2.20. A famous observation by Weyl states that the k^{th} eigenvalue under Dirichlet boundary conditions growth with power $k^{2/n}$. This amounts to the celebrated Weyl law which we state under minimal hypothesis as formulated by Rozenbljum:

Theorem 2.45 (Weyl Law, [92, 93, 112])**.** *Let $\Omega \subset \mathbb{R}^n$ be open with bounded volume, $\operatorname{vol}_n \Omega < \infty$. Then*

$$N_D(\Omega, t) = C_W^{(n)} \operatorname{vol}_n(\Omega) t^{n/2} + o\left(t^{n/2}\right), \qquad \text{as } t \to \infty,$$

where $C_W^{(n)} := (2\pi)^{-n}\omega_n$ and $\omega_n := \operatorname{vol}_n(B_0(1)) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the n-dimensional unit ball. Whenever Ω is a domain with vanishing essential spectrum of the corresponding Laplacian, this law also holds true under Neumann or mixed boundary conditions.

Weyl original proof involved a simple covering argument; a planar domain Ω is approximated with a disjoint union of squares and applying a Dirichlet-Neumann bracketing argument. Since the remainder term then depends on the discrepancy between this cover by squares and Ω, it appears natural to assume that the remainder term is governed by the geometry of the boundary of Ω. Based on this observation, in [113] Weyl conjectured that for sufficiently regular domains $\Omega \subset \mathbb{R}^n$, the asymptotic behaviour is given by

$$\begin{aligned} N_D(\Omega, t) = C_W^{(n)} \operatorname{vol}_n(\Omega) t^{n/2} - \frac{1}{4} C_W^{(n-1)} \operatorname{vol}_{n-1}(\partial\Omega) t^{(n-1)/2} \\ + o\left(t^{(n-1)/2}\right) \text{ as } t \to \infty. \end{aligned} \tag{2.6}$$

A similar conjecture was formulated for the Neumann counting function; it differs from the Dirichlet case only by the sign of the second term. In several steps (see for example [47, 97, 99, 100, 101, 106]) on proving Weyl's conjecture, Thm. 2.45 was extended and generalised to elliptic operators and manifolds. It is believed to be correct for all $C^\infty(\mathbb{R}^n)$-domains. Indeed, supposing that $\partial\Omega$ is smooth and additionally assuming that Billiard trajectories are almost never periodic, the Weyl conjecture was proved by Ivrii in 1980 in [46] (see also [12, 47, 48, 96]) for several boundary conditions including Dirichlet and Neumann type. It is conjectured that the Billard hypothesis is true for any domain with smooth boundary but it is proven only in a few special cases as studied for example by Vassiliev in [108]. However, a fundamental regularity assumption on $\partial\Omega$ is essential. In fact, if the boundary is highly irregular such a result cannot be expected and is wrong in general. In particular, the remainder term in (2.6) is meaningless whenever the boundary shows fractal structure with non-integer Hausdorff-dimension $\dim_H \partial\Omega > n-1$ as then $\operatorname{vol}_{n-1} \partial\Omega = \infty$. Notice that under Ivrii's conditions, the exponent in the second term coincides with half the topological dimension $\dim(\partial\Omega)/2 = (\dim(\Omega)-1)/2$ of the boundary $\partial\Omega$ and that the associated coefficient is linked to the domain's surface area. It is therefore natural to expect analogue substitutes of these quantities in case of non-integer dimensions.

Motivated by physical observations, in a first step in this direction, Berry conjectured that the remainder term would be linked to the Hausdorff dimension and the Hausdorff measure of the boundary in [9, 10]. This however turned out to be false as a number of counterexamples were found in [13, 72, 73] leading to the formulation of a series of modified Weyl-Berry conjectures by Lapidus et al.

Conjecture 2.46 ([65, 66])**.** Let $\Omega \subset \mathbb{R}^n$ be a sufficiently regular domain but with Minkowski dimension $\delta \in [n-1, n)$ and Minkowski content $M \in (0, \infty)$. Then there is a $C_{n,\delta}$ depending only on n and δ such that

$$N_D(\Omega,t) = C_W^{(n)} t^{n/2} + C_{n,\delta} M t^{\delta/2} + o\left(t^{\delta/2}\right) \qquad \text{as } t \to \infty.$$

Whenever $\sigma_{\text{ess}}(-\Delta_N) = \emptyset$, the same asymptotic law holds true for $N_N(\Omega,t)$ with a different constant $\widetilde{C}_{n,\delta}$.

A recent result by Frank-Larson in [30] shows that an averaged version of Berry's original conjecture holds true: For Lipschitz domains $\Omega \subset \text{Lip}(\mathbb{R}^n)$, the sum of the first eigenvalues defined by the trace $\text{Tr}(-\Delta_D - \lambda) := \sum_{\lambda_k \in \sigma(-\Delta_D):\lambda_k \leq \lambda} (\lambda - \lambda_k)$ has an asymptotic behavior that depends on the topological dimension of $\partial\Omega$. Namely, $\text{Tr}(-\Delta_D - \lambda) = \frac{n}{n+2} C_W^{(n)} \text{vol}_n(\Omega) \lambda^{1+\frac{n}{2}} - \frac{1}{4} C_W^{(\delta)} \mathcal{H}_\delta(\partial\Omega) \lambda^{1+\frac{\delta}{2}} + o\left(\lambda^{1+\frac{\delta}{2}}\right)$, where $\delta := \dim_{\text{Top}}(\partial\Omega)$.

In case of Dirichlet boundary conditions and $\Omega \subset \mathbb{R}^1$, Lapidus and Pomerance verified the modified Weyl-Berry conjecture in [68] by proving the existence of a c_δ for which $N_D(\Omega,t) = C_W^{(1)} \text{vol}_1(\Omega) t^{1/2} - c_\delta \mathcal{M}_\delta(\partial\Omega) t^{\delta/2} + o(t^{\delta/2})$ as $t \to \infty$. However it was subsequently shown that the analogue is incorrect in higher dimensions in general in [69] by providing counter examples. In arbitrary dimensions, Lapidus proved the following asymptotic law.

Theorem 2.47 ([65])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with upper inner Minkowski content $\overline{M}_\delta(\partial\Omega,\Omega) \in (0,\infty)$ for $\delta \in [n-1,n)$. Then*

$$N_D(\Omega,t) = C_W^{(n)} \text{vol}_n(\Omega) t^{n/2} + \mathcal{O}\left(t^{\delta/2}\right).$$

If $\overline{M}_\delta(\partial\Omega,\mathbb{R}^n) \in (0,\infty)$ for $\delta \in [n-1,n)$, then

$$N_D(\Omega,t) = C_W^{(n)} \text{vol}_n(\Omega) t^{n/2} + \mathcal{O}\left(t^{\delta/2}\right).$$

There is a notable difference of conditions for both kinds of boundary conditions, i.e. existence of the upper inner Minkowski content at Dirichlet boundary conditions versus the existence of "full" upper Minkowski content at Neumann boundary conditions.

Of particular relevance to the present work is the article by Netrusov-Safarov in [80], where an exact rather than an asymptotic bound on $N_N(\Omega, t) - C_W^{(n)} \operatorname{vol}_n(\Omega) t^{n/2}$ was obtained for domains whose boundary locally is a graph of a BV-function. This will be discussed in more detail in Sec. 4.2.

Chapter 3

Estimates of the second Neumann eigenvalue

This chapter is partially based on [61] by Kombrink and the author. The main goal is to provide lower bounds of the first non-trivial the eigenvalue λ_2^N of the Laplacian with respect to Neumann boundary conditions for a given domain D of a class of domains (defined in Def. 3.3) that is not restricted to domains with cone conditions or domains whose boundary is locally a graph. We obtain a lower bound for the first non-trivial eigenvalue of such domains in Lem. 3.8. This is achieved with the introduction of specific 1-foliations (Sec. 3.2). The construction is designed to naturally include p-Rohde snowflakes and related snowflake-like domains. Afterwards we discuss several examples including Koch snowflakes. We start by recalling a result for C^0-domains.

Definition 3.1 (Def. 1.1-1.2 in [80]). Let $\Omega' \subset \mathbb{R}^{n-1}$ be a domain and let $f \in C^0(\overline{\Omega'})$. We define the *oscillation of f on Ω'* as

$$\operatorname{Osc}(f,\Omega') := \frac{1}{2}\left(\sup_{x\in\Omega'} f(x) - \inf_{x\in\Omega'} f(x)\right).$$

Theorem 3.2 (Lem. 2.6 in [80]). *Let $\delta > 0$ and $Q^c_{n-1} := (0,c)^{n-1} \times \{0\} \subset \mathbb{R}^n$ be an $(n-1)$-dimensional cube parallel to the coordinate axes and of side length $c \leq \delta$. Let $f \in C^0(Q^c_{n-1})$ with $\operatorname{Osc}(f, Q^c_{n-1}) \leq \delta/2$ and $b = \inf f - c$. Let $D = \Gamma_{f,b}$ be the graph-set* *of the C^0-function f in the following sense:*

$$\Gamma_{f,b} := \{x \in \mathbb{R}^n \,:\, \exists y \in Q_{n-1}^c \text{ s. t. } x = (y,t) \text{ and } t \in (b, f(y))\}.$$

Then $N_N(\Gamma_{f,b}, t^2) = 1$ *if* $t \leq \left(1 + (2\pi)^{-2}\right)^{-1/2} \delta^{-1}$.

A key idea of the proof of Thm. 3.2 is to observe that $\int_D g(x)dx = \int_{Q_{n-1}^c} \int_b^{f(y)} g(y,t)dt$ for $g \in H^1(D)$. In order to generalise this, we introduce a family of paths γ that should reflect some kind of expected self-similarity of the boundary.

Suppose Ω satisfies the following property: For any sufficiently small $\epsilon > 0$ there is b such that $\Omega_{-\epsilon}$ can be covered by domains $\{D_i\}_{i\in I}$ of the form $\Gamma_{f,b}$ with $\mathrm{Osc}(f, \mathrm{dom}(f)) \leq \epsilon/2$. Thm. 3.2 then shows that $N_N(\Omega_{-\epsilon}, t) \leq \#I$ whenever t is sufficiently small, namely $t < \left(1 + (2\pi)^{-2}\right)^{-1} \epsilon^{-2}$. This idea will later be used to prove upper bounds for Neumann counting functions in Thm. 4.15. Within their work (cf. [79, 80]), Netrusov-Safarov raised the question about extending this result to domains with boundary of different nature. In this section we define a framework with which one can extend this result to a substantial class of domains that are not C^0-domains. A prototypical example is the Koch snowflake.

3.1 Well-foliated domains and bounds on eigenvalues

The below definition will play a crucial role throughout this work since it allows for a direct application to snowflake-like domains.

Definition 3.3 (Well-foliated domain)**.** Let $X, D \subset \mathbb{R}^n$ be domains. Suppose $\varphi : X \to D$ is a homeomorphism with continuous extension to $\overline{X}$ that is differentiable almost everywhere following the rule for change of variables under integration (see also Prop. 3.5). Suppose that $I_0 \times (-r, 0) \subset X \subset \overline{I_0} \times [-r, L]$ for $r, L \in (0, \infty)$ where I_0 is a convex domain $\subset \mathbb{R}^{n-1}$ and suppose that $\varphi(q, 0) = q \in \overline{D}$. A *fibre through* $q := (q_1, \cdots, q_{n-1}) \in I_0$ is defined as the map $\gamma_q : t \mapsto \varphi(q, t - r)$ where t is assumed to be positive and sufficiently small. We write $dx_1 \cdots dx_n = \beta(q,t)dqdt$ for the change of variables (with $q \in I_0$) and set $E := \varphi(I_0 \times (-r, 0))$. The domain D is called *well-foliated* with parameters $(r, L, \mathcal{I}_\beta, \beta_{\mathrm{inf}}, E)$

if the following properties are satisfied:

(i). Any fibre through a point $q \in I_0$ satisfies $|\gamma_q'(t)| = 1$ almost everywhere in D. We then denote the length of the fibre by $\operatorname{len}\gamma_q$.

(ii). The quantity $\beta(q,t)$ as introduced above satisfies $\sup_{q\in I_0}\int_0^{\operatorname{len}\gamma_q}\beta(q,t-r)dt < \infty$ and $\operatorname{ess\,inf}_{(q,t)\in X}\beta(q,t) > 0$.

The set of all paths of a well-foliated domain will be denoted by $\Gamma := \{\gamma_q : q \in I_0\}$. For any measurable $S \subset D$, we write $\beta_{\inf}(S) := \operatorname{ess\,inf}_{x\in S}\beta(x)$ and define $\mathcal{I}_\beta(S) := \sup_{q\in I_0}\int_0^{\operatorname{len}\gamma_q} 1_S(\gamma_q(t))\beta(\gamma_q(t))dt$. We write $I_D := \varphi(I_0 \times \{-r\})$ (called *base*) for the set of starting points of paths in the foliated domain.

Note that we can identify such path γ_q with its intersection point $\gamma_q(r) = q \in I_0 \times \{0\}$. One has a straightforward gluing property for such domains by concatenating paths or joining bases.

Proposition 3.4 (Gluing property of well-foliated domains)**.** *Let D and D' be two disjoint well-foliated domains with bases I_D and $I_{D'}$. Let $Y' \subset \partial D' \setminus I_{D'}$ be the set of all end points of paths in D'. If $I_0 \subset Y'$ when we allow the set $E \subset D$ to be empty, then* $\operatorname{int}\overline{D' \cup D}$ *is well-foliated with base $I_{D'}$.*

Proof. To any $q \in I_D$ there is a unique $q' \in I_{D'}$ with $\lim_{t\to\operatorname{len}\gamma_{q'}}\gamma_{q'}(t) = q$. Concatenating any such two pairs of fibres to $\widetilde{\gamma_{q'}}$ gives fibres of finite length $\operatorname{len}\widetilde{\gamma_{q'}} = \operatorname{len}\gamma_q + \operatorname{len}\gamma_{q'}$. All properties of well-foliated domains are then inherited. □

We notice that sufficiently regular maps preserve the structure of a well-foliated domain. This follows directly from the fact that one may extend the classical calculus of change of variables to certain non-differentiable maps that include locally Lipschitz maps as shown in [39]. More precisely we have the following result. Recall that a function $f : X \to \mathbb{R}$ with open $X \subset \mathbb{R}^n$ is *approximately totally differentiable* at $x_0 \in X$ if there is some $L = (L_1, \cdots, L_n) \in \mathbb{R}^n$ for which for every $\epsilon > 0$ the set $A(f,\epsilon) := \{x \in X : \frac{|f(x)-f(x_0)-L(x-x_0)|}{|x-x_0|} < \epsilon\}$ has a density point at x_0, i.e $\lim_{\delta\to 0}\frac{\operatorname{vol}_n(A(f,\epsilon)\cap B_\delta(x_0))}{\operatorname{vol}_n(B_\delta(x_0))} = 1$.

Proposition 3.5 ([39]). *Let $X \subset \mathbb{R}^n$ be a domain and $f : X \to \mathbb{R}^n$ be an injective map that is approximately totally differentiable Lebesgue-almost everywhere in X. Then for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ one has $\int_X (u \circ f)|J_f|dx = \int_{f(X)} u(y)dy$, where J_f is the Jacobian containing the approximate differentials L_i.*

In this context we recall Rademacher's celebrated theorem and show that Lipschitz maps preserve well-foliatedness.

Theorem 3.6 (Rademacher, [88]). *Let X, X' be open subsets of a Euclidean space and $f : X \to \mathbb{R}^m$ be Lipschitz. Then f is differentiable Lebesgue-almost everywhere. If f is injective, the rule for change of variables applies to f.*

Proposition 3.7. *If $f : D \to \mathbb{R}^n$ is an injective Lipschitz map and D is a well-foliated domain as in Def. 3.3, then $f(D)$ is also well-foliated. The same is true for any differentiable bi-conformal map.*

Proof. By Rademacher's theorem, f is differentiable almost everywhere. In the notation of Def. 3.3, this shows that $f \circ \varphi$ is an almost everywhere differentiable homeomorphism. As f is Lipschitz, the length of each fibre $f(\gamma_q)$ in $f(D)$ is finite so that a reparametrisation by arc length is possible fulfilling (i) and similarly also (ii) by change of variables applied to a Lipschitz map. Any conformal map on D has differential bounded away from 0 and ∞. □

The concept of well-foliated domains allows for a generalisation of a result by Netrusov-Safarov in [80] to certain non-graph domains. The following main result of this chapter will enable us to apply this result in order to obtain estimates of spectral counting functions of domains that have a related foliation property introduced later in Def. 4.1. We will recover the result of Netrusov-Safarov from [80] for bounded variation domains in Sec. 4.2.

Lemma 3.8. *Let $D \subset \mathbb{R}^n$ be a domain that is well-foliated with a family of fibres $\{\gamma\}_{\gamma \in \Gamma}$ in the notation of Def. 3.3. Setting $\Lambda_2^N(X) := \min\{\lambda_2^N(X), \inf \sigma_{\text{ess}}(-\Delta_N(X))\} = \inf\{\lambda \in \sigma(-\Delta_N(X)) : \lambda > 0\}$ to be the infimum over all positive elements in the spectrum of the*

Neumann Laplacian defined on X for $X \in \{D, E\}$,

$$\Lambda_2^N(D) \geq \Lambda_2^N(E) \left(1 + \mathcal{I}_\beta(D \setminus E) \left(\frac{1}{\sqrt{r\beta_{\inf}(E)}} + \sqrt{\frac{L\lambda_2^N(E)}{\beta_{\inf}(D)}}\right)^2\right)^{-1}$$

Proof. Let $u \in H^1(D) \cap 1^\perp$ and set $\overline{u_E} := \frac{1}{\operatorname{vol} E} \int_E u dx$. Then $\|u\|^2_{L^2(D)} \leq \|u - \overline{u_E}\|^2_{L^2(D)}$. In particular, by Cor. 2.6, it suffices to show the following Poincaré-Wirtinger inequality

$$\|u - \overline{u_E}\|^2_{L^2(D)} \leq C\|\nabla u\|^2_{L^2(D)}, \tag{3.1}$$

with C^{-1} given by the right hand side of the claim. To this end, we subdivide the expression

$$\|u - \overline{u_E}\|^2_{L^2(D)} = \underbrace{\int_E |u(y) - \overline{u_E}|^2 dy}_{:=I_1} + \underbrace{\int_{D\setminus E} |u(x) - \overline{u_E}|^2 dx}_{:=I_2},$$

and estimate I_1 and I_2 separately. Applying Cor. 2.6 to E one has

$$I_1 \leq \Lambda_2^N(E)^{-1}\|\nabla u\|^2_{L^2(E)} \leq \Lambda_2^N(E)^{-1}\|\nabla u\|^2_{L^2(D)}. \tag{3.2}$$

Recall that for any $a, b, c \in \mathbb{R}$ and $\alpha > 0$, there is the following inequality: $|a - b|^2 \leq (1 + \alpha)|a - c|^2 + (1 + \alpha^{-1})|b - c|^2$.[1] We now consider

$$\begin{aligned} I_2 = \int_{D\setminus E} |u(x) - \overline{u_E}|^2 dx &= \int_{I_0} \int_r^{\operatorname{len}\gamma_q} |u(\gamma_q(t)) - \overline{u_E}|^2 \beta(q,t) dt dq, \\ &= \frac{1}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r |u(\gamma_q(t)) - \overline{u_E}|^2 \beta(q,t) dt' dt dq \\ &\leq \underbrace{\frac{1+\alpha}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r |u(\gamma_q(t')) - \overline{u_E}|^2 \beta(q,t) dt' dt dq}_{=:I_2'} \\ &\quad + \underbrace{\frac{1+\alpha^{-1}}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r |u(\gamma_q(t)) - u(\gamma_q(t'))|^2 \beta(q,t) dt' dt dq}_{=:I_2''}, \end{aligned}$$

[1] This can be seen for example by minimising the function $h(c) := A|a - c|^2 + B|b - c|^2 - |a - b|^2$ for fixed a, b in dependence of A, B. This shows $A = 1 + \alpha$ and $B = 1 + \alpha^{-1}$ for some α. Finally the condition $\alpha > 0$ then follows from $h(c) \geq 0$.

which holds for any $\alpha > 0$. By assumption, $\int_r^{\operatorname{len}\gamma_q} \beta(q,t)dt \leq \mathcal{I}_\beta(D\backslash E)$ for all $q \in I_0$. Hence

$$\begin{aligned} I_2' &= \frac{1+\alpha}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r |u(\gamma_q(t')) - \overline{u_E}|^2 \beta(q,t) dt' dt dq \\ &\leq \frac{1+\alpha}{r} \mathcal{I}_\beta(D\backslash E) \int_{I_0} \int_0^r |u(\gamma(t')) - \overline{u_E}|^2 dt' dq \\ &\leq \frac{1+\alpha}{r} \frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(E)} \int_{I_0} \int_0^r |u(\gamma_q(t')) - \overline{u_E}|^2 \beta(q,t') dt' dq \\ &= \frac{1+\alpha}{r} \frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(E)} \|u - \overline{u_E}\|^2_{L^2(E)} \leq \Lambda_2^N(E)^{-1} \frac{1+\alpha}{r} \frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(E)} \|\nabla u\|^2_{L^2(D)}. \end{aligned}$$

Using Jensen's inequality and $|\nabla \gamma_q(s)|^2 = 1$, $|s| \leq 1$ and $|\partial_s u| = |\langle s, \nabla u \rangle| \leq |\nabla u|$,

$$\begin{aligned} |\gamma_q(t) - \gamma_q(t_0)|^2 = \left| \int_{t_0}^t \partial_s u(\gamma_q(s)) ds \right|^2 &\leq (t - t_0) \int_{t_0}^t |\partial_s u(\gamma_q(s))|^2 \, ds \\ &\leq (t-t_0) \int_{t_0}^t |\nabla u|^2 \, ds \leq (t-t_0) \int_0^{\operatorname{len}\gamma_q} \left| \nabla u|_{\gamma_q(s)} \right|^2 ds. \end{aligned}$$

Therefore

$$\begin{aligned} I_2'' &= \frac{1+\alpha^{-1}}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r |u(\gamma_q(t)) - u(\gamma_q(t'))|^2 \beta(q,t) dt' dt dq \\ &= \frac{1+\alpha^{-1}}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r \left| \int_{t'}^t \partial_s u(\gamma_q(s))|_{s=\sigma} d\sigma \right|^2 \beta(q,t) dt' dt dq \\ &\leq \frac{1+\alpha^{-1}}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r (t-t') \int_{t'}^t |\partial_s u(\gamma_q(s))|_{s=\sigma}|^2 \, d\sigma \beta(q,t) dt' dt dq \end{aligned}$$

Then since $t - t' \leq L$,

$$\begin{aligned} I_2'' &\leq \frac{1+\alpha^{-1}}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r (t-t') \int_{t'}^t \left| \nabla u(x)|_{x=\gamma_q(\sigma)} \right|^2 d\sigma \beta(q,t) dt' dt dq \\ &\leq \frac{1+\alpha^{-1}}{r} \int_{I_0} \int_r^{\operatorname{len}\gamma_q} \int_0^r L \int_0^{\operatorname{len}\gamma_q} \left| \nabla u(x)|_{x=\gamma_q(\sigma)} \right|^2 d\sigma \beta(q,t) dt' dt dq \\ &\leq (1+\alpha^{-1}) L \mathcal{I}_\beta(D\backslash E) \int_{I_0} \int_0^{\operatorname{len}\gamma_q} \left| \nabla u(x)|_{x=\gamma_q(\sigma)} \right|^2 d\sigma dq \\ &\leq (1+\alpha^{-1}) L \frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(D)} \int_{I_0} \int_0^{\operatorname{len}\gamma_q} \left| \nabla u(x)|_{x=\gamma_q(\sigma)} \right|^2 \beta(q,\sigma) d\sigma dq \end{aligned}$$

$$= (1+\alpha^{-1})L\frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\text{inf}}(D)}\|\nabla u\|^2_{L^2(D)}.$$

Maximising with respect to $\alpha > 0$ yields the claim. □

Importantly, Lem. 3.8 gives a substitute to the missing domain monotonicity from Rem. 2.21. For domains which are geometrically similar to a cuboid, it provides an estimate that plays a crucial role in estimates for the spectral function in Chap. 4.

Corollary 3.9 ("$\Lambda_2^N(D) \asymp \text{diam}(D)^{-2}$"). *Let D be a well-foliated domain with parameters $(r, L, \mathcal{I}_\beta, \beta_{\text{inf}}, E)$ and $\Lambda_2^N(X) := \min\left(\inf\left(\sigma_{\text{ess}}\left(-\Delta_N(X)\right)\right), \lambda_2^N(X)\right)$ be defined as in Thm. 2.5 for $X \in \{D, E\}$. Suppose $\Lambda_2^N(E) \asymp \epsilon^{-2}$, $r \asymp L \asymp \mathcal{I}_\beta \asymp \epsilon$ and $\beta_{\text{inf}} \asymp 1$. Then there is a C depending on $r, L, \mathcal{I}_\beta, \beta_{\text{inf}}, E$ and D such that*

$$\Lambda_2^N(D) \geq C\epsilon^{-2}$$

Proof. By on Lem. 3.8,

$$\Lambda_2^N(D) \geq \Lambda_2^N(E)\left(1+\mathcal{I}_\beta(D\setminus E)\left(\frac{1}{\sqrt{r\beta_{\text{inf}}(E)}}+\sqrt{\frac{L\lambda_2^N(E)}{\beta_{\text{inf}}(D)}}\right)^2\right)^{-1}.$$

By hypothesis there are constants $c_r^\pm, c_L^\pm, c_{\mathcal{I}}^\pm$ and $c_\Lambda^\pm$ such that for all sufficiently small $\epsilon > 0$

$$r\epsilon^{-1} \in [c_r^-, c_r^+], \quad L\epsilon^{-1} \in [c_L^-, c_L^+], \quad \mathcal{I}_\beta\epsilon^{-1} \in [c_{\mathcal{I}}^-, c_{\mathcal{I}}^+] \text{ and } \Lambda_2^N(E)\epsilon^2 \in [c_\Lambda^-, c_\Lambda^+].$$

These constants may be assumed positive since $r, L, \mathcal{I}_\beta, \Lambda_2^N(E) > 0$. Moreover β_{inf} does not depend on ϵ. Therefore

$$\Lambda_2^N(D) \geq c_\Lambda^-\epsilon^{-2}\left(1+c_{\mathcal{I}}^+\epsilon\left(\frac{1}{\sqrt{c_r^-\epsilon\beta_{\text{inf}}(E)}}+\sqrt{\frac{c_L^+\epsilon c_\Lambda^+\epsilon^{-2}}{\beta_{\text{inf}}(D)}}\right)^2\right)^{-1}.$$

Canceling all ϵ then gives $C = c_\Lambda^-\left(1+c_{\mathcal{I}}^+\left(\frac{1}{\sqrt{c_r^-\beta_{\text{inf}}(E)}}+\sqrt{\frac{c_L^+c_\Lambda^+}{\beta_{\text{inf}}(D)}}\right)^2\right)^{-1}$. □

Together with the notion introduced in Def. 4.1 this constant can be rewritten as in Sec. 4.4.

Regarding generalisations of Lem. 3.8 we notice that in fact many of the involved steps can be generalised to $W^{1,p}$-spaces. This is discussed in Sec. 6.2.1.

3.2 Construction of foliations for snowflake-like domains

In this section we give a method to construct well-foliated domains based on an idea by Brolin in [11] to construct paths in Julia sets. Similar ideas are well-known and are often used to study Julia sets, see for example Falconer's application to Siegel disks in [28, Ch. 14]. We begin by briefly summarising the original concept. This motivates an adaptation to Koch curves and, more generally, p-Rohde curves. After introducing this adjusted construction for p-Rohde curves, we generalise the setting even further. It will then become evident that foliations of the basic building components of the construction are necessary. These so-called *seed foliations* are introduced next. We combine the seed foliation and the generalised setting to obtain iterative procedures to construct well-foliated domains that cover a large class of snowflake-like domains including p-Rohde snowflakes.

Brolin construction. Focusing on domains that are topological disks bounded by a Julia set, a family of paths is constructed using the iterative nature of the Julia set. More precisely, let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map on the complex plane and let α be simple attractive fixed point with a bounded attractive basin $A(\alpha)$ (i.e. the largest connected component of points $z \in \mathbb{C}$ around α for which $f^{(k)}(z) \to \alpha$ as $k \to \infty$). Let $r > 0$ be such that $B_{2r}(\alpha) \subset A(\alpha)$ and define $B_0 := B_r(\alpha)$ and $B_k := f^{-1}B_{k-1}$ for $k \in \mathbb{N}$. Notice that $A(\alpha) = \lim_{k\to\infty} B_k$ in the Hausdorff sense. Then there is a bi-holomorphic map G (called uniformisation) such that $G(B_1 \setminus B_0)$ is an annulus of radii r_1, r_2 with $r_1 < r_2$. Then one defines the family of radial paths in $G(B_1 \setminus B_0)$ given by $\gamma_\theta : (r_1, r_2) \to G(B_1 \setminus B_0)$ via $t \mapsto te^{i\theta}$ for fixed $\theta \in [0, 2\pi)$. Now $G^{-1}\gamma_\theta : (r_1, r_2) \to B_1 \setminus B_0$ and $B_1 \setminus B_0 = \bigsqcup_{\theta\in[0,2\pi)} \text{image}_{t\in(r_1,r_2)} G^{-1}\gamma_\theta(t)$. Letting f^{-1} act on each such γ_θ we obtain a family of paths in $B_2 \setminus B_1$. Iteratively applying f^{-1} then yields a family of disjoint paths covering each $B_k \setminus B_{k-1}$. For any path $\gamma^{(k)} := f^{-k}G^{-1}\gamma_\theta$ in $B_k \setminus B_{k-1}$ there is a unique path $f^{-k+1}G^{-1}\gamma_{\theta'}$ in $B_{k-1} \setminus B_{k-2}$ whose endpoint match the starting point of $\gamma^{(k)}$. Connecting any such two paths yields a family of disjoint paths covering $B_k \setminus B_0$. As $k \to \infty$ this gives a family of paths connecting ∂B_0 with any point in $\operatorname{int} A(\alpha)$. We define a projection π_k by mapping each endpoint of a path in $B_k \setminus B_{k-1}$ to its starting point.

Example 3.10. We describe the unit disk $B_1(0)$ as the set bounded by the Julia set $S^1 \subset \mathbb{C}$ applying the above framework to the original construction by Brolin. Suppose $f(z) = z^2 + c$ with $c = 0$ and let $B_0 := B_{1/2}(0)$ be a disk. Let $\Gamma_0 = \partial B_0$ and generally $\Gamma_n := (f^{-1})^n \partial B_0$ with $B_n \overset{\mathrm{H}}{\to} S^1$. In this case the uniformisation $G = \mathrm{id}_{\mathbb{C}}$ is trivial. The projection π_k then acts on Γ_k as $\pi : \Gamma_k \ni x = 2^{-2^{1-k}} e^{i\theta} \mapsto 2^{-2^{-k}} e^{i\theta}$, that is, $\pi_k : z \mapsto 2^{-2^{-k}} z$. We have $I_D = [0, 2\pi)$ corresponding to ∂B_0 and the corresponding $\beta(\theta, |f^{-k}(1/2)|)$ then reads

$$\beta(\theta, |f^{-k}(1/2)|) = 2^{(1-2^{-k})} = \frac{|f^{-k}\left(\frac{1}{2}e^{i\theta}\right)|}{|\frac{1}{2}e^{i\theta}|}$$

showing that β is radially symmetric and grows linearly with the radius as is expected from the foliation by straight radial paths in the disk.

We now adapt the above construction to settings motivated by the Koch snowflake K or, more precisely, to the open (disconnected) bounded set whose boundary is given by the union of a Koch curve C and its base interval $[0, 1]$. After this first adjustment, the generalisation to p-Rohde curves is natural.

Construction in Koch curves and p-Rohde curves. Recall that C is the limit set of a self-similar IFS given by the four maps $\phi_1[\frac{1}{3}], \cdots, \phi_4[\frac{1}{3}]$ as given in Def. 2.33 and Ex. 2.34. We replace f^{-1} and B_0 from the setting above with $\Phi := \bigcup_i \phi_i$, and the domain whose boundary is given by the union of $I_0 := [0, 1]$ and ΦI_0, respectively. This domain (the largest prong of the Koch curve) is given by a equilateral triangle of length $\frac{1}{3}$. Since the elements of the sequence $I_n := \Phi^n I_0$ are not disjoint, a bi-holomorphic map as described above does not exist. Instead we manually introduce a *seed foliation* from the domain with boundary $I_0 \cap I_1$ (cf. Fig. 3.2(a)) as set of vertical paths from any point in I_0 to I_1. Now acting Φ on this seed foliation yields a family of paths between I_1 and I_2. Iterating to infinity and concatenating endpoints of images of the seed foliation then yields a family of paths from I_0 to C.

More generally, for any p-Rohde curve we have the following construction: Let $\Omega \subset \mathbb{R}^2$ be the domain whose boundary is given by the union of any p-Rohde curve and $I_0 := [0, 1]$. As before, put $\Phi_n := \bigcup_{i=1}^{4} \phi_i^{(n)}$. Depending on the p-Rohde curve, $\phi_i^{(n)}$ is either $\phi_i[p]$ or $\phi_i[\frac{1}{4}]$. One defines a seed foliation for $p \neq 1/4$ and the trivial foliation linking I_0 to I_0 if $p = \frac{1}{4}$. In higher dimensions, this can be done analogously.

Based on the above construction we now propose a further generalised setting. We maintain the idea of a domain with boundary given by the limit set of a system of iterated maps but we allow additional liberty in the composition of the maps. As in the previous construction, this needs a seed foliation. This seed foliation will be introduced immediately afterwards as a basic step of a foliation based on which a full foliation can be constructed iteratively.

Generalised setup for the construction of well-foliated domains. Let $I_0 := D_{n-1} \subset \mathbb{R}^{n-1} \times \{0\}$ with D_k being a k-dimensional hyperplanar polyhedron[2] and let Σ be a (potentially infinite) index set of homeomorphic and bi-Lipschitz maps $(\phi_i)_{i\in\Sigma} : \mathbb{R}^n \to \mathbb{R}^n$ with a composition rule $A : \left(\{\emptyset\} \cup \Sigma^{\text{fin}}\right) \times \Sigma \to \{0, 1\}$, i.e. two maps ϕ_w, ϕ_i can be composed iff wi is admissible, i.e. $A(w, i) = 1$, where we set $\phi_\emptyset := \text{id}$. The set Σ_A^* is comprised of all admissible finite and infinite words in this sense. We assume that A is "$\overline{\sigma}$-complete", i.e. $w \in \Sigma_A^k \Rightarrow \overline{\sigma}w \in \Sigma_A^{k-1}$ for any $k \geq 1$.

For any finite word $w = (w_1 w_2 \cdots w_k) \in \Sigma^{\text{fin}}$, we set $\phi_w := \phi_{w_1}\phi_{w_2} \cdots \phi_{w_k}$ as before. Suppose the family of maps $\{\phi_i\}_{i\in\Sigma}$ satisfies the following properties:

(i). (Orientation). $\phi_w(I_0) \subset \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ for all $w \in \Sigma_A^*$.

(ii). (Separation). For any two words $w \neq v \in \Sigma_A^k$ of same finite length k we have

[2] More generality is possible here, but does not seem to provide much further insight. For example I_0 could be any compact and 1-connected subset of D_{n-1}. One could also replace I_0 with a 1-connected (meaning there is a single chart to D_{n-1}) $n-1$-dimensional manifold with boundary.

$\phi_w(\operatorname{int} I_0) \cap \phi_v(\operatorname{int} I_0) = \emptyset$ for all $k \in \mathbb{N}$, where we denote the interior in $\mathbb{R}^{n-1}$ of I_0 by $\operatorname{int} I_0$. One may also see this as a form of injectivity of a parametrisation by points in I_0.

(iii). (Bounding domain). For any finite word $w \in \Sigma_A^k$ the sets $S_w := \bigcup_{i:A(w,i)=1} \phi_i I_0$ and $S_k := \bigcup_{w\in\Sigma_A^k} \phi_w(I_0)$ are $n-1$-dimensional topological planes. The boundary of I_0 is contained in the boundaries of S_w, S_k, $\partial I_0 \subset \partial S_w$ and $\partial I_0 \subset \partial S_k$. There are unique unbounded connected components Y_w and Y_k of $\mathbb{R}^n \setminus (I_0 \cup S_w)$ and $\mathbb{R}^n \setminus (I_0 \cup S_k)$, respectively. The interior of their complements are denoted by X_w and X_k. We define $\Omega_k := \operatorname{int}(\mathbb{R}^n \setminus \operatorname{int} Y_k)$ and $D_w := \phi_w X_w$.

(iv). (Nestedness). The open sets $\{\Omega_k\}_{k\in\mathbb{N}}$ are nested, i.e. $\Omega_k \subset \Omega_{k+1}$ and all Ω_k are contained in some common sufficiently large set. As a consequence (see for example [42]) $\Omega_k \overset{\text{H}}{\to} \overline{\bigcup_{k\geq 1} \Omega_k}$. We then define $\Omega := \operatorname{int}\overline{\bigcup_{k\geq 1} \Omega_k}$.

Seed foliation. Let $k \in \mathbb{N}$ and $w \in \Sigma_A^k$ be arbitrary. We manually construct a family of paths $\{\gamma^w\}_{\gamma^w\in\Gamma_0(w)}$ differentiable up to at most finitely many points and of finite length $\operatorname{len}\gamma^w \leq T$ (independent of w) and parametrised by arc-length foliating X_w. We require from this family of paths that for all paths γ^w one has $\gamma^w(t) \in I_0$ iff $t = 0$ and either $\gamma^w(t) \in \partial X_w \setminus I_0$ iff $t = \operatorname{len}\gamma^w$ or $\operatorname{len}\gamma^w = 0$. The set $\{\Gamma_0(w) : w \in \Sigma_A^*\}$ will be called *seed foliation* from hereon. We define $\pi : \bigcup_w X_w \to I_0$ that maps points in a path $\gamma \in \Gamma_0$ to its starting point in I_0. We will see in Prop. 3.14 that in the case of a p-Rohde snowflake with $p < (\sqrt{3}-1)/2$, the set of X_w to be considered contains only two fundamental sets. Perhaps the most obvious kind of seed foliation and also the one mostly used in this work will be called *equidistant seed foliation*. It is defined for planar sets by the condition that the ratio of the distances between the endpoints and the starting points of any two fibres in the seed foliation is constant for any connected component of the foliated set.

We finish the construction with an iteration procedure that produces a set of paths as requested in Def. 3.3 based on a seed foliation in the above generalised setup.

Iteration procedure. Let $\Omega \ni x$ be arbitrary. Then, by (iv) in the general setting, either there is a minimal k with $x \in \Omega_{k+1}$ but $x \notin \Omega_k$ or $x \in \Omega_1$. In other words, x lies in a set bounded by $\left(\bigcup_{w\in\Sigma_A^{k+1}} \phi_w I_0\right) \cup \left(\bigcup_{w\in\Sigma_A^k} \phi_w I_0\right)$ for some $k \geq 0$. But

$$\left(\bigcup_{w\in\Sigma_A^{k+1}} \phi_w I_0\right) \cup \left(\bigcup_{w\in\Sigma_A^k} \phi_w I_0\right) = \bigcup_{w\in\Sigma_A^k} \partial D_w$$

so that there is a $w \in \Sigma_A^k$ with $x \in D_w$. Then, by definition, $\phi_w^{-1}x \in X_w$ and the path in $\Gamma_0(w)$ starting at $x_w := \pi\phi_w^{-1}x$, denoted by $\gamma_{x_w}^w$, runs through $\phi_w^{-1}x$. Then $\phi_w\gamma_{x_w}^w$ is a path in D_w through x. Its starting point $\phi_w x_w$ lies in $\phi_{\overline{\sigma}w}I_0$ by assumption on the seed foliation. But then there is a path $\gamma_{x_{\overline{\sigma}w}}^{\overline{\sigma}w}$ in $\Gamma_0(\overline{\sigma}w)$ through $\phi_{\overline{\sigma}w}^{-1}(\phi_w x_w)$ which starts at $x_{\overline{\sigma}w} := \pi\phi_{\overline{\sigma}w}^{-1}(\phi_w x_w)$. Therefore there the concatenated path $\phi_w\gamma_{x_w}^w * \phi_{\overline{\sigma}w}\gamma_{x_{\overline{\sigma}w}}^{\overline{\sigma}w}$ goes from x via $\phi_w x_w$ through $\phi_{\overline{\sigma}w}x_{\overline{\sigma}w}$ in $D_w \cup D_{\overline{\sigma}w}$. In general one defines $x_{\overline{\sigma}^\ell w} := \pi\phi_{\overline{\sigma}^\ell w}^{-1}x_{\overline{\sigma}^{\ell-1}w}$ and the concatenated path $\gamma_{q(x)} := \phi_w\gamma_{x_w}^w * \phi_{\overline{\sigma}w}\gamma_{x_{\overline{\sigma}w}}^{\overline{\sigma}w} * \cdots * \phi_{\overline{\sigma}^k w}\gamma_{x_{\overline{\sigma}^k w}}^{\overline{\sigma}^k w}$ traverses Ω from x to a base point $q(x) \in I_0$. Conjugating the projection map π from the seed foliation with ϕ_w, i.e. $\pi_w := \phi_w\pi\phi_w^{-1}$, results in a map $\pi_w : D_w \to \phi_w I_0$ mapping points $x \in D_w$ to the starting point of the path through x within D_w. The construction above yields a family of paths Γ covering Ω sharing the same properties as the seed foliation covering Ω_1 up to differentiability; the global paths are differentiable almost everywhere.

Each path in the set Γ of such constructed paths (from hereon called *fibre*), traverses a unique point $q \in I_0$. Therefore one may parameterise Ω by $\gamma_q(t)$ for $t \in (0, \operatorname{len}\gamma_q)$ and $q \in I_0$. We let $\varphi : (q,t) \mapsto \gamma_q(t)$ denote the associated change of coordinates, let $q \in \operatorname{int} I_0$ be fixed and $t > 0$ be such that $x := \gamma_q(t) \in \operatorname{int} D_w$ for some $w \in \Sigma_A^k$. Further, we let $t_w < t$ denote the curve parameter of γ_q of the intersection point q_w of $\phi_w(I_0)$ with γ_q. To simplify notation, we set $\xi := \varphi|_{\varphi^{-1}(D_\emptyset)} : (q,t) \mapsto \gamma_q(t) \in D_\emptyset$. Analogously to the construction of π_w we define ξ_w mapping (q_w, t) with $q_w \in \phi_w(I_0)$ and small enough t to the point in D_w whose fibre runs through q_w and has a length t from q_w. Then $\varphi(q,t) = \xi_w(\pi_{\overline{\sigma}w}^{-1}\pi_{\overline{\sigma}^2 w}^{-1}\cdots\pi_{\overline{\sigma}^k w}^{-1}q, t - t_w)$. Notice that by definition $\pi = \pi_\emptyset$.

By Prop. 3.5 and Thm. 3.6, the density function $\beta := |\det D\varphi|$ of the change of coordinates is defined almost everywhere and satisfies

$$\beta(q,t) = \left|\det\left(\frac{\partial}{\partial q}\left(\xi_w(\pi^{-1}_{\overline{\sigma}w}\circ\cdots\circ\pi^{-1}_{\overline{\sigma}^k w}q, t-t_w)\right), e_\alpha\right)\right| \tag{3.3}$$

with a unit vector e_α that lies tangential to the curve γ_q at curve parameter t. Whenever $\{\phi_i\}_{i\in\Sigma}$ are similarity maps, one may further simplify this based on the canceling of $D\phi_w(x)$ and $D\phi_w^{-1}(y)$ even if $x \neq y$. In the case of triangulations (see Fig. 3.1) when $\{\partial_{q_i}\varphi(q,t)\}_{i=1,\cdots,n-1}$ spans an $(n-1)$-dimensional polygon P_{n-1} and $e_\alpha := \partial_t\varphi(q,t)$ are locally constant in q and t, then $\beta(q,t) = \mathrm{vol}_{n-1}(P_{n-1})\sin\alpha$, where α is the angle between the plane P_{n-1} and e_α.

This quantity is readily computed for triangles, which will play a crucial role later (see also Sec. 6.1.2). An example is illustrated in Fig. 3.1.

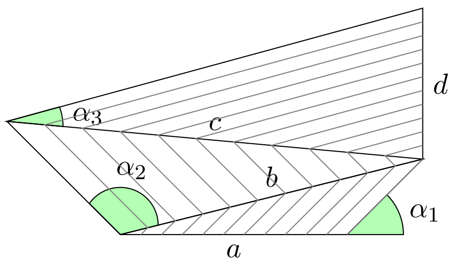


Figure 3.1: Basic case of a 'natural' foliation in a sequence of triangles. In such a situation an elementary calculation reveals that the density function β is given by $\beta_1 = \sin\alpha_1$ in the bottom triangle, $\beta_2 = \frac{b}{a}\sin\alpha_2$ in the middle triangle and $\beta_3 = \frac{c}{a}\sin\alpha_3$ in the top triangle.

The collection of fibres thus constructed gives a partition of the domain into rectified curves that are differentiable almost everywhere. We glue $E := I_0 \times (-r, 0)$ to the set Ω and extend each fibre γ_q through E by setting $\gamma_q(t) := q + e_n$ for $-r < t < 0$ where $e_n \perp I_0$ using Prop. 3.4. We adjust the parametrisation by $t \mapsto t + r$ so that any such extended path γ is parameterised by an interval $(0, \mathrm{len}\,\gamma)$. After the gluing of E, $\sup_{q\in I_0} \mathrm{len}\,\gamma_q > 0$.

Remark 3.11. By Thm. 3.6, bi-Lipschitz maps satisfy the necessary differentiability conditions above. However it was shown by Hata in [40] that α-Hölder maps are in general not almost everywhere differentiable. Actually for any $\alpha < 1$ there is an α-Hölder continuous map that is not Lebesgue-almost everywhere differentiable.

Definition 3.12. A set as constructed above with $E = I_0 \times (-r, 0)$ is called *regular well-foliated domain* if (i) and (ii) in Def. 3.3 are fulfilled.

We finish this section by noticing that in many cases the lengths of the paths constructed above are finite. This is of great importance in later applications to eigenvalues. This importance is visible for example from the proof of Lem. 3.8 as the length of corresponding paths must be finite in such cases.

Proposition 3.13. *Let ϕ_i be Lipschitz maps with Lipschitz constants L_i for $i \in \Sigma$. If $L_i \leq L < 1$ for all $i \in \Sigma$, all thus constructed paths have finite length.*

Proof. Let the length of paths in Γ_0 foliating Ω_1 be bounded from above by T. Let $x \in D_w$ for some finite word $x \in \Sigma_A^k$ and write γ for the path through x. Then

$$\operatorname{len} \gamma := \sum_{\ell=0}^{k} \operatorname{len}\left(\phi_{\overline{\sigma}^\ell w} \gamma_{x_{\overline{\sigma}^\ell w}}^{\overline{\sigma}^\ell w}\right) \leq T \sum_{\ell=0}^{k} L^{k-\ell} \leq \frac{T}{1-L}.$$

Since this bound is independent of the length of the word, the length of any path foliating Ω is also bounded like this. □

3.3 Application to domains bounded by p-Rohde curves

In this section we give a more in-depth discussion of regular well-foliated domains with a foliation as introduced in Sec. 3.2 whose boundary is given by the union of I_0 and a p-Rohde curve. It will later be shown in Thm. 4.7 and Prop. 4.9 that this setting covers a reasonably large class of domains for our studies. We discuss a result applicable to p-Rohde snowflakes whenever $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$ showing that domains whose boundary is given by the union of some I_0 and a p-Rohde curve are regular well-foliated domains.

Proposition 3.14. *Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $\partial\Omega$ consists of a unit interval $[0,1] \times \{0\} \subset \mathbb{R}^2$ and a p-Rohde curve with $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$. Then there is a foliation of Ω making it regular well-foliated.*

Proof. We explicitly construct a suitable foliation and check its properties. As before let $\{\phi_i^p\}_{i=1,\cdots,4}$ be the four self-similar maps for the Koch snowflake with parameter $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$ and let $\phi_i(x) := \phi_{i-4}^{1/4}$ for $i \in \{5, \cdots, 8\}$. Then for $\Sigma := \{1, \cdots, 8\}$, any fixed p-Rohde curve has a composition rule A such that the limit domain Ω has a boundary given by the union of $I_0 = [0, 1]$ and the p-Rohde curve. By definition the maps are bi-Lipschitz with Lipschitz constant given by $p \leq \frac{\sqrt{3}-1}{2} < 1$ and satisfy conditions (i)-(iv) of the general setting in Sec. 3.2. Next, we define a seed foliation. To this end, notice that there are at most two distinct X_w's to be taken into consideration.

(i). If $\exists w \in \Sigma_A^k$ for which $\{i \in \Sigma : A(w, i) = 1\} = \{1, \cdots, 4\}$, we use vertical lines as paths. In other words, we define $\pi : X_w \to I_0$ as the usual projection to $\mathbb{R} \times \{0\}$, see Fig. 3.2(a).

(ii). If $\exists w \in \Sigma_A^k$ for which $\{i \in \Sigma : A(w, i) = 1\} = \{5, \cdots, 8\}$, the foliation to be constructed is forcedly trivial as no path of non-zero length is possible. Therefore one has $\pi : X_w \to I_0$ by $x \mapsto x$. One may also identify this with the usual projection of the aforementioned case.

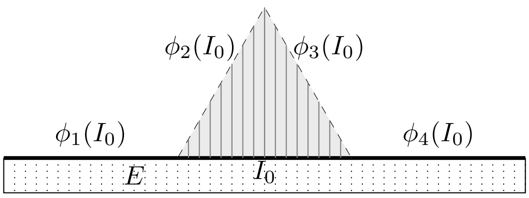


Figure 3.2(a) Example of a seed foliation in $X_\emptyset$ in the case of a $p = \frac{1}{3}$-Rohde curve whose first construction uses the maps $\{\phi_i[p]\}_{i\in\{1,\cdots,4\}}$. It connects I_0 (a unit interval) and $\Phi(I_0)$ (the first iteration of the Koch curve, dashed) given by its four maps $\{\phi_i[p]\}_{i=1,\ldots,4}$ by straight lines. Additionally a box E has been added underneath I_0 as needed in Lem. 3.8 and the fibres have been extended (in dotted lines) through E.

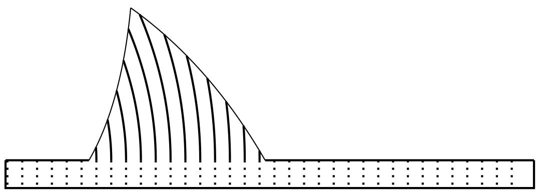


Figure 3.2(b) Example of a seed foliation of the conformally distorted Koch curve described in Prop. 3.7 with $I_0 = [2, 17]$ under the conformal map $f : \mathbb{R}^2 \setminus B_1(0) \to \mathbb{R}^2 \setminus B_1(0)$, $(x, y) \mapsto (x^2 - y^2, 2xy)$.

By construction $\operatorname{ess\,inf} \beta = 1$, each fibre γ_q is parametrised by arc length and by Prop. 3.13 the total length of any path is uniformly bounded. Finally, Def. 3.3(ii) is verified as follows: Since the constructed foliation is piecewise parallel, β is piecewise constant. Any fibre γ constructed as above can be split into fibres $\phi_w \gamma_{x_w}^w, \phi_{\overline{\sigma} w} \gamma_{x_{\overline{\sigma} w}}^{\overline{\sigma} w}, \cdots, \phi_{\overline{\sigma}^k w} \gamma_{x_{\overline{\sigma}^k w}}^{\overline{\sigma}^k w}$ through

$D_w, D_{\overline{\sigma}w}, \cdots, D_{\overline{\sigma}^k w}$ for some $k \in \mathbb{N}$ and

$$\sup_{\gamma\in\Gamma}\int_0^{\operatorname{len}\gamma}\beta dt = \sup_{q\in I_0}\int_0^{\operatorname{len}\gamma_q}\beta(q,t)dt \leq \sup_{w\in\Sigma_A^*}\sum_{\ell=1}^{\infty}\underbrace{\left(\sup_{\varphi(q,t)\in D_{w|_\ell}}\beta(q,t)\right)}_{\leq\left(\frac{2p}{1-2p}\right)^\ell}\underbrace{\left(\sup_{q\in I_0}\operatorname{len}(\phi_{w|_\ell}(\gamma_q))\right)}_{\leq p^\ell\sqrt{p-\frac{1}{4}}}$$
$$\ll \sum_{\ell\geq 1}\left(\frac{2p^2}{1-2p}\right)^\ell < \infty,$$

by hypothesis on p. □

Remark 3.15. There is a correspondence between expansions of $q \in I_0$ to certain bases and the geometry of a fibre γ in a p-Rohde snowflake: To each starting point $q \in I_0 = [0,1]$ there are at most two expansions coming from the dynamical system given by

$$\begin{cases} f^p : [0,1] \to [0,1] \quad \text{via} \quad x \mapsto f_i^p(x) \quad \text{for } i \in \{1,\cdots,4\}, \text{ with } f_i^p : J_i^p \to [0,1], \\ f^{1/4} : [0,1] \to [0,1] \quad \text{via} \quad x \mapsto f_i^{1/4}(x) \quad \text{for } i \in \{5,\cdots,8\}, \text{ with } f_i^{1/4} : J_i^{1/4} \to [0,1], \end{cases}$$

where $J_1^p = [0,p), J_2^p = [p,1/2), J_3^p = [1/2,1-p), J_4^p = [1-p,1]$ and $f_i^p(x) = \frac{x-\inf J_i}{|J_i|}$. The resulting expansion code for a $q \in I_0$ may be written as $w(q) = (a_1a_2a_3\cdots)$ for $a_i \in \{1,\cdots,8\}$. An immediate piece of information about q given in the code is that the number of left and right turns corresponds to the number of 2s (left) and 3s (right): $\#(\text{left resp. right turns}) = \max\{0, \#(i : a_i = 2 \text{ resp. } 3) - 1\}$. Moreover, if $a_j \in \{2,3\}$, the number of consecutive digits $\notin \{2,3\}$ corresponds to the power of the scaling factor of the next-level triangle. In the examples in Fig. 3.3 this means that the fibre with $q_{\text{red}} = \frac{55}{108}$ with an expansion code of $w(q_{\text{red}}) = (3113)$ goes up, "stays still" for two iterations to then proceed straight up to terminate in the first equilateral triangle at the right side of size 3^{-4}. Similarly, $q_{\text{blue}} = \frac{238}{405}$ with an expansion code of $w(q_{\text{blue}}) = (33124\overline{2})$. This implies that the fibre through q_{blue} turns right into the first equilateral triangle to the right of size 3^{-1}, "stays still" for one iteration to then go up into the a equilateral triangle of size 3^{-4}. From there it turns left, rests for one iteration and then turns left for every next iteration for ever. Finally, the orange path corresponding to $q_{\text{orange}} = \frac{97}{162}$ with an expansion code of

$w(q_{\text{orange}}) = (3332)$ runs straight into the next smaller equilateral triangle twice and then terminates on its left side.

The given code for each path γ_q given by its base point $q \in I_0$ also equals the code of its final point $\gamma_q(\ell_\gamma) \in \partial K$ in the sense of the IFS. This also gives a quick method to find q for special paths.

Example 3.16. Let γ_q be the path in the fully homogeneous p-Rohde snowflake as foliated in Prop. 3.14 that goes upwards and then turns left into an infinite left spiral. It corresponds to a base point $q \in I_0$ with the code $\overline{2}$ and therefore is given by the intersection of $\mathrm{id}_{I_0} : x \mapsto x$ and $f_2 : x \mapsto \frac{x-p}{\frac{1}{2}-p}$. So $q = \frac{2p}{2p+1} \in I_0$.

In contrast to Prop. 3.14 we show that the construction in the proof of Prop. 3.14 does not necessarily fulfil the condition of Def. 3.3(ii). For ease of notation, we denote by $\overline{2}$ the infinite word consisting of a recurring 2 in an alphabet Σ and so on.

Proposition 3.17. *Let $p \in \left[\frac{\sqrt{3}-1}{2}, \frac{1}{2}\right)$ and suppose the composition rule A of a p-Rohde curve is such that there is a finite word w_0 with $w_0\overline{\kappa}$ with $\kappa \in \{2,3\}$. Then the construction of Prop. 3.14 violates Def. 3.3(ii). Therefore the domain obtained by gluing a rectangle E as before to the open set whose boundary is given by the union of that p-Rohde curve and $I_0 = [0,1]$ is not well-foliated.*

Proof. Suppose without loss of generality $w_0\overline{2} \in \Sigma_A^*$ and $w_0 = \emptyset$. Otherwise consider $\phi_{w_0}^{-1}\Omega$ and reflect to interchange $2 \leftrightarrow 3$. Consider the fibre starting at $Q := \frac{2p}{2p+1} \in I_0$. By a simple geometric argument in Ex. 3.16, this fibre runs through $D_\emptyset, D_2, D_{22}, D_{222}, \cdots$. Since β is locally constant with value $\left(\frac{2p}{1-2p}\right)^\ell$ along every sub-fibre $\phi_{\overline{2}|_\ell}\left(\gamma_{x_{\overline{2}|_\ell}}^{\overline{2}|_\ell}\right)$ for the recurring word $\overline{2}$, one has

$$\sup_{q \in I_0} \int_0^{\operatorname{len}\gamma_q} \beta(q,t)dt \geq \int_0^{\operatorname{len}\gamma_Q} \beta(Q,t)dt = \sum_{\ell=1}^{\infty} \left(\frac{2p}{1-2p}\right)^\ell \underbrace{\left(\operatorname{len}(\phi_{\overline{2}|_\ell}(\gamma_Q))\right)}_{=p^\ell\left(\frac{\sqrt{2p-1}(p+1)}{2(p-2)(2p-1)}\right)}$$

$$\asymp \sum_{\ell=0}^{\infty} \left(\frac{2p^2}{1-2p}\right)^\ell = \infty$$

as $p > \frac{\sqrt{3}-1}{2}$. □

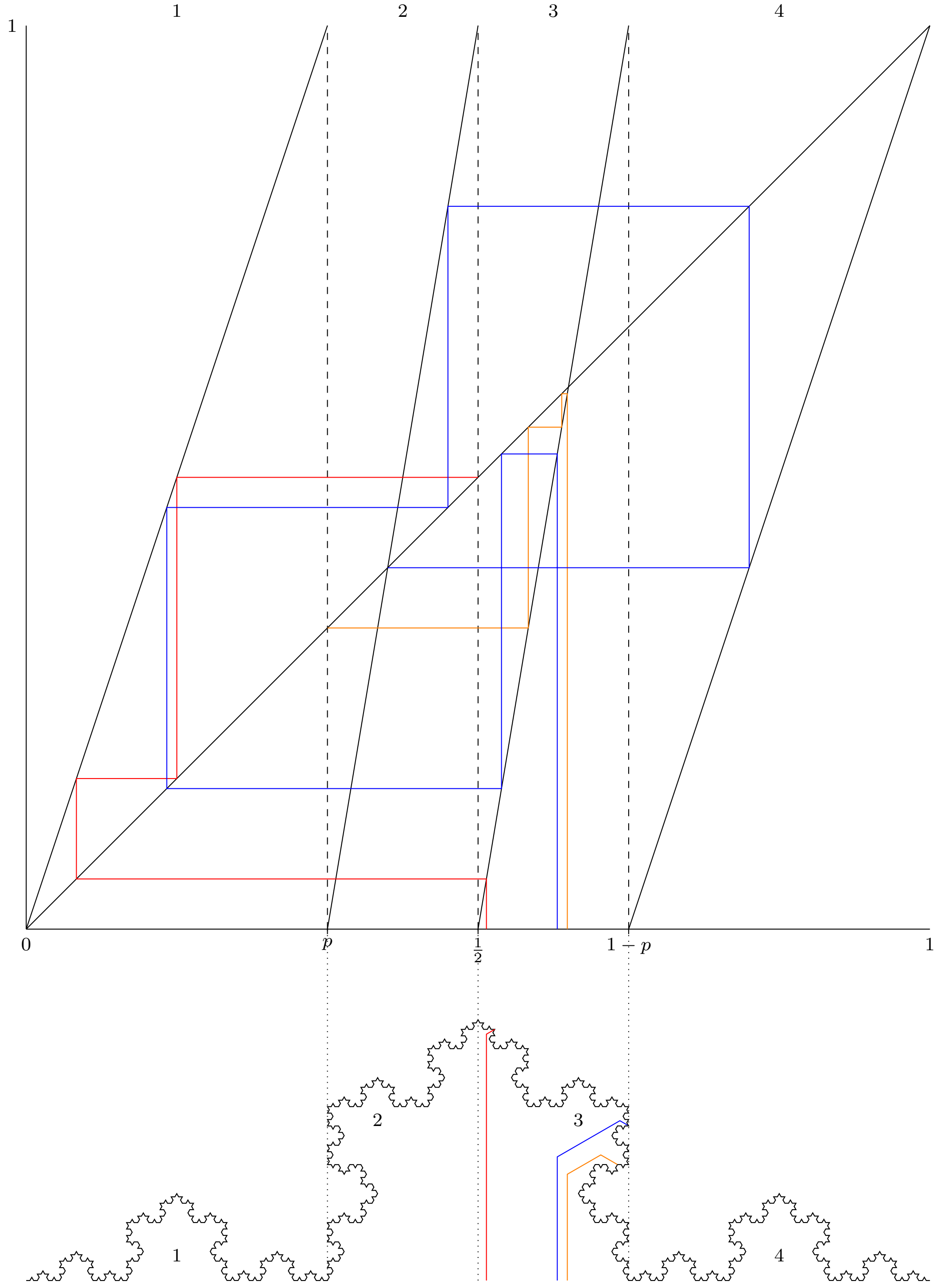


Figure 3.3: Decompositions corresponding to three different paths (cf. Fig. 4.5) of a classical Koch snowflake as fully homogeneous $\frac{1}{3}$-Rohde snowflake. The decomposition of the base points of the red line then read $w\left(q_{\text{red}} = \frac{55}{108}\right) = (3113)$. For the blue curve, one has $w\left(q_{\text{blue}} = \frac{238}{405}\right) = \left(33124\overline{2}\right)$ and finally the code of the orange leave reads $w\left(q_{\text{orange}} = \frac{97}{162}\right) = (3332)$.

This shows that there are p-Rohde snowflakes, in particular with dimensions approaching 2 for which the seed foliation proposed in Prop. 3.14 does not lead to a well-foliated domain. The rest of this section is dedicated to a few instructive examples.

Example 3.18 (Square snowflakes). Let $\tau \in \left(0, \frac{1}{3}\right]$. We consider a class of square Koch curves $SK(\tau)$ depending on τ (see Fig. 3.4) with an equidistant seed foliation and show that this construction allows for a well-foliation iff a snowflake bounded by four of these curves is a quasidisk, cf. Fig. 3.6. More precisely, we show that Def. 3.3(ii) is violated

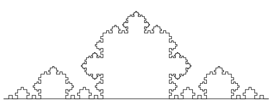


Figure 3.4(a) $3\tau = 0{,}75$.

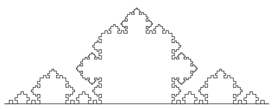


Figure 3.4(b) $3\tau = 0{,}8$.

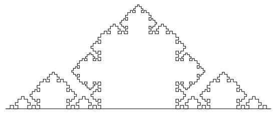


Figure 3.4(c) $3\tau = 0{,}85$.

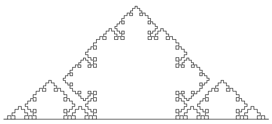


Figure 3.4(d) $3\tau = 0{,}9$.

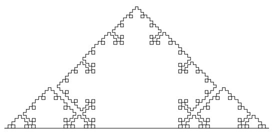


Figure 3.4(e) $3\tau = 0{,}95$.

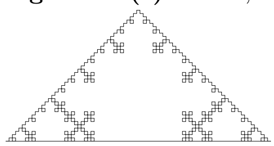


Figure 3.4(f) $3\tau = 1$.

Figure 3.4: 4^{th} iteration of sets bounded by square snowflakes curves $SK(\tau)$ and $[0,1]$ in Ex. 3.18 at various instances of $\tau \in (0, 1/3]$ approaching $\frac{1}{3}$. Notice that $SK(1/3)$ is not a part of a boundary of a quasidisk.

in the case of an open set $SQ(\tau)$ bounded by $SK(\tau)$ and $I_0 := [0,1]$ (see Fig. 3.5) with a equidistant seed foliation precisely for $\tau = \frac{1}{3}$. The sets $SK(\tau)$ are constructed as limit set of a self-similar IFS: Let $\Sigma := \{1, \cdots, 5\}$ and let $\{\phi_i\}_{i \in \Sigma}$ be the following bi-Lipschitz contractions:

$$\phi_1(x) := \frac{1-\tau}{2}x, \quad \phi_2(x) := \tau R_{\pi/2}(x) + \begin{pmatrix} \frac{1}{2}(1-\tau) \\ 0 \end{pmatrix}, \quad \phi_3(x) := \tau x + \begin{pmatrix} \frac{1}{2}(1-\tau) \\ \tau \end{pmatrix},$$
$$\phi_4(x) := \tau R_{-\pi/2}(x) + \begin{pmatrix} \frac{1}{2}(1+\tau) \\ \tau \end{pmatrix}, \quad \phi_5(x) = \phi_1(x) + \begin{pmatrix} \frac{1+\tau}{2} \\ 0 \end{pmatrix}.$$

where R_α is the usual rotation matrix on $\mathbb{R}^2$ for angle $\alpha \in [0, 2\pi)$. We define an *equidistant seed foliation* by defining paths as shown in Fig. 3.5 so that $\phi_2(I_0), \phi_3(I_0)$ and $\phi_4(I_0)$ are each reached by paths intersecting I_0 in $\left[\frac{1-\tau}{2}, \frac{1+\tau}{2}\right]$. We write $SQ(\tau)$ for the resulting open set Ω in the notation of (iv). Now consider the central path γ_Q with starting point

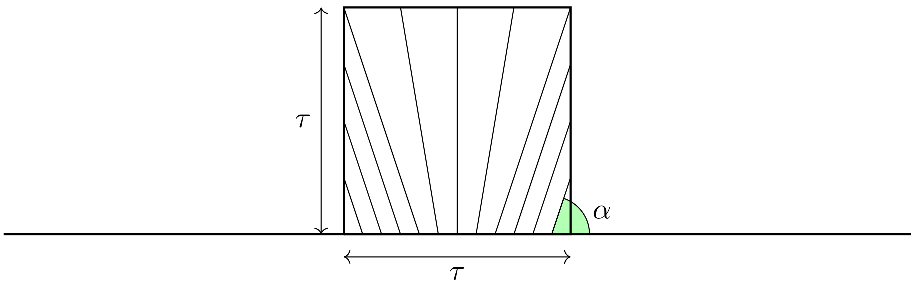


Figure 3.5: Equidistant seed foliation for a square snowflake where any measurable $X \subset \bigcup_{i=2}^{4} \phi_i I_0$ is covered by a section of I_0 with Lebesgue measure $\lambda_1(X)/3$, while $\phi_1 I_0 \cup \phi_5 I_0$ coincide with a subset of I_0 and are therefore covered trivially by paths of length 0. The construction is independent of τ and $\alpha = \arctan 3$ is the minimal angle of any path with I_0.

$Q = 1/2 \in I_0$, which traverses $D_\emptyset, D_3, D_{33}, D_{333}, \ldots$ (notation of (iii)). Along this path, β as defined in (3.3) changes by a factor 3 with each iteration step and is monotonic in between any two iteration steps. Since the considered path has a length of τ^{k+1} in the k^{th} iteration step, we arrive at

$$\sup_{q \in I_0} \int_0^{\operatorname{len} \gamma_q} \beta(q,t)dt \geq \int_0^{\operatorname{len} \gamma_Q} \beta(Q,t)dt = \sum_{\ell=1}^{\infty} 3^\ell \underbrace{\left(\operatorname{len}\left(\phi_{\overline{3}|_\ell}(\gamma_Q)\right)\right)}_{=\tau^{\ell+1}},$$

which diverges iff $\tau = \frac{1}{3}$. Independent of τ, $\beta(q,t) \in \left(3^k \sin\arctan 3, 3^{k+1}\left(1 + \frac{1}{\sin\arctan 3}\right)\right)$ at a point in $D_{w|_k}$ for any word w so that $\operatorname{ess\,inf} \beta > 0$. Likewise we have an upper bound for $\mathcal{I}_\beta$ whenever $\tau < \frac{1}{3}$. Let Γ_0 be the set of paths in the seed foliation as before. Then

$$\sup_{q \in I_0} \int_0^{\operatorname{len} \gamma_q} \beta(q,t)dt \leq \sum_{k=0}^{\infty} \left(3^{k+1}\left(1 + \frac{1}{\sin\arctan 3}\right)\tau^k \sup_{\gamma \in \Gamma_0} \operatorname{len}(\gamma)\right)$$

which diverges iff $\tau = \frac{1}{3}$. Similarly to Ex. 2.17) one can show that $SK(\tau)$ satisfies the Ahlfors arc criterion in Prop. 2.16 iff $\tau < \frac{1}{3}$. This shows that the sets $SQ(\tau)$ are well-foliated whenever their boundary is part of a quasicircle.

A different kind of curve was studied by Knopp who showed that curves such as the Koch curve belong to a wider class of Koch-like curves which sometimes have non-trivial Lebesgue measure.

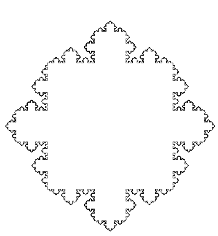


Figure 3.6(a) $3\tau = 0{,}75$.

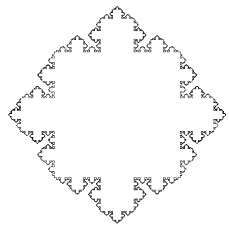


Figure 3.6(b) $3\tau = 0{,}8$.

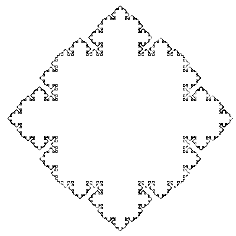


Figure 3.6(c) $3\tau = 0{,}85$.

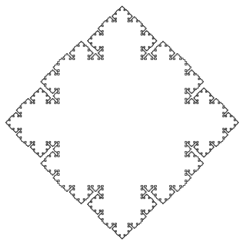


Figure 3.6(d) $3\tau = 0{,}9$.

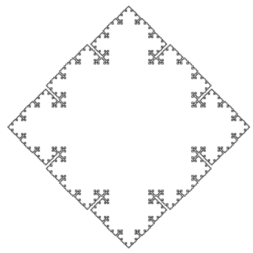


Figure 3.6(e) $3\tau = 0{,}95$.

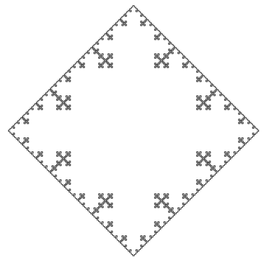


Figure 3.6(f) $3\tau = 1$.

Figure 3.6: 4^{th} iteration of square snowflakes domains (i.e. the domain bounded by four copies of $SK(\tau)$) at the same instances of $\tau \in (0, 1/3]$ as shown in Fig. 3.4. Notice that the domain for $\tau = \frac{1}{3}$ is not a quasidisk.

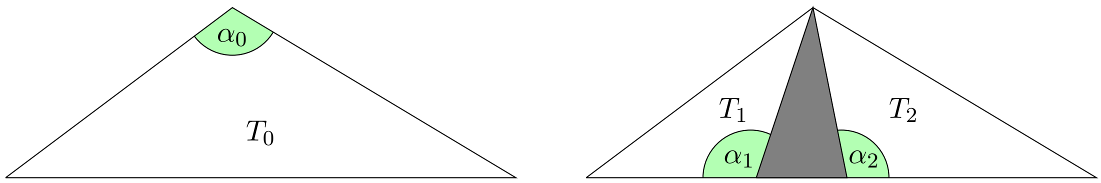


Figure 3.7: First two construction steps of Knopp curves, as described in [56].

Knopp curves. We study a class of curves introduced by Knopp in [56] as a further instance of snowflake-like curves that may be addressed in the construction in Sec. 3.2. Since the original reference is not commonly available, we give a brief summary of the result. Consider a curve constructed in the following way: Let T_0 be an obtuse triangle of angle $\alpha_0 > 90°$ and base interval $I_0 = [0, 1]$. Chose a ratio θ_1 and a sub-interval of the base side of T_0 corresponding to a sub-triangle T' to be removed from T_0 with $\mathrm{vol}_2(T') = \theta_1 \,\mathrm{vol}_2(T_0)$ as shown in Fig. 3.7 such that $T_0 \setminus T'$ consists of two obtuse triangles T_1, T_2 both of which have angles α_1, α_2 with $90° < \alpha_i \leq \alpha_0$ for $i = 1, 2$. Now choose a ratio θ_2 and repeat this procedure with T_1 and T_2 (using the same θ_2 for both T_1 and T_2) resulting in four triangles $T_{11}, T_{12}, T_{21}, T_{22}$ and so on, giving a family of triangles indexed by finite words in $\{1, 2\}^n$

for any $n \in \mathbb{N}$. Since the triangles are all measurable sets that are disjoint up to 2 points, one may compute the area taken away at every iteration by

$$A_0 = \mathrm{vol}_2\, T_0\,, \quad A_{n+1} := A_n \cdot (1 - \theta_n),$$

so that $\lim_{n\to\infty} A_n = \mathrm{vol}_2(T_0) \prod_{n=1}^{\infty}(1-\theta_n)$. In other words, $\lim_{n\to\infty} A_n$ has non-trivial volume if $\sum_n \theta_n$ converges. Indeed, in this setting Knopp showed the following.

Theorem 3.19 ([56]). *The limit set $K := \bigcup_{w\in\{0,1\}^\infty} \bigcap_{n\in\ltimes} T_{w|_n}$ is a nowhere differentiable Jordan curve. Moreover it is an Osgood curve (i.e. $\mathrm{vol}_2(K) \neq 0$) iff $\sum_{n\in\mathbb{N}} \theta_n$ converges.*

Now Prop. 2.18 implies that a Knopp curve is not a quasicircle if $\sum_n \theta_n = \infty$. More precisely, we formulate a simple criterion for Knopp curves (cf. also Ex. 2.17).

Proposition 3.20. *Let K be a Knopp curve to a sequence θ_i of ratios as above. Then K is part of the boundary of a quasidisk if $\inf_i \theta_i > 0$.*

The complement of the second iteration (cf. Fig. 3.7) is given by a triangle $T' = T_0 \setminus (T_1 \cup T_2)$. The fourth iteration than additionally removes four triangles that are connected either to I_0 or to T'. More generally the union of all triangles that are removed at an even iteration step of this procedure gives a snowflake-like set whose boundary is given by the union of the base interval I_0 and the Knopp curve. Therefore this class of Knopp curves provides additional examples for the construction of well-foliated domains as discussed in Sec. 3.2. In this context, the above Prop. 3.20 guarantees emptiness of the essential Neumann spectrum of any domain whose boundary consists of unions of such Knopp curves.

Chapter 4

Weyl asymptotics of spectral counting functions

In this chapter we introduce a class of domains (Def. 4.1) whose inner ϵ-parallel neighbourhood can be covered by well-foliated domains in a controlled manner allowing estimates of the spectral counting function of the Neumann problem on that domain. We study the conditions under which this framework covers p-Rohde snowflakes and show how the framework relates to a related approach by Netrusov-Safarov [80] in Sec. 4.2. Next we prove our main result, Thm. 4.15 and apply this to snowflakes.

4.1 Well-covered domains

Definition 4.1. A domain $\Omega \subset \mathbb{R}^n$ is called *well-covered* if there exists $\mu \in \mathbb{N}$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ there is a volume cover of $\Omega_{-\epsilon}$ by well-foliated domains $\{D_i^\epsilon \subset \Omega\}_{i \in I_\epsilon}$ with parameters $(r_i^\epsilon, L_i^\epsilon, \mathcal{I}_{\beta_i}^\epsilon, \beta_{\inf,i}^\epsilon, E_i^\epsilon)$ as in Def. 3.3 and multiplicity of the volume cover uniformly bounded from above by μ such that

(i). $\inf_{0<\epsilon\leq\epsilon_0} \inf_{i\in I_\epsilon} \beta_{\inf}(D_i^\epsilon) > 0$.

(ii). There is a c_E independent of $i \in I_\epsilon$ and $\epsilon \in (0, \epsilon_0]$ such that any E_i^ϵ has first non-trivial Neumann eigenvalue $\lambda_2^N(E_i^\epsilon) \geq c_E\,(r_i^\epsilon)^{-2}$.

The second condition, (ii), is often satisfied automatically by the below result of Payne-Weinberger, [83] (see also [1]).

Theorem 4.2 ([83])**.** *Let $E \subset \mathbb{R}^2$ be convex and let $\lambda_2^N(E)$ be the first non-trivial Neumann eigenvalue of E. Then*

$$\lambda_2^N(E) \geq \left(\frac{\pi}{\operatorname{diam} E}\right)^2 .$$

Remark 4.3. The reverse inequality in Def. 4.1.(ii) is often automatically satisfied. More precisely, based on a result by Szegő in [105], Weinberger showed a Faber-Krahn-type isoperimetric inequality in [110]:

$$\lambda_2^N(\Omega) \leq \lambda_2^N(\Omega^*) = \frac{p_n^2}{\left(\frac{1}{2}\operatorname{diam}\Omega^*\right)^2} = p_n^2\left(\frac{\omega_n}{\operatorname{vol}_n \Omega}\right)^{\frac{2}{n}},$$

where Ω^* is the ball with $\operatorname{vol}_n(\Omega^*) = \operatorname{vol}_n(\Omega)$, ω_n is the volume of the n-dimensional unit ball and p_n is the first positive zero of $\left(x^{1-\frac{n}{2}} J_{\frac{n}{2}}(x)\right)'$, i. e. the smallest positive solution of $J_{\frac{n}{2}}(x) = xJ_{1+\frac{n}{2}}(x)$ where $J_m(x)$ is the m-th spherical Bessel function.

The cardinality of the cover $\{D_i^\epsilon\}_{i\in I_\epsilon}$, denoted by $\#I_\epsilon$ in Def. 4.1 will play an important role in Thm. 4.15. Next we show that it is closely related to upper inner Minkowski measurability of Ω.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain for which there are $r_\pm > 0$ and $v_\pm > 0$ and $\mu \in \mathbb{N}$ such that for every sufficiently small $\epsilon > 0$ there is a finite volume cover $\bigcup_{i\in I_\epsilon} D_i^\epsilon$ of $\Omega_{-\epsilon}$ with multiplicity $\leq \mu$ and*

$$\Omega_{-\epsilon r_-} \subset \operatorname{int}\overline{\bigcup_{i\in I_\epsilon} D_i^\epsilon} \subset \Omega_{-\epsilon r_+} \qquad \textit{and} \qquad \operatorname{vol}_n(D_i^\epsilon)\epsilon^{-n} \in [v_-, v_+]. \tag{4.1}$$

Then Ω has finite upper inner Minkowski content at dimension δ iff one has $\#I_\epsilon \asymp \epsilon^{-\delta}$ for all sufficiently small ϵ.

We will later show in Thm. 4.7 that these conditions are satisfied when Ω is a p-Rohde snowflakes with $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$. More generally:

Proposition 4.5. *Let Ω be a quasidisk. Then there are $r_\pm > 0$, $v_\pm > 0$ and for any sufficiently small $\epsilon > 0$ there is a volume cover $\{D_i^\epsilon\}$ of $\Omega_{-\epsilon}$ as in defined in Prop. 4.4.*

We will prove Prop. 4.5 after Thm. 4.7.

Proof of Prop. 4.4. By Def. 2.29 the upper inner Minkowski content at dimension δ exists and is non-zero, if $\operatorname{vol}_n(\Omega_{-\epsilon}) \asymp \epsilon^{n-\delta}$. Suppose $\#I_\epsilon \asymp \epsilon^{-\delta}$. Then

$$\operatorname{vol}_n(\Omega_{-\epsilon}) \le \sum_{i\in I_{\epsilon/r_-}} \operatorname{vol}_n(D_i^{\epsilon/r_-}) \le \#I_{\epsilon/r_-} v_+ (\epsilon/r_-)^n \ll \epsilon^{n-\delta}$$

gives an upper bound for the upper inner Minkowski content of Ω at dimension δ. A lower bound can be obtained as follows.

$$\operatorname{vol}_n(\Omega_{-\epsilon}) \ge \mu^{-1} \sum_{i\in I_{\epsilon/r_+}} \operatorname{vol}_n(D_i^{\epsilon/r_+}) \ge \mu^{-1} \#I_{\epsilon/r_-} v_- (\epsilon/r_+)^n \gg \epsilon^{n-\delta}$$

showing that Ω has finite and non-zero upper inner Minkowski content at dimension δ. Conversely, if the upper inner Minkowski content exists and is non-zero at dimension δ, then

$$\#I_\epsilon \epsilon^n \ge v_+^{-1} \sum_{i\in I_\epsilon} \operatorname{vol}_n(D_i^\epsilon) \ge v_+^{-1} \operatorname{vol}_n(\Omega_{-\epsilon r_-}) \gg \epsilon^{n-\delta}$$

and

$$\#I_\epsilon \epsilon^n \le v_-^{-1} \sum_{i\in I_\epsilon} \operatorname{vol}_n(D_i^\epsilon) \le \mu v_-^{-1} \operatorname{vol}_n(\Omega_{-\epsilon r_+}) \ll \epsilon^{n-\delta}$$

showing $\#I_\epsilon \asymp \epsilon^{-\delta}$. □

We consider well-covered domains whose volume covering domains D_i^ϵ have uniformly comparable geometry in the following sense.

Definition 4.6. A family of volume covers $\{\{D_i^\epsilon\}_{i\in I_\epsilon}\}_\epsilon$ of $\Omega_{-\epsilon}$ for a common domain Ω is said be of *uniform shape* (or *uniform* for short) if $\left(\lambda_2^N(E_i^\epsilon)\right)^{-1/2}, \mathcal{I}_\beta(D_i^\epsilon \setminus E_i^\epsilon), r_i^\epsilon$ and L_i^ϵ of each element D_i^ϵ (in the notation of Def. 3.3) are uniformly comparable in ϵ, i.e. if there are constants independent of i and ϵ with

$$\left(\lambda_2^N(E_i^\epsilon)\right)^{-1/2} \asymp \mathcal{I}_\beta(D_i^\epsilon \setminus E_i^\epsilon) \asymp r_i^\epsilon \asymp L_i^\epsilon \asymp \epsilon.$$

If all D_i^ϵ are regular well-foliated domains, we say that Ω is *regular well-covered*.

By Cor. 3.9, for any uniformly well-covered domain Ω (that is a well-covered by a uniform volume cover) there is a constant $C_1(\Omega)$ (cf. Sec. 4.4) with $\lambda_2^N(D_i^\epsilon) \geq C_1(\Omega)\epsilon^{-2}$ for all $i \in I_\epsilon$ and all sufficiently small ϵ.

4.1.1 Coverings of p-Rohde snowflakes by well-foliated domains

Let $p \in \left[\frac{1}{4}, \frac{1}{2}\right)$. Let Ω be a p-Rohde snowflake and $\epsilon > 0$ be small enough. We construct a cover by well-foliated domains $\{D_i^\epsilon\}_{i\in I_\epsilon}$ of $\Omega_{-\epsilon}$ where the growth behaviour of $\#(I_\epsilon)$ is linked to the upper inner Minkowski content. We show that this property is preserved under certain bi-Lipschitz transformations. The following result may be seen as the analogue to Propositions 3.7 and 3.14 in the context of well-covered domains.

Theorem 4.7. *Let $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$ and $\Omega \subset \mathbb{R}^2$ be a p-Rohde snowflake. Then Ω is uniformly regular well-covered and the conditions of Prop. 4.4 are met. If Ω is homogeneous, it is also upper inner Minkowski measurable.*

Proof. We define a volume cover of $\Omega_{-\epsilon}$ by regular well-foliated domains in the following way. We define the contraction ratios of ϕ_i in the usual way:

$$q(i) := \begin{cases} p & \text{if } i \in \{1, \cdots, 4\} \\ \frac{1}{4} & \text{if } i \in \{5, \cdots, 8\} \end{cases}$$

Recall the notation of a composition rule of well-foliated domains from Sec. 3.2 and Prop. 3.14. For any finite word $w \in \Sigma_A^{\text{fin}}$, we define $\epsilon(w) := \epsilon_0 \prod_{\ell=1}^{|w|} q(w_\ell)$ and $J(w) := (\epsilon(w), \epsilon(\overline{\sigma} w)]$. The intervals $J(w)$ agree with an approach by Lapidus-Pearse in [67] up a constant prefactor. Let K be one of the four p-Rohde curves bounding Ω and consider the domain D whose boundary is given by the union of K and $I_0 = [0,1]$. By construction (cf. Def. 2.33), D consists of sets $\{D_w\}_{w\in\Sigma_A^{\text{fin}}}$. For any such $D_w = \phi_w X_w$ we consider the union of D_w with all its higher iterations,

$$G_w := \operatorname{int}\left(\overline{D_w \cup \bigcup_{i\in\Sigma:A(w,i)=1} D_{wi} \cup \bigcup_{j\in\Sigma:A(wi,j)=1}\ \bigcup_{i\in\Sigma:A(w,i)=1} D_{wij} \cup \cdots}\right)$$

Then $\phi_w^{-1}G_w$ is a domain whose boundary is given by the union of I_0 and a p-Rohde curve. Now let $\epsilon > 0$. Then following the foliation procedure as in Sec. 3.2 we obtain a collection of regular well-foliated domains $G_w(\epsilon) := \operatorname{int}\overline{G_w \cup \varphi_w(I_0 \times [-\epsilon, 0])}$. We introduce three classes of regular well-foliated domains used for the uniform cover of $\Omega_{-\epsilon}$ by domains containing $G_w(\epsilon)$:

(i). fringed rectangles, denoted by $\mathrm{fR}_w(\epsilon)$, see Fig. 4.1(a). If w ends with 2, resp. 24 and w' ends with 3, resp. 31, we construct one domain covering both $G_w(0)$ and $G_{w'}(0)$: Take an isosceles triangle with legs of length $\prod_{\ell=1}^{|w|} q(w_\ell)$ and base I_0' of length $(1-2p)\prod_{\ell=1}^{|w|} q(w_\ell)$. We equip this triangle with an equidistant foliation (as defined in the proof of Prop. 3.14) and glue a rectangle $E_{\mathrm{fR}} = I_0' \times (-\epsilon(w), 0)$ giving a well-foliated domain T_w. We then glue G_w with the left leg and $G_{w'}$ with the right leg. If w ends with 24, resp. 28 and w' ends with 35, resp. 31, we replace the isosceles triangle with one with two legs of lengths $l_w = \prod_{\ell=1}^{|w|} q(w_\ell)$ and $l_{w'} = \prod_{\ell=1}^{|w'|} q(w'_\ell)$ and adapt the base of the triangle accordingly to length $c = \sqrt{l_w^2 + l_{w'}^2 + 2l_w l_{w'}\cos(2\alpha_p)}$. Moreover we replace E_{fR} from before with a parallelogram of height $\epsilon(w)$ and angle $\delta_o := \alpha_p - \arcsin\frac{\sin 2\alpha_p}{l_{w'}c}$, see Fig. 4.1(b). By Thm. 4.2 this satisfies 3.3(ii) with $\lambda_2^2(E_{\mathrm{fR}}) \geq \pi^2(\epsilon(w)^2 + c^2 + 2\epsilon(w)c\cos\delta_p)^{-1} \asymp \epsilon^{-2}$.

(ii). short rectangles, denoted by $\mathrm{sR}_w(\epsilon)$, see Fig. 4.2(a). If w ends with $11, 21, 34, 44$, we take the usual regular well-foliated domain consisting of $G_w(\epsilon(w))$.

(iii). long rectangles, denoted by $\mathrm{lR}_w(\epsilon)$, see Fig. 4.2(b). If w ends with $14, 41$, take $G_w(0)$ with base $\phi_w I_0$ and glue it to $E' := (0, \mathrm{vol}_1\,\phi_w I_0 + \epsilon(w)\sin\alpha_p) \times (-\epsilon(w), 0)$, where $\sin\alpha_o = \frac{\sqrt{4p-1}}{2p}$. Identify the base of $G_w(0)$ as $(0, \mathrm{vol}_1\,\phi_w I_0) \subset (0, \mathrm{vol}_1\,\phi_w I_0 +$

$\epsilon \sin \alpha_p)$ or $(\epsilon(w) \sin \alpha_p, \text{vol}_1 \phi_w I_0 + \epsilon(w) \sin \alpha_p) \subset (0, \text{vol}_1 \phi_w I_0 + \epsilon(w) \sin \alpha_p)$ and glue accordingly.

(iv). If w ends with $5, 6, 7, 8$, we proceed as with short rectangles (sR) but with p replaced with $\frac{1}{4}$.

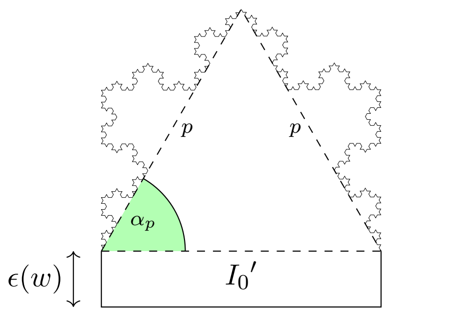


Figure 4.1(a) Construction of $\text{fR}_w(\epsilon)$ for $p = \frac{1}{3}$ as used in the proof of Thm. 4.7.

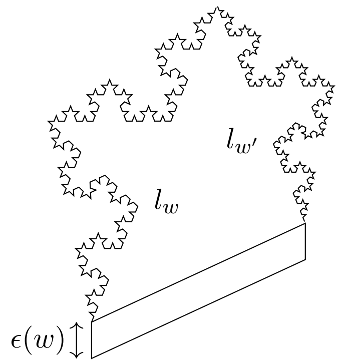


Figure 4.1(b) Extract from a p-Rohde snowflake with $p = \frac{1}{2(1+\frac{2}{\sqrt{29}})} \lesssim \frac{\sqrt{3}-1}{2}$ to illustrate the construction of $\text{fR}_w(\epsilon)$ in the event of asymmetric iterations, i.e. if w ends with 24 but w' ends with 15.

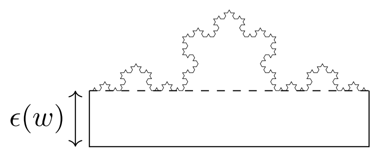


Figure 4.2(a) Construction of $sR_w(\epsilon)$ for $p = \frac{1}{3}$.

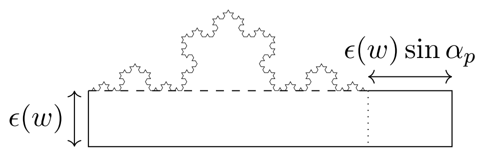


Figure 4.2(b) Construction of $\text{lR}_w(\epsilon)$ for $p = \frac{1}{3}$.

By Prop. 3.14, all four types of domains are regular well-foliated domains. For a sufficiently small given $\epsilon > 0$, we select all domains fR, lR, sR as shown in Fig. 4.3 with $\epsilon \in J(w)$. We are left to show that this cover is uniform and to verify (4.1) in Prop. 4.4. For any finite word w, the given objects cover one or two $G_w(\epsilon)$ for selected instances of w. In any case of covering domain one has $\beta_{\text{inf}} = 1$ and $\epsilon(w) = r \le L \le \mathcal{I}_\beta \le \sum_{\ell=0}^{\infty} \left(\frac{2p}{1-2p}\right)^{1+\ell} \text{len}\left(\phi_w \left(\phi_2\right)^\ell \gamma_{1/2}\right) = \prod_{\ell=0}^{|w|} q(w_\ell) \sum_{m=1}^{\infty} \left(\frac{p^2}{1-2p}\right)^m \frac{\sqrt{4p-1}}{2} = \prod_{\ell=0}^{|w|} q(w_\ell) \frac{1-2p}{1-2p-2p^2} \frac{\sqrt{4p-1}}{2} = \frac{\epsilon(w)}{\epsilon_0} \frac{1-2p}{1-2p-2p^2} \frac{\sqrt{4p-1}}{2}$ for the path $\gamma_{1/2}$ from the foliation of the isosceles triangle as this is the longest path within the isosceles triangle and also within the seed foliation of the domain whose boundary is given by the union of I_0 and $\bigcup_{i=1}^4 \phi_i I_0$.

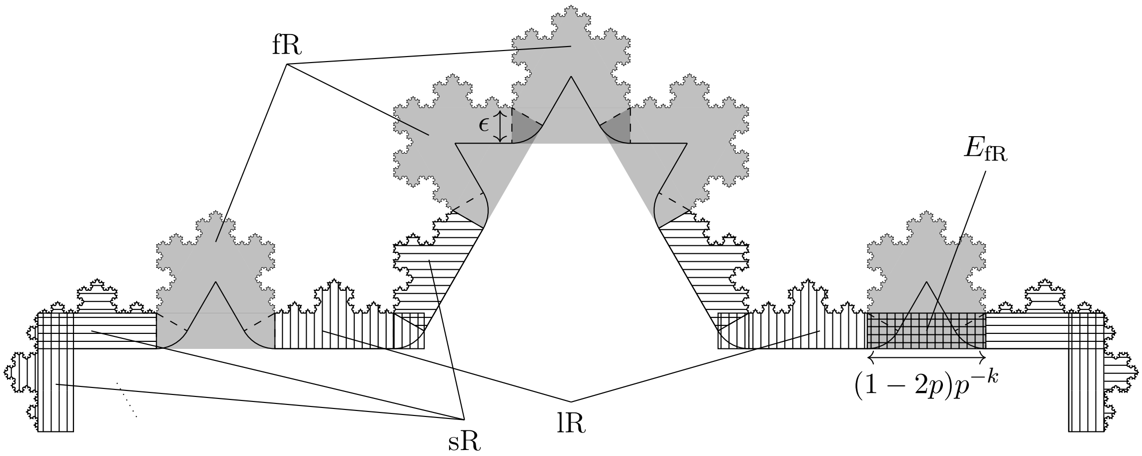


Figure 4.3: Example of a cover of the top side of $\Omega_{-\epsilon}$ by well foliated domains where Ω is a (fully homogeneous) p-Rohde snowflake. The overlaps of domains is shown in dark gray. Notice that the multiplicity of the cover of $\mu = 2$ remains true for any such cover. The domains are labeled up to symmetry.

While images of $\gamma_{1/2}$ are not within one path, each image is a sharp upper bound to the length of the path at each iteration level. Finally, since $\epsilon \in J(w)$, $\epsilon(w) \leq \epsilon \leq \epsilon(\overline{\sigma} w)$ and $\epsilon(w)/\epsilon(\overline{\sigma} w) \in \{\frac{1}{4}, p\}$ showing $\epsilon \asymp \epsilon(w)$. By construction of the covering domains fR, lR and sR we observe that $r_- = \min(\frac{1}{4}, p) = \frac{1}{4}$ and $r_+ = \max\{4, p^{-1}, \sqrt{1+\sin^2\alpha_p}\}$. Moreover, any covering well-foliated domain with corresponding E as defined in (i)-(iv) has a volume bounded from above by $\mathrm{vol}_2(E_{\mathrm{lR}}(w)) + \mathrm{vol}_2(T_w) + 2\sum_{\ell=1}^{\infty} 2^{\ell-1} p^{\ell} \mathrm{vol}_2(T_0)$ and from below by $\mathrm{vol}_2 E$, where T_0 is an isosceles triangle of base length $1-2p$ and height $\frac{1}{2}\sqrt{4p-1}$. The first non-trivial Neumann eigenvalues of the corresponding domains are given by $\pi^2\left(\frac{1}{\epsilon(w)^2} + \frac{1}{b^2}\right)$ and for all domains, fR, sR and lR, $b \asymp \epsilon(w)$.

If Ω is homogeneous, the above discussion can be drastically simplified as only one word in $s \in \{0,1\}^{\mathbb{N}}$ fully describes the construction of a homogeneous p-Rohde snowflake. This is done by identifying $\{5, \cdots, 8\}$ with the digit 0 and $\{1, \ldots, 4\}$ with the digit 1. The following dynamical system as $k \to k+1$ dictates the number of covering domains. If $s_{k+1} = 0$, one replaces a domain sR or lR with 4 copies of sR rescaled by $\frac{1}{4}$ and each fR by 6 copies of sR and one copy of fR rescaled by $\frac{1}{4}$. Analogously, if $s_{k+1} = 1$, one replaces each sR or lR with two copies of sR or lR and one copy of fR both rescaled by p.

Each fR is replaced with two copies of sR and three copies of fR rescaled py p. Writing sR/lR $=: \binom{1}{0}$ and fR $=: \binom{0}{1}$, this replacement procedure of n copies of sR/lR and m copies of fR at k-th iteration corresponds to $B(s_k)\binom{n}{m}$ with $B(0) := \left(\begin{smallmatrix}4 & 6\\ 0 & 1\end{smallmatrix}\right)$ and $B(1) := \left(\begin{smallmatrix}2 & 2\\ 1 & 3\end{smallmatrix}\right)$. Setting $B(s|_k) := \prod_{\ell=1}^{k} B(s_k)$ there are $C(s|_k) := \left\|B(s|_k)\binom{4}{0}\right\|_1$ well-foliated domains with $r = \epsilon(s|_k)$ covering $\Omega_{-\epsilon(s|_k)}$. Next, we obtain an expression of $C(s|_k)$ depending on the word s. We write $v := \binom{4}{0}$, $x := \binom{2}{-1}$, $A := B(0)$ and $B := B(1)$ for brevity. Notice that x is an eigenvector of both A and B to eigenvalue 1 and moreover

$$[B^p, A^q]\, B^k v = -\frac{4^{k+1}}{3}\,(4^q - 1)\,(4^p - 1)\,x,$$

Then for $p_0 + \sum_{\ell=1}^{m} q_\ell + p_\ell = k$ and $p_i, q_i \geq 0$

$$\begin{aligned}
C(s|_k) &= (1,1)A^{p_0}B^{q_1}A^{p_1}B^{q_2}A^{p_2}\cdots B^{q_m}A^{p_m}v\\
&= (1,1)A^{p_0}B^{q_1}A^{p_1}B^{q_2}A^{p_2}\cdots B^{q_{k-1}}A^{p_{k-1}}\left(A^{p_k}B^{q_k}v - \frac{1}{3}\,(4^{q_k}-1)\,(4^{p_k}-1)\,x\right)\\
&= (1,1)A^{p_0}B^{q_1}A^{p_1}B^{q_2}A^{p_2}\cdots B^{q_{k-1}}A^{p_{k-1}+p_k}B^{q_k}v - \frac{4}{3}\,(4^{q_k}-1)\,(4^{p_k}-1)\\
&= (1,1)A^{p_0}B^{q_1}A^{p_1}B^{q_2}A^{p_2}\cdots A^{p_{k-1}+p_k}B^{q_{k-1}+q_k}v\\
&\qquad - \frac{4}{3}\,(4^{q_k}-1)\,(4^{p_k}-1) - \frac{4^{q_k+1}}{3}\,(4^{q_{k-1}}-1)\left(4^{p_{k-1}+p_k}-1\right)\\
&= \cdots\\
&= 4^{k+1} - \frac{4^{k+1-p_0}}{3} + \frac{4^{k+1-p_0-q_1}}{3} - \frac{4^{k+1-p_0-q_1-p_1}}{3} \pm \cdots + \frac{4^{p_m+1}}{3}\\
&\in \left[\frac{2\cdot 4^{k+1}}{3}, \frac{10\cdot 4^{k+1}}{9}\right).
\end{aligned}$$

As in Rem. 2.39, $\partial\Omega$ can be understood as the union of four copies of R where R is a Moran set as in Def. 2.37. Indeed, with the definition of ϵ, one has $4^k = \epsilon(s|_k)^{-\delta_k}$, where δ_k is the unique root of $t \mapsto 1 - \prod_{\ell=1}^{k} 4q(s_\ell)^t$ showing that $\#I_\epsilon \asymp \epsilon(s|_k)^{-\delta_k}$. It can be shown (cf. [38], Prop. 2.38) that $\delta := \limsup_{k\to\infty} \delta_k$ is the upper Minkowski dimension of R and hence of $\partial\Omega$. □

Notice that the above argument for Minkowski measurability holds true if $p \geq \frac{\sqrt{3}-1}{2}$. In the case of a non-homogeneous p-Rohde snowflake Ω, each point $z \in \partial\Omega$ has an individual word $s(z) \in \Sigma_A^\infty$. Using the above covering technique and the theorem of Rohde, we can now prove Prop. 4.5 in a relatively constructive manner.

Proof of Prop. 4.5. Let $F : \Omega_p \to \Omega$ be bi-Lipschitz with a p-Rohde snowflake Ω_p for some $p \in \left[\frac{1}{4}, \frac{1}{2}\right)$. Then we cover the region near the boundary of Ω_p as in the proof of Thm. 4.7. Up to the condition in Def. 3.3(ii), the properties of this volume cover do not change if $p \in \left[\frac{\sqrt{3}-1}{2}, \frac{1}{2}\right)$ so that any p-Rohde snowflake satisfies the condition of Prop. 4.4. Let us denote this volume covering set by $\{D^\epsilon(w) : \epsilon \in J(w)\}$ and we write the base of each $D^\epsilon(w)$ as $I^\epsilon(w)$. We define the inner boundary of the volume cover as $A := \partial \overline{\bigcup D^\epsilon(w)} \setminus \partial\Omega$. Then $FD^\epsilon(w)$ is a finite volume cover of a region near the boundary of Ω. Since F is bi-Lipschitz (see for example Wakin-Eftekhari in [25]), $\mathrm{vol}_2(FD^\epsilon(w)) \asymp \mathrm{vol}_2(D^\epsilon(w))$. Let $r_- := \inf_{x\in\partial\Omega} \mathrm{dist}(x, A)$ and $r_+ := \sup_{x\in\partial\Omega} \mathrm{dist}(x, A)$. Then $r_\pm$ satisfy the conditions of Prop. 4.4 for Ω_p and since F is bi-Lipschitz, $r_-^F := \inf_{x\in\partial F\Omega} \mathrm{dist}(x, FA) = \inf_{x\in\partial\Omega} \mathrm{dist}(Fx, FA) \asymp r_-$ and analogously for r_+^F. □

Similarly to the argument above, under a technical assumption (called *E-property*) Thm. 4.7 can be extended to certain quasidisks.

Definition 4.8. Let $X, X' \subset \mathbb{R}^n$ be two domains. Then a bi-Lipschitz map $f : X \to X'$ is said to have the *E-property* if there is $N \in \mathbb{N}, \alpha_0 > 0$ and $\kappa \in \mathbb{R}_+$ such that for any sufficiently small $\epsilon > 0$ and any cuboid of side length ϵ, denoted by $E_\epsilon \subset X$, with parallel fibres $\{\gamma_q : q \mapsto q + te_n\}$ that are parallel to one side of E_ϵ, the image fE_ϵ is covered with at most N $(n-1)$-dimensional hyperplanar convex domains $\{P_j\}$ with piecewise smooth boundary and normal ν_j so that any fibre $f \circ \gamma_q$ traverses through at least one P_j and all fibres through one P_j traverse it at an angle $> \alpha_0$ and $P_j(\epsilon') := P_j + (0, \epsilon\kappa)\nu_j \subset \Omega$.

Clearly any locally affine map has the E-property. If a quasicircle is bi-Lipschitz to a p-Rohde snowflake as in Prop. 4.7 with E-property, it can be uniformly well-covered by applying the bi-Lipschitz map to the uniform regular well-covering of the corresponding p-Rohde snowflake. More generally:

Proposition 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega' \to \Omega$ be bi-Lipschitz with E-property and a (uniformly) well-covered Ω'. Then Ω is (uniformly) well-covered.*

Proof. Let Ω' be well-covered by $\{D_i^\epsilon\}_{i \in I_\epsilon}$ with corresponding $\{E_i^\epsilon\}$. Then with Prop. 3.7, consider fD_i^ϵ as a volume cover of $\Omega_{-\epsilon'}$ with $\epsilon' := \min\{\epsilon/L, L\epsilon\}$ where f has Lipschitz constant L. By the E-property, $f(E_i^\epsilon)$ contains at most $M \leq N$ convex $(n-1)$-dimensional regions $\{P_j\}_{j=1,\cdots,M}$ with normal ν_j that intersect every fibre in fE_i^ϵ. Then $P_j(\kappa\epsilon(w)) \subset \Omega$ is convex and has $\operatorname{diam} P_j(\kappa\epsilon(w)) \leq \sqrt{(\operatorname{diam} E)^2 + \kappa^2\epsilon(w)^2}$. Then by the theorem of Funano (Thm. 2.22), it satisfies Def. 4.1(ii) since for any j there is a sphere of radius $\asymp \epsilon$ containing P_j. □

4.2 Bounded Variation domains as well-covered domains

In this section we show that the class of well-covered domains contains the class of domains of bounded variation. For the class of domains of bounded variation bounds on Laplace eigenvalue counting functions have been found in [80]. Recall the definition of $\operatorname{Osc}(f, Q_n)$ in Def. 3.1

Definition 4.10 (Def. 1.1-1.2 in [80])**.** Let Q_n be an n-dimensional cube with arbitrary size whose edges are parallel to the axes.

(i). Let $f : Q_n \to \mathbb{R}$ be bounded. Then for any $\delta > 0$, we define $\mathcal{V}_\delta(f, Q_n)$ as the maximal number of disjoint open cubes $Q_{n,i} \subset Q_n$ whose edges are parallel to coordinate axes with $\operatorname{Osc}(f, Q_{n,i}) \geq \delta$ for all i. If $\operatorname{Osc}(f, Q_n) < \delta$, we set $\mathcal{V}_\delta(f, Q_n) := 1$.

(ii). Let $\tau : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be non-decreasing. The space spanned by the continuous functions $f : \overline{Q_n} \to \mathbb{R}$ defined on some cube Q_n such that $\mathcal{V}_{1/t}(f, Q_n) \leq \tau(t)$ for all t, is called the set of functions with (τ, ∞)-bounded variation and denoted by $\mathrm{BV}_{\tau,\infty}(Q_n)$.

For a domain $\Omega \subset \mathbb{R}^n$, we write $\Omega \in \mathrm{BV}_{\tau,\infty}$ iff for any point $p \in \partial\Omega$ there is an open neighbourhood $U_p \subset \mathbb{R}^n$ such that (up to an $SO(n)$), $U_p \cap \Omega$ is the set of points below the graph of some $f \in BV_{\tau,\infty}(Q_{n-1})$ and a constant function $b < \inf f$.

Theorem 4.11. *Let $\tau(t) \asymp t^{\delta}$ for some $\delta \geq 1$. Then the set of well-covered domains contains* $\mathrm{BV}_{\tau,\infty}$ *as a proper subset.*

Proof. Let $\Omega \in \mathrm{BV}_{\tau,\infty}$ be a domain in $\mathbb{R}^n$ and let $p \in \partial\Omega$ with corresponding neighbourhood U_p. Then $U_p \cap \Omega$ has boundary given by the union of a graph Γ_f of a function $f : \overline{Q_{n-1}} \to \mathbb{R}$ from above and a constant $b < \inf f$ from below. Hence there is a foliation $\{\gamma \in \Gamma\}$ by straight lines: For any $x \in Q_{n-1}$, let $\gamma_x : t \mapsto (x_1, x_2, \cdots, x_{n-1}, b) + t(0, \cdots, 0, 1)$ be so that $\operatorname{len} \gamma_x = f(x) - b > 0$. Such foliations satisfy conditions (i)-(ii) in Def. 3.3 trivially, as the corresponding density β is 1 everywhere. Then for some $r \in (b, \inf f)$, we set $E := \{x \in U_p : \exists \gamma \text{ s. t. } x = \gamma(r)\}$ showing that U_p is well-foliated for all $p \in \partial\Omega$.

Since $\partial\Omega$ is compact (being closed and bounded), there is a finite set of open neighbourhoods U_p corresponding to some $f \in \mathrm{BV}_{\tau,\infty}$ that cover $\partial\Omega$. By [80, Thm. 3.5, Cor. 3.8], for any $\epsilon > 0$ small enough, each of these can be covered by a family of well-foliated domains of size $\asymp \epsilon$ with $r = \epsilon$ and of cardinality $\ll \mathcal{V}_{\epsilon/2}(f, Q_{n-1}) + \epsilon^{-n} \operatorname{vol}_n(\Omega_{-\epsilon}) + \epsilon^{-1} \int_1^{1/\epsilon} t^{-2} \tau(t) dt \ll \tau(1/\epsilon) = \epsilon^{-\delta}$.

Since the Koch snowflake K is not locally a graph, $K \notin \mathrm{BV}_{\tau,\infty}$ while K is well-covered as shown in Sec. 4.3.1. □

Remark 4.12. Suppose there is a bi-Lipschitz function $f : \Omega \to \Omega'$ with Lipschitz constant L, such that $\Omega' \in \mathrm{BV}_{\tau,\infty}$ with $\tau(t) \sim t^{-\delta}$ for some $\delta > 1$. Then Ω is well-covered. Such domains are studied in [80] but in many practical cases, it appears to be easier to show a domain is well-covered instead of showing it is $\mathrm{BV}_{\tau,\infty}$ up to a bi-Lipschitz transformation. Moreover, Thm. 4.11 shows that Thm. 4.15 covers a larger class than $\mathrm{BV}_{\tau,\infty}$ addressing the open problem raised in [80, Sec. 5.4.2].

4.3 Asymptotic behaviour of spectral counting functions of well-covered domains

In this section we obtain an estimate on remainder terms of $N_N(\Omega, t)$ in particular when Ω is a snowflake-like domain. Before we prove the main result, Thm. 4.15, we recall the aforementioned result from [80] in Thm. 4.13 and Cor. 4.14. This result is restricted to bounded variation domains which were briefly discussed in Sec. 4.2. We then show how our setup enables us to study more general cases based on Lem. 3.8 which allows domains whose boundary is not necessarily locally a graph. Afterwards we concentrate our attention in particular on p-Rohde snowflakes in Sec. 4.3.1 stressing the construction in Sec. 3.2.

Theorem 4.13 (Cor. 1.5 in [80]). *Let $\Omega \in \mathrm{BV}_{\tau,\infty}(\mathbb{R}^n)$. Then there is $C_\Omega \in \mathbb{R}$ such that*

$$|N_N(\Omega,t) - C_W \operatorname{vol}_n(\Omega)\lambda^{n/2}| \leq C_\Omega \lambda^{(n-1)/2} \int_{C_\Omega^{-1}}^{C_\Omega\sqrt{\lambda}} \left(\frac{1}{t} + t^{-n}\tau(t)\right) dt.$$

With Thm. 4.11 this gives the following result:

Corollary 4.14. *For bounded variation domains Ω one has $N_N(\Omega,t) = C_W^{(n)} t^{n/2} + \mathcal{O}(t^{\delta/2})$, if $\delta := \overline{\dim_M}(\partial\Omega,\Omega) > n-1$ is the upper inner Minkowski dimension.*

We are now in the position to prove an analogous asymptotic result on $N_N(\Omega,t)$ for more general domains, with focus on domains with fractal boundary, thus generalising the results and partially resolving a question raised in [80, Sec. 5.4.2] in Thm. 4.11 and Rem. 4.12. Key for obtaining such an asymptotic result for $N_N(\Omega,t)$ is the construction of a foliation in $\Omega_{-\epsilon}$ paired with Lem. 3.8.

Theorem 4.15. *Let $\Omega \subset \mathbb{R}^n$ be uniformly well-covered by $\{D_i^\epsilon\}_{i\in I_\epsilon}$ with $\#I_\epsilon \leq C\epsilon^{-\delta}$ for fixed C,δ. Then there is a C_M depending only on Ω such that*

$$N_N(\Omega,t) \leq C_W \operatorname{vol}_n(\Omega) t^{n/2} + C_M t^{\delta/2}.$$

In particular, if $p \in \left[\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$, this applies to the cases where $\Omega \subset \mathbb{R}^2$ is bi-Lipschitz to an upper inner Minkowski measurable p-Rohde snowflake with E-property or if Ω is bi-Lipschitz to a homogeneous p-Rohde snowflake with E-property.

Proof. Let $\{Q\}_{Q\in\mathcal{W}}$ denote a Whitney cover of Ω (see Lem. 2.41). For $\epsilon > 0$ we restrict $\{Q\}_{Q\in\mathcal{W}}$ to the smallest subset $\mathcal{W}_\epsilon \subset \mathcal{W}$ of cubes volume covering $\Omega\backslash\Omega_{-\epsilon}$, i. e. those cubes that are sufficiently far away from $\partial\Omega$ and hence are sufficiently large. Suppose that ϵ is small enough so that $\Omega_{-\epsilon}$ is uniformly volume covered by well-foliated domains $\{D_i^\epsilon\}_{i\in I_\epsilon}$. Let $J_\epsilon \subset I_\epsilon$ be such that $D_i^\epsilon \cap \left(\Omega \setminus \bigcup_{\mathcal{W}_\epsilon} Q\right) \neq \emptyset$ for $i \in J_\epsilon$ and set $U_\epsilon := \{D_i^\epsilon\}_{i\in J_\epsilon}$. Then $\Omega = \bigcup_{i\in J_\epsilon} D_i^\epsilon \cup \bigcup_{\mathcal{W}_\epsilon} Q$ and this volume cover can be split into two disjoint volume subcovers. Let $T_\epsilon := \{Q \in \mathcal{W}_\epsilon : Q \cap \bigcup_{i\in J_\epsilon} D_i^\epsilon \neq \emptyset\}$ and $U_\epsilon^{\mathcal{W}} := U_\epsilon \cup T_\epsilon$ be the set of all well-foliated domains D_i^ϵ together with all cubes in $\mathcal{W}_\epsilon$ that intersect some D_i^ϵ. Next, we restrict the Whitney cover further to the minimal set of cubes necessary to volume cover Ω given the volume cover $\{U\}_{U\in U_\epsilon^{\mathcal{W}}}$ by defining $\widetilde{\mathcal{W}_\epsilon} := \{Q \in \mathcal{W}_\epsilon : Q \cap U = \emptyset \ \forall U \in U_\epsilon^{\mathcal{W}}\}$. These volume covers are disjoint and have multiplicity $\mu(U_\epsilon^{\mathcal{W}}) \leq \mu + 1$ and $\mu(\widetilde{\mathcal{W}_\epsilon}) = 1$ so that $\operatorname{vol}_n \Omega = \operatorname{vol}_n \left[\left(\bigcup_{U_\epsilon^{\mathcal{W}}} U\right) \sqcup \left(\operatorname{int} \bigcup_{\widetilde{\mathcal{W}_\epsilon}} \overline{Q}\right)\right]$. Therefore

$$N_N(\Omega, t) \leq \underbrace{N_N\Bigg(\bigcup_{U_\epsilon^{\mathcal{W}}} U, t\Bigg)}_{:=S_1^\epsilon(t)} + \underbrace{N_N\Bigg(\operatorname{int}\bigcup_{\widetilde{\mathcal{W}_\epsilon}} \overline{Q}, t\Bigg)}_{:=S_2^\epsilon(t)}$$

and we estimate both terms separately.

$S_1^\epsilon(t)$. By construction one can estimate the first non-trivial eigenvalue of any Whitney cube $Q \in U_\epsilon^{\mathcal{W}}$ as

$$\lambda_2^N(Q) = \left(\frac{\sqrt{n}\pi}{\operatorname{diam} Q}\right)^2 \geq \left(\frac{\sqrt{n}\pi}{\operatorname{dist}(Q, \partial\Omega)}\right)^2 \geq \left(\frac{\sqrt{n}\pi}{r_+\epsilon}\right)^2,$$

where r_+ is taken from Prop. 4.4. For large enough λ we can choose ϵ such that $(\mu+1)\lambda = C_2\epsilon^{-2}$. Any $Q \in T_\epsilon$ satisfies $\operatorname{dist}(Q, \partial\Omega) \geq \epsilon - \operatorname{diam}(Q)$ (i. e. $\operatorname{diam} Q \geq \frac{\epsilon}{5}$) and $\operatorname{diam} Q \leq \operatorname{dist}(Q, \partial\Omega) \leq r_+\epsilon$. Then, by Prop. 2.42,

$$\begin{aligned} S_1^\epsilon(\lambda) &\leq \sum_{U\in U_\epsilon^{\mathcal{W}}} N_N\left(U, (\mu+1)\lambda\right) \leq \#I_\epsilon + \sum_{k_-\leq k\leq k_+} \#\mathcal{W}_k \leq \#I_\epsilon + 2^{k_-\delta} \sum_{k=0}^{\lfloor 2+\log_2(5r_+)\rfloor} \mathfrak{M}_\Omega 2^{k\delta} \\ &\leq C(\Omega)\epsilon^{-\delta} + \mathfrak{M}_\Omega \left(\frac{\sqrt{n}}{r_+}\right)^\delta \frac{(8\cdot 5r_+)^\delta - 1}{2^\delta - 1}\epsilon^{-\delta} \leq C_3(\Omega)\epsilon^{-\delta} = \frac{C_3(\Omega)(\mu+1)^{\delta/2}}{C_2(\Omega)^{\delta/2}}\lambda^{\delta/2}, \end{aligned}$$

where $k_- := \left\lfloor \log_2 \frac{\sqrt{n}}{r_+\epsilon} \right\rfloor$ and $k_+ := \left\lceil \log_2 \frac{5\sqrt{n}}{\epsilon} \right\rceil$ and hence $k_+ - k_- \leq 2 + \log_2(5r_+)$.

$S_2^\epsilon(t)$. Notice that $\operatorname{int} \bigcup_{Q\in\widetilde{\mathcal{W}_\epsilon}} \overline{Q}$ is a polygon whose volume is bounded by $\operatorname{vol}_n \Omega$. By

Prop. 2.43, Prop. 2.44 and a result on two-term asymptotics for polygons (see Sec. 4.5 and Thm. 7.4.11 in [48] or [108]),

$$\limsup_{t\to\infty} \frac{S_2^\epsilon(t)}{C_W^{(n)}\operatorname{vol}_n(\Omega)t^{n/2} + \frac{C_W^{(n-1)}}{4}A_\Omega \epsilon^{n-1-\delta}t^{(n-1)/2}} \leq 1.$$

The estimates for $S_1^\epsilon(t)$ and $S_2^\epsilon(t)$ together show that for large enough t

$$N_N(\Omega,t) \leq C_W^{(n)}\operatorname{vol}_n(\Omega)t^{n/2} + M_\Omega t^{\delta/2}. \qquad \square$$

The above theorem can be used to obtain explicit bounds for the spectral counting function, see Rem. 5.8(i)-(ii). We shall make use of the following simple observation about the spectrum of cubes based on a result by Pólya in [86].

Lemma 4.16. *The Neumann counting function of a unit cube $I^n = (0,1)^n \subset \mathbb{R}^n$ has an upper bound of $N_N(I^n,t) \leq N_D(I^n,(\sqrt{t}+2\pi\sqrt{n})^2) \leq C_W^{(n)}(\sqrt{t}+2\pi\sqrt{n})^n$.*

Proof. $N_N(I^n,t) = \#\{k = (k_1,\cdots,k_n) \in \mathbb{N}_0^n : |k|_2^2 \leq t/\pi^2\}$ and $N_D(I^n,t) = \#\{k = (k_1,\cdots,k_n) \in \mathbb{N}^n : |k|_2^2 \leq t/\pi^2\}$. Defining $\widetilde{N_N}(r) := \#\{k \in \mathbb{N}_0^n : |k|_2^2 \leq r^2\}$ one observes $N_N(I^n,t) = \widetilde{N_N}(\sqrt{t}/\pi)$ and analogously for N_D. Notice that $\widetilde{N_N}(r) \leq \widetilde{N_D}(r+2\sqrt{n})$. Thus $N_N(I^n,t) = \widetilde{N_N}(\sqrt{t}/\pi) \leq \widetilde{N_D}(\sqrt{t}/\pi+2\sqrt{n}) = N_D(I^n,(\pi(\sqrt{t}/\pi+2\sqrt{n})^2)$. Finally $N_D(I^n,t) \leq C_W^{(n)}t^{n/2}$ by Pólya's result. $\square$

Remark 4.17. In the proof of Thm. 4.15 a couple of adaptations offer interesting variations. Further observations are mentioned in Sec. 6.2.2.

(i). The minimal value of t for which the estimate of Thm. 4.15 holds true depends entirely on the maximal value of ϵ in Def. 4.1. More precisely, if Ω is well-covered for all $\epsilon < \epsilon_0$, an upper bound for $N_N(\Omega,t)$ for all $t > t_0 := \frac{C_2\epsilon_0^{-2}}{\mu+1}$ is presented below. This variant of the above estimate involves estimating $S_2^\epsilon(t) \leq \sum_{Q\in\widetilde{\mathcal{W}}_\epsilon} N_N(Q,t)$.

More precisely, setting $k'_- := \lfloor -\log_2 \frac{\operatorname{diam}\Omega}{\sqrt{n}} \rfloor$, one has

$$S_2^\epsilon(t) \le \sum_{Q \in \widetilde{\mathcal{W}}_\epsilon} N_N(Q,t) \le \sum_{k=k'_-}^{k_+} \#\mathcal{W}_k N_N(2^{-k} I^n, t)$$

While more accurate estimates are possible, this is sufficient to provide an upper bound in terms of counting functions of cubes. Applying Lem. 4.16 to the cubes in $\mathcal{W}_\epsilon$ yields an explicit upper bound of $S_2^\epsilon(t)$ that is not only asymptotic but holds for all $t > t_0$.

$$\begin{aligned} S_2^\epsilon(t) &\le C_W^{(n)} \sum_{k=k'_-}^{k_+} \#\mathcal{W}_k \left(2^{-k}\sqrt{t} + 2\pi\sqrt{n}\right)^n \\ &\le C_W^{(n)} \left(\operatorname{vol}(\Omega) t^{n/2} + \mathfrak{M}_\Omega \sum_{k=k'_-}^{k_+} 2^{k\delta} \left[\left(2^{-k}\sqrt{t} + 2\pi\sqrt{n}\right)^n - \left(2^{-k}\sqrt{t}\right)^n \right] \right). \end{aligned}$$

With $S_2^\epsilon(t) \le S_2^\epsilon(t_0)$ if $t \le t_0$ this extends to a global estimate of $N_N(\Omega,t)$ and the coefficient of such an error term estimate will be denoted by M_K^{abs}. Since the estimate of S_1^ϵ is also correct for all t, this gives an absolute estimate of the remainder term of the Neumann counting function. In the regular case of $\delta = n-1$, this estimate then only provides a weaker estimate of $t^{(n-1)/2}\log t$ instead of the boundary term $t^{(n-1)/2}$. Similarly such an asymptotic term is also obtained if one uses the regular asymptotic law of the counting functions $N_N(2^{-k}I^n, t)$ as shown for example by Lapidus in [65] whenever the upper Minkowski content agrees the upper inner Minkowski content.

(ii). Let $\Omega \subset \mathbb{R}^n$ be any open set with finite volume such that $\operatorname{vol}_n(\Omega_{-\epsilon}) \le \widetilde{C}\epsilon^{n-\delta}$ for some $\delta \in (n-1, n)$ and all sufficiently small $\epsilon > 0$. Then, using Whitney covers and an estimate of the form of Prop. 2.42, one can find lower bounds for $N_D(\Omega,t)$ as was shown by van den Berg and Lianantonakis in [107]. Indeed, for any such domain Ω one has

$$N_D(\Omega,t) \ge C_W^{(n)} \operatorname{vol}_n(\Omega) t^{n/2} - \frac{5\widetilde{C}}{(n-\delta)(\delta+1-n)} t^{\delta/2},$$

if $\delta > n-1$ and $t > 0$, and

$$N_D(\Omega,t) \geq C_W^{(n)} \operatorname{vol}_n(\Omega) t^{n/2} - 3\widetilde{C} t^{(n-1)/2} \left(\log\left((2\operatorname{vol}_n \Omega)^{2/n} t\right)\right),$$

if $\delta = n-1$ and $t > \frac{4}{\sqrt[n]{\operatorname{vol}_n(\Omega)^2}}$. Together with Thm. 4.15, this gives explicit upper and lower bounds for the remainder term of counting functions (for Neumann, Dirichlet and mixed boundary conditions) of well-covered domains whenever $\delta > n-1$: Since $\operatorname{vol}_n(\Omega_{-\epsilon}) \leq \operatorname{vol}_n\left(\bigcup_{i\in I_\epsilon} D_i^\epsilon\right) \leq c_{\mathrm{vol}}^+ C(\Omega)\epsilon^{n-\delta}$, we may set $\widetilde{C} := c_{\mathrm{vol}}^+ C(\Omega)$ to obtain an estimate for all $t > t_0$ for $|N_N(\Omega,t) - C_W^{(n)} \operatorname{vol}_n \Omega t^{n/2}| \leq \max\{M_K^{\mathrm{abs}}, \frac{5\widetilde{C}}{((n-\delta)(\delta+1-n))}\} t^{\delta/2}$ with M_K^{abs} taken from (i) above. This constant will be denoted by $\widetilde{M_K}$. It plays a central role in the companion paper [62]. In [62] these estimates are used to deduce high order asymptotic terms for $N_D(U,t)$ for self-similar sprays U whose generators are finite unions of pairwise disjoint Koch snowflakes.

(iii). Combining Thm. 1.10 of [80] with Thm. 4.11 it follows that for any n there is a well-covered domain $\Omega \subset \mathbb{R}^n$ and a $t_\Omega \in \mathbb{R}_+$ for which $N_N(\Omega,t) - C_W^{(n)} \operatorname{vol}_n \Omega t^{n/2} \geq t_\Omega^{-1} t^{\delta/2}$ for all $t > t_\Omega$. In this sense, the result of Thm. 4.15 is order-sharp.

(iv). One may consider only non-trivial eigenvalues, thus disregarding the trivial zero eigenvalue of each D_i^ϵ that comes with Neumann boundary conditions. In this case one can adapt the proof above by setting $S_2^\epsilon(t) = 0$ essentially re-obtaining existing results (for example in [65]) for Dirichlet boundary conditions.

While it is sensible to restrict the attention to extension domains, we did not make use of the existence of such an extension or, for that matter, the emptiness of the essential spectrum of the Neumann Laplacian at any point in our argument. Since the above result implies the finiteness of the spectral counting function, we obtain the following result.

Corollary 4.18. *Let $\Omega \subset \mathbb{R}^n$ be a well-covered domain with positive and finite upper inner Minkowski content. Then $\sigma_{\mathrm{ess}}(-\Delta_N) = \emptyset$.*

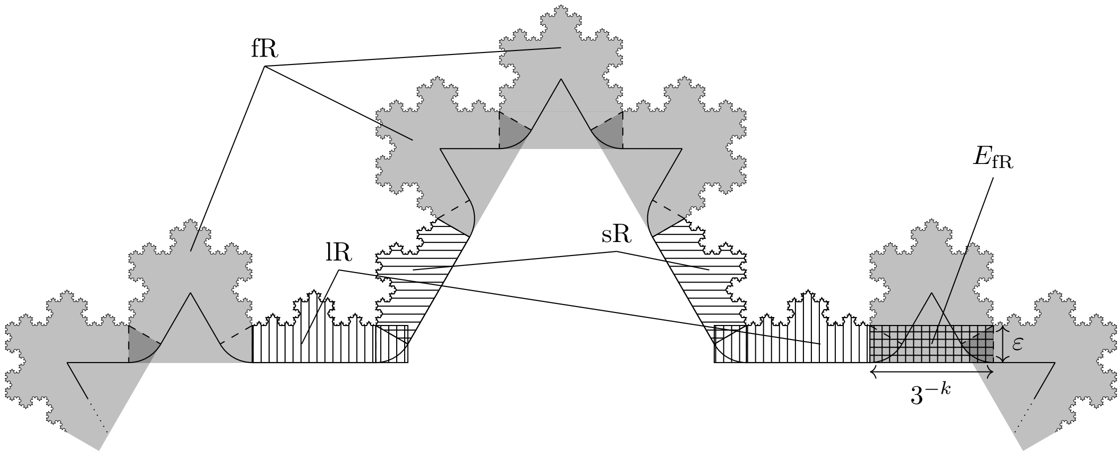


Figure 4.4: Instance of the cover of an inner ϵ-neighbourhood $K_{-\epsilon}$ of the classical Koch snowflake K and $\epsilon \in J_2^{(1/3)}$. In difference to the cover suggested for p-Rohde snowflakes, both end segments of this curve are parts of additional fR's, as the snowflake is built out of three equilateral Koch curves.

4.3.1 Application to snowflakes

For snowflake domains and more generally any domain whose boundary has a self-similar structure, the values of $\beta(\gamma, t)$ as in Def. 3.3 can often be easily understood: They increase in discrete steps according to a power-law as t approaches $\operatorname{len}\gamma$ (cf. Fig. 4.5). For the Koch snowflake for example, one might say, that β doubles in value each time a fibre passes through an image of I_0 under some ϕ_w for a $w \in \{1, \ldots, 4\}^*$.

The remainder of this section is devoted to finding an explicit estimate for the remainder term of the eigenvalue counting function of the Neumann Laplacian for fully homogeneous p-Rohde snowflakes R_p, the classical Koch snowflake K and more generally Koch snowflakes with a parameter p (that is a snowflake bounded by three fully homogeneous p-Rohde curves) denoted by K_p. To some extent this is a direct application of Thm. 4.7.

Fix $p \in \left(\frac{1}{4}, \frac{\sqrt{3}-1}{2}\right)$ and let $\delta := \overline{\dim}_M \partial R_p = \overline{\dim}_M \partial K_p = -\log_p 4$ denote the upper inner Minkowski dimension of the boundary of R_p and K_p. As shown in Thm. 4.7, K_p and R_p are well-covered. Let $\epsilon \in J_k^{(p)} := \left(p^{k+1} \frac{1-2p}{\sqrt{4p-1}}\,, p^k \frac{1-2p}{\sqrt{4p-1}}\right]$. In [67], it was shown that K has finite upper inner Minkowski measure. We cover the classical Koch snowflake and R_p as depicted in Fig. 4.4 and Fig. 4.3. Following the proof of Thm. 4.7, the regular uniform well-covering of $K_{p_{-\epsilon}}$ and $R_{p_{-\epsilon}}$ consists of three types of objects: For ease of notation we replace $\epsilon(w)$ with ϵ in this application since both the Koch snowflake and the Rohde snowflake are fully homogeneous.

(A) Fringed rectangles (fR) $(1-2p)p^{k-1} \times \epsilon$ with fractal top of height $p^{k-1}\frac{\sqrt{4p-1}}{2}$.

(B) Short rectangles (sR) $p^k \times \epsilon$ with a fractal top of height $p^k\frac{\sqrt{4p-1}}{2}$.

(C) Long rectangles (lR) $\left(p^k + \frac{\epsilon\sqrt{4p-1}}{2p}\right) \times \epsilon$ and a fractal top of height $p^k\frac{\sqrt{4p-1}}{2}$.

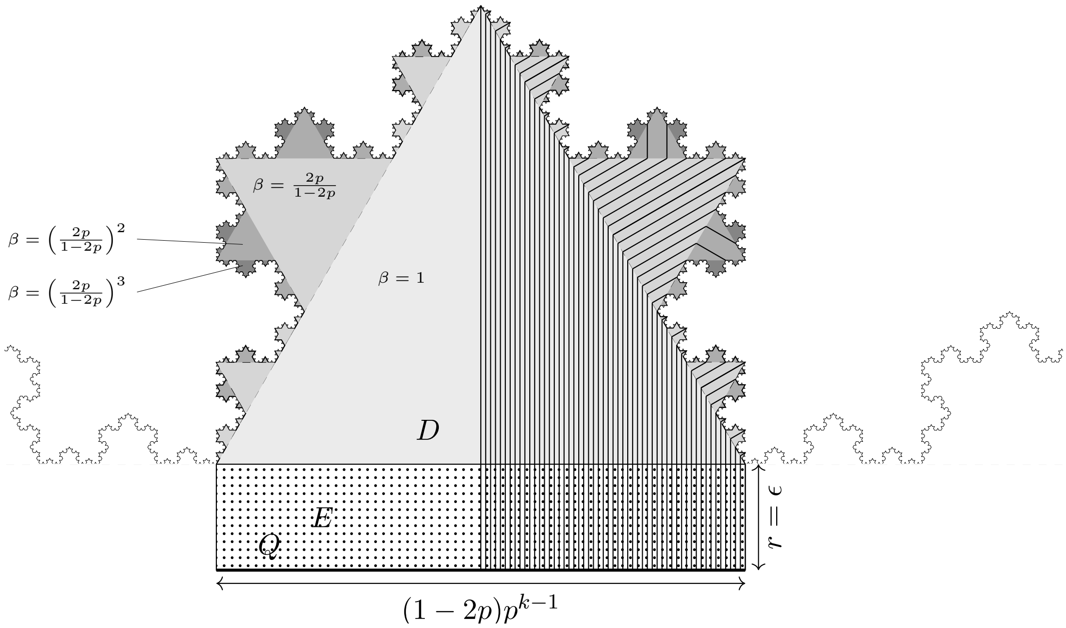


Figure 4.5: Systematic construction of the foliation of sets near the boundary of the p-snowflake of Rohde type in the case of a "fringed rectangle (fR)" (see Sec. 4.3.1(A)). The well-foliated domain is the dotted rectangle (the set E) and the section of the snowflake above it. The gray areas within the snowflake indicate areas of constant $\beta(\gamma, t)$. One notices that in the particular case of $p = 1/3$, a connection between the ternary expansion of $\gamma(0) \in \mathbb{R}$ and the shape of the curve (and in particular its number of turns) is evident.

We now obtain numeric values for the estimates from Thm. 4.15: By the iteration dynamics, $R_{p_{-\epsilon}}$ is covered by $\frac{4}{3}(4^k - 1)$ instances of fR and $\frac{4}{3}(4^k + 2)$ copies of sR or lR. In total this amounts to a cover of cardinality $\#I_\epsilon = \frac{4}{3}(2 \cdot 4^k + 1) \leq C(R_p)\epsilon^{-\delta}$ with $C(R_p) = 3\left(\frac{1-2p}{\sqrt{4p-1}}\right)^\delta$. Analogously one can chose $C(K_p) = \frac{3}{4}C(R_p)$. For the classical Koch snowflake $K := K_{1/3}$, the cardinality of the cover is $2 \cdot 4^k - 2 \leq C(K)\epsilon^{-\delta}$ with

$C(K) = 1$. By construction $r = \epsilon(w)$ for any finite word w, so that $c_r^{\pm} = 1$. The length L of a fibre satisfies $\epsilon = r \leq L < r + \sum_{\ell=0}^{\infty} p^{\ell+k-1}\frac{\sqrt{4p-1}}{2} \leq \left(1 + \frac{4p-1}{p^2(2-6p+4p^2)}\right)\epsilon$ implying $c_L^- \geq 1$ and $c_L^+ \leq 1 + \frac{4p-1}{p^2(2-6p+4p^2)}$. From our previous estimate on $\mathcal{I}_\beta$ we infer $c_{\mathcal{I}}^- \geq 1$ and $c_{\mathcal{I}}^+ \leq 1 + \frac{4p-1}{2p^2(1-2p-2p^2)}$. The diameters of the covering domains (A)-(C) above are found to be bounded from above by $\sqrt{[(1-2p)p^{k-1}]^2 + \left(\epsilon + p^{k-1}\frac{\sqrt{4p-1}}{2}\right)^2}$ so that $c_{\text{diam}}^+ \leq \sqrt{\frac{4p-1}{p^4} + \left(1 - \frac{4p-1}{2p^2(1-2p)}\right)^2}$. Moreover, $c_{\text{diam}}^- \geq 1$.

To (i) and (ii): By minimising $\lambda_2^N(E_{\text{lR}})\epsilon^2$ as a function of $\epsilon \in J_k^{(p)}$ we obtain $c_E(R_p) = \min\left(1, \frac{4(1-2p)^2p^2}{(3-2p)^2(4p-1)}\right)\pi^2$. Using the above constants we compute $C_1(R_{p_{-\epsilon}})$ as given in Cor. 3.9 so that $\lambda_2^N(D_i^\epsilon) \geq \epsilon^{-2}C_1(R_{p_{-\epsilon}})$. See Fig. 4.6(a) for a plot of $C_1(R_{p_{-\epsilon}})$. In

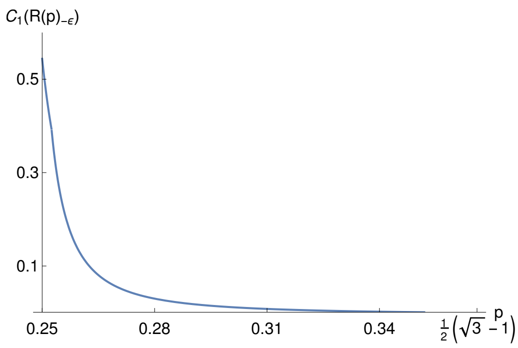


Figure 4.6(a) Values of $C_1(R_{p_{-\epsilon}})$ for values for $p \in (\frac{1}{4}, \frac{\sqrt{3}-1}{2})$ as in Cor. 3.9. The values approach 0 as $p \nearrow \frac{\sqrt{3}-1}{2}$.

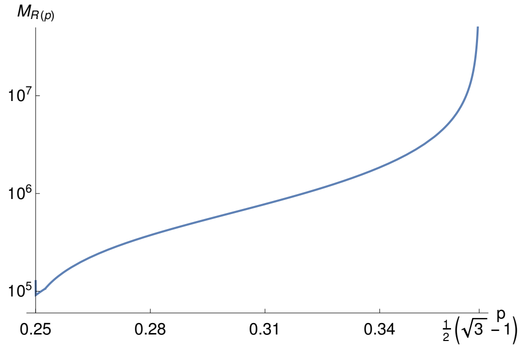


Figure 4.6(b) Values of $M_{K(p)}$ for values for $p \in (\frac{1}{4}, \frac{\sqrt{3}-1}{2})$ as given by Thm. 4.15.

particular in the case of the classical Koch snowflake one finds $\lambda_2^N(D) \geq 0{,}0031\epsilon^{-2}$ for any element of a cover of $K_{-\epsilon}$ as given by (A)-(C) above.

Based on the above considerations, one now obtains expressions for M_{R_p} as well as coefficients following Rem. 4.17.(i). As expected, the coefficients M_{R_p} diverge as $p \nearrow \frac{\sqrt{3}-1}{2}$ and remain regular as $p \searrow 1/4$, see Fig. 4.6(b).

In the case of the classical Koch snowflake K, a different better estimate is possible based on an estimate of the area of the inner ϵ-parallel neighbourhood of K by Lapidus and Pearse in [67]:

$$\text{vol}_2(K_{-\epsilon}) \leq 3\left[\epsilon^{2-\delta}4^{-\{x\}}\left(\frac{3\sqrt{3}}{40}9^{\{x\}} + \frac{\sqrt{3}}{2}3^{\{x\}} + \frac{1}{6}\left(\frac{\pi}{3} - \sqrt{3}\right)\right) - \frac{\epsilon^2}{3}\left(\frac{\pi}{3} + 2\sqrt{3}\right)\right],$$

where $\delta = \log_3 4 = \overline{\dim}_M(\partial K, K)$ and $\{x\}$ is the fractional part of $x := -\log_3(\epsilon\sqrt{3})$. Therefore,

$$\overline{\mathcal{M}}_\delta(\partial K, K) \leq \frac{1}{480}\left(723\sqrt{3} + 20\pi\right).$$

Moreover, the monotonicity of the above upper bound shows that $\epsilon' \leq 0$ in our estimate and hence $\mathfrak{M}_K \leq 11.61$. Based on this, $C_3(K) \approx 1354$ and $S_1^\epsilon(t) \leq 104282\, t^{\delta/2}$. Computing M_K we finally obtain an asymptotic upper bound for the remainder term of the Neumann counting function of the Koch snowflake of

$$N_N(K,t) - C_W^{(2)} \operatorname{vol}_2(K)t \leq 104325.5\, t^{\delta/2}.$$

Alternatively, an absolute upper bound can be found following Rem. 4.17.(i):

$$N_N(K,t) - C_W^{(2)} \operatorname{vol}(K)t \leq C_W^{(2)}\left(3.537 \cdot 10^6\, t^{\delta/2} - 353 - 911\sqrt{t}\right). \tag{4.2}$$

Since $\epsilon < 1/9$ is more than sufficient to satisfy Def. 4.1, this holds true for all $t \geq 0.1$. Some limited computational results on the Neumann (or Dirichlet) spectrum of the Koch Snowflake can be found in [81, 104]. A few estimations in the above application were deliberately chosen sub-optimally for the benefit of the presentation so that some improvements in these constants are expected to be possible.

4.4 Constants

The following constants are used within this document. Let Ω be well-covered by $\{D_i^\epsilon\}_{i\in I_\epsilon}$ with corresponding c_E as in Def. 4.1(ii) and $E_i^\epsilon, r_i^\epsilon, L_i^\epsilon, \mathcal{I}_{\beta,i}^\epsilon, \beta_{\inf,i}^\epsilon$ to each D_i^ϵ in the cover and indexed accordingly. Based on the conditions in Cor. 3.9 we define $c_r^\pm, c_L^\pm, c_\mathcal{I}^\pm$ via

$$r_i^\epsilon \epsilon^{-1} \in [c_r^-, c_r^+], \quad L_i^\epsilon \epsilon^{-1} \in [c_L^-, c_L^+], \quad \mathcal{I}_{\beta,i}^\epsilon \epsilon^{-1} \in [c_\mathcal{I}^-, c_\mathcal{I}^+].$$

$\mathfrak{M}_\Omega$ and A_Ω are defined in Prop. 2.42 and Prop. 2.43, respectively and r_+ is as in the proof of Thm. 4.15.

$$
\begin{aligned}
C_W^{(n)} &:= \frac{(4\pi)^{-n/2}}{\Gamma(1+\frac{n}{2})} \\
C_1(\Omega) &:= c_E (c_r^+)^{-2} \left[1 + c_{\mathcal{I}}^+ \left(\frac{1}{\sqrt{c_r^- \inf_{i,\epsilon} \beta_{\inf}(E_i^\epsilon)}} + \sqrt{\frac{c_L^+ c_E c_r^+}{\inf_{i,\epsilon} \beta_{\inf}(D_i^\epsilon)}} \right)^2 \right]^{-1} \\
C_2(\Omega) &:= \min \left\{ C_1(\Omega), \frac{1}{2} \left(\sqrt{n}\pi / r_+\right)^2 \right\} \\
C_3(\Omega) &:= C(\Omega) + \mathfrak{M}_\Omega \frac{(40\sqrt{n})^\delta}{2^\delta - 1} \\
M_\Omega &:= C_3(\Omega) \left(\frac{\mu+1}{C_2(\Omega)} \right)^{\delta/2} + \frac{C_W^{(n-1)}}{4} A_\Omega \left(\frac{\mu+1}{C_2(\Omega)} \right)^{\frac{\delta-(n-1)}{2}} .
\end{aligned}
$$

Chapter 5

Renewal theory and fractal sprays

In this chapter we combine renewal theoretic results with the above to study Dirichlet counting functions on fractal sprays. We first discuss necessary background of graph directed systems and cite a result by Kombrink on asymptotic expansions of renewal functions in Thm. 5.6. In Sec. 5.2 we apply the results on snowflakes from the previous chapter to a setting of fractal sprays whose gaps have fractal boundary and compare the asymptotic expansion of the spectral counting function to the one of the volume of the inner ϵ-parallel neighbourhood. Section 5.2 is largely based on [61] by Kombrink and the author.

5.1 Graph directed systems

In this section the basic notion of a graph directed system (GDS) is defined, following the approach in [74]. Let (V, Σ, i, t) be a directed multi-graph with vertex set V and an alphabet Σ as the set of edges. Let $i, t : \Sigma \to V$ be the initial and terminal map on Σ indicating start and end point of an edge such that $\Sigma \ni e : i(e) \to t(e)$. Now to each $v \in V$ a compact complete metric space X_v is assigned and to each edge e a bijective contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with Lipschitz constant L_e is assigned. This amounts to a graph directed system (sometimes also called graph directed scheme) in the following way.

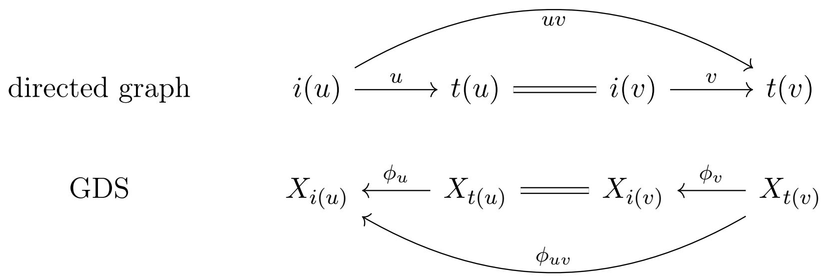
Definition 5.1. Let V be a finite set and Σ be at most countable. Then a *graph directed system (GDS)* $(\{X_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ consists of a family of compact subsets $X_v \subset \mathbb{R}^n$ for each vertex $v \in V$ and Lipschitz continuous maps $\phi_e : X_{t(e)} \to X_{i(e)}$ along each edge $E \ni e = (i(e), t(e))$ such that $\sup_{e \in \Sigma} L_e < 1$, where L_e is the Lipschitz constant of the map ϕ_e.

Therefore this definition is contravariant in that the maps in the GDS point backwards compared to the edge in the directed graph. The *incidence matrix* A of a GDS is then defined analogously to the composition rule in the construction of well-foliated domains in Sec. 3.2 via

$$A : \Sigma \times \Sigma \to \{0,1\} \text{ via } A(u,v) := \begin{cases} 1 & \text{if } t(u) = i(v) \\ 0 & \text{else} \end{cases}$$

For any word $w = w_1 w_2 w_3 \cdots$ the *(left) shift map* is defined via $\sigma w := w_2 w_3 \cdots$ and $w|_m := w_1 w_2 w_3 \cdots w_m$.

Following the definition of the incidence matrix of a GDS, two maps ϕ_u, ϕ_v for $u, v \in \Sigma$ can be composed as $\phi_u \phi_v = \phi_{uv}$ iff $uv \in \Sigma_A^2$:



This explains how the composition of words is related to the composition of maps. For $w = w_1 w_2 \cdots w_m \in \Sigma_A^{\text{fin}}$, the map ϕ_w is given by $\phi_w := \phi_{w_1} \phi_{w_2} \cdots w_m$ as defined before. For any $\xi : \Sigma_A^\infty \to \mathbb{C}$ the *n-th Birkhoff sum* is defined as

$$S_n \xi := \begin{cases} \sum_{k=0}^{n-1} \xi \circ \sigma^k & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

For any word w with $\operatorname{len} w \geq n$ the subword of length n is denoted by $w|_n := w_1 w_2 \cdots w_n$.

Any word $w \in \Sigma_A^n$ gives rise to a *cylinder of length* n defined by $[w] := \{x \in \Sigma_A^\infty \text{ s.t. } x|_n = w\}$.

Let $(\{X_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ be a GDS as in Def. 5.1 and $\Sigma_A^\infty \ni w$. Then the sequence $(w|_n)_{n\in\mathbb{N}}$ gives rise to a sequence of contractions with the common codomain $X_{i(w_1)}$ and Lipschitz constant $\leq (\sup_{i\in\mathbb{N}} L_{w_i})^n$ bounded away from 1. Therefore the sequence $\left(\phi_{w|_n}\left(X_{t(w_n)}\right)\right)_{n\in\mathbb{N}}$ of compact subsets of $X_{i(w_1)}$ is nested and, by Cantor's Intersection and the following intersection is non-empty:

$$\bigcap_n \phi_{w|_n}(X_{t(w_n)}) \neq \emptyset.$$

Analogously to the case of an IFS the diameter of this intersection has an arbitrarily small upper bound given by

$$\operatorname{diam} \bigcap_n \phi_{w|_n}(X_{t(w_n)}) \leq \lim_n (\sup L_e)^n \operatorname{diam}(X_{t(w_n)}) = 0,$$

so that this intersection contains exactly one point, $\bigcap_n \phi_{w|_n}(X_{t(w_n)}) = \{x\}$. This point is defined as the projection (also called *coding map*) π from the word w to a subset of $X_{i(w_1)}$ and by the above considerations this gives rise to the *limit set of a GDS* $(\{X_v\}_{v\in V}, \{\phi_e\}_{e\in\Sigma})$, defined as

$$J(F) := \pi\left(\Sigma_A^\infty\right).$$

Definition 5.2. Let A be the incidence matrix of some GDS. Then A is called

(i). *finitely irreducible* if there is a finite subset $\Lambda \subset \Sigma_A^{\text{fin}}$ such that for any two edges i, j there is a path $w \in \Lambda$ such that iwj is an admissible word.

(ii). *finitely primitive* if there is a $p \in \mathbb{N}$ and a finite subset $\Lambda \subset \Sigma_A^p$ such that for any two edges i, j there is a word $w \in \Lambda$ such that iwj is admissible.

A GDS is called *conformal* if all maps are conformal and additionally all $X_v = \overline{\operatorname{int} X_v}$ satisfy the cone condition in Def. 2.7 and the open set condition (see Thm. 2.36) is satisfied.

In this case the limit set is called *self-conformal*.

Theorem 5.3 (Moran-Hutchinson formula for limits sets of GDS, [45, 74, 78]). *Let J be the limit set of a conformal GDS in the same notation as above with finitely primitive incidence matrix and let $\xi(y) := \log|\phi'_{y_1}\pi\sigma y|$. Then the Hausdorff dimension of J is*

$$\dim_H J = \inf\left\{t \in \mathbb{R} : P(t\xi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\operatorname{len} w = n} \sup_{\tau\in[w]} |\phi'_\tau|^t < 0\right\}$$

5.1.1 Motivation for renewal theory

We motivate the general idea behind a *Laplacian off a fractal* by the Sierpiński Gasket (See Fig. 5.1).

Definition 5.4. Let $G \subset \mathbb{R}^n$ be any compact set. Then its *bounded complement* G^{bc} is defined as the union of all bounded connected components of the complement.

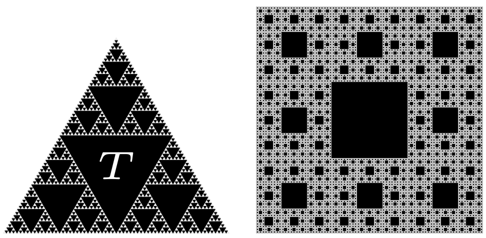


Figure 5.1: (Left): The Sierpiński gasket S_G with $\dim_M(S_G) = \log_2 3$. The bounded complement is filled with black. (Right): The Sierpiński carpet S_C with $\dim_M(S_C) = \log_3 8$. The bounded complement is filled with black. The base triangle is labeled with T.

Let S_G be the Sierpiński gasket: Let V be a singleton and $\Sigma = \{1,2,3\}$ with ϕ_i given by affine maps of the form $x \mapsto \frac{1}{2}x + q_i$ for fixed $q_1 = (0,0), q_2 = (1/2, \sqrt{3}/2)$ and $q_3 = (1,0)$. Let $\Omega := {S_G}^{\mathrm{bc}}$ as the prototypical example. Then

$$\Omega = T \sqcup \bigsqcup_{i\in\Sigma} \phi_i\Omega,$$

where T is the largest inner equilateral triangle (also called *gap*). Iterating this yields

$$\begin{aligned}\Omega &= T \sqcup \bigsqcup_{i_1\in\Sigma} \phi_{i_1}\left(T \sqcup \bigsqcup_{i_2\in\Sigma}\phi_{i_2}\Omega\right) = T \sqcup \bigsqcup_{i\in\Sigma}\phi_i T \sqcup \bigsqcup_{w\in\Sigma^2}\phi_w\Omega \\ &\vdots \\ &= \bigsqcup_{n=0}^{M}\left(\bigsqcup_{w\in\Sigma^n}\phi_w T\right) \sqcup \bigsqcup_{w\in\Sigma^{M+1}}\phi_w\Omega \qquad \forall M\in\mathbb{N} \qquad (5.1)\\ &= \underbrace{\left(\bigsqcup_{n=0}^{M}\bigsqcup_{u\in\Sigma^n}\bigsqcup_{w\in\Sigma^m}\phi_u\phi_w T\right)}_{:=\Omega_m^{\text{fin}}} \sqcup \underbrace{\left(\bigsqcup_{n=0}^{m-1}\bigsqcup_{w\in\Sigma^n}\phi_w T\right)}_{:=\Omega_m^{\text{rest}}} \qquad (5.2)\\ &\qquad \sqcup \bigsqcup_{w\in\Sigma^{M+m+1}}\phi_w\Omega \qquad \forall M,m\in\mathbb{N}.\end{aligned}$$

Since any Dirichlet eigenvalue is positive, $\forall\lambda_0 \geq 0 \exists M_0 \in \mathbb{N}$ such that $\forall M \geq M_0, \forall\lambda \leq \lambda_0$ with $N_D\left(\bigsqcup_{w\in\Sigma^M}\phi_w\Omega,\lambda\right) = 0$. This implies that the counting function of Dirichlet eigenvalues of the Laplace operator on Ω, $N_D(\Omega,\lambda)$ (resp. $N_N(\Omega,\lambda)$ satisfies the following renewal equation:

$$\begin{aligned}N_D(\Omega,\lambda) &= \sum_{n\in\mathbb{N}_0}\sum_{w\in\Sigma^n} N_D(\phi_w T,\lambda) \\ &= \sum_{n\in\mathbb{N}_0}\sum_{w\in\Sigma^n} N_D\left(T,\left(\phi_w'\right)^2\lambda\right) = \sum_{n\in\mathbb{N}_0} 3^n N_D\left(T,4^{-n}\lambda\right), \qquad (5.3)\end{aligned}$$

since $|\phi_e'| = 1/2$ for any $e \in \Sigma$. Eigenvalues of the occurring gaps in this example were extensively studied in [77]. Such relations are studied in Renewal Theory. The above argument immediately generalises to the following result.

Proposition 5.5 (Renewal Equation)**.** *Let $t : \Sigma_A^\infty \to \mathbb{R}$ be such that for any $w \in \Sigma_A^\infty$*

$$\lim_{M\to\infty}\sum_{y\in\Sigma_A^\infty:\sigma^M y=w} S_n t(w) = \infty.$$

Let $c : \Sigma_A^\infty \to \mathbb{R}_{>0}$ and $g : \Sigma_A^\infty \times \mathbb{R} \to \mathbb{R}$ be arbitrary. Let $N : \Sigma_A^\infty \times \mathbb{R} \to \mathbb{R}$ be such that *there is a* $\overline{t}$ *with* $\operatorname{supp}(t \mapsto N(w,t)) \subset [\overline{t}, \infty)$ *for all* $w \in \Sigma_A^\infty$ *and suppose* N *satisfies*

$$N(w,t) = g(w,t) + \sum_{x \in \Sigma_A^\infty : \sigma x = w} c_x N(x, t - t_x).$$

Then there are $\xi, \eta : \Sigma_A^\infty \to \mathbb{R}$ *such that*

$$N(w,t) = \sum_{n=0}^{\infty} \sum_{y \in \Sigma_A^\infty : \sigma^n y = w} g(y, t - S_n \xi(y)) e^{S_n \eta(y)}.$$

This last expression is addressed in Renewal Theory.

Prop. 5.5 can be used to translate (5.3) to a renewal equation with $N(t) := N_D(\Omega, e^t)$ and $g(t) := N_D(T, e^t)$:

$$N(t) = \sum_{n \geq 0} \sum_{y \in \Sigma_A^n} g(t - S_n \xi(y)),$$

where $\xi(y) := -2 \log \phi'_{y_1}$. Based in particular on works by Kesseböhmer-Kombrink, Mauldin-Urbańksi, Pollicott [60, 74, 84, 85] there are upper bounds for spectral counting functions on the bounded complement of limit sets of GDS with finitely primitive incidence matrix whenever there are such bounds for the spectral counting functions of the gaps.

5.1.2 Renewal theoretical asymptotic behaviour of spectral counting functions

Write $\mathrm{Lip}_\theta(\Sigma_A^\infty, \mathbb{R})$ for the space (equipped with the metric $\|f\|_\infty + \sup_{n \in \mathbb{N}} \sup_{w \in \Sigma_A^n} \{|f(x) - f(y)| \text{ s.t. } x, y \in [w]\} \theta^{-n}$) of bounded Lipschitz continuous maps with respect to the metric $d_\theta(x,y) := \theta^{\max(k \in \mathbb{N} : x|_k = y|_k)}$ in Σ_A^∞. For any continuous (with respect to d_θ) map $f : \Sigma_A^\infty \to \mathbb{R}$ with $\sum_{e \in \Sigma} e^{\sup\{f|_{[e]}\}} < 1$ we define the *Ruelle-Perron-Frobenius Operator* (also called *transfer operator*) as

$$\mathcal{L}_f : \mathrm{Lip}_\theta(\Sigma_A^\infty, \mathbb{R}) \to \mathrm{Lip}_\theta(\Sigma_A^\infty, \mathbb{R}) \qquad \text{via} \qquad \mathcal{L}_f(g)(x) := \sum_{y \in \Sigma_A^\infty : \sigma y = x} e^{f(y)} g(y).$$

Let $f_x : \mathbb{R} \to \mathbb{R}$ for each $x \in \Sigma_A^\infty$ and suppose $\xi \in \mathrm{Lip}_\theta(\Sigma_A^\infty, \mathbb{R})$ is a-lattice, i. e. $\exists \xi', h \in \mathrm{Lip}_\theta(\Sigma_A^\infty, \mathbb{R})$ such that $\xi(y) = \xi'(y) + h(\sigma y) - h(y)$ with lattice constant given by $a := \sup\left\{a' > 0 : \xi'(\Sigma_A^{\mathbb{N}}) \subset a'\mathbb{Z}\right\} > 0$. For any $\beta \in [0, \alpha)$ the *two-sided Fourier-Laplace transformation shifted by* h is defined as

$$\widehat{f}_{x\beta}(z) := \sum_{\ell\in\mathbb{Z}} e^{a\ell z} f_x(a\ell + \beta - h(x))$$

The maximal strip of absolute convergence of this series is denoted by

$$\begin{aligned} \text{abs-conv}(f_x) &:= \{z \in \mathbb{C} | \widehat{f}_{x\beta}(z) \text{ converges absolutely } \forall \beta \in [0, a)\} \\ &= \{z \in \mathbb{C} | s^{f_x,-}_{\text{abs-conv}} \leq \Re(z) \leq s^{f_x,+}_{\text{abs-conv}}\}, \end{aligned} \tag{5.4}$$

Of course the boundaries $s^{f_x,\pm}_{\text{abs-conv}}$ of this strip can be infinite. From the definition it follows that $\widehat{f}_{x\beta}$ is holomorphic on abs-conv(f_x) for all β. Finally let $\widehat{f}_{x\beta}^{\,\text{mero}}$ be the maximal meromorphic extension of $\widehat{f}_{x\beta}$ and denote its domain by abs-conv$^{\text{mero}}(f_x)$. Since ξ is a-lattice $\exists \xi' \sim \xi$ such that $\forall y \in \Sigma_A^\infty\ \exists n_y \in \mathbb{Z} : \xi'(y) = n_y a$ so that

$$S_n\xi(y) = S_n\xi'(y) + S_n\left(h(\sigma(y)) - h(y)\right) = \underbrace{\sum_{k=0}^{n-1} n_{\sigma^k y} a}_{\in a\mathbb{Z}} + h(x) - h(y).$$

Let $\beta \in [0, a)$,

$$N_x(t) = \sum_{n\in\mathbb{N}} \sum_{y\in\Sigma_A^\infty : \sigma^n y = x} f_y(t - S_n\xi(y)) \tag{5.5}$$

and let $z \in \text{abs-conv}(N_x) \cap \bigcap_{y\in\Sigma_A^\infty} \text{abs-conv}(f_y)$. Moreover we request $\|\mathcal{L}_{z\xi'}\|_{\text{op}} < 1$. Then by the definition of the two-sided Fourier-Laplace transformation

$$\begin{aligned} \widehat{N}_{x\beta}(z) &:= \sum_{\ell\in\mathbb{Z}} e^{a\ell z} \sum_{n\in\mathbb{N}_0} \sum_{\sigma^n y = x} f_y(a\ell + \beta - h(x) - S_n\xi(y)) \\ &= \sum_{n\in\mathbb{N}_0} \sum_{\sigma^n y = x} \sum_{\ell\in\mathbb{Z}} e^{a\ell z} f_y(a\ell + \beta - h(y) - S_n\xi'(y)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{N}_0} \sum_{\sigma^n y = x} e^{S_n\xi'(y)z} \sum_{\widetilde{\ell}\in\mathbb{Z}} e^{a\widetilde{\ell}z} f_y(a\widetilde{\ell} + \beta - h(y)) \\
&= \sum_{n\in\mathbb{N}_0} \sum_{\sigma^n y = x} e^{S_n\xi'(y)z} \widehat{f_{y}}_{\beta}(z) \\
&= \sum_{n\in\mathbb{N}_0} \sum_{\sigma y^{(n-1)} = x} \cdots \sum_{\sigma y^{(1)} = y^{(2)}} \sum_{\sigma y = y^{(1)}} e^{z\xi'(\sigma^{n-1}y)} \cdots e^{z\xi'(\sigma y)} e^{z\xi'(y)} \widehat{f_{y}}_{\beta}(z) \\
&= \sum_{n\in\mathbb{N}_0} \sum_{\sigma y^{(n-1)} = x} \cdots \sum_{\sigma y^{(1)} = y^{(2)}} \sum_{\sigma y = y^{(1)}} e^{z\xi'(y^{(n-1)})} \cdots e^{z\xi'(y^{(1)})} e^{z\xi'(y)} \widehat{f_{y}}_{\beta}(z) \\
&= \sum_{n\in\mathbb{N}_0} \underbrace{\sum_{\sigma y^{(n-1)} = x} e^{z\xi'(y^{(n-1)})} \cdots \underbrace{\sum_{\sigma y^{(1)} = y^{(2)}} e^{z\xi'(y^{(1)})} \underbrace{\sum_{\sigma y = y^{(1)}} e^{z\xi'(y)} \widehat{f_{y}}_{\beta}(z)}_{=\mathcal{L}_{z\xi'}\left(w\mapsto \widehat{f_w}_\beta(z)\right)}}_{=\mathcal{L}_{z\xi'}\left(\mathcal{L}_{z\xi'}\left(w\mapsto \widehat{f_w}_\beta(z)\right)\right)}}_{=\mathcal{L}^n_{z\xi'}\left(w\mapsto \widehat{f_w}_\beta(z)\right)} \\
&= \sum_{n\in\mathbb{N}_0} \mathcal{L}^n_{z\xi'} \left(w \mapsto \widehat{f_w}_\beta(z)\right) \\
&= \left(\mathrm{id} - \mathcal{L}_{z\xi'}\right)^{-1} \left(w \mapsto \widehat{f_w}_\beta(z)\right)
\end{aligned}
$$

This motivates the analysis of the so-called *dynamical ζ-function* given by $\zeta : z \mapsto \left(\mathrm{id} - \mathcal{L}_{z\xi'}\right)^{-1}$ and, in particular, conditions on z for which $\mathcal{L}_{z\xi'}$ has 1 in its spectrum. This approach relies on perturbation theory of linear operators as presented by Kato in [53]. Based on spectral results on the transfer operator by Pollicott and Ruelle in [84, 85] and [94, Chap. 5], a result by Kato in [53, Chap. VII §1.3] shows that for an eigenvalue λ_z of $\mathcal{L}_{z\xi}$ the map $w \mapsto \lambda_{z+w}$ consists of branches of one or several (if multiplicity of the eigenvalue at z is $m_z > 1$) analytical functions with at most algebraic singularities at $w = 0$. For a non-branching eigenvalue λ_z we observe that $w \mapsto \lambda_{z+w}$ is holomorphic in a neighbourhood of 0 and either constant or $\neq \lambda_z$ in a small neighbourhood of 0 since it would otherwise be non-constant with λ_z as accumulation point. In the latter case, an eigenvalue λ_z is called *strongly regular*. For a transfer operator with a strongly regular eigenvalue 1, Kesseböhmer-Kombrink (cf. [60]) showed the following asymptotic estimate for renewal functions based on earlier works by Lalley in [64]. A simplified version of the proof can be found in Thm. 5.10.

Theorem 5.6 (Renewal Theorem, [59])**.** *For any GDS with vertex set V, edge set Σ and incidence matrix A, define*

$$\mathcal{Z} := \{z \in \mathbb{C} : 1 \in \sigma(\mathcal{L}_{z\xi})\}.$$

Put $s^{\pm} := s^{f_x,\pm}_{\text{abs-conv}}$ as in (5.4) and assume $s^- < \delta < t^*$ with $t^* := \sup\{t \in \mathbb{R} : \sum_{e\in\Sigma} e^{\sup t\xi|_{[e]}} < \infty\}$. Let A be finitely irreducible and suppose that 1 is a strongly regular eigenvalue for all $z \in \mathcal{Z}$. Let N_x and f_y be related as in (5.5) with ξ being a-lattice with corresponding $\xi'(y) \in a\mathbb{Z}$ for all $y \in \Sigma_A^{\mathbb{N}}$ such that for any small enough $\epsilon > 0$ the estimate

$$\sum_{n\in\mathbb{N}_0}\sum_{\sigma^n y = x} |f_y(t - S_n\xi(y))| \in o\left(e^{-t(s^- - \epsilon)}\right)$$

holds true for all $t \in \mathbb{R}$ small enough. Let δ_0 be such that $(\mathrm{id} - \mathcal{L}_{z\xi})^{-1}$ is meromorphic in $\{z \in \mathbb{C} : \Re(z) < \delta_0\}$. Then for any $\epsilon > 0$ the sets

$$\mathcal{Z}(\epsilon) := \{z \in \mathcal{Z} : \Re(z) < \delta_0 - \epsilon, \Im(z) \in [-\pi/a, \pi, a)\},$$
$$\mathcal{P}_{x,\beta}(\epsilon) := \left\{ \begin{array}{c} z \in \text{abs-conv}^{\text{mero}}(f_x) : z \textit{ is a pole of } \widehat{f_{x\beta}}^{\text{mero}} \\ \textit{and } \Im(z) \in [-\pi/a, \pi/a)\} \cap \mathbb{C}_{<\delta_0 - \epsilon}\} \end{array} \right\}$$

are finite and there exist functions $A_{x,\beta}(s,\ell), B_{x,\beta}(s,\ell)$ of polynomial growth in ℓ such that

$$N_x(a\ell + \beta) = \sum_{z\in\mathcal{Z}(\epsilon)\setminus\mathcal{P}_{x,\beta}(\epsilon)} e^{-a\ell z} A_{x,\beta}(z,\ell) + \sum_{z\in\mathcal{P}_{x,\beta}(\epsilon)} e^{-a\ell z} B_{x,\beta}(z,\ell) + o\left(e^{-a\ell(\min(\delta,s^+)-\epsilon)}\right).$$

Remark 5.7. (i). Most renewal theorems contain a similar result for when ξ is non-lattice. However it may occur that the error terms grow faster in such cases.

(ii). The set $\mathcal{Z}$ of points at which the dynamical ζ-function is singular is a natural generalisation of the set of complex dimensions introduced by Lapidus-Frankenhuijsen in [70].

Remark 5.8.

(i). The error term given by Thm. 5.6 follows from the necessary convergence of a power series. The domain of absolute convergence of this sum depends on the asymptotic properties of f_x. Whenever the two-sided Fourier-Laplace transform of f_x has a global meromorphic extension, the error term vanishes.

(ii). This proof is constructive in the sense that the sequences of coefficients $A_{x,\beta}(z,\ell)$ and $B_{x,\beta}(z,\ell)$ can be explicitly computed. This is particularly easy if the set of common poles is empty since then the more involved effect of poles on poles is non-existent: Suppose there are no $z \in \mathbb{C}$ such that $\mathcal{L}_{z\xi'}$ has an eigenvalue 1 and $\widehat{f_{x\beta}}^{\text{mero}}$ has a pole at z.

(a) Let z be a pole of $\widehat{f_{x\beta}}^{\text{mero}}(z)$ and let $\widetilde{C}^k_{x,\beta}(z)$ be the corresponding residues as in the proof of Thm. 5.6. Then the coefficients $C^k_{x,\beta}(z)$ are given by

$$C^k_{x,\beta}(z) = \left(\mathrm{id} - \mathcal{L}_{z\xi'}\right)^{-1} \left(\widetilde{C}^k_{-,\beta}(z)\right)(x)$$

Whenever f_x does not depend on x, this can be solved trivially. In the notation of (5.6), this is the case if Y_v is independent of v or more generally if all Y_v are isospectral.

(b) Let now $z \in \mathcal{Z}(\epsilon)$. Let γ_w be the strongly regular eigenvalue of $\mathcal{L}_{w\xi}$ such that $\gamma_z = 1$ and denote the multiplicity of this eigenvalue with $m_w = m_z$ as before. Suppose that $\widehat{f_x}^{\text{mero}}$ is non-singular at z. Since γ_z is strongly regular, $w \mapsto (\mathrm{id} - \mathcal{L}_{w\xi})^{-1}$ is defined on a neighbourhood of z. Following the argument of Ch.I §5.1 in [53] and [59] one shows that the simple pole of the dynamical ζ-function corresponding to the simple eigenvalue 1 of $\mathcal{L}_{z\xi}$ at $z = \delta$ has a coefficient of

$$D^1_{x,\beta}(\delta) = a\frac{\int_{\Sigma_A^{\mathbb{N}}} \widehat{f_{-\beta}}^{\text{mero}}(\delta) d\nu_{\delta\xi}}{\int_{\Sigma_A^{\mathbb{N}}} \xi dm_{\delta\xi}}.$$

In the simple case where $f_x = f$ is independent of $x \in \Sigma_A^{\mathbb{N}}$, $A = \mathbf{1}$ and ξ is constant on cylinders of length 1 this simplifies to

$$D^1_{x,\beta}(\delta) = a\frac{\widehat{f}_{x\beta}^{\text{mero}}(\delta)}{\sum_{e\in\Sigma} \xi e^{\delta\xi}}.$$

In the case of common poles these two effects are combined. This is notationally cumbersome but does not involve any other effects.

5.1.3 Spectral counting functions on complements of limit sets of GDS

This section focuses on applications of the general results above to counting problems on complements of limits sets as motivated in Sec.5.1.1. The general aim is to find asymptotic expansions for the counting function $N_D(\Omega, t)$ where $\Omega = J^{\text{bc}}$ is the bounded complement of the limit set J of a GDS.

Let V, Σ and A be given for some GDS $(\{Y_v\}, \{\phi_e\})$. Then the bounded complement of some limit set is given by

$$\Omega := \bigsqcup_{v\in V} Y_v$$

for *gaps* given defined as $Y_v := X_v \backslash J$. These gaps satisfy

$$Y_v = \underbrace{X_v \backslash \left(\bigcup_{e\in\Sigma: i(e)=v} \phi_e X_{t(e)} \right)}_{:=\Omega_v} \cup \underbrace{\left(\bigcup_{e\in\Sigma: i(e)=v} \phi_e Y_{t(e)} \right)}_{:=\Phi_v^Y}. \tag{5.6}$$

It is clear from this expression that certain separation conditions can be used in order to achieve stronger results about the spectrum of gaps; see Fig. 5.2. Namely, if the images of the gaps under ϕ_e intersect, the counting function of their union is not related to the sum

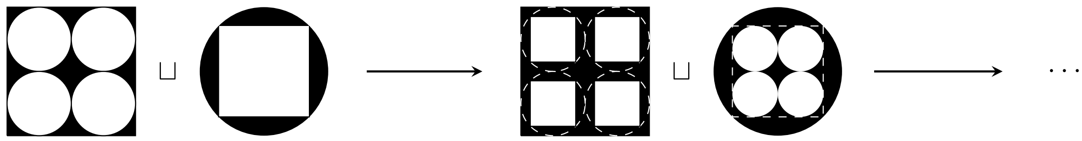


Figure 5.2: A GDS where images of contractions share boundary visualised by the first two iterations of the contractions. The "inner boundaries" are drawn dashed. The limit set of the left component is a Cantor dust. There are no self-replicating components in this case.

of counting functions of $\phi_e Y_{t(e)}$ anymore. Moreover, if Ω_v and Φ_v^Y intersect, their union will contain connected components that are not images under ϕ_e of another domain $Y_{v'}$. Since spectral counting functions are in general not well-behaved under union of domains, such intersections can cause non-trivial behaviour. A GDS is called *separating* if the family of images $\big(\phi_e(X_{t(e)})\big)_{e\in\Sigma}$ and $\big(\Omega_v, \Phi_v^Y\big)$ are almost disjoint (i.e. disjoint up to finitely many points) for any $v \in V$. If a GDS is not separating, the gaps are not self-replicating under the action of the contractions. In these cases, the spectral properties of the gaps can only be estimated using more general concepts. Let $R_v^\Phi(\lambda)$ be defined as

$$R_v^\Phi(t) := N_D(Y_v, t) - N_D(\Omega_v, t) - \sum_{e\in\Sigma:i(e)=v} N_D(\phi_e Y_{t(e)}, t).$$

Prop. 2.24 implies that $R_v^\Phi \geq 0$. In other words, R_v^Φ accounts for the eigenfunctions that vanish on the boundary but not on "inner boundaries". Clearly R_v^Φ vanishes identically if the unions in (5.6) are disjoint or if the GDS is separating and $R_v^\Phi(t)$ vanishes for small enough t. The counting function of each gap Y_v then satisfies

$$N_D(Y_v, \lambda) = N_D(\Omega_v, \lambda) + R_v^\Phi(\lambda) + \sum_{e\in\Sigma:i(e)=v} N_D(\phi_e Y_{t(e)}, \lambda). \tag{5.7}$$

Therefore R_v^Φ as above has a strong influence on the asymptotics. The asymptotic behaviour of R_v^Φ can be bounded with Cor. 2.25: Indeed Thm. 4.15 shows that $R_v^\Phi(t) = \mathcal{O}(t^{\delta_v/2})$, where

$$\delta_v := \max_{e:i(e)=v} \left\{ \overline{\dim}_M \partial Y_v, \overline{\dim}_M \partial \Omega_v, \overline{\dim}_M \partial \phi_e Y_{t(e)}, \overline{\dim}_M \partial \bigcup_{i(e)=v} \phi_e Y_{te} \right\} \tag{5.8}$$

if all Y_v, Ω_v, Φ_v^Y are well-covered and all ϕ_e have the E-property.

5.2 Fractal sprays generated by the Koch snowflake

The goal of this section is to show directly how further asymptotic terms may arise from specific function systems with a lattice property as already shown in Thm. 5.6. This section is based on [62] by Kombrink and the author.

Our class of examples is based on the construction shown in Fig. 5.3(a) (see also Fig. 5.4). More precisely, we consider the iterated function system (IFS) $\Phi := \Phi(0,0) :=$

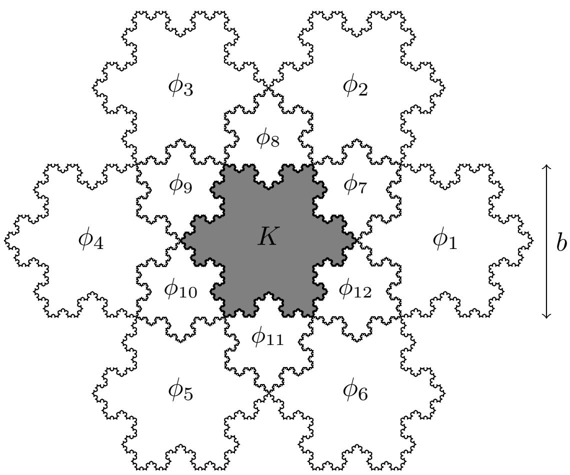


Figure 5.3(a) Depiction of the IFS of the fractal spray studied in Sec. 5.3 and Sec. 5.4. The *base length* b of the snowflake K is also shown.

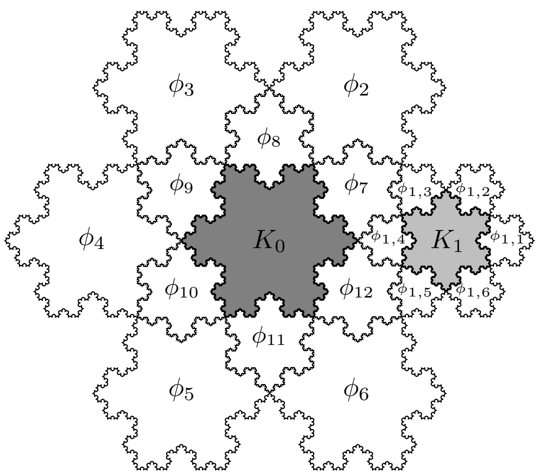


Figure 5.3(b) Depiction of the variant $\Omega(1,0)$ of the IFS of the fractal spray studied in Sec. 5.3 and Sec. 5.4. In this case, the map ϕ_1 was replaced by the six maps $\phi_{1,1}, \dots, \phi_{1,6}$ giving rise to an additional connected component of the generator $G = K_0 \cup K_1$. Analogously one can replace $\phi_7, \dots, \phi_{12}$.

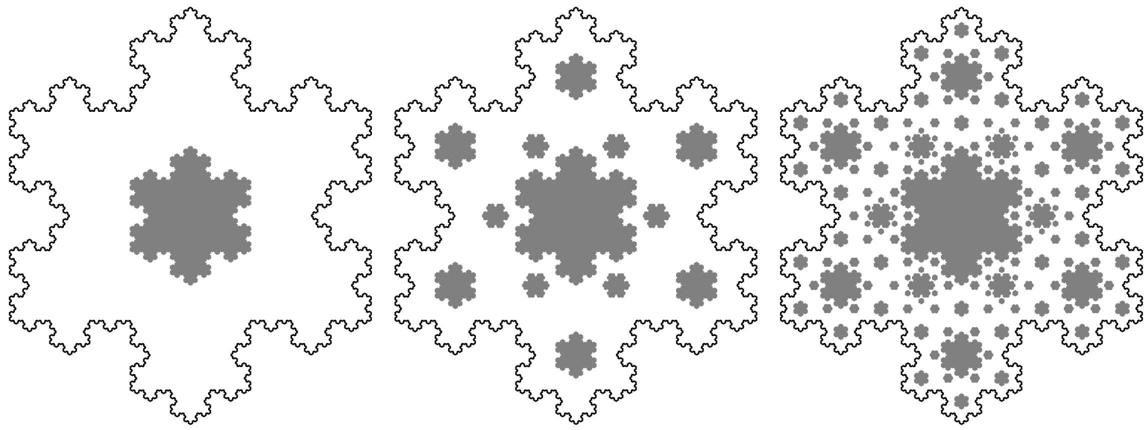


Figure 5.4: Example of the iterative construction of Ω as defined in (5.10). From left to right the 0^{th}, first and second iterations of the generator K under Φ are shown, rotated by 30°. The 0^{th} iteration (left) shows K. The first iteration (middle) shows $K \cup \Phi(K)$. The second iteration (right) then shows $K \cup \Phi(K) \cup \Phi^2(K)$.

$\{\phi_1, \ldots, \phi_{12}\}$ defined on $\mathbb{R}^2$ given by the maps

$$\phi_i(x) := \begin{cases} \frac{1}{3}x + \frac{2\sqrt{3}}{3}\begin{pmatrix} \cos[(i-1)\pi/3] \\ \\ \sin[(i-1)\pi/3] \end{pmatrix} & \text{if } i \in \{1, \ldots, 6\} \\ \frac{1}{3\sqrt{3}} R_{\pi/6}(x) + \frac{2}{3}\begin{pmatrix} \cos[\pi/6 + (i-7)\pi/3] \\ \\ \sin[\pi/6 + (i-7)\pi/3] \end{pmatrix} & \text{if } i \in \{7, \ldots, 12\} \end{cases}$$

where $R_{\pi/6}$ is a rotation by $\pi/6$ about the origin. Define the action of Φ on subsets of $\mathbb{R}^2$ by $\Phi A := \bigcup_{i \in \Sigma} \phi_i A$ with $i \in \{1, \ldots, 12\} =: \Sigma$. Further, let F denote the unique non-empty compact invariant set associated to the IFS, i. e. the set satisfying $F = \Phi F := \bigcup_{i \in \Sigma} \phi_i F$. Note that F is contained in the disk around the origin of radius $\sqrt{3}$, with $\binom{-\sqrt{3}}{0}$ and $\binom{\sqrt{3}}{0}$ belonging to F. As all ϕ_i are similarities, F is a self-similar set. From the definition of the maps it is evident that Φ consists of six contractions with contraction ratios $r_1, \ldots, r_6 = 1/3 = (\exp(-a))^2$ and six contractions with contraction ratios $r_7, \ldots, r_{12} = \sqrt{3}/9 = (\exp(-a))^3$, where $a := \log 3/2$ is known as the *lattice constant*. Note that by construction, $\mathbb{R}^2 \setminus F$ has got a unique unbounded connected component, which we denote by U. Further, we let $O := \mathbb{R}^2 \setminus \overline{U}$. Then O is open and satisfies $\phi_i O \subseteq O$ for $i \in \Sigma$ and

$\phi_i O \cap \phi_j O = \emptyset$ for $i \neq j \in \Sigma$. This implies that the open set condition (OSC) is satisfied and that O is a feasible open set for the OSC. Fig. 5.3(a) shows the images of $\overline{O}$ under the maps $\phi_1, \dots, \phi_{12}$.

For $m \in \mathbb{N}$ and $\omega = (\omega_1, \dots, \omega_m) \in \Sigma^m$, let $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_m}$ and

$$r_\omega := \exp(-a\nu_\omega) := \prod_{i=1}^{m} r_i \quad \text{with} \quad \nu_\omega := \sum_{i=1}^{m} \nu_{\omega_i} \in \mathbb{N}. \tag{5.9}$$

We define $K := O \setminus \overline{\Phi O}$, and note that K is the interior of a Koch snowflake with base length $b = 1$, see Fig. 5.3(a). A central object of our studies is

$$\Omega := \bigcup_{m=0}^{\infty} \Phi^m(K) := K \cup \bigcup_{m=1}^{\infty} \bigcup_{\omega \in \Sigma^m} \phi_\omega(K) \tag{5.10}$$

which is a countable union of disjoint open sets $\phi_w(K)$ and can be viewed as a fractal spray with *generator* K. Here, $\Phi^0(K)$ is understood to be K. The first three iterations of the construction of Ω in (5.10) are shown in Fig. 5.4.

We will moreover study the sets $\Omega(k_1, k_2)$ which result from modifications of the above construction as explained below. For $(k_1, k_2) \in \{0, \dots, 6\}^2$, we replace each of $\phi_1, \dots, \phi_{k_1}$ with six maps of contraction ratio $1/9$ and each $\phi_7, \dots, \phi_{k_2+6}$ with six maps with contraction ratio $1/(9\sqrt{3})$. The replacement of ϕ_i with six maps $\phi_{i,1}, \dots, \phi_{i,6}$ is done in such a way that $\bigcup_{k=1}^{6} \phi_{i,k} O \subset \phi_i O$, that $\phi_{i,k} O \cap \phi_{i,j} O = \emptyset$ for all $k \neq j$, and that $\partial \phi_i O \subset \partial \bigcup_{k=1}^{6} \phi_{i,k} O$. See Fig. 5.3(b) for an example of the replacement procedure. The corresponding IFS consisting of $12 + 5(k_1 + k_2)$ maps will be denoted by $\Phi(k_1, k_2)$ and the associated alphabet by $\Sigma(k_1, k_2)$. The generator $O \setminus \overline{\Phi(k_1, k_2)(O)}$, that we denote by G in this setting, has $1 + k_1 + k_2$ connected components. The fractal spray $\bigcup_{m=0}^{\infty} \Phi^m(k_1, k_2)(G)$ generated by G will be denoted by $\Omega(k_1, k_2)$. We write $k_1 = 0$ when no replacement is intended for $\phi_1, \dots, \phi_6$ (and correspondingly $k_2 = 0$) so that $\Omega(0, 0) := \Omega$.

5.2.1 Results for counting functions on the Koch snowflake

In the next section, Sec. 5.3, we prove asymptotic expansions for $N_D(\Omega(k_1, k_2), t)$ with $\Omega(k_1, k_2)$ as introduced above in Thm. 5.10. To this end an estimate for the spectral counting function of the gaps (given for example by K in Fig. 5.3(a) and by K_0 and K_1 in Fig. 5.3(b)) is necessary. We make use of our previous result in Sec. 4.3.1 to give numerical upper and lower bounds in Cor. 5.9. This result is also included in [61] by Kombrink and the author.

Corollary 5.9 (cf. [61])**.** *Let K be a Koch snowflake of base length 1 as defined in Fig. 5.3(a) and $\dim_M \partial K = \delta = \log_3 4$. Then*

$$C_-\lambda^{\delta/2} \leq N_N(K, \lambda) - \frac{1}{4\pi}\operatorname{vol}_2(K)\lambda \leq C_+\lambda^{\delta/2}.$$

for all $\lambda \geq 0.1$ with $C_- := -1481$ and $C_+ := 281.5 \cdot 10^3$.

5.3 Asymptotic behaviour of spectral counting functions

Let $\Omega(k_1, k_2)$ be the limit set described in Fig. 5.3(a)-5.3(b) and Sec. 5.2 with corresponding generator $K \subset \mathbb{R}^2$ being a Koch snowflake with base length 1. As above, let $\delta = \log 4/\log 3$ denote the Minkowski dimension of the boundary of K.

Theorem 5.10. *There is an asymptotic expansion of $N_D(\Omega(k_1, k_2), e^t)$ of the form*

$$N_D(\Omega(k_1, k_2), e^t) = \frac{1}{4\pi}\operatorname{vol}_2(\Omega(k_1, k_2))e^t - \sum_{\substack{z\in\mathcal{Z}_C,\\ \Re(z)<-\delta/2}} \widetilde{H_{\beta(t)}}(z)e^{-2a\lfloor\frac{t}{2a}\rfloor z} + o\left(e^{t(\delta/2+\gamma)}\right)$$

as $t \to \infty$ for any $\gamma > 0$. The absolute values of $\widetilde{H_{\beta(t)}}(z)$ are bounded from above according to Tab. 5.1. Here, $\beta(t) := 2a\{\frac{t}{2a}\}$, with $\{x\}$ denoting the fractional part of x, and $\mathcal{Z}_C := \left\{z \in \mathbb{C} \ : \ \sum_{i\in\Sigma} r_i^{-2z} = 1,\ \Im(z) \in \left[0, \frac{\pi}{a}\right)\right\}$.

Proof. We first consider $\Omega := \Omega(0, 0)$. We define $N(t) := N_D(\Omega, e^t)$ and $g(t) := N_D(K, e^t) = \frac{1}{4\pi}\operatorname{vol}_2(K)e^t + M(t)e^{t\delta/2}$ for an $M \in \mathcal{O}(1)$ as shown in Cor. 5.9. By (iii),(iv) in Sec. 2.2

and (5.10), $N(t)$ satisfies

$$\begin{aligned} N(t) &= \sum_{k\geq 0}\sum_{w\in\Sigma^k} N_D(\phi_w K, e^t) \\ &= \sum_{k\geq 0}\sum_{w\in\Sigma^k} g(t - 2\log r_{w_1} - 2\log r_{w_2} \cdots - 2\log r_{w_k}). \end{aligned}$$

Let $\ell_0 \in \mathbb{Z}$ be maximal such that $N(2a\ell_0) = 0$. Then for every $\beta \in [0, 2a)$ we define its two-sided Fourier-Laplace transform $\widehat{N}_\beta(z)$ for $\Re(z) < -1$ and rewrite this to isolate the poles of its maximal meromorphic extension. We make use of the fact that $\{-2\log r_i : i \in \Sigma\} \subset 2a\mathbb{Z}$ which follows from (5.9), so that for any $w \in \Sigma^k$ there is a $\nu_\omega \in \mathbb{Z}$ with $-2\log r_\omega = 2a\nu_\omega$, and perform an index shift $\ell \to \ell + \sum_{i=1}^k \nu_i = \widetilde{\ell}$.

$$\begin{aligned} \widehat{N}_\beta(z) &:= \sum_{\ell\in\mathbb{Z}} e^{2a\ell z} N(2a\ell+\beta) = \sum_{k\geq 0}\sum_{w\in\Sigma^k}\sum_{\ell\in\mathbb{Z}} e^{2a\ell z} g(2a\ell + \beta - 2\log r_{w_1} \cdots - 2\log r_{w_k}) \\ &= \sum_{k\geq 0}\sum_{w\in\Sigma^k} e^{z\left(-2\log r_{w_1} - \cdots - 2\log r_{w_k}\right)} \sum_{\widetilde{\ell}\in\mathbb{Z}} e^{2a\widetilde{\ell} z} g(2a\widetilde{\ell} + \beta) \\ &= \sum_{k\geq 0}\sum_{w\in\Sigma^k} {r_w}^{-2z} \left(\sum_{\ell\geq\ell_0} e^{2a\ell z} \frac{1}{4\pi} \operatorname{vol}_2(K) e^{2a\ell+\beta} + \sum_{\ell\geq\ell_0} e^{2a\ell z} M(2a\ell+\beta) e^{(2a\ell+\beta)\delta/2} \right) \\ &= \underbrace{\frac{1}{1 - \sum_{i\in\Sigma} {r_i}^{-2z}}}_{:=P(z)} \underbrace{\left(\frac{\frac{1}{4\pi}\operatorname{vol}_2(K) e^{\beta} e^{2a\ell_0(z+1)}}{1 - e^{2a(z+1)}} + M_\beta(z) \right)}_{:=H_\beta(z)} \end{aligned}$$

for some complex function M_β bounded in $\mathbb{C}_{-\delta/2} := \{z \in \mathbb{C} : \Re(z) < -\delta/2\}$ by

$$0 \leq |M_\beta(z)| \leq \left| \sum_{\ell\geq\ell_0} e^{2a\ell z} M(2a\ell+\beta) e^{(2a\ell+\beta)\delta/2} \right| \leq \left| \frac{\widetilde{M_K} e^{\beta\delta/2}}{1 - e^{2a(z+\delta/2)}} \right|,$$

where $\widetilde{M_K} := \max(C_-, C_+)$ is taken from Cor. 5.9. Therefore, $\widehat{N}_\beta$ can be meromorphically extended to $\mathbb{C}_{-\delta/2}$. Let

$$\mathcal{Z}_{\mathrm{C}} := \left\{ z \in \mathbb{C} \ : \ \sum_{i\in\Sigma} r_i^{-2z} = 1, \ \Im(z) \in \left[0, \frac{\pi}{a}\right) \right\}$$

Then, since all poles in $\mathcal{Z}_C$ are simple and $-1 \notin \mathcal{Z}_\mathrm{C}$,

$$\begin{aligned}
&\sum_{\ell \geq 0} e^{2a\ell s} N(2a\ell+\beta) - \sum_{z \in \mathcal{Z}_\mathrm{C} \cap \mathbb{C}_{-\delta/2}} \frac{-2aH_\beta(z)\operatorname{Res}_z P}{1-e^{2a(s-z)}} - \left(\frac{\frac{1}{4\pi}\operatorname{vol}_2(K)e^\beta}{1-\sum_{i\in\Sigma} {r_i}^2} \right) \frac{1}{1-e^{2a(s+1)}} \\
&= \sum_{\ell \geq 0} e^{2a\ell s} \left(N(2a\ell+\beta) + \sum_{z \in \mathcal{Z}_\mathrm{C} \cap \mathbb{C}_{-\delta/2}} \frac{-2aH_\beta(z)}{\sum_{i\in\Sigma} 2\log(r_i){r_i}^{-2z}} e^{-2a\ell z} - \frac{1}{4\pi} \cdot \frac{\operatorname{vol}_2(K)}{1-\sum_{i\in\Sigma} {r_i}^2} e^{2a\ell+\beta} \right) \\
&=: \sum_{\ell \geq 0} e^{2a\ell s} \cdot Y_\ell
\end{aligned}$$

is holomorphic in $\mathbb{C}_{-\delta/2}$. As a power series has a singularity on its radius of convergence, and the above series expansion is holomorphic in $\mathbb{C}_{-\delta/2}$ we deduce that $Y_\ell \in o(e^{2a\ell(\delta/2+\gamma)})$ for any $\gamma > 0$ as $\ell \to \infty$. Thus, for any $\gamma > 0$

$$N(2a\ell+\beta) = \frac{1}{4\pi} \cdot \frac{\operatorname{vol}_2(K)}{1-\sum_{i\in\Sigma} {r_i}^2} e^{2a\ell+\beta} - \sum_{z \in \mathcal{Z}_\mathrm{C} \cap \mathbb{C}_{-\delta/2}} \frac{-2aH_\beta(z)}{\sum_{i\in\Sigma} 2\log(r_i){r_i}^{-2z}} e^{-2a\ell z} + o(e^{a\ell(\delta+2\gamma)})$$

as $\ell \to \infty$. Notice that $\frac{\operatorname{vol}_2(K)}{1-\sum_{i\in\Sigma} {r_i}^2} = \operatorname{vol}_2(K) \sum_{\omega\in\Sigma^*} {r_\omega}^2 = \operatorname{vol}_2(\Omega)$ as expected. Writing $t = 2a\lfloor \frac{t}{2a} \rfloor + \beta(t)$ with $\beta(t) = 2a\{\frac{t}{2a}\}$, this implies an asymptotic expansion of $N(t)$ given by

$$N(t) = \frac{1}{4\pi}\operatorname{vol}_2(\Omega)e^t - \sum_{z \in \mathcal{Z}_\mathrm{C} \cap \mathbb{C}_{-\delta/2}} \widetilde{H_{\beta(t)}}(z) e^{-2a\lfloor \frac{t}{2a} \rfloor z} + o(e^{t(\delta/2+\gamma)}), \qquad (5.11)$$

with $\widetilde{H_{\beta(t)}}(z) := \frac{-2aH_{\beta(t)}(z)}{\sum_{i\in\Sigma} 2\log(r_i){r_i}^{-2z}}$. Since $t \mapsto H_{\beta(t)}(z)$ is bounded, the asymptotic expansion in (5.11) already contains all terms corresponding to a growth rate $e^{tz'}$ with $z' \in (\delta/2, 1)$.

This estimation remains correct in the case of $(k_1, k_2) \neq (0,0)$ after adapting $C_\pm$ from Cor. 5.9. By (iv) in Sec. 2.2, $N_D(\alpha\Omega, t) = N_D(\Omega, \alpha^2 t) \leq C_W \operatorname{vol}_2(\alpha\Omega)t + C_+\alpha^\delta t^{\delta/2}$. So we obtain an expression for an upper bound of the remainder term corresponding to $\Omega(k_1, k_2)$ if we replace $C_\pm$ with $C_\pm + 9^{-\delta}k_1C_\pm + (9\sqrt{3})^{-\delta}k_2C_\pm$. In Tab. 5.1, we provide approximates of the exponents and bounds on the coefficients corresponding to $z \in \mathcal{Z}_\mathrm{C} \cap (-1, -\delta/2)$ for three different allocations of $(k_1, k_2) \in \{0, \ldots, 6\}^2$ exhibiting different arrangements of the relevant poles. □

(k_1, k_2)	$z \in \mathcal{Z}_C$ with $\Re(z) < -\frac{\delta}{2} \mod \frac{2\pi i}{a}\mathbb{Z}$	Upper bound of $\left\lvert\widetilde{H_{\beta(t)}}(z)\right\rvert$
$(0,0)$	-0.952455	$1.68 \cdot 10^6$
$(0,6)$	-0.928326,	$1.81 \cdot 10^6$,
	$-0.71134 \pm 2.58082i$	$2.45 \cdot 10^5$
$(6,6)$	-0.888243,	$1.68 \cdot 10^6$,
	$-0.839089 \pm 1.34671i$,	$3.46 \cdot 10^5$,
	$-0.666227 \pm 2.8596i$	$2.92 \cdot 10^5$

Table 5.1: Upper bounds for coefficients of the asymptotic expansion at the relevant poles.

Remark 5.11. A critical problem occurs in the asymptotic expansion in the following sense: Suppose $N_D(\Omega, t) = (2\pi)^n V_n t^{n/2} + M(\log t)t^{\delta/2} + A(\log t)t^{\delta/2}$ with some bounded M and

$$A : t \mapsto \begin{cases} \frac{1}{\lfloor t/a \rfloor} & \text{if } \exists m \in \mathbb{N}_0 : \lfloor t/a \rfloor = 2^m \\ 0 & \text{else} \end{cases}$$

so that $A(\log t) \in o(1)$ and $g(t) = (2\pi)^n V_n e^{tn/2} + M(t)e^{t\delta/2} + A(t)e^{t\delta/2}$. Then this third term in $g(t)$ leads to the following term in the two-sided Fourier-Laplace transform $\widehat{g}_\beta(z)$:

$$\sum_{m \in \mathbb{N}_0} \frac{\left(e^{a(z+\delta/2)}\right)^{2^m}}{2^m}.$$

If this had a meromorphic extension beyond $\Re(z) < -\delta/2$, so would $\widehat{g}'_\beta(z)$. But $\widehat{g}'_\beta(z)$ contains a series of the form

$$\sum_{m \in \mathbb{N}_0} \left(e^{a(z+\delta/2)}\right)^{2^m},$$

which diverges whenever $e^{a(z+\delta/2)}$ is a root of unity of any power of 2. In other words, whenever $z = -\frac{\delta}{2} + \frac{2\pi i q}{a 2^p}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. However, this set lies dense in $-\frac{\delta}{2} + i\mathbb{R} \subset \mathbb{C}$. This shows that further information about the behaviour of the remainder-term of $N_D(\Omega, t)$ is necessary in order to ensure existence of a meromorphic extension of $\widehat{g}_\beta(z)$ that has a pole at $-\delta/2$.

5.4 Asymptotic behaviour of ϵ-parallel volumes

In this section, we will derive an asymptotic expansion of $\epsilon \mapsto \mathrm{vol}_2\left(\Omega(k_1,k_2)_{-\epsilon}\right)$ for the sets $\Omega(k_1,k_2)$ with $(k_1,k_2)\in\{0,\dots,6\}^2$ that we defined in Sec. 5.2.

Theorem 5.12. *Let $\Omega(k_1,k_2)$ be as in Sec. 5.2. Then for all $\beta\in[0,a)$ we have an asymptotic expansion of $\mathrm{vol}_2\left(\Omega(k_1,k_2)_{-e^{-(a\ell+\beta)}}\right)$ as $\ell\to\infty$ of the following form.*

$$\mathrm{vol}_2\left(\Omega(k_1,k_2)_{-e^{-(a\ell+\beta)}}\right) = Q_\beta(2)e^{-2a\ell} + Q_\beta\left(2-\frac{\log 4}{\log 3}\right)e^{-a\ell(2-\frac{\log 4}{\log 3})} + \sum_{z\in\mathcal{Z}_{\mathrm{P}}} e^{-a\ell z}Q_\beta(z) + o(e^{-a\ell\gamma}),$$

as $\ell\to\infty$ for any $\gamma>0$, with coefficients Q_β given in (5.16)-(5.18) *and evaluated in Tab. 5.2. Here, $\mathcal{Z}_P := \{z\in\mathbb{C}\ :\ \sum_{i\in\Sigma} r_i^{2-z}=1,\ \Im(z)\in\left[0,\frac{2\pi}{a}\right)\}$.*

(k_1,k_2)	Values of $Q_\beta(2)$	Values of $Q_\beta(2-\frac{\log 4}{\log 3})$
$(0,0)$	$-\frac{e^{-2\beta}}{22}\cdot\left[v\left(\frac{\beta}{2a}\right)+v\left(\frac{a+\beta}{2a}\right)\right]$	$-\frac{2e^{-\beta(2-\frac{\log 4}{\log 3})}}{5}\cdot\left[u\left(\frac{\beta}{2a}\right)+\widetilde{u}\left(\frac{a+\beta}{2a}\right)\right]$
$(0,6)$	$-\frac{e^{-2\beta}}{82}\cdot\left[v\left(\frac{\beta}{2a}\right)+v\left(\frac{a+\beta}{2a}\right)\right]$	$-\frac{10e^{-\beta(2-\frac{\log 4}{\log 3})}}{13}\cdot\left[u\left(\frac{\beta}{2a}\right)+\widetilde{u}\left(\frac{a+\beta}{2a}\right)\right]$
$(6,6)$	$-\frac{e^{-2\beta}}{142}\cdot\left[v\left(\frac{\beta}{2a}\right)+v\left(\frac{a+\beta}{2a}\right)\right]$	$-\frac{22e^{-\beta(2-\frac{\log 4}{\log 3})}}{19}\cdot\left[u\left(\frac{\beta}{2a}\right)+\widetilde{u}\left(\frac{a+\beta}{2a}\right)\right]$

Table 5.2: Exact values of the coefficients of the asymptotic expansion of the Lebesgue measure of an inner ϵ-parallel neighbourhood of $\Omega(k_1,k_2)$.

A key part of the proof of Thm. 5.12 relies on precise knowledge of the inner ϵ-parallel volume of the generator $G := O\setminus\overline{\Phi(k_1,k_2)O}$ for all $\epsilon>0$. As G is a disjoint union of Koch snowflakes of different sizes, a key step in the proof is to determine the inner ϵ-parallel volume of the Koch snowflake of base length 1 in the next lemma.

Lemma 5.13. *Let K denote the filled-in Koch snowflake with base-length* 1*. The map $\epsilon\mapsto\mathrm{vol}_2(K_{-\epsilon})$, defined on the positive reals, which maps ϵ to the area of the inner ϵ neighbourhood $K_{-\epsilon} := \{x\in K\ :\ \mathrm{dist}(x,\partial K)<\epsilon\}$, is continuous and can be evaluated as* *follows.*

$$\mathrm{vol}_2(K_{-\epsilon}) = \begin{cases} \frac{2\sqrt{3}}{5} & : \epsilon > \frac{1}{3} \\ \frac{7\sqrt{3}}{30} + \sqrt{\epsilon^2 - \frac{1}{36}} + 6\epsilon^2 \arcsin\left(\frac{1}{6\epsilon}\right) - \pi\epsilon^2 & : \frac{\sqrt{3}}{9} < \epsilon \leq \frac{1}{3} \\ \frac{8\sqrt{3}}{45} + \pi\epsilon^2 + 12 \cdot \mathrm{vol}_2(K_{-\epsilon} \cap \Gamma) & : \frac{1}{9} < \epsilon \leq \frac{\sqrt{3}}{9} \\ u \circ \alpha(\epsilon)\epsilon^{2-\log 4/\log 3} + v \circ \alpha(\epsilon)\epsilon^2 & : \epsilon \leq \frac{1}{9} \text{ and } \alpha(\epsilon) < \frac{1}{2} \\ \widetilde{u} \circ \alpha(\epsilon)\epsilon^{2-\log 4/\log 3} + v \circ \alpha(\epsilon)\epsilon^2 & : \epsilon \leq \frac{1}{9} \text{ and } \alpha(\epsilon) \geq \frac{1}{2}. \end{cases} \tag{5.12}$$

Here, $\{t\} := t - \lfloor t \rfloor$ *denotes the fractional part of a real number* t, $\alpha(\epsilon) := \left\{-\frac{\log \epsilon}{\log 3}\right\}$, Γ *is the equilateral filled-in triangle shown in Fig. 5.5(b) and defined in the proof of this lemma. Further,* $u, \widetilde{u}, v \colon [0, 1) \to \mathbb{R}$ *are given by*

$$\begin{aligned} u(t) &:= \left(\frac{9}{4}\right)^t \cdot \left[\frac{21\sqrt{3}}{40} + \frac{3}{4} \cdot \sqrt{3^{-2t} - \frac{1}{4}} + 81 \cdot \mathrm{vol}_2(K_{-3^{-t-2}} \cap \Gamma)\right] \\ &\qquad + \left(\frac{1}{4}\right)^t \cdot \left[\frac{3}{2} \cdot \arcsin\left(\frac{3^t}{2}\right) - \frac{\pi}{6}\right], \\ \widetilde{u}(t) &:= \left(\frac{9}{4}\right)^t \cdot \left[\frac{2\sqrt{3}}{5} + 27 \cdot \mathrm{vol}_2(K_{-3^{-t-1}} \cap \Gamma) + 81 \cdot \mathrm{vol}_2(K_{-3^{-t-2}} \cap \Gamma)\right] + \left(\frac{1}{4}\right)^t \cdot \frac{\pi}{3}, \\ v(t) &:= -\frac{\pi}{3} - 324 \cdot 9^t \cdot \mathrm{vol}_2(K_{-3^{-t-2}} \cap \Gamma). \end{aligned}$$

Remark 5.14. Note that for $\epsilon \leq 1/9$ the area of the inner ϵ neighbourhood $\mathrm{vol}_2(K_{-\epsilon})$ of the filled-in Koch snowflake K was determined in [67], where $u \circ \alpha(\epsilon)$, $\widetilde{u} \circ \alpha(\epsilon)$ and $v \circ \alpha(\epsilon)$ are expressed as infinite complex series. With Lem. 5.13 we provide a more geometric representation of $\mathrm{vol}_2(K_{-\epsilon})$ and an alternative and simpler proof of its scaling behaviour.

Proof of Lem. 5.13. Let F be the Koch curve that is generated by the four contractions $\psi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ for $i \in \{1, \ldots, 4\}$ given by

$$\psi_1(x) = \frac{1}{3}x, \ \psi_2(x) = \frac{1}{3}R_{\pi/3}(x) + \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

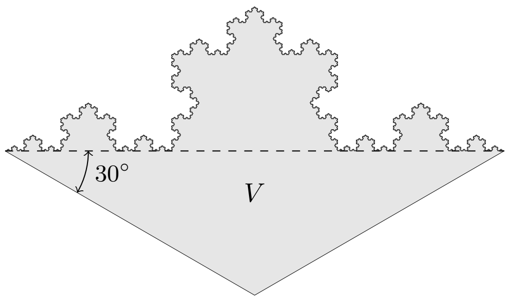


Figure 5.5(a) Visualisation of the set V.

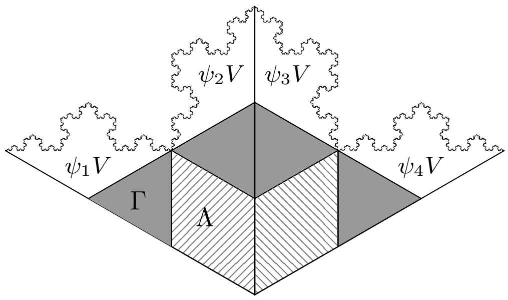


Figure 5.5(b) Decomposition of V into four congruent copies of Γ, two congruent copies of Λ and the sets $\psi_1(V), \dots, \psi_4(V)$.

$$\psi_3(x) = \frac{1}{3}R_{-\pi/3}(x) + \frac{1}{6}\begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}, \ \psi_4(x) = \frac{1}{3}x + \frac{1}{3}\begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

with R_α denoting the rotation matrix to the angle α about the origin. Further, let V denote the open region whose boundary is given by the union of F and the two line segments $\left\{t\binom{3}{-\sqrt{3}} : t \in \left[0, \frac{1}{6}\right]\right\}$ and $\left\{\binom{1}{0} - t\binom{3}{-\sqrt{3}} : t \in \left[0, \frac{1}{6}\right]\right\}$, see Fig. 5.5(a). Suppose without loss of generality that the position of K in $\mathbb{R}^2$ is so that $K = \overline{V \cup V^1 \cup V^2}$ with V^k denoting the image of V under the rotation around $\frac{1}{6}\binom{3}{-\sqrt{3}}$ by the angle $\frac{2\pi k}{3}$. Then

$$\operatorname{vol}_2(K_{-\epsilon}) = 3 \cdot \operatorname{vol}_2(K_{-\epsilon} \cap V). \tag{5.13}$$

Therefore, in the following, we focus on determining $\operatorname{vol}_2(K_{-\epsilon} \cap V)$. For this, let Γ denote the filled-in equilateral triangle with vertices $\frac{1}{18}\binom{3}{-\sqrt{3}}$, $\frac{1}{9}\binom{3}{-\sqrt{3}}$ and $\frac{1}{3}\binom{1}{0}$. Moreover, Λ will denote the filled-in rhombus with vertices $\frac{1}{3}\binom{1}{0}$, $\frac{1}{9}\binom{3}{-\sqrt{3}}$, $\frac{1}{6}\binom{3}{-\sqrt{3}}$ and $\frac{1}{18}\binom{9}{-\sqrt{3}}$. The sets Γ and Λ are depicted in Fig. 5.5(b). For large enough ϵ, the sets Λ, Γ and $\psi_i(V)$, $i \in \{1, \dots, 4\}$, are fully contained in $K_{-\epsilon}$. This changes when ϵ decreases (see Fig. 5.6(a)) and in the following we distinguish between different cases corresponding to ϵ, where this behaviour changes.

Case 1: If $\epsilon > \frac{1}{3}$ then

$$\operatorname{vol}_2(K_{-\epsilon} \cap V) = \operatorname{vol}_2(V) = \operatorname{vol}_2\left(V \cap \mathbb{R} \times \mathbb{R}^+\right) + \operatorname{vol}_2\left(V \cap \mathbb{R} \times \mathbb{R}^-\right) = \frac{\sqrt{3}}{20} + \frac{\sqrt{3}}{12} = \frac{2\sqrt{3}}{15}.$$

Case 2: If $\epsilon \in (\frac{\sqrt{3}}{9}, \frac{1}{3}]$ then

$$\begin{aligned}\mathrm{vol}_2\left(K_{-\epsilon} \cap V\right) &= \sum_{i=1}^{4} \mathrm{vol}_2\left(K_{-\epsilon} \cap \psi_i(V)\right) + 4\,\mathrm{vol}_2\left(K_{-\epsilon} \cap \Gamma\right) + 2\,\mathrm{vol}_2\left(K_{-\epsilon} \cap \Lambda\right) \\ &= 4\,\mathrm{vol}_2\left(\psi_1(V)\right) + 4\,\mathrm{vol}_2(\Gamma) + 2\,\mathrm{vol}_2\left(K_{-\epsilon} \cap \Lambda\right).\end{aligned}$$

With $\rho = \rho(\epsilon) = \frac{2\pi}{3} - 2\arccos\left(\frac{1}{6\epsilon}\right)$ being the angle that is shown in Fig. 5.6(b), we can evaluate $\mathrm{vol}_2\left(K_{-\epsilon} \cap \Lambda\right)$ as follows.

$$\begin{aligned}\mathrm{vol}_2\left(K_{-\epsilon} \cap \Lambda\right) &= \frac{\rho(\epsilon)}{2}\epsilon^2 + 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{9} \cdot \epsilon \cdot \sin\left(\frac{\pi}{6} - \frac{\rho(\epsilon)}{2}\right) \\ &= \frac{\pi}{3}\epsilon^2 - \epsilon^2 \arccos\left(\frac{1}{6\epsilon}\right) - \frac{\sqrt{3}}{108} + \frac{1}{6}\sqrt{\epsilon^2 - \frac{1}{36}}\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{vol}_2\left(K_{-\epsilon} \cap V\right) &= \frac{4}{9} \cdot \frac{2\sqrt{3}}{15} + 4 \cdot \frac{\sqrt{3}}{108} + \frac{2}{3}\pi\epsilon^2 - 2\epsilon^2 \arccos\left(\frac{1}{6\epsilon}\right) - \frac{\sqrt{3}}{54} + \frac{1}{3}\sqrt{\epsilon^2 - \frac{1}{36}} \\ &= \frac{7\sqrt{3}}{90} + \frac{2}{3}\pi\epsilon^2 - 2\epsilon^2 \arccos\left(\frac{1}{6\epsilon}\right) + \frac{1}{3}\sqrt{\epsilon^2 - \frac{1}{36}}.\end{aligned}$$

Using the identity $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$ the assertion follows.

Case 3: If $\epsilon \in (\frac{1}{9}, \frac{\sqrt{3}}{9}]$ then

$$\begin{aligned}\mathrm{vol}_2\left(K_{-\epsilon} \cap V\right) &= \sum_{i=1}^{4} \mathrm{vol}_2\left(K_{-\epsilon} \cap \psi_i(V)\right) + 4 \cdot \mathrm{vol}_2\left(K_{-\epsilon} \cap \Gamma\right) + 2 \cdot \mathrm{vol}_2\left(K_{-\epsilon} \cap \Lambda\right) \\ &= \frac{4}{9} \cdot \frac{2\sqrt{3}}{15} + 4\,\mathrm{vol}_2(K_{-\epsilon} \cap \Gamma) + \frac{\pi}{3}\epsilon^2.\end{aligned}$$

Case 4: Suppose that $\epsilon \leq \frac{1}{9}$. Let $W := V \setminus \bigcup_{i=1}^{4} \psi(V)$. Then

$$V = \bigcup_{k=0}^{\infty} \bigcup_{\omega \in \{1,\ldots,4\}^k} \psi_\omega(W) \cup \bigcap_{k=0}^{\infty} \bigcup_{\omega \in \{1,\ldots,4\}^k} \psi_\omega(V),$$

where all unions are disjoint. As $\bigcap_{k=0}^{\infty} \bigcup_{\omega \in \{1,\ldots,4\}^k} \psi_\omega(V) \subset \overline{\bigcap_{k=0}^{\infty} \bigcup_{\omega \in \{1,\ldots,4\}^k} \psi_\omega(V)} \subset F$, we know that $\mathrm{vol}_2(\bigcap_{k=0}^{\infty} \bigcup_{\omega\in\{1,\dots,4\}^k} \psi_\omega(V)) = 0$. Therefore,

$$\begin{aligned}\mathrm{vol}_2\left(K_{-\epsilon} \cap V\right) &= \sum_{k=0}^{\infty} \sum_{\omega\in\{1,\dots,4\}^k} \mathrm{vol}_2\left(K_{-\epsilon} \cap \psi_\omega(W)\right) \\ &= \sum_{k=0}^{\infty} \sum_{\omega\in\{1,\dots,4\}^k} \mathrm{vol}_2\left((\psi_\omega K)_{-\epsilon} \cap \psi_\omega(W)\right) \\ &= \sum_{k=0}^{\infty} \left(\frac{4}{9}\right)^k \mathrm{vol}_2\left(K_{-3^k\epsilon} \cap W\right).\end{aligned} \tag{5.14}$$

Now, using the decomposition of W into four congruent copies of Γ and two congruent copies of Λ, see Fig. 5.5(b), we obtain

$$\begin{aligned}&\mathrm{vol}_2\left(K_{-3^k\epsilon} \cap W\right) \\ &= 4\cdot \mathrm{vol}_2\left(K_{-3^k\epsilon} \cap \Gamma\right) + 2\cdot \mathrm{vol}_2\left(K_{-3^k\epsilon} \cap \Lambda\right) \\ &= \begin{cases} 4\cdot \frac{\sqrt{3}}{108} + 2\cdot \frac{\sqrt{3}}{54} & : k \geq \lfloor -\frac{\log\epsilon}{\log 3}\rfloor \\ 4\cdot \frac{\sqrt{3}}{108} + \frac{2\pi}{3}3^{2k}\epsilon^2 - 2\cdot 3^{2k}\epsilon^2 \arccos\left(\frac{1}{6\cdot 3^k\epsilon}\right) & \\ \qquad - \frac{\sqrt{3}}{54} + \frac{1}{3}\sqrt{3^{2k}\epsilon^2 - \frac{1}{36}} & : \lfloor -\frac{1}{2} - \frac{\log\epsilon}{\log 3}\rfloor \leq k < \lfloor -\frac{\log\epsilon}{\log 3}\rfloor \\ 4\cdot \mathrm{vol}_2\left(K_{-3^k\epsilon} \cap \Gamma\right) + \frac{\pi}{3}3^{2k}\epsilon^2 & : k < \lfloor -\frac{1}{2} - \frac{\log\epsilon}{\log 3}\rfloor. \end{cases}\end{aligned} \tag{5.15}$$

Combining (5.14) with (5.15) we can evaluate $\mathrm{vol}_2(K_{-\epsilon} \cap V)$, leading to (5.12). For this, note the following.

(i). With $\alpha(\epsilon) := \left\{-\frac{\log\epsilon}{\log 3}\right\}$ as defined in the statement of this Lemma, it is convenient to write $\lfloor -\frac{\log\epsilon}{\log 3}\rfloor = -\frac{\log\epsilon}{\log 3} - \alpha(\epsilon)$.

(ii). If $\alpha(\epsilon) < \frac{1}{2}$, then $\lfloor -\frac{1}{2} - \frac{\log\epsilon}{\log 3}\rfloor = \lfloor -\frac{\log\epsilon}{\log 3}\rfloor - 1$. If $\alpha(\epsilon) \geq \frac{1}{2}$, then $\lfloor -\frac{1}{2} - \frac{\log\epsilon}{\log 3}\rfloor = \lfloor -\frac{\log\epsilon}{\log 3}\rfloor$. Thus, the middle case of (5.15) occurs if and only if $\alpha(\epsilon) < \frac{1}{2}$.

(iii). Due to the self-similarity of the Koch curve, $\mathrm{vol}_2(K_{-\epsilon} \cap \Gamma) = 9\,\mathrm{vol}_2(K_{-\epsilon/3} \cap \Gamma)$ whenever $\epsilon \leq \frac{1}{9}$.

□

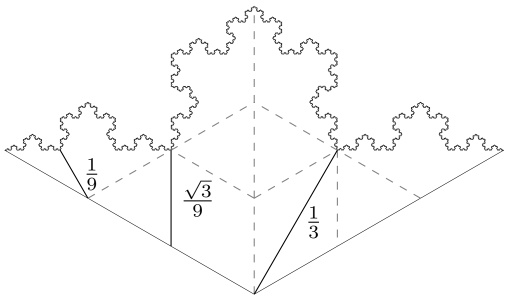


Figure 5.6(a) Visualisation of the lengths which lead to the different cases in the proof of Lem. 5.13: $\mathrm{vol}_2(K_{-\epsilon} \cap \Lambda) = \mathrm{vol}_2(\Lambda)$ if and only if $\epsilon \geq \frac{1}{3}$. $\mathrm{vol}_2(K_{-\epsilon} \cap \Gamma) = \mathrm{vol}_2(\Gamma)$ if and only if $\epsilon \geq \frac{\sqrt{3}}{9}$. $\mathrm{vol}_2(K_{-\epsilon} \cap \psi_1 V) = \mathrm{vol}_2(\psi_1 V)$ if and only if $\epsilon \geq \frac{1}{9}$.

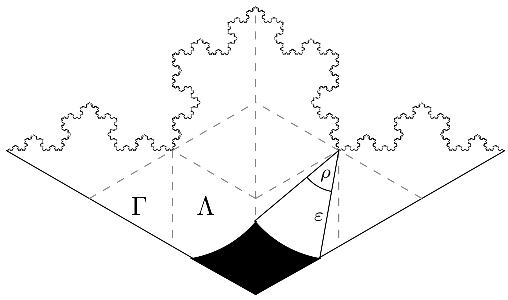


Figure 5.6(b) Example of an inner ϵ neighbourhood of K for $\epsilon \in (\frac{\sqrt{3}}{9}, \frac{1}{3}]$ as in Case 2 in the proof of Lem. 5.13. Here, $\psi_1 V$ and Γ are fully contained in $K_{-\epsilon}$, whereas Λ is not.

Proof of Thm. 5.12. In this proof we fix $(k_1, k_2) \in \{0, \ldots, 6\}^2$ and abbreviate the fractal spray $\Omega(k_1, k_2)$, the alphabet $\Sigma(k_1, k_2)$ and the IFS $\Phi(k_1, k_2)$ as denoted in Sec. 5.2 by Ω, Σ and Φ respectively. For $\beta \in [0, a)$, we define

$$N(a\ell + \beta) := \mathrm{vol}_2\left(\Omega_{-e^{-(a\ell+\beta)}}\right).$$

Recall from (5.10) that Ω is a disjoint union of open sets $\Omega = \bigcup_{k=0}^{\infty} \bigcup_{\omega \in \Sigma^k} \phi_\omega(G)$, where the generator $G := O \setminus \overline{\Phi O}$ can have several connected components, depending on (k_1, k_2), and where $\Sigma^0 := \{\emptyset\}$ and $\phi_\emptyset$ is the identity on $\mathbb{R}^2$. With the identity $\Omega_{-\epsilon} := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \epsilon\} = (\partial\Omega)_\epsilon \cap \Omega$, where $F_\epsilon := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, F) < \epsilon\}$ denotes the *ϵ-parallel set* of a set $F \subset \mathbb{R}^2$, we have that

$$\begin{aligned}
N(a\ell + \beta) &= \mathrm{vol}_2\Big((\partial\Omega)_{e^{-(a\ell+\beta)}} \cap \bigcup_{k=0}^{\infty} \bigcup_{\omega \in \Sigma^k} \phi_\omega G\Big) = \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} \mathrm{vol}_2\left((\partial\Omega)_{e^{-(a\ell+\beta)}} \cap \phi_\omega G\right) \\
&= \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} \mathrm{vol}_2\left((\phi_\omega(\partial\Omega))_{e^{-(a\ell+\beta)}} \cap \phi_\omega G\right) \\
&= \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} \mathrm{vol}_2\left(\phi_\omega\left((\partial\Omega)_{e^{-(a\ell+\beta-a\nu_\omega)}} \cap G\right)\right) \\
&= \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} e^{-2a\nu_\omega}\, \mathrm{vol}_2\left(G_{-e^{-(a(\ell-\nu_\omega)+\beta)}}\right).
\end{aligned}$$

In the last two equations (5.9) was used. Next, we consider the two-sided Fourier-Laplace transform $\widehat{N}_\beta$ dependent on $\beta \in [0, a)$, acting on $\mathbb{C}$ and given by

$$\widehat{N}_\beta(z) = \sum_{\ell=-\infty}^{\infty} e^{a\ell z} N(a\ell + \beta).$$

For $z \in \{z \in \mathbb{C} \mid 0 < \Re(z) < 2 - \dim_M(\partial\Omega)\}$, where $\dim_M(\partial\Omega)$ denotes the Minkowski dimension of $\partial\Omega$, the Fourier-Laplace transform $\widehat{N}_\beta(z)$ converges and the order of summation can be swapped, leading to the following conversion.

$$\begin{aligned}\widehat{N}_\beta(z) &= \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} e^{-2a\nu_\omega} \sum_{\ell=-\infty}^{\infty} e^{a\ell z} \operatorname{vol}_2\left(G_{-e^{-(a(\ell-\nu_\omega)+\beta)}}\right) \\ &= \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} e^{-a\nu_\omega(2-z)} \sum_{\ell=-\infty}^{\infty} e^{a\ell z} \operatorname{vol}_2\left(G_{-e^{-(a\ell+\beta)}}\right)\end{aligned}$$

In the last equality we have used an index shift, as $\nu_\omega \in \mathbb{Z}$ by (5.9). Depending on (k_1, k_2), the generator G may have several connected components $K^{(0)}, \ldots, K^{(k_1+k_2)}$, all of which are Koch snowflakes. Thus,

$$\operatorname{vol}_2\left(G_{-e^{-(a\ell+\beta)}}\right) = \sum_{j=0}^{k_1+k_2} \operatorname{vol}_2\left(K^{(j)}_{-e^{-(a\ell+\beta)}}\right) = \sum_{j=-k_1}^{k_2} b_j^2 \operatorname{vol}_2\left(K_{-e^{-(a\ell+\beta+\log b_j)}}\right),$$

with b_j denoting the base length of the Koch snowflake $K^{(j)}$. In our setting, we have $b_0 = 1 = e^{-0\cdot a}$, $b_j = \sqrt{3}/3 = e^{-1\cdot a}$ for $j < 0$ and $b_j = 1/3 = e^{-2\cdot a}$ for $j > 0$, implying

$$\widehat{N}_\beta(z) = \left(1 + k_1 e^{a(z-2)} + k_2 e^{2a(z-2)}\right) \sum_{k=0}^{\infty} \sum_{\omega \in \Sigma^k} e^{-a\nu_\omega(2-z)} \sum_{\ell=-\infty}^{\infty} \underbrace{e^{a\ell z} \operatorname{vol}_2\left(K_{-e^{-(a\ell+\beta)}}\right)}_{=:h(\ell)}.$$

We can evaluate the series with indices k and ℓ independently. For the series with index k we use that $e^{-a\nu_\omega} = r_\omega$ and that $\sum_{\omega\in\Sigma^k} r_\omega^{2-z} = \left(\sum_{i\in\Sigma} r_i^{2-z}\right)^k$. For the series with index ℓ, we use (5.12), and split the series in the following way. $\sum_{\ell=-\infty}^{\infty} h(\ell) = \sum_{\ell=-\infty}^{1} h(\ell) + h(2) + h(3) + \sum_{\ell=4}^{\infty} h(\ell)\mathbb{1}_{2\mathbb{Z}}(\ell) + \sum_{\ell=4}^{\infty} h(\ell)\mathbb{1}_{2\mathbb{Z}+1}(\ell)$.

$$\widehat{N}_\beta(z) \cdot \left(1 + k_1 e^{a(z-2)} + k_2 e^{2a(z-2)}\right)^{-1}$$

$$
\begin{aligned}
&= \frac{1}{1-\sum_{i\in\Sigma} r_i^{2-z}} \Bigg[\frac{e^{az}}{1-e^{-az}} \cdot \frac{2\sqrt{3}}{5} + \frac{e^{2az}}{3} \\
&\qquad \cdot \left(\frac{7\sqrt{3}}{10} + \sqrt{e^{-2\beta} - \frac{1}{4}} + 2e^{-2\beta} \arcsin\left(\frac{e^{\beta}}{2}\right) - \frac{\pi e^{-2\beta}}{3} \right) \\
&\qquad + e^{3az} \cdot \left(\frac{8\sqrt{3}}{45} + \frac{\pi e^{-2\beta}}{27} + 12\,\mathrm{vol}_2(K_{-e^{-\beta}\sqrt{3}^{-3}} \cap \Gamma) \right) \\
&\qquad + u\left(\frac{\beta}{2a}\right) e^{-\beta(2-\frac{\log 4}{\log 3})} \cdot \frac{e^{4a(z-2+\frac{\log 4}{\log 3})}}{1-e^{2a(z-2+\frac{\log 4}{\log 3})}} + v\left(\frac{\beta}{2a}\right) e^{-2\beta} \frac{e^{4a(z-2)}}{1-e^{2a(z-2)}} \\
&\qquad + \widetilde{u}\left(\frac{a+\beta}{2a}\right) e^{-\beta(2-\frac{\log 4}{\log 3})} \cdot \frac{e^{5a(z-2+\frac{\log 4}{\log 3})}}{1-e^{2a(z-2+\frac{\log 4}{\log 3})}} + v\left(\frac{a+\beta}{2a}\right) e^{-2\beta} \frac{e^{5a(z-2)}}{1-e^{2a(z-2)}} \Bigg] \\
&=: \frac{1}{1-\sum_{i\in\Sigma} r_i^{2-z}} \cdot L(z)
\end{aligned}
$$

The right-hand side has a meromorphic extension to $\mathbb{C}$ with simple poles at z in

$$
\mathcal{Z} := \left\{ z \in \mathbb{C} \,\middle|\, \sum_{i\in\Sigma} r_i^{2-z} = 1 \right\} \quad \cup \quad \mathcal{S} := \left\{ 0, 2-\log_3 4, 2 \right\}.
$$

Define

$$
Q_\beta(2) := \frac{e^{-2\beta}}{2(1-\#\Sigma)} \cdot \left[v\left(\frac{\beta}{2a}\right) + v\left(\frac{a+\beta}{2a}\right) \right], \tag{5.16}
$$

$$
Q_\beta\left(2-\frac{\log 4}{\log 3}\right) := \frac{(1+\frac{k_1}{2}+\frac{k_2}{4})e^{-\beta(2-\frac{\log 4}{\log 3})}}{2\big(1-\sum_{i\in\Sigma} r_i^{\log 4/\log 3}\big)} \cdot \left[u\left(\frac{\beta}{2a}\right) + \widetilde{u}\left(\frac{a+\beta}{2a}\right) \right] \quad \text{and} \tag{5.17}
$$

$$
Q_\beta(z) := -\frac{a(1+k_1 e^{a(z-2)} + k_2 e^{2a(z-2)})}{\sum_{i\in\Sigma} \log r_i \cdot r_i^{2-z}} \cdot L(z) \tag{5.18}
$$

for $z \in \mathcal{Z}$. Then

$$
H_\beta(s) := \widehat{N}_\beta(s) - \sum_{\ell=-\infty}^{-1} e^{a\ell s} N(a\ell+\beta) - \frac{Q_\beta(2)}{1-e^{a(s-2)}} - \frac{Q_\beta(2-\frac{\log 4}{\log 3})}{1-e^{a(s-2+\frac{\log 4}{\log 3})}} - \sum_{z\in\mathcal{Z}_{\mathrm{P}}} \frac{Q_\beta(z)}{1-e^{a(s-z)}}
$$

extends to a holomorphic function on $\mathbb{C}$, where $\mathcal{Z}_{\mathrm{P}} := \{z \in \mathcal{Z} : \Im(z) \in [0, \frac{2\pi}{a})\}$. On $\{z \in \mathbb{C} \mid 0 < \Re(z) < \min_{s\in\mathcal{Z}} \Re(s)\}$ each summand of H_β can be developed into a power series:

$$H_\beta(s) = \sum_{\ell=0}^{\infty} e^{a\ell s} \left[N(a\ell + \beta) - Q_\beta(2) e^{-2a\ell} - Q_\beta\left(2 - \frac{\log 4}{\log 3}\right) e^{-a\ell(2-\frac{\log 4}{\log 3})} \right.$$
$$\left. - \sum_{z \in \mathcal{Z}_{\mathrm{P}}} e^{-a\ell z} Q_\beta(z) \right]$$

As a power series has a singularity on its radius of convergence, H_β being holomorphic on $\mathbb{C}$ implies that as $\ell \to \infty$

$$N(a\ell + \beta) = Q_\beta(2)e^{-2a\ell} + Q_\beta\left(2 - \frac{\log 4}{\log 3}\right) e^{-a\ell(2-\frac{\log 4}{\log 3})} + \sum_{z \in \mathcal{Z}_{\mathrm{P}}} e^{-a\ell z} Q_\beta(z) + o(e^{-a\ell\gamma})$$

for any $\gamma > 0$. For a selection of choices of (k_1, k_2), the coefficients $Q_\beta(z)$ at $z \in \mathcal{S}$ take the values as shown in Tab. 5.2. $\square$

Chapter 6

Further developments and open questions

In this final chapter we collect several further directions to be explored and we discuss possible generalisations in particular of Lem. 3.8 and Thm. 4.15.

6.1 Generalisation of Thm. 4.15 to general quasidisks

Thm. 4.15 is restricted in two ways. First it only applies to sufficiently small p, i.e. $p < \frac{\sqrt{3}-1}{2}$ and secondly it requires an bi-Lipschitz map with E-property as in Def. 4.8 in order to translate to bi-Lipschitz equivalent maps.

6.1.1 Understand the E-property of a bi-Lipschitz map

Recall that by Rohde's theorem, any quasidisk has a bi-Lipschitz map onto a p-Rohde snowflake. It appears conceivable that any such bi-Lipschitz map has the E-property. Suppose a bi-Lipschitz map f is (piecewise) smooth (by Rademacher's theorem it is diffeomorphic almost everywhere). Then f maps (piecewise) smooth surfaces to piecewise smooth surfaces. For such a piecewise smooth bi-Lipschitz map f the image of a ϵ-sized cuboid E_ϵ with foliation $\{\gamma\}_{\gamma\in\Gamma}$ and normal ν, $\langle df(\nu), df(\gamma')\rangle$ is continuous.

6.1.2 Extension of Thm. 4.7 to $p \in \left[\frac{\sqrt{3}-1}{2}, \frac{1}{2}\right)$

Referring to the proof of Thm. 4.7, the proposed foliation does not lead to convergent integrals if $p \geq \frac{\sqrt{3}-1}{2}$, see also Prop. 3.17. One may consider alternative seed foliations such as described in Fig. 6.1. As indicated in the visualisation, it is constructed in the following way: One draws an isosceles triangle below each p-Rohde curve where each leg touches I_0 at angle α_0 (coloured in green). For the next iteration this construction is repeated but with new angles $\alpha_{01}, \alpha_{11} < \alpha_0$ (coloured in red and blue, respectively). This produces a domain that can easily be triangulated by dividing each of the two legs of the big isosceles triangles (with angle α_0) into four equidistant segments of which each is the basis of a foliation heading entirely to one of the four images of the next iteration of the p-Rohde curve. Within each triangle one then constructs a simple seed foliation parallel to one side. At higher iterations this is repeated with a strictly decreasing sequence of angles. It is expected that such a triangulated seed foliation allows adjusting angles so that $\beta_{\inf} > 0$ and $\mathcal{I}_\beta < \infty$. The key advantage of such a construction is that this foliation covers each cylinder of the p-Rohde curve with the same density. In comparison, the equidistant seed foliation discussed in Prop. 3.14 has a higher density in the central section of each iteration leading to a faster divergence of $\mathcal{I}_\beta$. Notice that $\beta_{\inf} \to \infty$ as $\alpha_0 \to 0$ so that this construction cannot be understood as a small perturbation of the original construction in Prop. 3.14.

6.2 Further generalisations

6.2.1 $W^{1,p}$-Poincaré-Wirtinger inequalities and p-Laplacians

In addition to elliptic operators, the so-called p-Laplacian attracted a considerable interest in recent years, see for example Lê's work in [71] and the references therein. Let $1 < p < \infty$) and define the *p-Laplacian* on a domain Ω by its energy functional $\mathcal{E}_p(u) := p^{-1}\|\nabla u\|^p_{L^p(\Omega)}$ on $W^{1,p}(\Omega)$. On smooth functions this operator then acts as $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ and its eigenvalue equation reads $-\Delta_p u = \lambda |u|^{p-2} u$. Notice that Δ_2 is the usual Laplacian.

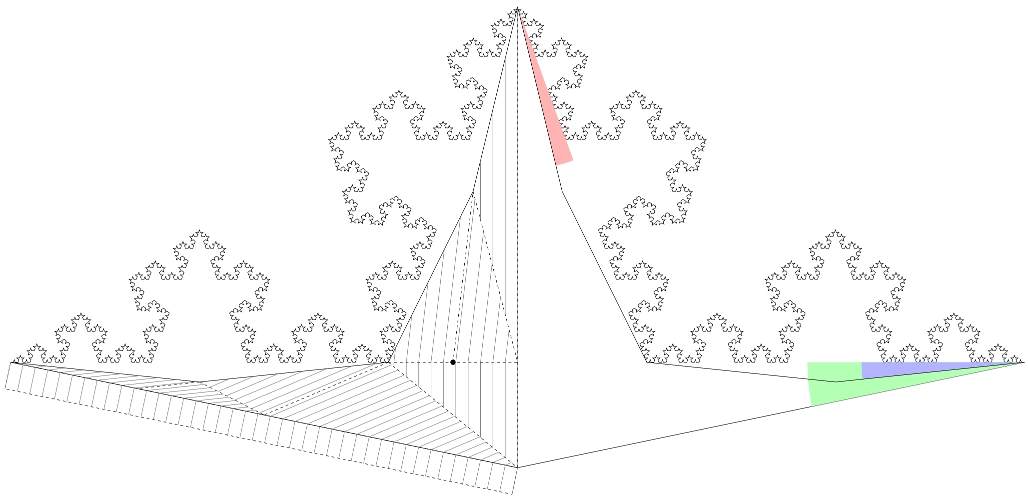


Figure 6.1: Example of a piece of boundary of a fully homogeneous p-Rohde snowflake with an alternative seed foliation for $p = \frac{1}{2+2\cos(70°)} > \frac{\sqrt{3}-1}{2}$. The coloured angles are explained in the construction in Sec. 6.1.2. For the sake of a better overview the angles are only shown in the right half while the triangulated seed foliation is only shown in the left half; the construction is symmetric.

While the spectral analysis of Δ_p for $p \neq 2$ is more involved if $p \neq 2$, a modification of the result in Lem. 3.8 still holds as a variant Poincaré-Wirtinger inequality. For any $u \in L^p(\Omega)$ with $1 < p < \infty$ the map $\mathbb{R} \ni t \mapsto \|u - t\|_{L^p(\Omega)}$ has a unique minimiser denoted by u_Ω. The uniqueness follows from the strict convexity of L^p-norms if $1 < p < \infty$: Suppose $t \neq t'$ are two minimisers. Then

$$\begin{aligned}\left\| u - \frac{t+t'}{2} \right\|_{L^p(\Omega)} &= \left\| \frac{u-t}{2} + \frac{u-t'}{2} \right\|_{L^p(\Omega)} \\ &< \frac{1}{2}\|u-t\|_{L^p(\Omega)} + \frac{1}{2}\|u-t'\|_{L^p(\Omega)} = \|u-t\|_{L^p(\Omega)}\end{aligned}$$

contradicting the minimising property of t, t'. For this minimiser we have the following inequality in $W^{1,p}(D)$ where D is well-foliated.

Proposition 6.1. *Let $D \subset \mathbb{R}^n$ be well-foliated with notation as in Def. 3.3. Suppose there is a C_E for $E \subset D$ such that $\|u - u_E\|_{L^p(E)} \le C_E \|\nabla u\|_{L^p(E)}$ for all $u \in W^{1,p}(E)$.*[1] *Then*

[1]Let $\iota : E \hookrightarrow D$ be the usual inclusion. Then $\iota^* W^{1,p}(D) \subset W^{1,p}(E)$ and it actually is sufficient if there exists a C_E for all $u \in \iota^* W^{1,p}(D) = \{u \in W^{1,p}(E) : \exists U \in W^{1,p}(D) : U|_E = u\}$.

for any $u \in W^{1,p}(D)$

$$\|u - u_D\|^p_{L^p(D)} \leq C_D \|\nabla u\|^p_{L^p(D)},$$

with $C_D = C_E + C_E \frac{2^{p-1}}{r} \frac{\mathcal{I}_\beta(D \backslash E)}{\beta_{\inf}(E)} + 2^{p-1} L \frac{\mathcal{I}_\beta(D \backslash E)}{\beta_{\inf}(D)}$.

As is seen immediately from the proof, we actually proves the slightly stronger inequality with u_D replaced with u_E. A classical Poincaré-Wirtinger inequality would be implied if u_E had the additional property that

$$\left\| u - \frac{1}{\operatorname{vol}_n D} \int_D u dx \right\|_{L^p(D)} \leq \|u - u_E\|_{L^p(D)} \ \forall u \in W^{1,p}(D).$$

Proof. Essentially this is analogous to the proof of Lem. 3.8 after replacing 2 with p wherever appropriate. Let $u \in W^{1,p}(D)$. Then $\|u - u_D\|^p_{L^p(D)} \leq \|u - u_E\|^p_{L^p(D)}$. We subdivide the expression

$$\|u - u_E\|^p_{L^p(D)} = \underbrace{\int_E |u(y) - u_E|^p dy}_{:=I_1} + \underbrace{\int_{D \backslash E} |u(x) - u_E|^p dx}_{:=I_2},$$

and estimate I_1 and I_2 separately. Then by definition of u_E one has

$$I_1 \leq C_E \|\nabla u\|^p_{L^p(E)} \leq C_E \|\nabla u\|^p_{L^p(D)}. \tag{6.1}$$

From convexity of $x \mapsto |x|^p$ it follows that for any $a, b, c \in \mathbb{R}$, there is the following inequality: $|a-b|^p = 2^p |(a-c)/2 - (b-c)/2|^p \leq 2^{p-1} \left(|a-c|^p + |b-c|^p\right)$. Then analogously to the proof of Lem. 3.8,

$$\begin{aligned} I_2 &= \int_{D \backslash E} |u(x) - u_E|^p dx = \int_{I_0} \int_r^{\operatorname{len} \gamma_q} |u(\gamma_q(t)) - u_E|^p \beta(q,t) dt dq, \\ &= \frac{1}{r} \int_{I_0} \int_r^{\operatorname{len} \gamma_q} \int_0^r |u(\gamma_q(t)) - u_E|^p \beta(q,t) dt' dt dq \end{aligned}$$

$$\leq \underbrace{\frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r |u(\gamma_q(t')) - u_E|^p \beta(q,t) dt' dt dq}_{=:I_2'}$$
$$+ \underbrace{\frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r |u(\gamma_q(t)) - u(\gamma_q(t'))|^p \beta(q,t) dt' dt dq,}_{=:I_2''}$$

And by assumption $\int_r^{\operatorname{len}\gamma_q}\beta(q,t)dt \leq \mathcal{I}_\beta(D\backslash E)$ for all $q \in I_0$. For I_2' one has

$$\begin{aligned}
I_2' &= \frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r |u(\gamma_q(t')) - u_E|^p \beta(q,t) dt' dt dq \\
&\leq \frac{2^{p-1}}{r}\mathcal{I}_\beta(D\backslash E)\int_{I_0}\int_0^r |u(\gamma(t')) - u_E|^p dt' d\gamma \\
&\leq \frac{2^{p-1}}{r}\frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(E)}\int_{I_0}\int_0^r |u(\gamma_q(t')) - u_E|^p \beta(q,t') dt' dq \\
&= \frac{2^{p-1}}{r}\frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(E)}\|u - u_E\|^p_{L^p(E)} \leq C_E \frac{2^{p-1}}{r}\frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(E)}\|\nabla u\|^p_{L^p(D)},
\end{aligned}$$

Using Jensen's inequality and $|\nabla\gamma_q(s)|^p = 1$, $|s| \leq 1$ and $|\partial_s u| = |\langle s, \nabla u\rangle| \leq |\nabla u|$,

$$\begin{aligned}
|u(x) - u(y)|^p = \left|\int_{t_0}^t \partial_s u(\gamma_q(s)) ds\right|^p &\leq (t - t_0)\int_{t_0}^t |\partial_s u(\gamma_q(s))|^p ds \\
&\leq (t - t_0)\int_{t_0}^t |\nabla u|^p ds \leq (t - t_0)\int_0^{\operatorname{len}\gamma_q} \left|\nabla u|_{\gamma_q(s)}\right|^p ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
I_2'' &= \frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r |u(\gamma_q(t)) - u(\gamma_q(t'))|^p \beta(q,t) dt' dt dq \\
&= \frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r \left|\int_{t'}^t \partial_s u(\gamma_q(s))|_{s=\sigma} d\sigma\right|^p \beta(q,t) dt' dt dq \\
&\leq \frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r (t - t')\int_{t'}^t |\partial_s u(\gamma_q(s))|_{s=\sigma}|^p d\sigma \beta(q,t) dt' dt dq
\end{aligned}$$

Then since $t - t' \leq L$,

$$I_2'' \leq \frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r (t - t')\int_{t'}^t \left|\nabla u(x)|_{x=\gamma_q(\sigma)}\right|^p d\sigma \beta(q,t) dt' dt dq$$

$$
\begin{aligned}
&\leq \frac{2^{p-1}}{r}\int_{I_0}\int_r^{\operatorname{len}\gamma_q}\int_0^r L\int_0^{\operatorname{len}\gamma_q}\left|\nabla u(x)|_{x=\gamma_q(\sigma)}\right|^p d\sigma\beta(q,t)dt'dtdq\\
&\leq 2^{p-1}L\mathcal{I}_\beta(D\backslash E)\int_{I_0}\int_0^{\operatorname{len}\gamma_q}\left|\nabla u(x)|_{x=\gamma_q(\sigma)}\right|^p d\sigma dq\\
&\leq 2^{p-1}L\frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(D)}\int_{I_0}\int_0^{\operatorname{len}\gamma_q}\left|\nabla u(x)|_{x=\gamma_q(\sigma)}\right|^p \beta(q,\sigma)d\sigma dq\\
&= 2^{p-1}L\frac{\mathcal{I}_\beta(D\backslash E)}{\beta_{\inf}(D)}\|\nabla u\|^p_{L^p(D)}. \qquad \square
\end{aligned}
$$

6.2.2 Multifractal structure of boundaries

Suppose the boundary $\partial\Omega$ of a domain Ω exhibits a multifractal structure, that is, the Minkowski dimension of an open neighbourhood of $x \in \partial\Omega$ depends on x. Extending Prop. 2.43 to such contexts together with (C) is expected to provide estimates of the counting function depending on the multifractal spectrum. Indeed one finds the following version of Thm. 4.15.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\delta, \mathfrak{M}, C, A : (i_0, i_1) \to [n-1, n]$, be integrable multifractal dimensional data (that is, local versions of dimensions and contents replacing upper inner Minkowski dimension and the constants in Prop. 2.42 and Prop. 2.43 with the following properties*

(i). Let $\mathcal{W} := \bigcup_{k\geq k_0}\mathcal{W}_k$ be a Whitney cover of Ω as constructed in Sec. 2.5. Then the k^{th} slice $\mathcal{W}_k$, i.e. the set of Whitney cubes in the Whitney cover of Ω of size 2^{-k}, has a cardinality bounded from above as

$$
\#\mathcal{W}_k \leq \int_{i_0}^{i_1}\mathfrak{M}(i)2^{k\delta(i)}di
$$

(ii). Ω is well-covered by $\{D_i^\epsilon\}_{i\in I_\epsilon}$ with

$$
\#I_\epsilon \leq \int_{i_0}^{i_1} C(i)\epsilon^{-\delta(i)}di
$$

(iii). The $(n-1)$-dimensional volume of the boundary of the inner Whitney cubes for some

ϵ in the notation of Prop. 2.43 is bounded from above by

$$\mathrm{vol}_{n-1}\,\partial\left(\overline{\bigcup_{Q\in\mathcal{W}_\epsilon} Q}\right) \leq \int_{i_0}^{i_1} A(i)\epsilon^{n-1+\delta(i)}di.$$

Then $N_N(\Omega,t) \leq C_W^{(n)}\,\mathrm{vol}_n\,\Omega t^{n/2} + \int_{i_0}^{i_1} M(i)t^{\delta(i)/2}di$ *with*

$$M(i) = \left(C(i) + \mathfrak{M}(i)\frac{(40\sqrt{n})^{\delta(i)}}{2^{\delta(i)}-1}\right)\left(\frac{\mu+1}{C_2(\Omega)}\right)^{\delta(i)/2} + \frac{C_W^{(n-1)}}{4}A(i)\left(\frac{\mu+1}{C_2(\Omega)}\right)^{\frac{\delta(i)-(n-1)}{2}}.$$

Proof. Analogously to the proof of Thm. 4.15 one has $N_N(\Omega,t) \leq S_1^\epsilon(t) + S_2^\epsilon(t)$ with identical construction. It follows that for $k_\pm$ with $k_+ - k_- \leq 2 + \log_2(5r_+)$,

$$\begin{aligned}
S_1^\epsilon(t) &\leq \#I_\epsilon + \sum_{k_-\leq k\leq k_+} \#\mathcal{W}_k \\
&\leq \int_{i_0}^{i_1} C(i)\epsilon^{-\delta(i)}di + \sum_{k_-\leq k\leq k_+}\int_{i_0}^{i_1}\mathfrak{M}(i)2^{k\delta(i)}di \\
&\leq \int_{i_0}^{i_1}\left(C(i) + \mathfrak{M}(i)\left(\frac{\sqrt{n}}{r_+}\right)^{\delta(i)}\frac{(40r_+)^{\delta(i)}-1}{2^{\delta(i)}-1}\epsilon^{-\delta(i)}\right)di
\end{aligned}$$

Likewise we obtain an analogous estimate for S_2^ϵ:

$$S_2^\epsilon(t) \leq C_W^{(n)}\,\mathrm{vol}_n\,\Omega t^{n/2} + \frac{C_W^{(n-1)}}{4}\int_{i_0}^{i_1} A(i)t^{\delta(i)/2}di. \qquad \square$$

6.2.3 Elliptic operators

This work restricts itself to the classical Laplacian operator throughout. There are however straightforward generalisations to positive uniform elliptic operators with constant leading coefficients, $L : \Omega \ni x \mapsto \sum_{|\alpha|\leq 2m} a_\alpha(x)\partial^\alpha$ with some C such that $(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha > C|\xi|^{2m}$ for all $\xi \in \mathbb{R}^n$ as was shown by Lapidus in [65]. Both Lem. 3.8 and Thm. 4.15 allow adaptations to such elliptic operators. By definition (ϵ,∞)-domains (and also quasidisks by Jones' characterisation of quasidisks in [49]), also admit higher extension operators $E : H^m(\Omega) \to H^m(\mathbb{R}^n)$. Since quasidisks, being (ϵ,∞)-domains, also admit higher extension operators $E : H^m(\Omega) \to H^m(\mathbb{R}^n)$, it remains to be studied under what conditions a p-Rohde snowflakes remains well-covered.

6.3 Applications on other quasidisks

6.3.1 Julia sets

The formulation of well-foliated domains can also be applied to Julia sets if based on the original construction by Brolin (cf. Sec. 3.2). The choice of a Julia set, say J_c to a map $f_c(z) := z^2 + c$ with $|c| > 0$ but sufficiently small, leads to specific uniformisation maps G_c depending on c. The foliation built in this manner is then governed by upper bounds on the derivatives of G_c and G_c^{-1} and therefore non-trivially on c.

6.3.2 Stochastic p-Rohde snowflakes

Suppose the p in the construction of a p-Rohde curve (cf. Def. 2.33) is not constant but varies according so a probabilistic distribution. Since the construction suggested in Prop. 3.14 does not need constant p at each iteration step of the construction, this allows for more general domains. In particular one may allow $p > \frac{\sqrt{3}-1}{2}$ occasionally as long as $\mathcal{I}_\beta < \infty$. It is to be expected that such stochastic p-Rohde snowflakes allow for analogues of Lem. 3.8 and Thm. 4.15.

6.3.3 Numerical approximation of optimal Poincaré-Sobolev constants for p-Rohde snowflakes

Rösler-Stepanenko showed "computability" (i.e. convergence of numerical approximations) of Dirichlet eigenvalues in [95]. Similarly Hinz-Rozanova-Pierrat-Teplyaev proved the following Mosco convergence result.

Definition 6.3 (Convergence of domains, Def. 2.2.8 and 2.2.3 in [42])**.** A sequence of bounded open sets $O_n \subset \mathbb{R}^d$ is said to *converge in Hausdorff sense* to some open set O if there is a compact set K with $O \cup \bigcup_n O_n \subset K$ for which the sequence $K \backslash O_n$ converges to

$K \backslash O$ in Hausdorff sense. This definition does not depend on the choice of K. The sequence O_n is said to *converge in the sense of characteristic functions* to some O if $\chi_{O_n} \to \chi_O$ in $L^p(\mathbb{R}^d)$-norm for all $1 \leq p < \infty$.

These two notions are indeed independent as can be seen through the following counter-examples.

- Let $C_n \subset \mathbb{R}^2$ be the n^{th} iteration of an iterative construction of a Knopp curve C over the unit interval $[0,1] \subset \mathbb{R}^2$ with non-vanishing Lebesgue volume. Both C_n and C are compact and $C_n \to C$ in Hausdorff sense. Then by definition $O_n := [0,1]^2 \setminus C_n$ converges to $O := [0,1]^2 \setminus C$ in Hausdorff sense. However, since $\mathrm{vol}_2(C_n) = 0$ for all n but $\mathrm{vol}_2(C) \neq 0$, $O_n \not\to O$ in the sense of characteristic functions.
- Conversely consider the open set $I_n := [0,1] \backslash \bigcup_{k=0}^{2^n} \left\{ \frac{k}{2^n} \right\} \subset \mathbb{R}$. Then $I_m \to (0,1)$ in the sense of characteristic functions but $I_n \to \emptyset$ in Hausdorff sense.

Theorem 6.4 (Mosco convergence, Thm. 6.1 in [43]). *Let $\{\Omega_m\}_{m \in \mathbb{N}}$ be a sequence of quasi-circles such that $\Omega_m \stackrel{H}{\to} \Omega$ and in the sense of characteristic functions (meaning $\mathrm{vol}_n(\Omega_m) \to \mathrm{vol}_n(\Omega)$). Then there is a subsequence of $(\Omega_{m_k})_k$ for which $\sigma(-\Delta_N(\Omega_{m_k})_k) \stackrel{H}{\to} \sigma(-\Delta_N(\Omega))$.*

Such results are known for snowflakes in particular in the Dirichlet case with numerical results from Neuberger-Sieben-Swift in [81] and Banjai-Boulton in [8]. It is therefore justified to employ numerical methods (for example as suggested by Carstensen-Gedicke and Hu-Huang-Ma in [17, 44]) on finite approximation levels of p-Rohde snowflakes and in particular to regular foliated domains. It is expected that the estimates presented in Lem. 3.8 for the Poincaré-Sobolev constants are far from optimal.

Bibliography

[1] G. Acosta and R. G. Durán, *An optimal Poincaré inequality in L^1 for convex domains*, Proceedings of the American Mathematical Society **132** (2004), no. 1, 195–202.

[2] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Academic Press, 1975.

[3] L. V. Ahlfors, *Quasiconformal reflections*, Acta Mathematica **109** (1963), 291–301.

[4] ______, *Lectures on Quasiconformal Mappings*, 2. ed., University Lecture Series, vol. 38, Amer. Math. Soc., 2006.

[5] W. Ahrendt, R. Nittka, W. Peter, and F. Steiner, *Mathematical analysis of evolution, information, and complexity*, ch. 1, Wiley VCH, 2009, W. Arendt and W. P. Schleich (eds.).

[6] K. Arfi and A. Rozanova-Pierrat, *Dirichlet-to-Neumann or Poincaré-Steklov operator on fractals described by d-sets*, Discrete and Continuous Dynamical Systems **12** (2019), no. 1, 1–26.

[7] R. Balka and T. Keleti, *Lipschitz images and dimensions*, Advances in Mathematics **446** (2024).

[8] L. Banjai and L. Boulton, *Computation of sharp estimates of the Poincaré constant on planar domains with piecewise self-similar boundary*, Journal of Fractal Geometry **8** (2021), no. 2, 153–188.

[9] M. V. Berry, *Distribution of modes in fractal resonators*, Structural Stability in Physics (1979), 51–53.

[10] ______, *Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes, diffractals*, Proc. Symp. Pure Math. **36** (1980), 13–38, in 'Geometry of the Laplace Operator'.

[11] H. Brolin, *Invariant sets under iteration of rational functions*, Arkiv för Matematik **6** (1965), no. 6, 103–144.

[12] M. Bronstein and V. Ivrii, *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. i: Pushing the Limits*, Commun. Part. Diff. Equat. **28** (2003), no. 1-2, 83–102.

[13] J. Brossard and R. Carmona, *Can one hear the dimension of a fractal?*, Comm. Math. Phys. **104** (1986), 103–122.

[14] V. I. Burenkov, *Sobolev Spaces on Domains*, 1. ed., Teubner-Texte zur Mathematik, vol. 137, Teubner, 1998.

[15] A.-P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc. Sympos. Pure Math. **IV** (1961), 33–49.

[16] G. Cardone and A. Khrabustovskyi, *Neumann spectral problem in a domain with very corrugated boundary*, Journal of Differential Equations **259** (2015), no. 6, 2333–2367.

[17] C. Carstensen and J. Gedicke, *Guaranteed lower bounds for eigenvalues*, Mathematics of Computation **83** (2014), 2605–2629.

[18] W. Chen and B. D. Sleemann, *On nonisometric isospectral connected fractal domains*, Revista Matemática Iberoamericana **16** (2000), no. 2, 351–361.

[19] Y. Colin de Verdière, *Construction de laplaciens dont une partie finie du spectre est donnée*, Annales scientifiques de l'École Normale Supérieure **20** (1987), no. 4, 599–615.

[20] R. Courant and D. Hilbert, *Methoden der mathematischen Physik I*, Die Grundlehren der mathematischen Wissenschaften, vol. XII, Julius Springer, 1924.

[21] ———, *Methods of mathematical physics*, vol. I, Interscience Publishers, Inc., 1953.

[22] E. B. Davies, *Spectral theory and differential operators*, Cambridge University Press, 1995.

[23] J. T. Dawgzone, *Mind Games*, 2525.

[24] A. Dekkers, A. Rozanova-Pierrat, and A. Teplyaev, *Mixed boundary valued problems for linear and nonlinear wave equations in domains with fractal boundaries*, Calculus of Variations and Partial Differential Equations **61** (2022), no. 75.

[25] A. Eftekhari and M. B. Wakin, *What happens to a manifold under a bi-Lipschitz map?*, Discrete & Computational Geometry **57** (2017), 641–673.

[26] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 2016.

[27] K. Falconer, *Techniques in fractal geometry*, 1. ed., Wiley, 1997.

[28] ———, *Fractal Geometry: Mathematical Foundations and Applications*, 2. ed., Wiley, 2003.

[29] N. D. Filonov and Yu. G. Safarov, *Asymptotic Estimates of the Difference Between the Dirichlet and Neumann Counting Functions*, Functional Analysis and Its Applications **44** (2010), no. 4, 286–294.

[30] R. L. Frank and S. Larson, *Two-term spectral asymptotics for the Dirichlet Laplacian in a Lipschitz domain*, Journal für die reine und angewandte Mathematik **766** (2020), 195–228.

[31] K. Funano, *A note on domain monotonicity for the Neumann eigenvalues of the Laplacian*, Illinois Journal of Mathematics **67** (2023), no. 4, 677–686.

[32] M. García-Bravo, T. Rajala, and J. Takanen, *A necessary condition for Sobolev extension domains in higher dimensions*, Nonlinear Analysis **240** (2024).

[33] F. Gehring, M. Gaven, and B. Palka, *An Introduction to the Theory of Higher-Dimensional Quasiconformal Mappings*, Mathematical Surveys and Monographs, vol. 216, American Mathematical Society, 2017.

[34] V. Gol'dshtein, V. Pchelintsev, and A. Ukhlov, *Integral estimates of conformal derivatives and spectral properties of the Neumann-Laplacian*, Journal of Mathematical Analysis and Applications **463** (2018), no. 1, 19–39.

[35] V. Gol'dshtein and A. G. Ramm, *Compactness of the embedding operators for rough domains*, Mathematical Inequalities & Applications **4** (2001), no. 1, 127–141.

[36] S. Goldstein, *Random walks and diffusions on fractals, Percolation theory and ergodic theory of infinite particle systems*, IMA Math. Appl. **8** (1987), 121–129.

[37] C. Gordon, D. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Inventiones mathematicae **110** (1992), no. 1, 1–22.

[38] Y. Gu and J. J. Miao, *Dimensions of a class of self-affine Moran sets*, Journal of Mathematical Analysis and Applications **513** (2022).

[39] P. Hajłasz, *Change of variables formula under minimal assumptions*, Colloquium Mathematicum **LXIV** (1993), no. 1, 93–101.

[40] M. Hata, *On continuous functions with no unilateral derivatives*, Annales de l'Institut Fourier **38** (1988), 43–62.

[41] R. Hempel, L. A. Seco, and B. Simon, *The Essential Spectrum of Neumann Laplacians on Some Bounded Singular Domains*, Journal of Functional Analysis **102** (1991), 448–483.

[42] A. Henrot and M. Pierre, *Shape Variation and Optimization*, Tracts in Mathematics, no. 28, EMS, 2018.

[43] M. Hinz, A. Rozanova-Pierrat, and A. Teplyaev, *Boundary value problems on non-Lipschitz uniform domains: stability, compactness and the existence of optimal shapes*, Asymptotic Analysis **134** (2023), no. 1-2, 25–61.

[44] J. Hu, Y. Huang, and R. Ma, *Guaranteed lower bounds for eigenvalues of elliptic operators*, Journal of Scientific Computing **67** (2016), 1181–1197.

[45] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[46] V. Ivrii, *Second term of the spectral asymptotic expansion of the Laplace-Beltrami operator on manifolds with boundary*, Functional Analysis and Its Applications **14** (1980), no. 2, 98–106, Magnitogorsk Mining-Metallurgical Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 14, No. 2, pp. 25-34, April-June, 1980. Original article submitted November 15, 1979.

[47] ———, *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. II: Domains with Boundaries and Degenerations*, Commun. Part. Diff. Equat. **28** (2003), no. 1-2, 103–128.

[48] ———, *Microlocal analysis, sharp spectral asymptotics and applications I*, Springer, 2019.

[49] P. W. Jones, *Quasiconformal Mappings and Extendability of Functions in Sobolev Spaces*, Acta Mathematica **147** (1981), no. 1, 71–88.

[50] J. Jost, *Partial Differential Equations*, 1. ed., Springer, 2002.

[51] ———, *Partial Differential Equations*, 2. ed., Springer, 2007.

[52] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), no. 4.

[53] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren der mathematischen Wissenschaften, vol. 132, Springer Verlag, 1995, Reprint of the 1980 version.

[54] J. Kigami, *A Harmonic Calculus on the Sierpinski Spaces*, Japan J. Appl. Math. **6** (1989), 259–290.

[55] J. Kigami and M. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on P.C.F. self-similar fractals*, Communications in Mathematical Physics **158** (1993), 93–125.

[56] K. Knopp, *Einheitliche Erzeugung und Darstellung der Kurven von Peano, Osgood und von Koch*, Archiv der Mathematik und Physik **26** (1917), 103–115.

[57] H. v. Koch, *Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire*, Arkiv för Matematik, Astronomi och Fysik **1** (1904), 681–702.

[58] S. Kombrink, *Renewal theorems for processes with dependent interarrival times*, Advances in Applied Probability **50** (2018), no. 4, 1193–1216.

[59] ———, *Asymptotic Expansions of Renewal Functions*, in preparation, 2024.

[60] S. Kombrink and M. Keßeböhmer, *A complex Ruelle-Perron-Frobenius theorem and for infinite alphabets with application to renewal theory*, Discrete Contin. Dyn. Syst. **10** (2017), no. 2, 335–352, Series S.

[61] S. Kombrink and L. Schmidt, *On bounds for the remainder term of counting functions of the Neumann Laplacian on domains with fractal boundary*, arxiv:2312.12308 (2023), preprint.

[62] ———, *Eigenvalue counting functions and parallel volumes for examples of fractal sprays generated by the Koch snowflake*, Recent Developments in Fractals and Related Fields **4** (to appear).

[63] A. Käenmäki, J. Lehrbäck, and M. Vuorinen, *Dimensions, Whitney Covers, and Tubular Neighborhoods*, Indiana University Mathematics Journal **62** (2013), no. 6, 1861–1889.

[64] S. P. Lalley, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, noneuclidean tessellations and their fractal limits*, **163** (1989), 1–55.

[65] M. L. Lapidus, *Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture*, Trans. Amer. Math. Soc. **325** (1991), 465–529.

[66] ———, *Vibrations of Fractal Drums, the Riemann Hypothesis, Waves in Fractal Media and the Weyl-Berry Conjecture*, Ordinary and Partial Differential Equations IV (Dundee, Scotland, UK) (B.D. Sleeman and R.J. Jarvis, eds.), Proc. Twelth International Conference on 'The Theory of Partial Differential Equations', vol. 289, Longman Scientific and Technical, 1993, pp. 126–209.

[67] M. L. Lapidus and E. P. J. Pearse, *A tube formula for the Koch snowflake curve, with applications to complex dimensions*, J. London Math. Soc **74** (2006), no. 2, 397–414.

[68] M. L. Lapidus and C. Pomerance, *The Riemann Zeta-Function and the One-Dimensional Weyl-Berry Conjecture for Fractal Drums*, Proceedings of the London Mathematical Society **s3-66** (1993), no. 1, 41–69.

[69] ———, *Counterexamples to the modified Weyl-Berry conjecture on fractal drums*, Mathematical Proceedings of the Cambridge Philosophical Society **119** (1996), 167–178.

[70] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions*, Springer Monographs in Mathematics, Springer, 2006.

[71] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Analysis: Theory, Methods & Applications **64** (2006), no. 5, 1057–1099.

[72] M. Levitin and M. van den Berg, *Functions of Weierstrass type and spectral asymptotics for iterated sets*, The Quarterly Journal of Mathematics **47** (1996), no. 4, 493–509.

[73] M. Levitin and D. Vassiliev, *Spectral Asymptotics, Renewal Theorem, and the Berry Conjecture for a Class of Fractals*, Proceedings of the London Mathematical Society **72** (1996), no. 3, 188–214.

[74] D. Mauldin and M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Tracts in Mathematics, Cambridge University Press, 2003.

[75] V. Maz'ya, *On Neumann's problem in domains with nonregular boundaries*, Sibirian Mathematical Journal **9** (1968), 990–1012, Translated from Russian to English, originally published in Sibirskii matematicheskii zhurnal **9** (1968), no. 6, 1322-1350.

[76] ———, *Sobolev Spaces*, 2. ed., Grundlehren der mathematischen Wissenschaften, vol. 342, Springer, 2011.

[77] B.J. McArtin, *Laplacian Eigenstructure of the Equilateral Triangle*, Hikari Ltd., 2011.

[78] P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Mathematical Proceedings of the Cambridge Philosophical Society **46** (1946), no. 1, 15–33.

[79] Yu. Netrusov, *Sharp remainder estimates in the Weyl formula for the Neumann Laplacian on a class of planar regions*, Journal of Functional Analysis **250** (2007), no. 1, 21–41.

[80] Yu. Netrusov and Yu. G. Safarov, *Weyl asymptotic formula for the Laplacian on domains with rough boundaries*, Communications in Mathematical Physics **253** (2005), 481–509.

[81] J. M. Neuberger, N. Sieben, and J. W. Swift, *Computing eigenfunctions on the Koch snowflake: A new grid and symmetry*, Journal of Computational and Applied Mathematics **191** (2006), no. 1, 126–142.

[82] W.-M. Ni and X. Wang, *On the first positive Neumann eigenvalue*, Discrete and Continuous Dynamical Systems **17** (2007), no. 1, 1–19.

[83] L. E. Payne and H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Archive for Rational Mechanics and Analysis **5** (1960), 286–292.

[84] M. Pollicott, *A complex Ruelle-Perron-Frobenius theorem and two counterexamples*, Ergodic Theory and Dynamical Systems **4** (1984), 135–146.

[85] ______, *Meromorphic extensions of generalised zeta functios*, Invent. math. **85** (1986), 147–164.

[86] G. Pólya, *On the Eigenvalues of vibrating Membranes*, Proceedings of the London Math. Soc. **s3-11** (1961), no. 1, 419–433.

[87] O. Post and J. Zimmer, *Graph-like spaces approximated by discrete graphs and applications*, Mathematische Nachrichten **243** (2021), no. 3.

[88] H. Rademacher, *Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale*, Mathematische Annalen **79** (1919), no. 4, 340–359.

[89] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, revised and enlarged edition ed., Academic Press, INC., 1980.

[90] F. Rellich, *Das Eigenwertproblem von* $\Delta u + \lambda u = 0$, Interscience Publishers, Inc., 1948, in Studies and Essays Presented to R. Courant.

[91] S. Rohde, *Quasicircles modulo bilipschitz maps*, Rev. Mat. Iberoam. **17** (2001), 643–659.

[92] G. V. Rosenbljum, *On the eigenvalues of the first boundary value problem in unbounded domains*, Math. USSR-Sb **18** (1972), 235–248.

[93] ______, *The computation of the spectral asymptotics for the laplace operator in domains of infinite measure*, J. Sov. Math. **6** (1976), 64–71.

[94] D. Ruelle, *Thermodynamik Formalism*, 2. ed., Cambridge Mathematical Library, 2004.

[95] F. Rösler and A. Stepanenko, *Computing eigenvalues of the Laplacian on rough domains*, Mathematics of Computation **93** (2023), no. 3.

[96] Yu. Safarov and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Differential Operators*, Spectral theory of PDEs, XIX Winter Mathematical School, Voronezh, Russia, Jan. 1987.

[97] ———, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, Translations of Mathematical Monographs, vol. 155, Amer. Math. Soc., 1996.

[98] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, 1. ed., Graduate Texts in Mathematics, vol. 265, Springer, 2011.

[99] R. Seeley, *A Sharp Asymptotic Remainder Estimate for the Eigenvalues of the Laplacian in a Domain of R^3*, Advances in Mathematics **29** (1978), no. 2, 244–269.

[100] ———, *An estimate near the boundary for the spectral function of the Laplace operator*, American Journal of Mathematics **102** (1980), no. 5, 869–902.

[101] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, vol. 105, Cambridge University Press, 1993.

[102] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univerity Press, 1970.

[103] R. S. Strichartz, *Differential Equations on Fractals*, Princeton University Press, 2006.

[104] R. S. Strichartz and S. C. Wiese, *Spectral Properties of Laplacians on Snowflake Domains and Filled Julia Sets*, Experimental Mathematics (2020).

[105] G. Szegő, *Inequalities for Certain Eigenvalues of a Membrane of Given Area*, Journal of Rational Mechanics and Analysis **3** (1954), 343–356.

[106] M. van den Berg, *A uniform bound on trace ($e^{t\Delta}$) for convex regions in R^n with smooth boundaries*, Comrnun. Math. Phys. **92** (1984), 525–530.

[107] M. van den Berg and M. Lianantonakis, *Asymptotics for the Spectrum of the Dirichlet Laplacian on Horn-shaped Regions*, Indiana University Mathematics Journal **50** (2001), no. 1, 299–333.

[108] D. Vassiliev, *Two-Term Asymptotics of the Spectrum of a Boundary Value Problem in the Case of a piecewise smooth Boundary*, Sov. Math. Dokl. **22** (1986), no. 1, 227–230, English translation from Uspekhi Mat. Nauk 40, no.5, p.233 (in Russian).

[109] S. K. Vodop'yanov, V. M. Gol'dshtein, and T. G. Latfullin, *Criteria of functions of the class* ${L_2}^1$ *from unbounded plane domains*, Siberian Mathematical Journal **20** (1979), 298–301, Translated from Russian to English, originally published in Sibirskii matematicheskii zhurnal **20** (1979), 416-419.

[110] H. F. Weinberger, *An isoperimetric inequality for the n-dimensional free membrane problem*, Arch. Ration. Mech. Anal. **5** (1956), 633–636.

[111] D. Werner, *Funktionalanalysis*, 4. revised ed., Springer, 2002.

[112] H. Weyl, *Ueber die asymptotische Verteilung der Eigenwerte*, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen (1911), 110–117, Mathematisch Physikalische Klasse.

[113] ______, *Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze*, Journal für die reine und angewandte Mathematik **143** (1913), 177–202.

[114] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the Amer. Math. Soc. **36** (1934), no. 1, 63–89.

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