PREDICTIONS IN OPEN FAN-JARVIS-RUAN-WITTEN THEORY VIA MIRROR SYMMETRY, MODULARITY AND WALL-CROSSING

by

ROB MAHER

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Abstract

In recent works of Buryak, Clader, Gross, Kelly and Tessler, genus zero open enumerative theories and their mirrors for the Landau-Ginzburg models $W_0 = x^r$ and $W_0 = x_1^{r_1} + x_2^{r_2}$ were constructed. We build on this work by defining an open mirror B-model with gravitational descendants for any Fermat, chain or loop polynomial. This is done by presenting a recursive algorithm for finding flat coordinates of Dubrovin's Frobenius manifold for a Landau-Ginzburg B-model. Furthermore, we present formulas for these coordinates for any ADE or elliptic singularity. Although these open Saito potentials do not yet have an enumerative interpretation, we present explicit formulas for these generating functions, together with modularity properties in the elliptic case. Finally, we find that the B-model exhibits a wall-crossing structure. We classify this structure by describing a Lie group action of wall-crossing transformations, which we prove is faithful and transitive in the rank 2 case.

CONTENTS

1 Introduction **4**

2 Closed FJRW Theory **16**

2.1 Landau-Ginzburg Models 16

2.1.1 Quasi-Homogeneous Polynomials 17

2.1.2 Groups of Diagonal Symmetries 19

2.2 W-Spin Curves 22

2.2.1 Recollections on Orbifold Structures. 22

2.2.2 Marked Orbicurves and Line Bundles. 23

2.3 Stacks of Stable W-Spin Curves 27

2.4 FJRW Cohomological Field Theory 30

2.4.1 The Chiral Ring 30

2.4.2 A Glimpse of Intersection Theory 31

3 Closed Saito-Givental Theory and Landau-Ginzburg Mirror Symmetry **34**

3.1 Saito Theory Via Deformations 35

3.1.1 Simple and Elliptic Singularities 35

3.1.2 The Local Algebra 40

3.1.3 Families of Frobenius Algebras 43

3.1.4 Hypercohomology and The Brieskorn Lattice. 45

3.1.5 The Gauss-Manin Vector Bundle 48

3.1.6 Primitive Forms 50

3.2 Saito Theory as Oscillatory Integrals 52
3.2.1 Primitive Forms Via Oscillatory Integrals 52
3.2.2 Oscillatory Integrals 54
3.2.3 Summary of Saito Theory for Landau-Ginzburg Models 59
3.2.4 Flat Coordinates of Simple and Elliptic Singularities 62
3.2.5 Period Integrals in Rank Two and Formulas for Flat Coordinates 67
3.3 Mirror Symmetry 72
3.3.1 A Review of the Krawitz Mirror Map 72
3.3.2 Closed Landau-Ginzburg Mirror Symmetry 76
3.3.3 Using Mirror Symmetry to Calculate FJRW Phases 78

4 Open Saito-Givental Theory 80
4.1 Open Saito-Givental Potentials 80
4.2 Primary Open Saito Potentials 86
4.2.1 ADE Singularities 87
4.2.2 Modularity and Elliptic Singularities 89
4.3 Flat Coordinates and Primitive Forms for Chains and Loops of Rank Two 94

5 Wall-Crossing 106
5.1 Motivation 106
5.2 The Wall-Crossing Group 111
5.2.1 Quasi-Homogeneity 113
5.2.2 Wall-Crossing for Elliptic Singularities 117
5.3 Properties of the Landau-Ginzburg Wall-Crossing Group 123
5.3.1 Faithfulness 123
5.3.2 Transitivity 125

6 Towards Open FJRW Theory 130
6.1 W-Spin Structures 130
6.1.1 Open r-Spin Disks 130

6.1.2 Open W-Spin Disks . . . 132

6.1.3 Selection Rules for Open FJRW Theory. . . . 133

6.1.4 Concavity . . . 137

6.1.5 Grading of W-Spin Disks. . . . 143

6.1.6 Open Mirror Symmetry . . . 144

6.2 Equivalences in Landau-Ginzburg Theories . . . 145

6.2.1 Equivalences in FJRW Theory . . . 146

6.2.2 Equivalences in Primary Saito Theories . . . 146

Bibliography **153**

A The Hypergeometric Equation for the Cubic Elliptic Curve **173**

A.1 Triangle Functions . . . 173

A.2 Solving Hypergeometric Equations Near Infinity . . . 177

A.3 Series Reversion and Modular Expansions . . . 180

B A Remark on Open WDVV Equations for Simple Singularities **183**

CHAPTER 1

INTRODUCTION

In a sequence of papers, [FJR08; FJR11b; FJR13] Fan, Jarvis and Ruan developed FJRW theory as a closed enumerative theory for Landau-Ginzburg A-models, inspired by physical work of Witten [Wit93a]. This was done as there is currently no complete theory of how to take a $\mathcal{N} = (2,2)$ supersymmetric Landau-Ginzburg action and perform an A-twist to obtain a topological field theory coupled to gravity whilst retaining Lorentz invariance. See sections 2 and 3.1 of [GS09] where Guffin and Sharpe discuss the difficulties of such a theory.

FJRW theory takes as input data (W_0, G) where $W_0 : \mathbb{C}^N \to \mathbb{C}$ is a weighted homogeneous polynomial and G is an admissible choice of subgroup of the automorphism of W_0. The polynomial W_0 is further restricted to be invertible and such superpotentials were classified by Kreuzer and Skarke [KS92] as Sebastiani-Thom sums of Fermat, chain and loop polynomials. One reason for this added condition of invertibility is that the corresponding hypersurfaces in weighted projective space are suitably well-behaved under the Landau-Ginzburg / Calabi-Yau correspondence [Wit93b].

Given this input data, the output of FJRW theory is an enumerative theory which enjoys the properties of a cohomological field theory, in the sense of Kontsevich and Manin [KM94]. By construction, the CohFT has an enumerative interpretation as a count of W-spin curves. In particular, for $W_0 = x^r$ and $G = G^{\max}_{W_0} = \mu_r$ where μ_r is the group of r^{th} roots of unity, the resulting FJRW theory is equivalent to the r-

spin construction of Abramovich, Chiodo, Jarvis, Kimura and Vaintrob [AJ03; Jar00; JKV01; Chi08]. See [FJR11a] for an exposition of the relation between FJRW theory and r-spin structures. Although the enumerative picture of closed FJRW theory is well understood, computations are notoriously difficult, especially if the invariants do not satisfy a condition known as concavity. For example, Guéré overcame the problem of non-concavity in [Gué16] by explicitly calculating the virtual fundamental cycle for FJRW theory via Givental's quantum Riemann-Roch formalism. On the other hand, [HLSW22] bypass such tricky calculations through the use of various reconstruction theorems and the WDVV equations.

Conversely, the mirror Landau-Ginzburg B-model is easy to compute with. As (twisted) topological field theories, these were investigated in [LVW89; Vaf91; Wit92]. These works also give motivation for the particular focus on the case of quasi-homogeneous potentials on $\mathbb{C}^N$, although we do not dwell on the details here. In this thesis, from the physical viewpoint, we consider a deformation $W_{\boldsymbol{s}}$ of W_0 by adding certain source terms to the superpotential and writing descent equations to compute correlation functions. From a mathematical viewpoint, the deformation $W_{\boldsymbol{s}}$ is a versal deformation of the singularity. Using primitive forms, the mathematical foundations of the closed B-model as a cohomological field theory were developed by Saito [Sai83a; Sai83b] for genus zero; Givental then provided a remarkable formula in [Giv96; Giv01] for the higher genus theory with descendants. This was subsequently completed by Teleman [Tel12] and applied to the Landau-Ginzburg case by Milanov [Mil14].

This modern formulation of the B-model, now called Saito-Givental theory in the literature, is dependent on the *local algebra* of the invertible singularity W_0 which plays the role of the B-model chiral ring. The local algebra has a flat Frobenius structure encoded in the flat coordinates $\{t_\mu\}_{\mu\in B}$ of Dubrovin's Frobenius manifolds [Dub96]. Here, the set $B \subseteq \mathbb{Z}^N_{\geq 0}$ is an index set for a unique choice of basis of the local algebra. This special choice of basis, $\{x^\mu := \prod_{i=1}^N x_i^{\mu_i} \,|\, \mu \in B\}$, is called the *standard good basis* in Definition 2.9 of [HLSW22]. Via the work of Givental, the flat coordinates can be extended to descendant parameters $t_{\mu,d}$ with the identification $t_{\mu,0} = t_\mu$.

Nevertheless, finding explicit formulas for t_μ in Saito-Givental theory is difficult. Our first result in this thesis is a calculation of flat coordinates for singularities whose type is either ADE or simple elliptic.

Flat Coordinates for the B-Model Using Oscillatory Integrals

Roughly speaking, the B-model involves period integrals of the form

$$\int_{\Xi_\mu} e^{W/\hbar} f d^N x.$$

Here, W is a deformation of W_0 and f is a regular function on $\mathbb{C}^N \times \mathcal{M}$ with $\mathcal{M}$ the parameter space of the deformation. Furthermore, if f is chosen so that $f \cdot d^N x$ is a primitive form, the manifold $\mathcal{M}$ can be endowed with a Frobenius structure with local coordinates (t_μ). See Definition 3.2.12. Moreover, $\hbar \in \mathbb{C}^*$ is an auxiliary parameter and the Ξ_μ form a special basis of non-compact cycles in $\mathbb{C}^N$, often known as Lefschetz thimbles in the literature. This basis is dual to the standard good basis of [HLSW22] under a perfect pairing defined by oscillatory integrals in Section 3.2.

There are numerous methods to calculate flat coordinates for simple and elliptic singularities. For A_r, the authors of [IN82] carry out computations via a generating function, whilst in [BGK16] Belavin, Gepner and Kononov find flat coordinates by directly solving the Gauss-Manin system. They obtain a conjecture for the integral representation of flat coordinates which we prove in Proposition 1.0.1 and Corollary 1.0.2 below. Computations for E_6 and E_8 are done in [Yan81] and [KW81] respectively.

The cases of D_r and E_7 are chain polynomials and are more computationally technical. The flat structure of E_7 is considered via E_7-invariant polynomials in [Abr09]. On the other hand, Klemm, Theisen and Schmidt derive potentials via topological conformal field theory in [KTS92]. The advantage of this physical perspective is that it also allows for computations of the flat structure of elliptic singularities. Such structures for the elliptic singularities $W_{E_6^{(1,1)}} = x_1^3 + x_2^3 + x_3^3$ and $W_{E_7^{(1,1)}} = x_1^4 + x_2^4$ are also derived in [Kat86; Sat93].

In [Nou84; NY98], there is a unifying systematic treatment of flat coordinates of simple and elliptic singularities, which includes D_r and $E_8^{(1,1)}$. This was done by solving the Riemann-Hilbert-Birkhoff problem of the Gauss-Manin system associated to the singularity. The details of this calculation are quite technical. Thus, a different approach to finding flat coordinates is valuable. In particular, we prove the following.

Proposition 1.0.1. Given an invertible singularity W_0, denote by $W_{\boldsymbol{s}}$ the versal deformation in the local coordinates $(s_\mu)_{\mu\in B}$ of $\mathcal{M}$:

$$W_{\boldsymbol{s}} = W_0 + \sum_{\mu\in B} s_\mu x^\mu.$$

For $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^B$, define $l(\boldsymbol{k}) = \sum_{\alpha\in B} k_\alpha \cdot \alpha$. Then

$$\int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} d^N x = \sum_{i\in\mathbb{Z}} \mathcal{J}_\mu^{(i)}(\boldsymbol{s})\hbar^{-i}, \quad \mathcal{J}_\mu^{(i)}(\boldsymbol{s}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^{|B|} \\ \deg \boldsymbol{s}^{\boldsymbol{k}} = \deg s_\mu - 1 + i}} \left(\frac{1}{\hbar^{q(\boldsymbol{k},i)}} \int_{\Xi_\mu} e^{W_0/\hbar} x^{l(\boldsymbol{k})} d^N x \right) \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}!}$$

where $q(\boldsymbol{k}, i)$ defined in equation (3.2.9) is the number of times integration by parts is used to reduce $x^{l(\boldsymbol{k})}$ to an element x^μ of the good basis. Furthermore, $\deg s_\mu$ is given in Definition 3.1.13.

Using this proposition, we prove the following generalisation of Theorem 1.1 of [NY98].

Corollary 1.0.2. Let W_0 be an ADE or simple elliptic singularity. A primitive form and flat coordinates for W_0 are given by

$$f d^N x = \frac{d^N x}{\mathcal{J}_0^{(0)}(\boldsymbol{s})}, \quad t_\mu(\boldsymbol{s}) = \frac{\mathcal{J}_\mu^{(1)}(\boldsymbol{s})}{\mathcal{J}_0^{(0)}(\boldsymbol{s})}.$$

For an ADE singularity, $\mathcal{J}_0^{(0)}(\boldsymbol{s}) \equiv 1$ and for a simple elliptic singularity, $\mathcal{J}_0^{(0)}(\boldsymbol{s})$ is a function of the marginal parameter s only. See Definition 3.2.9.

The proof follows by a calculation that uses only Taylor expansion and integration by parts, compared to the transcendental methods of Noumi and Yamada. It is exactly this simplification that allows us to easily compute flat coordinates of the elliptic singularities not considered in [NY98], such as the chain polynomial $\tilde{W}_{E_7^{(1,1)}} = x_1^4 + x_1 x_2^3$.

Furthermore, in the notation of Noumi and Yamada, we have

$$\begin{aligned}
\psi_\mu^{(i)} &:= \mathcal{J}_\mu^{(i)} \\
\mathrm{wt}(\boldsymbol{k}) &:= \deg \boldsymbol{s}^{\boldsymbol{k}} = \sum_{\alpha \in B} \deg s_\alpha \cdot k_\alpha, \\
c_\mu(l(\boldsymbol{k})) &:= \frac{1}{\hbar^{q(\boldsymbol{k},i)}} \int_{\Xi_\mu} e^{W_0/\hbar} x^{l(\boldsymbol{k})} d^N x.
\end{aligned}$$

Here, we use $\mathcal{J}_\mu^{(i)}$ instead of $\psi_\mu^{(i)}$ as we wish to emphasise the enumerative importance of these coefficients as related to a J function. The integrals $c_\mu(l(\boldsymbol{k}))$ can be calculated in terms of Γ functions and hypergeometric functions via the results in Section 3.2.4. This gives a concrete realisation, in the case of Landau-Ginzburg models, for the variations of semi-infinite Hodge structures developed by Barannikov [Bar01].

Open Saito-Givental Theory.

Proposition 1.0.1 and Corollary 1.0.2 are enough to specify the closed Landau-Ginzburg B-model invariants for ADE and simple elliptic singularities. Construction of topological Landau-Ginzburg models for *open* strings, however, is still not clear in many cases. There are perhaps three main interlinked approaches that are currently available.

The first uses ideas from topological gravity. Pandharipande, Solomon and Tessler [PST22] first constructed a moduli space of genus zero Riemann surfaces with boundary for the case of 2-spin structures. They furthermore conjectured an open analogue of Witten's conjecture in the A_1 case: that a generating function of intersection numbers on this moduli space is a τ-function of the open KdV hierarchy. Buryak [Bur15] interpreted this open KdV hierarchy as part of the Burgers-KdV hierarchy. The open Witten conjecture was subsequently proven in [BT17]. The main tool was the Kontsevich-Penner matrix model, formed from the Hermitian matrix model in [Kon92] by adding a logarithmic term to the potential. Based on Alexandrov's calculations in [Ale15a; Ale15b; Ale17] of $\mathcal{W}$-algebra constraints for the Kontsevich-Penner matrix model, Safnuk [Saf16] introduced a topological recursion procedure involving half integer genera on a spectral curve to compute open intersection numbers. Although the matrix model is very explicit, Basalaev and Buryak were able to generalise these potentials to the case of A_r and D_r singularities in [BB21]. This was done by considering open WDVV equations, solutions to which give rise to flat F-manifolds.

The second approach is homological. Kapustin and Li [KL03] argued that the correct category of D-branes for a Landau-Ginzburg model W_0 was the associated category of matrix factorisations of W_0. Results of Kontsevich [Kon95] and Costello [Cos07] hinted that this category should give rise to the chiral ring of a Landau-Ginzburg B-model. This was confirmed when Dyckerhoff proved that the Hochschild homology of this category was the local algebra of the singularity [Dyc11]. See also the work of Efimov [Efi18]. In [Orl09] Orlov subsequently proved that the category of matrix factorisations of W_0 is equivalent to (a Verdier quotient of) the derived category of coherent sheaves over a hypersurface defined by $W_0 = 0$. Orlov's equivalence was later interpreted as a B-model analogue for the Landau-Ginzburg / Calabi-Yau correspondence by Chiodo, Iritani and Ruan [CIR14]. More recently, Saito theory has been recast categorically: one may define a categorical primitive form as in [CLT21]. See also the results in [Tu21a]. Homological mirror theorems may then be proven via isomorphisms of the corresponding cohomological field theories. Indeed, by working with the category of matrix factorisations, He, Polishchuk, Shen and Vaintrob in [HPSV22] prove a mirror theorem with the additional removal of a technical assumption of [HLSW22, Theorem 1.2]. Furthermore, Tu [Tu21b] constructs a categorical Saito theory with a non-trivial choice of subgroup of $G_{W_0}^{\max}$. It is then demonstrated that this is mirror to the expected FJRW theory.

A third approach to open topological Landau-Ginzburg models is via mirror symmetry and enumerative geometry. This is the route we take in this thesis. An open r-spin theory has been constructed in [BCT22a; BCT22b; BCT23]; an open B-model, on the other hand, for $W_0 = x^r$ and $W_0 = x_1^{r_1} + x_2^{r_2}$ has been established in [GKT22a; GKT22b] by Gross, Kelly and Tessler. The main point of this thesis is the construction of an open enumerative B-model *for any* invertible singularity. In Section 3.1 we describe an iterative algorithm for the computation of primitive forms and flat coordinates for generic rank two polynomials. Our method is based on work by Gross, Kelly and Tessler [GKT22a; GKT22b] where these authors consider higher order deformations of the superpotentials $W_0 = x^r$ and $W_0 = x_1^{r_1} + x_2^{r_2}$. This is different from the perturbative primitive form expansions outlined in [LLS14; LLSS17], although the two approaches are related.

Similar to the rank two Fermat case, we find that there are non-uniqueness issues surrounding the computation of the flat coordinates for rank two chains and loops. Our main result is then Theorem 1.0.6 which provides structure for this non-uniqueness.

Let us now make this more precise.

Definition 1.0.3. Given a Landau-Ginzburg model W_0, let $\mathcal{I}$ be a multiset of $\mu \in B$. Consider the ring $A_{\mathcal{I},\text{sym}} := \mathbb{Q}[t_{\mu,d} \,|\, \mu \in B, d \in \mathbb{Z}_{\geq 0}]/\text{Ideal}(\mathcal{I})$ for an ideal, Ideal($\mathcal{I}$), generated by monomials of the form $\prod_d t_{\mu,d}^{n_d}$ so that $\sum n_d$ is greater than the number of times the element μ appears in $\mathcal{I}$. Given rational numbers $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}} \in \mathbb{Q}$ indexed by $k_i \geq 0$, a multiset I and descendant vector $\boldsymbol{d}$, an *open ancestor Saito potential* for W_0 is

$$W := \sum_{l\geq 0} \sum_{k_1,\ldots,k_N \geq 0} \sum_{A=\{(\mu_i,d_i)\}\in\mathcal{A}_l} (-1)^{l-1} \frac{\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}}{|\text{Aut}(A)|} \prod_{i=1}^{l} t_{\mu_i,d_i} \prod_{i=1}^{N} x_i^{k_i}$$

where $\mathcal{A}_l$ denotes a set whose elements A are cardinality l multisets of tuples $I := \{(\mu_i)\}_{i=1}^{l}$ together with the descendant vector $\boldsymbol{d}$.

An important special case of open Saito potentials are *primary* open Saito potentials where the descendant vector $\boldsymbol{d} = \boldsymbol{0}$. Given W_0, one may calculate the flat coordinates and change variable in the versal deformation $W_{\boldsymbol{s}}$ to write the s_μ parameters in terms of the flat coordinates t_μ. We identify the formal parameters $t_{\mu,0}$ of the open Saito potential with the flat coordinates t_μ via $t_{\mu,0} = t_\mu$. In Corollaries 4.3.7 and 4.3.12, we provide justification for this identification in rank two.

Definition 1.0.3 extends Definition 4.12 in [GKT22b] from rank two Fermat superpotentials to any invertible Landau-Ginzburg potential. The authors of that paper have shown that if the $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$ are enumerative for $W_0 = x^r$ or $W_0 = x_1^{r_1} + x_2^{r_2}$, then the open Saito potential is a natural choice of deformation of W_0. It is furthermore analogous to generating functions involved in toric Fano / Landau-Ginzburg mirror symmetry of [CO06; Gro10; FOOO10]. See [Gro11; FOOO12] for expositions. Crucially, in all of these works, the enumerative invariants are not uniquely defined in all cases as they exhibit a wall-crossing structure.

In Definition 4.18 of [GKT22b], a *wall-crossing group* is defined for $W_0 = x_1^{r_1}+x_2^{r_2}$ that classifies such wall-crossing transformations and thus what invariants can be obtained in open Saito-Givental theory. In this thesis, we propose a definition of the wall-crossing group, $G_{A,W_0} \leq \mathrm{Aut}_A(A[[x_1, \ldots, x_N]])$ depending on some ring A and *any* invertible W_0. More precisely, we extend the previous definition of the wall-crossing group given in [GKT22b] to cover the cases of chain and loop polynomials, which are defined in Theorem 2.1.6.

Definition 1.0.4. Let $A = A_{\mathcal{I},\mathrm{sym}}$. For a Landau-Ginzburg model W_0, the *Landau-Ginzburg wall-crossing group* $G_{A,W_0} \leq \mathrm{Aut}_A(A[[x_1, \ldots, x_N]])$ is the group of elements ψ such that the following axioms hold.

1. ψ must satisfy
$$\psi = \mathrm{id} \bmod \langle (t_{\mu,d}) |\, \mu \in B, d \in \mathbb{Z}_{\geq 0} \rangle .$$
2. ψ preserves the volume form $dx_1 \wedge \cdots \wedge dx_N$.
3. ψ preserves the ideal generated by $\prod_{i=1}^N x_i$.
4. Each monomial in ψW is a count of *balanced* $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$.

The *primary Landau-Ginzburg wall-crossing group* is defined by restricting G_{A,W_0} to $\boldsymbol{d} = \boldsymbol{0}$ with the additional *quasi-homogeneity* condition: ψ commutes with the grading operator,

$$\mathcal{E} := \sum_{\mu \in B} s_\mu \deg s_\mu \frac{\partial}{\partial s_\mu} + \sum_{i=1}^{N} x_i \deg x_i \frac{\partial}{\partial x_i}.$$

This definition of the wall-crossing group consists of four axioms, three of which appear in [GKT22b]. The first three are dealt with by considering G_{A,W_0} as a subgroup of a group, G_A, of vector fields. See Proposition 5.2.1. The final axiom is new: it ensures that open Saito potentials should match with an enumerative theory. These are combinatorial conditions that are referred to as *balancing conditions* or selection rules for $\nu_{k_1,\dots,k_N,I,\boldsymbol{d}}$. See Definition 4.1.1 for the definition of balanced $\nu_{k_1,\dots,k_N,I,\boldsymbol{d}}$.

To make the construction of the wall-crossing group more explicit, in Sections 4.2 and 4.3 we consider restrictions to primary open Saito potentials. As a simple consequence of this definition, we give a description of the primary wall-crossing group for any singularity with central charge $c \leq 1$. To state this, we first need some technical details. For W_0 an elliptic singularity, we define the *marginal basis element* x^{μ_c}, with $\mu_c \in B$, as the unique standard good basis element whose degree induced by $\mathcal{E}$ is equal to one. The *marginal flat coordinate* t_{μ_c} is then the flat coordinate associated to x^{μ_c} and correspondingly satisfies $\deg t_{\mu_c} = 0$. Finally, for $W_0 = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{r_{ij}}$, the *exponent matrix* is $E_{W_0} = (r_{ij})$.

Proposition 1.0.5. The primary wall-crossing group is trivial if and only if W_0 is simple. Moreover, let W_0 be an elliptic singularity in rank $N = 2$ or $N = 3$ with exponent matrix E_{W_0} and marginal flat coordinate $t := t_{\mu_c}$. If $g = e^V$ is an element of the primary Landau-Ginzburg wall-crossing group then V is of the form

$$V = \sum_{i=1}^{N} t^{k_i} (x_1 \partial_1 - x_i \partial_i)$$

where each k_i must satisfy

$$k_i \left(E_{W_0}^T\right)^{-1} \mu_c \in \mathbb{Z}^N. \tag{1.0.1}$$

Equation (1.0.1) is identical to equation (15) of [MS16], although in that paper it appears in the context of Picard-Fuchs equations for elliptic periods.

In analogy to Theorem 4.26 of [GKT22b] for rank 2 Fermat polynomials, our main result is the following theorem, which, provides justification, in rank two, for our choice of definition of the wall-crossing group.

Theorem 1.0.6. For a generic rank two chain or loop polynomial, the wall-crossing group acts faithfully and transitively on open ancestor Saito potentials of balanced $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$.

Mirror Symmetry and Open FJRW Theory

In order to prove Theorem 1.0.6, we take inspiration from mirror symmetry. Mirror theorems in the Landau-Ginzburg case were first developed in [FJR13], generalised by Krawitz in [Kra10], and subsequently proven for most cases in [HLSW22]. This was based on the early physical work of Berglund, Hübsch and Henningson [BH93; BH95]. In contrast to Calabi-Yau/Calabi-Yau models or toric/Landau-Ginzburg models, we remark that in this type of mirror symmetry, there are Landau-Ginzburg models on both A and B sides.

In this thesis, we consider open Saito potentials as a sum over coefficients $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$ whose indices are balanced. We refer the reader to Definition 4.1.1 for a more thorough introduction to balancing conditions and the notation. Briefly, however, we define $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$ to be balanced if:

1. For each $i = 1, \ldots, N$, the *integral degree* condition holds,

$$q_i^T(2|I| + \sum_{j=1}^{N} k_j - 1) - 2\sum_{\mu\in I}\Theta_i^{\mu} - \sum_{j=1}^{N} k_j\Theta_i^{x_j} - \Theta_j^{\text{root}} =: -1 - e_i \in \mathbb{Z}.$$

2. For each i, we have the *grading* condition

$$e_i = k_i \mod 2.$$

3. We have the *dimension* condition

$$\sum_{i=1}^{|I|} 2d_i + \sum_{i=1}^{N} e_i = 2|I| + \sum_{j=1}^{N} k_j - 2.$$

If any of these conditions are not satisfied, then we take $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}} = 0$. These three numeric conditions seem unpleasant at a first glance. However, these are precisely the selection rules that we predict will give non-vanishing open FJRW invariants in genus zero and where e_i plays the role of the rank of some vector bundle. This is inspired by work of Buryak, Clader and Tessler [BCT22a; BCT22b] which establishes an open FJRW theory for $W_0 = x^r$ whilst the same is done in [GKT22b] for $W_0 = x_1^{r_1} + x_2^{r_2}$. In both of these cases, Gross, Kelly and Tessler have proven mirror symmetry statements. We do not attempt here to develop an open FJRW theory for more general Landau-Ginzburg models: instead we produce results on wall-crossing from the open B-model and any such open FJRW theory should match with our results.

Summary - Structure of the Thesis

Chapter 2 contains no new results and is instead a review of closed FJRW theory as geometric motivation for the thesis. The selection rules in section 2.4 will become particularly important in later chapters.

Chapter 3 is a review of Saito theory from the perspective of deformations. Using integration by parts formulas for oscillatory integrals, in the case of any ADE and simple elliptic singularity we derive expressions for primitive forms and flat coordinates, some of which are new.

Chapter 4 defines open Saito-Givental theory for any Landau-Ginzburg model, with descendants, and generalises Definition 4.6 of [GKT22b]. Inspired by the selection rules of FJRW theory, we write the open Saito-Givental potentials using balancing conditions. Using the results of Chapter 3, we then construct new examples of open primary Saito potentials in the chain cases for D_4 and E_7. We also give new expressions for an open Saito potential of the cubic in $\mathbb{P}^2$ given as $W_{E_6^{(1,1)}} = x_1^3 + x_2^3 + x_3^3 = 0$. This is done both in terms of the flat marginal coordinate t and in terms of a q-expansion, where $q = e^{2\pi i t}$ is a modular parameter. We speculate that this may be useful for an open analogue of the Landau-Ginzburg / Calabi-Yau correspondence.

Finally, we perform a perturbative calculation of flat coordinates for any Landau-Ginzburg model in two variables. This is employed in the subsequent chapter.

Chapter 5 begins by illustrating how perturbative calculations of flat coordinates leads to non-uniqueness of open Saito potentials in the chain case of $W_{E_7^{(1,1)}} = x_1^4 + x_1 x_2^3$. We then generalise this structure, concluding with the main results of the thesis: Theorems 5.3.3 and 5.3.4. These are generalisations of Theorem 4.26 in [GKT22b].

Chapter 6 speculates on definitions for an open FJRW theory such that an analogous mirror theorem holds. We prove some new results concerning concavity of certain invariants. We also review and use the theory of Coxeter and elliptic Weyl groups to predict some equivalences in open Saito theories.

Appendix A gives a review of triangle functions necessary for understanding the modularity property of the open Saito potentials in Chapter 4. Moreover, we provide details of the computation for the modular expansion of the open Saito potential in this chapter.

Appendix B contains a minor correction to the proof of Theorem 6.1 in [BB21].

CHAPTER 2

CLOSED FJRW THEORY

We briefly review an enumerative construction of the Landau-Ginzburg A-model that was first proposed by Witten [Wit93a] and fully established by Fan, Jarvis and Ruan [FJR08; FJR11b; FJR13]. Roughly speaking, this is an intersection theory on the moduli space of solutions to Witten's equation,

$$\overline{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0.$$

Here, the Landau-Ginzburg superpotential W is a quasi-homogeneous polynomial and u_i are sections of an orbifold line bundle over an orbicurve. Let us now explain these notions in turn.

2.1 Landau-Ginzburg Models

Definition 2.1.1. A Landau-Ginzburg model is a triple (X, W, G) where X is a quasi-projective variety, W is a regular function on X and G is a group which acts on X such that W is G-invariant.

We now place several restrictions on the potential W and the group G.

2.1.1 Quasi-Homogeneous Polynomials

For the quasi-projective variety, X, we take $X = \mathbb{C}^N$ and the regular function W must thus be a polynomial in N variables. We place additional constraints on W so that it has certain homogeneity properties.

Definition 2.1.2. A polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is called *quasi-homogeneous* if there exist positive integers $c_1, \ldots, c_N, d$ such that $\gcd(c_1, \ldots, c_N) = 1$ and

$$W\Big(\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N\Big) = \lambda^d W(x_1, \ldots, x_N)$$

for all $\lambda \in \mathbb{C}$. The positive integers $c_1, \ldots, c_N$ are called *weights* while d is the *degree* of W. The rational numbers $q_i = c_i/d$ are called the *charges* of W.

Remark 2.1.3. In regards to Definition 2.1.1, the reason we have chosen to specialise to $X = \mathbb{C}^N$ and W a quasi-homogeneous polynomial is natural from supersymmetric field theory. Indeed, given a supersymmetric action in two dimensions with F-terms specified by a function $W : X \to \mathbb{C}$, it can be shown that W must be holomorphic so that supersymmetry is preserved. This further implies that the manifold X must be non-compact so that W is both non-constant and holomorphic. Furthermore, choosing X to be a (non-compact) Calabi-Yau variety yields a non-anomalous field theory after quantisation. Finally, to obtain a conformal field theory at fixed points of the renormalisation group flow, a sufficient condition is that W is quasi-homogeneous. These physical considerations justify the choice $X = \mathbb{C}^N$ and W a quasi-homogeneous polynomial.

Explicitly, we may write $W : \mathbb{C}^N \to \mathbb{C}$ via

$$W(x_1, \ldots, x_N) = \sum_{i=1}^{M} \alpha_i \prod_{j=1}^{N} x_j^{r_{ij}}.$$

We define the $M \times N$ exponent matrix E_W via $(E_W)_{ij} = r_{ij}$. The condition of quasi-homogeneity then implies

$$E_W \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} d.$$

Alternatively, written in terms of charges, this becomes

$$E_W \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.1.1}$$

One may view W as a polynomial in $\mathbb{C}^N$ so that $V(W)$ is a hypersurface in the weighted projective space

$$\mathbb{P}(c_1, \ldots, c_N) = (\mathbb{C}^N \setminus \{0\})/\mathbb{C}^* \tag{2.1.2}$$

where $\mathbb{C}^* \subset \mathbb{C}^N$ acts by a scaling $(x_1, \ldots, x_N) \mapsto (\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N)$ for $\lambda \in \mathbb{C}^*$.

Definition 2.1.4. A quasi-homogeneous polynomial W with charges $q_1, \ldots, q_N$ is called *non-degenerate* if the hypersurface $V(W)$ has exactly one singularity that is required to be isolated at the origin. A non-degenerate polynomial is called *admissible* if W contains no monomials of the form $x_i x_j$. An admissible polynomial is called *invertible* if the number of variables equals the number of monomials.

Remark 2.1.5. If W is invertible, then in the above notation, we have $M = N$ so that

$$W(x_1, \ldots, x_N) = \sum_{i=1}^{N} \alpha_i \prod_{j=1}^{N} x_j^{r_{ij}}.$$

Since there are N monomials, the $\alpha_i \in \mathbb{C}$ can be absorbed into the variables $x_1, \ldots, x_N$ by a rescaling. Hence, without loss of generality, we set $\alpha_i = 1$.

Non-degeneracy of W implies that $V(W)$ is quasi-smooth in $\mathbb{P}(c_1, \ldots, c_N)$. In other words, $V(W)$ has at worst finite quotient singularities so that it is a smooth Deligne-Mumford stack. Furthermore if W is invertible, the exponent matrix E_W is square. From equation (2.1.1), the charges are uniquely determined and lie in $(0, 1/2) \cap \mathbb{Q}$ if E_W is invertible.

The classification of invertible polynomials was completed by Kreuzer-Skarke [KS92] and is described in the following theorem.

Theorem 2.1.6. An admissible polynomial W is invertible if and only if it can be written, up to a rescaling and relabelling, as a Sebastiani-Thom sum of the following Fermat, chain and loop polynomials

$$W_{\text{Fermat}} = x^r, \tag{i}$$

$$W_{\text{Chain}} = x_1^{r_1}x_2 + x_2^{r_2}x_3 + \cdots + x_N^{r_N}, \tag{ii}$$

$$W_{\text{Loop}} = x_1^{r_1}x_2 + x_2^{r_2}x_3 + \cdots + x_N^{r_N}x_1. \tag{iii}$$

For details of the Sebastiani-Thom construction, see [Dim92, Chapter 3]. As a result of this classification theorem, these three types of polynomials are referred to as *atomic type*. In this thesis, the Landau-Ginzburg potential W will always be assumed invertible.

2.1.2 Groups of Diagonal Symmetries

In FJRW theory, one also has to make a choice of symmetry group $G \leq G_W^{\max}$ for a Landau-Ginzburg model W.

Definition 2.1.7. The *multiplicative group of maximal diagonal symmetries*, $G_W^{\max}$, of a Landau-Ginzburg model W is defined as

$$G_W^{\max} := \{(\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^\times)^N \,|\, W(\lambda_1 x_1, \ldots, \lambda_n x_N) = W(x_1, \ldots, x_N)\, \forall x_1, \ldots, x_N\}.$$

Via the following proposition given in [Kra10], it is straightforward to give of a description of $G_W^{\max}$.

Proposition 2.1.8. Let $W : \mathbb{C}^N \to \mathbb{C}$ be a Landau-Ginzburg model with exponent matrix E_W. Write

$$E_W^{-1} = \Big(\rho_1\Big|\cdots\Big|\rho_N\Big), \qquad \rho_k = \begin{pmatrix} \varphi_1^{(k)} \\ \vdots \\ \varphi_N^{(k)} \end{pmatrix}.$$

Then the ρ_k generate $G_W^{\max}$ under the action

$$\rho_k \cdot x_j = e^{2\pi i \varphi_j^{(k)}} x_j. \tag{2.1.3}$$

As a consequence of this proposition and the Kreuzer-Skarke classification, to characterise $G_W^{\max}$ one needs only calculate E_W^{-1} in the Fermat, chain and loop case. This was done by Kreuzer [Kre94] and Artebani, Boissière and Sabani [ABS14, Proposition 2].

Proposition 2.1.9. Let W be a Sebastiani-Thom sum of Fermat, chain and loop polynomials, $W = \sum_{k=1}^n W_k$. Then $G_W^{\max}$ is the direct product of the groups $G_{W_k}^{\max}$. Furthermore, if W is atomic type, $G_W^{\max}$ is cyclic of order $|\det E_W|$ and has the following generators.

1. If W is a Fermat polynomial, $W = x^r$, then $G_W^{\max}$ is generated by $e^{2\pi i/r}$.

2. For $W = x_1^{r_1} + \cdots x_{N-1}x_N^{r_N}$ a chain polynomial, E_W^{-1} is the lower triangular matrix given by

$$(E_W^{-1})_{ij} = (-1)^{i+j} \prod_{j \le l \le i} \frac{1}{r_l}$$

   for $j \le i$ and is zero otherwise. Thus $G_W^{\max}$ is generated by $\rho_1 = (\varphi_1, \ldots, \varphi_N) \in (\mathbb{C}^*)^N$ where

$$\varphi_j = \exp\Big(2\pi i \cdot (-1)^{N+j} \prod_{l=1}^{N} \frac{1}{r_l}\Big).$$

3. For $W = x_1^{r_1}x_2 + \cdots + x_{N-1}x_N^{r_N}$ a loop polynomial with

$$D = \det E_W = \prod_{i=1}^{N} r_i + (-1)^{N+1}$$

then E_W^{-1} is given by

$$(E_W^{-1})_{ij} = \begin{cases} \frac{(-1)^{N+j-i}}{D} \prod_{k=j+1}^{i-1} r_k, & i > j \\ \frac{(-1)^{j-i}}{D} \prod_{k=j+1}^{N} r_k \cdot \prod_{l=1}^{i-1} r_l, & i \leq j. \end{cases}$$

Here, we interpret the empty product as having a value of 1. Thus, $G_W^{\max}$ is generated by $\rho_1 = (\varphi_1, \ldots, \varphi_N) \in (\mathbb{C}^*)^N$ where

$$\varphi_1 = \exp\left(2\pi i \frac{(-1)^N}{D}\right), \text{ and } \varphi_i = \exp\left(2\pi i \frac{(-1)^{N+1-i}}{D} \cdot \prod_{l=1}^{i-1} r_l\right), \; i \geq 2.$$

This can be proven by Gaussian elimination with the inverse exponent matrix of chain polynomial particularly simple to calculate since the matrices are lower triangular in this case.

Admissibility. In the data of a Landau-Ginzburg model, $(\mathbb{C}^N, W, G)$, we consider choices of $G \leq G_W^{\max}$. Suppose W has weights $c_1, \ldots, c_n$. In this case, we have the quasi-smooth hypersurface

$$V(W) \subseteq \mathbb{P}(c_1, \ldots, c_N) = \frac{\mathbb{C}^N \setminus \{0\}}{\mathbb{C}^*} \tag{2.1.4}$$

where the $\mathbb{C}^*$ action is given by $\lambda \cdot (x_1, \ldots, x_N) = (\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N)$. For some choice of $G \leq G_W^{\max}$, one is often interested in the orbifold

$$V(W)/G \subseteq \mathbb{P}(c_1, \ldots, c_N)/G.$$

However, we see from the right hand side that there are constraints on the choice of G for this construction to make sense. In particular, we define

$$J := G_W^{\max} \cap \mathbb{C}^* = \{(\lambda^{c_1}, \ldots, \lambda^{c_N}) \in (\mathbb{C}^*)^N \mid W(\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N) = W(x_1, \ldots, x_N)\}$$

where the $\mathbb{C}^*$ is the same as that given in (2.1.4). Thus, in order to define FJRW theory, we require that the choice of subgroup G must contain J.

Definition 2.1.10. A subgroup $G \leq G_W^{\max}$ is called *admissible* if $J \leq G$.

Remark 2.1.11. In fact one can show that $J = \langle (e^{2\pi i q_1}, \dots, e^{2\pi i q_N}) \rangle$. Indeed, it is straightforward to verify that $(e^{2\pi i q_1}, \dots, e^{2\pi i q_N}) \in J$ while the converse inclusion is the content of Proposition 2.1.8.

2.2 W-Spin Curves

2.2.1 Recollections on Orbifold Structures.

In Satake's original definition in [Sat57], an orbifold structure on a topological space X gives an open cover $\mathscr{U}$ of X such that for each $U \in \mathscr{U}$ there is a smooth manifold V, a smooth finite group G that acts on V, and a continuous map $\pi : V \to U$ which factors through the quotient to induce a homeomorphism $U \cong V/G$. The collection (V, G, π) is called a *uniformising system* of U. The uniformising system must be compatible with the orbifold structure in the following sense.

1. If $U' \subset U$ are both elements in the open cover $\mathscr{U}$ of the orbifold structure on X, then there exists an injection $(V', G', \pi') \to (V, G, \pi)$ of uniformising systems which we define in a moment.

2. For any $p \in U_1 \cap U_2$ with $U_1, U_2 \in \mathscr{U}$, there exists $U_3 \in \mathscr{U}$ such that $p \in U_3 \subset U_1 \cap U_2$.

The definition of an injection of uniformising systems is given as follows. Let $U' \subset U$ be open subsets of X with uniformising systems (V', G', π') and (V, G, π) respectively. Then the embedding $i : U' \to U$ is said to induce an injection $(V', G', \pi') \to (V, G, \pi)$ of uniformising systems if there exists an injective $\tau : G' \to G$ that is an isomorphism on the kernel of the action, and there exists a τ-equivariant open embedding $\psi : V' \to V$ such that $i \circ \pi' = \pi \circ \psi$.

An orbifold bundle may be defined using a similar notion of uniformising system for bundles. However, much like ordinary vector bundles, it is often more convenient to describe them using transition functions. Indeed, let $\mathscr{U}$ be an open cover of X and let $U' \subset U$ with $U', U \in \mathscr{U}$. Given the embedding $i : U' \to U$ and any injection $(V', G', \pi') \to (V, G, \pi)$, a transition function for an orbifold bundle of rank k is a map $g_i : V \to \mathrm{Aut}(\mathbb{C}^k)$ such that $V' \times \mathbb{C}^k \to V \times \mathbb{C}^k$ given by $(x, v) \mapsto (i(x), g_i(x)v)$ is an open embedding and for any composition of injections $j \circ i$ we have

$$g_{j \circ i}(x) = g_j(i(x)) \circ g_i(x). \tag{2.2.1}$$

The data of the maps g_i can be used to reconstruct an orbifold bundle $\mathrm{pr} : E \to X$. Isomorphisms of orbifold bundles may also be given in terms of transition functions: two collections of transition functions $g^{(1)}$ and $g^{(2)}$ define isomorphic bundles if for any injection $i : (V', G', \pi') \to (V, G, \pi)$ there exist morphisms $\delta_V : V \to \mathrm{Aut}(\mathbb{C}^k)$ such that

$$g_i^{(2)}(x) = \delta_V(i(x)) \circ g_i^{(1)}(x) \circ (\delta_V(x))^{-1}.$$

A section of $\mathrm{pr} : E \to X$ is any continuous map s such that, for (V, G, π) a uniformising system, s is given locally by a G-equivariant function $s_V : V \to V \times \mathbb{C}^k$ and such that $\mathrm{pr} \circ s_V = \mathrm{id}_V$.

2.2.2 Marked Orbicurves and Line Bundles.

We now specialise to the case where $X = C$ is a one dimensional orbifold and $E = S$ is an orbifold line bundle. In FJRW theory and in [CR04], it is useful to define orbifold structures on C on the level of germs at points rather than open covers. In this case, for each $p \in C$ there is a uniformising system (V_p, G_p, π_p) with some compatibility conditions.

Definition 2.2.1. An *orbicurve* $(C, p_1, \ldots, p_n)$ is a Riemann surface C with an orbifold structure at each marked point p_i. More precisely, at each marked point there is a group G_{p_i} and a canonical isomorphism $G_{p_i} \cong \mathbb{Z}_{m_i}$.

Definition 2.2.2. An orbicurve is called *smooth* if the underlying coarse curve $|C|$ is smooth, where $|C|$ is obtained by forgetting the orbifold structure. We denote this forgetful morphism by $\varrho : C \to |C|$. Similarly, C is called *nodal* if $|C|$ is nodal.

Explicitly, the action of $\mathbb{Z}_m$ on an orbifold line bundle S is constructed as follows. At non-orbifold points $p \in C$ where G_p is trivial, there is nothing to be done. At each orbifolded point $p \in C$, suppose $\Delta \times \mathbb{C}$ is a chart of S with local coordinates (z, s). Then $1 \in G_p \cong \mathbb{Z}_m$ acts via

$$(z, s) \mapsto \left(e^{\frac{2\pi i}{m}} z, e^{\frac{2\pi i v}{m}} s\right) \tag{2.2.2}$$

with $e^{\frac{2\pi i v}{m}} \in U(1)$ and $v \in \{0, 1, \ldots, m-1\}$. In this way, at each orbifold point there is an induced representation $G_p \to \mathrm{Aut}(S) \cong U(1)$.

An important example of an orbifold line bundle is given in the following definition.

Definition 2.2.3. Let $\omega_{|C|}$ be the canonical bundle of $|C|$. The *log canonical bundle of* $|C|$ is defined as

$$\omega_{|C|,\log} := \omega_{|C|} \otimes \mathcal{O}(p_1) \otimes \cdots \otimes \mathcal{O}(p_n)$$

where $\mathcal{O}(p_i)$ is the sheaf of functions which may have simple pole at p_i. The *log canonical bundle of* C is defined as $\omega_{C,\log} := \varrho^* \omega_{|C|,\log}$.

As described in [FJR13], for smooth orbicurves C one may pushforward an orbifold line bundle S to the underlying curve $|C|$ by virtue of the fact that the sheaf of locally invariant sections of S is locally free and of rank one if C is smooth. Thus, we take the double dual of this sheaf and denote the resulting line bundle over $|C|$ as $|S|$. If C is nodal, one may consider $|S|$ as a line bundle on the normalisation of C. Furthermore, a cohomology theory of orbifolds is established in [CR04] and the following result is shown in Section 2.1.2 of [FJR13].

Lemma 2.2.4. The forgetful morphism $\varrho : C \to |C|$ induces an isomorphism on cohomologies,

$$H^i(C, S) \xrightarrow{\sim} H^i(|C|, |S|) \tag{2.2.3}$$

for $i = 0, 1$.

Proof. We first define a map $H^0(|C|, |S|) \to H^0(C, S)$. Let s be a holomorphic section of $|S|$ that has local representative $f(u)$. We thus have that $\left(z, z^v f(z^m)\right)$ is a local representative of a holomorphic section $\varrho^*(s)$ of S. At non-orbifolded points, $\varrho^*(s)$ and s can be identified via Equation (2.2.2). This process can be reversed so that a section of $|S|$ can be constructed from a section of S.

Similarly, we define a map $H^1(|C|, |S|) \to H^1(C, S)$. To do this, we first define the map of $(0, 1)$ forms $\Omega^{0,1}(|S|) \to \Omega^{0,1}(S)$ in the same way as above. Explicitly, if $f(u)d\overline{u}$ is a local section of $\Omega^{0,1}(|S|)$, then $z^v f(z^m) m\overline{z}^{m-1} d\overline{z}$ is a local section of $\Omega^{0,1}(S)$. By reversing this process and using the Dolbeault isomorphism, we have the desired isomorphism $H^1(|C|, |S|) \to H^1(C, S)$. □

Definition 2.2.5. Let $W = \sum_{i=1}^N \prod_{j=1}^N x_j^{r_{ij}}$ be an invertible polynomial. A *W-spin curve* is an orbicurve $(C, p_1, \ldots, p_n)$ together with orbifold line bundles $S_1, \ldots, S_N$, called *spin bundles*, and isomorphisms

$$\varphi_i : \bigotimes_{j=1}^N S_j^{r_{ij}} \xrightarrow{\sim} \omega_{C,\log}.$$

Furthermore, the S_j induce a representation $\Pi_{p_i} : G_{p_i} \to U(1)^N$ which we require to be faithful.

We shall explain the requirement of a faithful representation in a moment.

Definition 2.2.6. For each orbifold marked point, p_i, we denote

$$\gamma_i = \Pi_{p_i}(1) = \left(e^{2\pi i \Theta_1^{\gamma_i}}, \ldots, e^{2\pi i \Theta_N^{\gamma_i}}\right)$$

Such group elements are called *decorations at p_i* and the associated $\Theta^{\gamma_i} = \left(\Theta_1^{\gamma_i}, \ldots, \Theta_N^{\gamma_i}\right)$ are called *phases.*

In a moment we will give an example of the phases Θ^γ for the potential $W = x^r$. For more general potentials, we calculate the phases using mirror symmetry in Chapter 4. Furthermore, the phases may also be thought of as twists in the sense of the following result [FJR13, Proposition 2.1.24].

Proposition 2.2.7. Let W be a quasi-homogeneous singularity with matrix of exponents $E_W = (r_{ij})$. Given a W-spin curve on a smooth orbicurve $(C, p_1, \ldots, p_k)$, there are induced isomorphisms

$$\bigotimes_{j=1}^{N} |S_j|^{r_{ij}} \cong \omega_{|C|,\log} \otimes \mathcal{O}\Big(-\sum_{l=1}^{k}\sum_{j=1}^{N} r_{ij}\Theta_j^{\gamma_l} p_l\Big) \cong \omega_{|C|} \otimes \mathcal{O}\Big(-\sum_{l=1}^{k}\sum_{j=1}^{N}(r_{ij}\Theta_j^{\gamma_l} - 1)p_l\Big)$$

for each $i = 1, \ldots, N$.

Example 2.2.8. We will illustrate some of the ideas using the potential $W = x^r$. This ultimately reproduces r-spin theory of Jarvis, Kimura and Vaintrob [JKV01]. An x^r-structure is a choice of orbicurve $(C, p_1, \ldots, p_k)$ and isomorphism $S^{\otimes r} \to \omega_{C,\log}$. Furthermore, we have

$$\omega_{C,\log} = \omega_{|C|,\log}.$$

This is because given an orbifold point p with local group $G_p \cong \mathbb{Z}_m$ and local coordinate z, the sheaf $\omega_{C,\log}$ is generated by $\frac{dz}{z}$, while $\omega_{|C|,\log}$ is generated by $\frac{dx}{x}$ with $x = z^m$. However, we note that $\omega_{C,\log}$

$$\frac{dx}{x} = m\frac{dz}{z}$$

and that this is invariant under the $\mathbb{Z}_m$ action given in (2.2.2). Now since $\omega_{C,\log} \cong S^{\otimes r}$, from the second coordinate in the action (2.2.2) we have must have $\frac{v}{m} = \frac{l}{r}$ for some $l \in \{0, 1, \ldots, r-1\}$. Thus, we have that the phases for γ_l are given by

$$\Theta^{\gamma_l} = \frac{l}{r}.$$

Thus Proposition (2.2.7) gives an isomorphism

$$|S|^r \cong \omega_{|C|,\log} \otimes \mathcal{O}\Big(-\sum_{l=1}^{k} r\Theta^{\gamma_l} p_l \Big). \tag{2.2.4}$$

This isomorphism is the required data in the definition of r-spin curves.

Moreover, the following lemma is proven in Lemma 2.1.18 of [FJR13].

Lemma 2.2.9. Each decoration γ is an element of $G_W^{\max}$.

Proof. Recall that we have isomorphisms $\varphi_i : S_1^{r_i,1} \otimes \cdots \otimes S_N^{r_i,N} \to \omega_{C,\log}$. We have seen in the previous example the local groups act trivially on $\omega_{C,\log}$. However, $\gamma \in G_p$ acts on $S_1^{r_i,1} \otimes \cdots \otimes S_N^{r_i,N}$ as $\exp\Big(2\pi i \sum_j r_{ij}\Theta_j^{\gamma}\Big)$. Since the representation is faithful, we must have $\sum_j r_{ij}\Theta_j^{\gamma} \in \mathbb{Z}$. For $W = \sum_{i=1}^N \prod_{j=1}^N x_j^{r_{ij}}$, we thus have that $\gamma \in G_W^{\max}$. □

2.3 Stacks of Stable W-Spin Curves

To define a moduli space of W-spin curves, we need an appropriate notion of morphism of such curves.

Definition 2.3.1. Given two W-spin curve structures, $(C, p_1, \ldots, p_n, \{S_j\}, \{\varphi_j\})$ and $(C, p_1, \ldots, p_n, \{S_j'\}, \{\varphi_j'\})$, we note that morphisms $\xi_j : S_j \to S_j'$ induce morphisms

$$\Xi_i : \bigotimes_{j=1}^{N} S_j^{r_{ij}} \to \bigotimes_{j=1}^{N} (S_j')^{r_{ij}}$$

for each i since equation (2.2.1) behaves naturally under tensor products. An *isomorphism* of W-spin curves is a collection of orbifold bundle isomorphisms $\xi_i : S_j \to S_j'$ such that $\varphi_i = \varphi_i' \circ \Xi_i$.

Roughly speaking, to obtain a compact moduli space, one needs as usual to add stable orbicurves to the stack of smooth curves.

Definition 2.3.2. An orbicurve is called *stable* if the underlying coarse curve is stable.

Definition 2.3.3. A *genus g, n-pointed stable W-spin orbicurve over a base T* is a flat family of genus g, n-pointed stable orbicurves $C \to T$ with markings $\mathscr{P}_i \subset C$ and sections $\sigma_i : T \to \mathscr{P}_i$ together with orbifold line bundles $\{S_j\}$ on C and isomorphisms $\varphi_i : \bigotimes_{j=1}^N S_j^{r_{ij}} \xrightarrow{\sim} \omega_{C/T,\log}$ where $\omega_{C/T,\log}$ is the relative log-canonical bundle. We require that $(\{S_j\}, \{\varphi_j\})$ induce a W-spin structure on every fibre C_t.

Definition 2.3.4. We denote the stack of stable W-spin orbicurves by $\overline{\mathcal{M}}_{g,k}^W$.

Remark 2.3.5. There is an isomorphism $\overline{\mathcal{M}}_{g,k}^W \cong \overline{\mathcal{M}}_{g,k}^{1/r}$ for $W = x^r$ where $\overline{\mathcal{M}}_{g,k}^{1/r}$ is the stack of r-spin curves. This is shown in section 2.4 of [FJR11a] as a consequence of Proposition 2.2.7 applied on the smooth locus. The isomorphism also holds for nodal curves but the details are more tedious. For details, see section 4 of [AJ03].

A crucial component of the enumerative theory is the natural forgetful morphism $\mathrm{st} : \overline{\mathcal{M}}_{g,k}^W \to \overline{\mathcal{M}}_{g,k}$ defined by forgetting the spin bundles and orbifold structure. For $g = 0$, the map st is a bijection of sets, but not of stacks since W-spin have extra automorphisms. However, Fan, Jarvis and Ruan prove the following theorem.

Theorem 2.3.6. The morphism st is flat, proper and quasi-finite.

This ensures that $\overline{\mathcal{M}}_{g,k}^W$ is smooth and compact as an orbifold or a Deligne-Mumford stack, and that the dimensions of $\overline{\mathcal{M}}_{g,k}^W$ and $\overline{\mathcal{M}}_{g,k}$ coincide. The above theorem also ensures that the moduli space $\overline{\mathcal{M}}_{g,k}^W$ has a universal curve $C_{g,k} \to \overline{\mathcal{M}}_{g,k}^W$ that can be constructed from the universal curve of $\overline{\mathcal{M}}_{g,k}$. See Theorem 2.2.6 of [FJR13].

The orbifold structure along with each decoration γ_i is locally constant and is thus constant for each component of $\overline{\mathcal{M}}_{g,k}^W$. Thus, for $\gamma_1, \ldots, \gamma_k \in G_W^{\max}$ there is a decomposition into open and closed substacks,

$$\overline{\mathcal{M}}_{g,k}^W = \bigoplus_{\gamma_1,\ldots,\gamma_k \in G_W^{\max}} \overline{\mathcal{M}}_{g,k}^W(\gamma_1, \ldots, \gamma_k)$$

in which the decoration of the marked point p_i is γ_i.

Proposition 2.3.7. Fix an invertible polynomial W with charges $q_1, \dots, q_N$. Then $\overline{\mathcal{M}}^W_{g,k}(\gamma_1, \dots, \gamma_k)$ is non-empty if and only if

$$q_j(2g-2+k) - \sum_{l=1}^{k} \Theta_j^{\gamma_l} \in \mathbb{Z}$$

for each $j = 1, \dots, N$ and where $\gamma_1, \dots, \gamma_k \in G_W^{\max}$ with $\gamma_l = \left(e^{2\pi i \Theta_1^{\gamma_l}}, \dots, e^{2\pi i \Theta_N^{\gamma_l}}\right)$

The key idea of the proof here is that given the line bundles S_i of a W-spin curve, due to the isomorphism in the data of a W-spin curve, we have

$$\sum_{j=1}^{N} r_{ij} \deg |S_j| = q_j(2g-2+k) - \sum_{j=1}^{N}\sum_{l=1}^{k} r_{ij}\Theta_j^{\gamma_l}.$$

Since q_j satisfy $\sum_{j=1}^N r_{ij} q_j = 1$, we have

$$\deg |S_j| = q_j(2g-2+k) - \sum_{l=1}^{k} r_{ij}\Theta_j^{\gamma_l}.$$

As a degree of a line bundle, this must be an integer. The converse is slightly more involved and we refer the reader to [FJR13].

Example 2.3.8. Let us briefly illustrate these ideas in the r-spin case. To calculate the degree of the bundle $|S|$, we use the isomorphism of (2.2.4). Indeed equating degrees of both sides of this isomorphism, we have

$$r \cdot \deg |S| = \deg \omega_{|C|,\log} - r \sum_{l=1}^{k} \Theta^{\gamma_l}.$$

By a calculation using the Riemann-Roch theorem, we find

$$\deg \omega_{|C|,\log} = 2g - 2 + k.$$

Thus, we indeed find that

$$\deg|S| = \frac{1}{r}(2g-2+k) - \sum_{l=1}^{k} \Theta^{\gamma_l}$$

which, as a degree of a line bundle, must be an integer.

2.4 FJRW Cohomological Field Theory

2.4.1 The Chiral Ring

The input data for FJRW theory is a quasi-homogeneous, invertible and non-degenerate W and is defined for a choice of subgroup $G \leq G_W^{\max}$. In summary, the most general A-model state space is given by the following.

Definition 2.4.1. Let G be an admissible subgroup of $G_W^{\max}$. Define the *fixed locus* of $\gamma \in G$ by

$$\mathrm{Fix}(\gamma) := \{(a_1,\ldots,a_N) \in \mathbb{C}^N \mid \gamma(a_1,\ldots,a_N) = (a_1,\ldots,a_N)\}.$$

with dimension $N_\gamma := \dim_{\mathbb{C}} \mathrm{Fix}(\gamma)$. The *A-model state space* is defined as

$$\mathscr{H}_{W,G} := \bigoplus_{\gamma\in G} \Big(H^{N_\gamma}\Big(\mathrm{Fix}(\gamma), \Big(W|_{\mathrm{Fix}(\gamma)}\Big)^{\infty}; \mathbb{C}\Big)\Big)^G$$

where, for $R \gg 0$, we have defined

$$\Big(W|_{\mathrm{Fix}(g)}\Big)^{\infty} := \Big(\mathrm{Re}\Big(W|_{\mathrm{Fix}(g)}\Big)\Big)^{-1}(R,\infty).$$

In general, FJRW theory can be defined for any admissible subgroup $G \leq G_W^{\max}$. However, it is not currently known how to define a corresponding B-model if $G \neq G_W^{\max}$. Hence, in the following discussion, the choice of admissible subgroup will be $G_W^{\max}$.

In [CR04], an orbifold analogue of Poincaré duality is proven. Hence,

$$\mathscr{H}_{W,G_W^{\max}} \cong \bigoplus_{\gamma\in G_W^{\max}} \left(H_{N_\gamma}\left(\mathrm{Fix}(\gamma), \left(W|_{\mathrm{Fix}(\gamma)}\right)^{\infty}; \mathbb{C}\right)\right)^{G_W^{\max}}.$$

Thus, we will write the state space as

$$\mathscr{H}_{W,G_W^{\max}} = \bigoplus_{\gamma\in G_W^{\max}} \mathscr{H}_\gamma$$

with

$$\mathscr{H}_\gamma = \left(H_{N_\gamma}\left(\mathrm{Fix}(\gamma), \left(W|_{\mathrm{Fix}(\gamma)}\right)^{\infty}; \mathbb{C}\right)\right)^{G_W^{\max}}.$$

2.4.2 A Glimpse of Intersection Theory

In [FJR13], a virtual fundamental cycle is constructed

$$[\overline{\mathcal{M}}_{g,k}^{W}]^{\mathrm{vir}} \in H_*(\overline{\mathcal{M}}_{g,k}^{W}, \mathbb{C}) \otimes \prod_{i=1}^{k} \mathscr{H}_{\gamma_i}.$$

The following theorem is then proven.

Theorem 2.4.2. Let $\mathbb{L}_i$ denote the *tautological line bundles* over $\overline{\mathcal{M}}_{g,k}$ whose fibre at each point $(C, p_1, \ldots, p_k) \in \overline{\mathcal{M}}_{g,k}$ is the cotangent space $T^*_{p_i}C$. Denote $\psi_i = c_1(\mathbb{L}_i)$. Define $\Lambda^W_{g,k,\boldsymbol{d}} : \mathscr{H}^{\otimes k}_{W,G_W^{\max}} \to H^*(\overline{\mathcal{M}}_{g,k})$ by

$$\Lambda^W_{g,k,\boldsymbol{d}}(\alpha_1, \ldots, \alpha_k) = \frac{|G_W^{\max}|^g}{\deg(\mathrm{st})} \mathrm{PD}\ \mathrm{st}_*\left([\overline{\mathcal{M}}_{g,k}^{W}]^{\mathrm{vir}} \cap \prod_{i=1}^{k} \alpha_i\right) \cup \prod_{i=1}^{k} \psi_i^{d_i}$$

with $\alpha_i \in \mathscr{H}_{\gamma_i}$ and PD the Poincaré dual. Then the maps $\Lambda^W_{g,k,\boldsymbol{d}}$ define a cohomological field theory in the sense of Kontsevich-Manin [KM94].

As usual with enumerative theories, the technical details required to construct the virtual fundamental cycle make computations difficult. Fortunately, there exists a drastic simplification, under some fairly mild assumptions, which we presently explain.

Definition 2.4.3. Sectors of the state space where $\mathrm{Fix}(\gamma) = \{0\}$ are called *narrow*. Otherwise the sector is called *broad*.

Example 2.4.4. For the potential $W = x^r$ we have $G_W^{\max} = \mu_r$, the r^{th} roots of unity. We have already calculated the phases

$$\Theta^{\gamma_l} = \frac{l}{r}$$

for $l = 0, 1, \dots, r-1$. Hence $\gamma_l = e^{2\pi i \Theta^{\gamma_l}} \neq 1$ for $l \neq 0$. Hence, all insertions in r-spin theory with $l \neq 0$ are narrow.

The property of being narrow is important for the following reason in genus zero.

Theorem 2.4.5. Suppose each γ_i is narrow and let $(S_1, \dots, S_N)$ be the universal line bundles of the universal W-spin genus zero curve defined over the universal curve $\pi : C_{0,k} \to \overline{\mathcal{M}}_{0,k}^W$. Suppose $\pi_*\Big(\bigoplus_{i=1}^N S_i\Big) = 0$. Then

$$[\overline{\mathcal{M}}_{0,k}^W]^{\mathrm{vir}} = e\Big(\bigoplus_{i=1}^N (R^1\pi_* S_i)^\vee\Big) \cap [\overline{\mathcal{M}}_{0,k}^W]$$

where e represents the Euler class.

For convenience, we define the *Witten bundles* $\mathcal{W}_i := (R^1\pi_* S_i)^\vee$ so that the *total Witten bundle* is defined as

$$\mathcal{W} = \bigoplus_{i=1}^N \mathcal{W}_i.$$

Denote $\mathcal{W}(\gamma)$ to be the restriction of $\mathcal{W}$ to $\overline{\mathcal{M}}_{0,k}^W(\gamma_1, \dots, \gamma_k)$. Thus, given the hypotheses of Theorem 2.4.5, we may think of this intersection theory as an integral:

$$\langle \tau^{\gamma_1} \cdots \tau^{\gamma_k} \rangle := \int_{[\overline{\mathcal{M}}_{0,k}^W]} e\Big(\mathcal{W}(\gamma)\Big) = \int_{[\overline{\mathcal{M}}_{0,k}]} \mathrm{st}^* e\Big(\mathcal{W}(\gamma)\Big). \tag{2.4.1}$$

Remark 2.4.6. We may also restrict the virtual fundamental class to its components $[\overline{\mathcal{M}}_{0,k}^W(\gamma_1, \dots, \gamma_k)]^{\mathrm{vir}}$ so that the genus zero intersection numbers can be written more compactly,

$$\langle \tau^{\gamma_1} \cdots \tau^{\gamma_k} \rangle = \int_{[\overline{\mathcal{M}}^W_{0,k}(\gamma_1,\ldots,\gamma_k)]^{\text{vir}}} 1.$$

We have specialised to genus zero so that the Witten bundle is indeed a vector bundle with fibers

$$\bigoplus_{i=1}^{N} H^1(C_{0,k}, S)^\vee \cong \bigoplus_{i=1}^{N} H^1(|C_{0,k}|, |S|)^\vee$$

where we have used the isomorphism in equation (2.2.3) induced by the forgetful morphism $\varrho : C_{0,k} \to |C_{0,k}|$.

From dimension considerations in equation (2.4.1), we match the degree of $e(\mathcal{W})$ with the real dimension of $\overline{\mathcal{M}}^W_{0,k}$. We note that, as the top Chern class, the degree of $e(\mathcal{W})$ is $2 \text{ rk } \mathcal{W}$. Consequently, to define a non-vanishing FJRW invariant, we must have that

$$\text{rk } \mathcal{W} = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,k}.$$

This is an example of what we will call a *selection rule* later in the thesis.

CHAPTER 3

CLOSED SAITO-GIVENTAL THEORY AND LANDAU-GINZBURG MIRROR SYMMETRY

Saito theory was originally developed in the works of Saito, Sekiguchi and Yano [SSY80; Sai93]. These authors considered Frobenius structures of orbit spaces of finite Coxeter groups. Equivalently, one may consider Saito theory as endowing a singularity W with a Frobenius structure by deforming W. This is the approach that is useful from the perspective of mirror symmetry in this thesis. Indeed, much like the B-model that computes Gromov-Witten invariants of Fano varieties as summarised in [Gro11], Saito theory for quasi-homogeneous polynomials W is described by the following period integrals

$$\int_{\Xi} e^{W_{\boldsymbol{s}}/\hbar}\zeta.$$

Here, Ξ is a standard good basis element of the Landau-Ginzburg B-model state space, $W_{\boldsymbol{s}}$ is versal deformation of W_0 and ζ is a primitive form. We shall describe each of these notions in turn using Saito's original work on higher residues, after which we shall explain the modern interpretation using oscillatory integrals. Finally, we review the mirror symmetry construction of Berglund, Hübsch and Krawitz that interprets Saito theory as B-model that is mirror to FJRW theory.

3.1 Saito Theory Via Deformations

In this section, we give an exposition of Saito theory as established in [Sai81; Sai83a; Sai83b]. However, we use the modern formulation in terms of Frobenius manifolds and flat coordinates. For a more detailed exposition, we refer the reader to [Her02] and [Sab07].

3.1.1 Simple and Elliptic Singularities

We start by describing the most basic types of quasi-homogeneous singularity from Chapter 2. These are the simple singularities

$$\begin{aligned}
W_{A_r} &= x_1^{r+1}, \quad r \geq 1, \\
W_{D_r} &= x_1^{r-1} + x_1 x_2^2, \quad r \geq 4, \\
W_{E_6} &= x_1^3 + x_2^4, \\
W_{E_7} &= x_1^3 + x_1 x_2^3, \\
W_{E_8} &= x_1^3 + x_2^5,
\end{aligned}$$

and simple elliptic singularities

$$\begin{aligned}
W_{E_6^{(1,1)}} &= x_1^3 + x_2^3 + x_3^3, \\
W_{E_7^{(1,1)}} &= x_1^4 + x_2^4, \\
W_{E_8^{(1,1)}} &= x_1^6 + x_2^3.
\end{aligned}$$

We note that Theorem 2.1.5 implies that all these polynomials are invertible. One characterisation of simple and elliptic singularities is in terms of the central charge,

$$c_W := \sum_{i=1}^{N} (1 - 2q_i)$$

of a quasi-homogeneous polynomial W with $q_i = \deg x_i$. Simple singularities are characterised by $0 < c < 1$ while elliptic singularities are characterised by $c = 1$. We note, however, that this labelling of quasi-homogeneous polynomials by Dynkin diagrams is not one-to-one. We refer to the full list in Part II of [AGV12]. See also Table 1 of [MS16] for elliptic singularities. We illustrate this by briefly explaining a link between singularity theory and Lie algebras.

Blow Ups and Du Val Singularities.

Via blow ups, we first resolve the singular variety X defined by the vanishing of the du Val singularity W. In this way, we obtain a non-singular variety $\tilde{X}$. Subsequently, we consider an intersection graph of the corresponding exceptional divisors. It is instructive to do this for singularities of type A_r.

Example 3.1.1. Consider the A_1 singularity

$$W = z^2 + x^2 + y^2$$

with $X := V(W)$. Here, we have used a Morse stabilisation. We shall explain and justify this later. We now embed $X \subset \mathbb{C}^3$ so that the blow up is also embedded,

$$\tilde{X} \subset \mathrm{Bl}_0\mathbb{C}^3 = \{(x, y, z), [s : t : u] \in \mathbb{C}^3 \times \mathbb{P}^2 \,|\, xt = sy,\, xu = sz,\, tz = uy\}.$$

The blow up $\mathrm{Bl}_0\mathbb{C}^3$ comes with a projection map $\pi : \mathrm{Bl}_0\mathbb{C}^3 \to \mathbb{C}^3$. Recall that the proper transform, Y, of X is defined to be the Zariski closure of $\pi^{-1}(X \setminus \{0\})$ in $\mathrm{Bl}_0\mathbb{C}^3$. Now, Y is covered by the three standard open affines sets, with $s = 1$, $t = 1$ and $u = 1$ respectively. For the open affine where $u = 1$, we find $x = sz$ and $y = tz$ in $\mathrm{Bl}_0\mathbb{C}^3$. In these coordinates the A_1 singularity reads

$$z^2(s^2 + t^2 + 1) = 0.$$

There are now two irreducible components defined by $z = 0$ and $s^2 + t^2 + 1 = 0$. We recall the exceptional divisor is defined by $E := Y \cap \pi^{-1}(\{0\})$. The component $z = 0$ corresponds to $\pi^{-1}(0)$, while on the other hand, the proper transform Y is given by $s^2 + t^2 + 1 = 0$. For the A_1 singularity, E is a smooth curve in $\mathbb{P}^2$ defined by $s^2 + t^2 + u^2 = 0$. This is verified by considering the other affine opens where $t = 1$ and $s = 1$. Observe that in this case E is isomorphic to $\mathbb{P}^1$. Indeed, via the genus-degree formula, we find that the genus of E is zero. As a graph, we represent the A_1 singularity by a single vertex.

Consider now the A_2 singularity, $W = z^3 + x^2 + y^2$. With the same notation as before, we find

$$z^2(s^2 + t^2 + z) = 0$$

for the open affine where $u = 1$. To find the exceptional divisor E, we set $z = 0$ which implies $x = y = 0$. In this chart, E is therefore given by the line $L_a = \{(0, 0, 0), [a, \pm ia, 1] : a \in \mathbb{C}\}$. In the chart where $s = 1$ we have

$$x^2(1 + t^2 + xu^3) = 0.$$

Similar to the above, E in this chart is given by the line $L_b = \{(0, 0, 0), [1, \pm i, b] | b \in \mathbb{C}\}$. This is the same line as L_a viewed in this chart. There is no need to consider the chart where $t = 1$; by symmetry this will give the same result as $s = 1$. Considering the limit $b \to \infty$ or equivalently $b^{-1} \to 0$, we find that L_b becomes the point $\{(0, 0, 0), [0, 0, 1]|\}$. This is the same as taking $a = 0$ in L_a. Hence, we find that E is two projective lines intersecting at $\{(0, 0, 0), [0, 0, 1]|\}$. It is readily verified that this point is a non-singular point of the proper transform. Hence, we have found a resolution of the A_2 singularity. The intersection graph is then two vertices joined by a single edge.

For the A_r singularity, with $r > 2$, the intersection point $\{(0, 0, 0), [0, 0, 1]|\}$ is in fact a singular point of type A_{r-2}. One may blow up again and repeat the above procedure inductively to find the corresponding Dynkin diagram for A_r.

Example 3.1.2. We first consider the singularity $W = x^2 + y^3 + z^3$. In the same notation as above, in the chart where $u = 1$ we find the equations

$$\begin{aligned} x^2 + y^3 + z^3 &= 0 \\ xt &= sy \\ x &= sz \\ tz &= y \end{aligned}$$

which we solve to find $z^2(s^2 + t^3 z + z) = 0$. The proper transform is thus the singular surface

$$s^2 + z(t^3 + 1) = 0.$$

The exceptional divisor occurs when $z = 0$ which implies $s = 0$. This gives a copy of $\mathbb{P}^1$ which we call E_0. In the other two charts where $s = 1$ and where $t = 1$ respectively, we obtain

$$1 + x(u^3 + t^3) = 0, \quad s^2 + y(u^3 + 1) = 0.$$

Thus, we observe that the only singularities were in the first chart, where $u = 1$, $s = z = 0$ and $t^3 + 1 = 0$. We claim that these are A_1 singularities. Indeed, one may calculate the ideal generated by the partial derivatives for this singularity and compare to the A_1 singularity. We now blow up again since we wish to calculate the intersection graph. To this end, we blow up the curve

$$\tilde{x}^2 + \tilde{z}(\tilde{y}^3 + 1) = 0.$$

We let ζ satisfy $\zeta^3 = -1$. Define the translation $\tilde{y}_1 := \tilde{y} - \zeta$ so that the above surface becomes

$$\tilde{x}^2 + \tilde{z}(\tilde{y}_1^3 + 3\tilde{y}_1^2\zeta + 3\tilde{y}_1\zeta^2) = 0$$

and is now singular at the origin in these coordinates. Blowing up in the $\tilde{u} = 1$ chart we find

$$\tilde{s}^2 + \tilde{z}^2\tilde{t}^3 + 3\zeta\tilde{b}^2\tilde{z} + 3\zeta^2\tilde{t} = 0.$$

The exceptional divisor E_1 is given by the intersection of this with $\tilde{z} = 0$. Thus, E_1 is given by $\tilde{s}^2 + 3\zeta^2\tilde{t} = 0$ which is again a copy of $\mathbb{P}^1$. Intersecting E_0 and E_1 we find the point given by $\tilde{s} = \tilde{t} = \tilde{z} = 0$. There are three copies of E_1 corresponding to the three possible ζ values. The intersection graph of $W = x^2 + y^3 + z^3$ is shown in Figure 3.1 and coincides with the Dynkin diagram for D_4.

One may apply the same procedure to $W_{D_4} = x^2 + y^2z + z^3$, although we omit details here. The exceptional divisor E in this case is again a copy of $\mathbb{P}^1$. There is no intersection of E with the blow up in the chart with $s = 1$; for $t = 1$ we find

$$s^2 + yu + yu^3 = 0.$$

This is an A_1 singularity. For the chart where $u = 1$ we have

$$s^2 + zt^2 + z = 0.$$

This has two singular points, $s = 0, t = \pm i, z = 0$ which are both A_1 type.

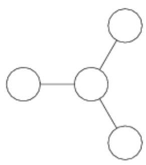

Figure 3.1: The intersection graph for both $(W_{D_4})_{\text{Fermat}} = x^2 + y^3 + z^3$ and $(W_{D_4})_{\text{Chain}} = x^2 + y^2z + z^3$. The middle vertex corresponds to the line E_0. This intersects three copies of the line E_1 corresponding to the three edges in the above Dynkin diagram.

One may repeat this procedure for each case and show that each intersection graph coincides with the corresponding Dynkin diagram. This procedure can be extended to simple elliptic singularities. We refer the reader to [Rei97, Chapter 4] and [Dim92, Appendix A].

3.1.2 The Local Algebra

Classifying generic singularities using Dynkin diagrams is in general a rather difficult problem. Thus, we return to Definition 2.4.1 where the chiral ring for the A-model of a singularity W was given as

$$\mathscr{H}_{W,G} := \bigoplus_{\gamma \in G} \Big(H^{N_\gamma}\Big(\mathrm{Fix}(\gamma), \Big(W|_{\mathrm{Fix}(\gamma)}\Big)^{\infty}; \mathbb{C}\Big)\Big)^{G}.$$

Currently, the full B-model of enumerative Saito-Givental theory is defined only for the trivial group $G = \{1\}$. Consequently, we make the following definition.

Definition 3.1.3. The B-model *chiral ring* is defined as

$$\mathscr{H}_W := H^N\Big(\mathbb{C}^N, W^{\infty}; \mathbb{C}\Big).$$

This defines the B-model chiral ring as a vector space. The main goal of this subsection is to endow $\mathscr{H}_W$ with the structure of a Frobenius algebra. We start by defining a pairing.

Lemma 3.1.4. There is a well defined perfect pairing

$$\eta : \mathscr{H}_W \otimes \mathscr{H}_W \to \mathbb{C}.$$

Proof. We first observe that there is a pairing

$$H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{C}) \otimes H_N(\mathbb{C}^N, W^{\infty}; \mathbb{C}) \to \mathbb{C} \tag{3.1.1}$$

given by the usual intersection pairing of cycles in homology. Here, we set $W^{-\infty} := (\mathrm{Re}W)^{-1}(-\infty, -R)$ with $R \gg 0$. We explicitly describe this pairing on basis elements. Such homology groups have a basis of Lefschetz thimbles $\Delta_i^{\pm} \in H_N(\mathbb{C}^N, W^{\pm\infty}; \mathbb{C})$. These cycles are constructed by deforming W to a holomorphic Morse function W_s and taking preimages of non-intersecting paths in $\mathbb{C}$ which start at critical values of W_s. Furthermore, we assume these paths are open and horizontal in the direction Re $W_s = \pm\infty$. The

Lefschetz thimbles satisfy

$$\Delta_i^- \cap \Delta_j^+ = \delta_{ij}.$$

Hence this pairing is perfect. We now show that

$$H_N(\mathbb{C}^N, W^\infty; \mathbb{C}) \cong H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{C}). \tag{3.1.2}$$

Indeed, if W has degree d and weights $c_1, \ldots, c_N$, choose any $\xi \in \mathbb{C}$ such that $\xi^d = -1$. Multiplication by the diagonal matrix $\operatorname{diag}(\xi^{c_1}, \ldots, \xi^{c_N})$ defines a map $\mathbb{C}^N \to \mathbb{C}^N$ sending W^∞ to $W^{-\infty}$. This map induces the required isomorphism (3.1.2) on the relative homology.

Thus, from (3.1.1) we obtain a pairing

$$H_N(\mathbb{C}^N, W^\infty; \mathbb{C}) \otimes H_N(\mathbb{C}^N, W^\infty; \mathbb{C}) \to \mathbb{C}.$$

Poincaré duality on Chen-Ruan cohomology [CR04, Propsoition 3.3.1] implies that the relative cohomology group $H^N(\mathbb{C}^N, W^\infty; \mathbb{C})$ is dual to $H_N(\mathbb{C}^N, W^\infty; \mathbb{C})$. Thus, by dualising, we obtain a perfect pairing as required. □

Although we have defined a pairing, we also require $H^N(\mathbb{C}^N, W^{-\infty}; \mathbb{C})$ to have a ring structure. We do this by giving a different presentation of the chiral ring.

Proposition 3.1.5. Let W be a Landau-Ginzburg model. Then the B-model state space $\mathscr{H}_W$ is isomorphic to the *local algebra* $\mathscr{D}_W$:

$$\mathscr{H}_W \cong \mathscr{D}_W := \frac{\mathbb{C}[x_1, \ldots, x_N]}{\left\langle \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N} \right\rangle} \cdot d^N x.$$

Here $d^N x = dx_1 \wedge \cdots \wedge dx_N$.

Proof. Consider the long exact sequence of relative homology of the pair $(\mathbb{C}^N, \mathbb{C}^N_t)$ where we define $\mathbb{C}^N_t := \mathbb{C}^N \cap W^{-1}(t)$ as the Milnor fibre. Since $\mathbb{C}^N$ is contractible, we have that

$H_{N+1}(\mathbb{C}^N;\mathbb{Z}) = H_{N-1}(\mathbb{C}^N;\mathbb{Z}) = 0$. In particular, we see that the connecting morphism $H_N(\mathbb{C}^N,\mathbb{C}^N_t;\mathbb{Z}) \to H_{N-1}(\mathbb{C}^N_t;\mathbb{Z})$ in the long exact sequence for relative homology is an isomorphism. Furthermore, the rescaling $(x_1,\ldots,x_N) \mapsto (\lambda^{c_1}x_1,\ldots,\lambda^{c_N}x_N)$ induces the isomorphism $H_N(\mathbb{C}^N,W^\infty;\mathbb{Z}) \cong H_N(\mathbb{C}^N,\mathbb{C}^N_t;\mathbb{Z})$. Then tensoring with $\mathbb{C}$ and using Poincaré duality on Chen-Ruan cohomology gives

$$H^N(\mathbb{C}^N,W^\infty;\mathbb{C}) \cong H^N(\mathbb{C}^N,\mathbb{C}^N_t;\mathbb{C}) \cong H^{N-1}(\mathbb{C}^N_t;\mathbb{C}).$$

Now, a theorem of Wall [Wal80a; Wal80b] shows that in fact,

$$H^{N-1}(\mathbb{C}^N_t;\mathbb{C}) \cong \frac{\Omega^N_{\mathbb{C}^N}}{dW\wedge\Omega^{N-1}_{\mathbb{C}^N}} = \Gamma\Big(\mathbb{C}^N, \operatorname{coker}(\Omega^{N-1}_{\mathbb{C}^N} \xrightarrow{dW\wedge} \Omega^N_{\mathbb{C}^N})\Big). \tag{3.1.3}$$

Here, we implicitly use the analytic topology so that $\Omega^p_{\mathbb{C}^N}$ is the sheaf of differential p-forms on $\mathbb{C}^N$. Observe that the above map given in the cokernel above is described in local coordinates $(x_1,\ldots,x_N)$ by

$$dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_N \mapsto \frac{\partial W}{\partial x_i}dx_1\wedge\cdots\wedge dx_N.$$

Thus, we find that the state space is given as

$$H^N(\mathbb{C}^N,W^\infty;\mathbb{C}) \cong \frac{\mathbb{C}[x_1,\ldots,x_N]}{\left\langle \frac{\partial W}{\partial x_1},\ldots,\frac{\partial W}{\partial x_N}\right\rangle}\cdot d^N x$$

as desired. □

The following result describing the perfect pairing η on the local algebra is originally due to Cecotti [Cec91] and is stated precisely in Proposition 2.12 of [FS13].

Lemma 3.1.6. Define a pairing

$$\mathscr{D}_W \otimes \mathscr{D}_W \to \mathbb{C},$$

given by the residue map

$$(f \cdot d^N x, g \cdot d^N x) \mapsto \operatorname*{Res}_{x_i=0}\left(\frac{fg}{\prod_{i=1}^N \partial_i W} \cdot d^N x\right) = \frac{1}{(2\pi i)^N}\int_C \frac{fg}{\prod_{i=1}^N \partial_i W} \cdot d^N x$$

where C is the boundary of a polydisc containing the origin in $\mathbb{C}^N$. This pairing is induced by the intersection pairing $\eta : \mathscr{H}_W \otimes \mathscr{H}_W \to \mathbb{C}$ via the isomorphism $\mathscr{H}_W \cong \mathscr{D}_W$.

The following result is then well-known. See [Dub96; Dub99] for example.

Proposition 3.1.7. The local algebra $\mathscr{D}_W$ with usual ring structure and residue pairing η is a Frobenius algebra.

3.1.3 Families of Frobenius Algebras

We start by using a deformation of quasi-homogeneous superpotentials.

Definition 3.1.8. Let W be a Fermat, chain or loop polynomial. Suppose B is an index set such that $\{x^\mu\}_{\mu\in B}$ is a basis of the local algebra. For $\boldsymbol{s} = (s_\mu)_{\mu\in B} \in \mathbb{C}^B$, where $|B| = \dim \mathscr{D}_W$, a *versal deformation* of such a polynomial is

$$W_{\boldsymbol{s}} = W + \sum_{\mu\in B} s_\mu x^\mu.$$

Remark 3.1.9. *A priori,* there are of course many different choices for the basis of the local algebra. Later, we will choose a specific basis according to [HLSW22] that is canonical for our purpose of mirror symmetry.

Example 3.1.10. Consider the potential $W = x^{r+1}$. The local algebra is

$$\mathscr{D}_W = \frac{\mathbb{C}[x]}{(x^r)} \cdot dx.$$

One choice of basis is $\{1, x, \ldots, x^{r-1}\}$ with corresponding versal deformation

$$W_{\boldsymbol{s}} = x^{r+1} + s_0 + s_1 x + \cdots + s_{r-1}x^{r-1}.$$

Proposition 3.1.11. There is an isomorphism of $\mathbb{C}[\boldsymbol{s}]$-modules

$$\bigoplus_{i=1}^{n} \mathbb{C}[\boldsymbol{s}] \frac{\partial}{\partial s_i} \to \frac{\mathbb{C}[\boldsymbol{s}][x_1, \ldots, x_N]}{\left\langle \frac{\partial W_{\boldsymbol{s}}}{\partial x_1}, \ldots, \frac{\partial W_{\boldsymbol{s}}}{\partial x_N} \right\rangle}.$$

Proof. The required isomorphism is defined on generators as $\frac{\partial}{\partial s_i} \mapsto \left[\frac{\partial W_{\boldsymbol{s}}}{\partial s_i} \right]$. □

The above map is called the Kodaira-Spencer isomorphism. One may define an analogous residue pairing

$$\eta_{W_{\boldsymbol{s}}} : \frac{\mathbb{C}[\boldsymbol{s}][x_1, \ldots, x_N]}{\left\langle \frac{\partial W_{\boldsymbol{s}}}{\partial x_1}, \ldots, \frac{\partial W_{\boldsymbol{s}}}{\partial x_N} \right\rangle}^{\otimes 2} \to \mathbb{C}[\boldsymbol{s}], \quad \eta_{W_{\boldsymbol{s}}}(f, g) = \frac{1}{(2\pi i)^N} \int_C \frac{fg}{\prod_{i=1}^N \partial_i W_{\boldsymbol{s}}} \cdot d^N x$$

where C is the boundary of a polydisc containing all the critical points $\{x \in \mathbb{C}^N : \frac{\partial W_{\boldsymbol{s}}}{\partial x_i} = 0 \text{ for each } i\}$.

In [Sai93], Saito found special parameters $t_1, \ldots, t_n$. such that the following result holds.

Theorem 3.1.12. There exists a change of coordinates $\boldsymbol{s} \mapsto \boldsymbol{t}$ such that

$$\eta_{W_{\boldsymbol{s}}} \left(\frac{\partial W_{\boldsymbol{s}(\boldsymbol{t})}}{\partial t_\mu}, \frac{\partial W_{\boldsymbol{s}(\boldsymbol{t})}}{\partial t_\nu} \right) \in \mathbb{C}.$$

Via the Kodaira-Spencer isomorphism, we move this flat pairing to $\bigoplus_{i=1}^n \mathbb{C}[\boldsymbol{s}] \frac{\partial}{\partial s_i}$. In this way, we obtain a flat family of Frobenius algebras parameterised by $\boldsymbol{s}$. The aim of the rest of section 3.1 is to more precisely interpret the t_μ as flat coordinates for a certain Frobenius manifold. Moreover, this Frobenius structure is *conformal*: in the case of Landau-Ginzburg models this means there is a grading induced by the quasi-homogeneous W which we explain in the following definition.

Definition 3.1.13. Let W be a quasi-homogeneous polynomial of degree 1 where x_i has degree q_i for $i = 1, \ldots, N$. Let $\{x^\mu\}_{\mu \in B}$ be a basis of the local algebra with versal deformation $W_{\boldsymbol{s}}$. We define a grading on $\mathbb{C}[x_1,\dots,x_N][\{s_\mu\}_{\mu\in B}]$ given by $\deg x_i := q_i$ and

$$\deg s_\mu := 1 - \deg x^\mu = 1 - \sum_{i=1}^N \mu_i \deg x_i$$

so that $W_{\boldsymbol{s}}$ is also quasi-homogeneous of degree one. The vector field given by

$$\mathcal{E} := \sum_{\mu\in B} s_\mu \deg s_\mu \frac{\partial}{\partial s_\mu} + \sum_{i=1}^N x_i \deg x_i \frac{\partial}{\partial x_i}$$

we call a *grading operator*. We may restrict $\mathcal{E}$ to $\mathbb{C}[\{s_\mu\}_{\mu\in B}]$ giving the *Euler vector field*,

$$E := \sum_{\mu\in B} s_\mu \deg s_\mu \frac{\partial}{\partial s_\mu}.$$

3.1.4 Hypercohomology and The Brieskorn Lattice.

There is another useful description of the state space that is given by *hypercohomology*. A standard introduction to hypercohomology and spectral sequences in terms of homological algebra is given in Chapters 15 and 17 of [CE99].

Proposition 3.1.14. We have the following isomorphism

$$\mathbb{H}^i\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, dW\wedge)\Big) = \begin{cases} \mathscr{D}_W, & i = N, \\ 0, & i \neq N. \end{cases}$$

Proof. We use the hypercohomology spectral sequence where one can show that

$$E_2^{p,q} := H^q(\mathbb{C}^N, \mathcal{H}^p(\Omega^\bullet_{\mathbb{C}^N}, dW\wedge)) \implies \mathbb{H}^{p+q}\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, dW\wedge)\Big)$$

where $\mathcal{H}^q$ is the cohomology of $(\Omega^\bullet_{\mathbb{C}^n}, dW\wedge)$ and H^p is sheaf cohomology. One can inductively show that $\mathcal{H}^p(\Omega_{\mathbb{C}^N}, dW\wedge) = 0$ for $p < N$. See Example 2.32 of [Gro10]. For $p = N$ we have already seen that

$$\mathcal{H}^N(\Omega^\bullet_{\mathbb{C}^N}, dW\wedge) \cong \mathscr{D}_W.$$

Since W has isolated critical points, the sheaf $\mathcal{H}^N(\Omega_{\mathbb{C}^N}, dW\wedge)$ is supported on a zero dimensional space. Given that $\mathcal{H}^p(\Omega_{\mathbb{C}^N}, dW\wedge)$ is only non-trivial for $p = N$ on which its support is zero dimensional, the hypercohomology spectral sequence degenerates at E_2. In other words, $E_2^{p,q} = 0$ for all $q > 0$. Thus, by Theorem 5.12 [CE99, Chapter 15], we have that $\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, dW\wedge)\Big) \cong E_2^{N,0}$. This isomorphism reads

$$\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, dW\wedge)\Big) \cong H^0(\mathbb{C}^n, \mathcal{H}^N(\Omega_{\mathbb{C}^N}, dW\wedge)) \cong \Gamma\Big(\mathbb{C}^N, \operatorname{coker}(\Omega^{N-1}_{\mathbb{C}^N} \xrightarrow{dW\wedge} \Omega^n_{\mathbb{C}^N})\Big)$$

while the other hypercohomologies $\mathbb{H}^p\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, dW\wedge)\Big)$ vanish for $p \neq N$. □

To obtain the full cohomological field theory, the Frobenius algebra pairing on the B-model chiral ring is not sufficient. Hence, by modifying the above hypercohomology group, we define a new pairing using the twisted de Rham complex $(\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)$.

Definition 3.1.15. Let $\hbar \in \mathbb{C}^*$. The Brieskorn lattice as a $\mathbb{C}[\hbar]$-module is defined as the hypercohomology group $\mathbb{H}^\bullet\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big)$.

Lemma 3.1.16. The Brieskorn lattice is only non-trivial for $i = N$ in which case we have the following isomorphism:

$$\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big) \cong H^N(\Gamma(\mathbb{C}^N, \Omega^\bullet_{\mathbb{C}^N}), \hbar d + dW\wedge).$$

Proof. We use another hypercohomology sequence

$$E_1^{p,q} = H^p(H^q(\mathbb{C}^N, \Omega^\bullet_{\mathbb{C}^N}), \hbar d + dW\wedge) \implies \mathbb{H}^{p+q}\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big).$$

Now, $\mathbb{C}^N$ is affine and $\Omega^r_{\mathbb{C}^N}$ is a coherent sheaf. Therefore, by Serre's vanishing theorem, we have that $H^q(\mathbb{C}^N, \Omega^\bullet_{\mathbb{C}^N}) = 0$ for $q > 0$. Thus, the spectral sequence degenerates at the E_1 term and so we have that $E_1^{p,0} \cong \mathbb{H}^p\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big)$. This is only non-trivial in the case $p = N$. □

Corollary 3.1.17. The Brieskorn lattice is isomorphic to

$$\mathscr{H}_W^{(0)} = \frac{\Omega^N_{\mathbb{C}^N}[[\hbar]]}{(\hbar d + dW\wedge)\Omega^{N-1}_{\mathbb{C}^N}[[\hbar]]}.$$

By setting $\hbar = 0$, we see that

$$\mathscr{D}_W \cong \mathscr{H}_W^{(0)}/\hbar\mathscr{H}_W^{(0)}.$$

The above corollary shows that, as sets, we may consider $\mathscr{D}_W \subseteq \mathscr{H}_W^{(0)}$. Later on, Lemma 3.2.2 shows that this containment holds as $\mathbb{C}[\hbar]$-modules.

Remark 3.1.18. Since the complexes $(\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)$ and $(\Omega^\bullet_{\mathbb{C}^N}, d + \hbar^{-1}dW\wedge)$ are quasi-isomorphic, we have that

$$\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big) \cong \mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, d + \hbar^{-1}dW\wedge)\Big) \cong \frac{\Omega^N_{\mathbb{C}^N}[[\hbar]]}{(d + \hbar^{-1}dW\wedge)\Omega^{N-1}_{\mathbb{C}^N}[[\hbar]]}.$$

Example 3.1.19. It is instructive to illustrate these ideas with the example of the singularity $W = x^r$. From the above corollary and remark, it is enough to compute the cohomology of the complex

$$0 \to \Omega^0_{\mathbb{C}}[[\hbar]] \xrightarrow{d+\hbar^{-1}dW\wedge} \Omega^1_{\mathbb{C}}[[\hbar]] \to 0$$

so that

$$\mathbb{H}^1\Big(\mathbb{C}, (\Omega^\bullet_{\mathbb{C}}, d + \hbar^{-1}dW\wedge)\Big) \cong \frac{\Omega^1_{\mathbb{C}}[[\hbar]]}{(d + \hbar^{-1}dW\wedge)\Omega^0_{\mathbb{C}}[[\hbar]]}.$$

We note that

$$(d + \hbar^{-1}dW) \cdot 1 = \hbar^{-1}rx^{r-1}dx.$$

We also observe that

$$(d + \hbar^{-1}dW) \cdot x^a = (ax^{a-1} + \hbar^{-1}rx^{a+r-1})dx$$

for $a > 0$. Thus, we find the relations

$$x^{r-1}dx = 0,$$
$$x^{a-1+r}dx = -\hbar\frac{a}{r}x^{a-1}dx.$$

The first of these is the usual relation arising in the local algebra for $W = x^r$ whilst the second is an 'integration by parts' formula.

Remark 3.1.20. The local algebra is invariant up to isomorphism if one rescales the holomorphic volume form $\omega = d^N x \mapsto C\omega$ for $C \in \mathbb{C}^*$. Thus the pairing defined above is also invariant if one changes the cycles accordingly.

3.1.5 The Gauss-Manin Vector Bundle

In this section, we define the Gauss-Manin vector bundle and equip it with a connection. We furthermore define higher residue pairings as this will be essential to be able to define a primitive form. We mostly follow the material presented in [LLS14].

Definition 3.1.21. Let W be a Landau-Ginzburg model with $W_{\boldsymbol{s}}$ a versal deformation and index set B. Suppose M is a small ball in $\mathbb{C}^B$. The *Gauss-Manin vector bundle* is given by the sheaf of $\mathcal{O}_M[\hbar]$-modules

$$\mathscr{H}^{(0)}_{W_{\boldsymbol{s}}} := \frac{\Omega^N_{\mathbb{C}^N\times M/M}[[\hbar]]}{(\hbar d + dW_{\boldsymbol{s}}\wedge)\Omega^{N-1}_{\mathbb{C}^N\times M/M}[[\hbar]]}$$

where $\Omega^N_{\mathbb{C}^N\times M/M}$ is a sheaf of relative differential N-forms.

One can show that $\mathscr{H}^{(0)}_{W_{\boldsymbol{s}}}$ is a locally free sheaf of rank n on $M \times D^*$ where D^* is a punctured disk in $\mathbb{C}$ with coordinate $\hbar$. More precisely, for an open $U \subset M \times D^*$, we have an isomorphism

$$\mathscr{H}^{(0)}_{W_{\boldsymbol{s}}}|_U \cong \bigoplus_{\mu\in B} \mathcal{O}_{M\times D^*}|_U \cdot x^\mu d^N x. \tag{3.1.4}$$

On this bundle we have the Gauss-Manin connection, $\nabla : T(M \times D^*) \times \mathscr{H}^{(0)}_{W_s} \to \mathscr{H}^{(0)}_{W_s}$ where $T(M \times D^*)$ is the holomorphic tangent bundle of $M \times D^*$. For ζ a section of $\mathscr{H}^{(0)}_{W_s}$, the connection ∇ is defined via

$$\nabla_{\partial_{s_\mu}} \zeta = \frac{\partial \zeta}{\partial s_\mu} + \frac{1}{\hbar} \frac{\partial W_s}{\partial s_\mu} \zeta, \quad \nabla_{\partial_\hbar} \zeta = \frac{\partial \zeta}{\partial \hbar} - \frac{1}{\hbar^2} W_s \zeta.$$

Higher Residues. We present a simplified theory of the higher residue pairing originally considered in [Sai83c].

Let

$$\eta_W : \mathscr{H}_W \otimes_{\mathbb{C}} \mathscr{H}_W \to \mathbb{C}$$

be the classical residue pairing. There is a pairing $\langle -, - \rangle : \Omega^N_{\mathbb{C}^N}[[\hbar]] \times \Omega^N_{\mathbb{C}^N}[[\hbar]] \to \mathbb{C}$ given by

$$\langle f(\hbar)\chi_1, g(\hbar)\chi_2 \rangle = \operatorname*{Res}_{\hbar=0} \Big(f(\hbar) g(-\hbar) d\hbar \Big) \eta_W([\chi_1], [\chi_2])$$

where $[\chi]$ is the class of χ in $\mathscr{D}_W$.

Lemma 3.1.22. For $\alpha \in \Omega^{N-1}_{\mathbb{C}^N}$, the following equation holds

$$\langle \hbar d\alpha + dW \wedge \alpha, - \rangle = 0.$$

Proof. One can work in local coordinates to use the residue pairing and Cauchy's theorem to show explicitly that the above equation is true. □

As an alternative, simpler proof, see Lemma 3.2.11. From the above lemma, the pairing $\langle -, - \rangle$ descends to $\mathscr{H}^{(0)}_W$. Moreover, η_W is symmetric and non-degenerate, while the residue in $\hbar$ provides skew-symmetry. Hence, we have in fact a symplectic pairing

$$\langle -, - \rangle : \mathscr{H}^{(0)}_W \times \mathscr{H}^{(0)}_W \to \mathbb{C}.$$

In a similar way, one may define higher residues.

Definition 3.1.23. The *higher residue* is the map $K_W : \mathscr{H}_W^{(0)} \times \mathscr{H}_W^{(0)} \to \hbar^N \mathbb{C}[[\hbar]]$ defined via the following formula,

$$K_W\Big(f(\hbar)\chi_1, g(\hbar)\chi_2\Big) = \hbar^N f(\hbar) g(-\hbar) \eta_W([\chi_1], [\chi_2]).$$

This is indeed well defined via the same argument in Lemma 3.1.22.

Remark 3.1.24. Observe that K_W and the symplectic pairing $\langle -, - \rangle$ are related via

$$\langle -, - \rangle = \operatorname*{Res}_{\hbar=0}\Big(\hbar^{-N} K_W(-,-) d\hbar\Big).$$

One may extend the higher residue to the versal deformation $W_{\boldsymbol{s}}$. This is the map $K_{W_{\boldsymbol{s}}} : \mathscr{H}_{W_{\boldsymbol{s}}}^{(0)} \times \mathscr{H}_{W_{\boldsymbol{s}}}^{(0)} \to \hbar^N \mathcal{O}_M[[\hbar]]$ given by

$$K_{W_{\boldsymbol{s}}}\Big(f(\hbar)\chi_1, g(\hbar)\chi_2\Big) = \hbar^N f(\hbar) g(-\hbar) \eta_{W_{\boldsymbol{s}}}([\chi_1], [\chi_2]).$$

Here, $\eta_{W_{\boldsymbol{s}}}$ is the classical residue pairing with respect to $W_{\boldsymbol{s}}$.

Remark 3.1.25. In [LLS14], the triple $(\mathscr{H}_{W_{\boldsymbol{s}}}^{(0)}, K_{W_{\boldsymbol{s}}}, \nabla)$ is called a variation of semi-infinite Hodge structure. This was originally studied in the case of Calabi-Yau varieties by Barannikov [Bar01].

3.1.6 Primitive Forms

In this subsection we define a primitive form. This is the crucial step in endowing the ball $M \subset \mathbb{C}^B$ with the structure of a Frobenius manifold.

Definition 3.1.26. A *primitive form* is a section ζ of $\mathscr{H}_{W_{\boldsymbol{s}}}^{(0)}$ such that

1. (Primitivity) The element ζ induces an $\mathcal{O}_M$-module isomorphism

$$\phi : TM \to \mathscr{H}_{W_{\boldsymbol{s}}}^{(0)} / \hbar \mathscr{H}_{W_{\boldsymbol{s}}}^{(0)}, \qquad V \mapsto \hbar \nabla_V \zeta.$$

2. (Homogeneity) For the Euler vector field

$$E = \sum_{\mu \in B} s_\mu \deg s_\mu, \partial_{s_\mu}$$

the following equation is satisfied

$$\Big(\nabla_{\hbar\partial_\hbar} + \nabla_E\Big)\zeta = \Big(\sum_{i=1}^{N} \deg x_i\Big)\zeta$$

where ∇ is the Gauss-Manin connection.

3. (Orthogonality) For any two vector fields V_1, V_2 on M we have

$$K_{W_{\boldsymbol{s}}}\Big(\nabla_{V_1}\zeta, \nabla_{V_2}\zeta\Big) \in \hbar^{N-2}\mathcal{O}_M.$$

4. (Holonomicity) For any three vector fields V_1, V_2, V_3 on M, we have

$$K_{W_{\boldsymbol{s}}}\Big(\nabla_{V_1}\nabla_{V_2}\zeta, \nabla_{V_3}\zeta\Big) \in \hbar^{N-3}\mathcal{O}_M \oplus \hbar^{N-2}\mathcal{O}_M$$
$$K_{W_{\boldsymbol{s}}}\Big(\nabla_{\hbar\partial_\hbar}\nabla_{V_1}\zeta, \nabla_{V_3}\zeta\Big) \in \hbar^{N-3}\mathcal{O}_M \oplus \hbar^{N-2}\mathcal{O}_M.$$

The isomorphism ϕ, called the period mapping, in the primitivity condition is critical for the following reason. Roughly speaking, one may construct and solve a Riemann-Hilbert-Birkhoff problem on the bundle $\mathscr{H}^{(0)}_{W_{\boldsymbol{s}}}/\hbar\mathscr{H}^{(0)}_{W_{\boldsymbol{s}}}$. Via ϕ^{-1}, one may carry over this structure to TM, thus endowing M with a Frobenius manifold structure. This is the content of Saito's original paper [Sai83a]. We refer the reader interested in the Frobenius structure of M to [Sab07].

3.2 Saito Theory as Oscillatory Integrals

The previous definition of a primitive form is slightly unwieldy. Indeed, it is not clear how to calculate them from the data $(\mathscr{H}^{(0)}_{W_s}, K_{W_s}, \nabla)$, even for simple singularities. We therefore review a useful characterisation of primitive forms in terms of oscillatory integrals. Subsequently, we use this to calculate the flat Frobenius structure for the ADE and simple elliptic singularities.

3.2.1 Primitive Forms Via Oscillatory Integrals

Work of Barannikov [Bar01] established the use of oscillatory integrals in the B-model, thereby avoiding the technical details of variations of semi-infinite Hodge structure. This was then later fully developed by Li, Li, Saito and Shen [LLS14; LLSS17], the contents of which we partially summarise here.

We first extend the definition of the Brieskorn lattice $\mathscr{H}^{(0)}_W$. Define

$$\mathscr{H}^{(0)}_W((\hbar)) := \frac{\Omega^N_{\mathbb{C}^N}((\hbar))}{(\hbar d + dW\wedge)\Omega^{N-1}_{\mathbb{C}^N}((\hbar))}.$$

The higher residue pairing extends to a pairing

$$K_W : \mathscr{H}^{(0)}_W((\hbar)) \otimes \mathscr{H}^{(0)}_W((\hbar)) \to \mathbb{C}((\hbar)).$$

Recall that we have a symplectic pairing $\langle -, - \rangle$ on $\mathscr{H}^{(0)}_W$. This can naturally be extended to $\mathscr{H}^{(0)}_W((\hbar))$ via

$$\langle -, - \rangle : \mathscr{H}^{(0)}_W((\hbar)) \otimes \mathscr{H}^{(0)}_W((\hbar)) \to \mathbb{C}, \quad \langle -, - \rangle = \operatorname*{Res}_{\hbar=0}\Big(\hbar^{-N} K_W(-,-) d\hbar\Big).$$

Definition 3.2.1. Let $\mathcal{L}$ be a splitting of $\mathscr{H}^{(0)}_W((\hbar))$ with $\mathscr{H}^{(0)}_W((\hbar)) = \mathscr{H}^{(0)}_W \oplus \mathcal{L}$. If $\mathcal{L}$ satisfies $\hbar^{-1}\mathcal{L} \subset \mathcal{L}$ and is also an isotropic subspace with respect to $\langle -, - \rangle$, then $\mathcal{L}$ is called an *opposite filtration*.

As the next lemma shows, the choice of an opposite filtration is equivalent to the choice of a basis of the local algebra $\mathscr{D}_W$.

Lemma 3.2.2. Let $A := \mathscr{H}_W^{(0)} \cap \hbar\mathcal{L}$. Then $A \cong \mathscr{D}_W$ and we can identify

$$\mathscr{H}_W^{(0)} = A[[\hbar]], \quad \mathscr{H}_W^{(0)}((\hbar)) = A((\hbar)), \quad \mathcal{L} = \hbar^{-1}A[\hbar^{-1}].$$

Proof. Introduce the semi-infinite Hodge filtration on $\mathscr{H}_W^{(0)}((\hbar))$ by $\mathscr{H}_W^{(-k)} := \hbar^k \mathscr{H}_W^{(0)}$ for $k \in \mathbb{Z}_{\geq 0}$. Since $\mathcal{L}$ is a splitting such that $\hbar^{-1}\mathcal{L} \subset \mathcal{L}$, we find

$$\mathscr{H}_W^{(0)}((\hbar)) = \hbar^k \mathscr{H}_W^{(0)}((\hbar)) = \mathscr{H}_W^{(-k)} \oplus \hbar^k \mathcal{L}.$$

This implies

$$\mathscr{H}_W^{(-k)} = \mathscr{H}_W^{(-k-1)} \oplus (\mathscr{H}_W^{(-k)} \cap \hbar^{k+1}\mathcal{L}).$$

In particular, we have

$$\mathscr{H}_W^{(0)}/\mathscr{H}_W^{(-1)} \cong \mathscr{H}_W^{(0)} \cap \hbar\mathcal{L} = A.$$

But

$$\mathscr{H}_W^{(0)}/\mathscr{H}_W^{(-1)} = \mathscr{H}_W^{(0)}/\hbar\mathscr{H}_W^{(0)} = \mathscr{D}_W.$$

The identifications then follow. □

To define the notion of a good opposite filtration, we introduce a $\mathbb{Q}$-grading on $\mathscr{H}_W^{(0)}((\hbar))$ with grading operator

$$\mathbb{E} := \hbar\partial_\hbar + \sum_{i=1}^{N} x_i \deg x_i \partial_{x_i}.$$

Definition 3.2.3. An opposite filtration $\mathcal{L}$ of $\mathscr{H}_W^{(0)}((\hbar))$ is called *good* if it preserves the $\mathbb{Q}$-grading or more precisely, if $\mathbb{E}\mathcal{L} \subset \mathcal{L}$. Thus, given a good opposite filtration $\mathcal{L}$, we call the corresponding basis a *good basis* of the local algebra $\mathscr{D}_W$.

Remark 3.2.4. It is shown in Proposition 5.11 of [LLS14] that the condition $\mathbb{E}\mathcal{L} \subset \mathcal{L}$ is equivalent to $\nabla_{\hbar\partial_\hbar}\mathcal{L} \subset \mathcal{L}$ where ∇ is the Gauss-Manin connection of (3.1.4).

The following theorem, proven by Li, Li and Saito [LLS14], characterises primitive forms in terms of good opposite filtrations.

Proposition 3.2.5. Given a good opposite filtration $\mathcal{L}$, there is a unique section ζ of $\mathscr{H}^{(0)}_{W_{\boldsymbol{s}}}$ up to scalar multiplication, such that

$$[e^{\frac{W_{\boldsymbol{s}}-W}{\hbar}}\zeta] \in [1 \cdot \omega] + \mathcal{L}[[\boldsymbol{s}]] \tag{3.2.1}$$

as elements in $\mathscr{H}^{(0)}_W((\hbar))$. Any ζ such that the above holds is a primitive form. Furthermore, let

$$\mathcal{M} := \{\text{primitive forms}\}/\sim$$

where $\zeta_1 \sim \zeta_2$ if and only if $\zeta_1 = k\zeta_2$ for some $k \in \mathbb{C}^*$. Then there is a well defined bijection of sets

$$\{\text{good opposite filtrations of } \mathscr{H}^{(0)}_W((\hbar))\} \to \mathcal{M}, \quad \mathcal{L} \to \zeta.$$

Hence, to describe primitive forms, we need only describe the set of good opposite filtrations, or equivalently, the set of good bases of $\mathscr{D}_W$. We do this in the next section.

3.2.2 Oscillatory Integrals

Based on the previous section, we now reformulate Saito theory that is more transparent in its use of integration. We first give the definition of the *standard good basis* for the local algebra associated to a general invertible polynomial. In particular, we review the definitions and results in section 2.2 of [HLSW22].

Definition 3.2.6. Let W be a Fermat, chain or loop polynomial as in Theorem 2.1.6. In multi-index notation, $x^\mu := \prod_{j=1}^N x_j^{\mu_j}$, the *standard good bases* indexed by a set B is $\left\{x^\mu \middle| \mu \in B\right\}$ where B is given by the following.

i. If W is a Fermat polynomial, then $B = B_{\text{Fermat}} = \{0, 1, \dots, r-2\}$.

ii. If W is a chain polynomial, then $B = B_{\text{Chain}}$ consists of those $(\mu_1, \dots, \mu_N) \in \mathbb{Z}^N_{\geq 0}$ such that $\mu_i \leq r_i - 1$ and $(\mu_1, \dots, \mu_N)$ is not of the form $(*, *, \dots, *, k, r_{N-2l} - 1, 0, r_{N-2l+2} - 1, \dots, 0, r_{N-2} - 1, 0, r_N - 1)$ with $k \geq 1$.

iii. If W is a loop polynomial, then $B = B_{\text{Loop}} = \{(\mu_1, \dots, \mu_N) \in \mathbb{Z}^N_{\geq 0} \big| \mu_i \leq r_i - 1\}$.

That this standard basis is indeed good in the sense of Definition 3.2.3 is the content of Theorem 2.10 of [HLSW22].

Remark 3.2.7. *A priori*, have simply chosen a basis of the local algebra. We have seen that a good basis corresponds to different good splittings of the Hodge filtration associated to the singularity. Among these good splittings, or equivalently good bases, He, Li Shen and Webb in [HLSW22] then make a further unique choice of good basis that they call standard which is Definition 3.2.6 presented here. In that paper, this choice is necessary in order to identify the A and B model Frobenius cohomological field theories. From a physical perspective, this choice is related to the holomorphic anomaly equation [BCOV94].

Example 3.2.8. Let us consider the chain polynomial $W = x_1^3 + x_1 x_2^3$. This is the E_7 singularity. The standard good basis of $\mathscr{D}_W$ is given by

$$\{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_1^2 x_2\} \cdot dx_1 dx_2.$$

Concerning the grading, we observe that $\deg x_1 = 1/3$ and $\deg x_2 = 2/9$. Hence, using Definition 3.1.13 we find the grading on the vesral deformation parameters $\boldsymbol{s}$ as

$$\deg s_{00} = 1, \quad \deg s_{01} = \frac{7}{9}, \quad \deg s_{10} = \frac{2}{3},$$
$$\deg s_{02} = \frac{5}{9}, \quad \deg s_{11} = \frac{4}{9}, \quad \deg s_{20} = \frac{1}{3}, \quad \deg s_{21} = \frac{1}{9}.$$

Consider also the chain polynomial $W = x_1^4 + x_1 x_2^3$ that represents the elliptic singularity,

$E_7^{(1,1)}$. The standard basis is now given by

$$\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^2x_2, x_1^3, x_1^3x_2\} \cdot dx_1dx_2.$$

Concerning the grading, we first observe that $\deg x_1 = \deg x_2 = 1/4$. Hence we find

$$\deg s_{00} = 1, \quad \deg s_{10} = \deg s_{01} = \frac{3}{4}, \quad \deg s_{20} = \deg s_{11} = \deg s_{02} = \frac{1}{2}$$
$$\deg s_{21} = \deg s_{30} = \frac{1}{4}, \quad \deg s_{31} = 0.$$

In this example, for $E_7^{(1,1)}$ we observe that there is a unique parameter with $\deg s = 0$, or equivalently, a unique standard basis element x^μ with $\deg x^\mu = 1$. This is in fact a characteristic trait of all simple elliptic singularities.

Definition 3.2.9. Let W be a simple elliptic singularity. The basis element of the local algebra with degree one is denoted x^{μ_c}. The corresponding parameter $s := s_{\mu_c}$ of degree zero we call the *marginal parameter*.

Remark 3.2.10. For a generic invertible singularity W, the central charge defined in Section 3.1.1,

$$c_W = \sum_{i=1}^{N}(1 - 2\deg x_i)$$

is in fact also equal to

$$c_W = \max_{\mu \in B}\{\deg x^\mu\}.$$

By definition, we immediately see that simple elliptic singularities have a central charge of one.

Having now given a concrete basis of volume forms for the local algebra $\mathscr{D}_W$, we now proceed to integrate them.

We recall from Section 3.1.2 that in the relative homology group $H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{C})$ we allow cycles which are unbounded in the direction $\operatorname{Re} W \to -\infty$. Integrating over elements in this group, as opposed to $H_N(\mathbb{C}^N, W^{\infty}; \mathbb{C})$, leads to much better convergence properties. Hence, we have a pairing with the Brieskorn lattice defined by oscillatory integrals

$$H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{C}) \otimes \mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, d + \hbar^{-1} dW \wedge)\Big) \to \mathbb{C}, \qquad (\Xi, \chi) \mapsto \int_\Xi e^{W/\hbar}\chi \quad (3.2.2)$$

where χ is a representative of an element in $\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, d + \hbar^{-1} dW \wedge)\Big)$.

Lemma 3.2.11. The pairing above is well-defined.

Proof. We show that this pairing does not depend on the choice of representative. Indeed, observe that for α an $N-1$ form on $\mathbb{C}^N$ we have

$$d(e^{W/\hbar}\alpha) = e^{W/\hbar}(d\alpha + \hbar^{-1} dW \wedge \alpha).$$

Using Stokes' theorem, we find

$$\int_\Xi e^{W/\hbar}\chi = \int_\Xi e^{W/\hbar}\chi + d(e^{W/\hbar}\alpha) = \int_\Xi e^{W/\hbar}(\chi + d\alpha + \hbar^{-1} dW \wedge \alpha). \quad (3.2.3)$$

Consequently, we indeed find the integral does not depend on the choice of representative in the hypercohomology $\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, d + \hbar^{-1} dW \wedge)\Big)$. Furthermore, we were able to apply Stokes' theorem to set the boundary terms to zero precisely because $\Xi \in H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{C})$. □

Given the standard good basis $\{x^\mu\}_{\mu \in B}$ of $\mathscr{D}_W$, we denote the corresponding dual basis of cycles by $\{\Xi_\mu\}_{\mu \in B}$. By considering $\mathscr{D}_W$ as contained in the Brieskorn lattice, this dual basis has the defining property

$$\int_{\Xi_\mu} d^N x \, e^{W/\hbar} x^\lambda = \delta_{\mu\lambda}$$

under the pairing (3.2.2).

Definition 3.2.12. Let W be an invertible, non-degenerate quasi-homogeneous polynomial and let $W_{\boldsymbol{s}}$ be its versal deformation with respect to the standard good basis $\{x^\mu\}_{\mu \in B}$. Denote Ξ_0 to be the basis element that is dual to $1 \cdot d^N x \in \mathscr{D}_W$. Let $f \in \mathbb{C}[[(s_\mu)_{\mu \in B}, (x_i)_{i=1}^N]]$. Consider the oscillatory integral,

$$\int_{\Xi_\mu} e^{W/\hbar} f d^N x = \sum_{i \in \mathbb{Z}} \mathcal{J}_\mu^{(i)}(\boldsymbol{s}) \hbar^{-i}. \tag{3.2.4}$$

where the above series is considered formally. If $\mathcal{J}_\mu^{(i)}(\boldsymbol{s}) \equiv 0$ for all $i < 0$ and $\mathcal{J}_\mu^{(0)}(\boldsymbol{s}) = \delta_{\mu,0}$, then $f d^N x$ is called a *primitive form*. In this case, we define the *flat coordinates* as

$$t_\mu(\boldsymbol{s}) = \mathcal{J}_\mu^{(1)}(\boldsymbol{s}).$$

That these are, in fact, the same flat coordinates of Saito's Frobenius manifold is proven in [LLSS17]. Using this definition, one may show that for any Fermat, chain or loop polynomial,

$$t_\mu = s_\mu + \mathcal{O}(s_*^2). \tag{3.2.5}$$

The flat coordinates further satisfy

$$E t_\mu(\boldsymbol{s}) = (\deg t_\mu) t_\mu$$

where $\deg t_\mu = \deg s_\mu$. Hence, the Euler vector field can be written as

$$E = \sum_{\mu \in B} t_\mu \deg t_\mu \frac{\partial}{\partial t_\mu}.$$

3.2.3 Summary of Saito Theory for Landau-Ginzburg Models

Let us summarise the important notions in Saito theory that we have discussed.

- For the purposes of the B-models in this thesis, a *Landau-Ginzburg model* is a quasi-homogeneous polynomial $W : \mathbb{C}^N \to \mathbb{C}$ where a polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is called quasi-homogenous of rank N if there exist $q_1, \ldots, q_N \in \mathbb{Q}$ such that

$$W\Big(\lambda^{q_1} x_1, \ldots, \lambda^{q_N} x_N\Big) = \lambda W(x_1, \ldots, x_N)$$

  for all $\lambda \in \mathbb{C}$. The rational numbers q_i are called the *charges* of W.

- A quasi-homogeneous polynomial W is called *non-degenerate* if W has precisely one singularity that is isolated at the origin. A non-degenerate polynomial is called *invertible* if the number of variables equals the number of monomials.

- A non-degenerate, quasi-homogeneous polynomial W is invertible if and only if it can be written, up to a rescaling and relabelling, as a Sebastiani-Thom sum of the following Fermat, loop and chain polynomials

$$W_{\text{Fermat}} = x^r, \tag{i}$$

$$W_{\text{Chain}} = x_1^{r_1} x_2 + x_2^{r_2} x_3 + \cdots + x_N^{r_N}, \tag{ii}$$

$$W_{\text{Loop}} = x_1^{r_1} x_2 + x_2^{r_2} x_3 + \cdots + x_N^{r_N} x_1. \tag{iii}$$

- The B-model chiral ring is given as the local algebra

$$\mathscr{D}_W = \frac{\mathbb{C}[x_1, \ldots, x_N]}{\left\langle \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N} \right\rangle} \cdot d^N x \cong \frac{\Omega^N_{\mathbb{C}^N}}{dW \wedge \Omega^{N-1}_{\mathbb{C}^N}}.$$

- There is a relative homology group $H_N(\mathbb{C}^N, \mathrm{Re}\, W/\hbar \ll 0; \mathbb{C})$ where $\hbar \in \mathbb{C}^*$ is an auxiliary parameter and we allow cycles which are unbounded in the direction $\mathrm{Re}\, W/\hbar \to -\infty$.

- We consider the twisted de Rham complex $(\Omega^\bullet_{\mathbb{C}^N}, \hbar d+dW\wedge)$. There is a well-defined perfect pairing defined by oscillatory integrals

$$H_N(\mathbb{C}^N, \text{Re } W/\hbar \ll 0; \mathbb{C}) \otimes \mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d+dW_0\wedge)\Big) \to \mathbb{C}, \qquad (\Xi, \chi) \mapsto \int_\Xi e^{W/\hbar}\chi,$$

where $\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big)$ is a hypercohomology group.

- There is an isomorphism between this hypercohomology group and the Brieskorn lattice

$$\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW\wedge)\Big) \cong \frac{\Omega^N_{\mathbb{C}^N}[[\hbar]]}{(\hbar d + dW\wedge)\Omega^{N-1}_{\mathbb{C}^N}[[\hbar]]}.$$

By setting $\hbar = 0$, we may consider $\mathscr{D}_W$ as contained in this hypercohomology group. More precisely, there is an isomorphism

$$\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW_0\wedge)\Big) \cong \hbar\mathbb{H}^N\Big(\mathbb{C}^N, (\Omega^\bullet_{\mathbb{C}^N}, \hbar d + dW_0\wedge)\Big) \oplus \mathscr{D}_{W_0}$$

so that we may consider $\mathscr{D}_{W_0}$ as contained in this hypercohomology group.

- In multi-index notation, $x^\mu := \prod_{j=1}^N x_j^{\mu_j}$, the *standard good bases* indexed by a set B is $\Big\{x^\mu \Big| \mu \in B\Big\}$ where B is given by the following.

  i. If W is a Fermat polynomial, then $B = B_{\text{Fermat}} = \{0, 1, \dots, r-2\}$.

  ii. If W is a chain polynomial, then $B = B_{\text{Chain}}$ consists of those $(\mu_1, \dots, \mu_N) \in \mathbb{Z}^N_{\geq 0}$ such that $\mu_i \leq r_i - 1$ and $(\mu_1, \dots, \mu_N)$ is not of the form $(*, *, \dots, *, k, r_{N-2l} - 1, 0, r_{N-2l+2} - 1, \dots, 0, r_{N-2} - 1, 0, r_N - 1)$ with $k \geq 1$.

  iii. If W is a loop polynomial, then $B = B_{\text{Loop}} = \{(\mu_1, \dots, \mu_N) \in \mathbb{Z}^N_{\geq 0} \Big| \mu_i \leq r_i - 1\}$.

- Given the standard good basis $\{x^\mu\}_{\mu \in B}$ of $\mathscr{D}_W$ together with the perfect pairing, we denote the corresponding dual basis of cycles by $\{\Xi_\mu\}_{\mu \in B}$. This has the defining duality property

$$\int_{\Xi_\alpha} e^{W/\hbar} x^\beta d^N x = \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta}$ is 1 if $\alpha = \beta$ and 0 otherwise.

- Fix a versal deformation of W parameterised by coordinates $(s_\mu)_{\mu \in B}$,

$$W_{\boldsymbol{s}} := W + \sum_{\mu \in B} s_\mu x^\mu.$$

We define a grading on $\mathbb{C}[x_1, \ldots, x_N][\{s_\mu\}_{\mu \in B}]$ given by $\deg x_i := q_i$ and

$$\deg s_\mu := 1 - \deg x^\mu = 1 - \sum_{i=1}^{N} \mu_i \deg x_i$$

so that $W_{\boldsymbol{s}}$ is also quasi-homogeneous of degree one.

- The vector field given by

$$\mathcal{E} := \sum_{\mu \in B} s_\mu \deg s_\mu \frac{\partial}{\partial s_\mu} + \sum_{i=1}^{N} x_i \deg x_i \frac{\partial}{\partial x_i}$$

we call a *grading operator.*

- For simple elliptic singularities with standard good basis fixed, there is a unique highest degree basis element. We write this element as x^{μ_c} and it satisfies $\deg x^{\mu_c} = 1$. The corresponding parameter s_{μ_c} has degree zero and is called *marginal.*

- Denote by Ξ_0 the basis element that is dual to $1 \cdot d^N x \in \mathscr{D}_W$. Let $f : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{C}$ be a polynomial function. Consider the oscillatory integral,

$$\int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} f d^N x = \sum_{i \in \mathbb{Z}} \mathcal{J}_\mu^{(i)}(\boldsymbol{s}) \hbar^{-i}$$

where the above series is considered formally. If $\mathcal{J}_\mu^{(i)}(\boldsymbol{s}) \equiv 0$ for all $i < 0$ and $\mathcal{J}_\mu^{(0)}(\boldsymbol{s}) = \delta_{\mu,0}$, then $f d^N x$ is called a *primitive form.*

- In this case, we define the *flat coordinates* as

$$t_\mu(\boldsymbol{s}) = \mathcal{J}_\mu^{(1)}(\boldsymbol{s}).$$

- Using this definition, one may show that for any Fermat, chain or loop polynomial,

$$t_\mu = s_\mu + \mathcal{O}(s_*^2).$$

- The flat coordinates satisfy

$$\mathcal{E} t_\mu(\boldsymbol{s}) = (\deg t_\mu) t_\mu$$

where $\deg t_\mu = \deg s_\mu$. Hence, the grading operator can be written as

$$\mathcal{E} = \sum_{\mu \in B} t_\mu \deg t_\mu \frac{\partial}{\partial t_\mu} + \sum_{i=1}^{N} x_i \deg x_i \frac{\partial}{\partial x_i}.$$

3.2.4 Flat Coordinates of Simple and Elliptic Singularities

In practice, how one calculates primitive forms and flat coordinates is to expand the integral $\int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} d^N x$ perturbatively. One can then read off what the primitive form $f d^N x$ is. In the case of ADE and simple elliptic singularities, one may obtain closed expressions for primitive forms. To this end, we define the following quantity.

Definition 3.2.13. Let W be an ADE or simple elliptic singularity. Write $W_{\boldsymbol{s}}$ for the versal deformation with respect to the standard good basis. Define the *deformed flat coordinates*

$$\tilde{t}_\mu := \int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} \, d^N x \tag{3.2.6}$$

with $\Xi_\mu \in H_N(\mathbb{C}^N, \operatorname{Re} W/\hbar \ll 0; \mathbb{C})$ for each $\mu \in B$.

Remark 3.2.14. The deformed flat coordinates have geometric significance as the flat coordinates of a deformation of the Levi-Civita connection for the corresponding Frobenius manifold. See Lecture 2 of [Dub99] for more details. Here, however, we simply view the $\tilde{t}_\mu$ as a useful computational tool.

Proposition 3.2.15. Let W be a Fermat, chain or loop polynomial that is either ADE or simple elliptic type. For $\boldsymbol{k} \in \mathbb{Z}^B_{\geq 0}$ denote

$$l(\boldsymbol{k}) := \sum_{\alpha \in B} k_\alpha \cdot \alpha \in \mathbb{N}^N, \quad \deg \boldsymbol{s}^{\boldsymbol{k}} := \sum_{\alpha \in B} k_\alpha \cdot \deg s_\alpha \in \mathbb{Q}^N$$

where B is the index set for the standard good basis. The deformed flat coordinates are

$$\tilde{t}_\mu = \sum_{i \in \mathbb{Z}} \frac{\mathcal{J}^{(i)}_\mu(\boldsymbol{s})}{\hbar^i},$$

where

$$\mathcal{J}^{(i)}_\mu(\boldsymbol{s}) = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^B_{\geq 0} \\ \deg \boldsymbol{s}^{\boldsymbol{k}} = \deg s_\mu - 1 + i}} c_\mu(l(\boldsymbol{k})) \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}!}$$

and

$$c_\mu(l(\boldsymbol{k})) = \frac{1}{\hbar^{q(\boldsymbol{k},i),i)}} \int_{\Xi_\mu} e^{W/\hbar} x^{l(\boldsymbol{k})} \, d^N x.$$

Here, $q(\boldsymbol{k}, i)$ is defined in equation (3.2.9).

Proof. To prove this, we write the deformed flat coordinates as

$$\tilde{t}_\mu = \int_{\Xi_\mu} e^{W/\hbar} e^{\frac{1}{\hbar} \sum_{\alpha \in B} s_\alpha x^\alpha} \, d^N x = \int_{\Xi_\mu} e^{W/\hbar} \sum_{m=0}^{\infty} \frac{1}{\hbar^m m!} \Big(\sum_{\alpha \in B} s_\alpha x^\alpha \Big)^m \, d^N x$$

where we have expanded the exponential. We now further expand the multinomial to obtain

$$\tilde{t}_\mu = \int_{\Xi_\mu} e^{W/\hbar} \sum_{m=0}^{\infty} \sum_{\substack{k_1, \ldots, k_{|B|} \geq 0 \\ \sum k_\beta = m}} \frac{1}{\hbar^{\sum_\beta k_\beta} m!} \binom{m}{k_1, \ldots, k_{|B|}} \prod_{\alpha \in B} s_\alpha^{k_\alpha} x^{k_\alpha \alpha} \, d^N x.$$

Employing multi-index notation and cancelling the $m!$ in the multinomial coefficient, we find

$$\tilde{t}_\mu = \int_{\Xi_\mu} e^{W/\hbar} \sum_{m=0}^{\infty} \sum_{\substack{k_1, \ldots k_{|B|} \geq 0 \\ \sum k_\beta = m}} \frac{1}{\hbar^{\sum_\beta k_\beta}} \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}!} x^{l(\boldsymbol{k})} \, d^N x.$$

Since we consider this as a formal series, we formally interchange the sum with the integral to obtain

$$\tilde{t}_\mu = \sum_{\boldsymbol{k} \in \mathbb{Z}^B_{\geq 0}} \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}! \hbar^{\sum k_\alpha}} \int_{\Xi_\mu} e^{W/\hbar} x^{l(\boldsymbol{k})} \, d^N x. \tag{3.2.7}$$

Let us now organise the terms so that

$$\tilde{t}_\mu = \sum_{i \in \mathbb{Z}} \frac{\mathcal{J}^{(i)}_\mu(\boldsymbol{s})}{\hbar^i} \tag{3.2.8}$$

where $\mathcal{J}^{(i)}_\mu(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$ does not depend on $\hbar$. For each $\boldsymbol{k} \in \mathbb{Z}^B_{\geq 0}$ and $i \in \mathbb{Z}$, define the integer $q(\boldsymbol{k}, i)$ by

$$\sum_{\alpha=1}^{|B|} k_\alpha - q(\boldsymbol{k}, i) = i. \tag{3.2.9}$$

Comparing (3.2.8) to (3.2.7), we find

$$\mathcal{J}^{(i)}_\mu(\boldsymbol{s}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^B_{\geq 0}} \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}! \hbar^{q(\boldsymbol{k},i)}} \int_{\Xi_\mu} e^{W/\hbar} x^{l(\boldsymbol{k})} \, d^N x. \tag{3.2.10}$$

The integer $q(\boldsymbol{k}, i)$ is the total number of times integration by parts is used to reduce $x^{l(\boldsymbol{k})}$ to an element of the standard good basis of the local algebra.

We now prove that a necessary condition for a summand of $\mathcal{J}^{(i)}_\mu(\boldsymbol{s})$ to be non-zero is $\deg \boldsymbol{s}^{\boldsymbol{k}} = \deg s_\mu - 1 + i$. We use a grading where $\deg \hbar = 1$. Since $W/\hbar$ has degree zero, the element $e^{W/\hbar}$ also has a well-defined degree of zero. Moreover, under such a grading, the 'integration by parts' map $\hbar d + dW\wedge : \Omega^\bullet_{\mathbb{C}^N}[[\hbar]] \to \Omega^\bullet_{\mathbb{C}^N}[[\hbar]]$ is a homogeneous morphism of graded vector spaces. Combining these two observations, we find that the deformed flat coordinates also have a well defined degree. This implies $\deg \tilde{t}_\mu = \deg s_\mu - 1$ since $\tilde{t}_\mu$ contains the term $\frac{s_\mu}{\hbar}$ via equation (3.2.5). Matching degrees on both sides of (3.2.8) implies that $\mathcal{J}^{(i)}_\mu(\boldsymbol{s})$ are also quasi-homogeneous with degree

$$\deg \mathcal{J}^{(i)}_\mu(\boldsymbol{s}) = \deg s_\mu - 1 + i.$$

On the other hand, since $\mathcal{J}_\mu^{(i)}$ does not depend on $\hbar$, we observe that

$$\deg \mathcal{J}_\mu^{(i)}(\boldsymbol{s}) = \deg \boldsymbol{s}^{\boldsymbol{k}}$$

via equation (3.2.10). Equating the two above equations gives the desired result. $\square$

Remark 3.2.16. In the proof we assigned $\hbar$ a degree of 1. It is perhaps also natural, nonetheless, to define $\deg \hbar = 2$. Indeed, this degree is chosen by Costello [Cos09] and Căldăraru-Costello-Tu [CCT20] using topological conformal field theory, as well as in the description of Airy structures by Kontsevich-Soibelman [KS18]. If we assign $\deg \hbar = 2$ instead, the proof of Proposition 3.2.15 is readily modified. For example, we may shift indices in the expansion of $\tilde{t}_\mu$ to write

$$\tilde{t}_\mu = \sum_{i \in 2\mathbb{Z}+\frac{1}{2}} \frac{\mathcal{J}_\mu^{(i)}(\boldsymbol{s})}{\hbar^{\frac{i}{2}+\frac{1}{2}}}.$$

In this case we have

$$\deg \tilde{t}_\mu = \deg s_\mu - 2$$

since $\tilde{t}_\mu$ contains the term $\frac{s_\mu}{\hbar}$. This now implies that

$$\deg \mathcal{J}_\mu^{(i)} = \deg s_\mu - 2 + 2\left(\frac{i}{2} + \frac{1}{2}\right) = \deg s_\mu - 1 + i$$

as before. Furthermore, to ensure that $W/\hbar$ has degree zero, we may modify the values of $\deg x_i$ so that W is quasi-homogeneous of degree two. Since the exact value for $\deg \hbar \neq 0$ is unimportant for our purposes, we keep $\deg \hbar = 1$.

Corollary 3.2.17. The flat coordinates of a simple singularity are given by

$$t_\mu(\boldsymbol{s}) = \mathcal{J}_\mu^{(1)}(\boldsymbol{s})$$

The flat coordinates for an elliptic singularity with marginal parameter s are given by

$$t_\mu(\boldsymbol{s}) = \frac{\mathcal{J}_\mu^{(1)}(\boldsymbol{s})}{\mathcal{J}_0^{(0)}(s)}.$$

Proof. Case I: ADE Singularities. One may prove this by numerically expanding out the relevant integral for each simple singularity to show that

$$\int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} d^N x = \delta_{\mu,0} + \hbar^{-1} t_\mu(\boldsymbol{s}) + O(\hbar^{-2}) \tag{3.2.11}$$

where Ξ_0 is the cycle that is dual to the element $d^N x$ in the local algebra. Alternatively, one may use the following degree counting argument. Recall from Proposition 3.2.15 that

$$\tilde{t}_\mu = \int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} d^N x = \sum_{i\in\mathbb{Z}} \frac{\mathcal{J}_\mu^{(i)}(\boldsymbol{s})}{\hbar^i}$$

where

$$\mathcal{J}_\mu^{(i)}(\boldsymbol{s}) = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^B \\ \deg \boldsymbol{s}^{\boldsymbol{k}} = \deg s_\mu - 1 + i}} c_\mu(l(\boldsymbol{k})) \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}!}. \tag{3.2.12}$$

However, we observe that for an ADE polynomial,

$$\deg s_\mu > 0 \tag{3.2.13}$$

for all $\mu \in B$. Since $\deg \boldsymbol{s}^{\boldsymbol{k}} \geq 0$ for $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^B$, we find that $\mathcal{J}_\mu^{(i)}(\boldsymbol{s}) \equiv 0$ for all $i < 0$. This implies that $\tilde{t}_\mu$ contains no positive powers of $\hbar$. For similar reasons, fixing $i = 0$ implies that the only contributing term is when $\mu = 0$ and $\boldsymbol{k} = 0$. From the definition of c_μ, we find that the coefficient is indeed 1.

Case II: Elliptic Singularities. The marginal parameter s satisfies $\deg s = 0$. In the expansion (3.2.12) we find $\mathcal{J}_\mu^{(i)}(\boldsymbol{s}) \equiv 0$ for all $\mu \in B$ and $i < 0$. However, for $i = 0$ we find that there are non-trivial contributions, but for $\mu = 0$ only. Since $\deg s_{\mu=0} = 1$, we observe that $\mathcal{J}_0^{(0)}$ is a function of s only. Hence, we find the expansion

$$\int_{\Xi_\mu} e^{W_{\boldsymbol{s}}/\hbar} d^N x = \mathcal{J}_0^{(0)}(s)\delta_{\mu,0} + \hbar^{-1}\mathcal{J}_\mu^{(1)}(\boldsymbol{s}) + O(\hbar^{-2}).$$

One choice of a primitive form is therefore $\frac{d^N x}{\mathcal{J}_0^{(0)}(s)}$ implying that the flat coordinates are

$$t_\mu(\boldsymbol{s}) = \frac{\mathcal{J}_\mu^{(1)}(\boldsymbol{s})}{\mathcal{J}_0^{(0)}(s)}.$$

as desired. □

3.2.5 Period Integrals in Rank Two and Formulas for Flat Coordinates

Throughout the thesis, we need explicit formulas for oscillatory integrals. To calculate such integrals, we use the following lemmas.

Lemma 3.2.18. For $W = x^r$, we have

$$\int_{\Xi_\mu} x^k e^{W/\hbar} dx = (-\hbar)^m \frac{\Gamma\left(\frac{k+1}{r} + m\right)}{\Gamma\left(\frac{\mu+1}{r}\right)} \delta_{k,rm+\mu}$$

for some $m \in \mathbb{Z}_{\geq 0}$.

Proof. We use integration by parts to find

$$\int_{\Xi_\mu} x^k e^{W/\hbar} dx = -\hbar\left(\frac{k+1}{r} - 1\right) \int_{\Xi_\mu} x^{k-r} e^{W/\hbar}\, dx = -\hbar \frac{\Gamma\left(\frac{k+1}{r}\right)}{\Gamma\left(\frac{k+1}{r} - 1\right)} \int_{\Xi_\mu} x^{k-r} e^{W/\hbar}\, dx$$

where we used functional equation $\Gamma(z+1) = z\Gamma(z)$. Repeated application of integration by parts m times yields cancellation of Γ functions. Hence,

$$\int_{\Xi_\mu} x^k e^{W/\hbar}\, dx = (-\hbar)^m \frac{\Gamma\left(\frac{k+1}{r}\right)}{\Gamma\left(\frac{k+1}{r} - m\right)} \int_{\Xi_\mu} x^{k-mr} e^{W/\hbar}\, dx = (-\hbar)^m \frac{\Gamma\left(\frac{k+1}{r} + m\right)}{\Gamma\left(\frac{\mu+1}{r}\right)} \delta_{k,rm+\mu}$$

as desired. $\square$

Corollary 3.2.19. For $W = x_1^{r_1} + x_2^{r_2}$ we have

$$\int_{\Xi_\mu} x_1^{k_1} x_2^{k_2} e^{W/\hbar}\, dx_1 dx_2 = (-\hbar)^{m_1+m_2} \frac{\Gamma\left(\frac{k_1+1}{r_1} + m_1\right)}{\Gamma\left(\frac{\mu_1+1}{r_1}\right)} \frac{\Gamma\left(\frac{k_2+1}{r_2} + m_2\right)}{\Gamma\left(\frac{\mu_2+1}{r_2}\right)}$$

if $k_i = r_i m_i + \mu_i$ for some $m_i \in \mathbb{Z}_{\geq 0}$ and the integral is zero otherwise.

Proof. For the Fermat singularity in rank 2, $W = x_1^{r_1} + x_2^{r_2}$, the x_1 and x_2 variables decouple and integration by parts can be used independently on each variable. Therefore, we need only apply the previous lemma to each variable in this case. $\square$

Lemma 3.2.20. For the rank 2 chain $W = x_1^{r_1} + x_1 x_2^{r_2}$ we have

$$\int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar)^{m_1+m_2} \frac{\Gamma\left(\frac{\mu_2+1}{r_2} + m_2\right)}{\Gamma\left(\frac{\mu_2+1}{r_2}\right)} \frac{\Gamma\left(\frac{\mu_1+1}{r_1} - \frac{\mu_2+1}{r_1 r_2} + m_1\right)}{\Gamma\left(\frac{\mu_1+1}{r_1} - \frac{\mu_2+1}{r_1 r_2}\right)}$$

if $k_1 = r_1 m_1 + \mu_1 + m_2$ and $k_2 = r_2 m_2 + \mu_2$ for some $m_i \in \mathbb{Z}_{\geq 0}$ and the integral is zero otherwise.

Proof. Performing integration by parts on the x_2 variable we find

$$\int e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = -\hbar \left(\frac{k_2+1}{r_2} - 1\right) \int e^{W/\hbar} x_1^{k_1-1} x_2^{k_2-r_2}\, dx_1 dx_2$$

$$= -\hbar \frac{\Gamma\left(\frac{k_2+1}{k_2}\right)}{\Gamma\left(\frac{k_2+1}{r_2} - 1\right)} \int e^{W/\hbar} x_1^{k_1-1} x_2^{k_2-r_2}\, dx_1 dx_2$$

Similarly, for the x_1 variable,

$$\int e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = -\hbar\Big(\frac{k_1+1}{r_1} - \frac{k_2+1}{r_1 r_2} - 1\Big) \int e^{W/\hbar} x_1^{k_1-r_1} x_2^{r_2}\, dx_1 dx_2$$

$$= -\hbar \frac{\Gamma\Big(\frac{k_1+1}{r_1} - \frac{k_2+1}{r_1 r_2}\Big)}{\Gamma\Big(\frac{k_1+1}{r_1} - \frac{k_2+1}{r_1 r_2} - 1\Big)} \int e^{W/\hbar} x_1^{k_1-r_1} x_2^{k_2}\, dx_1 dx_2.$$

Hence, repeated application of integration by parts yields the desired formulas. □

Lemma 3.2.21. For the rank 2 loop singularity $W = x_1^{r_1} x_2 + x_1 x_2^{r_2}$, we have

$$\int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar)^{m_1+m_2} \frac{\Gamma\Big(\frac{r_2 k_1 + r_2 - k_2 - 1}{r_1 r_2 - 1} + m_1\Big)}{\Gamma\Big(\frac{r_2 \mu_1 + r_2 - \mu_2 - 1}{r_1 r_2 - 1}\Big)} \frac{\Gamma\Big(\frac{r_1 k_2 + r_1 - k_1 - 1}{r_1 r_2 - 1} + m_2\Big)}{\Gamma\Big(\frac{r_1 \mu_2 + r_1 - \mu_1 - 1}{r_1 r_2 - 1}\Big)}$$

if $k_1 = r_1 m_1 + \mu_1 + m_2$ and $k_2 = r_2 m_2 + \mu_2 + m_1$ for some $m_i \in \mathbb{Z}_{\geq 0}$ and the integral is zero otherwise.

Proof. Performing integration by parts on the x_1 variable we find

$$\int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar) \frac{r_2 k_1 + r_2 - k_2 - r_1 r_2}{r_1 r_2 - 1} \int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1 - r_1} x_2^{k_2 - 1}\, dx_1 dx_2$$

so that

$$\int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar) \frac{\Gamma\Big(\frac{r_2 k_1 + r_2 - k_2 - 1}{r_1 r_2 - 1}\Big)}{\Gamma\Big(\frac{r_2 k_1 + r_2 - k_2 - 1}{r_1 r_2 - 1} - 1\Big)} \int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1 - r_1} x_2^{k_2 - 1}\, dx_1 dx_2$$

For the x_2 variable we have

$$\int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar) \frac{r_1 k_2 + r_1 - k_1 - r_1 r_2}{r_1 r_2 - 1} \int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1 - 1} x_2^{k_2 - r_2}\, dx_1 dx_2$$

so that

$$\int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar) \frac{\Gamma\left(\frac{r_1k_2+r_1-k_1-1}{r_1r_2-1}\right)}{\Gamma\left(\frac{r_1k_2+r_1-k_1-1}{r_1r_2-1} - 1\right)} \int_{\Xi_\mu} e^{W/\hbar} x_1^{k_1-1} x_2^{k_2-r_2}\, dx_1 dx_2$$

Repeated application of integration by parts gives the desired result. □

It is a straightforward exercise to generalise this to rank 3. We will, however, omit these calculations as they are largely irrelevant for our purposes.

Example 3.2.22. Let us now apply these formulas to the simple elliptic singularity $W = x_1^4 + x_1 x_2^3$. The basis element of highest degree is $x_1^3 x_2$. The marginal parameter is s_{31}. Remarkably, $\mathcal{J}_0^{(0)}(s)$ and $\mathcal{J}_{31}^{(1)}(s)$ can be identified with a certain hypergeometric function. Recall that the hypergeometric functions are defined through

$$_2F_1(\alpha, \beta, \gamma; z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{z^m}{m!}$$

where $(\cdot)_m$ is the Pochhammer symbol. The identifications are

$$\mathcal{J}_0^{(0)}(s) = {}_2F_1\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; -\frac{4s^3}{27}\right) = \sum_{m=0}^{\infty} \frac{(1/12)_m (7/12)_m}{(2/3)_m} \frac{(-1)^m 4^m s^{3m}}{27^m m!} \tag{3.2.14}$$

and

$$\mathcal{J}_{31}^{(1)}(s) = s \cdot {}_2F_1\left(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; -\frac{4s^3}{27}\right).$$

We will only give explicit details for $\mathcal{J}_0^{(0)}$ as $\mathcal{J}_{31}^{(1)}$ is similar. By definition of $\mathcal{J}_0^{(0)}$ we have

$$\mathcal{J}_0^{(0)}(\boldsymbol{s}) = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^9 \\ \deg(\boldsymbol{s}^{\boldsymbol{k}})=0}} c_0(l(\boldsymbol{k})) \frac{\boldsymbol{s}^{\boldsymbol{k}}}{\boldsymbol{k}!}.$$

As we have discussed before, we have $\boldsymbol{k} = (0, 0, \ldots, 0, i)$ for $i \in \mathbb{Z}_{\geq 0}$. For such a k, we have by definition of $l(\boldsymbol{k})$

$$l(k) = (3i, i).$$

Thus we have

$$\mathcal{J}_0^{(0)}(\boldsymbol{s}) = \sum_{i\in\mathbb{Z}_{\geq 0}} c_0\Big((3i,i)\Big)\frac{s^i}{i!}.$$

To calculate $c_0\Big((3i,i)\Big)$ we use Proposition 3.2.20. In that notation we find $\lambda_1 = \lambda_2 = 0$. Hence, we obtain the equations

$$3i = 4m_1 + m_2, \quad i = 3m_2.$$

This implies that $m_1 = 2m_2$. Relabelling $m_2 = m$ we have

$$c_0\Big((3i,i)\Big) = (-1)^m\left(\frac{1}{3}\right)_m\left(\frac{1}{6}\right)_{2m}.$$

Thus,

$$\mathcal{J}_0^{(0)}(s) = \sum_{m=0}^{\infty}(-1)^m\left(\frac{1}{3}\right)_m\left(\frac{1}{6}\right)_{2m}\frac{s^{3m}}{(3m)!}.$$

However, we observe that

$$(3m)! = (3^3)^m m!\cdot\prod_{j=1}^{m}\Big(j-\frac{1}{3}\Big)\Big(j-\frac{2}{3}\Big) = 27^m m!\left(\frac{2}{3}\right)_m\left(\frac{1}{3}\right)_m.$$

Hence to complete the identification given in (3.2.14), we must show that

$$4^m\left(\frac{1}{12}\right)_m\left(\frac{7}{12}\right)_m = \left(\frac{1}{6}\right)_{2m}.$$

One may proceed by induction. Indeed, this equation is trivially true for $m = 0$. Assuming that it is true for some m, observe that

$$\left(\frac{1}{6}\right)_{2m+2} = \Big(\frac{1}{6}+2m\Big)\Big(\frac{1}{6}+2m+1\Big)\left(\frac{1}{6}\right)_{2m} = 4\Big(\frac{1}{12}+m\Big)\Big(\frac{7}{12}+m\Big)\left(\frac{1}{6}\right)_{2m}.$$

Hence, with the induction hypothesis we have

$$\left(\frac{1}{6}\right)_{2m+2} = 4\Big(\frac{1}{12}+m\Big)\Big(\frac{7}{12}+m\Big)\cdot 4^m\left(\frac{1}{12}\right)_m\left(\frac{7}{12}\right)_m = 4^{m+1}\left(\frac{1}{12}\right)_{m+1}\left(\frac{7}{12}\right)_{m+1}$$

as claimed. Consequently, the primitive form we have found may be written as

$$\zeta = \frac{d^2x}{{}_2F_1\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; -\frac{4s^3}{27}\right)}.$$

This agrees with the original computation in [Sai81] and generalises the calculation of Noumi and Yamada [NY98]. Thus, the expression for the marginal flat coordinate $t := t_{31}$ of $W = x_1^4 + x_1x_2^3$ is

$$t = \frac{s \cdot {}_2F_1(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; -\frac{4s^3}{27})}{{}_2F_1(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; -\frac{4s^3}{27})}.$$

3.3 Mirror Symmetry

In previous sections we have alluded to the existence of mirror relations between FJRW theory and Saito theory. In this section, we review how to precisely construct the mirror Landau-Ginzburg model W^T given a Landau-Ginzburg model W. This construction was first proposed by Berglund, Henningson and Hübsch [BH93; BH95] and Krawitz [Kra10] subsequently provided the explicit mirror map as a Frobenius algebra isomorphism for the chiral rings.

3.3.1 A Review of the Krawitz Mirror Map

We first recall the definition of the state space of the orbifolded Landau-Ginzburg model given in section 2.3. To this end, let $\mathscr{Q}_W$ be the local algebra. We choose a subgroup $G \leq G_W^{\max}$. Recall that for the full A-model to be defined, FJRW theory requires that G be admissible. Similarly, the full B-model is currently only defined for the choice $G = \{1\}$. Here, however, the orbifolded A and B-model chiral rings can in principle be defined for any admissible subgroup G.

For $g \in G$, denote $W_g := W|_{\mathrm{Fix}(g)}$ and let e_g be the standard volume form on $\mathrm{Fix}(g)$.

Then the state space of the Landau-Ginzburg model $(\mathbb{C}^N, W, G)$ is

$$\mathscr{D}_{W,G} := \bigoplus_{g\in G} \Big(\mathscr{D}_{W_g} \cdot e_g\Big)^G$$

where $\mathscr{D}_{W_g}$ is the local algebra of the restriction W_g. We write the elements of $\mathscr{D}_{W,G}$ as $p(x_1, \ldots, x_N)e_g$ where e_g is the standard volume form on $\mathrm{Fix}(g)$. For example if $g = \mathrm{id}$, then $\mathrm{Fix}(g) = \mathbb{C}^N$ and $e_g = dx_1 \wedge \cdots dx_N$. If $\mathrm{Fix}(g) = \{0\}$, we put $e_g = 1$.

Roughly speaking, the Krawitz mirror map interchanges elements of the local algebra with elements of the symmetry group.

More precisely, for a superpotential W, let E_W be the exponent matrix. Recall that the mirror polynomial W^T is defined via the exponent matrix $E_{W^T} := E_W^T$. Denote

$$E_W^{-1} = \Big(\rho_1|\cdots|\rho_N\Big), \text{ where } \rho_k = \begin{pmatrix} \varphi_1^{(k)} \\ \vdots \\ \varphi_N^{(k)} \end{pmatrix}.$$

We recall from Section 2.1.2 that the group $G_W^{\max}$ is generated by the ρ_k. These act on the variables as

$$\rho_k x_j = \exp\Big(2\pi i \varphi_j^{(k)}\Big) x_j$$

and acts trivially on constants. On the other hand, letting

$$E_W^{-1} = \begin{pmatrix} — & \overline{\rho}_1 & — \\ & \vdots & \\ — & \overline{\rho}_N & — \end{pmatrix},$$

we observe that the group $G_{W^T}^{\max}$ is generated by the $\overline{\rho}_k$. We recall that there exists a dual group $G^T \leq G_{W^T}^{\max}$ such that $(G^T)^T = G$ and $(G_W^{\max})^T = \{1\}$. See section 2.7 of [Kra10] for details of this construction.

For $g \in G$, denote the set of fixed indices by F_g. That is to say, g can be written in the form $g = (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_N})$ with $\theta_j \in [0, 1)$ and so we put $F_g = \{j \,|\, \theta_j = 0\}$. The following proposition will allow us to define the Krawitz mirror map.

Proposition 3.3.1. Let $h \in G_W^{\max}$. Write

$$H = \prod_{j \in F_h} x_j^{\mu_j}$$

so that

$$H \cdot e_h \in \mathscr{D}_{W_h} \cdot e_h.$$

Denote $\gamma_H = \prod_{j \in F_h} \overline{\rho}_j^{\mu_j+1} \in G_{W^T}^{\max}$. Then if W is of Fermat or chain type, there is a unique element

$$\Gamma_h = \prod_{j \in F_{\gamma_H}} x_j^{r_j}$$

with

$$\Gamma_h \cdot e_{\gamma_H} \in \mathscr{D}_{W_{\gamma_H}^T} \cdot e_{\gamma_H}$$

such that $h = \prod_{j \in F_{\gamma_H}} \rho_j^{r_j+1}$. If W is a loop potential, there is a unique choice of Γ_h as above, unless $\gamma_H = 1$.

In the above notation, if W is a loop potential with $\gamma_H = 1$, Krawitz gives a prescription for choosing r_j. For details and a proof of the above proposition more generally, see Lemma 2.2 of [Kra10].

Definition 3.3.2. The unprojected mirror map

$$\bigoplus_{h \in G} \mathscr{D}_{W_h} \cdot e_h \to \bigoplus_{\gamma \in G^T} \mathscr{D}_{W_{\gamma_H}^T} \cdot e_{\gamma_H}$$

is defined on generators as

$$H \cdot e_h \mapsto \Gamma_h \cdot e_{\gamma_H}.$$

Given this definition, we have the following two results due to Krawitz [Kra10, Theorem 2.3, Theorem 2.4]

Theorem 3.3.3. The unprojected mirror map is a vector space isomorphism.

Corollary 3.3.4. There is a vector space isomorphism

$$\mathscr{Q}_{W,G} \cong \mathscr{Q}_{W^T,G^T}.$$

The proof Theorem 3.3.3 is to simply replace G with G^T and recall that $(G^T)^T = G$. The proof of Corollary 3.3.4 is to restrict to the G-invariant part of the unprojected mirror map. The dual group G^T is defined so that the projected mirror map is well defined. The vector space isomorphism given in the above corollary is called the *Krawitz mirror map.*

Remark 3.3.5. An alternative shorthand notation that Krawitz introduces is to write $\xi = m\,|g\rangle \in \mathscr{Q}_{W,G}$ for a monomial $m \in \mathscr{Q}_{W_g}$. Here the volume form is omitted in the notation as it is fixed once the element g is prescribed. The Krawitz mirror map $\Psi : \mathscr{Q}_{W^T} \to \mathscr{Q}_{W,G}$ can then be written as

$$\Psi(x_i) = \begin{cases} x_i\,|1\rangle & \text{if } x_i \text{ is a loop variable with } a_i = N = 2 \\ 1\,\left|\rho_i \prod_{j=1}^N \rho_j\right\rangle & \text{otherwise} \end{cases}$$

and furthermore,

$$\Psi\left(\prod_{j=1}^{N} x_j^{\alpha_j}\right) \in \mathscr{Q}_{W_g}, \text{ where } g = \prod_{j=1}^{N} \rho_j^{\alpha_j+1}. \tag{3.3.1}$$

Example 3.3.6. Let us first consider the case of the ungauged Landau Ginzburg model $(W, G) = (x_1^3 + x_1x_2^3, \{1\})$ corresponding to the E_7 singularity. Denote the chiral ring as $\mathscr{Q}_{W,1} = \mathscr{Q}_W$ whereby the mirror is

$$\mathscr{Q}_{W^T,G^{\max}_{W^T}} = \bigoplus_{g\in G^{\max}_{W^T}} \left(\mathscr{Q}_{W^T_g} \cdot e_g\right)^{G^{\max}_{W^T}}.$$

Since G is the trivial group we take $h = \mathrm{id}$. Note that $F_h = \{1, 2\}$. For the first basis element of $\mathscr{Q}_W$ we take $H = 1 = x_1^0x_2^0$ i.e. $\mu_1 = \mu_2 = 0$. We have

$$E_W^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{9} & \frac{1}{3} \end{pmatrix}.$$

Thus, we have $\overline{\rho}_1 = \left(e^{2\pi i/3}, 1\right)$ and $\overline{\rho}_2 = \left(e^{-2\pi i/9}, e^{2\pi i/3}\right)$. Define $\gamma_H = \overline{\rho}_1^1\overline{\rho}_2^1 = \left(e^{4\pi i/9}, e^{2\pi i/3}\right)$. Notice that $F_{\gamma_H} = \varnothing$. Hence, under the mirror map we find

$$1 \cdot e_{\text{id}} \mapsto e_{\overline{\rho}_1^1\overline{\rho}_2^1}.$$

Similarly,

$$\begin{aligned} x_1 \cdot e_{\text{id}} &\mapsto e_{\overline{\rho}_1^2\overline{\rho}_2^1}, \\ x_2 \cdot e_{\text{id}} &\mapsto e_{\overline{\rho}_1^1\overline{\rho}_2^2}, \\ x_1^2 \cdot e_{\text{id}} &\mapsto e_{\overline{\rho}_1^3\overline{\rho}_2^1}, \\ x_2^2 \cdot e_{\text{id}} &\mapsto x_1^3 e_{\overline{\rho}_1^1\overline{\rho}_2^3}, \\ x_1x_2 \cdot e_{\text{id}} &\mapsto e_{\overline{\rho}_1^2\overline{\rho}_2^2}, \\ x_1^2x_2 \cdot e_{\text{id}} &\mapsto e_{\overline{\rho}_1^3\overline{\rho}_2^2}. \end{aligned}$$

We note that for each good basis element except x_2^2, the set F_{γ_H} was empty. Only for x_2^2 did we have $\text{Fix}(\gamma_H) = \text{Fix}(\overline{\rho_1}^1\overline{\rho_2}^3) = \text{Fix}(\text{id}) = \mathbb{C}^2 \neq \{0\}$. This is related to the notion of *concavity* that we shall explain in Chapter 6.

3.3.2 Closed Landau-Ginzburg Mirror Symmetry

The main result of [Kra10] is that, if all charges $q_i < \frac{1}{2}$, it is possible to upgrade the vector space isomorphism of state spaces to a Frobenius algebra isomorphism. The condition $q_i < \frac{1}{2}$ ensures that all algebra generators are narrow; this is a mild assumption that only excludes the A_1 singularity and chain polynomials with $a_N = 2$. The Frobenius algebra isomorphism can be further upgraded to an isomorphism of Frobenius manifolds as done in [HLSW22].

Theorem 3.3.7. Let $W : \mathbb{C}^N \to \mathbb{C}$ be an invertible polynomial where no variable has charge $q_i = 1/2$. Then the Frobenius manifold from FJRW theory of W and $G = G_W^{\max}$, and the Frobenius manifold from Saito theory of W^T, are isomorphic.

The strategy for the proof of the above theorem is as follows. One computes the three point correlation functions and a subset of four point correlators. These are matched with the mirror FJRW invariants. Then in [LLSS17; HLSW22], the authors use the WDVV equations and topological recursion relations to match both A and B-model invariants.

The three and four point correlation functions of the B-model are computed via the method developed in [LLSS17]. For a simple singularity, it is possible to uniquely rewrite s_μ in terms of the flat coordinates t_μ. Under this change of variables, we can rewrite $\mathcal{J}_\mu^{(r)} \in \mathbb{C}[\boldsymbol{s}]$ as $\mathcal{J}_\mu^{(r)} \in \mathbb{C}[\boldsymbol{t}]$. The polynomials $\mathcal{J}_\mu^{(2)} \in \mathbb{C}[\boldsymbol{t}]$ in fact give rise to genus zero enumerative invariants in the following way. On the state space $\mathscr{D}_W$, recall that we have a pairing η. We let $\eta_{\mu\nu}$ denote the matrix of η with respect to the standard basis. It can be calculated that $\eta_{\mu,\nu}$ is the anti-diagonal matrix with 1's along the anti-diagonal. From Saito's Frobenius manifold, we have a Frobenius potential F satisfying the WDVV equations. It is shown further in [LLSS17] that F satisfies

$$\partial_{t_\mu} F(\boldsymbol{t}) = \sum_{\nu \in B} \eta_{\mu\nu} \mathcal{J}_\mu^{(2)}(\boldsymbol{t}).$$

One then integrates these equations to obtain F. From the Frobenius potential F, we define the B-model correlation functions,

$$\langle \tau_{a_1}, \cdots, \tau_{a_k} \rangle := \left. \frac{\partial^k F}{\partial t_{a_1} \cdots \partial t_{a_k}} \right|_{\boldsymbol{t}=0}.$$

This gives a concrete method of calculating the Frobenius potential F.

Example 3.3.8. Consider the E_7 singularity $W = x_1^3 + x_1 x_2^3$. The $\hbar^{-2}$ terms of the J function, up to quadratic order in s, are given by

$$(s_{00}s_{02} + \frac{1}{2}s_{01}^2)x_1^2 + (s_{00}s_{21} + s_{01}s_{20} + s_{10}s_{11} - \frac{3}{2}s_{02}^2)x_1x_2^2 + (s_{00}s_{11} + s_{01}s_{10})x_1x_2$$
$$+ s_{00}s_{01}x_1 + (s_{00}s_{20} + \frac{1}{2}s_{10}^2 - 3s_{01}s_{02})x_2^2 + s_{00}s_{10}x_2 + \frac{1}{2}s_{00}^2.$$

For the cubic terms, we may simply replace $t_\mu = s_\mu$ as all other terms will be of higher order. After tedious calculations we find that the cubic part of the Frobenius potential is

$$F|_{\text{cubic terms}} = t_{01}t_{20}t_{00} + t_{10}t_{11}t_{00} + \frac{1}{2}t_{21}t_{00}^2 + \frac{1}{2}t_{10}^2t_{01} - \frac{3}{2}t_{02}^2t_{00} - \frac{3}{2}t_{01}^2t_{02}.$$

To obtain the full semi-simple cohomological field theory, one may use Givental's formula in [Giv96; Giv01] for the higher genus theory with descendants. Hence, the Frobenius manifold isomorphism of the Theorem 3.3.7 is enough to imply the mirror symmetry for the full cohomological field theory.

3.3.3 Using Mirror Symmetry to Calculate FJRW Phases

Recall Definition 2.2.6 where we defined the FJRW phases $0 \leq \Theta_i^\gamma < 1$ for an internal marked point with decoration $\gamma \in G_W^{\max}$. Let us now calculate the FJRW phases of internal marked points for any Fermat, chain or loop polynomial in rank N.

Definition 3.3.9. Let W be a Fermat, chain or loop polynomial in rank N. Let W^T be the BHK mirror polynomial with charges $q_1^T, \dots, q_N^T$ and exponent matrix $E_{W^T} = (r_{ij})$. Given a W^T-spin curve, the FJRW phase Θ^μ of a marked point with twist μ is given by

$$\Theta^\mu = E_{W^T}^{-1}\begin{pmatrix}\mu_1\\ \vdots\\ \mu_N\end{pmatrix} + \begin{pmatrix}q_1^T\\ \vdots\\ q_N^T\end{pmatrix} \tag{3.3.2}$$

where x^μ is the unique standard basis element of the B-model local algebra such that $\Psi(x^\mu) \in \mathscr{D}_{W_g^T}$ with $g = e^{2\pi i \Theta^\mu}$.

Remark 3.3.10. The significance of this definition is that the good basis elements on the B-model in fact determine the FJRW phases and twists of the A-model. Indeed, we first observe that $g = e^{2\pi i\Theta^\mu}$, with Θ^μ given by (3.3.2), is the group element sector that is the image of x^μ under the Krawitz mirror map Ψ. Recalling that

$$E_{W^T}\begin{pmatrix} q_1^T \\ \vdots \\ q_N^T \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

we find that (3.3.2) is equivalent to

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = E_{W^T}\begin{pmatrix} \Theta_1^\mu \\ \vdots \\ \Theta_N^\mu \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

or equivalently

$$\mu_i = \sum_{j=1}^{N} r_{ij}\Theta_j^\mu - 1$$

in agreement with the twist in Proposition 2.2.7. Thus if we assume that the index μ of the standard good basis element is equal to the FJRW twist of a marked point, we find that Θ^μ given in (3.3.2) coincides with the FJRW phase.

For convenience, we present two corollaries which give the explicit expressions for FJRW phases in the case of a chain and loop polynomial in generic rank.

Corollary 3.3.11. Let W be a chain polynomial of rank N. Then the j^{th} entry of the phase Θ^μ is

$$\Theta_j^\mu = \sum_{j\leq i\leq N} (-1)^{i+j}(\mu_i + 1) \prod_{j\leq l\leq i} \frac{1}{r_l}.$$

Corollary 3.3.12. Let W be a loop polynomial of rank N. Then the j^{th} entry of the phase Θ^μ is

$$\Theta_j^\mu = \sum_{1\leq i<j} (-1)^{N+i+j}\frac{\mu_i+1}{D}\prod_{k=i+1}^{j-1} r_k + \sum_{j\leq i\leq N} (-1)^{i+j}\frac{\mu_i+1}{D}\prod_{k=i+1}^{N} r_k \cdot \prod_{l=1}^{j-1} r_j.$$

Both corollaries follow from Definition 3.3.9 and Proposition 2.1.9.

CHAPTER 4

OPEN SAITO-GIVENTAL THEORY

In this section, we review and extend the work of Gross-Kelly-Tessler [GKT22a; GKT22b] by defining open Saito potentials for any chain or loop polynomial. To be able to handle any chain and loop polynomial of rank two, we define B-model selection rules that are expected from the mirror open FJRW theory. We then give examples of primary open potentials for ADE and simple elliptic singularities. These are new examples of open potentials and are constructed in Examples 4.2.2, 4.2.4 and 4.2.9 using results from Chapter 3. In the case of simple elliptic singularities, we further explore the modularity properties. Finally, we present a perturbative algorithm for the computation of open Saito potentials. To this end, Theorems 4.3.4 and 4.3.9 and their corollaries are the important results used in Chapter 5.

4.1 Open Saito-Givental Potentials

In this section, we review and extend the open Landau-Ginzburg B-model of Gross, Kelly, and Tessler [GKT22a; GKT22b]. We change notation slightly and denote an invertible polynomial by W_0 rather than W. For the following definition, recall that for $W_0 = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{r_{ij}}$, the *exponent matrix* is $E_{W_0} = (r_{ij})$. We also recall that the mirror polynomial W_0^T is defined by the transposed exponent matrix $E_{W_0^T} := (E_{W_0})^T$.

Definition 4.1.1. Let W_0 be an invertible polynomial with exponent matrix $E_{W_0} = (r_{ij})$.

Let W_0^T be the mirror polynomial with charges $q_1^T, \dots, q_N^T$. For each $\mu = (\mu_1, \dots, \mu_N) \in B \subseteq \mathbb{Z}^N_{\geq 0}$, define the *internal phase*

$$\Theta^\mu = \begin{pmatrix} \Theta_1^\mu \\ \vdots \\ \Theta_N^\mu \end{pmatrix} := (E^T_{W_0})^{-1} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} + \begin{pmatrix} q_1^T \\ \vdots \\ q_N^T \end{pmatrix} \in \mathbb{Q}^N. \tag{4.1.1}$$

Define the *boundary phase* $\Theta^{x_i} \in \mathbb{Q}^N$ as the vector with components

$$\Theta_j^{x_i} := \delta_{ij} + q_j^T - 2(E^{-1}_{W_0})_{ij} \tag{4.1.2}$$

and the *root phase* as

$$\Theta_j^{\text{root}} := 1 - q_j^T. \tag{4.1.3}$$

Let I be a multiset containing elements $\mu \in B$. Let $\boldsymbol{d} \in \mathbb{Z}^I_{\geq 0}$. We say that $(k_1, \dots, k_N) \in \mathbb{Z}^N_{\geq 0}$ is *balanced* with respect to I and $\boldsymbol{d}$ if the following conditions hold:

$$q_i^T(2|I| + \sum_{j=1}^N k_j - 1) - 2\sum_{\mu \in I} \Theta_i^\mu - \sum_{j=1}^N k_j \Theta_i^{x_j} - \Theta_j^{\text{root}} =: -1 - e_i \in \mathbb{Z}, \tag{4.1.4}$$

$$e_i = k_i \mod 2, \tag{4.1.5}$$

$$\sum_{i=1}^{|I|} 2d_i + \sum_{i=1}^N e_i = 2|I| + \sum_{j=1}^N k_j - 2. \tag{4.1.6}$$

Taken together, we refer to these three conditions as *selection rules.* We write $(k_1, \dots, k_N) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$ to denote $(k_1, \dots, k_N)$ that is balanced with respect to I and $\boldsymbol{d}$.

Remark 4.1.2. For the case of the rank two Fermat polynomial, the conditions (4.1.4)-(4.1.6) provide enumerative constraints for the mirror FJRW theory. See Proposition 2.30 of [GKT22b].

To illustrate the selection rules, we write them out explicitly for chain and loop polynomials in rank two in the following examples. These will be important examples for the rest of the paper.

Example 4.1.3. Let $W_0 = x_1^{r_1} + x_1 x_2^{r_2}$ which implies $W_0^T = x_1^{r_1} x_2 + x_2^{r_2}$. To write the selection rules, we first calculate the phases and twists of each marked point. The charges of W_0^T are

$$q_1^T = \frac{r_2 - 1}{r_1 r_2}, \quad q_2^T = \frac{1}{r_2}.$$

Fix a multiset I whose elements are $\mu_n = (a_n, b_n)$ and we denote the corresponding internal phases by Θ^{μ_n}. Via equation (4.1.1) we have

$$\Theta^{\mu_n} = \Big(\frac{r_2 a_n - b_n}{r_1 r_2} + \frac{r_2 - 1}{r_1 r_2}, \frac{b_n + 1}{r_2}\Big).$$

The root and boundary phases are

$$\Theta^{\text{root}} = \Big(1 - \frac{r_2 - 1}{r_1 r_2}, 1 - \frac{1}{r_2}\Big), \quad \Theta^{x_1} = \Big(1 + \frac{r_2 - 1}{r_1 r_2} - \frac{2}{r_1}, \frac{1}{r_2}\Big), \quad \Theta^{x_2} = \Big(\frac{r_2 - 1}{r_1 r_2} + \frac{2}{r_1 r_2}, 1 - \frac{1}{r_2}\Big).$$

The integral degree constraint (4.1.4) now gives

$$q_j^T (2|I| + k_1 + k_2 - 1) - \Theta_j^{\text{root}} - k_1 \Theta_j^{x_1} - k_2 \Theta_j^{x_2} - 2 \sum_{n=1}^{|I|} \Theta_j^{\mu_n} = -1 - e_j \in \mathbb{Z}.$$

For $j = 1$, this simplifies to

$$e_1 = k_1 + \frac{2(k_2 - r_2 k_1 + \sum_{n=1}^{|I|} r_2 a_n - b_n)}{r_1 r_2} \in \mathbb{Z}.$$

The graded condition (4.1.5) is

$$e_1 = k_1 \mod 2.$$

By combining the above two equations and setting

$$r_1(I) := \sum_{n=1}^{|I|} a_n, \quad r_2(I) := \sum_{n=1}^{|I|} b_n$$

we find

$$r_2k_1 - k_2 \equiv r_2r_1(I) - r_2(I) \mod r_1r_2.$$

Similarly, for the second integral degree constraint where we take $j = 2$, we ultimately find

$$k_2 \equiv r_2(I) \mod r_2.$$

The dimension condition, given as equation (4.1.6), reads

$$\sum_{i\in I} 2d_i + k_1 + k_2 + \frac{2(k_2 - r_2k_1 + r_2r_1(I) - r_2(I))}{r_1r_2} + \frac{2(r_2(I) - k_2)}{r_2} = 2|I| + k_1 + k_2 - 2.$$

To simplify notation, we set

$$m(I, \boldsymbol{d}) := r_1r_2 + r_2r_1(I) + r_1r_2(I) - r_2(I) + \sum_{i\in I} r_1r_2(d_i - 1). \tag{4.1.7}$$

Hence, for $W_0 = x_1^{r_1} + x_1x_2^{r_2}$, we have found that $(k_1, k_2) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$ if and only if

$$r_2k_1 - k_2 \equiv r_2r_1(I) - r_2(I) \mod r_1r_2, \quad k_2 \equiv r_2(I) \mod r_2, \quad r_2k_1 + r_1k_2 - k_2 = m(I, \boldsymbol{d}).$$

Example 4.1.4. Let $W_0 = x_1^{r_1}x_2 + x_1x_2^{r_2}$. The charges of $W_0^T = W_0$ are

$$q_1 = \frac{r_2 - 1}{r_1r_2 - 1}, \quad q_2 = \frac{r_1 - 1}{r_1r_2 - 1}.$$

Fix a multiset I whose elements are $\mu_n = (a_n, b_n)$. The corresponding internal phases are

$$\Theta^{\mu_n} = \Big(\frac{(r_2 - 1)(a_n + 1) - b_n}{r_1r_2 - 1}, \frac{(r_1 - 1)(b_n + 1) - a_n}{r_1r_2 - 1}\Big).$$

The root and boundary phases are

$$\Theta^{\text{root}} = \Big(1 - \frac{r_2 - 1}{r_1r_2 - 1}, 1 - \frac{r_1 - 1}{r_1r_2 - 1}\Big),$$

$$\Theta^{x_1} = \Big(1+\frac{r_2-1}{r_1r_2-1}-2\frac{r_2}{r_1r_2-1}, \frac{r_1+1}{r_1r_2-1}\Big), \quad \Theta^{x_2} = \Big(\frac{r_2+1}{r_1r_2-1}, 1+\frac{r_1-1}{r_1r_2-1}-2\frac{r_1}{r_1r_2-1}\Big),$$

One may perform similar calculations to the previous example to find the integral degree constraints as

$$r_2k_1-r_2 \equiv r_2r_1(I)-r_2(I) \mod (r_1r_2-1), \quad r_1k_2-r_1 \equiv r_1r_2(I)-r_1(I) \mod (r_1r_2-1).$$

Furthermore, the dimension condition gives

$$r_1k_2 + r_2k_1 - k_1 - k_2 = m(I, \boldsymbol{d}).$$

where

$$m(I, \boldsymbol{d}) = r_1r_2(I) + r_2r_1(I) - r_1(I) - r_2(I) + (r_1r_2-1) + \sum_{i\in I}(r_1r_2-1)(d_i-1) \quad (4.1.8)$$

In order to define open Saito potentials, we show that $\mathbb{Q}$-Bal$(I, \boldsymbol{d})$ is not always empty.

Proposition 4.1.5. For each $i = 1, \dots, N$, denote by $\hat{e}_{x_i}$ as the vector with one in the i^{th} place and zeroes otherwise. For $I_i = \{\hat{e}_{x_i}\}$, we have $(\delta_{i,1}, \dots, \delta_{i,N}) \in \mathbb{Q}$-Bal$(I_i, d = 0)$.

Proof. Fix $i = 1, \dots, N$. We show that these sets satisfy the 3 conditions (4.1.4), (4.1.5) and (4.1.6). Indeed, the left hand side of the integral degree condition reads

$$2q_j^T - \Theta_j^{\text{root}} - \Theta_j^{x_i} - 2\Theta_j^{\mu=\hat{e}_{x_i}}$$

for each $j = 1, \dots, N$. After substituting the phases, we find that the above expression is equal to $-1-\delta_{ij}$ which is certainly an integer. Furthermore, the grading condition is immediately satisfied.

Finally, for the left hand side of the dimension condition with $d = 0$, we have

$$\sum_{j=1}^{N} \delta_{ij} = 1.$$

On the other hand, since $|I_i| = 1$, the right hand side becomes

$$2 + \sum_{j=1}^{N} \delta_{ij} - 2 = 1$$

as desired.

$\square$

Corollary 4.1.6. The set $\mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$ is non-empty for some $I \neq \varnothing$ and $\boldsymbol{d} = \mathbf{0}$.

With this corollary, we now make the following definition which extends Definitions 4.10 and 4.12 in [GKT22b] to the case of a rank N polynomial.

Definition 4.1.7. Let $\mathcal{I}$ be a multiset containing elements $\mu \in B$. Given W_0, let $\text{Ideal}(\mathcal{I})$ be an ideal of $\mathbb{Q}[t_{\mu,d} \,|\, \mu \in B, d \in \mathbb{Z}_{\geq 0}]$ that is generated by monomials of the form $\prod_d t_{\mu,d}^{n_d}$ so that $\sum n_d$ is greater than the multiplicity of the element $\mu \in \mathcal{I}$. Consider the ring $A_{\mathcal{I},\text{sym}} := \mathbb{Q}[t_{\mu,d} \,|\, \mu \in B, d \in \mathbb{Z}_{\geq 0}]/\text{Ideal}(\mathcal{I})$. Given $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}} \in \mathbb{Q}$ such that $(k_1, \ldots, k_N) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$, we define an *open ancestor Saito-Givental potential*

$$W^{\nu,\text{sym}} := \sum_{l \geq 0} \sum_{A=(I,\boldsymbol{d}) \in \mathcal{A}_l} \sum_{\substack{k_1,\ldots,k_N \geq 0, \\ k_1,\ldots,k_N \in \mathbb{Q}\text{-Bal}(I,\boldsymbol{d})}} (-1)^{l-1} \frac{\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}}{|\text{Aut}(A)|} \prod_{i=1}^{l} t_{\mu_i,d_i} \prod_{i=1}^{N} x_i^{k_i} \in A_{\mathcal{I},\text{sym}}[[x_1, \ldots, x_N]]$$

where $\mathcal{A}_l$ denotes the set whose elements are cardinality l multisets of tuples $I = \{(\mu_i)\}_{i=1}^{l}$, together with the descendant vector $\boldsymbol{d}$ and $\text{Aut}(A)$ is the automorphism group of the multiset A. Moreover, we require that $W^{\nu,\text{sym}}$ is quasi-homogeneous of degree one:

$$\mathcal{E} W^{\nu,\text{sym}} = W^{\nu,\text{sym}}.$$

Abusing terminology, we also say that $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$ is balanced if $(k_1, \ldots, k_N) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$.

We will comment on the finiteness of this sum in Remark 4.3.2.

4.2 Primary Open Saito Potentials

An important special case of open Saito potentials are *primary* open Saito potentials where the descendant vector $\boldsymbol{d} = \mathbf{0}$. Given W_0, one may calculate the flat coordinates and change variable in the versal deformation $W_{\boldsymbol{s}}$ to write the s_μ parameters in terms of the flat coordinates t_μ. We identify the formal parameters $t_{\mu,0}$ of the open Saito potential with the flat coordinates t_μ via $t_{\mu,0} = t_\mu$. In Corollaries 4.3.7 and 4.3.12, we provide justification for this identification in rank two. We write

$$W_{\boldsymbol{t}} := \sum_{l\geq 0} \sum_{k_1,\ldots,k_N\geq 0} \sum_{A=(\mu_i,0)\in\mathcal{A}_l} (-1)^{l-1} \frac{\nu_{k_1,\ldots,k_N,I,\mathbf{0}}}{|\mathrm{Aut}(A)|} \prod_{i=1}^{l} t_{\mu_i,0} \prod_{i=1}^{N} x_i^{k_i} \in A_{\mathcal{I},\mathrm{sym}}[[x_1,\ldots,x_N]].$$

Using Proposition 4.1.5 and equation (3.2.5), the first order terms in $W_{\boldsymbol{s}(\boldsymbol{t})}$ are balanced. Moreover, we note that since $W_{\boldsymbol{s}}$ is quasi-homogeneous of degree one with respect to the grading operator $\mathcal{E}$, primary open Saito potentials $W_{\boldsymbol{s}(\boldsymbol{t})}$ must also be quasi-homogeneous of degree one by equation (3.2.5).

Observation 4.2.1. The rational coefficients $\nu_{k_1,\ldots,k_N,I,\mathbf{0}}$ for small l are given by the following reasoning. In a primary open Saito potential $W_{\boldsymbol{t}}$, for $l = 0$ we must have $A = \varnothing$ and $|\mathrm{Aut}(A)| = 1$. Since there are no t variables in these terms, this must agree with W_0 and hence $\nu = -1$. Similarly, for $l = 1$, we must have $\nu = 1$ via equation (3.2.5).

We note that the change of variables s_μ to t_μ cannot necessarily be done uniquely. This means that primary open Saito potentials W are not unique in general. In other words, the coefficients $\nu_{k_1,\ldots,k_N,I,\mathbf{0}}$ are not uniquely defined. We say that they exhibit a *wall-crossing* structure. This is the key point of the paper that is discussed in this chapter and the next.

To illustrate the non-uniqueness issue, we perform concrete calculations of primary open Saito potentials and chamber indices for certain simple and elliptic singularities.

4.2.1 ADE Singularities

In this subsection, we give explicit formulas for the open Saito potentials in the case of D_4 and E_7 singularities. These are less well studied in the literature as these are chain polynomials.

Example 4.2.2. Using Corollary 3.2.17, we see that $dx_1 \wedge dx_2$ is a primitive form for the D_4 singularity $(W_0)_{D_4} = x_1^3 + x_1 x_2^2$. Explicitly writing out the Γ functions using Lemma 3.2.20, we find the flat coordinates are

$$\begin{aligned} t_{20} &= s_{20} \\ t_{01} &= s_{01} \\ t_{10} &= s_{10} - \frac{1}{12} s_{20}^2 \\ t_{00} &= s_{00} - \frac{1}{6} s_{10} s_{20} + \frac{7}{216} s_{20}^3. \end{aligned}$$

These may be uniquely inverted to give

$$\begin{aligned} s_{20} &= t_{20} \\ s_{01} &= t_{01} \\ s_{10} &= t_{10} + \frac{1}{12} t_{20}^2 \\ s_{00} &= t_{00} + \frac{1}{6} t_{10} t_{20} - \frac{1}{54} t_{20}^3. \end{aligned}$$

We thus find

$$W_{\boldsymbol{t}} = x_1^3 + x_1 x_2^2 + t_{20} x_1^2 + t_{01} x_2 + \left(t_{10} + \frac{1}{12} t_{20}^2\right) x_1 + t_{00} + \frac{1}{6} t_{10} t_{20} - \frac{1}{54} t_{20}^3. \tag{4.2.1}$$

Example 4.2.3. The D_4 singularity is also given by $(\tilde{W}_0)_{D_4} = x_1^3 + x_2^3$. We find the flat coordinates for $(\tilde{W}_0)_{D_4}$ are

$$t_{11} = s_{11}$$
$$t_{01} = s_{01}$$
$$t_{10} = s_{10}$$
$$t_{00} = s_{00} + \frac{1}{54}s_{11}^3.$$

Therefore, the open Saito generating potential is

$$W_{\boldsymbol{t}} - W_0 = t_{11}x_1^2 + t_{01}x_2 + t_{10}x_1 + t_{00} - \frac{1}{54}t_{11}^3. \tag{4.2.2}$$

We will later define a notion of equivalence of open Saito theories that will clarify these calculations.

Example 4.2.4. For the E_7 singularity $W_0 = x_1^3 + x_1x_2^3$, the open Saito potential is given by

$$\begin{aligned} W_{\boldsymbol{t}} - W_0 = {} & t_{21}x_1^2x_2 + \left(t_{20} - \frac{4t_{21}^3}{81}\right)x_1^2 + \left(t_{11} + \frac{4t_{21}t_{20}}{9} - \frac{t_{21}^4}{243}\right)x_1x_2 + t_{02}x_2^2 \\ & + \left(t_{10} + \frac{t_{02}t_{21}}{3} - \frac{5t_{11}t_{21}^2}{54} + \frac{5t_{20}^2}{18} - \frac{5t_{20}t_{21}^3}{243} + \frac{t_{21}^6}{19683}\right)x_1 \\ & + \left(t_{01} + \frac{t_{10}t_{21}}{9} + \frac{t_{11}t_{20}}{9} - \frac{t_{11}t_{21}^3}{486} + \frac{t_{20}^2t_{21}}{54} - \frac{t_{20}t_{21}^4}{4374}\right)x_2 \\ & + \left(t_{00} - \frac{t_{01}t_{21}^2}{27} + \frac{t_{02}t_{11}}{3} + \frac{4t_{02}t_{20}t_{21}}{27} - \frac{t_{02}t_{21}^4}{729} + \frac{2t_{10}t_{20}}{9} - \frac{t_{11}^2t_{21}}{27} - \frac{t_{11}t_{20}t_{21}^2}{81} \right. \\ & \left. + \frac{4t_{20}^3}{243} - \frac{t_{20}^2t_{21}^3}{729} + \frac{t_{20}t_{21}^6}{118098} - \frac{t_{21}^9}{28697814}\right). \end{aligned}$$

Remark 4.2.5. It is not obvious how to identify the open Saito potential for $(W_0)_{D_4} = x_1^3 + x_1x_2^2$ found in this thesis with the open Saito potential $F^o_{D_4}$ found in Example 5.5 of [BB21]. We will comment on this discrepancy in Appendix B.

4.2.2 Modularity and Elliptic Singularities

In this subsection, we present computations of open Saito potentials for some elliptic singularities.

The following result characterises the primitive forms for elliptic singularities given originally in [Sai83a].

Proposition 4.2.6. Let W be an elliptic singularity of rank 3, using Morse stabilisation if necessary. Let s the marginal versal deformation parameter corresponding to the standard good basis element x^{μ_c}. That is, $\deg x^{\mu_c} = 1$ and $\deg s = 0$. Consider the family of elliptic curves $E_s = \{W_0 + sx^{\mu_c} = 0\} \subset \mathbb{P}^2$. Let

$$z(1-z)\frac{d^2\pi}{dz^2} + \Big(\gamma - (1+\alpha+\beta)z\Big)\frac{d\pi}{dz} - \alpha\beta\pi = 0 \tag{4.2.3}$$

be the corresponding Picard-Fuchs equation for the periods of E_s with $\alpha, \beta, \gamma \in \mathbb{C}$ and $z = Cs^{\frac{1}{1-\gamma}}$ for a constant C given in Lemma 2.1 of [MS16]. Then a primitive form for W_0 is

$$\zeta = \frac{d^3x}{{}_2F_1(\alpha, \beta, \gamma; z)}.$$

For a proof, see Lemma 2.1 of [MS16]. The following proposition is then proven in [NY98].

Proposition 4.2.7. Let W_0 be an elliptic singularity of Fermat type. In the notation of Proposition 4.2.6, for an elliptic singularity W_0 the marginal flat coordinate is

$$t = \frac{s \cdot {}_2F_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; Cs^{\frac{1}{1-\gamma}})}{{}_2F_1(\alpha, \beta, \gamma; Cs^{\frac{1}{1-\gamma}})} \tag{4.2.4}$$

for the same parameters α, β, γ and variables z, s given in the Picard-Fuchs equation (4.2.3)for E_s.

Using the formulas presented in the previous chapter, one may check case-by-case that the above proposition extends to any invertible simple elliptic singularity. This proposition now has important consequences for the corresponding primary open Saito potentials.

Proposition 4.2.8. Consider the primary open Saito potentials $W_{\boldsymbol{t}}$ for an elliptic singularity W_0 of Fermat type. Suppose all non-marginal flat coordinates are set to zero. There exists a primary open Saito potential $W_{\boldsymbol{t}}$ that is a (meromorphic) modular function of weight zero.

Proof. We note that the marginal flat coordinate t given in (4.2.4) is a ratio of hypergeometric functions. This has an inverse given by a Schwarz triangle function $s(t)$. This implies that a primary open Saito potential is

$$W_t = W_0 + s(t)x^{\mu_c}.$$

Modularity then immediately follows from the fact $s(t)$ is modular invariant and W_0 is constant in t. □

See Chapter VI of [Neh75] for an expansive survey of the theory of Schwarz triangle functions.

In light of equation (3.2.22), the above two propositions extend to the chain polynomial $W_0 = x_1^4 + x_1x_2^3$. It is natural to expect that these results extend to any simple elliptic singularity; one may check this by exhaustive direct calculation of flat coordinates for such singularities. However, we postpone a discussion until Chapter 5.

We now focus on the $E_6^{(1,1)}$ singularity since this is of geometric interest as the cubic elliptic curve in $\mathbb{P}^2$.

Example 4.2.9. Consider the elliptic singularity $W_0 = x_1^3 + x_2^3 + x_3^3$. The versal deformation is

$$W = W_0 + sx_1x_2x_3 + s_{110}x_1x_2 + s_{101}x_1x_3 + s_{011}x_2x_3 + s_{100}x_1 + s_{010}x_2 + s_{001}x_3 + s_{000}$$

The marginal flat coordinate is

$$t(s) = \frac{s \cdot {}_2F_1(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; -\frac{1}{27}s^3)}{{}_2F_1(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; -\frac{1}{27}s^3)}. \tag{4.2.5}$$

We expand this around $s = 0$ and use series reversion (see Appendix A) to find one possible $s(t)$. Furthermore, we consider the pencil of elliptic curves $W_0 + s(t)x_1x_2x_3 = 0$ by setting all non-marginal variables to zero. In this case, we find that an open Saito potential is

$$\begin{aligned} W_t - W_0 = x_1x_2x_3\Big(t &+ \frac{1}{162}t^4 + \frac{37}{918540}t^7 + \frac{47}{223205220}t^{10} + \frac{50233}{51707721265200}t^{13} \\ &+ \frac{50131}{11727311182947360}t^{16} + \frac{207140851}{11505811592979441738000}t^{19} \\ &+ \frac{376053259}{5219036138575474772356800}t^{22} + \frac{2977848681889}{10695370758782720450990790240000}t^{25} \\ &+ \frac{59515502138947}{56761616061350951359862242698508800}t^{28} + O(t^{31})\Big). \end{aligned}$$

We note that series reversion only provides one possible inverse to (4.2.5) and so $s(t)$ is not uniquely defined. Thus, the open Saito potential is also not unique.

To see modularity, we change variable $s = -3w^{1/3}$ in (4.2.5) where we choose the branch of the cube root such that $w^{1/3} \in \mathbb{R}$ if $w \in \mathbb{R}_{\geq 0}$. Thus we find

$$t(w) = -3f(w)$$

where

$$f(w) = \frac{w^{1/3} \cdot {}_2F_1(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; w)}{{}_2F_1(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; w)}$$

defines a mapping $f : \mathbb{H} \to \Delta$. Here, we denote the upper half plane as $\mathbb{H}$ and Δ is the hyperbolic triangle enclosed by three circular arcs with angles $\{\frac{\pi}{3}, 0, 0\}$ at the respective vertices $\{f(0), f(\infty), f(1)\}$. We denote by $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ the group generated by pairs of reflections across the sides of Δ. One may analytically continue f through the real axis:

we obtain in this way a map from $\mathbb{P}^1$ with a branch cut to the set $\Delta \cup g \cdot \Delta$ where $g \in \Gamma$. Iterating, we obtain an infinitely many-valued function on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. It is well known that the inverse $w(t)$ is a single-valued meromorphic function that is modular invariant and also automorphic with respect to Γ. In particular, $s(t)$ is also modular invariant; this is stated in Lemma 9 of [Str12]. For a discussion of triangle groups, see Appendix A.

Remark 4.2.10. Due to modularity, one may calculate the q-series expansion. Indeed, following Lehner [Leh54] details of which are in Appendix A, we calculate the q-expansion:

$$w(t) = \sum_{n=-1}^{\infty} = q^{-1} + \frac{5}{9} + \frac{2}{27}q - \frac{76}{19683}q^2 - \frac{1}{2187}q^3 + \frac{44}{531441}q^4 - \frac{1384}{387420489}q^5 - \frac{4}{14348907}q^6 + O(q^7)$$

where $q = e^{\frac{2\pi i t}{3}}$. Furthermore, in section 3 of [Leh54], the asymptotics are extracted as

$$w(t) = \sum_{n=-1}^{\infty} a_n q^n, \quad a_n \sim \frac{2\pi}{3\sqrt{n}} J_1\left(\frac{4\pi\sqrt{n}}{3}\right), \; n \to \infty$$

where J_1 is the Bessel function of the first kind [1]. As expected from above, this is meromorphic: there is a simple pole at the 'large volume limit' $q = 0$, or equivalently $t = i\infty$. The resulting Fourier coefficients in the above expansion, and in the expansion of $s(t) = -3w^{1/3}(t)$, may be of interest in open orbifold Gromov-Witten theory of $[X/G]$ where $X = V(W_0) \subset \mathbb{P}^2$ and $G = \mathbb{Z}_3^3/\mathbb{Z}_3$. In particular, it would be interesting to compare these results to that of [LZ15] and also [AL23; PSW08]. It is currently an open problem to find an analogue of the Landau-Ginzburg / Calabi-Yau correspondence for open FJRW and Gromov-Witten theory. See [CIR14; CR10; PS16; Wit93b] for more details of this correspondence in the closed case.

Explicitly, the full set of the versal deformation parameters are given by

$$s(t) = -3z^{\frac{1}{3}}(t)$$

[1] We have successfully run a machine learning program, the LSTM recurrent neural net, to learn the Fourier coefficients which would suggest the existence of a recursion relation or closed form expression as presented here for these coefficients.

and

$$s_{110}(t) = (1-z)^{\frac{2}{3}}{}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)t_{110},$$

$$s_{101}(t) = (1-z)^{\frac{2}{3}}{}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)t_{101}$$

$$s_{011}(t) = (1-z)^{\frac{2}{3}}{}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)t_{011}$$

$$s_{001}(t) = (1-z)^{\frac{1}{3}}{}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)t_{001} + O(t^2),$$

$$s_{010}(t) = (1-z)^{\frac{1}{3}}{}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)t_{010} + O(t^2),$$

$$s_{100}(t) = (1-z)^{\frac{1}{3}}{}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)t_{100} + O(t^2),$$

$$s_{000}(t) = t_{000} + z^{\frac{2}{3}}(1-z){}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)\Big({}_2F_1\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};z\Big)\Big)'(t_{001}t_{110} + t_{010}t_{101} + t_{100}t_{011}) + O(t^3)$$

where $O(t^2)$ and $O(t^3)$ are quadratic and cubic in the non-marginal variables. This computation can be found in [NY98]. It is an interesting question as to whether one may consider this full set of parameters and still have modularity. In the case of the cubic in $\mathbb{P}^2$, the answer is affirmative as found by Verlinde and Warner [VW91]. In particular, the full open primary Saito potential $W_{\boldsymbol{t}} - W_0$ is modular invariant. The corresponding transformations are

$$\begin{aligned} t &\mapsto \frac{at+b}{ct+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}), \\ t_\mu &\mapsto \frac{t_\mu}{ct+d}, \quad \mu \neq (1,1,1),(0,0,0) \\ t_{000} &\mapsto t_{000} + \frac{1}{2}\frac{c}{ct+d}(t_{001}t_{110} + t_{010}t_{101} + t_{100}t_{011}). \end{aligned}$$

This is fully explained in terms of *modular Frobenius manifolds* in [MS11; MS12].

4.3 Flat Coordinates and Primitive Forms for Chains and Loops of Rank Two

For the remainder of this chapter, we specialise to the $N = 2$ case.

Definition 4.3.1. Fix a finite multiset $I \subseteq \mathbb{N}$. Define

$$A_I := \mathbb{Q}[\{u_{i,d} \,|\, i \in I, d \in \mathbb{Z}_{\geq 0}\}]/\langle \{u_{i,d}u_{i,d'} \,|\, i \in I,\ d, d' \in \mathbb{Z}_{\geq 0}\}\rangle.$$

Let $(k_1, k_2) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$. Then the potential associated to ν is

$$W^\nu := \sum_{\substack{J \subseteq I, \boldsymbol{d} \in \mathbb{Z}^J_{\geq 0} \\ (k_1,k_2) \in \mathbb{Q}\text{-Bal}(J,\boldsymbol{d})}} (-1)^{|J|-1} \nu_{k_1,k_2,J,\boldsymbol{d}} x_1^{k_1} x_2^{k_2} \prod_{i \in J} u_{i,d} \in A_I[[x_1, x_2]].$$

Remark 4.3.2. In the case of rank two Fermat, chain or loop polynomials, we observe that for a fixed k_1 and k_2, there are only a finite number of $J \subset I$ and $\boldsymbol{d} \in \mathbb{Z}^J_{\geq 0}$ such that $(k_1, k_2) \in \mathbb{Q}\text{-Bal}(J, \boldsymbol{d})$. Therefore, the coefficient of $x_1^{k_1} x_2^{k_2}$ is indeed polynomial.

We recall Definition 4.13, Lemma 4.14 and Corollary 4.15 of [GKT22b]. We define the map

$$\psi_I : A_{I,\text{sym}} \to A_I, \quad t_{(\alpha,\beta),d} \mapsto \sum_{i \in I : (a_i,b_i) = (\alpha_i,\beta_i)} u_{i,d}.$$

This induces a map $\psi_I : A_{I,\text{sym}}[[x_1, x_2]] \to A_I[[x_1, x_2]]$. Furthermore, this map is a well-defined and injective homomorphism such that for $A \in \mathcal{A}_l$,

$$\psi_I \left(\frac{\prod_{i=1}^l t_{(\alpha_i,\beta_i)}}{|\text{Aut}(A)|} \right) = \sum_{\substack{J \subseteq I \\ \boldsymbol{d}' \in S}} \prod_{j \in J} u_{j,d'_j}$$

and where $S \subset \mathbb{Z}^J_{\geq 0}$ is the set for which there exists a bijection $\phi : J \to \{1, 2, \ldots, l\}$ such that $(a_i, b_i) = (\alpha_{\phi(i)}, \beta_{\phi(i)})$ and $d'_j = d_{\phi(j)}$. For balanced ν with respect to I, this now implies that

$$\psi_I(W^{\nu,\text{sym}}) = W^\nu. \tag{4.3.1}$$

We perform calculations using W^ν in the variables $u_{i,d}$ thereby obtaining corollaries for $W^{\nu,\mathrm{sym}}$ in the variables $t_{\mu,d}$. We treat the chain and loop cases separately.

Chain Potentials.

Lemma 4.3.3. Define

$$d(J,\boldsymbol{d}) := \frac{r_1 r_2(J) + r_2 r_1(J) - r_2(J) - m(J,\boldsymbol{d})}{r_1 r_2} - 1.$$

If $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is balanced then $d(J,\boldsymbol{d}) < 0$. Conversely, if $d(J,\boldsymbol{d}) < 0$ then there exists a $(k_1,k_2) \in \mathbb{Z}^2_{\geq 0}$ such that $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is balanced.

Proof. Suppose that $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is balanced. Recall from Example 4.1.3 that $\nu_{k_1,k_2,\{(a_j,b_j)\}_{j\in J},\boldsymbol{d}}$ is balanced if and only if

$$k_2 \equiv r_2(J) \mod r_2, \quad r_2k_1 - k_2 \equiv r_2 \cdot r_1(J) - r_2(J) \mod r_1r_2, \quad r_2k_1 + r_1k_2 - k_2 = m(J,\boldsymbol{d}).$$

Since $k_1, k_2 \geq 0$ we find

$$k_2 = r_2(J) + p_2 r_2, \quad r_2 k_1 - k_2 = r_2 \cdot r_1(J) - r_2(J) + p_1 r_1 r_2 \tag{4.3.2}$$

for some $p_1, p_2 \in \mathbb{Z}_{\geq 0}$. Substituting this into the third balancing condition yields

$$r_1 r_2(J) + r_2 r_1(J) - r_2(J) + (p_1 + p_2) r_1 r_2 = m(J,\boldsymbol{d}).$$

For $p_1, p_2 \geq 0$, this is equivalent to

$$d(J,\boldsymbol{d}) = -p_1 - p_2 - 1 < 0 \tag{4.3.3}$$

as desired. For the converse direction, since $d(J,\boldsymbol{d}) < 0$, we define p_1, p_2 via (4.3.3) and thus define the desired k_1 and k_2 via (4.3.2). □

Theorem 4.3.4. If $W^\nu := x_1^{r_1} + x_1 x_2^{r_2} \mod \langle \{u_{i,d} \,|\, i \in I,\, d \in \mathbb{Z}_{\geq 0}\}\rangle$ then

$$\int_{\Xi_{(a,b)}} e^{W^\nu/\hbar}\Omega = \delta_{a,0}\delta_{b,0} + \sum_{\substack{J\subseteq I \\ J\neq\varnothing}} \sum_{\boldsymbol{d}\in\mathbb{Z}^J_{\geq 0}} (-1)^{|J|}(-\hbar)^{-d(J,\boldsymbol{d})-2}\mathcal{A}(J,\nu,\boldsymbol{d})\delta_{r_1(J),a}\delta_{r_2(J),b}\prod_{i\in J} u_{i,d_i} \tag{4.3.4}$$

where

$$\mathcal{A}(J,\nu,\boldsymbol{d}) = \sum_{h=1}^{|J|}\frac{1}{h!}\sum_{\substack{J_1\sqcup\cdots\sqcup J_h=J \\ \{k_1(j),k_2(j)\}_{j=1}^h\in\mathbb{Q}\text{-Bal}(J_j,\boldsymbol{d}|_{J_j})}} \Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h}\prod_{j=1}^h \nu_{k_1(j),k_2(j),J_j,\boldsymbol{d}|_{J_j}}.$$

with

$$\Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} = \frac{\Gamma\left(\frac{1+\sum_{j=1}^h k_2(j)}{r_2}\right)}{\Gamma\left(\frac{1+r_2(J)}{r_2}\right)}\frac{\Gamma\left(\frac{1+\sum_{j=1}^h k_1(j)}{r_1} - \frac{1+\sum_{j=1}^h k_2(j)}{r_1 r_2}\right)}{\Gamma\left(\frac{1+r_1(J)}{r_1} - \frac{1+r_2(J)}{r_1r_2}\right)}$$

and where $\{k_1(j),k_2(j)\}_{j=1}^h \in \mathbb{Q}\text{-Bal}(J_j,\boldsymbol{d}|_{J_j})$ means

$$k_2(j) = r_2(J_j) \mod r_2, \quad r_2k_1(j) - k_2(j) = r_2r_1(J_j) - r_2(J_j) \mod r_1r_2$$

and

$$r_1k_2(j) + r_2k_1(j) - k_2(j) = m(J_j,\boldsymbol{d}|_{J_j}). \tag{4.3.5}$$

Proof. Fix $J \subseteq I$. The case $J = \varnothing$ follows from the definition of a good basis and gives the contribution $\delta_{a,0}\delta_{b,0}$. Thus, consider $J \neq \varnothing$. Recall that $\nu_{k_1,k_2,\{(a_j,b_j)\}_{j\in J},\boldsymbol{d}}$ is balanced if and only if

$$k_2 \equiv r_2(J) \mod r_2, \quad r_2k_1 - k_2 \equiv r_2\cdot r_1(J) - r_2(J) \mod r_1r_2, \quad r_2k_1 + r_1k_2 - k_2 = m(J,\boldsymbol{d}).$$

By definition of $r_1(J)$ and $r_2(J)$, there exist non-negative integers $l_1(J)$ and $l_2(J)$ such that

$$\sum_{j\in J} a_j = r_1(J) + l_1(J)r_1, \quad \sum_{j\in J} b_j = r_2(J) + l_2(J)r_2.$$

We may write

$$\int_{\Xi_{(a,b)}} e^{W^\nu/\hbar}\Omega = \int_{\Xi_{(a,b)}} \left(\sum_{h=0}^{\infty} \frac{(W^\nu - x_1^{r_1} - x_1 x_2^{r_2})}{h!\hbar^h} \right) e^{(x_1^{r_1}+x_1x_2^{r_2})/\hbar}\Omega.$$

Noting that $W^\nu - x_1^{r_1} - x_1x_2^{r_2} \equiv 0 \mod \langle \{u_{i,d}u_{i,d'} \mid i \in I,\, d,d' \in \mathbb{Z}_{\geq 0}\}\rangle$ we expand the summation. If $J \neq \varnothing$ then the coefficient $u_J = \prod_{i\in J} u_{i,d_i}$ is

$$\Lambda_J = \int_{\Xi_{(a,b)}} \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{\substack{J_1\sqcup\cdots\sqcup J_h = J \\ \{k_1(j),k_2(j)\}_{j=1}^h}} \left(x_1^{\sum_j k_1(j)} x_2^{\sum_j k_2(j)} \hbar^{-h} \prod_{j=1}^{h} (-1)^{|J_j|-1} \nu_{k_1(j),k_2(j),J_j,\boldsymbol{d}|_{J_j}} \right) e^{(x_1^{r_1}+x_1x_2^{r_2})/\hbar}\Omega.$$

For convenience, we simplify the notation $\{k_1(j),k_2(j)\}_{j=1}^h \in \mathbb{Q}\text{-Bal}(J_j, \boldsymbol{d}|_{J_j})$ by assuming that all sums over $\{k_1(j),k_2(j)\}_{j=1}^h$ are of balanced indices. We recall from Lemma 3.2.20 that for the rank 2 chain $W_0 = x_1^{r_1} + x_1x_2^{r_2}$ we have

$$\int_{\Xi_\mu} e^{W_0/\hbar} x_1^{k_1} x_2^{k_2}\, dx_1 dx_2 = (-\hbar)^{m_1+m_2} \frac{\Gamma\left(\frac{\mu_2+1}{r_2} + m_2\right)}{\Gamma\left(\frac{\mu_2+1}{r_2}\right)} \frac{\Gamma\left(\frac{\mu_1+1}{r_1} - \frac{\mu_2+1}{r_1r_2} + m_1\right)}{\Gamma\left(\frac{\mu_1+1}{r_1} - \frac{\mu_2+1}{r_1r_2}\right)}$$

if $k_1 = r_1m_1 + \mu_1 + m_2$ and $k_2 = r_2m_2 + \mu_2$ for some $m_i \in \mathbb{Z}_{\geq 0}$ and the integral is zero otherwise. Thus, performing integration by parts on Λ_J we see that the integral is zero unless $(a,b) = (r_1(J), r_2(J))$. Integrating the monomial $x_1^{\sum_j k_1(j)} x_2^{\sum_j k_2(j)}$ produces a factor $\hbar^{n_1+n_2}$ where

$$\sum_{j\in J} k_1(j) = r_1(J) + n_1r_1 + n_2, \quad \sum_{j\in J} k_2(j) = r_2(J) + n_2r_2. \tag{4.3.6}$$

Again using Lemma 3.2.20 we find

$$\Lambda_J = \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{\substack{J_1\sqcup\cdots\sqcup J_h = J \\ \{k_1(j),k_2(j)\}_{j=1}^h}} (-1)^{|J|-1+n_1+n_2} \hbar^{n_1+n_2-h} \Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} \prod_{j=1}^{h} \nu_{k_1(j),k_2(j),J_j,\boldsymbol{d}|_{J_j}} \delta_{r_1(J),a}\delta_{r_2(J),b}.$$

Using the definition of $m(J,\boldsymbol{d})$ in (4.1.7), we obtain $\sum_j m(J_j, \boldsymbol{d}|_{J_j}) = (h-1)r_1r_2 + m(J,\boldsymbol{d})$.

Combining with (4.3.5) and (4.3.6) thus gives,

$$\begin{aligned} n_2 + n_1 &= \frac{\sum_j k_2(j) - r_2(J)}{r_2} + \frac{\sum_j k_1(j) - r_1(J)}{r_1} - \frac{\sum_j k_2(j) - r_2(J)}{r_1 r_2} \\ &= \frac{\sum_j m(J_j, \boldsymbol{d}|_{J_j}) - r_1 r_2(J) - r_2 r_1(J) + r_2(J)}{r_1 r_2} \\ &= \frac{(h-1) r_1 r_2 + m(J, \boldsymbol{d}) - r_1 r_2(J) - r_2 r_1(J) + r_2(J)}{r_1 r_2}. \end{aligned}$$

Hence,

$$(n_1 + n_2 - h + 1) r_1 r_2 = m(J, \boldsymbol{d}) - r_1 r_2(J) - r_2 r_1(J) + r_2(J) = (-d(J, \boldsymbol{d}) - 1) r_1 r_2.$$

Thus, we find

$$\Lambda_J = (-1)^{|J|} (-\hbar)^{-d(J,\boldsymbol{d})-2} \mathcal{A}(J, \nu, \boldsymbol{d}) \delta_{r_1(J),a} \delta_{r_2(J),b}$$

as desired. □

Corollary 4.3.5. For

$$W^{\nu,\mathrm{sym}} = x_1^{r_1} + x_1 x_2^{r_2} \mod \langle \{t_{\alpha,\beta,d}\} \rangle$$

we have

$$\int_{\Xi_{(a,b)}} e^{W^{\nu,\mathrm{sym}}/\hbar} \Omega = \delta_{a,0}\delta_{b,0} + \sum_{l \geq 1} \sum_{\{(\alpha_i,\beta_i,d_i)\} \in \mathcal{A}_l} (-1)^l (-\hbar)^{-d(J,\boldsymbol{d})-2} \mathcal{A}(J, \nu, \boldsymbol{d}) \delta_{r_1(J),a} \delta_{r_2(J),b} \left(\frac{\prod_{i=1}^l t_{\alpha_i,\beta_i,d_i}}{|\mathrm{Aut}(A)|} \right).$$

Proof. This follows from the previous theorem and equation (4.3.1). □

Corollary 4.3.6. For a rank two chain polynomial, there exists $(\nu_{k_1,\ldots,k_N,J,\boldsymbol{d}}) \in \mathbb{Q}$ such that, if $d(J, \boldsymbol{d}) < 0$ and $|J| \geq 2$, then

$$\mathcal{A}(J, \nu, \boldsymbol{d}) = 0.$$

Proof. Recall that

$$\mathcal{A}(J,\nu,\boldsymbol{d}) = \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{J_1 \sqcup \cdots \sqcup J_h = J} \sum_{\substack{\{k_1(j),k_2(j)\}_{j=1}^h \\ (k_1(j),k_2(j))}} \Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} \prod_{j=1}^h \nu_{k_1(j),k_2(j),J_j,\boldsymbol{d}|_{J_j}}$$

where

$$\Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} = \frac{\Gamma\left(\frac{1+\sum_{j=1}^h k_2(j)}{r_2}\right)}{\Gamma\left(\frac{1+r_2(J)}{r_2}\right)} \frac{\Gamma\left(\frac{1+\sum_{j=1}^h k_1(j)}{r_1} - \frac{1+\sum_{j=1}^h k_2(j)}{r_1 r_2}\right)}{\Gamma\left(\frac{1+r_1(J)}{r_1} - \frac{1+r_2(J)}{r_1 r_2}\right)}.$$

We proceed by induction. For the base case, fix $J = \{j_1, j_2\}$. We split $\mathcal{A}(J,\nu,\boldsymbol{d})$ into $h = 1$ and $h = 2$ terms. For $h = 2$ terms, partition J into $J_1 = \{j_1\}$ and $J_2 = \{j_2\}$, or $J_1 = \{j_2\}$ and $J_2 = \{j_1\}$. Thus

$$\begin{aligned}
\mathcal{A}(J,\nu,\boldsymbol{d}) = &\sum_{k_1,k_2 \in \mathbb{Q}\text{-Bal}(J,\boldsymbol{d})} \Gamma_{k_1,k_2,J} \nu_{k_1,k_2,J,\boldsymbol{d}} \\
&+ \frac{1}{2}\left(\sum_{\substack{k_1(1),k_2(1) \in \mathbb{Q}\text{-Bal}(\{j_1\},\boldsymbol{d}|_{\{j_1\}}) \\ k_1(2),k_2(2) \in \mathbb{Q}\text{-Bal}(\{j_2\},\boldsymbol{d}|_{\{j_2\}})}} \Gamma_{k_1(j),k_2(j),\{j_1\},\{j_2\}} \nu_{k_1(1),k_2(1),\{j_1\},\boldsymbol{d}_{j_1}} \nu_{k_1(2),k_2(2),\{j_2\},\boldsymbol{d}_{j_2}} \right. \\
&+ \left. \sum_{\substack{k_1(1),k_2(1) \in \mathbb{Q}\text{-Bal}(\{j_2\},\boldsymbol{d}|_{j_2}) \\ k_1(2),k_2(2) \in \mathbb{Q}\text{-Bal}(\{j_1\},\boldsymbol{d}|_{j_1})}} \Gamma_{k_1(j),k_2(j),\{j_1\},\{j_2\}} \nu_{k_1(1),k_2(1),\{j_2\},\boldsymbol{d}_{j_2}} \nu_{k_1(2),k_2(2),\{j_1\},\boldsymbol{d}_{j_1}} \right).
\end{aligned}$$

We now choose a collection $(\nu_{k_1,k_2,J,\boldsymbol{d}})$ such that only one $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is non-zero in the first summation. Since $d(J,\boldsymbol{d}) < 0$, using Lemma 4.3.3 we are able to find such a non-vanishing $\nu_{k_1,k_2,J,\boldsymbol{d}}$. Hence, given $\nu_{k_1(m),k_2(m),\{j_m\},\boldsymbol{d}_{j_m}}$ on the singletons $\{j_m\}$ for $m = 1, 2$ we see that the choice of

$$\begin{aligned}
\nu_{k_1,k_2,J,\boldsymbol{d}} = &\frac{-1}{2\Gamma_{k_1,k_2,J}} \left(\sum_{\substack{k_1(1),k_2(1) \in \mathbb{Q}\text{-Bal}(\{j_1\},\boldsymbol{d}|_{\{j_1\}}) \\ k_1(2),k_2(2) \in \mathbb{Q}\text{-Bal}(\{j_2\},\boldsymbol{d}|_{\{j_2\}})}} \Gamma_{k_1(j),k_2(j),\{j_1\},\{j_2\}} \nu_{k_1(1),k_2(1),\{j_1\},\boldsymbol{d}_{j_1}} \nu_{k_1(2),k_2(2),\{j_2\},\boldsymbol{d}_{j_2}} \right. \\
&+ \left. \sum_{\substack{k_1(1),k_2(1) \in \mathbb{Q}\text{-Bal}(\{j_2\},\boldsymbol{d}|_{\{j_2\}}) \\ k_1(2),k_2(2) \in \mathbb{Q}\text{-Bal}(\{j_1\},\boldsymbol{d}|_{j_1})}} \Gamma_{k_1(j),k_2(j),\{j_1\},\{j_2\}} \nu_{k_1(1),k_2(1),\{j_2\},\boldsymbol{d}_{j_2}} \nu_{k_1(2),k_2(2),\{j_1\},\boldsymbol{d}_{j_1}} \right)
\end{aligned}$$

forces $\mathcal{A}(J,\nu,\boldsymbol{d}) = 0$.

Assume the result is true for all $J' \subset J$ with $2 \leq |J'| < |J|$. For such a J with $|J| > 2$, $\mathcal{A}(J, \nu, \boldsymbol{d})$ reads

$$\begin{aligned}\mathcal{A}(J, \nu, \boldsymbol{d}) = & \sum_{k_1, k_2 \in \mathbb{Q}\text{-Bal}(J, \boldsymbol{d})} \Gamma_{k_1, k_2, J} \nu_{k_1, k_2, J, \boldsymbol{d}} \\ & + \sum_{h=2}^{|J|} \frac{1}{h!} \sum_{J_1 \sqcup \cdots \sqcup J_h = J} \sum_{\substack{\{k_1(j), k_2(j)\}_{j=1}^h \\ (k_1(j), k_2(j)) \in \mathbb{Q}\text{-Bal}(J_j, \boldsymbol{d}|_{J_j})}} \Gamma_{\{k_1(j), k_2(j), J_j\}_{j=1}^h} \prod_{j=1}^h \nu_{k_1(j), k_2(j), J_j, \boldsymbol{d}|_{J_j}}.\end{aligned}$$

We again choose (ν) such that only one $\nu_{k_1, k_2, J, \boldsymbol{d}}$ is non-zero in the first summation. The ν terms in the $h \geq 2$ summation are defined on proper subsets of J and so are chosen in the inductive step. Thus, we choose

$$\nu_{k_1, k_2, J, \boldsymbol{d}} = -\frac{1}{\Gamma_{k_1, k_2, J}} \sum_{h=2}^{|J|} \frac{1}{h!} \sum_{J_1 \sqcup \cdots \sqcup J_h = J} \sum_{\substack{\{k_1(j), k_2(j)\}_{j=1}^h \\ (k_1(j), k_2(j)) \in \mathbb{Q}\text{-Bal}(J_j, \boldsymbol{d}|_{J_j})}} \Gamma_{\{k_1(j), k_2(j), J_j\}_{j=1}^h} \prod_{j=1}^h \nu_{k_1(j), k_2(j), J_j, \boldsymbol{d}|_{J_j}}$$

which forces $\mathcal{A}(J, \nu, \boldsymbol{d}) = 0$. □

Corollary 4.3.7. There exists $(\nu_{k_1, \ldots, k_N, J, \boldsymbol{d}}) \in \mathbb{Q}$ such that $\Omega = dx_1 \wedge dx_2$ is a primitive form and $t_{\alpha, \beta, 0}$ are flat coordinates.

Proof. From Observation 4.2.1, we see that $W^{\nu, \text{sym}}$ satisfies the hypothesis of Theorem 4.3.4. Moreover, since $\nu_{k_1, \ldots, k_N, J, \boldsymbol{d}} = 0$ if $(k_1, \ldots, k_N) \notin \mathbb{Q}\text{-Bal}(J, \boldsymbol{d})$, there is no contribution of the form $(-\hbar)^{-d(J, \boldsymbol{d}) - 2} \mathcal{A}(J, \nu, \boldsymbol{d})$ with $d(J, \boldsymbol{d}) < 0$ and $|J| \geq 2$. However, for $J = \{(a, b)\}$ and $\boldsymbol{d} = 0$ so that $d(J, \boldsymbol{d}) = -1$, we indeed find the stated formula for the coefficient of $\hbar^{-1}$ using Observation 4.2.1 and equation (4.3.4). □

In Section 5.1 we give an example of how to calculate flat coordinates for $W_0 = x_1^4 + x_1 x_2^3$ using this method.

Loop Potentials.

Lemma 4.3.8. Define

$$d(J, \boldsymbol{d}) := \frac{r_1 r_2(J) + r_2 r_1(J) - r_1(J) - r_2(J) - m(J, \boldsymbol{d})}{r_1 r_2 - 1} - 1.$$

If $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is balanced then $d(J, \boldsymbol{d}) < 0$. Conversely, if $d(J, \boldsymbol{d}) < 0$ then there exists a $(k_1, k_2) \in \mathbb{Z}^2_{\geq 0}$ such that $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is balanced.

Proof. Suppose that $\nu_{k_1,k_2,J,\boldsymbol{d}}$ is balanced. Recall from Example 4.1.4 that $\nu_{k_1,k_2,\{(a_j,b_j)\}_{j\in J},\boldsymbol{d}}$ is balanced if and only if

$$r_1 k_2 - k_2 \equiv r_1 r_2(J) - r_1(J) \mod r_1 r_2 - 1, \quad r_2 k_1 - k_2 \equiv r_2 \cdot r_1(J) - r_2(J) \mod r_1 r_2 - 1$$

and

$$r_2 k_1 + r_1 k_2 - k_1 - k_2 = m(J, \boldsymbol{d})$$

Upon calculating $d(J, \boldsymbol{d})$, we find that for some $p_1, p_2 \geq 0$, we have

$$d(J, \boldsymbol{d}) = -p_1 - p_2 - 1 < 0$$

as desired. The converse direction is proven similarly to Lemma 4.3.3. □

Theorem 4.3.9. If $W^\nu := x_1^{r_1} x_2 + x_1 x_2^{r_2} \mod \langle \{u_{i,d} u_{i,d'} \mid i \in I, d, d' \in \mathbb{Z}_{\geq 0}\} \rangle$ then

$$\int_{\Xi_{(a,b)}} e^{W^\nu/\hbar} \Omega = \delta_{a,0}\delta_{b,0} + \sum_{\substack{J \subseteq I \\ J \neq \varnothing}} \sum_{\boldsymbol{d} \in \mathbb{Z}^J_{\geq 0}} (-1)^{|J|} (-\hbar)^{-d(J,\boldsymbol{d})-2} \mathcal{A}(J, \nu, \boldsymbol{d}) \delta_{r_1(J),a} \delta_{r_2(J),b} \prod_{i \in J} u_i \tag{4.3.7}$$

where

$$\mathcal{A}(J, \nu, \boldsymbol{d}) = \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{\substack{J_1 \sqcup \cdots \sqcup J_h = J \\ \{k_1(j), k_2(j)\}_{j=1}^h \in \mathbb{Q}\text{-Bal}(J_j, \boldsymbol{d}|_{J_j})}} \Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} \prod_{j=1}^h \nu_{k_1(j),k_2(j),J_j \boldsymbol{d}|_{J_j}}$$

with

$$\Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} := \frac{\Gamma\left(\frac{r_2\sum k_1(j)+r_2-\sum k_2(j)-1}{r_1r_2-1}\right)}{\Gamma\left(\frac{r_2r_1(J)+r_2-r_2(J)-1}{r_1r_2-1}\right)}\frac{\Gamma\left(\frac{r_1\sum k_2(j)+r_1-\sum k_1(j)-1}{r_1r_2-1}\right)}{\Gamma\left(\frac{r_1r_2(J)+r_1-r_1(J)-1}{r_1r_2-1}\right)}$$

and where $\{k_1(j), k_2(j)\}_{j=1}^h \in \mathbb{Q}\text{-Bal}(J_j, \boldsymbol{d}|_{J_j})$ means

$$r_1k_2(j)-k_1(j) = r_1r_2(J_j)-r_1(J) \mod r_1r_2-1, \quad r_2k_1(j)-k_2(j) = r_2r_1(J_j)-r_2(J_j) \mod r_1r_2-1$$

and

$$r_2k_1(j) + r_1k_2(j) - k_1(j) - k_2(j) = m(J_j, \boldsymbol{d}|_{J_j}). \tag{4.3.8}$$

Proof. Fix $J \subseteq I$. The case $J = \varnothing$ follows from the definition of a good basis and gives the contribution $\delta_{a,0}\delta_{b,0}$. Thus, consider $J \neq \varnothing$. Recall that $\nu_{k_1,k_2,\{(a_j,b_j)\}_{j\in J},\boldsymbol{d}}$ is balanced if and only if

$$r_1k_2 - k_1 \equiv r_1 \cdot r_2(J) - r_1(J) \mod (r_1r_2-1), \quad r_2k_1 - k_2 \equiv r_2 \cdot r_1(J) - r_2(J) \mod (r_1r_2-1),$$
$$r_2k_1 + r_1k_2 - k_1 - k_2 = m(J, \boldsymbol{d}).$$

By definition of $r_1(J)$ and $r_2(J)$, there exist integers $l_1(J)$ and $l_2(J)$ such that

$$\sum_{j\in J} a_j = r_1(J) + l_1(J)r_1, \quad \sum_{j\in J} b_j = r_2(J) + l_2(J)r_2.$$

We may write

$$\int_{\Xi_{(a,b)}} e^{W^\nu/\hbar}\Omega = \int_{\Xi_{(a,b)}} \left(\sum_{h=0}^{\infty} \frac{(W^\nu - x_1^{r_1}x_2 - x_1x_2^{r_2})}{h!\hbar^h}\right) e^{(x_1^{r_1}x_2+x_1x_2^{r_2})/\hbar}\Omega.$$

Noting that $W^\nu - x_1^{r_1}x_2 - x_1x_2^{r_2} \equiv 0 \mod \langle\{u_i^2 \,|\, i \in I\}\rangle$ we expand the summation. If

$J \neq \varnothing$ then the coefficient $u_J = \prod_{i \in J} u_i$ is

$$\Lambda_J = \int_{\Xi_{(a,b)}} \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{\substack{J_1 \sqcup \cdots \sqcup J_h = J \\ \{k_1(j), k_2(j)\}_{j=1}^h}} \left(x_1^{\sum_j k_1(j)} x_2^{\sum_j k_2(j)} \hbar^{-h} \prod_{j=1}^{h} (-1)^{|J_j|-1} \nu_{k_1(j), k_2(j), J_j \boldsymbol{d}|_{J_j}} \right) e^{\frac{x_1^{r_1} x_2 + x_1 x_2^{r_2}}{\hbar}} \Omega.$$

For convenience, we simplify the notation $\{k_1(j), k_2(j)\}_{j=1}^h \in \mathbb{Q}\text{-Bal}(J_j, \boldsymbol{d}|_{J_j})$ by assuming that all sums over $\{k_1(j), k_2(j)\}_{j=1}^h$ are of balanced indices. We recall from Lemma 3.2.21 that for the rank 2 loop singularity $W_0 = x_1^{r_1} x_2 + x_1 x_2^{r_2}$, we have

$$\int_{\Xi_\mu} e^{W_0/\hbar} x_1^{k_1} x_2^{k_2} \, dx_1 dx_2 = (-\hbar)^{m_1+m_2} \frac{\Gamma\left(\frac{r_2 k_1 + r_2 - k_2 - 1}{r_1 r_2 - 1} + m_1\right)}{\Gamma\left(\frac{r_2 \mu_1 + r_2 - \mu_2 - 1}{r_1 r_2 - 1}\right)} \frac{\Gamma\left(\frac{r_1 k_2 + r_1 - k_1 - 1}{r_1 r_2 - 1} + m_2\right)}{\Gamma\left(\frac{r_1 \mu_2 + r_1 - \mu_1 - 1}{r_1 r_2 - 1}\right)}$$

if $k_1 = r_1 m_1 + \mu_1 + m_2$ and $k_2 = r_2 m_2 + \mu_2 + m_1$ for some $m_i \in \mathbb{Z}_{\geq 0}$ and the integral is zero otherwise. Thus performing integration by parts on Λ_J, we see that the integral is zero unless $(a, b) = (r_1(J), r_2(J))$. Integrating the monomial $x_1^{\sum_j k_1(j)} x_2^{\sum_j k_2(j)}$ produces a factor $\hbar^{n_1+n_2}$ where

$$\sum_{j \in J} k_1(j) = r_1(J) + n_1 r_1 + n_2, \quad \sum_{j \in J} k_2(j) = r_2(J) + n_2 r_2 + n_1. \tag{4.3.9}$$

Again using Lemma 3.2.21 we find

$$\Lambda_J = \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{\substack{J_1 \sqcup \cdots \sqcup J_h = J \\ \{k_1(j), k_2(j)\}_{j=1}^h}} (-1)^{|J|-1+n_1+n_2} \hbar^{n_1+n_2-h} \Gamma_{\{k_1(j), k_2(j), J_j\}_{j=1}^h} \prod_{j=1}^{h} \nu_{k_1(j), k_2(j), J_j, \boldsymbol{d}|_{J_j}} \delta_{r_1(J), a} \delta_{r_2(J), b}.$$

From the definition of $m(J, \boldsymbol{d})$ in (4.1.8) we find $\sum_j m(J_j, \boldsymbol{d}|_{J_j}) = (h-1)(r_1 r_2 - 1) + m(J, \boldsymbol{d})$. Combining with (4.3.8) and (4.3.9) we obtain

$$\begin{aligned} n_1 + n_2 &= \frac{r_1 \sum k_2(j) - \sum k_1(j) - r_1 r_2(J) + r_1(J)}{r_1 r_2 - 1} + \frac{r_2 \sum k_1(j) - \sum k_2(j) - r_2 r_1(J) + r_2(J)}{r_1 r_2 - 1} \\ &= \frac{\sum_j m(J_j, \boldsymbol{d}|_{J_j}) - r_1 r_2(J) - r_2 r_1(J) + r_2(J)}{r_1 r_2 - 1} \\ &= \frac{(h-1)(r_1 r_2 - 1) + m(J, \boldsymbol{d}) - r_1 r_2(J) - r_2 r_1(J) + r_1(J) + r_2(J)}{r_1 r_2 - 1}. \end{aligned}$$

Hence,

$$(n_1+n_2-h+1)(r_1r_2-1) = m(J,\boldsymbol{d})-r_1r_2(J)-r_2r_1(J)+r_1(J)+r_2(J) = (-d(J,\boldsymbol{d})-1)(r_1r_2-1).$$

Thus, we find

$$\Lambda_J = (-1)^{|J|}(-\hbar)^{-d(J,\boldsymbol{d})-2}\mathcal{A}(J,\nu,\boldsymbol{d})\delta_{r_1(J),a}\delta_{r_2(J),b}$$

as desired. □

Corollary 4.3.10. For

$$W^{\nu,\mathrm{sym}} = x_1^{r_1}x_2 + x_1x_2^{r_2} \mod \langle\{t_{\alpha,\beta,d}\}\rangle$$

we have

$$\int_{\Xi_{(a,b)}} e^{W^{\nu,\mathrm{sym}}/\hbar}\Omega = \delta_{a,0}\delta_{b,0}+\sum_{l\geq 1}\sum_{\{(\alpha_i,\beta_i,d_i)\}\in\mathcal{A}_l}(-1)^l(-\hbar)^{-d(J,\boldsymbol{d})-2}\mathcal{A}(J,\nu,\boldsymbol{d})\delta_{r_1(J),a}\delta_{r_2(J),b}\left(\frac{\prod_{i=1}^l t_{\alpha_i,\beta_i,d_i}}{|\mathrm{Aut}(A)|}\right).$$

Proof. This follows from the Theorem 4.3.9 and equation (4.3.1). □

Corollary 4.3.11. For a rank two loop polynomial, there exists $(\nu_{k_1,\ldots,k_N,J,\boldsymbol{d}}) \in \mathbb{Q}$ such that, if $d(J,\boldsymbol{d}) < 0$ and $|J| \geq 2$, then

$$\mathcal{A}(J,\nu,\boldsymbol{d}) = 0.$$

Proof. This is proved in exactly the same way as Corollary 4.3.6, except now we use Lemma 4.3.8 and a different set of Γ functions given by

$$\Gamma_{\{k_1(j),k_2(j),J_j\}_{j=1}^h} := \frac{\Gamma\left(\frac{r_2\sum k_1(j)+r_2-\sum k_2(j)-1}{r_1r_2-1}\right)}{\Gamma\left(\frac{r_2r_1(J)+r_2-r_2(J)-1}{r_1r_2-1}\right)}\frac{\Gamma\left(\frac{r_1\sum k_2(j)+r_1-\sum k_1(j)-1}{r_1r_2-1}\right)}{\Gamma\left(\frac{r_1r_2(J)+r_1-r_1(J)-1}{r_1r_2-1}\right)}.$$

□

Corollary 4.3.12. There exists $(\nu_{k_1,\ldots,k_N,J,\boldsymbol{d}}) \in \mathbb{Q}$ such that, $\Omega = dx_1 \wedge dx_2$ is a primitive form and $t_{\alpha,\beta,0}$ are flat coordinates for a rank two loop polynomial.

Proof. This is proved in the same way as the chain case in Corollary 4.3.7. □

This method of calculating flat coordinates could be extended to Fermat, chain and loop polynomials in rank $N > 2$, although the details are tedious to write down.

In the rest of this thesis, we assume that any collection $(\nu_{k_1,k_2,I,\boldsymbol{d}})$ in rank two is suitably chosen such that $dx_1 \wedge dx_2$ is a primitive form and $t_{\alpha,\beta,0}$ are flat coordinates.

CHAPTER 5

WALL-CROSSING

In the previous chapter we defined a primary open Saito potential as the versal deformation written in terms of the flat coordinates $t_\mu(\boldsymbol{s})$. However, we have already noted that the change of variables $\boldsymbol{s} \mapsto \boldsymbol{t}$ cannot necessarily be done uniquely. In this chapter, we start by giving an example of Corollary 4.3.7 in the case that $W_0 = x_1^4 + x_1x_2^3$ by explicitly constructing an open Saito potential given that $dx_1 \wedge dx_2$ is a primitive form. We then describe the set of possible open ancestor Saito-Givental potentials by defining a Lie group of wall-crossing transformations. We conclude by proving that this group acts faithfully and transitively on such potentials, whereby the main results of the thesis are Theorems 5.3.3 and 5.3.4. Corollary 5.3.5 is a direct consequence of these results and explains the computational result given for $W_0 = x_1^4 + x_1x_2^3$ in Section 5.1.

5.1 Motivation

To motivate this section, we consider the example of the elliptic singularity $W_0 = x_1^4 + x_1x_2^3$. In Chapter 3, we have seen that dx_1dx_2 is not a primitive form. In particular, for the expression

$$\int e^{W_{\boldsymbol{s}}/\hbar} dx_1 dx_2$$

contains $\hbar^0$ terms of the form $1 + O(s_{31}^3)$ when integrated over a cycle Ξ_{00} and where s_{31} is the marginal parameter.

To overcome this issue, we consider an open Saito potential $W \in A[x_1, x_2]$ where $A = \mathbb{C}\Big[\{t_{ij}\}_{(i,j)\in B}\Big]/\mathcal{I}$ with $\mathcal{I}$ an ideal in A and B is the index set for the standard good basis which in this case reads,

$$B = \{(0,0),(0,1),(1,0),(2,0),(1,1),(0,2),(2,1),(3,0),(3,1)\}.$$

Suppose we choose the ideal $\mathcal{I}$ that is generated by (t_{ij}) for $(i,j) \in B \setminus \{(3,1)\}$. One open Saito potential at first order in t_{31} is

$$W = x_1^4 + x_1 x_2^3 + t_{31} x_1^3 x_2.$$

Now suppose we consider higher order terms and define

$$W^{(3)} := x_1^4 + x_1 x_2^3 + t_{31} x_2 x_1^3 + \lambda_1 t_{31}^3 x_2^3 x_1 + \mu_1 t_{31}^3 x_1^4. \tag{5.1.1}$$

One may view the addition of the terms $\lambda_1 t_{31}^3 x_2^3 x_1$ and $\mu_1 t_{31}^3 x_1^4$ as a scaling of the original potential. Another reason for perturbing by these terms is that, viewed as sections of the Gauss-Manin vector bundle, we have

$$[x_1^4] = -\frac{\hbar}{6}[1], \quad [x_1 x_2^3] = -\frac{\hbar}{3}[1].$$

Hence, the addition of these terms corresponds to a point in the moduli space of primitive forms for $E_7^{(1,1)}$ defined in Section 5.4 of Li, Li and Saito [LLS14].

The factor of t_{31}^3 in the perturbation is a judicious choice for the following reason. Suppose we chose the ideal $\mathcal{I}$ to be the generated by the same parameters but with t_{31}^4 as an additional generator. Then working over this ring we have

$$\int_{\Xi_{00}} e^{W^{(3)}/\hbar} dx_1 dx_2 = \int_{\Xi_{00}} e^{W/\hbar} \sum_{k=0}^{\infty} \frac{(t_{31} x_2 x_1^3 + \lambda_1 t_{31}^3 x_2^3 x_1 + \mu_1 t_{31}^3 x_1^4)^k}{k!\hbar^k}$$

where we have expanded the exponential. Modulo t_{31}^4 we find that this is equal to

$$\int_{\Xi_{00}} e^{W_0/\hbar}\Big(1 + \frac{t_{31}^3}{3!\hbar^3}x_2^3x_1^9 + \lambda_1\frac{t_{31}^3}{\hbar}x_2^3x_1 + \mu_1\frac{t_{31}^3}{\hbar}x_1^4\Big).$$

There are no t_{31}^2 terms since these vanish using integration by parts. To calculate the other terms, we use the by parts formula in Lemma 3.2.20,

$$\int_{\Xi_{00}} dx_1dx_2\, x_2^ix_1^je^{W_0/\hbar} = (-\hbar)^{m_2+m_1}\frac{\Gamma\Big(\frac{1}{3}+m_2\Big)}{\Gamma\Big(\frac{1}{3}\Big)}\frac{\Gamma\Big(\frac{1}{6}+m_1\Big)}{\Gamma\Big(\frac{1}{6}\Big)}$$

which is valid for $i = 3m_2$ and $j = 4m_1 + m_2$. This yields the expression

$$\int_{\Xi_{00}} e^{W^{(3)}/\hbar}dx_1dx_2 = 1 - \frac{t_{31}^3}{6}\Big(\frac{7}{108} + 2\lambda_1 + \mu_1\Big) + O(t_{31}^6).$$

Choosing λ_1 and μ_1 such that

$$\frac{7}{108} + 2\lambda_1 + \mu_1 = 0 \tag{5.1.2}$$

then ensures that the appropriate $\hbar^0$ terms vanish at $O(t_{31}^3)$. One may repeat this procedure perturbatively to all orders to give a primitive form.

Remark 5.1.1. This matches the spirit of the Gross-Siebert program [GPS10; GS11b] whereby degenerations are constructed order by order using scattering diagrams and interpreted as instanton corrections. See [Aur09; GS11a; GOR15] for an exposition. Despite this similarity, it is not obvious how to incorporate Landau-Ginzburg mirror symmetry into the Gross-Siebert program: there is no target space for a Landau-Ginzburg model and hence no corresponding SYZ fibration. Therefore, the combinatorial machinery of the Gross-Siebert program is currently out of reach for us.

Rather than calculate these linear constraints directly, we will describe a wall-crossing group for Landau-Ginzburg potentials. This is analogous to the construction of the wall-crossing group of Gross-Kelly-Tessler. The wall crossing does in fact contain information on these relations.

More precisely, consider again $W_0 = x_1^4 + x_1 x_2^3$. Let V be the vector field defined by

$$V = C \cdot t_{31}^3 (x_1 \partial_1 - x_2 \partial_2)$$

where C is a constant. The wall crossing group is then generated by e^V and where the group multiplication is given by the Baker-Campbell-Hausdorff formula.

The action is given by the following lemma.

Lemma 5.1.2. With V as above, we have

$$x_1 \mapsto e^V x_1 = e^{C t_{31}^k} x_1, \quad x_2 \mapsto e^V x_2 = e^{-C t_{31}^k} x_2. \tag{5.1.3}$$

Proof. By virtue that $x_1 \partial_{x_1}$ and $x_2 \partial_{x_2}$ commute, we observe that

$$e^V = e^{C t_{31}^k x_1 \partial_{x_1}} \cdot e^{-C t_{31}^k x_2 \partial_{x_2}}.$$

Hence, since $\alpha z \partial_z$ is the generator of multiplication by e^α, we find

$$e^V x_1 = e^{C t_{31}^k x_1 \partial_{x_1}} x_1 = e^{C t_{31}^k} x_1, \quad e^V x_2 = e^{-C t_{31}^k x_2 \partial_{x_2}} x_2 = e^{-C t_{31}^k} x_2$$

as desired. □

Thus, taking $W^{(3)}$ as in equation (5.1.1), we find that, modulo t_{31}^4 terms, $W^{(3)}$ transforms as

$$W^{(3)} \mapsto \tilde{W}^{(3)} = x_1^4 + x_1 x_2^3 + t_{31} x_2 x_1^3 + t_{31}^3 x_1 x_2^3 (\lambda_1 - 2C) + t_{13}^3 (\mu_1 + 4C) x_1^4.$$

Comparing to (5.1.1), the wall crossing relations on λ_1 and μ_1 are therefore

$$\lambda_1 \mapsto \lambda_1 - 2C, \quad \mu_1 \mapsto \mu_1 + 4C. \tag{5.1.4}$$

We note that this transformation leaves the linear relation (5.1.2) invariant.

To further emphasise this point, we repeat this procedure up to t_{31}^6 terms. We choose the maximal ideal $\mathcal{I}$ to be generated by (t_{ij}) for $(i,j) \in B \setminus \{(3,1)\}$ and additionally t_{31}^7. The potential is now

$$W^{(6)} = x_1^4 + x_1 x_2^3 + t_{31} x_2 x_1^3 + \lambda_1 t_{31}^3 x_2^3 x_1 + \mu_1 t_{31}^3 x_1^4 + \gamma t_{31}^4 x_2 x_1^3 + \lambda_2 t_{31}^6 x_2^3 x_1 + \mu_2 t_{31}^6 x_1^4.$$

Acting via e^V yields the wall crossing formulas

$$\lambda_2 \mapsto \lambda_2 - 2C\lambda_1 + 2C^2, \quad \mu_2 \mapsto \mu_2 + 4C\mu_1 + 8C^2, \quad \gamma \mapsto \gamma + 2C \tag{5.1.5}$$

with λ_1 and μ_1 transforming the same way as before. It can also be calculated that at $O(t_{31}^6)$ the integral we have is

$$\int \Big(\frac{\lambda_1^2 x_2^6 x_1^2}{2} + \frac{\mu_1^2 x_1^8}{2} + \lambda_2 x_2^3 x_1 + \mu_2 x_1^4 + \frac{\lambda_1 x_2^6 x_1^{10}}{6} + \frac{\mu_1 x_2^3 x_1^{13}}{6} + \lambda_1 \mu_1 x_2^3 x_1^5 + \frac{\gamma x_2^3 x_1^9}{2} + \frac{x_2^6 x_1^{18}}{720} \Big) t_{31}^6 e^{W_0/\hbar} dx_1 dx_2.$$

Repeated application of the by parts formula yields the linear relation

$$\frac{2}{9}\lambda_1^2 + \frac{7}{72}\mu_1^2 - \frac{1}{3}\lambda_2 - \frac{1}{6}\mu_2 + \frac{7}{486}\lambda_1 + \frac{91}{3888}\mu_1 + \frac{1}{18}\lambda_1\mu_1 - \frac{7}{216}\gamma + \frac{1729}{2916} = 0.$$

Again, we find that this relation is invariant under the wall crossing transformations in equations (5.1.4) and (5.1.5).

5.2 The Wall-Crossing Group

We now define the Landau-Ginzburg wall-crossing G_{A,W_0} group for a general Landau-Ginzburg model W_0. To define G_{A,W_0}, we construct a larger group G_A considered in Section 4.3 of [GKT22b] of which G_{A,W_0} is a subgroup.

Proposition 5.2.1. Let $A = A_{\mathcal{I},\mathrm{sym}}$. Fix a maximal ideal $\mathfrak{m} \subset A$. Then the vector space

$$\mathfrak{g}_{A,K} := \bigoplus_{\substack{(a_1,\ldots,a_N)\in\mathbb{Z}^N_{\geq 0}\\ \sum a_i \leq K}} \sum_{i=1}^{N} \mathfrak{m} \cdot \left(x_1^{a_1} \cdots x_N^{a_N} \Big((a_i+1)x_1\partial_{x_1} - (a_1+1)x_i\partial_{x_i} \Big) \right). \tag{5.2.1}$$

can be endowed with the structure of a graded nilpotent Lie algebra. Furthermore, denote

$$G_A := \lim_{\substack{\leftarrow \\ K}} \exp(\mathfrak{g}_{A,K}).$$

Then $\psi \in G_A \leq \mathrm{Aut}_A(A[[x_1,\ldots,x_N]])$ satisfies the following properties:

1. ψ has the form
$$\psi = \mathrm{id} \bmod \left\langle \{t_{\mu,d}\} \middle|\, \mu \in B, d \in \mathbb{Z}_{\geq 0} \right\rangle.$$

2. ψ preserves the volume form $dx_1 \wedge \cdots \wedge dx_N$.

3. ψ preserves the ideal generated by $\prod_{i=1}^N x_i$.

Proof. Define

$$R_K = A[x_1,\ldots,x_N]/(x_1,\ldots,x_N)^{K+1}$$

so that the power series ring $A[[x_1,\ldots,x_N]]$ is the inverse limit of R_K. Let

$$D_K = \bigoplus_{i=1}^{N} R_K \partial_{x_i}$$

be the A-module of derivations of R_K equipped with the usual commutator as the Lie bracket. The vector space $\mathfrak{g}_{A,K}$ is a subspace of D_K and we claim that it is closed under the commutator. Indeed, set

$$V_a = \sum_{i=1}^{N} g_i \cdot \left(x_1^{a_1} \cdots x_N^{a_N} \Big((a_i+1)x_1\partial_{x_1} - (a_1+1)x_i\partial_{x_i} \Big) \right)$$

and

$$V_b = \sum_{j=1}^{N} g_j \cdot \left(x_1^{b_1} \cdots x_N^{b_N} \Big((b_j+1)x_1\partial_{x_1} - (b_1+1)x_j\partial_{x_j} \Big) \right)$$

for some $g_i, g_j \in \mathfrak{m}$ with $i, j = 1, \ldots, N$. We calculate

$$[V_a, V_b] = \sum_{i,j=1}^{N} g_i g_j \left(\prod_{l=1}^{N} x_l^{a_l+b_l} \right) \left(\Big((a_i+b_i+1)\alpha + (a_j+b_j+1)\beta \Big) x_1\partial_{x_1} - (a_1+b_1+1)(\alpha x_i\partial_{x_i} + \beta x_j\partial_{x_j}) \right) \tag{5.2.2}$$

with

$$\alpha = \frac{-(b_j+1)(a_1+1)a_1 + (b_1+1)(a_1+1)a_j}{a_1+b_1+1}, \quad \beta = \frac{(a_i+1)(b_1+1)b_1 - (a_1+1)(b_1+1)b_i}{a_1+b_1+1}$$

which implies closure under the Lie bracket. We define a grading on $\mathfrak{g}_{A,K}$ as the one induced by the grading on $A[x_1, \ldots, x_N]$. In order for (5.2.1) to be a decomposition into homogeneous pieces, we require that each $g_i \in \mathfrak{m}$ in the summation is homogeneous and of the same degree. Compatibility of the Lie bracket and the grading then follows from equation (5.2.2). As $\mathfrak{g}_{A,K}$ is nilpotent, we form the Lie group $G_{A,K} := \exp(\mathfrak{g}_{A,K})$ upon which we take the inverse limit. As sets, we identify G_A with the inverse limit of $\mathfrak{g}_{A,K}$, which we write as $\mathfrak{g}_A$. We note that for $V \in \mathfrak{g}_A$, the group element $\exp(V)$ acts via

$$f \mapsto \sum_{n=0}^{\infty} \frac{V^n(f)}{n!}. \tag{5.2.3}$$

The exponential ensures that elements of $G_{A,K}$ are the identity modulo the ideal linear in $t_{\mu,d}$. Thus property 1 is satisfied. Property 3 can be seen by acting elements of $\mathfrak{g}_{A,K}$ on $\prod_{i=1}^{N} x_i$. Finally, we show that elements of G_A preserve the volume form. This is equivalent to showing that $\mathcal{L}_V(dx_1 \wedge \cdots \wedge dx_N) = 0$ for $V = x_1^{a_1} \cdots x_N^{a_N} \Big((a_i+1)x_1\partial_{x_1} - (a_1+1)x_i\partial_{x_i} \Big)$ and where $\mathcal{L}_V$ is the Lie derivative. Applying the homotopy formula for the Lie derivative, we find

$$\mathcal{L}_V(dx_1 \wedge \cdots \wedge dx_N) = d(\iota(V)dx_1 \wedge \cdots \wedge dx_N)$$

since $dx_1 \wedge \cdots \wedge dx_N$ is closed. The right hand side of the above equation reads

$$\begin{aligned}
&d\Big(x_1^{a_1}\cdots x_N^{a_N}\big((a_i+1)x_1 dx_2 \wedge \cdots dx_N - (-1)^{i-1}(a_1+1)dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_N\big)\Big)\\
&= x_1^{a_1}\cdots x_N^{a_N}\big((a_1+1)(a_i+1) - (a_i+1)(a_1+1)\big)dx_1 \wedge \cdots \wedge dx_N\\
&= 0
\end{aligned}$$

as desired. □

Remark 5.2.2. We remark that the vector fields in $\mathfrak{g}_A$ are not the only ones that preserve the volume form. Indeed, vector fields of the form $f(x_1, \ldots, \widehat{x_i}, \ldots, x_N)\partial_{x_i}$ also preserve $dx_1 \wedge \cdots \wedge dx_N$. However, these vector fields are irrelevant for our purposes in relation to enumerative theories. See Definition 3.18 of strongly positive in [GKT22b].

Definition 5.2.3. For a Landau-Ginzburg model W_0, the *Landau-Ginzburg wall-crossing group* $G_{A,W_0} \leq G_A$ is the group of elements ψ such that ψW is still an open Saito potential i.e. each monomial in ψW with non-zero coefficient $\nu_{k_1,\ldots,k_N,I,\boldsymbol{d}}$ must satisfy $(k_1, \ldots, k_N) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$. The *primary Landau-Ginzburg wall-crossing group* is defined by restricting G_{A,W_0} to $\boldsymbol{d} = \boldsymbol{0}$ with the additional *quasi-homogeneity condition*: ψ commutes with the grading operator $\mathcal{E}$.

In the rest of this section, we use Definition 5.2.3 to obtain necessary conditions so that the exponential of $V \in \mathfrak{g}_A$ is an element of G_{A,W_0}.

5.2.1 Quasi-Homogeneity

The quasi-homogeneity condition is necessary since it implies that ψ preserves the quasi-homogeneity of primary open Saito potentials W. Indeed, assuming that $\mathcal{E}\psi = \psi\mathcal{E}$ on primary open Saito potentials, we observe

$$\mathcal{L}_{\mathcal{E}}(\psi W) = \mathcal{E}(\psi W) = \psi(\mathcal{E}W) = \psi(\mathcal{L}_{\mathcal{E}}W) = \psi W$$

since W is quasi-homogeneous of degree one.

Lemma 5.2.4. Fix $V_{a_1,\ldots,a_N} \in \mathfrak{g}_A$ where

$$V_{a_1,\ldots,a_N} = \sum_{i=1}^{N} g_i \cdot \left(x_1^{a_1} \cdots x_N^{a_N} \Big((a_i+1)x_1\partial_{x_1} - (a_1+1)x_i\partial_{x_i} \Big) \right)$$

for some choice of homogeneous elements $g_i \in \mathfrak{m}$ with degree $\deg g_i = c \in \mathbb{Q}$ for each i. Then

$$[\mathcal{E}, V_{a_1,\ldots,a_N}] = (\deg V_{a_1,\ldots,a_N}) \cdot V_{a_1,\ldots,a_N}$$

where

$$\deg V_{a_1,\ldots,a_N} = c + \sum_{i=1}^{N} a_i \deg x_i.$$

Proof. One may see this via a direct calculation. As an alternative proof, one recalls the Lie derivative on vector fields,

$$\mathcal{L}_{\mathcal{E}} V_{a_1,\ldots,a_N} := [\mathcal{E}, V_{a_1,\ldots,a_N}].$$

Since $\deg g_i = c$ is the same for each i, we have

$$\mathcal{L}_{\mathcal{E}} g_i = c \cdot g_i.$$

This is sufficient to imply $V_{a_1,\ldots,a_N}$ is homogeneous. More precisely, there is a well-defined $\deg V_{a_1,\ldots,a_N} \in \mathbb{Q}$ such that

$$\mathcal{L}_{\mathcal{E}} V_{a_1,\ldots,a_N} = \deg V_{a_1,\ldots,a_N} \cdot V_{a_1,\ldots,a_N}.$$

Indeed, to calculate $\deg V_{a_1,\ldots,a_N}$, we treat ∂_{x_i} as having degree $-\deg x_i$ since $\mathcal{L}_{\mathcal{E}}\partial_{x_i} =$

$[\mathcal{E}, \partial_{x_i}] = -\deg x_i \partial_{x_i}$. Noting that $x_k \partial_k$ thus has degree 0 for each k, we find using the Leibniz rule for the Lie derivative

$$\begin{aligned}\mathcal{L}_{\mathcal{E}} V_{a_1,\ldots,a_N} &= \sum_{i=1}^{N} \Big(\mathcal{L}_{\mathcal{E}} g_i \cdot x_1^{a_1} \cdots x_N^{a_N} + g_i \mathcal{L}_{\mathcal{E}}(x_1^{a_1} \cdots x_N^{a_N}) \Big) \cdot \Big((a_i+1) x_1 \partial_{x_1} - (a_1+1) x_i \partial_{x_i} \Big) \\ &= \Big(c + \sum_{j=1}^{N} a_j \deg x_j \Big) \cdot V_{a_1,\ldots,a_N}\end{aligned}$$

as desired. □

Proposition 5.2.5. Fix $V_{a_1,\ldots,a_N} \in \mathfrak{g}_A$ such that $\psi = \exp(V_{a_1,\ldots,a_N}) \in G_A$ is a corresponding element of the primary wall-crossing group. Then the quasi-homogeneity condition

$$\mathcal{E}\psi = \psi\mathcal{E}$$

is equivalent to

$$[\mathcal{E}, V_{a_1,\ldots,a_N}] = 0.$$

Proof. We first observe that the quasi-homogeneity condition is equivalent to $[\mathcal{E}, \psi] = 0$. Assuming that this holds, we now show that $[\mathcal{E}, V_{a_1,\ldots,a_N}] = 0$ by calculating the commutator $[\mathcal{E}, \psi]$. By (5.2.3), we have

$$[\mathcal{E}, \psi] = [\mathcal{E}, \exp(V_{a_1,\ldots,a_N})] = \sum_{k=0}^{\infty} \frac{1}{k!} \Big[\mathcal{E}, V_{a_1,\ldots,a_N}^k \Big] = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r=1}^{k} V_{a_1,\ldots,a_N}^{r-1} [\mathcal{E}, V_{a_1,\ldots,a_N}] V_{a_1,\ldots,a_N}^{k-r}. \tag{5.2.4}$$

Using Lemma 5.2.4 yields

$$[\mathcal{E}, \psi] = \deg V_{a_1,\ldots,a_N} \cdot \sum_{k=0}^{\infty} \frac{k}{k!} V_{a_1,\ldots,a_N}^k = \deg V_{a_1,\ldots,a_N} \cdot V_{a_1,\ldots,a_N} \cdot \exp(V_{a_1,\ldots,a_N}).$$

Hence, if $[\mathcal{E}, \psi] = 0$, we find

$$\deg V_{a_1,\ldots,a_N} = 0$$

which, employing Lemma 5.2.4, implies $[\mathcal{E}, V_{a_1,\ldots,a_N}] = 0$. The converse direction is deduced from equation (5.2.4). □

Corollary 5.2.6. The primary wall-crossing group is trivial if and only if W_0 is an ADE singularity.

Proof. Let W_0 be an ADE singularity. For a contradiction, assume that the primary wall-crossing group is non-trivial. Hence, $V \in \mathfrak{g}_A$ is of the form

$$\sum_{i=1}^{N} g_i \cdot x_1^{a_1} \cdots x_N^{a_N} \Big((a_i+1)x_1\partial_{x_1} - (a_1+1)x_i\partial_{x_i} \Big).$$

where there is at least one $i = 2, \ldots, N$ such that $g_i \neq 0$. From the previous proposition, we have the condition

$$\deg V = \deg g_i + \sum_{j=1}^{N} a_j \deg x_j = 0 \tag{5.2.5}$$

for each $i = 2, \ldots, N$. The a_j are non-negative and the $\deg x_j$ are positive. Thus, for equation (5.2.5) to hold, we must have $\deg g_i = 0$ and $a_i = 0$ for each i. This provides the contradiction: simple singularities are characterised by their flat coordinates all having strictly positive degree so that $\deg g_i > 0$ if $g_i \neq 0$. Thus, $g_i = 0$ for each i and the wall-crossing group is trivial. The converse is proven similarly. □

Corollary 5.2.7. For simple elliptic singularities with unique marginal flat coordinate t of degree zero, the primary Landau-Ginzburg wall-crossing group is generated, as a Lie group, by vector fields of the form

$$V = \sum_{i=1}^{N} C_i t^{k_i} \Big(x_1\partial_1 - x_i\partial_i \Big)$$

where $C_i \in \mathbb{C}$ and for some $k_i \in \mathbb{Z}_{>0}$.

Proof. Denote a generator of the primary Landau-Ginzburg wall-crossing group by $V \in \mathfrak{g}_A$. Due to the additive structure of $\mathfrak{g}_A$, without loss of generality, we assume that the g_i's are monomials. Applying the same reasoning as before, we find that $\deg g_i = 0$ for each $i = 2, \ldots, N$. Simple elliptic singularities, however, are characterised by the fact that all such g_i have non-negative degrees and the unique flat coordinate with vanishing degree is the marginal coordinate t. Thus, we must have $g_i = C_i t^{k_i}$ implying that V is of the required form. □

Remark 5.2.8. For generic singularities, one may have $\deg g < 0$ and the previous reasoning fails.

The final step for the primary wall-crossing group of elliptic singularities is thus to determine k_i. To do this, we return to Definition 5.2.3.

5.2.2 Wall-Crossing for Elliptic Singularities

In Definition 5.2.3, we enforce the condition of ψW is an open Saito potential as a necessary condition for the existence of a mirror to an expected open FJRW theory. For simple elliptic singularities, this is sufficient to completely describe the primary wall-crossing group. The algorithm that we propose for finding V applies to any simple elliptic singularity in Fermat, chain or loop form in either rank two or three.

Firstly, the following lemma is useful.

Lemma 5.2.9. Let W_0 be an elliptic singularity in rank three. Fix

$$V := t^{k_2}(x_1\partial_1 - x_2\partial_2), \quad V' := t^{k_3}(x_1\partial_1 - x_3\partial_3)$$

as two elements of $\mathfrak{g}_A$. If e^V and $e^{V'}$ are both elements of the wall-crossing group G_{A,W_0}, then k_2 and k_3 are not coprime.

Proof. Let $x_1^{m_1}x_2^{m_2}x_3^{m_3}$ be a monomial in an open Saito potential of a simple elliptic singularity W_0. Under V the monomial transforms as

$$e^V x_1^{m_1}x_2^{m_2}x_3^{m_3} = e^{(m_1-m_2)t^{k_2}} x_1^{m_1}x_2^{m_2}x_3^{m_3}. \tag{5.2.6}$$

Similarly, we observe that

$$e^{V'} x_1^{m_1}x_2^{m_2}x_3^{m_3} = e^{(m_1-m_3)t^{k_3}} x_1^{m_1}x_2^{m_2}x_3^{m_3}. \tag{5.2.7}$$

After expanding the exponentials in (5.2.6) and (5.2.7), the transformed open Saito potentials contain terms proportional to $t^{k_2}x_1^{m_1}x_2^{m_2}x_3^{m_3}$ and $t^{k_3}x_1^{m_1}x_2^{m_2}x_3^{m_3}$ respectively. Applying Definition 5.2.3, we find the same form of balancing conditions that constrain k_2 also constrain k_3. Although the k_i are not necessarily uniquely determined from these conditions, we deduce that k_2 and k_3 must have a common factor greater than one. □

Proposition 5.2.10. Fix W_0 an elliptic singularity in rank $N = 2$ or $N = 3$. Denote by t the marginal parameter corresponding to the standard good basis element x^{μ_c}. If $g = e^V$ is an element of the primary Landau-Ginzburg wall-crossing group then V is of the form

$$V = \sum_{i=1}^{N} t^{k_i}(x_1\partial_1 - x_i\partial_i)$$

with each k_i satisfying

$$k_i\left(E_{W_0}^T\right)^{-1}\mu_c \in \mathbb{Z}^N. \tag{5.2.8}$$

Proof. In the following we consider the wall-crossing in three cases: the Fermat term $x_i^{r_i}$, the chain term $x_{i-1}x_i^{r_i}$ and the loop term $x_i^{r_i}x_{i+1}$ for fixed i in the open Saito potential. Without loss of generality, we assume that $k_i = k$ for each i and for some k. Indeed, we choose k as the smallest among the set $\{k_i\}_{i=1}^N$; since k and each k_i have a common factor greater than one via Lemma 5.2.9, any selection rule which is true for k is also true for k_i. The strategy for each term is then the same: we act via e^V on each term and expand the exponential. To first order, this has the effect of adding $t^k x_i^{r_i}$ to the Fermat, $t^k x_{i-1}x_i^{r_i}$ to the chain, and $t^k x_i^{r_i}x_{i+1}$ to the loop terms in the open Saito potential. Each of these terms must be balanced according to Definition 5.2.3. We thus write down the selection rules for each of these terms which will constrain the values of k.

Case (i): Terms of the Form $t^k x_i^{r_i}$. The Fermat term $t^k x_i^{r_i}$ in the open Saito potential must satisfy the integral degree condition (4.1.4). The left hand side of this condition reads

$$q_j^T(2k + r_i - 1) - \Theta_j^{\text{root}} - r_i\Theta_j^{x_i} - 2k\Theta_j^{\mu_c}$$

for each j. The integral degree condition is that this expression must be an integer. Substituting the values of the phases, we find that the above selection rule becomes the following constraint on k,

$$(r_i - 2)\delta_{ij} + 2k(E^{-1}_{W_0^T}\mu_c)_j \in \mathbb{Z} \tag{5.2.9}$$

for each $1 \leq j \leq N$. Although the first term appears to be redundant as it is an integer already, we temporarily keep this term. We may further strengthen constraint (5.2.9) by imposing the grading condition of Definition 4.1.1. Indeed, we find that (4.1.5) reads

$$(r_i - 2)\delta_{ij} + 2k(E^{-1}_{W_0^T}\mu_c)_j = r_i\delta_{ij} \mod 2$$

for each j. Hence, we indeed find

$$kE^{-1}_{W_0^T}\mu_c \in \mathbb{Z}^N.$$

Case (ii): Terms of the Form $t^k x_{i-1} x_i^{r_i}$. Consider the chain term $t^k x_i^{r_i} x_{i-1}$. The integral degree condition for this term is

$$q_j^T(2k + r_i) - \Theta_j^{\text{root}} - r_i\Theta_j^{x_i} - \Theta_j^{x_{i-1}} - 2k\Theta_j^{\mu_c} \in \mathbb{Z}.$$

Substituting the values of the twists we find

$$(r_i - 2)\delta_{ij} - \delta_{i-1,j} - 2k(E^{-1}_{W_0^T}\mu_c)_j \in \mathbb{Z}$$

similar to (5.2.9). As before, we impose the grading condition which yields

$$(r_i - 2)\delta_{ij} + \delta_{i-1,j} + 2k(E^{-1}_{W_0^T}\mu_c)_j = r_i\delta_{ij} + \delta_{i-1,j} \mod 2.$$

Hence

$$k(E_{W_0^T}^{-1}\mu_c)_j \in \mathbb{Z}.$$

Case (iii): Terms of the Form $t^k x_i^{r_i} x_{i+1}$. This is essentially the same as Case (ii). The integral degree condition for this loop term is

$$q_j^T(2k + r_i) - \Theta_j^{\text{root}} - r_i\Theta_j^{x_i} - \Theta_j^{x_{i+1}} - 2k\Theta_j^{\mu_c} \in \mathbb{Z}.$$

for each j. Substituting the values of the phases gives the constraint

$$(r_i - 2)\delta_{ij} + \delta_{i+1,j} + 2k(E_{W_0^T}^{-1}\mu_c)_j \in \mathbb{Z}.$$

as desired. Imposing the grading condition,

$$(r_i - 2)\delta_{ij} + \delta_{i+1,j} + 2k(E_{W_0^T}^{-1}\mu_c)_j = r_i\delta_{ij} + \delta_{i+1,j} \mod 2,$$

one finds the same desired result as above. □

Remark 5.2.11. The above proposition shows that the wall-crossing group is not always the trivial group. Furthermore, equation (5.2.8) is identical to equation (15) of [MS16] which arises in the related context of Picard-Fuchs equations for elliptic curves. The tuple (μ_c, k) satisfying (5.2.8) is known as the *charge vector*.

Example 5.2.12. Consider the singularity $W_0 = x_1^3 + x_2^3 + x_3^3$. Consider the vector field V given by

$$V = t^k(x_1\partial_1 - x_2\partial_2) + t^k(x_1\partial_1 - x_3\partial_3).$$

In the case of x_3^3, this can be used to find conditions for k given that $i = 3$ and $r_3 = 3$, together with $\mu_c = (1,1,1)$ and

$$E^{-1}_{W_0^T} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Thus the equation (5.2.8) of the above proposition yields

$$kE^{-1}_{W_0^T}\mu_c = \begin{pmatrix} \frac{1}{3}k \\ \\ \frac{1}{3}k \\ \\ \frac{1}{3}k \end{pmatrix} \in \mathbb{Z}^3.$$

Hence, k must be divisible by 3. This agrees with the primitive form calculation in section 6 of [LLS14]. Therefore, the action of the wall-crossing group is given by $e^V \cdot W$ where

$$V = t^3\Big((x_1\partial_1 - x_2\partial_2) + (x_1\partial_1 - x_3\partial_3)\Big).$$

Using equation (5.2.6) we calculate that

$$e^V x_1x_2x_3 = x_1x_2x_3.$$

As in Proposition 4.2.8, we restrict W so that all non-marginal parameters vanish. We choose $s(t)$ to be the inverse of the Schwarz triangle function. We refer the reader to Appendix A.1 for details of these functions. Thus, we have

$$W(t) - W_0 = s(t)x_1x_2x_2$$

we find that $W(t)$ is a fixed point of the wall-crossing group action. Thus, since $s(t)$ is modular, modularity is preserved by the action of e^V for the cubic in $\mathbb{P}^2$. However, this will fail for generic simple elliptic singularities. For example, consider the singularity $W = x_1^4 + x_1 x_2^3$. The versal deformation is given by $W(t) - W_0 = s(t) x_1^3 x_2$. Under the wall crossing group generated by

$$V = t^3(x_1\partial_1 - x_2\partial_2)$$

this transforms as

$$e^V\Big(W(t) - W_0\Big) = e^V\Big(s(t)x_1^3x_2\Big) = e^{2t^3}s(t)x_1^3x_2.$$

However, $e^{2t^3}s(t)$ is not modular invariant.

Example 5.2.13. Consider $W_0 = x_1^4 + x_2^4$. We find

$$E^{-1}_{W_0^T} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Thus we find

$$kE^{-1}_{W_0^T}\mu_c = \begin{pmatrix} \frac{k}{2} \\ \frac{k}{2} \end{pmatrix} \in \mathbb{Z}^2.$$

Thus, k must be divisible by 2.

Remark 5.2.14. We remark that the open Saito theory of a singularity W_0 is equivalent to the open Saito theory of its Morse stabilisation $\tilde{W}_0$. This can be seen for an elliptic singularity from the viewpoint of wall-crossing. Indeed, for an elliptic singularity W_0 consider the vector field

$$V = C_2t^k(x_1\partial_1 - x_2\partial_2) + C_3t^k(x_1\partial_1 - x_3\partial_3).$$

We have that

$$\tilde{W}_0 = W_0(x_1, x_2) + x_3^2 \mapsto e^V W_0(x_1, x_2) + e^{-2C_3s^k}x_3^2.$$

However, the presence of the term $t^k x_3^2$ places no extra constraint on k. Indeed, we have that

$$kE_{W_0^T}^{-1}\mu_c = k\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} * \\ * \\ 0 \end{pmatrix}.$$

Thus, the third component of $kE_{W_0^T}^{-1}\mu_c$ vanishes and so is automatically an integer.

5.3 Properties of the Landau-Ginzburg Wall-Crossing Group

To show that we have made the correct choices when constructing the Landau-Ginzburg wall-crossing group, in this section we prove that the corresponding group action of G_{A,W_0} is faithful and transitive on open ancestor Saito potentials.

5.3.1 Faithfulness

The main result in this subsection is the following.

Proposition 5.3.1. Let W_0 be a chain or loop polynomial in rank two and let $A = A_I$ or $A_{I,\mathrm{sym}}$. Suppose G_{A,W_0} is the Landau-Ginzburg wall-crossing group. The action of G_{A,W_0} on the open ancestor Saito potential is faithful.

Proof. Let $e^V \in G_{A,W_0}$ so that V is contained in the Lie algebra $\mathfrak{g}_{A,W_0,K}$. Let $K \geq 2$ be the smallest integer such that $V = 0 \mod \mathfrak{m}^K$ but $V \neq 0 \mod \mathfrak{m}^{K+1}$. Then working modulo $\mathfrak{m}^{K+1}$ we find

$$W \mapsto e^V W = W + V(W).$$

Hence, to show that this action is faithful it suffices to check that $VW = 0$ modulo $\mathfrak{m}^{K+1}$ implies $V = 0$.

The element V is a linear combination of vector fields given in $\mathfrak{g}_{A,W_0,K}$. Choose a minimal a_1 so that V has a non-zero summand in the (a_1, a_2) sector of the direct sum

(5.2.1) for some a_2. Choose a_2 such that it is minimal among choices that fix a_1. In particular, we consider the minimal pair (a_1, a_2) occurring in $V \mod \mathfrak{m}^{K+1}$ with respect to the lexicographic ordering so that V' is a summand of V. Thus we may write V' as

$$V' = C \cdot x_1^{a_1} x_2^{a_2} \Big((a_2 + 1) x_1 \partial_{x_1} - (a_1 + 1) x_2 \partial_{x_2} \Big).$$

Here, $C \in \mathfrak{m}^K / \mathfrak{m}^{K+1}$. Now, for the chain polynomial $W = x_1^{r_1} + x_1 x_2^{r_2}$ we find

$$VW = C \cdot \Big(r_1 (a_2 + 1) x_1^{a_1 + r_1} x_2^{a_2} + (a_2 + 1 - a_1 r_2 - r_2) x_1^{a_1 + 1} x_2^{a_2 + r_2} \Big).$$

Note that the first term is always non-zero. However, by minimality of (a_1, a_2), there is no cancellation of such terms in $VW = 0$. This implies $C = 0$ so that $V = 0 \mod \mathfrak{m}^{k+1}$ contradicting the definition of k. Thus $V = 0$.

For the loop polynomial $W = x_1^{r_1} x_2 + x_1 x_2^{r_2}$ a similar calculation yields

$$V'W = C \cdot \Big((a_2 r_1 + r_1 - a_1 - 1) x_1^{a_1 + r_1} x_2^{a_2 + 1} + (a_2 + 1 - a_1 r_2 - r_2) x_1^{a_1 + 1} x_2^{a_2 + r_2} \Big).$$

It suffices to show that at least one of these terms is always non-zero, since the same reasoning as the chain polynomial will then apply. Hence, assume that

$$a_2 r_1 + r_1 - a_1 - 1 = a_2 + 1 - a_1 r_2 - r_2 = 0.$$

Solving for a_2 and substituting yields the equation

$$(a_1 + 1)(r_1 r_2 - 1) = 0.$$

This cannot be true since $r_1, r_2 \geq 1$ by assumption that W_0 is non-degenerate and $a_1 \geq 0$ by definition of the wall-crossing group. This completes the proof for loop polynomials. □

Remark 5.3.2. The proof of faithfulness of the wall-crossing group action for rank 3 elliptic singularities is immediate. Indeed, the action is given by multiplication which is certainly faithful.

5.3.2 Transitivity

The main result of the paper consists of the following two theorems. Via Corollaries 4.3.7 and 4.3.12, we assume that any collection $(\nu_{k_1,k_2,I,\boldsymbol{d}})$ in rank two is chosen such that $dx_1 \wedge dx_2$ is a primitive form and $t_{\alpha,\beta,0}$ are flat coordinates.

Theorem 5.3.3. The Landau-Ginzburg wall-crossing group G_{A,W_0} acts transitively on open Saito potentials for $W_0 = x_1^{r_1} + x_1 x_2^{r_2}$.

Proof. We first consider the non-symmetric case and work with the ring A_I. Let $\nu_{k_1,k_2,I,\boldsymbol{d}}$ and $\nu'_{k_1,k_2,I,\boldsymbol{d}}$ be two coefficients of open Saito potentials. We use induction and prove that for any positive integer k, there exists $g \in G_{A,W_0}$ such that $g(\nu_{k_1,k_2,I,\boldsymbol{d}}) = \nu'_{k_1,k_2,I,\boldsymbol{d}}$ for any $(k_1,k_2) \in \mathbb{Q}\text{-Bal}(I,\boldsymbol{d})$ with $r_2k_1 + r_1k_2 - k_2 < k$. The base case $k = 0$ is immediate since we take $g = \text{id}$. Define an ideal $\mathcal{I}_k = \langle x_1^{k_1} x_2^{k_2} \in A_I[[x_1,x_2]] \mid r_2k_1 + r_1k_2 - k_2 \geq k \rangle$.

For the induction hypothesis, assume that there is a g which for a given k satisfies the transitivity property. By replacing $\nu_{k_1,k_2,I,\boldsymbol{d}}$ with $g(\nu_{k_1,k_2,I,\boldsymbol{d}})$, we assume that $W^{\nu} \equiv W^{\nu'}$ mod $\mathcal{I}_k$. Now let $J \subseteq I$ be a set with $(k_1,k_2) \in \mathbb{Q}\text{-Bal}(J,\boldsymbol{d})$ and $r_2k_1 + r_1k_2 - k_2 = k$. Consider the coefficient of u_J in $W^{\nu'} - W^{\nu}$,

$$(-1)^{|J|-1} \sum_{(k_1,k_2) \in \mathbb{Q}\text{-Bal}(J,\boldsymbol{d})} (\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}}) x_1^{k_1} x_2^{k_2}.$$

Recall that we have a complete classification of balanced conditions for rank 2 chain polynomials in Example 4.1.3. Namely, we must have

$$r_2k_1 - k_2 \equiv r_2r_1(J) - r_2(J) \mod r_1r_2, \quad k_2 \equiv r_2(J) \mod r_2.$$

Equivalently, k_1 and k_2 can be written in terms of an integer

$$p = 0, \dots, N-1, N = \frac{\sum_{i\in J} a_i - r_1(J)}{r_1} + \frac{\sum_{i\in J} b_i - r_2(J)}{r_2} - |J| + 1$$

with

$$k_1(p) = r_1(J) + pr_1 + N - p, \quad k_2(p) = r_2(J) + (N-p)r_2.$$

Note that $r_2k_1 + r_1k_2 - k_2$ is independent of p and hence is always k.

Thus, we write the coefficient u_J of $W^{\nu'} - W^{\nu}$ as

$$(-1)^{|J|-1}\sum_{p=0}^{N}(\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}})x_1^{r_1(J)+pr_1+N-p}x_2^{r_2(J)+(N-p)r_2}.$$

Let $p' \geq 0$ be the smallest p such that $\nu'_{k_1,k_2,J,\boldsymbol{d}} \neq \nu_{k_1,k_2,J,\boldsymbol{d}}$. Consider the vector field

$$V = (-1)^{|J|}u_J\frac{\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}}}{k_2(p'+1)+1-r_2k_1(p')}x_1^{k_1(p')-1}x_2^{k_2(p'+1)}\Big((k_2(p'+1)+1)x_1\partial_1 - k_1(p')x_2\partial_2\Big).$$

Observe that this is an element of $\mathfrak{g}_{A_I}$. Modulo $\mathcal{I}_{k+1}$, we observe that e^VW^{ν} differs from W^{ν} only in the coefficients of $u_Jx_1^{k_1(p')}x_2^{k_2(p')}$ and $u_Jx_1^{k_1(p'+1)}x_2^{k_2(p'+1)}$. This implies that e^V is indeed an element of the wall-crossing group. Moreover, the coefficient of $u_Jx_1^{k_1(p')}x_2^{k_2(p')}$ in e^VW^{ν} coincides with the coefficient of the same monomial in $W^{\nu'}$. Hence, by replacing W^{ν} with e^VW^{ν} and repeating inductively on p', we assume that the coefficient of u_J in $W^{\nu'} - W^{\nu}$ is

$$(-1)^k(\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}})x_1^{r_1(J)+Nr_1}x_2^{r_2(J)}.$$

Therefore, there is at most one element $(k_1,k_2) \in \mathbb{Q}\text{-Bal}(J,\boldsymbol{d})$ with $\nu_{k_1,k_2,J,\boldsymbol{d}} \neq \nu'_{k_1,k_2,J,\boldsymbol{d}}$. Furthermore, we inductively assumed $\nu_{k'_1,k'_2,J',\boldsymbol{d}'} = \nu'_{k'_1,k'_2,J',\boldsymbol{d}'}$ when $J' \subset J$. From the selection rules, we calculate

$$m(J,\boldsymbol{d}) = r_2k_1+r_1k_2-k_2 = r_2r_1(J)+r_1r_2(J)-r_2(J)+Nr_1r_2 \geq r_2r_1(J)+r_1r_2(J)-r_2(J).$$

Hence, if $|J| \geq 2$, by definition of ν and the balancing conditions, we have $\mathcal{A}(J, \boldsymbol{d}, \nu) = \mathcal{A}(J, \nu', \boldsymbol{d}) = 0$. For $|J| = 1$, we also have the same equality. Unravelling these equations, we observe that they coincide apart from the contributions from the terms $\nu_{k_1,k_2,J,\boldsymbol{d}}$ and $\nu'_{k_1,k_2,J,\boldsymbol{d}}$. Hence, these two terms must be equal.

The symmetric case $A = A_{I,\mathrm{sym}}$ is similar. □

Theorem 5.3.4. The Landau-Ginzburg wall-crossing group G_{A,W_0} acts transitively on open Saito potentials for $W_0 = x_1^{r_1}x_2 + x_1x_2^{r_2}$.

Proof. We first consider the non-symmetric case and work with the ring A_I. Let $\nu_{k_1,k_2,I,\boldsymbol{d}}$ and $\nu'_{k_1,k_2,I,\boldsymbol{d}}$ be two coefficients of open Saito potentials. We use induction and prove that for any positive integer k, there exists $g \in G_{A,W_0}$ such that $g(\nu_{k_1,k_2,I,\boldsymbol{d}}) = \nu'_{k_1,k_2,I,\boldsymbol{d}}$ for any $(k_1, k_2) \in \mathbb{Q}\text{-Bal}(I, \boldsymbol{d})$ with $r_2k_1 + r_1k_2 - k_1 - k_2 < k$. The base case $k = 0$ is immediate since we take $g = \mathrm{id}$. Define an ideal $\mathcal{I}_k = \langle x_1^{k_1}x_2^{k_2} \in A_I[[x_1, x_2]] \mid r_2k_1 + r_1k_2 - k_1 - k_2 \geq k \rangle$.

For the induction hypothesis, assume that there is a g which for a given k satisfies the transitivity property. By replacing $\nu_{k_1,k_2,I,\boldsymbol{d}}$ with $g(\nu_{k_1,k_2,I,\boldsymbol{d}})$, we assume that $W^\nu \equiv W^{\nu'}$ mod $\mathcal{I}_k$. Now let $J \subseteq I$ be a set with $(k_1, k_2) \in \mathrm{Bal}(J, \boldsymbol{d})$ with $r_2k_1 + r_1k_2 - k_1 - k_2 = k$. Consider the coefficient of u_J in $W^{\nu'} - W^\nu$. That is to say,

$$(-1)^{|J|-1} \sum_{(k_1,k_2)\in \mathrm{Bal}(J,\boldsymbol{d})} (\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}})x_1^{k_1}x_2^{k_2}.$$

Recall that we have a complete classification of balanced conditions for rank 2 loop polynomials in Example 4.1.4. Namely, we must have

$$r_2k_1 - k_2 \equiv r_2r_1(J) - r_2(J) \mod r_1r_2 - 1, \quad r_1k_2 - k_1 \equiv r_2(J) \mod r_1r_2 - 1$$

Equivalently, k_1, k_2 can be written in terms of an integer

$$p = 0, \ldots, N-1, N = \frac{\sum_{i\in J} a_i - r_1(J)}{r_1} + \frac{\sum_{i\in J} b_i - r_2(J)}{r_2} - |J| + 1$$

so that

$$k_1(p) = r_1(J) + pr_1 + N - p, \quad k_2(p) = r_2(J) + (N-p)r_2 + p$$

Observe that $r_2k_1 + r_1k_2 - k_1 - k_2$ is independent of p and hence is always k. Thus, we write the coefficient u_J of $W^{\nu'} - W^{\nu}$ as

$$(-1)^{|J|-1}\sum_{p=0}^{N}(\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}})x_1^{r_1(J)+pr_1+N-p}x_2^{r_2(J)+(N-p)r_2+p}.$$

Let $p' \geq 0$ be the smallest p such that $\nu'_{k_1,k_2,J,\boldsymbol{d}} \neq \nu_{k_1,k_2,J,\boldsymbol{d}}$. Consider the vector field

$$V = (-1)^{|J|}u_J\frac{\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}}}{k_2(p'+1) - r_2k_1(p')}x_1^{k_1(p')-1}x_2^{k_2(p'+1)-1}\Big(k_2(p'+1)x_1\partial_1 - k_1(p')x_2\partial_2\Big).$$

Observe that this is an element in $\mathfrak{g}_{A_I}$. Modulo $\mathcal{I}_{k+1}$, we note that $e^V W^{\nu}$ differs from W^{ν} only in the coefficients of $u_Jx_1^{k_1(p')}x_2^{k_2(p')}$ and $u_Jx_1^{k_1(p'+1)}x_2^{k_2(p'+1)}$. This implies that e^V is indeed an element of the wall-crossing group. Moreover, the coefficient of $u_Jx_1^{k_1(p')}x_2^{k_2(p')}$ coincides with the coefficient of the same monomial in $W^{\nu'}$. Hence, by replacing W^{ν} with e^VW^{ν} and repeating inductively on p', we assume that the coefficient of u_J in $W^{\nu'} - W^{\nu}$ is

$$(-1)^k(\nu'_{k_1,k_2,J,\boldsymbol{d}} - \nu_{k_1,k_2,J,\boldsymbol{d}})x_1^{r_1(J)+Nr_1}x_2^{r_2(J)+N}.$$

Therefore, there is at most one element $(k_1,k_2) \in \mathbb{Q}\text{-Bal}(J,\boldsymbol{d})$ with $\nu_{k_1,k_2,J,\boldsymbol{d}} \neq \nu'_{k_1,k_2,J,\boldsymbol{d}}$. Furthermore, we inductively assumed $\nu_{k'_1,k'_2,J',\boldsymbol{d}'} = \nu'_{k'_1,k'_2,J',\boldsymbol{d}'}$ when $J' \subset J$. From the selection rules, we calculate

$$\begin{aligned} m(J,\boldsymbol{d}) = r_2k_1 + r_1k_2 - k_1 - k_2 &= r_2r_1(J) + r_1r_2(J) - r_1(J) - r_2(J) + Nr_1r_2 \\ &\geq r_2r_1(J) + r_1r_2(J) - r_1(J) - r_2(J). \end{aligned}$$

Hence, if $|J| \geq 2$, by the definition of ν and the balancing conditions, we find $\mathcal{A}(J,\boldsymbol{d},\nu) = \mathcal{A}(J,\nu',\boldsymbol{d}) = 0$. For $|J| = 1$, we also have the same equality. Unravelling these equations, we notice that they coincide apart from the contribution from the terms ν and ν'. Hence, these two terms must be equal.

The symmetric case $A = A_{I,\mathrm{sym}}$ is similar. □

Using these results we now clarify the computational result from the motivation section at the start of this chapter.

Corollary 5.3.5. Let W_0 be a chain or loop polynomial in rank two. Let I be a finite marking set. Suppose that ν and ν' are balanced chamber indices with respect to I. Then

$$\mathcal{A}(J, \boldsymbol{d}, \nu) = \mathcal{A}(J, \boldsymbol{d}, \nu')$$

Proof. By the previous theorems on transitivity, there exists a $g \in G_{A,W_0}$ such that $g\nu = \nu'$. Viewing g as a map on the coordinates $x_1, \ldots, x_N$, we find via the projection formula,

$$\int_\Xi e^{W^\nu/\hbar} dx_1 \wedge \cdots \wedge dx_N = \int_{g^{-1}(\Xi)} g^*\Big(e^{W^\nu/\hbar} dx_1 \wedge \cdots \wedge dx_N\Big).$$

However, from the definition of the wall crossing group, we know that g is identity modulo an ideal linear in the t variables. Thus, the integral is unchanged if we integrate over $g^{-1}(\Xi)$ or Ξ. Again by the definition of the wall crossing group, we know that g preserves the volume form; for $g = e^V$ this implies

$$g^*(dx_1 \wedge \cdots \wedge dx_N) = \mathcal{L}_V(dx_1 \wedge \cdots \wedge dx_N) = 0.$$

Since g acts on W^ν via $g \cdot W^\nu = W^{g\nu} = W^{\nu'}$, putting this all together we find

$$\int_\Xi e^{W^\nu/\hbar} dx_1 \wedge \cdots \wedge dx_N = \int_\Xi e^{W^{\nu'}/\hbar} dx_1 \wedge \cdots \wedge dx_N.$$

Comparing coefficients using Corollaries 4.3.7 and 4.3.12, we find the desired result.

□

CHAPTER 6

TOWARDS OPEN FJRW THEORY

In Chapter 2 we began by reviewing closed FJRW theory as the geometric motivation for the study of Landau-Ginzburg models. In subsequent chapters, we then reviewed and extended open Saito theory. The advantage of this approach is that we can make predictions about enumerative open FJRW theory, whilst bypassing the technicalities of such a theory. This allows for the prediction of the existence of modular FJRW generating functions in the elliptic case. This is stated more precisely in Conjecture 6.2.21. Given the results of Chapters 4 and 5, we predict, moreover, the definitions of balanced W-spin disks so that the open Landau-Ginzburg mirror conjecture holds. This is stated in Conjecture 6.1.14.

6.1 W-Spin Structures

6.1.1 Open r-Spin Disks

To motivate the main definitions, we review some constructions of open r-spin theory given in [BCT22a].

We recall that an r-spin disk is defined, roughly, by the data of an orbicurve C together with an involution $\phi : C \to C$ so that the coarse underlying curve $|C|$ is given by gluing two curves Σ and $\overline{\Sigma}$ along the common boundary $\partial\Sigma$ and where $\overline{\Sigma}$ is obtained by reversing the complex structure. In particular, if $z_1, \ldots, z_l \in \Sigma \setminus \partial\Sigma$ are internal marked points, then the conjugates $\overline{z}_i = \phi(z_i)$ are internal marked points in $\overline{\Sigma} \setminus \partial\Sigma$ for some internal marking set I with $l = |I|$. We also let $w_1, \ldots, w_k \in \partial\Sigma$ be boundary marked points for some boundary marking set $\mathcal{B}$ with $k = |\mathcal{B}|$.

The r-spin structure also contains the data of a spin bundle S which is an orbifold line bundle over C together with an isomorphism of orbifold line bundles

$$\varphi : S^{\otimes r} \to \omega_{C,\log} := \rho^* \omega_{|C|,\log} = \rho^* \Big(\omega_{|C|} \otimes \mathcal{O}\Big(\sum_{n=1}^{|I|} [z_n] + \sum_{n=1}^{|I|} [\overline{z}_n] + \sum_{m=1}^{|\mathcal{B}|} [w_m] \Big) \Big) \tag{6.1.1}$$

where $[z_p], [\overline{z}_p], [w_q]$ are the corresponding divisors on C and $\rho : C \to |C|$ is defined by forgetting the orbifold structure on C. It can be shown that the underlying coarse bundle $|S| = \rho_* S$ satisfies

$$|S|^r \cong \omega_{|C|} \otimes \mathcal{O}\Big(- \sum_{n=1}^{|I|} a_n [z_n] - \sum_{n=1}^{|I|} a_n [\overline{z}_n] - \sum_{m=1}^{|\mathcal{B}|} b_m [w_m] \Big) \tag{6.1.2}$$

for some choice of *twists* a_n and b_m,

$$a_n \in \{0, 1, \ldots, r-2\}, \quad b_m = r - 2. \tag{6.1.3}$$

These twists correspond to the standard good basis elements of $\mathscr{D}_{W_0}$ for $W_0 = x^r$ and in Example 6.1.10 we shall, explain, via mirror symmetry, the origin of the twist $r-2$ for each boundary point.

Remark 6.1.1. Based on work of Chiodo [Chi08] and Norbury [Nor23], the authors of [CGG23] establish a version of r-spin theory where one considers instead the log anti-canonical bundle $\omega_{|C|}^{\vee}$. The authors subsequently show that, for $r = 2$ and $r = 3$, a generating function for this enumerative theory is the so-called Brézin-Gross-Witten τ-function [BG80; GW80] of the r-KdV hierarchy. By computing $\mathcal{W}$-algebra constraints as evidence, it is conjectured that this also holds for all r. This τ-function is a continuation of Kontsevich's matrix model [Kon92] to negative values of r. For further details, see [MMS96]. It may be interesting to construct an analogous 'open negative r-spin' theory, with corresponding structures of topological recursion relations and integrable hierarchies.

6.1.2 Open W-Spin Disks

The case of r-spin disks is a specific case of W-spin disks for the choice the A_{r-1} singularity. For a proof of this fact, see [AJ03; FJR11a]. Here, we generalise this to any Sebastiani-Thom sums of Fermat, chain or loop polynomial following [FJR13] and [GKT22b]. We denote such a polynomial by W whose exponent matrix is $E_W = (r_{ij})$. The following is a simplification of the usual notion of marked W-spin disk but it is sufficient for our purposes. We say that a marked W-spin disk is a genus zero W-spin curve with boundary,

$$(C, \{w_m\}_{m=1}^{k}, \{z_n\}_{n=1}^{l}, \{\overline{z}_n\}_{n=1}^{l}, \phi, \{S_i\}_{i=1}^{N}, \{\varphi_i\}_{i=1}^{N})$$

where

1. C is an orbicurve
2. $\phi : C \to C$ is an anti-holomorphic involution that realises $|C|$ as a union of curves Σ and $\overline{\Sigma}$ with conjugate complex structures and common boundary $\partial\Sigma = \partial\overline{\Sigma}$ which is the fixed locus of $|\phi|$.
3. $z_n, \overline{z}_n$ are internal marked points such that $\phi(z_n) = \overline{z}_n$,
4. w_j are marked points on the boundary of C so that $\phi(w_m) = w_m$,
5. S_i are orbifold line bundles over C such that the maps

$$\varphi_i : \bigotimes_{j=1}^{N} S_j^{r_{ij}} \xrightarrow{\sim} \omega_{C,\log}$$

   are isomorphisms.

To introduce the twists, recall that the spin bundles S_j induce a representation $\Pi_x : G_x \to (G_W^{\max})^N$ of the local group G_x at each orbifold point x. For each orbifold marked point, x, we denote

$$\gamma_x = \Pi_x(1) = \left(e^{2\pi i \Theta_1^{\gamma_x}}, \dots, e^{2\pi i \Theta_N^{\gamma_x}}\right) \in (G_W^{\max})^N.$$

We recall that such group elements are called decorations at x and the associated $\Theta^{\gamma_x} = \left(\Theta_1^{\gamma_x}, \dots, \Theta_N^{\gamma_x}\right)$ are called FJRW phases. Given (6.1.1) in the r-spin case, we expect that there is an isomorphism,

$$\bigotimes_{j=1}^{N} |S_j|^{r_{ij}} \cong \omega_{|C|} \otimes \mathcal{O}\Big(- \sum_{n=1}^{l} \sum_{j=1}^{N} (r_{ij}\Theta_j^{\gamma_n} - 1)([z_n] + [\overline{z}_n]) - \sum_{m=1}^{k} \sum_{j=1}^{N} (r_{ij}\Theta_j^{\gamma_m} - 1)[w_m]\Big). \tag{6.1.4}$$

It will therefore be important to calculate and give explicit expressions for the phases of the corresponding internal and boundary marked points as this allows one to deduce the twists in (6.1.4). The main tool we use here are selection rules that arise from integral degree constraints of spin bundles and the dimension axiom for non-vanishing FJRW invariants.

6.1.3 Selection Rules for Open FJRW Theory.

In this section, with the absence of an open FJRW theory for general potentials, we *define* balanced W-spin disks as those disks which satisfy certain numerical conditions: these conditions are precisely the selection rules for the non-vanishing of invariants that would be expected from such an enumerative theory. This is already well-known for r-spin disks as described in [BCT22a], although it is instructive to review this here to set up notation.

Example 6.1.2. Let us consider these consider these integrality and balancing calculations from the point of view of open r-spin theory. Suppose that C is an orbicurve with $|I|$ internal marked points of phases $\Theta^i = \frac{a_i+1}{r}$ and $|\mathcal{B}|$ boundary marked points of phases

$\Theta^j = \frac{b_j+1}{r}$. Here we take $a_i, b_j \in \mathbb{Z}_r$ and we include the root marked point in the set $\mathcal{B}$ of boundary marked points. Given the antiholomorphic involution, we consider that there are $2|I| + |\mathcal{B}|$ marked points. Hence, given a r-spin disk with spin bundle S, the degree of $|S|$ is given by

$$\deg |S| = \frac{1}{r}(2|I| + |J| - 2) - 2\sum_{n=1}^{|I|} \frac{a_n+1}{r} - \sum_{m=1}^{|\mathcal{B}|} \frac{b_m+1}{r}.$$

This simplifies to

$$\deg |S| = -\frac{1}{r}\Big(2 + 2\sum_{n=1}^{|I|} a_n + \sum_{m=1}^{|\mathcal{B}|} b_m\Big).$$

This quantity must be an integer and we will impose this selection rule in a moment. Since $\deg |S| < 0$, there cannot be any non-trivial global sections of $|S|$. This implies $H^0(C, |S|) = 0$. By definition of the Witten bundle, we have

$$\text{rk } \mathcal{W} = \dim_{\mathbb{C}} H^1(C, |S|) = -\deg |S| - 1 = \frac{2 - r + 2\sum_{i=1}^{|I|} a_i + \sum_{j=1}^{|\mathcal{B}|} b_j}{r}$$

where in the second equality we have used the Riemann-Roch theorem. Thus to have non-vanishing FJRW invariants we must have that this rank is an integer. Hence, we recover equation (2.8) of [BCT22a].

We will now generalise the above example for any quasi-homogeneous superpotential W.

Definition 6.1.3. Let $C_{0,k,l}$ be a genus zero W-spin disk with k boundary points and l internal points and where W is an invertible polynomial with charges $q_1, \ldots, q_N$. We say that $C_{0,k,l}$ is *balanced* if the following two conditions hold.

1. For each $i = 1, \ldots, N$,

$$q_i(2l + k - 2) - 2\sum_{n=1}^{l} \Theta_i^{\gamma_n} - \sum_{m=1}^{k} \Theta_i^{\gamma_m} =: -1 - e_i \in \mathbb{Z} \tag{6.1.5}$$

   where $\Theta_i^{\gamma_p}, \Theta_i^{\gamma_q}$ are the i^{th} components of the phases of internal and boundary marked points respectively.

2. Furthermore, we have
$$\sum_{n=1}^{l} 2d_n + \sum_{i=1}^{N} e_i = 2l + k - 3. \tag{6.1.6}$$

The conditions (6.1.5) and (6.1.6) coincide with (4.1.4) and (4.1.6). The following observation reveals the geometric origins of these conditions.

Observation 6.1.4. The anti-holomorphic involution contained in the data of a W-spin disk doubles the number of internal marked points. Hence the degree of S_i is

$$\deg S_i = q_i(2l + k - 2) - 2\sum_{n=1}^{l} \Theta_i^{\gamma_n} - \sum_{m=1}^{k} \Theta_i^{\gamma_m}.$$

Thus, equation (6.1.5) is obtained by requiring $\deg S_i \in \mathbb{Z}$ for each i. For simplicity we now define

$$\deg S_i := -1 - e_i$$

for some $e_i \in \mathbb{Z}$. Supposing that the i^{th} Witten bundle $\mathcal{W}_i$ has been constructed over the open W-spin disk C in an analogous way to Section 2.4.2, the e_i will play the role of the degree of the Euler class of $\mathcal{W}_i$. We also note that since $\deg S_i < 0$, we must have $H^0(C, S_i) = 0$. This observation will be useful for Riemann-Roch calculations.

To obtain (6.1.6), we first suppose that the analogous Witten bundle together with the tautological line bundles $\mathbb{L}_n^{d_n}$, as defined in Chapter 2, has been constructed over a W-spin disk. We then match the rank of the Witten bundle and the dimension of the moduli space of genus zero disks with k boundary marked points and l boundary marked points. We denote such a moduli soace as $\overline{\mathcal{M}}_{0,k,l}$. For more details, see for example [PST22]. Indeed, we have that

$$\text{rk}\ \mathcal{W}_i = h^1(S_i) = h^1(|S_i|).$$

Assuming $|C_{0,k,l}|$ is smooth, $h^1(|S_i|)$ can be calculated via the Riemann-Roch theorem.

Indeed, as before we have $h^0(|S_i|) = 0$ for reasons of degree. Thus, we find

$$\text{rk } \mathcal{W}_i = h^1(|S_i|) = -\deg |S_i| - 1 = e_i.$$

Thus,

$$\text{rk } \mathcal{W} = \sum_{i=1}^{N} \text{rk } \mathcal{W}_i = \sum_{i=1}^{N} e_i.$$

We note that the degree of $e(\mathcal{W})$ is 2 rk $\mathcal{W}$. We now suppose there exists a moduli space, $\overline{\mathcal{M}}^W_{0,k,l}$, of W-spin disks with k boundary marked points and l internal marked points. This moduli space has been constructed for $W = x^r$ and $W = x^r + y^s$ in [BCT22a; GKT22b]. To calculate $\dim_{\mathbb{R}} \overline{\mathcal{M}}^W_{0,k,l}$ we consider $\overline{\mathcal{M}}_{0,k,l}$, the moduli space of disks with k boundary points and l internal marked points. Suppose further that there is an quasi-finite and flat morphism st: $\overline{\mathcal{M}}^W_{0,k,l} \to \overline{\mathcal{M}}_{0,k,l}$ defined by forgetting the W-spin structure. We conjecture that such a morphism exists in the case of disks as it already exists in the case of closed curves as explained in Theorem 2.3.6 and Theorem 2.2.6 of [FJR13]. By pushing forward through st we have

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}^W_{0,k,l} = \dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,k,l} = 2 \dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,k,l} = 2(2l + k - 3).$$

The first equality holds via the fact that st is assumed quasi-finite and flat. Consequently, to define a non-vanishing FJRW invariant, we must have that

$$\text{rk } \mathcal{W} + \sum_{n=1}^{l} 2d_n = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,k,l}$$

from dimension considerations when integrating. Hence, we find

$$\sum_{i=1}^{N} e_i + \sum_{n=1}^{l} 2d_n = 2l + k - 3$$

as desired.

6.1.4 Concavity

Recall that we already defined a narrow and broad insertion in the framework of FJRW theory of Chapter 2. However, we now recast this definition into the B-model using Krawitz mirror symmetry in order to predict the concavity of boundary marked points.

Closed Invariants.

Definition 6.1.5. Consider a term in the closed Saito potential of the form $\prod_{i=1}^{l} t_{\mu_i}$ for a potential W_0. We say that the insertion t_{μ_i} is narrow if the corresponding phase Θ^{μ_i} given in (4.1.1) has no integer components. Otherwise, the insertion is called broad.

The following lemmas are proven in Lemmas 2.1 and 2.2 of [Kra10, section 2.8], although we provide more explicit detail here.

Lemma 6.1.6. For the rank N chain polynomial $W = x_1^{r_1} + \cdots + x_{N-1}x_N^{r_N}$, the variable t_μ in a Saito potential is a broad insertion if and only if the components of μ satisfy $\mu_j = \delta_{N-j}^{\text{even}}(r_j - 1)$ where $s \leq j \leq N$ for some s.

Proof. For an insertion t_μ to be non-concave, we must have that the associated phase Θ^μ contains identity elements in some components. By permuting indices if necessary, assume that these components are $\{s, s+1, \ldots, N\}$. In other words, $\gamma_\mu = e^{2\pi i \Theta^\mu}$ fixes the variables $x_s, x_{s+1}, \ldots, x_N$. Using the explicit expression for the phases Θ^μ in Corollary 3.3.11, for the component $j = N$ we find the equation

$$\Theta_N^\mu = \frac{\mu_N + 1}{r_N} \in \mathbb{Z}.$$

This integer must be 1 since $0 \leq \mu_N \leq r_N - 1$. Thus we find $\mu_N = r_N - 1$. We can then solve the other equations iteratively. Indeed, for the $j = N-1$ component, we find

$$\Theta_{N-1}^\mu = \frac{\mu_{N-1} + 1}{r_{N-1}} - \frac{\mu_N + 1}{r_{N-1} r_N} \in \mathbb{Z}$$

again via Corollary 3.3.11. For $\mu_N = r_N - 1$ we thus have

$$\frac{\mu_{N-1}}{r_{N-1}} \in \mathbb{Z}.$$

Since $0 \leq \mu_{N-1} \leq r_{N-1} - 1$ we find $\mu_{N-1} = 0$. One may repeat this procedure to find that if the insertion τ_μ is non-concave then μ has components $\mu_j = \delta^{\text{even}}_{N-j}(r_j - 1)$ for some $s \leq j \leq N$. Conversely, such a μ gives a non-concave insertion by definition. □

Lemma 6.1.7. Suppose the loop potential $W = x_1^{r_1}x_2 + x_2^{r_2}x_3 + \cdots + x_N^{r_N}x_1$ satisfies $r_i \geq 2$ for each i. If N is even, then there are exactly two broad insertions corresponding to $\mu_j = \delta^{\text{odd}}_j(r_j - 1)$ and $\mu_j = \delta^{\text{even}}_j(r_j - 1)$. If N is odd, all insertions are narrow.

Proof. Suppose that t_μ is a non-concave insertion. The only symmetry with non-trivial fixed locus is the identity. Thus for a broad insertion, the group element that x^μ maps to is the identity. However, by Lemma 1.2 of [Kra10], this can only happen for N even. Therefore, all insertions for loops of odd rank are concave.

For N even, we claim that t_μ is broad if and only if $\mu_j = \delta^{\text{odd}}_j(r_j - 1)$ or $\mu_j = \delta^{\text{even}}_j(r_j - 1)$. Indeed, for an insertion t_μ to be broad, we must have that the associated phase Θ^μ contains identity elements in some components. However, when solving $\Theta^\mu_j \in \mathbb{Z}$ for some j^{th} component, the equations are identical modulo some permutation of the indices. Hence it is sufficient to consider the $j = N$ component which is given in Corollary 3.3.12. For N odd, we note that this component satisfies,

$$0 < \frac{1}{D}\Big((\mu_1 + 1) - (\mu_2 + 1)r_1 + \cdots + (\mu_N + 1)r_1 \cdots r_{N-1}\Big) \leq 1 \qquad (6.1.7)$$

since $0 \leq \mu_i \leq r_i - 1$ and where $D = \det E_{W_0}$. Hence, a non-concave invariant must satisfy

$$(\mu_1 + 1) - (\mu_2 + 1)r_1 + \cdots + (\mu_N + 1)r_1 \cdots r_{N-1} = D = r_1 \cdots r_N + 1. \qquad (6.1.8)$$

Since $\mu_i \leq r_i - 1$ and $r_i \geq 2$ we compare the coefficients of $r_1 \cdots r_N$ and 1 to find

$\mu_N = r_N - 1$ and $\mu_1 = 0$ respectively. However, since N is assumed odd, there is at least one equation that cannot be satisfied if $\mu_i \leq r_i - 1$. Hence for loops of odd rank, all invariants are concave, as already proven above. For N even, a similar analysis shows that there are two non-concave insertions when $\mu = (0, r_2 - 1, 0, r_4 - 1, \ldots, 0, r_N - 1)$ and $\mu = (r_1 - 1, 0, r_3 - 1, \ldots, r_{N-1} - 1, 0)$. □

Open Invariants.

Analogously to the closed case, we define narrow and broad insertions for boundary marked points.

Definition 6.1.8. Consider a term of the open Saito potential of the form $\prod_{i=1}^{l} t_{\mu_i} \prod_{i=1}^{N} x_i^{k_i}$ for an invertible Landau-Ginzburg model W. We say that the boundary insertion x_j is narrow if the corresponding phase Θ^{x_i} has no integer components. Otherwise, the insertion is called broad.

Observation 6.1.9. Via Definition 3.3.9, we see that Θ having no integer components is equivalent to the corresponding twist μ in equation (3.3.2) being a narrow insertion.

To illustrate the main ideas in calculating the FJRW phases of boundary marked points and whether they are narrow, we first discuss the open r-spin case.

Example 6.1.10. In [GKT22a], the primary open Saito potential for $W = x^r$ is given by fixing the versal deformation

$$W_s - W_0 = \sum_{j=0}^{r-2} s_j x^j \tag{6.1.9}$$

with an appropriate change of coordinates

$$s_j = t_j + O(t^2).$$

There are two types of boundary marked point to consider: the boundary x-points and the root. To calculate the phase of the root, we consider the term $s_0 \sim t_0$ in (6.1.9). At leading order, this corresponds to an r-spin disk with one root boundary point and two internal marked points. Thus, the selection rule (6.1.5) reads

$$\frac{1}{r} - 2\Theta^0 - \Theta^{\text{root}} =: m \in \mathbb{Z}$$

where Θ^0 is the phase of the two internal marked points. The value of Θ^0 is given in (4.1.1) with $\mu = 0$. The integer m is chosen such that $0 \leq \Theta^{\text{root}} < 1$. In this case, we choose $m = -1$ yielding

$$\Theta^{\text{root}} = 1 - \frac{1}{r}.$$

To calculate the phase Θ^x of the x-boundary point, we consider the term $s_1 x \sim t_1 x$ in the open Saito potential. This corresponds to a count over r-spin disks with one x-boundary point, two internal marked points of phase Θ^1, and the root point. The selection rule is

$$\frac{2}{r} - 2\Theta^1 - \Theta^x - \Theta^{\text{root}} \in \mathbb{Z}.$$

Proceeding similar to the above, we solve for Θ^x to find

$$\Theta^x = 1 - \frac{1}{r}.$$

Thus, we have that the twist for the x boundary point and root are

$$r\Theta^x - 1 = r - 2 = r\Theta^{\text{root}} - 1$$

in agreement with the expression for b_m in (6.1.3). Since the twist $r-2$ corresponds to the standard good basis element x^{r-2} which is a narrow insertion, the root and x boundary marked points are also narrow via observation 6.1.9. In other words, Θ^x and Θ^{root} are never integers for integer $r \geq 2$.

The above example is perhaps convoluted since we already stated in Chapter 2 that all insertions are in fact narrow for $W = x^r$. Nevertheless, we can now extend this argument to the case of a general Sebastiani-Thom sum of Fermat, chain and loop polynomials.

Proposition 6.1.11. Let W be a Sebastiani-Thom sum of Fermat, chain and loop polynomials. A boundary insertion of any W-spin disk is narrow.

Proof. Let W be any invertible polynomial with exponent matrix $E_W = (r_{ij})$. Let W^T be the BHK mirror polynomial with charges $q_1^T, \ldots, q_N^T$. Let Θ^{root}, $\Theta^{x_i} \in \mathbb{Q}^N$ be the phases of the root and x_i boundary point respectively. We first calculate the phases of root and x_i boundary points. Indeed, recall that the we denote the phases of internal marked points are given by Θ^{μ}.

The standard good basis of the local algebra for any W always contains the element $\mu = 0$. Hence, there is always the term t_0 in the open Saito potential since $s_0 \sim t_0$ to leading order. This corresponds to a W-spin disk, with two internal marked points of phase $\Theta^{\mu=0}$ and a root boundary point. Hence the phase of the root is given by the selection rule (6.1.5),

$$q_j^T - \Theta_j^{\text{root}} - 2\Theta^{\mu=0} \in \mathbb{Z}.$$

As in the previous example, $\Theta^{\mu=0}$ is given in (4.1.1) which implies

$$\Theta_j^{\text{root}} = 1 - q_j^T$$

since $\Theta_j^{\text{root}} \in [0, 1)$.

Similarly, the standard good basis contains the element $\mu = \hat{e}_{x_i}$ for any W where $\hat{e}_{x_i}$ is the vector whose only non-vanishing component is unity in the i^{th} place. Thus, the open Saito potential always contains the term $t_{\hat{e}_{x_i}} x^{\hat{e}_{x_i}}$. The two internal marked points corresponding to $t_{\hat{e}_{x_i}}$ each have phase $\Theta^{\mu=\hat{e}_{x_i}}$. The selection rules therefore read

$$2q_j^T - \Theta_j^{\text{root}} - \Theta_j^{x_i} - 2\Theta_j^{\mu=\hat{e}_{x_i}} =: -1 - e_j^{(i)} \in \mathbb{Z}$$

for each $j = 1, \ldots, N$. The integers $e_j^{(i)}$ are chosen so that $0 \leq \Theta_j^{x_i} < 1$. This implies

$$\Theta_j^{x_i} = e_j^{(i)} + q_j^T - 2(E_{W^T}^{-1}\hat{e}_{x_i})_j = e_j^{(i)} + q_j^T - 2(E_W^{-1})_{ij}.$$

To determine $e_j^{(i)}$, we use the balancing condition (6.1.6) for the term $t_\mu x^\mu$ in the versal deformation. This corresponds to a W-spin disk with total number of marked points $3 + \sum_{i=1}^N \mu_i$. The selection rule is

$$q_j^T(1 + \sum_{i=1}^N \mu_i) - \sum_{i=1}^N \mu_i \Theta_j^{x_i} - \Theta_j^{\text{root}} - 2\Theta^\mu \in \mathbb{Z}.$$

Writing $\mu = \sum_{i=1}^N \mu_i \hat{e}_{x_i}$ we find the left hand side becomes

$$-1 - \sum_{i=1}^N \mu_i e_j^{(i)} \tag{6.1.10}$$

which is an indeed an integer. The dimension condition (6.1.6) reads

$$\sum_{i=1}^N \sum_{j=1}^N \mu_i e_j^{(i)} = \sum_{i=1}^N \mu_i. \tag{6.1.11}$$

Since (6.1.11) must hold for each μ, we find $e_j^{(i)} = \delta_{ij}$. In summary, we have calculated

$$\Theta_j^{\text{root}} = 1 - q_j^T, \quad \Theta_j^{x_i} = \delta_{ij} + q_j^T - 2(E_W^{-1})_{ij}. \tag{6.1.12}$$

We now substitute both these phases into (3.3.9) to find that the j^{th} component of the twist of the root boundary point is

$$\mu_j = \sum_{i=1}^N r_{ij} - 2$$

and the j^{th} component of the twist of the x_i boundary point is

$$\mu_j^{x_i} = r_{ij} - 2\delta_{ij}.$$

Via lemmas (6.1.6) and (6.1.7), these correspond to narrow insertions. This completes the proof. □

Remark 6.1.12. We remark that equation (6.1.10) can be interpreted as degree of a certain spin bundle S_j; that the degree is also negative implies that $h^0(S_j) = 0$ and therefore, the balancing condition (6.1.6) is expected to hold in such a theory.

6.1.5 Grading of W-Spin Disks.

Now that we have calculated the twists of the root and x_i boundary points, we may impose a grading on the boundary points following [GKT22b].

Definition 6.1.13. Let W be an invertible polynomial with charges $q_1, \ldots, q_N$ and whose exponent matrix has diagonal elements $(r_j)_{j=1}^N$. For a boundary marking w on a W-spin disk, define

$$\mathrm{alt}_i(w) = \begin{cases} 1 & \text{if } \mu_i^w = r_i - 2 \text{ or } r_i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Denote $\mathrm{alt}(w) := (\mathrm{alt}_i(w))_{i=1}^N$. Then a graded, balanced W-spin disk is a genus zero W-spin disk $C_{0,k,l}$ such that the following two conditions hold.

1. For each $i = 1, \ldots, N$,

$$q_i(2l + k - 2) - 2\sum_{n=1}^{l} \Theta_i^{\gamma_n} - \sum_{m=1}^{k} \Theta_i^{\gamma_m} =: -1 - e_i \in \mathbb{Z}$$

   where $\Theta_i^{\gamma_n}, \Theta_i^{\gamma_m}$ are the i^{th} components of the phases of internal and boundary marked points respectively.

2. If all insertions are narrow,

$$e_i = -1 + \sum_{m=1}^{k} \mathrm{alt}_i(w_m) \mod 2. \tag{6.1.13}$$

3. We have

$$\sum_{i=1}^{N} e_i = 2l + k - 3.$$

We observe that for an x_i boundary point we have $\mathrm{alt}_j(x_i) = \delta_{ij}$, while for a root boundary point we have $\mathrm{alt}_j(\mathrm{root}) = 1$. Hence, equations (4.1.5) and (6.1.13) coincide.

6.1.6 Open Mirror Symmetry

We start this subsection with the open Landau-Ginzburg mirror conjecture.

Conjecture 6.1.14. Given any quasi-homogeneous invertible and non-degenerate polynomial W_0, there exists an open FJRW theory for W_0^T, such that the open Saito-Givental theory of W_0 computes the open FJRW theory of W_0^T.

Work of Buryak-Clader-Tessler [BCT22a; BCT22b] establishes an open FJRW theory for $W_0 = x^r$ whilst the same is done in [GKT22b] for $W_0 = x_1^{r_1} + x_2^{r_2}$. In both of these cases, Conjecture 6.1.14 has been proven by Gross-Kelly-Tessler. In this thesis we have not attempted to develop a open FJRW theory for more general superpotentials; instead we produced results on explicit computations of potentials wall-crossing from the open B-model. Any such open FJRW theory should match with our results.

The major difficulty with the open FJRW theory considered in [GKT22b] is the construction of an orientable moduli space, together with a universal curve, of open W-spin disks that one may integrate over. Indeed, these moduli spaces are typically real orbifolds with corners. An example is shown below.

Remark 6.1.15. These moduli spaces have an intersting connection to A_∞-structures: the original definition of A_∞-spaces given by Stasheff in [Sta63] involves associahedra. These associahedra correspond precisely to moduli spaces of disks with no internal marked points.

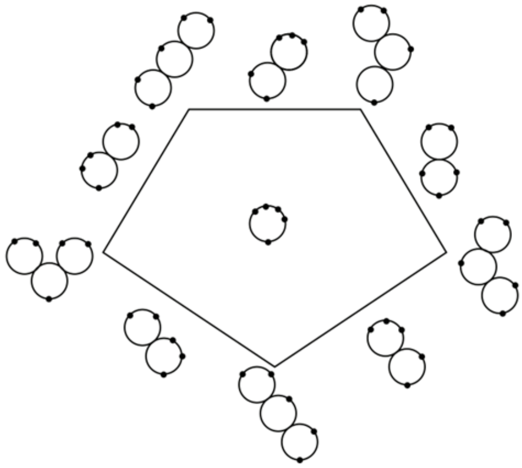

Figure 6.1: The moduli space of stable disks with 5 boundary marked points.

6.2 Equivalences in Landau-Ginzburg Theories

The main question we wish to tackle in this section concerns defining when two FJRW theories, or two Saito theories, are equivalent. We also discuss how one may practically determine equivalence. This turns out to be a slightly easier problem for open Saito theories than FJRW theories. Under the common theme in the thesis of using Saito theory to avoid open FJRW theory, we discuss equivalences in open Saito theories. For simplicity, we restrict attention to simple singularities, given typically by the following

$$\begin{aligned}
W_{A_r} &= x_1^{r+1}, \quad r \geq 1,\\
W_{D_r} &= x_1^{r-1} + x_1 x_2^2, \quad r \geq 4,\\
W_{E_6} &= x_1^3 + x_2^4,\\
W_{E_7} &= x_1^3 + x_1 x_2^3,\\
W_{E_8} &= x_1^3 + x_2^5,
\end{aligned}$$

and the simple elliptic singularities given typically as the Fermat polynomials

$$
\begin{aligned}
W_{E_6^{(1,1)}} &= x_1^3 + x_2^3 + x_3^3, \\
W_{E_7^{(1,1)}} &= x_1^4 + x_2^4, \\
W_{E_8^{(1,1)}} &= x_1^6 + x_2^3.
\end{aligned}
$$

6.2.1 Equivalences in FJRW Theory

This subsection is entirely taken from section 2 of [FHS23].

Definition 6.2.1. We say that two closed FJRW theories are *equivalent* if there is a Frobenius algebra isomorphism of the chiral rings that induces a coordinate transformation between ancestor FJRW generating functions.

Since the mirror to an FJRW theory with maximal admissible group is the semi-simple Saito theory of the mirror polynomial, via Givental's formula it suffices to consider the primary theories. We have the following theorem proven in [FHS23, Theorem 2.8].

Theorem 6.2.2. Let $\widetilde{W}$ be a quasi-homogeneous invertible polynomial in two variables with central charge strictly less than one. Then the closed FJRW theory of any admissible pair $(\widetilde{W}, G)$ is equivalent to either $(x^2 + y^2, \langle J \rangle)$ or $(W, G_W^{\max})$ where W is an ADE polynomial given as above.

It is hoped that one may prove similar results in the case of open FJRW theory, if one defines such a theory correctly. In light of the open Landau-Ginzburg mirror symmetry conjecture, equivalences in open Saito theory may be a tool for such results.

6.2.2 Equivalences in Primary Saito Theories

Definition 6.2.3. Let W and $\tilde{W}$ be two simple singularities. We say that two primary Saito theories of W and $\tilde{W}$ are *equivalent* if there is a Frobenius algebra isomorphism of the chiral rings that induces a coordinate transformation between Saito potentials.

Remark 6.2.4. We restrict to primary theories of ADE polynomials for simplicity; the Saito potentials are uniquely defined in this case and so there is no added technicality of wall-crossing.

Example 6.2.5. It is straightforward to see that the open Saito theory of a Morse stabilisation of a singularity W is equivalent to the original open Saito theory of W. Indeed, the local algebras are certainly isomorphic, via a direct calculation of partial derivatives. On the other hand, the standard good bases of W and a Morse stabilisation of W are also identical. Consequently, the flat coordinates and open Saito potentials are identical.

Right Equivalence

It is a classical problem to find the right notion of isomorphism of singularities in order to have a classification of ADE and simple elliptic singularities. For example, recall that the D_4 singularity may be obtained by either of the following two singularities

$$(W_{D_4})_{\text{Chain}} = x_1^3 + x_1 x_2^2, \quad (W_{D_4})_{\text{Fermat}} = x_1^3 + x_2^3 \tag{6.2.1}$$

Similarly, a full list of presentations for the elliptic singularities are given in Table 1 of [MS16].

This classification problem was solved Benson, Mather and Yau [MY82; BY90], which we briefly review.

Definition 6.2.6. Two singularities are called *right equivalent* if they are equal up to a biholomorphic change of variables. Furthermore, two singularities are called *$\mathscr{D}$-equivalent* if the corresponding local algebras are isomorphic.

Theorem 6.2.7. Two singularities are right equivalent if and only if they are $\mathscr{D}$-equivalent.

A proof of this theorem is the main result of [MY82]. See also Theorem 3.3 of [BY90].

Thus, to show that the local algebras are isomorphic, we need only find a transformation between the singularities themselves. Moreover, we have the following theorem due to Saito [Sai87, Theorem 2].

Theorem 6.2.8. If a non-degenerate, invertible, quasi-homogeneous polynomial W has a central charge strictly less than one. Then W is right equivalent to a simple singularity. If a non-degenerate, invertible, quasi-homogeneous polynomial W has a central charge equal to one, then W is equivalent to a simple elliptic singularity.

Orbit Spaces of Coxeter Groups

The second part of Definition 6.2.3 is to find a transformation of Saito potentials. This means finding an isomorphism of Frobenius manifold that induces a transformation on the flat coordinates. For this identification, we use the theory of reflection groups and invariant polynomials. We briefly review this here and we follow the exposition of [Her02].

Definition 6.2.9. Suppose V is a real dimensional vector space with an inner product $(\cdot,\cdot)$. For $u, \alpha \in V$, a *reflection* is a linear operator $r_\alpha : V \to V$ given by

$$s_\alpha(u) = u - 2\frac{(u,\alpha)}{(\alpha,\alpha)}\alpha.$$

Definition 6.2.10. A finite set $\mathcal{R}$ of non-zero vectors is called a *root system* if $r_\alpha \mathcal{R} \subset \mathcal{R}$ for every $\alpha \in \mathcal{R}$ and the only vectors colinear to $\alpha \in \mathcal{R}$ are $\pm\alpha$.

Definition 6.2.11. Given a root system $\mathcal{R}$, we say it is crystallographic if

$$2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$$

for all $\alpha, \beta \in \mathcal{R}$. The corresponding group $W := \langle r_\alpha : \alpha \in \mathcal{R}\rangle$ is then called the *Weyl group*.

It is well-known that W is a finite group. Indeed, letting $S_\mathcal{R}$ be the symmetric group on $\mathcal{R}$, we define the injective group homomorphism $\phi : W \to S_{\mathcal{R}}$ by sending $r \in W$ to the element of $S_{\mathcal{R}}$ that permutes the roots in the identical way to r.

Theorem 6.2.12. There exists a set $\Delta \subset \mathcal{R}$ such that the Weyl group is generated by the set $S = \{r_\alpha : \alpha \in \Delta\}$ with relations

$$(r_\alpha r_\beta)^{m_{\alpha,\beta}} = 1$$

for $\alpha, \beta \in \Delta$, $m_{\alpha,\beta} \in \mathbb{Z}_{\geq 0}$ and $m_{\alpha,\alpha} = 1$.

See [Cox34, Theorem 8] for a proof, or see [Her02, Section 5.3] for a summary. The pair (W, S) is called a Coxeter system and the collection $m_{\alpha,\beta}$ determine W up to isomorphism. The Coxeter diagram is constructed by assigning a vertex to each element $\alpha \in \Delta$ and an edge is drawn between $\alpha, \beta \in \Delta$ if $m_{\alpha,\beta} \geq 3$. The edge acquires a label if $m_{\alpha,\beta} \geq 4$, although for our purpose we content ourselves with the unlabelled case where $m_{\alpha,\beta} \leq 3$.

Definition 6.2.13. A Coxeter system (W, S) is said to be *irreducible* if the Coxeter diagram is connected.

An equivalent way of saying this is that the generating S cannot be partitioned into two commuting subsets. This means that the root system $\mathcal{R}$ cannot be decomposed as $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ such that $(\mathcal{R}_1, \mathcal{R}_2) = 0$. Irreducible Coxeter systems have been classified; in the ADE case they coincide with the simply laced Dynkin diagrams.

Let us now discuss the associated Frobenius structure. Denote $S(V^*)\otimes\mathbb{C} \cong \mathbb{C}[s_1, \ldots, s_n]$ as the complexified symmetric algebra of the dual space. This has a natural graded ring structure. We define an action of W on $S(V^*)$ by

$$(w \cdot f)(u) = f(w^{-1}u)$$

for $w \in W, f \in V^*$ and $u \in V$.

The following two theorems are then results of Dubrovin, Saito, Sekiguchi and Yano [Dub98; SSY80; Sai93].

Theorem 6.2.14. Let $\mathcal{M}_W$ denote the orbit space $(V \otimes \mathbb{C})/W$. Suppose $\mathcal{D}$ is the collection of irregular orbits. That is to say, orbits whose cardinality is different from $|W|$. Then $\mathcal{M}_W \setminus \mathcal{D}$ can be endowed with the structure of a Frobenius manifold.

Theorem 6.2.15. Consider the ring of invariants $S(V \otimes \mathbb{C})^W$ where W is a Weyl group W and V is a vector space. Then there exists a basis of homogeneous polynomials $\{t_1, \ldots, t_n\} \subset S(V)^W$ such that

$$S(V \otimes \mathbb{C})^W \cong \mathbb{C}[t_1, \ldots, t_n]$$

and the t_i are flat coordinates of the Frobenius manifold $\mathcal{M}_W \setminus \mathcal{D}$.

The existence of such homogeneous polynomials inducing the isomorphism above is sometimes called *Chevalley's Theorem.*

A result of Hertling [Her02] then says the following.

Theorem 6.2.16. Any Frobenius manifold that has a polynomial solution to the WDVV equations is isomorphic to a Frobenius manifold induced by the above construction from Coxeter groups.

Let us illustrate Hertling's theorem using the example of the A_r singularity.

Example 6.2.17. For $W = x^{r+1}$ we write the versal deformation as

$$W_{\boldsymbol{s}} = x^{r+1} + s_0 + s_1 x + \cdots + s_{r-1} x^{r-1} = \prod_{i=1}^{r+1} (x - a_i)$$

for some a_i such that $\sum_{i=1}^{r+1} a_i = 0$. In other words, the s_μ are elementary symmetric polynomials of degree $r+1-\mu$ in the a_i variables. We observe that the space of parameters a_i,

$$\mathfrak{h} := \Big\{ (a_1, \ldots, a_{r+1}) \in \mathbb{C}^{r+1} \Big| \sum_{i=1}^{r+1} a_i = 0 \Big\}$$

is a Cartan subalgebra of type A_r. Moreover, Chevalley's theorem gives an isomorphism of complex manifolds

$$\mathbb{C}^r \cong \mathfrak{h}/S_{r+1}.$$

Here, S_{r+1} is the symmetric group but can also be considered as the Weyl group of $\mathrm{SL}(r+1;\mathbb{C})$. In fact, the Frobenius structure arising from deformation theory are isomorphic to those arising from invariant polynomials. Indeed, let $\mathcal{D} \subset \mathbb{C}^r$ be the *discriminant locus*; that is to say, where $W_{\boldsymbol{s}} = 0$ has multiple roots. Then for $\boldsymbol{s} \in \mathbb{C}^r \setminus \mathcal{D}$, the lattice $H_1(\mathbb{C}, W_{\boldsymbol{s}}^{-1}(0);\mathbb{Z})$ is isomorphic to the root lattice with type A_r. By taking the covering space $\widetilde{\mathbb{C}^r \setminus \mathcal{D}}$ and a primitive form dx, the period mapping is given by

$$\begin{aligned} \phi : \widetilde{\mathbb{C}^r \setminus \mathcal{D}} &\to \mathfrak{h} = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}), \\ \boldsymbol{s} &\mapsto \left(\gamma_{ij} \mapsto \int_{a_j(\boldsymbol{s})}^{a_i(\boldsymbol{s})} dx\right) \end{aligned}$$

where $\gamma_{ij} \in H_1(\mathbb{C}, W_{\boldsymbol{s}}^{-1}(0);\mathbb{Z})$ is a path from a_j to a_i.

Elliptic Root Systems

One may generalise the above discussion to *elliptic root systems.* These were first introduced by Saito [Sai85; Sai90; ST97; SY00] to describe the versal deformation of simple elliptic singularities. Work of Bertola [Ber00a; Ber00b] and Satake [Sat10] has shown that, in certain 'codimension one cases', the quotient space can again be given the structure of a Frobenius manifold. Here, 'codimension one' means that there is a unique flat coordinate of highest degree. For our purpose, this is the marginal coordinate for the singularities $E_6^{(1,1)}, E_7^{(1,1)}$ and $E_8^{(1,1)}$. In such cases, the ring of invariants is no longer a polynomial ring, but a ring of modular functions instead. It is currently an open problem to classify these modular solutions to the WDVV equations, in analogy to Hertling's theorem for Coxeter groups.

Remark 6.2.18. There are other generalisations in addition to elliptic Weyl groups. For example, one may consider extended affine Weyl groups where solutions to the WDVV equations are trigonometric. See [DZ98; BG22] for example.

Equivalences in Open Saito Theories

Let us comment on the two open Saito potentials we have calculated for D_4.

Example 6.2.19. Recall that we have calculated the open Saito potentials for the presentations of the D_4 singularity given in equation (6.2.1). Firstly, the singularities $(W_{D_4})_{\text{Chain}} = 0$ and $(W_{D_4})_{\text{Fermat}} = 0$ are a union of 3 lines intersecting at the origin. Thus, there must be a linear transformation between them, proving that they are right equivalent. This implies that their local algebras are isomorphic using Theorem 6.2.7. To show that there is an isomorphism of Frobenius manifolds, we know that the flat coordinates of both are polynomials. Hertling's theorem then implies the Frobenius manifolds of these du Val singularities arise as Frobenius manifolds from the Coxeter group D_4. Hence, there must be a coordinate transformation between the flat coordinates. Since we have shown in Chapter 5 that the primary wall crossing group is trivial for the primary theory of simple singularities, this is enough to imply equivalence of the primary open Saito theory.

It is hoped that the above example may be straightforwardly generalised to the open Saito theory of any simple singularity. In light of the open Landau-Ginzburg mirror conjecture, one may hope to prove a result analogous to Theorem 6.2.2, once open FJRW has been fully defined.

One may try, furthermore, to extend the above reasoning to elliptic singularities.

Example 6.2.20. Consider the example

$$(W_{E_7^{(1,1)}})_{\text{Chain}} = x_1^4 + x_1 x_2^3, \quad (W_{E_7^{(1,1)}})_{\text{Fermat}} = x_1^4 + x_2^4$$

Similar to before, the vanishing of these polynomials both define a union of 4 lines intersecting at the origin, thus proving $\mathscr{D}$-equivalence. It is known that they both correspond to the Dynkin diagram of $E_7^{(1,1)}$ and from Section 4.2.2 we know that both admit open primary Saito potentials which are modular.

Generalising the above example and combining with the open Landau-Ginzburg mirror conjecture, we obtain the following conjecture. This is a generalisation of Proposition 4.2.8 to all simple elliptic singularities of either Fermat, chain or loop type.

Conjecture 6.2.21. Consider the primary open Saito potentials W for a simple elliptic singularity W_0. Suppose all non-marginal flat coordinates are set to zero. Then there exists a primary open Saito potential such that $W(t)$ is a meromorphic modular function of weight zero. Furthermore, there is a transformation involving all flat coordinates that leaves the primary open Saito potential invariant. Combining this with the open Landau-Ginzburg mirror symmetry conjecture, the corresponding mirror open FJRW potential should exhibit the same modularity properties.

Although one could prove the modularity parts of conjecture via exhaustive calculation of open Saito potentials, a slicker proof may result from understanding the equivalences between Saito theories of simple elliptic singularities.

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APPENDIX A

THE HYPERGEOMETRIC EQUATION FOR THE CUBIC ELLIPTIC CURVE

A.1 Triangle Functions

In this appendix we follow [Neh75, Section 5, Chapter 6]. Consider a conformal map from the upper half complex plane to a curvilinear triangle. By the Riemann mapping theorem, such a mapping exists. The following result is then well known.

Theorem A.1.1. Suppose that $z = f(w)$ is a map from $\{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ to the interior of curvilinear triangle $\triangle$ with angles $\pi\theta, \pi\phi, \pi\psi$ corresponding to the vertices $f(0), f(\infty)$ and $f(1)$. Then f is of the form

$$f(w) = \frac{y_1(w)}{y_2(w)}$$

where y_1 and y_2 are linearly independent solutions to the following hypergeometric equation:

$$w(1-w)\frac{d^2y}{dw^2} + [\gamma - (\alpha+\beta+1)w]\frac{dy}{dw} - \alpha\beta y = 0$$

with

$$\alpha = \frac{1}{2}(1+\phi-\theta-\psi), \quad \beta = \frac{1}{2}(1-\theta-\phi-\psi), \quad \gamma = 1-\theta.$$

For details of this result, we refer the reader to [Neh75, Section 7, Chapter 5]. However, what we are most interested in is the determination of existence of an inverse function, which we denote by $w = S(z)$. We call such a function a *Schwarz triangle function.*

To fix the position of the curvilinear triangle in the z plane, we fix the vertex at the angle $\pi\theta$ so that it is at the origin and the two edges of the triangle are linear. We claim that this is always possible if $\theta \neq 0$ using a Möbius transformation. Indeed, suppose C_1 and C_2 are the two circles of the triangle $\triangle$ that meet at $z = A$ for the angle $\pi\theta$. Since $\theta \neq 0$, there exists another intersection point, $z = B$ say, of C_1 and C_2. The desired Möbius transformation is then

$$z \mapsto \frac{z - A}{z - B}.$$

This transformation maps C_1 and C_2 to linear segments through the origin, while the third edge forms an arc of a circle. We denote this circle by Γ.

Definition A.1.2. An *orthogonal circle* C to the curvilinear triangle $\triangle$ is a circle that perpendicularly intersects Γ (that is, the tangent lines of C and Γ at their points of intersection are perpendicular).

Lemma A.1.3. Orthogonal circles exist for $\theta + \phi + \psi < 1$.

Proof. We use the above transformation to bring the vertex with angle $\pi\theta$ to be positioned at the origin. We use an additional rotation so that the vertex with angle $\pi\psi$ lies on the positive real axis. Then any circle centred at the origin is orthogonal to the linear sides. Let Γ be the circle for the third side. If $\theta + \phi + \psi < 1$, then the origin is outside Γ. Thus, we draw a tangent line from the origin to a point P on Γ. The circle C centred at the origin passing through P is then the desired orthogonal circle. □

The following result is not new and can be found in [Neh75, Section 5, Chapter 6], but we reproduce it here for completeness.

Theorem A.1.4. The Schwarz triangle function $w = S(z)$ for the triangle $\triangle$ with angles $\pi\theta, \pi\phi$ and $\pi\psi$ with $\theta + \phi + \psi < 1$ is single valued in the interior of the orthogonal circle

C if and only if

$$\theta = \frac{1}{m}, \quad \phi = \frac{1}{n}, \quad \psi = \frac{1}{p} \tag{A.1.1}$$

where $m, n, p \in \mathbb{Z}_{>0}$. Furthermore, it is not possible to continue S beyond the boundary of C.

Proof. For the first direction, suppose that the angles satisfy equation (A.1.1). Since the map w maps two lines with angle $\pi\theta$ onto the real axis, $w = S(z)$ must have the form

$$S(z) = z^{\frac{1}{\theta}} S_1(z)$$

where S_1 is single-valued around $z = 0$. Since $\theta = \frac{1}{m}$, we must have that

$$S(z) = z^m S_1(z)$$

is also single-valued around $z = 0$. By transforming the angles $\pi\phi$ or $\pi\psi$ to the origin, the same reasoning applies to $\phi = \frac{1}{n}$ and $\psi = \frac{1}{p}$.

Conversely, suppose that $w = S(z)$ is single valued in the orthogonal circle C of $\triangle$. We claim that $S(z)$ in its domain of definition has no singularities other than poles of order n. Indeed, all analytic continuations of the original domain $\triangle$ of $S(z)$ can be obtained by iterative reflections through the sides of $\triangle$ in the z-plane. Such a transformation maps circles to circles, hence preserving the magnitude of the angles while reversing orientation. The corresponding reflections in the upper half w-plane are reflections through the real axis. Thus, the boundaries of the triangles in the z plane are mapped to segments of the real axis. From the reasoning above, $S(z)$ is single valued near the vertices of the reflected triangles if θ, ϕ, ψ satisfy condition (A.1.1). Moreover, such vertices are the only possible singularities of S and, again due to the form of S, can only be singular at the vertices corresponding to the angle $\pi\phi$. This proves the claim.

To prove the theorem, we will show that the domain of existence coincides with the orthogonal circle C. In particular, this domain is simply connected and the result in the theorem will follow from the monodromy theorem. To show that the domain coincides with C, we first observe that any inverted triangle lies within C. Indeed, any reflected triangle is still orthogonal to the image of C under this reflection, since the reflection preserves angles. However, this image is precisely C. Thus, all inverted triangles lie within the same orthogonal circle as the original domain $\triangle$.

Conversely, for the reverse inclusion, we show that any point in C can be obtained by reflections. Indeed, suppose that all reflections have been performed to reach a point in C. At the domain boundary, there are no circles whose radii are positive as one could then carry out another reflection; this would be in contradiction to maximality of the domain. All such circular arcs of radius tending to zero must be orthogonal to C. This implies that they also intersect C. Due to the vanishing of the radii, the points of the circular arcs must therefore also be contained in C. This shows that the domain coincides with C and completes the proof. $\square$

Corollary A.1.5. If the conditions of the above theorem are satisfied, then $S(z)$ is an automorphic function in the domain $|z| < 1$.

Proof. From the proof of the above theorem, we know that an even number of reflections preserves both magnitude and orientation of the angles. This induces a conformal map. Since reflections preserve circles, this reflection must be equivalent to a Möbius transformation $\gamma(z)$ in the z-plane where γ is the map

$$\gamma \cdot z = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{C}$. However, in the w-plane, an the corresponding even number of reflections in the real axis returns the original point. Thus, we find that

$$S\Big(\gamma \cdot z\Big) = S(z).$$

We now describe the automorphy group which we call the triangle group. It is well known that the set of such γ form a group which can be generated by three elements corresponding to an even number of reflections across the sides of the original triangle $\triangle$. Moreover, reflections, and thus actions of the triangle group, leave the orthogonal circle invariant. However, normalising S with the requirement that C mist have radius 1, we know that any such action of the triangle group must be of the form

$$z \mapsto \delta\left(\frac{a-z}{1-\bar{a}z}\right) \tag{A.1.2}$$

for $|\delta| = 1$ and $|a| < 1$. Thus, $S(z)$ must be a single valued automorphic form with respect to the triangle group action given by (A.1.2). □

A.2 Solving Hypergeometric Equations Near Infinity

We note that the normalisation of S in the above proof is not the one we use for the hypergeometric equation. Indeed, the solution we use is

$$y_1(w) = w^{1-\gamma}{}_2F_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; w), \quad y_2(w) = {}_2F_1(\alpha, \beta, \gamma; w), \quad f(w) = \frac{y_1(w)}{y_2(w)}.$$

This is a solution around $w = 0$.

For the case of $W_{E_6^{(1,1)}} = x_1^3 + x_2^3 + x_3^3 + s x_1 x_2 x_3$, the marginal flat coordinate t is

$$t = -3\frac{w^{1/3} \cdot {}_2F_1(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; w)}{{}_2F_1(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; w)}.$$

where $w = -\frac{1}{27}s^3$. That is to say, t is proportional to the a ratio of linearly independent solutions around to the hypergeometric equation,

$$w(1-w)\frac{d^2y}{dw^2} + \left(\frac{2}{3} - \frac{5}{3}w\right)\frac{dy}{dw} - \frac{1}{9}y = 0. \tag{A.2.1}$$

The resulting open Saito potentials are expansions around $t = 0$.

However, we may also solve the above equation as w or equivalently t become large. This will allow us to deduce q expansions of Saito potentials in the next subsection.

To study the solution to equation (A.2.1) as $w \to \infty$, we use the substitution $w = x^{-1}$. Thus we obtain the differential equation

$$(x^3 - x^2)\frac{d^2y}{dx^2} + \left(\frac{4}{3}x^2 - \frac{1}{3}x\right)\frac{dy}{dx} - \frac{1}{9}y = 0. \tag{A.2.2}$$

It can be checked that the point $x = 0$ is a regular singular point and thus we use the method of Frobenius to solve (A.2.2).

We use the ansatz

$$y = \sum_{r=0}^{\infty} a_r x^{r+c} \tag{A.2.3}$$

with $a_0 \neq 0$. Substituting (A.2.3) into (A.2.2) we find

$$(x^3 - x^2)\sum_{r=0}^{\infty} a_r(r+c)(r+c-1)x^{r+c-2} + \left(\frac{4}{3}x^2 - \frac{1}{3}x\right)\sum_{r=0}^{\infty} a_r(r+c)x^{r+c-1} - \frac{1}{9}\sum_{r=0}^{\infty} a_r x^{r+c} = 0.$$

Expanding the terms and changing indices in the sums, the above equation becomes

$$\sum_{r=1}^{\infty} a_{r-1}(r+c-1)(r+c-2)x^{r+c} - \sum_{r=0}^{\infty} a_r(r+c)(r+c-1)x^{r+c} + \frac{4}{3}\sum_{r=1}^{\infty} a_{r-1}(r+c-1)x^{r+c}$$
$$- \frac{1}{3}\sum_{r=0}^{\infty} a_r(r+c)x^{r+c} - \frac{1}{9}\sum_{r=0}^{\infty} a_r x^{r+c} = 0.$$

Employing linear independence, we equate coefficients of x^k to zero for all k. For x^c, we find

$$a_0\Big(-c(c-1) - \frac{c}{3} - \frac{1}{9}\Big) = 0.$$

Since $a_0 \neq 0$, we find c as the double root $c = \frac{1}{3}$. From the other terms, we ultimately obtain the recursion relation

$$a_r = \frac{\left(r+c-1\right)\left(r+c-\frac{2}{3}\right)}{\left(r+c-\frac{1}{3}\right)} a_{r-1} = \frac{(c)_r\left(c+\frac{1}{3}\right)_r}{\left(c+\frac{2}{3}\right)_r \cdot \left(c+\frac{2}{3}\right)_r} a_0.$$

Thus, our general solution has the form

$$y = a_0 \sum_{r=0}^{\infty} \frac{(c)_r\left(c+\frac{1}{3}\right)_r}{\left(c+\frac{2}{3}\right)_r \cdot \left(c+\frac{2}{3}\right)_r} x^{r+c}.$$

It is only the ratio of y_1 and y_2 that we are interested in. Thus, without loss of generality, we set $a_0 = 1$. Our two linearly independent solutions are given by

$$y_1 = y|_{c=1/3}, \quad y_2 = \left.\frac{\partial y}{\partial c}\right|_{c=1/3}.$$

With $x = w^{-1}$, we observe that

$$y_1 = x^{1/3}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; x\right) = w^{-1/3}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; w^{-1}\right).$$

To calculate the derivative in y_2, we have the following lemma.

Lemma A.2.1. Define

$$M_r = \frac{(c)_r\left(c+\frac{1}{3}\right)_r}{\left(c+\frac{2}{3}\right)_r \cdot \left(c+\frac{2}{3}\right)_r}.$$

Then

$$\frac{\partial M_r}{\partial c} = M_r \sum_{k=0}^{r-1} \left(\frac{1}{c+k} + \frac{1}{c+\frac{1}{3}+k} - \frac{2}{c+\frac{2}{3}+k}\right).$$

Proof. Observe that

$$\log M_r = \sum_{k=0}^{r-1} \log(c+k) + \log\left(c+\frac{1}{3}+k\right) - 2\log\left(c+\frac{2}{3}+k\right).$$

The result now follows from differentiating with respect to c. □

Thus, we find that

$$\frac{\partial}{\partial c}\Big(x^c\sum_{r=0} M_r x^r\Big) = x^c\Big(\log x\sum_{r=0}^{\infty} M_r x^r + \sum_{r=0}^{\infty} M_r\sum_{k=0}^{r-1}\Big(\frac{1}{c+k}+\frac{1}{c+\frac{1}{3}+k}-\frac{2}{c+\frac{2}{3}+k}\Big)x^r\Big).$$

Evaluating at $c=\frac{1}{3}$ and noting that $x=w^{-1}$ we find

$$y_2 = w^{-1/3}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{3}\right)_r\left(\frac{2}{3}\right)_r}{(1)_r\cdot(1)_r}\Big(\log w^{-1}+\sum_{k=0}^{r-1}\Big(\frac{1}{\frac{1}{3}+k}+\frac{1}{k+\frac{2}{3}+k}-\frac{2}{k+1}\Big)\Big)w^{-r}.$$

Thus we find that two linearly independent solutions are

$$\begin{aligned}
y_1 &= w^{-1/3}{}_2F_1\Big(\frac{1}{3},\frac{2}{3},1;w^{-1}\Big)\\
y_2 &= w^{-1/3}\Big[F\Big(\frac{1}{3},\frac{2}{3},1;w^{-1}\Big)-\log w\cdot{}_2F_1\Big(\frac{1}{3},\frac{2}{3},1;w^{-1}\Big)\Big]
\end{aligned}$$

where

$$F\Big(\frac{1}{3},\frac{2}{3},1;w^{-1}\Big)=\sum_{r=0}^{\infty}\frac{\left(\frac{1}{3}\right)_r\left(\frac{2}{3}\right)_r}{(1)_r\cdot(1)_r}\sum_{k=0}^{r-1}\Big(\frac{1}{\frac{1}{3}+k}+\frac{1}{k+\frac{2}{3}+k}-\frac{2}{k+1}\Big)w^{-r}.$$

A.3 Series Reversion and Modular Expansions

As an aside, we review the series reversion technique to obtain the open Saito potential for $E_6^{(1,1)}$ in section 4.1.2.

For a power series with no constant term

$$t = a_1 s + a_2 s^2 + \cdots \tag{A.3.1}$$

we wish to find an expansion of an inverse

$$s = A_1 t + A_2 t^2 + \cdots\ . \tag{A.3.2}$$

This can be done simply by substituting (A.3.2) into (A.3.1) so that

$$t = a_1(A_1 t + A_2 t^2 + \cdots) + a_2(A_1 t + A_2 t^2 + \cdots)^2 + \cdots .$$

This allows us to determine A_i recursively.

We now use this series reversion technique to find a Fourier series for the Schwarz triangle functions. Recall that we have two linearly independent solutions to the hypergeometric equation at infinity, we may calculate the Fourier series of the triangle function

$$w = S(z) = \sum_{n=-\infty}^{\infty} c_n q^n \tag{A.3.3}$$

where c_n are to be determined and $q = e^{2\pi i z}$.

To do this, we follow the algorithm proposed by Lehner [Leh54]. Indeed, recall

$$\begin{aligned} y_1 &= w^{-1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; w^{-1}\right) \\ y_2 &= w^{-1/3}\left[F\left(\frac{1}{3}, \frac{2}{3}, 1; w^{-1}\right) - \log w \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; w^{-1}\right)\right]. \end{aligned}$$

We take

$$-2\pi i z = -\frac{y_2}{y_1} = \log w - \frac{F\left(\frac{1}{3}, \frac{2}{3}, 1; w^{-1}\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; w^{-1}\right)}.$$

One may calculate a series expansion of the above expression so that

$$-2\pi i z = \log w + \frac{A_1}{w} + \frac{A_2}{w^2} + \cdots$$

where $A_k \in \mathbb{Q}$. Exponentiating gives another series expansion

$$q^{-1} = e^{-2\pi i z} = w \cdot \exp\left(\frac{A_1}{w} + \frac{A_2}{w^2} + \cdots\right) = w \cdot \left(1 + \frac{B_1}{w} + \frac{B_2}{w^2} + \cdots\right) \tag{A.3.4}$$

where $B_k \in \mathbb{Q}$.

Thus, in equation (A.3.3) we must have $c_n = 0$ for $n < -1$ and $c_{-1} = 1$. To find the higher order terms in

$$w = q^{-1}(1 + c_0 q + c_1 q^2 + \cdots) \tag{A.3.5}$$

we set

$$w^{-1} = q(1 + d_0 q + d_1 q^2 + \cdots) \tag{A.3.6}$$

where the d_k are calculated in terms of the c_k via the binomial theorem. This can also be done for w^{-n}. Then substituting A.3.5 and A.3.6 into A.3.4 we find

$$q^{-1} = q^{-1}(1+c_0 q+c_1 q^2+\cdots)\Big(1+B_1 q(1+d_0 q+d_1 q^2+\cdots)+B_2 q^2(1+d'_0 q+d'_1 q^2+\cdots)+\cdots\Big).$$

The above equation then allows us to determine c_k recursively.

APPENDIX B

A REMARK ON OPEN WDVV EQUATIONS FOR SIMPLE SINGULARITIES

As an aside, we return to Remark 4.2.5 where there is disagreement on the open Saito potentials for $W_{D_4} = x_1^3 + x_1x_2^3$ in the Gross-Kelly-Tessler construction and in the paper of Basalaev-Buryak. Worse still, it may seem that our results for $W_{D_4} = x_1^3 + x_1x_2^3$ contradicts Theorem 6.1 of [BB21]. Let us briefly comment on the discrepancy. In that paper, the authors take a different approach where they consider solutions to open WDVV equations with an extra parameter which we call σ here. Their approach comes with a significant advantage: much like solutions of the closed WDVV equations inducing a Frobenius manifold, solutions of open WDVV equations give a flat F manifold structure. However, the potential that is found for $W_{D_4} = x_1^3 + x_1x_2^2$ contains a pole along $\sigma = 0$. Whilst an analogue of open WDVV equations are unknown in approach of [GKT22b], the advantage of the Gross-Kelly-Tessler method is that the generating potential is a homogeneous polynomial and there is a proven corresponding mirror theorem in some cases.

To illustrate why the results of [BB21] do not contradict our expressions for the open Saito potential for $W_{E_7} = x_1^3+x_1x_2^2$, we give a brief survey of the Basalaev-Buryak method and consider homogeneous, polynomial solutions to the open WDVV equations for E_7. In doing so, we give a small correction to the proof of Theorem 6.1 of Basalaev-Buryak by using the correct good basis for $W_{E_7} = x_1^3 + x_1x_2^3$.

Throughout this section we employ Einstein summation convention.

Let W be a simple singularity. Let $\{t^\mu\}_{\mu\in B}$ be a system of flat coordinates for the Saito-Frobenius manifold. Recall that, locally, this Frobenius manifold is described by a solution of the WDVV equations. Let $F(t^0,\ldots,t^n)$ be a solution of this system of PDE's. There is another system of PDE's for a function $F^o(t^0,\ldots,t^n,\sigma)$ with σ an additional parameter that describes the corresponding open geometry. These equations are given as

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}\frac{\partial^2 F^o}{\partial t^\nu \partial\sigma}+\frac{\partial^2 F^o}{\partial t^\alpha\partial t^\beta}\frac{\partial^2 F^o}{\partial\sigma^2}=\frac{\partial^2 F^o}{\partial t^\alpha\partial\sigma}\frac{\partial^2 F^o}{\partial t^\beta\partial\sigma}. \tag{B.0.1}$$

In open Gromov-Witten theory and in open r-spin theory in [PST22; BCT22a], solutions of the open WDVV equations are considered subject to the condition

$$\frac{\partial^2 F^o}{\partial t^0\partial\sigma}=1. \tag{B.0.2}$$

We will impose the same condition here. Recall that the coordinate t^0 plays a special role, since the vector field $e=\frac{\partial}{\partial t^0}$ is the unit of the corresponding Frobenius algebra. Recall for a general simple singularity W, the Euler vector field E is given by

$$E=\sum_{\mu\in B}(\deg s_\mu)s_\mu\frac{\partial}{\partial s_\mu}.$$

The Euler field E may also be written in terms of the flat coordinates

$$E=\sum_{\mu=1}(\deg t_\mu)t_\mu\frac{\partial}{\partial t_\mu}.$$

This may be extended to the open case by declaring that the degree of σ is $\frac{1-\delta}{2}$. Here, we recall that δ is the conformal dimension of the Frobenius manifold. It is defined through the conformal Killing equation,

$$\mathcal{L}_E\eta=(2-\delta)\eta.$$

Equivalently, δ is given by

$$\delta = 1 - \frac{2}{h}$$

where h is the Coxeter number of the corresponding Lie algebra for a simple singularity.

Definition B.0.1. A polynomial $F^o(t, \sigma)$ is called *homogeneous* if there exists $\deg F^o \in \mathbb{Q}$ such that

$$\mathcal{L}_E F^o + \frac{1-\delta}{2}\sigma \frac{\partial F^o}{\partial \sigma} = (\deg F^o) F^o. \tag{B.0.3}$$

In the works of [PST22; BCT22a], homogeneous solutions of the WDVV equations were considered where $\deg F^o$ is necessarily given by $\frac{3-\delta}{2}$. Henceforth, we shall only consider such solutions.

The main result of this section is the following proposition, due to Basalaev-Buryak.

Proposition B.0.2. Consider the Frobenius manifold associated to the E_7 singularity $W = x_1^3 + x_1 x_2^3$. Let F be the corresponding Frobenius potential. Then there are no homogeneous polynomial solutions F^o to the open WDVV equations (B.0.1) subject to (B.0.2).

Before we can prove this result, we need a technical lemma.

Lemma B.0.3. We have

$$\left.\frac{\partial F^o}{\partial s_\mu}\right|_{s,\sigma=0} = 0, \quad \left.\frac{\partial^2 F^o}{\partial \sigma^2}\right|_{s,\sigma=0} = 0.$$

Proof. For E_7 we find $\delta = 8/9$. If $\left.\frac{\partial F^o}{\partial s_\mu}\right|_{s,\sigma=0}$ did not vanish, this would imply that F^o contains a linear term in v. However, given that $\deg s_\mu \neq \frac{3-\delta}{2}$ for all μ, this linear term does not obey the homogeneity condition (B.0.3). Hence we must have $\left.\frac{\partial F^o}{\partial s_\mu}\right|_{s,\sigma=0} = 0$. Similarly, if we assume that $\left.\frac{\partial^2 F^o}{\partial \sigma^2}\right|_{s,\sigma=0}$ did not vanish then F^o would contain a σ^2 term. However, we have that $\frac{3-\delta}{2} > 2 \cdot \frac{1-\delta}{2}$ and so F^o would not be homogeneous. ☐

Proof of Proposition B.0.2. Recall that the structure constants (c_t) of the Frobenius algebra are defined by

$$\frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta} = (c_t)^\mu_{\alpha\beta} \frac{\partial}{\partial t^\mu}.$$

They are related to the Frobenius potential via

$$(c_t)^\nu_{\alpha\beta} = (c_t)_{\alpha\beta\mu}\eta^{\mu\nu} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu}\eta^{\mu\nu}.$$

Thus we rewrite the open WDVV equations (B.0.1) as

$$(c_t)^\nu_{\alpha\beta}\frac{\partial^2 F^o}{\partial t^\nu \partial \sigma} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta}\frac{\partial^2 F^o}{\partial \sigma^2} = \frac{\partial^2 F^o}{\partial t^\alpha \partial \sigma}\frac{\partial^2 F^o}{\partial t^\beta \partial \sigma}. \tag{B.0.4}$$

We wish now to change variables from the flat coordinates $\boldsymbol{t}$ to the versal deformation parameters $\boldsymbol{s}$. Recall that the structure constants $(c_t)^\nu_{\alpha\beta}$ obey the following tensor transformation law

$$(c_t)^\nu_{\alpha\beta} = (c_s)^{\tilde{\nu}}_{\tilde{\alpha}\tilde{\beta}}\frac{\partial t^\nu}{\partial s^{\tilde{\nu}}}\frac{\partial s^{\tilde{\alpha}}}{\partial t^\alpha}\frac{\partial s^{\tilde{\beta}}}{\partial t^\beta}.$$

It is also useful to recall that

$$\frac{\partial t^\nu}{\partial s^{\tilde{\nu}}}\frac{\partial s^\mu}{\partial t^\nu} = \delta^\mu_{\tilde{\nu}}.$$

After some manipulation, one may rewrite (B.0.4) as

$$(c_s)^\mu_{\alpha\beta}\frac{\partial^2 F^o}{\partial s^\mu \partial \sigma} + \frac{\partial^2 F^o}{\partial s^\alpha \partial s^\beta}\frac{\partial^2 F^o}{\partial \sigma^2} + \frac{\partial t^{\tilde{\alpha}}}{\partial s^\alpha}\frac{\partial t^{\tilde{\beta}}}{\partial s^\beta}\frac{\partial^2 s_\mu}{\partial t^{\tilde{\alpha}} \partial t^{\tilde{\beta}}}\frac{\partial F^o}{\partial s_\mu}\frac{\partial F^o}{\partial \sigma^2} = \frac{\partial^2 F^o}{\partial s^\alpha \partial \sigma}\frac{\partial^2 F^o}{\partial s^\beta \partial \sigma}.$$

We now differentiate this equation with respect to s_γ and set $s, \sigma = 0$.

From the previous lemma, we calculate that

$$\frac{\partial}{\partial s_\gamma}\left(\frac{\partial t^{\tilde{\alpha}}}{\partial s^\alpha}\frac{\partial t^{\tilde{\beta}}}{\partial s^\beta}\frac{\partial^2 s_\mu}{\partial t^{\tilde{\alpha}} \partial t^{\tilde{\beta}}}\frac{\partial F^o}{\partial s_\mu}\frac{\partial F^o}{\partial \sigma^2}\right)\Big|_{s,\sigma=0} = 0.$$

Thus, we obtain the following equation

$$\frac{\partial}{\partial s_\gamma}\left((c_s)^\mu_{\alpha\beta}\frac{\partial^2 F^o}{\partial s^\mu \partial\sigma}\right)\Big|_{s,\sigma=0} + \frac{\partial}{\partial s_\gamma}\left(\frac{\partial^2 F^o}{\partial s^\alpha \partial s^\beta}\frac{\partial^2 F^o}{\partial \sigma^2}\right)\Big|_{s,\sigma=0} = \frac{\partial}{\partial s_\gamma}\left(\frac{\partial^2 F^o}{\partial s^\alpha \partial\sigma}\frac{\partial^2 F^o}{\partial s^\beta \partial\sigma}\right)\Big|_{s,\sigma=0}. \tag{B.0.5}$$

To prove the proposition, it is now sufficient to find indices α, β, γ such that

$$(c_s)^\mu_{\alpha\beta}|_{s=0} = 0, \quad \frac{\partial (c_s)^\mu_{\alpha\beta}}{\partial s_\gamma} = A\delta^{\mu,0},\ A \in \mathbb{C}^* \tag{B.0.6}$$

and

$$\frac{\partial^2 F^0}{\partial s_\alpha \partial s_\beta} = 0, \quad \frac{\partial}{\partial s_\gamma}\left(\frac{\partial^2 F^o}{\partial s^\alpha \partial\sigma}\frac{\partial^2 F^o}{\partial s^\beta \partial\sigma}\right) = 0 \tag{B.0.7}$$

hold. This is because equation (B.0.5) would then imply

$$\frac{\partial^2 F^o}{\partial s_0 \partial\sigma} = 0.$$

After changing variables back to the flat coordinates, this in turn implies

$$\frac{\partial^2 F^o}{\partial t_0 \partial\sigma} = 0.$$

This contradicts the constraint (B.0.2).

We claim that $\alpha = (1,0), \beta = (0,2)$ and $\gamma = (0,1)$ suffices. Indeed, the structure constants (c_s) are defined through the Frobenius algebra multiplication,

$$\frac{\partial}{\partial s^\alpha} \circ \frac{\partial}{\partial s^\beta} = (c_s)^\mu_{\alpha\beta}\frac{\partial}{\partial s^\mu}. \tag{B.0.8}$$

However, we may also define an equivalent algebra multiplication via the following construction. Let W_s be the versal deformation. Explicitly,

$$W_s = x_2^3 x_1 + x_1^3 + s_0 + s_{01}x_2 + s_{10}x_1 + s_{02}x_2^2 + s_{11}x_1x_2 + s_{20}x_1^2 + s_{21}x_2x_1^2.$$

In [BB21], the good basis that is required to define the Landau-Ginzburg B-model was not used. We therefore modify their calculation slightly. Let M be the Saito-Frobenius manifold for E_7 and let $\mathcal{T}_M$ be its tangent sheaf. Recall that $\mathcal{T}_M(\mathbb{C}^n)$ is the $\mathbb{C}[\boldsymbol{s}]$-module freely generated by $\left\{\frac{\partial}{\partial s_\mu}\right\}_{\mu\in I}$. We identify $\mathcal{T}_M(\mathbb{C}^n)$ as an algebra via the Kodaira-Spencer isomorphism

$$\Psi : \mathcal{T}_M(\mathbb{C}^n) \to \mathbb{C}[x_1, x_2, v]/\Big(\partial_{x_1} W_{\boldsymbol{s}},\, \partial_{x_2} W_{\boldsymbol{s}}\Big), \quad \Psi\Big(\frac{\partial}{\partial s_\mu}\Big) = [x^\mu].$$

Thus, $\mathcal{T}_M(\mathbb{C}^n)$ inherits an algebra structure via Ψ. We now calculate that

$$\Psi\Big(\frac{\partial}{\partial s_{10}} \circ \frac{\partial}{\partial s_{02}}\Big) = [x_2^2 x_1].$$

Observe that

$$\frac{\partial W_{\boldsymbol{s}}}{\partial x_2} = 3x_2^2 x_1 + s_{01} + 2s_{02}x_2 + s_{11}x_1 + s_{21}x_1^2.$$

Therefore, as an element in $\mathbb{C}[x_1, x_2, \boldsymbol{s}]/\Big(\partial_{x_1} W_{\boldsymbol{s}},\, \partial_{x_2} W_{\boldsymbol{s}}\Big)$ we have

$$[x_2^2 x_1] = -\frac{1}{3}\Big(s_{01} + 2s_{02}x_2 + s_{11}x_1 + s_{21}x_1^2\Big).$$

Under the isomorphism Ψ^{-1}, this element maps to

$$-\frac{1}{3}\Big(s_{01}\frac{\partial}{\partial s_0} + 2s_{02}\frac{\partial}{\partial s_{01}} + s_{11}\frac{\partial}{\partial s_{10}} + s_{21}\frac{\partial}{\partial s_{20}}\Big). \tag{B.0.9}$$

Comparing equations (B.0.9) and (B.0.8) we find that equation (B.0.6) is satisfied with $A = -1/3$. To show that (B.0.7) is true, one can argue in a similar way to Lemma B.0.3. The details here are the same whether one uses the standard good basis or the basis chosen by Basalaev-Buryak. See [BB21] for the explicit calculation. □

A similar conclusion can be found for D_4, and D_N more generally as in [BB21, Theorem 6.1].