FROM GRAPH MINOR THEORY TO 3-DIMENSIONS

by

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A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Mathematics
College of Engineering and Physical Sciences
The University of Birmingham
December 2023

Abstract

This thesis comprises of three projects from topological combinatorics and structural graph theory. Firstly, we extend Heawood's theorem on the colourability of plane triangulations to triangulations of 3-space by proving that a triangulation of 3-space can be edge coloured with three colours if and only if all edges have even degree. Next, we propose an open question that seeks to generalise the Four Colour Theorem from two to three dimensions and show that 12 instead of four colours are both sufficient and necessary to colour every 2-complex that embeds in a prescribed 3-manifold. However, our example of a 2-complex that requires 12 colours is not simplicial. We give bounds on this colouring number for simplicial 2-complexes. Lastly, we look at graphs that are not 1-tough and consider the set of minimal seperators of these graphs that "witness" the non-toughness of the graph.

To Grandad Grumps

ACKNOWLEDGEMENTS

I have been very fortunate to have Johannes Carmesin as my supervisor throughout my PhD. Thank you for your guidance and unwavering support and encouragement. And for the often helpful and always inspiring lunchtime discussions along with Jan, Will and Tim.

I extend my thanks to the staff and students of the CDT of Topological Design, especially the rest of my CDT cohort, Seb, Jay and Barn, thank you for the camaraderie and friendship.

Thanks to my fellow research students in the School of Mathematics for all of the fun memories, with a special mentions to Nikos for making sure I took breaks, Rosie for being my crafting buddy, and Berk for our walks home.

A special thank you goes to Chris for always believing in me, and for putting up with me and the guinea pigs invading your home in the final few months of my PhD. And a thank you to my guinea pigs for being very cute and forcing me to keep a routine.

I am very appreciative to my parents, Phil and Joanne, for raising me to have a love of learning and for providing me with the supportive environment and opportunities that allowed me to pursue higher eduation.

I would also like to thank my extended family who have always supported and encouraged me both academically and personally. In particular, Grandad Grumps, thank you for always wanting to hear about my research, your sincere interest in listening to me talk about a topic you only cared about because I'm interested in it spoke volumes of your love and support, and that meant the world.

CONTENTS

CHAPTER 1

INTRODUCTION

A major subject area in structural graph theory is graph minor theory, which sits in the intersection between combinatorics and topology.

A *graph* is defined as a pair $G = (V, E)$ where V is the set of *vertices of* G and E is a set of pairs of V called *edges of* G. A graph H is a *minor* of the graph G if it can be obtained by deleting and contracting edges of G.

One important avenue within graph minor theory has been characterising the embeddability of graphs in 2D surfaces using the minor relation. In 2004, Robertson and Seymour proved a profound theorem which states that the list of excluded minors for any minor-closed class of graphs is finite [26]. The proof of this theorem includes the Graph Minor Structure Theorem which establishes a fundamental connection between graph minors and topological embeddings of graphs. A well-known example of such an excluded minor characterisation is the result that was first proved by Kuratowski in 1930 [23]: a graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor. In 2017, Carmesin obtained a 3-dimensional analogue of Kuratowski's thereom [4, 5, 6, 7, 8] by characterising the 2-complexes that are embeddable in $\mathbb{S}^3$ in terms of forbidden minors.

This thesis considers problems which further explores ideas from graph minor theory in both two and three dimensions. Chapter 2 provides foundational background for the topics explored in the thesis. In Chapter 3, a 3-dimensional analogue of Heawood's theorem concerning the 3-colorability of plane triangulations is proved. Chapter 4 initiates the exploration of a 3-dimensional analogue of the four colour theorem. In Chapter 5, we will look at the separators of graphs that are not 1-tough.

Viewing graphs from a topological perspective we can replace the vertices with points and the edges with copies of the unit interval with 0 and 1 identified with the points corresponding to two endvertices of the original edge. These topological spaces are called *1-complexes*. Extending this to 3-dimensions, we get *2-complexes*, which are 1-complexes together with a set of faces. A 2-complex that embeds in $\mathbb{S}^3$ must have planar link graphs[1] at every vertex, so we can use what we know about planar graphs to help prove more global statements about 2-complexes.

Carmesin's 2017 paper series created the foundations for Chapter 3 and Chapter 4 where we look into extending colouring theorems to 2-complexes.

A *(vertex-)colouring* of a graph G is a labelling of the vertices of G such that two adjacent vertices get different colours. We say G is *k-colourable* if it can be coloured in this way using k colours.

A *(edge-)colouring* of a 2-complex C is a labelling of the edges of C such that two edges that are adjacent in a face of C get different colours. Note that if two edges are adjacent in a face of C, they correspond to two adjacent vertices in the link graph at their shared endvertex, so an edge-colouring of C induces vertex-colourings of each link graph.

In 1898, Heawood proved that a plane triangulation is 3-colourable if and only if all of the vertices have even degree [22]. In Chapter 3 we extend Heawood's theorem by showing that 3-colourings of the link graphs of a 2-complex C can be simultaneously extended to a global edge colouring of C, as follows:

Theorem 1.1. *A triangulation of 3-space[2] can be edge-coloured with three colours if and only if all edges have even degree.*

Chapter 3 is based on a paper [13] that is joint work with Johannes Carmesin and

[1]The *link graph* at a vertex v of a 2-complex C is the graph $L(v)$ on the edges incident with v in C, there is an edge between two vertices in $L(v)$ if they share a face at v in C.

[2]Here, a *triangulation of 3-space* would be a 2-dimensional simplicial complex where all of the chambers are tetrahedrons.

Bethany Saunders.

The four colour theorem is a famous problem in graph theory that can be traced back to the 1850's [3]. The graph theoretic version of the theorem states that every planar graph is 4-colourable. The following question seeks to generalise the four colour theorem to three dimensions.

Open Question 1.2. *Let M be a 3-manifold. What is the least integer k such that every 2-dimensional simplicial complex that embeds in M is k-colourable?*

In Chapter 4 we give bounds for this question by proving that every 2-dimensional simplicial complex is 12-colourable and giving an example of one which requires 5 colours. We then show that the answer is '$k = 12$' when *simplicial* is dropped from the question.

The second part of Chapter 4 is based on a note [24] that is joint work with Jan Kurkofka.

A graph G is *t-tough* if $G \backslash S$ has at most $\frac{|S|}{t}$ connected components, for every separator S of G. In 1973, Chvátal observed that all Hamiltonian graphs are 1-tough [14], the converse of this statement was disproved in 2000 by Bauer, Broersma and Veldman [2]. Although not all 1-tough graphs are Hamiltonian, studying the class of graphs that are *not* 1-tough could assist in understanding the properties of Hamiltonian graphs.

By definition, graphs that are not 1-tough must have a separator whose removal leads to more connected components than there were vertices in the separator. Such a separator that witnesses the non-toughness of a graph is called *feeble.*

In Chapter 5 we look at non-tough graphs and prove that if a non-tough graph has crossed feeble separators, then these are not minimal feeble separators. More formally:

Lemma 1.3. *Let G be a non-tough graph with two crossing feeble separators S_1, S_2 such that $|S_1| = |S_2| = s$ and $S_1 \cap S_2 = \emptyset$. Then G has a feeble separator S' with $|S'| < s$.*

CHAPTER 2

BACKGROUND

This chapter is based on Diestel's book on graph theory [15], Hatcher's book on algebraic topology [20], Oxley's book on matroid theory [25], and Carmesin's paper series on the 3-dimensional Kuratowksi embeddings [4, 5, 6, 7, 8].

2.1 Graph theory

A *graph* is a pair $G = (V, E)$ where V is the set of *vertices of* G and E is a multiset of 2-subsets of V called *edges of* G. If G is not clear from context, we write $V(G)$ and $E(G)$.

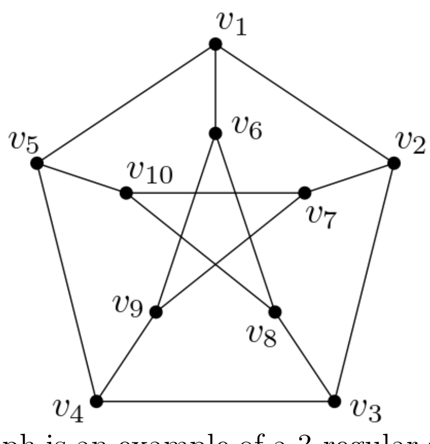


Figure 2.1: The Petersen graph is an example of a 3-regular simple graph with 10 vertices and 15 edges. The vertex v_1 has neighbours v_2, v_5, and v_6, and is incident with the edges v_1v_2, v_1v_5, and v_1v_6.

For $e = xy \in E$, we call x and y the *endvertices* of the edge e, and these vertices are *incident* with the edge e. The number of edges incident with a vertex v is the *degree of* v, denoted by $d(v)$. If all vertices have the same degree k, we say G is *(k-)regular*. The number $d(G)$ is the *average degree* of G. Two vertices x and y are *adjacent* or *neighbours* if $xy \in E$, we denote by $N(x)$ the set of neighbours, or *neighbourhood*, of the vertex x. Two edges with a common endvertex are *adjacent*.

If two edges have the same endvertices then we say they are *parallel edges*. An edge with only one endvertex is a *loop*. We call G *simple* if it contains no parallel edges or loops, and a *multigraph* otherwise.

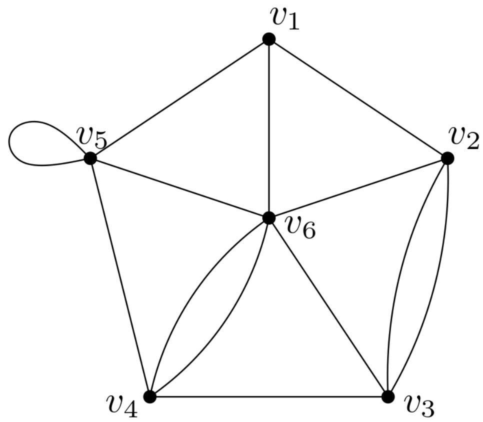


Figure 2.2: This is an example of a multigraph. The edge v_5v_5 is a loop. There are two sets of parallel edges; between v_4 and v_6, and between v_2 and v_3.

Two graphs G and G' are said to be *isomorphic* if there exists a bijection $\varphi\colon V(G) \to V(G')$ such that $xy \in E(G) \Leftrightarrow \varphi(x)\varphi(y) \in E(G')$ for all $x, y \in V(G)$.

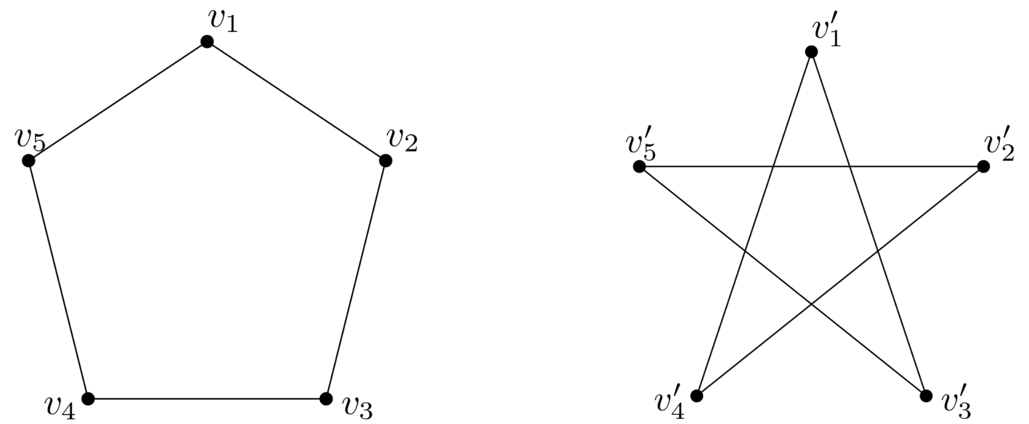
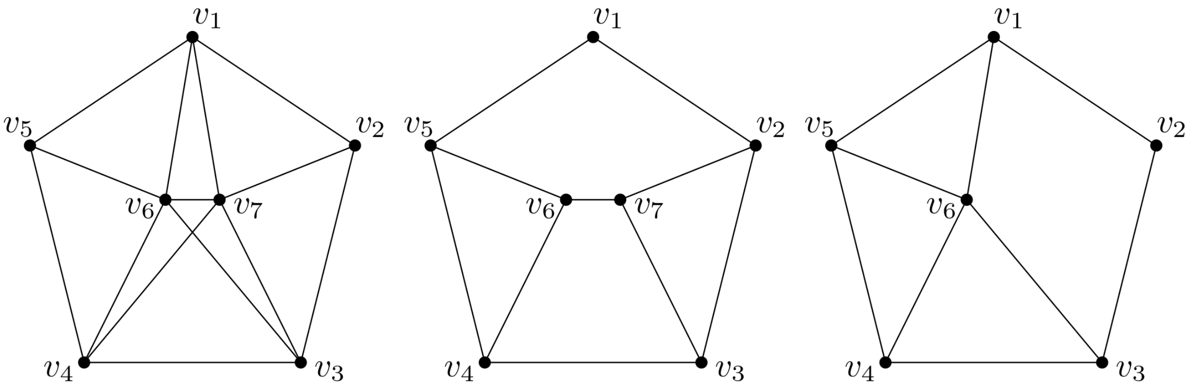


Figure 2.3: Two isomorphic graphs. The bijection between them is as follows: $\varphi(v_1) = v'_1$, $\varphi(v_2) = v'_3$, $\varphi(v_3) = v'_5$, $\varphi(v_4) = v'_2$, $\varphi(v_5) = v'_4$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a vertex subset $X \subseteq V$, if G' is the graph that has vertex set X and contains all of the edges $xy \in E$ such that $x, y \in X$, then G' is an *induced subgraph* of G, denoted $G[X]$. We denote by $G \backslash X$ the subgraph $G[V \backslash X]$, and write $G \backslash v$ if $X = \{v\}$.



(a) G (b) G_1 (c) G_2

Figure 2.4: The graph G has G_1 and G_2 as subgraphs. G_2 is an induced subgraph, but G_1 is not.

Given an edge-subset Y of G, the *edge induced subgraph* $G[Y]$ is the subgraph of G whose edge set is Y and vertex set is the set of all endvertices of edges in Y. Denote by $G - Y$ the subgraph $G[E \backslash Y]$, and write $G - e$ if $Y = \{e\}$.

A subgraph H of G is *spanning* if $V(H) = V(G)$.

A *walk* in G is an alternating sequence of vertices and edges $v_0 e_0 \ldots e_{n-1} v_n$ such that the vertices v_i and v_{i+1} are incident with the edge e_i for all $i < n$. If $v_0 = v_n$ the walk is *closed*. If G is simple, the walk can be defined just by its vertices. A *path* is a walk with no repeating vertices. A *cycle* is a closed walk with no repeating vertices aside from $v_0 = v_n$.

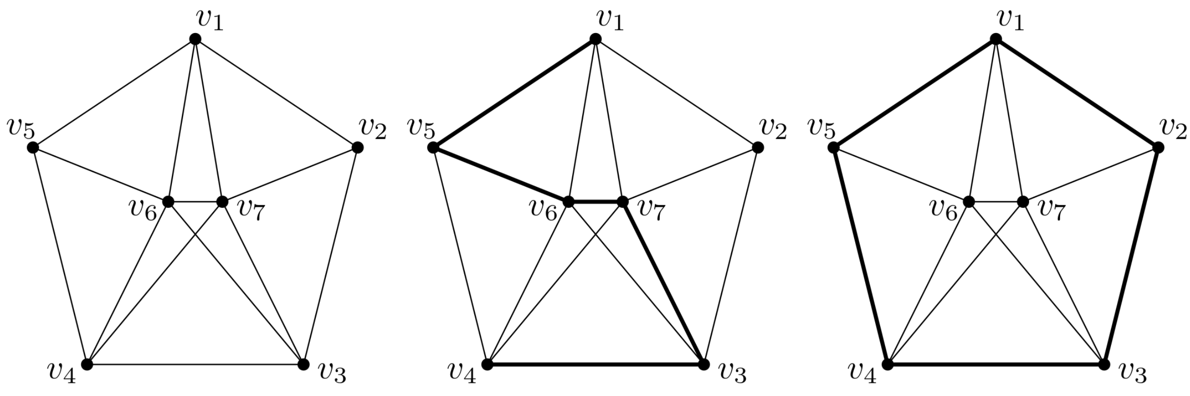


Figure 2.5: A path $P = v_1v_5v_6v_7v_3v_4$ and a cycle $C = v_1v_2v_3v_4v_5v_1$ in the graph G.

We say G is *connected* if there is a path between any two of its vertices, and *disconnected* otherwise. The maximal connected subgraphs of a disconnected graph G are called the *(connected) components* of G. We call G *k-connected* if $|V(G)| > k$ and $G \backslash X$ is connected for every $X \subseteq V$ with $|X| < k$.

We say G is a *tree* if it is connected and contains no cycles. G is a *forest* if it is the disjoint union of trees.

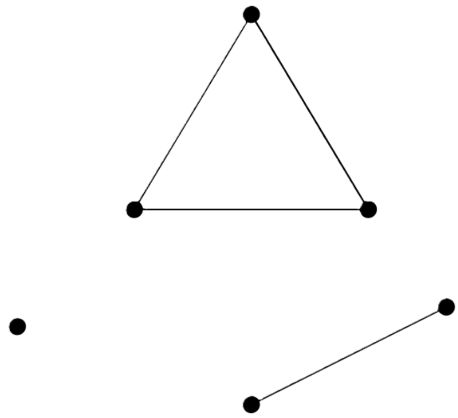


Figure 2.6: An example of a graph with three connected components.

We say that a set $S \subseteq V$ *separates* two vertices a and b if every path from a to b in G contains a vertex from S. A vertex v that separates two other vertices in the same component of G is called a *cutvertex*.

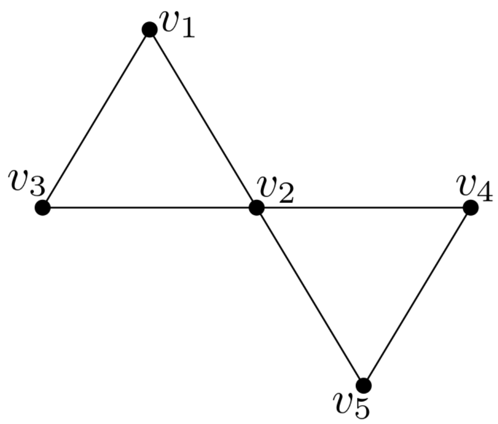


Figure 2.7: The vertex v_2 is a cutvertex.

Let A, B and S be vertex subsets of a connected graph G such that $A \cup B = V$ and $S = A \cap B$. We say S *separates* A and B if every $A - B$ path in G contains a vertex from S. We call S, with $|S| = k$, a *k-separator* if $G \backslash S$ is disconnected. The pair $\{A, B\}$ is called a *separation* of G; we refer to the sets A and B as the *sides* of the separation. If both $A \backslash B$ and $B \backslash A$ are non-empty, then the separation is *proper*.

Definition 2.1. We call G *t-tough* if for every separator S the graph $G \backslash S$ has at most $\frac{|S|}{t}$ connected components.

Example 2.2. The graph S_6 as in Figure 2.8 is not 1-tough. In fact, there are no 1-tough graphs with a cutvertex.

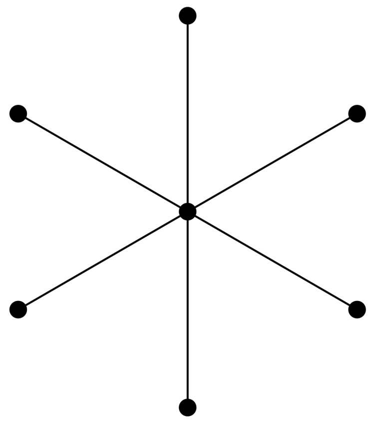


Figure 2.8: The star S_6 is not t-tough for any t.

Definition 2.3. Let S_1 and S_2 be two separators in G such that $S_1 \neq S_2$. We say they *cross* if S_i separates two vertices of S_{2-i} for each $i \in \mathbb{Z}_2$. Otherwise, they are *nested* if $S_1 \cap S_2 = \varnothing$ and S_i is entirely contained in one of the sides of S for each $i \in \mathbb{Z}_2$.

A set of separators of G is *totally nested* if all of its elements are pairwise nested.

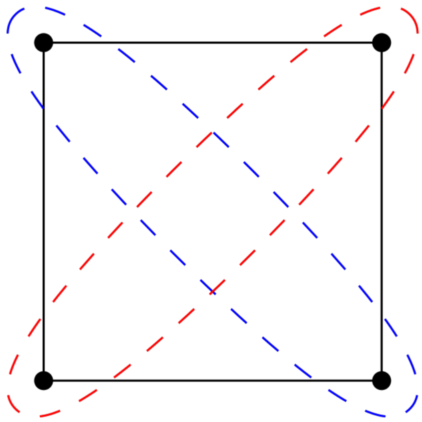


Figure 2.9: The two 2-separators of C_4 are crossed.

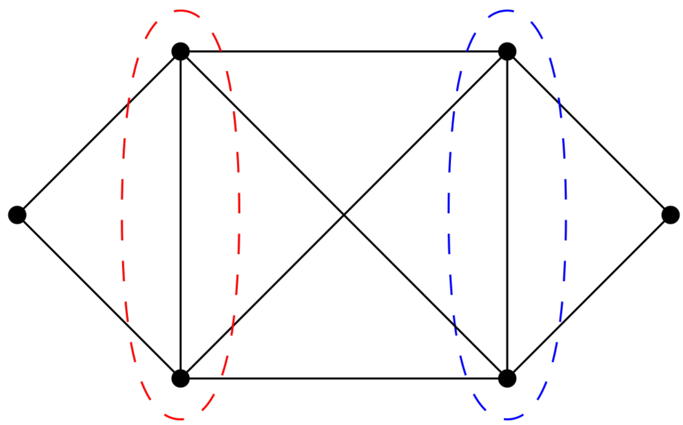


Figure 2.10: A graph with a set of totally nested 2-separators.

A *tree-decomposition* of G is a pair $(T, \mathcal{V})$ consisting of a tree T and a family $\mathcal{V} = (V_t)_{t \in V(T)}$ of vertex sets $V_t \subseteq V(G)$ such that $G = \bigcup_{t \in V(T)} G[V_t]$ and the vertex set $\{t \in V(T) : v \in V_t\}$ is connected in T for all $v \in V(G)$.

The edge t_1t_2 of T *induces* the separation (X_1, X_2) of G for $X_i = \bigcup_{t \in V(T_i)} V_t$, where T_i is the component of $T - t_1t_2$ that contains t_i.

Theorem 2.4 (Carmesin, Diestel, Hundertmark, and Stein [11])**.** *Every totally nested set*

N of proper separations of G defines a tree-decomposition of G, whose edges then induce the separations in N.

An *edge cut* of a connected G is a subset Y of edges of G such that $G - Y$ is disconnected. A *bond* is a minimal edge cut.

A *directed graph* (or *digraph*) D is one whose edges all have a direction. In a directed graph the edges xy and yx are not equal as they have opposite directions. When D is constructed by assigning directions to every edge of an undirected graph G, we call D an *orientation* of G. A *flow* on D is an assignment of real numbers to the edges of D $f\colon E \to \mathbb{R}$ such that $f(xy) = -f(yx)$ and $\sum_{y \in N(x)} f(xy) = 0$ for all $x \in V(G)$, apart from some *sink* and *source* vertices where the flow leaves and enters the network, respectively.

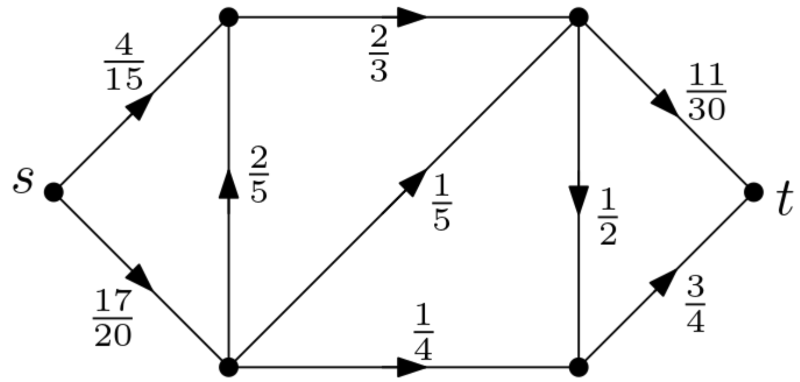
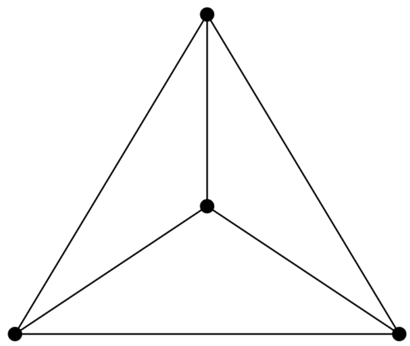


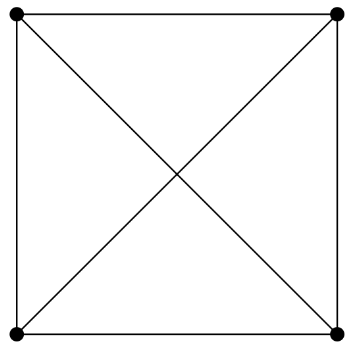
Figure 2.11: A directed graph with a flow. The vertex s is a source vertex, and t is a sink vertex.

The *complete graph on n vertices*, denoted K_n, is the graph where all vertices are pairwise adjacent. A simple graph G is called *bipartite* if V can be partitioned into two disjoint sets such that every edge has an endvertex from each vertex subset. The bipartite graphs that have every possible edge are called the *complete bipartite graphs* and are denoted $K_{n,m}$ for where n and m are the sizes of the two vertex subsets.

A graph G is *embedded* on a surface S if G is represented in S such that the vertices are distinct points in S and the edges of G are non-overlapping arcs between the points associated with its endvertices. A *plane graph* is a graph embedded in the *plane*, $\mathbb{R}^2$. We call G *planar* if it is isomorphic to a plane graph, which is called a *planar embedding* or *drawing* of G. The regions of $\mathbb{R}^2\backslash G$ are called the *faces* of G.



(a) G



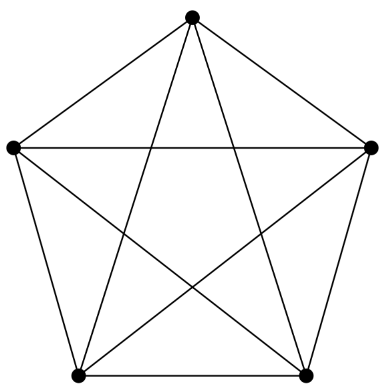
(b) H

Figure 2.12: Two representations of the complete graph K_4. G is a plane graph and is a planar embedding of K_4. H is isomorphic to G.

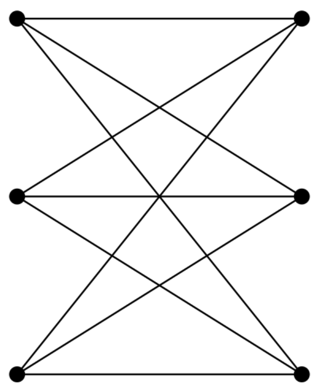
Theorem 2.5 (Euler's Formula). *Let G be a connected plane graph with v vertices, e edges and f faces, then $v - e + f = 2$.*

For $e = xy \in E$, we *subdivide* this edge by replacing e with a path between x and y where every other vertex on this path has degree 2. A *subdivision of G* is a graph that con be obtained by subdividing edges of G.

Theorem 2.6 (Kuratowski [23]). *A graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5 as a subgraph.*



(a) K_5



(b) $K_{3,3}$

Figure 2.13: The graphs K_5 and $K_{3,3}$ as in Theorem 2.6

For $e = xy \in E$, we *contract* this edge by removing e and identifying the endvertices x and y into one vertex v_e that is adjacent to the neighbours of x and y. We denote by G/e the graph obtained by contracting the edge e.

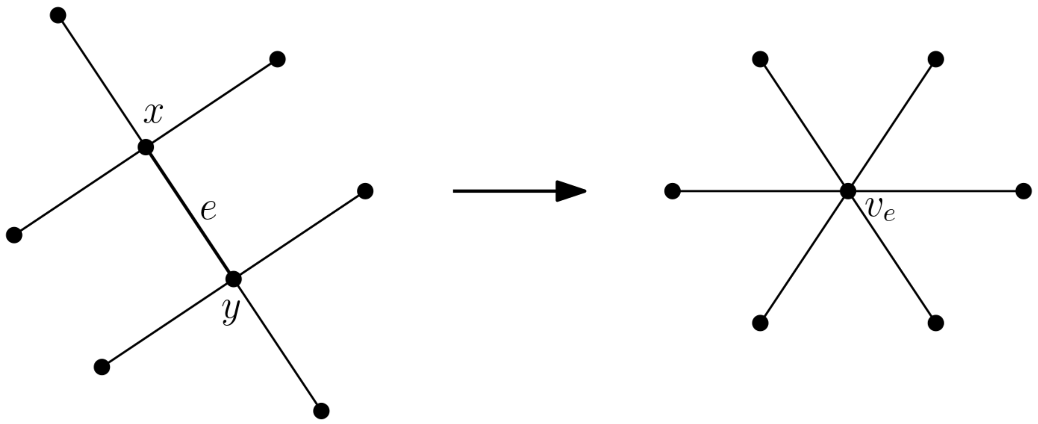


Figure 2.14: Contracting the edge e.

A graph H is a *minor* of G if we can obtain H from G by contracting edges, deleting edges, and deleting isolated vertices.

We can restate Theorem 2.6 in terms of minors:

Theorem 2.7 (Wagner [27]). *A graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor.*

For $v \in V$ of degree 2, we *suppress* this vertex by removing v and adding an edge between its two neighbours. A graph H is a *topological minor* of G if some subdivision of H is a subgraph of G, or equivalently, if we can obtain H from G by deleting edges, deleting isolated vertices and suppressing vertices of degree 2.

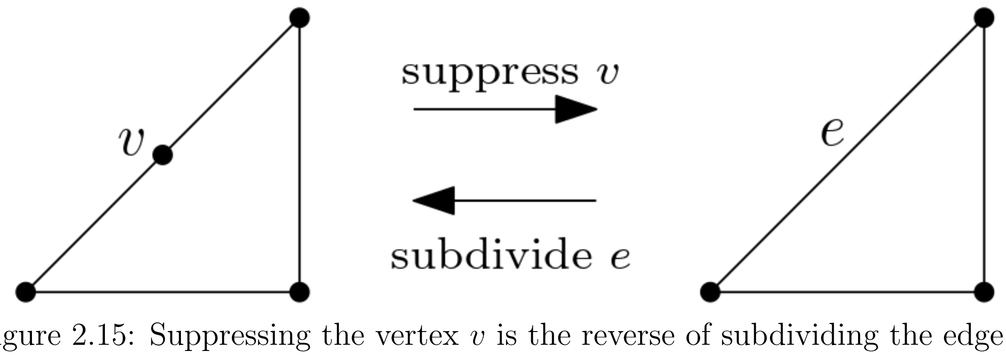


Figure 2.15: Suppressing the vertex v is the reverse of subdividing the edge e.

A relation $\leqslant$ on X is called a *well-quasi-ordering* if it is reflexive and transitive, plus every infinite sequence $x_0, x_1, \ldots$ in X contains a pair $x_i \leqslant x_j$ with $i < j$.

Theorem 2.8 (Robertson and Seymour [26])**.** *The finite graphs are well-quasi-ordered by the minor relation* $\preceq$.

Corollary 2.9. *For any minor-closed*[1] *set of graphs* X, *the set of minor-minimal graphs not in* X *is finite.*

Example 2.10. The minor-minimal non-planar graphs are exactly K_5 and $K_{3,3}$. In fact, for every surface S a graph being embeddable in S is minor-closed and so embeddability in any surface can be characterised by a finite list of minors.

Other minor-closed families of graphs include forests, outerplanar graphs and series-parallel graphs.

The *(plane) dual graph* G^* of a plane graph G is the plane graph defined by embedding a vertex inside each face of G and connecting two vertices by an edge when the faces share an edge in G, note that there exists an obvious bijection between $E(G^*)$ and $E(G)$.

Since G and G^* have the same edge set, deleting an edge from G will result in removing it from G^* in some way. In fact, it will result in the contraction of e in G^*, so $(G\backslash e)^* = G^*/e$. Similarly, $(G/e)^* = G^*\backslash e$.

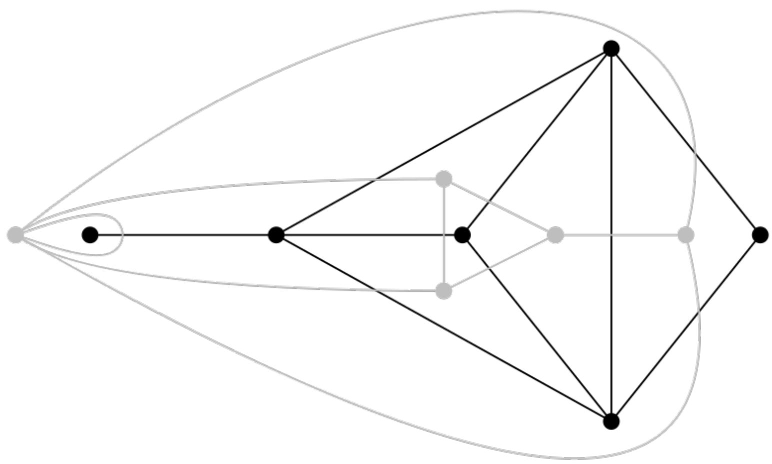


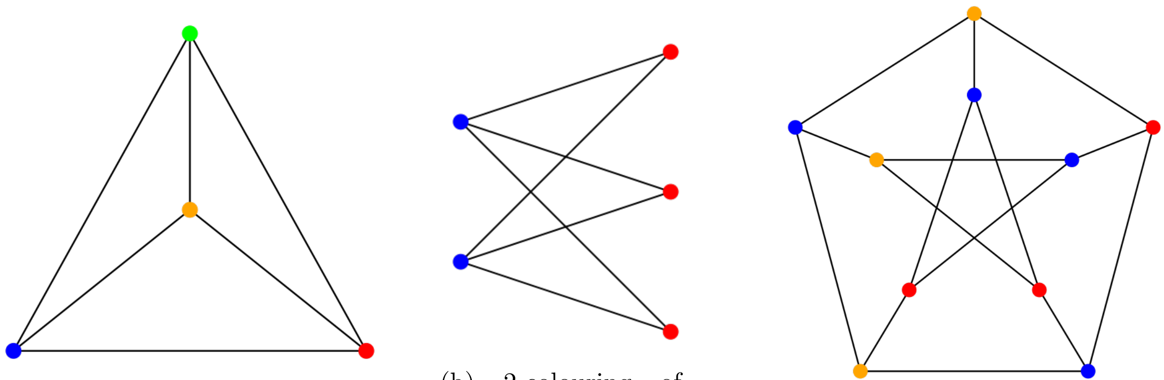
Figure 2.16: A plane graph and its dual.

[1]We say that a set of graphs X is *minor-closed* if $G \in X \implies G' \in X$ for all $G' \preceq G$.

Proposition 2.11. *For a plane graph G and its dual G^*, an edge subset $Y \subseteq E(G)$ is the edge set of a cycle in G if and only if it is the edge set of a bond in G^*.*

A *(vertex-)colouring* of G is a function $c \colon v \to X$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. We call the elements of X the *colours*. c is a *k-colouring* when $|X| = k$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that there exists a k-colouring of G. If $\chi(G) = k$ then G is k-chromatic, and we say that G is *k-colourable* when $\chi(G) \leqslant k$. The set X is a partition of the vertex set of G into independent sets, called *colour classes*.

Example 2.12. Graphs with no edges are the only 1-chromatic graphs, and the only 2-chromatic graphs are the non-empty bipartite graphs. The complete graph K_n is n-chromatic for every $n \in \mathbb{Z}$.



(a) 4-colouring of K_4.

(b) 2-colouring of $K_{2,3}$

(c) 3-colouring of the Petersen graph.

Figure 2.17: Vertex-colourings of some graphs.

Theorem 2.13 (Four colour theorem [1]). *Every planar graph is 4-colourable.*

2.2 Topology

Given a set X, a *topology* on X is a collection τ of subsets of X with the following properties:

i $\emptyset \in \tau, X \in \tau$;

ii the intersection of any two subsets in τ is in τ;

iii the union of any collection of subsets in τ is in τ.

A pair (X, τ) such that τ is a topology on X is called a *topological space*.

Example 2.14.

- The *discrete topology* on X consists of all subsets of X.
- Given a topological space (X, τ) and a subset $X' \subseteq X$, the *subspace topology* on X' is given by $\tau' = \{U \cap X' : U \in \tau\}$.

A function $f : X \to Y$ between two topological spaces, (X, τ) and (Y, υ), is *continuous* if for every set $V \in \upsilon$ the inverse image of V, $f^{-1}(V)$, is in τ.

Two topological spaces X and X' are said to be *homeomorphic*, denoted $X \cong X'$, if there exists a continuous bijection $f : X \to X'$ such that the inverse $f^{-1} : X' \to X$ is also continuous. Such a function is called a *homeomorphism*.

A *path* from a point x to a point y in X is a continuous function $\sigma : [0, 1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$. X is said to be *path connected* if, for every pair of points $x, x' \in X$, there is a path in X from x to x'.

A topological space X is called *simply connected* if it is path connected and every path between two points, x and y, in X can be continuously transformed into any other path between x and y in X.

A *simplicial complex* $K = (V, S)$ consists of a set V together with a set S of finite non-empty subsets, called *simplices* of V such that the following hold.

- $v \in V \implies \{v\} \in S$;

- $\sigma \in S$ and $\emptyset \neq \tau \subseteq \sigma \implies \tau \in S$.

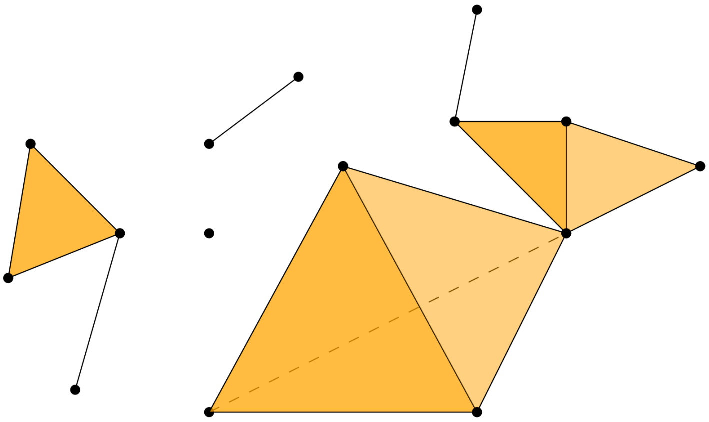


Figure 2.18: A 2-dimensional simplicial complex.

The *vertices* of K are the elements of V, and elements of S are the *faces* of K. If a simplex consists of $n+1$ elements of V, we call it a *n-simplex* and it has *dimension* n. If $\sigma \in S$ and $\tau \subseteq \sigma$, then we call τ a *face* of σ. The *dimension* of a simplicial complex K is the largest dimension of a simplex in K.

Simplices are n-dimensional generalisations of triangles: a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc.

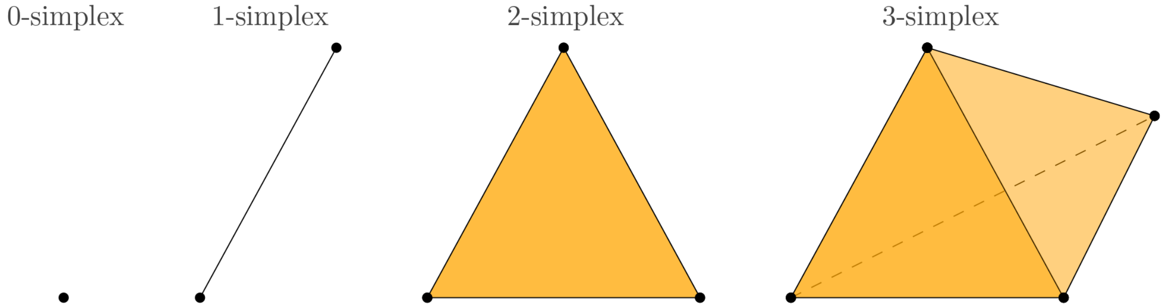


Figure 2.19: n-dimensional simplices for $n = 0, 1, 2, 3$.

The *underlying space* $|K|$ of a simplicial complex K is defined as $|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$ with the subspace topology.

A *triangulation* of a topological space X is a homeomorphism $h : |K| \to X$ from the underlying space of some simplicial complex K.

A *subcomplex* of K is a simplicial complex K' such that every face of K' is a face of K. The *d-skeleton* of K is the subcomplex consisting of all of the faces of K that have dimension at most d.

For $r \in \mathbb{Z}$, the *r-chain group* of K is the vector space $C_r(K)$ over a field $\mathbb{F}$ generated by treating the r-simplices of K as a basis.

The *boundary homomorphism* $d_r : C_r(K) \to C_{r-1}(K)$ is defined by

$$d_r(\sigma) = \sum_{i=0}^{r} (-1)^i (v_0, v_1, \ldots, \hat{v}_i, \ldots, v_r)$$

where the simplex $(v_0, v_1, \ldots, \hat{v}_i, \ldots, v_r)$ is the face of σ obtained by deleting the vertex v_i. The *r-cycle group* of K, denoted $Z_r(K)$, is the kernel of the boundary homomorphism. The image of the boundary homomorphism is called the *r-boundary group* of K, denoted $B_r(K)$.

The r^{th} *homology group* of K, denoted $H_r(K)$, is the quotient group $Z_r(K)/B_r(K)$.

A simplicial complex K is said to be *connected* when there exists a path along the edges of K between every pair of vertices of K.

Proposition 2.15. *If a simplicial complex K is connected, then $|K|$ is path connected. In this case, $H_0(K) \cong \mathbb{Z}$.*

2.3 Matroids

A *matroid* M is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E with the following properties:

(I1) $\emptyset \in \mathcal{I}$;

(I2) $I \in \mathcal{I}$ and $I' \subseteq I \implies I' \in \mathcal{I}$;

(I3) $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2| \implies$ there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

This matroid M is called a *matroid on* E. The members of $\mathcal{I}$ are the *independent sets* of M, and E is the *ground set.* We write $E(M)$ for E and $\mathcal{I}(M)$ for $\mathcal{I}$ when M is not obvious.

For any subset X of E, the set $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}\}$ defines the independent sets of a matroid $(X, \mathcal{I}|X)$ called the *restriction of* M *to* X.

Two matroids M_1 and M_2 are *isomorphic* if there exists a bijection φ from $E(M_1)$ to $E(M_2)$ such that $\varphi(X) \in \mathcal{I}(M_2)$ if and only if $X \in \mathcal{I}(M_1)$, for all $X \in E(M_1)$.

Let A be an $m \times n$ matrix over a field $\mathbb{F}$. Then $M = (E, \mathcal{I})$ is a matroid, where E is the set of columns of A and $\mathcal{I}$ is the set of linearly independent subsets X of E in the vector space $V(m, \mathbb{F})$. This is the *vector matroid* of A, denoted by $M[A]$.

Example 2.16. Let A be the following matrix over the field $\mathbb{R}$

$$\begin{array}{c} \\ A = \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The vector matroid of A has ground set $E = \{1, 2, 3, 4, 5, 6, 7\}$, and the set $\mathcal{I}$ of independent sets of $M[A]$ consists of all subsets of $E \backslash \{4\}$ with at most three elements, except for $\{1, 2, 7\}, \{1, 6, 7\}, \{2, 3, 5\}, \{3, 5, 6\}$ and any subset containing $\{2, 6\}$.

If M is isomorphic to the vector matroid of a matrix D over a field $\mathbb{F}$, then we say M is *representable (over $\mathbb{F}$)*, and D is a *representation for M (over $\mathbb{F}$)*.

A subset of E that is not in $\mathcal{I}$ is *dependent*. A minimal dependent set[2] of M is called a *circuit* of M, and we denote the set of circuits by $\mathcal{C}(M)$, or $\mathcal{C}$ if M is obvious.

A matroid is uniquely defined by its set of circuits, so we can restate the definition of a matroid to be $M = (E, \mathcal{C})$ with ground set E and a collection $\mathcal{C}$ of subsets of E called circuits with the following properties:

(C1) $\emptyset \notin \mathcal{C}$;

(C2) $C, C' \in \mathcal{C}$ and $C' \subseteq C \implies C' = C$;

(C3) $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, with $e \in C_1 \cap C_2 \implies$ there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \backslash e$.

If E is the set of edges of a graph G and $\mathcal{C}$ is the set of cycles of G, then $\mathcal{C}$ is the set of circuits of a matroid on E, called the *cycle matroid* of G and is denoted by $M(G)$. The independent sets of $M(G)$ are the edge sets $S \subseteq E$ that do not contain the edge set of a cycle as a subset i.e. the edge-induced subgraph $G[S]$ is a forest for all $S \subseteq E$.

The cycle matroid of a graph G can also be defined as the vector matroid of any incidence matrix[3] of G.

Example 2.17. Consider the graph G in Figure 2.20. The cycle matroid of G has the ground set $E = \{1, 2, 3, 4, 5, 6, 7\}$, and the set of circuits of $M(G)$ is $\mathcal{C} = \{\{4\}, \{2, 6\}, \{1, 2, 7\}, \{1, 6, 7\}, \{2, 3, 5\}, \{3, 5, 6\}, \{1, 3, 5, 7\}\}$. Note that $M(G) = M[A]$ from Example 2.16.

A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*.

All matroids on three or fewer elements are graphic, see [25] for a table of these matroids and a corresponding graph for each. These corresponding graphs are not necessarily unique, a graphic matroid can be isomorphic to the cycle matroids of multiple graphs.

---

[2]The *minimal dependent sets* are the dependent subsets of E whose proper subsets are all independent.

[3]The *incidence matrix* of a graph G is an $n \times m$ matrix B, where $n = |V(G)|$ and $m = |E(G)|$, such that $B_{i,j} = 1$ when v_i is an endvertex of the edge e_j, and $B_{i,j} = 0$ otherwise.

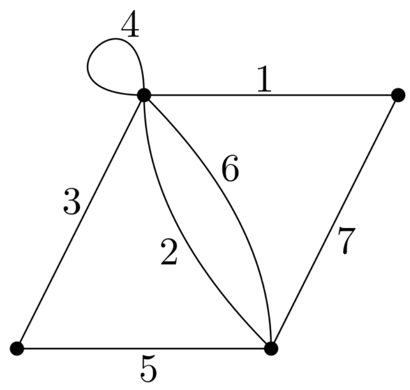


Figure 2.20: A planar graph G for which $M(G)$ is isomorphic to $M[A]$ from Example 2.16.

The maximal independent sets of M are called the *bases* of M, we denote the set of bases by $\mathcal{B}(M)$.

We can again restate the definition of a matroid in terms of bases. A matroid $M = (E, \mathcal{B})$ has ground set E and a collection $\mathcal{B}$ of subsets of E called bases with the following properties:

(B1) $\mathcal{B} \neq \emptyset$;

(B2) $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \backslash B_2 \implies$ there exists $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

Every element of $\mathcal{B}$ has the same cardinality which we call the *rank* of M, denoted $r(M)$. We can also define the rank $r(X)$ of any subset X of E to be the cardinality of a basis B of $M|X$.

Since the independent sets of cycle matroids are the sets $X \in E$ that do not contain cycles, the bases of $M(G)$ are exactly the spanning trees of G which have $|V(G)| - 1$ edges, so the rank of $M(G)$ is $|V(G)| - 1$.

The *dual (matroid)* of M is the matroid M^* whose ground set is $E(M)$ and whose bases are the complements of the bases of M, i.e. $\mathcal{B}(M^*) = \{E(M) \backslash B : B \in \mathcal{B}(M)\}$.

Example 2.18. Let M be the matroid from the previous two examples. The set of bases of M is

$$\mathcal{B} = \{\{1,2,3\},\{1,2,5\},\{1,3,5\},\{1,3,6\},\{1,3,7\},\{1,5,6\},\{1,5,7\},\{2,3,7\},$$
$$\{2,5,7\},\{3,5,7\},\{3,6,7\},\{5,6,7\}\}$$

So M^* is the matroid with ground set E and the set of bases of M^* is

$$\mathcal{B}(M^*) = \{\{4,5,6,7\},\{3,4,6,7\},\{2,4,6,7\},\{2,4,5,7\},\{2,4,5,6\},\{2,3,4,7\},\{2,3,4,6\},$$
$$\{1,4,5,6\},\{1,3,4,6\},\{1,2,4,6\},\{1,2,4,5\},\{1,2,3,4\}\}$$

We denote the dual of the cycle matroid of a graph G by $M^*(G)$, this is called the *bond matroid* of G.

Theorem 2.19 (Whitney's planarity criterion [28]). *A graph G is planar if and only if there exists a graph G' such that $M(G') = M^*(G)$.*

Unsurprisingly, if G is planar its bond matroid is actually the cycle matroid of its dual graph G^*, that is $M^*(G) = M(G^*)$.

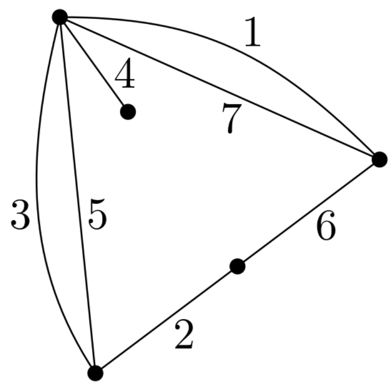


Figure 2.21: The dual graph G^* of the planar graph G from Figure 2.20.

Example 2.20. For G in Figure 2.20, the dual graph is G^* in Figure 2.21. It is easy to see that the cycle matroid of G^* is M^* as in Example 2.18

A *k-separation* of M is a bipartition (X, Y) of E such that $r(X)+r(Y)-r(M) = k-1$. We call M *k-connected* if it has no k'-separator for all $k' < k$. The k-separations of $M(G)$ (for G connected) correspond to the k-separations of G.

We can also define minor operations on matroids, as follows. The matroid obtained by *deleting* a subset $S \subseteq E$ from M has ground set $E \backslash S$ and independent sets are the subsets of $E \backslash S$ that are independent in M, this matroid is denoted by $M \backslash S$. Let M/S be the matroid obtained by *contracting* S from M, this is defined through deleting S in M^*. So $M/S = (M^* \backslash S)^*$.

This means that, as with graphs, contraction and deletion are dual. So we get that $(M \backslash e)^* = M^*/e$ and $(M/e)^* = M^* \backslash e$. In fact, performing these operations on a cycle matroid extends to performing the graph-theoretic minor operations on the graphs that generated the matroid. That is, $M(G \backslash S) = M(G) \backslash S$ and $M(G/e) = M(G)/e$.

A matroid N is a *minor* of M if we can obtain N from M by performing a sequence of deletions and contractions.

2.4 Embeddable 2-complexes

A *(topological) embedding* of a simplicial complex C into a topological space X is an injective continuous map from (the geometric realisation of) C into X.

A *2-complex* $C = (V, E, F)$ is a (directed multi-)graph[4] $G = (V, E)$ together with a set F of closed walks called *faces*. We call the graph G the *1-skeleton* of C. A *2-dimensional simplicial complex* is a 2-complex such that G is simple and each face is bounded by exactly three distinct edges, we refer to these as *simplicial 2-complexes*.

The *link graph* $L(v)$ of a simplicial 2-complex C at a vertex v is the graph whose vertex set is the set of edges incident with v. The edge set is the faces incident with v in C such that the face f corresponds to an edge in $L(v)$ with the edges of C incident with f and v as its endvertices. The definition of a link graph extends to non-simplicial 2-complexes by adding two vertices in $L(v)$ for each loop incident with v and adding an edge between two vertices in $L(v)$ if a face of C traverses the corresponding edges in C.

Note that, if C is an embeddable 2-complex, then its link graph must be planar, which gives rise to some of the obstructions for embeddability. However, there are non-embeddable 2-complexes that have planar link graphs, see Figure 2.22 for an example of such a 2-complex. A simplicial 2-complex is *locally k-connected* if all of its link graphs are k-connected.

Let G_1 and G_2 be two graphs such that there exists v in $V(G_1)$ and $V(G_2)$ with a bijection φ between the edges incident with v in G_1 and G_2. The *vertex-sum* of G_1 and G_2 over v with respect to φ is the graph $G = G_1 \oplus_v G_2$ obtained from the disjoint union of G_1 and G_2 by deleting v in both G_1 and G_2 and adding an edge between any pair of vertices (v_1, v_2) with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that (v, v_1) and (v, v_2) are mapped to each other by φ.

[4]In this section all graphs are directed and can have parallel edges and loops, unless stated otherwise. If we also have a orientation on every face of a simplicial 2-complex C we say that it is a *directed* 2-complex.

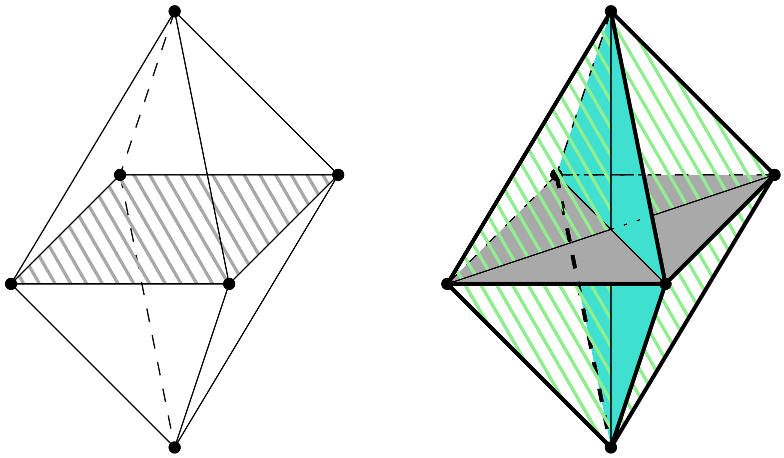


Figure 2.22: The 2-complex on the right is obtained from the octahedron with its eight triangular faces by adding three faces of size 4 orthogonal to the three axis. It is easy to see that all link graphs are planar. Note that adding only one of these three orthogonal faces gives the embeddable 2-complex on the left. There is space for a second face to be added on the outside of this embedding, but it can be shown that the 2-complex with all three 4-faces added is not embeddable. [4]

2.4.1 Space minors and rotation systems

Let e be a non-loop edge of C, we *contract* this edge by identifying the two endvertices of e, remove e from all incident faces, and then remove e. Denote by C/e the 2-complex obtained by contracting e in C. For a face f of C, if f has size two we *contract* f by deleting f and identifying its two incident edges. Otherwise, if f has size one we delete f and remove its incident edge from all faces, then remove this edge.

Let v be a vertex of C, we *split* v by replacing it by multiple copies in the following way, for each component of the link graph $L(v)$ there is a copy of v with the incidences inherited from this component of $L(v)$. We *topologically delete* an edge e of C by replacing it with a copy for every incident face such that this copy is incident only with that face where it takes the role of e.

A 2-complex D is a *space minor* of C if we can obtain it from C by contracting non-loop edges[5] and faces of size at most two, deleting faces, topologically deleting edges, and splitting vertices.

These space minor operations preserve embeddability in any 3-dimensional manifold.

[5]Contracting loops does not preserve embeddability in general.

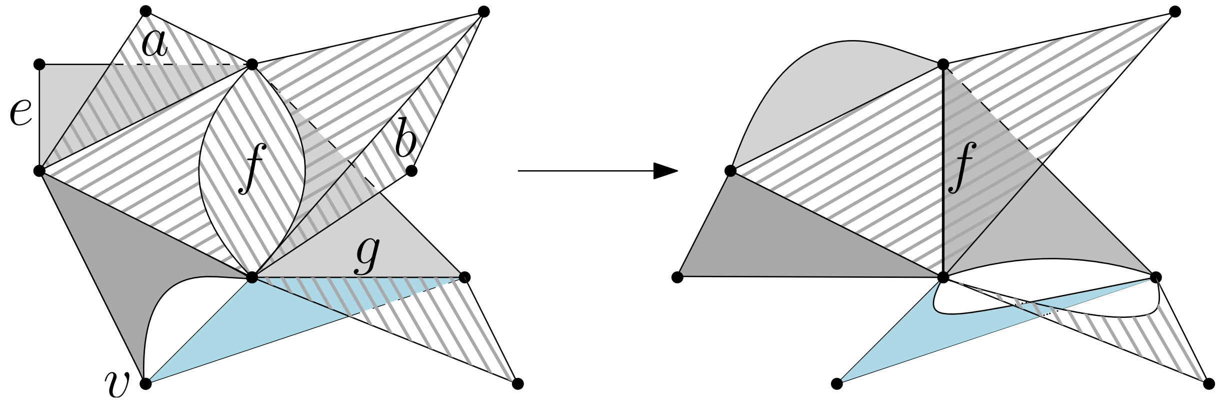


Figure 2.23: A graph and a space minor of that graph. We contract the edge e and the face f, delete the faces a and b, split the vertex v, and topologically delete the edge g. [4]

We can use space minors to see why the 2-complex on the right in Figure 2.22, call it O, does not embed into 3-space. Contract any one of the faces of size three to a single vertex in the following way. First contract an edge to make a face of size two, then contract this face to a single edge that can be contracted to a vertex. The link graph at this new vertex will be $K_{3,3}$ which is non-planar, so the new 2-complex cannot be embeddable in 3-space. Then we have that O cannot be embeddable since it has a non-embeddable minor.

A 2-complex is *3-bounded* if all its faces are incident with at most three edges. The closure of the class of simplicial 2-complexes under the space minor operations is the class of 3-bounded 2-complexes.

The *dual graph* of an embedding of C in $\mathbb{S}^3$ is the graph G whose vertices are the components of $\mathbb{S}^3 \backslash C$, called the *chambers* of the embedding, and for each face f of C add an edge with endvertices being the vertices corresponding to the two chambers f touches in the embedding. The *dual complex* of the embedding is the 2-complex D with 1-skeleton G and faces F such that there exists a bijection $f : E(C) \to F$, where each edge $e \in E(C)$ is mapped to the face in D that is incident to the edges of G that correspond to the faces that are incident to e in C. The *local surfaces* of an embedding of C in $\mathbb{S}^3$ are the boundaries of the chambers of the embedding.

A *rotation system* of a graph G is a family of cyclic orientations σ_v of the edges incident with each vertex v. Every embedding of a planar graph G induces a planar rotation system. A rotation system of G is *planar* if it is induced by a planar embedding of G. A *rotation system* of a 2-complex C is a family of cyclic orientations σ_e of the faces incident with each edge e. The orientations σ_v and σ_e are called *rotators*. A rotation system of C *induces* a rotation system of each link graph $L(v)$ by restricting to the edges incident with v, we take σ_e if e is directed towards v, and the reverse otherwise. A rotation system of C is *planar* if the induced rotation systems at every link graph are planar.

If we have a topological embedding of C in some oriented 3-manifold M, then the rotation system *induced by* this embedding is the one which corresponds to the embedding in the following way: for each edge e in C the cyclic orientation $\sigma(e)$ of the faces around e is the ordering in which they are embedded around e in the direction of the orientation of M.

Lemma 2.21 (Carmesin [4]). *If a 2-complex C has a planar rotation system, then any space minor of C also has a planar rotation system.*

To see that the space minor operations preserve the property of having a planar rotation system, consider what happens to the link graphs under each space minor operations.

Contracting a face f of size 2 in C contracts the edges corresponding to f in both of the link graphs at the two vertices incident with f.

The operation on the link graphs corresponding to contracting a face of size one is explained in Figure 2.24. Deleting a face f in C deletes the corresponding edges in the link graphs, along with deleting the vertices that correspond to the edges incident only with f in C.

Contracting a non-loop edge in C merges the two link graphs at its endvertices.

Splitting a vertex splits the link graph at that vertex into a separate link graph for each of its connected components.

Topologically deleting an edge e adds a copy of the vertex corresponding to e in the link graphs for each edge incident with the original vertex, this copy is only incident with that edge.

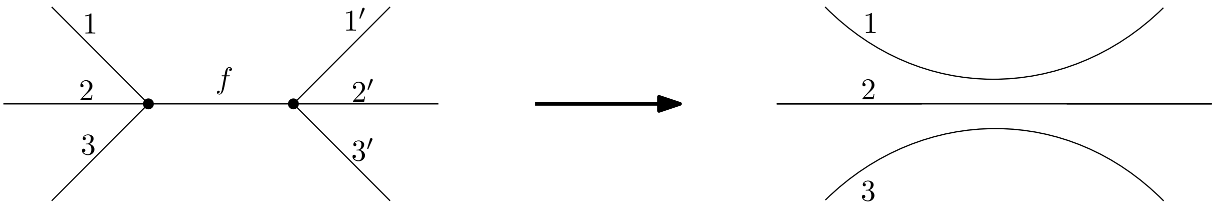


Figure 2.24: The operation on a link graph of a 2-complex C corresponding to contracting a face of size one in C. [4]

Theorem 2.22 (Carmesin [5])**.** *A simply connected simplicial 2-complex has an embedding in $\mathbb{S}^3$ if and only if it has a planar rotation system.*

2.4.2 A 3-dimensional Kuratowski characterisation

For G with no loops, the *cone* over G is the 2-complex with vertex set $V(G)$ plus one additional vertex, called the *top*, edge set $E(G)$ plus one edge for each vertex $v \in V(G)$ joining v with the top, and face set consisting of one face for each $e \in E(G)$ that is incident with e and the two edges from the endvertices of e to the top.

Note that if G had no parallel edges, then the cone over G will be a simplicial 2-complex. Also, the link graph at the top of the cone if isomorphic G.

Example 2.23. For G non-planar, the cone over G will have a non-planar link graph at its top and therefore cannot be embeddable in $\mathbb{S}^3$. In particular, K_5 and $K_{3,3}$ do not embed in $\mathbb{S}^3$.

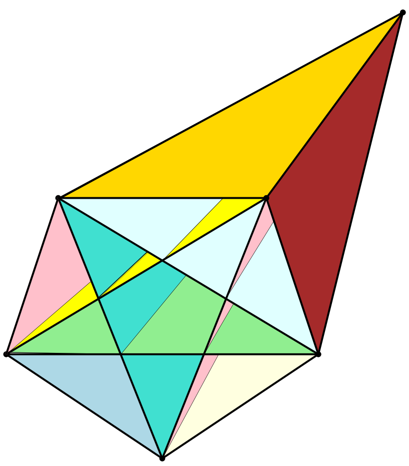


Figure 2.25: The cone over K_5. [4]

Lemma 2.24 (Carmesin [10])**.** *Let G and H be graphs without loops such that H is a minor of G. The cone over H is a space minor of the cone over G.*

Let C and D be two 2-complexes with a bijection f between two subsets of their vertices, we can obtain a 2-complex by *simplicial gluing* C and D via f in the following way: start with the disjoint union of C and D and identify vertices via the bijection f, we also identify the edges and faces that have all incident vertices in the image of f such that after identification of these vertices they have the same incident vertices. Gluings of simplicial complexes are also simplicial complexes.

The *simplicial combined cone* over the vertex-sum $G = G_1 \oplus_e G_2$ is obtained by gluing the cone over G_1 and the cone over G_2 via the vertex set containing the vertices in faces incident with the edge e in the cones.

Lemma 2.25 (Carmesin [10])**.** *Up to subdivision, there are exactly five vertex-sums $H = H_1 \oplus_e H_2$, where H is a subdivision of K_5 or $K_{3,3}$ and H_1 and H_2 both have at least three vertices of degree at least three. One such H is a subdivision of K_5, and the other four are subdivisions of $K_{3,3}$.*

For two 2-complexes C and D with isomorphic subcomplexes C' and D', respectively, we obtain a 2-complex from C by *gluing* D at C' in the following way: start with the disjoint union of C and D and identify C' with D' via their isomorphism.

Given a vertex-sum $G = G_1 \oplus_e G_2$, the *combined cone* over this vertex-sum is obtained from the cone over G_1 by gluing the cone over G_2 via the subcomplex consisting of the edge e and its incident edges in both cones such that in the resulting complex the tops of the cones are glued onto distinct endvertices of e.

A triangulation of the Möbius strip is *nice* if it has a central cycle of length three and all edges of face degree two are on the central cycle or have exactly one endvertex on the central cycle. A *Möbius obstruction* is obtained from a nice triangulation of the Möbius strip by attaching a face at the central cycle. These Möbius obstruction 2-complexes do not embed in 3-space.

Let $\mathcal{Z}$ be the set of graphs containing cones over subdivisions of K_5 and $K_{3,3}$, simplicial combined cones from the five families of Lemma 2.25, and Möbius obstructions.

Theorem 2.26 (Carmesin [10])**.** *Let C be a locally 3-connected simplicial 2-complex such that the first homology group $H_1(C, \mathbb{F}_p)$ is trivial for some prime p. The following are equivalent:*

- *C embeds in 3-space;*
- *C is simply connected and has no space restriction*[6] *from the explicit list $\mathcal{Z}$.*

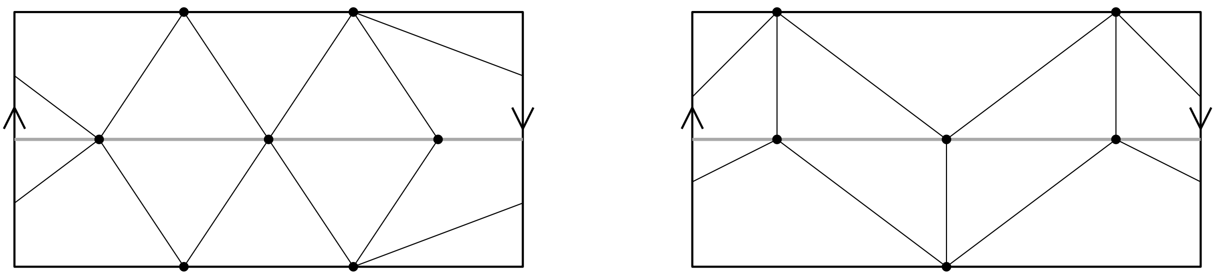


Figure 2.26: The only two nice space minor-minimal triangulations of the Möbius strip. [10]

Let $\mathcal{Z}'$ denote the list consisting of the cones over K_5 and $K_{3,3}$, the five combined cones over K_5 and $K_{3,3}$ from Lemma 2.25, and the Möbius obstructions obtained from the triangulations in Figure 2.26.

Lemma 2.27 (Carmesin [10])**.** *Every element of $\mathcal{Z}$ has a space minor in $\mathcal{Z}'$. The set $\mathcal{Z}'$ has 9 elements.*

In a simplicial 2-complex C, a *mega face* $F = (f_i | i \in \mathbb{Z}_n)$ is a cyclic orientation of faces f_i of C plus, for each $i \in \mathbb{Z}_n$, an edge e_i of C that is only incident with f_i and f_{i+1} such that $e_i \neq e_{i+1}$ and $f_i \neq f_{i+1}$ for all $i \in \mathbb{Z}_n$. A *boundary component* of a mega face F is a connected component of the 1-skeleton of C restricted to the faces f_i after we topologically delete the edges e_i.

Let the cycle o be a boundary component of a mega face F, we say that F is *locally monotone* at o if, for every edge e of o and each face f_i containing e, the next face face of

[6]A *space restriction* is a space minor obtained without contracting edges or faces.

F after f_i that contains an edge of o contains the unique edge of o that has an endvertex in common with e and e_{i+1}. Consider the number of indices i such that e is incident with f_i, this number is the same for each edge e of o when F is locally monotone at o and we call it the *winding number* of F at o.

A *torus crossing obstruction* is a simplicial complex C whose faces can be partitioned into two mega faces that both have the cycle o as a boundary component and are locally monotone at o but with different winding numbers. We call o the *base cycle*, and denote the set of torus crossing obstructions by $\mathcal{T}$.

Let G be a link graph of C at the vertex v that is incident with a single loop l. We say that G is *helicopter planar* if G has a plane embedding such that the rotators at the two vertices of G corresponding to l are reverse of one another. A *helicopter complex* is a 2-complex with a single loop l such that the link graph at the vertex v incident with l is not helicopter planar.

A helicopter complex is *nice* if its link graph at the vertex incident with the loop is a subdivision of a 3-connected graph.

Lemma 2.28. *There is a finite set $\mathcal{X}$ of helicopter complexes such that every nice helicopter complex has a space minor in $\mathcal{X}$.*

Now, denote by $\mathcal{Y}$ the list of the nine 2-complexes from $\mathcal{Z}'$ together with the finite list $\mathcal{X}$. This then leads to the refined Kuratowski characterisation.

Theorem 2.29 (Carmesin [8])**.** *Let C be a simplicial 2-complex such that the first homology group $H_1(C, \mathbb{F}_p)$ is trivial for some prime p. The following are equivalent:*

- *C has a topological embedding in 3-space;*
- *C is simply connected and has no stretching*[7] *that has a space minor in $\mathcal{Y} \cup \mathcal{T}$.*

[7]A 2-complex $\overrightarrow{C}$ is a *stretching* of C if it can be obtained from C by applying some *stretching operations*. The precise definition of a stretching of C can be found in [8].

2.4.3 A Whitney-type characterisation

The *incidence vector* of an edge e of C is the vector with an entry for every face f, this entry is 0 if e and f are not incident, 1 if the direction at e is positive in the orientation of f, and -1 otherwise. The matrix given by all incidence vectors is called the *edge/face adjacency matrix* of C.

Here, we assume that all 2-complexes are directed. However, for different directed simplicial 2-complexes with the same underlying simplicial 2-complex the dual matroids are isomorphic, so we omit the term 'directed'.

The following theorem is equivalent to Theorem 2.22.

Theorem 2.30 (Carmesin [10]). *Let C be a simplicial 2-complex embedded into $\mathbb{S}^3$. Then the edge/face incidence matrix of C represents (over any field) a matroid which is isomorphic to the cycle matroid of the dual graph of the embedding.*

This face inspires the definition of the dual matroid of a 2-complex C, which is the matroid represented by the edge/face adjacency matrix of C over the field $\mathbb{F}_3$[8]. More formally:

Definition 2.31. Given a field k, the *k-dual matroid* of a simplicial 2-complex C is the k-vector matroid whose ground set is the set of faces of C and its circuit space is generated by the incidence vectors of the edges of C. If k is clear by context, we call this the *dual matroid* of C. We denote the dual matroid of C by $M^*(C)$.

The 3-dimensional analogue of Whitney's planarity criterion is not as simple as a simplicial 2-complex is embeddable in 3-space if and only if its dual matroid is graphic. This is due to a few reasons, one of which is that the cone over K_5's dual matroid is trivially graphic as it is just a bunch of loops, however it does not embed into 3-space. To prevent this type of obstruction, we define the following.

A simplicial 2-complex C is *k-local* if for every vertex v we have that the dual of the cycle matroid of $L(v)$ is equal to $M^*(C)$ restricted to the faces incident with v.

[8]Here, $\mathbb{F}_3$ could be replaced with $\mathbb{F}_p$ for p any prime other than 2.

Also, we need to restrict to simply connected simplicial 2-complexes because dual matroids of triangulations of general homology spheres are graphic, but these triangulations do not embed into $\mathbb{S}^3$ in general.

This leads to the 3-dimensional analogue of Whitney's planarity criterion.

Theorem 2.32 (Carmesin [9])**.** *For every field k, a k-local simply connected simplicial 2-complex C embeds in 3-space if and only if its k-dual matroid is graphic; in this case k-dual matroids over different fields k are isomorphic.*

CHAPTER 3

3-COLOURABLE 3-DIMENSIONAL TRIANGULATIONS

In 1898, Heawood proved that a maximal plane triangulation is vertex colourable in three colours if and only if all its vertices have even degrees [22]. In this chapter we prove a 3-dimensional analogue of this theorem.

In fact there is quite a natural way to extend theorems about planar graphs to 3-space. Indeed, each 2-dimensional simplicial complex (which we will refer to as a *simplicial 2-complex* from now) embedded in 3-space has a planar link graph[1] at each of its vertices. Hence we are interested in global statements of the simplicial complex that project down to the theorem we are trying to extend in each of its link graphs, see [4, 5, 6, 7, 8] for details.

A *(proper) edge-colouring* of a 2-complex C is a labelling of each of the edges of C such that no two edges that share a face have the same label.[2] The *(face-)degree* of an edge e in a 2-complex C is the number of faces of C that e is incident with.

Intuitively speaking, Heawood's theorem says that local 3-colourings of the faces extend to global 3-colourings of plane triangulations. We extend this *Heawood principle* even further; these 3-colourings of the link graphs can be simultaneously extended to global edge-colourings of 2-complexes, as follows:

[1]The *link graph* at a vertex v of a 2-complex C is the graph $L(v)$ on the edges incident with v in C, there is an edge between two vertices in $L(v)$ if they share a face at v in C.

[2]Note that this definition of edge-colouring a 2-complex C is not the same as edge-colouring the 1-skeleton of C as a graph in the usual way.

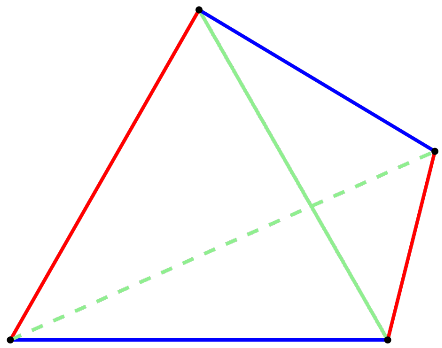


Figure 3.1: A 3-edge-colouring of a tetrahedron. This is an example of a spatial triangulation where all the edges have even degree.

Theorem. *A triangulation of 3-space*[3] *can be edge-coloured with three colours if and only if all edges have even degree.*

The n-dimensional version of this theorem was claimed without proof in the 70s [16], however we do not agree that it is as simple of a result as they believed. Indeed, another paper [19] claims to prove the 3-dimensional case, however their argument does not seem to work. For further details on this, see the concluding remarks.

For basics and background, refer to Diestel's book on graph theory [15], Hatcher's book on algebraic topology [20], and the paper series on the 3-dimensional Kuratowski embeddings [4, 5, 6, 7, 8].

3.1 3-Colourability of Eulerian Simplicial 2-Complexes

Definition 3.1. The *Spatial Line Graph* $SL(C)$ of a simplicial 2-complex C represents the face adjacencies of the edges of the simplicial 2-complex. The vertex set of $SL(C)$ consists of the edges of C, if two edges in C have a face in common, their corresponding vertices in $SL(C)$ are adjacent.

Definition 3.2. The *Directed Spatial Line Graph* $DL(C)$ of a simplicial 2-complex C whose faces have an orientation is a directed graph. The directed spatial line graph is

[3] Here, a *triangulation of 3-space* would be a simplicial 2-complex where all of the chambers are tetrahedrons.

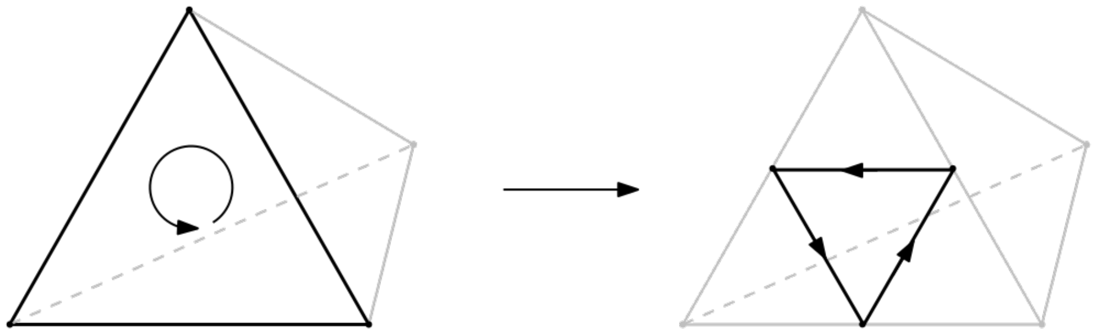


Figure 3.2: In a directed spatial line graph, for a face with orientation as in the left tetrahedron, we give the edges of the spatial line graph the directions as in the right tetrahedron.

the spatial line graph of C with the added property that the edge joining two vertices in $DL(C)$ is directed with the orientation of the face that the edges are incident to in C, as in Figure 3.2.

Definition 3.3. The *tetrahedron face cycles* in a spatial line graph of a simplicial 2-complex are the cycles whose vertices correspond to the three edges that bound a face on a tetrahedron in the simplicial 2-complex.

The *tetrahedron vertex cycles* in a spatial line graph of a simplicial 2-complex are the cycles whose vertices correspond to the three edges that are adjacent to a common vertex in a tetrahedron in the simplicial 2-complex.

We refer to the tetrahedron face cycles and the tetrahedron vertex cycles just as the *tetrahedron cycles.*

Definition 3.4. An orientation of the faces of a planar graph is *consistent* if each edge has opposite directions in the orientations chosen for each of its two incident faces.

The following lemma is proved in [7].

Lemma 3.5. *The incidence vectors of the edges of a simplicial 2-complex generate all cycles of the dual matroid*[4] *over any field, in particular over* $\mathbb{F}_2$.

Lemma 3.6. *A graph is bipartite if and only if all cycles have even length.*

[4]See Definition 2.31 for the definition of dual matroids.

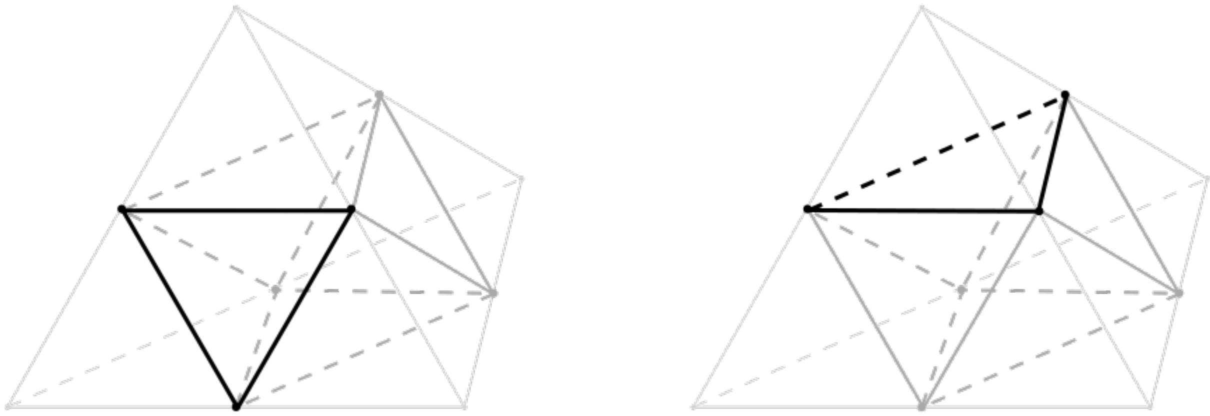


Figure 3.3: A tetrahedron face cycle (left) and a tetrahedron vertex cycle (right), they are both called tetrahedron cycles.

Lemma 3.7. *Given a simplicial 2-complex embedded in 3-space with all edges of even degree, then the dual graph is bipartite.*

Proof. Let C be a simplicial 2-complex embedded in 3-space with all edges having even degree. Consider the dual matroid M of C. For each edge e of C, take the column vector $\boldsymbol{e} \in \mathbb{F}_2^F$ where each face $f \in F$ in C corresponds to a row of $\boldsymbol{e}$, and there is a 1 in that row if e is incident to f, and 0 otherwise. These binary column vectors make up a generating set for the cycle space of the matroid M. Each of the vectors $\boldsymbol{e}$ will have even length since each edge in C has even degree.

Any cycle of M can be written as a sum of the elements in the cycle space. So because we have a generating set for the cycles with all elements having even length, this means that any cycle of M will have even length. To see this, consider the sum two binary vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. The length of $\boldsymbol{v} + \boldsymbol{w}$ is just the length of $\boldsymbol{v}$ plus the length of $\boldsymbol{w}$ minus 2 times the number of coordinates where $\boldsymbol{v}$ and $\boldsymbol{w}$ both equal 1.

Now we have that all cycles in M have even length, which means it is bipartite. □

Definition 3.8. The *effective length* of a cycle o in a weighted digraph D is the sum over the weights of all edges of o oriented in one direction, minus the sum over the weights of the edges of o oriented in the opposite direction.

Definition 3.9. Given an abelian group Γ, a Γ-*co-flow* is an assignment of elements of

Γ (referred to as weights) to the edges of a directed graph D such that every cycle o of D has effective length 0.

Lemma 3.10. *Given an abelian group Γ and a graph G, there is a colouring of the vertices of G with the elements of Γ such that adjacent vertices receive different colours if and only if there is a nowhere zero Γ-co-flow on the edges.*

Proof. This follows from Section 6.3 in [15]. $\square$

Corollary 3.11. *A weighted digraph with each cycle having effective length 0 mod k, where the label on each edge is $d \in \mathbb{Z}_k \backslash [0]_k$, is k-vertex-colourable.*

Proof. This is a special case of Lemma 3.10. $\square$

Lemma 3.12. *Let C be a simplicial 2-complex and $SL(C)$ the spatial line graph of C. If there exists a k-vertex-colouring of $SL(C)$ then a k-edge-colouring of C exists.*

Proof. Let C be a simplicial 2-complex with edge set E and set of faces F. Let $SL(C)$ be the spatial line graph of C, with vertex set V_L and edge set E_L.

Suppose there exists a k-vertex-colouring of $SL(C)$, a labelling $c : V_L \to X$ such that $|X| = k$ and for $x, y \in V_L$ we have that $c(x) \neq c(y)$ whenever $\{x, y\} \in E_L$. So no two adjacent vertices in $SL(C)$ share the same label. By the definition of a spatial line graph, two vertices in $SL(C)$ are adjacent if their corresponding edges in C are incident to a common face.

We define an edge-colouring of C as follows: $c' : E \to X'$ where $c'(x) = c(x)$ for $x \in E$. First notice that if $x \in E$ then $x \in V_L$ since the vertex set of $SL(C)$ is precisely the edge set of C by definition.

Consider $y, z \in E = V_L$ with $\{y, z\} \subseteq f$ for $f \in F$. So y and z are edges of C incident to a common face, by the definition their corresponding vertices in $SL(C)$ are adjacent. In other words $\{y, z\} \in E_L$. Hence $c'(y) = c(y) \neq c(z) = c'(z)$ whenever $\{y, z\} \subseteq f$ for some $f \in F$. So c' is a consistent edge-colouring of C.

Moreover, let $c'(x) \in X'$, then $c'(x) = c(x) \in X$ so we have that $X' \subseteq X$ and $|X'| \leqslant |X| = k$. Hence a k-edge-colouring of C exists. $\square$

Theorem 3.13. *A triangulation of 3-space can be edge-coloured with three colours if and only if all edges have even degree.*

Proof. First we show that a triangulation of 3-space that can be edge-coloured with three colours has the property that all edges have even degree.

Let C be a simplicial 2-complex embedded in 3-space with every chamber a tetrahedron and can be coloured using 3 colours. Suppose for contradiction that C has an edge e of odd degree.

Consider the link graph at v, one of the end vertices of e, the vertex that corresponds to e in this graph has odd degree. Since C is a triangulation of 3-space we must have that all of the link graphs are triangulations of the plane. Then by Heawood's theorem the link graph at v cannot be 3-colourable as it has a vertex of odd degree.

Now we prove the opposite direction; a triangulation of 3-space where all edges have even degree can be edge-coloured with three colours. Let C be a simplicial 2-complex embedded in 3-space with every chamber a tetrahedron and every edge even degree.

Sublemma 3.14. *There exists an orientation of all the faces of C such that the four faces of each tetrahedron form a consistent orientation of that tetrahedron.*

Proof. There are two consistent orientations for the faces in tetrahedrons, the left handed and right handed orientations.

Because the dual graph is bipartite by Lemma 3.7, the tetrahedra in C can be split into two parts such that if two tetrahedra are in the same set then they do not share a face. We can give all of the tetrahedra in one of the sets the left handed orientation and give the other set the right handed orientation.

When two tetrahedra meet at a face the two orientations will be mirrors of each other and so the orientation on that face will be the same. So we have an orientation for all of the faces in C that are on tetrahedrons which form consistent orientations on the tetrahedrons. □

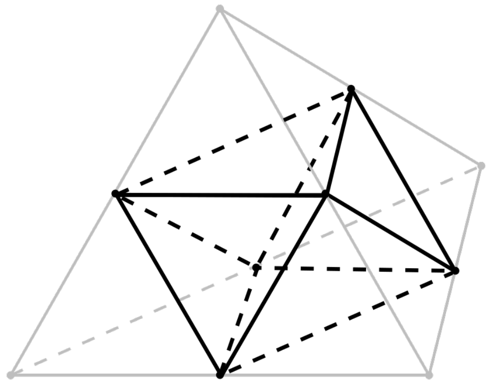


Figure 3.4: The octahedron (black) in a spatial line graph that is where the tetrahedrons of the original simplicial 2-complex (grey) were.

Sublemma 3.15. *There exists an orientation of $DL(C)$ such that every cycle has effective length 0 mod 3.*

Proof. Give the edges of $DL(C)$ the same orientation that the corresponding face has in C. That is, for a face in C with edges x, y, z and orientation xyz the corresponding edges in $DL(C)$ will have directions xy, yz, and zx.

It is trivial to see that the tetrahedron cycles of $DL(C)$ all have effective length 0 mod 3. To show that the other cycles in $DL(C)$ have effective length 0 mod 3 we need to build a simplicial 2-complex from $DL(C)$ such that all its faces are bounded by cycles of effective length 0 mod 3 and show it is simply connected. For a simply connected simplicial 2-complex, the face cycles generate all of the other cycles so our tetrahedron cycles having effective length 0 mod 3 implies that all cycles will have effective length 0 mod 3.

We build the simplicial complex D from $DL(C)$ by adding a face at every tetrahedron cycle.

In [5] it was proved that if C is a locally connected simplicial complex embedded in $\mathbb{S}^3$, then C being simply connected is equivalent to all the local surfaces being bounded by spheres. So it is enough to show that all local surfaces of D are bounded by spheres.

In D we have octahedrons where the tetrahedrons in $DL(C)$ were, see Figure 3.4, which are bounded by tetrahedrons of $DL(C)$ and so these are bounded by spheres. The only other local surfaces are the voids where the vertices of the tetrahedrons in $DL(C)$ were. These are bounded by the new faces we added and so are also bounded by spheres and we are done. □

Now we have that $DL(C)$ is a digraph with every cycle having effective length 0 mod 3. So by Corollary 3.11 $DL(C)$ is 3-colourable.

Then by Lemma 3.12 there exists a 3-colouring of C. □

Proposition 3.16. *The vertices of a triangulation of 3-space are 4-colourable iff the edges are 3-colourable.*

Proof. Take a triangulation of 3-space with a colouring on the vertices using 4 colours. Label the 4 colours with the vectors $\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}$ from the abelian group $\mathbb{F}_2 \times \mathbb{F}_2$. Then for each edge, the new colour would be the sum of the two colours on the end vertices, over $\mathbb{F}_2 \times \mathbb{F}_2$. This will give us 3 colours as the vector $\binom{0}{0}$ cannot appear; if it did it would mean that the two end vertices have the same colour which is not possible under a proper colouring. Also, it is easy to see that if the three vertices on a triangle have different colours then the edges will also get different colours, so this 3-colouring on the edges is proper.

Take a triangulation of 3-space with a 3-colouring on the edges. Replace the 3 colours on the edges with the vectors $\binom{0}{1}, \binom{1}{0}, \binom{1}{1}$. For any triangle we will have exactly one copy of all three vectors present so the sum of the vectors on the triangle cycle is zero. Since the triangulation is simply connected, we have that all of the cycles are generated by the face cycles, i.e. the triangles. So the vectors on every cycle sums to zero. Now we have shown that this is a nowhere zero co-flow, and we can use Lemma 3.10 to show that we have a 4-colouring on the vertices. □

We can now restate our theorem as follows.

Corollary 3.17. *For a triangulation of 3-space, the vertices are 4-colourable iff all edges have even degree.*

Proof. Combine Proposition 3.16 and Theorem 3.13. □

3.2 Concluding Remarks

Note that our proof actually proves the stronger statement with the 3-sphere replaced by a general homology sphere. However the generalisation of our theorem to any 3-manifold is false. To see this, consider the following example.

Example 3.18. Take a triangulation of 3-space with all edges even degree. All such triangulations have a unique 3-colouring up to permutation of colours. Take two tetrahedron that do not intersect with any common faces, edges, or vertices. Remove their interiors from the manifold and identify their boundaries in such a way that the colouring isn't compatible. Now we have a triangulation of a 3-manifold where all edges have even degree but isn't 3-colourable.

In the 70s a paper was published [19] that claims to prove Theorem 3.13. In their proof it is claimed that in a triangulation of a simply connected space every loop of 3-simplices is a sum of simple loops[5]. However, consider the loop of 3-simplices, as in Figure 3.5, obtained by gluing together 3-simplices in a linear way along faces and then identifying a vertex of the first 3-simplex with a vertex of the last. This is a loop of 3-simplices that could occur in a such a triangulation, but it is not clear how this is always a sum of simple loops or how one could extend a colouring of a simple loop to a colouring of this loop.

The d-dimensional version of Theorem 3.13 is open. The statement is as follows.

Conjecture 3.19. *Let C be a triangulation of $\mathbb{S}^d$. Then the 1-skeleton of C is colourable with $d+1$ colours if and only if all its $(d-2)$-faces are incident with an even number of $(d-1)$-faces.*

[5]A *simple loop* is 3-simplices around an interior edge.

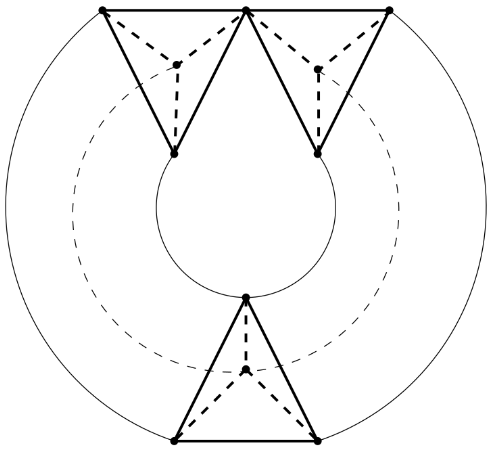


Figure 3.5: A loop of 3-simplices obtained by gluing together 3-simplices in a linear way along faces and then identifying a vertex of the first 3-simplex with a vertex of the last 3-simplex.

3.3 Further thoughts on the paper

The proof of Proposition 3.16 gives a method of constructing a 3-edge-colouring of a triangulation of 3-space given a 4-vertex colouring of the triangulation, Figure 3.6 shows an example of this.

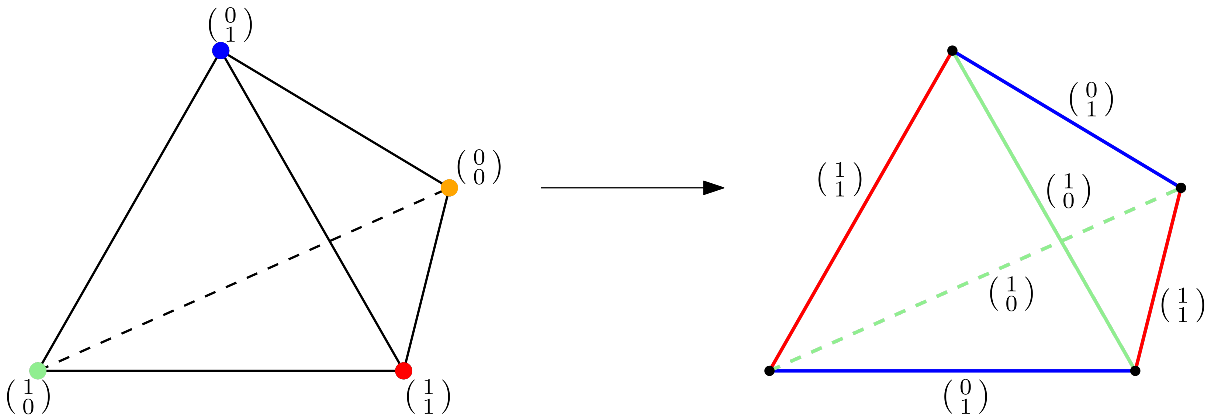


Figure 3.6: A 4-vertex colouring of a tetrahedron and its corresponding 3-edge-colouring, as in Proposition 3.16.

We will now give another counterexample for why Theorem 3.13 does not work in any 3-manifold. This example shows a triangulation of the 3-torus that has every edge of even degree, but is not 3-edge-colourable.

The motivation for this example was that Heawood's original theorem on the 3-colourability of maximal plane triangulations is not true on the torus. To see this, consider

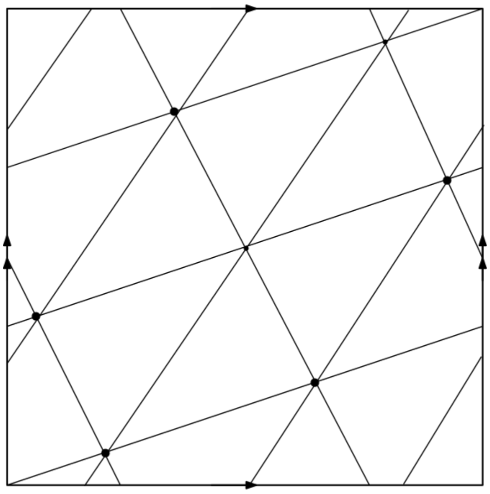
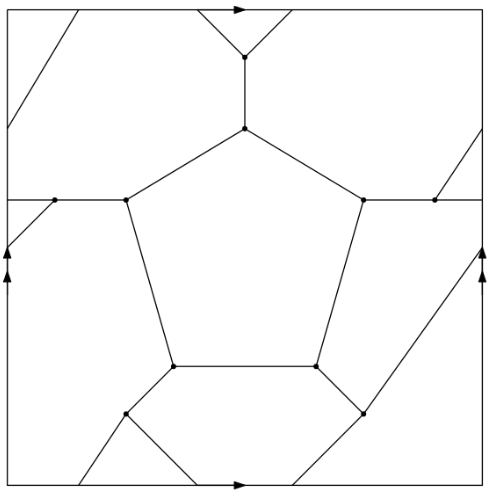


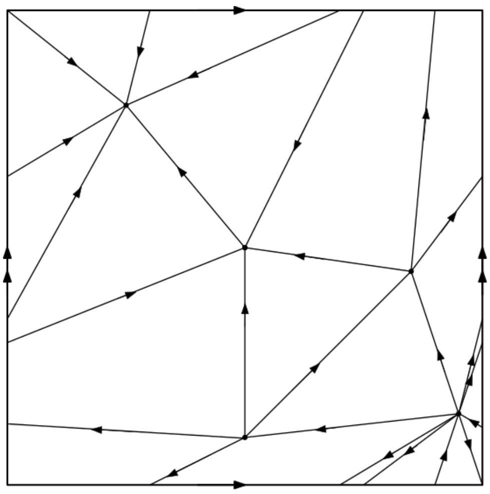
Figure 3.7: K_7 embedded on the torus.

K_7 embedded on the torus, as in Figure 3.7. In this graph every vertex has even degree as they all have degree 6, and the chromatic number of K_7 is 7, so obviously cannot be coloured using 3 colours.

Figure 3.8



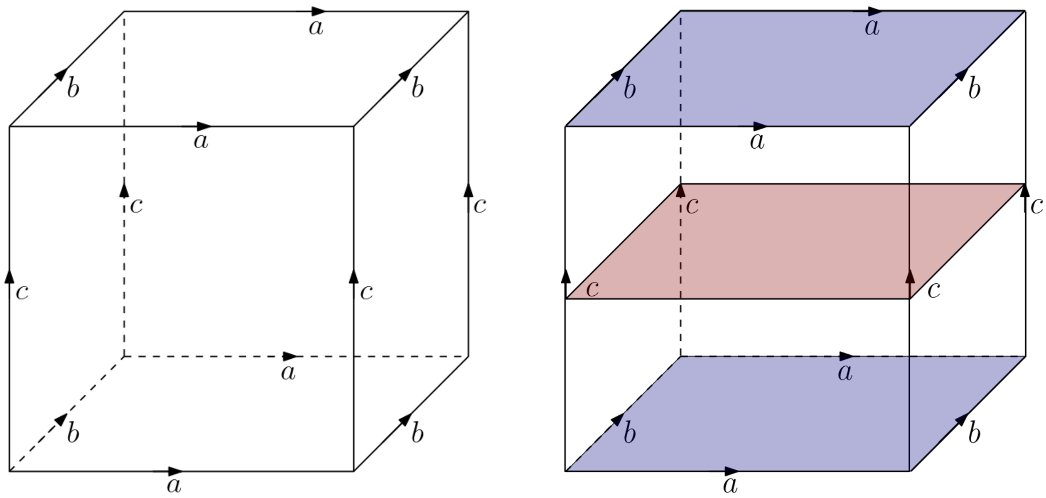
(a) The Petersen graph embedded on the torus.



(b) The dual of the Petersen graph embedded on the torus. An orientation has been given to the edges.

Example 3.20. Consider the Petersen graph embedded on the torus, as in Figure 3.8a. Now take G to be the dual of this embedding, as in Figure 3.8b. Consider the simplicial 2-complex that is a triangulation of the torus with G as its 1-skeleton, this simplicial 2-complex is not 3-edge colourable. To see this note that its spatial line graph is equivalent to the line graph of the Petersen graph.

Figure 3.9



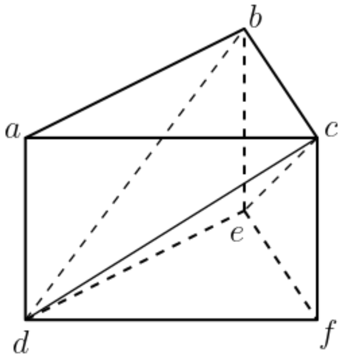
(a) A visualisation of the 3-torus as a cube.

(b) How to layer two copies of a graph inside the 3-torus

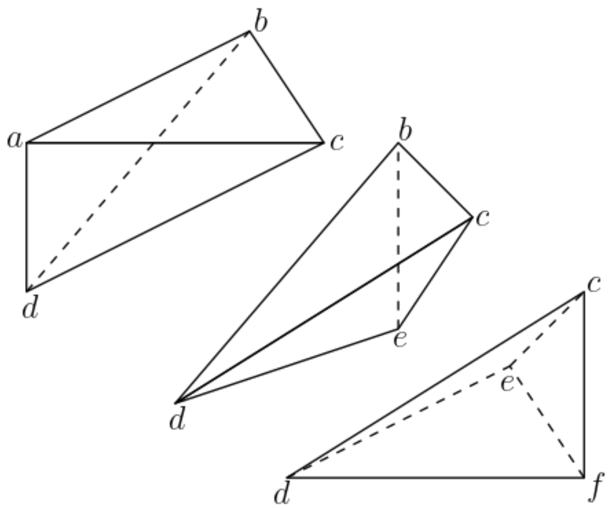
We need to build a simplicial 2-complex that is embedded into the 3-torus using this. To start, consider the 3-torus as a cube, where opposite faces are glued together, as in Figure 3.9a. On the cube of the 3-torus, embed the dual of the Petersen graph onto one of the faces, and once again as another layer inside the cube. Now we will have a cube with the dual graph on two of the outside faces which are identified, and once again as a layer in the middle, as in Figure 3.9b. To connect these faces, connect each triangle face to itself by attaching a copy of the triangulation of the triangular prism in Figure 3.10. Figure 3.8b also shows an orientation of the edges of the dual graph, each triangular face has 2 edges of one direction and one of the other. A way to place the prism triangulations in a way that is consistent is to orient them such that the diagonal edge on the outside is pointing downwards in the direction of the arrow. It's easy to see that each edge will still have even degree once we do this.

All of the edges in this simplicial 2-complex embedded in the 3-torus have even degree, but it does not have a 3-edge-colouring as the layers do not have a 3-edge-colouring.

Figure 3.10



(a) A triangulation of the triangular prism.



(b) The 3 separate tetrahedron inside the triangulation.

CHAPTER 4

THE CHROMATIC NUMBER OF EMBEDDABLE 2-COMPLEXES

Motivated by the Four Colour Theorem for planar graphs, we raise the following open question, which essentially seeks to generalise the Four Colour Theorem from two to three dimensions, using the same extension from planar graphs to 3-space as in the previous chapter. An (*edge-*)*colouring* of a 2-complex C assigns to every edge of C a colour such that two edges e and e' receive different colours whenever e and e' share an endvertex v and the boundary of some face of C enters and leaves v through e and e', respectively.

Open Question 4.1. *Let M be a 3-manifold. What is the least integer k such that every simplicial 2-complex that embeds in M is k-colourable?*

In this chapter, we will consider bounds for this question, and then discuss what the answer is when we remove *simplicial* from the question. That latter part of this chapter is based on [24].

4.1 Simplicial 2-complexes

In this section, we give an upperbound for k in Open Question 4.1.

Theorem 4.2. *Every simplicial 2-complex that embeds in a 3-manifold is 12-colourable.*

The proof of Theorem 4.2 given here is by induction on the number of edges of the simplicial 2-complex. This is motivated by a similar proof of the six colour theorem for planar graphs which uses the fact that the average degree of a planar graph is less than 6.

The 6-colour theorem is proved by induction on the number of vertices of a graph G and the main idea is to find a vertex v with degree less than 6, remove it, then the resulting planar graph is 6-colourable and there is a colour not used by the neighbourhood of v that we can use to colour v and get a 6-colouring of G. However, removing an edge from a simplicial 2-complex C does not necessarily result in a simplicial 2-complex with the same edge neighbourhoods. If we remove e and its incident faces to get the 2-complex C', this means that for every incident face the other two edges are no longer adjacent in C'.

To get around this problem, we introduce the notion of partially colouring the edge set of a simplicial complex.

Definition 4.3. Let C be a simplicial 2-complex. A *partial colouring* of an edge subset $D \in E(C)$ is a labelling of the edges in D with colours such that any two edges that are incident with a common face get different colours.

Proof of Theorem 4.2. Let C be a simplicial 2-complex that has e edges, v vertices and f faces. Let E be the set of all edges. For a partial colouring, take a subset $X \subseteq E$. If $|X| \leqslant 12$ then we obviously have a partial colouring of X using at most 12 colours.

For induction, assume that for every subset of size $n-1$ we can find a partial colouring using 12 or less colours. Now, let the subset $X \subseteq E$ have size n. For each $e \in X$, we have two endvertices v_1 and v_2. Consider $L(v_1)[X]$ and $L(v_2)[X]$, the induced subgraphs of the link graphs at v_1 and v_2 restricted to the vertices in $X \cap V(L(v_i))$.

Label e with the sum of the degrees of the vertex e in $L(v_1)[X]$ and $L(v_2)[X]$: this is the number of edges in X that e is adjacent to in a face of C. Since C is embeddable, every link graph is planar. The label on each edge of C is the sum of the degrees of two vertices from planar graphs, and every planar graph has average degree less than 6, so the average of the labels on the edges is less than 12 and we must have an edge $e_1 \in X$ that is labelled with a number less than 12. Remove this edge from X, then we have a subset with $n-1$ edges and we can find a partial colouring using 12 or less colours by the inductive hypothesis. Now, add e_1 back into X and since there are less than 12 edges that share a face with e_1 in X there is a colour leftover for us to use for e_1 to get a partial colouring of X using 12 colours. $\square$

We can not use the ideas behind the six colour theorem proof to prove 11-colourability of simplicial 2-complexes because there is no guarantee that the edge corresponding to the vertex with degree at most 5 in $L(v)$ has degree at most 5 in the link graph at its other end vertex.

This gives an upper bound of 12 for k in Open Question 4.1. This bound is not necessarily optimal; if we could find a simplicial 2-complex that has chromatic number 12, then it would be. So we also need to approach this question from below and find simplicial 2-complexes that require more colours for a proper edge-colouring.

First, take the cone over a 4-chromatic planar graph, this gives a 4-chromatic simplicial 2-complex, so 4 colours are necessary to colour every simplicial 2-complex. The next logical step is to find a simplicial 2-complex that needs 5 colours. The idea behind the following example is that K_5 can be embedded on the torus, as in Figure 4.1, so it is possible to build a simplicial 2-complex with K_5 as its 1-skeleton that can embed in 3-space. Then, since $\chi(K_5) = 5$, the simplicial 2-complex will also be 5-chromatic.

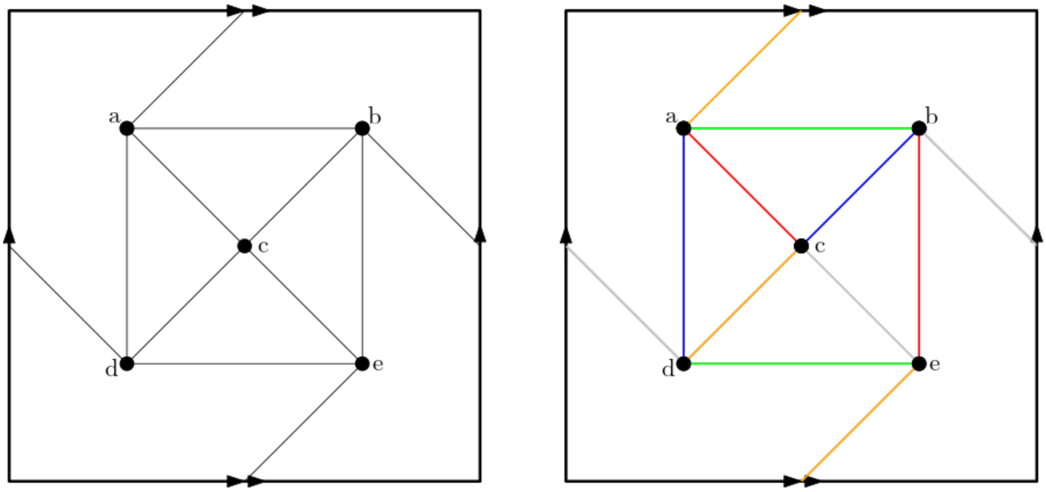
Example 4.4. Let C be a simplicial 2-complex with K_5 as its 1-skeleton and every possible 3-cycle as its faces. Using the vertex labelling as in Figure 4.1 the sets of the vertices V edges E and faces F of C are as follows.

$$V = \{a, b, c, d, e\}$$

$$E = \{ab, ac, ad, ae, bc, bd, be, cd, ce, de\}$$

$$F = \{abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde\}$$

Colour C in the same way as the 5-colouring of K_5 as shown in Figure 4.1. The colour classes are $\{ab, de\}$, $\{ac, be\}$, $\{ad, bc\}$, $\{ae, cd\}$, and $\{bd, ce\}$.

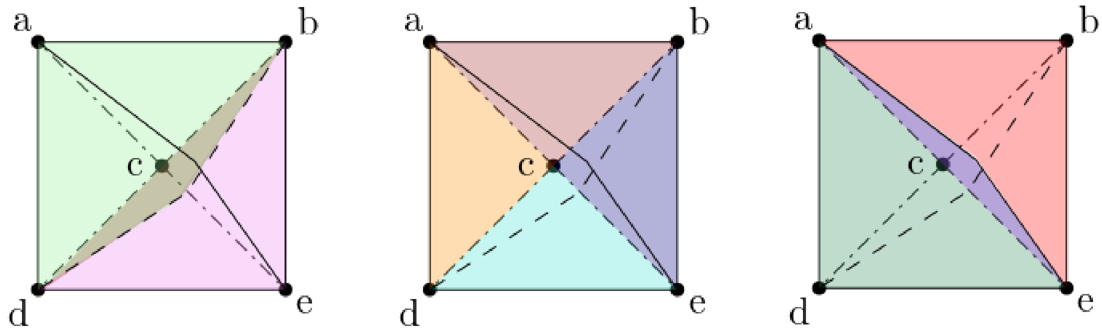


(a) K_5 embedded on the torus. (b) An edge-colouring of K_5.

Figure 4.1

To see that C is 5-chromatic, first assume for contradiction that C can be coloured with 4 or fewer colours. There will be a colour class with 3 or more edges in it, 2 of these edges must have a common endvertex as C only has 5 vertices. Since every possible 3-cycle is a face of C, then any two edges with a common endvertex are incident with a common face and therefore cannot be in the same colour class. So C has chromatic number 5.

To see that C is embedabble in 3-space, consider the layers of faces in Figure 4.2. Visualise C with the edge bd running below the complex and the edge ae above the complex. The figure then shows how the faces are embedded.



(a) (b) (c)

Figure 4.2: The faces of the simplicial 2-complex C embedded in 3-space. These three layers can be arranged as follows: put the three faces from (a) on the top, the four faces from (b) in the middle, and the three faces from (c) on the bottom.

4.2 On the edge-chromatic number of 2-complexes

In this section, we show that the answer is '$k = 12$' for every 3-manifold M if *simplicial* is dropped from Open Question 4.1:

Theorem 4.5.

(1) *Every 2-complex that embeds in a 3-manifold is 12-colourable.*

(2) *There is a 2-complex that embeds in $\mathbb{S}^3$ and which is not 11-colourable.*

This section is part of a project that aims to extend planar graph theory to three dimensions. Previously, the following results have been extended:

	2D	*3D*
• Kuratowski's theorem	[23, 15]	[10]
• Excluded-minors characterisation of outerplanar graphs	folklore	[12]
• Heawood's theorem on the colourability of plane triangulations	[22]	[13]
• Whitney's theorem on unique embeddings of 3-connected graphs	[28, 15]	[18]

4.2.1 Terminology

We use the terminology of [15]. In this note, graphs may have loops and parallel edges.

4.2.1.1 1-complexes

Let G be a graph with vertex-set V and edge-set E. We can obtain a topological space from G, called the *1-complex* of G and also denoted by G, as follows. The *0-skeleton* of G is V equipped with the discrete topology. For every edge $e \in E$, let $[0,1]_e$ be a copy of the unit interval, disjoint from V and from all other copies $[0,1]_{e'}$. Furthermore, arbitrarily fix a map $\varphi_e \colon \{0_e, 1_e\} \to V$ such that the image of φ_e is equal to the set of ends of e (so there are two choices for φ_e if e is not a loop, and only one choice if e is a loop). The *1-complex* of G is obtained from the 0-skeleton of G by adding all copies $[0,1]_e$ for all edges $e \in E$ and identifying 0_e and 1_e with their images under φ_e. Note that taking the quotient as above also defines a topology on the 1-complex. For convenience, we now change the notation $[0,1]_e$ to refer to $[0,1]_e$ after taking the quotient as above, so that we have $[0,1]_e \subseteq G$. We then call $[0,1]_e$ a *topological edge* of the 1-complex G, and write e for $[0,1]_e$ when there is no danger of confusion. The *third-edges* of G are the closed intervals $[0, \frac{1}{3}]_e$ and $[\frac{2}{3}, 1]_e$ of the topological edges $[0,1]_e$, where e ranges over all edges of the graph G.

4.2.1.2 2-complexes

A *2-complex* C is a topological space obtained from a 1-complex G by disjointly adding closed 2-dimensional discs D_i ($i \in I$), fixing a continuous *gluing map* $\varphi_i \colon \partial D_i \to G$ for each i, and identifying x with $\varphi_i(x)$ for all i and $x \in \partial D_i$. In this note, we will only need to consider 2-complexes whose gluing maps φ_i follow closed walks in G at constant nonzero speed. This will allow us to also view the gluing maps φ_i from a combinatorial perspective, through the closed walks they *correspond* to. The subspaces of C obtained from the discs D_i by gluing their boundaries to the 1-skeleton are the *2-cells* of C. The vertices and edges of C are the vertices and edges of its 1-skeleton. A 2-complex C is said to be *simplicial* if G is simple and each gluing map follows a closed walk that goes once around a triangle.

1-complexes and 2-complexes are instances of the more general cell complexes, colorredsee Hatcher's book on algebraic topology [20] for more information on cell complexes.

4.2.1.3 Link graphs

Let C be a 2-complex with 1-skeleton G and gluing maps φ_i $(i \in I)$ for its 2-cells. The *link graph* of C, which we denote by $L(C)$, is defined as follows. The vertices of $L(C)$ are the third-edges of G. For each i, we follow φ_i along the circle that is its domain (the direction does not matter), and we add an edge between two vertices I and J in $L(C)$ whenever I and J share a vertex v in G and φ_i first traverses I to reach v and then traverses J next (or vice versa). Hence the link graph $L(C)$ may contain parallel edges and loops, even if C only has one 2-cell.

A *pairing* of a set S, with $|S|$ even, is partition of S into classes of size two. A *paired graph* is a pair of a graph G and a pairing π of its vertex set. Every link graph has a *default pairing* in which every two third-edges that are included in the same topological edge form a class. When we view a link graph as a paired graph, we always use the default pairing.

4.2.1.4 Colourings of paired graphs and 2-complexes

A *pair-colouring* of a paired graph (G, π) is a colouring c of the pairs in π such that $c(p) \neq c(q)$ whenever G contains an edge joining a vertex in p to a vertex in q. The terms *pair-chromatic number* and *k-pair-colourable* are defined as expected.

An *(edge-)colouring* of a 2-complex C is a colouring c of the edges e of C such that c induces a pair-colouring of the link graph of C (which colours every vertex $I \subseteq [0,1]_e$ of $L(C)$ with the colour $c(e)$). The terms *edge-chromatic number* and *k-edge-colourable* are also defined as expected.

4.2.2 Proof of (1)

A *2-pire map* is a paired graph (G, π) where G is planar. We sometimes call G a 2-pire map when π is clear from context, or say that G is a 2-pire map with pairing π. Isomorphisms between 2-pire maps are required to respect their pairings. The name and definition are directly motivated by Heawood's *m-pire problem* [21], which was surveyed in [17].

Example 4.6. Since every 2-complex that embeds in a 3-manifold has planar link graphs, the link graphs of such 2-complexes, equipped with their default pairings, are examples of 2-pire maps.

Lemma 4.7 (Heawood [21]; surveyed in [17])**.** *Every 2-pire map is 12-pair-colourable.*

The *paired quotient* of a paired graph (G, π) is the graph G/π obtained from G by identifying every two vertices that are paired by π, keeping all edges.

Lemma 4.8. *Let C be a 2-complex, and let π denote the default pairing of the link graph $L(C)$. The following numbers are equal:*

(i) *the edge-chromatic number of the 2-complex C,*

(ii) *the pair-chromatic number of the link graph $L(C)$, and*

(iii) *the vertex-chromatic number of the paired quotient $L(C)/\pi$.* □

Proof of Theorem 4.5 (1)*.* We combine Example 4.6 with Lemma 4.7 and Lemma 4.8. □

4.2.3 Proof of (2)

Lemma 4.9. *[21] There exists a 2-pire map whose pair-chromatic number is equal to 12.*

Proof. For convenience, we have included Figure 4.3, which shows a 2-pire map whose paired quotient is a K_{12}, and whose pair-chromatic number is equal to 12 by Lemma 4.8. □

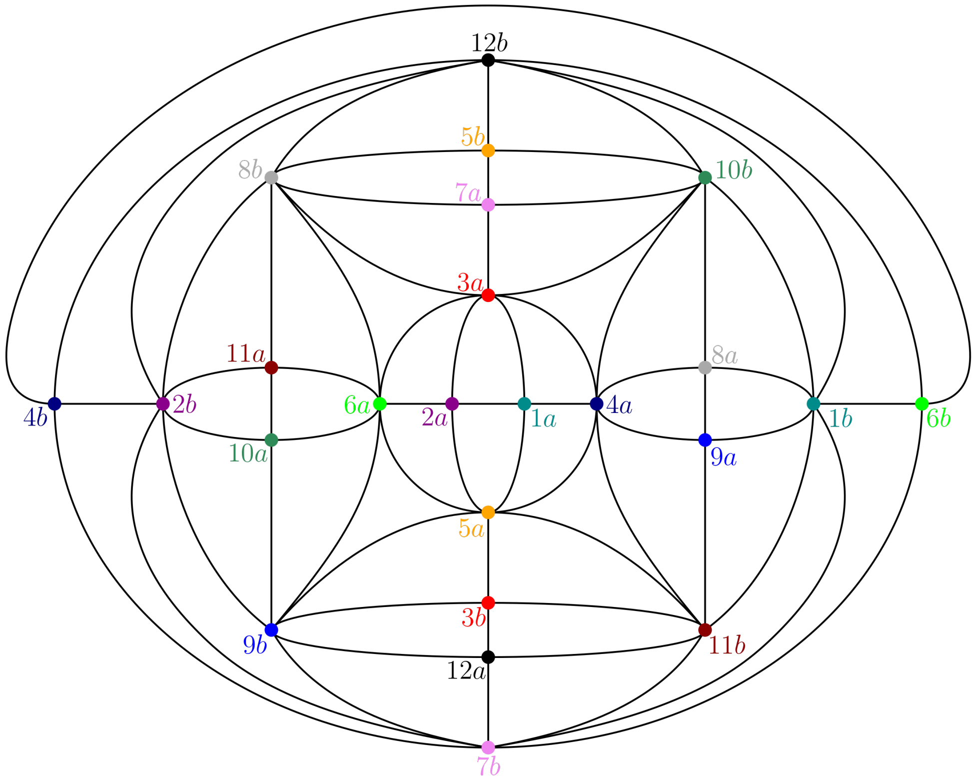


Figure 4.3: A 2-pire map whose pair-chromatic number is equal to 12. This was found by Kim Scott and published (in slightly different form) in [17]

To find a 2-complex C that is not 11-colourable but embeds in $\mathbb{S}^3$, it suffices by Lemma 4.8 to construct C so that its link graph with the default pairing contains a spanning copy of the 2-pire map as provided by Lemma 4.9 while simultaneously making sure that C embeds in $\mathbb{S}^3$. In the following, we will offer a 2-step construction that achieves just that. The first step will be Lemma 4.10 below.

Recall that the degree of a vertex v in a graph G is the number of edges of G that are incident with v, counting loops twice. The pairing π of a paired graph (G, π) is *degree-faithful* if every two paired vertices have the same degree in G.

We define *punctured 2-complexes* as follows. Let G be a 1-complex, and let F_i $(i \in I)$ be pairwise disjoint copies of the closed strip $\mathbb{S}^1 \times [0,1]$. For each i, let F_i^* denote the subspace of F_i that corresponds to the circle $\mathbb{S}^1 \times \{0\}$, and fix a continuous *gluing map* $\varphi_i \colon F_i^* \to G$. As for 2-complexes, we require the maps φ_i to follow closed walks in G at constant speed. The topological space D obtained from G and the closed strips F_i

by identifying x with $\varphi_i(x)$ for all $i \in I$ and $x \in F_i^*$ is a *punctured 2-complex*. The name is motivated by the fact that every punctured 2-complex can be obtained from a genuine 2-complex by 'puncturing' every 2-cell. The *link graph* of a punctured 2-complex is analogous to that of the link graph of a genuine 2-complex. In fact, the link graph of a 2-complex is invariant under 'puncturing'. The subspaces of D obtained from the closed strips F_i by gluing F_i^* to G are the *punctured 2-cells* of D.

The following definition is a variation of a similar definition in [18]. Let X be a set of points in $\mathbb{S}^3$. The *shadow* of X is the set of all points in $\mathbb{S}^3$ that lie on a straight line segment between the origin and some point in X.

Lemma 4.10. *For every 2-pire map G with a degree-faithful pairing π there exists a punctured 2-complex C such that the link-graph of C with the default pairing is isomorphic to (G, π) and C embeds in $\mathbb{S}^3$.*

Proof. Let B_r denote the closed ball of radius r around the origin in $\mathbb{S}^3$. Since G is planar, there is an embedding α of G (viewed as a 1-complex) in the boundary of the unit ball B_1.

Next, we construct a graph H together with an embedding β of H in $\mathbb{S}^3$, as follows. The graph H has only one vertex h, which β maps to the origin. For every pair $p = \{u, v\} \in \pi$, the pairing of the 2-pire map G, we add a loop e_p to H with end h and let β map the interior of e_p into $\mathbb{S}^3 - h$ so that the intersection of $\beta(e_p)$ with B_1 is equal to the shadow of $\{\alpha(u), \alpha(v)\}$. It is not hard to make sure that the images of distinct loops under β do not intersect except in h, for example as follows. We enumerate the pairs in π as $p_1, \ldots, p_n$. Then we let β map e_p for $p = p_i = \{u, v\}$ to the union of the following three subspaces of $\mathbb{S}^3$: the two straight line segments that link the origin to ∂B_{i+1} and pass through $\alpha(u)$ and $\alpha(v)$, respectively, plus one of the obvious arcs that links $\alpha(u)$ and $\alpha(v)$ in the boundary ∂B_{i+1}.

The graph H, viewed as a 1-complex, will be the 1-skeleton of the punctured 2-complex C, which we construct next. For every vertex v of G, let S_v denote the image under α of the union of all third-edges of G that contain v. Note that the subspaces S_v are pairwise disjoint, and that S_u is homeomorphic to S_v by a homeomorphism mapping

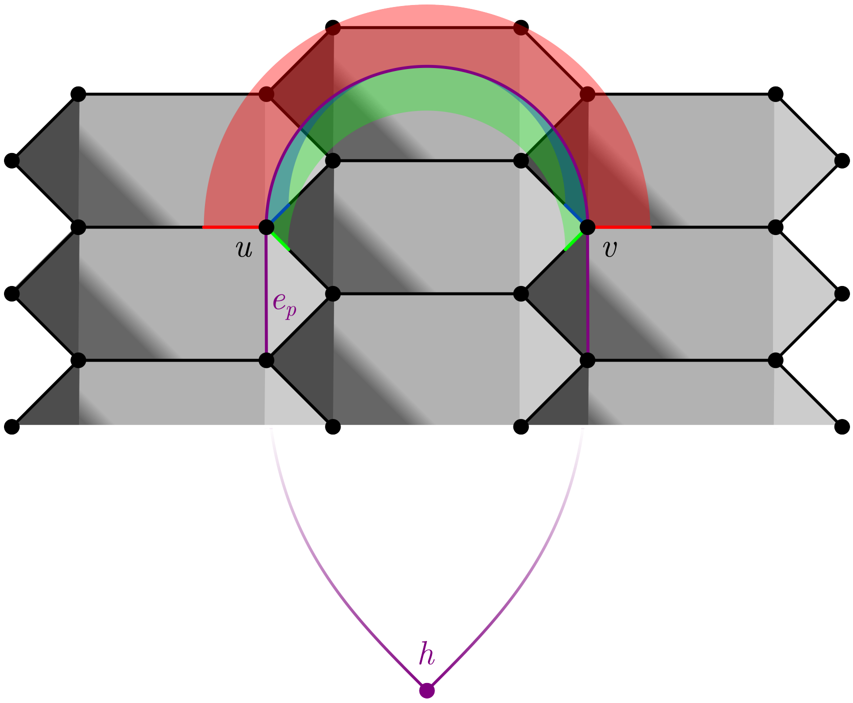


Figure 4.4: The image of ι_p for $p = \{u, v\}$. The black graph is $\alpha(G)$. The shadow of $\alpha(G)$ is indicated by the grey shaded areas.

$\alpha(u)$ to $\alpha(v)$ for all pairs $\{u, v\} = p$ since π is degree-faithful. For each pair $p = \{u, v\}$ in π, we informally link up S_u and S_v in $\mathbb{S}^3$ minus the interior of B_1 by embedding the space $S_u \times [0, 1] \cong S_v \times [0, 1]$ so that this follows the topological path $\beta(e_p)$, as shown in Figure 4.4. More precisely, we find an embedding ι_p of $S_u \times [0, 1]$ in $\mathbb{S}^3$ such that

- ι_p maps $S_u \times \{0\}$ to S_u and $S_u \times \{1\}$ to S_v;
- ι_p maps $\alpha(u) \times [0, 1]$ to $\beta(e_p) \setminus \mathring{B}_1$; and
- the image of ι_p avoids $B_1 \backslash (S_u \cup S_v)$.

We can greedily find the embeddings ι_p for all pairs $p \in \pi$ so that their images are pairwise disjoint: For example, if we construct β using the balls of distinct radii as outlined above, we could even write down an explicit description of ι_p, which we do not as it would be tremendously tedious, but it is possible.

Let C be the topological space obtained from the shadow of $\alpha(G)$ by adding the images of the embeddings ι_p for all pairs $p \in \pi$. The construction of C ensures that all connected components of $C\backslash\beta(H)$ are homeomorphic to $\mathbb{S}^1 \times (0, 1]$. Hence C is a punctured 2-complex with 1-skeleton $\beta(H)$. By construction, the link graph of C with its default pairing is homeomorphic to G with the pairing π. □

Proof of Theorem 4.5 item (2). By Lemma 4.9, there exists a 2-pire map G with pairing π such that the pair-chromatic number of G with regard to π is equal to 12. For every edge of G we add an edge in parallel, to make sure that all vertices of G have even degree, which then allows us to add loops to G so that π becomes degree-faithful.

By Lemma 4.10, there exists a punctured 2-complex C as a subspace of $\mathbb{S}^3$ such that the link-graph of C with the default pairing is isomorphic to G with the pairing π. Let $F_1, \dots, F_n$ be the punctured 2-cells of C and let $\varphi_1, \dots, \varphi_n$ be the corresponding gluing maps.

For each $i = 1, \dots, n$ we do the following. Let W_i denote the closed walk in H that φ_i traverses at constant speed. Let U_i denote the smallest initial segment of W_i that uses an edge. We define W_i' to be the closed walk $W_i' = W_i U_i U_i^- W_i^-$, where W^- denotes the reverse of a walk W and writing the walks in sequence means concatenation.

Figure 4.5: The replacement step

We obtain the 2-complex C' from C by replacing each punctured 2-cell F_i with a genuine 2-cell F_i' whose boundary we glue along W_i'. By following W_i' and working in close proximity to the punctured 2-cell $F_i \subseteq \mathbb{S}^3$, we can embed the interiors of the F_i' in $\mathbb{S}^3$ as depicted in Figure 4.5 so that we obtain an embedding of C' in $\mathbb{S}^3$. □

4.3 Further thoughts on the paper

In the paper, we defined 1- and 2-complexes as topological spaces, and showed how to obtain a 1-complex from a graph. We can also recover the underlying graph from a 1-complex G in the obvious way: let the set of vertices V be the 0-skeleton of G, and let two vertices v, v' be connected by an edge if there is a topological edge of G with v and v' at its ends. Defining 2-complexes in this way allows us to define the punctured 2-complexes we needed to help construct the embeddings.

Lemma 4.7 is a direct consequence of Heawood's proof of the $6m$-colourability of m-pire maps, which uses the same logic as the 6-colour theorem (as planar graphs are just 1-pire maps) and Theorem 4.2.

Also note that the paired quotient $L(C)/\pi$ is equivalent to the spatial line graph $SL(C)$ from Section 3.1.

CHAPTER 5

FEEBLE SEPARATORS OF NON-TOUGH GRAPHS

Trying to determine whether a graph is Hamiltonian is NP-hard. In 1973, Chvátal observed that all Hamiltonian graphs are 1-tough[1][14], however his conjecture that this connection goes both ways was disproved in 2000 by Bauer, Broersma and Veldman [2]. Although, studying the class of graphs that are not 1-tough could still give some insight into the properties of Hamiltonian graphs.

By definition, graphs that are not 1-tough must have a separator that breaks the graph into more connected components than there are vertices in the separator. Such a separator that witnesses the non-toughness of a graph is called *feeble*.

In this chapter, we will look at graphs that are not 1-tough, in particular we discuss the set of feeble separators of a graph that have minimal size.

5.1 Tools and Terminology

Definition 5.1. A separator $S \subseteq V(G)$ is a *feeble separator* if $G \setminus S$ has more than $|S|$ connected components.

If a graph G has a feeble separator then, by definition, it is not t-tough for any $t \geqslant 1$. We will call such graphs *non-tough*.

[1]See Definition 2.1 for the definition of a t-tough graph.

Example 5.2. For any star S_n, $n \geqslant 2$, the central vertex with degree equal to n is a feeble separator, as in Figure 5.1.

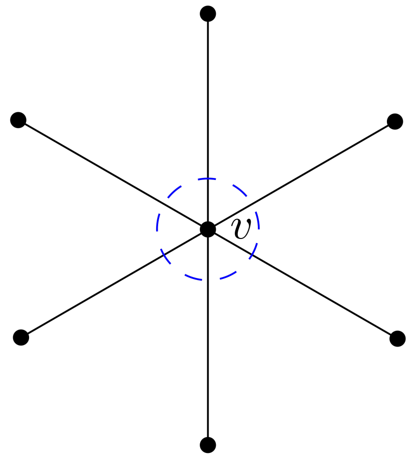


Figure 5.1: The central vertex v of the star S_6 is a feeble separator, $S_6 \backslash v$ has 6 connected components.

Definition 5.3. For two separators S_1 and S_2, a connected component A of $G\backslash S_i$ is called:

- *tiny* if there exists a connected component B of $G\backslash S_{2-i}$ such that $A\backslash S_{2-i} \subseteq B$.
- *large* if there exists a tiny component T of $G\backslash S_{2-i}$ such that $T\backslash S_{2-i} \subseteq A$.
- *good* if it is also a connected component of $G\backslash S_{2-i}$.

Note that if A is good, then it is both tiny and large. In this case, it is a connected component of $G\backslash(S_1 \cup S_2)$.

The *load* of a connected component C of $G\backslash S_i$ is the number of vertices of S_{2-i} that are contained in C, i.e. the load of C is equal to $|C \cap S_{2-i}|$.

Example 5.4. Consider the graph in Figure 5.2, call it G. Let $S_1 = \{v_1, v_5, v_7\}$ (the separator in red) and $S_2 = \{v_1, v_4, v_6\}$ (the separator in blue).

The subgraph $G[v_2]$ is a connected component of both $G\backslash S_1$ and $G\backslash S_2$, so it is a good component. The graph $G\backslash S_2$ has the subgraph $G[v_3]$ as a connected component, $G[v_3]$ is also contained in a connected component of $G\backslash S_1$ so it is a tiny component. The component of $G\backslash S_1$ that $G[v_3]$ is contained in is $G[v_3, v_6]$ which is a large component.

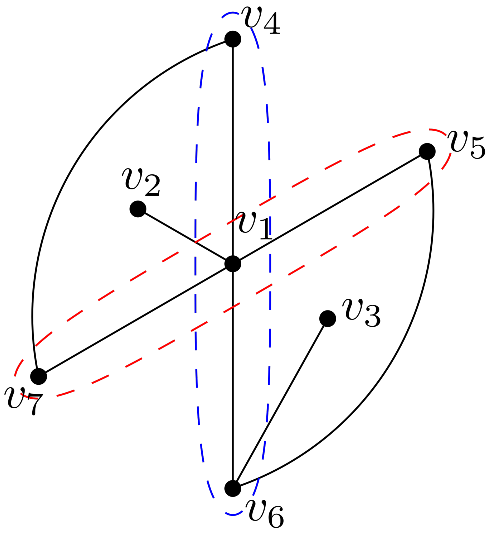


Figure 5.2: A graph with two 3-separators. The separator $S_2 = \{v_1, v_4, v_6\}$ is a feeble separator, but $S_1 = \{v_1, v_5, v_7\}$ is not feeble.

5.2 Nested Feeble Separators

One question we could ask if whether the set of minimal feeble separators of a non-tough graph G is totally nested. In other words, if two feeble separators cross[2] then they must not be of minimal size.

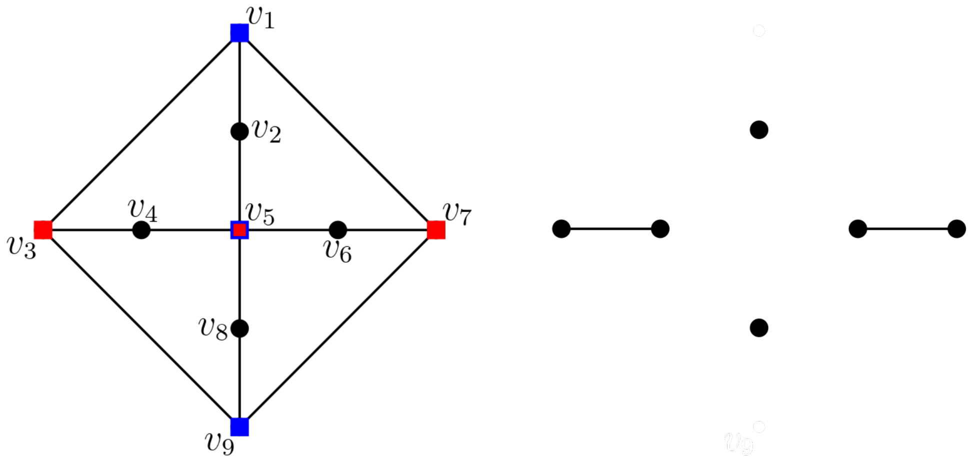
Conjecture 5.5. *Let G be a non-tough graph with two crossing feeble separators S_1, S_2 such that $|S_1| = |S_2| = s$. Then G has a feeble separator S' with $|S'| < s$.*

If this conjecture were true, this would mean that any non-tough graph G would have a totally-nested set of minimal feeble separators, this set of separators would then define a tree-decomposition of G. Unfortunately, this is not true. To see this, consider the following example.

Example 5.6. Let G be the graph in Figure 5.3, and let $S_1 = \{v_3, v_5, v_7\}$ and $S_2 = \{v_1, v_5, v_9\}$. We have $|S_1| = |S_2| = 3$, and the number of connected components in $G \backslash S_i$ is 4, for $i = 1, 2$, so the conditions in Conjecture 5.5 are satisfied.

[2]See Definition 2.3 for the definition of crossing separators.

Figure 5.3: A counterexample for the crossing lemma.

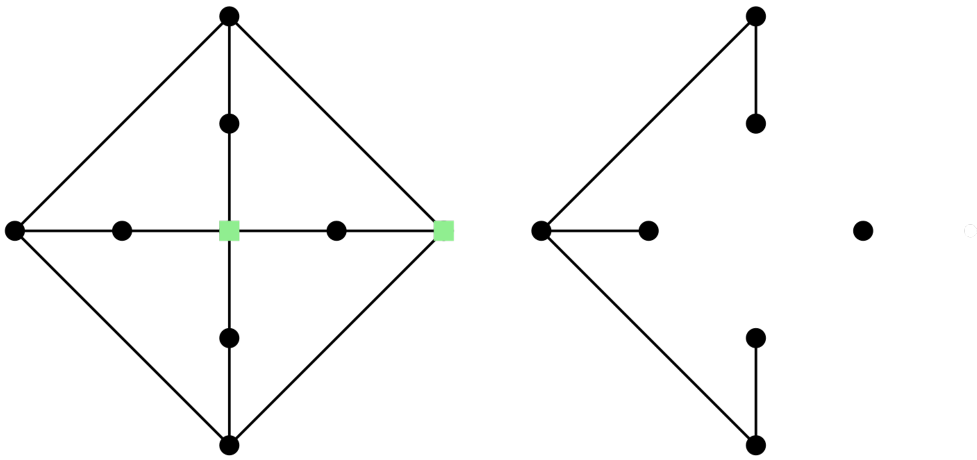


(a) The graph G with two feeble crossing separators S_1 (coloured in red) and S_2 (coloured in blue).

(b) The graph $G \backslash S_2$.

It is easy to see that the only separators of size less than 3 are those of the form $S' = \{v_5, v_j\}$, for $j \in \{1, 3, 7, 9\}$. However, the number of connected components in $G \backslash S'$ is 2, for all $j \in \{1, 3, 7, 9\}$, as seen in Figure 5.4a. So none of these separators are feeble, and there does not exist a feeble separator for G with size less than the size of the original crossing feeble separators.

Figure 5.4: A separator of size 2 in G.



(a) The graph G with S', a separator of size 2, coloured in green.

(b) The graph $G \backslash S'$.

To avoid such a counterexample, we add the condition that the intersection of any two crossing separators is empty.

Lemma 5.7 (Crossing Lemma). *Let G be a non-tough graph with two crossing feeble separators S_1, S_2 such that $|S_1| = |S_2| = s$ and $S_1 \cap S_2 = \emptyset$. Then G has a feeble separator S' with $|S'| < s$.*

Proof. Let S_1 and S_2 be two feeble separators of the graph G that cross.

Since $S_1 \cap S_2 = \emptyset$, there must be at least one connected component of $G \backslash S_i$ that intersects S_{2-i} for both $i \in \mathbb{Z}_2$. So there exists non-tiny components of $G \backslash S_i$, for all $i \in \mathbb{Z}_2$.

Assume that every connected component A of $G \backslash S_i$ contains more vertices of S_{2-i} than tiny components of $G \backslash S_{2-i}$. Every vertex of S_{2-i} is contained in some connected component of $G \backslash S_i$. So we get:

$$s \geqslant \sum_A (\text{ \#tiny components of } G \backslash S_{2-i} \text{ in } A + 1) > \text{ \#connected components of } G \backslash S_i$$

This is a contradiction to S_i being feeble. So there must be some connected component A of $G \backslash S_i$ that contains at least as many tiny components of $G \backslash S_{2-i}$ as vertices of S_{2-i}.

Since S_1 and S_2 cross the load of A must be less than s. Then let S' be the intersection of A with the feeble separator S_{2-i}. S' has size equal to the load of A. Consider the tiny components of $G \backslash S_{2-i}$ that are contained in A. No two of these can be in the same connected component of $G \backslash S'$ as they are not connected to any vertex of S_{2-i} that is not in A. So $G \backslash S'$ must have a connected component containing each of the tiny components, plus at least one more for the rest of the graph, and S' is feeble with $|S'| < s$. □

Corollary 5.8. *Let G be a non-tough graph such that the intersection of every pair of crossing feeble separators of the same size is empty. Then the set of minimal feeble separators of G is totally nested*[3].

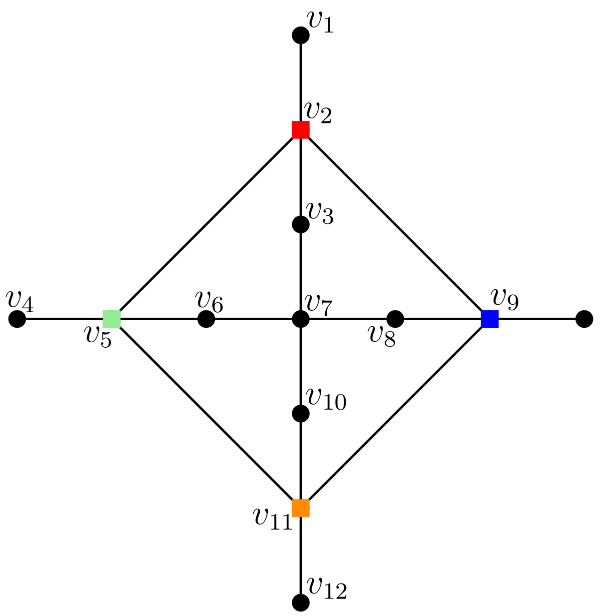
Proof. Let S_1 be a feeble separator of G with $|S| = k$ minimal. If there exists another feeble separator S_2 with $|S_2| = k$ that is crossed with S_1 then by Lemma 5.7 we can find another feeble separator S' with $|S'| < k$, contradicting the minimality of k. □

[3]Since non-intersecting separators are either crossed or nested.

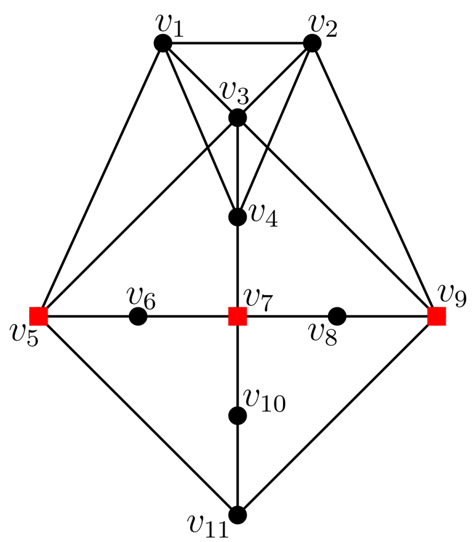
5.3 Concluding Remarks

A non-tough graph whose set of minimal feeble separators is totally nested (we will call such graphs *nested feeble* from now on) can have the graph G from Example 5.6 as a minor, see Figure 5.5.

Figure 5.5: Two nested feeble graphs that have G as a minor, with their minimal feeble separators highlighted.



(a) H_1 has four minimal feeble separators: $\{v_2\}, \{v_5\}, \{v_9\}, \{v_{11}\}$.



(b) H_2 has one minimal feeble separator: $\{v_5, v_7, v_9\}$.

Open Question 5.9. *Let H be any nested feeble graph with G as a minor. What does the set of minimal feeble separators of H look like?*

From Figure 5.5, the graph H_1 has minimal feeble separators of size 1 and the graph H_2 has only 1 minimal feeble separator, these two cases are not very interesting as they are automatically totally nested sets. Are there any nested feeble graphs with G as a minor with more interesting minimal feeble separators?

Another interesting question is how do we characterise nested feeble graphs?

Conjecture 5.10. *Let H be any non-tough graph without G as a minor. H is a nested feeble graph.*

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