

Limit shapes in one dimensional free fermion systems

by

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ABSTRACT

In this thesis we study rare fluctuations in one dimensional free fermion systems which take the form of space-time disturbances known as limit shapes. Limit shapes have a long, rich history, but substantially less has been said on what happens when limit shapes meet. The expectation is that they should combine to create some new shape. This work deals with finding a rigorous way to find and characterise these *limit shape transitions*. We are able to make considerable progress by mapping the free fermions to a class of models known as Coulomb gas models which can be readily solved. These Coulomb gas models exhibit phase transitions that we show correspond to transitions in the underlying limit shapes. As an extension, we are able to classify the limit shapes we find in an industrial way and better understand their fluctuations. Further, we present simple physical arguments that suggest that our results will be valid even in the strongly interacting regime, i.e. when the fermions are no longer free.

DEDICATION

For mum, dad and Charlotte.

ACKNOWLEDGEMENTS

This work would not be possible without the guidance of my supervisor, Dima. I hope to be able to follow his example in my future studies.

I am enormously grateful for the support of my friends and colleagues both inside the physics department and out. I couldn't have done it without you.

I also thank my collaborators on manuscripts: Alexander Abanov and Samuel Pickering.

'Highly unlikely but not impossible.'

Suzanne Collins, *Catching Fire*

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CHAPTER 1

INTRODUCTION

Humans tend to overestimate the likelihood of rare events, especially if their outcome is undesirable. However, we should not feel too bad, because this is also sometimes the case in mathematics. Conventional wisdom, based on some law of large numbers or central limit theorem (CLT), overestimates the likelihood of a rare event which is troublesome if the rare event is, for example, an earthquake [1]. Large deviation theory is an attempt to rectify this. Heuristically, it can be seen as a generalisation of the CLT to situations where a variable fluctuates drastically away from its mean value. This is the context in which most mathematicians encounter it [2, 3]. On the other hand, physicists, whether they know it or not, also regularly deal with it because of its deep ties to the saddle-point approximations used in statistical physics [4].

In this thesis we are interested in the rare events that occur in many-body free fermion systems. The rare event takes the form of a space-time disturbance caused by the fermions departing from their equilibrium configuration. Sometimes this departure can be so drastic that a homogeneously filled interval excavates entirely, something known as the emptiness formation probability (EFP) [5, 6]. The boundary traced out by the space-time disturbance often takes a characteristic shape, known as a limit shape.

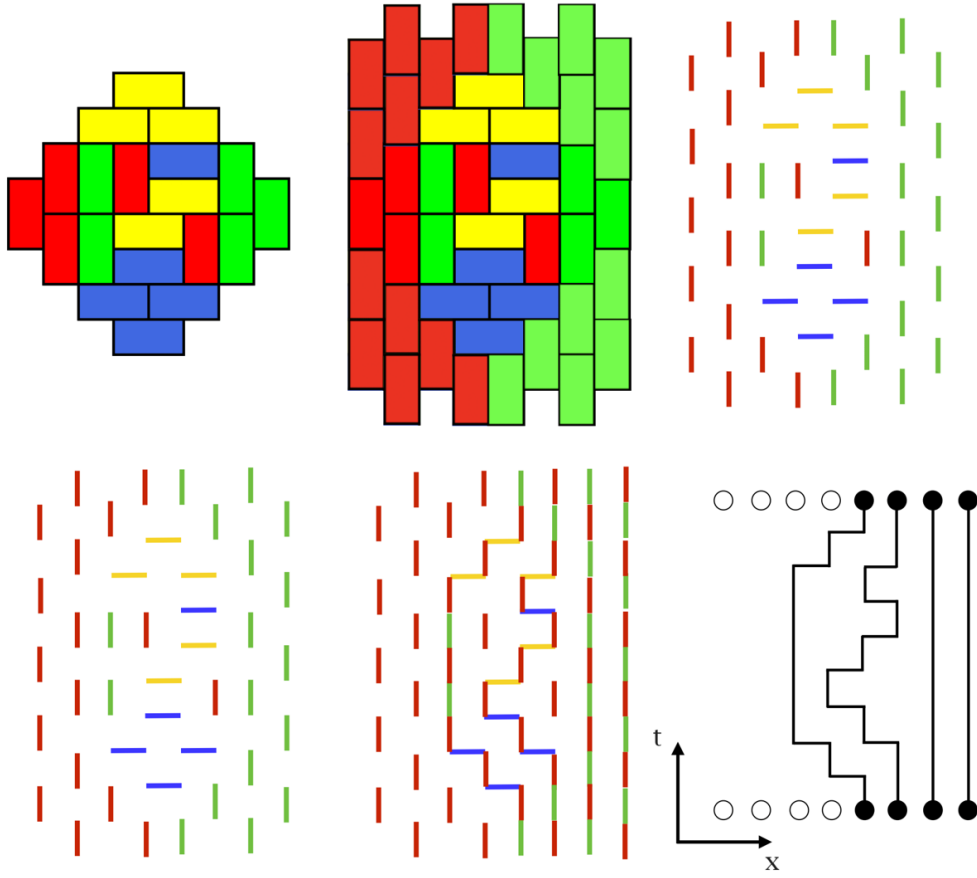
Limit shapes are fascinating objects in their own right, which frequently crop up in statistical physics. In this context, the 'limit shape' represents the most probable macroscopic

configuration of a system, one that arises in the thermodynamic limit. This configuration usually has a well-defined boundary demarcating 'liquid' and 'frozen' regions, that is regions where the thermodynamic quantity of interest (which in our case will be the number of fermions) is and is not fluctuating respectively. It has long been known [7] that some quantum mechanical problems can be mapped onto statistical mechanics ones by identifying the direction of time evolution as another spatial dimension, explaining why one expects to see limit shapes in the free fermion setting. Of particular note is the ability to map non-crossing, directed polymers (which are readily accessible experimentally) to free fermions [8, 9]. In this instance the length of these long polymers is identified with the direction of time evolution and the fact that they are non-crossing translates to the Pauli exclusion principle in the free fermion picture.

To help illustrate the limit shape phenomenon, we can look at a quintessential example: the dimer tilings of the Aztec diamond. Dimers, which are like tiles one can place in a domain, are objects that frequently crop up in statistical physics and we refer the reader to some good lectures on dimers in [10]. Fig.1.1 depicts the mapping between dimers and the imaginary time evolution of free fermions, relating the statistical and quantum mechanical in the manner discussed above.

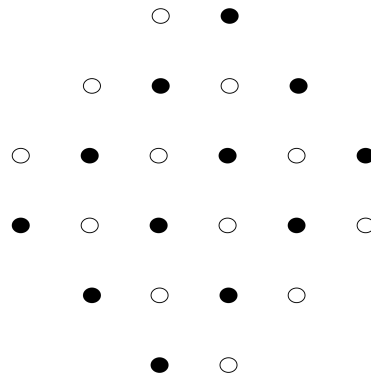
We begin by taking a bipartite lattice, shown in Fig.1.2, which has the shape of an Aztec Diamond. Tiles (dimers) are now placed randomly on the diamond, Fig.1.3. To get a grasp on the underlying structure, we colour the tiles according to how they are orientated; vertical tiles whose upper edge encloses a black (white) dot are coloured green (red) whilst horizontal tiles whose left edge encloses a black (white) dot are coloured blue (yellow). The newly coloured configuration is shown in Fig.1.4. Let us now take the continuum limit; sending L , the order of the diamond (or the number of fermions), to infinity, Fig.1.5. Miraculously, we find the appearance of two regions, separated by a distinct circular boundary. Outside this boundary we have the so-called 'frozen' region where the tiles have all ended up facing in the same directions. Inside the boundary we have the 'liquid' or 'fluctuating' region where tile orientation is random. That this

Figure 1.1: Dimers to free-fermions



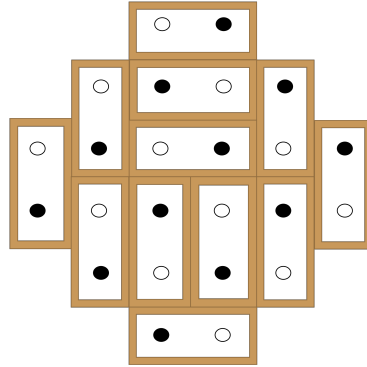
How to go from dimers to the worldlines of free fermions.

Figure 1.2: Aztec diamond



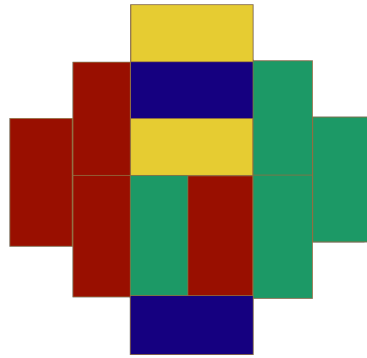
Blank Aztec Diamond lattice of order (the number of sites along the width) 6. Taken from MCA [11].

Figure 1.3: Tiled Aztec diamond



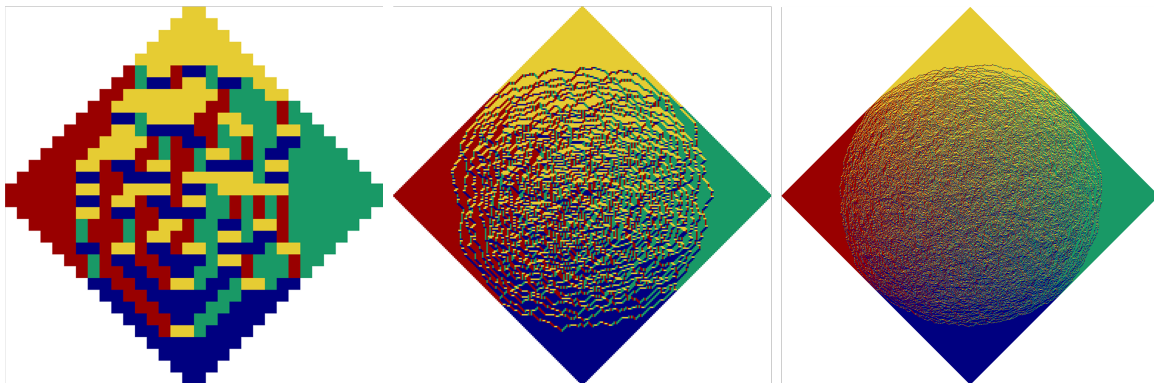
The lattice, randomly tiled [11].

Figure 1.4: Coloured Aztec diamond



The tiled lattice now coloured according to our rules [11].

Figure 1.5: Thermodynamic limit



Depiction of the thermodynamic limit, taken from [12]. From left to right we have that L , the order of the diamond, is taken to infinity.

circular boundary should appear in the thermodynamic limit is the content of a theorem: the 'Arctic Circle Theorem' [13] and the circle is called the 'limit shape'. For finite but large L the distinct boundary that characterises a limit shape will roughen as tiles (or fluctuations) bleed out into the frozen regions.

Despite being given a thorough treatment, little has been said on what happens when limit shapes meet with the exception of the intersection of two aztec diamonds [14, 15] where it was found a transition occurs and the shapes combine to form a new shape. What is missing, then, is a way to rigorously find these 'limit shape transitions' and thus it is the primary goal of this thesis to seek out and find new limit shapes and their transitions in free fermion systems. As a bonus, we find that by doing so we are able to better understand fluctuations in the limit shape systems as well.

Due to the thermodynamically large number of fermions implied by the limit shape phenomenon, in the past one particularly effective approach has been a hydrodynamical one [16]. In this approach, one tries not to describe each and every individual fermion, but rather their collective behaviour instead; characterised by a 'density' and a 'velocity' field both of which depend on space, x and time, t . Despite the simplicity inherent in the collective approach, the search for new limit shapes and their transitions has proven tough. In this work we present an alternative method of finding limit shapes based on the Coulomb gas formalism, to be discussed later, which concerns itself not with density and velocity fields that depend on space and time but a 'space' and a 'time' field that depend on density and velocity instead, in a manner very similar to [17]. We will show that in this approach we are able to find new limit shapes and their transitions in a rigorous fashion thus completing the project of this work.

1.1 Structure of Thesis

Chapter 2 offers a comprehensive theoretical background on the methods we will be employing. We introduce the central concept of this work, large deviation theory, before

reviewing the path integral, where the theory will be predominantly applied. The path integrals we consider range from those of single-particle quantum mechanics to Coulomb gas models to conformal field theories (CFT). In all cases, we will be interested in how large deviation theory can be applied to them.

Chapter 3 presents a manuscript intended for future publication [18] which acts as a miniature, self-contained overview of the thesis. It compiles all the limit shapes we have found into a compendium. This is supplemented by work on how to classify limit shapes and describe their fluctuations. We find that we can do this in a topological fashion by mapping the limit shapes to Riemann surfaces.

Chapter 4 presents one piece of published work [19], and another work on *arXiv* intended for future publication [20], which show in detail how we managed to find the limit shapes in the Compendium. The first deals with limit shapes that arise from the Gross-Witten-Wadia (GWW) matrix model [21], a statistical model that can be mapped to free fermions. We demonstrate how the model's representation as a Coulomb gas allows us to find new limit shapes and their transitions hidden within it. The second manuscript deals with limit shapes that arise in the large deviations of the Full-Counting Statistics (FCS) of free fermions. Both these models exhibit a third-order phase transition, corresponding to a transition in the underlying limit shapes, which we argue will persist even in the presence of interactions, i.e. when the fermions are no longer free.

In chapter 5 we conclude the work, summarising what has been achieved and discussing future avenues for research.

1.2 Originality and Publications

For concreteness, throughout this work we will indicate what the contributions of the author were. These will be explicitly indicated before any of the manuscripts in chapters 3 and 4. *Chapters 1 and 2 do not contain new results.* The published work is

- James S Pallister, Dimitri M Gangardt, and Alexander G Abanov. Limit shape phase transitions: a merger of arctic circles. *Journal of Physics A: Mathematical and Theoretical*, 55(30):304001, jul 2022

Work intended for future publication is

- James S. Pallister, Samuel H. Pickering, Dimitri M. Gangardt, and Alexander G. Abanov. Phase transitions in full counting statistics of free fermions and directed polymers, 2024
- James S. Pallister. Compendium of limit shapes

On top of these, in appendix [A](#), we include a brief, unpublished note *Quantum ripples in imaginary time* about a topic only tangentially related to the main body of the thesis. It is included for extra colour, available to the devoted reader, but its omission will not harm one's understanding of the rest of the text.

CHAPTER 2

BACKGROUND THEORY

2.1 Large deviation theory

A Limit shape is, by definition, the saddle point configuration of a statistical model, one that manifests itself in the thermodynamic limit. We thus need a mathematical framework to understand how this saddle point configuration emerges and that is provided by large deviation theory which we introduce here.

2.1.1 A simple example involving coin tosses

We begin with an illustrative example taken from [1]. Consider $N = 1000$ coins flipped simultaneously whilst keeping count of the number of heads shown. Over multiple trials, the probability distribution of getting M heads out of N , $P(M, N)$, can be found from the binomial theorem

$$P(M, N) = \frac{1}{2^N} \binom{N}{M}. \quad (2.1)$$

Simple reasoning or examining the plot of this function, Fig.2.1, tells us that the average value is of course $\overline{M} = N/2 = 500$. It is also not hard to find the variance: $\sigma^2 = Npq = N/4$ where $p = q$ is the probability of success/failure. Typical deviations away from $\overline{M}$

are therefore of the size $\sim \sqrt{N}$. This fact will be important later. As M is nothing but the sum of a large number of independent and identically distributed random variables (i.e. we could consider summing 0 for every tail or 1) the above information and the central limit theorem tell us that we can approximate $P(M, N)$ by the Gaussian

$$P(M, N) \sim \sqrt{\frac{2}{\pi N}} e^{-\frac{2(M-\bar{M})^2}{N}}. \quad (2.2)$$

A cursory glance at Fig. 2.1 shows a massive discrepancy between the actual result and the CLT outside of the typical regime. What went wrong? The answer lies in where the two disagree: they disagree for the rare or extreme events where the number of heads is very large or very small (the tails of the distribution). In these cases, the deviation from the mean is substantially greater than $\sqrt{N}$ and the CLT fails.

Just as maths got us into this predicament, it can also get us out whence we introduce large deviation theory. Let us set $M = cN$ where $c = O(1)$ such that the deviation is 'large'. We use Sterling's approximation in Eq. (2.1) $N! \sim \sqrt{2\pi N} \exp[N \log N - N]$ to find

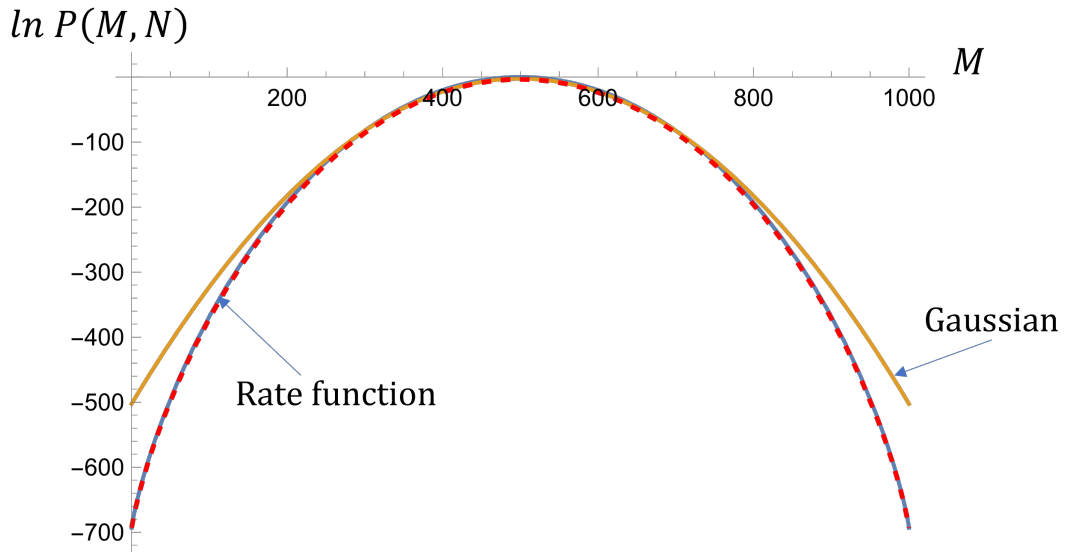
$$P(M = cN, N) \sim e^{-N\phi(c)}, \quad (2.3)$$

where

$$\phi(c) = c \log c + (1 - c) \log (1 - c) + \log 2, \quad (2.4)$$

known as the rate function. The plot of this shows excellent agreement with the actual result *even in the vicinity of the typical behaviour*, as can be verified by Taylor expanding $\phi(c)$ about $c = \bar{M}/N = 1/2$. Large deviation theory has come to the rescue.

Figure 2.1: Coin flip rate function



Rate function for the coin flip example. Eq. (2.1) depicted as the (red) dashed line shows excellent agreement with the rate function, the blue line, Eq. (2.3). Marked in orange is the prediction of the CLT.

2.1.2 The large deviation principle

Large deviation theory is not just well-equipped to describe coin tosses: it has been applied to finance and insurance [22], climate science [23] and even the early universe [24]. In our quest for new limit shapes we will be applying it to many-body systems of free fermions.

To formalise this ubiquity we introduce the *large deviation principle* that arises simply whenever a probability distribution satisfies the scaling $P_n \sim e^{-nS}$. This principle is to be understood in the large n limit, that is

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \ln P(R_n = r) = S(r), \quad (2.5)$$

which gives the probability for r incidences of the random variable, R_n . In the coin-flipping example we were interested in the random variable M occurring cN times where c was $O(1)$. S is the large deviation function or rate function from before. In the statistical physics setting, this rate function is like the free energy. For us, P will be the partition function of the system we are interested in, and we evaluate S by calculating

the saddlepoint of this path integral. This will now be explained.

2.2 Path Integrals

Large deviation theory readily lends itself to computations of path integrals as we will now show. We give an elementary example in terms of single particle, non-relativistic quantum mechanics, but our arguments readily generalise to field theories.

2.2.1 The path integral

We briefly review the path integral approach to quantum mechanics, following [25]. We can determine the properties of a quantum system from its Hamiltonian, which takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (2.6)$$

This operator defines the dynamics of the wavefunction $|\psi\rangle$ as

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle, \quad (2.7)$$

the Schrodinger equation. It is trivial to find the time dependence of the wavefunction

$$|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle, \quad (2.8)$$

where

$$\hat{U}(t) = \exp\left\{-\frac{i}{\hbar} \hat{H} t\right\}. \quad (2.9)$$

Formally, we would like to calculate the transition probabilities $\langle x_2(t) | \hat{U}(t) | x_1(0) \rangle$. Our work will, in fact, be concerned with the imaginary time evolution of the system, achieved by *Wick-rotation* $t \rightarrow -i\tau$. By doing so we notice something quite remarkable. The

transition probability is equivalent to the partition function of a thermal ensemble at inverse temperature $\beta = \tau/\hbar$

$$Z = \text{Tr} \left[e^{-\beta \hat{H}} \right] = \int dx \langle x | e^{-\beta \hat{H}} | x \rangle, \quad (2.10)$$

once we periodically identify the start and end points, i.e. $x_2 = x_1(\beta)$. To evaluate such a quantity we apply a *Trotter decomposition*, splitting the time evolution operator into a series of $N \gg 1$ operators, each time evolving for a duration $\epsilon = \beta\hbar/N$

$$Z = \int dx \langle x | e^{-\frac{\epsilon \hat{H}}{N}} \dots e^{-\frac{\epsilon \hat{H}}{N}} | x \rangle. \quad (2.11)$$

We now insert the resolution of unity for a set of momentum states

$$\mathcal{I} = \int \frac{dp_i}{2\pi\hbar} |p_i\rangle \langle p_i|, \quad i = 1, \dots, N, \quad (2.12)$$

to the left of every operator and the same for position states

$$\mathcal{I} = \int dx_i |x_i\rangle \langle x_i|, \quad i = 1, \dots, N, \quad (2.13)$$

to the right. These states satisfy the relations

$$\langle x | \hat{x} | x' \rangle = x \langle x | x' \rangle = x \delta(x - x'), \quad \langle x | \hat{p} | x' \rangle = -i\hbar \partial_x \langle x | x' \rangle = -i\hbar \partial_x \delta(x - x'). \quad (2.14)$$

We therefore consider a product of matrix elements

$$\begin{aligned} \langle x_{i+1} | p_i \rangle \langle p_i | e^{-\frac{\epsilon}{\hbar} \hat{H}} | x_i \rangle &= e^{ip_i x_{i+1}} \langle p + i | e^{-\frac{\epsilon}{\hbar} \hat{H} + \mathcal{O}(\epsilon^2)} | x_i \rangle \\ &= \exp \left\{ -\frac{\epsilon}{\hbar} \left[\frac{p_i^2}{2m} - ip_i \frac{x_{i+1} - x_i}{\epsilon} + V(x_i) + \mathcal{O}(\epsilon) \right] \right\}. \end{aligned} \quad (2.15)$$

The x -integration in the trace is taken care of by noticing that at the right end we have $\langle x_1|x \rangle = \delta(x_1 - x)$ and similarly on the left. We have

$$Z = \int \prod_{i=1}^N \frac{dx_i dp_i}{2\pi\hbar} \exp \left\{ -\frac{\epsilon}{\hbar} \sum_{j=1}^N \left[\frac{p_j^2}{2m} - ip_j \frac{x_{j+1} - x_j}{\epsilon} + V(x_j) + \mathcal{O}(\epsilon) \right] \right\}. \quad (2.16)$$

Sending $N \rightarrow \infty$ the corrections drop out and we represent the partition symbolically as

$$Z = \int_{x(\beta\hbar)=x(0)} \mathcal{D}x \mathcal{D} \left(\frac{p}{2\pi\hbar} \right) \exp \left\{ -\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \left[\frac{p(\tau)^2}{2m} - ip(\tau)\dot{x}(\tau) + V(x(\tau)) \right] \right\}. \quad (2.17)$$

More generally, for a system of many free particles with Hamiltonian H , not much of our argument changes, and we write

$$Z = \int_{x(\beta\hbar)=x(0)} \mathcal{D}x \mathcal{D} \left(\frac{p}{2\pi\hbar} \right) \exp \left\{ -\frac{1}{\hbar} \sum_i \int_0^{\beta\hbar} d\tau [H(x_i, p_i) - ip_i(\tau)\dot{x}_i(\tau)] \right\}, \quad (2.18)$$

where now $\mathcal{D}x, \mathcal{D}(p/2\pi\hbar)$ are to be understood as integrals over the trajectory of each and every one of the particles. We call $S = \sum_i \int_0^{\beta\hbar} d\tau [H(x_i, p_i) - ip_i(\tau)\dot{x}_i(\tau)]$ the action.

We would ideally like to evaluate the above exactly. This is generally quite difficult, unless the Hamiltonian is quadratic in x and p . Fortunately, if like us you are interested in large deviation theory, a simplification occurs. Let us set $\hbar = 1$, re-scale the imaginary time and position by β and send $\beta = R \rightarrow \infty$. We have

$$Z = \lim_{R \rightarrow \infty} \int_{x(1)=x(0)} \mathcal{D}x \mathcal{D} \left(\frac{p}{2\pi\hbar} \right) \exp \left\{ -R \sum_i \int_0^1 d\tau [H(x_i, p_i) - ip_i(\tau)\dot{x}_i(\tau)] \right\}. \quad (2.19)$$

Z is now going to be dominated by the set of x_i, p_i that minimises the action because all other contributions go to zero in the large R limit. These x_i, p_i must then obey Hamilton's equations

$$ip_i = -\frac{\partial H}{\partial x_i}, \quad i\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad (2.20)$$

known as the saddle-point configuration. We can then write the large deviation principle

$$\lim_{R \rightarrow \infty} -\frac{1}{R} \ln Z = S, \quad (2.21)$$

where S is the action evaluated on the saddle-point.

2.2.2 A single particle example

As an instructive example, consider the large deviations in the trajectory of a single particle with free Hamiltonian $H = p^2/2$. We want to calculate

$$Z = \lim_{R \rightarrow \infty} \int_{x(1)=x(0)} \mathcal{D}x \mathcal{D} \left(\frac{p}{2\pi\hbar} \right) \exp \left\{ -R \int_0^1 d\tau \left[\frac{p^2}{2} - ip(\tau)\dot{x}(\tau) \right] \right\}, \quad (2.22)$$

where the particle begins and ends at the same position $x(\tau = 0) = x(\tau = 1)$. At intermediate times we are allowed to impose additional geometrical constraints which drive and force the particle away from 'typical behaviour'; in this case sitting still at $x(0)$. These constraints amount to delta functions inserted into our path integral, e.g. $\delta(x_1 - x(\tau_1))$ which force our particle to be at a particular place at a particular time. Though the particle is free such constraints are enough to lead to some very interesting behaviour. In principle one can dream up all sorts of constraints, but the ones most relevant to this work will be those which fix a particle's position at a certain time or geometrical constraints. The problem is to minimise the action subject to these additional constraints. This area of physics goes by the name *geometric optics* [26, 27]. Hamilton's equations are

$$i\dot{p} = 0, \quad i\dot{x} = p. \quad (2.23)$$

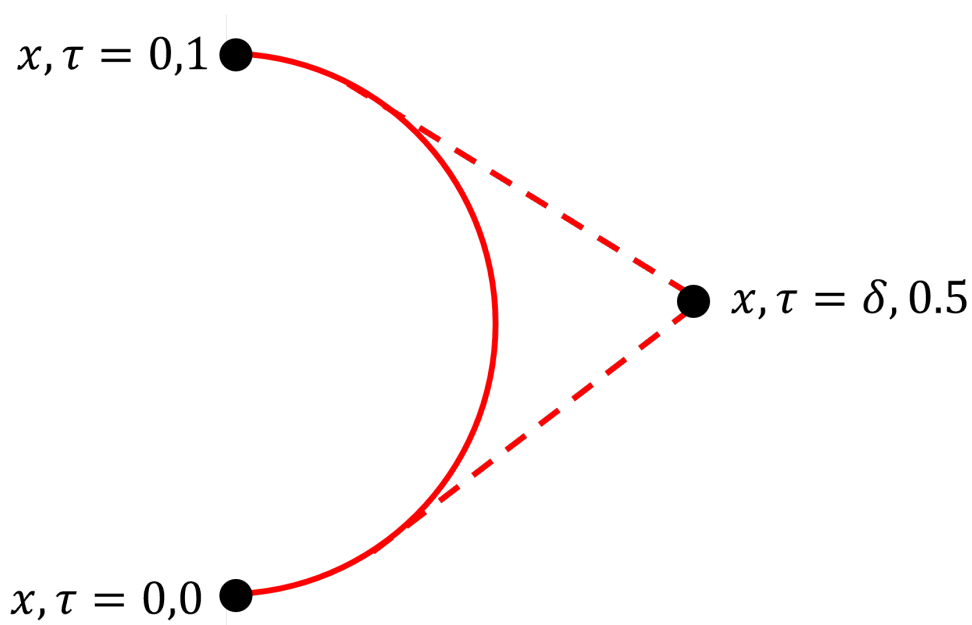
Thus $p(\tau) = v, x(\tau) = d - iv\tau$. On these equations we have

$$Z \sim e^{Rv^2/2} = e^{RL^2/2}, \quad (2.24)$$

where L is the length of the particle's trajectory. Minimising the action is thus equivalent to minimising this length (subject to any constraints).

Here's one example of solving this problem with a constraint taken from [26]. We ask that a particle avoids a semicircular boundary and deviates to a point $x = \delta$ away from the origin. As shown above, the action associated with this will be the length of the path the particle takes which must be weighted by the length of the undeviated trajectory, Figure 2.2.

Figure 2.2: Optimal fluctuation around a semi-circle



Optimal trajectory for a free particle to avoid a semi-circular boundary and deviate to a point a distance δ away. The particle initially follows the boundary, before shooting off tangentially along the dashed line subsequently returning again. When evaluating the action we normalise its value evaluated on the deviated path with that on the undeviated one.

The length of the undeviated path is simply $L_u = \pi$ for a contribution to the action $S_u = \pi^2/2$. All that is left is to evaluate the length of the deviated path and its action S_d to give the probability of the fluctuation

$$P_{\text{fluct.}} = e^{\frac{R}{2}(S_u - S_d)}. \quad (2.25)$$

The length of the deviated trajectory can be easily found by considering the right-angled triangle formed by joining the points $(0, 0.5)$, $(\delta, 0.5)$, (x_1, τ_1) where x_1, τ_1 is the point where the particle leaves the semi-circle. We find the length of the dashed segment to be $\sqrt{\delta^2 - \frac{1}{4}}$ and from this the curved arc leading to it to be $\frac{1}{2} \arccos \sqrt{\delta^2 - \frac{1}{4}}/\delta$. Thus

$$S_d = 2 \left[\sqrt{\delta^2 - \frac{1}{4}} + \frac{1}{2} \arccos \frac{\sqrt{\delta^2 - \frac{1}{4}}}{\delta} \right]. \quad (2.26)$$

We then have

$$P_{\text{fluct.}} = \exp \left\{ -4R \left(\left[\sqrt{\delta^2 - \frac{1}{4}} + \frac{1}{2} \arccos \frac{\sqrt{\delta^2 - \frac{1}{4}}}{\delta} \right]^2 - \frac{\pi^2}{16} \right) \right\}. \quad (2.27)$$

Of note is when $\delta - \frac{1}{2} \ll 1$, i.e. the particle is deviating only slightly away from the semi-circular boundary where we find

$$P_{\text{fluct.}} \sim \exp \left\{ -\frac{8\pi R}{3} \left(\delta - \frac{1}{2} \right)^{3/2} \right\}. \quad (2.28)$$

This should come as some surprise as one would expect that the small fluctuations away from the boundary would be Gaussian, with exponent 2 not 3/2 (perhaps by invoking some CLT type argument as we have done previously). The fact that this is not the case here is because the constrained motion we have forced the particle to undertake to begin with- following the semi-circular boundary- is already an atypical path and so fluctuations around this path should behave atypically as well, i.e. not fall into the expected Gaussian universality class. The new class we have stepped into is, in fact, called the KPZ universality class characterised by the tell-tale 3/2 exponent [28, 29]. This scaling is quite ubiquitous and can be found in one-dimensional surface growth models and some interacting particle systems. Its origin can usually be traced back to an underlying non-linear stochastic PDE known as the KPZ equation. As a reflection of this universality we point out that so long as the boundary is convex the KPZ scaling will still manifest near

its boundary [30, 31].

Though we have looked at particles, all the arguments we have presented above readily extend to fields, an example of which we will show in the next section.

2.3 Random matrices and the Coulomb gas

The central way we find new limit shapes in this work is by mapping free fermion problems onto random matrix models [32, 33, 34] which have a representation as a Coulomb gas [35, 36]. This section will explain what we mean by this. We will then show how large deviation theory can be applied to a particular Coulomb gas model. Though this model is simple, it contains all the features we will see in the rest of this work.

2.3.1 Random matrices

A random matrix is simply a matrix whose entries are sampled from some probability distribution. The investigation of the eigenvalues of such matrices and the field of random matrix theory (RMT) was initiated by Wigner in Nuclear physics [37] and has since exploded covering as diverse areas as quantum chaos [38], quantum gravity [39] and number theory [40]. A particularly simple class of random matrices to work with are Gaussian matrices, X , drawn from the probability distribution

$$P(X) \propto e^{-\frac{\beta}{2}(X, X)}. \quad (2.29)$$

Depending on the symmetries of the problem, the matrix X , the inner product (X, X) and β will take different forms. Because of its links to free fermions (discussed imminently) the Gaussian Unitary Ensemble (GUE) where X is an $N \times N$ hermitian matrix, $(X, X) = \text{Tr} X^\dagger X$ will be that studied here. The elements of the hermitian matrix are such that the diagonal elements are real $X_{ii} \sim \mathcal{N}(0, 1)$ and the off-diagonal elements are complex $X_{jk} = u_{jk} + i v_{jk}$ where $u_{jk}, v_{jk} \sim \mathcal{N}(0, 1/2)$. Then, each element of the matrix has the

probability density function (PDF)

$$Z = \int dX_{ij} e^{-X_{ij}^2}. \quad (2.30)$$

Multiplying these together we have the joint PDF

$$Z_N = \int \prod_{i \leq j} dX_{ij} e^{-2 \sum_{i < j} |X_{ij}^2| - \sum_i X_{ii}^2} = \int \prod_{i \leq j} dX_{ij} e^{-\text{Tr}(X^2)}. \quad (2.31)$$

Note that this corresponds to $\beta = 2$ (for those curious, $\beta = 1, 4$ are known as the Gaussian Orthogonal and Gaussian Symplectic ensembles respectively). As a step forward, we now diagonalise X , denoting its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$. The trace in the exponent is then simply the sum of these eigenvalues squared, but care must be taken with the measure of integration which transforms nontrivially, introducing a Jacobian, J such that $\int \prod_{i \leq j} dX_{ij} \rightarrow \int d^N \lambda |\det J|$. Calculation of this Jacobian can be found in [41]. The result is

$$|\det J| = \prod_{j < k} |\lambda_j - \lambda_k|^2. \quad (2.32)$$

We can then write the joint probability density function of the GUE [32]

$$Z_N = \int d^N \lambda \exp \left[- \sum_i \lambda_i^2 + \sum_{j < k} \ln (|\lambda_k - \lambda_j|^2) \right]. \quad (2.33)$$

2.3.2 Equivalence to free fermions

This joint PDF can be connected to free fermions in the following way. Consider N free fermions in the harmonic trap $V(x) = m\omega^2 x^2/2$. From hereon in we will set $m = \omega = \hbar = 1$. Their many-body ground state wave function is constructed from a Slater determinant

of single-particle orbitals, i.e.

$$\psi_0(x_1, x_2, \dots, x_N) = \det[\phi_i(x_j)] / \sqrt{N!}, \quad 0 \leq i \leq N-1, 1 \leq j \leq N. \quad (2.34)$$

These single particle orbitals satisfy the Schrodinger equation

$$i\partial_t \phi_j(x, t) = -\frac{1}{2} \partial_x^2 \phi_j(x, t) + \frac{x^2}{2} \phi_j(x, t). \quad (2.35)$$

The solution is given in terms of Hermite polynomials $H_n(x)$ times a Gaussian

$$\phi_n(x) \propto e^{-x^2/2} H_n(x). \quad (2.36)$$

The Slater determinant ensures the wavefunction is anti-symmetric under particle exchange and hence obeys the Pauli exclusion principle. For the ground state, we can write the many-body wavefunction explicitly by noting that $\det(H_i(x_j)) \propto \prod_{0 \leq i < j \leq N-1} (x_i - x_j)$, known as the Vandermonde determinant. As such the integral of the squared amplitude of the many-body ground state wavefunction is

$$Z_N = \int |\psi_0(x_1, x_2, \dots, x_N)|^2 d^N x = \int e^{-\sum_i x_i^2} \prod_{i < j} (x_j - x_i)^2 d^N x. \quad (2.37)$$

Exponentiating the product one recovers the joint PDF of the GUE if we identify the positions of the particles with the eigenvalues of the random matrix. Therefore, the ground state of a system of free fermions in a harmonic trap is a realisation of a RMT.

The joint PDF allows us to derive certain statistical properties of the GUE and hence the free fermions in a harmonic trap. For example we could be interested in the average density of eigenvalues, that is how many reside between λ and $\lambda + d\lambda$ per unit length, a quantity denoted $\rho(\lambda, N) = \sum_i^N \langle \delta(\lambda - \lambda_i) \rangle / N$ where expectation values are taken w.r.t.

the PDF. Wigner showed that in the large N limit

$$\rho(\lambda, N) = \rho_{\text{s.c.}}(\lambda, N) = \sqrt{\frac{2}{N\pi^2}} \left(1 - \frac{\lambda^2}{2N}\right)^{1/2}, \quad (2.38)$$

which is known as the Wigner semicircle law, owing to the density profile's shape Fig. 2.3. The law states that on average the eigenvalues, alternatively the free fermions in a harmonic trap, find themselves confined to a region $[-\sqrt{2N}, \sqrt{2N}]$.

The quantity that is of immediate interest to us is the probability that all the eigenvalues are less than some value, w , which can be found by evaluating the partition function or path integral $Z_N(w)$

$$Z_N(w) = \prod_i^N \int_{-\infty}^w d\lambda_i P(\lambda_1, \lambda_2, \dots, \lambda_N). \quad (2.39)$$

If $w < \sqrt{2N}$ one expects that the eigenvalues will be driven away from their equilibrium profile Eq. (2.38) and assume a different distribution.

2.3.3 Coulomb gas formulation

In order to evaluate the partition function we move to a collective description of the eigenvalues known as a Coulomb gas model. Let us introduce the fluctuating density field $\rho(\lambda) = \sum_i^N \delta(\lambda - \lambda_i)/N$. In the large N limit, the partition function can then be expressed

$$Z_N(w) = \int \mathcal{D}\rho dC \exp[-2N^2 E[\rho, C] + O(N)], \quad (2.40)$$

where

$$E[\rho, C] = \int_{-\infty}^w d\lambda \frac{\lambda^2}{2} \rho(\lambda) - \frac{1}{2} \int_{-\infty}^w d\lambda \int_{-\infty}^w d\lambda' \rho(\lambda) \rho(\lambda') \ln(|\lambda - \lambda'|) + C \left(\int_{-\infty}^w \rho(\lambda) d\lambda - 1 \right). \quad (2.41)$$

Note that we have re-scaled the potential by a factor of N . We have also scaled the eigenvalues by $\sqrt{N}$ such that $w \rightarrow \sqrt{N}w$. This is because we are interested in atypical and so large fluctuations of the eigenvalues. Specifically we want to look at large fluctuations of the eigenvalues away from the right edge of the Wigner semi-circle. As this edge is located at $\sqrt{2N}$ we need these fluctuations to be of a similar order to be large. When we make this scaling the whole action accumulates a factor of N^2 outside the front. In the large N limit, then, we are free to apply what we learned in the previous sections with regards to large deviation theory and a saddle-point analysis. The above can be interpreted as a gas of N charged particles interacting via a two dimensional Coulomb potential and an external quadratic potential. C is a Lagrange multiplier there to fix the total number of particles. The advantage of this formalism is that up to small ($O(N)$) corrections, we have reduced the problem to the dynamics of just two variables, $\rho(\lambda), C$. Their dynamics is fixed by their Euler-Lagrange equations

$$\frac{\delta E}{\delta \rho} = 0 \implies \frac{\lambda^2}{2} - \int \rho(\lambda') \ln(|\lambda - \lambda'|) d\lambda' + C = 0, \quad (2.42)$$

$$\frac{\partial E}{\partial C} = 0 \implies \int_{-\infty}^w \rho(\lambda) d\lambda = 1. \quad (2.43)$$

Solving these equations, we can determine the optimal $\rho(\lambda), C$, plug these back into the action and hence calculate the probability of the large deviation we are interested in. For the moment, however, let us focus on the typical behaviour of the system and so set $w = \infty$.

Taking the derivative of the first equation, we have

$$\lambda = \mathcal{P} \int \frac{\rho(\lambda')}{\lambda - \lambda'}, \quad (2.44)$$

where $\mathcal{P}$ denotes the principal part of the integral. There is then a particularly elegant way to solve these equations that revolves around 'loop equations' [42, 43, 44, 45, 46]. We

introduce the resolvent

$$g(\lambda) = \frac{1}{\pi} \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}. \quad (2.45)$$

We can then fix $\rho(\lambda)$ by demanding that the resolvent has the following properties. First, as a consequence of the Sokhotski–Plemelj theorem [47], we know that

$$g(\lambda \pm i0) = \frac{1}{\pi} \mathcal{P} \int \frac{\rho(\lambda')}{\lambda - \lambda'} \pm i\rho(\lambda). \quad (2.46)$$

Due to Eq. (2.44) the principal part has to be equal to λ . We also know from the normalisation constraint that as $\lambda \rightarrow \infty$

$$\lim_{\lambda \rightarrow \infty} g(\lambda) = \frac{1}{\pi\lambda}. \quad (2.47)$$

We define one more criterion, which we will relax later, that the density has support over just one region of the real line (known as a ‘one-cut’ solution) and does not diverge in this region. These restrictions are sufficient to uniquely specify the answer

$$g(\lambda) = \frac{\lambda}{\pi} - \frac{1}{\pi} \sqrt{\lambda^2 - 2}. \quad (2.48)$$

The density is thus

$$\rho(\lambda) = \sqrt{\frac{2}{\pi^2}} \left(1 - \frac{\lambda^2}{2}\right)^{1/2}, \quad (2.49)$$

which is the Wigner semi circle, Eq. (2.38) once all scalings are restored.

2.3.4 Large deviations in the GUE

In this section we follow closely the derivation found in [35]. We mentioned above that the density profile must be amended if $w < \sqrt{2}$. In this case, the free energy of the system must change as well, and so we say the Coulomb gas undergoes a phase transition. We

only have two ways in which we can alter the density profile: by relaxing the constraint that it does not diverge and by relaxing the constraint that it has singular support. Let us relax the former of these and try and guess the form of the answer. We expect the density to diverge at w in such a way that w acts like a 'hard wall' in position space against which the particles will push up. However, we also still want to ensure that for large λ the normalisation constraint is obeyed. This leads us to postulate the following *ansatz* for the new density profile

$$g(\lambda) = \frac{\lambda}{\pi} - \frac{(\lambda - a)}{\pi} \sqrt{\frac{\lambda - b}{\lambda - w}}. \quad (2.50)$$

This *ansatz* implements a divergence at w which is what we wanted and ensures that for large λ the normalisation constraint can be obeyed. a, b are then to be fixed subject to this constraint. Expanding for $\lambda \rightarrow \infty$ and fixing the coefficients we find that

$$a(w) = \frac{w - b(w)}{2}, \quad (2.51)$$

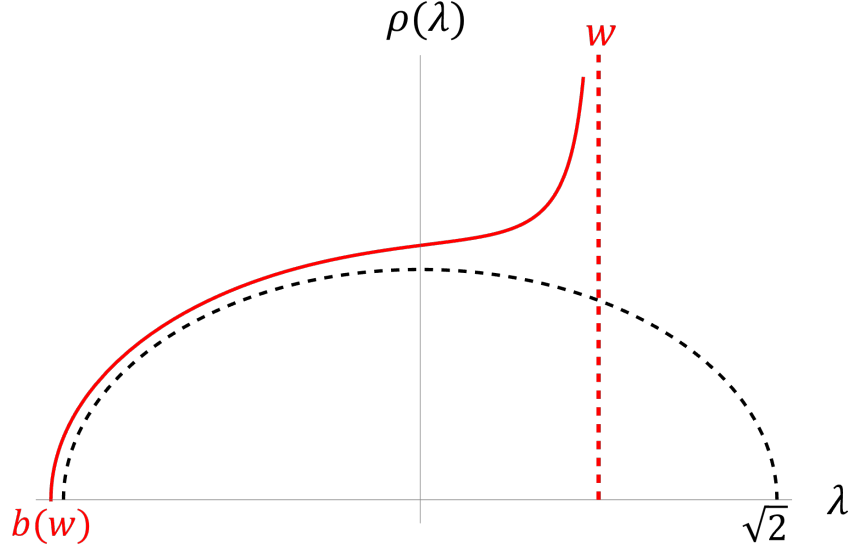
$$b(w) = \frac{1}{3} \left(w - 2\sqrt{6 + w^2} \right). \quad (2.52)$$

Therefore, the density profile is

$$\rho(\lambda) = \frac{\sqrt{\lambda - b(w)}}{2\pi\sqrt{w - \lambda}} [w - b(w) - 2\lambda], \quad (2.53)$$

This result is valid for $w < \sqrt{2}$. We plot the density configurations in Fig.2.3.

Figure 2.3: Large deviation of GUE



Pushed (solid line) and un-pushed Wigner semi-circle (dashed line) density profiles.

We now calculate the action or free-energy associated with this configuration. A shortcut to doing this which will be heavily exploited throughout this work is to leverage Eq. (2.42). On the saddlepoint, this equation implies

$$E[\rho, C] = -\frac{C}{2} + \frac{1}{4} \int_{b(w)}^w \lambda^2 \rho(\lambda) d\lambda. \quad (2.54)$$

The Lagrange multiplier is determined by Eq. (2.42) as long as we choose λ so that it lies in the support of the density profile. To do this we choose $\lambda = w$ which implies

$$C = -\frac{w^2}{2} + \int_{b(w)}^w \ln(|w - \lambda'|) \rho(\lambda') d\lambda'. \quad (2.55)$$

Therefore,

$$E[\rho, C] = E[w] = \frac{w^2}{4} - \frac{1}{2} \int_{b(w)}^w \ln(|w - \lambda'|) \rho(\lambda') d\lambda' + \frac{1}{4} \int_{b(w)}^w \lambda^2 \rho(\lambda) d\lambda. \quad (2.56)$$

Note that for $w = \sqrt{2}$ the above computation gives the free energy associated with the

equilibrium configuration. Plugging this into MATHEMATICA we find

$$E[w] = \frac{1}{108} \left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} + 27 \left(\ln 18 - 2 \ln \left(w + \sqrt{w^2 + 6} \right) \right) \right]. \quad (2.57)$$

Of note is its behaviour near the point of transition

$$E[w] \sim \frac{1}{6\sqrt{2}} \left(\sqrt{2} - w \right)^3, \quad w \rightarrow \sqrt{2}, \quad (2.58)$$

where w approaches the edge of the *unpushed* configuration, the Wigner semi-circle from before. There is then a transition from the pushed phase where there is an atypical behaviour of the particles which exhibit a diverging density around w and the typical, unpushed phase for $\sqrt{2} < w < \infty$. What we have found indicates that this transition is of the third order. This order is argued to be a ubiquitous feature of transitions involving a 'pushed' configuration [35, 48] and we will see its presence throughout this work. In the language of large deviation theory $E[w]$ is the rate function and the large deviation principle is

$$Z_N(w) \sim \exp \left[-2N^2 E(w) \right] \quad (2.59)$$

as $N \rightarrow \infty$.

2.4 Full Counting Statistics

Full-counting statistics (FCS) allows us to access the statistical behaviour of a system through knowledge of the generating function

$$\chi(s) = \langle e^{-sn} \rangle = \sum_{n=-\infty}^{\infty} P_n e^{-sn}, \quad (2.60)$$

for some probability distribution P_n . It contains information on the fluctuations of the variable n (later, we will be interested in the case when $n = N_A$ which gives fluctuations in the number of particles N in an interval A). A slightly more useful object to consider is found by taking the logarithm of the above

$$\log \chi(s) = \sum_{m=1}^{\infty} \frac{(-s)^m C_m}{m!}, \quad (2.61)$$

where the C_m are the cumulants of the distribution, i.e. $C_1, C_2, C_3 \dots$ represent the mean, variance, skewness...

FCS has been of great interest experimentally, with past work on one-dimensional magnetism [49, 50, 51], quantum noise [52, 53] and ultra-cold atoms [54, 55, 56]. It can also be linked to the entanglement entropy of many-body quantum systems thus giving us another means with which to probe them [57, 58, 59, 60, 61].

2.4.1 FCS of the Binomial distribution

As an example of FCS in action, we study a simple problem: the FCS of the binomial distribution

$$\text{Bin}(n, p; k) = \binom{n}{k} (1-p)^{n-k} p^k, \quad (2.62)$$

denoting the probability of k successes in n trials (later we will be interested in the probability of finding k particles in a region of size $n = 2L$). This has the generating function

$$\chi(s) = [1 + p(e^{-s} - 1)]^n. \quad (2.63)$$

We can then find the cumulants using a Taylor's series on $\log \chi(s)$, recovering the expected values of the mean $= C_1 = np$, the variance $= C_2 = np(1-p)$ and the skewness $= C_3 = np(1-p)(1-2p)$. Consider now the limit $n \rightarrow \infty$. We choose to re-scale

$s \rightarrow s/\sqrt{n}$. We recall from our previous discussion of large deviation theory that this is equivalent to studying *typical* deviations from the mean. We therefore expect the probability distribution to be a normal distribution in this limit with mean C_1 and variance C_2 . In this limit the generating function takes the form

$$\log \chi(s) \sim -sp\sqrt{n} + \frac{s^2}{2}p(1-p) + O(1/n^{3/2}), \quad (2.64)$$

and so only the first two cumulants survive in the large n limit. The probability distribution is the Fourier transform of $\chi(is)$

$$P(k) \sim \int \frac{ds}{2\pi} e^{i\sqrt{n}sk} \chi(is) \sim e^{-\frac{(k-C_1)^2}{2C_2}}, \quad (2.65)$$

taking the expected Gaussian form. What we have presented here are results about fluctuations found in the *typical* regime. In contrast, in the manuscript [20] we calculated FCS for a system in the large deviation regime. In this section, we review how to do this for the GUE, a calculation done before in [62].

2.4.2 FCS of the GUE

We wish to calculate

$$P(N_A) = \int \frac{ds}{2\pi} e^{isN_A} \chi(is), \quad (2.66)$$

where $\chi(s) = \langle e^{-s \int_a^b \rho(\lambda) d\lambda} \rangle$. This is the probability of finding N_A particles in a region $A : x \in (a, b)$. For simplicity we will take $b = -a = L$ where $L < \sqrt{2}$ such that the interval is within the bounds of the Wigner semicircle. The expectation value is taken w.r.t the partition function of the GUE such that

$$\chi(s) = \frac{1}{Z_N(w = \sqrt{2})} \int \mathcal{D}\rho dC \exp[-2N^2 E[\rho, C, s]], \quad (2.67)$$

where

$$E[\rho, C, s] = \int_{-\infty}^{\infty} d\lambda \frac{\lambda^2}{2} \rho(\lambda) - \frac{1}{2} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \rho(\lambda) \rho(\lambda') \ln(|\lambda - \lambda'|) \\ + C \left(\int_{-\infty}^{\infty} \rho(\lambda) d\lambda - 1 \right) + \frac{s}{2} \left(\int_{-L}^L \rho(\lambda) d\lambda \right), \quad (2.68)$$

We have re-scaled s by N , such that in the large N limit $P(N_A)$ satisfies a large deviation principle. Obviously if $s = 0$ we know the correct form for $\rho(\lambda)$, it is just the Wigner semi-circle. The question now is how does the constraint enforced by the FCS change this solution?

It is useful to define the average number of particles in the interval obtained by integrating the Wigner semi-circle function

$$\langle N_A \rangle = N \int_{-L}^L \rho_{\text{s.c.}}(\lambda) d\lambda = \frac{N}{\pi} \left(L\sqrt{2 - L^2} + 2 \arcsin \left(\frac{L}{\sqrt{2}} \right) \right). \quad (2.69)$$

We will restrict ourselves to the case where

$$\int_{-L}^L \rho(\lambda) d\lambda = N_A/N = k_A < \langle N_A \rangle/N = \overline{k_A}, \quad (2.70)$$

In fact, if one doesn't impose this restriction one can find a phase transition as k_A is varied from below to above $\overline{k_A}$ [62].

Like before we need to posit a suitable *ansatz* for the resolvent, which can be found in [62]. This *ansatz* must have the following properties. First, like always, it must satisfy the appropriate normalisation as $\lambda \rightarrow \infty$, i.e. it must behave like $1/\pi\lambda$. Next, let's look at the saddlepoint equation for this problem

$$\lambda + \frac{s}{2} (\delta(L - \lambda) - \delta(-L - \lambda)) = \mathcal{P} \int \frac{\rho(\lambda')}{\lambda - \lambda'}. \quad (2.71)$$

We observe that the term involving s in the action has introduced discontinuities at $\lambda = \pm L$. We can view this term as an additional potential applied to the charges on top

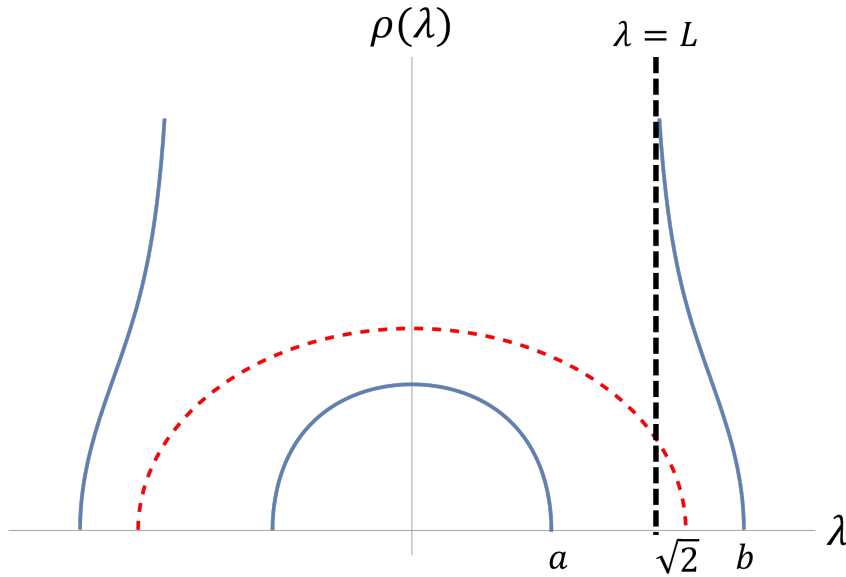
of the quadratic one. Because it is discontinuous we must assume that the eigenvalues have support over several distinct intervals (with their boundaries at $\pm L$). Thus the resolvent needs suitably placed branch points. All these considerations lead to the following *ansatz*

$$g(\lambda) = \frac{\lambda}{\pi} - \frac{1}{\pi} \sqrt{\frac{(\lambda^2 - a^2)(\lambda^2 - b^2)}{(\lambda^2 - L^2)}}, \quad (2.72)$$

plotted in Fig. 2.4. We can fix the zeros of this function by considering the limit $\lambda \rightarrow \infty$ where we find

$$a^2 + b^2 = 2 + L^2. \quad (2.73)$$

Figure 2.4: FCS of GUE



The solid blue is the plot of the density profile corresponding to the ansatz Eq. (2.72) whilst the dashed red curve is the Wigner semi-circle. Notice the appearance of disconnected regions in the density profile.

We are left with two further constraints from variation of the action w.r.t the Lagrange multipliers C, s . These constraints allow us to fix b, L assuming k_A is fixed (we could also allow k_A to vary and fix L). The resulting integrals, however, are quite unwieldy and the

computation of the action is not particularly instructive.

Instead, let us zoom in on the regime of typical fluctuations in particle number and see whether we can recover the expected gaussian behaviour. In this regime $\rho(\lambda)$ approaches the Wigner semi-circle which one can check from Eq. (2.72) corresponds to taking the limits $a \rightarrow L, b \rightarrow \sqrt{2}$ and $s \rightarrow 0$. Let us expand Eq. (2.68) about this point, writing $\rho(\lambda) = \rho_{\text{s.c.}}(\lambda) + h(\lambda)/N$ and $s = \delta s/N$. Schematically, we then have $E = E^{(0)} + E^{(1)} + E^{(2)}$. To leading order, we get just the energy associated with the Wigner semicircle.

$$E^{(0)} = E[\rho_{\text{s.c.}}, C] \quad (2.74)$$

Remembering that $P(N_A)$ is normalised against this contribution, this term is cancelled. To next order we find $E^{(1)} = 0$ because $\rho_{\text{s.c.}}$ is on the equations of motion. To second order we then have

$$N^2 E^{(2)}[h, \delta s] = -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' h(\lambda) h(\lambda') \ln(|\lambda - \lambda'|) + \frac{\delta s}{2} \left(\int_{-L}^L h(\lambda) d\lambda \right). \quad (2.75)$$

This action remains to be minimised. We can get a better handle on these fluctuations by using a trick. Consider the following field

$$\phi(z, \bar{z}) = \int d\lambda' \ln(|z - \lambda'|) h(\lambda'). \quad (2.76)$$

Its derivative behaves a lot like the resolvent. We have the useful formulas

$$\partial \phi(z) = \int d\lambda' \frac{h(\lambda')}{z - \lambda'}, \quad (2.77)$$

$$\partial \phi(\lambda \pm i0) = \mathcal{P} \int d\lambda' \frac{h(\lambda')}{\lambda - \lambda'} \mp i\pi h(\lambda). \quad (2.78)$$

We now claim that Eq. (2.75) is equivalent to

$$N^2 E^{(2)}[h(\phi), \delta s] = \frac{1}{2\pi} \int d^2 z |\partial \phi|^2 - \frac{\delta s}{2\pi i} (\phi(L) - \phi(-L)). \quad (2.79)$$

To see this, we work backward. Using Stokes' theorem we have

$$N^2 E^{(2)}[h(\phi)] = \frac{1}{2\pi} \int d^2 z |\partial\phi|^2 = \frac{1}{4\pi i} \oint dz \phi \partial\phi = \frac{1}{4\pi i} \int_{-\infty}^{\infty} d\lambda [\phi \partial\phi]_{\lambda-i0}^{\lambda+i0}. \quad (2.80)$$

The properties of our newly defined field then tell us

$$N^2 E^{(2)}[h] = -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda \phi(h) h = -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' h(\lambda) h(\lambda') \ln(|\lambda - \lambda'|). \quad (2.81)$$

The term involving δs is trivial. The action we have found is that of a Gaussian free field theory or two-dimensional CFT. This is a behemoth subject and has been extensively studied in the context of limit shapes, many body systems of free fermions and Quantum Generalised Hydrodynamics [17, 63, 64, 65, 66, 67, 68]. For an older reference relating directly to Coulomb gases see [69]. The important point is that the fluctuations are manifestly quadratic in nature and are in a known form. We will now use these facts as we calculate

$$\chi(s) \sim \int \mathcal{D}\phi \exp \left\{ - \int d^2 z \left[\frac{1}{\pi} |\partial\phi|^2 - J(z, L) \phi(z) \right] \right\}, \quad (2.82)$$

where $J(z, L) = \frac{\delta s}{\pi i} \delta^2(z - L) - \frac{\delta s}{\pi i} \delta^2(z + L)$. An elementary Gaussian integral yields

$$\chi(s) \sim e^{-\frac{\pi}{4} \int J(z, L) (\bar{\partial}\partial)^{-1} J(z', L) dz dz'}, \quad (2.83)$$

where $(\bar{\partial}\partial)^{-1}$ is to be understood as the Green function or propagator. For a 2D CFT the propagator satisfies Poisson's equation

$$\langle \phi(z') \bar{\partial}\partial\phi(z) \rangle = -\pi \delta^2(z - z'), \quad (2.84)$$

with a point charge as the source term. We use the standard fact that

$$\bar{\partial}\partial \log |z - z'|^2 = 2\pi \delta^2(z - z'), \quad (2.85)$$

and so $(\bar{\partial}\partial)^{-1} = -\log|z - z'|$. Putting it all together

$$\chi(is) \sim e^{-\frac{s^2}{2\pi} \log 2L}. \quad (2.86)$$

We then have

$$P(N_A) \sim e^{-\frac{\pi N_A^2}{2 \log 2L}}. \quad (2.87)$$

This result is in accordance with our intuition: the typical fluctuations are Gaussian or normally distributed similar to how they were for the binomial distribution. The fact that we have a variance proportional to the logarithm of the system size is also expected because in the 1D setup we are considering the particles do not have much room to spread out (compared to higher dimensions) and so their correlations grow *slower* than the system size, $2L$. As mentioned above, this variance can also be linked to the entanglement entropy which is known to scale like the logarithm of the system size in one dimension [57, 58, 59, 60, 61].

2.5 Summary

This section has provided an introduction to the fascinating area of large deviation theory. We have seen it arising in a simple coin-tossing example, in the imaginary time path integrals of quantum mechanics and Coulomb gas models. In addition, we have armed the reader with knowledge of FCS. These are all the techniques we will need in the rest of this work.

CHAPTER 3

COMPENDIUM OF LIMIT SHAPES

The manuscript we present in this section can be read as a self-contained miniature version of this thesis, highlighting all of its main results and some extensions. Equipped with the background theory this section should be eminently understandable.

3.1 *Compendium of free fermion limit shapes*

The following is the manuscript for *Compendium of free fermion limit shapes* written by myself. I thank my supervisor Dimitri Gangardt for some helpful revisions and clarification on some points.

This work contains no new results, though it does generalise several old ones and re-derives key results in a new way. These are indicated clearly in the work. This manuscript is intended for future publication.

Compendium of free fermion limit shapes

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Abstract. We present a compendium and review of free-fermion limit shapes.

1. Introduction

Large deviation theory is an attempt to model systems undergoing rare fluctuations [1, 2]. In the context of many-body systems of free fermions in (1+1)D, these rare fluctuations take the form of instanton-like configurations known as limit shapes. Essentially, if one waits long enough, the fermions have a chance to wildly depart from their equilibrium configuration and as they do so the spacetime trajectories they follow trace out boundaries that look like shapes. For example, one can look at the rare event in which a once homogeneous line of fermions leaves an entire region of space empty, known as the emptiness formation probability (EFP) [3]. It was shown that the boundary between occupied and unoccupied sites took the shape of an astroid. One can also look at the rare event where the fermions start in a domain wall state and return to that same state at a later time, where one finds a circular boundary with the limit shape dubbed an 'arctic circle' [4, 5].

Progress has been made on finding these free fermion limit shapes via a semi-classical hydrodynamic approach ‡. This approach focuses on collective fields, a density $\rho(x, \tau)$ and a velocity field $v(x, \tau)$ that encode all the long-wavelength dynamics of the system. Recently, an alternative approach was introduced to address what happens when two arctic circle limit shapes meet [8]. The problem was mapped to the Gross-Witten Wadia (GWW) matrix model [9], which had a representation as a Coulomb gas [10]. We showed that a third-order phase transition occurred when the limit shapes touch. The approach we used was focused not on the real-space hydrodynamics, but momentum-space instead, i.e. we introduced a space $x(\rho, v)$ and a time $\tau(\rho, v)$ field and then solved the problem in terms of those variables. In this manner it is very similar to the Generalised Hydrodynamics and Quantum Generalised Hydrodynamic approaches [11, 12, 13]. How our work differs is our focus on the *large* deviations in such systems, i.e. we study *atypical* fluctuations, specifically the formation of limit shapes and their phase transitions. Part of this work is to exploit this approach further to obtain results pertaining to not only old limit shapes but new ones as well. We emphasise how this new approach, in contrast to the old one, can deal with limit shape *transitions*. We present all these findings as a compendium.

We will also discuss a complimentary third formulation of the problem in terms of Riemann surfaces which are in one to one correspondence with the limit shapes we find. These Riemann surfaces are shown to solve a minimal surface problem offering yet another formulation of the problem. We show that it is best suited at describing both the perturbative and non-perturbative behaviour observed in limit shapes and can give a topological description to their transitions.

This paper is intended as an introduction to and review of free-fermion limit shapes and their transitions. As such no new results are presented, though we are able to

‡ Though we will be referring to free fermions throughout this work, bear in mind that numerous other systems such as directed Polymers [6] and random walkers with non-intersecting constraint [7] can be mapped onto them

generalize a few old ones. We hope that by forming all the work in a compendium the reader can get a better feel for this rich and interesting area of physics.

The layout is as follows: the first three sections introduce the different formulations of the limit shape problem with the rest of the work devoted to giving examples of limit shapes and discussing their features. Section 2 discusses the conventional real-space hydrodynamic formulation and explains why a different approach is needed. Section 3 introduces this approach which we later show is more useful in helping us find limit shapes and their transitions. In Section 4 we give yet another formulation of the problem in terms of minimal surfaces. This formulation is most useful for studying fluctuations in limit shape systems. With the different formalisms established, the bulk of the work is then given by the Compendium in Section 5 where we present known and new limit shapes. We then discuss their perturbative and non-perturbative behaviour in Section 6 and conclude in Section 7.

2. Hydrodynamic formulation

To motivate our new approach we begin with a review of the real-space approach to the hydrodynamics of free fermions. We begin with the microscopic action for N such free fermions, with dispersion $\varepsilon(k)$, which is given by

$$S[\{k(\tau)\}, \{x(\tau)\}] = R \int_{\tau_i=-\tau_f}^{\tau_f} d\tau \sum_l^N \left[ix_l \dot{k}_l + \varepsilon(k_l) \right] + \text{B.C.}(k_l, x_l), \quad (1)$$

where we have introduced some as yet unspecified boundary conditions, B.C.. In the spirit of large deviation theory we have scaled the imaginary time, τ , by R which will be taken to infinity. Physically, this means that we are waiting an infinite amount of time so that a rare fluctuation, like a limit shape, may occur. Mathematically, it means that if one were to evaluate the partition function

$$Z = \sum_P (-1)^P \int_{x_l(\tau_f)=x_{P_l}(\tau_i)} \mathcal{D}x \mathcal{D}k e^{-S}, \quad (2)$$

where $\mathcal{D}x = \prod_i dx_i$, $\mathcal{D}k = \prod_i \frac{dk_i}{2\pi}$, it will be dominated by the saddle-point configuration of the k_l 's and x_l 's which minimise the action. Notice that we have enforced the antisymmetry properties of the fermions in the path integral and, due to working in imaginary time, we have periodically identified configurations at $\tau_{f,i}$. It was found in [8, 14] that when we evaluate this partition function, to ensure we get a finite answer, we have to normalise it against some reference configuration of the fermions, typically one in which the limit shape is not present. This is because, implicit in our calculations of these large deviations is the idea that we have driven the particles away from their equilibrium and this density fluctuation has to be normalised against its unperturbed state. As such the N particles being referred to above are to be most accurately thought of as *quasi-particle* excitations on top of this reference configuration. These quasiparticles

are responsible for forming the limit shape. Without normalising against a background the path integral diverges. For simplicity's sake we will refer to the quasi-particles as fermions in the rest of this work but one should bear in mind this subtle point.

The variation of the action w.r.t. $x_l(\tau)$, $k_l(\tau)$ inside the time interval leads to the ballistic motion

$$k_l(\tau) = k_l(0) \equiv k_l \quad (3)$$

$$x_l(\tau) = x_l(0) - i\varepsilon'(k_l)\tau. \quad (4)$$

Knowing the precise form of the boundary conditions we could then, in principle, (numerically) solve the resulting equations exactly. Analytically, it is far easier to move to a collective description in terms of a density $\rho(x, \tau)$ and velocity field $v(x, \tau)$. In the limit $N \rightarrow \infty$ this description becomes exact because the fluctuations due to individual particles are washed out. This is the statement of hydrodynamics. To make this transition precise, we must first introduce the phase space distribution (Wigner function) of the free fermions $W(x, k, \tau)$. In the large N limit it is a function defined such that it takes the value 1 in the region filled with fermions and 0 in the region where they are absent (filled with holes). It defines the time-dependent boundary of a 'blob' in phase space, the Fermi surface Γ_τ shown in Fig.1. We should think of this 'blob' as a fluid whose motion is described by some hydrodynamical theory. $W(x, k, \tau)$ allows us to calculate both the momentum- and real-space densities in a symmetrical fashion

$$\sigma(k, \tau) = \frac{1}{2} \int dx W(x, k, \tau) = \frac{1}{2} [x_+(k, \tau) - x_-(k, \tau)], \quad (5)$$

$$\pi\rho(x, \tau) = \frac{1}{2} \int dk W(x, k, \tau) = \frac{1}{2} [k_+(x, \tau) - k_-(x, \tau)]. \quad (6)$$

With these considerations, in the large N limit, we may rewrite Eq. (1)

$$S = R^2 \int d\tau \int \frac{dx dk}{2\pi} (i(\partial_x)^{-1} \partial_\tau W(x, k, \tau) + \varepsilon(k)) W(x, k, \tau) + \text{B.C.}(W) \quad (7)$$

Note the factor of R^2 in front of the integral, arising from re-scaling the time and space coordinate. Using the properties of the Wigner function we can also write

$$S = R^2 \int d\tau \int dx \left\{ \frac{i}{4\pi} [k(\partial_x)^{-1} \partial_\tau k - \bar{k}(\partial_x)^{-1} \partial_\tau \bar{k}] + \int_{-\bar{k}}^k \frac{dk}{2\pi} \varepsilon(k) \right\} + \text{B.C.}(k, \bar{k}), \quad (8)$$

where we have introduced the complex fermi point $k(x, \tau) = \pi\rho(x, \tau) + iv(x, \tau)$. This action was found first in [3].

We now vary the above w.r.t $k, \bar{k}$ to find the equations of motion

$$\partial_\tau k - i\partial_x \varepsilon(k) = 0, \quad (9)$$

and conjugate. Separating this into real and imaginary parts, we arrive at Euclidean versions of the continuity and Euler equations. These describe the motion of our fermi-fluid shown in Fig.1.

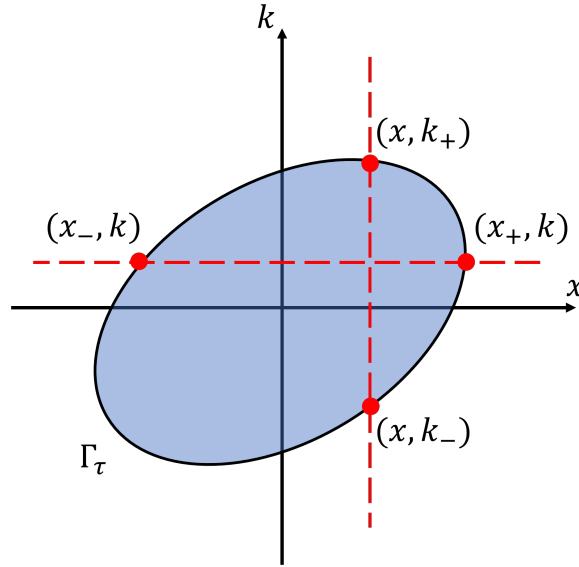


Figure 1. Depiction of a typical phase space and its boundary, Γ_τ , at some arbitrary time. We depict the two fermi points $k_\pm$, conversely $x_\pm$. $W(x, k, \tau) = 1$ inside the 'blob' and $W(x, k, \tau) = 0$ outside.

A formal way to solve Eq. (9) is by using what is known as the Hodograph transform [15], one can also use the method of characteristics however the former is more readily generalised to interacting systems, see e.g. [16]. Begin by applying the chain rule to write derivatives w.r.t $k, \bar{k}$ in terms of x, τ , which we may express by the following matrix equation

$$\begin{pmatrix} \partial_k \\ \partial_{\bar{k}} \end{pmatrix} = \begin{pmatrix} \partial_k x & \partial_k \tau \\ \partial_{\bar{k}} x & \partial_{\bar{k}} \tau \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_\tau \end{pmatrix}. \quad (10)$$

Inverting we have

$$\begin{pmatrix} \partial_x \\ \partial_\tau \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \partial_{\bar{k}} \tau & -\partial_k \tau \\ -\partial_{\bar{k}} x & \partial_k x \end{pmatrix} \begin{pmatrix} \partial_k \\ \partial_{\bar{k}} \end{pmatrix}, \quad (11)$$

$$J = \partial_k x \partial_{\bar{k}} \tau - \partial_k \tau \partial_{\bar{k}} x, \quad (12)$$

where J is the Jacobian, assumed to be non-zero. We can then eliminate the x, τ derivatives in Eq. (9) in favour of $k, \bar{k}$ and in doing so interchange the dependent and independent variables of the problem. We arrive at

$$\partial_{\bar{k}} x + i\varepsilon'(k) \partial_{\bar{k}} \tau = 0, \quad (13)$$

and conjugate who's formal solution is

$$x(k, \bar{k}) + i\varepsilon'(k) \tau(k, \bar{k}) = g(k), \quad (14)$$

and conjugate, for some analytic function $g(k)$ to be found from the boundary conditions. Note that Eq. (14) is the continuous analog of Eq. (4) with $g(k)$ playing the role of $x_l(0)$.

Writing the above as $F_0(x + i\varepsilon'(k)\tau = X, k) = X - g(k) = 0$ we have what is known as an *algebraic* or *spectral* curve [10, 17]. It contains all the information, semi-classically, about the distribution of the fermions. For example, we may solve it to find $\rho(x, \tau), v(x, \tau)$ thus accessing their density and velocity distributions. It is clear that we have now reduced the problem of free fermion hydrodynamics to finding the function $g(k)$.

One trivial way to do this is to have the boundary conditions specify $g(k)$ at $\tau = 0$ and then ballistically propagate to find the solution at all times, a setup we depict on the LHS of Fig.2. Essentially, we contrive the boundary conditions to fix $k = g^{-1}(x)$ at $\tau = 0$ such that $F_0(x, k) = x - g(k) = 0$. Via ballistic propagation we find the curve at all times: $F_0(X, k) = X - g(k) = 0$. In this work we would like to go beyond

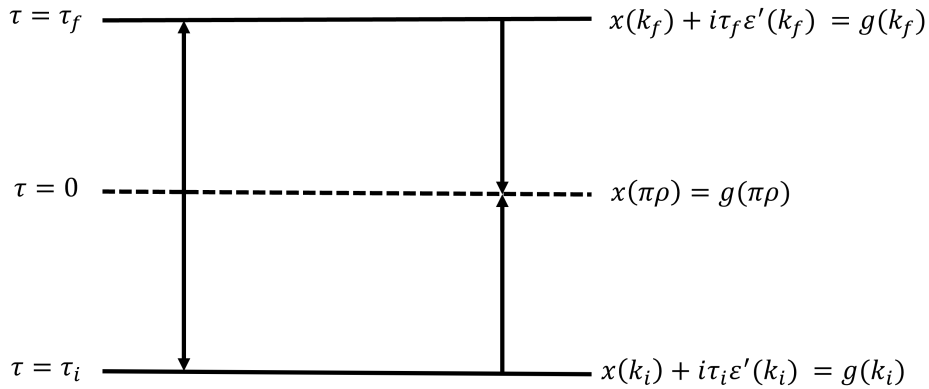


Figure 2. Depiction of ballistic propagation of algebraic curve. The double-headed arrow on the left represents the evolution in time of a given density profile from $\tau = 0$. The single arrows on the right show time evolution forwards/backwards from a given initial/final density profile.

such a simple example. Matytsin, in their analysis of the large N limit of the Itzykson-Zuber integral, pointed out the difficulty of doing this [18]. They came across similar equations to (9, 14) and tried to find $g(k)$ subject to *arbitrary* boundary conditions: specifying the distribution at the initial time, $\text{Re}(k(x, \tau_i)) = \pi\rho_i$, and then calculating the trajectory that would lead to a different distribution, $\text{Re}(k(x, \tau_f)) = \pi\rho_f$, at the final time. This more complex situation is depicted on the RHS of Fig.2. From the figure it is clear that the two situations: specifying $g(k)$ at $\tau = 0$, or finding it from boundary conditions imposed at $\tau_{i,f}$ are related via ballistic propagation. However, because for the initial/final conditions we have to propagate forward and backward in a self-consistent way such that the solutions match at $\tau = 0$, it turns out to be a much harder problem.

One example that has been worked out is if the initial and final conditions are the domain wall states: $\pi\rho_i = \pi\rho_f = \theta(x)$. For the choice of dispersion $\varepsilon(k) = -\cos k$ the correct answer is $g(k) = -\cos k$ so that the algebraic curve is [5, 19]

$$x + i \sin k\tau + \cos k = 0. \quad (15)$$

Plotting the resulting density profile, we get Fig.3. The boundary conditions have led to the formation of a famous limit shape, the arctic circle [4, 5]. One can see the

characteristic boundary, in this case a circle, separating the space between frozen, non-fluctuating regions where $\rho = 0, 1$ and a liquid interior where the particles are allowed to fluctuate and $\rho \in (0, 1)$. To find the solution, Eq. (15), in the hydrodynamic approach

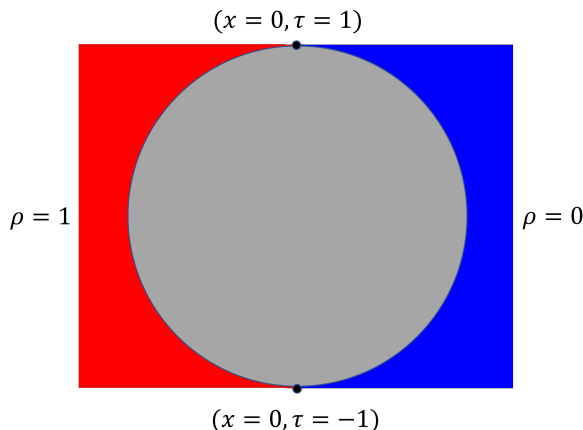


Figure 3. Arctic circle limit shape. We indicate the right and left 'frozen' regions where the density is fixed at $\rho = 0, 1$ and the grey, 'liquid' interior where $\rho \in (0, 1)$. For simplicity, we have chosen $\tau_f = -\tau_i = 1$. One can see the domain walls of fermions at $\tau = \pm 1$, centred at $x = 0$

takes a great deal of work because of the difficulty in finding a suitable $g(k)$, one that correctly implements the boundary conditions at $\tau = \pm 1$. In this case the setup is simple enough that one can arrive at Eq. (15) by observing that when $\tau = 1$ we have

$$x = -e^{i\pi\rho-v}, \quad (16)$$

and, because x is real, ρ must jump between 0 and 1 as it is varied (similar considerations apply at $\tau = -1$) and so it does correctly implement the domain wall or step function boundary conditions. But now, suppose we change the boundary conditions so that we demand that we have a 'top-hat' instead of a step function at either end, i.e. $\pi\rho_i = \pi\rho_f = \theta(x - \lambda) - \theta(x + \lambda)$. What happens now? Intuitively, we might say we would get two circles, assuming the top hat was wide enough, because it's the sum of two step functions. But what needs to change in our algebraic curve to incorporate both solutions? And what if the top hat was narrow and the two circles touched? It is not manifestly obvious what to do in these situations, nor does the hydrodynamic approach give you a recipe to find out for yourself. The more mathematical literature on limit shapes can accommodate more than one shape and even discusses limit shape transitions, see, e.g. [20], so it is possible, but it is not clear how to do that here.

These concerns motivate the need to find an approach in which $g(k)$ is given to us, removing the need to guess. This is what was done in [8] and will be the focus of the rest of this work. It was found that for the top-hat initial and final conditions described above that it was possible to determine $g(k)$ precisely without the need to guess by exploiting a novel mapping between the microscopic action Eq. (1) with the top-hat

domain wall boundary conditions and a 'Coulomb gas' model (Section 3) §, the latter of which amounts to a formulation of the problem in terms of the collective variable $\sigma(k)$ as opposed to $k(x, \tau)$. It was then further shown that the so-called loop equation [25, 26, 27, 28, 29] encoding the solution to the Coulomb gas model was equivalent to the algebraic curve $F_0(x, k)$. Recalling that if we know the form of the curve at $\tau = 0$ we can always ballistically propagate forward and backward to find the whole curve, we see that finding the loop equation is the same as finding $g(k)$.

3. Coulomb gas formulation

The limit shapes we find all derive from a particular class of matrix/Coulomb gas models who's action, in the large N limit, can be written

$$S[\sigma, \mu] = -\frac{1}{2} \int_{\gamma} \frac{dk}{2\pi} \frac{dk'}{2\pi} \sigma(k) \log \left| e^{ik} - e^{ik'} \right|^2 \sigma(k') - \int_{\gamma} \frac{dk}{2\pi} V(k) \sigma(k) + \sum_i \mu_i \int_{\gamma_i} \frac{dk}{2\pi} (\sigma(k) - \nu_i), \quad (17)$$

with partition function

$$Z = \int \mathcal{D}\sigma \prod_i d\mu_i \exp\{-4R^2 S\}. \quad (18)$$

Eq. (18) is understood to be equivalent to the partition function Eq. (2).

The above represents the energy of a configuration of 'charged' particles interacting via a logarithmic Coulomb potential and some external potential $V(k)$. The form of this potential varies depending on the setup of the problem. In the models considered in Sections 5.1, 5.2 (studied in [8]) the potential is the fermion's dispersion $V(k) = \varepsilon(k)$. For the model used in Sections 5.3, 5.4 (studied in [14]), $V(k) = \alpha\theta(k_F - |k|)$. This step-function potential constrains the fermions to lie in an interval of length 1 (in units of R), its strength being characterised by the control parameter α . The density of the particles is given by the counting function $\sigma(k) = (\pi/R) \sum_i \delta(k - k_i)$ and we assume they distribute themselves along some contours γ_i , who's union we denote γ . The total number of particles residing in each of these contours, ν_i , is fixed by variation of the action w.r.t the Lagrange multipliers μ_i and $\sum_i \nu_i = N/R$. As above, N is the number of particles participating in the limit shape or the number of quasi-particle excitations on top of the reference configuration. Formally, the equations of motion of the above are

$$\frac{\delta S[\sigma]}{\delta \sigma} = 0 \implies -V(k) - \int_{\gamma_i} \frac{dk'}{2\pi} \log \left| e^{ik} - e^{ik'} \right|^2 \sigma(k') + \mu_i = 0, \quad (19)$$

$$\frac{\partial S[\sigma]}{\partial \mu_i} = 0 \implies \int \frac{dk}{2\pi} \sigma(k) = \nu_i. \quad (20)$$

§ Coulomb gas models are intimately linked to matrix models which have attracted a great deal of attention in areas ranging from large N Yang-Mills theory to 2D quantum gravity [9, 21, 22, 23, 24].

We can represent their solution by doing the following. First, we can define a function

$$x(k) = 2 \int_{\gamma_i} \frac{dk'}{2\pi} \frac{e^{ik}}{e^{ik} - e^{ik'}} \sigma(k') - iV'(k). \quad (21)$$

This function obviously has the property that

$$x(k \pm i0) = x_{\pm}(k) = \pm \sigma(k), \quad (22)$$

which we can choose to represent as the algebraic curve or 'loop equation'

$$F_0(x, k) = x^2 - \sigma^2(k) = 0. \quad (23)$$

What form $\sigma(k)$ takes depends on the form of the potential involved and the choice of contours γ_i , [10] and [30] provide further detail on this and we will discuss a particular case below. The important thing to note is, like with the hydrodynamics, the solution can be represented by an algebraic curve. In fact, the choice of name for our function $x(k)$ is no coincidence. For, in [8, 14] it was shown that for some specific Coulomb gas models derived from a microscopic system of free fermions this function, $x(k)$, was indeed nothing but the same $x(k, \bar{k})$ appearing in the algebraic curves derived in the hydrodynamic, real-space approach, taken along the slice $v = 0$. Physically, then, $x_{\pm}(k)$ represent the to edges of the Fermi surface in Fig.1. The upshot is that the function we were after $g(k \pm i0) = \pm \sigma(k)$ and therefore the solutions to the Coulomb gas model provided solutions to the hydrodynamic one. This is most obvious with the following factorisation

$$F_0(x, k) = (x - \sigma(k))(x + \sigma(k)) = 0. \quad (24)$$

i.e. the product of two algebraic curves, both of which looking like a solution to the hydrodynamic equations of motion. It was further shown that ballistic evolution worked much the same in both cases and so could be introduced as

$$F_0(x + i\epsilon'(k)\tau = X, k) = X^2 - \sigma^2(k) = 0. \quad (25)$$

where k is now complex.

As an example of how this all plays out in practise, in [8] the authors considered a microscopic system of free fermions with top-hat initial and final conditions. We discussed above how we lacked intuition about this problem in the real-space hydrodynamics. It was found that the problem could be mapped to a Coulomb gas model, one that was known beforehand [9] and amounts to choosing $V(k) = \varepsilon(k) = -\cos k$ with one contour $\gamma_1 : k \in (-k_c, k_c)$ and $\nu_1 = \lambda$. As k_c is decreased from π this model was shown to exhibit a phase transition. The solution could be found by considering the equation $x_+(k) + x_-(k) = 0$ which one can check translates to the following

$$\int_{\gamma} \frac{dk'}{2\pi} \cot \frac{k - k'}{2} \sigma(k') = \sin k, \quad (26)$$

which is nothing but the derivative of Eq. (19). If one chooses $k_c = \pi$ such that $k' \in (-\pi, \pi)$ the left hand side is nothing but the definition of the Hilbert transform. The transform essentially amounts to replacing every incidence of a sine with a cosine and vice-versa. This immediately determines $\sigma(k) = a + \cos k$ where a is a constant. This constant can then be fixed by the normalisation constraint Eq. (20) $a = \lambda$. When one plots the density profile resulting from Eq. (25) one finds the appearance of two arctic circles (each one corresponding to a solution of the curve $\pm\sigma(k)$) Fig. 4. We will discuss even further this solution and many more like it later on in section 5.

The takeaway at this stage is that the Coulomb gas formalism has managed to provide a rigorous way of finding $g(k)$ given a set of boundary conditions. As discussed before, this is a very hard problem. The obvious limitations to this method as it stands are the fact that there is no guarantee for an arbitrary Coulomb gas model that the function we introduced, $x(k)$, can be identified with the same $x(k)$ coming from the hydrodynamics. Where this identification has been made it has always been done by starting from a microscopic configuration of free fermions, like Eq. (2) and working from there.

4. Minimal surface formulation

We now provide a third and final formulation of free fermion hydrodynamics in terms of minimal surfaces. This formulation will ultimately prove useful later on when we analyse fluctuations in limit shape systems. Ignoring for the moment the Lagrange multipliers, one may rewrite Eq. (17)

$$S[\chi_{\pm}] = \oint_{\gamma} \frac{dz}{4\pi i} (\chi_- \partial \chi_+ - \chi_+ \partial \chi_-) + \oint_{\gamma} \frac{dz}{2\pi i} (V_-(z) \partial \chi_+ - V_+(z) \partial \chi_-), \quad (27)$$

if we introduce a new field

$$\chi(z) = \int_{\gamma} \frac{dk}{2\pi} \log(z - e^{ik}) \sigma(k), \quad (28)$$

and split V and χ into their components $\pm$ analytic inside/outside the unit circle, i.e. $\chi_{\pm} = \chi(z \pm i0)$. Note that as $z \rightarrow \infty$

$$z \partial \chi(z) \sim \int \frac{dk}{2\pi} \sigma(k) = N/R, \quad (29)$$

and as a consequence of Eq. (19)

$$\partial \chi_{\pm} = -\partial V_{\pm} \mp i \frac{\sigma(z)}{z} = -\partial V_{\pm} + i \frac{g_{\mp}}{z}. \quad (30)$$

Employing Stoke's theorem

$$\int_D d^2 z (\bar{\partial} f + \partial \bar{f}) = \frac{1}{2i} \oint_{\partial D} (f dz - \bar{f} d\bar{z}),$$

we may write the action more compactly as

$$S[\chi] = \frac{1}{\pi} \int_D |\partial\chi + \partial V|^2 d^2z, \quad (31)$$

where $\partial D = \gamma$. A version of this action has been found before in [31]. Using Eq. (30) we find

$$S[g] = \frac{1}{\pi} \int_D |g|^2 d^2z. \quad (32)$$

One can choose to view $\frac{1}{2}|g(z)|^2 d^2z = dA$ as an area element defined on a Riemann surface, see the discussion towards the end of [Appendix A](#). Doing so allows us to make the rather succinct statement that on the saddlepoint solution we can write the action in terms of the classical area of the Riemann sheets, A_{cl} , something noted in [32, 33, 34]. Therefore

$$Z \sim e^{-8R^2 \left(\frac{A_{\text{cl}}}{\pi}\right)}. \quad (33)$$

Thus, viewed in this way, the logarithm of the probability of a limit forming *is proportional to the area of its Riemann surface*. This description in terms of Riemann surfaces is a particularly fruitful one. Using it as a jumping off point one can begin to study entanglement entropy following methods described in [31, 35, 36, 37] which consider partition functions taken over Riemann surfaces. Later, we will be using it to study fluctuations in limit shape systems, like what was done in [19], and will be employing very similar techniques to [12, 13, 38, 39, 40] as well.

What we have found is compelling evidence that the hydrodynamical problem can be reformulated as a minimal surface one, where we must find the function $g(z)$ which minimises the area of the Riemann surface subject to a set of constraints. What remains is to not only identify these constraints but make sure they uniquely specify a solution to the problem.

One obvious constraint to impose is that $g(z)$ must have a certain number, s , of branch cuts which are the contours γ_i whose endpoints are either poles or zeroes (for simplicity we always assume they reside along the real axis, $v = 0$). The union of these contours, $\gamma = \prod_i \cup \gamma_i$ are dictated by the number of cuts the solutions should have. Another constraint is imposed by the potential, $V(z)$, which influences the form of $g(z)$ via the asymptotics of Eq. (30). At large z we find that

$$i \frac{g(z)}{z} \underset{z \rightarrow \infty}{\sim} \frac{\sum_i \nu_i}{z} + V'(z) + O(1/z^2). \quad (34)$$

We assume we are also given the total number of particles $N/R = \sum_i \nu_i$.

It turns out that given the s cuts, the potential and the total number of particles we have enough information to uniquely specify $g(z)$. To see this, let us call the degree of $zV'(z)$, also the degree of $V(z)$, d . Eq. (34) then gives us access to $d + 1$ equations, one from the normalisation using the fact that we know the total number of particles

and d from knowing the form of the potential. We know that we have s number of endpoints to find. Therefore a unique solution can be found if $s \leq d + 1$. If $s > d + 1$ then we would need to either start specifying the endpoints of the cuts explicitly or start specifying the filling fractions, ν_i . Let's call the number of extra equations we introduce f . We find that there is a unique solution if $s \leq d + 1 + f$.

In corollary, the hydrodynamic problem can be reformulated as a minimal surface one where we must find the unique minimal surface $g(z)$ that satisfies the following constraints:

- $g(z)$ has s number of branch cuts whose endpoints are either poles or zeroes.
- The asymptotics of $g(z)$ are fixed by the total particle number and the form of the potential $V(z)$ as

$$i \frac{g(z)}{z} \underset{z \rightarrow \infty}{\sim} \frac{\sum_i \nu_i}{z} + V'(z). \quad (35)$$

- $s \leq d + 1 + f$ where d is the degree of $V(z)$ and f the number of endpoints plus filling fractions we specify.

Together, these conditions amount to a *completely equivalent* formulation of the hydrodynamic problem we began with.

5. Compendium of limit shapes

In what follows, we will illustrate several instances where we have explicitly shown a link between the (ballistically-evolved) loop equations from Coulomb gas models and the algebraic curves from hydrodynamics. The algebraic curves we find are all special because they have all been shown to lead to the formation of limit shapes, in much the same way as the algebraic curve Eq. (15) was shown to lead to the Arctic Circle. Essentially, we use the Coulomb gas formalism as an especially powerful tool to study large deviations in free fermion systems. We want to stress that only for the correspondences shown below has this link been proven. There is no *a priori* guarantee that any given solution to a Coulomb gas model will correspond to a limit shape or indeed that its density $\sigma(k)$ will correspond to a physically reasonable $g(k)$. Indeed, for the models where it has been proven, the starting point has always been a microscopic system of free fermions with specific constraints put in place which one would expect would then lead to the formation of a limit shape, e.g. the domain wall constraint of the single arctic circle limit shape discussed before.

As a useful aid, we will draw the limit shapes in both the (x, τ) and (ρ, v) planes. This is easily done by solving $F_0(X, k) = 0$ to express (x, τ) in terms of (ρ, v) or *vice-versa*. Bearing in mind their domains, $\rho \in (0, 1)$ and $v \in (-\infty, \infty)$, the limit shapes drawn in the (ρ, v) plane will manifest themselves as infinite Riemann sheets (for more information on the link between algebraic curves and Riemannian geometry see [Appendix A](#)). These Riemann sheets can be joined via their branch cuts to form the Riemann surfaces mentioned in the previous section.

5.1. Separated arctic limit shapes

We have discussed above the generalisation of the arctic circle problem where now $\pi\rho_i = \pi\rho_f = \theta(x - \lambda) - \theta(x + \lambda)$, creating a 'top-hat' domain wall. Our naive (and ultimately correct) expectation was that there will be two arctic circles, one at each end of the wall. We remind that the equivalent Coulomb gas model was found in [8] and corresponds to the choice $V(k) = \varepsilon(k) = -\cos k$ and $\gamma = \gamma_1 : k \in (-\pi, \pi)$. The associated algebraic curve or loop-equation is

$$F_0(x, k) = x^2 - (\lambda + \varepsilon'_H(k))^2 = 0, \quad (36)$$

i.e. $g(k) = (\lambda + \varepsilon'_H(k))$. We use $\varepsilon'_H(k)$ to denote the Hilbert transform of $\varepsilon'(k)$. It should be clear that such a result will be valid for other dispersions as well, not just $\varepsilon(k) = -\cos k$, however with other dispersions one must be careful of so-called 'crazy regions' [41]: regions where the density diverges and falls outside the range $(0, \pi)$. The parameter λ is related to the number of particles participating in the limit shape: $\lambda = N/R$. One can check that by including it in the answer we satisfy the normalisation constraint. In the paper we showed that by ballistic evolution of Eq. (36) we in fact generated solutions to the hydrodynamical problem with the 'top hat' domain wall boundary conditions, i.e.

$$F_0(X, k) = X^2 - (\lambda + \varepsilon'_H(k))^2 = (x + i\varepsilon'(k)\tau - g(k))(x + i\varepsilon'(k)\tau + g(k)) = 0, \quad (37)$$

with $g(k) = \lambda + \varepsilon'_H(k)$. To substantiate our arguments, we plot the density profile and mapping $(\rho, v) \leftrightarrow (x, \tau)$ of this solution in Fig.4. The two Riemann sheets in the (ρ, v) plane manifest themselves in the (x, τ) plane as two arctic circle limit shapes exactly as expected. Intuitively, so long as the shapes are separated, incorporating multiple shapes amounts to multiplying the algebraic curves of the individual shapes together. By doing so we essentially build higher and higher order algebraic curves e.g. $X^3, X^4 \dots X^m$ which can always then be factorised into a product of m first order curves with each curve describing one limit shape.

We may also recover the single arctic limit shape from Eq. (37). Shifting the origin $x \rightarrow x - \lambda$ the curve becomes

$$F_0(x, k) = (x - \varepsilon'_H(k))(x + \varepsilon'_H(k) + 2\lambda) = 0. \quad (38)$$

Dividing by λ and sending it to infinity we have

$$F_0(x, k) = x - \varepsilon'_H(k) = 0, \quad (39)$$

for the one limit shape. Sending $\lambda \rightarrow \infty$ merely reflects the fact that for one circle the reservoir of particles/holes to the left/right of the domain wall is infinite. In Fig.4 we can imagine dragging the left circle to infinity, elongating the filled fermi-sea at the top and bottom an infinite amount leaving just the rightmost circle on its own.

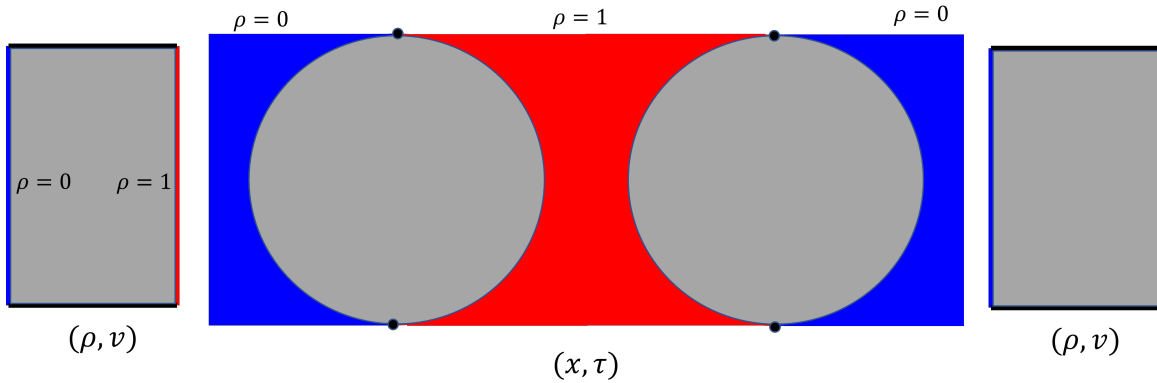


Figure 4. Mapping between the (ρ, v) plane and the (x, τ) plane for the two arctic circle problem with dispersion $\varepsilon(k) = -\cos k$. The Riemann sheet *topology* remains unchanged for other dispersions. The mapping provided by (36) takes the two sheets residing in the (ρ, v) plane and maps them into the (x, τ) plane with each sheet corresponding to a circle. The entirety of the left and right edges of the sheets - which for this rest of this work are always located at $\pi\rho = 0, \pi$ respectively- are mapped to the unoccupied and fully-occupied frozen regions of fermions whilst $v = \pm\infty$ (the top and bottom lines of the sheets) maps to the single black points. These figures draw inspiration from one found in [19].

5.2. Merged arctic limit shapes

For the critical value of $\lambda = \lambda_c = |\varepsilon'_H(\pi)|$ simple geometrical arguments tell us that the two limit shapes will touch. In the hydrodynamics we can say that the solutions to Eq. (9) become multi-valued, though it is not too clear how to remedy this multi-valuedness. From the coulomb gas or Riemann sheet perspective it is quite straightforward: we must introduce a branch cut. Such a branch cut will connect the two Riemann sheets and in (x, τ) space this manifests itself as the limit shapes merging, creating a new shape. This solution will be referred to as a 'one-cut' solution: $\gamma = \gamma_1 : k \in (-k_c, k_c)$. The Coulomb gas model and hence loop equation/algebraic curve for $\varepsilon(k) = -\cos k$ was found in [8]. In fact, a generalisation to arbitrary dispersion can be found in [42, 43]. Labelling $\varepsilon(k) = \sum_{l=0}^n \alpha_l \cos(lk)$ we find

$$F_0(X, k) = X^2 - \left(2\sqrt{\sin^2 \frac{k_c}{2} - \sin^2 \frac{k}{2}} \sum_{l=1}^{\infty} Q_l \cos \left(\left(l - \frac{1}{2} \right) k \right) \right)^2 = 0, \quad (40)$$

for $-k_c < k < k_c$ and $X(k) = 0$ otherwise with

$$Q_l = \sum_{m=0}^n \frac{\alpha_{l+m}(l+m)}{\lambda} P_m(\cos(k_c)). \quad (41)$$

P_m are the Legendre polynomials and k_c is determined by the criterion $Q_1 = Q_0 + 2$. Solutions exist if one can find real k_c satisfying the above. For example, for $n = 1$,

$\alpha_1 = 1$, equivalent to the GWW matrix model [9], one finds $Q_0 = \frac{q}{\lambda} \cos k_c$, $Q_1 = \frac{1}{\lambda}$, $\sin^2 \frac{k_c}{2} = \lambda$ and so

$$F_0(X, k) = X^2 - \left(2 \cos \frac{k}{2} \sqrt{\sin^2 \frac{k_c}{2} - \sin^2 \frac{k}{2}} \right)^2 = 0. \quad (42)$$

We present the $(\rho, v) \leftrightarrow (x, \tau)$ mapping of the solution Eq. (42) in Fig.5. The limit shape is shaped like a nephroid.

This novel 'limit shape transition' that we have uncovered has a clear interpretation in the Coulomb gas model where it undergoes a third order phase transition when $\lambda = \lambda_c$. That is, its free energy, $-4R^2S$ exhibits a discontinuity in its third derivative w.r.t. λ . Calculation of this free energy can be found in [8].

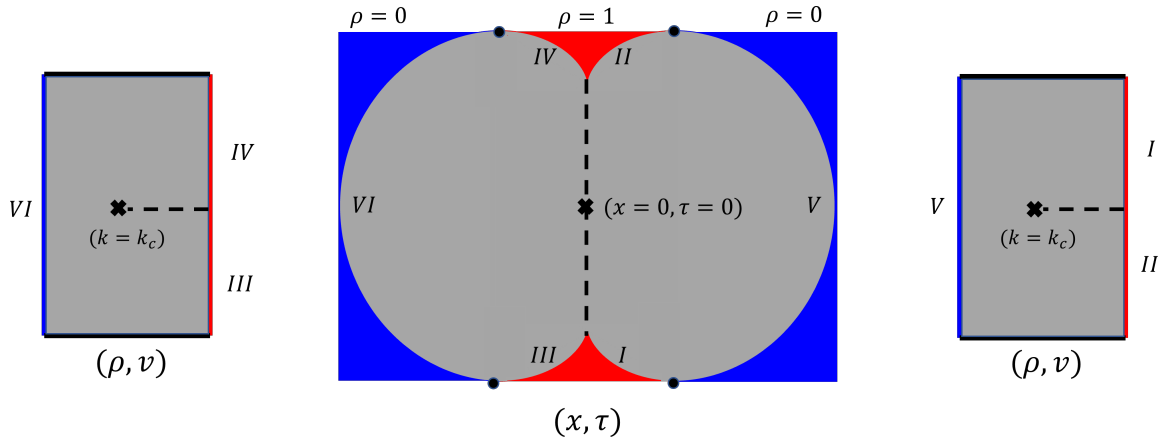


Figure 5. Merged arctic circle limit shapes shown for $n = 1, \alpha_1 = 1$. The topology of the Riemann sheets does not change with other choices of dispersion (there remains just the one branch cut). The sheets meet along the branch cut indicated by the horizontal dashed lines at $v = 0$. This branch cut corresponds to the vertical dashed line in the x, τ plane. The branch point is represented by the cross and the labels I, II, III, IV, V and VI indicate which parts of the boundary curve map to which parts of the Riemann sheets.

5.3. Separated EFP limit shapes

We now impose a new type of boundary condition, the so-called emptiness condition [3]. That is that $\pi \rho_i = \pi \rho_f = 0$ for $x \in (-R/2, R/2)$. With the present scalings we set $R = 1$ and the solutions then encode information about the full counting statistics (FCS) of free fermions: telling us the likelihood of finding a given number of fermions within an interval of size 1. The equivalent Coulomb gas model was found in [14]. It corresponds to choosing $V(k) = \alpha \theta(k_F - |k|)$ which enforces the emptiness constraint by penalising particles for residing in this interval, with a strength α . As this strength is tuned to infinity no particles reside in the interval, which results in the EFP [3] mentioned in the

introduction. For arbitrary potential some particles are allowed to reside in the empty interval and there was found to be a phase transition between the two scenarios. In the former case we end up with just one contour- $\gamma = \gamma_1 : k \in (-k_F, k_F)$ whilst in the latter we have the union of two contours $\gamma = \gamma_1 \cup \gamma_2 : k \in (-k_c, k_c) \cup (-k_F, k_F)$ and so have a 'two-cut' solution. The loop equation/algebraic curve in this case was found to be

$$F_0(X, k) = X^2 - \frac{\sin^2 \frac{k}{2} - \sin^2 \frac{k_c}{2}}{\sin^2 \frac{k}{2} - \sin^2 \frac{k_F}{2}} = 0. \quad (43)$$

We graph the $(\rho, v) \leftrightarrow (x, \tau)$ mapping in Fig.6 for the dispersion $\varepsilon(k) = -\cos k$ where we find the appearance of limit shapes shaped like deltoids. We note the presence of 'hard edges', the dots in Fig.6 located at $k = \pi \pm i\infty$, where this density diverges. Hard edges are a ubiquitous feature of Coulomb gas models, see e.g. [30]. In contrast to the nephroid this is a so-called 'two-cut' solution of the Coulomb gas model because of the two sets of branch points $(\pm k_c, \pm k_F)$.

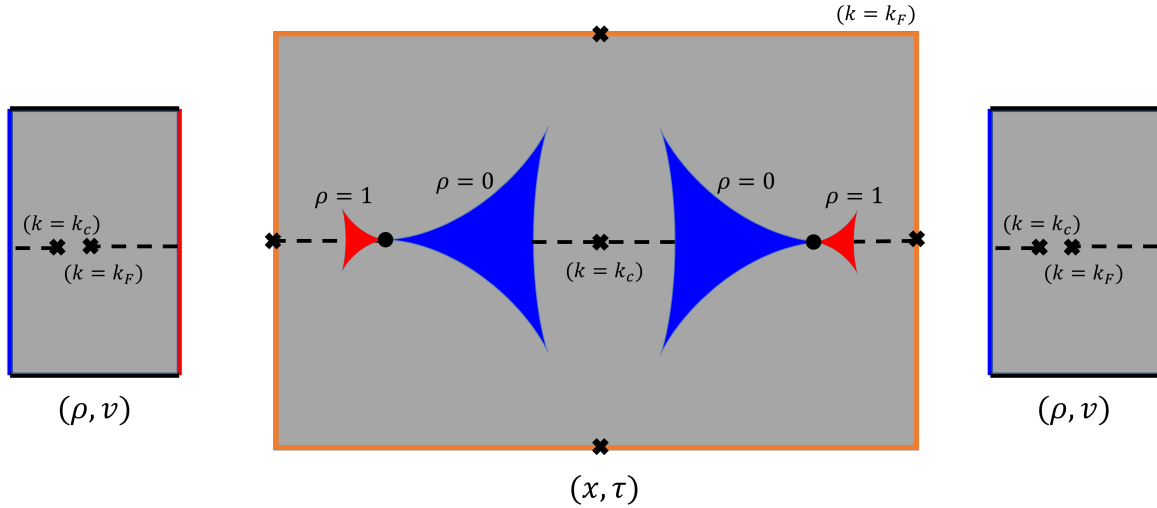


Figure 6. Depiction of the unmerged EFP limit shapes, which present as four deltoids. The branch points (also poles) at $k = k_F$ map to spacetime infinity. The left hand edges of the Riemann sheets map to the two larger central deltoids residing at $\rho = 0$ whilst the right hand edges present as two smaller, out deltoids at $\rho = 1$. The tops and bottoms of the sheets at $v = \pm\infty$ map to the black dots where they are 'hard edges'. The branch point at k_c which resides at $x = \tau = 0$ is indicated with the cross in the centre.

5.4. Merged EFP limit shapes

In the limit $k_c \rightarrow 0$ the above two-cut solution transitions to a one-cut solution and it was shown that this correlated with a third order phase transition in the Coulomb gas model. As the cut disappears the shapes merge producing an astroid whose algebraic

curve is

$$F_0(X, k) = X^2 - \frac{\sin^2 \frac{k}{2}}{\sin^2 \frac{k}{2} - \sin^2 \frac{k_F}{2}} = 0, \quad (44)$$

a result first found in [3]. We present the $(\rho, v) \leftrightarrow (x, \tau)$ mapping in Fig.7.

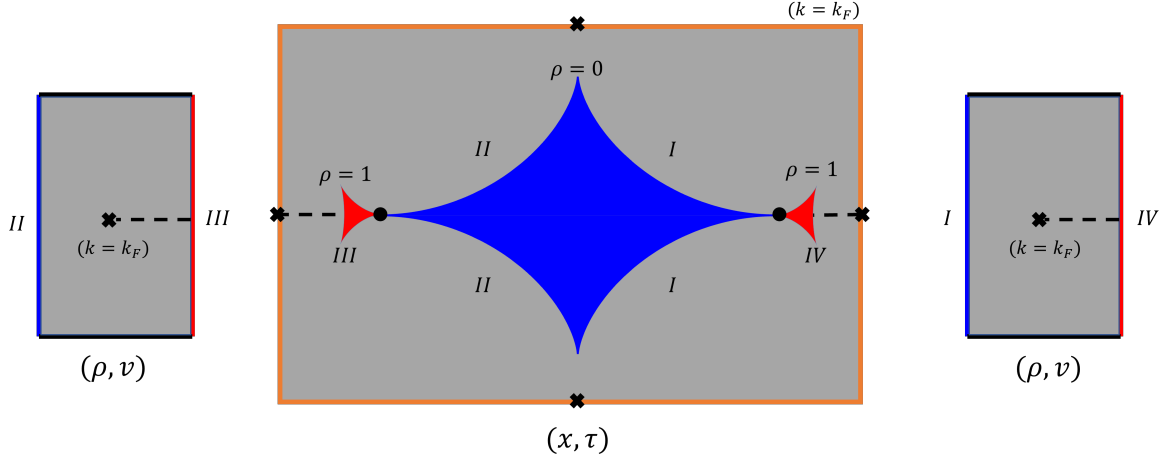


Figure 7. The merged EFP limit shape, an astroid. Compared to the unmerged case there is the absence of the branch cut at k_c .

6. Perturbative and non-perturbative behaviour of limit shape systems

Limit shapes, as we have understood them so far, are defined by their strict boundaries which separate the fluctuating and non-fluctuating regions. This picture is accurate in the thermodynamic, large N , limit but for finite N it suffers from non-perturbative corrections that take the form of instantons. These instantons are nothing but fermions which have bled into the frozen regions, blurring the once strict boundary between fluctuating and not. They represent *atypical* fluctuations in the system wherein the eigenvalues/charges/fermions have departed from their dominant, saddlepoint configuration.

Besides instantons, we may consider the perturbative or typical fluctuations of the limit shapes as well. These are the quantum fluctuations that exist 'on top of' the saddlepoint configurations and represent a collective motion of the underlying eigenvalues/charges/fermions. We find that these fluctuations can be described by a Gaussian free field theory that lives on the Riemann surfaces we have described in the same vein as many other, related works dealing with the conformal field theory of a single arctic circle, Gaussian free field theories of many body systems of free fermions and Quantum Generalised Hydrodynamics [12, 13, 19, 38, 39, 40]. With regards to the latter, these fluctuations are nothing but fluctuations of the time dependent Fermi surface exactly like those seen in [12] (though we focus on an (x, τ) or $(k, \bar{k})$ parameterisation

as opposed to (x, k)). To see this, remember that $\sigma(k)$ is directly related to $X(k)$ which describes the boundary of the Fermi surface. Fluctuations of $\sigma(k)$ are, then, fluctuations of the Fermi surface.

We will discuss both types of behaviour in this section using the minimal surface formulation outlined at the start.

6.1. Fluctuations on Riemann surfaces

First, let's focus on the perturbative behaviour of limit shape systems. To proceed we expand Eq. (32) about the saddle-point solution by writing $g = g_{\text{s.p.}} + \delta\tilde{g}/N$ and find

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots = S^{(0)} + \frac{1}{\pi} \int_D d^2z |\delta g|^2 + \dots, \quad (45)$$

where $\delta g = \delta\tilde{g}/N$. We have used the fact that on the saddlepoint the first order term in this expansion vanishes. To next order we find the quadratic action $S^{(2)}$ with higher orders suppressed at large N . We remind that the domain this action is defined on, D , is fixed and that $S^{(0)}$ is proportional to its area, Eq. (32). *À la* bosonisation we introduce the height field $\partial h = \delta g$

$$S^{(2)} = \frac{1}{\pi} \int_D d^2z |\partial h|^2. \quad (46)$$

What we have found is very similar to the fluctuations in the real-space formulation, those associated with Eq. (8). They take the form

$$S^{(2)} = \int \frac{d^2x}{K(x, \tau)} \sqrt{g} g^{ij} \partial_i h \partial_j h, \quad (47)$$

Appendix B see also [44]. The *emergent metric* g_{ij} depends upon the saddle-point configurations of the system's density and velocity fields and $K(x, \tau)$ is known as the Luttinger parameter. At low energies a variety of systems take the form of Eq. (47), this is the Luttinger-Liquid paradigm. The most well-known of these systems are interacting fermions, but certain random surface models also fall into this class [47, 48]. Free fermions (and other systems with constant Luttinger parameter) are of particular interest to us because in these cases (46, 47) actually coincide, **Appendix C**. The upshot is that the ripples on the Riemann surface are equivalent to the motion of a scalar field propagating in the curved space $g_{ij}(x, \tau)$. This is a remarkable result, one that has been extensively studied, most relevantly in the context of Gaussian free field theories and Quantum Generalised Hydrodynamics [12, 13, 19, 38, 39, 40, 44, 49, 50]. Contrary to the usual bosonisation procedure, which is carried out on fermions with a constant linear dispersion, k , we have managed to bosonise a system of fermions with an inhomogeneous,

|| Emergent metrics are essential to the field called analogue gravity [45]. This field began with work by Unruh [46] who noticed that fluctuations in a barotropic, inviscid fluid looked like they propagated in a curved space. It has since blossomed as people seek to understand more exotic features of quantum gravity in tabletop experiments.

non-linear dispersion $\epsilon(k(x, \tau))$. The trick to getting it to work was to change coordinates from x, τ space to $k, \bar{k}$ space: when written in terms of these new coordinates, the dispersion appears homogeneous and linear. Put another way, conventional bosonisation implies a 'light-cone' spreading of fermions: $x(\tau) = x(0) \pm ik\tau$ where they propagate with constant velocity, k . We show in [Appendix D](#) that the boundaries of the limit shapes can be identified with these light cones and so in this instance the light cones are curved, which would seem to prohibit the bosonisation procedure. However, we know we can map the limit shapes to Riemann sheets via the mapping from x, τ to $k, \bar{k}$. When we do so, we find that the *curved* boundaries of the limit shapes map to the *straight* edges of the Riemann sheets. Thus, intuitively, in these new coordinates it should be possible for one to carry out the bosonisation procedure, which is what we have found by arriving at the quadratic action Eq. (46).

This action is an example of a Gaussian free field theory or Conformal Field Theory (CFT) [51, 52, 53]. The subject is well-studied and consequently fluctuations in limit shape systems are also well-studied, for example in [19, 38, 44, 54] which used CFT to calculate the entanglement entropy in a sub-region of a limit shape.

6.2. Topological phase transitions

Throughout this work we have seen multiple instances of phase transitions in Coulomb gas models. These phase transitions corresponded to the redistribution of the eigenvalues of an underlying matrix model and as such constitute a non-perturbative phenomena. When the eigenvalues redistribute, cuts appear and disappear on the Riemann surfaces. This changes their topology as we now discuss.

Consider the unmerged arctic limit shapes. They have no cuts on their surface. Following remarks in [Appendix A](#) they can be completed into spheres of genus 0 by introducing points at complex infinity. The merger transition corresponds to gluing the two spheres together along the newly formed branch cut to form another sphere, Fig.8. Though seemingly trivial, this transition has a physical consequence: it enables particles to travel from one sheet to the other without the need for any tunneling. It is as though a 'wormhole' of sorts has opened up between the two spheres, Fig.9.

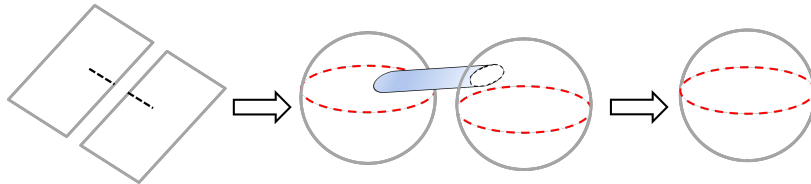


Figure 8. One cut solution being transformed into a sphere. The two Riemann sheets which represent two distinct spheres are glued along the cut to form just one sphere.

The deltoid to astroid transition is also quite interesting. The unmerged limit shapes can once again be completed into two spheres which are now connected to each other by two branch cuts. The branch cuts can then be expanded into tubes that connect

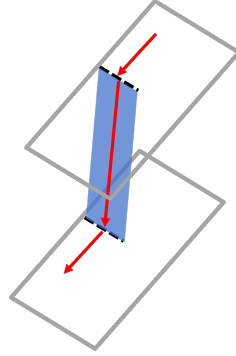


Figure 9. Cartoon sketch of the wormhole joining the two Riemann sheets. The arrows depict the trajectory a particle can take as it crosses from one sheet to the other.

the two spheres resulting in a torus, Fig.A1. Their merger transition corresponds to a pair of these branch cuts disappearing. As they do, it is like the ring of the torus is being pinched at a point. Eventually, this pinching cuts the torus resulting in a cylinder, which is topologically equivalent to the genus 0 sphere.

6.3. A Gauge/Gravity duality

We have shown that the charges of the Coulomb gas (alternatively the eigenvalues of the matrix model) distribute themselves along some one-dimensional contour, γ . In the large N limit these charges 'condense' to form a smooth, one-dimensional surface. The geometry of this surface is encoded by the function $g(z)$. As this function is holomorphic it may be analytically continued into the complex plane resulting in a two-dimensional geometry: the Riemann surfaces. In a sense these surfaces *emerge* from the collective behaviour of the eigenvalues of the matrix model. Unlike the emergent metric discussed above, this geometry has truly emerged in the sense that there were no pre-defined notions of 'coordinates' describing the surface until the large N limit was taken. Naturally, the idea of a truly emergent geometry is a tantalising one and has been discussed extensively [32, 33, 55, 56, 57, 58, 59, 60, 61, 62, 63]. In all cases the authors considered matrix/Coulomb gas models similar to the ones discussed here and found that their eigenvalues collectively defined the space on which a two dimensional string theory lives¶. This correspondence between matrix models and geometry is an example of the *gauge/gravity duality* [69, 70, 71]. Its central idea is that gravitational degrees of freedom should have some equivalent, quantum description in terms of a gauge theory. We have actually seen this correspondence in action for the arctic circles. On the gauge side of the duality we had the GWW matrix model of large N QCD [9], whilst on the gravitational side was the emergent Riemannian geometry (and its associated string

¶ More specifically, the Riemann sheets are the so-called worldsheets swept out by propagating strings, see [64] for some introductory notes on the subject. For some sources on the correspondence between matrix models, free fermions and strings see [65, 66, 67, 68].

theory).

6.4. Instanton effects

Instantons bleed into frozen regions; obscuring the once smooth boundary of a limit shape. The most likely way this can happen is if a fermion following the boundary of the limit shape 'shoots off' and escapes into the uncharted frozen region. With exponentially small probability, a fermion can cross this frozen region to rejoin the boundary of a neighbouring limit shape. In the event that there is no such shape to be found, the fermion is doomed to wander off, with a vanishing chance of being found at space-time infinity. We may ask the question: what is the probability that we observe a fermion a distance δ away from the limit shape boundary? To answer it, we assume that pulling one fermion or eigenvalue away from its desired trajectory does not disrupt the trajectories of the other eigenvalues. In other words, $N - 1$ of the fermions are fixed, forming the limit shape and the N th one is allowed to undergo an atypical deviation in which it 'escapes' the others. In the large N limit this is a reasonable assumption to make because the energy cost in re-configuring the $N - 1$ particles scales $\sim N^2$, whereas pulling just one particle out only incurs a penalty $\sim N$ making it energetically much more favourable. It is the probability of this atypical deviation we wish to calculate in this section.

Assume that for some $v^* = \text{Im } k^*$ a fermion following the boundary of a limit shape (residing at density $\text{Re } k = \pi\rho = \pi$) 'escapes' and shoots off along the tangent line it was momentarily following⁺. We wish to calculate the difference between the action associated to this trajectory, $S_{\text{inst.}}$, and one following the boundary of the limit shape, $S_{\text{cl.}}$. This difference tells us the probability of the fluctuation $P_{\text{fluct.}} \sim e^{-\delta S}$ where $\delta S = S_{\text{inst.}} - S_{\text{cl.}}$. We depict the setup in Fig.10. On the left we have the instanton trajectory and on the right we show the difference between the classical and instanton paths exploiting the symmetry of the curve. This approach of finding the difference between the classical and instanton path in order to study fluctuations also goes by the name *geometric optics* [73, 74]. To set about calculating this fluctuation we need a suitable action. This will just be the action for the single free fermion that we choose to be our instanton, residing at the boundary $k = \pi + iv$ such that

$$S = \int d\tau (-v\dot{x} - \varepsilon(iv)). \quad (48)$$

We are now in a position to calculate the difference between the classical and instanton

⁺ In fact, it is possible to use such considerations to find the limit shape itself using the *tangent method* [72]

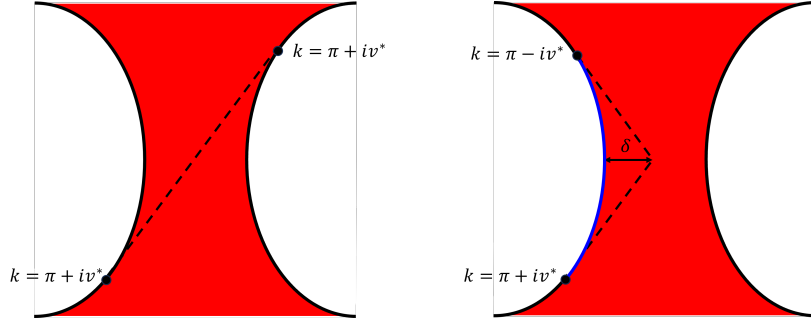


Figure 10. Left: depiction of instanton trajectory. Right: using the symmetry of the problem, the instanton trajectory can be folded back on itself. In blue is the difference between the instanton and the classical solution. It is the action associated with this path we must calculate.

path

$$\begin{aligned} \delta S = S_{\text{inst.}} - S_{\text{cl.}} &= \int_{-\tau^*}^{\tau^*} d\tau (-v^* \dot{x}|_{v^*} - \varepsilon(iv^*)) \\ &\quad - \int_{-\tau^*}^{\tau^*} d\tau (-v \dot{x} - \varepsilon(iv)), \end{aligned} \quad (49)$$

where v^*, τ^* are the velocity and time at which the particle escapes respectively. Although the above calculation is doable, it requires us to solve the algebraic curve explicitly in order to find both x and τ as functions of v . It would be easier if we could write things in terms of a known quantity like $g(v)$. It turns out that this is possible by exploiting a trick.

We write Eq. (48) in time-parameterisation-invariant form

$$S = \int dq [-v(q) \dot{x} - \varepsilon(iv(q)) \dot{\tau}], \quad (50)$$

by introducing a new time coordinate, q . Setting $\tau(q) = q$ we return to Eq. (48). It should be clear that we can choose q to be anything we like, including v itself. With this choice a drastic simplification in the calculation of Eq. (49) occurs. The first term only involves the constant v^* and is therefore independent of time. Therefore, if we change variable from τ to v this term will disappear. Subsequently, we are left with

$$\delta S = \int_{v^*}^{-v^*} dv (\dot{x}v + \dot{\tau} \varepsilon(iv)). \quad (51)$$

Integrating by parts and exploiting the symmetry of the instanton trajectory, we find

$$\delta S = 2 \int_0^{v^*} dv (x + \varepsilon'(iv) \tau) = 2 \int_0^{v^*} g(v) dv. \quad (52)$$

We discuss in [Appendix A](#) that δS can be interpreted as the length of a closed path on the Riemann surface of the limit shape.

The last quantity left to evaluate is v^* . We must choose it such that the two solutions to the curve, $\pm g(\pi + iv)$, meet because this then allows the instanton to travel between the two by 'jumping' from one branch of the solution to the other. Consequently, we set $g(\pi + iv^*) = 0$.

As an example, for the unmerged circles we find: $g(v^*) = \lambda - \cosh v^* = 0 \implies v^* = \operatorname{arcosh}(\lambda)$ and from this

$$\delta S = 2 \left(\lambda \log \left(\lambda - \sqrt{\lambda^2 - 1} \right) - \sqrt{\lambda^2 - 1} \right). \quad (53)$$

More generally, assume that $g(v)$ admits a quadratic expansion $g(v) = \delta - \frac{g''(0)v^2}{2}$, with δ indicated in Fig. 10. We find $v^* = \sqrt{2\delta/g''(0)}$ and thus

$$\delta S = \frac{2\sqrt{2}}{3} \left(\frac{\delta}{g''(0)^{1/3}} \right)^{3/2}. \quad (54)$$

Eq. (53) has actually been found before in the context of instantons in the GWW matrix model [75, 76]. What has been uncovered here, then, is a dynamical, spacetime interpretation of these instantons. Their contribution becomes $O(1)$ right when the boundaries of the limit shapes touch ($\delta = 0, \lambda = 1$) allowing these instantons to proliferate which can be thought of as inducing the phase transition.

It is also possible to obtain higher order corrections to this result, [77]. Consider

$$P_{\text{fluct.}} = \int_0^\infty dv' e^{-2 \int_0^{v'} dv g(v)}. \quad (55)$$

That is, we calculate an instanton trajectory that ends at *any* velocity $v' > 0$ and we sum over all possibilities. By saddlepoint the above is solved by $g(v = v^*) = 0$ in keeping with what was said above. We may continue our expansion around this solution to obtain higher order corrections; to second order for example

$$P_{\text{fluct.}} \sim \frac{1}{2} \sqrt{\frac{\pi}{g'(v^*)}} e^{-\delta S}. \quad (56)$$

For an extension to multi-instanton solutions see [78, 79].

7. Final remarks

In this work we have given an extensive account of limit shapes, their transitions and their fluctuations using a combination of hydrodynamic, Coulomb gas and minimal surface formalisms. Our work is by no means exhaustive with regards to particular instances of limit shapes. What we will claim, however, is that the classification of the different possible *topologies* of limit shapes using the Riemann sheet picture is much more extensive. As shown in Appendix A, if limit shapes can be mapped to Riemann surfaces and Riemann surfaces are uniquely specified by their moduli, then these moduli must provide a rigorous way to categorise all the possible limit shapes there are. Essentially,

there are only so many combinations of poles, branch cuts and branch points we can introduce onto a Riemann sheet and equivalent classes can be identified via the moduli parameters. Though the origins of the limit shapes we have seen vary, what unifies them is this description in terms of Riemann surface topology.

The keen reader may have spotted that the phase transition in both the arctic and the EFP limit shapes is correlated with the appearance of a branch cut. Simply put, the reason why is because the branch cut forces the particles to redistribute themselves as they attempt to minimise their free energy in the presence of the new constraint. That this transition should be of the *third* order has been argued for on general grounds in [80].

We see multiple areas for future work and highlight a couple of them here. First, we may ask about limit shapes where the algebraic curve is a polynomial of X of higher degree than two. Although one can find some examples by multiplying various one or two limit shape curves together, in the spirit of this work we should be able to rigorously find examples of these by considering some equivalent Coulomb gas or matrix model. There are in fact so-called multi-matrix models where there are multiple charge densities, $\sigma_1(k), \sigma_2(k) \dots$ resulting in loop equations in higher powers of X . For example, some two-matrix models have a cubic instead of a quadratic curve [81]. Potentially, then, these could provide us with new, more complex limit shape solutions.

Related to our discussion of the fluctuations it would also be interesting to find the Green function or propagator of the systems we have described. For free fermions such a function can be used to calculate all higher order correlation functions making it a quantity of eminent interest. Zooming in on different regions of the limit shapes this propagator is known to take various universal forms, discussed extensively in the literature. For example, one finds so-called Airy behaviour near the edges of the arctic circles [82], Pearcey behaviour at the cusp singularities where two limit shapes merge [83] and at exactly the point where the two circles first begin to touch we find tacnode behaviour [84]. What has not been discussed, is the behaviour of the propagator across a phase transition, like the arctic circle merger. In this case one would expect, intuitively, that the propagator be a sum of two terms; one representing correlations within the same limit shape (*intra*-) and one representing correlations between different limit shapes (*inter*-). The inter-shape contribution should be smaller than the intra-shape one if the shapes are suitably far apart. In fact, as $\lambda \rightarrow \infty$, it should tend to zero. As the shapes are brought closer we would expect the inter-shape contributions to grow in size until they become as dominant as the intra- ones right at the point of transition. What process, then, should mediate the size of these contributions? From Section 6.4 it is clear that instantons are the answer. After all, only they are capable of travelling from one shape to the other thus mediating correlations between the two. We have even seen that when the limit shapes are far apart their contribution is small, because they have to travel a large distance with an exponentially decaying probability. As the shapes are brought closer this probability increases until the point of transition where it becomes $O(1)$ and the intra- and inter- contributions match.

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Appendix A. The A's and B's of Riemann surfaces

What follows is a non-rigorous rundown of some basic concepts in Riemannian geometry. This area is vast, but two pertinent references are [10, 85].

Writing $z = e^{ik}$ we can assume $g^2(z)$ will take the following form

$$g^2(z) = \frac{P(z)}{Q(z)}, \quad (\text{A.1})$$

with P, Q polynomials in z . In general $g(z)$ will be multi-valued. As those familiar with complex variable theory will know, we can view a multi-valued function as a single-valued, holomorphic function that lives on the two-sheeted cover of the complex plane. Here the two sheets correspond to the two solutions of Eq. (A.1) and they are 'glued' together along their shared branch cuts. As an example, consider the setup with two branch cuts depicted in Fig. A1. The two sheets are joined along the two cuts. Since $g(z)$ is regular at infinity we can actually compactify the sheets into spheres by adding points at infinity. These spheres are then glued together by the two cuts, now tubes, resulting in a torus which has genus 1. We may extend this logic to sheets with multiple cuts with the end result that if $g(z)$ has s number of cuts the resulting Riemann surface will have genus $g = s - 1$.

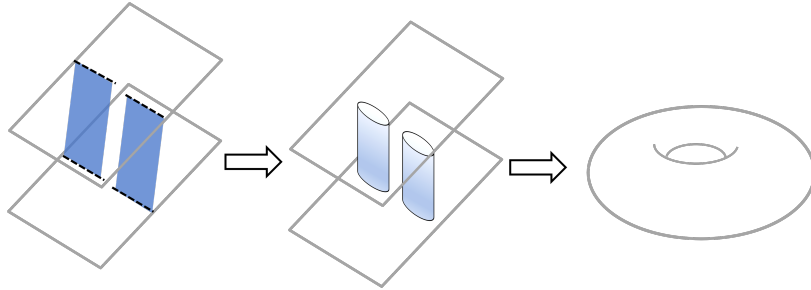


Figure A1. Two cut solution being transformed into a torus. One may take the branch cuts and expand them into tubes. Compactifying each sheet into a sphere by adding points at infinity yields a torus. This Figure is inspired by one in [10].

Riemann surfaces are characterised by a set of moduli or periods related to integrals taken around closed loops on their surface. Consider the two cuts as forming the union of contours $\cup_{i=1}^2 \gamma_i = \cup_{i=1}^2 [a_i, b_i]$. As long as $a_1 < b_1 < a_2 < b_2$, these end points can be the poles or zeros of $g^2(z)$. Then, two integration contours can be defined, termed the A - and B -cycles. The A -cycle, A_i , goes around the cut $[a_i, b_i]$, see figure A2. The B -cycle, B_i , runs between the cuts $[b_i, a_{i+1}]$, figure A2. We have the following key results

$$\frac{1}{4\pi i} \oint_{A_i} dz \partial \chi(z) = \int_{a_i}^{b_i} \frac{dk}{2\pi} \sigma(k) = \nu_i \quad (\text{A.2})$$

$$\text{Re} \oint_{B_i} dz \partial \chi(z) = \text{Re} \int_{b_i}^{a_{i+1}} \frac{dz}{z} g(z) = \tilde{\nu}_i. \quad (\text{A.3})$$

The first follows from the definition of χ in Eq. (28). ν_i is the filling fraction or the proportion of eigenvalues in a given cut. The second follows from the knowledge that $g(z)$ changes its sign as the contour goes between the two sheets, but $V(z)$ does not. Together $(\nu_i, \tilde{\nu}_i)$ are known as the periods or moduli of the Riemann surface and uniquely characterise it. Hence by calculating these moduli we have found a way to, in principle, classify limit shapes in a rigorous fashion.

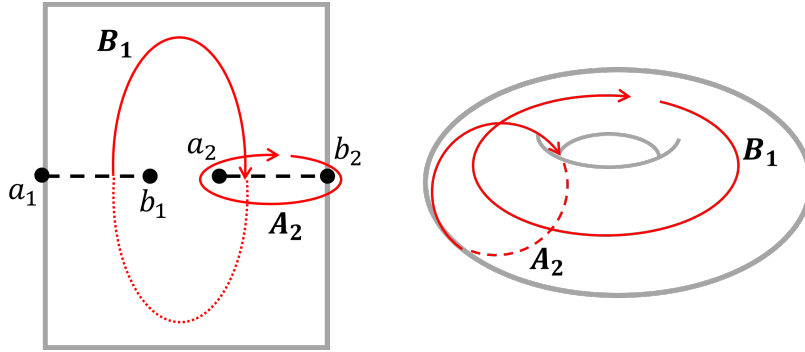


Figure A2. Depiction of A- and B- cycles. On the left, we have the picture from the perspective of a single Riemann sheet. The B-cycle goes between the cuts and by doing so crosses over to the other sheet (depicted by the dotted segment of the arrow). The A-cycle wraps around the cut. On the right we have the same picture but now we have compactified the two sheets into the torus Riemann surface. A and B are now the closed cycles on this surface.

Given a meromorphic function, like $|g(z)|^2$, one can use it to construct a quadratic differential

$$q = |g(z)|^2 d^2 z = G_{z\bar{z}} d^2 z, \quad (\text{A.4})$$

defining a conformally flat metric on the Riemann surface, $G_{z\bar{z}}$. It allows us to define notions of 'area' and 'length' as follows

$$A = \int \sqrt{|\det G|} d^2 z = \frac{1}{2} \int |g(z)|^2 d^2 z \quad (\text{A.5})$$

$$L = \int \frac{g(z)}{z} dz. \quad (\text{A.6})$$

We recognise the former as nothing but Eq. (32), up to a factor. In other words, the logarithm of the probability of a limit shape forming is proportional to the (minimised) area of its associated Riemann surface.

Now consider the latter and the quantity

$$L_{B_d} = \oint_{B_d} dL, \quad dL = \frac{g(z)}{z} dz. \quad (\text{A.7})$$

The path, B_d , is depicted in Fig. A3. This path is like a B -cycle in the sense that it goes from one Riemann sheet to the other. It differs because it does not straddle

two cuts in the process, just one. We dub it a *degenerate B-cycle* to reflect this fact. One then finds that Eq. (A.7) is equivalent to the instanton action, Eq. (52). In particular, as $v^* \rightarrow 0$ the length of this path goes to zero and hence the probability of the fluctuation $P_{\text{fluct.}} \sim e^{-L_{B_d}}$ goes to 1. Therefore, the instanton corrections to the limit shape formation probability are also equal to some geometric quantity: the length of the degenerate B-cycle. In this way, we find that many facets and questions to do with the hydrodynamics have answers in terms of purely geometric constructions.

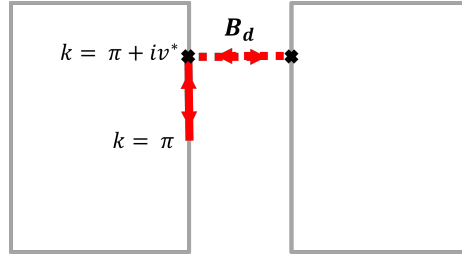


Figure A3. Depiction of instanton trajectory from the perspective of the Riemann sheets. The trajectory is a closed cycle that we dub a degenerate B-cycle. The dashed path has zero length, it is there to represent the instanton going between the two sheets via the branch point $k = \pi + iv^*$.

Appendix B. Deriving real-space fluctuations

To find the behaviour of the fluctuations in the system, we follow the methods used in [44, 38]. Let us consider the action

$$S = \int d^2x L(\rho, j), \quad (\text{B.1})$$

where ρ and j are constrained by the continuity equation

$$\partial_\tau \rho + \partial_x j = 0. \quad (\text{B.2})$$

Introducing a height field ϕ such that $\rho = \partial_x \phi, j = \partial_\tau \phi$ this constraint will be automatically satisfied. We now expand the action to second order. We write $\phi_{\text{cl}} + \delta\phi$, where ϕ_{cl} is the solution to the saddlepoint equation. Schematically, we have a series of terms

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots \quad (\text{B.3})$$

In reality the fluctuations in ϕ are suppressed by factors of N and so we terminate the above at second order. On the saddlepoint solution the contribution from S_1 vanishes and S_0 is just a constant independent of $\delta\phi$. The dynamics of the problem all reside at second order where we find

$$S^{(2)} = \frac{1}{2} \int d^2x \begin{bmatrix} \partial_x \delta\phi & \partial_\tau \delta\phi \end{bmatrix} \begin{bmatrix} L_{\rho\rho} & L_{\rho j} \\ L_{j\rho} & L_{jj} \end{bmatrix} \begin{bmatrix} \partial_x \delta\phi \\ \partial_\tau \delta\phi \end{bmatrix}. \quad (\text{B.4})$$

The square matrix is called the Hessian of L or $\text{Hess}(L)$. The above may be rewritten as

$$S^{(2)} = \int \frac{d^2x}{8\pi K} \sqrt{g} g^{ij} \partial_i \delta\phi \partial_j \delta\phi, \quad (\text{B.5})$$

if we identify $g_{ij} = (\text{Hess}(L))^{-1}$ and $K = \left(4\pi \sqrt{\det(\text{Hess}(L))}\right)^{-1}$. K in general can depend on space and time and is called the Luttinger parameter.

We can now specialise the above arguments to the hydrodynamic action Eq. (8)

$$S = \int d\tau \int dx \left\{ \frac{i}{4\pi} [k (\partial_x)^{-1} \partial_\tau k - \bar{k} (\partial_x)^{-1} \partial_\tau \bar{k}] + \int_{-\bar{k}}^k \frac{dk}{2\pi} \varepsilon(k) \right\}, \quad (\text{B.6})$$

and $\rho = \text{Re}(k)/\pi$ and $j = \text{Re}(\varepsilon(k))/\pi$. We find for the elements of the Hessian (see also [3])

$$L_{jj} = \frac{\pi}{\text{Re}(\varepsilon')}, \quad (\text{B.7})$$

$$L_{\rho j} = L_{j\rho} = -\frac{\pi \text{Im}(\varepsilon')}{\text{Re}(\varepsilon')}, \quad (\text{B.8})$$

$$L_{\rho\rho} = \frac{\pi |\varepsilon'|^2}{\text{Re}(\varepsilon')}. \quad (\text{B.9})$$

From these we get the metric (up to conformal factors)

$$ds^2 = dx^2 - 2 \text{Im}(\varepsilon'(k(x, \tau))) dx d\tau + |\varepsilon'(k(x, \tau))|^2 d\tau^2. \quad (\text{B.10})$$

We also find $K = 4\pi$. To match with the conventions of the main text we simply have to re-scale the fields and so identify $\delta\phi = h/2$. In this way the free fermion point is now when $K = \pi$.

Appendix C. Equivalence between fluctuations

Let us assume that the hydrodynamical system in x, τ space is just that for free fermions, i.e. described by Eq. (8) with $K(x, \tau) = \pi$ (not much of the following argument changes for other, constant values of K which can always be dealt with by re-scaling the fields). By changing basis to a conformal coordinate system,

$$\frac{1}{|\lambda|^2} (du^2 + dv^2) = g_{ij}(x, \tau) dx^i dx^j, \quad (\text{C.1})$$

the overall conformal factor in g^{ij} will cancel with that coming from the determinant of g_{ij} in Eq. (47) leaving something similar to Eq. (46). To prove that (46, 47) are in fact identical, it remains to be shown that a conformal coordinate system is one provided by $(z, \bar{z})$ or, equivalently, $(k, \bar{k})$ i.e.

$$\frac{1}{|\lambda|^2} d^2k = g_{ij}(x, \tau) dx^i dx^j. \quad (\text{C.2})$$

Let us assume this to be true and write the above using explicit knowledge of the metric Eq. (B.10)

$$\frac{1}{|\lambda|^2} d^2 k = dx^2 - 2 \operatorname{Im}(\varepsilon'(k(x, \tau))) dx d\tau + |\varepsilon'(k(x, \tau))|^2 d\tau^2. \quad (\text{C.3})$$

For brevity, we will drop the reminder of the dependence of k on x, τ from now on. Following a method outlined in [86], we proceed by factorising both sides like so

$$|\lambda|^2 \left(dx + (F + i\sqrt{d}) d\tau \right) \left(dx + (F - i\sqrt{d}) d\tau \right) = (dk)(d\bar{k}), \quad (\text{C.4})$$

with $d = G - F^2$, $F = -\operatorname{Im}(\varepsilon'(k))$, $G = |\varepsilon'(k)|^2$. Examining one of the factors we have

$$\lambda \left(dx + (F + i\sqrt{d}) d\tau \right) = \frac{\partial k}{\partial x} dx + \frac{\partial k}{\partial \tau} d\tau. \quad (\text{C.5})$$

Equating the coefficients of dx and $d\tau$ and then eliminating λ we arrive at

$$(F + i\sqrt{d}) \frac{\partial k}{\partial x} = \frac{\partial k}{\partial \tau}. \quad (\text{C.6})$$

Rearranging,

$$\frac{\partial k}{\partial \tau} - \left(F + i\sqrt{G - F^2} \right) \frac{\partial k}{\partial x} = 0, \quad (\text{C.7})$$

and then realising that

$$i\varepsilon'(k) = \left(F + i\sqrt{G - F^2} \right), \quad (\text{C.8})$$

the equation becomes

$$\frac{\partial k}{\partial \tau} - i\varepsilon'(k) \frac{\partial k}{\partial x} = 0. \quad (\text{C.9})$$

The process of diagonalising the 2D metric (in fact any 2D metric) is therefore equivalent to solving Eq. (9)*. Consequently, the coordinate system provided by $(k(x, \tau), \bar{k}(x, \tau))$ must be the one that makes the metric conformal and thus Eq. (47) and Eq. (46) are equivalent.

Appendix D. Limit shapes and light cones

Consider first the two unmerged arctic limit shapes with $V(k) = \varepsilon(k) = -\cos k$. As we have two limit shapes we will have two emergent metrics in the real-space. Solving Eq. (36) for $k(x, \tau)$ (alternatively, see Eq.(42) in [8]) and using Eq. (B.10) we can find the metrics up to a conformal factor

$$ds_{\mp}^2 = dx^2 - \frac{2(x \mp \lambda)\tau}{1 - \tau^2} dx d\tau + \frac{1 - (x \mp \lambda)^2}{1 - \tau^2} d\tau^2. \quad (\text{D.1})$$

* The conjugate equation can be found by repeating the above steps with the other part of the factorisation.

Setting $ds^2 = 0$ in (B.10) and rearranging for $dx/d\tau$ we have the trajectories of the null-geodesics

$$\frac{dx}{d\tau} = -\text{Im}(\varepsilon'(k)) \pm i\text{Re}(\varepsilon'(k)), \quad (\text{D.2})$$

$$= \frac{(x \mp \lambda)\tau}{1 - \tau^2} \pm i \frac{\sqrt{1 - (x \mp \lambda)^2 - \tau^2}}{1 - \tau^2}. \quad (\text{D.3})$$

x, τ are always purely real, as such the only way to satisfy the above is for the real part of the dispersion to vanish or, equivalently, for $k = 0 + iv, \pi + iv$. Therefore, the trajectories of the null-geodesics trace out the boundaries of the limit shapes. This is sensibly telling us that fluctuations on the Riemann surface are confined by its boundary or in other words the boundaries of the limit shapes are like lightcones for the free fermions. One can verify via direct substitution that $x(\tau) = \pm\lambda \pm \sqrt{1 - \tau^2}$ (where the two $\pm$ signs are taken independent of one another) are the null-geodesics and that they trace out four semi-circles which together form the boundaries of the two arctic circles.

For another example of an emergent metric we can look at the emptiness solution, Eq. (44). Finding the exact form of the metric is tricky but we can expand $F_0(X, k)$ in the vicinity of a hard edge and draw some conclusions from there. It takes the form

$$F_0(X, k) \sim X - \frac{b}{k^2} = 0, \quad (\text{D.4})$$

for some constant, b , which can be found by expanding Eq. (44) about $k = \infty$. Taking $\varepsilon'(k) \sim k$ in this region, rearranging and neglecting small terms (D.4) becomes

$$\tau v^3 - (3i\tau\pi\rho + x)v^2 + (2i\pi\rho x - 3\tau(\pi\rho)^2)v - b = 0. \quad (\text{D.5})$$

We find

$$\pi\rho = \sqrt{\frac{-27b\tau^2 - 4x^3}{27\tau^2x}}, \quad (\text{D.6})$$

$$v = \frac{2x}{3\tau}. \quad (\text{D.7})$$

Using (B.10) the emergent metric is

$$ds^2 = \frac{1}{|\lambda|^2} d^2k = dx^2 - \frac{4x}{3\tau} dx d\tau + \frac{x^3 - 27b\tau^2}{27\tau^2x} d\tau^2, \quad (\text{D.8})$$

The boundary of this limit shape, that is the equation of the null-geodesic of this metric, has the trajectory

$$x = -\frac{3b^{1/3}\tau^{2/3}}{2^{2/3}}, \quad (\text{D.9})$$

which is precisely the equation that describes the shape in the vicinity of the hard edge (the left cusp in Fig.7).

3.2 Summary

In this section we have provided a whistle-stop tour over the central results of this thesis. We have indeed shown that it is possible to get traction on the problem of finding limit shapes and their transitions via the introduction of a novel Coulomb gas model. In the following section we address these results in greater length.

Additionally, we have introduced a third formulation of the limit shape problem (besides the hydrodynamic and Coulomb gas approaches) in terms of minimal surfaces. This approach allowed us to easily analyse fluctuations in limit shape systems and helped classify the limit shapes themselves. We found agreement with several important results in the literature.

Armed with the Compendium and the background section, the reader can safely proceed to see in detail how the calculations and results in the Compendium were obtained.

CHAPTER 4

RESULTS

The following contains two manuscripts which were referenced in the Compendium. These go in depth into the limit shapes that were found in the Compendium. The first manuscript deals with the arctic circles and their merger, which we generalised in the Compendium. The second manuscript deals with the deltoid to astroid transition which is related to the FCS of free fermions. The two manuscripts (plus appendix [A](#)) constitute all the new results to be found in this thesis.

4.1 *Limit shape phase transitions: a merger of arctic circles*

The following is the manuscript for the publication *Limit shape phase transitions: a merger of arctic circles* produced by myself, my supervisor Dimitri Gangardt and our collaborator Alexander. Abanov. As first author, I performed the majority of the calculations and generated the figures for the paper. I made the link between the solutions to the Coulomb gas and hydrodynamical formulations. A version of this manuscript has been published in J.Phys. A: Math. Theor. [\[19\]](#).

Limit shape phase transitions: a merger of arctic circles.

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Abstract. We consider a free fermion formulation of a statistical model exhibiting a limit shape phenomenon. The model is shown to have a phase transition that can be visualized as the merger of two liquid regions – arctic circles. We show that the merging arctic circles provide a space-time resolved picture of the phase transition in lattice QCD known as Gross-Witten-Wadia transition. The latter is a continuous phase transition of the third order. We argue that this transition is universal and is not spoiled by interactions if parity and time-reversal symmetries are preserved. We refer to this universal transition as the Merger Transition.

1. Introduction

A *limit shape phenomenon* in statistical mechanics is the appearance of a most probable macroscopic state. This state is usually characterized by a well-defined boundary separating frozen and fluctuating (liquid) spatial regions. Other, macroscopically different states are exponentially suppressed in the thermodynamic limit. The formation of the limit shape is enforced by domain-wall-type boundary conditions. Well-known examples of limit shape phenomenon are the problems of *emptiness formation probability* in free fermions [1], and the formation of a frozen region in dimer models [2]. In the former, the macroscopic empty region formed in a system of free one-dimensional fermions has the shape of an astroid [1]. In the example of dimer coverings of the Aztec diamond lattice, the curve separating frozen from liquid regions has a shape of a circle and is a subject of the *Arctic Circle Theorem* [3]. Remarkably, due to the existence of an exact mapping between lattice fermion models and dimer systems, one can study both emptiness and arctic boundary formations using the same set of tools and techniques from quantum field theory.

The limit shape phenomenon has a long history with earlier works on representation theory [4] and crystal surfaces [5, 6, 7]. Over the years, it attracted a lot of interest from physicists, mathematicians, and computer scientists. We refer the reader for recent reviews on various aspects of limit shape phenomena to Refs. [2, 8].

In this work, we focus on the arctic limit shape problem in a free fermion description. We consider a one-parameter family of arctic shapes. As a function of a control parameter, λ , two liquid regions merge and form a single liquid domain surrounded by frozen regions. We identify this transition with the transition known in lattice QCD, a statistical model of large random matrices. This model undergoes a third-order phase transition in the distribution of eigenvalues of the Wilson loop operator that occurs as a function of the t'Hooft coupling λ in the large- N limit [9, 10, 11]. Such a link between QCD and a free-fermion chain has been considered before by Pérez-García and Tierz [12] though they did not consider it in the context of the arctic circle phenomenon. In this context, we present a spatially resolved picture of the transition, computing the exact arctic limiting shape at all values of the coupling λ , which is simply an aspect ratio parameter determined by “arctic” boundary conditions. We refer to the phase transition associated with the merger of liquid regions such as arctic circles as the Merger Transition.

The merger of two liquid region has been considered before both in the arctic curve problem for dimer covering of the Aztec diamond lattice [13] and for random walkers with nonintersecting constraint [14], where the name “tacnode process” was coined to reflect the shape of the fluctuating region. The random walkers description used in the latter work is practically identical to the free fermion formulation used in this work. In contrast to Refs. [13, 14] we focus here on the phase transition corresponding to the merger of liquid regions. A problem of tiling of the Aztec diamond lattice with cut-out corner has been recently considered in Ref. [15]. The third order phase transition

has been observed in that setting as a function of the size of the cut-out region and parallels with Gross-Witten-Wadia [9, 10] and Douglas-Kazakov [11] phase transitions has been made [15]. Though these parallels have been drawn, the transition found in [15] is not the same as the Gross-Witten-Wadia one we describe here. The cut-out region produces a diverging density in [15] whereas here we find no such divergence. Finally, we remark that the Coulomb gas technique used in this work allows us to connect the above mentioned phase transition to a more general phenomenon of phase transitions in the constrained Coulomb gases. The latter have been argued to be universally of the third order [16].

This paper is organized as follows. In Section 2 we start by formulating the limit shape problem of interest as a partition function of free, one-dimensional, lattice fermions evolving in imaginary time. Evaluating the partition function in the thermodynamic limit by the saddle point method is reduced to finding the momentum distribution problem equivalent to finding the equilibrium distribution of eigenvalues in Ref. [9], and we reproduce it here. We extend the collective description to space-time resolved configurations and propose a method to obtain complete space-time dependence of density and velocity of fermions in Section 3. We compute and illustrate solutions for these fields in the separated and merged phases in Section 4. In Section 5 we analyze the phase transition corresponding to the merger of two arctic circles and provide arguments for its universality in Section 6. Discussion of the results and further arguments in favour of the universality of the phase transition are outlined in Section 7.

2. Modeling the limit shape: free fermions on a lattice and Gross-Witten-Wadia model

We start with formulating a model for limit shape phase transitions. Let us consider the following matrix element of the imaginary time propagator of free, one-dimensional, lattice fermions

$$Z_N(R) = \langle N | e^{-2RH} | N \rangle. \quad (1)$$

It is well known (see, e.g., Ref. [8]) that this matrix element represents the partition function of dimer coverings subject to extreme boundary conditions. The latter are encoded by the initial and final state shown schematically in Fig. 1

$$|N\rangle = |2L\rangle = \prod_{x \in I_L} c_x^\dagger |0\rangle \quad (2)$$

corresponding to fully occupied half-integer sites between the positions $-L$ and L of two domain walls, $I_L = [-L + \frac{1}{2}, \dots, L - \frac{1}{2}]$. The fermionic operators have the following

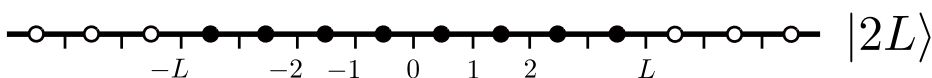


Figure 1. Initial state $|2L\rangle$.

anti-commutation relations

$$\{c_x, c_{x'}^\dagger\} = \delta_{x,x'}, \quad c_x = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikx} c(k). \quad (3)$$

The boundary state $|2L\rangle$ is normalized so that $Z_{2L}(0) = 1$ and can be viewed as a real space “Fermi sea”. The dynamics of the fermions is governed by the tight-binding Hamiltonian

$$H = \frac{1}{2} \sum_x c_{x+1}^\dagger c_x + \text{h.c.} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \varepsilon(k) c^\dagger(k) c(k) \quad (4)$$

characterized by the dispersion relation $\varepsilon(k) = -\cos k$. The fermionic operators thus have the following time dependence

$$\begin{aligned} c(k, \tau) &= e^{\tau H} c(k) e^{-\tau H} = e^{-\tau \varepsilon(k)} c(k), \\ c^\dagger(k, \tau) &= e^{\tau H} c^\dagger(k) e^{-\tau H} = e^{\tau \varepsilon(k)} c^\dagger(k). \end{aligned} \quad (5)$$

We proceed calculating the propagator Eq. (9) by inserting the complete set of eigenstates

$$Z_N(R) = \frac{1}{N!} \int_{-\pi}^{\pi} \prod_{i=1}^N \frac{dk_i}{2\pi} \langle N | \{k\} \rangle e^{-2R \sum_l \varepsilon(k_l)} \langle \{k\} | N \rangle, \quad (6)$$

where $|\{k\}\rangle = c^\dagger(k_1) \dots c^\dagger(k_N) |0\rangle$ and

$$\langle \{k\} | N \rangle = \langle 0 | \prod_{i=1}^N c(k_i) \prod_{x \in I_L} c_x^\dagger | 0 \rangle = e^{-i(L-1/2) \sum_{i=1}^N k_i} \Delta(e^{ik}). \quad (7)$$

Here $\Delta(e^{ik})$ is the Vandermonde determinant defined as

$$\Delta(e^{ik}) = \det_{i,j=1}^N e^{i(i-1)k_j} = \prod_{i < j} (e^{ik_i} - e^{ik_j}). \quad (8)$$

Combining (6) with (7) we express the partition function as a multiple integral over N quasimomenta:

$$Z_N(R) = \frac{1}{N!} \int \frac{d^N k}{(2\pi)^N} |\Delta(e^{ik})|^2 e^{-2R \sum_l \varepsilon(k_l)}. \quad (9)$$

The crucial observation is that Eq. (9) coincides exactly with the Gross-Witten-Wadia (GWW) model [9, 10] of lattice QCD. This model was shown to undergo a third order phase transition in the thermodynamic limit $N, R \rightarrow \infty$ with fixed ratio $2\lambda = N/R$. The transition occurs for critical value of the ratio $\lambda = L/R = 1$.

In the large N, R limit the statistical sum Eq. (9) can be rewritten as a sum over configurations of collective quasimomenta distribution $\sigma(k) = (\pi/R) \sum_i \delta(k - k_i)$:

$$Z_N(R) = e^{-4R^2 f(\lambda)} = \int \mathcal{D}\sigma e^{-4R^2 s[\sigma; \mu]}, \quad (10)$$

where

$$s[\sigma; \mu] = \int \frac{dk}{2\pi} \sigma(k) \varepsilon(k) - \frac{1}{2} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \sigma(k) \log \left| e^{ik} - e^{ik'} \right|^2 \sigma(k') - \mu \left(\int \frac{dk}{2\pi} \sigma(k) - \lambda \right) \quad (11)$$

contains the chemical potential μ to ensure the normalisation

$$\int \frac{dk}{2\pi} \sigma(k) = \lambda. \quad (12)$$

Here we deviate from the standard notations and scale the action with R rather than N [We also use $\lambda = N/2R$, while $\lambda = N/R$ in Ref. [9]. To compare the results one has to replace λ by $\lambda/2$ in our formulae.]. These scaling and notations are natural for the space-time picture we are about to present below. In particular, the free energy

$$f(\lambda) = \begin{cases} -\frac{1}{4}, & \lambda > 1 \\ -\lambda + \frac{3\lambda^2}{4} - \frac{\lambda^2}{2} \log \lambda, & \lambda < 1 \end{cases} \quad (13)$$

becomes a constant for $\lambda > 1$ corresponding to $\ln Z_N(R) = R^2$ independent of N .

The result (13) is obtained semiclassically by considering the saddle point configuration $\sigma(k)$ minimizing the action Eq. (11). This approximation is controlled by the large parameter R^2 and the optimal configuration was found in Ref. [9]. Our main goal is to make the connection between optimal $\sigma(k)$ and the large scale space-time structure of the fermionic configuration dominating the statistical sum (9). For this we need some technical details about the minimization procedure which we outline below.

The action (11) has to be minimized with respect to $\sigma(k)$ and μ . Therefore, in addition to Eq. (12), we obtain the equilibrium condition

$$\varepsilon(k) - \mu - \int_{I_k} \frac{dk'}{2\pi} \log \left| e^{ik} - e^{ik'} \right|^2 \sigma(k') = 0, \quad (14)$$

valid for k belonging to the interval(s) I_k where $\sigma(k) > 0$. As was discussed in Ref. [9], Eq. (14) is equivalent to the electrostatic problem of finding the equilibrium configuration of charges with density $\sigma(k)$ interacting among themselves via a 2D logarithmic Coulomb potential and subject to an external field represented by $\varepsilon(k)$. It was found that the distribution $\sigma(k)$ undergoes a qualitative change: for $\lambda > 1$ the quasimomenta fill the whole Brillouin zone $I_k = [-\pi, \pi]$, while for $\lambda < 1$ a gap appears near $k = \pm\pi$ and $I_k = [-k_c, k_c]$.

Here we present this solution using the method of semiclassical loop equations developed in Random Matrix Theory [17, 18, 19, 20, 21] (see also Ref. [22] for applications to GWW model). We define a function

$$X(k) = \lambda - i\varepsilon'(k) - 2 \int \frac{dk'}{2\pi} \frac{e^{ik}}{e^{ik} - e^{ik'}} \sigma(k'), \quad (15)$$

analytic in the complex plane of k cut along an interval on the real axis which represents the support of $\sigma(k)$, so that

$$X(k \pm i0) \equiv X_{\pm}(k) = \pm\sigma(k). \quad (16)$$

The solution of Eq. (14) can be represented by the implicit relation

$$F_0(X, k) = X^2 + m^2 - (\lambda + \cos k)^2 = 0, \quad (17)$$

where m plays the role of order parameter and has the following behavior across the GWW phase transition

$$m = \begin{cases} 0, & \lambda > 1 \\ 1 - \lambda, & \lambda < 1 \end{cases}. \quad (18)$$

The implicit relation Eq. (17) is a *complex curve* of the Random Matrix Model behind GWW model which provides full information about the optimal distribution of eigenvalues e^{ik_l} . Despite the fact that the leading order behaviour of the fermionic propagator Eq. (9) is obtained using this mean-field solution, little can be said about the dynamics of underlying fermionic model. In the next sections we extract the full space-time picture of the semiclassical dynamics of the free fermions. This space-time resolved picture will be used to explore beautiful hidden features of the GWW third order transition which we reinterpret in the context of the limit shape phenomenon.

3. Dynamics of free fermions in GWW model

The solution of GWW model presented in the previous section is essentially static and while it gives the large- R behaviour of the fermionic propagator Eq. (9) the dynamics of the fermions leading to this result is buried under rather technical Coulomb gas calculations.

In order to reveal the dynamical picture we need an extra time dimension to label the time-dependent configurations of dynamical variables. We use the standard approach and split the evolution operator e^{-2RH} into a product of infinitesimal evolutions representing the propagator Eq. (1) as the path integral

$$Z(R, L) = \int D[\sigma, \vartheta] e^{-4R^2 S[\sigma, \vartheta]} \quad (19)$$

over time-dependent momentum distribution $\sigma(k, \tau)$ and its conjugate field $\vartheta(k, \tau)$ labeled by dimensionless time τ defined on the interval $-1 < \tau < 1$. The field $\sigma(k)$ of the previous section corresponds to the time slice $\tau = 0$ of the imaginary time dynamics with fields for $\tau \neq 0$ integrated out.

The action in Eq. (19) can be naturally represented as a sum of two terms, the dynamical term

$$S_d = \frac{1}{2} \int_{-1}^1 d\tau \int \frac{dk}{2\pi} [\vartheta(k, \tau) \partial_{\tau} \sigma(k, \tau) + (\varepsilon(k) - \mu) \sigma(k, \tau)] \quad (20)$$

corresponding to the evolution operator and the boundary term which originates from the matrix elements $\langle \{k\}^{f/i} | N \rangle \rightarrow \langle \sigma^{f/i} | N \rangle$ and its complex conjugate in Eq. (6) between the fields $\sigma^{f/i}(k) = \sigma(k, \pm 1)$ on the boundaries of the time interval,

$$S_b = -\log \langle N | \sigma^f \rangle - \log \langle \sigma^i | N \rangle , \quad (21)$$

where

$$\log \langle \sigma | N \rangle = \log \langle N | \sigma \rangle^* = -\frac{1}{2} \int \frac{dk}{2\pi} \frac{dk'}{2\pi} \sigma(k) \log(e^{ik} - e^{ik'}) \sigma(k') . \quad (22)$$

Variation of the action gives

$$\begin{aligned} \delta S = \delta S_d + \delta S_b = & \frac{1}{2} \int d\tau \int \frac{dk}{2\pi} [\delta \vartheta (\partial_\tau \sigma) + \delta \sigma (-\partial_\tau \vartheta + \varepsilon(k) - \mu)] \\ & - \frac{1}{2} \int \frac{dk}{2\pi} [(\vartheta^i + \theta^i) \delta \sigma^i - (\vartheta^f - \bar{\theta}^f) \delta \sigma^f] , \end{aligned} \quad (23)$$

where

$$\theta^{f/i}(k) = \int \frac{dk'}{2\pi} \log(e^{ik} - e^{ik'})^2 \sigma^{f/i}(k') \quad (24)$$

are complex potentials induced by the boundary distributions $\sigma^{f/i}(k)$. The variation w.r.t. fields inside the time interval gives

$$\partial_\tau \sigma(k, \tau) = 0 , \quad (25)$$

$$\partial_\tau \vartheta(k, \tau) = \varepsilon(k) - \mu . \quad (26)$$

These equations hold independently for every quasimomentum k due to the translational invariance of the Hamiltonian. Their solution is

$$\sigma(k, \tau) = \sigma(k) , \quad (27)$$

$$\vartheta(k, \tau) = \vartheta(k, 0) + (\varepsilon(k) - \mu)\tau . \quad (28)$$

The variation w.r.t. boundary fields $\delta \sigma_\pm$ fixes

$$\vartheta(k, 0) - \varepsilon(k) + \mu = -\theta(k) , \quad (29)$$

$$\vartheta(k, 0) + \varepsilon(k) - \mu = \bar{\theta}(k) , \quad (30)$$

since for a time-independent $\sigma(k, \tau) = \sigma_\pm(k) = \sigma(k)$ we can use $\theta^{f/i}(k) = \theta(k)$ in Eq. (24). The difference of these equation reproduces exactly the electrostatic equilibrium condition, Eq. (14). This equation also mixes different quasimomenta as the boundary conditions break the translational invariance. The sum of these equations yields $\vartheta(k, 0) = 0$ for $\sigma(k) = \sigma(-k)$, consistent with the time-reversal symmetry of the problem.

These time-dependent configurations dominating the path integral Eq. (19) are related to dynamics of free fermions and, ultimately, to the limit shape emerging as the

dominant contribution to the statistical sum Eq. (9). This relation is quite non-trivial as we show below. Let us introduce the complex combinations of the time-dependent fields

$$X_{\pm}(k, \tau) = \pm\sigma(k, \tau) - i\partial_k\vartheta(k, \tau). \quad (31)$$

On the equations of motion (27,28) they evolve ballistically

$$X_{\pm}(k, \tau) = X_{\pm}(k) - i\varepsilon'(k)\tau, \quad (32)$$

from the initial value $X_{\pm}(k, 0) = X_{\pm}(k) = \pm\sigma(k)$ obtained from the complex curve Eq. (17), so that

$$F_{\tau}(x, k) = F_0(x + i\varepsilon'(k)\tau, k) = (x + i\tau \sin k)^2 + m^2 - (\lambda + \cos k)^2 = 0, \quad (33)$$

where m is given by (18). The time-dependent spatial configuration dominating the statistical sum Eq. (9) is fully determined by this equation. Indeed, we show in Appendix A that $X(k)$ is a position (in units of R) of a particle with momentum k at $\tau = 0$, so the ballistic motion Eq. (32) describes (imaginary) time dynamics of free fermions. Our approach is one of the collective field theory [23, 24]. Below we use the method of fermionic droplets in phase space (see Ref. [25] and references therein) extending it to imaginary time dynamics.

Given k_l are momenta of fermions, we can determine the distribution of their positions x_l in the large R limit using a semiclassical approach. To do this, we consider the phase space distribution (Wigner function) $W_0(x, k)$, which for free fermions has a very simple structure: $W_0(x, k) = 1$ in the region filled with particles and $W_0(x, k) = 0$ in the region filled with holes, so one only needs to know the shape of the boundary separating these regions. The latter is nothing but the solution of Eq. (17) for real x and k , *i.e.* this boundary is a real section of the complex curve. It is natural to postulate the Wigner function in the form

$$W_0(x, k) = \frac{1}{\pi} \text{Im} \log F_0(x, k). \quad (34)$$

The normalisation (12) implies that x is the coordinate of fermions measured in units of R . For real but sign changing function $F_0(x, k)$ the implicit equation Eq. (17) gives the boundary of the occupied portion of the phase space.

Generally speaking, the boundary separating particles from holes in phase space may consist of a number of curves encircling disconnected regions in the phase space, but we do not consider this possibility here. In our case, there is only one connected region in the phase space, and Eq. (17) defines a two-valued analytic function $X(k)$ with a single branch cut along the interval I_k . The distribution $\sigma(k)$ is given by the jump across the cut:

$$\sigma(k) = \frac{1}{2} [X_+(k) - X_-(k)], \quad (35)$$

where $X_{\pm}(k) = X(k \mp i\epsilon)$. Physically, these two solutions represent the distance between the most remote right and left fermions. As particles fill uniformly the region between $X_-(k)$ and $X_+(k)$, the above expression for $\sigma(k)$ is equivalent to

$$\sigma(k) = \frac{1}{2} \int_{X_-}^{X_+} W_0(x, k) dx. \quad (36)$$

Conversely, resolving Eq. (17) for $k = K(x)$ gives again a two-valued function with branches $K_{\pm}(x)$ which can be interpreted as local Fermi points, so their difference is proportional to the real space density of fermions,

$$\rho(x) = \int_{K_-}^{K_+} \frac{dk}{2\pi} W_0(x, k) = \frac{1}{2\pi} [K_+(x) - K_-(x)]. \quad (37)$$

These considerations are illustrated in Fig. 2.

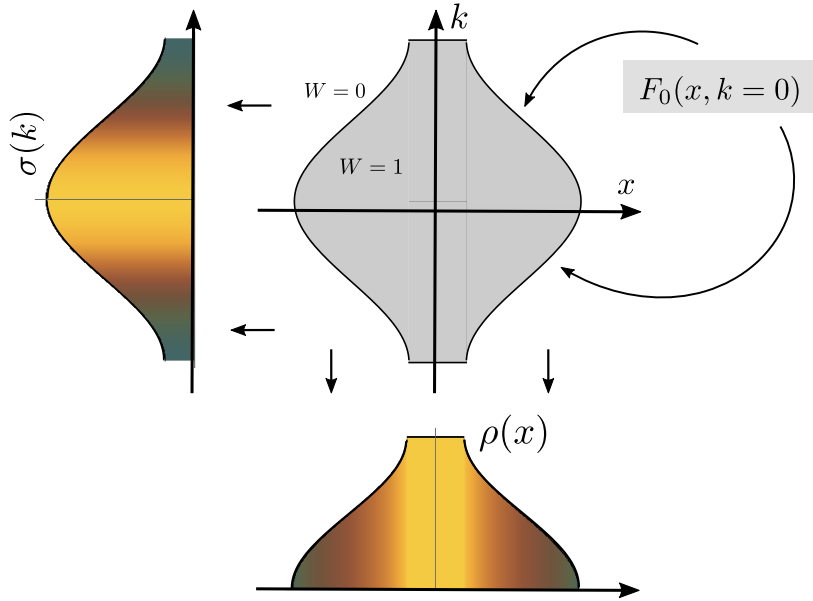


Figure 2. Wigner function for free fermions. The quasimomentum distribution $\sigma(k)$ and real space density $\rho(x)$ are obtained by integrating $W(x, k)$ along the x and k respectively.

This semiclassical picture can be extended to arbitrary τ by making the ballistic shift Eq. (33) in the complex curve. In contrast to Eq. (17), the function $F_\tau(x, k)$ is complex, so the requirement of vanishing of its real and imaginary parts constitute a system of two equations for two unknowns: the density $\rho(x, \tau) = (k + \bar{k})/2\pi$ and the velocity $v(x, \tau) = (k - \bar{k})/2i$. Solving Eq. (33) for fixed real x, τ gives space-time information about density and velocity configurations which can otherwise be viewed as a mapping between space-time (x, τ) and a surface parametrised by local coordinates

(ρ, v) . Finding the shape of such a surface can be formulated as a minimisation problem leading to complex Burgers equations for combinations $k, \bar{k} = \pi\rho \pm iv$ of the hydrodynamic fields. In the presence of nontrivial boundary conditions imposed at the endpoints of the time interval, the limit shape phenomenon is expected [26]. In the next section, we obtain and characterize the limit shapes across GWW phase transition.

4. Arctic shapes: fluctuating regions and their boundaries

We turn to the solutions of Eq. (33) for various values of the control parameter λ . In terms of the variable $z = e^{ik} = e^{i\pi\rho - v}$ the equation $F_\tau(x, k) = 0$ is a quartic equation and has four solutions. Moreover, owing to the symmetry of the spectral curve Eq. (33) if z is a solution then $\bar{z}$ is also a solution, thus the solutions are either real corresponding to $\rho = 0, 1$ or there is a complex conjugate pair of solutions corresponding to $k = \pi\rho + iv$ and $-\bar{k} = -\pi\rho + iv$ with $0 < \rho < 1$. The former, real case, corresponds to the frozen regions while in the latter case the point (x, τ) lies within the fluctuating (aka liquid) region.

As the point (x, τ) approaches the boundary of the fluctuating region two solutions $z = e^{ik}, \bar{z} = e^{-i\bar{k}}$ move toward each other and coalesce on the real axis when the boundary is reached. If this happens for $\text{Re } z > 0$ then $k = -\bar{k} = iv$ and the point (x, τ) is on the boundary of the empty frozen region, $\rho = 0$. For two solutions meeting on the negative segment of the real axis, $\text{Re } z < 0$, the point (x, τ) reaches the boundary with the fully occupied frozen region, $\rho = 1$. The condition to have degenerate solutions is

$$\partial_k F_\tau(x, k) = 0. \quad (38)$$

Imposing it in addition to Eq. (33) and solving for $k = iv$ or $k = \pi + iv$ leads to the parametric form of the boundary of a frozen region,

$$x - V(v)\tau = G(v), \quad (39)$$

$$-\partial_v V(v)\tau = \partial_v G(v), \quad (40)$$

where for the boundary of the empty region $V(v) = -i\varepsilon'(iv) = \sinh v$ and $G(v) \equiv X(iv)$ is a solution of Eq. (17) with $k = iv$. The boundary of the fully occupied frozen region, $\rho = 1$, is obtained from Eqs. (39),(40) by substituting $G(v) = X(\pi + iv)$, $V(v) = -i\varepsilon'(\pi + iv) = -\sinh v$. This transformation $k \rightarrow k + \pi$ amounts to the *particle-hole conjugation*. Since Eq. (17) has two solutions for X , the boundary of fluctuating regions consists of two outer curves separating it from the frozen empty regions and two inner boundaries separating it from the full frozen region.

Eqs. (39,40) determine boundaries of liquid regions (see Fig. 3). It was shown that this method known as the “tangent method” allows to find the shape of liquid regions even in the presence of short-range interactions [27]. The essence of this method is the fact that the boundary curve $x(\tau)$ and the function $G(v)$ are related by the generalised

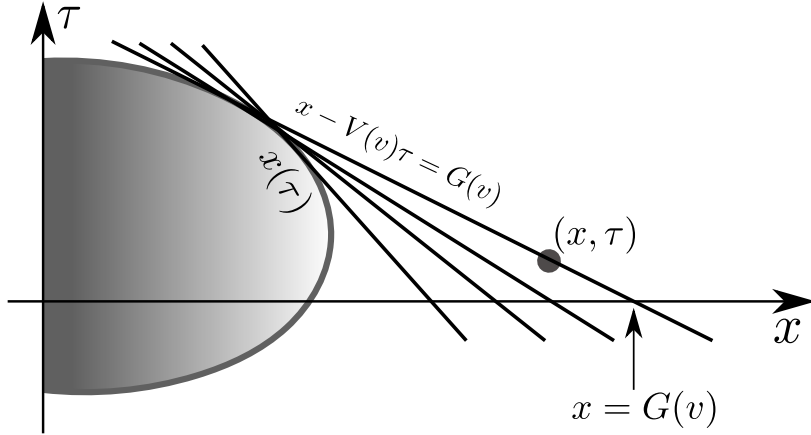


Figure 3. One parameter family of lines given by Eq. (39) tangent to the boundary of fluctuating region $x(\tau)$. The point of tangency for given value of parameter v is determined by Eq. (40).

Legendre transform

$$G(v) = x - V(v)\tau, \quad \frac{dx}{d\tau} = V(v). \quad (41)$$

For free fermions this allows to recover the function $G(v)$ and, consequently, the function $F_0(x, k)$ from the explicit form of the boundaries $x(\tau)$.

By fixing a point (x, τ) inside the frozen region one can solve Eq. (39) geometrically by finding the slope $v(x, \tau)$ of the line tangential to the frozen boundary passing through (x, τ) as shown in Fig. 3. Since there are typically more than one such lines, the function $v(x, \tau)$ is multivalued (quadruple-valued in our case). For (x, τ) close enough to the boundary, two points of tangency approach each other, and so do their slopes. Once (x, τ) reaches the boundary, the two points of tangency coalesce, and their slopes degenerate. This geometric scenario is equivalent to our earlier statement about coalescing roots.

As one moves inside the fluctuating region, two solutions $z = e^{ik}, e^{-i\bar{k}}$ leave the real axis symmetrically and ρ begins to evolve from its frozen value. Solving $F_\tau(x, k) = 0$ for k at fixed (x, τ) provides the density and velocity profiles. Below we illustrate the boundaries of the fluctuating regions and calculate the density and velocity profiles for $\lambda > 1$ and $\lambda < 1$, which in the standard nomenclature of Refs. [9, 10] are called strong and weak coupling phases, respectively.

From our geometric perspective, a more intuitive designation for these phases is more appropriate. Indeed, looking at density profiles shown in Fig. 4 one sees that the main feature of these solutions is that for $\lambda > 1$, the liquid regions are completely separated by the frozen boundary, while for $\lambda < 1$ the liquid regions merge. In the following, we use the names of separated and merged phases for $\lambda > 1$ and $\lambda < 1$, respectively. The critical value $\lambda_c = 1$ of the control parameter corresponds to the Merger Transition.

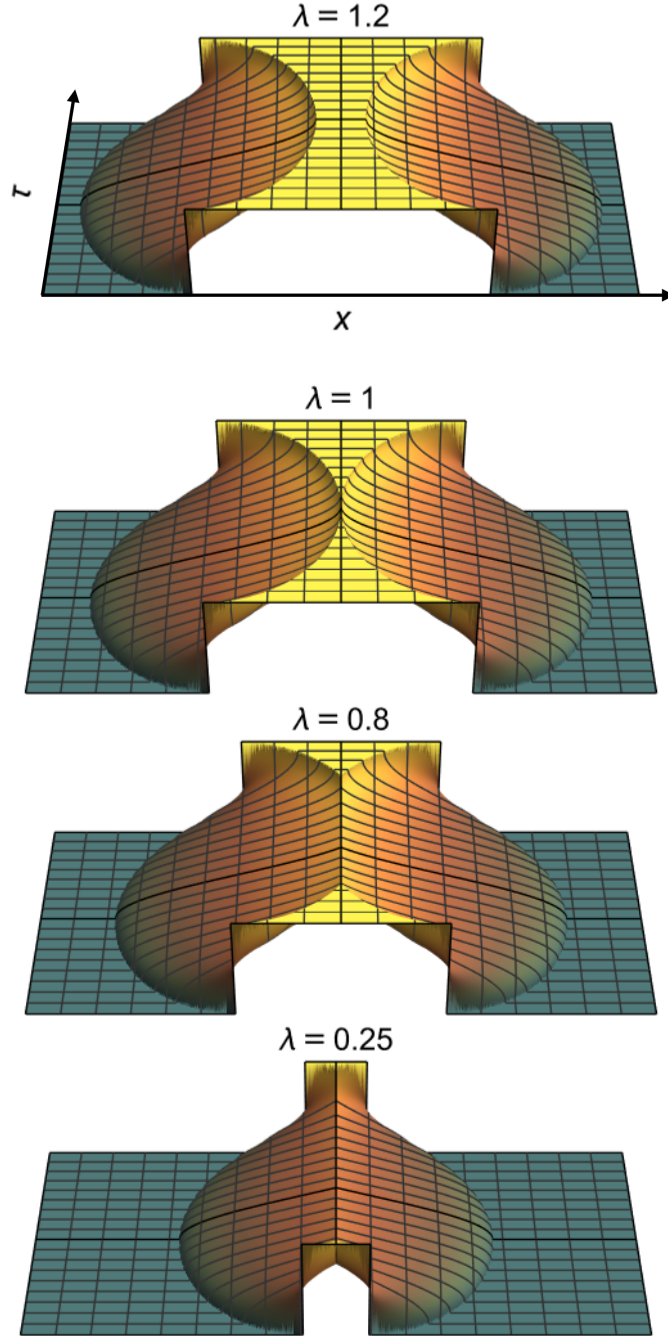


Figure 4. 3D plot space-time density profile $\rho(x, \tau)$ across the Merger Transition.

4.1. Strong coupling $\lambda > 1$: separated arctic circles

In this regime equation $F_\tau(x, k) = 0$ is readily solved since $m = 0$ and the function $F_\tau(x, k)$ in Eq. (33) factorizes. Substituting $k = iv$ we get $G(v) = \pm(\lambda + \cosh v)$ to be used in Eqs. (39),(40) for the boundaries of the empty frozen region. This leads to $\tau = \mp \tanh v$, $x = \pm(\lambda + 1/\cosh v)$ which is the parametric equation of the outer half circles centered at $\pm\lambda$. The particle-hole transformation gives the inner half circles

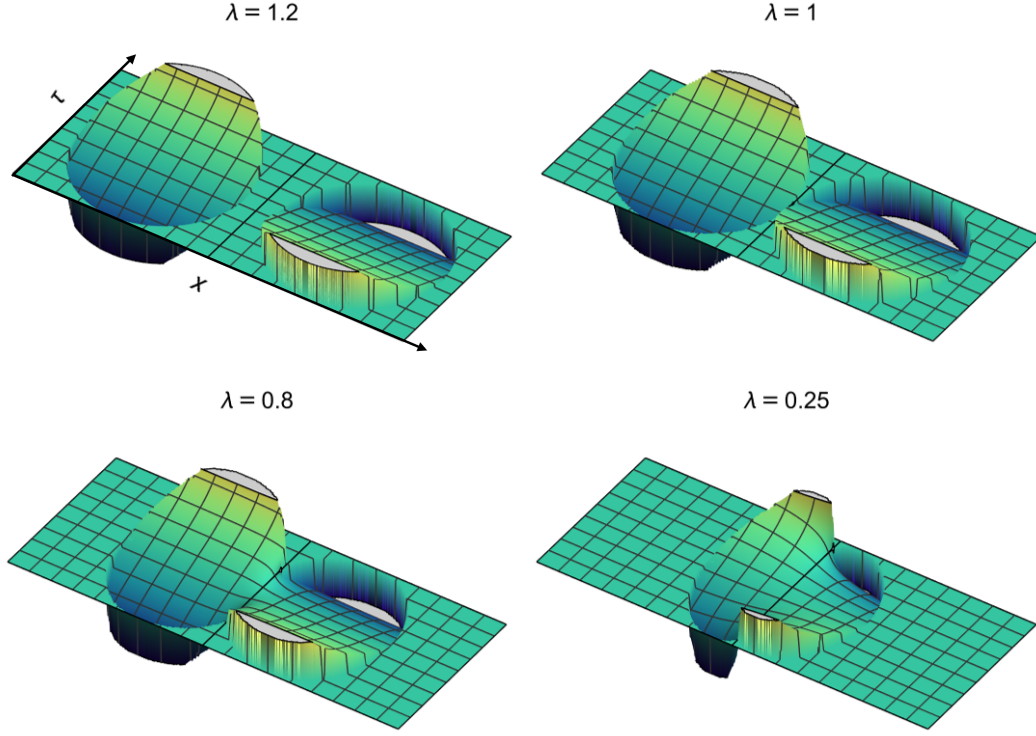


Figure 5. 3D plot space-time velocity profile $v(x, \tau)$ in the separated, $\lambda > 1$, and the merged, $\lambda < 1$, phases.

$\tau = \mp \tanh v$, $x = \pm(\lambda - 1/\cosh v)$ describing the boundaries of the fully occupied frozen region. The overall shape consists of two arctic circles separated by the full frozen region. As the circles are centered at $\pm\lambda$ and have unit radii the width of the frozen region between them is $2\lambda - 2$.

The density and velocity fields inside the fluctuating circular regions are obtained from the same equation allowing $k = \pi\rho + iv$ to be complex. Inside the right circle, $(x - \lambda)^2 + \tau^2 < 1$ we have

$$\rho(x, \tau) = \frac{1}{\pi} \arccos \frac{x - \lambda}{\sqrt{1 - \tau^2}}, \quad (42)$$

$$v(x, \tau) = \frac{1}{2} \log \frac{1 - \tau}{1 + \tau}, \quad (43)$$

which are the known density and velocity profiles for one arctic circle [28, 8] (shifted by λ in the positive x -direction). The density and velocity profiles inside the left circle, $(x + \lambda)^2 + \tau^2 < 1$ are obtained from the above expression by reflection $x \rightarrow -x$, $\tau \rightarrow -\tau$, corresponding to the reflected boundary conditions ($\rho = 0$ on the left hand side and $\rho = 1$ on the right hand side). The velocity field “freezes” at its boundary value and stays constant as one sweeps across the fluctuating region at fixed value of τ . Inverting Eq. (42) for $\tau = 0$ yields the momentum distribution of GWW model in the strong

coupling phase

$$\sigma(k) = \lambda + \cos k \quad (44)$$

positive everywhere in the Brillouin zone $-\pi < k < \pi$.

4.2. Weak coupling $\lambda < 1$: merged arctic circles

For $\lambda < 1$ the critical parameter $m = 1 - \lambda > 0$ and Eq. (17) gives $G(v) = \pm\sqrt{(\lambda + \cosh v)^2 - m^2}$ for the outer boundary and $G(v) = \pm\sqrt{(\lambda - \cosh v)^2 - m^2}$ for the inner one. Eliminating v from Eq. (40) leads to a full quartic equation and its analysis becomes somewhat cumbersome. While it might be solved analytically the resulting expressions are not very useful. Some analytic progress can be made in the regime of small v , including the interesting central region, which we explore in more detail in the next section. Here we just mention that while for the outer boundaries, adjacent to the empty frozen region, $\rho = 0$, the function $\tau(v)$ remains smooth, the solution for the inner boundary curve leads to $\tau(v)$ undergoing a discontinuity at $v = 0$. This produces two characteristic cusps in the inner boundary with tips at $(0, \pm\sqrt{m})$ and the arctic shape consists of a single simply connected region, which we refer to as “merged arctic circles”. We present the density and velocity profiles obtained by solving Eqs. (39,40) numerically using Mathematica[®] in Figs. 4, 5.

We can glean some insight on the solution in the weak coupling regime by analytically solving Eq. (17) along the lines $\tau = 0$ and $x = 0$. In the former case we see that we have the momentum distribution

$$\sigma(k) = 2 \cos \frac{k}{2} \sqrt{\cos^2 \frac{k}{2} - m}, \quad (45)$$

which vanishes in a finite interval in the Brillouin zone around $k = \pi$. Outside of this interval it develops square root singularities at $\pm k_c$, where $k_c = 2 \arccos \sqrt{m}$. This corresponds to a single interval of fluctuating density,

$$\rho(x) = \frac{1}{\pi} \arccos \left(\sqrt{x^2 + m^2} - \lambda \right), \quad (46)$$

which attains its maximum value $\rho_{\max} = k_c/2\pi$ at $x = 0$.

Putting $x = 0$ in Eq. (17) we get the density along the central “ridge”,

$$\rho(0, \tau) = \frac{1}{2} + \frac{1}{\pi} \arcsin \sqrt{\frac{\lambda}{1 - \tau^2}}. \quad (47)$$

valid for $|\tau| < \sqrt{m}$. We observe that this expression (up to a constant and overall normalization) coincide with the equilibrium density profile (with τ playing the role of coordinate) of discrete log-gas [29] with density $\sqrt{\lambda}$. For $\lambda = 1$ the cusps of the frozen boundary touch, the ridge shrinks to a point and the frozen boundary looks as two circles touching at $(x, \tau) = (0, 0)$.

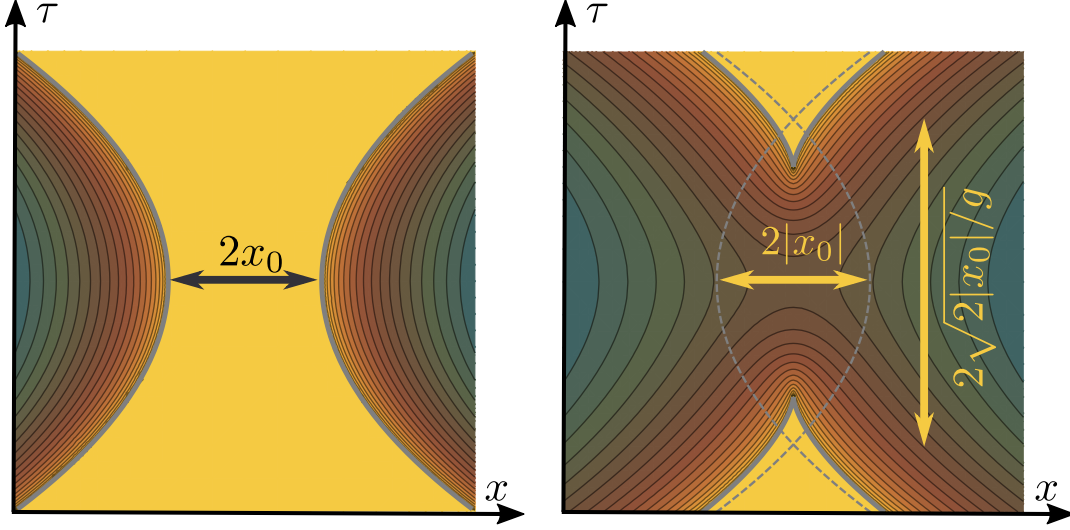


Figure 6. Central fluctuating region and its boundaries in the separated, $x_0 > 0$, and merged, $x_0 < 0$, phases. Dashed lines are boundaries of naively shifted separated arctic curves, Eq. (50).

5. Merger Transition

We have observed that the main feature of the phase transition is the merger of two arctic circles at $x = 0$ via disappearance of the central isthmus and emergence of characteristic cusps in the frozen boundary. We call it the Merger Transition and study it by expanding Eq. (33) in the vicinity of $(x, \tau) = (0, 0)$ in small deviations $q = \pi - k = \pi\delta\rho + iv \ll 1$. It is convenient at this stage to restore the macroscopic length and time scales, which is equivalent to rescaling $x \rightarrow x/R$, $\tau \rightarrow \tau/R$ in Eq. (33). The approximate complex curve (33)

$$(x - i\tau q)^2 = \left(x_0 + \frac{q^2}{2g}\right)^2 - m^2 R^2 \quad (48)$$

depends on two parameters with the geometric meaning discussed below. The density profile obtained from this equation is shown in Fig. 6. Below we discuss the prominent features of this solution.

We consider the frozen boundary first. In the separated phase $m = 0$ and substituting $\delta\rho = 0$ into Eq. (48) we have the boundary in the parametric form

$$x = \pm \left(x_0 + \frac{v^2}{2g}\right), \quad \tau = \pm v/g. \quad (49)$$

Eliminating v leads to the boundary consisting of two parabolae,

$$x = \pm \left(x_0 + \frac{g\tau^2}{2}\right). \quad (50)$$

Close to the boundaries the density deviates from 1 as square root of the distance to the boundary into the liquid phase, *i.e.* for $\tau = 0$ we have for the right region ($x > x_0$)

$$\delta\rho = \frac{\sqrt{2g}}{\pi}(x - x_0)^{1/2}, \quad x \gtrsim x_0, \quad (51)$$

and velocity keeps its value $v = -g\tau$ away from the boundary.

The parameters x_0 and g have the clear geometric meaning of half the distance between the frozen boundaries, and their curvature. In terms of the right hand side of Eq. (39) $x_0 = RG(0) = L - R$ and $g^{-1} = RG''(0) = R$. The shape (50) is valid for $x_0 > 0$ and describes the boundaries of the central frozen isthmus between two fluctuating regions. Decreasing x_0 leads to the isthmus thinning out until its two boundaries touch at the transition $x_0 = 0$.

For $x_0 < 0$ the curve Eq. (50) corresponds to the unphysical situation of overlapping liquid regions with double valued density profile. To rectify this situation one has to modify the right hand side of Eq. (48) by positive order parameter $m = |x_0|/R$. This gives

$$x + v\tau = \mp \sqrt{\left(x_0 - \frac{v^2}{2g}\right)^2 - m^2 R^2} = \mp v\tau_0 \sqrt{1 + \left(\frac{v}{v_0}\right)^2}, \quad (52)$$

where $\tau_0 = \sqrt{|x_0|/g}$ is the characteristic time and $v_0 = 2|x_0|/\tau_0$ is the characteristic velocity scale. The degeneracy condition, Eq. (40), leads to

$$\tau = \mp \tau_0 \frac{1 + 2\left(\frac{v}{v_0}\right)^2}{\sqrt{1 + \left(\frac{v}{v_0}\right)^2}}. \quad (53)$$

This equation does not have real solutions for $-\tau_0 < \tau < \tau_0$. This is the interval in which there is no separation of the two liquid regions and its endpoints $(0, \pm\tau_0)$ correspond to the boundary cusps. The presence of such a liquid region between arctic curves is the characteristic feature of the merged phase. Tuning x_0 to zero from below makes the cusps approach each other so that the width of the liquid “channel” behaves as $|x_0|^{1/2}$. At the Merger Transition, $x_0 = 0$, the cusps touch each other, and the channel connecting the liquid regions closes.

The solution of Eq. (53) for $|\tau| > \tau_0$ gives the boundary velocity

$$v(\tau) = \pm v_0 \Upsilon\left(\frac{\tau}{\tau_0}\right) \quad (54)$$

in terms of the universal function

$$\Upsilon(s) = \frac{1}{2} \sqrt{\frac{s^2}{2} - 2} + \sqrt{2s^2 + \frac{s^4}{4}}, \quad (55)$$

which has a square root singularity as s approaches 1 from above. Substitution of (54) into (52) gives the universal shape of the boundary below the Merger Transition

$$x(\tau) = \pm 2|x_0| \frac{\Upsilon^3\left(\frac{\tau}{\tau_0}\right)}{\sqrt{1 + \Upsilon^2\left(\frac{\tau}{\tau_0}\right)}}, \quad (56)$$

which behaves near the tips of the boundary cusps as

$$x(\tau) = \pm \frac{8|x_0|}{3\sqrt{6}} \left(\frac{\tau}{\tau_0} - 1\right)^{3/2}. \quad (57)$$

This behaviour is identical to the standard singularity of a mapping of a two-dimensional surface onto a plane known as cusp in Whitney classification [30, 31]. Expanding the right hand side of Eq. (52) for $v \ll v_0$ gives, for the positive sign choice, the canonical form at the cusp singularity

$$\frac{x}{|x_0|} = \left(\frac{v}{v_0}\right)^3 + 2\left(1 - \frac{\tau}{\tau_0}\right) \left(\frac{v}{v_0}\right). \quad (58)$$

Here one regards (v, x, τ) as three dimensional coordinates of a two-dimensional surface projected along the v -direction onto the plane (x, τ) . The choice of the negative sign in Eq. (52) leads to another surface with another Whitney cusp at $\tau = -\tau_0$. For $|\tau| > \tau_0$ the boundaries $x(\tau)$ are folds in Whitney classification [30].

In the limit $\tau \gg \tau_0$ one recovers the parabolic shape Eq. (50) of the boundary. Note that as x_0 is negative, the large τ asymptotics are reproduced by the “naive” shift of the frozen boundary, namely Eq. (50) with $x_0 < 0$, shown by dashed lines in Fig. 6. The complex curve (52) and the resulting function (55) interpolate nicely between the cusp and fold singularities.

In the merged phase the density profile for $\tau = 0$ becomes

$$\delta\rho = \frac{\sqrt{2g}}{\pi} \left(\sqrt{x_0^2 + x^2} + |x_0|\right)^{1/2}, \quad (59)$$

which has minimum value $\delta\rho_{\min} = 2\sqrt{g|x_0|}/\pi$. Across the narrow strait between the two macroscopic liquid regions, $x = 0$, $-\tau_0 < \tau < \tau_0$, the density depletion has a semicircle shape,

$$\delta\rho = \frac{2g}{\pi} \sqrt{\frac{|x_0|}{g} - \tau^2}. \quad (60)$$

6. Universality of the Merger Transition

We expect the above scenario of the Merger Transition, the shape of the frozen boundary, and the density profile of the liquid regions near the point of merger to be universal.

The expectation is based on the fact that for any interacting model with short-range interactions, the boundary can be obtained from dynamics of free fermions (particles or holes) leading to the universal complex curve Eq. (48). The situation is similar to the universal free-fermionic physics near boundaries of interacting systems [32]. Below we provide a more detailed argument based on the symmetries of our solution and scaling considerations.

6.1. Complex curve at the phase transition

Let us consider the complex curve (33) close to the merger phase transition and close to the space-time point where the merger occurs. In that region the curve (33) becomes (48) which we reproduce here as

$$X^2 = \left(x_0 + \frac{q^2}{2g}\right)^2 - m^2 R^2. \quad (61)$$

Here $X = x - i\tau q$. The boundary conditions we used as well as the dispersion of fermions have time reversal $\mathcal{T}$ and space reflection (or parity) $\mathcal{P}$ symmetries. We summarize these symmetries as

$$\mathcal{P}: \quad x \rightarrow -x, \quad q \rightarrow -q, \quad X \rightarrow -X, \quad (62)$$

$$\mathcal{T}: \quad \tau \rightarrow -\tau, \quad q \rightarrow \bar{q}, \quad X \rightarrow \bar{X}. \quad (63)$$

It is straightforward to check that the curve (61) has symmetries (62,63). In particular, the time reversal symmetry guarantees the reality of the coefficients in the rhs of (61) and the parity symmetry makes sure that the rhs is a function of q^2 . Therefore, based only on the analyticity of $X(q)$ and the symmetries (62,63) we could write

$$X^2 = a(q^2 + \delta_1)(q^2 + \delta_2) \quad (64)$$

as the most general algebraic solution up to order q^4 . Here, a is real and $\delta_{1,2}$ are either both real or form a complex conjugate pair $\delta_1 = \bar{\delta}_2$.

Let us now attempt to describe the most general transition at which two liquid regions merge, keeping the symmetries (62,63). On one side of the phase transition we should have separate liquid regions which is equivalent to the conditions $a > 0$, $\delta \equiv \delta_1 = \delta_2 > 0$. In this case we have $X = \pm\sqrt{a}(q^2 + \delta)$ describing two separate liquid regions corresponding to the choice of sign (see the left panel of Fig. 6). When $\delta = 0$ the boundaries of liquid regions touch. For the merged phase, one can consider the density and current profiles corresponding to (64). Since the flux of particles $j \sim \rho v = \text{Im } q^2$ must be zero for $x = 0$ by symmetry, the spectral curve should have real roots for all τ . In addition $\rho \sim \text{Re } q$ should vanish for $\tau > \tau_0$. The only possibility for this to happen is the one corresponding to $\delta_1 = \delta < 0$ and $\delta_2 = 0$ (and the identical solution with $\delta_{1,2}$ exchanged). That is $X^2 = aq^2(q^2 + 2\delta)$ (see the right panel of Fig. 6). We summarize

all cases as

$$X^2/a = \begin{cases} (q^2 + \delta)^2, & \delta > 0, \\ q^4, & \delta = 0, \\ q^2(q^2 + 2\delta), & \delta < 0. \end{cases} \quad (65)$$

In the particular model considered in this paper $a = 1/4$, $\delta = 2(\lambda - 1)$.

The phase transition is driven by the real parameter δ . During the transition, the type of complex curve changes as summarized in (65). As we explained above, the transition- in the shape of the curve is essentially fixed by requirements of (i) parity and time-reversal symmetries of the fermion dispersion (Hamiltonian) and boundary conditions (ii) the requirement of the least possible number of singularities. The latter requirement forces (65) to be the quadratic polynomial in q^2 with either degenerate roots or having one of the roots at 0.

We remark that we expect (65) to capture the universal dependence of density and velocity of fermions near the merging point. Indeed, assume that there is an interaction between fermions which preserves $\mathcal{P}$ and $\mathcal{T}$ symmetries of the Hamiltonian (for example, fermions corresponding to XXZ spin chain). Then, near the point of the merger of two liquid regions, the density of fermions is small (or close to 1). In this limit, the interaction is negligible, and the solution of hydrodynamic equations is still captured by a complex curve $X(q)$. Symmetries will then fix the form of (65) as a universal curve that changes across the phase transition.

6.2. Scaling estimates at the Merger Transition

One can estimate the action (i.e., free energy) cost of the merged configuration using the following geometric argument. For sufficiently small $|x_0|$, $x_0 < 0$ the overlap of two arctic curves computed naively has x dimension $2|x_0|$ and τ dimension $2\sqrt{2|x_0|/g}$ with the area of the overlap scaling as $A \sim g^{-1/2}|x_0|^{3/2}$, see Fig. 6. The depletion of the density at point $(0,0)$ is equal to $\delta\rho = 2(g|x_0|)^{1/2}/\pi$. The change of the energy (per unit length) because of the depletion is $\delta E \sim \delta\rho \times \delta\rho^2 \sim \delta\rho^3 \sim g^{3/2}|x_0|^{3/2}$. An estimate of the action cost of the overlap region gives

$$\delta S \sim A \times \delta E \sim (g^{-1/2}|x_0|^{3/2}) \times (g^{3/2}|x_0|^{3/2}) = g|x_0|^3. \quad (66)$$

This explains why the phase transition is of the third order. The exact expression for the free fermion problem, Eq. (13) gives

$$\delta S = R^2 \times \frac{2}{3}m^3 = \frac{2}{3} \frac{|x_0|^3}{R}, \quad (67)$$

in full accordance with the estimate (66) for $g = 1/R$. Similarly to the situation at the edge of an interacting system [32] the precise relation between the parameter g controlling the scale of typical boundary fluctuations and the global parameters of the limiting shape is model specific and depends on interaction details.

7. Discussion

In this work, we considered an example of a phase transition of the third order in a system characterized by a limit shape phenomenon. The transition is a merger of two liquid (fluctuating) regions controlled by a geometric parameter defined by boundary conditions. We used a free fermion formulation of the problem and mapped the transition to the well-known transition in the Gross-Witten-Wadia model [9, 10].

In the Coulomb gas picture of the transition developed in [9, 10] the transition is the change of the support of the density of the Coulomb gas. In a complex plane, the support changes from the whole unit circle to just an arc of the circle. [Compare to similar Coulomb gas transitions described in the Ref. [16].] Using the path integral of 1d lattice fermions propagating in imaginary time, we extended this picture to space-time. The optimal space-time configuration of fermions dominating the path integral exhibits the limit shape phenomenon. Namely, the space-time is divided into liquid regions, where fermion density is between 0 and 1, and frozen regions with $\rho = 0, 1$. At the change of the geometric control parameter entering through the boundary conditions, two liquid regions (arctic circles) separated by a frozen region merge into a single connected liquid region (see Fig. 4). At the value of the geometric parameter corresponding to the transition, the liquid regions touch. Within the mean-field approximation valid in the thermodynamic limit when the sizes of liquid regions are much bigger than the lattice size, we compute the optimal space-time distributions of the density and velocity of fermions at both sides of the transition. We note that for a finite size (finite N), the transition is somewhat smeared by statistical fluctuations. The character of these fluctuations near the boundaries of liquid regions and, in particular, near the point of the merger is a subject of intense studies (see [14, 13] and reference therein).

It is well known that hydrodynamic solutions of free fermion problems can be described by an analytic function (complex curve) [1, 2]. Zooming into the vicinity of the merger, one can describe the transition as a change of the complex curve. Namely, we find that the curve changes according to (65) when the control parameter δ passes through its critical value $\delta = 0$. We argued that this change of the curve is universal, assuming that additional symmetries (parity and time-reversal) are present. The universal density profiles on both sides of the transition are shown in Fig. 6. Using the mapping to the GWW model, in the considered free fermion problem, one can also compute the free energy exactly and identify the phase transition as the third-order. We finally remark that as $j = 0$ at $x = 0$, found from the symmetries of the problem, we could reformulate our problem equivalently as a single arctic circle which comes up against a boundary.

A natural question one can ask is whether the features of the merger phase transition described above are universal in the presence of interactions between fermions. The answer depends on the symmetries of the system. To be more specific, let us consider an example of XXZ quantum spin chain. The Jordan-Wigner transformation maps the XXZ spin chain to fermions with short-range interactions, and one can consider the

amplitude (1) for these fermions as a function of L . The qualitative picture of the merger of two liquid regions stays the same, at least for small anisotropy parameter Δ when the interaction between fermions is weak (see Ref.[8]). The symmetries of the problem include parity, time-reversal, and particle-hole symmetry. Near the point of merger, the fermion density is small or close to 1. If the fermion density is close to 1, it can be mapped to small density using particle-hole symmetry of the problem. In the limit of small density, the fermions with short-range interactions become indistinguishable from free fermions. Therefore, we expect that in the neighborhood of the space-time point where the merger occurs, one can still use a complex curve to describe the fermion density and velocity profiles. [In fact one can use complex curves to describe the whole segments of the boundary of the frozen region [27].] Then one can use parity and time-reversal symmetry to argue for the universal shape of the curve (65). This means that only the parameters such as the effective mass in $X = x + iq\tau/m$, the relationship of the control parameter δ to parameters of boundary conditions, and overall scales will depend on details of the model. Up to the values of these parameters, the density and velocity profiles near the merger will be determined by the universal curve (65).

One more feature that needs a discussion is the order of the phase transition. In the presence of interactions, there is no free fermion description (or Coulomb gas picture) describing the field distributions in the liquid region globally. This is why only the boundary of the region [27] and the vicinity of the merger point allowed an analytical treatment so far. However, we argued in Sec. 6.2 that the contribution of the vicinity of the merger to the free energy has a discontinuity in the third derivative with respect to the control parameter δ . Assuming that the contribution of the regions far from the merger point is analytic in δ , we conjecture that the phase transition is of the third order even in the presence of finite interactions.

The above arguments can be summarized by a conjecture that there is a (possibly symmetry-protected) universality class of the third-order phase transitions occurring at the merger of liquid regions in systems exhibiting limit shape phenomena. To confirm (or disprove) this conjecture, further analytical and numerical studies of systems such as XXZ spin chains or six-vertex models are needed.

In conclusion we add that the real-time analog of (1) has been considered in [33]. In this context, the quantity (1) is referred to as quantum return probability (aka Loschmidt echo). The discontinuity or change of behaviour of the quantum return probability found in [33] can be attributed to the merging arctic circles phase transition considered in this work.

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Appendix A. Dynamic picture for discrete momenta

We start with Eq. (6) and insert the unity

$$\hat{I} = \sum_{\{x\}} \int \frac{d^N k}{(2\pi)^N} |\{x\}\rangle \langle \{x\} | \{k\}\rangle \langle \{k\} | \quad (\text{A.1})$$

at every intermediate time slice labeled by τ . Here $|\{k\}\rangle = |k_1, k_2, \dots, k_N\rangle$ are fermionic states labeled by the ordered set of momenta, $k_1 \leq k_2 \leq \dots \leq k_N$ and similarly for $|\{x\}\rangle$. This leads to the path integral with the following action

$$\begin{aligned} S[\{k(\tau)\}, \{x(\tau)\}] &= R \int_{-1}^1 d\tau \sum_l \left[i x_l \dot{k}_l + \varepsilon(k_l) \right] - \log \Delta^*(e^{ik^f}) - \log \Delta(e^{ik^i}) \\ &\quad - i \left(L - \frac{1}{2} \right) \sum_l (k_l^f - k_l^i) \end{aligned} \quad (\text{A.2})$$

in terms of the phase space trajectories $k_l(\tau)$, $x_l(\tau)$. The terms depending explicitly on the momenta $k_l^{f/i} = k_l(\pm 1)$ at the end points of the time interval arise from the Eq. (7).

The variation of the action w.r.t. $x_l(\tau)$, $k_l(\tau)$ inside the time interval leads to the ballistic motion

$$k_l(\tau) = k_l(0) \equiv k_l \quad (\text{A.3})$$

$$x_l(\tau) = x_l(0) - i\varepsilon'(k_l)\tau, \quad (\text{A.4})$$

while the variation w.r.t. $k_l^{f/i}$ fixes N momenta and N initial coordinates $x_l(0)$ as

$$x_l(0) - \lambda - i\varepsilon'(k_l) = -\frac{i}{R} \frac{\partial}{\partial k_l} \log \Delta^*(e^{ik}) = -\frac{1}{R} \sum_{j \neq l} \frac{e^{-ik_l}}{e^{-ik_l} - e^{-ik_j}}, \quad (\text{A.5})$$

$$x_l(0) - \lambda + i\varepsilon'(k_l) = \frac{i}{R} \frac{\partial}{\partial k_l} \log \Delta(e^{ik}) = -\frac{1}{R} \sum_{j \neq l} \frac{e^{ik_l}}{e^{ik_l} - e^{ik_j}}. \quad (\text{A.6})$$

The second equation is nothing but the discrete analogue of Eq. (15) if the collective field $X(k)$ is defined so that $X(k_l) = x_l(0)$.

4.2 *Large deviations of the full counting statistics of free fermions on an interval*

The following is the manuscript for *Large deviations of the full counting statistics of free fermions on an interval* produced by myself, Samuel. H. Pickering, my supervisor Dimitri Gangardt and our collaborator Alexander Abanov. As first author, I wrote the first draft including generating some figures for the paper, came up with the key idea that there would be a phase transition between partial and full emptiness in the limit shapes and found the corresponding solution. I also computed the free energy and other related quantities and found the experimental link to polymers. The collaborators revised the work, clarifying the link to hydrodynamics in the appendix and Samuel crucially helped clarify the steps mapping the FCS of free fermions to a Coulomb gas model.

Phase transitions in full counting statistics of free fermions and directed polymers

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We consider directed polymers in 1+1 spatial dimension under action of an external repulsive potential along a line. Using the exact mapping onto imaginary time evolution of free fermions we find that for sufficiently strong potential the system of polymers undergoes a continuous configurational phase transition. The transition corresponds to merging empty regions in the dominant limit shape.

Introduction Limit shapes, *i.e.* the emergence of dominant configurations with sharp boundaries in fluctuating statistical systems, are spectacular phenomena playing an important role in studying universal macroscopic behaviour in statistical physics [1, 2]. Recently, this phenomenon was discovered in directed polymers or vicious walkers [3] where it was shown that the strong mutual repulsion between polymers gives rise to emergent long-range entropic forces that combine with rigid topological constraints to produce sharp features of the coarse-grained polymer density.

In quantum many-body systems limit shape configurations were reported in studies of the so-called Emptiness Formation Probability (EFP) [4–8], the probability $P_R(0)$ that a macroscopic (on the scale of inter-particle separation) interval $I = [-R/2, R/2]$ is completely devoid of particles. The quantum hydrodynamic approach [7] has elucidated the relationship between EFP and a limit shape – the optimal density profile with an astroid-shaped boundary in euclidean space-time. In recent works similar limit shapes were found for weakly interacting bosonic liquids [9] and polytropic gases [10].

It is known since the work of de Gennes [11] that non-crossing directed polymers are exactly mapped to world-lines of noninteracting fermions propagating in imaginary time. Thus, the combination of the Pauli Principle and the emptiness constraint leads naturally to essentially the same limit shape phenomenon.

EFP is a particular case of Full Counting Statistics (FCS), defined via the generating function

$$\chi_R(s) = \langle e^{-sN_R} \rangle = \sum_{N=0}^{\infty} P_R(N) e^{-sN}. \quad (1)$$

FCS is defined as the probability $P_R(N)$ that the number of polymers crossing the interval $N_R = N$. EFP is obtained by taking $s \rightarrow +\infty$ limit in (1).

FCS was extensively studied in the context of quantum noise for mesoscopic condensed matter [12, 13], ultra-cold atomic systems [14–16], and one-dimensional magnetism [17–19]. The cumulants of FCS are universally related to quantum entanglement entropy [20–23]. Recently, the generating function Eq. (1) appeared naturally in the

context of continuously measured quantum systems [24]. It was proposed in Ref. [19] to use analytical properties of the FCS generating function to characterise the thermodynamic phases and to probe nonlocal correlations in many-body quantum systems.

In these works FCS was studied in the regime of *typical* fluctuations where the corresponding probability distribution $P_R(N)$ is almost Gaussian with higher cumulants suppressed by inverse powers of R [8, 25]. In contrast, EFP represents a rare event in which the density of particles in the interval deviates macroscopically from its average value $\nu_{\infty} = \langle N_R \rangle / R$. The corresponding optimal instanton configuration in 1+1 dimensions has a peculiar astroid shape with typical dimensions $R \times R$. This motivates our study of FCS in the large deviation form

$$\chi_R(s) \sim e^{-R^2 f(s/R)}, \quad (2)$$

in terms of the rescaled parameter $\alpha = s/R$. In this Letter we show that the transition from typical to rare fluctuations is accompanied by a structural change of the dominant limit shape.

Model. We consider a statistical ensemble of non-intersecting paths $\{x_i(\tau)\}$, $x_i(\tau) < x_{i+1}(\tau)$ representing directed polymers. The polymers are parameterized by the continuous longitudinal coordinate τ , $\tau \in [-\beta/2, \beta/2]$ and lateral coordinates $x_i \in \mathbb{Z}$ which take values on a one-dimensional lattice and obey $x_i(\beta/2) = x_i(-\beta/2)$. Their density is $\nu(x, \tau) = \sum_i \delta_{x, x_i(\tau)}$, where $\delta_{x, y}$ is the Kronecker delta. In terms of the polymers, FCS is the distribution of the observable N_R equal to the number of polymers with $|x_i(0)| < R/2$.

Directed polymers can be mapped onto the world-lines of non-interacting fermions hopping between nearest neighbor sites on a one dimensional lattice propagating in imaginary time τ [3, 11, 26]. Their evolution is governed by the tight-binding Hamiltonian H given by Eq. (11) below. In terms of this Hamiltonian, the generating function of FCS can then be represented as the many-body quantum expectation value

$$\chi_R(s) = \langle e^{-sN_R} \rangle = \frac{\text{Tr}(e^{-sN_R} e^{-\beta H})}{\text{Tr}(e^{-\beta H})}. \quad (3)$$

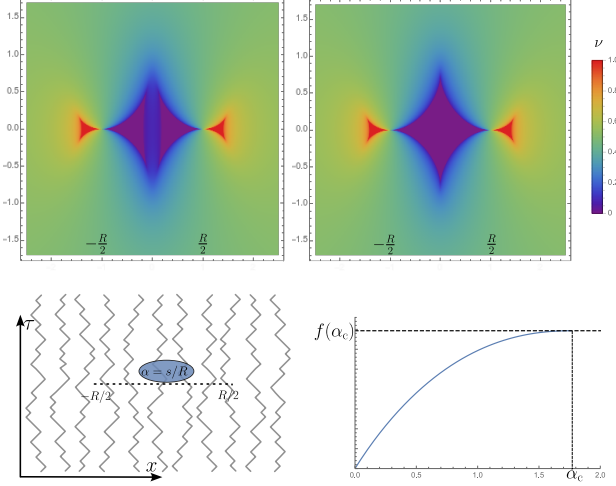


Figure 1. Top: the dominant density configuration $\nu(x, \tau)$ is shown for $\alpha < \alpha_c$ (left) and $\alpha \geq \alpha_c$ (right) is shown for $\nu_\infty = 1/2$. Bottom left: a sketch of polymers' configuration. Each polymer passing through the interval $I = [-R/2, R/2]$ is penalised by weight $e^{-\alpha R}$. Bottom right: the rate function $f(\alpha)$, see Eq. (2).

which is the main object of our study.

Results. The main result is a configurational transition of directed polymers at the critical value α_c , illustrated in Fig. 1 by plotting the typical density $\nu(x, \tau)$. The latter is obtained by solving

$$x + i\tau \sin k = Rg(e^{ik}), \quad (4)$$

for the real part of the complex momentum $k(x, \tau) = \pi\nu + i\nu$ at fixed x, τ . The function $g(z)$ is given by

$$g(z) = \frac{1}{2} \sqrt{\frac{(z - e^{ik_0})(z - e^{-ik_0})}{(z - e^{ik_F})(z - e^{-ik_F})}}. \quad (5)$$

Here $k_F = \pi\nu_\infty$ is proportional to the asymptotic density ν_∞ at $x, \tau \rightarrow \infty$ and k_0 is controlled by α via

$$\alpha = 2\text{Re} \int_{-1}^1 \frac{g(z)dz}{z}, \quad k_0 > 0 \quad (6)$$

and is related by $k_0 = \pi\nu(0, 0)$ to the density in the center of the interval I .

For moderate α , the optimal polymer density consists of two deltoid-shaped empty ($\nu = 0$) and two deltoid-shaped full ($\nu = 1$) regions, as shown in the left panel of Fig. 1. The number of polymers in the interval I is macroscopically depleted with average density

$$\nu_I = f'(\alpha) = \frac{1}{\pi} \text{Im} \int_{e^{-ik_F}}^{e^{ik_F}} \frac{g(z)dz}{z}. \quad (7)$$

The value $k_0 = 0$ marks the onset of the transition to a macroscopically empty interval, $\nu_I = 0$. At this point

the integral (6) reaches its maximum value α_c given by

$$\cosh \frac{\alpha_c}{2} \cos \frac{k_F}{2} = 1. \quad (8)$$

The right panel of Fig. 1 shows the empty regions having merged into one astroid-shaped empty region which includes the whole interval I . This situation, where polymers avoid the interval completely, persists for $\alpha > \alpha_c$ with the constant rate function given by the EFP result of Ref. [7],

$$f(\alpha) = f(\alpha_c) = -\log \cos \frac{k_F}{2}, \quad \alpha > \alpha_c. \quad (9)$$

Just below the transition $\alpha \lesssim \alpha_c$ the behaviour of the rate function is found to be

$$f(\alpha_c - \delta) = f(\alpha_c) + \frac{\delta^2}{2 \log \delta}, \quad \delta \ll 1. \quad (10)$$

Thus the transition is continuous, featuring a divergent third derivative of the rate function similar to the transitions reported earlier in [27–29].

Derivation. To generate the configurations of the polymers via fermionic evolution we use the tight-binding second quantized Hamiltonian

$$H = \sum_{xy} (h)_{xy} c_x^\dagger c_y \quad (11)$$

expressed via creation/annihilation operators $c_x^\dagger, c_x$ obeying standard anti-commutation relations $\{c_x, c_{x'}^\dagger\} = \delta_{xx'}$. The nearest neighbor hopping matrix

$$h_{xy} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(x-y)} (-\cos k - \mu), \quad (12)$$

contains the tight-binding dispersion $\varepsilon(k) = -\cos k$ and a chemical potential μ fixing the equilibrium density. The operator of particle number in an interval I is

$$N_R = \sum_{xy} (n_I)_{xy} c_x^\dagger c_y = \sum_{x \in I} c_x^\dagger c_x, \quad (13)$$

where $(n_I)_{xy} = \delta_{xy}$, for $x \in I$ and zero otherwise. It is the one-particle projector, $n_I = n_I^2$, on the interval I . Due to the particle-hole symmetry, the FCS obeys $\chi_R(s, \beta, \mu) = e^{-sR} \chi_R(-s, \beta, -\mu)$ so in what follows we focus on $s \geq 0$.

To find the generating function Eq. (3) we express, following [30], the traces in many-particle Fock space as determinants of operators in one-particle space:

$$\chi_R(s) = \frac{\det(1 + e^{-sn_I} e^{-\beta h})}{\det(1 + e^{-\beta h})}. \quad (14)$$

The fact that n_I projects into R lattice sites allows us to rewrite $\chi_R(s)$ as a determinant of $R \times R$ Toeplitz matrix with the elements

$$(1 + (e^{-s} - 1)n_F)_{xy} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(x-y)} e^{-sv(k)}, \quad (15)$$

where $n_F = (1 + e^{\beta h})^{-1}$ is the Fermi occupation operator.

We consider the zero temperature limit, $\beta \rightarrow \infty$ corresponding to infinitely long polymers. In this limit n_F becomes the projector on the single interval in the momentum space, $|k| < k_F$ with $\mu = -\cos k_F$. The so-called Toeplitz symbol $e^{-sv(k)}$ becomes singular since

$$v(k) = -\frac{1}{s} \log \frac{1 + e^{-s} e^{-\beta h(k)}}{1 + e^{-\beta h(k)}} \rightarrow \theta(k_F - |k|). \quad (16)$$

Although the theory of Toeplitz determinants with singular symbols is well developed, see, *e.g.* Ref. [31], it is not directly applicable in our case where the parameter s scales with the size of the matrix R . Here we use an alternative approach based on the exact representation of the generating function as the R -dimensional integral,

$$\chi_R(s) = \frac{1}{R!} \int \frac{d^R k}{(2\pi)^R} |\Delta(e^{ik})|^2 \prod_{l=1}^R e^{-sv(k_l)}, \quad (17)$$

which is a configurational integral of a classical two-dimensional Coulomb gas of R charges [32]. In addition to a repulsive interaction given by the logarithm of Vandermonde determinant, $\Delta(e^{ik}) = \prod_{1 \leq m < n \leq R} (e^{ik_n} - e^{ik_m})$, the charges on the interval $|k| < k_F$ are subject to a constant electrostatic repulsive potential of magnitude $s = \alpha R$. This scaling makes the potential comparable to the interparticle repulsion, both terms being proportional to R^2 justifying the scaling in Eq. (2).

Due to the presence of large parameter R the integral (17) is dominated by a stationary configuration of charges. Their positions, $z_l = e^{ik_l}$, are encoded in the poles of the resolvent

$$g(z) + \frac{1}{2} = \frac{1}{R} \sum_{l=1}^R \frac{z}{z - e^{ik_l}} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{z \sigma(k)}{z - e^{ik}}. \quad (18)$$

Here $\sigma(k) = \frac{1}{R} \sum_{l=1}^R \delta(k - k_l)$ is the density of the poles, which becomes the smooth function, Eq. (5), for $R \rightarrow \infty$. In this limit the charges are distributed along the branch cuts of $g(z)$, those being the two disjoint arcs of unit circle $z = e^{ik}$, $|k| < k_0$ and $k_F < |k| < \pi$.

Introducing the complex potential, $g(z) + 1/2 = z \partial_z \varphi(z)$, the stationarity condition can be expressed as the electrostatic equilibrium of the charges: the real part of $\varphi(z)$ is constant along each of the branch cuts. The parameter α then plays the role of the voltage between the cuts, hence the condition (6) fixing the parameter ν_0 . Conversely, the imaginary part of the potential counts the number of charges and leads to Eq. (7). The density corresponding to the resolvent $g(z)$ in Eq. (5) is

$$\sigma(k) = 2 \operatorname{Re} g(e^{ik}) = \sqrt{\frac{\sin^2 \frac{k}{2} - \sin^2 \frac{k_0}{2}}{\sin^2 \frac{k}{2} - \sin^2 \frac{k_F}{2}}}. \quad (19)$$

and is shown in the left panel of Fig. 2. The calculations leading to these results are standard and can be found in Supplemental Material [32].

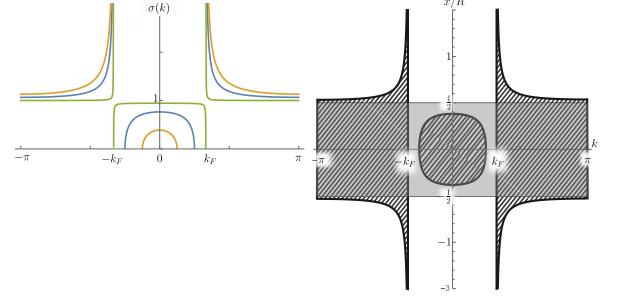


Figure 2. Left: Quasiparticle momentum distribution corresponding to the solution of Coulomb Gas problem Eq. (19) for $k_F/\pi = 1/3$ and α corresponding to $k_0/\pi = 0.332, 0.25, 0.125$. Right: quasiparticle Wigner function $f(x, k)$ for $k_0/\pi = 0.25$. The striped areas correspond to regions where $f(x, k) = 1$. The light grey rectangle represents regions where the quasiparticle vacuum ($s = 0$) Wigner function $f_{\text{qp}}^{\text{vac}}(x, k) = 1$.

Hydrodynamic interpretation The solution of the Coulomb Gas model contains information about the *state* of the system at $\tau = 0$ which, using the hydrodynamic evolution of fermions, allows one to extract full information about the real space configurations of constrained directed polymers. This can be traced back to interpreting Eq. (3) as an imaginary time path integral with the source term proportional to the counting parameter s acting at $\tau = 0$ and affecting R sites of the interval I .

The configurations contributing to the generating function Eq. (1) can be parametrised by degrees of freedom of R fermionic quasiparticles propagating in imaginary time with $R\sigma(k)$ playing the role of their momentum distribution at $\tau = 0$.

It is well known [7] that the large scale dynamics of such fermionic quasiparticles is captured by the hydrodynamic approach. In this approach one introduces two hydrodynamic fields, the density $\nu(x, \tau)$ and velocity $v(x, \tau)$ combined into complex momentum $k(x, \tau) = \pi\nu + iv$. The complex momenta satisfy the complex Burger's equation

$$i\partial_\tau k + \partial_x \varepsilon(k) = 0, \quad \tau \neq 0 \quad (20)$$

equivalent to hydrodynamic continuity and Euler equations after separation into real and imaginary parts.

As a result of time-reversal symmetry $k(x, -\tau) = \bar{k}(x, \tau)$ the density $\nu(x, \tau)$ and velocity $v(x, \tau)$ fields are even/odd functions of τ correspondingly. The initial condition $k_0(x) = k(x, 0)$ is, therefore, real. The inverse function $x_0(k)$ such that $x_0(k_0(x)) = x$ parametrises the implicit solution to Eq. (20)

$$x + i\varepsilon'(k)\tau = x_0(k). \quad (21)$$

Below we argue that the right hand side is given by the rescaled solution of the Coulomb gas problem as

$$x_0(k) = R\sigma(k)/2 = Rg(e^{ik}). \quad (22)$$

Indeed, for $s = 0$ the solutions of hydrodynamic Eq. (20) are given by the constant configuration $k(x, \tau) = \bar{k}(x, \tau) = k_F = \pi\nu_\infty$. For $s > 0$ the density is non-uniform and we write

$$\nu(x, 0) - \nu_\infty = \int \frac{dk}{2\pi} (f(x, k) - f^{\text{vac}}(x, k)). \quad (23)$$

Here we represented the density fluctuation as a sum of contributions from different momenta, using the semiclassical Wigner function $f(x, k)$. The vacuum contribution $f^{\text{vac}}(x, k)$ corresponding to uniform density ν_∞ at $s = 0$, is subtracted as $\nu - \nu_\infty = 0$ for $s = 0$. The function $f(x, k)$ has the form of a local Fermi distribution, $f(x, k) = 1$ for $|k| < k_0(x)$ and zero outside this interval, see Fig. 2. The branches $\pm k_0(x)$ are thus the local ‘‘Fermi momenta’’ of the fermionic particles.

The same boundary can be represented using the inverse function $x_0(k)$. In terms of the Wigner function we have

$$2x_0(k) = \sum_x [f(x, k) - f^{\text{vac}}(x, k) + f_{\text{qp}}^{\text{vac}}(x, k)]. \quad (24)$$

Here we added the Wigner function $f_{\text{qp}}^{\text{vac}}(x, k) = \theta(R/2 - |x|)$ representing the vacuum ($s = 0$) distribution of *quasiparticles* as shown in Fig. 2. The sum in the r.h.s. of Eq. (24) is nothing but the momentum distribution of the quasiparticles $R\sigma(k)$. This establishes the important link, Eq. (22), between the Coulomb Gas charge density and the fluid dynamics.

The solution $k_0(x)$ of Eq. (24) at $\tau = 0$, shows that for $0 < \alpha < \alpha_c$ there is a non-zero density in some interval around $x = 0$. As one moves towards the ends of the interval the density drops to zero at $\pm x_- = Rg(1)$. At $x = \pm R/2$ the density jumps to unity and stays saturated up to $\pm x_+ = Rg(-1)$. Beyond this point the density decays to its asymptotic value $\nu_\infty = k_F/\pi$.

Solving Eq. (21) away from $\tau = 0$ provides the full space-time profile of density and velocity in terms of the complex function $k(x, \tau)$. For $0 < \alpha < \alpha_c$ there is always a central fluctuating region between the deltoid-shaped frozen regions which evolve from the frozen intervals with $\nu = 0, 1$ at $\tau = 0$. The boundary of the frozen regions is given in the parametric form

$$x/R = G_\pm(v) - \tanh v G'_\pm(v), \quad (25)$$

$$\tau/R = \mp G'_\pm(v) / \cosh v, \quad (26)$$

which is obtained by substituting $k_+ = iv, k_- = \pi + iv$ into Eq. (4) and defining $G_\pm(v) = g(e^{ik_\pm}) = g(\pm e^{-v})$ for the boundary with empty/full frozen region.

When α is increased from zero the parameter ν_0 decreases monotonically from $\nu_0 = \nu_\infty$. The transition corresponds to disappearance of the cut, $\nu_0 = 0$, at the critical value α_c . Calculation of α_c , Eq. (8) and the corresponding value of free energy, Eq. (9) are outlined in Supplemental Material [32].

To establish the critical behaviour close to the transition, $\nu_0 \rightarrow 0$, we consider Eqs. (6) and (7). The density $\sigma(k)$ is zero everywhere in $[-k_F, k_F]$ except for a small interval $[-k_0, k_0]$ where it can be approximated by the Wigner semicircle. The standard calculation gives

$$\nu_I = \frac{k_0}{2 \sin \frac{k_F}{2}} \int_{-k_0}^{k_0} \frac{dk}{2\pi} \sqrt{1 - \left(\frac{k}{k_0}\right)^2} = \frac{k_0^2}{8 \sin \frac{k_F}{2}}. \quad (27)$$

Calculating α is more involved. One can show [32] that the corresponding integral contains a logarithmic non-analyticity:

$$\alpha - \alpha_c = -\frac{k_0^2}{8 \sin \frac{k_F}{2}} \left[1 + \log \left(\frac{64}{k_0^2} \tan^2 \frac{k_F}{2} \right) \right]. \quad (28)$$

Thus in the leading order $\delta = \alpha_c - \alpha \simeq \nu_I \log(1/\nu_I)$, which can be inverted as $\nu_I = \delta / \log(1/\delta)$. Integrating $f'(\alpha) = \nu_I$ from α_c to $\alpha = \alpha_c - \delta$ one gets Eq. (10).

Conclusions In this work we have computed the full counting statistics of lattice fermions within a macroscopic interval in the previously unexplored regime of large deviations and found the presence of a continuous phase transition between zero and near zero occupancy of the interval. The mapping to Coulomb gas makes our findings in agreement with findings of Refs. [33].

We obtained an analytic form for the rate function associated with the large deviation using a mapping between the FCS generating function and a Coulomb gas model. In contrast to the previous studies [34–36] the Coulomb Gas model is naturally formulated in momentum space of R quasiparticles leading to the existence of the following duality. It is convenient to rewrite (14) as

$$\begin{aligned} \chi_R(s) &= \det(1 - n_F + e^{-sn_I} n_F) \\ &= \det(1 - (1 - e^{-s}) n_I n_F). \end{aligned} \quad (29)$$

In the limit of zero temperature, $\beta \rightarrow \infty$, the Fermi occupation operator n_F also becomes a projector on the negative energy states and the expression (29) is completely symmetric with respect to the interchange $n_I \leftrightarrow n_F$. One has a choice of computing (29) in the momentum basis where it becomes the Fredholm determinant with the sine-kernel or in the coordinate basis, where it becomes the Toeplitz determinant of the size $R \times R$. This change corresponds to switching between the original fermions and the quasiparticles used for the hydrodynamic description (see Fig. 2).

At zero temperature FCS is, as expected, insensitive to the exact form of the fermion dispersion as long as a single interval of momenta is occupied in the ground state. This has to be contrasted with the space-time configurations of the hydrodynamic fields depending explicitly on the form of the dispersion, see *e.g.* Eq. (21). Moreover, for a dispersion leading to crossing of the worldlines of the fermions the hydrodynamic interpretation may even fail

as some configurations would have negative Boltzmann weights [37]. These subtleties disappear in the continuous limit $k_F, k_0 \rightarrow 0$ with k_0/k_F fixed, in which $\epsilon(k) \sim k^2$ and Galilean symmetry is restored.

Finally, the momentum space representation used here can be extended to finite temperatures, by modifying the effective potential $v(k)$, Eq. (16) in the Coulomb Gas model Eq. (17).

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Supplemental material: Phase transitions in full counting statistics of free fermions and directed polymers

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 (Dated: May 22, 2024)

I. TOEPLITZ DETERMINANT AS A COULOMB GAS

The equivalence between the finite-size Toeplitz determinant and the Coulomb gas partition function is well known. It appeared already in works of Heine and Szegő (see for example, the formula (2.2.11) in Ref. [1]). In Ref. [2] it is referred to as a "well-known integral representation of a Toeplitz determinant". Written as an integral over unitary group it is sometimes referred to as the Heine-Szegő formula, see section 2 of Ref. [3]. In the latter reference the formula was generalized to Toeplitz minors. For a recent work on further generalizations including other groups, see Ref. [4]. To make the paper more self-contained we present here the derivation of that equivalence.

Let us consider the Toeplitz determinant $\det(T)$ of the $R \times R$ matrix defined by the Toeplitz symbol $t(k)$ as

$$T_{mn} = \int \frac{dk}{2\pi} t(k) e^{ik(m-n)}, \quad (\text{S.1})$$

and $n, m = 1, \dots, R$. Using Andréief's integration formula [5], we obtain

$$\det(T) = \frac{1}{R!} \int \frac{dk_1}{2\pi} \dots \int \frac{dk_R}{2\pi} \det(t(k_j) e^{ik_j(m-1)})_{j,m=1}^R \det(e^{-ik_j(n-1)})_{j,n=1}^R,$$

and then using Vandermonde formula

$$\det(T) = \frac{1}{R!} \int \frac{d^R k}{(2\pi)^R} |\Delta(e^{ik})|^2 \prod_{j=1}^R t(k_j), \quad (\text{S.2})$$

where the Vandermonde determinant is defined as

$$\Delta(e^{ik}) \equiv \prod_{1 \leq n < m \leq R} (e^{ik_m} - e^{ik_n}). \quad (\text{S.3})$$

Exponentiating the integrand in (S.2) we obtain a partition function for the Coulomb gas

$$\det(T) = \frac{1}{R!} \int \frac{d^R k}{(2\pi)^R} e^{-F(e^{ik})}, \quad (\text{S.4})$$

$$F(e^{ik}) = - \sum_{1 \leq n < m \leq R} \log |e^{ik_n} - e^{ik_m}|^2 - \sum_{n=1}^R \log(t(k_n)).$$

In this representation, the logarithm of the Toeplitz symbol enters as an external potential applied to the Coulomb gas on a unit circle.

We will apply the general Coulomb-gas representation of the Toeplitz determinant (S.4) to the FCS generating function.

II. TOEPLITZ DETERMINANT REPRESENTATION OF FCS AND SOLUTION OF THE CORRESPONDING COULOMB GAS PROBLEM

For FCS generating function we get the Toeplitz determinant and the corresponding Coulomb gas partition function with the symbol

$$t(k) = \frac{1 + e^{-s} e^{-\beta h(k)}}{1 + e^{-\beta h(k)}} = e^{-sv(k)}, \quad (\text{S.5})$$

For finite β this symbol is smooth, but at zero temperature, $1/\beta = 0$ it develops step-like singularities, resulting in $v(k) = 1$ in the interval $-k_F < k < k_F$ and 0 otherwise, that is $v(k) = \theta(k_F - |k|)$. Asymptotics of the corresponding Toeplitz determinant can be obtained by standard analysis [6] via Szegő theorem and Hartwig-Fischer conjecture. The situation becomes more complicated in the scaling limit $s = \alpha R$. This is where the Coulomb gas representation

$$\chi_R(s) = \frac{1}{R!} \int \frac{d^R k}{(2\pi)^R} |\Delta(e^{ik})|^2 \prod_{l=1}^R e^{-sv(k_l)}, \quad (\text{S.6})$$

becomes indispensable. This integral (S.6) is dominated by stationary configurations of charges. To find it we assume charges are smoothly distributed on support γ with density

$$\sigma(k) = \frac{2\pi}{R} \sum_i \delta(k - k_i), \quad \int_{\gamma} \frac{dk}{2\pi} \sigma(k) = 1. \quad (\text{S.7})$$

and represent Eq. (S.6) as the path integral

$$\chi_R(\alpha R) = e^{-R^2 f(\alpha)} = \int D[\sigma] d\mu e^{-R^2 S[\sigma]}, \quad (\text{S.8})$$

over configurations $\sigma(k)$ weighted by the action

$$S[\sigma] = -\frac{1}{2} \int_{\gamma} \frac{dk}{2\pi} \int_{\gamma} \frac{dk'}{2\pi} \sigma(k) \log |e^{ik} - e^{ik'}|^2 \sigma(k') + \alpha \int_{\gamma} \frac{dk}{2\pi} \sigma(k) v(k) - \mu \left(\int_{\gamma} \frac{dk}{2\pi} \sigma(k) - 1 \right). \quad (\text{S.9})$$

The chemical potential μ acts as a Lagrange multiplier to ensure the normalisation in Eq. (S.7). The measure of integration $D[\sigma]$ is chosen so that $\chi_R(0) = 1$, or, equivalently, $f(0) = 0$. Moving from the microscopic momenta $\{k_i\}$ to the macroscopic density σ generates $\mathcal{O}(R)$ entropic terms in the exponent. These are subdominant in our large deviation scaling limit and thus neglected.

Taking variation of the action w.r.t. $\sigma(k)$ we arrive to the saddle-point equation

$$\mu = \alpha v(k) - \int_{\gamma} \log |e^{ik} - e^{ik'}|^2 \sigma(k') \frac{dk'}{2\pi}. \quad (\text{S.10})$$

valid in the support $k \in \gamma$.

We introduce the complex potential

$$\varphi(z) = \int_{\gamma} \frac{dk}{2\pi} \log(z - e^{ik}) \sigma(k), \quad (\text{S.11})$$

analytic in the whole complex plane of z apart from the support $z = e^{ik}$, $k \in \gamma$. As a result of the discontinuity of the potential $v(k)$ we assume the support consists of two disjoint intervals $\gamma = \gamma_+ \cup \gamma_-$. This is illustrated in the left panel of Fig. 1. Choosing two points $k_- = \pi$, $k_+ = 0$ on the support γ such that $v(k_-) = 0$ and $v(k_+) = 1$, Eq. (S.10) becomes equivalent to

$$2\text{Re } \varphi(-1) = -\mu, \quad (\text{S.12})$$

$$2\text{Re } \varphi(+1) = \alpha - \mu. \quad (\text{S.13})$$

Additionally, as a consequence of the normalisation condition, Eq. (S.7), $\varphi(z) = \log z$ for $z \rightarrow \infty$. This condition together with Eqs. (S.12), (S.13) fully determine the optimal $\varphi(z)$.

The support γ and the density $\sigma(k)$ are obtained from the non-analyticity of the resolvent $g(z) + 1/2 = z \partial_z \varphi(z)$. The density of charges is given by the jump across the cuts, $\sigma(k) = g_+(e^{ik}) - g_-(e^{ik}) = 2\text{Re } g(e^{ik})$. Taking the derivative of both sides of Eq. (S.10) we obtain the condition

$$g_+(e^{ik}) + g_-(e^{ik}) = 0, \quad k \in \gamma, k \neq \pm k_F. \quad (\text{S.14})$$

Allowing for possible empty intervals, $\sigma(k) = 0$, leads to $g_+^2(z) = g_-^2(z)$, so $g^2(z)$ is analytic in the whole complex plane apart from poles at $z = e^{\pm i k_F}$ and $z = \infty$, where it behaves as $g^2(z) = 1/4 + 1/z + \mathcal{O}(z^{-2})$ due to normalisation. This fixes completely the form of the resolvent,

$$g^2(z) = \frac{1}{4} + \frac{zd}{(z - e^{ik_F})(z - e^{-ik_F})}. \quad (\text{S.15})$$

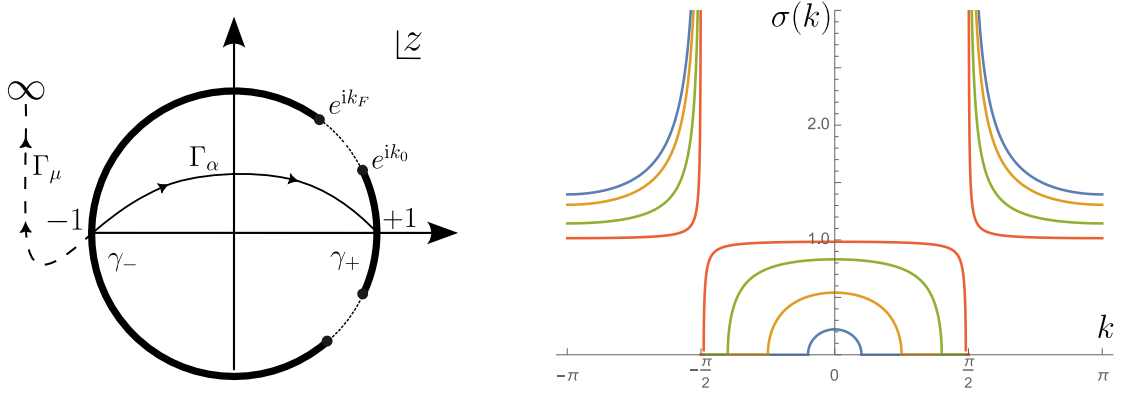


Figure 1. Left: Complex plane structure for Coulomb gas problem, $z = e^{ik}$. The charges reside in two cuts specified by z_F and z_α . Complete emptiness corresponds to $z_\alpha = 1$. Contours of integration in Eqs. (S.18), (S.30) are shown. Right: density of charges $\sigma(k)$, Eq. (S.17) for $\nu_0 = k_0/\pi = 0.1, 0.25, 0.4, 0.49$.

The parameter d is written in the form $d = \sin^2 \frac{k_0}{2} - \sin^2 \frac{k_F}{2}$ so that

$$g(z) = \frac{1}{2} \sqrt{\frac{(z - e^{ik_0})(z - e^{-ik_0})}{(z - e^{ik_F})(z - e^{-ik_F})}}, \quad (\text{S.16})$$

coincides with Eq. (4) in the main text. The corresponding density of charges is

$$\sigma(k) = \sqrt{\frac{\sin^2 \frac{k}{2} - \sin^2 \frac{k_0}{2}}{\sin^2 \frac{k}{2} - \sin^2 \frac{k_F}{2}}}, \quad (\text{S.17})$$

and is shown in the right panel of Fig. 1 for several values of $k_0 = \pi\nu_0$.

Using Eqs. (S.13) and (S.12) and $\partial_z \phi = (g(z) + 1/2)/z$ we obtain

$$\alpha = 2\text{Re} (\varphi(+1) - \varphi(-1)) = 2\text{Re} \int_{\Gamma_\alpha} \frac{g(z)dz}{z} = 2\text{Re} \int_{-1}^{+1} \frac{g(z)dz}{z}, \quad (\text{S.18})$$

$$\mu = -2\text{Re} \varphi(-1) = 2\text{Re} \int_{\Gamma_\mu} \frac{g(z)dz}{z} = 2\text{Re} \int_{-1}^{\infty} \frac{g(z)dz}{z}. \quad (\text{S.19})$$

The integration countours $\Gamma_{\alpha, \mu}$ are shown in the left panel of Fig. 1. They can be deformed as long as their end-points remain on the same intervals $\gamma_\pm$ of the support γ . The integral for μ diverges logarithmically at infinity and must be regularised by subtracting $\mu_0 = \int_{-1}^{\infty} dz/z$ corresponding to $k_0 = k_F$ and $\alpha = 0$.

Taking variation of the action, Eq. (S.9), w.r.t. α and using the saddle-point equation, Eq. (S.10) we get

$$f'(\alpha) = \frac{\partial S}{\partial \alpha} = \int_{-k_F}^{k_F} \frac{dk}{2\pi} \sigma(k) = \nu_I, \quad (\text{S.20})$$

which is equivalent to Eq. (6) in the main text as for $z = e^{ik}$ we have $\sigma(k)dk = 2 \text{Im} g(z)dz/z$.

The phase transition happens when only one branch cut is left for $\nu_0 = 0$. The integral in Eq. (S.18) is elementary for $g(z)$ in (S.16) at $k_0 = 0$ providing the critical value, Eq. (7) in the main text.

Similarly, using the normalisation Eq. (S.7) and the saddle-point equations Eq. (S.10), to eliminate the dependence on the log term in Eq. (S.9) in favour of α, μ we get

$$f(\alpha) = \frac{\mu}{2} + \frac{\alpha\nu_I}{2}. \quad (\text{S.21})$$

yielding the correct normalisation $f(0) = 0$. For $\alpha \geq \alpha_c$ we have $\nu_I = 0$ and $f = \mu/2$. For this regime the chemical potential, Eq. (S.19), becomes an elementary integral and we find $\mu = -\log \cos^2 \frac{k_F}{2}$, leading to Eq. (8) in the main text.

III. EVALUATION OF INTEGRALS IN THE CRITICAL REGIME

We obtain the leading behaviour of α , Eq. (S.18), in the limit $k_0 \rightarrow 0$. Defining $a = \sin \frac{k_0}{2}$, $b = \sin \frac{k_F}{2}$ and using $z = \mp e^{-v}$ in the negative/positive part of the interval $[-1, 1] = [-1, 0] \cup [0, 1]$, Eq. (S.18) becomes the sum of two contributions $\alpha = I(\sqrt{1-a^2}, \sqrt{1-b^2}) - I(a, b)$, where subtracting one for convergence

$$I(a, b) = \int_0^\infty dv \left(\sqrt{\frac{\sinh^2 \frac{v}{2} + a^2}{\sinh^2 \frac{v}{2} + b^2}} - 1 \right) \quad (\text{S.22})$$

The critical value α_c corresponds to $a = 0$, which gives

$$\begin{aligned} \alpha_c &= I(1, \sqrt{1-b^2}) - I(0, b) = \int_0^\infty dv \left(\frac{\cosh \frac{v}{2}}{\sqrt{\cosh^2 \frac{v}{2} - b^2}} - \frac{\sinh \frac{v}{2}}{\sqrt{\sinh^2 \frac{v}{2} + b^2}} \right) \\ &= 2 \operatorname{arcosh} \frac{1}{\sqrt{1-b^2}}, \end{aligned} \quad (\text{S.23})$$

identical to Eq. (7) in the main text. We proceed with calculating $\alpha - \alpha_c$ by combining the terms

$$\begin{aligned} \alpha - \alpha_c &= \left[I(\sqrt{1-a^2}, \sqrt{1-b^2}) - I(1, \sqrt{1-b^2}) \right] \\ &\quad - [I(a, b) - I(0, b)] = I_1 - I_2. \end{aligned} \quad (\text{S.24})$$

The first term can be approximated to the leading order,

$$\begin{aligned} I_1 &= I(\sqrt{1-a^2}, \sqrt{1-b^2}) - I(1, \sqrt{1-b^2}) = \int_0^\infty dv \frac{\sqrt{\cosh^2 \frac{v}{2} - a^2 - \cosh \frac{v}{2}}}{\sqrt{\sinh^2 \frac{v}{2} + 1 - b^2}} \\ &\simeq -\frac{a^2}{2} \int_0^\infty \frac{dv}{\cosh \frac{v}{2} \sqrt{\sinh^2 \frac{v}{2} + 1 - b^2}} = -a^2 \int_0^\infty \frac{du}{\cosh^2 u - b^2 \sinh^2 u} = -\frac{a^2}{b} \operatorname{artanh} b, \end{aligned} \quad (\text{S.25})$$

with the change of variables $\sinh \frac{v}{2} = \sqrt{1-b^2} \sinh u$, $\cosh \frac{v}{2} dv = 2\sqrt{1-b^2} \cosh u du$.

To evaluate the second term we divide the integration domain into two intervals,

$$\begin{aligned} I_2 &= I(a, b) - I(0, b) = \left(\int_0^\lambda + \int_\lambda^\infty \right) dv \frac{\sqrt{\sinh^2 \frac{v}{2} + a^2 - \sinh \frac{v}{2}}}{\sqrt{\sinh^2 \frac{v}{2} + b^2}} \\ &\simeq \frac{1}{b} \int_0^\lambda dv \left(\sqrt{\left(\frac{v}{2} \right)^2 + a^2} - \frac{v}{2} \right) + \frac{a^2}{2} \int_\lambda^\infty \frac{dv}{\sinh \frac{v}{2} \sqrt{\sinh^2 \frac{v}{2} + b^2}}, \end{aligned} \quad (\text{S.26})$$

with the corresponding approximations valid provided we chose $a \ll \lambda \ll b$. The first integral is elementary and after change of variables $v = 2ax$ becomes

$$\begin{aligned} \frac{2a^2}{b} \left[\int_0^{\lambda/2a} dx \sqrt{x^2 + 1} - \frac{1}{2} \left(\frac{\lambda}{2a} \right)^2 \right] &= \frac{a^2}{b} \left[\frac{\lambda}{2a} \sqrt{1 + \left(\frac{\lambda}{2a} \right)^2} - \left(\frac{\lambda}{2a} \right)^2 + \operatorname{arsinh} \frac{\lambda}{2a} \right] \\ &\simeq \frac{a^2}{b} \left[\frac{1}{2} + \log \frac{\lambda}{a} \right]. \end{aligned} \quad (\text{S.27})$$

The second integral is evaluated similarly to Eq. (S.25) by changing variable $\cosh \frac{v}{2} = \sqrt{1-b^2} \cosh u$,

$$\begin{aligned} \frac{a^2}{2} \int_\lambda^\infty \frac{dv}{\sinh \frac{v}{2} \sqrt{\sinh^2 \frac{v}{2} + b^2}} &= -\frac{a^2}{b^2} \int_{\lambda'}^\infty \frac{du}{\cosh^2 u - b^{-2} \sinh^2 u} \\ &= -\frac{a^2}{b} \left(\operatorname{artanh} b - \operatorname{artanh} \frac{b}{\tanh \lambda'} \right). \end{aligned} \quad (\text{S.28})$$

Here $\cosh \lambda' = \cosh(\lambda/2)/\sqrt{1-b^2}$, so that $1-b/\tanh \lambda' \simeq (1-b^2)\lambda^2/8b^2$ and $2 \operatorname{artanh}(b/\tanh \lambda') \simeq \log((16/\lambda^2)b^2/(1-b^2))$. Using this we get the answer

$$\begin{aligned} \alpha - \alpha_c &\simeq -\frac{a^2}{2b} \left[1 + 2 \log \frac{\lambda}{a} - \log \left(\frac{1-b^2}{b^2} \frac{\lambda^2}{16} \right) \right] \\ &= -\frac{a^2}{2b} \left[1 + \log \left(\frac{16}{a^2} \frac{b^2}{1-b^2} \right) \right], \end{aligned} \quad (\text{S.29})$$

independent of the cut-off λ .

To calculate the chemical potential we use Eq. (S.12). Subtracting the result for $\alpha = 0$ containing the same logarithmic divergence, we have

$$\mu - \mu_0 = 2 \operatorname{Re} \int_{e^{ik_0}}^{\infty} \left(\frac{g(z)}{z} - \frac{1}{2z} \right) dz. \quad (\text{S.30})$$

Taking $k_0 = \pi$ and using $z = -e^v$ this integral becomes

$$\mu - \mu_0 = I \left(\sqrt{1-a^2}, \sqrt{1-b^2} \right) - I(1, 1). \quad (\text{S.31})$$

For $a = 0$ the integral is elementary

$$\mu - \mu_0 = \int_0^{\infty} \left(\frac{\cosh \frac{v}{2} dv}{\sqrt{\sinh^2 \frac{v}{2} + 1 - b^2}} - 1 \right) = \log \frac{1}{1-b^2}. \quad (\text{S.32})$$

Using the resolvent, Eq. (4), the integrals in Eqs. (S.18), (S.20) can be evaluated for general values of a, b as well in terms of the incomplete and complete elliptic integral of the third kind $\Pi(n; \phi|m)$ and $\Pi(n|m) = \Pi(n; \frac{\pi}{2}|m)$. We obtain

$$\alpha = -\frac{2a^2}{b\sqrt{1-a^2}} \Pi \left(\frac{1}{1-a^2} \middle| \frac{1-\frac{a^2}{b^2}}{1-a^2} \right), \quad (\text{S.33})$$

$$\begin{aligned} 1 - \nu_I &= \int_{k_F}^{2\pi-k_F} \frac{dk}{2\pi} \sigma(k) = \frac{2}{\pi} \frac{(1-a^2)}{\sqrt{b^2-a^2}} \Pi \left(\frac{1-b^2}{a^2-b^2} \middle| \frac{1-b^2}{1-\frac{b^2}{a^2}} \right), \\ \mu(a, b) &= \frac{2(1-a^2)}{a\sqrt{1-b^2}} \Pi \left(\frac{1}{a^2}; \arcsin a \middle| \frac{1-\frac{b^2}{a^2}}{1-b^2} \right), \end{aligned} \quad (\text{S.34})$$

where $a = \sin \frac{k_0}{2}$, $b = \sin \frac{k_F}{2}$.

IV. HYDRODYNAMICS AND COULOMB GAS SOLUTION

It is useful to paramterize the hydrodynamical solution via the complex potential $\chi(x, \tau)$ such that

$$\partial_x \chi(x, \tau) = k(x, \tau) = \pi \nu(x, \tau) + i v(x, \tau). \quad (\text{S.35})$$

and represent it via Legendre transform

$$\chi(x, \tau) = kx - G(k) + i\varepsilon(k)\tau \quad (\text{S.36})$$

$$x + i\varepsilon'(k)\tau = G'(k) \quad (\text{S.37})$$

This way the hydrodynamic equations of motion Eq. (20) is automatically satisfied. Note that the time τ is treated as an external parameter.

From (S.35) it follows that the density of polymers is given by

$$\nu(x, \tau) = \frac{1}{\pi} \operatorname{Re} \partial_x \chi(x, \tau) \quad (\text{S.38})$$

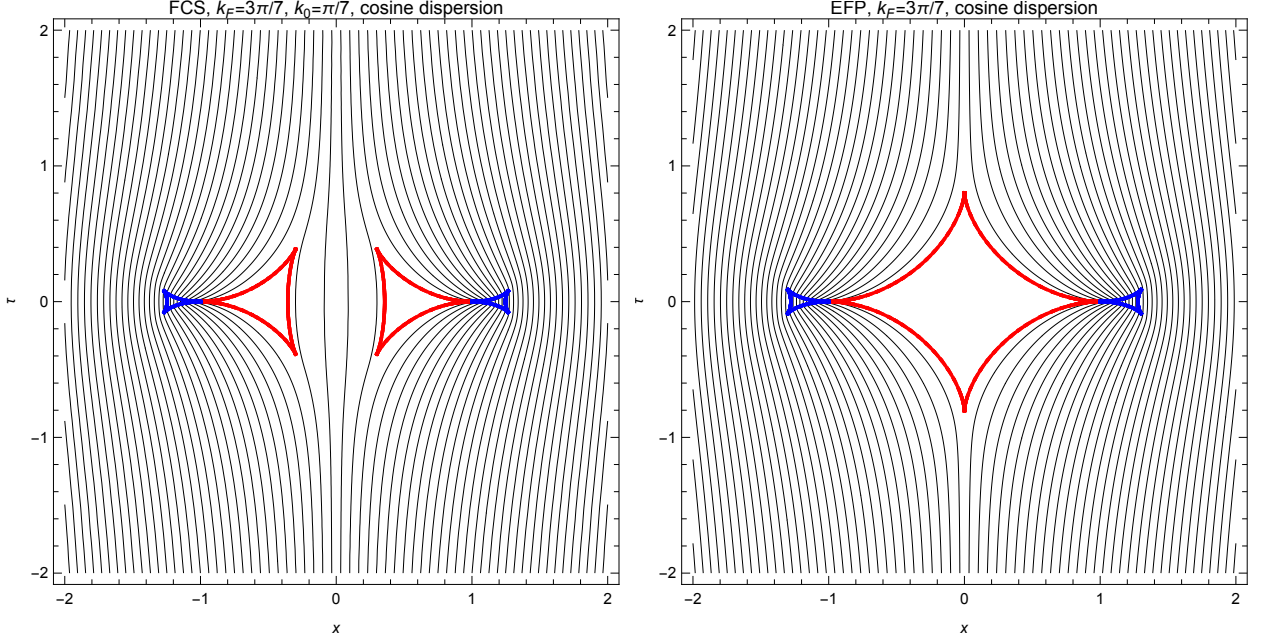


Figure 2. Left: Polymer configuration is shown for $k_F = 3\pi/7$, $k_0 = \pi/7$. the boundaries of completely empty regions ($\nu = 0$) and regions with maximal density ($\nu = 1$) are shown in red and blue, respectively. Right: Polymer configuration is shown for $k_F = 3\pi/7$ and $\alpha > \alpha_c$.

and, therefore, one can write the following function

$$n(x, \tau) = \int_0^x \nu(x, \tau) dx = \frac{1}{\pi} \text{Re} \left(\chi(x, \tau) - \chi(0, \tau) \right) \quad (\text{S.39})$$

as a counting function of polymers so that the solution $x_l(\tau)$ of the equation $n(x, \tau) = l$ gives the trajectory of the polymer number l . We show some of those trajectories in Fig. 2.

Focusing on initial state at $\tau = 0$ we have $kx = \chi(x) + G(k)$. Interpreting the left hand side of this equation as phase-space volume occupied by n particles with $k = k_n$ and $x = x_n$. Increasing the number of particles by one, $n \rightarrow n + 1 = n + dn$ we compute

$$2\pi dn = d(kx) = d\chi(x) + dG(k) = \pi\nu(x)dx + G'(k)dk. \quad (\text{S.40})$$

As $\nu(x) = dn/dx$ we must have $G'(k) = dn/dk = \sigma(k)/2$ [7] which is yet another way to see that the relation between initial positions of particles with momentum k and momentum distribution, Eq. (22) in the main text.

Practically, to plot the trajectories, it is more convenient to obtain from (S.37)

$$\tau = -i \frac{G' - \bar{G}'}{\varepsilon + \bar{\varepsilon}'}, \quad (\text{S.41})$$

$$x = \frac{G'\bar{\varepsilon}' + \bar{G}'\varepsilon'}{\varepsilon' + \bar{\varepsilon}'} \quad (\text{S.42})$$

and then

$$n = \frac{1}{\pi} \text{Re} \chi = \frac{1}{\pi} \text{Re}(kx - G + i\varepsilon\tau) = \frac{1}{\pi} \text{Re} \chi = \frac{1}{\pi} \text{Re} \left(k \frac{G'\bar{\varepsilon}' + \bar{G}'\varepsilon'}{\varepsilon' + \bar{\varepsilon}'} - G + \varepsilon \frac{G' - \bar{G}'}{\varepsilon + \bar{\varepsilon}'} \right). \quad (\text{S.43})$$

The latter can be rewritten as

$$n = \frac{1}{\pi} \text{Re} \left[kG' - G - (k\varepsilon' - \varepsilon) \frac{G' - \bar{G}'}{\varepsilon' + \bar{\varepsilon}'} \right]. \quad (\text{S.44})$$

The equations (S.41,S.42,S.44) give the function $n(x, \tau)$ in a parametric form with two real parameters $\pi\nu$ and ν so that $k, \bar{k} = \pi\nu \pm i\nu$. The configurations of the polymers are obtained as the lines given by $n = \text{const}$ with

$$\begin{aligned} G(k) &= \frac{1}{2} \int^k dk' \sqrt{\frac{\sin^2 \frac{k'}{2} - \sin^2 \frac{k_0}{2}}{\sin^2 \frac{k'}{2} - \sin^2 \frac{k_F}{2}}} \\ &= \frac{2(1-a^2)}{\sqrt{b^2-a^2}} \left(\Pi \left(\frac{b^2-1}{b^2-a^2} \middle| \frac{a^2(b^2-1)}{b^2-a^2} \right) - \Pi \left(\frac{b^2-1}{b^2-a^2}; \phi \middle| \frac{a^2(b^2-1)}{b^2-a^2} \right) \right) \end{aligned} \quad (\text{S.45})$$

given in terms of the incomplete and complete elliptic functions $\Pi(n; \phi|m)$, $\Pi(n|m)$ with $a = \sin \frac{k_0}{2}$, $b = \sin \frac{k_F}{2}$ and

$$\phi = \arcsin \frac{\sqrt{1-u^2}}{\sqrt{u^2-a^2}} \frac{\sqrt{b^2-a^2}}{\sqrt{1-b^2}}, \quad u = \sin \frac{k}{2}. \quad (\text{S.46})$$

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4.3 Summary

In this chapter we have presented the central results of this thesis. That is, we have found two instances of limit shape transitions and uncovered how they occur via mapping the free fermion problem to a Coulomb gas model. The transitions have led to the identification of new limit shapes: the nephroid from the merged arctic circles and the deltoids from the FCS. We have thus achieved the central aim of this thesis which was to find new shapes and their transitions in a rigorous fashion.

In the first of the Coulomb gas models considered, equivalent to the GWW matrix model, we explored the merger of arctic circle limit shapes which resulted in a new, nephroid shape. The system underwent a phase transition as the dominant configuration of eigenvalues in the underlying Coulomb gas changed. We argued that this transition would persist even in the presence of interactions, which means that our results have a wide realm of applicability.

In the second of the models we found a similar phase transition, this one induced by an emptiness constraint placed on the free fermion system. Crucially, in this manuscript we were able to explore the FCS of free fermions (with no external potential) in the regime of large deviations for the first time wherein we found new limit shapes, shaped like deltoids.

CHAPTER 5

CONCLUSION

In this thesis we have successfully managed to find new limit shapes and their transitions in free fermion systems in a rigorous fashion. This was achieved by mapping the fermionic problem onto a Coulomb gas one. As an addition, in the Compendium we gave a complementary description to the Coulomb gas in terms of Riemann surfaces which allowed us to make substantial progress on classifying and characterising limit shapes and their fluctuations. We have thus achieved what we set out to do in this thesis.

In Chapter 2 we extensively covered the relevant background methods required to understand our results. We introduced large deviation theory with a simple coin flipping example and then stated the large deviation principle, which allows one to explore the atypical behaviour of statistical systems. We explained the details behind a method important to all our results, the path integral, and showed that it was amenable to a large deviation principle. Illustratively, we worked out the large deviations associated with a particle conditioned to avoid a semicircle. After this, we discussed Coulomb gas models. Most of the results in this thesis revolved around being able to map these models to free fermions. In particular in the papers [19, 20]. We stress that when the fermions are no longer free this mapping no longer holds, however as was found in [19] some of the results regarding the nature of the phase transitions uncovered do. Essentially, near the edges of interacting systems, assuming the interactions are short range, the density of fermions is

so low that their separation is greater than the interaction range and they are essentially free. As the phase transitions uncovered in this work take place at the edges of the limit shapes where they touch and so precisely where this argument applies, one would expect phase transitions and limit shape merging even in the presence of interactions and despite the fact that the Coulomb gas method fails. For a more thorough discussion of some of these arguments, see [70]. We then studied large deviations in Coulomb gas models, and hence large deviations in free-fermion systems, paving the way to use Coulomb gases to analyse limit shapes. We rounded out the section by looking at yet another application of large deviation theory and Coulomb gases in the FCS of free fermions.

Chapter 3 comprised of the work *Compendium of free fermion limit shapes*, intended for publication. In this work, we collected all the limit shapes we had found and described some of their more general features. We were even able to extend some results obtained previously in [19]. The latter half of the work introduced a complimentary description in terms of Riemann surfaces that was better equipped for studying fluctuations in limit shape systems. It enabled us to re-derive several key results in the literature like the non-perturbative role that instantons play in some matrix models. Such a description also allowed us to classify limit shapes in a more formal way.

Chapter 4 presented the main results of this work in which we gave an in depth discussion of the new limit shapes and transitions we had found. In the first of the manuscripts *Limit shape phase transitions: a merger of arctic circles* we detailed the merger transition of the arctic circles producing the brand new nephroid limit shape. We argued for the ubiquity of this transition even in the presence of interactions. In the work *Large deviations of the full counting statistics of free fermions on an interval* we extended the FCS of free fermions to the previously unexplored regime of large deviations and found another new limit shape, the deltoid and their phase transition.

There are a couple of areas that we would have liked to explore, but did not have time. The first is finding the Green function or propagator for the arctic circles across their merger transition. As argued for at the end of the Compendium, we expect this

Green function to be comprised of a sum of two terms, one representing correlations between the circles and one representing the correlations within them. These correlations should not be given equal weights. Only as the circles are brought closer should the correlations between circles begin to match those within them. We believe that this weighting is provided by instanton-like considerations.

We would also have liked to explore more complicated limit shapes where their algebraic curves involved higher powers of $X(k)$ than just two. These, we believe, should be equivalent to some multi-matrix model.

Lastly, what is perhaps missing most is a look at how limit shapes may be experimentally realised. The mapping to directed polymers touched upon in the FCS manuscript is a promising route. By doing so, we would have a much better understanding of this fascinating area of physics.

APPENDIX A

QUANTUM RIPPLES IN IMAGINARY TIME

The following is a brief note *Quantum ripples in imaginary time* produced by myself, my supervisor Dimitri Gangardt and our collaborator Alexander Abanov. As first author I came up with the idea and wrote most of the work, with helpful revisions provided by the other authors.

The note discusses in more detail the Wigner function, which was touched upon in the Compendium. It uses the Wigner function to connect two seemingly disparate ideas: 'crazy regions' in limit shapes [71] and 'quantum ripples' in hydrodynamic systems [72]. It is included in the appendix as supplemental material for anyone who wishes to understand the larger role the Wigner function plays in limit shapes and their fluctuations but is not required reading to understand the rest of thesis.

Quantum ripples in imaginary time

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It was noticed by Bocini that in the collective, bosonised description of the imaginary time dynamics of free fermions there were regions of spacetime where their collective density field diverged. Such regions were dubbed 'crazy regions'. Separately, Bettelheim observed that in the real-time dynamics of free fermions once quantum corrections were accounted for this smooth density field displayed prominent 'quantum ripples'. In this brief note we link these two ideas and show that Bocini's 'crazy regions' are nothing but Bettelheim's 'quantum ripples' that have gotten out of hand in the imaginary time dynamics.

One-dimensional many-body systems have always been particularly amenable to long-wavelength descriptions such as those given by hydrodynamics. This is largely due to the necessity of collective excitations in such systems which invite the use of collective variables, such as a density $\rho(x, t)$ and velocity $v(x, t)$ field which describe the system as a fluid. In this work, we will be considering a fluid of free fermions and their imaginary time dynamics.

The Wigner function, $W(x, k, t)$, enters the picture because semi-classically, i.e. in the long wavelength limit, it can be interpreted as the distribution in phase space of the fermions. As such it contains vital information on their dynamics. It defines a curve in phase space, Γ_t (Fig. 1), in such a way that within the curve it is 1 indicating that there are fermions residing there and outside of it it is 0. For a system with just one fermi point, k_F ,

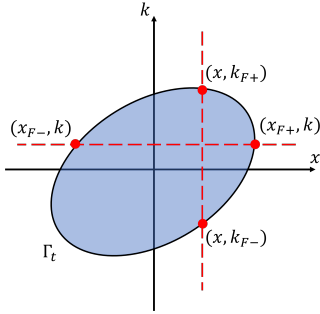


Figure 1. Depiction of phase space and its boundary, Γ_t as indicated by the Wigner function. The fermi points are the boundaries of the phase space for fixed x ($k_{F\pm}$) or alternatively fixed k ($x_{F\pm}$). In principle more elaborate geometries of multiple 'blobs' in phase space are allowed but we will not consider them here.

a natural *ansatz* for the Wigner function is

$$W(x, k, t) = \theta(k + k_F(x, t)). \quad (1)$$

In this work, we choose to write this instead as

$$W(x, k, t) = \theta(x + x_F(k, t)), \quad (2)$$

exploiting the freedom the Wigner function gives us to parameterise phase space in terms x or k . With this we

can recover the density profile of the fermions

$$\rho(x, t) = \int \frac{dk}{2\pi} W(x, k, t). \quad (3)$$

The Wigner function obeys the evolution equation

$$\partial_t W - \varepsilon'(k) \partial_x W - O(\hbar) = 0, \quad (4)$$

where $\varepsilon(k)$ is the dispersion of the free fermions. Despite corrections $O(\hbar)$ we may work in the semiclassical limit $\hbar \rightarrow 0$ where the above becomes simpler to solve (for quadratic dispersion, there are in fact no higher order corrections). One can apply the method of characteristics to Eq. (4) to find that the Wigner function and hence the curve it implies in phase space will propagate ballistically from some given initial configuration $W(x, k, 0)$

$$W(x, k, t) = W(x + \varepsilon'(k)t, k, 0). \quad (5)$$

We can also go one step further and consider quantum corrections to the Wigner function. These quantum corrections are controlled by the semiclassically small parameter $\hbar$. It is convention in the hydrodynamics to use the number of particles, N , instead which is large in the semiclassical limit. As pointed out in e.g. [1] the local density approximation (LDA) ensures these two limits are the same, i.e. $N \sim 1/\hbar$. The quantum corrections can then be found from the *exact* expression for the Wigner function

$$W(x, k, 0) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} G\left(k + \frac{q}{2}, k - \frac{q}{2}\right) e^{iqx}, \quad (6)$$

given in terms of the Fourier transform of the momentum space Green's function $G(k', k)$. Using this form, Bettelheim [2, 3] observed that the quantum-corrected Wigner function was

$$W(x, k, t) \sim \text{Ai}_1\left(\tilde{F}(x, k, t)\right), \quad (7)$$

where $\tilde{F}(x, k, t) = 2^{2/3} |\kappa(k, t)| F(x, k, t)$, $F(x, k, t) = x + x_F(k, t)$, $\kappa(k, t) = (\partial_k^2 x_F(k, t)/2N^2)^{-1/3}$ and

$$\text{Ai}_1(s) = \int \frac{dq}{2\pi i q} e^{i\left(qs + \frac{q^3}{3}\right)}. \quad (8)$$

In the large N limit the above function monotonically transitions from 0 to 1 thus recovering the step function Eq. (1). Introducing a further integral over k they studied the density

$$\rho(x, t) = \int \frac{dk}{2\pi} \text{Ai}_1 \left(2^{2/3} |\kappa(k, t)| (x + x_F(k, t)) \right). \quad (9)$$

By expanding about the fermi surface, i.e. writing $x_F(k, t) = x_F(k(t), t) = x_F(t)$ it was found that the quantum corrections manifested themselves as 'ripples' on top of the smooth, classical density profile.

This picture of the semiclassical real-time dynamics of free fermions is clearly very well understood. What is far less well understood are the dynamics in *imaginary* time. Such dynamics occur when one considers instanton-like configurations of fermions like the merging arctic limit shape problem or emptiness formation [4, 5]. Due to the numerous mappings between imaginary time free fermions and various statistical models like Directed Polymers, viscous walkers or dimers [6–8] the problem is of eminent interest in statistical physics as well.

In this work, we derive the following results pertaining to the imaginary time dynamics of free fermions. The first is that one should replace Eq. (1) with

$$W(x, k, \tau) = \frac{1}{\pi} \text{Im} (\log F(x, k, \tau)), \quad (10)$$

whilst they should replace Eq. (7) with

$$W(x, k, \tau) \sim \text{Ai}_1 \left(\tilde{F}(x, k, \tau) \right), \quad (11)$$

where all functions are as they were defined above with $t \rightarrow i\tau$ if F is bounded away from the negative real-axis else it is divergent. Essentially, in the imaginary time dynamics, the function F becomes complex and its argument dictates what saddlepoints dominate in a stationary phase approximation to the integral. The upshot is that the Wigner function is not well-defined everywhere and in certain regions it, and the resulting density profile diverge exponentially.

For the limit shape class of problems these divergences are known and were given the name 'crazy regions' by Bocini [9]. They studied the density of free fermions represented by

$$\rho(x, \tau) = \int_{-\pi}^{\pi} \frac{dK}{2\pi} \int_{-\pi}^{\pi} \frac{dQ}{2\pi} \frac{e^{i \int_K^Q dk F(x, k, \tau)}}{2i \sin \left(\frac{Q-K}{2} \right)}, \quad (12)$$

and then went on to evaluate the above via stationary phase and found that depending on the choice of saddlepoints the density profile could diverge.

This manuscript discusses the link between Bettelheim's work and crazy regions. We show that it is possible to go from Eq. (12) to Eq. (9) (written in imaginary time $t \rightarrow i\tau$). Therefore Bettelheim's calculations and

Bocini's are in fact virtually identical up to a Wick rotation. As a result the divergences Bocini noticed must be nothing but Bettelheim's quantum ripples which have gotten out of hand in the imaginary time dynamics. In other words in some regions the ripples cease being totally oscillatory and acquire an exponentially increasing factor.

To derive our results we consider, like Bettelheim, the exact Wigner function with no approximation Eq. (6). We need the momentum space propagator which we calculate from bosonisation. For the sake of argument we will consider a system with just two fermi points, a left L and a right R and consider excitations around just the right. If one likes they can think of this as equivalent to us sending the left point off to infinity where it can have no effect. We will discuss multiple fermi-points briefly at the end. Introducing the momentum space occupation function

$$\hat{\sigma}^{(R)}(k) = \psi^{(R)\dagger}(k) \psi^{(R)}(k), \quad (13)$$

and the coherent states

$$|\Psi\rangle = e^{i \int \Phi(k) \hat{\sigma}^{(R)}(k) dk} |0\rangle, \quad (14)$$

for free fermions on a lattice the desired propagator is

$$\begin{aligned} G(k', k) &= \langle \Psi | \psi^\dagger(k') \psi(k) | \Psi \rangle \\ &= \frac{e^{i\Phi(k') - i\Phi(k)}}{2i \sin \left(\frac{k-k'}{2} \right)}. \end{aligned} \quad (15)$$

As such the Wigner function is

$$W(x, k, 0) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{e^{iqx - i \int_{k-q/2}^{k+q/2} dq' \Phi'(q')}}{2i \sin \left(\frac{q}{2} \right)}. \quad (16)$$

Note that using $Q = k + q/2, K = k - q/2$ and then introducing an integral over K we have Eq. (12). Ballistically propagating in imaginary time then gives us the fully space-time resolved Wigner function

$$W(x, k, \tau) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{e^{iqx - i \int_{k-q/2}^{k+q/2} dq' \Phi'(q') - \tau \int_{k-q/2}^{k+q/2} dq' \varepsilon'(q')}}{2i \sin \left(\frac{q}{2} \right)}. \quad (17)$$

To progress we now perform what is known as a gradient expansion. We exploit the large semiclassical parameter N by scaling such that $q \rightarrow q/N, x \rightarrow xN, \tau \rightarrow \tau N, \Phi \rightarrow \Phi N$. This allows us to then approximate $\int_{k-q/2}^{k+q/2} dq' \Phi'(q') \approx \Phi(k)q + \Phi'''(k) \frac{q^3}{24N^2}$ and similarly for $\varepsilon'(k)$. In the same limit we can take the denominator in the small angle approximation such that

$$W(x, k, \tau) = \int_{-\infty}^{\infty} \frac{dq}{2\pi i q} e^{iqF(x, k, \tau) + \frac{iq^3}{24N^2} \partial_k^2 x_F(k, \tau)}, \quad (18)$$

where we identified $x_F(k, \tau) = -\Phi'(k) + i\tau\varepsilon'(k)$. Rescaling q we can write

$$W(x, k, \tau) \sim \int \frac{dq}{2\pi i q} e^{i(q\tilde{F} + \frac{q^3}{3})}. \quad (19)$$

We end up back at Eq. (8) but with complex argument. This makes all the difference. By introducing a further integral over k we have Eq. (9) formally establishing the link between Eq. (12) and Eq. (9).

Consider the case when $N \rightarrow \infty$. In this limit κ is large and the first term in the exponent dominates. Despite this apparent simplification we must proceed carefully. Whilst it is true that at $\tau = 0$ what is left is the integral representation of the step function

$$W(x, k, 0) = \Theta(F(x, k, 0)) = \Theta(x - \Phi'(k)), \quad (20)$$

we cannot do the same for $\tau \neq 0$ as the exponent is complex and the step function is not well-defined for complex argument. Instead, we propose the alternate formula

$$W(x, k, \tau) = \frac{1}{\pi} \text{Im}(\log F(x, k, \tau)), \quad (21)$$

as the only sensible continuation of the Wigner function to imaginary time. We exploit the fact that the argument of the logarithm changes from 0 to π when one is outside or inside the curve $F(x, k, \tau)$ which is exactly the property we wished the Wigner function to have. Eq. (21) is one of the key results of this work which extends the Wigner function to imaginary time dynamics.

Let us now reintroduce the sub-leading cubic dependency in the exponent, which amounts to including quantum corrections. The discussion of these quantum corrections or quantum ripples in *real time* has been given in [2]. We now discuss them in imaginary time.

Returning to Eq. (19) we have what looks like $W(x, k, \tau) = \text{Ai}_1(\tilde{F}(x, k, \tau))$, we then recover Eq. (11). However, this is only true if F is sufficiently well behaved. To explain, let us calculate the asymptotics (large but finite N) of the Wigner function. We have the stationary phase equation $q^2 + \tilde{F} = 0$ and thus $q = \pm i\sqrt{\tilde{F}}$. Inserting either solution into Eq. (19) it becomes obvious that if the real part of the exponent diverges, then the Wigner function must also. Since the density profile can be obtained by just a further k integral over the Wigner function it will also diverge. This is precisely the 'crazy region' phenomena studied by Boccini.

It was noticed by Boccini that for the specific case of the arctic circle limit shape problem [10] wherein $\varepsilon'(k) = \sin k$ and $x_F(k, \tau) = i\tau \sin k - \cos k$ that there were no crazy regions at all. This can be explained easily by what we have just learnt. Let us take the Wigner function and introduce a further integral over k to get the density profile. Following the remarks after Eq. (9) we must expand around the fermi-surface or boundary of

the limit shape, writing $x_F(k(\tau), \tau) = x_F(\tau)$. This time dependent boundary must be described by a real function and so $F(x, k(\tau), \tau) = x + x_F(\tau)$ must also be real. Now observe the coincidence that in our gradient expansion $\partial_k^2 x_F(k, \tau)|_{k(\tau)} = -x_F(k(\tau), \tau)$ making this double derivative real as well. Consequently $\tilde{F}$ is purely real and so our saddlepoint analysis will be well-behaved. In this specific case there is really no difference between the real and imaginary time dynamics beyond just replacing t with $i\tau$ in all the relevant formulas. We stress that it is this pure coincidence that means there are no divergences, in general one expects to find them [9].

To finish off, we briefly address what happens when we include both fermi points. The propagator will now be a sum of four combinations corresponding to correlations between the LL, RR, RL and LR points. In the regime where the two fermi points $x_R = x_{F+}$, $x_L = x_{F-}$ are far apart the RL, LR contributions will be negligible because a particle would have to tunnel from one fermi surface to the other, exponentially suppressing its contribution. Therefore, to good approximation, we are left with the sum over the RR and LL movers

$$G(k, k') = \frac{e^{i\Phi(k) - i\Phi(k')} - e^{i\Phi(k') - i\Phi(k)}}{2i \sin\left(\frac{k - k'}{2} - i0\right)}. \quad (22)$$

For large κ it is then simple to see that

$$W(x, k, \tau) = \frac{1}{\pi} (\text{Im}(\log F_+) - \text{Im}(\log F_-)), \quad (23)$$

where $F_{\pm}(x, k, \tau) = x + x_{F\pm}(k, \tau)$. If we wish, we can continue back to real time where we find that

$$W(x, k, \tau) = \Theta(x + x_{F+}(k, t)) - \Theta(x + x_{F-}(k, t)). \quad (24)$$

As the two fermi points are brought closer together the contribution from the RL, LR points increases until at some stage it should presumably be comparable to that from LL, RR. In this case all four contributions would need to be taken into account. We defer this more complex situation to future work.

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