SPECIAL TOPICS IN CONE COMPLEMENTARITY

by

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Abstract

In this thesis, the concept of Monotone Extended Second Order Cone (MESOC), which is a new generalisation of second order cone, has been present. We discussed the fundamental properties of MESOC and demonstrated the positive operator, the Lyapunov-like transformation as well as the reducibility of this cone. The value Lyapunov rank has also been provided. We also investigated the isotonicity property of MESOC, and we showed the cylinder is the only isotonic projection set with respect to MESOC in the ambient space. Then we present the mixed complementarity problem on a general close, and convex cone can be solved by using an iterative method based on the isotonicity property of MESOC. Meanwhile, a numerical example has been illustrated to show the applicability of MESOC. We also investigated the formulas to show how to project onto MESOC. In the most general case, the formula we obtained is dependent on an equation for one real variable. The linear complementarity problem on the MESOC has also been studied. We have demonstrated that the linear complementarity problem on the MESOC can be converted to a mixed complementarity problem on the nonnegative orthant. The algorithms are discussed and numerical examples are also present. Moreover, we present an application of the MESOC, which is a portfolio optimisation problem with an analytical solution. At last, we studied the gradient projection method on the sphere. We showed that this method could be used in discussing the solvability of the complementarity problem and checking the copositivity of an operator with respect to cones. The numerical experiments which illustrate the copositivity of operators are also provided.

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CHAPTER 1

CONES AND COMPLEMENTARITY PROBLEMS

1.1 Introduction

Karush introduced the concept of complementarity in [49], and then the complementarity problem was considered for the non-negative orthant by Dantzig and Cottle in their technical report [19]. Meanwhile, Variational Inequality (VI), which is recognized as a generalization of the nonlinear complementarity problem (NCP), has been implemented in the research of optimisation as well as equilibrium problems. In the mid-1960s, the finite-dimensional NCP, as well as VI have been considered by researchers systematically. These two problems have played an important role not only in different fields of mathematics but also in engineering, economics and finance (see [10, 33, 43]). In 1964, Cottle pointed out the relationship between nonlinear complementarity problems and linear programming in his PhD thesis [18]. He observed that this relationship could be identified as a nonlinear complementarity problem; Cottle was named the father of linear complementarity. Almost at the same time, the first constructive method was developed by Scarf in 1967 (see [74]). As a result, the whole family of fixed point methods appeared following his research, which made great contributions to equilibrium programming. It refers to a wide field of mathematics programming, such as analysis, modelling and computation of equilibria. According to the research finished by Pang in [28], the complementarity problem has a close relationship with the equilibrium problem. More specifically, in [75], Scarf and

Hansen have shown for the equilibrium problems, the problems of finding their solutions by using fixed-point methods are CPs or VIs. Throughout the whole research history of the cone complementarity problem, in the beginning, even though the researchers' works are focused on the theory development based on the general cone, they only made progress in the applications based on the complementarity problem on the non-negative orthant. For example, some well-known applications are listed as follows. In the financial market, Jaillet, Lamberton and Lapeyre developed the algorithm, which was formulated to evaluate the price of the American options by using VI [48]. In the topics related to economics, Markov perfect equilibria, perfect competition equilibrium and Walrasian price equilibrium models have been developed. In engineering, the complementarity problem also played an important role in analysing the elastoplastic structure and solving the frictional contact problem. In recent years, rather than the complementarity problems based on the non-negative orthant, researchers have considered more applications of the complementarity problem based on more sophisticated cones, such as second order cones and positive semi-definite cones. By using the Karush-Kuhn-Tucker conditions on the second order conic optimisation problems, researchers developed applications based on the theory of second order cone complementarity problem in robotics [3] and robust game theory [54]. Their research results emphasise that the complementarity problem can be treated as a cross-cutting problem, which means it not only can play a powerful role in the research of optimisation and equilibrium problems but also can make great contributions to the applications in a wide range of disciplines.

From the previous research, we can see that second order cone programming has played a significant role in complementarity problems. Thus, the concept of extended second order cone (ESOC) is introduced by Németh and Zhang in [62], which could be recognised as a natural extension of the concepts of second order cone. Sznajder obtained the value of the Lyapunov rank (or bilinearity rank) of ESOC in [79] and proved the irreducibility of the ESOC. His conclusions pointed out that the ESOC is a numerically good cone, and is worth further investigation. Ferreira and Németh found a numerical way to solve the problem of how to project onto the ESOC [32]. Until now, Németh and Zhang [62] have shown that the ESOC can be implemented as a tool not only in solving the mixed complementarity problem based on a general cone but also in finding the solution of the variational inequalities based on a general convex set [63]. Németh and Xiao also demonstrated how to solve the linear complementarity problem based on the ESOC [61]. Their research not only pointed out the relationship between the solution of the linear complementarity problem based on ESOC and the solution of the mixed complementarity problem based on the non-negative orthant but also introduced an application to the optimisation problem of portfolio allocation, which is called the mean-ℓ^2 norm (ML2N) model. They have demonstrated the advantages of the mean-ℓ^2 norm (ML2N) model over the well-known mean-variance model (MV), which was developed by Markowitz in [56], and the mean-absolute deviation model (MAD), which was introduced in [52].

The problems related to isotone projection mapping and complementarity problems were considered by Isac and A.B. Németh in [58], some properties of isotone projection cones in Euclidean, as well as Hilbert space, have been developed by them in [44]. From the practical point of view, according to the research in recent years, the importance of the class of isotone projection cones in applications, such as monotone cone [42] and monotone nonnegative cone [64], have been acknowledged. Nishimura and Ok [64] also demonstrated that the isotonicity property of a projection could be implemented not only in finding the solvability of variational inequalities but also in solving the equilibrium problems which are related to these cones.

In conclusion, the previous research demonstrated the importance of both extended second order cone and ordered vector spaces in investigating the equilibrium problems, such as in economics, finance and traffic equilibrium. Thus, we will give another nautral extension of the concept of the second order cone based on ESOC, which is called the monotone extended second order cone (MESOC). Although there is a trivial relationship between ESOC and MESOC, we will show the difference between them. The previous research and applications on ESOC motivate us to investigate the complementarity problem based on the MESOC. In this thesis, we also developed an application related to the portfolio optimisation problem based on the monotone extended second order cone, which we present in Chapter 6.

This thesis is organised as follows: In the remaining part of this chapter, the main terminology and concepts mentioned in this work, which includes basic definitions, properties and basic examples and results related to cones and complementarity problems have been introduced and explained.

Then, in Chapter 2, we introduced the monotone extended second order cone and present its basic properties. We showed that this monotone extended second order cone is proper and is numerically good, which is worth investigating. Furthermore, we also studied the properties of the positive operators and Z-transformations (or Z-properties) of this cone, respectively. We found the necessary condition of sufficient condition for a matrix to be a positive operator or Z-transformation of the monotone extended second order cone, respectively, and we also developed the necessary and sufficient condition for a block diagonal matrix to be a positive operator on this cone. In this case, the corresponding Z-transformation can be easily obtained. Moreover, the Lyapunov-like transformation of the monotone extended second order cone is computed, and the Lyapunov rank is present.

In Chapter 3, we showed that the projection mapping onto a cylinder is an isotonic projection set with respect to MESOC. Then by using the isotonicity property of the monotone extended second order cone, we generated a fixed point iteration scheme, which is convergent to a solution of the mixed complementarity problem on a general closed and convex cone. The convergence of this iteration is order-based rather than based on a usual contraction mapping principle. A numerical example has also been demonstrated at the end of this chapter. By using the above iteration method, we showed the existence of a solution in exact numbers. Even this iteration scheme can also be implemented to solve some general mixed complementarity problems by using ESOC-isotonicity, we proved that for the mixed complementarity problems, which can be solved iteratively by using the MESOC-isotonicity, the same iterative scheme can not be used via ESOC.

In Chapter 4, we discuss the problem of how to project onto the monotone extended second order cone. We prove some initial results on the MESOC complementarity set. By using the relationship between the complementarity set of the monotone extended second order cone and the complementarity set of the monotone nonnegative cone, together with the Moreau decomposition theorem, we have reduced the projection onto the MESOC to two isotonic regressions in neighbouring dimensions, which can be solved efficiently by pool-adjacent-violators algorithm.

In Chapter 5, we investigate the linear complementarity problems defined on the monotone extended second order cones. We have demonstrated that the linear complementarity problem defined on the MESOC can be reduced to a mixed complementarity problem on the nonnegative orthant in a neighbourhood dimension. We also found that any point is a solution to the converted problem if this point is a solution to the Fischer– Burmeister complementarity function. We also show that the semi-smooth Newton method could be used to solve the converted mixed complementarity problem and provide a numerical example. Other algorithms are also discussed, such as the FB linear search method, which can be implemented in finding the solution to the merit function associated with the Fischer– Burmeister complementarity function, and this solution is also a solution to the converted mixed complementarity problem. Finally, we develop an application for the monotone extended second order cone, which is a portfolio optimisation problem defined on this cone. The explicit solution to the portfolio optimisation problem has also been derived at the end.

In Chapter 6, we have presented the gradient projection method on the sphere and provided the algorithms. We have demonstrated that the algorithm we found can be used to solve the constraint optimisation problem defined on the intersection of a cone and a sphere. We apply this algorithm to show the solvability of the complementarity problem or to check the copositivity of an operator (or matrix) with respect to a cone, as the relationship between these problems has been discussed. As far as we know, this is the first numerical method implemented to check operators' copositivity with respect to the positive semidefinite cone. We provide the numerical results obtained by implementing our algorithms and provide with the discussions on the copositivity of operators.

1.2 Preliminaries

The notations and auxiliary results which we use in this dissertation are presented in this section. First, let us consider some basic definitions related to cones.

Denote the canonical unit vectors of $\mathbb{R}^n$ by $e^1, \ldots, e^n$ and let $e = e^1 + \cdots + e^n$. Any vector $z \in \mathbb{R}^n$ is considered to be a column vector and can be uniquely written as $z = (z_1, \ldots, z_n)^\top := z_1 e^1 + \cdots + z_n e^n$. In particular $e = (1, \ldots, 1)^\top$.

The definition of canonical *inner product* of any two vectors $x, y \in \mathbb{R}^n$, where $\mathbb{R}^n$ represents the Euclidean space, is given by

$$\langle x, y \rangle := x^\top y = x_1 y_1 + \cdots + x_n y_n,$$

and the corresponding *norm* $\| \cdots \|$ is given as

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

We also have another formula for the inner product, which is called as *Abel's partial summation formula*, that is

$$\langle x, y \rangle = \sum_{i=1}^{p-1} (x_i - x_{i+1}) \sum_{j=1}^{i} y_j + x_p \sum_{i=1}^{p} y_i, \qquad \forall x, y \in \mathbb{R}^p.$$

Let p, q be positive integers, we identify $\mathbb{R}^p \times \mathbb{R}^q$ with $\mathbb{R}^{p+q}$ through $(x, y) = (x^\top, y^\top)^\top$. Let $(x, u) \in \mathbb{R}^p \times \mathbb{R}^q \ni (y, v)$, then the scalar product in $\mathbb{R}^p \times \mathbb{R}^q$ is given as

$$\langle (x, u)^\top, (y, v)^\top \rangle = \langle x, y \rangle + \langle u, v \rangle.$$

The set

$$\mathcal{H}(u,a) := \{x \in \mathbb{R}^n : \langle x-a, u\rangle = 0\}$$

is called an *affine hyperplane* through $a \in \mathbb{R}^m$ with the normal $u \in \mathbb{R}^n \setminus \{0\}$ and the corresponding sets

$$\mathcal{H}_-(u,a) := \{x \in \mathbb{R}^n : \langle x-a, u\rangle \leq 0\},$$

$$\mathcal{H}_+(u,a) := \{x \in \mathbb{R}^n : \langle x-a, u\rangle \geq 0\},$$

are called *closed half-spaces*. An *affine hyperplane* through the origin will be simply called *hyperplane*.

For an arbitrary set K, if it satisfy $\lambda v \in K$, for any $\lambda \in \mathbb{R}$ and $v \in K$, then K is called a *cone*. Moreover, a cone K is called *pointed* if we cannot find a straight line through the origin in K, which is equivalent to

$$K \cap -K \subseteq \{0\}.$$

Let $K \subseteq \mathbb{R}^n$ be a cone, if for any arbitrary $a, b > 0$, and $u, v \in K$, we have

$$au + bv \in K,$$

then cone K is a *convex cone*. Note that the convex cone is also a convex set.

If K is a convex cone as well as a closed set, then we call K as a *closed convex cone*. A cone K is called a *proper cone* if K is pointed, as well as a closed and convex cone with a nonempty interior.

A cone K in $\mathbb{R}^m$ is called as *reducible* if it can be expressed as a sum $K = K_1 + K_2$, where $K_1, K_2 \neq \{0\}$ are cones with $\operatorname{span}(K_1) \cap \operatorname{span}(K_2) = \{0\}$. Otherwise, it is called *irreducible*.

Denote K^* be the *dual cone* of K, then the formula for K^* is given as

$$K^* = \{v \in \mathbb{R}^n : \forall u \in K : \langle u, v\rangle \geq 0\}.$$

If $K \subseteq K^*$, then the cone K is *subdual*, if $K^* \subseteq K$, it is called *superdual*, otherwise, if $K = K^*$, the cone K is *self-dual*. Moreover, a cone $K \subseteq \mathbb{R}^n$ is a *simplicial cone* if there is a basis $\{u^i : 1 \leq i \leq n\}$ of $\mathbb{R}^n$ such that

$$K = \left\{\alpha_1 u^1 + \cdots + \alpha_n u^n \,:\, \alpha_i \geq 0,\, 1 \leq i \leq n\right\}.$$

The vectors u^i, $1 \leq i \leq m$ are called the *generators* of K. The simplicial cone has a property, which is nice and well-known. That is, the dual of a simplicial cone is also a simplicial cone.

Let $P_C(x)$ be the metric projection of the point x onto a closed and convex set C, then the problem of finding the projection of x onto the set C is equivalent to the following constrained optimisation problem

$$\mathbb{R}^n \ni x \mapsto P_C(x) := \operatorname{argmin}\{\|y - x\| \,:\, y \in C\}.$$

Moreover, the point $P_C(x)$ is unique, followed by the convexity of set C.

Necessarily, $P_C(x)$ is a point-to-point mapping which is well defined from $\mathbb{R}^n$ to C. We also indicate that the projection P_C is *nonexpansive*(see [69]), i.e., for any $x, y \in \mathbb{R}^m$,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|. \tag{1.1}$$

Definition 1.2.1. *If there exists an arbitrary vector $u \in \mathbb{R}^k_+$ and a matrix $T \in \mathbb{R}^{k\times k}$ such that*

$$Tu \geq 0,$$

then we called matrix T a S_0 matrix.

1.3 Definition of cone and complementarity problem

After the introductions of the fundamental concepts of vectors and cones, we will give the definition of the complementarity problem as well as some examples of cones which have been used in this dissertation.

Definition 1.3.1. *Denote $K \in \mathbb{R}^n$ be a nonempty closed and convex cone, and K^* be the dual cone of K, then the set*

$$C(K) = \{(u, v) \in K \times K^* : \langle u, v\rangle = 0\}$$

is called the complementarity set *of K.*

Then, we will give some examples of the complementarity set of cones which will be implemented in the proof of some theorems in this thesis.

Example 1.3.1. *The definition of the* monotone cone $\mathbb{R}^n_{\geq}$ *is given as*

$$\mathbb{R}^n_{\geq} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

It is easy to check that its dual cone $(\mathbb{R}^n_{\geq})^*$ is

$$(\mathbb{R}^n_{\geq})^* = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{j} y_j \geq 0,\ j = 1, 2, \ldots, n-1,\ \sum_{i=1}^{n} y_i = 0 \right\}.$$

It is an important object, also known as the Schur cone *(see [77], Example 7.4) since it induces the so-called* Schur ordering, *which plays an important role in the theory of optimisation, see [66].*

From the formula of $\mathbb{R}^n_{\geq}$ and $(\mathbb{R}^n_{\geq})^*$, we get the complementarity set $C(\mathbb{R}^n_{\geq})$ of the cone $\mathbb{R}^n_{\geq}$, which is given by

$$C(\mathbb{R}^n_{\geq}) = \left\{ (x, y) : x \in \mathbb{R}_{\geq}, y \in (\mathbb{R}^n_{\geq})^*,\ (x_i - x_{i+1}) \sum_{j=1}^{i} y_j = 0,\ \forall i = 1, 2, \ldots, n-1 \right\}.$$

Example 1.3.2. *Let us define the* monotone nonnegative cone $\mathbb{R}^n_{\geq +}$ *as:*

$$\mathbb{R}^n_{\geq +} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

Its dual cone is given by:

$$(\mathbb{R}^n_{\geq +})^* = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{j} y_i \geq 0,\ j = 1, 2, \ldots, n \right\},$$

and the complementarity set of $\mathbb{R}^n_{\geq +}$ *is*

$$C(\mathbb{R}^n_{\geq +}) = \Bigg\{ x \in \mathbb{R}^n_{\geq +}, y \in (\mathbb{R}^n_{\geq +})^* : \left(x_j = x_{j+1} \ \textit{or} \ \sum_{i=1}^{j} y_j = 0,\ \forall j = 1, 2, \ldots, n-1 \right)$$
$$\textit{and} \left(x_n = 0 \ \textit{or} \ \sum_{i=1}^{n} y_i = 0 \right) \Bigg\}.$$

Following the formula of $\mathbb{R}^n_{\geq +}$ *and* $(\mathbb{R}^n_{\geq +})^*$ *and the definition of the simplicial cone, we get both* $\mathbb{R}^n_{\geq +}$ *and* $(\mathbb{R}^n_{\geq +})^*$ *are simplicial cones.*

Example 1.3.3. *Németh and Zhang [62] defined the extended second order cone* (ESOC) *as follows*

$$\text{ESOC} = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\| e\},$$

and its dual cone is given as

$$(\text{ESOC})^* = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e \rangle \geq \|u\|,\ x \geq 0\},$$

where p and q are nonnegative integers.

To the end of this chapter, for a given convex closed cone K, we define concepts of variational inequality, complementarity problem, and mixed complementarity problem with respect to cone K.

Definition 1.3.2. *Let $C \subseteq \mathbb{R}^n$ be a closed and convex set and $F : C \to \mathbb{R}^n$ be a mapping. The variational inequality,* VI*(F,C), is to find a vector $x \in C$, such that for any $y \in C$, we have*

$$(y - x)^\top F(x) \geq 0.$$

Definition 1.3.3. *Let cone K be a closed and convex cone and K^* be its dual cone, and $F : K \to \mathbb{R}^n$ be a mapping, then the complementarity problem* (CP) *defined by K and F is to find $x \in \mathbb{R}^n$ such that the following conditions*

$$K \ni x \perp F(x) \in K^*$$

are satisfied, where $x \perp y$ denotes the perpendicular relation, i.e., $\langle x, y \rangle = 0$. The definition above is equivalent to

$$\mathrm{CP}(F, K) := \begin{cases} \textit{find a } x \in \mathbb{R}^n, \ \textit{such that} \\ (x, F(x)) \in C(K). \end{cases}$$

When $C = K$ is in the previous definition, it is well known and easy to see that $\mathrm{CP}(F, K)$ *is equivalent to* $\mathrm{CP}(F, C)$. *We denote the solution set of the complementarity problem* $\mathrm{CP}(F, K)$

$$\mathrm{SolCP}(F, K) = \{x \in \mathbb{R}^n : (x, \, F(x)) \in C(K)\}.$$

Definition 1.3.4. *The complementarity problem* $\mathrm{CP}(K, F)$ *will be denoted by* $\mathrm{NCP}(K, F)$, *which is the nonlinear complementarity problem when $K = \mathbb{R}^n_+$—the nonnegative orthant.*

Definition 1.3.5. *The mixed complementarity problem* MiCP *is a generalization of* NCP. *Let m, n be two positive integers, denote two mappings as: $F_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $F_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and let K be a nonempty closed and convex cone, then the mixed* *complementarity problem* MiCP *defined by* K, F_1 *and* F_2 *is given as*

$$\operatorname{MiCP}(F_1, F_2, K) := \begin{cases} \text{find a } (u,v)^\top \in \mathbb{R}^m \times \mathbb{R}^n, \text{ such that} \\ F_2(u,v) = 0 \text{ and } \langle u, F_1(u,v)\rangle = 0. \end{cases}$$

Denote the solution to the mixed complementarity problem $\operatorname{MiCP}(F_1, F_2, K)$ *as* $\operatorname{SolMiCP}(F_1, F_2, K)$, *we have*

$$\operatorname{SolMiCP}(F_1, F_2, K) = \left\{(u,v)^\top \in \mathbb{R}^{m\times n} : (u,\ F_1(u,v)) \in C(K) \text{ and } F_2(u,v) = 0\right\}$$

Definition 1.3.6. *Let* n *be a positive integer, denote* $T \in \mathbb{R}^n \times \mathbb{R}^n$ *to be an arbitrary matrix, and let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be an arbitrary linear mapping such that* $F(x) = Tx + q$, *where* $q \in \mathbb{R}^n$ *is an arbitrary vector. Let* K *be a nonempty closed and convex cone and* K^* *be its dual cone, then the definition of the linear complementarity problem* LCP *associated with* K *and* F *is*

$$\operatorname{LCP}(T, q, K) := \begin{cases} \text{find a } x \in \mathbb{R}^n, \text{ such that} \\ F(x) \in K^* \text{ and } \langle x, F(x)\rangle = 0. \end{cases}$$

And the solution to the linear complementarity problem $\operatorname{LCP}(T, q, K)$ *is given by*

$$\operatorname{SolLCP}(T, q, K) = \{x \in K : Tx + q \in K^* \text{ and } \langle x, Tx + q\rangle = 0 \text{ for } T \in \mathbb{R}^n \times \mathbb{R}^n \text{ and } q \in \mathbb{R}^n\}.$$

CHAPTER 2

MONOTONE EXTENDED SECOND ORDER CONE

In this Chapter, we introduce the definition of the monotone extended second order cone and some of its important properties. Chapters 2 and 3 are based on my joint work [37].

2.1 Basic concepts and properties of MESOC

First, let us introduce the definition of the *monotone extended second order cone* (MESOC).

Definition 2.1.1. *Let n, p and q be nonnegative integers such that $n = p + q$. MESOC is the following set in $\mathbb{R}^n$:*

$$L := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : x_1 \geq x_2 \geq \cdots \geq x_p \geq \|u\|\}. \tag{2.1}$$

Proposition 2.1.1. *The monotone extended second order cone is a proper cone, which means it is a pointed, closed and convex cone with a non-empty interior.*

Proposition 2.1.2. *For the monotone extended second order cone L defined in (2.1), we have $L = L_1 + L_2$, where*

$$L_1 := \text{cone}\left\{(\underbrace{1, \ldots, 1}_{p}, m_1, \ldots, m_q) : m_1^2 + \cdots + m_q^2 \leq 1\right\} \text{ and}$$

$$L_2 := \text{cone}\left\{(\underbrace{1,0,\ldots,0}_{p},\underbrace{0,\ldots,0}_{q}),(\underbrace{1,1,\ldots,0}_{p},\underbrace{0,\ldots,0}_{q}),\ldots,(\underbrace{1,1,\ldots,1,0}_{p},\underbrace{0,\ldots,0}_{q})\right\}.$$

By using the definition of the reducibility of the cone, we conclude that the monotone extended second order cone L is reducible.

Proof. First, we show the inclusion $L \subseteq L_1 + L_2$.

An arbitrary element $(x_1,\ldots,x_p,u_1,\ldots,u_q) \in L$, by the definition of L, can be represented as $(\sum_{i=1}^p a_i,\ldots,a_1+a_2,a_1,u_1,\ldots,u_q)$, where $a_i \geq 0$ for $i=2,\ldots,p$ and $a_1 \geq \|(u_1,\ldots,u_q)\|$. Hence,

$$\begin{aligned}
&(x_1,\ldots,x_p,u_1,\ldots,u_q)\\
&=\left(\sum_{i=1}^{p} a_i,\sum_{i=1}^{p-1} a_i,\ldots,a_1,u_1,\ldots,u_q\right)\\
&=(a_1,\ldots,a_1,u_1,\ldots,u_q)+(\underbrace{a_2,\ldots,a_2,0}_{p},\underbrace{0,\ldots,0}_{q})+\cdots+(\underbrace{a_p,0,\ldots,0}_{p},\underbrace{0,\ldots,0}_{q})\\
&=(a_1,\ldots,a_1,u_1,\ldots,u_q)+a_2(\underbrace{1,\ldots,1,0}_{p},\underbrace{0,\ldots,0}_{q})+\cdots+a_p(\underbrace{1,0,\ldots,0}_{p},\underbrace{0,\ldots,0}_{q}).
\end{aligned}$$

Obviously, $a_2(\underbrace{1,\ldots,1,0}_{p},\underbrace{0,\ldots,0}_{q})+\cdots+a_p(\underbrace{1,0,\ldots,0}_{p},\underbrace{0,\ldots,0}_{q}) \in L_2$.

Now, we will show that $(a_1,\ldots,a_1,u_1,\ldots,u_q) \in L_1$. It is trivial when $a_1=0$, so we assume that $a_1>0$. Thus, we have $(a_1,\ldots,a_1,u_1,\ldots,u_q)=a_1\left(1,\ldots,1,\frac{u_1}{a_1},\ldots,\frac{u_q}{a_1}\right)$. As $a_1 \geq \|(u_1,\ldots,u_q)\|$, we get

$$a_1 \geq \sqrt{u_1^2+\cdots+u_p^2} \equiv 1 \geq \sqrt{\left(\frac{u_1}{a_1}\right)^2+\cdots+\left(\frac{u_q}{a_1}\right)^2},$$

which, by the definition of L_1, gives that $(a_1,\ldots,a_1,u_1,\ldots,u_q) \in L_1$. Hence, we conclude that for arbitrary element $(x_1,\ldots,x_p,u_1,\ldots,u_q) \in L$, it can be represented as a sum of two elements in L_1 and L_2 respectively, that is

$$(a_1,\ldots,a_1,u_1,\ldots,u_q) \in L_1$$

and

$$a_2(\underbrace{1,\ldots,1,0}_{p},\underbrace{0,\ldots,0}_{q}) + \cdots + a_p(\underbrace{1,0,\ldots,0}_{p},\underbrace{0,\ldots,0}_{q}) \in L_2\,.$$

Now, let us consider the inverse case, which is $L_1 + L_2 \subseteq L$. From the definition of L, L_1 and L_2, it is obvious that $L_1 \subseteq L$ and $L_2 \subseteq L$. Then, by using convexity of the cone L, it follows that $L_1 + L_2 \subseteq L + L = L$. It concludes the proof of the equality $L = L_1 + L_2$. Obviously, the cones $L_1, L_2 \neq \{0\}$ and $\operatorname{span}(L_1) \cap \operatorname{span}(L_2) = \{0\}$. Thus, the monotone extended second order cone L is reducible. □

Proposition 2.1.3. *The dual cone of a monotone extended second order cone L defined above is given by*

$$M := \left\{ (x,u) \in \mathbb{R}^p \times \mathbb{R}^q : \sum_{i=1}^{j} x_i \geq 0, \forall j \in \{1,\ldots,p-1\}, \sum_{i=1}^{p} x_i \geq \|u\| \right\}, \qquad (2.2)$$

that is, $M = L^*$.

Proof. In order to prove $L^* = M$, we will show that $M \subseteq L^*$ first. Let $(x,u) \in L$ and $(y,v) \in M$. By using Abel's summation formula, we have

$$\langle (x,u),(y,v)\rangle = x^\top y + u^\top v = \sum_{i=1}^{p-1}(x_i - x_{i+1})\sum_{j=1}^{i} y_j + x_p \sum_{i=1}^{p} y_i + u^\top v \geq \|u\|\|v\| + u^\top v \geq 0.$$

So, we have $M \subseteq L^*$. Now, we show the converse inclusion. Let $(y,v) \in L^*$ and $e = (1,1,\ldots,1) \in \mathbb{R}^p$. Obviously, we have $(\|v\|e, -v) \in L$. Suppose $v \neq 0$, then

$$\langle (\|v\|e,-v),(y,v)\rangle \geq 0 \Leftrightarrow \|v\| \sum_{i=1}^{p} y_i - \|v\|^2 \geq 0.$$

Hence, $\sum_{i=1}^{p} y_i \geq \|v\|$. When $v = 0$, then $(e,0) \in L$ and $(y,0) \in L^*$ imply that $\sum_{i=1}^{p} y_i \geq 0 = \|v\|$. □

Proposition 2.1.4. *The monotone extended second order cone is a sub-dual cone when* $p \geq 2$*, and it is a self-dual cone if and only if* $p = 1$*. Moreover, when* $p = 1$*, the monotone* *extended second order cone is the same as the Lorentz cone in* $\mathbb{R} \times \mathbb{R}^q$.

Proof. When $p \geq 2$, from the definition of MESOC and the formula for the dual cone of MESOC in Proposition 2.1.3, we have $L \subseteq M$. Thus, MESOC is a sub-dual cone. Moreover, when $p = 1$, it is trivial that MESOC is the same as Lorentz cone, and then we have MESOC is self-dual when $p = 1$. □

After finding the dual cone of the monotone extended second order cone, the complementarity set of this cone will be established. In order to do so, we need to use the following result below.

Lemma 2.1.5. *For any arbitrary* $(x, u) \in L$ *and* $(y, v) \in M$*, we have*

$$\langle x, y \rangle \geq \|u\| \sum_{i=1}^{p} y_i \geq \|u\| \|v\| .$$

Proof. First, we show the validity of the first inequality, which is $\langle x, y \rangle \geq \|u\| \sum_{i=1}^{p} y_i$. Since $(x, u) \in L, (y, v) \in M$, by using Definition 2.1 and Definition 2.2, we have

$$x_1 \geq x_2 \geq \cdots \geq x_p \geq \|u\|$$

and

$$\sum_{i=1}^{j} y_j \geq 0, \text{ for all } j \in \{1, \ldots, p-1\}, \sum_{i=1}^{p} y_i \geq \|v\| \geq 0.$$

Thus, by using the backward induction,

$$\sum_{i=1}^{p} y_i = y_1 + y_2 + \ldots + y_p \geq 0$$

$$\Longrightarrow (x_p - \|u\|)\sum_{i=1}^{p-1} y_i + (x_p - \|u\|)y_p \geq 0$$

$$\Longrightarrow (x_{p-1} - \|u\|)\sum_{i=1}^{p-2} y_i + (x_{p-1} - \|u\|)y_{p-1} + (x_p - \|u\|)y_p \geq 0$$

$$\Longrightarrow (x_{p-2} - \|u\|)\sum_{i=1}^{p-3} y_i + (x_{p-2} - \|u\|)y_{p-2} + (x_{p-1} - \|u\|)y_{p-1}$$

$$+ (x_p - \|u\|)y_p \geq 0$$

$$\ldots$$

$$\Longrightarrow (x_1 - \|u\|)y_1 + (x_2 - \|u\|)y_2 + \cdots + (x_p - \|u\|)y_p \geq 0$$

$$\Longleftrightarrow \langle x, y\rangle \geq \|u\| \sum_{i=1}^{p} y_i \,.$$

Finally, since $\langle x, y\rangle \geq \|u\| \sum_{i=1}^{p} y_i$ and $\sum_{i=1}^{p} y_i \geq \|v\|$, we have

$$\langle x, y\rangle \geq \|u\| \sum_{i=1}^{p} y_i \geq \|u\|\|v\| \,.$$

□

Next, we introduce the complementarity set of the monotone extended second order cone by using Lemma 2.1.5.

Proposition 2.1.6. *Let $(x, y, u, v) \in C(L)$, where $(x, u) \in L$ and $(y, v) \in M$. If $u \neq 0, v \neq 0$, then the complementarity set of the monotone extended second order cone L is given by*

$$C(L) = \Bigg\{(x, u, y, v) : (x, u) \in L,\ (y, v) \in M,\ \langle x, y\rangle = \|u\| \sum_{i=1}^{p} y_i,\ \sum_{i=1}^{p} y_i = \|v\|, \text{ and } \exists\lambda > 0 \text{ such that } v = -\lambda u\Bigg\}$$

or equivalently,

$$C(L) = \Bigg\{(x,u,y,v) : (x,u) \in L,\ (y,v) \in M,\ (x_i - x_{i+1}) \sum_{j=1}^{i} y_j = 0\,,$$
$$\forall\, i = 1,\ldots,p-1, x_p = \|u\|,\ \sum_{i=1}^{p} y_i = \|v\|, \mathrm{and} \exists \lambda > 0\, \mathrm{such that v} = -\lambda \mathrm{u}\Bigg\}\,.$$

Proof. Let us define the following set

$$S := \Bigg\{(x,u,y,v) : (x,u) \in L,\ (y,v) \in M,\ \langle x, y\rangle = \|u\| \sum_{i=1}^{p} y_i,\ \sum_{i=1}^{p} y_i = \|v\|,\ \text{and}$$
$$\exists \lambda > 0 \text{ such that } v = -\lambda u\Bigg\}\,.$$

Now, we need to show that $C(L) = S$. First, we need to prove that $C(L) \subseteq S$. For an arbitrary $(x,u,y,v) \in C(L)$, by using Lemma 2.1.5, we have

$$\begin{aligned} 0 = \langle (x,u),(y,v)\rangle &= \langle x,y\rangle + \langle u,v\rangle \\ &\geq \|u\| \sum_{i=1}^{p} y_i + \langle u,v\rangle \\ &\geq \|u\|\|v\| + \langle u,v\rangle \geq 0\,. \end{aligned}$$

Hence, all the inequalities above must be equalities, and then all the equations and inequalities above will become

$$\begin{aligned} 0 = \langle x,y\rangle + \langle u,v\rangle &= \|u\| \sum_{i=1}^{p} y_i + \langle u,v\rangle \\ &= \|u\|\|v\| + \langle u,v\rangle = 0\,. \end{aligned}$$

Thus,

$$\langle x,y\rangle = \|u\| \sum_{i=1}^{p} y_i = \|u\|\|v\|\,. \tag{2.3}$$

Therefore,

$$\|u\| \sum_{i=1}^{p} y_i = \|u\|\|v\|$$

and

$$\|u\|\|v\| + \langle u, v\rangle = 0\,. \tag{2.4}$$

From (2.3) we get $\langle x, y\rangle = \|u\|\sum_{i=1}^{p} y_i$ and, subsequently, $\sum_{i=1}^{p} y_i = \|v\|$. From the equality case in the Cauchy-Schwarz inequality, equation (2.4) implies that $\exists\lambda > 0, v = -\lambda u$. Thus, $C(L) \subseteq S$. Now, we will show the satisfaction for the converse inclusion, which is $S \subseteq C(L)$. We have: $\forall(x, u, y, v) \in S$, $\exists\lambda > 0$ such that $v = -\lambda u$, $(x, u) \in L, (y, v) \in M, x^\top y = \|u\|\sum_{i=1}^{p} y_i$ and $\sum_{i=1}^{p} y_i = \|v\|$. Thus

$$\langle (x, u), (y, v)\rangle = \langle x, y\rangle + \langle u, v\rangle = \|u\|\|v\| + \langle u, v\rangle = 0\,.$$

Therefore, $(x, u, y, v) \in C(L)$. Hence, $S \subseteq C(L)$.

Finally, we have

$$C(L) = \Big\{(x, u, y, v) : (x, u) \in L,\ (y, v) \in M,\ \langle x, y\rangle = \|u\|\textstyle\sum_{i=1}^{p} y_i,\ \sum_{i=1}^{p} y_i = \|v\|, \\ \text{and } \exists\lambda > 0 \text{ such that } v = -\lambda u\Big\}. \tag{2.5}$$

Moreover,

$$\begin{aligned}\|u\|\sum_{i=1}^{p} y_i &= \langle x, y\rangle \\ &= y_1(x_1 - x_2) + (y_1 + y_2)(x_2 - x_3) + \cdots + (y_1 + y_2 + \cdots + y_{p-1})(x_{p-1} - x_p) \\ &+ (y_1 + y_2 + \cdots + y_p)x_p\end{aligned}$$

if and only if

$$\begin{aligned}&(\|u\| - x_p)\sum_{i=1}^{p} y_i \\ &= y_1(x_1 - x_2) + (y_1 + y_2)(x_2 - x_3) + \cdots + (y_1 + y_2 + \cdots + y_{p-1})(x_{p-1} - x_p)\,.\end{aligned}$$

In the equation above, it is obvious that the LHS ≤ 0 and the RHS ≥ 0, where LHS and RHS denote left hand side and right hand side, respectively. Thus, both the LHS and the RHS must be equal to 0. Since the components of the sum in the RHS are all nonnegative, each component must be equal to 0. Hence, from equation (2.5), it follows that

$$C(L) = \Bigg\{ (x,u,y,v) : (x,u) \in L,\ (y,v) \in M,\ (x_i - x_{i+1}) \sum_{j=1}^{i} y_j = 0\,,$$

$$\forall\, i = 1, \dots, p-1, x_p = \|u\|,\ \sum_{i=1}^{p} y_i = \|v\|,\ \text{and } \exists \lambda > 0 \text{ such that } v = -\lambda u \Bigg\}.$$

□

2.2 Positive operators and Z-transformations on the MESOC

Research into positive operators and the Z-transformations on cones, particularly on the second-order cone and on the extended second-order cone, remains a highly captivating area of study within Hilbert spaces and cone theory. Burns, Fiedler and Haynsworth developed the concepts of positive operator on polyhedral cones in [17]. Loewy and Schneider found the necessary and sufficient conditions for a linear mapping to be a positive operator on the n-dimensional Lorentz cone. Such a mapping, represented by a matrix, must satisfy a positive semidefinite condition in [53]. Tam [80, 81] discussed the properties of the structure of cones of positive operators and investigated the properties of respectively the positive operator on polyhedral cones and simplicial cones.

Meanwhile, it is also important to study the Z-transformation(Z-property) on the monotone extended second order cone. Berman and Plemmons illustrate the importance of Z-transformation in many aspects, not only in optimisation but also in economics, dynamical systems, and differential equations in [7]. In their Introduction of [40], Gowda and Tao studied the properties of Z-transformations on proper and symmetric cones and emphasised the significance of Z-transformations in the context of the linear complementarity problem. Németh and Gowda evaluated the connections between the positive operators and Z-transformations in [60].

Motivated by the aforementioned research, our interest lies in finding the differences between the properties of the positive operators on the monotone extended second order cone and those on general second order cones. Consequently, we also discuss the characteristics of the positive operators and Z-transformations on the monotone extended second order cone because even the monotone extended second order cone has some nice properties which have been introduced in the earlier part of this chapter, but at the same time, it lacks some good properties, notably self-duality.

First, we introduce some basic concepts of the positive operator.

Definition 2.2.1. *The set $\Gamma(K)$ of positive operators on a cone $K \subseteq \mathbb{R}^m$ is defined by*

$$\Gamma(K) = \{A \in \mathbb{R}^{m\times m} : \, Ax \in K \,\text{for any}\, x \in K\}.$$

The set of positive operators is a cone in $\mathbb{R}^{m\times m}$.

We also have the following proposition for the positive operator.

Proposition 2.2.1. *Matrix A represents a positive operator on K if and only if $A^\top$ represents a positive operator on K^*.*

Proposition 2.2.2. *Denote matrix A to be a positive operator of cone K, then AK is a closed and convex cone if K is a closed and convex cone.*

Proposition 2.2.3. *Denote matrix A be a positive operator of a closed and convex cone K, then AK is a polyhedral cone if and only if K is a polyhedral cone.*

Definition 2.2.2. *Denote $K \in \mathbb{R}^n$ to be a cone. A linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is a Z-transformation ($L \in Z(K)$), if*

$$(x, y) \in C(K) \Rightarrow \langle Lx, y\rangle \leq 0.$$

Proposition 2.2.4 (see [76]). *Let $K \in \mathbb{R}^n$ be a proper cone, then matrix $A \in \mathbb{R}^{n\times n}$ is a Z-transformation of cone K if and only if for any $t \geq 0$, we have e^{-tA} is a positive operator of cone K.*

Proposition 2.2.5. *If $K \in \mathbb{R}^n$ is a proper cone, then the Z-operator of K can be obtained by $Z(K) = \mathrm{cl}\,(LL(\mathrm{K}) - \Gamma(\mathrm{K}))$, where $LL(K)$ is the Lyapunov rank of K and $\Gamma(K)$ is the positive operator of K, respectively.*

Then, we will derive the conditions that a linear transformation is a positive operator on the MESOC. The theorem below illustrates the necessary condition for a linear operator to be a positive operator on the dual cone of the MESOC, and by using the Proposition 2.2.1, we will have the necessary condition for a linear operator to be a positive operator on the MESOC.

Theorem 2.2.6 (Necessary conditions for positive operators on $M(p,q)$). *Let p, q be two integers such that $p \geq 1$ and $q \geq 1$.If $A^{(p+q)\times(p+q)}$ is a positive operator on $M(p,q)$, then we have the following statements:*

(i) Let A_i be the i-th row of matrix A, then we have

$$\sum_{i=1}^{j} A_i^\top \in L(p,q), \qquad \text{for } j = 1, 2, \ldots, p.$$

(ii) Let A^i be the i-th column of matrix A, we have

$$A^i \in M(p,q), \qquad \text{for } i = 1, 2, \ldots, p$$

and

$$A^i - A^j \in M(p,q), \qquad \text{for any } i < j \leq p.$$

(iii) For an arbitrary vector $u = (u_1, u_2, \ldots, u_q)^\top \in \mathbb{R}^q$ with $\|u\| = 1$, we have

$$A^i + \sum_{j=1}^{q} u_j A^{p+j} \in M(p,q),$$

for $i = 1, 2, \ldots, p$*.*

(*iv*) *By adding any column of* A *from the first* p *columns to any column from the last* q *columns, we obtain a vector in* $M(p, q)$*.*

Proof. (*i*) Suppose that A is a positive operator on $M(p, q)$.

Then, for any $y = (y_1, y_2, \ldots, y_p, v)^\top \in M(p, q)$ with $y_i \in \mathbb{R}$ and $v \in \mathbb{R}^q$, we have $Ay \in M$, which is equivalent to

$$\begin{pmatrix} \langle A_1^\top, y \rangle \\ \langle A_2^\top, y \rangle \\ \vdots \\ \langle A_{p+q}^\top, y \rangle \end{pmatrix} \in M(p, q),$$

hence by using (2.2), we have

$$\sum_{i=1}^{j} \langle A_i^\top, y \rangle \geq 0, \, \forall j \in \{1, \ldots, p-1\} \text{ and } \sum_{i=1}^{p} \langle A_i^\top, y \rangle \geq \| \left(\langle A_{p+1}^\top, y \rangle, \ldots, \langle A_{p+q}^\top, y \rangle \right) \|.$$

Which implies that

$$\sum_{i=1}^{j} A_i^\top \in L(p, q), \qquad \text{for } j = 1, 2, \ldots, p.$$

(*ii*) By Proposition 2.2.1, the matrix $A^\top$ is a positive operator on $L(p, q)$. Hence, by using the fact $(A^i)^\top$ is the i-th row of $A^\top$, for any $x = (x_1, x_2, \ldots, x_p, u) \in L(p, q)$ with $x_i \in \mathbb{R}$

and $u \in \mathbb{R}^q$ we have that $A^\top x \in L$, which is equivalent to

$$\begin{pmatrix} \langle A^1, x \rangle \\ \langle A^2, x \rangle \\ \vdots \\ \langle A^p, x \rangle \\ \langle A^{p+1}, x \rangle \\ \vdots \\ \langle A^{p+q}, x \rangle \end{pmatrix} \in L(p,q).$$

Then, by using (2.1) we have

$$\langle A^1, x \rangle \geq \langle A^2, x \rangle \geq \ldots \geq \langle A^p, x \rangle \geq \|(\langle A^{p+1}, x \rangle, \ldots, \langle A^{p+q}, x \rangle)\| \geq 0. \tag{2.6}$$

Thus, $A^i \in M(p,q)$, for $i = 1, 2, \ldots, p$, or in other words the first p columns of A will be in $M(p,q)$.

Moreover, by using (2.6), we also have $\langle A^i - A^j, x \rangle \geq 0$ for any $1 \leq i < j \leq p$, which implies that

$$A^i - A^j \in M(p,q) \text{ for any } 1 \leq i < j \leq p.$$

(iii) For any $x \in L(p,q)$ and $i = 1, 2, \ldots, p$, by using the Cauchy inequality, we have

$$\begin{aligned} \langle x, a_i + \sum_{j=1}^q u_j a_{p+j} \rangle &= \langle x, a_i \rangle + \langle x, \sum_{j=1}^q u_j a_{p+j} \rangle \\ &= \langle x, a_i \rangle + \sum_{j=1}^q u_j \langle x, a_{p+j} \rangle \\ &\geq \langle x, a_i \rangle - \sqrt{\sum_{j=1}^q u_j^2} \sqrt{\sum_{j=1}^q \langle x, a_{p+j} \rangle^2} \\ &\geq \langle x, a_i \rangle - \sqrt{\sum_{j=1}^q \langle x, a_{p+j} \rangle^2} \end{aligned} \tag{2.7}$$

Since A is a positive operator on $M(p,q)$, $A^\top$ is a positive operator on $L(p,q)$, which is equivalent to $A^\top x \in L(p,q)$. By using the definition of $L(p,q)$, we have

$$\langle x, a_i\rangle \geq \sqrt{\sum_{j=1}^{q}\langle x, a_{p+j}\rangle^2}.$$

Hence, bearing in mind equation (2.7), we get

$$\langle x, a_i + \sum_{j=1}^{q} u_j a_{p+j}\rangle \geq \langle x, a_i\rangle - \sqrt{\sum_{j=1}^{q}\langle x, a_{p+j}\rangle^2} \geq 0.$$

Since we have $x \in L(p,q)$, we obtain that $a_i + \sum_{j=1}^{q} u_j a_{p+j} \in M(p,q)$, for any $i = 1, 2, \ldots, p$.

(iv) It follows from (iii) by letting $u = e_j$, where $j = 1, 2, \ldots, q$. □

Lemma 2.2.7. *For any positive integers p, q, m such that $m = p + q$, consider the matrix $J \in \mathbb{R}^{m\times m}$, where*

$$J = \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix}$$

where $e = (1, \ldots, 1)^\top \in \mathbb{R}^p$ and I_q represent the $q \times q$ identity matrix. For any $k = 1, 2, \ldots, p$, denote by $e^k \in \mathbb{R}^p$ the vector whose first k elements are equal to 1 and the rest are equal to 0. Then, we have

$$M(p,q) := \left\{w = (u,v) \in \mathbb{R}^p \times \mathbb{R}^q : \, w^\top J w \geq 0 \text{ and } \langle u, e^k\rangle \geq 0 \text{ for } k = 1, 2, \ldots, p\right\}.$$

Proof. Define the set S such that

$$S := \left\{w = (u,v) \in \mathbb{R}^p \times \mathbb{R}^q : \, w^\top J w \geq 0 \text{ and } \langle u, e^k\rangle \geq 0 \text{ for } k = 1, 2, \ldots, p\right\}.$$

In order to prove that $S = M(p,q)$, first we will show that $S \subseteq M(p,q)$.

For any $w = (u, v) \in S$, $w^\top Jw \geq 0$ yields

$$\langle u, e\rangle^2 - \|v\|^2 = u^\top ee^\top u - v^T v = w^\top Jw \geq 0. \tag{2.8}$$

From $\langle u, e^k\rangle \geq 0$ we obtain

$$\sum_{i=1}^{k} u_i \geq 0, \qquad \text{for } k = 1, 2, \ldots, p. \tag{2.9}$$

Combining (2.8) and (2.9) we conclude that for any $n = (u, v) \in S$,

$$\sum_{i=1}^{k} u_i \geq 0, \text{ for } k = 1, 2, \ldots, p-1 \text{ and } \sum_{i=1}^{p} u_i \geq \|v\|.$$

Thus, $w = (u, v) \in M$ and we have $S \subseteq M$.

Conversely, for any $w = (u, v) \in M$,

$$\sum_{i=1}^{p} u_i \geq \|v\| \implies \langle u, e\rangle \geq \|v\| \implies \langle u, e\rangle^2 - \|v\|^2 \geq 0 \implies w^\top Jw \geq 0. \tag{2.10}$$

Meanwhile,

$$\sum_{i=1}^{k} u_i \geq 0 \implies \langle u, e^k\rangle \geq 0, \text{ for any } k = 1, 2, \ldots, p-1. \tag{2.11}$$

By using (2.10) and (2.11), we conclude that $M \subseteq S$. Thus,

$$M(p, q) = \left\{w = (u, v) \in \mathbb{R}^p \times \mathbb{R}^q : w^\top Jw \geq 0 \text{ and } \langle u, e^k\rangle \geq 0 \text{ for any } k \in (1, 2, \ldots, p)\right\}$$

□

By using the lemma above, the theorem below illustrates the sufficient condition for a linear operator to be a positive operator on the dual cone of the MESOC, and by using the Proposition 2.2.1, we get the sufficient condition for a linear operator to be a positive operator on the MESOC.

Theorem 2.2.8 (Sufficient conditions for positive operators on $M(p,q)$). *Denote by $A \in \mathbb{R}^{m\times m}$ an arbitrary matrix, where p,q,m are positive integers such that $m = p+q$. Then, we have the following statements:*

(i) If there exist a $\lambda \geq 0$ such that $A^\top JA - \lambda J$ is positive semi-definite, and

$$\left(\sum_{i=1}^{j} A_i^{[1:p]}, 0^{[1:q]}\right)^\top \in L(p,q), \qquad \text{for } j = 1,2,\ldots,p,$$

where $A_i^{[1:p]} \in \mathbb{R}^p$ is a vector containing the first p elements of A_i and $0^{[1:q]} = (0,0,\ldots,0) \in \mathbb{R}^q$, then A is a positive operator on $M(p,q)$.

(ii) If matrix A is defined as $A = (e^{tT})^\top$, where $t \in \mathbb{R}$ and T is a matrix with the structure given in (2.22), then A is a positive operator on $M(p,q)$.

(iii) If T is Lyapunov-like on $M(p,q)$, then for any $t \in \mathbb{R}$, the matrix $A = e^{tT}$ is a positive operator on $M(p,q)$.

Proof. (i) We need to prove that for any $n = (y,v) \in M(p,q)$, we have $An \in M(p,q)$. Denote $An = (y^*, v^*)$ where $y^* \in \mathbb{R}^p$ and $v^* \in \mathbb{R}^q$. By using Lemma 2.2.7, we need to show that

$$(An)^\top J(An) \geq 0 \text{ and } \langle y^*, e^k\rangle \geq 0 \text{ for any } k \in (1,2,\ldots,p).$$

We have $\langle y^*, e^k\rangle \geq 0$ if and only if

$$y^* \in (\mathbb{R}^p_{\geq +})^*,$$

which is equivalent to

$$\begin{pmatrix} \langle (A_1^{[1:p]})^\top, y\rangle \\ \langle (A_2^{[1:p]})^\top, y\rangle \\ \vdots \\ \langle (A_p^{[1:p]})^\top, y\rangle \end{pmatrix} \in (\mathbb{R}^p_{\geq +})^* \iff \left(\sum_{i=1}^{j} A_i^{[1:p]}, 0^{[1:q]}\right)^\top \in L(p,q), \qquad \text{for } j = 1,2,\ldots,p.$$

Thus, since we assumed that

$$\left(\sum_{i=1}^{j} A_i^{[1:p]}, 0^{[1:q]}\right)^\top \in L(p,q), \qquad \text{for } j = 1, 2, \ldots, p,$$

from the above chain of equivalences, we get

$$\langle y^*, e^k \rangle \geq 0 \text{ for any } k = 1, 2, \ldots, p.$$

Moreover, since we have also assumed that there exists a $\lambda \geq 0$, such that $A^\top JA - \lambda J$ is positive semi-definite, it follows that for any arbitrary $n \in M(p,q)$, we have

$$n^\top (A^\top JA - \lambda J)n = (An)^\top J(An) - \lambda n^\top Jn \geq 0.$$

Since $n \in M(p,q)$, we have $n^\top Jn \geq 0$, which from the above implies $(An)^\top J(An) \geq \lambda n^\top Jn \geq 0$, hence for any $n \in M(p,q)$ we have $An \in M(p,q)$. Thus, A is a positive operator on $M(p,q)$.

(ii) Since we have shown T is Lyapunov matrix of $L(p,q)$ in Theorem 2.3.1, we have

$$e^{tT} \in Aut(L(p,q)) \text{ for any } t \in \mathbb{R}.$$

Since any automorphism is a positive operator, e^{tT} is a positive operator of $L(p,q)$. By using Proposition 2.2.1, we conclude that if $A = (e^{tT})^\top$, then A is a positive operator of $M(p,q)$.

(iii) If matrix T is Lyapunov-like on $M(p,q)$, then we have

$$e^{tT} \in Aut(M(p,q)) \text{ for any } t \in \mathbb{R}.$$

Thus, if $A = e^{tT}$, then A is automorphism on $M(p,q)$, furthermore, A is a positive operator on $M(p,q)$. $\square$

In Theorem 2.2.6 and Theorem 2.2.8, we have shown a necessary condition and a sufficient condition for a linear operator to be a positive operator on the MESOC, respectively. In the following theorem, we will provide a general result of the necessary and sufficient condition of a linear operator to be a positive operator on the MESOC.

Theorem 2.2.9. *Let $T \in \mathbb{R}^{(p+q)\times(p+q)}$ with*

$$T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

where $A \in \mathbb{R}^{p\times p}$, $D \in \mathbb{R}^{q\times q}$. Then T is a positive operator on L if and only if

$$\sum_{j=1}^{p}(a_{mj} - a_{nj}) \geq 0 \text{ for any } 1 \leq m < n \leq p,\ 1 \leq i, k \leq p,$$

and

$$\sum_{j=1}^{p} a_{ij} \geq \|D\|,$$

where $\|D\|$ denotes the operator norm of D.

Proof. First, we prove the sufficiency. For an arbitrary $(x, u) \in L$, let $(y, v) := T(x, u) = (Ax, Du)$, we will have

$$y_i = \sum_{j=1}^{p} a_{ij}x_j \geq \|u\| \sum_{j=1}^{p} a_{ij} \geq \|D\|\|u\| \geq \|v\|.$$

Moreover, for any $1 \leq m < n \leq p$, we have

$$y_m - y_n = \sum_{j=1}^{p}(a_{mj} - a_{nj})x_j \geq 0$$

Thus, $(y, v) \in L$ and then A is a positive operator on L.

Next, we prove the necessity. Suppose T is a positive operator on L, then for any $(x, u) \in L$

we have $(y, v) = T(x, u) = (Ax, Du) \in L$. Since $(y, v) \in L$, we have

$$y_m - y_n = \sum_{j=1}^{p} (a_{mj} - a_{nj}) x_j \geq 0 \text{ for any } 1 \leq m < n \leq p$$

which in the special case $(x, u) = (e, 0) \in L$ yields

$$\sum_{j=1}^{p} (a_{mj} - a_{nj}) \geq 0 \text{ for any } 1 \leq m < n \leq p,$$

Last, we will show that $\sum_{j=1}^{p} a_{ij} \geq \|D\|$. Suppose to the contrary that $\sum_{j=1}^{p} a_{ij} < \|D\|$. Let u^* be a nonzero vector such that $\|Du^*\| = \|D\|\|u^*\|$, we have $(\|u^*\|e, u^*) \in L$ and then $z = (\|u^*\|Ae, Du^*) \in L$. Note that $z_i = \sum_{j=1}^{p} a_{ij}\|u^*\|$, for $i = 1, 2, \ldots, p$. Then, $z_i < \|D\|\|u^*\| = \|Du^*\|$, which contradicts $z \in L$. Thus, $\sum_{j=1}^{p} a_{ij} \geq \|D\|$. □

After investigating the conditions for a linear operator to be a positive operator on the MESOC, we will study the property of Z-transformation on the MESOC.

Definition 2.2.3. *The set of all Z-transformation on a cone $K \subseteq \mathbb{R}^m$ is given as*

$$Z(K) := \{T \in \mathbb{R}^{m \times m} : \langle Tx, y \leq 0 \rangle \text{ for all } (x, y) \in C(K)\}.$$

Proposition 2.2.10. *For any positive integers p, q, n such that $p + q = n$. Let T be a linear mapping with the following block form:*

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q,$$

where $A \in \mathbb{R}^{p \times p}, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{q \times p}$, and $D \in \mathbb{R}^{q \times q}$. Then T is an arbitrary element in $Z(L)$ if and only

$$T_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

is a Z-transformation of cone L, where L is defined by (2.1)*, and matrix B and C satisfy*

$$\langle Bu, y\rangle + \langle Cx, v\rangle = 0,$$

where $(x, u, y, v) \in C(L)$.

Proof. Take any $(x, u, y, v) \in C(L)$. Suppose $T \in Z(L)$, we have

$$\langle Ax, y\rangle + \langle Bu, y\rangle + \langle Cx, v\rangle + \langle Du, v\rangle \leq 0, \tag{2.12}$$

and

$$\langle Ax, y\rangle - \langle Bu, y\rangle - \langle Cx, v\rangle + \langle Du, v\rangle \leq 0, \tag{2.13}$$

where the latter equation comes from the former one by substituting $-u$ for u and $-v$ for v. By using (2.12) and (2.13), we get

$$\langle Ax, y\rangle + \langle Du, v\rangle \leq 0. \tag{2.14}$$

Which implies that

$$T_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in Z(L).$$

We also conclude that $\langle Bu, y\rangle + \langle Cx, v\rangle = 0$, since if $\langle Bu, y\rangle + \langle Cx, v\rangle \neq 0$, when the equality holds in (2.14), either they will contradict inequality (2.12) or inequality (2.13). Conversely, if $T_1 \in Z(L)$, then for any $(x, u, y, v) \in C(L)$, we have

$$\langle Ax, y\rangle + \langle Du, v\rangle \leq 0.$$

Meanwhile, by using $\langle Bu, y\rangle + \langle Cx, v\rangle = 0$, we have

$$\langle Ax, y\rangle + \langle Bu, y\rangle + \langle Cx, v\rangle + \langle Du, v\rangle \leq 0,$$

and

$$\langle Ax, y\rangle - \langle Bu, y\rangle - \langle Cx, v\rangle + \langle Du, v\rangle \leq 0.$$

These two inequalities above indicate that $T \in Z(L)$.

□

Indeed, the problem of presenting a general formula for the Z-transformation of the MESOC is difficult as we need to know the structure of matrix A and D respectively. We can find that matrix A is a Z-transformation of the monotone nonnegative cone by letting $u = v = (0, \ldots, 0)$. To the best of our knowledge, from the previous literature, no one provides the structure of the Z-transformation of the monotone nonnegative cone. Thus, in order to find the structure of the Z-transformation of MESOC, we need to use some techniques. Recall Proposition 2.2.5, since $L \in \mathbb{R}^n$ is a proper cone, then the Z-transformation of L can be obtained by $Z(L) = \mathrm{cl}\,(\mathrm{LL}(L) - \Gamma(L))$, where $\mathrm{LL}(L)$ is the Lyapunov-like transformation of L and $\Gamma(L)$ is the positive operator of L, respectively. We already found the necessary conditions for a linear mapping to be a positive operator on the MESOC for a block diagonal case in Theorem 2.2.9. Thus, if we can find the necessary and sufficient condition for a liner operator to be a Lyapunov-like transformation of L, then by using the relationship of the Z-transformation given above, we can summarise the condition that a linear mapping to be a Z-transformation on the MESOC. In the next section we investigate the formula of Lyapunov-like transformation on the MESOC as well as find the Lyapunov rank of MESOC.

2.3 Lyapunov-like transformation and Lyapunov rank of MESOC

It is well known that the complementarity set illustrates the connection between the optimal conditions of conic programming and the complementarity theory. Rudolf introduced the definition of *bilinearity rank* in [71] and emphasised the importance of the value of bilinearity rank. The motivation why he was interested in investigating the value of bilinearity rank is that he wanted to build a system of n or more independent equations $\langle F_i(u), v\rangle$ by using every single equation $\langle u, v\rangle$ in the complementarity set $C(K)$, for a closed and convex cone $K \in \mathbb{R}^n$. The value of bilinearity rank could be used to quantify the possibility of the solubility of the system he build by using the existing algorithm. Recall that A matrix $A \in \mathbb{R}^{n\times n}$ is called *Lyapunov-like transformation* (or *Lyapunov-like matrix*) on K, if

$$\langle Ax, y\rangle = 0, \ \forall (x, y) \in C(K). \tag{2.15}$$

The relationship between the bilinearity rank of a cone K and its Lyapunov-like transformations was figured out by Gowda and Tao in [41]. They demonstrated that the value of the dimension of the space of all Lyapunov-like transformations (matrices) on K is indeed the value of bilinearity rank of K. That is, if we define a vector space LL(K) as the set of all Lyapunov-like transformations (matrices) on K and denote its dimension as $\beta(K)$, then the value of $\beta(K)$ is called as the *Lyapunov rank* (or *bilinearity rank*) of K. They also found the connection between Lie algebra and Lyapunov-like transformations. Previous research inspired us to find the Lyapunov rank of MESOC. Meanwhile, Sznajder [79] found the value of the Lyapunov rank of the extended second order cone. We are also interested in finding the similarities as well as differences between MESOC and ESOC. In order to find the value of the Lyapunov rank of MESOC, we will introduce the following lemma, which will help us calculate the value of the Lyapunov rank of MESOC.

Lemma 2.3.1. *Let $A \in \mathbb{R}^{p\times p}$, where p is a positive integer. Then, $A \in \mathrm{LL}(\mathbb{R}^{\mathrm{p}}_{\geq +})$ if and* *only if it is of the form*

$$A = \begin{bmatrix} a - \sum_{i=2}^{p} a_i & a_2 & a_3 & \cdots & \cdots & a_p \\ & a - \sum_{i=3}^{p} a_i & a_3 & \cdots & \cdots & a_p \\ & & a - \sum_{i=4}^{p} a_i & \cdots & \cdots & a_p \\ & & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & & a - a_p & a_p \\ & & & & & a \end{bmatrix}, \tag{2.16}$$

where $a, a_2, a_3 \ldots, a_p \in \mathbb{R}$ are arbitrary.

Proof. For $1 \leq i \leq p$, denote $e^i \in \mathbb{R}^p$ be the canonical unit vectors in $\mathbb{R}^p$ and e^{p+1} be the zero vector in $\mathbb{R}^p$. Let $u^i := \sum_{k=1}^{i} e^k \in \mathbb{R}^n_{\geq +}$ and $v^i := e^i - e^{i+1} \in (\mathbb{R}^n_{\geq +})^*$, for $1 \leq i \leq p$ (see Example 1.3.2). Then, $\langle u^i, v^j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, that is, $\delta_{ii} = 1$ and $\delta_{ij} = 0$, for $i \neq j$. When $i \neq j$, we will have $(u^i, v^j) \in C(\mathbb{R}^n_{\geq +})$, (as it can be seen from Example 1.3.2, too). Hence, if $A \in \mathrm{LL}(\mathbb{R}^n_{\geq +})$ and $i \neq j$, then

$$\langle Au^i, v^j \rangle = \sum_{k=1}^{i} (a_{jk} - a_{j+1,k}) = 0, \tag{2.17}$$

where we set $a_{p+1,k} := 0$. By using equation (2.17), we get

$$\sum_{\ell=j}^{p} \langle Au^i, v^\ell \rangle = \sum_{k=1}^{i} a_{jk} = 0, \quad \text{if} \quad j > i. \tag{2.18}$$

By equation (2.18) we get

$$a_{ji} = \sum_{k=1}^{i} a_{jk} - \sum_{k=1}^{i-1} a_{jk} = 0, \quad \text{if} \quad j > i. \tag{2.19}$$

By using again equation (2.17), we get

$$a_{ji} - a_{j+1,i} = \sum_{k=1}^{i} (a_{jk} - a_{j+1,k}) - \sum_{k=1}^{i-1} (a_{jk} - a_{j+1,k}) = 0, \quad \text{if} \quad j + 1 < i. \tag{2.20}$$

Thus, by using Equations (2.17), (2.19) and (2.20), we conclude that A is of the form (2.16). Conversely, suppose that A is of the form (2.16). Since for any element $(x, y) \in C(\mathbb{R}^n_{\geq +})$, by using Example 1.3.2, we have

$$(x, y) = \left(\sum_{i \in I} \alpha_i u^i, \sum_{j \in J} \beta_j v^j \right); \quad \alpha_i, \beta_j \geq 0, \tag{2.21}$$

for some $I, J \subseteq \{1, 2, \ldots, n\}$ with $I \cup J = \{1, 2, \ldots, n\}$ and $I \cap J = \varnothing$, because $\{u^i : 1 \leq i \leq j\} \subseteq \mathbb{R}^n_{\geq +}$ and $\{v^i : 1 \leq i \leq j\} \subseteq (\mathbb{R}^n_{\geq +})^*$ are generators of the simplicial cones $\mathbb{R}^n_{\geq +}$ and $(\mathbb{R}^n_{\geq +})^*$, respectively, and $x \perp y$. As $\langle Au^i, v^j \rangle = 0$, by considering the derivation of equations (2.17), (2.18), (2.19) and (2.20) above in the reverse order, equation (2.21) implies that $\langle Ax, y \rangle = 0$. Hence, $A \in \mathrm{LL}(\mathbb{R}^n_{\geq +})$. □

Then, we will show the value of the Lyapunov rank of the MESOC.

Theorem 2.3.2. *For the monotone extended second order cone* (2.1)*, any Lyapunov-like transformation T is of the form*

$$T = \left[\begin{array}{cccccc|ccc}
a - \sum_{j=2}^p a_j & a_2 & a_3 & \cdots & \cdots & a_p & c_1 & \cdots & c_q \\
 & a - \sum_{j=3}^p a_j & a_3 & \cdots & \cdots & a_p & c_1 & \cdots & c_q \\
 & & a - \sum_{j=4}^p a_j & \cdots & \cdots & a_p & c_1 & \cdots & c_q \\
 & & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & \mathbf{0} & & & a - a_p & a_p & c_1 & \cdots & c_q \\
 & & & & & a & c_1 & \cdots & c_q \\
\hline
 & & & & & c_1 & a & & * \\
 & \mathbf{0} & & & & \vdots & & \ddots & \\
 & & & & & c_q & -* & & a
\end{array}\right], \tag{2.22}$$

where $a, a_2, a_3, \ldots, a_p, c_1, \ldots, c_q \in \mathbb{R}$ are arbitrary. Hence, its Lyapunov rank is given by

$$\beta(L) = p + \frac{q(q+1)}{2}.$$

Proof. Recall that the complementarity set for the monotone extended second order cone

L is

$$C(L) = \{((x,u),(y,v)) \in L \times M : (x,u) \perp (y,v)\}.$$

We partition the above set in the following way:

$$C(L) := C_1(L) \cup C_2(L) \cup C_3(L) \cup C_4(L),$$

where

$$\begin{aligned}
C_1(L) &:= \{(x,0,y,0) \in C(L)\},\\
C_2(L) &:= \{(x,0,y,v) \in C(L) : v \neq 0\},\\
C_3(L) &:= \{(x,u,y,v) \in C(L) : u \neq 0 \neq v\},\\
C_4(L) &:= \{(x,u,y,0) \in C(L) : u \neq 0\}.
\end{aligned}$$

Since $x = 0 \Rightarrow u = 0$ and $y = 0 \Rightarrow v = 0$, for any Lyapunov-like transformation on L we only need to consider the case of $x \neq 0 \neq y$. Let T be any element of LL(L), so it has the following block form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q,$$

where $A \in \mathbb{R}^{p\times p}, B \in \mathbb{R}^{p\times q}, C \in \mathbb{R}^{q\times p}$, and $D \in \mathbb{R}^{q\times q}$. Take any $(x,u,y,v) \in C(L)$. Then (2.15) implies

$$\langle Ax, y\rangle + \langle Bu, y\rangle + \langle Cx, v\rangle + \langle Du, v\rangle = 0,$$

$$\langle Ax, y\rangle - \langle Bu, y\rangle - \langle Cx, v\rangle + \langle Du, v\rangle = 0,$$

where the latter equation comes from the former one by substituting $-u$ for u and $-v$ for v. By adding and subtracting the above equations, we get

$$\begin{aligned}
\langle Ax, y\rangle + \langle Du, v\rangle &= 0,\\
\langle Bu, y\rangle + \langle Cx, v\rangle &= 0.
\end{aligned} \tag{2.23}$$

By using an element $(x, 0, y, 0) \in L \times M$ in $C_1(L)$, with $x \in \mathbb{R}^p_{\geq +}$ and $y \in (\mathbb{R}^p_{\geq +})^*$, we get $\langle Ax, y \rangle = 0$, which implies that $A \in \mathrm{LL}(\mathbb{R}^{\mathrm{p}}_{\geq +})$.

Now, we will determine the structures of matrices B and C. By using elements in $C_2(L)$, from the second equation in (2.23), we get

$$\langle Cx, v \rangle = \langle B0, y \rangle + \langle Cx, v \rangle = 0.$$

Suppose that $Ca^i \neq 0$ for some $i < p$ and let $v := \frac{Ca^i}{\|Ca^i\|}$, and $y := e^j, (j > i)$, thus, $\langle y, e^j \rangle = 1 = \|v\|$. Hence, $(a^i, 0, e^j, v) \in C_2(L)$. Then $0 = \langle Cx, v \rangle = \langle Ca^i, v \rangle = \|Ca^i\|$, which leads to a contradiction. Hence, $Ca^i = 0$. Then, for certain $c_1, \ldots, c_q \in \mathbb{R}$ we have

$$C = \left[\begin{array}{c|c} & c_1 \\ \mathbf{0} & \vdots \\ & c_q \end{array}\right]_{q \times p}.$$

If $C = 0$, the second equation in (2.23) demonstrates that $\langle Bu, y \rangle = 0$ for all $(x, u, y, v) \in C_3(L)$. It is easy to verify that $(e, -v, e^i, v) \in C_3(L)$, where v is an arbitrary unit vector in $\mathbb{R}^q$. Hence, $\langle B(-v), e^i \rangle = 0$, for all $1 \leq i \leq p$, thus $Bv = 0$. In consequence, $B = 0$.

If $C \neq 0$, first we need to find the structure of matrix B. We have $\langle Bu, y \rangle = 0$ for any $(x, u, y, 0) \in C_4(L)$.

Let u^i denote the standard (canonical) unit vector in $\mathbb{R}^q$ and for any $n > m$, let $y^{m,n} := e^m - e^n \in \mathbb{R}^p$. Since $(e, u^i, y^{m,n}, 0) \in C_4(L)$,

$$\langle Bu^i, y^{m,n} \rangle = 0.$$

Therefore,

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_q \\ \vdots & \vdots & & \vdots \\ b_1 & b_2 & \cdots & b_q \end{bmatrix}_{p \times q}.$$

For $i = 1, \ldots, q$ and $j = 1, \ldots, p$, we have $(e, u^i, e^j, -u^i) \in C_3(L)$ and subsequently,

$$\langle Bu^i, e^j \rangle + \langle Ce, -u^i \rangle = 0.$$

It readily implies $b_i = c_i$. Hence,

$$B = \begin{bmatrix} c_1 & c_2 & \cdots & c_q \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_q \end{bmatrix}_{p\times q}.$$

As $(e, u, \frac{1}{p}e, -u) \in C_3(L)$ for all u with $\|u\| = 1$, by using (2.23), we have

$$\left\langle Ae, \frac{1}{p}e \right\rangle + \langle Du, -u \rangle = 0. \tag{2.24}$$

Let $a := \dfrac{\langle Ae, e\rangle}{p}$. Then (2.24) implies

$$\left\langle \left(\frac{D + D^T}{2} - aI\right) u, u \right\rangle = 0,$$

and hence

$$D + D^T = 2aI. \tag{2.25}$$

Obviously, $(e, -u^1, e^1, u^1) \in C(L)$ and using the first equation in (2.23) gives

$$\langle Ae, e^1 \rangle - \langle Du^1, u^1 \rangle = 0,$$

which implies that $d_{11} = \sum_j a_{1j}$. Thus, (2.25) implies that $d_{11} = a$ and hence, $\sum_{j=1}^p a_{1j} = a$.

By changing e^1 to e^2 (yes, we can), we have $\sum_{j=2}^p a_{2j} = d_{22} = a$. By following this process, we obtain that $d_{ii} = \sum_{j=1}^p a_{ij} = a$, for all $1 \le i \le p$.

Therefore, by equation (2.25), $A \in \mathrm{LL}(\mathbb{R}^{\mathrm{n}}_{\geq+})$ (shown above) and Lemma 2.3.1, any

Lyapunov-like transformation on L has the form (2.22).

Now, we want to show that any transformation T, which can be represented in the form (2.22), is Lyapunov-like on L, so let T be given as above. Then we have

$$\langle T(x,u),(y,v)\rangle = \langle Ax,y\rangle + \langle Du,v\rangle + \langle Bu,y\rangle + \langle Cx,v\rangle. \tag{2.26}$$

We wish to show that for any $(x,u,y,v) \in C(L)$, the RHS in the above equation is zero. We will perform a case-by-case analysis.

Case 1. For any $(x,u,y,v) := (x,0,y,0) \in C_1(L)$, the RHS of (2.26) is equal to zero, as $(x,y) \in \mathcal{C}\left(\mathbb{R}^n_{\geq +}\right)$ and we have already shown that $A \in \mathrm{LL}(\mathbb{R}^{\mathrm{n}}_{\geq +})$, hence it is enough to use Lemma 2.3.1 again.

Case 2. For any $(x,u,y,v) := (x,0,y,v) \in C_2(L)$, the RHS of (2.26) is $(c_1v_1+...+c_qv_q)x_p$. Suppose that $x_p \neq 0$. Then, since $(x,y) \in C\left(\mathbb{R}^n_{\geq +}\right)$, from Example 1.3.2 we get $y_1 + \cdots + y_p = 0$. Hence, $(y,v) \in M$ and (2.2) implies $v = 0$, which contradicts $(x,0,y,v) \in C_2(L)$. Thus, $x_p = 0$ and, therefore, the RHS of (2.26) is zero.

Case 3. Take an arbitrary $(x,u,y,v) \in C_3(L)$. Proposition 2.1.6 indicates that for some $\lambda > 0$ one has $v = -\lambda u$, thus

$$\begin{aligned}
\langle Ax,y\rangle + \langle Du,v\rangle &= \langle Ax,y\rangle + \left\langle \frac{D+D^T}{2}u, v\right\rangle \\
&= \langle Ax,y\rangle + a\langle u,v\rangle \\
&= \langle z,y\rangle + a\langle u,v\rangle \\
&= \sum_{i=1}^{p-1}\left[(z_i - z_{i+1})\sum_{j=1}^{i} y^j\right] + z_p\sum_{i=1}^{p} y^i + a\langle u,v\rangle,
\end{aligned} \tag{2.27}$$

where $z_i := \sum_{j=1}^{i} a_j x_i + \sum_{k=i+1}^{p} a_k x_k$, for any $1 \leq i \leq p-1$ and $z_p = \sum_{k=1}^{p} a_k x_p$. Then for any $1 \leq i \leq p-1$, it is easy check that $z_i - z_{i+1} = \sum_{j=1}^{i} a_j(x_i - x_{i+1})$. By inserting these equalities and the formula for z_p into equation (2.27), and by using Proposition 2.1.6, we obtain $\langle Ax, y\rangle + \langle Du, v\rangle = 0$. We will show that

$$\langle Bu, y\rangle + \langle Cx, v\rangle = 0.$$

By using the full power of Proposition 2.1.6, including $v = -\lambda u$ for some $\lambda > 0$, we have

$$\langle Bu, y\rangle = \sum_{i=1}^{q}(c_i u_i) \cdot \sum_{i=1}^{p} y_i = \|v\| \sum_{i=1}^{q}(c_i u_i)$$

and

$$\langle Cx, v\rangle = x_p \sum_{i=1}^{q}(c_i v_i) = \|u\| \sum_{i=1}^{q}(c_i v_i).$$

Then

$$\begin{aligned}\langle Bu, y\rangle + \langle Cx, v\rangle &= \|v\| \sum_{i=1}^{q}(c_i u_i) + \|u\| \sum_{i=1}^{q}(c_i v_i)\\ &= \lambda\|u\| \sum_{i=1}^{q}(c_i u_i) - \lambda\|u\| \sum_{i=1}^{q}(c_i u_i)\\ &= 0.\end{aligned}$$

Case 4. For any $(x, y, u, v) := (x, u, y, 0) \in C_4(L)$, the RHS of (2.26) is $(c_1 u_1 + ... + c_q u_q)(y_1 + ... + y_q)$. Suppose that $y_1 + \cdots + y_p \neq 0$. Then, since $(x, y) \in C(\mathbb{R}^n_{\geq+})$, from Example 1.3.2 we get $x_p = 0$. Hence, $(x, u) \in M$ and (2.2) implies $u = 0$, which contradicts $(x, u, y, 0) \in C_4(L)$. Thus, $y_1 + \cdots + y_p = 0$ and therefore, the RHS of (2.26) is zero.

In conclusion, the RHS of (2.26) is zero for any $(x, u, y, v) \in C_1(L) \cup C_2(L) \cup C_3(L) \cup C_4(L) = C(L)$ Therefore, $T \in$ LL(L). Following the definition of the Lyapunov rank, its value for the cone L equals the number of independent parameters in (2.22), which is $p + \frac{q(q+1)}{2}$. □

Moreover, Orlitzky and Gowda [68] present that a cone $K \in \mathbb{R}^n$ is numerical good if $\beta(K) \geq n$, since the complementarity set of cone K can be formulated by using n linearly-independent Lyapunov-like matrices when $\beta(K) \geq n$ holds.

In our case, $\beta(L) = p + \frac{q(q+1)}{2} \geq n = p + q$ will always hold when $q \geq 1$. Thus, we conclude that the monotone extended second order cone is a numerically good cone. Meanwhile, Orlitzky and Gowda also introduced the upper bound of $\beta(K)$ equals to $(n-1)^2$ when $K \in \mathbb{R}^n$. Moreover, in [67], Orlitzky showed that the upper bound of $\beta(K)$ equals to $\frac{n^2-n}{2}+1$ when $K \in \mathbb{R}^n$ is a proper cone. Considering the Lyapunov rank of MESOC, we found the upper bound is tight for $L \in \mathbb{R}^{1+2}$. Since in this case, $\beta(L) = p + \frac{q(q+1)}{2} = 1 + \frac{2(2+1)}{2} = 4 = (3-1)^2 = (p+q-1)^2$ and $\beta(L) = p + \frac{q(q+1)}{2} = 1 + \frac{2(2+1)}{2} = 4 = \frac{3^2-3}{2} + 1 = \frac{(p+q)^2-(p+q)}{2} + 1$. It is easy to find this is the only case for the MESOC, such that its Lyapunov rank is tight, for any other case for (p, q) when $p > 1$, the upper-bound is not tight for $L \in \mathbb{R}^{p+q}$.

2.4 Conclusions and comments

In this chapter, we study the properties of the monotone extended second order cone (MESOC). This cone is different from both the traditional Lorentz cone (second order cone), since the Lorentz cone is self-dual while the MESOC is not, and the previously introduced extended second-order cone, as the MESOC is reducible while the extended second order cone is irreducible [79]. We showed that the MESOC is a proper cone in Proposition 2.1.1 and found the formula of the dual cone of the MESOC and the complementarity set of MESOC in Proposition 2.1.3 and Proposition 2.1.6, respectively. Furthermore, we have developed some conditions for a positive operator on the MESOC. In Theorem 2.2.9, we illustrated a necessary and sufficient condition for a linear mapping with a block diagonal form to be a linear operator on the MESOC. We also demonstrated that the Z-transformation formula can be found using Proposition 2.2.5, which provided a relationship between the positive operator, Z-transformation and Lyapunov-like transformation on a proper cone. Finally, the Lyapunov rank (bilinearity rank) of a monotone extended second order cone defined as $L \in \mathbb{R}^{p+q}$ has been determined in Theorem 2.3.2, which is $\beta(L) = p + \frac{q(q+1)}{2}$. The formula for a linear mapping to be a Lyapunov-like transformation of MESOC is also provided in this theorem. We also concluded that the monotone extended second order cone is a numerically good cone.

CHAPTER 3

ISOTONICITY PROPERTY AND MIXED COMPLEMENTARITY PROBLEM

As we mentioned in the introduction, Isac and A.B.Németh in [44] have shown that the isotone projection cones have advantages in solving the corresponding complementarity problems since the ordering defined by cones in the iterative method is very useful. In this chapter, we demonstrate an isotonicity property of the monotone extended second order cone. Then, we use this isotonicity property of MESOC to solve a general mixed complementarity problem.

3.1 Isotonicity property of MESOC

In this chapter, we call a closed convex set C as an isotone projection set with respect to an arbitrary proper cone K if the projection onto the set C is isotone with respect to the order defined by the cone K. In order to derive the isotonicity property of MESOC, we will introduce some fundamental concepts and results related to the isotone projection cone first.

Definition 3.1.1. *A closed and convex cone K is an isotone projection cone if for arbitrary two vectors $u, v \in K$, the following implication holds:*

$$u \leq_K v \Rightarrow P_K(u) \leq_K P_K(v).$$

Theorem 3.1.1. *(see [58]) The closed convex set $C \subset \mathbb{R}^m$ with nonempty interior is a K-isotone projection set if and only if it is of the form*

$$C = \bigcap_{i \in \mathbb{N}} \mathcal{H}y_-(u^i, a^i),$$

where each affine hyperplane $\mathcal{H}y(u^i, a^i)$ is tangent to C and it is a K-isotone projection set.

The following two lemmas are from [62].

Lemma 3.1.2. *Let $K \subset \mathbb{R}^m$ be a closed convex cone and $\mathcal{H}y \subset \mathbb{R}^m$ be a hyperplane with a unit normal vector $a \in \mathbb{R}^m$. Then, $\mathcal{H}y$ is a K-isotone projection set if and only if*

$$\langle x, y \rangle \geq \langle a, x \rangle \langle a, y \rangle,$$

for any $x \in K$ and $y \in K^*$.

Lemma 3.1.3. *Let $z \in \mathbb{R}^m$, $K \subset \mathbb{R}^m$ be a closed convex cone, and $C \subset \mathbb{R}^m$ be a nonempty closed convex set. Then, C is a K-isotone projection set if and only if $C+z$ is a K-isotone projection set.*

Finally, by using the above three results, we derive an isotonicity property of MESOC, which we will use to solve complementarity problems on this cone.

Theorem 3.1.4. *Let L be the MESOC defined in $\mathbb{R}^n$ corresponding to the dimensions p and q, with $q > 1$ and $n = p + q$. The closed convex set with nonempty interior $K \subseteq \mathbb{R}^p \times \mathbb{R}^q$ is an L-isotone projection set if and only if $K = \mathbb{R}^p \times C$, for some closed convex set $C \subseteq \mathbb{R}^q$ with nonempty interior.*

Proof. First, suppose that $K = \mathbb{R}^p \times C$, where $C \subseteq \mathbb{R}^q$ is a nonempty closed convex set with nonempty interior. Let $(x, u), (y, v) \in \mathbb{R}^p \times \mathbb{R}^q$ be such that $(x, u) \leq_L (y, v)$, thus $(y - x, v - u) \in L$, i.e.,

$$y_1 - x_1 \geq y_2 - x_2 \geq \cdots \geq y_p - x_p \geq \|v - u\|. \tag{3.1}$$

Since C is a closed and convex set in $\mathbb{R}^q$, by the nonexpansivity (1.1) of P_C, we have

$$\|v - u\| \geq \|P_C v - P_C u\|,$$

which together with (3.1) yields

$$y_1 - x_1 \geq y_2 - x_2 \geq \cdots \geq y_p - x_p \geq \|P_C v - P_C u\|.$$

Thus,

$$(y, P_C v) - (x, P_C u) \in L$$

and therefore, we have

$$P_K(x, u) = (x, P_C u) \leq_L (y, P_C v) = P_K(y, v).$$

In conclusion, K is an L-isotone project set.

Conversely, suppose that the closed convex set $K \subseteq \mathbb{R}^p \times \mathbb{R}^q$ with nonempty interior is an L-isotone project set. If $p = 1$, then in [58], it has been proved that $K = \mathbb{R}^p \times C$, where C is a nonempty, closed and convex subset with nonempty interior of $\mathbb{R}^q$. Therefore, assume that $p > 1$. By Theorem 3.1.1 and Lemma 3.1.3, we need to show that for any tangent hyperplane $\mathcal{H}y$ of K with unit normal $\gamma = (a, u)$, we have $a = 0$. From Lemma 3.1.2, we have

$$\langle \zeta, \xi \rangle \geq \langle \gamma, \zeta \rangle \langle \gamma, \xi \rangle, \tag{3.2}$$

for any $\zeta := (x, v) \in L$ and $\xi := (y, w) \in L^*$. By Lemma 3.1.2, condition (3.2) holds. Let $x \in \mathbb{R}^p_+$ and $v \in \mathbb{R}^q$. Then, by using equation (2.1), and Proposition 2.1.3, obviously, we will have $\zeta := (\|v\|e, v) \in L$, $\xi := (\|v\|x, -\langle e, x\rangle v) \in L^*$ and $\langle \zeta, \xi \rangle = 0$. Then, by using condition (3.2), we have

$$0 \geq (\langle a, e\rangle \|v\| + \langle u, v\rangle)(\langle a, x\rangle \|v\| - \langle e, x\rangle \langle u, v\rangle). \tag{3.3}$$

In the inequality (3.3), suppose that $v \neq 0$ such that $\langle u, v\rangle = 0$, let $x = e$, then $0 \geq \langle a, e\rangle^2 \|v\|^2$, and we will have $\langle a, e\rangle = 0$. Thus, the inequality (3.3) will be equivalent as

$$0 \geq \langle u, v\rangle(\langle a, x\rangle \|v\| - \langle e, x\rangle\langle u, v\rangle). \tag{3.4}$$

First, let us consider the case when $u \neq 0$. Let $v^k \in \mathbb{R}^q$ be a sequence of points such that $\|v^k\| = 1$, $\langle u, v^k\rangle > 0$ and $\lim_{n\to+\infty}\langle u, v^k\rangle = 0$. Denote an arbitrary positive integer by n. In the inequality (3.4), if we choose $\lambda > 0$, let λ sufficiently large and $x := a + \lambda e \geq 0$ as well as $v = v^k$, we will have

$$(0 \geq \langle u, v^k\rangle(\|a\|^2 - \lambda p\langle u, v^k\rangle),) \text{ or equivalently } \|a\|^2 \leq \lambda p\langle u, v^k\rangle.$$

Then, in the last inequality, let $k \to +\infty$, we will have

$$\|a\|^2 \leq 0, \text{ or equivalently } a = 0.$$

Next, let us consider the case when $u = 0$. Since (a, u) is a unit vector, it follows that $a \neq 0$. Let $(x, y) \in C(\mathbb{R}^p_{\geq +})$ and $w \in \mathbb{R}^q$ such that $\langle y, e\rangle \geq \|w\|$. Then, by using the definition of MESOC in(2.1) and Proposition 2.1.3, we will have $\zeta := (x, 0) \in L$, $\xi := (y, w) \in L^*$ and $\langle \zeta, \xi\rangle = 0$. Hence, by using the inequality (3.2), we have

$$0 \geq \langle a, x\rangle\langle a, y\rangle,$$

for any $(x, y) \in C(\mathbb{R}^p_{\geq +})$ with $\langle x, y\rangle = 0$. From Example 1.3.2, we can choose $x = e^1 + \cdots + e^r$ and $y = e^s - e^{s+1}$, where $r, s \in \{1, \ldots, p\}$, and we set $e^{p+1} := 0$. Hence, $(a_1 + \cdots + a_r)(a_s - a_{s+1}) \leq 0$, where we set $a_{p+1} := 0$. Take now $r = 1$ and for $s = 1, \ldots, p$, add the inequalities $a_1(a_1 - a_2) \leq 0, \ldots, a_1(a_p - a_{p+1}) \leq 0$, to obtain (by the telescoping effect) $a_1 \cdot a_1 \leq 0$, which gives $a_1 = 0$. Similarly, for $r = 2$ and $s = 2, \ldots, p$, add the inequalities $(0 + a_2)(a_2 - a_3) \leq 0, \ldots, a_2(a_p - a_{p+1}) \leq 0$, to get $a_2 = 0$. Acting similarly (with $r = 3$, and so on), we get $a_3 = 0$, up to $a_p = 0$. Thus, $a = 0$. But this contradicts $a \neq 0$, so the case $u = 0$ cannot hold. □

3.2 Mixed complementarity problem

Facchinei and Pang defined the mixed complementarity problem (MiCP) on the non-negative orthant (see Subsection 9.4.2 in [28]). It is not only equivalent to a linearly constrained *variational inequality problem* (this relationship is also known as the *Karush-Kuhn-Tucker (KKT)* system of the variational inequality) but it can also be viewed as an NCP for a particular non-pointed cone. Németh and Zhang considered the MiCP defined on an arbitrary closed and convex cone. In Proposition 3.1.4, we have already shown that the projection mapping onto a cylinder is an isotonic projection set with respect to MESOC. Then, the importance of solving the mixed complementarity problem, which has been pointed out in the previous research, motivated us to consider using the isotonicity on MESOC we have obtained as a tool to solve the MiCP.

Let us first introduce the fundamental background which we have used in solving the mixed complementarity problem.

It is well-known that for the nonlinear complementarity problem $\mathrm{NCP}(F, K)$, x^* is its solution if and only if x^* is a fixed point of the mapping $K \ni x \mapsto P_K(x - F(x))$. For an arbitrary sequence $\{x^n\}$ generated by the fixed point iteration process

$$x^{n+1} = P_K(x^n - F(x^n)), \tag{3.5}$$

if the mapping F is continuous and the sequence $\{x^n\}$ is convergent to $x^* \in K$, then x^* is a fixed point of the mapping $K \ni x \mapsto P_K(x - F(x))$, hence x^* is a solution of the nonlinear complementarity problem $\mathrm{NCP}(F, K)$.

The nonlinear complementarity problem defined on a general closed and convex cone considered by Németh and Zhang is defined in the following lemma.

Lemma 3.2.1. *(Lemma 4 in [63]) Let* $K = \mathbb{R}^p \times C$*, where* C *is an arbitrary nonempty closed and convex cone in* $\mathbb{R}^q$*. Denote mapping* $G : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^p$*, mapping* $H : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^q$ *and mapping* $F = (G; H) : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^p \times \mathbb{R}^q$*. Then the nonlinear complementarity problem* $\mathrm{NCP}(F, K)$ *is equivalent to the mixed complementarity problem* $\mathrm{MiCP}(G, H, C, p, q)$ *defined as*

$$G(x, u) = 0, \;\; C \ni u \perp H(x, u) \in C^*.$$

Proof. It is standard and follows from the definition of the nonlinear complementarity problem $\mathrm{NCP}(F, K)$, by noting that $K^* = \{0\} \times C^*$. □

Thus, by using the notations of Lemma 3.2.1, the fixed point iteration (3.5) is equivalent to:

$$\begin{cases} x^{n+1} = x^n - G(x^n, u^n), \\ u^{n+1} = P_C(u^n - H(x^n, u^n)). \end{cases} \tag{3.6}$$

Meanwhile, Németh and Zhang also present sufficient conditions for the solvability of this problem:

Proposition 3.2.2. *(Proposition 2 in [63]) Let* $L \subseteq \mathbb{R}^m$ *be a pointed closed convex cone,* $K \subseteq \mathbb{R}^m$ *be a closed convex cone and* $F : \mathbb{R}^m \to \mathbb{R}^m$ *be a continuous mapping. Consider sequence* $\{x^n\}_{n\in\mathbb{N}}$ *which is defined by using (3.5).*
Assume that the mappings P_K *and* $I - F$ *are* $L-isotone$*, and* $x^0 = 0 \leq_L x^1$*. Let*

$$\Omega := K \cap L \cap F^{-1}(L) = \{x \in K \cap L : F(x) \in L\}$$

and

$$\Gamma := \{x \in K \cap L : P_K(x - F(x)) \leq_L x\}.$$

Then $\varnothing \neq \Omega \subset \Gamma$ *and the sequence* $\{x^n\}$ *is convergent to* x^**, which is a solution of* $\mathrm{NCP}(F, K)$*. Moreover,* x^* *is a lower L-bound of* Ω *and the L-least element of* Γ*.*

Thus, by using the previous lemma and proposition, we obtain the following theorem, which provides sufficient conditions for the solvability of $\mathrm{MiCP}(G, H, C, p, q)$ by using MESOC.

Theorem 3.2.3. *Let L be the monotone extended second order cone corresponding to p and q. For an arbitrary cone $K = \mathbb{R}^p \times C$, where C be a closed convex cone, denote its dual cone by K^*. Let $F = (G; H) : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^p \times \mathbb{R}^q$, such that $I - F$ is L-isotone, where I denotes the identical mapping, $G : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^p$ and $H : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^q$ are two continuous mappings. Consider a sequence $\{(x^n, u^n\}_{n\in\mathbb{N}}$ defined by (3.6), where $x^0 = 0 \in \mathbb{R}^p$ and $u^0 = 0 \in \mathbb{R}^q$. Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Suppose that the system of inequalities*

$$y_i - x_i \geq y_{i+1} - x_{i+1} \geq \|v - u\|; \quad 1 \leq i \leq p-1$$

implies the system of inequalities

$$\begin{aligned} y_i - x_i - (G(y,v)_i - G(x,u)_i) &\geq y_{i+1} - x_{i+1} - (G(y,v)_{i+1} - G(x,u)_{i+1}) \\ &\geq \|v - u - (H(y,v) - H(x,u))\|; \end{aligned}$$

$1 \leq i \leq p-1$, and that $x_i^1 \geq x_{i+1}^1 \geq \|u^1\|$; $1 \leq i \leq p-1$ (in particular, this holds when $-G(0,0)_i \geq -G(0,0)_{i+1} \geq \|H(0,0)\|$; $1 \leq i \leq p-1$). Let

$$\Omega := \{(x,u) \in \mathbb{R}^p \times C : x_1 \geq \cdots \geq x_p \geq \|u\|, G(x,u)_1 \geq \cdots \geq G(x,u)_p \geq \|H(x,u)\|\}$$

and

$$\begin{aligned} \Gamma := \{(x,u) \in \mathbb{R}^p \times C : x_1 \geq \cdots \geq x_p \geq \|u\|, G(x,u)_1 \geq \cdots \geq G(x,u)_p \\ \geq \|u - P_C(u - H(x,u))\|\}. \end{aligned}$$

Then $\varnothing \neq \Omega \subseteq \Gamma$, the sequence $\{(x^n, u^n)\}$ is convergent, and its limit (x^*, u^*) is a solution of $\mathrm{MiCP}(G, H, C, p, q)$. Moreover, (x^*, y^*) is also a lower L-bound of Ω as well as is the L-least element of Γ.

3.3 Numerical example

In this section, we will give a numerical example to show how to solve the mixed complementarity problem by using the isotonicity property of MESOC.

Let L be the monotone extended second order cone, then suppose that $K = \mathbb{R}^2 \times C$ where $C = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 \geq u_2 \geq 0\}$. Let $f_1(x, u) = \frac{1}{10}x_1 - \frac{1}{20}x_2 + \frac{1}{20}\|u\| + 1$ and $f_2(x, u) = \frac{1}{5}x_1 - \frac{3}{20}x_2 + \frac{1}{20}\|u\| - \frac{3}{5}$. Obviously, $f_1(x, u)$ and $f_2(x, u)$ are L-monotone. Define $\omega^1 := (2, 1, \frac{1}{3}, \frac{1}{6})$ and $\omega^2 := (2, 1, \frac{1}{6}, \frac{1}{3})$; it is easy to find out that $\omega^1, \omega^2 \in L$. Then, for two arbitrary vectors $(x, u), (y, v) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $(x, u) \leq_L (y, v)$, by using the definition of the MESOC, we have that $y_1 - x_1 \geq y_2 - x_2 \geq \|v - u\| \geq \|u\| - \|v\|$. Hence,

$$f_1(y, v) - f_1(x, u) = \frac{1}{10}(y_1 - x_1) - \frac{1}{20}(y_2 - x_2) - \frac{1}{20}(\|u\| - \|v\|) \geq 0,$$

$$f_2(y, v) - f_2(x, u) = \frac{1}{5}(y_1 - x_1) - \frac{3}{20}(x_2 - y_2) - \frac{1}{20}(\|u\| - \|v\|) \geq 0.$$

Since $\omega^1, \omega^2, (y, v) - (x, u) \in L$, by using the convexity of L, if we have $(x, u) \leq_L (y, v)$, then

$$(f_1(y, v) - f_1(x, u))\omega^1 + (f_2(y, v) - f_2(x, u))\omega^2 \in L,$$

which is equivalent to the following inequality:

$$f_1(x, u)\omega^1 + f_2(x, u)\omega^2 \leq_L f_1(y, v)\omega^1 + f_2(y, v)\omega^2.$$

Thus, the mapping $f_1\omega^1 + f_2\omega^2$ is L-isotone. Now, we define functions G and H as follows:

$$G(x, u) := \left(\frac{2}{5}x_1 + \frac{2}{5}x_2 - \frac{1}{5}\|u\| - \frac{4}{5}, -\frac{3}{10}x_1 + \frac{6}{5}x_2 - \frac{1}{10}\|u\| - \frac{2}{5}\right),$$

$$H(x, u) := \left(u_1 - \frac{1}{15}x_1 + \frac{1}{24}x_2 - \frac{1}{40}\|u\| - \frac{7}{30}, u_2 - \frac{1}{12}x_1 + \frac{7}{120}x_2 - \frac{1}{40}\|u\| + \frac{1}{30}\right).$$

Hence, we get

$$(x - G, u - H) = f_1\omega^1 + f_2\omega^2 = \left(2f_1 + 2f_2, f_1 + f_2, \frac{1}{3}f_1 + \frac{1}{6}f_2, \frac{1}{6}f_1 + \frac{1}{3}f_2\right)$$

is $L-$ isotone. Then, we check that all the conditions in Theorem 3.2.3 are satisfied. Let us start with the initial condition. We have,

$$-G(0,0,0,0) = \left(\frac{4}{5}, \frac{2}{5}\right) \quad \text{and} \quad \|H(0,0,0,0)\| = \sqrt{\left(-\frac{7}{30}\right)^2 + \left(\frac{1}{30}\right)^2} = \frac{\sqrt{2}}{6}.$$

Evidently, $-G(0,0,0,0)_1 \geq -G(0,0,0,0)_2 \geq \|H(0,0,0,0)\|$. Now, consider a vector $(\hat{x}, \hat{u}) := (30, 12, 4, 3) \in K$, which yields

$$G(\hat{x}, \hat{u}) = \left(15, \frac{9}{2}\right) \quad \text{and} \quad H(\hat{x}, \hat{u}) = \left(\frac{257}{120}, \frac{133}{120}\right).$$

Moreover, we have that $G(\hat{x}, \hat{u})_1 \geq G(\hat{x}, \hat{u})_2 \geq \|H(\hat{x}, \hat{u})\|$, which implies that $(\hat{x}, \hat{u}) \in \Omega$. Hence, $\Omega \neq \varnothing$. Next, we solve the mixed complementarity problem MiCP(G, H, C, p, q). For an arbitrary element (x, y), if it is a solution of MiCP(G, H, C, p, q), then

$$x - G(x, u) = (2f_1 + 2f_2, f_1 + f_2) \text{ where } f_i = f_i(x, u), i = 1, 2,$$

and $G(x, u) = 0$. Thus, we have $x_1 = 2f_1 + 2f_2$ and $x_2 = f_1 + f_2$. Moreover,

$$x_1 = \frac{1}{3}\|u\| + \frac{4}{3} \text{ and } x_2 = \frac{1}{6}\|u\| + \frac{2}{3}. \tag{3.7}$$

Meanwhile, we have $u \perp H(x, u)$, which implies

$$\langle u, H(x, u)\rangle = u_1\left(u_1 - \frac{1}{3}f_1 - \frac{1}{6}f_2\right) + u_2\left(u_2 - \frac{1}{6}f_1 - \frac{1}{3}f_2\right) = 0.$$

Then,

$$\|u\|^2 = u_1^2 + u_2^2 = \left(\frac{1}{3}u_1 + \frac{1}{6}u_2\right)f_1 + \left(\frac{1}{6}u_1 + \frac{1}{3}u_2\right)f_2. \tag{3.8}$$

We will figure out all the nonzero solutions on the boundary of C. For the first case, without loss of generality, suppose that $u_1 = u_2 > 0$, so we have $\|u\| = \sqrt{2}u_1 = \sqrt{2}u_2$ and, by using (3.8),

$$u_1 = u_2 = \frac{1}{4}(f_1 + f_2).$$

By using the definition of f_1 and f_2 as well as (3.7), we get

$$u_1 = u_2 = \frac{48 + 2\sqrt{2}}{287}.$$

Thus, the solution of MiCP(G, H, C, p, q) is

$$(x, u) = \left(\frac{384 + 16\sqrt{2}}{287}, \frac{192 + 8\sqrt{2}}{287}, \frac{48 + 2\sqrt{2}}{287}, \frac{48 + 2\sqrt{2}}{287}\right).$$

For the second case, we consider $u_2 = 0$, which implies that $\|u\| = u_1$. Hence, the equation (3.8) is equivalent to

$$u_1^2 - \left(\frac{1}{3}f_1 + \frac{1}{6}f_2\right)u_1 = 0.$$

Since $u_1 \neq 0$, we have

$$u_1 = \frac{1}{3}f_1 + \frac{1}{6}f_2.$$

By using the definition of f_1 and f_2, and (3.7) again, we have $u_1 = \frac{212}{691}$, which implies that $u = \left(\frac{212}{691}, 0\right)$. Thus,

$$(x, u) = \left(\frac{992}{691}, \frac{496}{691}, \frac{212}{691}, 0\right).$$

Consider $(0, 0, 0, 0)$ as a starting point in the fixed point iteration process (3.6). We have

$$\begin{cases} x_{n+1} = x^n - G(x^n, u^n) \\ \qquad = (2f_1(x^n, u^n) + 2f_2(x^n, u^n), f_1(x^n, u^n) + f_2(x^n, u^n)), \\ u_{n+1} = P_C(u^n - H(x^n, u^n)) \\ \qquad = P_C\left(\frac{1}{3}f_1(x^n, u^n) + \frac{1}{6}f_2(x^n, u^n), \frac{1}{6}f_1(x^n, u^n) + \frac{1}{3}f_2(x^n, u^n)\right). \end{cases} \tag{3.9}$$

From the above equations we get $x_1^{n+1} \geq x_2^{n+1}$. Moreover, since as the starting point, we set $(0,0,0,0)$, then for any arbitrary $i \in \mathbb{N}$, we have that $x_1^i \geq x_2^i \geq 0$. Define the set S as follows:

$$S := \left\{(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2 : 0 \leq x_1 < \frac{992}{691}, 0 \leq x_2 < \frac{496}{691}, 0 \leq u_1 < \frac{212}{691}, u_2 = 0\right\}.$$

We want to show that for any $n \in \mathbb{N}$, we have $(x^n, u^n) \in S$. We will prove it by induction. First, we have $(x^0, u^0) \in S$. Suppose next $0 \leq x_1^n < \frac{992}{691}$, $0 \leq x_2^n < \frac{496}{691}$, $0 \leq u_1^n < \frac{212}{691}$ and $u_2 = 0$, which is equivalent to $\|u^n\| = u_1^n$. Since $x_1^n \geq x_2^n$, we have

$$0 < x_1^{n+1} = 2f_1(x^n, u^n) + 2f_2(x^n, u^n) = \frac{3}{5}x_1^n - \frac{2}{5}x_2^n + \frac{1}{5}u_1^n + \frac{4}{5}$$

$$< \frac{3}{5} \cdot \frac{992}{691} - \frac{2}{5} \cdot \frac{496}{691} + \frac{1}{5} \cdot \frac{212}{691} + \frac{4}{5} = \frac{992}{691}.$$

Similarly,

$$0 < x_2^{n+1} = f_1(x^n, u^n) + f_2(x^n, u^n) = \frac{3}{10}x_1^n - \frac{1}{5}x_2^n + \frac{1}{10}u_1^n + \frac{2}{5}$$

$$< \frac{3}{10} \cdot \frac{992}{691} - \frac{1}{5} \cdot \frac{496}{691} + \frac{1}{10} \cdot \frac{212}{691} + \frac{2}{5} = \frac{496}{691}.$$

Meanwhile, we also have

$$u^n - H(x^n, u^n) = \left(\frac{1}{3}f_1(x^n, u^n) + \frac{1}{6}f_2(x^n, u^n), \frac{1}{6}f_1(x^n, u^n) + \frac{1}{3}f_2(x^n, u^n)\right).$$

Obviously, $(u^n - H(x^n, u^n))_1 > 0$, then

$$(u^n - H(x^n, u^n))_1 \quad < \quad \frac{1}{3}\left(\frac{1}{10} \cdot \frac{992}{691} - \frac{1}{20} \cdot \frac{496}{691} + \frac{1}{20} \cdot \frac{212}{691} + 1\right)$$

$$+\frac{1}{6}\left(\frac{1}{5} \cdot \frac{992}{691} - \frac{3}{20} \cdot \frac{496}{691} + \frac{1}{20} \cdot \frac{212}{691} - \frac{3}{5}\right) = \frac{212}{691}$$

and $0 < (u^n - H(x^n, u^n))_2$. It is easy to check that the projection of it onto C such that $0 \leq u_1^{n+1} < \frac{691}{212}$ and $u_2^{n+1} = 0$, must be given on the ray $\{(u_1, u_2) : u_1 \geq 0, u_2 = 0\}$. It is equivalent to

$$u^{n+1} = (u_1^{n+1}, u_2^{n+1}) = P_C\left(u^n - H(x^n, u^n)\right) = \left(\frac{1}{3}f_1(x^n, u^n) + \frac{1}{6}f_2(x^n, u^n), 0\right).$$

Thus, the system of equations (3.9) is equivalent to

$$\begin{cases} x_1^{n+1} = \frac{3}{5}x_1^n - \frac{2}{5}x_2^n + \frac{1}{5}u_1^n + \frac{4}{5}, \\ x_2^{n+1} = \frac{3}{10}x_1^n - \frac{1}{5}x_2^n + \frac{1}{10}u_1^n + \frac{2}{5}, \\ u_1^{n+1} = \frac{1}{15}x_1^n - \frac{1}{24}x_2^n + \frac{1}{40}u_1^n + \frac{7}{30}. \end{cases} \tag{3.10}$$

Moreover, we have $x_1^n = 2x_2^n$, so (3.10) is equivalent to

$$\begin{cases} x_1^{n+1} = 2x_2^{n+1}, \\ x_2^{n+1} = \frac{2}{5}x_2^n + \frac{1}{10}u_1^n + \frac{2}{5}, \\ u_1^{n+1} = \frac{11}{120}x_2^n + \frac{1}{40}u_1^n + \frac{7}{30}. \end{cases} \tag{3.11}$$

The last two lines in (3.11) can be aggregated as follows

$$\begin{bmatrix} x_2^{n+1} \\ \\ u_1^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{10} \\ \\ \frac{11}{120} & \frac{1}{40} \end{bmatrix} \begin{bmatrix} x_2^n \\ \\ u_1^n \end{bmatrix} + \begin{bmatrix} \frac{2}{5} \\ \\ \frac{7}{30} \end{bmatrix}.$$

One easily verifies that the above 2×2 matrix has both (real) eigenvalues whose absolute values are less than 1, so it is a convergent matrix. Hence, the above process is convergent to the unique fixed point $[x_2^* \; u_1^*]'$ of the above equation, regardless of a starting point

$[x_2^0\ u_1^0]' \in \mathbb{R}^2$. Explicitly,

$$\begin{bmatrix} x_2^* \\ \\ u_1^* \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{10} \\ \\ -\frac{11}{120} & \frac{39}{40} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{2}{5} \\ \\ \frac{7}{30} \end{bmatrix} = \begin{bmatrix} \frac{496}{691} \\ \\ \frac{212}{691} \end{bmatrix}.$$

Bearing in mind that $x_1^{n+1} = 2x_2^{n+1}$ and $u_2^n = 0$, we have the convergence:

$$(x^n, u^n) = (x_1^n, x_2^n, u_1^n, u_2^n) \to (x_1^*, x_2^*, u_1^*, 0) = \left(\frac{992}{691}, \frac{496}{691}, \frac{212}{691}, 0\right),$$

which is the same as one solution we have obtained on the boundary.

Remark 3.3.1. *We remark that $f_1\omega^1 + f_2\omega^2$ is not ESOC-isotone. Indeed, if we assume that $f_1\omega^1 + f_2\omega^2$ is L-isotone, then for any $(x, u) \leq_L (y, v)$ and ω^1, $\omega^2 \in L$, we have*

$$f_1(x, u)\omega^1 + f_2(x, u)\omega^2 \leq_L f_1(y, v)\omega^1 + f_2(y, v)\omega^2 \tag{3.12}$$

and it is equivalent to

$$(f_1(y, v) - f_1(x, u))\omega^1 + (f_2(y, v) - f_2(x, u))\omega^2 \in L.$$

For arbitrary $(x^*, u^*), (y^*, v^*) \in \mathbb{R}^p \times \mathbb{R}^q$, such that $y_1^* - x_1^* = \|u^* - v^*\| = \|u^*\| - \|v^*\| > 0$, $y_2^* - x_2^* = 2\|u^* - v^*\| = 2(\|u^*\| - \|v^*\|) > 0$, it is obvious that $(x^*, u^*) \leq_{\text{ESOC}} (y^*, v^*)$. Since $f_1(x, u) = \frac{1}{10}x_1 - \frac{1}{20}x_2 + \frac{1}{20}\|u\| + 1$ and $f_2(x, u) = \frac{1}{5}x_1 - \frac{3}{20}x_2 + \frac{1}{20}\|u\| - \frac{3}{5}$,

$$\begin{aligned} f_1(y^*, v^*) - f_1(x^*, u^*) &= \frac{1}{10}(y_1^* - x_1^*) - \frac{1}{20}(y_2^* - x_2^*) - \frac{1}{20}(\|u^*\| - \|v^*\|) \\ &= \frac{1}{10}(\|u^*\| - \|v^*\|) - \frac{2}{20}(\|u^*\| - \|v^*\|) - \frac{1}{20}(\|u^*\| - \|v^*\|) \\ &= -\frac{1}{20}(\|u^*\| - \|v^*\|) < 0, \end{aligned}$$

$$
\begin{aligned}
f_2(y^*,v^*)-f_2(x^*,u^*) &= \frac{1}{5}(y_1^*-x_1^*)-\frac{3}{20}(x_2^*-y_2^*)-\frac{1}{20}(\|u^*\|-\|v^*\|)\geq 0\\
&= \frac{1}{5}(\|u^*\|-\|v^*\|)-\frac{6}{20}(\|u^*\|-\|v^*\|)-\frac{1}{20}(\|u^*\|-\|v^*\|),\\
&= -\frac{3}{20}(\|u^*\|-\|v^*\|)<0,
\end{aligned}
$$

contradicting (3.12), so $f_1\omega^1+f_2\omega^2$ *is not ESOC-isotone. Let us recall that both* f_1 *and* f_2 *are MESOC-monotone (which has been proved in the numerical example) and not ESOC-monotone, which implies that* $f_1(y,v)-f_1(x,u)$ *and* $f_2(y,v)-f_2(x,u)$ *will not be nonnegative for all* $(x,u)\leq_{\text{ESOC}}(y,v)$. *Since both* $f_1(y^*,v^*)-f_1(x^*,u^*)$ *and* $f_2(y^*,v^*)-f_2(x^*,u^*)$ *are negative, then by using convexity of ESOC, since* ω^1, $\omega^2\in$ *MESOC* $\subseteq$ *ESOC, we have*

$$-(f_1(y^*,v^*)-f_1(x^*,u^*))\omega^1-(f_2(y^*,v^*)-f_2(x^*,u^*))\omega^2\in ESOC.$$

Meanwhile, if $f_1\omega_1+f_2\omega_2$ *were ESOC-isotone, then*

$$(f_1(y^*,v^*)-f_1(x^*,u^*))w^1+(f_2(y^*,v^*)-f_2(x^*,u^*))w^2\in ESOC.$$

Since ω^1 *and* ω^2 *are linearly independent, it contradicts the property of the pointedness of ESOC. Thus,* $f_1\omega^1+f_2\omega^2$ *is not ESOC-isotone.*

3.4 Conclusions and comments

In this chapter, we first derived the isotonicity property of the monotone extended second order cone(MESOC). We demonstrated that the cylinders defined as $\mathbb{R}^p\times C$, where C is an arbitrary closed convex set with nonempty interior in $\mathbb{R}^q$, are isotone projection sets with respect to the MESOC in Theorem 3.1.4. The isotonicity property is another difference between the MESOC and ESOC. As for the ESOC, any isotone projection set with respect to MESOC is such a cylinder, while for the ESOC, it is indicated that ESOC has a large family of ESOC − *isotone* mappings in [62]. Theorem 3.2.3 is the main theorem in this chapter. In this theorem, we used the MESOC-isotonicity of the projection onto the cylinder to solve general mixed complementarity problems. We illustrated the corresponding iterative method by using a numerical example with exact numbers. Although the iteration principle for the MESOC is similar to the corresponding one for ESOC, we remark that there are mixed complementarity problems which can be solved iteratively by MESOC, but the same iterative scheme cannot be used via ESOC because it does not satisfy the corresponding ESOC-isotonicity condition (merely the MESOC-isotonicity). This is due to the fact that although MESOC is a subset of ESOC, the MESOC-isotonicity of mappings does not imply their ESOC-isotonicity. This idea is underlined in the preceding section.

CHAPTER 4

HOW TO PROJECT ONTO MESOC

In the previous chapter, we presented the concepts and properties of the monotone extended second order cone. Meanwhile, we developed an iterative technique to show that the mixed complementarity problem could be solved by using the isotonicity property of the monotone extended second order cone. But from the optimisation point of view, we need to find an easy method to project onto this cone such that we could say this cone is very useful in optimisation. The results in this chapter have already been published in [30].

Generally speaking, the problem of projecting onto the monotone extended second order cone is basically a conic optimisation problem with respect to MESOC. Moreover, for an arbitrary point $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, we can project this point onto the monotone extended second order cone, and the corresponding second order conic optimisation problem will be formulated as

$$\min\{\|x - u\| + \|y - v\| : (x, y) \in \mathbb{R}^p \times \mathbb{R}^q, (u, v) \in L\}, \tag{4.1}$$

where L is the monotone extended second order cone in $\mathbb{R}^p \times \mathbb{R}^q$. If we use the method above to solve the problem of how to project onto MESOC, we will lose some useful properties of MESOC. Motivated by the aim of reducing the complexity of the problem, we found a more elegant way to show how to solve the problem of projecting onto MESOC by using the properties of MESOC and the Moreau's decomposition theorem as follows.

Theorem 4.0.1. *Denote $K \subset \mathbb{R}^n$ to be a closed and convex cone, and let K^* be its dual cone, and z be an arbitrary point in $\mathbb{R}^n$. Then, for any arbitrary points $x \in K$ and $y \in K^*$, we hold the equivalence for the two arguments as follows*

(i) *$z = x - y$ and $(x, y) \in C(K)$,*

(ii) *$x = P_K(z)$ and $y = P_{K^*}(-z)$.*

Moreover, from theorem 4.0.1 we also have the following conclusions

$$P_K(z) \perp P_{K^*}(-z), \qquad z = P_K(z) - P_{K^*}(-z).$$

After the introduction of the Moreau's decomposition theorem, we will define some notations, which will be implemented in this section.

For an arbitrary $z \in \mathbb{R}^p$, let $z = (z_1, \ldots, z_p)^\top$. Denote the nonnegative orthant in $\mathbb{R}^p$ by $\mathbb{R}^p_+ = \{x \in \mathbb{R}^p : x \geq 0\}$. The nonnegative orthant is a proper cone and also self-dual, i.e., $\mathbb{R}^p_+ = (\mathbb{R}^p_+)^*$. For an arbitrary real number $\alpha \in \mathbb{R}$, let $\alpha^+ := \max(\alpha, 0)$ and $\alpha^- := \max(-\alpha, 0)$. Then for any arbitrary vector $z \in \mathbb{R}^p$, let $z^+ := (z_1^+, \ldots, z_p^+)$, $z^- := (z_1^-, \ldots, z_p^-)$ and $|z| := (|z_1|, \ldots, |z_p|)$. Then we will have

$$z^+ = P_{\mathbb{R}^p_+}(z),\ z^- = P_{\mathbb{R}^p_+}(-z),\ z = z^+ - z^- \text{ and } |z| = z^+ + z^-. \tag{4.2}$$

4.1 Properties of the complementarity set for MESOC

For the next step, we will develop another property of the monotone extended second order cone. From the definition and the formula of the complementarity set of monotone extended second order cone in (2.1), as well as Proposition 2.1.6 and the monotone nonnegative cone in Example 1.3.2 respectively, we demonstrated the connection between the complementarity set of these two cones in the following propositions.

Proposition 4.1.1. *Denote $L \in \mathbb{R}^p \times \mathbb{R}^q$ be the monotone extended second order cone and L^* be its dual cone, then for arbitrary points $(x, u), (y, v) \in \mathbb{R}^p \times \mathbb{R}^q$. we have* *(i)* $(x,u) \in L$ *if and only if* $x - \|u\|e \in \mathbb{R}^p_{\geq+}$.

(ii) $(y,v) \in L^*$ *if and only if* $y - \|v\|e^p \in (\mathbb{R}^p_{\geq+})^*$,

where e^i *is the canonical unit vectors of* $\mathbb{R}^p$ *for* $i = 1, 2, \ldots, p$ *and* $e = e^1 + \cdots + e^p$.

Proof. Following the conclusions of the monotone nonnegative cone in Example 1.3.2, the definition of MESOC in (2.1) and Proposition 2.1.3, we will have the results above. □

Proposition 4.1.2. *Let* $x, y \in \mathbb{R}^p$, *and* $u, v \in \mathbb{R}^q \setminus \{0\}$.

(i) $(x, 0, y, v) \in C(L)$ *if and only if* $x_p = 0, \sum_{i=1}^p y_i \geq \|v\|$ *and* $(x,y) \in C(\mathbb{R}^p_{\geq+})$.

(ii) $(x, u, y, 0) \in C(L)$ *if and only if* $x_i \geq \|u\|$ *for all* i, $\sum_{i=1}^p y_i = 0$ *and* $(x,y) \in C(\mathbb{R}^p_{\geq+})$.

(iii) $(x, y, u, v) \in C(L)$ *if and only if* $x_p = \|u\|$, $\langle y, e\rangle = \|v\|$, $\langle u, v\rangle = -\|u\|\|v\|$, *and* $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+})$.

Proof. (i) First, we prove that if $(x, 0, y, v) \in C(L)$, then $x_p = 0, (x,y) \in C(\mathbb{R}^p_{\geq+})$. Since $(x, 0, y, v) \in C(L)$, then $\sum_{i=1}^p y_i \geq \|v\|$ and

$$
\begin{aligned}
0 &= \langle (x,0), (y,v)\rangle \\
&= y_1(x_1 - x_2) + (y_1 + y_2)(x_2 - x_3) + \cdots + (y_1 + y_2 + \cdots + y_{p-1})(x_{p-1} - x_p) \\
&+ (y_1 + y_2 + \cdots + y_p)x_p \\
&\geq 0
\end{aligned}
$$

Since $\sum_{i=1}^p y_i \geq \|v\| > 0$, then $\langle (x,0), (y,v)\rangle = 0$ implies $x_p = 0$. Meanwhile, we have $\langle (x,0), (y,v)\rangle = \sum_{i=1}^p x_i y_i = 0$. The inequalities $x_1 \geq x_2 \geq \ldots \geq x_p \geq 0$ imply $x \in \mathbb{R}^p_{\geq+}$, meanwhile, $\sum_{i=1}^j y_i \geq 0$, for all $j = 1, 2, \ldots, p-1$ and $\sum_{i=1}^p y_i \geq \|v\| > 0$ imply that $y \in (\mathbb{R}^p_{\geq+})^*$. Hence, $(x,y) \in C(\mathbb{R}^p_{\geq+})$.
Second, we need to prove that if for all $(x,y) \in C(\mathbb{R}^p_{\geq+})$, $x_p = 0$ and $\sum_{i=1}^p y_i \geq \|v\|$, then $(x,y) \in C(L)$.

From $(x,y)\in C(\mathbb{R}^p_{\geq+})$, we get $x\in\mathbb{R}^p_{\geq+}, y\in(\mathbb{R}^p_{\geq+})^*$ and $\langle x,y\rangle=0$. Thus, $x\in\mathbb{R}^p_{\geq+}$ and $x_p=0$ imply $(x,0)\in L$, meanwhile $y\in(\mathbb{R}^p_{\geq+})^*$, $\sum_{i=1}^p y_i\geq\|v\|$ and $\langle x,y\rangle=0$ imply $(y,v)\in M$ and $\langle(x,0),(y,v)\rangle=\langle x,y\rangle=0$. Thus, $(x,0,y,v)\in C(L)$.

(ii) First, we need to show that if $(x,u,y,0)\in C(L)$, then $x_i\geq\|u\|, \sum_{i=1}^p y_i=0$ and $(x,y)\in C(\mathbb{R}^p_{\geq+})$.

Since $(x,u,y,0)\in C(L)$, we get

$$\begin{aligned}
0&=\langle(x,u),(y,0)\rangle\\
&=\sum_{i=1}^p x_iy_i\\
&=y_1(x_1-x_2)+(y_1+y_2)(x_2-x_3)+\cdots+(y_1+y_2+\cdots+y_{p-1})(x_{p-1}-x_p)\\
&+(y_1+y_2+\cdots+y_p)x_p\\
&\geq 0
\end{aligned}$$

Thus $x_p\geq\|u\|>0$ implies $\sum_{i=1}^p y_i=0$. Meanwhile, it is obvious that $x\in\mathbb{R}^p_{\geq+}$, $y\in(\mathbb{R}^p_{\geq+})^*$. Thus $0=\langle(x,u),(y,0)\rangle=x^Ty$ implies $(x,u,y,0)\in C(\mathbb{R}^p_{\geq+})$

Second, we need to prove that if $x_i\geq\|u\|, \sum_{i=1}^p y_i=0, (x,y)\in C(\mathbb{R}^p_{\geq+})$, then we have $(x,u,y,0)\in C(L)$.

Since $(x,y)\in C(\mathbb{R}^p_{\geq+})$, we have $x\in\mathbb{R}^p_{\geq+}, y\in(\mathbb{R}^p_{\geq+})^*$ and $\sum_{i=1}^p x_iy_i=0$. Thus, $x\in\mathbb{R}^p_{\geq+}$ and $x_i\geq\|u\|$ imply $(x,u)\in L$. On the other hand, $y\in(\mathbb{R}^p_{\geq+})^*$ and $\sum_{i=1}^p y_i=0$ implies $(y,0)\in M$. Furthermore, $\sum_{i=1}^p x_iy_i=0$ implies $\langle(x,u),(y,0)\rangle=0$. Hence, $(x,u,y,0)\in C(L)$.

(iii) Take $(x,u,y,v)\in C(L)$. Then following the definition of $C(L)$, we get $(x,u)\in L$, $(y,v)\in L^*$ and $\langle(x,u),(y,v)\rangle=0$. Following the conclusions in Proposition 4.1.1, since $(x,u)\in L$ and $(y,v)\in L^*$, then we have $x-\|u\|e\in\mathbb{R}^p_{\geq+}$ and $y-\|v\|e^p\in(\mathbb{R}^p_{\geq+})^*$. Moreover, by using the Cauchy inequality, the condition of $\langle(x,u),(y,v)\rangle=0$, and the conclusion in Lemma 2.1.5, we have

$$0=\langle x,y\rangle+\langle u,v\rangle\geq\|u\|\langle y,e\rangle+\langle u,v\rangle\geq\|u\|\|v\|+\langle u,v\rangle\geq 0.$$

Thus, $\langle x, y\rangle = \|u\|\langle y, e\rangle$, $\|u\|\langle y, e\rangle = \|u\|\|v\|$ and $\langle u, v\rangle = -\|u\|\|v\|$. Moreover, consider we have supposed that $u \neq 0$, then we also have $\langle y, e\rangle = \|v\|$. Hence, by using the Abel's partial summation formula in (1.2), we have the following inequalities

$$0 \geq (\|u\| - x_p)\|v\| = (\|u\| - x_p)\langle y, e\rangle = \langle x, y\rangle - x_p \sum_{i=1}^{p} y_i = \sum_{i=1}^{p-1}(x_i - x_{i+1}) \sum_{j=1}^{i} y_j \geq 0.$$

Hence, all the terms in the inequalities above must equal to 0. In particular, we got $(\|u\| - x_p)\|v\| = 0$. Since we have supposed that $v \neq 0$, then we can conclude that $x_p = \|u\|$. On the other hand, consider the following equation,

$$\langle x - \|u\|e, y - \|v\|e^p\rangle = \langle x, y\rangle - \|u\|\langle y, e\rangle - x_p\|v\| + \|u\|\|v\|,$$

since we have already got the conclusions of $\langle x, y\rangle = \|u\|\langle y, e\rangle$ and $x_p = \|u\|$, then we can obtain that $\langle x - \|u\|e, y - \|v\|e^p\rangle = 0$. Hence, $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+})$, then we finished that proof of necessity.

Then we will consider the proof of sufficient. Suppose we have the following conditions, which are

$$x_p = \|u\|,\ \langle y, e\rangle = \|v\|,\ \langle u, v\rangle = -\|u\|\|v\|,\ \text{and } (x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+}).$$

Thus, $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+})$ implies that $x - \|u\|e \in \mathbb{R}^p_{\geq+}$ and $y - \|v\|e^p \in (\mathbb{R}^p_{\geq+})^*$. Then by using Proposition 4.1.1, we have $(x, u) \in L$ and $(y, v) \in L^*$. Moreover, from $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+})$, we have $\langle x - \|u\|e, y - \|v\|e^p\rangle = 0$ and it is equivalent to

$$\langle x, y\rangle - \|u\|\langle y, e\rangle - x_p\|v\| + \|u\|\|v\| = 0.$$

Thus, $x_p = \|u\|$ implies that $\langle x, y\rangle = \|u\|\langle y, e\rangle$.

Hence, by using the conditions of $\langle u, v\rangle = -\|u\|\|v\|$ and $\langle y, e\rangle = \|v\|$, we have

$$\langle (x,u),(y,v)\rangle = \langle x,y\rangle + \langle u,v\rangle = \|u\|\langle y,e\rangle - \|u\|\|v\| = \|u\| \left(\langle y,e\rangle - \|v\|\right) = 0.$$

Therefore, $(x, u, y, v) \in C(L)$. $\square$

4.2 Projection onto MESOC

After finding the relationships between the complementarity set of the monotone extended second order cone and the complementarity set of the monotone nonnegative cone, in order to show the formula which shows how to project onto the monotone extended second order cone, we need to introduce the following lemma.

Lemma 4.2.1. *For an arbitrary $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$, if $P_L(z, w) = (x, u)$ and $P_{L^*}(-z, -w) = (y, v)$, then the following conclusions hold:*

(i) $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle \geq \|w\|$ if and only if $u = 0$;

(ii) $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$ if and only if $v = 0$.

(iii) $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle < \|w\|$ and $P_{\mathbb{R}^p_{\geq +}}(z)_p < \|w\|$ if and only if $u \neq 0$ and $v \neq 0$.

Proof. First, let us prove the item (i). Suppose that $u = 0$, then we have $P_L(z, w) = (x, 0)$ and $P_{L^*}(-z, -w) = (y, v)$. By using the Moreau's decomposition theorem, we have

$$(x, 0) \in L,\ (y, v) \in L^*,\ \langle (x, 0), (y, v)\rangle = 0 \text{ and } (z, w) = (x, 0) - (y, v).$$

Hence, we have $x \in \mathbb{R}^p_{\geq +}$, $y \in (\mathbb{R}^p_{\geq +})^*$, $\langle y, e\rangle \geq \|v\|$, $\langle x, y\rangle = 0$, $z = x - y$ and $w = -v$. Hence, the Moreau's decomposition theorem for $\mathbb{R}^p_{\geq +}$ implies that

$$x = P_{\mathbb{R}^p_{\geq +}}(z) \text{ and } y = P_{(\mathbb{R}^p_{\geq +})^*}(-z).$$

Since $w = -v$ and $\langle y, e\rangle \geq \|v\|$, we have that $\langle P_{(\mathbb{R}^p_{\geq+})^*}(-z), e\rangle \geq \|w\|$. On the other side, let $\langle P_{(\mathbb{R}^p_{\geq+})^*}(-z), e\rangle \geq \|w\|$. Since $(P_{\mathbb{R}^p_{\geq+}}(z), 0) \in L$ and $\langle P_{(\mathbb{R}^p_{\geq+})^*}(-z), e\rangle \geq \|w\|$, we obtain that $(P_{(\mathbb{R}^p_{\geq+})^*}(-z), -w) \in L^*$. Moreover, we will have $(P_{\mathbb{R}^p_{\geq+}}(z), 0, P_{(\mathbb{R}^p_{\geq+})^*}(-z), -w) \in C(L)$ and $(z, w) = (P_{\mathbb{R}^p_{\geq+}}(z), 0) - (P_{(\mathbb{R}^p_{\geq+})^*}(-z), -w)$. Hence, by using the Moreau's decomposition theorem for L, we conclude that $P_L(z, w) = (P_{\mathbb{R}^p_{\geq+}}(z), 0)$ and $P_{L*}(-z, -w) = (P_{(\mathbb{R}^p_{\geq+})^*}(-z), -w)$. Thus, $u = 0$.

In the next step, we will show the satisfaction of item (ii). Let us first consider $v = 0$. Since we have $P_L(z, w) = (x, u)$ and $P_{L^*}(-z, -w) = (y, 0)$, Then by using Moreau's decomposition theorem for L, we have

$$(x, u) \in L,\ (y, 0) \in L^*,\ \langle (x, u), (y, 0)\rangle = 0 \text{ and } (z, w) = (x, u) - (y, 0).$$

Thus, we have

$$x \in \mathbb{R}^p_{\geq+},\ y \in (\mathbb{R}^p_{\geq+})^*,\ x_p \geq \|u\|,\ \langle x, y\rangle = 0,\ z = x - y \text{ and } w = u.$$

Then, by applying the Moreau's decomposition theorem for $\mathbb{R}^p_{\geq+}$ we conclude that

$$x = P_{\mathbb{R}^p_{\geq+}}(z) \text{ and } y = P_{(\mathbb{R}^p_{\geq+})^*}(-z).$$

Moreover, by using $w = u$ and $x_p \geq \|u\|$, we obtain that $P_{\mathbb{R}^p_{\geq+}}(z)_p \geq \|w\|$.

Conversely, suppose we have $P_{\mathbb{R}^p_{\geq+}}(z)_p \geq \|w\|$. Since $(P_{(\mathbb{R}^p_{\geq+})^*}(-z), 0) \in L^*$ and $P_{\mathbb{R}^p_{\geq+}}(z)_p \geq \|w\|$, we will have $(P_{\mathbb{R}^p_{\geq+}}(z), w) \in L$. Moreover, we have

$$(P_{\mathbb{R}^p_{\geq+}}(z), w, P_{(\mathbb{R}^p_{\geq+})^*}(-z), 0) \in C(L) \text{ and } (z, w) = (P_{\mathbb{R}^p_{\geq+}}(z), w) - (P_{(\mathbb{R}^p_{\geq+})^*}(-z), 0).$$

Hence, by applying Moreau's decomposition theorem for the monotone extended second order cone, we have

$$P_L(z, w) = (P_{\mathbb{R}^p_{\geq+}}(z), w) \text{ and } P_{L^*}(-z, -w) = (P_{(\mathbb{R}^p_{\geq+})^*}(-z), 0).$$

Therefore, $v = 0$.

The conclusion in Item (iii) is an immediate consequence of items (i) and (ii). $\square$

In order to simplify the notations in the formula of how to project into the monotone extended second order cone, let us first define the following functions. Let $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$ be an arbitrary given point, then we define

$$\phi(\lambda) := \left\langle P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)), e \right\rangle, \qquad f(\lambda) := z - \frac{1}{1+\lambda}\|w\|e + \frac{\lambda}{1+\lambda}\|w\|e^p. \tag{4.3}$$

Then we will show how to project into MESOC.

Theorem 4.2.2. *Let $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$, then the following statements hold:*

(1) *If $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle \geq \|w\|$, then*

$$P_L(z, w) = (P_{\mathbb{R}^p_{\geq +}}(z), 0), \qquad P_{L^*}(-z, -w) = (P_{(\mathbb{R}^p_{\geq +})^*}(-z), -w);$$

(2) *If $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$, then*

$$P_L(z, w) = (P_{\mathbb{R}^p_{\geq +}}(z), w), \qquad P_{L^*}(-z, -w) = (P_{(\mathbb{R}^p_{\geq +})^*}(-z), 0);$$

(3) *If $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle < \|w\|$ and $P_{\mathbb{R}^p_{\geq +}}(z)_p < \|w\|$, then the following equation*

$$\phi(\lambda) = 0, \tag{4.4}$$

has a unique positive solution $\lambda > 0$ and

$$P_L(z, w) = \left(P_{\mathbb{R}^p_{\geq +}}(-f(\lambda)) + \frac{1}{1+\lambda}\|w\|e, \frac{1}{1+\lambda}w\right),$$

$$P_{L^*}(-z, -w) = \left(P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) + \frac{\lambda}{1+\lambda}\|w\|e^p, -\frac{\lambda}{1+\lambda}w\right).$$

Proof. Let $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$. In order to prove this theorem, we need to find $(x, u) \in L$ and $(y, v) \in L^*$ such that

$$P_L(z, w) = (x, u), \qquad P_{L^*}(-z, -w) = (y, v). \tag{4.5}$$

To prove item (1), suppose we have $\langle P_{(\mathbb{R}^p_{\geq+})^*}(-z), e\rangle \geq \|w\|$. Then by using the conclusion in the item (i) of Lemma 4.2.1, we must have $u = 0$. Since $P_L(z, w) = (x, 0)$ and $P_{L^*}(-z, -w) = (y, v)$, then by using the Moreau's decomposition theorem for L, we have

$$(x, 0) \in L,\ (y, v) \in L^*,\ \langle (x, 0), (y, v)\rangle = 0 \text{ and } (z, w) = (x, 0) - (y, v).$$

Thus,

$$x \in \mathbb{R}^p_{\geq+},\ y \in (\mathbb{R}^p_{\geq+})^*,\ \langle x, y\rangle = 0,\ z = x - y \text{ and } v = -w.$$

Now, applying the Moreau's decomposition theorem for $\mathbb{R}^p_{\geq+}$, we conclude that $x = P_{\mathbb{R}^p_{\geq+}}(z)$ and $y = P_{(\mathbb{R}^p_{\geq+})^*}(-z)$. Then by using (4.5), $u = 0$ and $v = -w$, we finished the proof for item (1).

Then we will prove the item (2). Since $P_{\mathbb{R}^p_{\geq+}}(z)_p \geq \|w\|$, then from item (ii) of Lemma 4.2.1 we have $v = 0$. Since $P_L(z, w) = (x, u)$ and $P_{L*}(-z, -w) = (y, 0)$, then by applying the Moreau's decomposition theorem for the cone L, we have

$$(x, u) \in L,\ (y, 0) \in L^*,\ \langle (x, u), (y, 0)\rangle = 0 \text{ and } (z, w) = (x, u) - (y, 0).$$

Hence, we have

$$x \in \mathbb{R}^p_{\geq+},\ y \in (\mathbb{R}^p_{\geq+})^*,\ \langle x, y\rangle = 0,\ z = x - y \text{ and } u = w.$$

Then by using the Moreau's decomposition theorem for $\mathbb{R}^p_{\geq+}$, we conclude that $x =$

$P_{\mathbb{R}^p_{\geq+}}(z)$ and $y = P_{(\mathbb{R}^p_{\geq+})^*}(-z)$. Then by using (4.5), $v = 0$ and $u = w$, we have

$$P_L(z,w) = (P_{\mathbb{R}^p_{\geq+}}(z), w), \qquad P_{L^*}(-z,-w) = (P_{(\mathbb{R}^p_{\geq+})^*}(-z), 0).$$

Finally, we will show the satisfaction of item (3). We first note that by using $\langle P_{(\mathbb{R}^p_{\geq+})^*}(-z), e\rangle < \|w\|$, $P_{\mathbb{R}^p_{\geq+}}(z)_p < \|w\|$ and item (iii) from Lemma 4.2.1, we obtain that $u \neq 0$ and $v \neq 0$. Moreover, it follows from the Moreau's decomposition theorem, (4.5) is equivalent to

$$(x,u,y,v) \in C(L) \qquad (z,w) = (x,u) - (y,v). \tag{4.6}$$

Since we have $u \neq 0$, $v \neq 0$ and (4.6), then by applying Proposition 4.1.2 we have the following equivalent conditions

$$x_p = \|u\|, \qquad \langle y, e\rangle = \|v\|, \qquad \langle u, v\rangle = -\|u\|\|v\|, \qquad (x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+}), \tag{4.7}$$

and

$$z = x - y, \qquad w = u - v. \tag{4.8}$$

Since we have $\langle u, v\rangle = -\|u\|\|v\|$, $u \neq 0$ and $v \neq 0$, there exists $\lambda > 0$ such that $v = -\lambda u$. Then from the second equality in (4.8) we have

$$u = \frac{1}{1+\lambda}w, \qquad v = -\frac{\lambda}{1+\lambda}w. \tag{4.9}$$

Meanwhile, the second equality in (4.7) implies $\langle y, e\rangle = \|v\|$. Then, we obtain

$$\langle y, e\rangle = \frac{\lambda}{1+\lambda}\|w\|. \tag{4.10}$$

From (4.7) we have $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq+})$, then by applying the Moreau's decomposition theorem for $\mathbb{R}^p_{\geq+}$ we obtain

$$x - \|u\|e = P_{\mathbb{R}^p_{\geq+}}\left(x - \|u\|e - y + \|v\|e^p\right)$$

and

$$y - \|v\|e^p = P_{(\mathbb{R}^p_{\geq+})^*}\left(-x + \|u\|e + y - \|v\|e^p\right).$$

Thus, by using the first equality in (4.7) and (4.9) we obtain after some calculations that

$$x = P_{\mathbb{R}^p_{\geq+}}\left(z - \frac{1}{1+\lambda}\|w\|e + \frac{\lambda}{1+\lambda}\|w\|e^p\right) + \frac{1}{1+\lambda}\|w\|e; \tag{4.11}$$

$$y = P_{(\mathbb{R}^p_{\geq+})^*}\left(-z + \frac{1}{1+\lambda}\|w\|e - \frac{\lambda}{1+\lambda}\|w\|e^p\right) + \frac{\lambda}{1+\lambda}\|w\|e^p. \tag{4.12}$$

Hence, combining (4.5) with (4.9), (4.11) and (4.12) and taking into account second equality (4.3) we will have

$$P_L(z,w) = \left(P_{\mathbb{R}^p_{\geq+}}(-f(\lambda)) + \frac{1}{1+\lambda}\|w\|e, \frac{1}{1+\lambda}w\right)$$

and

$$P_{L^*}(-z,-w) = \left(P_{(\mathbb{R}^p_{\geq+})^*}(-f(\lambda)) + \frac{\lambda}{1+\lambda}\|w\|e^p, -\frac{\lambda}{1+\lambda}w\right).$$

The equation (4.4) will be obtained by using (4.10), (4.12) and second equality (4.3). The uniqueness of $\lambda > 0$ which satisfies (4.4) follows from the uniqueness of $P_L(z,w)$ and $P_{L^*}(-z,-w)$. □

Remark 4.2.1. *If $p = 1$, then the problem of projecting onto MESOC will be the same as the problem of projecting into the second order cone, then the projection formulas in Theorem 4.2.2 could be found in exercise 8.3 (c) in [13].*

Theorem 4.2.2 demonstrated the relationship between projecting into MESOC and projecting onto the monotone nonnegative cone. It is important to know how to compute a projection onto the cones $\mathbb{R}^p_{\geq+}$. Thus, we will introduce the following theorem, which states that in order to compute a projection onto the cone $\mathbb{R}^p_{\geq +}$, we need to know how to compute a projection onto the cones $\mathbb{R}^p_{\geq}$ and $\mathbb{R}^p_{+}$, its proof can be found in [57].

Theorem 4.2.3. *For any arbitrary* $x \in \mathbb{R}^p$, *we have* $P_{\mathbb{R}^p_{\geq +}}(x) = P_{\mathbb{R}^p_{\geq}}(x)^+ = P_{\mathbb{R}^p_{+}}(P_{\mathbb{R}^p_{\geq}}(x))$, *where* $P_{\mathbb{R}^p_{\geq}}(u)^+ = \max\{P_{\mathbb{R}^p_{\geq}}(u), 0\}$, *and* $\mathbb{R}^p_{+}$ *denotes the nonnegative orthant.*

In [8,55], some efficient numerical methods to compute projection onto the cones $\mathbb{R}^p_{\geq}$ have be introduced. Meanwhile, for computing how to project onto the cones $\mathbb{R}^p_{+}$, we have the well-known formula in (4.2). Meanwhile, we also developed the following lemmas which have shown the relationship between projection onto MESOC, projecting onto the monotone nonnegative cone and projecting onto the monotone cone.

Lemma 4.2.4. *Let* $\lambda > 0$ *be a real number, then* λ *is a solution of the equation* $\phi(\lambda) = 0$ *if, and only if,*

$$P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) = P_{(\mathbb{R}^p_{\geq})^*}(-f(\lambda))$$

or

$$P_{\mathbb{R}^p_{\geq +}}(f(\lambda)) = P_{\mathbb{R}^p_{\geq}}(f(\lambda)).$$

Proof. Suppose that $\phi(\lambda) = 0$. Since $P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) \in (\mathbb{R}^p_{\geq +})^*$, by using the formula of the dual cone of the monotone nonnegative cone which has been illustrated in Example 2.2, we have

$$\left\langle P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)), e^{1:j} \right\rangle = \sum_{i=1}^{j} P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda))_i \geq 0, \qquad j = 1, 2, \ldots, p.$$

Thus, due to $\phi(\lambda) = 0$, then by using the definition in (4.3), we have $\left\langle P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)), e \right\rangle = 0$. Since the dual cone of $\mathbb{R}^p_{\geq}$ is given by

$$(\mathbb{R}^p_{\geq})^* = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{j} y_j \geq 0,\ j = 1, 2, \ldots, p-1,\ \sum_{i=1}^{p} y_i = 0 \right\}.$$

Thus, $P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) \in (\mathbb{R}^p_{\geq})^*$.

On the other hand, since we have $(\mathbb{R}_{\geq})^* \subseteq (\mathbb{R}_{\geq +})^*$, then

$$\min\{\| - f(\lambda) - x\| \,, x \in (\mathbb{R}^p_{\geq +})^*\} \leq \min\{\| - f(\lambda) - x\| \,, x \in (\mathbb{R}^p_{\geq})^*\}.$$

Hence, by considering that $P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) = \operatorname{argmin}\{\| - f(\lambda) - x\| \,, x \in (\mathbb{R}^p_{\geq +})^*\}$, the projection onto a closed convex set is unique and that $P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) \in (\mathbb{R}^p_{\geq})^*$, we get

$$P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) = P_{(\mathbb{R}^p_{\geq})^*}(-f(\lambda)).$$

Moreover, the second equality is an immediate consequence of the Moreau's decomposition theorem for $\mathbb{R}^p_{\geq}$.

Reciprocally, assume that $P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) = P_{(\mathbb{R}^p_{\geq})^*}(-f(\lambda))$. Thus, the result follows by combining definitions of the dual cone of the monotone cone in Example 1.3.1 and the dual cone of the monotone nonnegative cone in Example 1.3.2 and (4.3). □

Lemma 4.2.5. *The real number $\lambda > 0$ is a solution of the equation $\phi(\lambda) = 0$ if, and only if, $P_{\mathbb{R}^p_+}(P_{\mathbb{R}^p_{\geq}}(f(\lambda))) = P_{\mathbb{R}^p_{\geq}}(f(\lambda))$, or equivalently $P_{\mathbb{R}^p_{\geq}}(f(\lambda)) \in \mathbb{R}^p_+$.*

Proof. The proof follows by combining Theorem 4.2.3 with Lemma 4.2.4. □

4.3 Conclusions and comments

In this chapter, we first present the properties of the complementarity set on the MESOC in Proposition 4.1.1 and Proposition 4.1.2. Then, we have illustrated that the problem of projection onto the monotone extended second order cone can be reduced to an isotone regression problem in neighbouring dimensions by using the Moreau's decomposition theorem and the properties of the complementarity set of MESOC in Theorem 4.2.2, which is the main theorem of this chapter. The isotone regression problem can be efficiently solved by using the pool-adjacent-violater algorithm [8, 23]. The complexity of the projection method proposed in the main theorem is considerably lower than the reformulation of this problem, which is to convert the projection problem onto a conic optimisation problem on the second order cone. The reformulation of the problem of projection has been present at the beginning of this chapter in (4.1). We expect some other conic optimisation problems will be easily solved by using this projection method than by transforming them into second order conic optimisation problems.

The projection onto the MESOC can be a useful ‘ingredient’ of projection methods for the latter problem. We will study a portfolio optimisation problem on the MESOC in the next chapter, which is an important application of the MESOC and indicates that the MESOC is useful in portfolio selection, see [56, 83]. Furthermore, we also predict more direct applications of the projection onto MESOC to practical problems. These applications would be regressions with respect to a set of points whose distance (more generally a ‘cost’) from a source point is expected to decrease, and only the position of the point closest to the source is important.

CHAPTER 5

THE LINEAR COMPLEMENTARITY PROBLEM ON THE MESOC

In the previous chapters, we introduced the monotone extended second order cone (MESOC) and studied the basic properties of this cone. We calculated the Lyapunov rank and presented the formula of Lyapunov-like transformation and positive operator of this cone. We also showed that the MESOC can be used as a tool in finding the solutions to the mixed complementarity problem on general cones by using the isotonicity properties of this cone. We expect that the problems based on the MESOC can be solved in a more efficient way by using the inner structure of the MESOC. Indeed, in the last chapter, we have illustrated such a particular problem of how to project onto the MESOC, which is much easier to solve directly by using the properties of the complementarity set of the MESOC rather than solving the reformulated second order conic optimisation problem.

In recent years, several applications based on the complementarity problems have been defined on different cones, such as extensions of second order cones, positive semidefinite cones, or direct product of these cones have emerged. Those applications based on the complementarity problems defined on the cones mentioned above are in robotics [3], robust game theory [54, 65], and elastoplasticity [84, 85]. All these applications come from the Karush–Kuhn–Tucker conditions of second order conic optimisation problems. The applications mentioned above for different cones indicated the importance of investigating different cones and the complementarity problems on cones, and the importance of cones and complementarity motivated us to consider the complementarity problems based on the MESOC and practical applications based on this cone.

The results in this chapter are mainly from my joint work with my supervisor, S.Z. Németh [38]. In this chapter, we studied the linear complementarity problems on the monotone extended second order cone. We demonstrated that the linear complementarity problem on the monotone extended second order cone can be converted into a mixed complementarity problem on the non-negative orthant defined in a neighbouring dimension. We proved that any point satisfying equation related to the Fischer–Burmeister (FB) complementarity function is a solution to the converted problem. We also showed that the semi-smooth Newton method could be used to solve the converted problem and provided a numerical example as well. Finally, we formulated a portfolio optimisation problem based on the monotone extended second order cone, which indicates that the MESOC is useful in portfolio selection. We also derived the explicit solution to the portfolio optimisation problem mentioned above. Note that in this section, as we use L to denote the lower triangular matrix, then, in order to avoid messing up notations, we will denote $\mathcal{L}$ to be the MESOC and $\mathcal{M}$ to be the dual cone of MESOC.

5.1 Problem formulation

Recall that in Definition 1.3.6, we have presented the definition of the linear complementarity problem on general cones. For the sake of completeness, we will first formulate the linear complementarity problem on the monotone extended second order cone.

For two arbitrary positive numbers p,q and n such that $p+q=n$, let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ be an arbitrary nonsingular block matrix matrix such that $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times p}$ and $D \in \mathbb{R}^{q\times q}$ are fixed matrices. Let (x,u), (y,v) be two arbitrary vectors such that $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. The formulation of the linear complementarity problem defined on the monotone extended second order cone $\mathcal{L} \in \mathbb{R}^n$ with a fixed vector $r \in \mathbb{R}^n$

and a linear mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(x, u) = T(x, u) + r$ is as follows:

$$\mathrm{LCP}(T, r, \mathcal{L}) := \begin{cases} \text{find a } (x, u)^\top \in \mathcal{L}, \text{ such that} \\ F(x, u) \in \mathcal{M} \text{ and } \langle (x, u)^\top, F(x, u) \rangle = 0, \end{cases}$$

where $\mathcal{M}$ is the dual cone of MESOC.

Then, based on Proposition 2.1.6, the following theorem is developed. It indicates the relationship between the linear complementarity problem on the MESOC with several other different complementarity problems defined on various cones.

Theorem 5.1.1. *Let (x, u), (y, v) be arbitrary vectors with $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q$. Consider the nonsingular block matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times p}$ and $D \in \mathbb{R}^{q\times q}$ are constant matrices. Then, for arbitrary vectors z and r, such that $z = (x, u)$ and $r = (y, v)$. In this theorem, to avoid messing up the notations, all the notations with a form as x' denote a variable. Let $\mathcal{L} \in \mathbb{R}^{p+q}$ be the monotone extended second order cone. The following statements hold:

- *(i) Let $u = 0$. Then, z is a solution of $LCP(T, r, \mathcal{L})$ if and only if x is a solution of $LCP(A, y, \mathbb{R}^p_{\geq+})$, $x_p = 0$ and $\sum_{i=1}^p (Ax_i + y_i) \geq \|Cx + v\|$.*
- *(ii) Let $Cx + Du + v = 0$. Then, z is a solution of $LCP(T, r, \mathcal{L})$ if and only if x is a solution of $MiCP(G, H, \mathbb{R}^p_{\geq+})$, $x_i \geq \|u\|$, and $\sum_{i=1}^p (Ax + Bu + v)_i = 0$, where G and H are defined by the formulas $G(x', u') = Ax' + Bu' + y$ and $H(x', u') = 0$.*
- *(iii) Let $u \neq 0 \neq Cx + Du + v$. Then, z is a solution of $LCP(T, r, \mathcal{L})$ if and only if z is a solution of $MiICP(G, H, F, \mathbb{R}^p_{\geq+})$, where F, G and H are defined by the formulas*

$$F(x', u') = x' - \|u'\|e, \; G(x', u') = Ax' + Bu' + y - \|Cx' + Du' + v\|e^p,$$

and

$$H(x', u') = u' e^\top (Ax' + Bu' + y) + \|u'\|(Cx' + Du' + v).$$

(iv) *Let* $\bar{z} = (\bar{x}, u) = (x - \|u\|e, u)$ *and* $u \neq 0 \neq Cx + Du + v$. *Then,* z *is a solution of* $LCP(T, r, \mathcal{L})$ *if and only if* $\bar{z}$ *is a solution of* $MiCP(\bar{G}, \bar{H}, \mathbb{R}^p_{\geq +})$, *where* $\bar{G}$ *and* $\bar{H}$ *are defined by the formulas*

$$\bar{G}(x', u') = A(x' + \|u'\|e) + Bu' + y - \|Cx' + \|u'\|e) + Du' + v\|e^p,$$

and

$$\bar{H}(x', u') = u' e^\top (A(x' + \|u'\|e) + Bu' + y) + \|u'\|(C(x' + \|u'\|e) + Du' + v).$$

(v) *When* $u \neq 0 \neq Cx + Du + v$, *the problem of finding a solution* $z = (x, u)$ *of the linear complementarity problem* $LCP(T, r, \mathcal{L})$ *is converted to a problem of finding a vector* $z = (x, u)$ *such that* $(\alpha, \beta) \in C(\mathbb{R}^p_+)$, *where*

$$\alpha = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{p-1} - x_p \\ x_p - \|u\| \end{pmatrix} \text{ and } \beta = \begin{pmatrix} (Ax + Bu + y)_1 \\ \sum_{i=1}^{2} (Ax + Bu + y)_i \\ \vdots \\ \sum_{i=1}^{p-1} (Ax + Bu + y)_i \\ \sum_{i=1}^{p} (Ax + Bu + y)_i \end{pmatrix}.$$

Moreover, let

$$x'_i(w') = \sum_{j=i}^{p-1} w'_j + x'_p = \sum_{j=i}^{p-1} w'_j + \|u'\|,$$

for any $i = 1, 2, \ldots, p-1$ *and any* $x', w' \in \mathbb{R}^p$, $u' \in \mathbb{R}^q$. *Let* $x'_p(w') = \|u'\|$. *Then, the problem of finding a vector* $z = (x, u)$ *such that* $(\alpha, \beta) \in C(\mathbb{R}^p_+)$ *is equivalent to* *the problem of finding a solution of* $MiCP(\hat{G}, \hat{H}, \mathbb{R}^{p-1}_+)$*, where*

$$\hat{G}(w', u') = \begin{pmatrix} (Ax'(w') + Bu' + y)_1 \\ \sum_{i=1}^{2}(Ax'(w') + Bu' + y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax'(w') + Bu' + y)_i \end{pmatrix}$$

and

$$\hat{H}(w', u') = u'e^\top(Ax'(w') + Bu' + y) + \|u'\|(Cx'(w') + Du' + v)$$

(vi) Let $t = \|u\|$*. Then,* z *is a solution of* $LCP(T, r, \mathcal{L})$ *if and only if* x *is a solution of* $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_+)$*, where* $\tilde{G}$ *and* $\tilde{H}$ *are defined by the formulas*

$$\tilde{G}(w', u', t') = \begin{pmatrix} (Ax'(w', t') + Bu' + y)_1 \\ \sum_{i=1}^{2}(Ax'(w', t') + Bu' + y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax'(w', t') + Bu' + y)_i \end{pmatrix} \in \mathbb{R}^{p-1}_+,$$

$$\tilde{H}(w', u', t') = \begin{pmatrix} u'e^\top(Ax'(w', t') + Bu' + y) + t'(Cx'(w', t') + Du' + v) \\ t'^2 - \|u'\|^2 \end{pmatrix}$$

and

$$x'(w', t') = \begin{pmatrix} w'_1 + w'_2 + \ldots + w'_{p-1} + t' \\ w'_2 + \ldots + w'_{p-1} + t' \\ \vdots \\ w'_{p-1} + t' \\ t' \end{pmatrix}.$$

Proof.

(i) By the definition of the linear complementarity problem, $z = (x, 0)$ is a solution of $LCP(T, r, \mathcal{L})$ if and only if $(x, 0, Ax + y, Cx + v) \in C(\mathcal{L})$, which, by using item (ii) in

Proposition 4.1.2, is equivalent to $x_p = 0$, $\sum_{i=1}^{p}(Ax_i + y_i) \geq \|Cx + v\|$ and $(x, Ax + y) \in C(\mathbb{R}^p_{\geq+})$. Finally, that is further equivalent to x being a solution of $LCP(A, y, \mathbb{R}^p_{\geq+})$.

(ii) Let $Cx + Du + v = 0$. By the definition of the linear complementarity problem, $z = (x, u)$ is a solution of $LCP(T, r, \mathcal{L})$ if and only if $(x, u, Ax+Bu+y, 0) \in C(\mathcal{L})$, which, by using item (iii) of Proposition 4.1.2, is equivalent to $x_i \geq \|u\|$, $e^\top(Ax + Bu + y) = 0$ and $(x, Ax+Bu+y) \in C(\mathbb{R}^p_{\geq+})$. We conclude that $z = (x, u)$ is a solution of $LCP(T, r, \mathcal{L})$ if and only if $z = (x, u)$ is a solution of $MiCP(G, H, \mathbb{R}^p_{\geq+})$.

(iii) By using the definition of linear complementarity problem, if $z = (x, u)$ is a solution of $LCP(T, r, \mathcal{L})$, then we have $(x, u, Ax + Bu + y, Cx + Du + v) \in C(\mathcal{L})$. Then, from item (iv) of Proposition 4.1.2 and the equality case in the Cauchy inequality, we have that $(x, u, Ax + Bu + y, Cx + Du + v) \in C(\mathcal{L})$ is equivalent to the existence of a $\lambda > 0$ such that the following equations hold:

$$x_p = \|u\|,$$

$$Cx + Du + v = -\lambda u, \tag{5.1}$$

$$e^\top(Ax + Bu + y) = \|Cx + Du + v\| = \lambda\|u\| \tag{5.2}$$

and

$$(x - \|u\|e, Ax + Bu + y - \|Cx + Du + v\|e^p) \in C(\mathbb{R}^p_{\geq+}). \tag{5.3}$$

By using (5.3), we conclude that

$$(F(x, u), G(x, u)) \in C(\mathbb{R}^p_{\geq+}).$$

By using equation (5.1) and (5.2), we have

$$H(x, u) = ue^\top(Ax + Bu + y) + \|u\|(Cx + Du + v) = 0.$$

Thus, z being a solution of $LCP(T, r, \mathcal{L})$ is equivalent to z being a solution of

$MiICP(G, H, F, \mathbb{R}^p_{\geq +})$.

(iv) Let $\bar{z} = (\bar{x}, u) = (x - \|u\|e, u)$ then by using the notations and conclusion in (iii), we have $F(x, u) \perp G(x, u)$. We also have that

$$F(x, u) = x - \|u\|e = \bar{x}$$

and

$$\begin{aligned} G(x, u) &= Ax + Bu + y - \|Cx + Du + v\|e^p \\ &= A(\bar{x} + \|u\|e) + Bu + y - \|C(\bar{x} + \|u\|e) + Du + v\|e^p \\ &= \bar{G}(\bar{x}, u). \end{aligned}$$

Thus, $\bar{x} \perp \bar{G}(\bar{x}, u)$.

From the proof of (iii), we get

$$\begin{aligned} 0 = H(x, u) &= ue^\top (Ax + Bu + y) + \|u\|(Cx + Du + v) \\ &= ue^\top (A(\bar{x} + \|u\|e) + Bu + y) + \|u\|(C(\bar{x} + \|u\|e) + Du + v) \\ &= \bar{H}(\bar{x}, u) \end{aligned}$$

Hence, $z = (x, u)$ being a solution of $LCP(T, r, \mathcal{L})$ is equivalent to $\bar{z} = (x - \|u\|e, u)$ being a solution of $MiCP(\bar{G}, \bar{H}, \mathbb{R}^p_{\geq +})$.

(v) If $z = (x, u)$ is a solution of the linear complementarity problem $LCP(T, r, \mathcal{L})$ we have

$$\mathcal{L} \ni \begin{pmatrix} x \\ u \end{pmatrix} \perp \begin{pmatrix} Ax + Bu + y \\ Cx + Du + v \end{pmatrix} \in \mathcal{M}.$$

From $(x,u) \in \mathcal{L}$ and $(Ax+Bu+y, Cx+Du+v) \in \mathcal{M}$, we have

$$\begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{p-1} - x_p \\ x_p - \|u\| \end{pmatrix} \in \mathbb{R}^p_+ \qquad \text{and} \qquad \begin{pmatrix} (Ax+Bu+y)_1 \\ \sum_{i=1}^{2}(Ax+Bu+y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax+Bu+y)_i \\ \sum_{i=1}^{p}(Ax+Bu+y)_i \end{pmatrix} \in \mathbb{R}^p_+.$$

We also note that, from Proposition 2.1.6 it follows that for an arbitrary vector $(x,u,y,v) \in C(\mathcal{L})$, we have the conditions

$$\begin{aligned} & x_p = \|u\|, \\ & \sum_{i=1}^{p} y_i = \|v\|, \\ & (x_i - x_{i+1}) \sum_{j=1}^{i} y_j = 0, \ \forall i = 1, \dots, p-1. \\ & v = -\lambda u. \end{aligned}$$

Then, in our case, since $(x, u, Ax+Bu+y, Cx+Du+v) \in C(\mathcal{L})$, we have

$$\mathbb{R}^p_+ \ni \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{p-1} - x_p \\ x_p - \|u\| \end{pmatrix} = \alpha \perp \beta = \begin{pmatrix} (Ax+Bu+y)_1 \\ \sum_{i=1}^{2}(Ax+Bu+y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax+Bu+y)_i \\ \sum_{i=1}^{p}(Ax+Bu+y)_i \end{pmatrix} \in \mathbb{R}^p_+. \tag{5.4}$$

where

$$x_p = \|u\|, \ Cx+Du+v = -\lambda u, \ \text{and} \ \sum_{i=1}^{p}(Ax+Bu+y)_i = \|Cx+Du+v\|.$$

Then the problem of finding a solution $z = (x,u)$ of the linear complementarity problem

$LCP(T, r, \mathcal{L})$ is converted to a problem of finding a vector $z = (x, u)$ such that $(\alpha, \beta) \in C(\mathbb{R}^p_+)$.

Moreover, let $w \in \mathbb{R}^p$ such that $w_i = x_i - x_{i+1}$ for any $i = 1, 2, \ldots, p-1$ and $w_p = x_p - \|u\| = 0$. Then we have $x = x(w)$ where $x_i(w) = \sum_{j=i}^{p-1} w_j + x_p = \sum_{j=i}^{p-1} w_j + \|u\|$ for any $i = 1, 2, \ldots, p-1$ and $x_p(w) = \|u\|$. Thus, (5.4) is equivalent to

$$\mathbb{R}^p_+ \ni \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{p-1} \\ w_p \end{pmatrix} = \alpha \perp \beta = \begin{pmatrix} (Ax(w) + Bu + y)_1 \\ \sum_{i=1}^{2}(Ax(w) + Bu + y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax(w) + Bu + y)_i \\ \sum_{i=1}^{p}(Ax(w) + Bu + y)_i \end{pmatrix} \in \mathbb{R}^p_+. \tag{5.5}$$

We also have from the solution of (iv) that

$$\hat{H}(w, u) = ue^\top(Ax(w) + Bu + y) + \|u\|(Cx(w) + Du + v) = 0.$$

Hence, the solution of (5.5) is equivalent to the solution of $MiCP(\hat{G}, \hat{H}, \mathbb{R}^{p-1}_+)$.

(vi) Note that the function $\hat{H}(w, u)$ is a semi-smooth function and it is not differentiable at $u = 0$. Thus, we need to reformulate this function to make sure it can be differentiable everywhere. Let $t = \|u\|$. Then, similarly to the proof of (v), for any $(x, u, Ax + Bu + y, Cx + Du + v) \in C(\mathcal{L})$, we have

$$\mathbb{R}^p_+ \ni \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{p-1} - x_p \\ x_p - t \end{pmatrix} = \alpha \perp \beta = \begin{pmatrix} (Ax + Bu + y)_1 \\ \sum_{i=1}^{2}(Ax + Bu + y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax + Bu + y)_i \\ \sum_{i=1}^{p}(Ax + Bu + y)_i \end{pmatrix} \in \mathbb{R}^p_+. \tag{5.6}$$

where

$$x_p = t,\ Cx + Du + v = -\lambda u,\ \text{and}\ \sum_{i=1}^{p}(Ax + Bu + y)_i = \|Cx + Du + v\|.$$

Next, let $\hat{w} \in \mathbb{R}^p$ such that $\hat{w}_i = x_i - x_{i+1}$ for any $i = 1, 2, \ldots, p-1$ and $\hat{w}_p = x_p - t = 0$. Then, we have $x = x(\hat{w}, t)$, where $x_i(\hat{w}) = \sum_{j=i}^{p-1} \hat{w}_j + x_p = \sum_{j=i}^{p-1} \hat{w}_j + t$, for any $i = 1, 2, \ldots, p-1$ and $x_p(w) = \|u\|$. Thus, (5.6) is equivalent to

$$\mathbb{R}^{p-1}_+ \ni \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_{p-1} \end{pmatrix} = \hat{\alpha} \perp \hat{\beta} = \begin{pmatrix} (Ax(\hat{w}, t) + Bu + y)_1 \\ \sum_{i=1}^{2}(Ax(\hat{w}, t) + Bu + y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax(\hat{w}, t) + Bu + y)_i \end{pmatrix} \in \mathbb{R}^{p-1}_+. \tag{5.7}$$

We also have from the solution of (v) that

$$\tilde{H}(\hat{w}, u, t) = \begin{pmatrix} ue^\top(Ax(\hat{w}, t) + Bu + y) + t(Cx(\hat{w}, t) + Du + v) \\ t^2 - \|u\|^2 \end{pmatrix} = 0$$

Hence, the solution of (5.7) is equivalent to the solution of $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_+)$.

□

5.2 Fischer–Burmeister function and the associated merit function

Theorem 5.1.1 shows that the linear complementarity problem on the monotone extended second order cone can be converted to a mixed complementarity problem defined on the non-negative orthant (defined by Facchinei and Pang, see Subsection 9.4.2 in [28]). By using this transformation scheme, we can study the linear complementarity problem on the MESOC by finding the solution to the converted problem, which is the mixed complementarity problem on the nonnegative orthant. Moreover, in [34, 35], Fischer introduced the Fischer–Burmeister complementarity function, which is a useful tool in finding the solution to the mixed complementarity problem on the nonnegative orthant. For arbitrary numbers a and b, the Fischer–Burmeister (FB) complementarity function (C-function) is defined as follows

$$\phi(a,b) = \sqrt{a^2+b^2} - (a+b).$$

From the definition of the FB C-function, we can conclude the following property

$$\phi(a,b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0 \text{ and } ab = 0.$$

We also note that $\phi(a,b)$ is a continuously differentiable function on $\mathbb{R}^2 \setminus O$. By using the function above, for any continuously differentiable function $\tilde{G}$, where $\tilde{G} = (\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_{p-1})$. Then by using the notations given in item (vi) in Theorem 5.1.1, the mixed complementarity problem, $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$ is equivalent to the following root finding problem for $\Phi(\hat{\omega}, u, t) = 0$, where

$$\Phi(\hat{\omega}, u, t) = \begin{pmatrix} \phi(\hat{\omega}_1, \tilde{G}_1(\hat{\omega}, u, t)) \\ \phi(\hat{\omega}_2, \tilde{G}_2(\hat{\omega}, u, t)) \\ \vdots \\ \phi(\hat{\omega}_{p-1}, \tilde{G}_{p-1}(\hat{\omega}, u, t)) \\ \tilde{H}(\hat{\omega}, u, t) \end{pmatrix}.$$

Since the mixed complementarity problem $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$ is generated by using the FB function, then we have a very useful property, that is, if there exists a point $(\hat{\omega}^*, u^*, t^*)$, such that the function

$$\Phi(\hat{\omega}^*, u^*, t^*) = 0,$$

then we have that the point $(\hat{\omega}^*, u^*, t^*)$ is a solution to the mixed complementarity problem $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$. Note that the last equality is semi-smooth. Thus, it can be solved by using the semi-smooth Newton method, which means we need to formulate the generalised Jacobian for the function $\Phi(\hat{\omega}, u, t)$, which in the sequel we will call generalised Jacobian.

Meanwhile, there is another way to reformulate the semi-smooth function to a smooth function. Note that for the FB C-function $\phi(a, b)$, we have $\phi(a, b)^2$ is continuously differentiable in $\mathbb{R}^2$. Then, consider the associated natural merit function of $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$

$$\Psi(\hat{\omega}, u, t) := \frac{1}{2}\Phi(\hat{\omega}, u, t)^\top \Phi(\hat{\omega}, u, t).$$

The function $\Psi(\hat{\omega}, u, t)$ is continuously differentiable. Note that the natural merit function equals zero at a point x^* if and only if x^* is a solution of $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$. Meanwhile, we have $\Psi(\hat{\omega}, u, t) \geq 0$ as it is a quadratic function. Thus, if the point $(\hat{\omega}^*, u^*, t^*)$ is a solution to $\Psi(\hat{\omega}, u, t) = 0$, then it is also a global minimiser of $\Psi(\hat{\omega}, u, t)$. On the other hand, if the point $(\hat{\omega}^*, u^*, t^*)$ is the global minimiser of $\Psi(\hat{\omega}, u, t)$ such that $\Psi(\hat{\omega}^*, u^*, t^*) = 0$, then the point $(\hat{\omega}^*, u^*, t^*)$ is a solution to the mixed complementarity problem $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$. Thus, the problem of finding a solution to $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$ is equivalent to the problem of finding the stationary point $(\hat{\omega}^*, u^*, t^*)$ of the unconstrained problem $\{\min \Psi(\hat{\omega}, u, t)\}$. Then in Section 5.3, we discuss the case where the Fischer-Burmeister function was implemented, and in Section 5.4, we focus on how to find the minimiser of the associated natural merit function of $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$.

5.3 Generalized Newton method for semismooth function

In this section, the formula of the generalised Jacobian for the FB-C function will be derived, and the corresponding semismooth Newton method will also be presented.

First, let us define the following matrix. Let $D_1 = diag(d_{11}(\hat{\omega}, u, t), \ldots, d_{p-1,p-1}(\hat{\omega}, u, t))$

and $D_2 = diag(d'_{11}(\hat{\omega}, u, t), \ldots, d'_{p-1,p-1}(\hat{\omega}, u, t))$, where

$$d_{ii} = \frac{\hat{\omega}_i}{\sqrt{\hat{\omega}_i^2 + (\tilde{G})_i^2(\hat{\omega}, u, t)}} - 1$$

and

$$d'_{ii} = \frac{\tilde{G}_i(\hat{\omega}, u, t)}{\sqrt{\hat{\omega}_i^2 + (\tilde{G})_i^2(\hat{\omega}, u, t)}} - 1$$

when $\hat{\omega}_i \neq 0$ or $(\tilde{G})_i \neq 0$. For the case when $\hat{\omega}_i = 0 = (\tilde{G})_i$, we have,

$$(d_{ii}, d'_{ii}) \in \{(y, z) : (y+1)^2 + (z+1)^2 = 1\}.$$

Then the generalised Jacobian of the FB C-function is the set given by

$$\partial\Phi(\hat{\omega}, u, t) \subseteq \begin{pmatrix} D_1 + D_2 J_{\hat{\omega}} \tilde{G}(\hat{\omega}, u, t) & D_2 J_{(u,t)} \tilde{G}(\hat{\omega}, u, t) \\ J_{\hat{\omega}} \tilde{H}(\hat{\omega}, u, t) & J_{(u,t)} \tilde{H}(\hat{\omega}, u, t) \end{pmatrix}.$$

Thus, for an arbitrary element in the set of generalised Jacobian

$$\mathbb{G} \in \partial\Phi(\hat{\omega}, u, t)$$

when $\hat{\omega}_i \neq 0 \neq \tilde{G}_i(\hat{\omega}, u, t)$, we have

$$\begin{aligned}(\mathbb{G})_i(\hat{\omega}, u, t) &= d_{ii} e^i + d'_{ii} J_{\hat{\omega}}(\tilde{G}_i(x, u, t)) \\ &= \left(\frac{\hat{\omega}_i}{\sqrt{\hat{\omega}_i^2 + (\tilde{G})_i^2(\hat{\omega}, u, t)}} - 1 \right) e^i + \left(\frac{(\tilde{G})_i}{\sqrt{\hat{\omega}_i^2 + (\tilde{G})_i^2(\hat{\omega}, u, t)}} - 1 \right) J_{\hat{\omega}} \tilde{G}_i(x, u, t).\end{aligned}$$

Then for the case when $\hat{\omega}_i = 0 = (\tilde{G})_i$, note that the FB C-function is semismooth at the point $(0, 0)$, then the generalised Jacobian can be obtained by using the generalised gradient on a composite function with $\partial\|(0, 0)\| = \mathbf{B}((0, 0), 1)$, where $\mathbf{B}(\mathbf{x}, 1)$ denotes a closed ball centred at a point $\mathbf{x}$ with the radius of 1. Thus, for the case when $\hat{\omega}_i = 0 =$

$\tilde{G}_i(\hat{\omega}, u, t)$, we have

$$(\mathbb{G})_i(\hat{\omega}, u, t) = \left\{ \left(d_{11} e^i + d'_{11} J_{\hat{\omega}} \tilde{G}_i(\hat{\omega}, u, t) : \, (d_{11}, d'_{11}) \in \mathbf{B}((-1,-1),1) \right) \right\}.$$

After obtaining the generalised Jacobian formula for the FB C-function, the conceptual version of the semismooth Newton method is given as follows.

Algorithm 1 Newton's method for the semismooth systems

Step 0: Set an initial point $z_0 = (\hat{\omega}, u, t) \in \mathbb{R}^{p+q}$, set $k = 0$

Step 1: Unless we have $\Phi(z_k) = 0$, solve the following system

$$\mathbb{G}(z_k) d_k = -\Phi(z_k)$$

and get the value d_k, where $\mathbb{G}(z_k)$ is an arbitrary element picked in $\partial\Phi(z_k)$

Step 2: Set $z_{k+1} = z_k + d_k$, $k = k + 1$ and go back to Step 1.

Then, by using Algorithm 1, we can obtain the solution $z^* = (\hat{\omega}^*, u^*, t^*)$ to the equation $\Phi(\hat{\omega}, u, t) = 0$. Hence, $z^* = (\hat{\omega}^*, u^*, t^*)$ is a solution to the mixed complementarity problem $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$. Thus, by using the following backtracking method derived from item (vi) in Theorem 5.1.1, we obtain the solution (x^*, u^*) to the linear complementarity problem $LCP(T, r, \mathcal{L})$.

Algorithm 2 Backtracking method

Step 0: Set a null vector $x \in \mathbb{R}^p$, set $k = p - 1$

Step 1: Let $x_p = t$,

Step 2: Stop until $k = 0$, let

$$x_k = x_{k+1} + \hat{\omega}_k.$$

Step 3: Set $k = k - 1$ and go back to Step 2.

Note that in Algorithm 1, we can only obtain the step-size d_k when the generalised Jacobian of the FB C-function is non-singular. Then, we need to illustrate the condition of the generalised Jacobian. For the sake of completeness, the definition of the P_0 matrix is given below.

Definition 5.3.1 (P_0 **matrix**). *[see [20]] Denote $A \in \mathbb{R}^{n\times n}$ be a matrix, then the matrix A is a P_0 matrix if all the principle minors of matrix A are non-negative. Moreover, A is a P-matrix if all the principle minors of A are positive.*

Next, we define a quadruple of disjoint index sets and a sufficient condition for the generalised Jacobian to be nonsingular. For any $i = 1, 2, \ldots, k$, denote

$$\alpha = \left\{ i : \hat{\omega}_i = 0 < \tilde{G}_i(\hat{\omega}, u, t) \right\}$$
$$\beta = \left\{ i : \hat{\omega}_i = 0 = \tilde{G}_i(\hat{\omega}, u, t) \right\}$$
$$\gamma = \left\{ i : \hat{\omega}_i > 0 = \tilde{G}_i(\hat{\omega}, u, t) \right\}$$
$$\delta = \{1, 2, \ldots, k\} \setminus (\alpha \cup \beta \cup \gamma)$$

Then the following proposition holds:

Proposition 5.3.1 (Proposition 9.4.2, [28]). *If $\tilde{G}(\hat{\omega}, u, t)$ and $\tilde{H}(\hat{\omega}, u, t)$ are two continuously differentiable function, let $\bar{\gamma} := \alpha \cup \beta \cup \delta$. Then, if the following two conditions hold, then the Jacobian is nonsingular.*

- *The following matrices are nonsingular for any $\bar{\gamma}$ satisfied $\gamma \subseteq \bar{\gamma} \subseteq \gamma \cup \beta$:*

$$\begin{pmatrix} J_{(u,t)} \tilde{H}(\hat{\omega}, u, t) & J_{\hat{\omega}_{\bar{\gamma}}} \tilde{H}(\hat{\omega}, u, t) \\ J_{(u,t)} \tilde{G}_{\bar{\gamma}}(\hat{\omega}, u, t) & J_{\hat{\omega}_{\bar{\gamma}}} \tilde{G}_{\bar{\gamma}}(\hat{\omega}, u, t) \end{pmatrix}$$

- *The Schur complement of the matrix*

$$\begin{pmatrix} J_{(u,t)} \tilde{H}(\hat{\omega}, u, t) & J_{\hat{\omega}_{\bar{\gamma}}} \tilde{H}(\hat{\omega}, u, t) \\ J_{(u,t)} \tilde{G}_{\bar{\gamma}}(\hat{\omega}, u, t) & J_{\hat{\omega}_{\bar{\gamma}}} \tilde{G}_{\bar{\gamma}}(\hat{\omega}, u, t) \end{pmatrix}$$

  in

$$\begin{pmatrix} J_{(u,t)} \tilde{H}(\hat{\omega}, u, t) & J_{\hat{\omega}_{\bar{\alpha}}} \tilde{H}(\hat{\omega}, u, t) \\ J_{(u,t)} \tilde{G}_{\bar{\gamma}}(\hat{\omega}, u, t) & J_{\hat{\omega}_{\bar{\alpha}}} \tilde{G}_{\bar{\alpha}}(\hat{\omega}, u, t) \end{pmatrix}$$

  is a P_0 matrix, where $\bar{\alpha} = \beta \cup \gamma \cup \delta$.

It is known (see Theorem 7.5.3 in [28]) that if the generalised Jacobian of the FB C-function is non-singular, then the convergence rate of the semismooth Newton method for the mixed complementarity problem $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_+)$ is Q-quadratic. In order to solve the mixed complementarity problem $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$, one may use some other algorithms, which have been discussed in [9, 28, 82], such as Levenberg-Marquardt algorithm or semismooth inexact Newton method. However, as the convergence rates for both the Levenberg-Marquardt algorithm and the semismooth inexact Newton method are at least linear, but when the generalised Jacobian is singular, the semismooth inexact Nowton method cannot be implied while the Levenberg-Marquardt algorithm can be used, see [82].

5.4 Finding the minimiser of the merit function

In this section, we will discuss the other approach we considered in the previous section when solving the mixed complementarity problem. Recall that $(\hat{\omega}^*, u^*, t^*)$ is a solution to the mixed complementarity problem if it is a solution of the merit function $\Psi(\hat{\omega}, u, t) = 0$. Since the merit function $\Psi(\hat{\omega}, u, t)$ is a quadratic function, then if $(\hat{\omega}^*, u^*, t^*)$ is a solution of $\Psi(\hat{\omega}, u, t) = 0$, then $(\hat{\omega}^*, u^*, t^*)$ is a global minimiser of the function $\Psi(\hat{\omega}, u, t)$. Since the merit function is continuously differentiable, then it is convex, then it is not difficult to find its global minimiser. Otherwise, if the merit function is nonconvex, the problem becomes difficult as we lack tools to work with a nonconvex case. In order to address this problem, we introduce the definition of a stationary point.

Definition 5.4.1. *A point $(\hat{\omega}^*, u^*, t^*)$ is called as a stationary point of a merit function $\Psi(\hat{\omega}, u, t)$ if the following inequality holds for any $(\hat{\omega}, u, t) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}$*

$$\left\langle (\hat{\omega} - \hat{\omega}^*, u - u^*, t - t^*)^\top, \nabla\Psi(\hat{\omega}^*, u^*, t^*) \right\rangle \geq 0.$$

The above problem is a variational inequality problem. But we can find that a point $(\hat{\omega}^*, u^*, t^*)$ is a stationary point of $\Psi(\hat{\omega}, u, t)$ cannot guarantee that it is also a global minimiser of $\Psi(\hat{\omega}, u, t)$. Thus, the definition of FB-regular will be introduced. First, let us consider the following index sets that have been introduced in [28],

$$\mathcal{C} = \{i : v_i \geq 0,\ H_i(u,v) \geq 0,\ v_i H_i(u,v) = 0\}, \qquad \text{(complementarity indices)}$$

$$\mathcal{R} = \{1,2,\ldots,n\} \setminus \mathcal{C}, \qquad \text{(residual indices)}$$

$$\mathcal{P} = \{i \in \mathcal{R} : v_i > 0,\ H_i(u,v) > 0\}, \qquad \text{(positive indices)}$$

$$\mathcal{N} = \{1,2,\ldots,q\} \setminus (\mathcal{C} \cup \mathcal{P}), \qquad \text{(negative indices)},$$

and for any arbitrary vector z, denote $z_{\mathcal{S}}$ to be the i-th coordinate of z, where i is an arbitrary integer such that $i \in \mathcal{S}$ and $\mathcal{S} \in \{\mathcal{C}, \mathcal{P}, \mathcal{N}\}$.

First, we will include the definition of a point to be FB-regular

Definition 5.4.2. ***(FB-regular)****[Definition 9.1.13, [28]] For the general formula of mixed complementarity problem $MiCP(G,H)$, for arbitrary $x \in \mathbb{R}^p$, $u \in \mathbb{R}^q$ and $t \in \mathbb{R}$, denote $\tilde{u} = (u,t)^\top$, then a point (x,u,t),is called FB-regular if $J_{\tilde{u}}H(x,u,t)$ is non-singular and if for any non-zero vector $z \in \mathbb{R}^q$ such that*

$$z_{\mathcal{C}} = 0, \qquad z_{\mathcal{P}} > 0, \qquad z_{\mathcal{N}} < 0,$$

there exists a non-zero vector $w \in \mathbb{R}^q$ such that

$$w_{\mathcal{C}} = 0, \qquad w_{\mathcal{P}} \geq 0, \qquad w_{\mathcal{N}} \leq 0$$

and

$$z^\top \left(M(x,u,t)/J_{\tilde{u}}H(x,u,t)\right) w \geq 0$$

where

$$M(x,u,t) = \begin{pmatrix} J_x G(x,u,t) & J_{\tilde{u}} G(x,u,t) \\ J_x H(x,u,t) & J_{\tilde{u}} H(x,u,t) \end{pmatrix} \in \mathbb{R}^{(p+q+1)\times(p+q+1)}$$

and $M(x,u,t)/J_{\tilde{u}}H(x,u,t)$ is the Schur complement of $J_{\tilde{u}}H(x,u,t)$ in $M(x,u,t)$.

Thus, in our case, for the mixed complementarity problem $MiCP\left(\tilde{G}(\hat{w},u,t),\tilde{H}(\hat{w},u,t)\right)$, the Jacobian of $\tilde{G}$ and $\tilde{H}$ are given as

$$J\tilde{G}(\hat{w},u,t) = \left(J_{\hat{w}}\tilde{G}(\hat{w},u,t), J_{\tilde{u}}\tilde{G}(\hat{w},u,t)\right) = \begin{pmatrix}\tilde{A} & \tilde{B}\end{pmatrix},$$
$$J\tilde{H}(\hat{w},u,t) = \left(J_{\hat{w}}\tilde{H}(\hat{w},u,t), J_{\tilde{u}}\tilde{H}(\hat{w},u,t)\right) = \begin{pmatrix}\tilde{C} & \tilde{D}\end{pmatrix},$$

where

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{11}+a_{12} & \dots & a_{11}+a_{12}+\dots+a_{1,p-1} \\ \sum_{i=1}^{2} a_{i1} & \sum_{i=1}^{2}(a_{i1}+a_{i2}) & \dots & \sum_{i=1}^{2}(a_{i1}+a_{i2}+\dots+a_{i,p-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{p} a_{i1} & \sum_{i=1}^{2}(a_{i1}+a_{i2}) & \dots & \sum_{i=1}^{2}(a_{i1}+a_{i2}+\dots+a_{i,p-1}) \end{pmatrix} = L_I A_{p-1,p-1} U_I,$$

where

$$L_I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

$$U_I = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

and $A_{i,j}$ is a sub-matrix of A, where

$$A_{i,j} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{pmatrix},$$

.

$$\tilde{B} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} & a_{11} + a_{12} + \dots + a_{1p} \\ \sum_{i=1}^{2} b_{i1} & \sum_{i=1}^{2} b_{i2} & \dots & \sum_{i=1}^{2} b_{iq} & \sum_{i=1}^{2} (a_{i1} + a_{i2} + \dots + a_{ip}) \\ \vdots & & & & \vdots \\ \sum_{i=1}^{p-1} b_{i1} & \sum_{i=1}^{p-1} b_{i2} & \dots & \sum_{i=1}^{p-1} b_{iq} & \sum_{i=1}^{p-1} (a_{i1} + a_{i2} + \dots + a_{ip}) \end{pmatrix} = \begin{pmatrix} L_I B & L_I A_{p-1,p} e \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} tC^* + ue^\top A^* \\ 0 \end{pmatrix},$$

where

$$A^* = \begin{pmatrix} a_{11} & a_{11} + a_{12} & \dots & \sum_{i=1}^{p-1} a_{1i} \\ a_{21} & a_{21} + a_{22} & \dots & \sum_{i=1}^{p-1} c_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p1} + a_{p2} & \dots & \sum_{i=1}^{p-1} a_{pi} \end{pmatrix}$$

and

$$C^* = \begin{pmatrix} c_{11} & c_{11} + c_{12} & \dots & \sum_{i=1}^{p-1} c_{1i} \\ c_{21} & c_{21} + c_{22} & \dots & \sum_{i=1}^{p-1} c_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q1} & c_{q1} + c_{q2} & \dots & \sum_{i=1}^{p-1} c_{qi} \end{pmatrix}$$

or equivalent to

$$\tilde{C} = \begin{pmatrix} tCU_I + ue^\top AU_I \\ 0 \end{pmatrix}.$$

Moreover

$$\tilde{D} = \begin{pmatrix} tD + ue^\top B + (Ax(\hat{w}, t) + Bu + y)^\top e I_{q \times q} & Du + v + Cx(\hat{w}, t) + tCe + ue^\top Ae \\ -2u^\top & 2t \end{pmatrix}$$

where

$$x(\hat{w}, t) = \begin{pmatrix} \hat{w}_1 + \hat{w}_2 + \ldots + \hat{w_{p-1}} + t \\ \hat{w}_2 + \ldots + \hat{w_{p-1}} + t \\ \vdots \\ \hat{w_{p-1}} + t \end{pmatrix}.$$

Then, if the Jacobian $\tilde{D} = J_{\tilde{u}} \tilde{H}(\hat{w}, u, t)$ is non-singular, the Schur complement of $\tilde{D}$ of the matrix $M(\hat{w}, u, y)$ is

$$\left(M/\tilde{D}\right) = \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$$

The following theorem was introduced by Facchinei and Pang. For the sake of completeness, we quote Theorem 9.4.4 in [28] and provide detailed proof here.

Theorem 5.4.1. *For arbitrary vectors $\hat{w} \in \mathbb{R}^{p-1}$, $u \in \mathbb{R}^q$ and $t \in \mathbb{R}$, we have $z = (\hat{w}, u, t)$ is a solution of $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_+)$ if and only if z is a FB-regular point of $\Psi(x)$ as well as a stationary point of $\Phi(x)$.*

Proof. First, suppose $z = (\hat{w}, u, t)$ is a solution to $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_+)$. Then we have $z = (\hat{w}, u, t)$ as a stationary point as well as the global minimum of the associate merit function $\Phi(x)$. Moreover, $z = (\hat{w}, u, t)$ is a solution to $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_+)$, which implies that $(\hat{w}, \tilde{G}(z)) \in C(\mathbb{R}^{p-1}_+)$. Then we have $\hat{w} = w_c$. Thus, the FB-regularity holds for $\hat{w}$ and $\mathcal{P} = \emptyset = \mathcal{N}$.

Conversely, if $z = (\hat{w}, u, t)$ is a stationary point of the merit function $\Psi(x)$, then $\nabla\Psi(z) = 0$ which implies that

$$(\partial\Phi(\hat{w}, u, t))^\top \Phi(\hat{w}, u, t) = \begin{pmatrix} D_1 + \tilde{A}^\top D_2 & \tilde{C}^\top \\ \tilde{B}^\top D_2 & \tilde{D}^\top \end{pmatrix} \Phi(\hat{w}, u, t) = 0$$

Thus, for any arbitrary vector $x \in \mathbb{R}^{p+q}$, we have

$$x^\top \begin{pmatrix} D_1 + \tilde{A}^\top D_2 & \tilde{C}^\top \\ \tilde{B}^\top D_2 & \tilde{D}^\top \end{pmatrix} \Phi(\hat{w}, u, t) = 0. \tag{5.8}$$

For vector x, we have that

$$x_{\mathcal{C}} = 0, \qquad x_{\mathcal{P}} > 0, \qquad x_{\mathcal{N}} < 0.$$

Then if z is not a solution to $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_{+})$, we have $\{1, 2, \ldots, p+q\} \setminus \mathcal{C} \neq \emptyset$. Let $y := D_2\Phi(x)$, see [Equation (9.1.14), [28]], and we have

$$y_{\mathcal{C}} = 0, \qquad y_{\mathcal{P}} > 0, \qquad y_{\mathcal{N}} < 0.$$

By using the definitions of D_1 and D_2, we conclude that $D_1\Phi(x)$ and $D_2\Phi(x)$ have the same sign. Thus,

$$x^{\top}(D_1\Phi) = x_{\mathcal{C}}^{\top}(D_1\Phi)_{\mathcal{C}} + x_{\mathcal{P}}^{\top}(D_1\Phi)_{\mathcal{P}} + x_{\mathcal{N}}^{\top}(D_1\Phi)_{\mathcal{N}} > 0,$$

since $x_{\{1,2,\ldots,p+q\}\setminus\mathcal{C}} \neq 0$, and by using the regularity of $J\tilde{G}(z)^{\top}$, which is $A^{\top}$, we have

$$x^{\top}\tilde{A}^{\top}(D_2\Phi) = x^{\top}\tilde{A}^{\top}y \geq 0.$$

Then, these two inequalities above together contradict the condition (5.8). Thus, we have set $\{1, 2, \ldots, p+q\} \setminus \mathcal{C} = \emptyset$ and z is a solution to $MiCP(\tilde{G}, \tilde{H}, \mathbb{R}^{p-1}_{+})$. $\square$

Note that the process of checking FB regularity, in general, is complex and more computationally expensive. The following algorithm is based on the Linear Newton Method and presented in [28], which can be used to find the solution to the mixed complementarity problem. Even though we do not implement this algorithm in the remaining chapter, for the sake of completeness, its conceptual version is given as follows.

Algorithm 3 FB line search algorithm

Step 0: Set an initial point $z_0 = (\hat{\omega}, u, t) \in \mathbb{R}^{(p-1)\times\mathbb{R}^q\times\mathbb{R}}$, set $l = 0$, $\rho > 0$, $n > 1$

Step 1: Stop if we have $\|\nabla\Psi(z_l)\| = 0$.

Step 2: Pick an arbitrary $\mathbb{G} \in \partial\Phi(\hat{\omega}, u, t)$, solve the following system for d_i

$$\mathbb{G}(z_l)d_l = -\Phi(z_l)$$

. Reset $d_l = -\nabla\Psi(z_l)$ if the system above is not solvable or if the following condition

$$-\rho\|d_l\|^n \geq \nabla\Psi(z_l)^\top d_l$$

is not satisfied.

Step 3: Find the smallest nonnegative integer i_l, such that if we set $i = i_l$, we have the following inequality

$$\Psi(z_l) + \frac{\gamma\nabla\Psi(z_l)^\top d_l}{2^i} \geq \Psi\left(z_l + \frac{d_l}{2^i}\right)$$

holds. Then set $\tau_l = \frac{1}{2^{i_l}}$

Step 4: Set $z_{l+1} = z_l + d_k\tau_l$, $l = l + 1$. Then go back to Step 1.

5.5 A Numerical Example

In this section, we will give a numerical example of the linear complementarity problem defined on the MESOC, which is a more general case and satisfies item (iii) in Proposition 4.1.2. Let us consider the linear complementarity problem on the MESOC $\mathcal{L} \subset \mathbb{R}^3 \times \mathbb{R}^2$. Then for any arbitrary point $z = (x, u) \in \mathbb{R}^3\times\mathbb{R}^2$, the aim of finding the solution to the linear complementarity problem is to find $z = (x, u) \in \mathbb{R}^3\times\mathbb{R}^2$ such that $(z, Tz + r) \in C(\mathcal{L})$. By using item (vi) in Theorem 5.1.1, the solution $z = (x, u)$ of the linear complementarity problem $LCP(T, r, \mathcal{L})$ is equivalent to the solution of the mixed complementarity problem

$MiCP(\tilde{G}, \tilde{H}, \mathbb{R}_+^{p-1})$ and we will have

$$\mathbb{R}_+^{p-1} \ni \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_{p-1} \end{pmatrix} \perp \tilde{G}(\hat{w}, u, t) = \begin{pmatrix} \tilde{G}_1(\hat{w}, u, t) \\ \tilde{G}_2(\hat{w}, u, t) \\ \vdots \\ \tilde{G}_{p-1}(\hat{w}, u, t) \end{pmatrix}$$
$$= \begin{pmatrix} (Ax(\hat{w}, t) + Bu + y)_1 \\ \sum_{i=1}^{2}(Ax(\hat{w}, t) + Bu + y)_i \\ \vdots \\ \sum_{i=1}^{p-1}(Ax(\hat{w}, t) + Bu + y)_i \end{pmatrix} \in \mathbb{R}_+^{p-1}$$

and

$$\tilde{H}(\hat{w}, u, t) = \begin{pmatrix} ue^\top(Ax(\hat{w}, t) + Bu + y) + t(Cx(\hat{w}, t) + Du + v) \\ t^2 - \|u\|^2 \end{pmatrix} = 0,$$

where

$$x(\hat{w}, t) = \begin{pmatrix} \hat{w}_1 + \hat{w}_2 + \ldots + \hat{w}_{p-1} + t \\ \hat{w}_2 + \ldots + \hat{w}_{p-1} + t \\ \vdots \\ \hat{w}_{p-1} + t \\ t \end{pmatrix}.$$

In order to find the solution to the mixed complementarity problem, we use the corresponding FB-based equation

$$\Phi(\hat{w}, u, t) = \begin{pmatrix} \phi(\hat{w}_1, \tilde{G}_1(\hat{w}, u, t)) \\ \phi(\hat{w}_2, \tilde{G}_2(\hat{w}, u, t)) \\ \vdots \\ \phi(\hat{w}_{p-1}, \tilde{G}_{p-1}(\hat{w}, u, t)) \\ \tilde{H}(\hat{w}, u, t) \end{pmatrix} = 0.$$

Let us consider the following example, where

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 3 \\ -2 & 6 & -1 & 0 & -1 \\ 1 & -3 & 0 & -1 & -2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix} \text{ and } r = \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 5 \end{pmatrix}$$

Since we have that matrices T, A and D are non-singular, then, if we set the tolerance $\tau = 10^{-7}$ and by using the semismooth Newton method provided in Algorithm 1, the sequence $\{z_k\} = \{(\hat{w}, u, t)_k\}$ will converge to a numerical solution to the mixed complementarity problem in 11 iterations.

Table 5.1: Numerical Results for the Mixed Complementarity Problem

Iteration	The value of $\Phi(z_k)$	Step-size d
0	2.17e+04	N/A
1	3.74e+03	87.2641
2	1.62e+03	15.3622
3	638.7488	9.21537
4	245.3569	4.68641
5	106.3955	2.09157
6	43.93762	1.12641
7	15.18293	0.50927
8	5.085132	0.22991
9	0.261361	0.03271
10	1.97e-03	5.1e-05
11	8.85e-08	1.2e-06

The solutions we got are

$$\hat{w}^* = \left(\frac{\sqrt{82 - 12\sqrt{46}}}{2}, 0 \right)^\top, \ t^* = \frac{\sqrt{82 - 12\sqrt{46}}}{2}$$

and

$$u^* = \left(\frac{-225 + 30\sqrt{46}}{82}, \frac{139 - 24\sqrt{46}}{82} \right)^\top.$$

Then, let us check whether this solution satisfies the condition of complementarity. We have

$$\hat{w}^* = \left(\frac{\sqrt{82-12\sqrt{46}}}{2}, 0\right)^\top \geq 0, \qquad \tilde{G}(\hat{w}^*, u^*, t^*) = \left(0, \frac{\sqrt{82-12\sqrt{46}}}{2}\right)^\top \geq 0.$$

Then

$$\mathbb{R}^2_+ \ni \hat{w}^* \perp \tilde{G}(\hat{w}, u, t) \in \mathbb{R}^2_+$$

Hence, we confirm that $\left(\hat{w}^*, \tilde{G}(\hat{w}, u, t)\right) \in C(\mathbb{R}^2_+)$.

By using the backtracking method provided in Algorithm 2, we have the solution to the linear complementarity problem, which is

$$\begin{aligned} z^* &= (x, u) \\ &= \left(\sqrt{82-12\sqrt{46}}, \frac{\sqrt{82-12\sqrt{46}}}{2}, \frac{\sqrt{82-12\sqrt{46}}}{2}, \frac{-225+30\sqrt{46}}{82}, \frac{139-24\sqrt{46}}{82}\right)^\top. \end{aligned}$$

By using the definition of the monotone extended second order cone, we have $z^* \in \mathcal{L}$, and

$$Tx + q = \begin{pmatrix} 1 & 0 & -2 & 1 & 3 \\ -2 & 6 & -1 & 0 & -1 \\ 1 & -3 & 0 & -1 & -2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{82-12\sqrt{46}} \\ \frac{\sqrt{82-12\sqrt{46}}}{2} \\ \frac{\sqrt{82-12\sqrt{46}}}{2} \\ \frac{-225+30\sqrt{46}}{82} \\ \frac{139-24\sqrt{46}}{82} \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{178-21\sqrt{46}}{41} \\ \frac{24\sqrt{46}+41\sqrt{82+12\sqrt{46}}+107}{82} \\ \frac{78\sqrt{46}-41\sqrt{82-12\sqrt{46}}-421}{82} \\ \frac{-36+54\sqrt{46}}{82} \\ \frac{324+6\sqrt{46}}{82} \end{pmatrix}$$

Then by using the definition of the dual cone of the monotone extended second order cone, we have $Tx + q \in \mathcal{M}$ and $\langle x, Tx + q\rangle = 0$. Thus, z^* is a solution to the linear complementarity problem.

5.6 Example for portfolio optimisation

As Facchinei and Pang summarized in [28], the Fischer-Burmeister function and the generalized Newton method can be used to solve both the linear and the nonlinear complementarity problems. In this section, we consider implementing this algorithm to solve a specific nonlinear complementarity problem, which is an application of a portfolio optimisation problem related to the monotone extended second order cone.

The classical mean-variance (MV) model was developed by Markowitz in [56]. Suppose we build a portfolio by using n arbitrary assets. Let $w \in \mathbb{R}^n$ denote the weights of the assets, $r \in \mathbb{R}^n$ represent the return of assets, and $\Sigma \in \mathbb{R}^n \times \mathbb{R}^n$ be the covariance matrix. Then, the two traditional and equivalent MV models could be given as:

$$\min_{w} \left\{ w^\top \Sigma w : \, r^\top w \geq \alpha, \, e^\top w = 1 \right\}$$

and

$$\max_{w} \left\{ r^\top w : \, w^\top \Sigma w \leq \beta, \, e^\top w = 1 \right\},$$

where α is the minimum profit that the investor demands and β is the maximal risk that the investor wants to tolerate. These are typical quadratic optimisation problems with higher computational complexity.

In order to reduce the complexity of solving the portfolio optimisation problem based on the traditional mean-variance model, lots of models have been introduced, such as the MAD model (see [52]), which has reduced the computational complexity significantly [50, 51]. But in [12], Bowen and Wentz also pointed out that the MAD model cannot provide an analytical solution while the analytic solution to the MV model can be present.

For $j = 1, \ldots, T$, define $U = (U_1, \ldots, U_T)^\top$, where $U_j = R^j - r$. Let y_j denote the upper bound of disturbance of return at day j , and $n \in \mathbb{R}$ is the number of assets in the portfolio. By using the Cauchy inequality, we also have $|U_j^\top w| \leq \|U_j\| \|w\|$ for any j. Denote f_j - the probability distribution of the rates of returns of assets, that is

$$f_j = \text{Probability} \left\{ (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n)^\top = \left(R_1^j, R_2^j, \ldots, R_n^j \right)^\top \right\} \in [0, 1],$$

where $j = 1, 2, \ldots, T$ denotes T different scenarios. Here $\hat{r} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n)^\top$ is a vector of returns of n different assets, and they are supposed to to be distributed over $R^j = (R_1^j, R_2^j, \ldots, R_n^j)$. Moreover, the sequence of $\{\hat{r}_j\}_{j=1,2,\ldots,T}$, $\{R_j\}_{j=1,2,\ldots,T}$ and $\{f_j\}_{j=1,2,\ldots,T}$ can be obtained by using the historical data of assets and some projection techniques to forecast the future behaviour of the assets. Meanwhile, since f_j is a probability vector, we have $e^\top f = 1$. Thus,

$$r = \mathbb{E}[\hat{r}] = \sum_{j=1}^{T} f_j R^j$$

The traditional MAD model can be given as follows,

$$\begin{aligned} \min_{y,w} \quad & c_0 f^\top y - r^\top w \\ \text{s.t.} \quad & y \geq \|U_j^\top w\|, \\ & e^\top w = 1, \\ & j = 1, 2, \ldots, T. \end{aligned}$$

Note that the constraint optimisation problem from the MAD model can be converted to a complementarity problem on the Lorentz cone. In reality, the uncertainty of the returns of the assets will increase with the increase of the investment horizon, as it becomes more difficult to predict the behaviour of the asset or the market. Thus, it is meaningful to optimize the MAD model to make it more in line with real-world market behaviour. Thus, we can provide a modified MAD model by giving the market risk an increasing trend. The modified MAD model, which is defined based on the MESOC, was proposed in our published paper [30]. It is defined as

$$\begin{aligned} \min_{y,w} \quad & c_0 f^\top y - r^\top w \\ \text{s.t.} \quad & y_T \geq y_{T-1} \geq \ldots \geq y_1 \geq \|U_{j^*}\|\|w\|, \\ & e^\top w = 1, \\ & j = 1, 2, \ldots, T. \end{aligned}$$

where $j^* = \operatorname{argmin}_j |U_j^\top w|$, for $j = 1, \ldots, T$ can also be obtained by using the historical data of assets, and $c_0 > 0$ is the Arrow-Pratt absolute risk-aversion index. Note that the vector

$$\left(\frac{y_T}{\|U_{j^*}\|}, \frac{y_{T-1}}{\|U_{j^*}\|}, \ldots, \frac{y_1}{\|U_{j^*}\|}, w\right)^\top$$

belongs to the monotone extended second order cone $\mathcal{L}_{T,n}$. Thus, the last problem is equivalent to the following conic optimisation problem:

$$\begin{aligned} \min_{y,w} \quad & c_0 f^\top y - r^\top w \\ \text{s.t.} \quad & e^\top w - 1 = 0, \\ & \left(\frac{y_T}{\|U_{j^*}\|}, \frac{y_{T-1}}{\|U_{j^*}\|}, \ldots, \frac{y_1}{\|U_{j^*}\|}, w\right)^\top \in \mathcal{L}_{T,n}, \end{aligned} \tag{5.9}$$

The KKT conditions of the problem (5.9) can be converted to the following complementarity problem

$$\mathcal{L} \ni \begin{pmatrix} \frac{y_T}{\|U_{j^*}\|} \\ \frac{y_{T-1}}{\|U_{j^*}\|} \\ \vdots \\ \frac{y_2}{\|U_{j^*}\|} \\ \frac{y_1}{\|U_{j^*}\|} \\ w \end{pmatrix} \perp \begin{pmatrix} c_0 f_T - \frac{\theta_T}{\|U_{j^*}\|} \\ c_0 f_{T-1} + \frac{\theta_T - \theta_{T-1}}{\|U_{j^*}\|} \\ \vdots \\ c_0 f_2 + \frac{\theta_3 - \theta_2}{\|U_{j^*}\|} \\ c_0 f_1 + \frac{\theta_2 - \theta_1}{\|U_{j^*}\|} \\ -r + \frac{\theta_1 w}{\|w\|} + \beta e \end{pmatrix} \in \mathcal{M}.$$

Note that this complementarity problem is a non-linear complementarity problem on the MESOC. By using Proposition 2.1.6 and Proposition 4.1.2, we have the following properties:

Proposition 5.6.1. *If* $-r + \frac{\theta_1 u}{\|u\|} + \beta e \neq 0$,

(i) There exists a $\lambda > 0$, *such that* $-r + \frac{\theta_1 w}{\|w\|} + \beta e = -\lambda w$.

(ii) $c_0 \sum_{i=1}^T f_i - \frac{\theta_1}{\|U_{j^*}\|} = \left\| -r + \frac{\theta_1 w}{\|w\|} + \beta e \right\|$.

(iii) $y_1 = \|w\| \|U_{j^*}\|$.

Since we have $\frac{y_T}{\|U_{j^*}\|} \geq \frac{y_{T-1}}{\|U_{j^*}\|} \geq \ldots \geq \frac{y_1}{\|U_{j^*}\|} \geq \|w\| > 0$, then the complementarity condition implies that $c_0 \sum_{i=1}^{T} f_i - \frac{\theta_1}{\|U_{j^*}\|} = 0$, which contradicts to the statement (ii) in Proposition 5.6.1. Meanwhile, the condition $e^\top w = 1$ imply that $w \neq 0$. Thus, in Proposition 4.1.2, items (i) and (iii) are inapplicable in the Problem 5.9 while item (ii) is. Then from item (ii) in Proposition 4.1.2, we have $-r + \frac{\theta_1 w}{\|w\|} + \beta e = 0$, which is equivalent to

$$w = -\frac{\|w\|}{\theta_1} \left(\beta e - r\right). \tag{5.10}$$

Meanwhile, by using $e^\top w = 1$ with (5.10) and let the number of assets be n, we have

$$1 = e^\top w = -\frac{\|w\|}{\theta_1} \left(n\beta - \langle r, e \rangle\right). \tag{5.11}$$

Let $\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i$, from (5.10) and (5.11), we have

$$w = \frac{\beta e - r}{n\beta - n\bar{r}}. \tag{5.12}$$

Note that because of the nature of the assets, we will hold that $\beta e \neq r$. Thus, $n\beta - n\bar{r} \neq 0$. Meanwhile, by using item (ii) in Proposition 4.1.2 again, we have

$$c_0 \sum_{i=1}^{T} f_i - \frac{\theta_1}{\|U_{j^*}\|} = \left\| -r + \frac{\theta_1 w}{\|w\|} + \beta e \right\| = 0, \tag{5.13}$$

which implies that $\theta_1 = c_0 \|U_{j^*}\| \sum_{i=1}^{T} f_i$. If we substitute this equality and (5.12) into (5.10), we have

$$\|\beta e - r\| = \left\| c_0 \|U_{j^*}\| \sum_{i=1}^{T} f_i \right\|,$$

which is equivalent to

$$n\beta^2 - 2n\bar{r}\beta + \|r\|^2 = \left(c_0 \|U_{j^*}\| \sum_{i=1}^{T} f_i \right)^2. \tag{5.14}$$

Note that following the definition of f_j, we hold that $\sum_{i=1}^{T} f_j = 1$. Then, by solving (5.14) for β, we have

$$\beta = \bar{r} \pm \sqrt{\frac{n\bar{r}^2 - \|r\|^2 + \left(c_0\|U_{j^*}\|\right)^2}{n}}. \tag{5.15}$$

Thus, following the existence of β, for any arbitrary solution (y, u) to the optimisation problem, we get the following condition

$$n\bar{r}^2 - \|r\|^2 + \left(c_0\|U_{j^*}\|\right)^2 \geq 0,$$

otherwise, β cannot be a real number. If we substitute the formula for β (5.15) into the denominator of (5.12), we have

$$w = \pm \frac{\beta e - r}{\sqrt{(n\bar{r})^2 - n\|r\|^2 + n\left(c_0\|U_{j^*}\|\right)^2}}.$$

By item (ii) in Proposition 4.1.2 we have $-r + \frac{\theta_1 w}{\|w\|} + \beta e = 0$. Meanwhile, by using (5.13) with $\sum_{i=1}^{T} f_j = 1$, we have $\theta_1 = c_0\|U_{j^*}\|$. Since we also have $c_0 > 0$, we conclude that $\mathbf{sgn}(w_i) = -\mathbf{sgn}(\beta - r_i)$, for $i = 1, 2, \ldots, n$. Thus, we have

$$w = -\frac{\left(\bar{r} - \sqrt{\frac{n\bar{r}^2 - \|r\|^2 + \left(c_0\|U_{j^*}\|\right)^2}{n}}\right) e - r}{\sqrt{(n\bar{r})^2 - n\|r\|^2 + n\left(c_0\|U_{j^*}\|\right)^2}}.$$

We can conclude that the weights of assets are related to the return of assets r, risk in the market $\|U_{j^*}\|$ and the risk preference from the investor c_0.

5.7 Conclusions and comments

In this chapter, we studied the linear complementarity problem on the monotone extended second order cone. Firstly, in the Theorem 5.1.1, which is the main theorem in this chapter, we have shown that the linear complementarity problem on the monotone extended second order cone can be converted to several kinds of complementarity problems. Most importantly, it can be converted to a mixed complementarity problem on the non-negative orthant. This transformation of the problem is very meaningful because, on the one hand, it can help reducing the complexity of finding the solutions to the original linear complementarity problem. On the other hand, various algorithms can be used to solve the mixed complementarity problem on the nonnegative orthant. In contrast, for the linear complementarity problem, we lack the tools to solve this kind of problem, especially when it is defined on the cones which are not self-dual. Since the process of solving the mixed complementarity problem is well studied, we can determine its solution by using either the FB-C function and the semismooth Newton method or by using the merit function associated with the FB-C function with the FB line search method. We also provide a numerical example showing by using the semismooth Newton method, that we can obtain the solution to the linear complementarity problem. In the end, an application based on the monotone extended second order cone is discussed, which is a portfolio optimisation problem. Unlike the traditional MAD model, the modified one based on the MESOC has an analytical solution, which has been present.

CHAPTER 6

GRADIENT PROJECTION METHOD ON THE SPHERE, COMPLEMENTARITY PROBLEMS AND COPOSITIVITY

6.1 Introduction

In this chapter, we study the connections between the existence of the solution to the cone complementarity problem and the copositivity of the operator (or matrix) with respect to the corresponding cone. As we mentioned in the previous chapters, the cone complementarity problem is an important topic with a variety of applications in many aspects. Some previous research, see [45–47],has already demonstrated that some important theorems in the cone complementarity problems can be converted to an optimisation problem defined on the intersection of the cone and sphere. Thus, in order to solve the constraint optimisation problem, we will discuss the gradient projection method in this chapter. Indeed, the intrinsic version of the gradient projection method and its variants will be proposed. Note that the gradient projection method can also be implemented in testing the copositivity of an operator with respect to $\mathcal{K}$ as well as checking the solvability of the complementarity problem defined on the cone $\mathcal{K}$. In this chapter, the cone $\mathcal{K}$ we discussed will be a proper cone. By using the algorithm, we conclude that the operator is not copositive with respect to the cone if we get a negative value. However, if we get a nonnegative value of the output of the algorithm, we are not completely sure that the operator is copositive with respect to the cone. To address this problem, two techniques can be applied. One is to use different starting points when setting this algorithm, and another way is to implement the algorithm several times. Both of the techniques will increase the likelihood that the algorithm will converge at a global minimiser. We also discussed the convergence analysis for the gradient projection method on the sphere. The results have already been resubmitted to the Journal of Global Optimization.

This chapter has been organized as follows. In Section 6.2 and Section 6.3, the concepts related to the cone complementarity problem and copositivity have been presented, as well as the relationship between $\mathcal{K}$-complementarity problems, problems of testing $\mathcal{K}$-copositivity and the constraint optimisation problem on the intersection of the cone $\mathcal{K}$ and a sphere. In order to consider a unified description of different cones, in these sections our work is in Euclidean vector spaces, including the case when $\mathcal{K}$ is the cone of positive semidefinite matrices. In Section 6.3, we provide examples for an operator to be $\mathcal{K}$-copositive when $\mathcal{K}$ is Lorentz cone. In Section 6.4, we present the concepts and basic results related to the gradient projection method and the closed spherically convex set, which are used in analysing this method in the later section. In Section 6.5, we discuss the problem of projection onto a closed spherically convex set and several properties obtained for this projection have also been present. In Section 6.6, in order to solve a general constrained optimisation problem, the gradient projection method on the sphere has been analysed. We also discussed two new variants of the gradient projection method. One is for the case when the Lipschitz constant cannot be easily obtained. The other one is for several special cases, such as when the cone $\mathcal{K}$ the nonnegative orthant $\mathbb{R}^n_+$, the Lorentz cone $\mathcal{L}^n$, and the cone of positive semidefinite matrices $\mathcal{S}^n_+$, respectively. Section 6.8 presents the numerical results we obtained by using the introduced algorithms. These results demonstrate how the gradient projection method on the sphere can be implemented to check the $\mathcal{K}$-copositivity of the linear mappings. In the last section, we provide some concluding remarks.

6.2 Complementarity problems and optimisation problem on the sphere

In this section, we illustrate the relationship between complementarity problems and an optimisation problem constrained to a suitable subset of the sphere. This relationship gives us a motivation for investigation in this chapter. For the sake of completeness, let us first recall some concepts.

Definition 6.2.1. ***(Inversion)*** *[[59], Definition 5] Let k be an arbitrary positive number, let i be an operator such that*

$$i : \mathbb{R}^k \setminus \{0\} \to \mathbb{R}^k \setminus \{0\};\ i(x) := \frac{x}{\|x\|^2}.$$

Then we call the operator i an inversion (of pole 0).

From the definition of the inversion, it is easy to check that the operator i is a one-to-one mapping, and we have that $i = i^{-1}$.

Definition 6.2.2. ***(Inversion of Mapping)*** *[[59], Definition 6] Denote $\mathbb{R}^n$ to be a finite n-dimensional real vector space and $F : \mathbb{R}^n \to \mathbb{R}^n$ to be an arbitrary mapping. The inversion of mapping of F is the mapping $\mathcal{I}(F) : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mathcal{I}(F)(x) := \begin{cases} \|x\|^2 F\left(\dfrac{x}{\|x\|^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We can also write the inversion mapping $\mathcal{I}(F)$ as

$$\mathcal{I}(F)(x) := \begin{cases} \|x\|^2 \left(F \circ i\right)(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We can also find that, following the definition of the inversion of mapping, we have $\mathcal{I}(\mathcal{I}(F)) = F$, or equivalent, $\mathcal{I}^{-1} = \mathcal{I}$.

Definition 6.2.3. ***(Lower scalar derivative)*** *[[47], Definition 1.6] Denote k be an arbitrary positive integer, let $\mathcal{K} \subseteq \mathbb{R}^k$ be a cone and $F : \mathcal{K} \to \mathbb{R}^k$ be a mapping, then we call*

$$\underline{F^{\#}}(x^*, \mathcal{K}) = \liminf_{x \to x^*,\, x - x^* \in \mathcal{K}} \frac{\langle F(x) - F(x^*), x - x^* \rangle}{\|x - x^*\|^2}$$

is the lower scalar derivative of F at x^* in the direction of $\mathcal{K}$.

Moreover, if the mapping F is Fréchet differentiable at x, Németh found another explicit way to show the formula of the lower scalar derivative in the following theorem.

Theorem 6.2.1. *[[59], Theorem 18] Denote k be an arbitrary positive integer, let x be an interior point of $\mathcal{K}$, where $\mathcal{K}$ is a proper cone in $\mathbb{R}^k$. We have*

$$\underline{F^{\#}}(x, \mathcal{K}) = \min_{\|u\|=1, u \in \mathcal{K}} \langle dF(x)u, u \rangle,$$

where $dF(x)$ is the differential of mapping F at point x.

Recall that the formula of the complementarity problem is given by

$$CP(F, \mathcal{K}) = \begin{cases} \text{Find } x^* \in \mathcal{K} \text{ such that} \\ F(x^*) \in \mathcal{K}^* \text{ and } \langle x^*, F(x^*) \rangle = 0. \end{cases}$$

In [46], Isac and Németh proved the following theorem.

Theorem 6.2.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary mapping and denote $\mathcal{K} \subset \mathbb{R}^n$ be a cone and $\mathcal{K}^*$ be the dual cone of $\mathcal{K}$. If we hold the following inequality*

$$\liminf_{x \to 0} \frac{\langle F(x) - F(0), x \rangle}{\|x\|^2} > 0,$$

then we conclude that the complementarity problem $CP(F, \mathcal{K})$ given above has a solution.

By using Definition 6.2.3, Theorem 6.2.1 and Theorem 6.2.2, if we denote $d\mathcal{I}(F)(0)$ as the *differential of the inversion of mapping* $\mathcal{I}(F)$ at 0, then we can conclude that whenever the mapping $\mathcal{I}(F)$ is differentiable at 0, we have

$$\liminf_{x\to 0} \frac{\langle F(x) - F(0), x\rangle}{\|x\|^2} = \min_{\|u\|=1, u\in\mathcal{K}} \langle d\mathcal{I}(F)(0)u, u\rangle. \tag{6.1}$$

Thus, the constrained optimisation problem (6.1) can present a sufficient condition to show the existence of a solution to the complementarity problem $CP(F, \mathcal{K})$, which is concluded in the following corollary.

Corollary 6.2.3. *Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a mapping such that the inversion of mapping of F, which is $\mathcal{I}(F)$ is differentiable at 0. Let $\mathcal{K} \subset \mathbb{R}^k$ be a cone. Then if the following inequality*

$$\min_{\|u\|=1, u\in\mathcal{K}} \langle d\mathcal{I}(F)(0)u, u\rangle > 0, \tag{6.2}$$

holds true, then the solution to the complementarity problem $CP(F, \mathcal{K})$ exists.

This corollary illustrates that the problem of showing the existence of the solution to a complementarity problem $CP(F, \mathcal{K})$ can be converted to a problem of proving the minimisation of a quadratic function on the intersection between the cone $\mathcal{K}$ and the sphere is positive.

The following theorem provides a class of mappings F such that the associated inversion of mapping $\mathcal{I}(F)$ is differentiable at 0.

Theorem 6.2.4. *Let n be an arbitrary positive integer, P_i, Q_i be two polynomial functions with k_i-the degree of the function P_i and m_i-the degree of the function Q_i, such that $m_i + 1 \geq k_i$. Suppose for all $x \in \mathbb{R}^n$ we have $Q_i(x) \neq 0$ for any $i \in \{1, 2, \ldots, r\}$, where $r \in \mathbb{N}$ is an arbitrary integer, and*

$$Q_i\left(\frac{x}{\|x\|^2}\right) \neq 0, \qquad \forall x \in \mathbb{R}^n,\ x \neq 0.$$

Let $q \in \mathbb{R}^n$ and denote $L : \mathbb{R}^n \to \mathbb{R}^n$ to be a linear mapping. Consider the following *mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ defined as*

$$F(x) = \sum_{i=1}^{r} \frac{P_i(x)}{Q_i(x)} e^i + Lx + q, \tag{6.3}$$

where e^i denotes the canonical unit vector. We conclude that the mappings F and the corresponding inversion of mapping $\mathcal{I}(F)$ are differentiable. Moreover, we conclude that $d\mathcal{I}(F)(0) = L$.

Proof. Since L, $Q_i(x) \neq 0$ and P_i, Q_i are differentiable, it follows that F is differentiable. Next, we prove the differentiability of $\mathcal{I}(F)$. For similar reasons as before, $\mathcal{I}(F)$ is differentiable for any $x \in D \setminus \{0\}$. Since $\mathbb{V} \ni x \mapsto \mathcal{I}(L+q)(x) = L(x) + q\|x\|^2$ is differentiable, it is enough to check the differentiability of $\mathcal{I}(P_i/Q_i)$ at 0 for an arbitrary i. Fix such an i and denote $g := \mathcal{I}(P_i/Q_i)$. Let $x \neq 0$. After some algebra, by using the homogeneity of the involved functions, we get

$$g(x) = \frac{\sum_{j=0}^{k_i} \|x\|^{2-2j} P_{ij}(x)}{\sum_{j=0}^{m_i} \|x\|^{-2j} Q_{ij}(x)} = \frac{\sum_{j=0}^{k_i} \|x\|^{2m_i-2j+2} P_{ij}(x)}{\sum_{j=0}^{m_i} \|x\|^{2m_i-2j} Q_{ij}(x)}, \tag{6.4}$$

where P_{ij} and Q_{ij} are the monomial terms of degree j in P_i and Q_i, respectively. We have $g(0) = 0$. Indeed, let us first consider the case $m_i + 1 = k_i$. Then, we have $P_{ik_i}(0) = 0$, because P_{ik_i} is homogeneous of degree $k_i > 0$, and the powers of $\|x\|$ in the remaining terms of the nominator of (6.4) are positive. Hence, $g(0) = 0$. If $m_i + 1 > k_i$, then $g(0) = 0$, because the powers of $\|x\|$ in all terms of the nominator of (6.4) are positive. In order to show that g is differentiable at 0, it is enough to prove that the directional derivative

$$\frac{\partial g}{\partial h}(0) = \lim_{t \to 0} \frac{g(th) - g(0)}{t} = \lim_{t \to 0} \frac{g(th)}{t}$$

exists and it is linear with respect to $h \in \mathbb{V}$. Since $Q_{im_i}(0) \neq 0$, it follows that $Q_{im_i}(v) \neq 0$ if v is sufficiently close to the origin. For such a v, by using (6.4) and again the homogeneity of the involved functions, we obtain

$$\frac{g(tv)}{t} = \frac{\displaystyle\sum_{j=0}^{k_i} t^{m_i-j+2}\|v\|^{m_i-j+2}P_{ij}(v)}{\displaystyle\sum_{j=0}^{m_i} t^{m_i-j}\|v\|^{m_i-j}Q_{ij}(v)},$$

which, after some algebraic manipulations, implies that

$$\frac{g(tv)}{t} = \frac{t\left(\|v\|^{m_i+1-k}P_{ik}(v)t^{m_i+1-k} + \cdots + \|v\|^{m_i+1}P_{i0}(v)t^{m_i+1}\right)}{Q_{im_i}(v) + \|v\|tQ_{i,m_i-1}(h)t + \cdots + \|v\|^{m_i}Q_{i0}(v)t^{m_i}}.$$

Since the nominator of the RHS of the second equality above is t multiplied by a polynomial of t (because the powers of t inside the bracket are nonnegative), it follows that $(\partial g/\partial v)(0) = 0$ if v is close enough to the origin. Hence, by using the positive homogeneity of the directional derivative, we obtain

$$\mathbb{V} \ni h \mapsto \frac{\partial g}{\partial h}(0) = 0,$$

which is linear. □

Then by using Theorem 6.2.4 and Corollary 6.2.3, we arrive at the following result.

Corollary 6.2.5. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone. If F is given as in* (6.3)*, then the complementarity problem $CP(F,\mathcal{K})$ has a solution if* $\min_{\|u\|=1, u\in\mathcal{K}}\langle Lu, u\rangle > 0$.

Moreover, we can also find that if all polynomial functions P_i in Corollary 6.2.5 reduce to the null function, then $CP(F,\mathcal{K})$ becomes a linear complementarity problem. Hence, Corollary 6.2.5 extends a well-known result from linear complementarity problem, which states that any positive definite matrix is a Q-matrix (see [20]).

By using [72], we obtain the following example, which gives a relationship between the complementary problem and a cone-constrained optimisation problem.

Example 6.2.1. *Denote $\mathcal{K} \subset \mathbb{R}^n$ be a cone and denote its dual cone by $\mathcal{K}^*$. Let $\varphi : \mathbb{R}^n \to$*

$\mathbb{R}$ *be a differentiable function and its gradient is denoted by* $D\varphi$. *Then, for the constrained optimisation problem*

$$\min_{x\in\mathcal{K}} \varphi(x) \tag{6.5}$$

the KKT conditions of it can be given as follows

$$D\varphi(x) = y,\ x \in \mathcal{K},\ y \in \mathcal{K}^*,\ \langle x, y\rangle = 0. \tag{6.6}$$

Hence, the KKT conditions of the optimisation problem (6.5) are equivalent to a complementarity problem $CP(F, \mathcal{K})$. *Let us consider an example. Suppose we have a function* $\varphi : \mathbb{R}^4 \to \mathbb{R}$ *defined by*

$$\varphi(x) = \frac{x_1^2 + x_3}{x_2^4 + x_4^4 + 1} + x_1^2 + x_3^2 + 3x_1x_3 + 2x_2x_4 + 5x_1 + 3x_3 + 4x_4,$$

where $x := (x_1, x_2, x_3, x_4)$. *Define the mapping* $F : \mathbb{R}^4 \to \mathbb{R}^4$ *by* $F(x) := D\varphi(x)$, *where* $D\varphi$ *represents the gradient of* φ. *Let us consider the mapping* F *given by the following formula*

$$F(x) = \sum_{i=1}^{4} \frac{P_i(x)}{Q_i(x)} e^i + Lx + q, \tag{6.7}$$

then we have $P_1(x) := 2x_1$, $Q_1(x) := x_2^4 + x_4^4 + 1$, $P_2(x) := -4x_2^3(x_1^2 + x_3)$, $Q_2(x) := (x_2^4 + x_4^4 + 1)^2$, $P_3(x) := 1$, $Q_3(x) := x_2^4 + x_4^4 + 1$, $P_4(x) := -4x_4^3(x_1^2 + x_3)$, $Q_4(x) := (x_2^4 + x_4^4 + 1)^2$ *and*

$$L := \begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \qquad q := \begin{pmatrix} 5 \\ 0 \\ 3 \\ 4 \end{pmatrix}. \tag{6.8}$$

Thus, by using Corollary 6.2.5, the solvability of the KKT conditions given in (6.6) *with* F *given by* (6.7) *depended on the strict copositivity of the matrix* L *given by* (6.8).

It should be mentioned that the complementarity problems with functions given in (6.3) can be considered as extensions of linear complementarity problems [20]. In this case, the KKT optimality conditions of quadratic optimisation problems can be converted to linear complementarity problems. If we consider extensions of quadratic optimisation problems which optimise the sum of special fractional polynomial functions and quadratic functions, then by writing the optimality conditions to the optimisation problem, we can have the complementarity problems with functions of type (6.3).

6.3 Copositivity with respect to cones

In this section, for the sake of completeness, we first introduce the basic concepts of $\mathcal{K}$-copositive and copositive. Then the relationship between the copositivity of a linear operator and the optimisation problem, which is constrained to the intersection of cone and sphere, is discussed.

Definition 6.3.1. ***($\mathcal{K}$-copositive)*** *Denote $\mathbb{R}^k$ be a n-dimensional real vector space, where k is an arbitrary positive integer. Let $\mathcal{K} \subseteq \mathbb{R}^k$ be a cone. A matrix $A \in \mathbb{R}^{k\times k}$ (or equivalently an operator $A : \mathbb{R}^k \to \mathbb{R}^k$) is called $\mathcal{K}-$copositive if for any $x \in \mathcal{K}$, we have*

$$\langle Ax, x\rangle \geq 0.$$

Moreover, if for any $x \in \operatorname{int} \mathcal{K}$, we have

$$\langle Ax, x\rangle > 0,$$

then we call A is strictly $\mathcal{K}$-copositive.

In particular, when $\mathcal{K}$ is the nonnegative orthant, i.e., when $\mathcal{K} = \mathbb{R}^k_+$, we have the following definition.

Definition 6.3.2. ***(Copositive)*** *A matrix $A \in \mathbb{R}^{k\times k}$ (or equivalent, a operator $A : \mathbb{R}^k \to$*

$\mathbb{R}^k$) *is called copositive if for any* $x \in \mathbb{R}^n_+$, *we have*

$$\langle Ax, x \rangle \geq 0.$$

Some researchers may use other names when they introduce the same definition. For example, in [27], the same concepts are called copositive with respect to set $\mathcal{K}$ or $\mathcal{K}$-semidefinite. In the research of combinatorial and non-convex quadratic optimisation, testing copositivity of different matrices is an important topic, but it is also difficult to test copositivity of a specifically given matrix as this problem is a co-NP-complete problem. From the previous research, in [4,14–16,24,78], we can find some algorithms which can be used in testing the copositivity of matrices. In [27], Eichfelder and Jahn showed that when $\mathcal{K}$ is a polyhedral cone, the problem of testing K-copositivity of matrices with respect to $\mathcal{K}$ can be reduced to the problem of testing classical copositivity. More importantly, in [6], the researchers found a way to test the classical copositivity of matrices by using the projection onto the intersection of a general cone and sphere. The next lemma illustrates the relationship between the copositivity and a quadratic programming (QP) problem on the intersection of cone and sphere, it is an immediate result of the Definition 6.3.1, and we will omit the proof.

Lemma 6.3.1. *Let* $A \in \mathbb{R}^{k\times k}$ *be an arbitrary matrix (or equivalent* $A : \mathbb{R}^k \to \mathbb{R}^k$ *be an operator) and* $\mathcal{K} \subseteq \mathbb{R}^k$ *be an arbitrary cone. Denote* $\bar{x}$ *be a (global) minimal solution to the quadratic optimisation problem (QP) defined as follows:*

$$\begin{aligned} \textbf{\textit{QP:}}\ \min f(x) &:= \frac{1}{2}\langle Ax, x\rangle \\ \langle x, x\rangle &= 1 \\ x \in \mathcal{K}, & \end{aligned} \tag{6.9}$$

Then we have the following conclusions:

(i) A *is* $\mathcal{K}$*-copositive if and only if* $f(\bar{x}) \geq 0$,

(ii) A is $\mathcal{K}$-strictly copositive if and only if $f(\bar{x}) > 0$.

(iii) A is not $\mathcal{K}$-copositive if and only if there exists a feasible x' such that $f(x') < 0$.

(iv) A is not $\mathcal{K}$-strictly copositive if there exists a feasible x' such that $f(x') = 0$.

Then, by using Corollary 6.2.5, we have the following result indicating the relationship between the copositivity and the existence of a solution to the corresponding cone complementarity problem.

Corollary 6.3.2. *Denote $\mathcal{K} \subseteq \mathbb{R}^k$ to be an arbitrary cone, and $I : \mathbb{R}^k \to \mathbb{R}^k$ be the identity mapping. Let F be defined by the equation (6.3), which is*

$$F(x) = \sum_{i=1}^{r} \frac{P_i(x)}{Q_i(x)} e^i + Lx + q.$$

Then, if there exists an $\alpha > 0$ such that $L - \alpha I$ is $\mathcal{K}$-copositive, we conclude that the complementarity problem $CP(F, \mathcal{K})$ has a solution.

Then, by Corollary 6.2.3, Corollary 6.3.2 and Lemma 6.3.1, we can conclude that the problem of the existence results for the complementarity problem $CP(F, \mathcal{K})$, where F is defined by equation (6.3), can be reduced to a problem of finding the global solution of problem (6.9) by testing the $\mathcal{K}$-copositivity of matrix A. Besides the nonnegative orthant, the conclusion from Corollary 6.3.2 also gives us a trigger to consider the copositivity with respect to other cones, such as the Lorentz cone $\mathcal{L}^n$ and the positive semidefinite cone. Recall the definition of Lorentz cone.

Definition 6.3.3. *For any $n > 1$, the Lorentz cone (also known as the ice-cream cone) in the Euclidean space $\mathbb{R}^n$ is defined as:*

$$\mathcal{L}^n = \left\{ (x, t)^\top \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leq t \right\}.$$

We also have that the Lorentz cone is symmetric cone.

Loewy and Schneider propose the following proposition in [53].

Proposition 6.3.3. *Let $\mathcal{L}^n$ be the Lorentz cone in dimension $\mathbb{R}^n$, let $A \in \mathcal{S}^n$ be an arbitrary $n \times n$ symmetric matrix, by denoting $J := \operatorname{diag}(-1, -1, \ldots, -1, 1)$ an $n \times n$ diagonal matrix, the matrix A is copositive with respect to $\mathcal{L}^n$ if and only if there exists a $\mu \in \mathbb{R}_+$ such that the matrix $A - \mu J$ is positive semidefinite.*

This proposition illustrates the necessary and sufficient conditions for a symmetric matrix to be copositive with respect to the Lorentz cone $\mathcal{L}^n$. Next, we state the homogeneous S-Lemma. Its proof can be found in [70] and will be omitted. The homogeneous S-Lemma, together with the conclusion from Proposition 6.3.3, provides a sufficient condition for a matrix to be $\mathcal{L}^n$-copositive.

Lemma 6.3.4. *(The Homogeneous S-Lemma). Suppose there exists x^* such that $(x^*)^\top Ax^* > 0$. If $x^\top Ax \geq 0$ implies $x^\top Bx \geq 0$ for any $x \in \mathbb{R}$, then there exists a $\mu \in \mathbb{R}_+$ such that $B - \mu A \succeq 0$, where $A \succeq 0$ is the notation of the positive semidefiniteness of matrix A.*

By using the results above, we have the following theorem.

Theorem 6.3.5. *Let F be defined by the equation (6.3) and denote $\mathcal{L}^n$ be the n-dimensional Lorentz cone. Suppose that there exists $\lambda, \mu > 0$, such that $L + \lambda I - \mu J$ is positive semidefinite. Then, the complementarity problem $CP(F, \mathcal{L}^n)$ has a solution.*

Proof. By using that $L + \lambda I - \mu J$ is positive semidefinite and Proposition 6.3.3, it follows that $L + \lambda I$ is copositive with respect to $\mathcal{L}^n$. Since

$$\liminf_{x \to 0} \frac{\langle F(x) - F(0), x \rangle}{\|x\|^2} = \min_{\|u\|=1, u \in \mathcal{L}^n} \langle Lu, u \rangle = \lambda > 0,$$

by using Theorem 6.2.2, we conclude that the complementarity problem $CP(F, \mathcal{L})$ has a solution. □

Let us define the cone of n-dimensional positive semidefinite matrices $\mathcal{S}^n_+$, also called a positive semidefinite cone, as

$$\mathcal{S}^n_+ = \{A \in \mathcal{S}^n : \ A \succeq 0\}.$$

There are several previous works on checking the copositivity with respect to a cone $\mathcal{K}$, such as [21, 27, 39]. But until now, there is no research on characterisation of the copositivity of a matrix with respect to the cone of positive semidefinite matrices $\mathcal{S}^n_+$. The results above can also be used to test the copositivity of matrices with respect to the cone of positive semidefinite matrices $\mathcal{S}^n_+$ when $n = 2$, as we can find that $\mathcal{S}^2_+$ is isomorphic to $\mathcal{L}^3$.

In the following section, in order to solve the quadratic optimisation problem (6.9), we introduce the gradient projection method on the sphere. This method can also be used to test the copositivity of operators with respect to different cones. We will also give some examples in the latter chapter and analyze the copositivity of operators with respect to the positive semidefinite cone.

6.4 Basic concepts and results

In this section, the basic concepts related to the gradient projection method on the sphere that can be used to solve the constraint optimisation problem on the sphere are introduced. Before the introduction of the basic concepts, we first give a more explicit form for the constraint optimisation problem we investigate in this section. Let $\mathbb{S}^n$ be the n-dimensional sphere in the Euclidean vector space, i.e.

$$\mathbb{S}^n := \left\{ p \in \mathbb{R}^{n+1} : \|p\| = 1 \right\}. \tag{6.10}$$

Let $f : \mathbb{S}^n \to \mathbb{R}$ be differentiable mapping and $\mathcal{C} \subseteq \mathbb{S}^n$ is closed and spherically convex (see Definition 6.4.1). Then the constraint optimisation problem we consider in this section is given as follows

$$\min\{f(p) : p \in \mathcal{C}\}. \tag{6.11}$$

Note that the problem (6.1) is a specific problem of (6.11). Thus, we can implement the gradient projection method to test the copositivity of operators with respect to cones

$\mathcal{K} \subseteq \mathbb{R}^{n+1}$. Meanwhile, by using Corollary 6.2.3, we can also use the gradient projection method to analyse the solvability of complementarity problems. In order to solve problem (6.11), some basic results about the sphere (6.10) are needed. After that, we show how to intrinsically project onto the spherically closed convex set $\mathcal{C} \subseteq \mathbb{S}^n$. The gradient projection method we implemented to solve problem (6.11) as well as the convergence analysis, will be introduced at the end.

First, for the sake of completeness, let us recall and introduce some general definitions and basic geometric properties of the sphere in Euclidean vector spaces, while the details of these definitions and properties have been introduced in [11, 25, 26, 73].

The *tangent hyperplane* at an arbitrary point $p \in \mathbb{S}^n$ is the *tangent hyperplane* at a point $p \in \mathbb{S}^n$ and is denoted by

$$T_p\mathbb{S}^n := \left\{ v \in \mathbb{R}^{n+1} \,:\, \langle p, v \rangle = 0 \right\}. \tag{6.12}$$

Then we have the corresponding *projection mapping* onto it, which is denoted by $\mathrm{Proj}_p : \mathbb{R}^{n+1} \to T_p\mathbb{S}^n$, is given by

$$\mathrm{Proj}_p\, x := x - \langle p, x \rangle p. \tag{6.13}$$

For any two arbitrary arbitrary points $p, q \in \mathbb{S}^n$, the *intrinsic distance on the sphere* between them is defined by

$$d(p, q) := \arccos\langle p, q \rangle. \tag{6.14}$$

A *geodesic segment* on the sphere joining two points $p, q \in \mathbb{S}^n$ is obtained by the intersection of a plane through these points and the origin of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$. The *arc length* of a geodesic segment ω is denoted by $\ell(\omega)$. If for a geodesic segment $\omega : [a, b] \to \mathbb{S}^n$ we have $\ell(\omega) := \arccos\langle \omega(a), \omega(b) \rangle$, then this geodesic segment is said to be *minimal.* For any two arbitrary points $x^*, y^* \in \mathbb{S}^n$, there exists a unique segment of minimal geodesic from x^* to y^* if we hold that $y^* \neq \pm x^*$.

Definition 6.4.1 (Spherically convex set). *A set* $\mathcal{C} \subseteq \mathbb{S}^n$ *is called* spherically convex *if for any two points $x^*, y^* \in \mathcal{C}$, all minimal geodesic segments joining them are contained in $\mathcal{C}$.*

We will give some examples of the spherically convex sets below, and there are more examples in [29].

Example 6.4.1. *Let $\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} : \ \|p\|^2 = 1\}$ is a sphere in the space $\mathbb{R}^{n+1}$. Thus, the set defined by $\mathcal{C}_1 = \mathbb{R}^{n+1}_+ \cap \mathbb{S}^n$ is spherically convex set. Moreover, $\mathcal{C}_2 = \mathbb{R}^{n+1}_+ \cap \mathbb{S}^n$ and $\mathcal{C}_3 = \{p \in \mathcal{S}^n_+ : \|p\| = 1\}$ are also spherically convex sets.*

Denote $\omega_{p,v}$ to be a geodesic, which is defined by its initial position p with velocity v at p. Consider the *exponential mapping* $\exp_p : T_p\mathbb{S}^n \to \mathbb{S}^n$ which is defined as $\exp_p v := \omega_{p,v}(1)$, and we have the following formula for the exponential mapping

$$\exp_p v := \begin{cases} \cos(\|v\|)\, p + \sin(\|v\|)\, \dfrac{v}{\|v\|}, & v \in T_p\mathbb{S}^n/\{0\}, \\ p, & v = 0. \end{cases} \tag{6.15}$$

Following the definition of geodesic, we can find out that, for any $t \in \mathbb{R}$, we get $\omega_{p,tv}(1) = \omega_{p,v}(t)$. Then we can find out that for any $t \in \mathbb{R}$, we hold $\omega_{p,v}(t) = \exp_p tv$ and by using (6.15) we have

$$\exp_p tv := \begin{cases} \cos(t\|v\|)\, p + \sin(t\|v\|)\, \dfrac{v}{\|v\|}, & v \in T_p\mathbb{S}^n/\{0\}, \\ p, & v = 0. \end{cases} \tag{6.16}$$

Note that (6.16) can also be used to represent the formula of geodesic starting at point $p \in \mathbb{S}^n$ with the velocity of $v \in T_p\mathbb{S}^n$ at point p.

Then, we will introduce the concepts of the inverse of the exponential mapping. Denote $\exp_p^{-1} : \mathbb{S}^n \to T_p\mathbb{S}^n$ be the *inverse of the exponential mapping*, then we have the formula of it, which is

$$\exp_p^{-1} q := \begin{cases} \frac{d(p,q)}{\sqrt{1-\langle p,q\rangle^2}} \operatorname{Proj}_p q, & q \notin \{p, -p\}, \\ 0, & q = p. \end{cases} \tag{6.17}$$

Then following (6.14) and (6.17) together with some calculation, we can obtain that

$$d(p,q) = \|\exp_q^{-1} p\|, \qquad p, q \in \mathbb{S}^n. \tag{6.18}$$

In the remainder of this chapter, denote $\Omega \subseteq \mathbb{S}^n$ be an open set, considering a mapping $f : \Omega \to \mathbb{R}$ as a differentiable function. Let $Df(p) \in \mathbb{V}^{n+1}$ be the usual gradient (Euclidean gradient) of function f at a point $p \in \Omega$. Then the formula of the *gradient on the sphere* of f at point $p \in \Omega$ is given by

$$\operatorname{grad} f(p) = \operatorname{Proj}_p Df(p). \tag{6.19}$$

Moreover, like the Hessian defined in the Euclidean space, if function f is twice differentiable, let $D^2 f(p) : \mathbb{V}^{n+1} \to \mathbb{V}^{n+1}$ be the usual Hessian operator or Euclidean Hessian of the function f at point p. Then we can obtain the *Hessian on the sphere* at an arbitrary point $p \in \Omega$, which can be defined as an operator $\operatorname{Hess} f(p) : T_p\mathbb{S}^n \to T_p\mathbb{S}^n$, with the formula as follows

$$\operatorname{Hess} f(p)u := \operatorname{Proj}_p \left(D^2 f(p)u - \langle Df(p), p\rangle u \right). \tag{6.20}$$

Moreover, the operator norm of the Hessian operator on the sphere will be given as:

$$\|\operatorname{Hess} f(p)\| := \sup_{\|u\|=1} |\langle \operatorname{Hess} f(p)u, u\rangle| = \sup_{\|u\|=1} \|\operatorname{Hess} f(p)u\|. \tag{6.21}$$

Then we give an example for the gradient and Hessian on the sphere.

Example 6.4.2. *Let us consider a particular special case. Consider* $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *to be a linear operator. Let us define the following function* $f(p) := \langle Ap, p\rangle$*. Then, by using* (6.13) *and* (6.19)*, for any* $u \in T_p\mathbb{S}^n$*, we have*

$$\langle \operatorname{grad} f(p), u\rangle = \langle Ap, u\rangle + \langle p, Au\rangle - 2\langle Ap, p\rangle\langle p, u\rangle.$$

Meanwhile, by using (6.13) *and* (6.20)*, for any* $u \in T_p\mathbb{S}^n$*, we get*

$$\langle \operatorname{Hess} f(p)u, u\rangle = 2\,\langle Au, u\rangle - 2\langle Ap, p\rangle\,\langle u, u\rangle\,.$$

If we denote $\lambda_{\max}(A) := \max_{\|u\|=1}\langle Au, u\rangle$ *and* $\lambda_{min}(A) := \min_{\|u\|=1}\langle Au, u\rangle$*. we can obtain the following inequalities:* $\|\operatorname{grad} f(p)\| \leq 2(\lambda_{\max}(A) - \lambda_{\min}(A))$ *and* $\|\operatorname{Hess} f(p)\| \leq 2(\lambda_{\max}(A) - \lambda_{\min}(A))$*.*

In order to introduce the definition of Lipschitz continuity, we first give the concept and formula of parallel transport. After that, we introduce the definition of Lipschitz continuous for a function on the spherically convex set.

For each $p, q \in \mathbb{S}^n$ such that $q \neq -p$, we denote the geodesic segment joining p and q by $[0, 1] \ni t \mapsto \omega_{pq}(t) := \exp_p t(\exp_p^{-1} q)$. Then if we denote the *parallel transport* from p to q along he geodesic segment ω_{pq} by $P_{pq} \colon T_p\mathbb{S}^n \to T_q\mathbb{S}^n$. We have the formula for P_{pq} is given by

$$P_{pq}(v) := v - \frac{1}{1 + \langle p, q\rangle}\langle q, v\rangle(p + q).$$

Definition 6.4.2. *Let* $\mathcal{C} \subset \mathbb{S}^n$ *be a spherically convex set. If for any two arbitrary points* $p, q \in \mathcal{C}$ *and some positive constant* $L \geq 0$*, we have the following inequality holds*

$$\|P_{pq} \operatorname{grad} f(p) - \operatorname{grad} f(q)\| \leq L d(p, q)$$

then we say that the gradient vector field of f *is* Lipschitz continuous *on the spherically closed convex set* $\mathcal{C}$ *with the constant* L

The following three lemmas illustrate the properties held by a Lipschitz continuous function on the spherically close and convex set.

Lemma 6.4.1. *The gradient vector field of f is Lipschitz continuous with constant $L \geq 0$ on $\mathcal{C}$ if and only if there exists $L \geq 0$ such that $\|Hess\, f(p)\| \leq L$, for all $p \in \mathcal{C}$.*

We omit the proof of Lemma 6.4.1 here, as it is similar to the proof of [11, Proposition 10.43].

By using Lemma 6.4.1 and Example 6.4.2, we have the following lemma.

Lemma 6.4.2. *Let $f : \Omega \to \mathbb{R}$ be given by $f(p) = \langle Ap, p \rangle$ and $\mathcal{C} \subseteq \Omega$ be a spherically convex set. Then, f is Lipschitz continuous with constant $L = 2(\lambda_{max}(A) - \lambda_{min}(A))$ on $\mathcal{C}$.*

Proof. By using Example 6.4.2, we have that $\|\operatorname{Hess} f(p)\| \leq 2(\lambda_{max}(A) - \lambda_{min}(A))$. Then, by using the last equality in (6.21) and Lemma 6.4.1, we can conclude that f is Lipschitz continuous with constant $L = 2(\lambda_{max}(A) - \lambda_{min}(A))$ on the spherically convex set $\mathcal{C}$. □

We will also omit the proof of the next lemma, as it is just a straight forward application result of [11, Corollary 10.54].

Lemma 6.4.3. *Suppose that the gradient of a function f on the sphere, which is* grad f*, is Lipschitz continuous on a convex set $\mathcal{C} \subseteq \Omega$ with constant $L \geq 0$. Then, we hold the following inequalities*

$$f(q) \leq f(p) + \langle \operatorname{grad} f(p), exp_p^{-1} q \rangle + \frac{L}{2} d^2(p, q), \qquad \forall p, q \in \mathcal{C}.$$

The last lemma we recall in this chapter is the well-known cosine law for triangles on the sphere. Since its proof is a straight forward application of (6.17), we will give the detailed proof here for the sake of completeness.

Lemma 6.4.4. *Suppose we have* $\hat{q}, \tilde{q}, \bar{q} \in \mathbb{S}^n$ *such that* $\tilde{q}, \bar{q} \notin \{\hat{q}, -\hat{q}\}$. *Denote* $\theta_{\hat{q}}$ *be the angle between the vectors* $exp_{\hat{q}}^{-1}\tilde{q}$ *and* $exp_{\hat{q}}^{-1}\bar{q}$. *Then, the following equation holds*

$$\cos d(\tilde{q}, \bar{q}) = \cos d(\hat{q}, \bar{q}) \cos d(\hat{q}, \tilde{q}) + \sin d(\hat{q}, \bar{q}) \sin d(\hat{q}, \tilde{q}) \cos \theta_{\hat{q}}.$$

Proof. First, by using(6.18), we have that $\langle \exp_{\hat{q}}^{-1}\tilde{q}, \exp_{\hat{q}}^{-1}\bar{q} \rangle = d(\hat{q}, \tilde{q}) d(\hat{q}, \bar{q}) \cos \theta_{\hat{q}}$. Meanwhilie, by using (6.17), we get

$$\left\langle \exp_{\hat{q}}^{-1}\tilde{q}, \exp_{\hat{q}}^{-1}\bar{q} \right\rangle = \frac{d(\hat{q}, \tilde{q})}{\sqrt{1 - \langle \hat{q}, \tilde{q} \rangle^2}} \frac{d(\hat{q}, \bar{q})}{\sqrt{1 - \langle \hat{p}, \bar{q} \rangle^2}} \left\langle \mathrm{Proj}_{\hat{q}} \tilde{q}, \mathrm{Proj}_{\hat{q}} \bar{q} \right\rangle .$$

Then by combining these two equations and using some algebraic manipulations, we have

$$\cos \theta_{\hat{q}} = \frac{1}{\sqrt{1 - \langle \hat{q}, \tilde{q} \rangle^2}} \frac{1}{\sqrt{1 - \langle \hat{p}, \bar{q} \rangle^2}} \left(\langle \tilde{q}, \bar{q} \rangle - \langle \hat{q}, \bar{q} \rangle \langle \tilde{q}, \hat{q} \rangle \right) . \tag{6.22}$$

Recall that $\langle \tilde{q}, \bar{q} \rangle = \cos d(\tilde{q}, \bar{q})$, $\langle \hat{q}, \bar{q} \rangle = \cos d(\hat{q}, \bar{q})$ and $\langle \tilde{q}, \hat{q} \rangle = \cos d(\tilde{q}, \hat{q})$. Thus, we have

$$\cos d(\tilde{q}, \bar{q}) = \cos d(\hat{q}, \bar{q}) \cos d(\hat{q}, \tilde{q}) + \sin d(\hat{q}, \bar{q}) \sin d(\hat{q}, \tilde{q}) \cos \theta_{\hat{q}}.$$

□

6.5 Projection onto a closed spherically convex set and its properties

In this section we focus on introducing the concepts related to the projection onto a closed spherically convex set and develop some new properties for the spherically convex set. We will start by recalling some concepts regarding the projection onto a closed convex set. Some proofs will be omitted as the details can be found in [29, 31]. Recall that in Definition 6.4.1, we indicate the condition that a set $\mathcal{C} \subseteq \mathbb{S}^n$ is a closed spherically convex set. In the remainder of this chapter, for the purpose of convenience, we assume that all the closed spherically convex sets are nonempty proper subsets of the sphere. Then for any set $\mathcal{C} \subseteq \mathbb{S}^n$, define the set $\mathcal{K}_\mathcal{C} \subseteq \mathbb{R}^{n+1}$ as following

$$\mathcal{K}_\mathcal{C} := \{tp \,:\, p \in \mathcal{C},\ t \in [0, +\infty)\}. \tag{6.23}$$

We can find out that set $\mathcal{K}_\mathcal{C}$ is a cone spanned by the closed spherically convex set $\mathcal{C}$ and it is also the smallest cone that contains the set $\mathcal{C}$. The following proposition provides another necessary and sufficient condition for a set to be closed spherically convex.

Proposition 6.5.1. *The closed set $\mathcal{C} \subseteq \mathbb{S}^n$ is spherically convex if and only if $\mathcal{K}_\mathcal{C} \subseteq \mathbb{R}^{n+1}$ is a pointed and convex cone.*

Consider the formula of the projection onto the closed spherically convex set $\mathcal{C}$, which is given by

$$\begin{aligned} \mathcal{P}_\mathcal{C}(p) &:= \{\bar{p} \in \mathcal{C} :\ d(p, \bar{p}) \leq d(p, q), \forall\, q \in \mathcal{C}\} \\ &= \{\bar{p} \in \mathcal{C} : \langle p, q\rangle \leq \langle p, \bar{p}\rangle, \forall\, q \in \mathcal{C}\}. \end{aligned} \tag{6.24}$$

We will demonstrate the main property of the projection onto a closed and spherically convex set in the following proposition.

Proposition 6.5.2. *Let $p \in \mathbb{S}^n$ and $\bar{p} \in \mathcal{C}$ such that $\langle p, \bar{p}\rangle > 0$. Then, $\bar{p} \in \mathcal{P}_\mathcal{C}(p)$ if and only if for any arbitrary $q \in \mathcal{C}$, we have that $\left\langle \mathrm{Proj}_{\bar{p}}\, p,\ \mathrm{Proj}_{\bar{p}}\, q\right\rangle \leq 0$. Moreover, $\mathcal{P}_\mathcal{C}(p)$ is a singleton.*

Remark 6.5.1. *In Proposition 6.5.2, the condition $\left\langle \mathrm{Proj}_{\bar{p}}\, p,\ \mathrm{Proj}_{\bar{p}}\, q\right\rangle \leq 0$ is equivalent to $\langle exp_{\bar{p}}^{-1} p, exp_{\bar{p}}^{-1} q\rangle \leq 0$.*

The following proposition indicates that for the problem of projecting a point onto a closed and spherically convex set, it is sufficient to project this point onto the cone spanned by this set. As we mentioned at the beginning of this section, the proof of this proposition can be found in [31].

Proposition 6.5.3. *Denote $\mathcal{C} \subseteq \mathbb{S}^n$ be a set which is nonempty, closed, and convex. If for an arbitrary point $p \in \mathbb{S}^n$, we have $P_{\mathcal{K}_\mathcal{C}}(p) \neq 0$. If we denote $P_{\mathcal{K}_\mathcal{C}}(p)$ be the usual orthogonal projection onto the cone $\mathcal{K}_\mathcal{C}$, then we have*

$$\mathcal{P}_\mathcal{C}(p) = \frac{P_{\mathcal{K}_\mathcal{C}}(p)}{\|P_{\mathcal{K}_\mathcal{C}}(p)\|}.$$

Then we provide two examples for Proposition 6.5.2.

Example 6.5.1. *Consider a proper set $\mathcal{C}_+ = \{p \in \mathbb{S}^n : \ p \in \mathbb{R}^{n+1}_+\}$. Then it is well known that the cone spanned by set $\mathcal{C}$ is the nonnegative orthant, that is $\mathcal{K}_{\mathcal{C}_+} = \mathbb{R}^{n+1}_+$. Note that the nonnegative orthant is a proper cone, which means it is pointed. Then if we denote the Euclidean projection onto the nonnegative orthant by $P_{\mathbb{R}^{n+1}_+}(z)$, by using the result in* (4.2)*, we will have $z^+ = P_{\mathbb{R}^{n+1}_+}(z)$. Thus, by using Proposition 6.5.3, for any points $p \in \mathbb{S}^n$ with $p^+ \neq 0$, we obtain the formula of the projection onto set $\mathcal{C}_+$ as*

$$\mathcal{P}_{\mathcal{C}_+}(p) = \frac{p^+}{\|p^+\|}.$$

Example 6.5.2. *Consider a proper set $\mathcal{C}_1 = \{p \in \mathbb{S}^n : \ p \in \mathcal{L}^n\}$. Then we can find out that the cone spanned by set $\mathcal{C}_1$ is the Lorentz cone, i.e. $\mathcal{K}_{\mathcal{C}_1} = \mathcal{L}^n$. Since the Lorentz is a pointed cone, then for any point $p = (x, t) \in \mathbb{S}^n$ with $t > 0$, by using Proposition 6.5.2, the formula of the projection onto set $\mathcal{C}_1$ is given by*

$$\mathcal{P}_{\mathcal{C}_1}(p) = \frac{P_{\mathcal{L}^n}(p)}{\|P_{\mathcal{L}^n}(p)\|},$$

where

$$P_{\mathcal{L}^n}(x,t) = \begin{cases} \frac{1}{2}\Big(\left[(t+\|x\|)^+ - (t-\|x\|)^+\right] \frac{x}{\|x\|}, (t-\|x\|)^+ + (t+\|x\|)^+ \Big), & x \neq 0, \\ \\ \left(t^+, 0\right), & x = 0. \end{cases}$$

The formula for the projection onto the Lorentz cone is provided in [36, Proposition 3.3]. Moreover, in [5, Proposition 3.3], another formula for the projection onto the Lorentz cone is given by

$$P_{\mathcal{L}^n}(x,t) = \begin{cases} (x,t), & t \geq \|x\|, \\ \frac{1}{2}\left(1+\frac{1}{\|x\|}\right)(x,\|x\|), & -\|x\| < t < \|x\|. \end{cases}$$

Example 6.5.3. *Recall that we have defined $\mathcal{S}^n$ to be the vector space of symmetric matrices over the real numbers $\mathbb{R}$ and $\mathcal{S}^n_+ = \{X \in \mathcal{S}^n : \ X \succeq 0\}$ to be the cone of n-dimensional positive semidefinite matrices earlier this chapter. It is well-known that the inner product of two matrices $X, Y \in \mathbb{S}^n$ can be defined as $\langle X, Y\rangle = tr(YX) = tr(XY)$, where tr denotes the trace of a matrix. Let $X \in \mathbb{S}^n$ and $\{v^1, v^2, \ldots, v^n\}$ be an orthonormal system of eigenvectors of the matrix X corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Thus, by using the spectral decomposition of X, we have*

$$X = \sum_{i=1}^{n} \lambda_i v^i (v^i)^\top.$$

Consider the case that the closed convex set $\mathcal{C} = \{X \in \mathbb{S}^n : \ X \in \mathcal{S}^n_+\}$. In this case, the cone spanned by set $\mathcal{C}$ is $K_{\mathcal{C}} = \mathcal{S}^n_+$. Then, the projection of $X \in \mathbb{S}^n$ onto $\mathcal{S}^n_+$ can be given by

$$P_{\mathcal{S}^n_+}(X) = \sum_{i=1}^{n} \lambda_i^+ v_i v_i^T,$$

where $\lambda_i^+ := \max\{\lambda_i, 0\}$. Then, by using Proposition 6.5.3, we conclude that for all matrices $X \in \mathbb{S}^n$ with $\mathcal{P}_{\mathcal{S}^n_+}(X) \neq 0$, we have

$$\mathcal{P}_{\mathcal{C}}(X) = \frac{P_{\mathcal{S}^n_+}(X)}{\|P_{\mathcal{S}^n_+}(X)\|}.$$

In the remainder of this section, we introduce some lemmas to discuss some new properties of the projection onto a closed spherically convex set. These new properties are useful in analysing the gradient projection method. Denote $\mathcal{C} \subset \mathbb{S}$ to be a nonempty closed spherically convex set.

Lemma 6.5.4. *Let $p, q \in \mathbb{S}^n$ and $\bar{\theta} > 0$ such that $\bar{\theta} < \pi/2$. Suppose we have $d(p,q) \leq \bar{\theta}$ and $p \in \mathcal{C}$. Then, we have the following inequality*

$$\cos(\bar{\theta}) d^2(p, \mathcal{P}_{\mathcal{C}}(q)) \leq \left\langle exp_p^{-1} q, exp_p^{-1} \mathcal{P}_{\mathcal{C}}(q) \right\rangle . \tag{6.25}$$

Proof. By applying Lemma 6.4.4 with $\hat{q} = p$, $\tilde{q} = q$ and $\bar{q} = \mathcal{P}_{\mathcal{C}}(q)$, we conclude that

$$\cos d(q, \mathcal{P}_{\mathcal{C}}(q)) = \cos d(p, \mathcal{P}_{\mathcal{C}}(q)) \cos d(p,q) + \sin d(p, \mathcal{P}_{\mathcal{C}}(q)) \sin d(p,q) \cos \theta_p. \tag{6.26}$$

Now, by using Lemma 6.4.4 with $\hat{q} = \mathcal{P}_{\mathcal{C}}(q)$, $\tilde{q} = q$ and $\bar{q} = p$, we obtain that

$$\cos d(q,p) = \cos d(\mathcal{P}_{\mathcal{C}}(q), p) \cos d(\mathcal{P}_{\mathcal{C}}(q), q) + \sin d(\mathcal{P}_{\mathcal{C}}(q), p) \sin d(\mathcal{P}_{\mathcal{C}}(q), q) \cos \theta_{\mathcal{P}_{\mathcal{C}}(q)}.$$

Since we have $\langle \exp_p^{-1} q, \exp_p^{-1} \mathcal{P}_{\mathcal{C}}(q) \rangle = d(p,q) d(p, \mathcal{P}_{\mathcal{C}}(q)) \cos \theta_p$, by using Proposition 6.5.2 and Remark 6.5.1, we get $\cos \theta_{\mathcal{P}_{\mathcal{C}}(q)} \leq 0$. Meanwhile, since we have $d(\mathcal{P}_{\mathcal{C}}(q), p) \leq \pi$ and $d(\mathcal{P}_{\mathcal{C}}(q), q) \leq \pi$, then the last equality we have can be converted to the following inequality

$$\cos d(q,p) \leq \cos d(\mathcal{P}_{\mathcal{C}}(q), p) \cos d(\mathcal{P}_{\mathcal{C}}(q), q). \tag{6.27}$$

Then, by adding equality (6.26) and inequality (6.27), after some algebraic manipulations, we have the following inequality

$$\begin{aligned} & \left(\cos d(q, \mathcal{P}_{\mathcal{C}}(q)) + \cos d(q,p)\right) \left(1 - \cos d(\mathcal{P}_{\mathcal{C}}(q), p)\right) \\ \leq \;& \sin d(p, \mathcal{P}_{\mathcal{C}}(q)) \sin d(p,q) \cos \theta_p. \end{aligned} \tag{6.28}$$

By using $\langle \exp_p^{-1} q, \exp_p^{-1} \mathcal{P}_{\mathcal{C}}(q) \rangle = d(p,q) d(p, \mathcal{P}_{\mathcal{C}}(q)) \cos \theta_p$ again, the inequality (6.28) is equivalent to

$$d^2(p,\mathcal{P}_{\mathcal{C}}(q))\frac{d(q,p)}{\sin d(p,q)}\left(\cos d(q,\mathcal{P}_{\mathcal{C}}(q))+\cos d(q,p)\right)\frac{1-\cos d(\mathcal{P}_{\mathcal{C}}(q),p)}{d(p,\mathcal{P}_{\mathcal{C}}(q))\sin d(p,\mathcal{P}_{\mathcal{C}}(q))} \leq \langle \exp_p^{-1} q, \exp_p^{-1}\mathcal{P}_{\mathcal{C}}(q)\rangle.$$

Since $d(q,\mathcal{P}_{\mathcal{C}}(q)) \leq d(q,p) \leq \pi$, then $\cos d(q,p) \leq \cos d(q,\mathcal{P}_{\mathcal{C}}(q))$. Thus, the last inequality can be converted to

$$\begin{aligned} & d^2(p,\mathcal{P}_{\mathcal{C}}(q))\frac{2d(q,p)\cos d(q,p)}{\sin d(p,q)}\frac{1-\cos d(\mathcal{P}_{\mathcal{C}}(q),p)}{d(p,\mathcal{P}_{\mathcal{C}}(q))\sin d(p,\mathcal{P}_{\mathcal{C}}(q))} \\ \leq\ & \langle \exp_p^{-1} q, \exp_p^{-1}\mathcal{P}_{\mathcal{C}}(q)\rangle. \end{aligned} \tag{6.29}$$

For any arbitrary $x \in (0, \frac{\pi}{2})$ we have $\frac{x}{\sin(x)} > 1$. Meanwhile, function $\cos(x)$ is monotone decreasing when $x \in (0, \frac{\pi}{2})$. Note we have $0 \leq d(p,q) \leq \bar{\theta} < \pi/2$. Thus, we will obtain

$$\cos(\bar{\theta}) \leq \cos d(q,p) \leq \frac{d(q,p)\cos d(q,p)}{\sin d(p,q)}. \tag{6.30}$$

On the other hand, we have $\frac{1-\cos(x)}{x\sin(x)} > \frac{1}{2}$ when $x \in (0, \frac{\pi}{2})$. Thus, by considering $d(q,\mathcal{P}_{\mathcal{C}}(q)) \leq d(q,p)$ and $d(p,q) \leq \bar{\theta} < \pi/2$, we have

$$\frac{1}{2} \leq \frac{1-\cos d(\mathcal{P}_{\mathcal{C}}(q),p)}{d(p,\mathcal{P}_{\mathcal{C}}(q))\sin d(p,\mathcal{P}_{\mathcal{C}}(q))}. \tag{6.31}$$

Therefore, by combining (6.29) with (6.30) and (6.31), inequality (6.25) follows. □

In order to simplify the notations, in the remaining part of this section, we take $\theta > 0$ such that

$$\bar{\theta} := \arccos(\theta) < \frac{\pi}{2}. \tag{6.32}$$

Let f be a function that is defined in (6.11), and we will have the following Lemma.

Lemma 6.5.5. *Suppose we have two constants θ and $\bar{\theta}$ which are satisfying* (6.32)*, let $p \in \mathcal{C}$ such that* $\operatorname{grad} f(p) \neq 0$*. Assume that we have another constant $\alpha \in \mathbb{R}$ such that* *the inequalities* $0 < \alpha \leq \bar{\theta}/\|\operatorname{grad} f(p)\|$ *hold. Then, we obtain the following inequality*

$$\begin{aligned} & \left\langle \operatorname{grad} f(p), exp_p^{-1}\mathcal{P}_{\mathcal{C}}\left(\exp_p\left(-\alpha \operatorname{grad} f(p)\right)\right)\right\rangle \\ \leq \; & -\frac{\theta}{\alpha} d^2\left(p, \mathcal{P}_{\mathcal{C}}\left(\exp_p\left(-\alpha \operatorname{grad} f(p)\right)\right)\right). \end{aligned} \tag{6.33}$$

Proof. To simplify the notations, since we have $p \in \mathcal{C}$ and $\alpha > 0$, we define

$$v := \operatorname{grad} f(p), \qquad q(\alpha) := \exp_p(-\alpha \operatorname{grad} f(p)). \tag{6.34}$$

Since $\operatorname{grad} f(p) \neq 0$, we have $\mathcal{P}_{\mathcal{C}}(q(\alpha)) \neq p$. By using (6.34), $\bar{\theta} < \pi/2$ and $\alpha \leq \bar{\theta}/\|\operatorname{grad} f(p)\|$ we conclude that $d(p, q(\alpha)) = \alpha\|\operatorname{grad} f(p)\| \leq \bar{\theta} < \pi/2$. Thus, we have $-\alpha v = \exp_p^{-1} q(\alpha)$. Then, by applying Lemma 6.5.4 with $q = q(\alpha)$, we will have the following inequality

$$\theta d^2(p, \mathcal{P}_{\mathcal{C}}(q(\alpha))) \leq \left\langle \exp_p^{-1} q(\alpha), \exp_p^{-1}\mathcal{P}_{\mathcal{C}}(q(\alpha))\right\rangle.$$

Thus, by substitute $-\alpha v = \exp_p^{-1} q(\alpha)$ into the last inequality, we have

$$\langle v, \exp_p^{-1}\mathcal{P}_{\mathcal{C}}(q(\alpha))\rangle \leq -\frac{\theta}{\alpha} d^2(p, \mathcal{P}_C(q(\alpha)).$$

By using (6.34), the last inequality is equivalent to (6.33). □

After the demonstration of the two important inequalities above, we will introduce another important concept.

Denote $\mathcal{C} \subseteq \Omega$ to be a closed spherically convex set. Let the point $\bar{p} \in \mathcal{C}$ to be the solution to the constrained optimisation problem (6.11), then we have

$$\left\langle Df(\bar{p}), \operatorname{Proj}_{\bar{p}} p\right\rangle = \left\langle \operatorname{grad} f(\bar{p}), \operatorname{Proj}_{\bar{p}} p\right\rangle = \langle \operatorname{grad} f(\bar{p}), p\rangle \geq 0, \quad \forall\, p \in \mathcal{C}. \tag{6.35}$$

Then for any point which is satisfying (6.35), we call the point a *stationary point* of the problem (6.11).

In the next corollary, two important properties of the projection onto the closed spherically convex set are presented, and these two properties are related to the definition of a stationary point, which is illustrated above.

Corollary 6.5.6. *Suppose we have two constants* θ *and* $\bar{\theta}$ *which are satisfying* (6.32)*, let* $p \in \mathcal{C}$ *such that* $\operatorname{grad} f(p) \neq 0$*. Assume that we have another constant* $\alpha \in \mathbb{R}$ *such that the inequalities* $0 < \alpha \leq \bar{\theta}/\|\operatorname{grad} f(p)\|$ *hold. The following two statements are true.*

(i) The point $\bar{p}$ *is stationary for the problem* (6.11) *if and only if*

$$\bar{p} = \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right).$$

(ii) If $\bar{p}$ *is a nonstationary point of the problem* (6.11)*, then*

$$\left\langle \operatorname{grad} f(\bar{p}), exp_{\bar{p}}^{-1}\mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)\right\rangle < 0. \tag{6.36}$$

Equivalently, if there exists $\bar{\alpha} \in \mathbb{R}$ *such that* $0 < \bar{\alpha} \leq \bar{\theta}/\|\operatorname{grad} f(\bar{p})\|$ *and*

$$\left\langle \operatorname{grad} f(\bar{p}), exp_{p}^{-1}\mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\bar{\alpha} \operatorname{grad} f(\bar{p})\right)\right)\right\rangle \geq 0, \tag{6.37}$$

then $\bar{p}$ *is stationary point of the problem* (6.11)*.*

Proof. First, let us prove item (i). Suppose that point $\bar{p}$ is stationary for the problem (6.11), by using (6.35) we have

$$\left\langle \operatorname{grad} f(\bar{p}), \operatorname{Proj}_{\bar{p}} p\right\rangle \geq 0, \quad \forall\, p \in \mathcal{C}. \tag{6.38}$$

Suppose we have

$$\bar{p} \neq \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right). \tag{6.39}$$

We will prove this by contradiction. From (6.13), we have $\operatorname{Proj}_{\bar{p}} p = p - \langle \bar{p}, p\rangle \bar{p}$, then by using the formula of the inverse of the exponential mapping, which is given in (6.17), along with (6.38) and (6.39), we have

$$\left\langle \operatorname{grad} f(\bar{p}), \exp_{\bar{p}}^{-1} \mathcal{P}_{\mathcal{C}} \left(\exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right) \right) \right\rangle \geq 0. \tag{6.40}$$

Then by using the inequality given in Lemma 6.5.5, we have

$$d(\bar{p}, \mathcal{P}_{\mathcal{C}}(\exp_{\bar{p}}(-\alpha \operatorname{grad} f(\bar{p})))) \leq 0. \tag{6.41}$$

Note that equation (6.41) contradicts equation (6.39). Thus,

$$\bar{p} = \mathcal{P}_{\mathcal{C}} \left(\exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right) \right).$$

Next, on the other hand, suppose we have $\bar{p} = \mathcal{P}_{\mathcal{C}} \left(\exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right) \right)$. From Proposition 6.5.2 and Remark 6.5.1, by substituting into $q = \exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right)$ we have

$$\langle \exp_{\bar{p}}^{-1} \exp_{\bar{p}} \left(-\alpha \operatorname{grad} f(\bar{p}) \right), \exp_{\bar{p}}^{-1} p \rangle \leq 0, \qquad \forall\, p \in \mathcal{C},$$

which can be reduced to

$$\langle \alpha \operatorname{grad} f(\bar{p}), \exp_{\bar{p}}^{-1} p \rangle \geq 0$$

for any $p \in \mathcal{C}$.

Since we have $\alpha > 0$, then by using the formula of the inverse of exponential mapping in (6.17) again, the last inequality implies that

$$\left\langle \operatorname{grad} f(\bar{p}), \operatorname{Proj}_{\bar{p}} p \right\rangle \geq 0, \quad \forall\, p \in \mathcal{C},$$

and we conclude that the point $\bar{p}$ is a stationary point for problem (6.11).

Therefore, the sufficient and necessary condition for a point to be stationary for problem (6.11) have been proved.

Next, we will prove item (ii). Suppose that the point $\bar{p}$ is not a stationary point of the problem (6.11), then by using the conclusion in item (i), we get

$$\bar{p} \neq \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right).$$

Thus, by applying Lemma 6.5.5, since the two constants we have, which are α, θ, are positive, we get

$$\begin{aligned} 0 &< \frac{\theta}{\alpha} d^2\left(\bar{p}, \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)\right) \\ &\leq -\left\langle \operatorname{grad} f(\bar{p}), \exp_{\bar{p}}^{-1} \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)\right\rangle, \end{aligned}$$

Thus,

$$\left\langle \operatorname{grad} f(\bar{p}), \exp_{\bar{p}}^{-1} \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\alpha \operatorname{grad} f(\bar{p})\right)\right)\right\rangle < 0.$$

Therefore, the first statement from item (ii) has been proved. Since the second statement in item (ii) is the contrapositive of the first statement in item (ii).
Thus, we can just conclude that if there exists $\bar{\alpha} \in \mathbb{R}$ such that $0 < \bar{\alpha} \leq \bar{\theta}/\|\operatorname{grad} f(\bar{p})\|$ and

$$\left\langle \operatorname{grad} f(\bar{p}), \exp_{p}^{-1} \mathcal{P}_{\mathcal{C}}\left(\exp_{\bar{p}}\left(-\bar{\alpha} \operatorname{grad} f(\bar{p})\right)\right)\right\rangle \geq 0,$$

then $\bar{p}$ is stationary point of problem (6.11). □

6.6 Gradient projection method on the sphere and algorithms

In this section, in order to solve the constraint optimisation problem (6.11), we will present the gradient projection method with the algorithms. At the beginning, we will introduce some new notations first. Denote a nonempty set $\mathcal{C}' \neq \varnothing$ be the solution set to the constraint optimisation problem (6.11) and denote the optimal value of f by f^*, where $f^* := \inf_{p \in \mathcal{C}} f(p)$. Then in the remaining section, we suppose that $\operatorname{grad} f$ is Lipschitz continuous on $\mathcal{C} \subseteq \mathbb{S}^n$ with a nonnegative constant $L \geq 0$. Moreover, we define another constant $\zeta \in \mathbb{R}$ to be an upper bound for the gradient of the function f

$$\max_{p \in \mathcal{C}} \| \operatorname{grad} f(p) \| \leq \zeta < +\infty. \tag{6.42}$$

The following example will demonstrate a particular case of a function with the upper bound of the gradient of the function.

Example 6.6.1. *Let $g : \mathbb{S}^n \to \mathbb{R}$ be a function such that $g(p) = p^\top Ap$. Then, by using the result in Example 6.4.2, we get $\| \operatorname{grad} g(p) \| \leq 2(\lambda_{\max}(A) - \lambda_{\min}(A))$. Thus, the upper bound of the gradient of the function $g(p)$ can be obtained by just letting $\zeta = 2(\lambda_{\max}(A) - \lambda_{\min}(A))$.*

Now we have all the concepts we need to demonstrate the gradient projection method and the algorithms. Next, the conceptual version of the gradient projection method, which is used to solve problem (6.11), is given in the Algorithm 1 below.

Algorithm 1: Gradient projection method on $\mathbb{S}^n$ with constant stepsize

Step 0. Take the constants $\zeta > 0$ satisfying (6.42), θ and $\bar{\theta} > 0$ satisfying (6.32) and $\alpha \in \mathbb{R}$ such that

$$0 < \alpha < \min \left\{ \frac{2\theta}{L}, \frac{\bar{\theta}}{\zeta} \right\}. \tag{6.43}$$

Take an initial point $p_0 \in \mathcal{C}$ and set $k = 0$;

Step 1. If $\operatorname{grad} f(p_k) = 0$, then **stop** and return p_k; otherwise, set the next iterate p_{k+1} as follows

$$p_{k+1} := \mathcal{P}_{\mathcal{C}} \left(\exp_{p_k} \left(-\alpha \operatorname{grad} f(p_k) \right) \right); \tag{6.44}$$

Step 2. Update $k \leftarrow k + 1$ and go to **Step 1**.

The remark below will discuss the value of the constants that we will choose when implementing Algorithm 1.

Remark 6.6.1. *Since the sphere $\mathbb{S}^n$ is a compact set and f is a differentiable function, then the minimiser of the function f exists. Let us denote it by $\bar{q} \in \mathbb{S}^n$ and we have* $\operatorname{grad} f(\bar{q}) = 0$. *Consequently, if* $\operatorname{grad} f$ *is Lipschitz continuous on $\mathbb{S}^n$, then by using Definition 6.4.2, for any $p \in \mathbb{S}^n$, we have* $\|\operatorname{grad} f(p)\| \leq Ld(p, \bar{q})$. *Hence,* $\|\operatorname{grad} f(p)\| \leq \pi L$, *for all $p \in \mathbb{S}^n$, and the value of constant ζ defined in* (6.42) *can be taken by letting $\zeta = \pi L$. Moreover, we would like to find the biggest interval for the step-size α, which is defined in* (6.43). *Thus, by considering the definition we set for θ and $\bar{\theta}$ in* (6.32), *we must take* $0 < \theta < 1$ *such that $\theta = \bar{\theta}$, and that leads to $\bar{\theta} > 0.7$. Therefore, by using inequality* (6.43), *we will set* $0 < \alpha < 0.7/(\pi L)$. *Moreover, we find out in Section 6.7 that this interval can be set bigger for a special case when $f(p) = p^\top Ap$.*

Then, in the next proposition, we show that Algorithm 1 is well defined.

Proposition 6.6.1. *Algorithm 1 is well defined and will generate a sequence $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$.*

Proof. Let $p_0 \in \mathcal{C}$ to be the starting point. Then without loss of generality, we just assume that we have $p_k \in \mathcal{C}$. From (6.42) and (6.43), we have $\max_{p\in\mathcal{C}} \|\operatorname{grad} f(p)\| \leq \zeta < +\infty$ and $0 < \alpha < \min\left\{\frac{2\theta}{L}, \frac{\bar{\theta}}{\zeta}\right\}$, which leads to

$$d(p_k, \exp_{p_k}(-\alpha \operatorname{grad} f(p_k))) = \alpha\|\operatorname{grad} f(p_k)\| < \bar{\theta}.$$

By using the definition of $\bar{\theta}$ we have $\bar{\theta} < \pi/2$, and subsequently we get

$$d(p_k, \exp_{p_k}(-\alpha \operatorname{grad} f(p_k))) < \pi/2.$$

Hence, due to $p_k \in \mathcal{C}$, by using (6.24), the last inequality implies that

$$d\big(\mathcal{P}_{\mathcal{C}}(\exp_{p_k}(-\alpha \operatorname{grad} f(p_k))), \exp_{p_k}(-\alpha \operatorname{grad} f(p_k))\big) < \frac{\pi}{2}.$$

Thus, by using Proposition 6.5.2, we have that $\mathcal{P}_{\mathcal{C}}(\exp_{p_k}(-\alpha \operatorname{grad} f(p_k)))$ is a singleton. Therefore, the point p_{k+1} belongs to the set $\mathcal{C}$, we conclude that Algorithm 1 is well defined and generates a sequence $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$. □

In the next lemma, we present a new inequality which is essential in analysing the sequence $(p_k)_{k\in\mathbb{N}}$ generated by Algorithm 1.

Lemma 6.6.2. *For the sequence $(p_k)_{k\in\mathbb{N}}$ that is generated by using Algorithm 1, we have the following inequality*

$$f(p_{k+1}) \leq f(p_k) - \left(\frac{2\theta - \alpha L}{2\alpha}\right) d^2(p_k, p_{k+1}), \qquad k = 0, 1, \ldots . \tag{6.45}$$

In particular, the sequence $(f(p_k))_{k\subseteq\mathbb{N}}$ is non-increasing and convergent.

Proof. By using Proposition 6.6.1, since $p_0 \in \mathcal{C}$, we have $(p_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}$. Let $\Omega = S^n$, $p = p_k$ and $q = p_{k+1}$, by applying Lemma 6.4.3, we get

$$f(p_{k+1}) \leq f(p_k) + \langle \operatorname{grad} f(p_k), \exp^{-1}_{p_k} p_{k+1}\rangle + \frac{L}{2} d^2(p_k, p_{k+1}).$$

From (6.44), we have $p_{k+1} := \mathcal{P}_{\mathcal{C}}\left(\exp_{p_k}\left(-\alpha \operatorname{grad} f(p_k)\right)\right)$. Then, by using Lemma 6.5.5 with $p = p_k$, we get

$$\begin{aligned} f(p_{k+1}) &\leq f(p_k) - \frac{\theta}{\alpha} d^2(p_k, p_{k+1}) + \frac{L}{2} d^2(p_k, p_{k+1}) \\ &= f(p_k) - \left(\frac{2\theta - \alpha L}{2\alpha}\right) d^2(p_k, p_{k+1}), \end{aligned} \tag{6.46}$$

which implies the inequality (6.45). By using (6.43), we have that $(2\theta - \alpha L)/2\alpha > 0$, which leads to $f(p_{k+1}) \leq f(p_k)$. Thus, we conclude that the sequence $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing. Hence, since $-\infty < f^*$ and $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing, the convergence of sequence $(f(p_k))_{k\in\mathbb{N}}$ follows. □

In the next theorem, we demonstrate that any cluster point of the sequence $(f(p_k))_{k\in\mathbb{N}}$ is a solution to the constraint optimisation problem (6.11).

Theorem 6.6.3. *If $\bar{p} \in \mathcal{C}$ is a cluster point of the sequence $(p_k)_{k\in\mathbb{N}}$, then $\bar{p}$ is also a stationary point of the constraint optimisation problem* (6.11).

Proof. For the cluster point $\bar{p} \in \mathcal{C}$, if we have $\operatorname{grad} f(\bar{p}) = 0$, then by using (6.35), we can find out that $\bar{p} \in \mathcal{C}$ is a stationary point of the constraint optimisation problem (6.11). Now, let us consider the case that $\operatorname{grad} f(\bar{p}) \neq 0$. Note that inequality (6.45) is equivalent to

$$d^2(p_k, p_{k+1}) \leq \frac{2\alpha}{2\theta - \alpha L} \left(f(p_k) - f(p_{k+1}) \right), \qquad k = 0, 1, \ldots. \tag{6.47}$$

Recall that in the Lemma 6.6.2, we indicate the sequence $(f(p_k))_{k\in\mathbb{N}}$ is non-increasing and converges. Thus, by using the property of convergence of sequence $(f(p_k))_{k\in\mathbb{N}}$ and (6.47), we have that $\lim_{k\to+\infty} d(p_k, p_{k+1}) = 0$.

Denote $\bar{p}$ to be a cluster point of sequence $(p_k)_{k\in\mathbb{N}}$ and let sequence $(p_{k_j})_{j\in\mathbb{N}}$ represents a subsequence of $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty} p_{k_j} = \bar{p}$. Since we have already proved that $\lim_{k\to+\infty} d(p_k, p_{k+1}) = 0$, and $(p_{k_j})_{j\in\mathbb{N}}$ is a subsequence of $(p_k)_{k\in\mathbb{N}}$, we obtain that $\lim_{j\to+\infty} d(p_{k_j+1}, p_{k_j}) = 0$. Hence, $\lim_{j\to+\infty} p_{k_j+1} = \bar{p}$.

Meanwhilie, by using (6.44) again, we have $p_{k_j+1} = \mathcal{P}_{\mathcal{C}}(\exp_{p_{k_j}}\left(-\alpha \operatorname{grad} f(p_{k_j})\right))$, for $j = 0, 1, \ldots$. Then by using Proposition (6.5.2), we have

$$\left\langle \operatorname{Proj}_{p_{k_j+1}} \left(\exp_{p_{k_j}}(-\alpha \operatorname{grad} f(p_{k_j})) \right), \operatorname{Proj}_{p_{k_j+1}} q \right\rangle \leq 0, \quad \forall\, q \in \mathcal{C}.$$

Then, since function $\operatorname{grad} f$ is continuous, then let $j \to +\infty$, by taking the limit, we have

$$\left\langle \operatorname{Proj}_{\bar{p}} \left(\exp_{\bar{p}}(-\alpha \operatorname{grad} f(\bar{p})) \right), \operatorname{Proj}_{\bar{p}} q \right\rangle \leq 0, \quad \forall\, q \in \mathcal{C},$$

which can be simplified as $\left\langle \exp_{\bar{p}}(-\alpha \operatorname{grad} f(\bar{p})), \operatorname{Proj}_{\bar{p}} q \right\rangle \leq 0$, for any $q \in \mathcal{C}$. Hence, in order to simplify the notation, let $v = -\alpha \operatorname{grad} f(\bar{p})$. Since from (6.43), we have $\alpha > 0$ and we have assumed $\operatorname{grad} f(\bar{p}) \neq 0$ at the beginning of this proof, then $v \neq 0$. Thus, by using (6.15), we can obtain

$$0 \geq \left\langle \cos(\|v\|)\bar{p} + \sin(\|v\|)\frac{v}{\|v\|}, \operatorname{Proj}_{\bar{p}} q \right\rangle = \frac{\sin(\|v\|)}{\|v\|} \left\langle v, \operatorname{Proj}_{\bar{p}} q \right\rangle, \quad \forall\, q \in \mathcal{C}. \tag{6.48}$$

Then by using the equations (6.42), (6.43), and (6.32), we have an inequality as follows

$$\|v\| = \alpha\|\operatorname{grad} f(\bar{p})\| \le \alpha\zeta \le \bar{\theta} < \frac{\pi}{2}.$$

Thus, $\sin(\|v\|) \ge 0$. Then inequality (6.48) is equivalent to

$$0 \ge \langle v, \operatorname{Proj}_{\bar{p}} q\rangle, \qquad \forall\, q \in \mathcal{C}.$$

By substituting $v = -\alpha \operatorname{grad} f(\bar{p})$ back to the above inequality, since $\alpha > 0$, we get

$$\big\langle \operatorname{grad} f(\bar{p}), \operatorname{Proj}_{\bar{p}} q \big\rangle \ge 0, \quad \forall\, q \in \mathcal{C}.$$

Finally, by using the definition of the stationary point, we have that $\bar{p} \in \mathcal{C}$ is a stationary point for problem (6.11). □

From item (i) in Corrolary 6.5.6, we have that if $p_k = \mathcal{P}_{\mathcal{C}}(\exp p_k(-\alpha_k \operatorname{grad} f(p_k)))$, then p_k is a stationary point of problem (6.11). Moreover, from (6.44) we have $p_{k+1} := \mathcal{P}_{\mathcal{C}}\left(\exp_{p_k}\big(-\alpha \operatorname{grad} f(p_k)\big)\right)$. Then,

$$d(p_k, p_{k+1}) = d\left(p_k, \mathcal{P}_{\mathcal{C}}(\exp p_k(-\alpha_k \operatorname{grad} f(p_k)))\right).$$

Thus, the value of $d(p_k, p_{k+1})$ can be treated as a measure of the stationarity of the sequence p_k. In the next theorem, an iteration-complexity bound for the measure $d(p_k, p_{k+1})$ will be provided. In order to simplify the notations in the statement, let us define $\beta := (2\theta - \alpha L)/(2\alpha) > 0$.

Theorem 6.6.4. *For any $N \in \mathbb{N}$, we have*

$$\min\left\{d(p_k, p_{k+1}) : \ k = 0, 1, \dots, N\right\} \le \sqrt{\frac{f(p_0) - f^*}{\beta}} \frac{1}{\sqrt{N+1}},$$

Proof. Let us substitute the formula of β defined above into (6.45), we obtain that

$$d^2(p_k, p_{k+1}) \leq \frac{f(p_k) - f(p_{k+1})}{\beta}, \tag{6.49}$$

for any $k = 0, 1, 2, \ldots$. Recall that at the beginning of this section, we have defined f^* as the optimal value of f. Note that in Lemma 6.6.2, we have also shown that sequence p_k is non-increasing. Then for any $k = 0, 1, 2, \ldots$, we have $f^* \leq f(p_k)$. By simply adding up the inequality (6.49) from $k = 0$ to $k = N$, we have

$$\sum_{k=0}^{N} d^2(p_{k+1}, p_k) \leq \frac{1}{\beta} \sum_{k=0}^{N} (f(p_k) - f(p_{k+1})) \leq \frac{1}{\beta}(f(p_0) - f^*). \tag{6.50}$$

And (6.50) implies that

$$(N+1) \min \left\{ d^2(p_k, p_{k+1}) : \ k = 0, 1, \ldots, N \right\} \leq (f(p_0) - f^*)/\beta$$

which is equivalent to

$$\min \left\{ d(p_k, p_{k+1}) : \ k = 0, 1, \ldots, N \right\} \leq \sqrt{\frac{f(p_0) - f^*}{\beta}} \frac{1}{\sqrt{N+1}}$$

□

For Algorithm 1, in order to make sure that it can be used to solve the problem (6.9), we must have the Lipschitz constant for $\operatorname{grad} f$ and an upper bound for $\| \operatorname{grad} f \|$ on $\mathcal{C} \subseteq \mathbb{S}^n$. However, we can not always know these constants in advance. These constants also are not always computable. For example, when we need to deal with some large-scale problems or in the case when we deal with positive semidefinite cones, this quantity is not easily computable. In this case, another variant of Algorithm 1 will be demonstrated below, with a backtracking stepsize rule approximating the Lipschitz constant. The new algorithm can also be illustrated to accumulate at stationary points as well. The conceptual version of the new algorithm with a backtracking stepsize rule to solve the problem (6.11) is given as follows:

Algorithm 2: Gradient projection method on $\mathbb{S}^n$ with backtracking stepsize rule

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Step 0. Take constants $\rho, \beta, \gamma_0 \in (0,1)$, $\bar{\alpha} > 0$, θ and $\bar{\theta} > 0$ satisfying (6.32). Choose initial point $p_0 \in \mathcal{C}$. Set $k = 0$;

Step 1. If $\operatorname{grad} f(p_k) = 0$, then **stop** and return p_k; otherwise take

$$0 < \alpha_k < \min\left\{ \frac{\bar{\theta}}{\| \operatorname{grad} f(p_k)\|}, \bar{\alpha} \right\}, \tag{6.51}$$

Step 2. Compute

$$y_k := \exp_{p_k}\left(-\alpha_k \operatorname{grad} f(p_k)\right), \qquad z_k := \mathcal{P}_{\mathcal{C}}\left(y_k\right), \tag{6.52}$$

$$\ell_k := \min\left\{ \ell \in \mathbb{N} : \ \beta^\ell \gamma_k \le \frac{\bar{\theta}}{\| \operatorname{grad} f(p_k)\|}, \ f\big(q(\beta^\ell \gamma_k)\big) \le f(p_k) + \rho\big(\beta^\ell \gamma_k\big)\big\langle \operatorname{grad} f(p_k), \exp^{-1}_{p_k} z_k \big\rangle \right\},$$

where $q(\tau) := \exp_{p_k}\left(\tau \exp^{-1}_{p_k} z_k\right)$ denotes the geodesic segment joining p_k to z_k;

Step 3. Set $p_{k+1} := q(\beta^{\ell_k}\gamma_k)$ and $\gamma_{k+1} := \beta^{\ell_k - 1}\gamma_k$;

Step 4. Update $k \leftarrow k+1$ and go to **Step 1**.

6.7 Special case

In this section, Algorithm 1 will be implemented to demonstrate how to solve the following constrained optimisation problem

$$\min\{f(p) := \langle Ap, p\rangle \ : \ p \in \mathcal{C}\}, \tag{6.53}$$

where $\mathcal{C} \subseteq \mathbb{S}^n$ and $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a linear operator such that $\lambda_{\min}(A) \neq \lambda_{\max}(A)$. Recall that in Lemma 6.4.2, we introduced the constant $L = 2(\lambda_{\max}(A) - \lambda_{\min}(A))$. Similarly, by considering (6.42) and Example 6.6.1, we define $\zeta = 2(\lambda_{\max}(A) - \lambda_{\min}(A))$. If we take $\theta = 0.7$, then we have $\hat{\theta} = \arccos 0.7$. Since $0.7 < \arccos(0.7)$, by using (6.43), we can find an $\alpha \in \mathbb{R}$ such that

$$0 < \alpha < \frac{0.35}{\lambda_{\max}(A) - \lambda_{\min}(A)}.$$

Recall that in Proposition 6.5.3, denote $P_{\mathcal{K}_\mathcal{C}}(p)$ to be the usual orthogonal projection onto the cone $\mathcal{K}_\mathcal{C}$, if for $p \in \mathbb{S}^n$ we have $P_{\mathcal{K}_\mathcal{C}}(p) \neq 0$, then the formula of the projection onto $\mathcal{C}$ will be given as

$$\mathcal{P}_\mathcal{C}(p) = \frac{P_{\mathcal{K}_\mathcal{C}}(p)}{\|P_{\mathcal{K}_\mathcal{C}}(p)\|}.$$

In this case, an adapted version of Algorithm 1 will be given as follows:

Algorithm 3: Gradient projection method on $\mathbb{S}^n$ to solve problem (6.53)

Step 0. Take $0 < \alpha < 0.35/(\lambda_{max}(A) - \lambda_{min}(A))$ and an initial point $p_0 \in \mathcal{C}$. Set $k = 0$;

Step 1. Set $v_k := Ap_k - \langle Ap_k, p_k \rangle p_k$. If $v_k = 0$, then **stop** and return p_k. Otherwise, set p_{k+1} as follows

$$p_{k+1} := \frac{P_{\mathcal{K}_\mathcal{C}}(q_k)}{\|P_{\mathcal{K}_\mathcal{C}}(q_k)\|}, \qquad q_k := \cos(\alpha\|v_k\|)\, p + \sin(\alpha\|v_k\|)\, \frac{v_k}{\|v_k\|}. \tag{6.54}$$

Step 2. Update $k \leftarrow k+1$ and go to **Step 1**.

In the following examples, we will illustrate the explicit formula of the first equation in (6.54) for the cases when the cone $\mathcal{K}_\mathcal{C}$ generated by $\mathcal{C} \subseteq \mathbb{S}^n$ is the nonnegative orthant $\mathbb{R}^{n+1}_+$, the Lorentz cone $\mathcal{L}^n$ or the positive semidefinite cone $\mathcal{S}^n_+$ respectively.

Example 6.7.1. *First, when we have $\mathcal{C} = \mathbb{R}^{n+1}_+ \cap \mathbb{S}^n$, in this case, $\mathcal{K}_\mathcal{C} = \mathbb{R}^{n+1}_+$. Thus, by using the result which is shown in Example 6.5.1, the first equation in* (6.54) *becomes*

$$p_{k+1} = \frac{q_k^+}{\|q_k^+\|}.$$

Example 6.7.2. *In this example, we will consider the case when $\mathcal{C} = \mathcal{L}^n \cap \mathbb{S}^n$, then we will have that $\mathcal{K}_\mathcal{C} = \mathcal{L}^n$. Let us define $q_k := (x_k, t_k) \in \mathbb{R}^n \times \mathbb{R}$, by using the second result* *in Example 6.5.2, the first equation in* (6.54) *is equivalent to*

$$p_{k+1} = \begin{cases} (x_k, t_k), & t_k \geq \|x_k\|, \\ \frac{1}{\sqrt{2}\|x_k\|}\left(x_k, \|x_k\|\right), & -\|x_k\| < t_k < \|x_k\|. \end{cases}$$

Example 6.7.3. *Now consider the case when $\mathcal{C} = \mathcal{S}^n_+ \cap \mathbb{S}^n$ and the cone spanned by $\mathcal{C}$ is the cone of positive semidefinite matrices $\mathcal{K}_\mathcal{C} = \mathcal{S}^n_+$. Let $q_k \in \mathcal{C}$ and $\{v^{1k}, v^{2k}, \ldots, v^{nk}\}$ be an orthonormal system of eigenvectors of the matrix q_k corresponding to the eigenvalues $\lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk}$, respectively. By using the spectral decomposition of q_k, we have*

$$q_k = \sum_{i=1}^{n} \lambda_{ik} v^{ik} (v^{ik})^\top.$$

Thus, by using the result in Example 6.5.3, we conclude that the first equation in (6.54) *will become*

$$p_{k+1} := \frac{\sum_{i=1}^n (\lambda_{ik})^+ v^{ik} (v^{ik})^T}{\| \sum_{i=1}^n (\lambda_{ik})^+ v^{ik} (v^{ik})^T \|}.$$

6.8 Numerical experiments

In this section, we will illustrate the numerical result we obtained by using the algorithm presented in the previous section to test the copositivity of operators with respect to different cones. We implemented the algorithms using Matlab 2018b. From the conclusion in the previous section, we get that the operator is not copositive with respect to the cone $\mathcal{K}$ if the output we got from the algorithm is negative. On the other hand, we can just guess that the operator may be copositive with respect to $\mathcal{K}$ with a higher probability if we get a positive result each time when we run the algorithm several times by using different starting points each time.

The cones we considered are the nonnegative orthant and the Lorentz cone. In order to implement the algorithm on the nonnegative orthant and the Lorentz cone, respectively, we use the function $f(p) = p^\top Ap$ and consider the optimisation problem given in (6.11).

Example 6.8.1. *Consider the following matrix, which is not copositive. See [14]:*

$$A_1 = \begin{pmatrix} 1 & -0.72 & -0.59 & 1 \\ -0.72 & 1 & -0.6 & -0.46 \\ -0.59 & -0.6 & 1 & -0.6 \\ 1 & -0.46 & -0.6 & 1 \end{pmatrix}. \tag{6.55}$$

Firstly, we consider the matrix A_1 given in (6.55) and $\mathcal{K}$-the nonnegative orthant in problem (6.11)*. The Algorithm 3 is implemented with the starting point $p_0 = [0.5\ 0.5\ 0.5\ 0.5]$, the result we obtain is $f^* = -0.2756 < 0$. Thus, we have the result and conclude that the matrix A_1 is not copositive. Now we consider problem* (6.11) *with respect to the Lorentz cone $\mathcal{L}^4$. In this case, if we implement Algorithm 3 with a starting point $p_0 = [\frac{\sqrt{3}}{6}\ \frac{\sqrt{3}}{6}\ \frac{\sqrt{3}}{6}\ \frac{3}{4}]$, the result we have is $f^* = -0.0545 < 0$. Thus, we can conclude that A_1 is also not copositive with respect to the Lorentz cone $\mathcal{L}^4$. Note that, by using Proposition 6.3.3, the copositivity with respect to the Lorentz cone can also be verified. In this case, suppose that A_1 is copositive with respect to the Lorentz cone $\mathcal{L}^4$. Therefore, by using Proposition 6.3.3, we conclude that there exists a $\mu \in \mathbb{R}_+$ such that matrix $A_1 - \mu J$ is positive semidefinite. Then let us consider one of its principal minors of matrix $A_1 - \mu J$, which is*

$$\begin{vmatrix} 1+\mu & 1 \\ 1 & 1-\mu \end{vmatrix} \geq 0,$$

and this inequality is equivalent to $\mu \leq 0$. Hence, we conclude that $\mu = 0$. Thus, if the matrix A_1 is copositive with respect to the Lorentz cone $\mathcal{L}^4$, then the matrix A_1 itself is positive semidefinite. But it is easy to check that the matrix A_1 has one negative eigenvalue $\lambda = -0.275649$, which implies that matrix A_1 is not positive semidefinite. Thus, we can conclude that A_1 is not copositive with respect to the Lorentz cone $\mathcal{L}^4$, and it is the same result we got by using Algorithm 3.

Example 6.8.2. *The second example we considered is the well-known Horn matrix:*

$$A_2 = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}. \tag{6.56}$$

First, consider the case when the cone $\mathcal{K}$ is the nonnegative orthant, i.e. $\mathcal{K} = \mathbb{R}^5_+$. By applying Algorithm 3 with different starting points, we obtain that $f^* = 1 > 0$. We can only conclude that the Horn matrix given in (6.56) *might be strictly copositive with respect to the nonnegative orthant $\mathbb{R}^5_+$. On the other hand, if we consider the case when $\mathcal{K} = \mathcal{L}^5$, that is when $\mathcal{K}$ is a Lorentz cone. In this case, by implying Algorithm 3 by using the starting point $p_0 = [0\ 0\ 0\ 0\ 1]$, the result we obtained is $f^* = -1.2018 < 0$. Therefore, we conclude that the Horn matrix A_2 is not copositive with respect to the Lorentz cone. This conclusion can also be verified by using the Proposition 6.3.3. Suppose A_2 is copositive with respect to the Lorentz cone $\mathcal{L}^5$, then we can find one $\mu \in \mathbb{R}_+$ such that the matrix*

$$A_2 - \mu J = \begin{pmatrix} 1-\mu & -1 & 1 & 1 & -1 \\ -1 & 1-\mu & -1 & 1 & 1 \\ 1 & -1 & 1-\mu & -1 & 1 \\ 1 & 1 & -1 & 1-\mu & -1 \\ -1 & 1 & 1 & -1 & 1+\mu \end{pmatrix}.$$

is positive semidefinite. Then, all the principal minors of this matrix must be nonnegative. Thus, we have

$$\begin{vmatrix} 1+\mu & -1 \\ -1 & 1-\mu \end{vmatrix} \geq 0,$$

which is equivalent to $1 - \mu^2 - 1 \geq 0$, this leads to $\mu^2 \leq 0$. Hence, we have $\mu = 0$. Thus, *if the Horn matrix A_2 is copositive with respect to $\mathcal{L}^5$, we have that matrix A_2 is positive semidefinite. But we can check once again that the Horn matrix A_2 has two negative eigenvalues, which are $\lambda_1 = \lambda_2 = 1 - \sqrt{5}$. This implies that the matrix A_2 is not positive semidefinite. Therefore, the Horn matrix A_2 is not copositive with respect to the Lorentz cone $\mathcal{L}^5$.*

The third example we investigated is the Hoffmann-Pereira matrix:

Example 6.8.3. *Consider the following Hoffmann-Pereira matrix, which is a copositive matrix, see [14]:*

$$A_3 = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}, \tag{6.57}$$

For the case of nonnegative orthant, when $\mathcal{K} = \mathbb{R}^7_+$, if we run Algorithm 3 with different starting points, we obtain that $f^* = 1 > 0$. Then we conclude that the Hoffmann-Pereira matrix A_3 is strictly copositive with respect to the nonnegative orthant $\mathbb{R}^7_+$. However, if we consider the case when $\mathcal{K}$ is Lorentz cone, that is $\mathcal{K} = \mathcal{L}^7$, with the starting point $p_0 = [0\ 0\ 0\ 0\ 0\ 0\ 1]$, the result we have is that $f^* = -0.6519 < 0$. Therefore, the Hoffmann-Pereira matrix A_3 is not copositive with respect to the Lorentz cone $\mathcal{L}^7$. This conclusion can also be obtained by using Proposition 6.3.3. Since the matrix $A_3 - \mu J$ has the same principal minor

$$\begin{vmatrix} 1+\mu & -1 \\ -1 & 1-\mu \end{vmatrix}$$

as $A_2 - \mu J$. then by using the result we obtained in Example 6.8.2, we conclude that $\mu = 0$, which implies that if the Hoffmann-Pereira matrix A_3 is copositive with respect to the Lorentz cone $\mathcal{L}^7$, then the Hoffmann-Pereira matrix A_3 is positive semidefinite. But *the Hoffmann-Pereira matrix A_3 is not a positive semidefinite matrix as it has 4 negative eigenvalues. Thus, we get again that the Hoffmann-Pereira matrix A_3 is not copositive with respect to the Lorentz cone $\mathcal{L}^7$.*

In [14,22], there are some other examples of copositive matrices that have been considered by researchers, which are a set of matrices related to the maximum clique problem from the DIMACS collection [1]. For those matrices, the real status of copositivity is known by construction. The justification is given in [14] and those matrices can be accessed at [2]. Then Algorithm 3 was implemented to test the copositivity of most matrices given in [2] with respect to the nonnegative orthant. In these cases, we have chosen 1000 randomly generated starting points and run the algorithm 1000 times. The results we got showed that Algorithm 3 has correctly detected the copositivity status of the tested matrices. From Table 6.1, we can see the average number of iterations in some of the cases. In the other cases, the behaviour of the algorithm is similar.

Table 6.1: Results of Testing the Copositivity of Generated Matrices

	order	copositive	not copositive	average nr. of iterations
Hamming4-4-not-COP	16		yes	6.28
Johnson6-2-4-not-COP	15		yes	31.69
Johnson6-4-4-not-COP	15		yes	31.92
Keller2-not-COP	16		yes	13.54
sanchis22-Not-COP	22		yes	141.68
Hamming4-4-in-Interior	16	yes		77.62
Johnson6-2-4-in-Interior	15	yes		56.48
Johnson6-4-4-in-Interior	15	yes		56.66
Keller2-in-Interior	16	yes		66.73
sanchis22-in-Interior	22	yes		140.5

Finally, we will give some examples of an operator that is copositive with respect to the positive semidefinite cones. Recall that we have defined $\mathcal{S}^n$ as a vector space of the $n \times n$ symmetric matrices earlier in this chapter. We define $\mathbb{S}^n = \{p \in \mathcal{S}^{n+1} : \ \|p\|^2 = 1\}$ to be a sphere. Denote $A : \mathcal{S}^{n+1} \to \mathcal{S}^{n+1}$ be a linear operator and $\mathcal{C} = \mathcal{S}^n_+$ be the positive semidefinite cone. Let us consider the following nonlinear programming problem

$$\min_{p \in \mathcal{C}} f(p) := \operatorname{tr}(pAp), \tag{6.58}$$

and we will implement Algorithm 2 to check the copositivity of matrix A with respect to the cone $\mathcal{C}$ in two particular examples from problem (6.58).

Example 6.8.4. *Let $a \in \mathcal{S}^n$ be a symmetric matrix and denote $A : \mathcal{S}^n \to \mathcal{S}^n$ be a linear operator that is defined by $Ap = apa$. In this case, we have $A^* = A$. Then by using (6.19), we get*

$$\operatorname{grad} f(p) = 2Ap - \operatorname{tr}((2Ap)p)p. \tag{6.59}$$

If we consider the case when matrix $a = A_1$, the matrix that is given in (6.55), and when $\mathcal{K}$ is the positive semidefinite cone, then by using Algorithm 2 with different starting points, we can get the result, that is $f^* \simeq 3 \cdot 10^{-6}$. Hence, the result we got by using Algorithm 2 represents that the operator $Ap = apa$ might be copositive with respect to the positive semidefinite cone.

Example 6.8.5. *Let $a \in \mathcal{S}^n$ be a symmetric matrix and denote $A : \mathcal{S}^n \to \mathcal{S}^n$ be a linear operator that is defined by defined by $Ap = pa + ap$. In this case, similar to the example above, we still hold that $A^* = A$ and we also have*

$$\operatorname{grad} f(p) = 2Ap - \operatorname{tr}((2Ap)p)p.$$

Considering the case when we have $a = A_2$, where the matrix A_2 is given in (6.56). Then, by applying Algorithm 2, the result we have is $f^* = -2.4721 < 0$. Since f^* is negative, we find that the operator $Ap = pa + ap$ with the Horn matrix A_2 is not copositive with respect to the positive semidefinite cone.

6.9 Conclusions and comments

In this chapter, in order to solve constrained optimisation problems on the sphere in finite-dimensional vector spaces, the gradient projection algorithm has been illustrated. In Corollary 6.2.5, we have demonstrated that the problem of finding the existence of the solution to the nonlinear complementarity problem can be reduced to a problem of proving that the solution to the corresponding constraint optimisation problem defined on the intersection of the sphere and a corresponding cone is positive. After that, by using the relationship between the copositivity and a quadratic programming (QP) problem defined on the intersection of a cone and sphere is shown in Lemma 6.3.1, Then the problem of analysing the solvability of the complementarity problem can also be converted to the problem of testing the cone copositivity of the considered linear operator via the introduced algorithm. The convergence analysis of this method is provided in Lemma 6.6.2. Moreover, two variants of the algorithm have also been introduced, one with a backtracking stepsize rule approximating the Lipschitz constant, while another one can be implemented for some special cases. As far as we know, this is the first numerical method introduced to check the copositivity of operators with respect to the positive semidefinite cone. Furthermore, several computational results have been provided in the last section of this chapter, which includes the numerical study of the cone copositivity of operators.

CHAPTER 7

FINAL REMARKS AND DISCUSSIONS

In this thesis, we studied some special topics in the cone complementarity. In this final chapter, we will review the results obtained in this thesis, as well as summarise the contributions we made and give some possible direction for future research.

We start with introducing a new extension of the second order cone, which is the monotone extended second order cone (MESOC). In the majority part of this thesis we are working on topics related to the properties of the MESOC and the complementarity problem of the MESOC. Even though there exist some similarities, we still could show that this cone is different from both the second order cone and the extended second order cone.

First, for the second order cone (or Lorentz cone), it is well known that it is a symmetric cone in $\mathbb{R}^p \times \mathbb{R}$, while, in Chapter 2, we found that the monotone extended second order cone is symmetric only when $p = 1$. Otherwise, it is a sub-dual cone and not symmetric. Furthermore, for the extended second order cone (ESOC), even MESOC is a subset of ESOC, but they are indeed different. Sznajder showed in [79] that ESOC is irreducible, while we found that MESOC is reducible. The value of the Lyapunov rank of MESOC has been obtained, and not only the value of the Lyapunov rank is different from ESOC, but also typical Lyapunov transformations are different.

In Chapter 3, we first discussed the isotonicity property of MESOC. Note that in [62], Németh and Zhang have pointed out that ESOC has a very wide class of isotone projection sets. However, for MESOC, we showed that the cylinders are the only isotone projection set with respect to the MESOC. Then, by using the MESOC-isotonicity of the projection onto the cylinder, we developed an iterative scheme which can be implemented to solve general mixed complementarity problems. Although the iterative scheme is similar to the iterative method for ESOC, which has been presented in [62]. However, we have shown that the mixed complementarity problem that can be solved by using MESOC-isotonicity cannot be solved by using ESOC. Even though MESOC is a subset of ESOC, from the MESOC-isotonicity of a mapping, we cannot obtain its ESOC-isotonicity. Thus, bears with similarities, MESOC is still an interesting cone which is worthy of investigation in a future research.

In Chapter 4, we demonstrated that the problem of projection onto the monotone extended second order cone can be reduced to an isotone regression problem in neighbouring dimensions by using the Moreau's decomposition theorem and the properties of the complementarity set of MESOC. We also presented the formulas to show how to project onto this cone; these formulas depend only on an equation for one real variable. For the view of practical applications, it is interesting and also very important to develop the numerical methods to figure out the solution of the equation (4.4); the solution could be found since the isotone regression problem can be efficiently solved by using the pool-adjacent-violater algorithm.

In Chapter 5, we studied the linear complementarity problem on the MESOC. We showed that this problem could be converted to a mixed complementarity problem on the nonnegative orthant. Thus, a variety of existing algorithms could be applied to find the solution to the converted problem. Two approaches have been given: one by using the Fischer–Burmeister (FB) C-function, and another by reformulating this problem to an unconstrained optimisation problem by using the merit function derived from FB C-function. The algorithms corresponding to these two approaches have also been provided. We provide a numerical example at the end of this chapter, which is based on the first approach. In the last section of this chapter, we introduced a portfolio optimisation problem, which is an application based on the MESOC. By using the earlier results of Chapter 2 and this chapter, we present an analytical solution to the portfolio optimisation problem in the end.

In Chapter 6, we represented the relationship between the existence of solutions to a cone complementarity problem and the optimisation of a quadratic function on the intersection of the sphere and the corresponding cone. We also demonstrated that the problem of checking the copositivity of an operator with respect to a cone could be converted to a problem of finding the minimisation of a quadratic function defined on the intersection of the sphere and the corresponding cone. Several algorithms, computational results, and the numerical results obtained in checking the cone copositivity of operators we presented at the end of the chapter. The results obtained from the algorithm cannot acknowledge that an operator is definitely copositive with respect to a cone. However, some other techniques can be applied to increase the likelihood, such as inputting different initial points and implementing this algorithm a large number of times.

After the final remarks for this thesis, we will outline some contributions we have:

1. We introduced another extension of the second order cone, which is the monotone extended second order cone. The properties of this cone have been discussed. The formula for the Lyapunov-like transformation of MESOC and the value of the Lyapunov rank have been presented. Moreover, we illustrated that MESOC can be used in solving the general mixed complementarity problem and provided a numerical example. These results above have been published in our paper [37].

2. We have shown that the problem of projection onto the MESOC can be reduced to an isotone regression problem in the neighbouring dimensions. The formulas illustrating how to project onto this cone have been presented, and the formulas for projecting onto the MESOC only depend on an equation for one real variable. These results have been published in our paper [30].

3. We studied the linear complementarity problem on the MESOC and found the

solution to the problem. The corresponding algorithms and numerical examples have also been presented. Moreover, we introduced an application to the MESOC, which is a portfolio optimisation problem and found the analytical solution to this problem.

4. We demonstrated that the gradient projection method on the sphere could be used both in checking the copositivity of an operator with respect to a cone and in analysing the existence of the solutions to the cone complementarity problem. The algorithms have been presented with convergence analysis and numerical experiments in checking the copositivity of operators with different cones. To the best of our knowledge, this is the first numerical method that has been implemented to check the copositivity of operators with respect to the positive semidefinite cone. The conclusions have been submitted in the Journal of Global Optimization.

At last, we will discuss some potential future research directions.

1. Can we consider other types of complementarity problems on the monotone extended second order cone? For example, in the real financial market, the objective function for some portfolio optimisation problems is defined based on some stochastic process (normally, the stochastic process will be the standard normal distribution). Let $\mathcal{F}$ be a set of events and $\mathcal{P} : \mathcal{F} \to [0,1]$ be a mapping from event to possibility. Denote $n \in \mathbb{R}$ to be the number of assets in a portfolio, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight of assets in the portfolio, $r_i = (r_{i,1}, r_{i,2}, \ldots, r_{i,T})$ be the logarithm return of the assets i from time 1 to time T, then it is well known that $r_{i,j} \sim N(0,1)$. For example, denote $\psi(\omega, r)$ to be the loss function which measures the loss of the portfolio, and consider the two most popular objective functions which have been developed to measure the risk of the portfolio, that are Value-at-Risk (VaR) and Conditional Value at Risk (CVaR). Value-at-Risk (VaR) is defined as follows,

$$\mathrm{VaR}_\alpha\left(\psi(\omega, r)\right) = \min\{\Gamma \in \mathbb{R} : \mathcal{P}\left(\psi(\omega, r) \geq \Gamma\right) \leq \alpha\}$$

where $\alpha \in (0,1)$ represents the confidence interval (quantifies the probability that the worst case happens). Conditional Value at Risk (CVaR) is defined by using VaR. We have

$$\mathrm{CVaR}_\alpha\left(\psi(\omega, r)\right) = -\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_s\left(\psi(\omega, r)\right)\, ds.$$

The investors would like to minimise the risk of a portfolio; we will have to face the following two optimisation problems.

$$\min_\omega \mathrm{VaR}_\alpha\left(\psi(\omega, r)\right) \qquad \text{and} \qquad \min_\omega \mathrm{CVaR}_\alpha\left(\psi(\omega, r)\right). \tag{7.1}$$

Since we have $r \sim N(0,1)$, one of our possible future research is if we can build a stochastic complementarity problem based on the monotone extended second order cone from an optimisation problem with the objective function given in (7.1) and find its solution. Moreover, will it be possible to find another application based on the monotone extended second order cone?

2. In Chapter 6, the gradient projection method on the sphere has been used in testing the copositivity of an operator with respect to certain cones. However, we also acknowledged that this algorithm cannot indicate for sure whether the operator is copositive with respect to the given cones if we got a positive result. Thus, another possible future research direction emerges whether we can amend this algorithm or find another algorithm that can be used to fill this gap. Moreover, can we find an algorithm that can be implemented to check the copositivity of an operator with respect to the extensions of the second order cone, such as the extended second order cone and the monotone extended second order cone?

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