QUALITATIVE THEORY FOR NONLINEAR NON-LOCAL REACTION-DIFFUSION EQUATIONS

by

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Abstract

This Thesis is concerned with the qualitative properties of nonlinear non-local reaction diffusion equations. We begin by establishing minimum and comparison principles for solutions to inequalities involving the integro-differential operators

$$P[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + cu + dJu - \partial_t u, \quad \text{on } \Omega_T,$$

and

$$Q[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + f(\cdot, \nabla u, u, Ju) - \partial_t u, \quad \text{on } \Omega_T,$$

respectively, with Ju denoting the convolution of u with an appropriately chosen integral kernel φ. The minimum and comparison principles are established under a variety of assumptions on the coefficients a_{ij}, b_i, c, d and growth/decay rates of u. Next, we demonstrate that the Cauchy problem associated with

$$\partial_t u = \Delta u + f(u, Ju), \quad \text{on } \Omega_T,$$

is well-posed, locally in time, when the nonlinear non-local term f is locally Lipschitz continuous. Additionally, we prove the existence of solutions when f is locally Hölder continuous (obtaining the existence of maximal and minimal solutions when f is assumed to be non-decreasing with respect to Ju). Afterwards we treat the non-local analogue to a problem arising from fractional-order autocatalysis (by taking $f(u, Ju) = (Ju)_+^p$, for $p \in (0,1)$) and show its well-posedness (locally in time). We accompany our analysis with numerical simulations, demonstrating the conditional converges of the finite difference scheme, and large-t asymptotics. Finally, we consider potential generalisations and extensions of the results presented in the text.

DECLARATION

This document contains a selection of results established by the author under the supervision Dr. John Christopher Meyer from October 2019 through September 2023. It was prepared subject to the regulations set by the University of Birmingham and, unless stated otherwise, the results, or methods used to acquire them are original.

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LIST OF SYMBOLS

The following list describes the notation that is used within the body of this document. Here, X denotes a subset of an $n-$dimensional Euclidean space, with its elements denoted by $(x_1, x_2, \ldots, x_n)$, unless stated otherwise.

$\lvert x \rvert$	$\left(\sum_{i=1}^n \lvert x_i \rvert^2\right)^{1/2}$ for $x \in \mathbb{R}^n$, unless specified otherwise.
$\nabla u, \Delta u$	Let Ω be an open domain in $\mathbb{R}^n$. If $u \in C^1(\Omega)$ then $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$. If additionally $u \in C^2(\Omega)$, then $\Delta u = \partial_{x_1 x_1} u + \cdots + \partial_{x_n x_n} u$.
$\lVert u \rVert_{L^\infty(X)}$	$\operatorname{ess\,sup}\{\lvert u \rvert\}$, for $u \in L^\infty(X)$.
$\lVert u \rVert_{L^p(X)}$	$\left(\int_X \lvert u \rvert^p\right)^{\frac{1}{p}}$, for $p \in [1, \infty)$ and $u \in L^p(X)$.
$\lVert u \rVert_{W^{k,\infty}(X)}$	$\sum_{\lvert\alpha\rvert \leq k} \lVert \partial_x^\alpha u \rVert_{L^\infty(X)}$, for $u \in W^{k,\infty}(X)$.
$\lVert u \rVert_{W^{k,p}(X)}$	$\left(\sum_{\lvert\alpha\rvert \leq k} \lVert \partial_x^\alpha u \rVert_{L^p(X)}^p\right)^{\frac{1}{p}}$, for $p \in [1, \infty)$ and $u \in W^{k,p}(X)$.
$\partial_x^\alpha u$	For $u \in W^{k,p}(X)$, with $p \in [1, \infty]$, $k \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_n = \lvert\alpha\rvert \leq k$, and $\alpha_i \in \mathbb{N}_0$ (for all $i = 1, \ldots, n$), then, $\partial_x^\alpha u = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u$.
$\mathbb{R}(X)$	The space of all functions with domain X and codomain $\mathbb{R}$.
$B_{x_0}^R$ (or B^R)	The open ball centred at x_0 (or $0_{\mathbb{R}^n}$) with radius R.
$C(X)$	The space of all continuous functions in $\mathbb{R}(X)$.
$L^\infty(X)$	The subspace of $\mathbb{R}(X)$ containing all Lebesgue measurable functions with bounded essential supremum and infimum.
$L^p(X)$	The subspace of $\mathbb{R}(X)$ containing all Lebesgue measurable functions $u \in \mathbb{R}(X)$ such that $\int_X \lvert u \rvert^p < \infty$, for $p \in [1, \infty)$.
$W^{k,\infty}(X)$	The subspace of $\mathbb{R}(X)$ containing all functions such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\lvert\alpha_1 + \cdots + \alpha_n\rvert \leq k$, the mixed

partial derivative $\partial_x^\alpha u$ exists, in the weak sense, with bounded essential supremum and infimum.

$W^{k,p}(X)$	The subspace of $\mathbb{R}(X)$ containing all functions such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\lvert\alpha_1 + \cdots + \alpha_n\rvert \leq k$, the mixed partial derivative $\partial_x^\alpha u$ exists, in the weak sense, in $L^p(X)$, i.e. $\lVert\partial_x^\alpha u\rVert_{L^p(X)} < \infty$, for $p \in [1, \infty)$.
$C^{l,m}(X)$	The subspace of $C(X)$ such that $\partial_{x_i}^j u(x,t)$ and $\partial_t^k u(x,t)$ exist and are continuous on X for all $j \leq l$ and $k \leq m$, with $j, k \in \mathbb{N}$ and $i = 1, \ldots, n$.
$C^l(X)$	The subspace of $C(X)$ such that $\partial_{x_i}^j u(x)$ exist and are continuous on X for all $j \leq l$ with $j \in \mathbb{N}$ and $i = 1, \ldots, n$.

CHAPTER 1

INTRODUCTION

Reaction-diffusion equations have been extensively used for modelling population densities as far back as the 1930s following the works of Fisher and Kolmogorov-Petrovskii-Piskunov (see [KPP91]). Even though models based on these FKPP equations have been instrumental in providing insight on population dynamics, they do not always capture the full picture. The study of non-local nonlinear second order parabolic partial differential equations, has attracted much attention over the past 35 years. In the pioneering works of Ei [Ei87] and Britton [Bri90, Bri89], evolution equations containing a convolution term $\varphi * u$ (to account for temporal, spatial or spatio-temporal delay) were considered in the context of population densities which evolve with observable dependence on the local population density.

To motivate our consideration on such reaction diffusion equations, the focus of this Thesis, we begin by formally deriving a population model (in manner extensively based on that presented in [GVA06, Vol11] and [BKUV22]). Different instances of this model will be will be subsequently recalled in later chapters both as examples for applications to the theory illustrated therein and as to highlight the limitations of our results. Next, we briefly describe other types of non-local interactions present in the literature that are not considered herein. Finally, we provide the structure of this Thesis with comments on the novelty of the results presented, contrasting them with classical results and in particular results that hold for local reaction diffusion equations.

1.1 A derivation of a non-local model

Consider a 2-dimensional spatial region with euclidean coordinates, $(x, y) \in \mathbb{R}^2$. Let $t \geq 0$ denote the time difference from an initial value set at $t = 0$. Now, let u denote the density of individuals at the point $(x, y, t) \in \mathbb{R}^2 \times [0, \infty)$. Suppose individuals move randomly in any direction, without bias for any particular direction. Set $p(r)dr$ to be the probability of an individual displacing themselves by a distance between r and $r + dr$, from (x, y) during a generation which we denote by $\delta t > 0$. We denote ρ_i to be the i-th moment of $p(r)$, namely

$$\rho_i = \int_0^\infty r^i p(r)dr, \text{ for } i \in \mathbb{N},$$

and we assume that for $i \leq 4$ that ρ_i is bounded; note that $\rho_0 = 1$; denote $\sqrt{\rho_2}$ as the mean square displacement of the individuals during one generation.

Now, assume that the displacement of individuals, within a generation, is independent of the rate of their birth and death. Then the local change of the population density u at (x, y) from t to $t + \delta t$ is given by

$$\delta u(x, y, t) = \left(\int_{\mathbb{R}^2} u(x', y', t) \frac{p(r)}{2\pi r} dx'dy' - u(x, y, t) + F(u, x, y, t) \right) \delta t, \tag{1.1}$$

with $r = \sqrt{(x - x')^2 + (y - y')^2}$ and F denoting the local rate of growth/decay in the population. By further assuming that u is sufficiently smooth, we perform a Taylor expansion of $u(x', y', t)$, centred at (x, y, t), to obtain

$$\begin{aligned} u(x', y', t) = u(x, y, t) &+ (x' - x)u_x(x, y, t) + (y' - y)u_y(x, y, t) \\ &+ \frac{(x' - x)^2}{2} u_{xx}(x, y, t) + \frac{(y' - y)^2}{2} u_{yy}(x, y, t) \\ &+ (x - x')(y - y')u_{xy}(x, y, t) + ... \end{aligned} \tag{1.2}$$

where we also consider third and fourth order terms. On substitution of $u(x', y', t)$ in (1.2), into (1.1), we note that the first and third order terms integrate to 0 via symmetry.

Moreover, provided that ρ_2 is much larger than ρ_4 we can neglect fourth order terms in (1.2) and, after integrating, obtain

$$\partial_t u = \frac{\rho_2}{4}\Delta u + F(u,\cdot) \quad \text{on } \mathbb{R}^2 \times (0,\infty). \tag{1.3}$$

The function F, for simplicity, is often considered to be independent of (x,y,t). To incorporate core features concerning the creation and removal of the population, F can be expressed in the form

$$F(u) = (B(u) - D(u))u^s, \tag{1.4}$$

where B denotes a 'birth' function, D a 'death' function, and u^s, for $s = 1,2$, is a reproduction factor[1].

Note that by considering $s = 1$, $B(u) = b$, the constant birth rate, and $D(u) = d + ku$ (with d denoting the constant natural death rate and k the increase in deaths due to competition), (1.3) becomes the local FKKP equation, as discussed in [Vol11, GVA06].

More generally; suppose that individuals consume resources in some neighbourhood of the point they are located, then D can be expressed as

$$D(u,x,y,t) = b + k\int_0^t \int_{\mathbb{R}^2} \psi(x-x',y-y',t-t')u(x',y',t')dx'dy'dt',$$

where the function ψ shows how individuals at (x',y',t') influence resources at the point (x,y,t). Recalling that the timescales of interest (e.g. reproduction, death etc.) are much larger than the characteristic timescales of the renewal of resources, we may assume that the influence of ψ is concentrated at $t' = t$ and write

$$\psi(x-x',y-y',t-t') = \varphi(x-x',y-y')\delta(t-t'),$$

[1]The values of s denote asexual and sexual reproduction respectively.

where δ denotes the Dirac δ function. Substituting ψ, into D we obtain

$$D(u,x,y,t) = d + k\int_{\mathbb{R}^2} \varphi(x-x', y-y')u(x',y',t)dx'dy'$$
$$:= d + kJu(x,y,t), \tag{1.5}$$

where φ denotes a non-negative summable function that takes into account the movement of individuals around (x, y) in order to consume resources[1]. Assuming again that $B(u) = b$, upon substitution of (1.5) into (1.3)-(1.4), we obtain

$$\partial_t u = d\Delta u + u^s(c + kJu), \quad \text{on } \mathbb{R}^2 \times (0,\infty). \tag{1.6}$$

For $s = 1$, (1.6) is the non-local FKPP equation (see Remark 2.4.4) and for $s = 2$ a generalised FKKP type equation (see end of Section 3.3).

Concerning the birth term, alternatively, we may take into account the search for mating partners in a localised region by considering a non-local reproduction rate function, given as a function Ju.

To consider more general birth and death terms, as well as more general semi-linear terms, we focus our attention on equation (1.3), throughout this Thesis, with F being a general function of (u, Ju), subject to regularity requirements.

We note that non-local reaction-diffusion equations have also been used to model many propagation phenomena that echo the situation present in the non-local FKPP equation (see for instance ([VP09, HR14, BNPR09]). Examples include: modelling the evolution of species and viruses (see [BBM+20, BRBV18] and the references therein); epidemiological models (see [AH05, Rua07] and the references therein); cell migration and tumour growth (e.g [CPSZ19, RTSG21]); Lotka-Volterra diffusion competition system (see [HWD20, NB23] and the references therein); and many more.

[1] Often, φ is assumed to be probability density function over the entirety of the spatial domain, centred at zero with bounded variation.

1.2 Non-local differential equations of other type & the scope of this Thesis

Within the wider mathematical literature, there exist many instances where the term 'non-local' is used. Often those instances arise in the context of boundary value problems for ordinary differential equations or partial differential equations. A non-exhaustive list includes: problems with $p-$Laplacian terms which are used to model non-local diffusion (eg. [AMRT08]); delay differential equations (eg. [GSW04]); differential equations with spatially non-local non-linear terms where the non-locality is represented as average of the function taken over the entire domain (eg. [KS18]); and, non-local Branching Markov Processes (eg. [HK23]). We note that these topics are not considered herein.

Here, we consider non-local problems where we take spatial averages of the solution to a reaction-diffusion equation in a neighbourhood of a point (frequently this is represented by the convolution of the solution u with an appropriately chosen integral kernel φ). Concerning the non-local theory we treat in this Thesis, we refer the reader to [Fre98, GSW04, TV20] and references therein as well as the concluding remarks of [Vol11] for an in depth discussion of the historical development of this research area.

Moreover, we do not consider real-world applications of non-local reaction diffusion equations herein. For instance, a very reasonable and frequent assumption on the integral kernel φ is to be of compact support with $\|\varphi\|_{L^1} = 1$ (see eg. [TLZ17, BN22]). This often arises from a modelling standpoint (eg. resources can only be gathered from a bounded region within a time interval), and in this case φ usually denotes a probability density function, centred at zero, with finite variance (assumptions one expects to hold in a biological setting since φ represents the interactions of individuals close to their neighbourhood). We will typically avoid such assumptions here as we aim to establish results without unnecessary restrictions. As such, we use the already established non-local reaction diffusion equations as a starting point to formulate a general theory on the subject and illustrate it's limitations.

1.3 Motivation and structure of the Thesis

From a mathematical standpoint, a general theory for non-local nonlinear reaction diffusion equations is, at present, incomplete. In many cases, solutions to Cauchy problems involving non-local reaction diffusion equations, can be obtained by adapting arguments present in [Fri08] and [LSU68] and other methods that can be utilised to obtain solutions for boundary value problems for non-local reaction-diffusion equations are available in [Vol11]. However, the lack of maximum/minimum and comparison principles in the general case, make global bounds on solutions and global well-posedness results significantly more difficult to establish when non-local rather than local interaction terms are included. This motivated the investigation conducted in Chapter 2, concerning minimum and comparison principles, and the theory concerning the well-posed of non-local Cauchy problems that follows.

For reference, general local well-posedness results, using semigroup methods for the existence and uniqueness of mild, strong and classical solutions, for non-local reaction-diffusion equations, are given in [Bys91, BMT19] for problems posed on bounded spatial domains[1] and finite time blow-up phenomena were considered in [Sou98]. Existence and uniqueness of steady state solutions are discussed in [Yam15].

Before we describe the structure of this Thesis, we note that, in the introductory sections of Chapters 2-4, we provide a detailed discussion of the results presented therein, and, in the concluding sections, we discuss direct implications and extensions of the main results, as well as, place the results in a wider context. The structure of this Thesis follows.

In Chapter 2, we focus on establishing minimum and comparison principles for differential inequalities associated with the integro-differential operators

$$P[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + cu + dJu - \partial_t u \quad \text{on } \Omega_T, \tag{1.1}$$

[1] These results motivate the analysis with non-local interactions on unbounded domains with weaker assumptions on the nonlinear non-local term using fundamental solutions present in Chapter 3.

and

$$Q[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + f(\cdot, \nabla u, u, Ju) - \partial_t u \quad \text{on } \Omega_T, \tag{1.2}$$

with

$$Ju(x,t) = \int_{\Omega} \varphi(x-y)u(y,t)dy, \quad \forall (x,t) \in \Omega_T, \tag{1.3}$$

for $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $u : \Omega_T \to \mathbb{R}$ appropriately chosen so that Ju is well-defined, and a_{ij}, b_i, c, d and f appropriate functions. We start by establishing a conditional minimum principle for integro-differential operators given by (1.1), utilising suitable weight functions, which allow for distinct growth/decay rates on the coefficients. We then demonstrate how to obtain minimum principles under certain assumptions, such as boundedness of u and summability of the second moment of the (non-negative) integral kernel φ. These principles are established without regularity assumptions on the coefficients in the integro-differential operator. Following this, an alternative approach is presented, which relies on integral representations of solutions to integro-differential inequalities. This approach establishes a minimum principle with weaker conditions on the non-local term, but imposes regularity and boundedness restrictions on certain coefficients of the integro-differential operator. These minimum principles are then used to establish comparison principles for super-solutions and sub-solutions for a semi-linear integro-differential operator, under appropriate regularity assumptions. The results present here generalise previous minimum and comparison principles established in [Vol11] (by considering a broader class of coefficients, having weaker assumptions on φ, and generalising the result to $\mathbb{R}^n$) and extend to the non-local setting results and techniques present in [MN14]. At the same time, they explore the interplay between the regularity of the coefficients of the operators involved, the growth/decay of the solution and the integrability of the integral kernel. It should also be noted that for all the minimum principles established in this chapter, the condition $d \geq 0$, i.e. the coefficient of the non-local term in (1.1), is shown to be necessary.

Consequently, for the comparison principles established, a necessary condition for f in (1.2), is to be non-decreasing with respect to Ju. Examples are provided to illustrate the applications and limitations of those principles. All the results presented in this chapter are novel.

In Chapter 3, qualitative properties of solutions to the Cauchy problem for the n-dimensional heat equation are established. These are used as motivation to establish similar qualitative properties for solutions to Cauchy problems for semi-linear non-local reaction diffusion equations of the type

$$\partial_t u = \Delta u + f(u, Ju) \quad \text{on } \Omega_T, \tag{1.4}$$

for f being a locally Lipschitz continuous function. Notably, we establish that the Cauchy problem associated with (1.4) is well-posed locally in time, assuming appropriate conditions for both the initial data and the nonlinear non-local interaction term given by f. In particular, the existence and uniqueness of solutions to the Cauchy problem associated with (1.4) follow from standard techniques that are utilised in the local setting (see for instance [Fri08, LSU68]) and we present them here for completeness. The local-in-time continuous dependence of solutions with respect to initial data *and* with respect to integral kernels "close enough in the L^1-norm" is a novel addition in the literature for classical solutions. These results are utilised in Chapter 4 where we generalise our assumptions on f in (1.4) to be locally Hölder continuous. We additionally highlight how different classes of regularity for the initial data affect a priori bounds on solutions to the aforementioned Cauchy problems, and their derivatives. The chapter concludes by providing higher order derivative bounds for solutions to Cauchy problems for nonlinear non-local reaction diffusion equations that are subsequently utilised for the conditional convergence of the numerical scheme presented in Chapter 4. It should also be noted, that no assumptions are made in this chapter on the monotonicity of f.

In Chapter 4, we first consider the Cauchy problem for nonlinear non-local reaction diffusion equations, with non-local nonlinear terms given by locally Hölder continuous functions[1]. We establish the local existence of solutions to these problems and show that solutions can be extended indefinitely, or, until a finite-time T_{MAX}, where blow-up occurs. These solutions are constructed from the limits of sequences of Cauchy problems where the nonlinear terms are Lipschitz continuous and tend to the Hölder continuous non-linear term in the limit. Our approach generalises the results for the local case in [MN15a] and require no monotonicity conditions on f. Moreover, since solutions to the Cauchy problem are constructed via our approach, they can subsequently be utilised to obtain maximal and minimal solutions to the Cauchy problem. Namely, under the additional assumption that the nonlinear, non-local term is non-decreasing with respect to Ju, comparison principles established in Chapter 2 can be applied. When comparison principles can be applied, the local existence of maximal and minimal solutions is established (these can then potentially be utilised to show global existence of solutions). Next, we consider an analogue to the source problem arising in isothermal autocatalytic chemical kinetics (see [NK93]), with the localised nonlinear term replaced by a non-local Hölder continuous nonlinear term of power-type, i.e.

$$\partial_t u = \Delta u + \max\{0, (Ju)\}^p, \quad \text{on } \Omega_T, \tag{1.5}$$

for $p \in (0, 1)$. We establish that the related Cauchy problem is locally well-posed in time, and has a unique global solution for all $T > 0$. Since $\max\{0, (Ju)\}^p$ in not Lipschitz continuous for $p \in (0, 1)$, uniqueness of solutions in this case is not straightforward. We establish the uniqueness of solutions via a problem-specific comparison principle, under sufficient assumptions on the initial data and the integral kernel (based on the approach in [AE87] and a similar analysis in [MN15a]). We compliment this analysis with a finite difference approximation to the above problem. This finite difference approximation is shown to satisfy appropriate bounds; and, be conditionally convergent as the mesh spacing tends to 0. We utilise the numerical scheme to illustrate the consistency of a concise

[1] I.e. (1.4), with f being locally Hölder continuous.

large-t asymptotic approximation for solutions to the Cauchy problem associated with (1.5). The large-t approximation is based upon a linearisation around the lower bound for solutions to the Cauchy problem and the method is further used to infer conditions for the global well-posedness for the Cauchy problem associated with (1.5). From the large-t approximation we infer information pertinent to global in time well-posedness of the Cauchy problem. All results in this chapter are novel.

In Chapter 5, we conclude with a discussion of potential extensions to the results established throughout Chapters 2-4, that were not considered therein.

CHAPTER 2

COMPARISON PRINCIPLES FOR A NONLINEAR INTEGRO-DIFFERENTIAL OPERATOR

2.1 Introduction

In this chapter we consider integro-differential operators, of second order linear parabolic partial differential type, with nonlinear terms of semilinear type, which include non-local zeroth order quantities; which we will henceforth refer to as 'integro-differential operators' for brevity.

We begin by providing a conditional minimum principle, conditional on the existence of any suitable *weight* function, which is then used to establish minimum principles with growth/decay rates on solutions of integro-differential inequalities involving (2.2.4). These extend the results in [MN14] by taking into account non-local terms and more general bounds on coefficients. This is achieved with the use of appropriate weight functions that allow for distinct growth/decay rates on the coefficients in (2.2.4) and φ in (2.2.1). Next, we demonstrate how one can obtain minimum principles under the assumptions that u is bounded and the second moment of φ is summable, extending the result in [Vol11, Ch. 9]. These minimum principles are established without regularity assumptions on the coefficients in the integro-differential operator (2.2.4). Using an alternative approach, relying on integral representations of solutions to integro-differential inequalities, we establish a minimum principle, where the condition on the non-local term is reduced to the kernel being summable, albeit at the cost of imposing regularity and boundedness restrictions on a_{ij}, b_i and c in (2.2.4). This complements the aforementioned results.

Using these minimum principles we establish associated comparison principles for super-solutions and sub-solutions for the operator in (2.2.5), under appropriate assumptions on the regularity of f. We then contextualise our results with two examples, which demonstrate: the limits of; and how to apply, the minimum principles and comparison principles discussed herein. These minimum and comparison principles, as presented here, are utilised in Chapters 3 and 4 to obtain bounds on solutions to reaction-diffusion equations that are then used to prove the local in time well-posedness of the Cauchy problems associated with them. To conclude, we place our results in a wider context, and, comment on potential extensions.

2.2 Preliminaries

Let Ω be an unbounded domain of $\mathbb{R}^n$. In relation to Ω, for any $T \in (0, \infty)$, we denote the following sets:

$$\Omega_T = \Omega \times (0, T] \quad \text{and} \quad \partial\Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times (0, T)).$$

The open ball in Ω of radius $R > 0$, with respect to the euclidean norm, centred at $x_0 \in \Omega$ ($x_0 = 0_{\mathbb{R}^n}$) is denoted by $B_{x_0}^R$ (B^R). We denote the closure of Ω_T as $\overline{\Omega}_T$. Here, $(x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times [0, T]$ denotes an $(n+1)-$dimensional vector. Moreover, we denote $\langle \cdot, \cdot \rangle$ to be the euclidean inner product in $\mathbb{R}^n$ and $|\cdot|$ to be its induced norm.

For $\alpha \in (0, 1]$, and $X \subseteq \mathbb{R}^n \times [0, \infty)$, the set $H_\alpha(X)$ denotes the set of all $u \in C(X)$ that satisfy the spatial Hölder condition

$$|u(x_1, t) - u(x_2, t)| \leq k_\alpha |x_1 - x_2|^\alpha \text{ for all } (x_1, t), (x_2, t) \in X$$

for some constant $k_\alpha \in [0, \infty)$.

We also define the following sets and operators that will be used throughout this chapter. For a measurable function $\theta : \Omega_T \to (0,\infty)$ we define

$$E_\theta(\Omega_T) = \{u : \Omega_T \to \mathbb{R} : u\theta \in L^\infty(\Omega_T)\}.$$

Moreover, for any *summable* $\varphi : \mathbb{R}^n \to [0,\infty)$ we define the integral operator[1] $J_\theta : E_\theta(\Omega_T) \to L^\infty(\Omega_T)$ given by

$$J_\theta u(x,t) = \int_\Omega \varphi(x-y)\theta(y,t)u(y,t)dy \quad \forall (x,t) \in \Omega_T \text{ and } u \in E_\theta(\Omega_T). \tag{2.2.1}$$

Furthermore, let $L : C^{2,1}(\Omega_T) \to \mathbb{R}(\Omega_T)$ be a second order linear parabolic partial differential operator given by

$$L[u] = \sum_{i,j=1}^n a_{ij}\partial_{x_i x_j}u + \sum_{i=1}^n b_i \partial_{x_i}u + cu - \partial_t u \quad \text{on } \Omega_T \tag{2.2.2}$$

for all $u \in C^{2,1}(\Omega_T)$ with $a_{ij}, b_i, c : \Omega_T \to \mathbb{R}$ such that:

$$A_{\min} |\eta|^2 \le \sum_{i,j=1}^n a_{ij}\eta_i\eta_j \quad \text{on } \Omega_T,\ \forall \eta \in \mathbb{R}^n, \tag{2.2.3}$$

for some constant $A_{\min} \in [0,\infty)$. Following (2.2.1) and (2.2.2) we define a linear non-local integro-differential operator $P_\theta : C^{2,1}(\Omega_T) \cap E_\theta(\Omega_T) \to \mathbb{R}(\Omega_T)$ to be

$$P_\theta[u] = \sum_{i,j=1}^n a_{ij}\partial_{x_i x_j}u + \sum_{i=1}^n b_i \partial_{x_i}u + cu + dJ_\theta u - \partial_t u := L[u] + dJ_\theta u \quad \text{on } \Omega_T, \tag{2.2.4}$$

for all $u \in C^{2,1}(\Omega_T) \cap E_\theta(\Omega_T)$ with $d : \Omega_T \to [0,\infty)$. When $\theta \equiv 1$ we adopt the conventions:

$$J_1 u = Ju \quad \text{and} \quad P_1[u] = P[u] \quad \text{on } \Omega_T.$$

The motivation of using this auxiliary function θ is to obtain the contradictions neces-

[1] Note that, if $\Omega = \mathbb{R}^n$, then $J_\theta u$ denotes the spatial convolution product φ with (θu).

sary to establish minimum principles. Moreover, we note that with the exception of the conditional minimum principle in Lemma 2.3.1, the function θ is only used in the proofs and not in the theorem statements and plays no role in the statement of the results. However, we include θ here, in the operator in (2.2.2), so that the domain of definition of the operator, is at all points clear.

For comparison principles we will also consider the semi-linear analogue of P_θ, namely

$$Q_\theta[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + f(\cdot, \nabla u, u, J_\theta u) - \partial_t u \quad \text{on } \Omega_T \tag{2.2.5}$$

for $u \in C^{2,1}(\Omega_T) \cap E_\theta(\Omega_T)$, with $f : \Omega_T \times \mathbb{R}^{n+2} \to \mathbb{R}$. The function f satisfies a *constrained local Lipschitz condition in u* (and analogously in v or w) on $\Omega_T \times \mathcal{K}$ if for any compact set $\mathcal{K} \subseteq \mathbb{R}^{n+2}$, there exist constants $k_{\mathcal{K}} \in [0, \infty)$ and $\beta \in [0, 1)$ such that

$$|f(x, t, w, u_1, v) - f(x, t, w, u_2, v)| \leq \frac{k_{\mathcal{K}}}{t^\beta}|u_1 - u_2|, \tag{2.2.6}$$

for all $(x, t) \in \Omega_T$ and $(w, u_1, v), (w, u_2, v) \in \mathcal{K}$. Moreover the function f satisfies a *constrained local upper Lipschitz condition in u* on $\Omega_T \times \mathcal{K}$ if for any compact set $\mathcal{K} \subseteq \mathbb{R}^{n+2}$, there exist constants $k_{\mathcal{K}} \in [0, \infty)$ and $\beta \in [0, 1)$ such that

$$f(x, t, w, u_1, v) - f(x, t, w, u_2, v) \leq \frac{k_{\mathcal{K}}}{t^\beta}(u_1 - u_2), \tag{2.2.7}$$

for all $(x, t) \in \Omega_T$ and $(w, u_1, v), (w, u_2, v) \in \mathcal{K}$ with $u_1 \geq u_2$. If $\beta = 0$ in (2.2.6) or (2.2.7) we say that f satisfies a *local Lipschitz condition in u* or *upper local Lipschitz condition in u* (or v, w) respectively.

2.3 Minimum principles

Maximum and minimum principles are widely used to establish qualitative properties of solutions to boundary value problems involving second order linear parabolic partial differential operators (e.g. uniqueness, continuous dependence, and regularity). For a historical development of this area, see [PW84].

2.3.1 Conditional minimum principle

Here we present a lemma that will be utilised throughout this section to prove minimum principles under various assumptions on the growth (or decay) rate of solutions to the integro-differential inequality

$$P[u] \leq 0 \quad \text{on } \Omega_T. \tag{2.3.1}$$

We adopt an approach used in [Wal70] and [MN14] to establish a conditional minimum principle, which is conditional on the existence of a suitable weight function.

Lemma 2.3.1 (Conditional minimum principle)**.** *Let $\theta \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ be positive, $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$, Ju be well-defined on Ω_T, and P be an operator defined as in (2.2.4). Furthermore, suppose:*

$$P[u] \leq 0 \quad \text{on } \Omega_T; \tag{2.3.2}$$

$$u \geq 0 \quad \text{on } \partial\Omega_T; \tag{2.3.3}$$

$$\lim_{|x|\to\infty} \frac{u(x,t)}{\theta(x,t)} = 0 \text{ uniformly with respect to } t \in [0,T]; \tag{2.3.4}$$

$$\frac{P[\theta]}{\theta} \text{ is bounded above on } \Omega_T. \tag{2.3.5}$$

Then $u \geq 0$ on $\overline{\Omega}_T$.

Proof. We define the auxiliary function $v : \overline{\Omega}_T \to \mathbb{R}$ to be

$$v(x,t) = \frac{u(x,t)}{\theta(x,t)} \quad \forall (x,t) \in \overline{\Omega}_T,$$

and note that $v \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$. Via (2.3.3) it follows that $v \geq 0$ on $\partial\Omega_T$ and via

(2.3.4) it follows that

$$\lim_{|x|\to\infty} v(x,t) = 0, \tag{2.3.6}$$

uniformly with respect to $t \in [0,T]$. Furthermore, via (2.3.2), v satisfies the linear non-local integro-differential inequality

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_ix_j}v + \sum_{i=1}^{n} \bar{b}_i\partial_{x_i}v + \bar{c}v + \bar{d}J_\theta v - \partial_t v \le 0 \quad \text{on } \Omega_T, \tag{2.3.7}$$

with

$$\bar{b}_i = b_i + \sum_{j=1}^{n} (a_{ij} + a_{ji}) \frac{\partial_{x_j}\theta}{\theta} \quad \text{on } \Omega_T;$$

$$\bar{c} = \frac{L[\theta]}{\theta} \quad \text{on } \Omega_T; \tag{2.3.8}$$

$$\bar{d} = \frac{d}{\theta} \quad \text{on } \Omega_T. \tag{2.3.9}$$

Now, let $\omega : \overline{\Omega}_T \to \mathbb{R}$ be given by

$$\omega(x,t) = v(x,t)e^{-\nu t} \quad \forall (x,t) \in \overline{\Omega}_T, \tag{2.3.10}$$

for some non-negative constant ν, which exists via (2.3.5), for which

$$\frac{P[\theta]}{\theta} < \nu \quad \text{on } \Omega_T. \tag{2.3.11}$$

Note that via (2.3.10) and (2.3.6) it follows that

$$\lim_{|x|\to\infty} \omega(x,t) = 0, \tag{2.3.12}$$

uniformly with respect to $t \in [0,T]$. Moreover, via (2.3.7), ω satisfies the linear non-local integro-differential inequality

$$\bar{P}_\theta[\omega] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j}\omega + \sum_{i=1}^{n} \bar{b}_i \partial_{x_i}\omega + (\bar{c}-\nu)\omega + \bar{d}J_\theta\omega - \partial_t\omega \leq 0 \quad \text{on } \Omega_T. \tag{2.3.13}$$

Upon inspection of (2.3.8)-(2.3.9), via (2.3.5) and (2.3.11), it follows that

$$\bar{P}_\theta[1] = \bar{c} - \nu + \bar{d}J_\theta 1 = \frac{P[\theta]}{\theta} - \nu < 0 \quad \text{on } \Omega_T. \tag{2.3.14}$$

Next, for any $\varepsilon > 0$, via (2.3.13)-(2.3.14), the function $w : \overline{\Omega}_T \to \mathbb{R}$ given by $w = \omega + \varepsilon$ on $\overline{\Omega}_T$ satisfies

$$\bar{P}_\theta[w] = \bar{P}_\theta[\omega] + \varepsilon \bar{P}_\theta[1] < 0 \quad \text{on } \Omega_T. \tag{2.3.15}$$

Moreover, via (2.3.3), $w \geq \varepsilon$ on $\partial\Omega_T$. Furthermore, via (2.3.12) it follows that

$$\lim_{|x|\to\infty} w(x,t) = \varepsilon$$

uniformly with respect to $t \in [0,T]$. Therefore, there exists a sufficiently large constant $R > 0$ such that

$$w > 0, \quad \text{on } (\Omega \setminus B^R) \times (0,T].$$

It remains to establish that $w > 0$ in $(\Omega \cap B^R) \times (0,T]$. For a contradiction, suppose that there exists $(x,t) \in (\Omega \cap B^R) \times (0,T]$ such that $w(x,t) < 0$. Since $w > 0$ on $\partial(\Omega \cap B^R) \times [0,T]$ and $w \in C(\overline{\Omega}_T)$, there exists $(x^*,t^*) \in (\Omega \cap B^R) \times (0,T]$ such that $w(x^*,t^*) = 0$ and $w > 0$ on $(\Omega \cap B^R) \times [0,t^*)$. Additionally, $\nabla w(x^*,t^*) = 0$, $\partial_t w(x^*,t^*) \leq 0$ and the Hessian matrix $D^2 w(x^*,t^*) = [\partial_{x_i x_j} w(x^*,t^*)]$ is positive semi-definite. Consequently, from (2.2.3) with the Schur product theorem (see, for example, [HJ85, Theorem 7.5.3]) we obtain that

$$\bar{P}[w](x^*,t^*) = \sum_{i,j=1}^{n} a_{ij}\partial_{x_ix_j} w(x^*,t^*) + \bar{d}(x^*,t^*)J_\theta w(x^*,t^*) - \partial_t w(x^*,t^*) \geq 0$$

which contradicts (2.3.15). Therefore, $w > 0$ on Ω_T. Letting $\varepsilon \to 0$ establishes that $\omega \geq 0$ on Ω_T, and hence, $u \geq 0$ on $\overline{\Omega}_T$, as required. □

For functions $u : \overline{\Omega}_T \to \mathbb{R}$ we consider a variety of conditions on their growth (and decay) as $|x|$ tends to infinity, detailed in the following definition.

Definition 2.3.2. Let $\alpha, \lambda \in (-\infty, 0]$. We denote that $u \in E_{\alpha,\lambda}(\Omega_T)$ if $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ and there exist $k_1, k_2 > 0$ such that

$$|u(x,t)| \leq k_1 \exp\Big\{-k_2\left(1+|x|^2\right)^{|\alpha|}\left(1+\ln\big(1+|x|^2\big)\right)^{|\lambda|}\Big\} \quad \forall (x,t) \in \overline{\Omega}_T. \tag{2.3.16}$$

Similarly, for $\alpha, \lambda \in [0,\infty)$, we denote that $u \in E_{\alpha,\lambda}(\Omega_T)$ if $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ and there exist $k_1, k_2 > 0$ such that

$$|u(x,t)| \leq k_1 \exp\Big\{k_2\left(1+|x|^2\right)^{\alpha}\left(1+\ln\big(1+|x|^2\big)\right)^{\lambda}\Big\} \quad \forall (x,t) \in \overline{\Omega}_T. \tag{2.3.17}$$

Note that the following inclusions hold: for $\alpha_1 \leq \alpha_2$, $E_{\alpha_1,\lambda}(\Omega_T)$ is a subspace of $E_{\alpha_2,\lambda}(\Omega_T)$; for $\lambda_1 \leq \lambda_2$, $E_{\alpha,\lambda_1}(\Omega_T)$ is a subspace of $E_{\alpha,\lambda_2}(\Omega_T)$. Also note that $E_{0,0}(\Omega_T) = C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T)$.

Remark 2.3.3. In the remainder of this section we will frequently use a class of functions, presented in [Cos80] and [MN14, Section 3], to highlight a connection between the restrictions placed upon u in (2.3.16)-(2.3.17) with the conditions on the coefficients of P in (2.2.4) in associated minimum principles. For completeness we present them here. Let $\xi : [1,\infty) \to [1,\infty)$ be given by

$$\xi(s) = s^{|\alpha|}(1+\ln s)^{|\lambda|} \quad \forall s \in [1,\infty), \tag{2.3.18}$$

for $\alpha, \lambda \in \mathbb{R}$. Provided that α and λ are not both zero, it follows that

$$\xi'(s) > 0; \tag{2.3.19}$$

$$s\xi'(s) \leq (|\alpha| + |\lambda|)\xi(s); \tag{2.3.20}$$

$$s\xi''(s) \leq (|\alpha| + |\lambda|)\xi(s)\xi'(s), \tag{2.3.21}$$

for all $s \in [1, \infty)$.

We now present three minimum principles concerning solutions $u : \overline{\Omega}_T \to \mathbb{R}$ to the linear integro-differential inequality (2.3.1) with P as in (2.2.4). We separate those results with respect to conditions on u as follows: u grows as in (2.3.17), as $|x| \to \infty$ (see Proposition 2.3.4); u decays unconditionally, as $|x| \to \infty$ (see Proposition 2.3.5); u decays, as in (2.3.16), as $|x| \to \infty$ (see Proposition 2.3.6). The following propositions highlight complementary assumptions on the coefficients of P as well as the integral kernel φ that are sufficient in order to obtain such minimum principles. We establish these results by applying the conditional minimum principle, with suitable weight functions.

Proposition 2.3.4. *Let $(\alpha, \lambda) \in [0, \infty)^2 \setminus \{(0,0)\}$, $u \in E_{\alpha,\lambda}(\Omega_T)$ and P be an operator as in (2.2.4) (i.e. $\theta \equiv 1$). Furthermore, suppose:*

$$P[u] \leq 0 \quad \text{on } \Omega_T; \tag{2.3.22}$$

$$u \geq 0 \quad \text{on } \partial\Omega_T; \tag{2.3.23}$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \leq \frac{A}{t^\beta}\left(1 + |x|^2\right)^{1-\alpha}\left(1 + \ln\left(1 + |x|^2\right)\right)^{-\lambda}|\eta|^2 \quad \forall (x,t) \in \Omega_T,\ \eta \in \mathbb{R}^n; \tag{2.3.24}$$

$$\sum_{i=1}^{n} b_i(x,t)x_i \leq \frac{B}{t^\beta}\left(1 + |x|^2\right) \quad \forall (x,t) \in \Omega_T; \tag{2.3.25}$$

$$c(x,t) \leq \frac{C}{t^\beta}\left(1 + |x|^2\right)^\alpha\left(1 + \ln\left(1 + |x|^2\right)\right)^\lambda \quad \forall (x,t) \in \Omega_T; \tag{2.3.26}$$

$$\operatorname{supp}\varphi \subseteq B^R; \tag{2.3.27}$$

$$d(x,t) \leq \frac{D}{t^\beta}\frac{e^{k\xi(1+|x|^2)}}{e^{2k\xi(1+(|x|+R)^2)}}\left(1 + |x|^2\right)^\alpha\left(1 + \ln\left(1 + |x|^2\right)\right)^\lambda \quad \forall (x,t) \in \Omega_T, \tag{2.3.28}$$

for constants $A, B, C, D \geq 0$, $R > 0$, $\beta \in [0, 1)$, *and* $k > k_2$ *for* k_2 *as in* (2.3.17), *and with* ξ *as in* (2.3.18) . *Then,* $u \geq 0$ *on* $\overline{\Omega}_T$.

Proof. Since $u \in C(\overline{\Omega}_T)$, via (2.3.27), Ju is well-defined on Ω_T. Now, let $\theta : \overline{\Omega}_{T_0} \to \mathbb{R}$ be given by

$$\theta(x,t) = e^{k\xi(1+|x|^2)e^{\mu t^{1-\beta}}} \quad \forall (x,t) \in \overline{\Omega}_{T_0}, \tag{2.3.29}$$

with

$$T_0 = \left(\frac{\ln(2)}{\mu} \right)^{1/(1-\beta)},$$

ξ given by (2.3.18), and

$$\mu = \frac{4A(2k+1)(\alpha+\lambda)^2 + 2(nA+B)(\alpha+\lambda) + C/k + D||\varphi||_{L^1(\mathbb{R}^n)}/k + 1}{(1-\beta)}. \tag{2.3.30}$$

We observe that $\theta \in C(\overline{\Omega}_{T_0}) \cap C^{2,1}(\Omega_{T_0})$. We next define the auxiliary function $v : \overline{\Omega}_{T_0} \to \mathbb{R}$ to be

$$v(x,t) = \frac{u(x,t)}{\theta(x,t)} \quad \forall (x,t) \in \overline{\Omega}_{T_0}. \tag{2.3.31}$$

Upon substituting (2.3.23) into (2.3.31), it follows that

$$v \geq 0 \quad \text{on } \partial\Omega_{T_0}.$$

Moreover, since $u \in E_{\alpha,\lambda}(\Omega_T)$, via (2.3.17) and (2.3.31), it follows that

$$\lim_{|x|\to\infty} v(x,t) = 0,$$

uniformly with respect to $t \in [0, T_0]$. For θ and ξ given by (2.3.29) and (2.3.18) respectively, we have

$$\partial_t \theta(x,t) = \theta(x,t) k e^{\mu t^{1-\beta}} \left(\xi(1+|x|^2)(1-\beta)t^{-\beta}\mu \right); \tag{2.3.32}$$

$$\partial_{x_i}\theta(x,t) = \theta(x,t)ke^{\mu t^{1-\beta}}\left(2\xi'(1+|x|^2)x_i\right); \tag{2.3.33}$$

$$\partial_{x_ix_j}\theta(x,t) = \theta(x,t)ke^{\mu t^{1-\beta}}\Big(4ke^{\mu t^{1-\beta}}(\xi'(1+|x|^2))^2x_ix_j$$
$$+4\xi''(1+|x|^2)x_ix_j + 2\delta_{ij}\xi'(1+|x|^2)\Big), \tag{2.3.34}$$

for all $(x,t) \in \Omega_{T_0}$. Now, via (2.3.27), it follows that

$$\begin{aligned} J\theta(x,t) &\leq \int_{\mathbb{R}^n} \varphi(x-y)\theta(y,t)dy \\ &= \int_{B^R} \varphi(y)\theta(x-y,t)dy \\ &\leq \sup_{y\in B^R}\{\theta(x-y,T_0)\}\int_{\mathbb{R}^n}\varphi(y)dy \\ &\leq \sup_{y\in B^R}\{e^{2k\xi(1+|x-y|^2)}\}\|\varphi\|_{L^1(\mathbb{R}^n)} \\ &\leq \|\varphi\|_{L^1(\mathbb{R}^n)}e^{2k\xi(1+(|x|+R)^2)} \end{aligned} \tag{2.3.35}$$

for all $(x,t) \in \overline{\Omega}_{T_0}$. Substituting (2.3.32)-(2.3.35) into $P : C^{2,1}(\Omega_T) \to \mathbb{R}(\Omega_T)$, as in (2.2.4), and using the bounds in (2.3.24)-(2.3.28) and (2.3.30), it follows that

$$\frac{P[\theta]}{\theta} < 0 \quad \text{on } \Omega_{T_0}. \tag{2.3.36}$$

By an application of Lemma 2.3.1 it now follows that $u \geq 0$ on $\overline{\Omega}_{T_0}$. By replacing v in (2.3.31) with

$$v(x,t) = \frac{u(x,t+T_0)}{\theta(x,t)} \quad \forall (x,t) \in \overline{\Omega}_{T_0}, \tag{2.3.37}$$

it follows, via the argument as above, that $u \geq 0$ on $\overline{\Omega}_{2T_0}$. Repeating this argument finitely many times establishes that $u \geq 0$ on $\overline{\Omega}_T$, as required. □

Proposition 2.3.5. *Let $u \in E_{0,0}(\Omega_T)$ and P be an operator defined as in (2.2.4) (i.e.*

$\theta \equiv 1$). *Furthermore, suppose:*

$$P[u] \leq 0 \quad \text{on } \Omega_T; \tag{2.3.38}$$

$$u \geq 0 \quad \text{on } \partial\Omega_T; \tag{2.3.39}$$

$$\lim_{|x|\to\infty} u(x,t) = 0 \text{ uniformly with respect to } t \in [0,T]; \tag{2.3.40}$$

$$c(x,t) \leq \frac{C}{t^\beta} \quad \forall (x,t) \in \Omega_T; \tag{2.3.41}$$

$$d(x,t) \leq \frac{D}{t^\beta} \quad \forall (x,t) \in \Omega_T, \tag{2.3.42}$$

for constants $C, D \geq 0$ *and* $\beta \in [0,1)$. *Then* $u \geq 0$ *on* $\overline{\Omega}_T$.

Proof. Since $u \in L^\infty(\overline{\Omega}_T)$ and $\varphi \in L^1(\mathbb{R}^n)$, it follows that Ju is well-defined on Ω_T. Now, let $\theta : \overline{\Omega}_T \to \mathbb{R}$ be given by

$$\theta(x,t) = e^{\mu t^{1-\beta}} \quad \forall (x,t) \in \overline{\Omega}_T, \tag{2.3.43}$$

with

$$\mu = \frac{C + D\|\varphi\|_{L^1(\Omega)}}{1-\beta}. \tag{2.3.44}$$

We observe that $\theta \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$. Via (2.3.40) it follows that

$$\lim_{|x|\to\infty} \frac{u(x,t)}{\theta(x,t)} = 0,$$

uniformly with respect to $t \in [0,T]$. Utilising (2.3.41), (2.3.42) and for θ as in (2.3.43), it follows that

$$\frac{P[\theta]}{\theta} = \frac{1}{\theta}\left(c\theta - \partial_t\theta + dJ\theta\right) \leq 0 \quad \text{on } \Omega_T.$$

By an application of Lemma 2.3.1 it follows that $u \geq 0$ on $\overline{\Omega}_T$, as required. □

Proposition 2.3.6. *Let* $(\alpha, \lambda) \in (-\infty, 0]^2 \setminus \{(0,0)\}$, $u \in E_{\alpha,\lambda}(\Omega_T)$ *and* P *be an operator* *as in* (2.2.4) *(i.e.* $\theta \equiv 1$*). Furthermore, suppose:*

$$P[u] \leq 0 \quad on\ \Omega_T; \tag{2.3.45}$$

$$u \geq 0 \quad on\ \partial\Omega_T; \tag{2.3.46}$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \leq \frac{A}{t^\beta}\left(1+|x|^2\right)^{1-|\alpha|}\left(1+\ln\left(1+|x|^2\right)\right)^{-|\lambda|}|\eta|^2 \quad \forall (x,t) \in \Omega_T,\ \eta \in \mathbb{R}^n; \tag{2.3.47}$$

$$\sum_{i=1}^{n} b_i(x,t)x_i \geq -\frac{B}{t^\beta}\left(1+|x|^2\right) \quad \forall (x,t) \in \Omega_T; \tag{2.3.48}$$

$$c(x,t) \leq \frac{C}{t^\beta}\left(1+|x|^2\right)^{|\alpha|}\left(1+\ln\left(1+|x|^2\right)\right)^{|\lambda|} \quad \forall (x,t) \in \Omega_T; \tag{2.3.49}$$

$$d(x,t) \leq \frac{De^{-k\xi(1+|x|^2)}}{t^\beta}\left(1+|x|^2\right)^{|\alpha|}\left(1+\ln\left(1+|x|^2\right)\right)^{|\lambda|} \quad \forall (x,t) \in \Omega_T, \tag{2.3.50}$$

for constants $A, B, C, D \geq 0$ *and* $\beta \in [0,1)$ *and* $0 < k < k_2$*, for* k_2 *as in* (2.3.16)*. Then,* $u \geq 0$ *on* $\overline{\Omega}_T$*.*

Proof. Since $u \in L^\infty(\overline{\Omega}_T)$ and $\varphi \in L^1(\mathbb{R}^n)$, it follows that Ju is well-defined on Ω_T. Now, let $\theta : \overline{\Omega}_{T_0} \to \mathbb{R}$ be defined to be

$$\theta(x,t) = e^{-k\xi(1+|x|^2)e^{-\mu t^{1-\beta}}} \quad \forall (x,t) \in \overline{\Omega}_{T_0}, \tag{2.3.51}$$

with

$$T_0 = \left(\frac{\ln(2)}{\mu}\right)^{1/(1-\beta)},$$

ξ given by (2.3.18), and

$$\mu = \frac{4A(|\alpha|+|\lambda|)^2 + 2B(|\alpha|+|\lambda|) + 2C/k + 2D||\varphi||_{L^1(\mathbb{R}^n)}/k + 1}{(1-\beta)}. \tag{2.3.52}$$

We observe that $\theta \in C(\overline{\Omega}_{T_0}) \cap C^{2,1}(\Omega_{T_0})$. We next define the auxiliary function $v : \overline{\Omega}_{T_0} \to \mathbb{R}$ to be

$$v(x,t) = \frac{u(x,t)}{\theta(x,t)} \quad \forall (x,t) \in \overline{\Omega}_{T_0}. \tag{2.3.53}$$

Upon substituting (2.3.51) into (2.3.53), it follows that

$$v \geq 0 \quad \text{on } \partial\Omega_{T_0}.$$

Moreover, since $u \in E_{\alpha,\lambda}(\Omega_T)$, via (2.3.17) and (2.3.31), it follows that

$$\lim_{|x|\to\infty} v(x,t) = 0,$$

uniformly with respect to $t \in [0, T_0]$. For θ and ξ given by (2.3.51) and (2.3.18) respectively, we have

$$\partial_t \theta(x,t) = \theta(x,t) k e^{-\mu t^{1-\beta}} \left(\xi(1+|x|^2)(1-\beta)t^{-\beta}\mu \right); \tag{2.3.54}$$

$$\partial_{x_i} \theta(x,t) = -\theta(x,t) k e^{-\mu t^{1-\beta}} \left(2\xi'(1+|x|^2)x_i \right); \tag{2.3.55}$$

$$\begin{aligned} \partial_{x_i x_j} \theta(x,t) = \theta(x,t) k e^{-\mu t^{1-\beta}} \Big(& 4k e^{-\mu t^{1-\beta}} (\xi'(1+|x|^2))^2 x_i x_j \\ & -4\xi''(1+|x|^2)x_i x_j - 2\delta_{ij}\xi'(1+|x|^2) \Big), \end{aligned} \tag{2.3.56}$$

for all $(x,t) \in \Omega_{T_0}$. Also note that

$$\begin{aligned} \frac{J\theta(x,t)}{\theta(x,t)} &= \frac{1}{\theta(x,t)} \int_\Omega \varphi(x-y)\theta(y,t)dy \\ &\leq \frac{1}{\theta(x,t)} \|\varphi\|_{L^1(\mathbb{R}^n)} \\ &\leq \|\varphi\|_{L^1(\mathbb{R}^n)} e^{\frac{k}{2}\xi(1+|x|^2)} \end{aligned} \tag{2.3.57}$$

for all $(x,t) \in \Omega_{T_0}$. Substituting (2.3.54)-(2.3.57) into $P : C^{2,1}(\Omega_T) \to \mathbb{R}(\Omega_T)$, as in (2.2.4), and using the bounds in (2.3.47)-(2.3.50) and (2.3.52), it follows that

$$\frac{P[\theta]}{\theta} < 0 \quad \text{on } \Omega_{T_0}.$$

By an application of Lemma 2.3.1 it now follows that $u \geq 0$ on $\overline{\Omega}_{T_0}$. By replacing v in

(2.3.53) with

$$v(x,t) = \frac{u(x,t+T_0)}{\theta(x,t)} \quad \forall (x,t) \in \overline{\Omega}_{T_0},$$

it follows, via the argument as above, that $u \geq 0$ on $\overline{\Omega}_{2T_0}$. Repeating this argument finitely many times establishes that $u \geq 0$ on $\overline{\Omega}_T$, as required.

□

When a decay condition as $|x| \to \infty$ on bounded solutions of the non-local integro-differential inequality

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + cu + dJu - \partial_t u \leq 0 \quad \text{ on } \Omega_T,$$

is not imposed (e.g. if one relaxes (2.3.40)), and, an estimate on the regularity of the coefficients a_{ij}, b_i is not available, one can instead impose further restrictions on the integral kernel φ to establish a minimum principle, as illustrated in the following proposition.

Proposition 2.3.7 (Weak minimum principle)**.** *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T)$, P be an operator defined as in* (2.2.4) *(i.e. $\theta \equiv 1$) and $\psi : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$\psi(x) = \varphi(x)|x|^2 \quad \forall x \in \mathbb{R}^n. \tag{2.3.58}$$

Furthermore, suppose:

$$P[u] \leq 0 \quad \text{on } \Omega_T; \tag{2.3.59}$$

$$u \geq 0 \quad \text{on } \partial\Omega_T; \tag{2.3.60}$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \leq \frac{A}{t^\beta}(1+|x|^2)\,|\eta|^2 \quad \forall (x,t) \in \Omega_T,\ \eta \in \mathbb{R}^n; \tag{2.3.61}$$

$$\sum_{i=1}^{n} b_i(x,t)x_i \leq \frac{B}{t^\beta}(1+|x|^2) \quad \forall (x,t) \in \Omega_T; \tag{2.3.62}$$

$$c(x,t) \leq \frac{C}{t^\beta} \quad \forall (x,t) \in \Omega_T; \tag{2.3.63}$$

$$0 \leq d(x,t) \leq \frac{D}{t^\beta} \quad \forall (x,t) \in \Omega_T; \tag{2.3.64}$$

$$\psi \in L^1(\mathbb{R}^n), \tag{2.3.65}$$

for non-negative constants A, B, C and D and $\beta \in [0,1)$. Then, $u \geq 0$ on $\overline{\Omega}_T$.

Proof. Since $u \in L^\infty(\overline{\Omega}_T)$ and $\varphi \in L^1(\mathbb{R}^n)$, it follows that Ju is well-defined on Ω_T. Now, let $\theta : \overline{\Omega}_T \to \mathbb{R}$ be given by

$$\theta(x,t) = (1+|x|^2)e^{\mu t^{1-\beta}}, \quad \forall (x,t) \in \overline{\Omega}_T, \tag{2.3.66}$$

with

$$\mu = \frac{nA + 2B + C + 3D(||\varphi||_{L^1(\mathbb{R}^n)} + ||\psi||_{L^1(\mathbb{R}^n)}) + 1}{1-\beta}. \tag{2.3.67}$$

It follows from (2.3.66) that $\theta \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$, and

$$\lim_{|x|\to\infty} \frac{u(x,t)}{\theta(x,t)} \leq \frac{\|u\|_{L^\infty(\overline{\Omega}_T)}}{\theta(x,t)} = 0$$

uniformly with respect to $t \in [0,T]$. Moreover, via (2.3.61)-(2.3.63) and (2.3.67), for θ as given by (2.3.66) it follows that

$$\frac{L[\theta]}{\theta} < 0 \quad \text{on } \Omega_T. \tag{2.3.68}$$

Now, using (2.3.65) we demonstrate that $\frac{1}{\theta}J\theta$ is bounded on Ω_T. Observe that:

$$\begin{aligned}
\frac{1}{\theta(x,t)} J\theta(x,t) &= \frac{1}{\theta(x,t)} \int_\Omega \varphi(x-y)\theta(y,t)dy \\
&\leq \int_{\mathbb{R}^n} \varphi(y) \frac{|x-y|^2+1}{|x|^2+1} dy \\
&= \int_{\mathbb{R}^n} \varphi(y) \frac{|x|^2 - 2\langle x,y\rangle + |y|^2 + 1}{|x|^2+1} dy \\
&\leq \int_{\mathbb{R}^n} \varphi(y) \left[1 + |y|\left(\frac{2|x|}{|x|^2+1}\right) + |y|^2\left(\frac{1}{|x|^2+1}\right)\right] dy \\
&\leq 3(\|\varphi\|_{L^1(\mathbb{R}^n)} + \|\psi\|_{L^1(\mathbb{R}^n)}) \quad \forall (x,t) \in \Omega_T.
\end{aligned} \tag{2.3.69}$$

Now, via (2.3.67), (2.3.68), (2.3.64), and (2.3.69), it follows that

$$\frac{P[\theta]}{\theta} < 0 \quad \text{on } \Omega_T.$$

By an application of Lemma 2.3.1 it now follows that $u \geq 0$ on $\overline{\Omega}_T$, as required. □

Remark 2.3.8. Note that by the argument above we have relaxed the restriction on φ in [DMV11, Theorem 5], where $\text{supp}(\varphi)$ is compact. Here, we allow φ to decay as $|x| \to \infty$, albeit with the decay rate constrained by the integrability of ψ. In addition, upon comparing Proposition 2.3.7 with 2.3.5 we see that the assumption on the decay on u can be "transferred" on the integral kernel φ.

By combining Proposition 2.3.7 with a strong minimum principle for second order linear parabolic partial differential inequalities (see, for instance, [Fri08, Chapter 2]), we obtain

Corollary 2.3.9 (Strong minimum principle)**.** *Suppose that the conditions of Proposition 2.3.7 are satisfied. Furthermore, suppose that for any* $x_0 \in \Omega$ *and* $R > 0$, $a_{ij}, b_i, c \in L^\infty(B^R_{x_0} \times (0,T])$. *Then, either* $u \equiv 0$ *on* $\overline{\Omega}_T$ *or* $u > 0$ *on* Ω_T.

Proof. By Proposition 2.3.7, $u \geq 0$ on Ω_T. Moreover, since $\varphi \geq 0$ on Ω we have

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + cu - \partial_t u \leq -dJu \leq 0 \quad \text{on } \Omega_T. \tag{2.3.70}$$

Therefore, for any $R > 0$, the inequality (2.3.70) holds on $B^R_{x_0} \times (0,T]$. Via the strong minimum principle for linear parabolic partial differential inequalities [Fri08, Chapter 2] the result follows, as required. □

Remark 2.3.10. By applying the standard strong minimum principle for linear parabolic partial differential inequalities, the assumption on c of Corollary 2.3.9 cannot be relaxed to the assumption on c in Proposition 2.3.7 (see [NM15, Section 3]). However we note that one can establish a sharper strong minimum principle than Corollary 2.3.9, requiring alternative conditions on c, using the regularised distance functions constructed in [Lie85], within a standard strong minimum principle argument.

2.3.2 Integral case

In Section 2.3.1 we showed that, without requiring assumptions on the regularity of coefficients of the linear non-local integro-differential operator P, we were able to establish minimum principles. Notably, we imposed conditions on the behaviour of φ or u as $|x| \to \infty$ instead of requiring regularity on coefficients of P. In this section, utilising a different approach (similar to that presented in [Fri08, Ch.2]) we will show that by assuming additional regularity as well as boundedness on the coefficients of P, we can establish a minimum principle for P without any further restrictions placed on φ, or on u as $|x| \to \infty$. For simplicity, we restrict attention to $\Omega = \mathbb{R}^n$, and discuss the case when Ω is an unbounded domain in $\mathbb{R}^n$ in the concluding remarks.

Let $L : C^{2,1}(\Omega_T) \to \mathbb{R}(\Omega_T)$ be given by (2.2.2). We suppose that for some $\alpha \in (0, 1]$, the coefficients $a_{ij}, b_i, c : \overline{\Omega}_T \to \mathbb{R}$ in L are such that

$$a_{ij}, b_i, c \in L^\infty(\overline{\Omega}_T) \cap H_\alpha(\overline{\Omega}_T). \tag{2.3.71}$$

Furthermore, $a_{ij} = a_{ji}$ on Ω_T, and

$$A_{min}|\eta|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \leq A_{max}|\eta|^2 \quad \forall (x,t) \in \overline{\Omega}_T,\ \eta \in \mathbb{R}^n, \tag{2.3.72}$$

for some constants $A_{min}, A_{max} \in \mathbb{R}_+$, and

$$|a_{ij}(x_1,t_1) - a_{ij}(x_2,t_2)| \leq k_\alpha(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}) \tag{2.3.73}$$

for all $(x_1,t_1), (x_2,t_2) \in \overline{\Omega}_T$ for some constant $k_\alpha \in \mathbb{R}_+$. Note that, in order to apply the theory presented in [Fri08], we require the coefficients to be continuously extendable onto $\partial\Omega_T$. Alternative conditions on a_{ij}, b_i and c to those presented here (which are sufficient to establish the existence of fundamental solutions for L) are discussed in [LSU68, p.356-414] and [BS23, ZC22]. We now state

Lemma 2.3.11. *Let L be an operator as in (2.2.2) and suppose that (2.3.71)-(2.3.73) are satisfied. Then, there exists a fundamental solution for L denoted by $\Gamma(x,t;\xi,\tau):\Omega^\Gamma\to\mathbb{R}$ with*

$$\Omega^\Gamma=\{(x,t;\xi,\tau)\in\overline{\Omega}_T\times\overline{\Omega}_T:0\leq\tau<t\leq T\}.$$

Specifically, for any fixed (ξ,τ), as a function of (x,t), Γ satisfies:

- *$L[\Gamma]=0$ on $\Omega\times(\tau,T]$.*
- *For every $f\in C(\overline{\Omega})\cap L^\infty(\overline{\Omega})$*

$$\lim_{t\searrow\tau}\int_\Omega\Gamma(x,t;\xi,\tau)f(\xi)d\xi=f(x)\quad\forall x\in\Omega.$$

Proof. See [Fri08, Theorem 10, p.23]. □

To proceed, we also require the existence of a fundamental solution for the adjoint operator of L, denoted by $L^*:C^{2,1}(\Omega_T)\to\mathbb{R}(\Omega_T)$. To define L^*, we require that the coefficients of L also satisfy

$$\partial_{x_kx_l}a_{ij},\ \partial_{x_k}a_{ij},\ \partial_{x_k}b_i\in L^\infty(\overline{\Omega}_T)\cap H_\alpha(\overline{\Omega}_T),\tag{2.3.74}$$

for $i,j,k,l=1,\dots,n$. Specifically, the adjoint operator of L is given by

$$L^*[v]=\sum_{i,j=1}^n\partial_{x_ix_j}(a_{ij}v)-\sum_{i=1}^n\partial_{x_i}(b_iv)+cv+\partial_tv\quad\text{on }\Omega_T,$$

for all $v\in C^{2,1}(\Omega_T)$. We can now state

Lemma 2.3.12. *Let L be the operator defined as in (2.2.2) and suppose that (2.3.71)-(2.3.73) and (2.3.74) are satisfied. Then, there exists a fundamental solution for L^*,* *denoted by* $\Gamma^* : \Omega^{\Gamma^*} \to \mathbb{R}$ *with*

$$\Omega^{\Gamma^*} = \{(x,t;\xi,\tau) \in \overline{\Omega}_T \times \overline{\Omega}_T : 0 \le t < \tau \le T\}.$$

Specifically, for any fixed (ξ,τ)*, as a function of* (x,t)*,* Γ^* *satisfies:*

- $L^*[\Gamma^*] = 0$ *on* $\Omega \times (0,\tau)$.
- *For every* $f \in C(\Omega) \cap L^\infty(\Omega)$

$$\lim_{t \nearrow \tau} \int_\Omega \Gamma^*(x,t;\xi,\tau) f(\xi) d\xi = f(x) \quad \forall x \in \Omega.$$

Additionally,

$$\Gamma(x,t;\xi,\tau) = \Gamma^*(\xi,\tau;x,t) \quad \forall (x,t;\xi,\tau) \in \Omega^\Gamma. \tag{2.3.75}$$

Proof. See [Fri08, Theorems 14 and 15, p.27-28]. □

We also require the following qualitative properties of Γ and Γ^*, stated in

Lemma 2.3.13. *For all* $(x,t;\xi,\tau) \in \Omega^\Gamma$*,* Γ *satisfies:*

$$0 < \Gamma(x,t;\xi,\tau) \le \kappa(t-\tau)^{-\frac{n}{2}} \exp\left(-\frac{\lambda|x-\xi|^2}{4(t-\tau)}\right), \tag{2.3.76}$$

$$|\partial_{x_i}\Gamma(x,t;\xi,\tau)| \le \kappa(t-\tau)^{-\frac{(n+1)}{2}} \exp\left(-\frac{\lambda|x-\xi|^2}{4(t-\tau)}\right), \tag{2.3.77}$$

and for all $(x,t;\xi,\tau) \in \Omega^{\Gamma^*}$*,* Γ^* *satisfies:*

$$0 < \Gamma^*(x,t;\xi,\tau) \le \kappa(\tau-t)^{-\frac{n}{2}} \exp\left(-\frac{\lambda|x-\xi|^2}{4(\tau-t)}\right), \tag{2.3.78}$$

$$|\partial_{x_i}\Gamma^*(x,t;\xi,\tau)| \le \kappa(\tau-t)^{-\frac{(n+1)}{2}} \exp\left(-\frac{\lambda|x-\xi|^2}{4(\tau-t)}\right), \tag{2.3.79}$$

for some constants $\kappa \in (0,\infty)$ *and* $\lambda \in (0, A_{min})$.

Proof. For the upper bounds in (2.3.76)-(2.3.79) see [Fri08, p.24 and p.28]. For the positivity of Γ and Γ^* see [Fri08, Theorem 11, p.44] and (2.3.75). $\square$

We now establish a weak minimum principle for P as in (2.2.4), with $\theta \equiv 1$.

Proposition 2.3.14 (Weak minimum principle)**.** *Let $u \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T) \cap L^\infty(\overline{\Omega}_T)$ and P be an operator as in (2.2.4) (i.e. $\theta \equiv 1$). Suppose that the coefficients $a_{ij}, b_i : \overline{\Omega}_T \to \mathbb{R}$ satisfy (2.3.71)-(2.3.73) and (2.3.74) and $c, d : \Omega_T \to [0, \infty)$ are bounded functions.*

Furthermore, suppose

$$P[u] \leq 0 \quad \text{on } \Omega_T; \tag{2.3.80}$$

$$u \geq 0 \quad \text{on } \partial\Omega_T. \tag{2.3.81}$$

Then, $u \geq 0$ on $\overline{\Omega}_T$.

Proof. We rewrite (2.3.80) as

$$L[u] + cu + dJu \leq 0 \quad \text{on } \Omega_T, \tag{2.3.82}$$

with L denoting the second order linear parabolic partial differential operator given by

$$L[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u - \partial_t u \quad \text{on } \Omega_T, \tag{2.3.83}$$

for $u \in C^{2,1}(\Omega_T)$. Now, for $R \in \mathbb{N}$ we define $\Gamma_R^* : \Omega^{\Gamma^*} \to \mathbb{R}$ to be

$$\Gamma_R^*(y, s; x, t) = \Gamma^*(y, s; x, t) H_R(y - x) \quad \forall (y, s; x, t) \in \Omega^{\Gamma^*}, \tag{2.3.84}$$

with Γ^* the fundamental solution for the adjoint operator of L in (2.3.83). The existence of Γ^* is guaranteed since the conditions of Lemma 2.3.12 are satisfied. The function $H_R \in C^2(\Omega)$ used to define Γ_R^* in (2.3.84) satisfies the following properties:

$$H_R(y) = \begin{cases} 1, & y \in B^R; \\ 0, & y \in \Omega \setminus B^{R+1}; \end{cases} \tag{2.3.85}$$

$$0 \le H_R(y) \le 1 \quad \forall y \in \Omega; \tag{2.3.86}$$

$$\sum_{i=1}^{n} |\partial_{y_i} H_R| + \sum_{i,j=1}^{n} |\partial_{y_i y_j} H_R| \le M \quad \text{on } B^{R+1}, \tag{2.3.87}$$

for a constant $M \ge 0$ independent of $R \in \mathbb{N}$. Upon multiplying (2.3.82) evaluated at (y,s), with $\Gamma_R^*(y,s;x,t)$, and integrating over $\Omega \times [\varepsilon_1, t-\varepsilon_2]$ with $\varepsilon_1, \varepsilon_2 \in (0, t/2)$, we obtain

$$\begin{aligned} 0 &\ge \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \Gamma_R^*(y,s;x,t)(L[u] + cu + dJu)(y,s)dyds \\ &= \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \Gamma_R^*(y,s;x,t) L[u](y,s)dyds \\ &\quad + \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \Gamma_R^*(y,s;x,t)(cu + dJu)(y,s)dyds. \end{aligned} \tag{2.3.88}$$

Via Green's identity for L and L^* [Fri08, p.27], the divergence theorem, and Lemma 2.3.12, the first integral in (2.3.88) is given by

$$\begin{aligned} &\int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \Gamma_R^*(y,s;x,t) L[u](y,s)dyds \\ &= \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} (uL^*[\Gamma_R^*] - \partial_s(u\Gamma_R^*))(y,s;x,t)dyds \\ &\quad + \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \left(\sum_{i=1}^{n} \partial_{y_i} \sum_{j=1}^{n} \left[\Gamma_R^* a_{ij} (\partial_{y_j} u) - u a_{ij} (\partial_{y_j} \Gamma_R^*) - u \Gamma_R^* (\partial_{y_j} a_{ij}) \right] \right) (y,s;x,t)dyds \\ &\quad\quad + \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \left(\sum_{i=1}^{n} \partial_{y_i} (b_i u \Gamma_R^*) \right) (y,s;x,t)dyds \end{aligned} \tag{2.3.89}$$

$$= \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1}} \Lambda(y,s;x,t) u(y,s)dyds - \int_{B_x^{R+1}} \Gamma_R^*(y,s;x,t)u(y,s)dy \Big|_{s=\varepsilon_1}^{t-\varepsilon_2}, \tag{2.3.90}$$

where, via the divergence theorem and (2.3.85)-(2.3.87), the second and third integrals in (2.3.89) vanish for all $(x,t) \in \Omega_T$ and $\Lambda : \Omega_t \to \mathbb{R}$ is given by

$$\begin{aligned}\Lambda(\cdot,\cdot;x,t) = &\sum_{i,j=1}^{n} (\partial_{y_j}(a_{ij}\Gamma^*)\partial_{y_i}H_R + \partial_{y_i}(a_{ij}\Gamma^*)\partial_{y_j}H_R) \\ &+ \sum_{i,j=1}^{n} a_{ij}\Gamma^*(\partial_{y_iy_j}H_R) - \sum_{i=1}^{n} b_i\Gamma^*(\partial_{y_i}H_R),\end{aligned} \tag{2.3.91}$$

on Ω_t for all $(x,t) \in \Omega_T$. Via (2.3.85)-(2.3.86) we have

$$\int_{\varepsilon_1}^{t-\varepsilon_2}\int_{B_x^{R+1}} \Lambda(y,s;x,t)u(y,s)dyds = \int_{\varepsilon_1}^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} \Lambda(y,s;x,t)u(y,s)dyds.$$

Using (2.3.71)-(2.3.73), (2.3.74), (2.3.87), (2.3.91) and Lemma 2.3.13, it follows that there exists a sufficiently large constant $C > 0$, independent of ε_1 and R, such that

$$\begin{aligned}&\left|\int_{\varepsilon_1}^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} \Lambda(y,s;x,t)u(y,s)dyds\right| \\ &\leq \|u\|_{L^\infty(\overline{\Omega}_T)}\int_{\varepsilon_1}^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} |\Lambda(y,s;x,t)|\,dyds \\ &\leq M\|u\|_{L^\infty(\overline{\Omega}_T)}\int_{\varepsilon_1}^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} \left|\sum_{i,j=1}^{n} \partial_{y_j}(a_{ij}(y,s)\Gamma^*(y,s;x,t)) + \partial_{y_i}(a_{ij}(y,s)\Gamma^*(y,s;x,t))\right| \\ &\qquad + \left|\sum_{i,j=1}^{n} a_{ij}(y,s)\Gamma^*(y,s;x,t)\right| + \left|\sum_{i=1}^{n} b_i(y,s)\Gamma^*(y,s;x,t)\right| dyds \\ &\leq C\int_0^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} \left|\sum_{i=1}^{n} \partial_{y_i}\Gamma^*(y,s;x,t)\right| + |\Gamma^*(y,s;x,t)|\,dyds.\end{aligned} \tag{2.3.92}$$

Via Lemma 2.3.13, the Fubini-Tonelli theorem and several changes of variables, we obtain

$$\begin{aligned}&\int_0^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} \left|\sum_{i=1}^{n} \partial_{y_i}\Gamma^*(y,s;x,t)\right| + |\Gamma^*(y,s;x,t)|\,dyds \\ &\leq \kappa\int_0^{t-\varepsilon_2}\int_{B_x^{R+1}\backslash B_x^R} n(t-s)^{-\frac{n+1}{2}}\exp\left(-\frac{\lambda|x-y|^2}{4(t-s)}\right) + (t-s)^{-\frac{n}{2}}\exp\left(-\frac{\lambda|x-y|^2}{4(t-s)}\right)dyds \\ &= \kappa\int_{\varepsilon_2}^{t}\int_{B_x^{R+1}\backslash B_x^R}\left(1+\frac{n}{\sqrt{\tau}}\right)\frac{1}{\tau^{\frac{n}{2}}}\exp\left(-\frac{\lambda|x-y|^2}{4\tau}\right)dyd\tau\end{aligned}$$

$$
\begin{aligned}
&= \kappa \int_{\varepsilon_2}^{t} \left(1+\frac{n}{\sqrt{\tau}}\right) \frac{1}{\tau^{\frac{n}{2}}} \int_{R}^{R+1} \exp\left(-\frac{\lambda r^2}{4\tau}\right) |s_n| r^{n-1} dr d\tau \\
&= \kappa |s_n| \int_{\varepsilon_2}^{t} \left(1+\frac{n}{\sqrt{\tau}}\right) \frac{1}{\tau^{\frac{n}{2}}} \int_{\frac{\lambda R^2}{4\tau}}^{\frac{\lambda (R+1)^2}{4\tau}} e^{-z} \left(\frac{4\tau z}{\lambda}\right)^{\frac{n-1}{2}} \left(\frac{\tau}{\lambda z}\right)^{\frac{1}{2}} dz d\tau \\
&= \frac{\kappa |s_n|}{2} \left(\frac{4}{\lambda}\right)^{\frac{n}{2}} \int_{\varepsilon_2}^{t} \left(1+\frac{n}{\sqrt{\tau}}\right) d\tau \int_{\frac{\lambda R^2}{4\tau}}^{\frac{\lambda (R+1)^2}{4\tau}} e^{-z} z^{\frac{n}{2}-1} dz \\
&< \frac{\kappa |s_n|}{2} \left(\frac{4}{\lambda}\right)^{\frac{n}{2}} \int_{\varepsilon_2}^{t} \left(1+\frac{n}{\sqrt{\tau}}\right) d\tau \int_{\frac{\lambda R^2}{4T}}^{\infty} e^{-z} z^{\frac{n}{2}-1} dz \\
&< \frac{\kappa |s_n|}{2} \left(\frac{4}{\lambda}\right)^{\frac{n}{2}} \left(T + 2n\sqrt{T}\right) \Gamma_{up}\left(\frac{n}{2}, \frac{\lambda R^2}{4T}\right) \qquad (2.3.93)
\end{aligned}
$$

where $\lambda > 0$, $|s_n|$ denotes the surface area of the unit $n-$sphere and $\Gamma_{up} : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ denotes the upper incomplete Gamma function (see for instance [AS64, p.260]). Upon substituting (2.3.93) into (2.3.92) it follows that

$$
\int_{\varepsilon_1}^{t-\varepsilon_2} \int_{B_x^{R+1} \setminus B_x^R} \Lambda(y,s;x,t) u(y,s) dy ds \to 0 \quad \text{as } R \to \infty, \qquad (2.3.94)
$$

uniformly for all $(x,t) \in \Omega_T$ and $\varepsilon_1 \in (0, \frac{t}{2})$. Thus, via (2.3.90), (2.3.94) and letting $R \to \infty$, the differential inequality in (2.3.88) reduces to

$$
0 \geq - \int_{\Omega} \Gamma^*(y,s;x,t) u(y,s) dy \Big|_{s=\varepsilon_1}^{t-\varepsilon_2} + \int_{\varepsilon_1}^{t-\varepsilon_2} \int_{\Omega} \Gamma^*(y,s;x,t)(cu + dJu)(y,s) dy ds. \qquad (2.3.95)
$$

Via Lemma 2.3.12 and by letting $\varepsilon_1, \varepsilon_2 \to 0^+$ in (2.3.95) we obtain

$$
u(x,t) \geq \int_{\Omega} \Gamma^*(y,0;x,t) u(y,0) dy + \int_{\Omega_t} \Gamma^*(y,s;x,t)(cu + dJu)(y,s) dy ds \qquad (2.3.96)
$$

for all $(x,t) \in \Omega_T$. Via (2.3.81), the first integral in (2.3.96) is non-negative and thus we have

$$-u(x,t) \leq \int_{\Omega_t} \Gamma^*(y,s;x,t)c(y,s)(-u)(y,s)dyds + \int_{\Omega_t} \Gamma^*(y,s;x,t)d(y,s)J(-u)(y,s)dyds \tag{2.3.97}$$

for all $(x,t) \in \Omega_T$. Define $\psi : \overline{\Omega}_T \to \mathbb{R}$ to be

$$\psi = -u \quad \text{on } \overline{\Omega}_T, \tag{2.3.98}$$

and additionally, define $\psi_\infty^+ : [0,T] \to \mathbb{R}$ to be

$$\psi_\infty^+(t) = \sup_{y\in\Omega}\{\max\{\psi(y,t),0\}\} \quad \forall t \in [0,T].$$

We note that $\psi_\infty^+ \in L^1([0,T])$ (for details, see [MN15a, Ch. 7]). Thus, via (2.3.78), (2.3.98), the boundedness of c, d and the integrability of φ, it follows that inequality (2.3.97) becomes

$$\begin{aligned}
\psi(x,t) &\leq \int_{\Omega_t} \Gamma^*(y,s;x,t)c(y,s)\max\{\psi(y,s),0\}dyds \\
&\quad + \int_{\Omega_t} \Gamma^*(y,s;x,t)d(y,s)(J\max\{\psi,0\})(y,s)dyds \\
&\leq \int_{\Omega_t} \Gamma^*(y,s;x,t)\left(\|c\|_{L^\infty(\Omega_T)}\psi_\infty^+(s) + \|d\|_{L^\infty(\Omega_T)}\int_\Omega \varphi(z-y)\psi_\infty^+(s)dz\right)dyds \\
&\leq \kappa\left(\|c\|_{L^\infty(\Omega_T)} + \|d\|_{L^\infty(\Omega_T)}\|\varphi\|_{L^1(\mathbb{R}^n)}\right)\int_{\Omega_t}\frac{\psi_\infty^+(s)}{(t-s)^{n/2}}\exp\left(-\frac{\lambda|y-x|^2}{4(t-s)}\right)dyds \\
&\leq \kappa\left(\|c\|_{L^\infty(\Omega_T)} + \|d\|_{L^\infty(\Omega_T)}\|\varphi\|_{L^1(\mathbb{R}^n)}\right)\left(\frac{2\sqrt{\pi}}{\sqrt{\lambda}}\right)^n\int_0^t \psi_\infty^+(s)ds \\
&= D\int_0^t \psi_\infty^+(s)ds
\end{aligned} \tag{2.3.99}$$

for all $(x,t) \in \Omega_T$, with the constant D given by

$$D = \kappa\left(\|c\|_{L^\infty(\Omega_T)} + \|d\|_{L^\infty(\Omega_T)}\|\varphi\|_{L^1(\mathbb{R}^n)}\right)\left(\frac{2\sqrt{\pi}}{\sqrt{\lambda}}\right)^n.$$

Since the right hand side of (2.3.99) is independent of $x \in \Omega$ and non-negative, it follows that

$$\psi_\infty^+(t) \leq D \int_0^t \psi_\infty^+(s)ds \quad \text{on } [0,T].$$

Via (2.3.81) and (2.3.98) it also follows that

$$\psi_\infty^+(0) = 0.$$

Therefore, via the Bellman-Grönwall inequality [MN15a, Proposition 5.6], we have that

$$\psi_\infty^+ \equiv 0 \quad \text{on } [0,T]. \tag{2.3.100}$$

Thus, from (2.3.98) and (2.3.100) we obtain that $u \geq 0$ on $\overline{\Omega}_T$, as required. □

2.4 Associated comparison principles

A natural application of the weak minimum principles established in Section 2.3 are comparison principles for solutions to semi-linear integro-differential inequalities involving the operator Q_θ, defined in (2.2.5). To utilise the Lipschitz properties discussed in Section 2.2, we introduce the following notation. For $\underline{u}, \overline{u} \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T)$, we denote

$$\mathcal{K}_0 = \left[\inf_{\overline{\Omega}_T} \min\{\overline{u}, \underline{u}\}, \sup_{\overline{\Omega}_T} \max\{\overline{u}, \underline{u}\}\right] \times \left[\inf_{\overline{\Omega}_T} \min\{J\overline{u}, J\underline{u}\}, \sup_{\overline{\Omega}_T} \max\{J\overline{u}, J\underline{u}\}\right]. \tag{2.4.1}$$

Further assuming that $\underline{u}, \overline{u} \in W^{1,\infty}(\overline{\Omega}_T)$ we also define

$$\mathcal{K}_1 = \left[\inf_{\overline{\Omega}_T} \min_{i=1,\dots,n} \{\partial_{x_i}\overline{u}, \partial_{x_i}\underline{u}\}, \sup_{\overline{\Omega}_T} \max_{i=1,\dots,n} \{\partial_{x_i}\overline{u}, \partial_{x_i}\underline{u}, \}\right]^n \tag{2.4.2}$$

and

$$\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_0. \tag{2.4.3}$$

We now establish a comparison principle based on an application of Proposition 2.3.7, specifically, we have

Theorem 2.4.1 (Comparison principle I)**.** *Let* $\underline{u}, \overline{u} \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T) \cap W^{1,\infty}(\overline{\Omega}_T)$*,* Q *be as in* (2.2.5) *(with* $\theta \equiv 1$*),* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be given by* (2.3.58)*, and* $\mathcal{K}$ *as in* (2.4.3)*. Furthermore, suppose:*

$$Q[\underline{u}] \geq Q[\overline{u}] \quad \text{on } \Omega_T; \tag{2.4.4}$$

$$\underline{u} \leq \overline{u} \quad \text{on } \partial\Omega_T; \tag{2.4.5}$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \leq \frac{A}{t^\beta}(1 + |x|^2)\,|\eta|^2 \quad \forall (x,t) \in \Omega_T,\ \eta \in \mathbb{R}^n; \tag{2.4.6}$$

$$\psi \in L^1(\mathbb{R}^n), \tag{2.4.7}$$

for non-negative constants A *and* $\beta \in [0,1)$*. Moreover, suppose that* $f(x,t,\nabla u, u, Ju)$*:*

$$\text{satisfies a constrained local Lipschitz condition in } \nabla u \text{ on } \Omega_T \times \mathcal{K}; \tag{2.4.8}$$

$$\text{satisfies a constrained local upper Lipschitz condition in } u \text{ on } \Omega_T \times \mathcal{K}; \tag{2.4.9}$$

$$\text{satisfies a constrained local Lipschitz condition in } Ju \text{ on } \Omega_T \times \mathcal{K}; \tag{2.4.10}$$

$$\text{is non-decreasing with respect to } Ju \text{ on } \Omega_T \times \mathcal{K}. \tag{2.4.11}$$

Then, $\underline{u} \leq \overline{u}$ *on* $\overline{\Omega}_T$*.*

Proof. Let $w : \overline{\Omega}_T \to \mathbb{R}$ be given by

$$w(x,t) = \overline{u}(x,t) - \underline{u}(x,t) \quad \forall (x,t) \in \overline{\Omega}_T. \tag{2.4.12}$$

Then $w \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T) \cap L^\infty(\overline{\Omega}_T) \cap W^{1,\infty}(\overline{\Omega}_T)$, $w \geq 0$ on $\partial\Omega_T$, and w satisfies the integro-differential inequality

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_ix_j}w + (f(\cdot,\nabla\overline{u},\overline{u},J\overline{u}) - f(\cdot,\nabla\underline{u},\underline{u},J\underline{u})) - \partial_t w \leq 0 \quad \text{on } \Omega_T. \tag{2.4.13}$$

Also note that, there exist $b_1,\dots,b_n,c,d:\Omega_T\to\mathbb{R}$ such that

$$f(\cdot,\nabla\overline{u},\overline{u},J\overline{u}) - f(\cdot,\nabla\underline{u},\underline{u},J\underline{u}) = \sum_{i=1}^{n} b_i\partial_{x_i}w + cw + dJw \quad \text{on } \Omega_T, \tag{2.4.14}$$

with $b_1,\dots,b_n,c$ and d given by:

$$b_i(x,t) = \begin{cases} \left.\dfrac{f(\cdot,\cdot,\overline{w}_i,\overline{u},J\overline{u}) - f(\cdot,\cdot,\underline{w}_i,\overline{u},J\overline{u})}{\partial_{x_i}\overline{u}-\partial_{x_i}\underline{u}}\right|_{(x,t)}, & \partial_{x_i}\overline{u}(x,t)\neq\partial_{x_i}\underline{u}(x,t),\\ 0, & \partial_{x_i}\overline{u}(x,t)=\partial_{x_i}\underline{u}(x,t),\end{cases}$$

$$c(x,t) = \begin{cases} \left.\dfrac{f(\cdot,\cdot,\nabla\underline{u},\overline{u},J\overline{u}) - f(\cdot,\cdot,\nabla\underline{u},\underline{u},J\overline{u})}{\overline{u}-\underline{u}}\right|_{(x,t)}, & \overline{u}(x,t)\neq\underline{u}(x,t),\\ 0, & \overline{u}(x,t)=\underline{u}(x,t),\end{cases} \tag{2.4.15}$$

$$d(x,t) = \begin{cases} \left.\dfrac{f(\cdot,\cdot,\nabla\underline{u},\underline{u},J\overline{u}) - f(\cdot,\cdot,\nabla\underline{u},\underline{u},J\underline{u})}{J\overline{u}-J\underline{u}}\right|_{(x,t)}, & J\overline{u}(x,t)\neq J\underline{u}(x,t),\\ 0, & J\overline{u}(x,t)=J\underline{u}(x,t),\end{cases} \tag{2.4.16}$$

for all $(x,t)\in\Omega_T$, where $\overline{w}_i,\underline{w}_i:\Omega_T\to\mathbb{R}$ are given by

$$\overline{w}_i(x,t) = \begin{cases} \partial_{x_j}\underline{u}(x,t), & \text{for } j<i,\\ \partial_{x_j}\overline{u}(x,t), & \text{for } j\geq i,\end{cases} \quad ; \quad \underline{w}_i(x,t) = \begin{cases} \partial_{x_j}\underline{u}(x,t), & \text{for } j\leq i,\\ \partial_{x_j}\overline{u}(x,t), & \text{for } j>i,\end{cases}$$

for all $j=1,\dots,n$ and $(x,t)\in\Omega_T$. Since f satisfies (2.4.8)-(2.4.11) and all the points where f is evaluated lie in $\Omega_T\times\mathcal{K}$, it follows that $b_1,\dots,b_n,c$ and d satisfy the conditions (2.3.62)-(2.3.64). Therefore, w in (2.4.12) and the linear integro-differential operator obtained from substitution of (2.4.14) into (2.4.13) satisfy the conditions of Proposition 2.3.7. Therefore, $w\geq 0$ on $\overline{\Omega}_T$ and hence $\underline{u}\leq\overline{u}$ on $\overline{\Omega}_T$, as required. □

Analogously to Theorem 2.4.1, utilising Proposition 2.3.5 instead of Proposition 2.3.7, we also obtain

Theorem 2.4.2 (Comparison principle II)**.** *Let* $\underline{u}, \overline{u} \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T) \cap W^{1,\infty}(\overline{\Omega}_T)$*,* Q *be as in* (2.2.5) *(with* $\theta \equiv 1$*) and* $\mathcal{K}$ *as in* (2.4.3)*. Furthermore, suppose:*

$$Q[\underline{u}] \geq Q[\overline{u}] \quad \text{on } \Omega_T;$$
$$\underline{u} \leq \overline{u} \quad \text{on } \partial\Omega_T;$$
$$\lim_{|x|\to\infty} \underline{u}(x,t) = \lim_{|x|\to\infty} \overline{u}(x,t) = 0 \text{ uniformly with respect to } t \in [0,T],$$

and conditions (2.4.9)-(2.4.11) *are satisfied. Then,* $\underline{u} \leq \overline{u}$ *on* $\overline{\Omega}_T$*.*

Proof. The proof follows the same approach as that of Theorem 2.4.1, and hence, is omitted. □

A companion comparison principle to Proposition 2.3.14 with the non-local, non-linear function f present in (2.2.5) is obtainable[1] albeit, for ease of illustration we restrict attention to f without a dependence on ∇u, i.e. $f(\cdot, u, Ju)$. We now introduce the non-local operator $\tilde{Q} : C^{2,1}(\Omega_T) \cap L^\infty(\Omega_T)$ given by

$$\tilde{Q}[u] = \sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + f(\cdot, u, Ju) - \partial_t u \quad \text{on } \Omega_T \tag{2.4.17}$$

and obtain the following comparison principle.

Theorem 2.4.3 (Comparison principle III)**.** *Let* $\underline{u}, \overline{u} \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T)$*,* $\tilde{Q}$ *be as in* (2.4.17) *and* $\mathcal{K}_0$ *as in* (2.4.1)*. Suppose that the coefficients* $a_{ij}, b_i : \overline{\Omega}_T \to \mathbb{R}$ *satisfy* (2.3.71)-(2.3.73) *and* (2.3.74)*. Moreover, suppose that* $f(x,t,u,Ju)$*:*

$$\text{satisfies Lipschitz conditions in } u \text{ and } Ju \text{ on } \Omega_T \times \mathcal{K}_0; \tag{2.4.18}$$
$$\text{is non-decreasing with respect to } Ju \text{ on } \Omega_T \times \mathcal{K}_0. \tag{2.4.19}$$

[1] As long as an assumption on f similar to (2.4.8) for the term involving ∇u is made, so that the regularity conditions on b_i in (2.3.74) are satisfied.

Furthermore, suppose that

$$\tilde{Q}[\underline{u}] \geq \tilde{Q}[\overline{u}] \quad on\ \Omega_T; \tag{2.4.20}$$

$$\underline{u} \leq \overline{u} \quad on\ \partial\Omega_T. \tag{2.4.21}$$

Then, $\underline{u} \leq \overline{u}$ *on* $\overline{\Omega}_T$.

Proof. Let $w : \overline{\Omega}_T \to \mathbb{R}$ be given by

$$w = (\overline{u} - \underline{u})\theta \quad \text{on } \overline{\Omega}_T, \tag{2.4.22}$$

with $\theta : [0, T] \to (0, \infty)$ given by

$$\theta(t) = e^{kt} \quad \forall t \in [0, T],$$

where k is a Lipschitz constant for f on $\Omega_T \times \mathcal{K}_0$, guaranteed to exist by (2.4.18). It follows immediately that $w \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \cap L^\infty(\overline{\Omega}_T)$ and, from (2.4.21) that

$$w \geq 0 \quad \text{on } \partial\Omega_T.$$

Via (2.4.17) and (2.4.20), w satisfies

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} w + \sum_{i=1}^{n} b_i \partial_{x_i} w + (f(\cdot, \overline{u}, J\overline{u}) - f(\cdot, \underline{u}, J\underline{u}))\theta + kw - \partial_t w \leq 0 \quad \text{on } \Omega_T. \tag{2.4.23}$$

Analogously to (2.4.15)-(2.4.16), via (2.4.18)-(2.4.19), it follows that there exist functions $c, d \in L^\infty(\Omega_T)$ such that

$$(f(\cdot, \overline{u}, J\overline{u}) - f(\cdot, \underline{u}, J\underline{u}))\theta + kw = cw + dJw \quad \text{on } \Omega_T. \tag{2.4.24}$$

Moreover, via (2.4.18)-(2.4.19) and a sufficiently large choice for k, it follows that

$$c, d \geq 0 \quad \text{on } \Omega_T.$$

Therefore, we may rewrite (2.4.23) as

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} w + \sum_{i=1}^{n} b_i \partial_{x_i} w + cw + dJw - \partial_t w \leq 0 \quad \text{on } \Omega_T. \tag{2.4.25}$$

Observe that w, given by (2.4.22), and the non-local integro-differential operator defined by (2.4.25) satisfy the conditions of Proposition 2.3.14. Therefore, $w \geq 0$ on $\overline{\Omega}_T$ and hence, $\overline{u} \geq \underline{u}$ on $\overline{\Omega}_T$, as required. □

Remark 2.4.4. A condition analogous to (2.4.19) is illustrated to be required in Theorem 2.4.3 via the following initial-boundary value problem. The initial-boundary value problem was chosen due to the pathological behaviour of travelling wave solutions of (2.4.29), as illustrated in [Bil20] and [NBLM23].

Let $\Omega = \mathbb{R}$ and $P : C^{2,1}(\Omega_T) \cap L^\infty(\Omega_T) \to \mathbb{R}(\Omega_T)$ be given by

$$P[u] = D\partial_{xx} u + f(u, Ju) - \partial_t u \quad \text{on } \Omega_T, \tag{2.4.26}$$

for all $u \in C^{2,1}(\Omega_T) \cap L^\infty(\Omega_T)$ with: D a positive constant; $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(u, Ju) = u(1 - Ju) \quad \forall (u, Ju) \in \mathbb{R}^2; \tag{2.4.27}$$

and with

$$\varphi \in L^1(\mathbb{R}) \cap C(\mathbb{R}), \ \varphi \geq 0, \ \varphi \text{ is even on } \mathbb{R}, \text{ and } \|\varphi\|_{L^1(\mathbb{R})} = 1. \tag{2.4.28}$$

The initial-boundary value problem is given by:

$$P[u] = 0 \quad \text{on } \Omega_T; \tag{2.4.29}$$

$$u(x,0) = u_0(x) \quad \forall x \in \Omega, \tag{2.4.30}$$

with P as in (2.4.26)-(2.4.28); and $u_0 \in C^2(\Omega)$ such that

$$u_0(x) = \begin{cases} 1, & x \in (-\infty, 0); \\ \eta(x), & x \in [0,1]; \\ 0, & x \in (1, \infty), \end{cases} \tag{2.4.31}$$

with $\eta : [0,1] \to [0,1]$ a sufficiently smooth decreasing function. Now consider $\overline{u} : \overline{\Omega}_T \to \mathbb{R}$ given by $\bar{u} \equiv 1$ on $\overline{\Omega}_T$ and $\underline{u} : \overline{\Omega}_T \to \mathbb{R}$ to be the unique solution to (2.4.29)-(2.4.31). Since f is locally Lipschitz continuous in $\mathbb{R}^2$ the existence of $\underline{u}$ is established by Theorem 3.3.16. It follows, from the smoothness of the initial data (see Propositions 3.3.6, 3.3.8 and Remark 3.3.10) that $\underline{u} \in C^{2,1}(\overline{\Omega}_T)$. Now we have:

$$P[\underline{u}] = 0 = 1 - \|\varphi\|_{L^1(\mathbb{R})} = P[\overline{u}] \quad \text{ on } \Omega_T,$$

and

$$\underline{u} \leq \overline{u} \quad \text{ on } \partial\Omega_T.$$

Further note that $\partial_{Ju} f(1,1) = -1 < 0$. Thus, $P, \underline{u}$ and $\overline{u}$ satisfy all of the conditions of Theorem 2.4.3, except the non-decreasing condition in (2.4.19). However, since $u \in C^{2,1}(\overline{\Omega}_T)$, and $Ju(0,0) \in (0,1)$, it follows that

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \left(D\partial_{xx}\underline{u} + \underline{u}(1 - J\underline{u}) - \partial_t \underline{u} \right)\big|_{(x,t)=(0,\varepsilon)} \\ &= D\partial_{xx}\underline{u}(0,0) + 1(1 - J\underline{u}(0,0)) - \partial_t \underline{u}(0,0) \\ &> -\partial_t \underline{u}(0,0) \end{aligned}$$

which implies that

$$\partial_t \underline{u}(0,0) > 0.$$

Therefore, there exists $(x^*, t^*) \in \mathbb{R} \times (0,T)$ such that $\underline{u}(x^*,t^*) > 1 = \overline{u}(x^*,t^*)$, violating the conclusion of Theorem 2.4.1.

We now provide an application of the comparison principles stated above.

Remark 2.4.5. Consider the initial-boundary value problem (2.4.26), (2.4.28), (2.4.29) and (2.4.30) with $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(u, Ju) = \max\{(Ju)(1-u), 0\} \quad \forall (u, Ju) \in \mathbb{R}^2, \tag{2.4.32}$$

and $u_0 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $0 \leq u_0 \leq 1$ on $\mathbb{R}$. Note that f is locally Lipschitz continuous in $\mathbb{R}^2$, as in Remark 2.4.4, there exists a unique solution $u \in C^{2,1}(\mathbb{R} \times [0,T]) \cap L^\infty(\mathbb{R} \times [0,T])$, for a sufficiently small $T > 0$, to this initial-boundary value problem. Since

$$\partial_t u - D\partial_{xx} u = f(u, Ju) \geq 0 \quad \text{on } \Omega_T, \tag{2.4.33}$$

it follows from the minimum principle for the heat equation that

$$u \geq 0 \quad \text{on } \overline{\Omega}_T. \tag{2.4.34}$$

Now, for any $T > 0$, define $\underline{u} = u$ on $\overline{\Omega}_T$, $\overline{u} \equiv 1$ on $\overline{\Omega}_T$ and $g : \mathbb{R} \to \mathbb{R}$ to be

$$g(v) = \|u\|_{L^\infty(\overline{\Omega}_T)} \max\{1 - v, 0\} \quad \forall v \in \mathbb{R}.$$

Then:

$$\partial_t \underline{u} - D\partial_{xx} \underline{u} = \max\{J\underline{u}(1-\underline{u}), 0\} \leq g(\underline{u}) \quad \text{on } \Omega_T; \tag{2.4.35}$$

$$\partial_t \overline{u} - D\partial_{xx} \overline{u} = 0 \geq g(\overline{u}) \quad \text{on } \Omega_T, \tag{2.4.36}$$

and

$$\underline{u} \le \overline{u} \quad \text{on } \partial\Omega_T. \tag{2.4.37}$$

Thus, via (2.4.35)-(2.4.37), it follows from Theorem 2.4.3 with $\varphi \equiv 0$ and $f = g$, that $\underline{u} \le \overline{u}$ on $\overline{\Omega}_T$ and hence, on recalling (2.4.34), we conclude that

$$0 \le u \le 1 \quad \text{on } \overline{\Omega}_T. \tag{2.4.38}$$

This a priori bound on the solution of the initial-boundary value problem provides sufficient information to conclude that the initial-boundary value problem has a global solution $u : \overline{\Omega}_\infty \to \mathbb{R}$ (i.e., the solution is extendable on $\overline{\Omega}_T$ for any $T > 0$), for each $u_0 \in C^2(\Omega) \cap L^\infty(\Omega)$.[1] Since solutions to the initial-boundary value problem are unique and bounded between 0 and 1, f in (2.4.32) can be replaced by $f(u, Ju) = Ju(1-u)$ for all $(u, Ju) \in \mathbb{R}^2$. Notably, $f \in C^1(\mathbb{R}^2)$ and $\partial_{Ju} f \ge 0$ on $[0,1]^2$.

Now, consider two cases of the the initial-boundary value problem, with respective initial data $u_0^1, u_0^2 : \Omega \to \mathbb{R}$, that satisfy

$$u_0^1 \le u_0^2 \quad \text{on } \Omega.$$

Then, we may apply Theorem 2.4.3, to conclude that the corresponding solutions $u^i : \overline{\Omega}_\infty \to \mathbb{R}$, for $i = 1, 2$, to the initial-boundary value problem satisfy

$$u^1 \le u^2 \quad \text{on } \overline{\Omega}_\infty.$$

[1]We will expand upon this topic in Chapter 3.

2.5 Conclusion

It is widely known that solutions to boundary value problems for second order linear parabolic partial differential equations in unbounded domains, need not be unique if growth restrictions (as $|x| \to \infty$) are not imposed. Indeed, even for the homogeneous heat equation in $\mathbb{R}^n \times [0,T]$, uniqueness fails if one does not assume a boundedness condition similar to

$$\int_0^T \int_{\mathbb{R}^n} |u(x,t)| \exp\left(-\mu|x|^2\right) dxdt < \infty,$$

for any constant $\mu > 0$ (see [Fri08, p.29-31] and [Hay78]). In this regard, the conditions in the minimum principle Proposition 2.3.4 complement these uniqueness and non-uniqueness results. It should also be noted that weak minimum principles, with slightly augmented conditions to those presented here, can be obtained for particular cases of (α, λ) if one uses a different ξ, as discussed in [MN14, Section 3].

For the initial-boundary value problem for the heat equation in $\mathbb{R}^n \times [0,T]$, sharp growth conditions on u as $|x| \to \infty$, for which uniqueness/non-uniqueness of solutions applies, are known (see [Hay78] and the references the therein). Such non-uniqueness results highlight conditions, which if violated, preclude maximum/minimum principles and comparison principles. Counter-examples that highlight the limitations of maximum/minimum principles for semilinear parabolic differential operators (as those discussed in [MN14, Section 3]), are not readily available in the non-local setting discussed here. Construction of such counter-examples for this non-local case would provide further clarity on the limitations of comparison theory for nonlinear non-local integro-differential operators.

In Proposition 2.3.6 we note that conditions can be improved on d via a more refined estimate on the integral in (2.3.57), by using the decay of θ as $|y| \to \infty$.

In Section 2.3.2 we assumed that $\Omega = \mathbb{R}^n$. However, we may also consider Ω to be an unbounded domain which is a strict subset of $\mathbb{R}^n$, with $\partial\Omega$ sufficiently smooth. The existence of derivative estimates for fundamental solutions to second order parabolic partial differential equations on such domains are discussed in [LSU68, Chapter IV] and [Fri08, Chapter 1]. Unfortunately, such derivative estimates are not established therein. To establish minimum principles using the approach in Section 2.3.2 would require these derivative estimates to bound derivative terms as $|x| \to \infty$. For such domains, a theorem similar to Proposition 2.3.14 (and the associated Theorem 2.4.3) can be established via the same argument. This statement applies, under the proviso that, compatibility conditions for $w = (\overline{u} - \underline{u})\theta$ and Γ^*, in a neighbourhood of $\partial\Omega \times [0, T]$, are specified so that the application of the divergence theorem in (2.3.89), yields

$$\int_{\varepsilon_1}^{t-\varepsilon_2} \int_S F \cdot \hat{n} dS dt \geq 0, \tag{2.5.1}$$

for all $(x, t) \in \Omega_T$, with $S = \partial\Omega \cap B_x^R$. Here, the $i-$th component of $F : \Omega_T \to \mathbb{R}^n$ is given by

$$\sum_{j=1}^{n} \left[\Gamma^* a_{ij} \left(\partial_{y_j} w\right) - w a_{ij} \left(\partial_{y_j} \Gamma^*\right) - w \Gamma^* \left(\partial_{y_j} a_{ij}\right)\right] + (b_i w \Gamma^*) \qquad \text{on } \Omega_T,$$

for $i = 1, \ldots, n$, and $\hat{n}$ is the outward normal vector to $S = \partial\Omega \cap B_x^R$. Moreover, the error term arising from the integral over $\partial\Omega \cap (B_x^{R+1} \setminus B_x^R)$ is required to decay sufficiently rapidly as $R \to \infty$.

We note that, the results in Section 2.4, namely the comparison principles for semi-linear integro-differential operators, can be extended, using the approach adopted by Hopf (see [PW84] or [Wal70]) to establish comparison principles for fully nonlinear second order parabolic partial differential inequalities, with non-local zeroth order quantities. Additionally, minimum principles with integrability conditions on u, rather than the point-wise bounds on functions in $E_{\alpha,\lambda}(\Omega_T)$, can in principle, also be established (see for example [AB66] and references therein). Moreover, one can accommodate a degree of coefficient blow-up in the interior of Ω_T and still establish minimum principles (see [Mey22] where these type of blow-up conditions are considered in the elliptic case). We note that the focus in results presented in this chapter concerned conditions on the coefficients as $|x| \to \infty$, and for brevity, these additional technicalities discussed above were not presented.

A natural application for the theorems in Section 2.4 is to establish uniqueness for solutions of initial-boundary value problems. However, as Remark 2.4.4 demonstrates, the solution to an initial-boundary value problem for a nonlinear non-local integro-differential equation, can be unique, without the operator satisfying a comparison principle. Results concerning uniqueness, and continuous dependence with respect to initial data, are often established via the Bellman-Grönwall inequality, see for example Lemma 3.3.3, Corollary 3.3.4 and Proposition 3.3.5. This approach to establish uniqueness does not require a monotonicity condition, such as condition (2.4.11), but merely regularity conditions on f.

Finally, we note that, depending on the conditions placed upon solutions, the regularity of the non-local nonlinearity f and the actual coefficients of the integro-differential operator, we have demonstrated that there are a variety of options available, if minimum or comparison principles are required. However, for the non-local reaction-diffusion equations we most frequently encounter, we work with $u \in L^\infty(\overline{\Omega}_T)$, uniformly parabolic integro-differential operators with coefficients of class C^∞, no other a priori assumptions on the growth/decay rate of u, and integral kernel $\varphi \in L^1(\mathbb{R}^n)$. This makes Proposition 2.3.14, and the associated comparison principle Theorem 2.4.3 the related results we most often utilise.

CHAPTER 3

WELL POSEDNESS OF THE NONLINEAR NON-LOCAL REACTION DIFFUSION EQUATION

3.1 Introduction

In this chapter we illustrate qualitative properties of solutions to the Cauchy problems for the $n-$dimensional heat equation (HE) and the nonlinear non-local reaction diffusion equation (CP). In Section 3.2 we show that (HE) is well-posed (globally) in the Hadamard sense by specifying appropriate initial data. Specifically, we prove that if the initial data is continuous and bounded, then there exists a unique solution to (HE) that is of class $C^{2,1}$ on Ω_T, which is continuous with respect to the initial data. Furthermore, we provide derivative bounds for the solution u of (HE) and highlight the differences in qualitative results if the initial data $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ and $u_0 \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$.

This is done as a motivation and exposition for the techniques that are utilised in the following section, where we shift our focus to the Cauchy problem for the non-local reaction-diffusion equation, with the nonlinear non-local term satisfying a local Lipschitz condition. In particular, in Section 3.3, utilising the results of Section 3.2, we establish that (CP) is well-posed (locally in time) in the Hadamard sense by specifying appropriate conditions for both the initial data and the nonlinear term f. Additionally, we highlight the differences in qualitative results if the initial data u_0 and the nonlinearity f are of different classes of regularity. We conclude this section by providing higher order derivative bounds for the solution u to (CP). In both cases we utilise the fundamental solution to the heat equation to represent our solutions.

Results established in this chapter are utilised in Section 4.2 in the construction of solutions when the nonlinearity f satisfies a local Hölder condition. Moreover the derivative estimates established here are instrumental in proving the convergence of the numerical scheme described in Section 4.4. Lastly we note that the continuous dependence results are used in Section 4.4 to enhance the discussion concerning the computations.

In this chapter, unless otherwise specified, $\Omega = \mathbb{R}^n$ and no monotonicity restrictions are placed on the nonlinearity f.

3.2 The $n-$dimensional heat equation

We consider the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u, & \text{on } \Omega_T; \\ u(x,0) = u_0(x), & \forall x \in \Omega; \\ u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T), \end{cases} \tag{HE}$$

with prescribed initial data $u_0 \in L^\infty(\Omega) \cap C(\Omega)$. Henceforth, unless stated otherwise, when we refer to (HE), we assume all the conditions mentioned above. We will show that (HE) is well-posed, in the Hadamard sense (see [Had07]), on Ω_T for any $T > 0$. The results presented in this section have been previously established (see for instance [Eva10]) and are proved here in the context of our work.

Theorem 3.2.1 (Uniqueness)**.** (HE) *has at most one solution on* $\overline{\Omega}_T$ *for any* $T > 0$.

Proof. Let $u_1, u_2 : \overline{\Omega}_T \to \mathbb{R}$ be solutions to (HE) with the same initial data u_0. Define $w : \overline{\Omega}_T \to \mathbb{R}$ as

$$w = u_1 - u_2 \quad \text{on } \overline{\Omega}_T.$$

It follows that $w_t - \Delta w = 0 \geq 0$ on Ω_T and $w \equiv 0$ on $\partial\Omega_T$. Therefore, via the weak minimum principle (see Proposition 2.3.7),

$$w \geq 0 \quad \text{on } \overline{\Omega}_T.$$

Via a symmetrical argument, it also follows that

$$-w \geq 0 \quad \text{on } \overline{\Omega}_T.$$

Thus, $w \equiv 0$ on $\overline{\Omega}_T$, or equivalently, $u_1 = u_2$ on $\overline{\Omega}_T$, as required. □

For what follows, we provide a definition for the free-space fundamental solution to the heat equation. Let $D_G = \{(x,t;y,s) \in \overline{\Omega}_T \times \overline{\Omega}_T : 0 \leq s < t \leq T\}$, and $G : D_G \to \mathbb{R}$ be the function given by

$$G(x,t;y,s) = \frac{1}{(2\sqrt{\pi})^n (t-s)^{n/2}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right), \tag{3.2.1}$$

for all $(x,t;y,s) \in D_G$. The function G is the fundamental solution for the heat equation in $\mathbb{R}^n \times [0,T]$ and is consistent with the definition provided in the previous chapter. Now, we consider the function $u : \overline{\Omega}_T \to \mathbb{R}$ given by

$$u(x,t) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} G(x,t;y,0)u_0(y)dy, & (x,t) \in \Omega_T; \\ u_0(x), & (x,t) \in \partial\Omega_T. \end{cases} \tag{3.2.2}$$

Since $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ we also have the following alternative representation for u obtained via a change of variables

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x + 2\sqrt{t}z)\frac{e^{-z^2}}{\sqrt{\pi}^n} dz \quad \forall (x,t) \in \overline{\Omega}_T, \tag{3.2.3}$$

with $z^2 = z \cdot z$, for all $z \in \mathbb{R}^n$.

Now we establish the following standard result.

Theorem 3.2.2 (Global existence and uniqueness)**.** (HE) *has a unique solution on* $\overline{\Omega}_T$ *for any* $T > 0$*. This solution is given by* $u : \overline{\Omega}_T \to \mathbb{R}$ *in* (3.2.2) *and* (3.2.3)*.*

Proof. First note that, for each $y \in \mathbb{R}^n$, $\partial_t(G(x,t;y,0)u_0(y))$ and $\Delta(G(x,t;y,0)u_0(y))$ exist and are continuous on Ω_T. By direct calculations[1] it follows that $u \in C^{2,1}(\Omega_T)$ and:

$$u_t(x,t) = \int_{\mathbb{R}^n} G(x,t;y,0)u_0(y)\frac{|x-y|^2 - 2nt}{4t^2}dy = \operatorname{div}(\nabla u(x,t)) = \Delta u(x,t)$$

for all $(x,t) \in \Omega_T$. Additionally, since $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ it follows from the uniform convergence of the integral in (3.2.3) and the continuity of u_0, that

$$\begin{aligned}
\lim_{t\to 0^+} u(x,t) &= \lim_{t\to 0^+} \int_{\mathbb{R}^n} u_0(x + 2\sqrt{t}z)\frac{e^{-z^2}}{\sqrt{\pi}^n}dz \\
&= \int_{\mathbb{R}^n} \lim_{t\to 0^+} u_0(x + 2\sqrt{t}z)\frac{e^{-z^2}}{\sqrt{\pi}^n}dz \\
&= \int_{\mathbb{R}^n} u_0(x)\frac{e^{-z^2}}{\sqrt{\pi}^n}dz \\
&= u_0(x) \qquad (3.2.4)
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Also, since $u_0 \in L^\infty(\Omega) \cap C(\Omega)$, applying Hölder's inequality to (3.2.3) yields

$$\|u\|_{L^\infty(\overline{\Omega}_T)} \le \|u_0\|_{L^\infty(\Omega)} \quad \text{on } \overline{\Omega}_T. \qquad (3.2.5)$$

Thus, $u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ for any $T > 0$ and solves (HE). Since $T > 0$ is arbitrary, the solution u given by (3.2.2)-(3.2.3), with (3.2.1), can be extended onto $\overline{\Omega}_\infty$. The uniqueness of u follows from Theorem 3.2.1. □

Theorem 3.2.3 (Continuous dependence)**.** *Let* $u, v : \overline{\Omega}_T \to \mathbb{R}$ *be solutions to* (HE) *with initial data* $u_0, v_0 \in L^\infty(\Omega) \cap C(\Omega)$ *respectively. Then, for every* $\varepsilon > 0$ *there exists*

[1] This follows from the continuity and uniform convergence as $|x| \to \infty$ of the derivatives of all integrands arising from (3.2.2). Derivative formulae then follow from the Leibniz rule.

$\delta = \delta(\varepsilon) > 0$ *such that if*

$$\|u_0 - v_0\|_{L^\infty(\Omega)} < \delta,$$

then

$$\|u - v\|_{L^\infty(\overline{\Omega}_T)} < \varepsilon.$$

Proof. Via (3.2.3) we have that

$$\begin{aligned}|u(x,t) - v(x,t)| &\le \int_{\mathbb{R}^n} |u_0(x + 2\sqrt{t}z) - v_0(x + 2\sqrt{t}z)| \frac{e^{-z^2}}{\sqrt{\pi}^n} dz \\ &\le \|u_0 - v_0\|_{L^\infty(\Omega)} < \delta \quad \forall (x,t) \in \overline{\Omega}_T.\end{aligned}$$

By setting $\delta = \varepsilon$ the result follows, as required. □

To establish derivative estimates on $\overline{\Omega}_T$, we assume that $u_0 \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$, and begin our analysis by noting that (following differentiation under the integral sign in (3.2.3), a change of variables, integration by parts and another change of variables) the following identities hold:

$$\partial_{x_i} u(x,t) = \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} \partial_{x_i} u_0(x + 2\sqrt{t}z) dz = \int_{\mathbb{R}^n} G(x,t;y,0) \partial_{y_i} u_0(y) dy, \tag{3.2.6}$$

for all $(x,t) \in \Omega_T$ and

$$\partial_{x_i x_j} u(x,t) = \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} \partial_{x_i x_j} u_0(x + 2\sqrt{t}z) dz = \int_{\mathbb{R}^n} G(x,t;y,0) \partial_{y_i y_j} u_0(y) dy, \tag{3.2.7}$$

for all $(x,t) \in \Omega_T$. Following this observation and noting that $\|G(x,t;\cdot,0)\|_{L^1(\mathbb{R}^n)} = 1$ for any $(x,t) \in \Omega_T$, we infer the following result.

Theorem 3.2.4 (Derivative estimates)**.** *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the unique solution to (HE). Further assume that $u_0 \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$ and positive constants m_0, M_0, m_0', M_0', m_0'' and M_0'' exist such that*

$$m_0 \le u_0 \le M_0, \quad m_0' \le \partial_{x_i} u_0 \le M_0', \quad m_0'' \le \partial_{x_i x_j} u_0 \le M_0'' \quad \text{on } \Omega,$$

for $i, j = 1, \ldots, n$. *Then,*

$$m_0 \leq u \leq M_0, \quad m_0' \leq \partial_{x_i} u \leq M_0', \quad m_0'' \leq \partial_{x_i x_j} u \leq M_0'' \quad \text{on } \overline{\Omega}_T,$$

for $i, j = 1, \ldots, n$. *Furthermore,*

$$n m_0'' \leq u_t \leq n M_0'' \quad \text{on } \overline{\Omega}_T.$$

Proof. Since u is a solution to (HE) this follows from (3.2.6) and (3.2.7). □

By assuming no more than $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ on the initial data, the derivative estimates in Theorem 3.2.4 are adapted in the following propositions.

Proposition 3.2.5. *Let* $u : \overline{\Omega}_T \to \mathbb{R}$ *be the solution to* (HE). *Then,*

$$|\partial_{x_i} u(x,t)| \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi t}} \quad \forall (x,t) \in \Omega_T, \tag{3.2.8}$$

for $i = 1, \ldots, n$.

Proof. By (3.2.2), by differentiating under the integral sign and following changes of variables we have:

$$\begin{aligned} \partial_{x_i} u(x,t) &= \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4t}}}{(2\sqrt{\pi t})^n} \left(\frac{-(x_i - y_i)}{2t} \right) u_0(y) dy \\ &= -\frac{1}{\sqrt{t}} \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} z_i u_0(x + 2\sqrt{t} z) dz, \end{aligned}$$

for all $(x,t) \in \Omega_T$. Via the triangle inequality and Hölder's inequality we obtain

$$\begin{aligned} |\partial_{x_i} u(x,t)| &\leq \frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{t}} \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} |z_i| \, dz \\ &= \frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi t}} \int_0^\infty \partial_{z_i}(-e^{-z_i^2}) dz_i \\ &= \frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi t}} \qquad \forall (x,t) \in \Omega_T, \end{aligned} \tag{3.2.9}$$

as required. ∎

Proposition 3.2.6. *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (HE). Then,*

$$|\partial_{x_i x_j} u(x,t)| \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{t}, \quad \forall (x,t) \in \Omega_T, \tag{3.2.10}$$

for $i, j = 1, \ldots, n$.

Proof. By (3.2.2), for all $(x,t) \in \Omega_T$, by differentiating under the integral sign, following changes of variables and via the triangle inequality and Hölder's inequality we have:

$$\begin{aligned}
|\partial_{x_i x_j} u(x,t)| &\leq \frac{\|u_0\|_{L^\infty(\Omega)}}{t} \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} \left| |z_i||z_j| - \frac{\delta_{ij}}{2} \right| dz \\
&\leq \frac{\|u_0\|_{L^\infty(\Omega)}}{t} \left(\int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} |z_i||z_j| dz + \frac{1}{2} \right) \qquad \forall (x,t) \in \Omega_T.
\end{aligned}$$

with δ_{ij} denoting the Kronecker delta notation[1]. Bounding the remaining integral, as in (3.2.9) yields (3.2.10), as required. ∎

3.3 Non-local reaction diffusion equations

In this section, we consider the following Cauchy problem

$$\begin{cases}
\partial_t u = \Delta u + f(u, Ju), & \text{on } \Omega_T; \\
u(x,0) = u_0(x), & \forall x \in \Omega; \\
u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T),
\end{cases} \tag{CP}$$

with prescribed initial data $u_0 \in L^\infty(\Omega) \cap C(\Omega)$, with $Ju : L^\infty(\Omega_T) \to L^\infty(\Omega_T)$ as in (2.2.1) given by the convolution product (with argument u),

$$Ju(x,t) = \int_{\mathbb{R}^n} \varphi(x-y) u(y,t) dy \quad \forall (x,t) \in \Omega_T,$$

[1] I.e. $\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$

with (prescribed) integral kernel $\varphi \in L^1(\mathbb{R}^n)$, and (prescribed) nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$ which is locally Lipschitz continuous, i.e. for any compact $U \subseteq \mathbb{R}^2$ there exist $L_U \geq 0$ such that

$$|f(u_1, v_1) - f(u_2, v_2)| \leq L_U(|u_1 - v_1| + |u_2 - v_2|),$$

for all $(u_1, v_1), (u_2, v_2) \in U$. Henceforth, unless stated otherwise, when we refer to (CP), we assume all the conditions mentioned above. We further define $F : \Omega_T \to \mathbb{R}$ to be

$$F(x,t) = f(u(x,t), Ju(x,t)) \quad \forall (x,t) \in \Omega_T. \tag{3.3.1}$$

We begin by showing that (CP) has a local solution.

Theorem 3.3.1 (Local existence)**.** (CP) *has a solution on* $\overline{\Omega}_\delta$*, for*

$$\delta = \frac{1}{2K + 2|f(0,0)| + 1} \tag{3.3.2}$$

with

$$K = L_U(1 + \|\varphi\|_{L^1(\mathbb{R}^n)}) \tag{3.3.3}$$

and $L_U \geq 0$ *is a Lipschitz constant for* f *on*

$$U = [-(1 + \|\varphi\|_{L^1(\mathbb{R}^n)})(2\|u_0\|_{L^\infty(\Omega)} + 1), (1 + \|\varphi\|_{L^1(\mathbb{R}^n)})(2\|u_0\|_{L^\infty(\Omega)} + 1)]^2.$$

Proof. Let B be the closed and bounded subspace of $L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta)$, equipped with the supremum norm $\|\cdot\|_{L^\infty(\overline{\Omega}_\delta)} = \|\cdot\|_\infty$,[1] so that

$$B = \{v : \overline{\Omega}_\delta \to \mathbb{R} : v \in L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta) \text{ and } \|v\|_\infty \leq 2\|u_0\|_{L^\infty(\Omega)} + 1\}. \tag{3.3.4}$$

[1] In this proof we use this shorthand notation for ease of presentation.

We note that $(B, \|\cdot\|_\infty)$ is a Banach space (given that it is a closed and bounded subspace of $L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta)$). By applying a Duhamel principle to (CP), we obtain the mapping $S : B \to L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta)$ such that

$$\begin{aligned} S(v)|_{(x,t)} = &\int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} u_0(x + 2\sqrt{t}z)dz \\ &+ \int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} f(v(x + 2\sqrt{t-s}z, s), Jv(x + 2\sqrt{t-s}z, s))dzds, \end{aligned}$$

for all $v \in B$ and for all $(x,t) \in \overline{\Omega}_\delta$. First we will show that $\text{Im}(S) \subseteq B$. For $v \in B$, via (3.3.2) - (3.3.4) it follows that

$$\begin{aligned} \|S(v)\|_\infty &\leq \|u_0\|_{L^\infty(\Omega)} + \delta\|f(v, Jv)\|_\infty \\ &= \|u_0\|_{L^\infty(\Omega)} + \delta\|f(v, Jv) + f(0,0) - f(0,0)\|_\infty \\ &\leq \|u_0\|_{L^\infty(\Omega)} + \delta(L_U(\|v\|_\infty + \|\varphi\|_{L^1(\mathbb{R}^n)}\|v\|_\infty) + |f(0,0)|) \\ &= \|u_0\|_{L^\infty(\Omega)} + \delta(K\|v\|_\infty) + |f(0,0)|) \\ &\leq \|u_0\|_{L^\infty(\Omega)} + \delta(K(2\|u_0\|_{L^\infty(\Omega)} + 1) + |f(0,0)|) \\ &= \|u_0\|_{L^\infty(\Omega)} + \frac{K(2\|u_0\|_{L^\infty(\Omega)} + 1) + |f(0,0)|}{2K + 2|f(0,0)| + 1} \\ &< \|u_0\|_{L^\infty(\Omega)} + \frac{2\|u_0\|_{L^\infty(\Omega)} + 1}{2} + \frac{1}{2} \\ &= 2\|u_0\|_{L^\infty(\Omega)} + 1. \end{aligned}$$

Therefore, $S : B \to B$. Next we show that S is a contraction mapping. For $v, w \in B$ and K as in (3.3.3), via (3.3.2) we have,

$$\begin{aligned} \|S(v) - S(w)\|_\infty = &\sup_{(x,t)\in\overline{\Omega}_\delta} \left\{ \int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} |f(v(x + 2\sqrt{t-s}z, s), Jv(x + 2\sqrt{t-s}z, s)) \right. \\ &\left. - f(w(x + 2\sqrt{t-s}z, s), Jw(x + 2\sqrt{t-s}z, s))|dzds \right\} \\ &\leq K \sup_{(x,t)\in\overline{\Omega}_\delta} \left\{ \int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} |v(x + 2\sqrt{t-s}z, s) \right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - w(x+2\sqrt{t-s}z,s)|dzds\Big\} \\
&\le K\|v-w\|_\infty \int_0^\delta \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} dzds \\
&= K\delta \|v-w\|_\infty \\
&= \frac{K}{2K+2|f(0,0)|+1}\|v-w\|_\infty \\
&\le \frac{1}{2}\|v-w\|_\infty.
\end{aligned}
$$

Therefore,

$$
\|S(v)-S(w)\|_\infty \le \frac{1}{2}\|v-w\|_\infty,
$$

for all $v, w \in B$. Hence, via the Banach fixed-point theorem[1], there exists $u^* \in B$ such that

$$
\begin{aligned}
u^*(x,t) &= \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} u_0(x+2\sqrt{t}z)dz \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} f(u^*(x+2\sqrt{t-s}z,s), Ju^*(x+2\sqrt{t-s}z,s))dzds \\
&= \int_{\mathbb{R}^n} G(x,t;y,0)u_0(y)dy \\
&\quad + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) f(u^*(y,s), Ju^*(y,s))dyds \quad \forall (x,t) \in \overline{\Omega}_\delta.
\end{aligned}
$$

To complete the proof, from the regularity of f and u^*, by using the results in [Fri08, Ch. 1], it follows that $u^* \in L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta) \cap C^{2,1}(\Omega_\delta)$, and moreover, that u^* is a solution to (CP) on $\overline{\Omega}_\delta$, as required. □

Remark 3.3.2 (Global existence)**.** Now, following the approach described in [MN15a, Theorem 6.4], and by assuming that a solution to (CP) is *a priori* bounded on $\overline{\Omega}_T$[2], we may extend u^* up to any $t \le T$ i.e, $u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ as given by

$$
u(x,t) = \int_{\mathbb{R}^n} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) f(u(y,s), Ju(y,s))dyds \tag{3.3.5}
$$

[1] For details see [Sau06, Theorem 1.19].

[2] I.e, there exists a uniform bound for a solution to (CP) for all $t \le T$.

for all $(x,t) \in \overline{\Omega}_T$, is a solution to (CP). If u is a priori bounded on $\overline{\Omega}_T$ for all $T > 0$, then there exists a solution to (CP) on $\overline{\Omega}_\infty$. We further note that although comparison theory is widely used to establish a priori bounds for the local instance of (CP), it is not applicable in the general (non-local) instance of (CP) we consider. However, if f is non-decreasing with respect to Ju with $\varphi \geq 0$, then the comparison theory developed in the previous chapter can be applied to establish the uniqueness of solutions to (CP).

For the remainder of this section we assume that solutions to (CP) exist on $\overline{\Omega}_T$. We now establish a result which will yield uniqueness and continuous dependence for (CP).

Lemma 3.3.3. *Let $u, v : \overline{\Omega}_T \to \mathbb{R}$ be solutions to (CP) with initial data u_0, v_0, and integral kernels $\varphi, \tilde{\varphi}$ respectively, and identical nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$. Then,*

$$|u(x,t) - v(x,t)| \leq (\|u_0 - v_0\|_{L^\infty(\Omega)} + C_1 t)e^{C_2 t} \quad \forall (x,t) \in \overline{\Omega}_T, \tag{3.3.6}$$

with $C_1 = L_U \|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)} \|u\|_{L^\infty(\overline{\Omega}_T)}$, $C_2 = L_U(1 + \|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)})$ and L_U a Lipschitz constant for f on $U = U_1 \times U_2$ with

$$U_1 = [-\max\{\|u\|_{L^\infty(\overline{\Omega}_T)}, \|v\|_{L^\infty(\overline{\Omega}_T)}\}, \max\{\|u\|_{L^\infty(\overline{\Omega}_T)}, \|v\|_{L^\infty(\overline{\Omega}_T)}\}]$$

and

$$U_2 = [-\max\{||Ju||_{L^\infty(\overline{\Omega}_T)}, ||\tilde{J}v||_{L^\infty(\overline{\Omega}_T)}\}, \max\{||Ju||_{L^\infty(\overline{\Omega}_T)}, ||\tilde{J}v||_{L^\infty(\overline{\Omega}_T)}\}].$$

Proof. Via (3.3.5) we have that

$$u(x,t) = \int_{\mathbb{R}^n} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) f(u(y,s), Ju(y,s))dyds;$$
$$v(x,t) = \int_{\mathbb{R}^n} G(x,t;y,0)v_0(y)dy + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) f(v(y,s), \tilde{J}v(y,s))dyds,$$

for all $(x,t) \in \overline{\Omega}_T$. Therefore

$$\begin{aligned} &|u(x,t) - v(x,t)| \\ &\leq \|u_0 - v_0\|_{L^\infty(\Omega)} \\ &\quad + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)|f(u(y,s), Ju(y,s)) - f(v(y,s), \tilde{J}v(y,s))|dyds, \end{aligned} \tag{3.3.7}$$

for all $(x,t) \in \overline{\Omega}_T$. Since f is locally Lipschitz continuous, and $J, \tilde{J}$ are linear operators. it follows from (3.3.7) that

$$\begin{aligned} &|u(x,t) - v(x,t)| \\ &\leq \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)L_U(|(u(y,s) - v(y,s)| \\ &\qquad + |Ju(y,s) - \tilde{J}v(y,s)|)dyds \\ &\leq \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)L_U(|u(y,s) - v(y,s)| \\ &\qquad + |Ju(y,s) - \tilde{J}u(y,s)| + |\tilde{J}u(y,s) - \tilde{J}v(y,s)|)dyds \\ &= \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)L_U(|u(y,s) - v(y,s)| \\ &\qquad + |(J - \tilde{J})u(y,s)| + |\tilde{J}(u(y,s) - v(y,s))|)dyds \\ &\leq \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)L_U(\|u(\cdot,s) - v(\cdot,s)\|_{L^\infty(\Omega)} \\ &\qquad + \|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u(\cdot,s)\|_{L^\infty(\Omega)} + \|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u(\cdot,s) - v(\cdot,s)\|_{L^\infty(\Omega)})dyds \\ &\leq \|u_0 - v_0\|_{L^\infty(\Omega)} + \int_0^t L_U(\|u(\cdot,s) - v(\cdot,s)\|_{L^\infty(\Omega)} + \|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\overline{\Omega}_T)} \\ &\qquad + \|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u(\cdot,s) - v(\cdot,s)\|_{L^\infty(\Omega)})ds, \end{aligned} \tag{3.3.8}$$

for all $(x,t) \in \overline{\Omega}_T$, where (3.3.8) follows from Hölder's inequality and

$$||G(x,t,\cdot,s)||_{L^1(\Omega)} = 1.$$

Now, we define $\psi : [0,T] \to [0,\infty)$ to be

$$\psi(t) = \|u(\cdot,t) - v(\cdot,t)\|_{L^\infty(\Omega)} \quad \forall t \in [0,T]. \tag{3.3.9}$$

On taking the supremum over all $x \in \mathbb{R}^n$ in the inequality (3.3.8), using (3.3.9), we obtain,

$$\psi(t) \leq \|u_0 - v_0\|_{L^\infty(\Omega)} + L_U\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\overline{\Omega}_T)}t + L_U(1 + \|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)})\int_0^t \psi(s)ds, \tag{3.3.10}$$

for all $t \in [0,T]$. Via an application of the Bellman-Grönwall inequality to (3.3.10) we have

$$\psi(t) \leq (\|u_0 - v_0\|_{L^\infty(\Omega)} + L_U\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\overline{\Omega}_T)}t)e^{L_U(1+\|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)})t} \quad \forall t \in [0,T],$$

which implies (3.3.6), as required. □

Corollary 3.3.4 (Uniqueness). *For any $T > 0$ there exists at most one solution to (CP).*

Proof. Let $u, v : \overline{\Omega}_T \to \mathbb{R}$ be solutions to (CP) with $v_0 = u_0$ on Ω, $\tilde{\varphi} = \varphi$ on $\mathbb{R}^n$, and identical nonlinearity f. Then, via Lemma 3.3.3, it follows that $|u(x,t) - v(x,t)| = 0$ for all $(x,t) \in \overline{\Omega}_T$. Therefore, $u = v$ on $\overline{\Omega}_T$, as required. □

Proposition 3.3.5 (Continuous dependence). *Suppose that $u : \overline{\Omega}_T \to \mathbb{R}$ is a solution to (CP) for fixed $T > 0$. Then, for each $\varepsilon > 0$ there exists $\delta > 0$, that only depends on $\varepsilon, T, \|u\|_{L^\infty(\overline{\Omega}_T)}, \|\varphi\|_{L^1(\mathbb{R}^n)}$ and a Lipschitz constant for f, such that, if for initial data v_0 and integral kernel $\tilde{\varphi}$ satisfying the conditions of (CP), we have*

$$\|u_0 - v_0\|_{L^\infty(\Omega)} < \delta \quad \text{and} \quad \|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)} < \delta, \tag{3.3.11}$$

then, there exists a solution $v : \overline{\Omega}_T \to \mathbb{R}$ to (CP) with initial data v_0, integral kernel $\tilde{\varphi}$

and nonlinearity f, such that

$$\|u - v\|_{L^\infty(\overline{\Omega}_T)} < \varepsilon. \tag{3.3.12}$$

Proof. Since $u : \overline{\Omega}_T \to \mathbb{R}$ is a solution (CP) it follows that $\|u\|_{L^\infty(\overline{\Omega}_T)} < \infty$. Without loss of generality, suppose that $\varepsilon < 1$. Let $U \subseteq \mathbb{R}^2$ be given by $U = [-a, a] \times [-b, b]$ with

$$a = \|u\|_{L^\infty(\overline{\Omega}_T)} + 1 \text{ and } b = (\|\varphi\|_{L^1(\mathbb{R}^n)} + 1)(\|u\|_{L^\infty(\overline{\Omega}_T)} + 1).$$

We define $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\tilde{f}(u, w) = \begin{cases} f(u, w), & \text{on } U; \\ f(u', w'), & \text{on } \mathbb{R}^2 \setminus U, \end{cases} \tag{3.3.13}$$

where $(u', w') \in \partial U$ is the closest point to $(u, w) \in \mathbb{R}^2 \setminus U$ (with respect to the Euclidean norm on $\mathbb{R}^2$)[1]. Note that, since f is locally Lipschitz continuous, by construction, $\tilde{f}$ is Lipschitz continuous on $\mathbb{R}^2$ and bounded, i.e. there exists $L \geq 0$ such that

$$\left|\tilde{f}(u_1, w_1) - \tilde{f}(u_2, w_2)\right| \leq L(|u_1 - u_2| + |w_1 - w_2|), \quad \forall (u_1, w_1), (u_2, w_2) \in \mathbb{R}^2,$$

and,

$$|\tilde{f}(u, w)| \leq \|f\|_{L^\infty(U)}, \quad \forall (u, w) \in \mathbb{R}^2. \tag{3.3.14}$$

Now, consider the problem (CP), denoted by ($\tilde{\text{CP}}$), with initial data v_0, kernel $\tilde{\varphi}$ and nonlinearity $\tilde{f}$ given by (3.3.13) with $\|u_0 - v_0\|_{L^\infty(\Omega)} < 1$, and $\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)} < 1$. Note that, via Corollary 3.3.4, when $u_0 = v_0$ and $\varphi = \tilde{\varphi}$, the unique solution to ($\tilde{\text{CP}}$) on $\overline{\Omega}_T$ is given by u (the unique solution to (CP)).

[1] To visualise: we partition $\mathbb{R}^2$ in 9 'boxes', the central, bounded box being U and 8 infinite boxes for each point of the compass. In the N, E, S and W boxes, $\tilde{f}$ is constant on lines perpendicular to the adjacent edge of U. In the NE, SE, SW and NW boxes, $\tilde{f}$ is constant with value equal to f evaluated at the adjacent corner of U.

Additionally, for any solution $v : \overline{\Omega}_T \to \mathbb{R}$ to ($\tilde{\text{CP}}$), such that v_0 and $\tilde{\varphi}$ satisfy the hypotheses of (CP), it follows from Remark 3.3.2, (3.3.13) and (3.3.14) that

$$\begin{aligned}\|v\|_{L^\infty(\overline{\Omega}_T)} &\leq \|v_0\|_{L^\infty(\Omega)} + T\|\tilde{f}\|_{L^\infty(\mathbb{R}^2)}\\ &\leq \|u_0\|_{L^\infty(\Omega)} + 1 + T\|f\|_{L^\infty(U)}.\end{aligned}$$

By applying Lemma 3.3.3 to two solutions of ($\tilde{\text{CP}}$), specifically, u in the hypothesis, the solution to ($\tilde{\text{CP}}$), denoted by v as above, we have

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^\infty(\Omega)} \leq (\|u_0 - v_0\|_{L^\infty(\partial\Omega_T)} + L\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\overline{\Omega}_T)}t)e^{L(1+\|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)})t},$$

for all $t \in [0,T]$, with L being a Lipschitz constant for f on U. Therefore, via (3.3.11)

$$\begin{aligned}\|u - v\|_{L^\infty(\overline{\Omega}_T)} &\leq (\|u_0 - v_0\|_{L^\infty(\partial\Omega_T)} + L\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\overline{\Omega}_T)}T)e^{L(1+\|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)})T}\\ &< \delta\left(1 + L\|u\|_{L^\infty(\overline{\Omega}_T)}T\right)e^{L(1+\delta+\|\varphi\|_{L^1(\mathbb{R}^n)})T}.\end{aligned}$$

Thus, if δ satisfies

$$0 < \delta < \varepsilon\left(\left(1 + L\|u\|_{L^\infty(\overline{\Omega}_T)}T\right)e^{L(2+\|\varphi\|_{L^1(\mathbb{R}^n)})T}\right)^{-1}, \tag{3.3.15}$$

then it follows that

$$\|u - v\|_{L^\infty(\overline{\Omega}_T)} < \varepsilon. \tag{3.3.16}$$

To verify that $v : \overline{\Omega}_T \to \mathbb{R}$, the arbitrary solution to ($\tilde{\text{CP}}$) in (3.3.16), is also a solution (CP), it suffices to show that $\tilde{f}(v, \tilde{J}v) = f(v, \tilde{J}v)$. Indeed, since $\varepsilon < 1$, it follows that

$$\|v\|_{L^\infty(\overline{\Omega}_T)} < \|u\|_{L^\infty(\overline{\Omega}_T)} + \varepsilon < \|u\|_{L^\infty(\overline{\Omega}_T)} + 1, \tag{3.3.17}$$

and

$$\left\|\tilde{J}v\right\|_{L^\infty(\overline{\Omega}_T)} \leq \|\tilde{\varphi}\|_{L^1(\mathbb{R}^n)}\|v\|_{L^\infty(\overline{\Omega}_T)} \leq (\|\varphi\|_{L^1(\mathbb{R}^n)}+1)(\|u\|_{L^\infty(\overline{\Omega}_T)}+1). \tag{3.3.18}$$

Hence, via (3.3.17), (3.3.18) and (3.3.13), $\tilde{f}(v,\tilde{J}v) = f(v,\tilde{J}v)$ on $\overline{\Omega}_T$. Therefore $v : \overline{\Omega}_T \to \mathbb{R}$ is a solution to (CP) with initial data v_0, integral kernel $\tilde{\varphi}$ and nonlinearity f. Finally, since δ in (3.3.15) is independent of the specific choice of v_0 and $\tilde{\varphi}$, the bound in (3.3.16) establishes (3.3.12), as required. □

For the remainder of this section we provide several results concerning derivative estimates for the solution of (CP) that depend on the regularity of the initial data u_0 and the nonlinearity f.

Proposition 3.3.6 ($\partial_{x_i}u$ estimates)**.** *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (CP) and suppose that $u_0 \in W^{1,\infty}(\Omega)\cap C^1(\Omega)$. Then*

$$|\partial_{x_i}u(x,t)| \leq \|\partial_{x_i}u_0\|_{L^\infty(\Omega)} + \frac{2\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}\sqrt{t}}{\sqrt{\pi}} \quad \forall (x,t)\in\overline{\Omega}_T, \tag{3.3.19}$$

for all $i = 1,\ldots,n$.

Proof. By differentiating under the integral sign in (3.3.5), permitted via the regularity of $f(u,Ju)$ and the integrability of G, and after a change of variables, via (3.3.1) we have:

$$\begin{aligned}
|\partial_{x_i}u(x,t)| &= \left|\int_{\mathbb{R}^n}\frac{e^{-z^2}}{\sqrt{\pi}^n}\partial_{x_i}u_0(x+2\sqrt{t}z)dz + \int_0^t\int_{\mathbb{R}^n}(\partial_{x_i}G(x,t;y,s))F(y,s)dyds\right| \\
&\leq \|\partial_{x_i}u_0\|_{L^\infty(\Omega)} + \int_0^t\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(2\sqrt{\pi})^n(t-s)^{n/2}}\frac{|y_i-x_i|}{2(t-s)}|F(y,s)|dyds \\
&\leq \|\partial_{x_i}u_0\|_{L^\infty(\Omega)} + \int_0^t\frac{1}{\sqrt{t-s}}\int_{\mathbb{R}^n}\frac{e^{-z^2}}{\sqrt{\pi}^n}|z_i||F(x+2\sqrt{t-s}z,s)|dzds \\
&\leq \|\partial_{x_i}u_0\|_{L^\infty(\Omega)} + \frac{2\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}\sqrt{t}}{\sqrt{\pi}}
\end{aligned}$$

for all $(x,t)\in\overline{\Omega}_T$, as required. □

Proposition 3.3.7. *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (CP). Then*

$$|\partial_{x_i} u(x,t)| \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi t}} + \frac{2\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}\sqrt{t}}{\sqrt{\pi}} \quad \forall (x,t) \in \Omega_T,$$

for all $i = 1, \ldots, n$.

Proof. The proof follows that of Proposition 3.2.5 for the term involving u_0 and that of Proposition 3.3.6 for the term involving the nonlinearity f. □

Proposition 3.3.8 ($\partial_{x_i x_j} u$ estimates)**.** *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (CP) and suppose that $u_0 \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$. Then*[1]

$$|\partial_{x_i x_j} u(x,t)| \leq \|\partial_{x_i x_j} u_0\|_{L^\infty(\Omega)} + 4I_{ij}K\|\nabla u\|_\infty \sqrt{t} \quad \forall (x,t) \in \overline{\Omega}_T, \tag{3.3.20}$$

for all $i, j = 1, \ldots, n$, with $K = L_U(1 + \|\varphi\|_{L^1(\mathbb{R}^n)})$, L_U a Lipschitz constant for f on

$$U = [-\|u\|_{L^\infty(\overline{\Omega}_T)}, \|u\|_{L^\infty(\overline{\Omega}_T)}] \times [-\|Ju\|_{L^\infty(\overline{\Omega}_T)}, \|Ju\|_{L^\infty(\overline{\Omega}_T)}]$$

and I_{ij} is given by

$$I_{ij} = \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} |z| \left| z_i z_j - \frac{\delta_{ij}}{2} \right| dz.$$

Proof. As in the proof of Proposition 3.3.6, by differentiating under the integral sign (3.3.5), and following a change of variables, we have:

$$\begin{aligned} |\partial_{x_i x_j} u(x,t)| &\leq \left| \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n} \partial_{x_i x_j} u_0(x + 2\sqrt{t}z) dz \right| \\ &\qquad + \left| \partial_{x_i x_j} \left(\int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) F(y,s) dy ds \right) \right| \\ &\leq \|\partial_{x_i x_j} u_0\|_{L^\infty(\Omega)} \\ &\qquad + \left| \partial_{x_i x_j} \left(\int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) F(y,s) dy ds \right) \right|, \quad \forall (x,t) \in \overline{\Omega}_T. \end{aligned} \tag{3.3.21}$$

[1]Here, we utilise the shorthand notation $||\nabla u||_\infty$ to mean $\max_{i=1,\ldots,n}\{||\partial_{x_i} u||_{L^\infty(\overline{\Omega}_T)}\}$.

The last term in (3.3.21) can be estimated as follows:

$$
\begin{aligned}
&\left|\partial_{x_i x_j}\left(\int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)F(y,s)dyds\right)\right| \\
&= \left|\int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n}\left(z_i z_j - \frac{\delta_{ij}}{2}\right)\frac{F(x+2\sqrt{t-s}z,s)}{t-s}dzds\right| \\
&\leq \left|\int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n}\left(z_i z_j - \frac{\delta_{ij}}{2}\right)\frac{F(x+2\sqrt{t-s}z,s)-F(x,s)}{t-s}dzds\right| \\
&\quad + \left|\int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n}\left(z_i z_j - \frac{\delta_{ij}}{2}\right)\frac{F(x,s)}{t-s}dzds\right| \\
&= \left|\int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n}\left(z_i z_j - \frac{\delta_{ij}}{2}\right)\frac{F(x+2\sqrt{t-s}z,s)-F(x,s)}{t-s}dzds\right| \\
&\leq \int_0^t \int_{\mathbb{R}^n} \frac{e^{-z^2}}{\sqrt{\pi}^n}\left|z_i z_j - \frac{\delta_{ij}}{2}\right|\left|\frac{F(x+2\sqrt{t-s}z,s)-F(x,s)}{t-s}\right|dzds \qquad (3.3.22)
\end{aligned}
$$

for all $(x,t) \in \overline{\Omega}_T$. Let $\xi_{ij} : \mathbb{R}^n \to \mathbb{R}_+$ be given by

$$
\xi_{ij}(z) = \frac{e^{-z^2}}{\sqrt{\pi}^n}\left|z_i z_j - \frac{\delta_{ij}}{2}\right| \quad \forall z \in \mathbb{R}^n \text{ and } i,j = 1,\ldots,n. \qquad (3.3.23)
$$

Recalling that f is locally Lipschitz continuous and J is linear with respect to u, it follows that (3.3.22) is bounded above by

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \xi_{ij}(z) L_U \left(\frac{|u(x+2\sqrt{t-s}z,s)-u(x,s)|}{t-s}\right. \\
&\qquad\qquad \left. + \frac{|J(u(x+2\sqrt{t-s}z,s)-u(x,s))|}{t-s}\right) dzds, \qquad (3.3.24)
\end{aligned}
$$

for all $(x,t) \in \overline{\Omega}_T$. Via Proposition 3.3.6, by applying the mean value theorem to the numerators of the fractions appearing in (3.3.24), we obtain

$$
|u(x+2\sqrt{t-s}z,s)-u(x,s)| \leq \|\nabla u\|_\infty 2\sqrt{t-s}|z| \qquad (3.3.25)
$$

and

$$|J(u(x+2\sqrt{t-s}z,s)-u(x,s))| \leq \int_{\mathbb{R}^n} \varphi(y)|u(x+2\sqrt{t-s}z-y,s)-u(x-y,s)|dy$$
$$\leq \|\varphi\|_{L^1(\mathbb{R}^n)}\|\nabla u\|_\infty |z| 2\sqrt{t-s}, \quad (3.3.26)$$

for all $(x,t) \in \overline{\Omega}_T$. Substituting (3.3.25)-(3.3.26) into (3.3.24) and recalling the definitions of K and I_{ij}, (3.3.22) is further bounded above by

$$\int_0^t I_{ij}\frac{2K\|\nabla u\|_\infty}{\sqrt{t-s}}ds = 4I_{ij}K\|\nabla u\|_\infty\sqrt{t} \quad \forall t \in [0,T]. \quad (3.3.27)$$

Finally, substituting (3.3.27) into (3.3.21) yields (3.3.20), as required. □

Proposition 3.3.9. *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (CP). Then*

$$|\partial_{x_i x_j} u(x,t)| \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{t} + \|u_0\|_{L^\infty(\Omega)}2\sqrt{\pi}I_{ij}K + 2\sqrt{\pi}I_{ij}K\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}t, \quad (3.3.28)$$

for all $(x,t) \in \Omega_T$, for all $i,j = 1,\dots,n$, with K, L_U, U and I_{ij} as in Proposition 3.3.8.

Proof. The first term of the right hand side of (3.3.28) is estimated as in (3.2.10). The second and third terms of the right hand side of (3.3.28) are estimated similarly to the terms of the right hand side of (3.3.20). In particular, via Proposition 3.3.7, recalling that f is locally Lipschitz continuous and J is linear with respect to u (following the same steps used to produce the bound in (3.3.27) for (3.3.21)) we have[1]

$$\left|\partial_{x_i x_j}\left(\int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)f(u(y,s),Ju(y,s))dyds\right)\right|$$
$$\leq \int_0^t \int_{\mathbb{R}^n} \xi_{ij}(z)\frac{K\|\nabla u(\cdot,s)\|_\infty |x+2\sqrt{t-s}z-x|}{t-s}dzds$$
$$\leq \int_0^t \int_{\mathbb{R}^n} \frac{2K\xi_{ij}(z)|z|}{\sqrt{t-s}}\left(\frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi s}} + \frac{2\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}\sqrt{s}}{\sqrt{\pi}}\right)dzds$$
$$= \frac{2I_{ij}K\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}ds + \frac{4I_{ij}K\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}}{\sqrt{\pi}}\int_0^t \sqrt{\frac{s}{t-s}}ds$$

[1]Here, $||\nabla u(\cdot,s)||_\infty$ denotes $\max_{i=1,\dots,n}\{||\partial_{x_i}u(\cdot,s)||_{L^\infty(\Omega)}\}$.

$$= \frac{2I_{ij}K\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi}}\mathrm{B}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{4I_{ij}K\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}}{\sqrt{\pi}}t\mathrm{B}\left(\frac{3}{2},\frac{1}{2}\right), \tag{3.3.29}$$

for all $(x,t) \in \Omega_T$, with $\xi_{ij} : \mathbb{R}^n \to \mathbb{R}$ as in (3.3.23) and $\mathrm{B} : \mathbb{R}_+^2 \to \mathbb{R}$ the Beta function defined as

$$\mathrm{B}(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}d\tau = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \forall x,y \geq 0.$$

Substituting the evaluations of Beta function in (3.3.29) yields (3.3.28), as required. □

Remark 3.3.10. Using Proposition 3.3.8 it follows that the solution $u : \overline{\Omega}_T \to \mathbb{R}$ to (CP), with $u_0 \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$, satisfies

$$|\partial_t u(x,t)| \leq \sum_{i=1}^n (||\partial_{x_i x_i} u_0||_{L^\infty(\Omega)} + 4I_{ii}K\|\nabla u\|_\infty \sqrt{t}) + \|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}, \quad \forall (x,t) \in \overline{\Omega}_T.$$

Remark 3.3.11. Using Proposition 3.3.9 it follows that the solution $u : \overline{\Omega}_T \to \mathbb{R}$ to (CP), satisfies

$$|\partial_t u(x,t)| \leq \sum_{i=1}^n \left(\frac{\|u_0\|_{L^\infty(\Omega)}}{t} + \|u_0\|_{L^\infty(\Omega)} 2\sqrt{\pi} I_{ii}K + 2\sqrt{\pi}I_{ii}K\|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}t\right)$$
$$+ \|f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)}, \quad \forall (x,t) \in \overline{\Omega}_T.$$

Remark 3.3.12. On examining the proofs of Proposition 3.3.6 and Proposition 3.3.8, it follows that if $u_0 \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$, then, for the solution $u : \overline{\Omega}_T \to \mathbb{R}$ to (CP), it follows that $\partial_{x_i} u$ and $\partial_{x_i x_j} u$ are continuous on $\overline{\Omega}_T$. Hence, as seen by the differential equation in (CP), u_t can be continuously extended from Ω_T onto $\overline{\Omega}_T$ and therefore

$$u \in L^\infty(\overline{\Omega}_T) \cap C^{2,1}(\overline{\Omega}_T).$$

Assuming higher regularity for both u_0 and f we acquire the following result.

Proposition 3.3.13. *Let $u : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (CP) and let $k \in \mathbb{N}$ satisfy $k \geq 3$. Further assume that $\partial_x^\alpha u_0 \in L^\infty(\Omega) \cap C(\Omega)$, for all multi-indexes α such that $|\alpha| \leq k$ and*

$f \in C^k(U)$, with

$$U = \left[\inf_{\overline{\Omega}_T} u, \sup_{\overline{\Omega}_T} u\right] \times \left[\inf_{\overline{\Omega}_T} Ju, \sup_{\overline{\Omega}_T} Ju\right].$$

Then, for all multi-indexes α such that $|\alpha| \le k+1$, $\partial_x^\alpha u$ exists on $\overline{\Omega}_T$. Moreover, for $|\alpha| \le k$ the derivatives satisfy the bounds

$$|\partial_x^\alpha u(x,t)| \le \|\partial_x^\alpha u_0\|_{L^\infty(\Omega)} + t\|\partial_x^\alpha f(u, Ju)\|_{L^\infty(\overline{\Omega}_T)} \quad \forall (x,t) \in \overline{\Omega}_T. \tag{3.3.30}$$

Furthermore, for α' such that $|\alpha'| = k+1$ and α such that $|\alpha| = k$, u satisfies the bound

$$|\partial_x^{\alpha'} u(x,t)| \le \frac{\|\partial_x^\alpha u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi t}} + \frac{2\|\partial_x^\alpha f(u, Ju)\|_{L^\infty(\overline{\Omega}_T)}\sqrt{t}}{\sqrt{\pi}} \quad \forall (x,t) \in \Omega_T. \tag{3.3.31}$$

Proof. We define $g : \Omega_T \to \mathbb{R}$ to be

$$g(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n} \quad \forall (x,t) \in \Omega_T.$$

Note that $g \in L^1(\Omega_T) \cap C^\infty(\Omega_T)$. Let $* : L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ be the standard convolution product on $\mathbb{R}^n$ given by

$$(a * b)(x) = \int_{\mathbb{R}^n} a(x-y)b(y)dy = \int_{\mathbb{R}^n} a(y)b(x-y)dy \quad \forall x \in \mathbb{R}^n.$$

Thus, we may write the solution $u : \overline{\Omega}_T \to \mathbb{R}$ to (CP) as

$$u(x,t) = (g(\cdot,t) * u_0(\cdot))(x) + \int_0^t (g(\cdot, t-s) * f(u(\cdot,s), Ju(\cdot,s))(x)ds \quad \forall (x,t) \in \overline{\Omega}_T.$$

Denoting $Ju = v$, following two applications of the chain rule, for any $i, j = 1, \dots, n$, it follows that

$$\begin{aligned} \partial_{x_i x_j} f(u,v) = &\partial_{uu} f(u,v)\partial_{x_i} u \partial_{x_j} u + \partial_u f(u,v)\partial_{x_i x_j} u + \partial_{uv} f(u,v)\partial_{x_i} v \partial_{x_j} u \\ &+ \partial_{uv} f(u,v)\partial_{x_i} u \partial_{x_j} v + \partial_{vv} f(u,v)\partial_{x_i} v \partial_{x_j} v + \partial_v f(u,v)\partial_{x_i x_j} v, \end{aligned} \tag{3.3.32}$$

on Ω_T. Since $\partial_{x_i}u, \partial_{x_ix_j}u \in L^\infty(\Omega_T)\cap C(\Omega_T)$, it follows that $\partial_{x_i}v = J\partial_{x_i}u$ and $\partial_{x_ix_j}v = J\partial_{x_ix_j}u$, and furthermore $J\partial_{x_i}u, J\partial_{x_ix_j}u \in L^\infty(\Omega_T)\cap C(\Omega_T)$. Therefore, via (3.3.32), $f(u,v)\in L^\infty(\Omega_T)\cap C(\Omega_T)$. Consequently, since

$$\partial_{x_ix_j}u(x,t) = (g(\cdot,t)*\partial_{x_ix_j}u_0(\cdot))(x) + \int_0^t (g(\cdot,t-s)*\partial_{x_ix_j}f(u(\cdot,s),Ju(\cdot,s))(x)ds, \tag{3.3.33}$$

for all $(x,t)\in\Omega_T$, it follows from $\partial_{x_i}g \in L^1(\Omega_T)$ for $i=1,...,n$, for $|\alpha|=3$, that $\partial_x^\alpha u$ exists on $\overline{\Omega}_T$, and moreover, that $\partial_x^\alpha u \in L^\infty(\overline{\Omega}_T)\cap C(\overline{\Omega}_T)$.

Now, suppose that, for some α with $|\alpha| = 3,\dots,k-1$, we have $\partial_x^\alpha u\in L^\infty(\overline{\Omega}_T)\cap C(\overline{\Omega}_T)$. Repeated applications of the chain rule show that $\partial_x^\alpha f(u,v)$ is a multivariate polynomial in $\partial_x^\beta f(u,v)|_{u,Ju}$, $\partial_x^\beta u$ and $\partial_x^\beta v = J\partial_x^\beta u$ (for a multi-index β such that $|\beta| = 1,\dots,|\alpha|$). This implies that $\partial_x^\alpha f(u,v)\in L^\infty(\overline{\Omega}_T)\cap C(\overline{\Omega}_T)$. Recalling that $g\in L^1(\Omega_T)\cap C^\infty(\Omega_T)$, following a similar argument to that used to produce (3.3.33), we have

$$\int_0^t (g(\cdot,t-s)*\partial_x^\alpha f(u(\cdot,s),Ju(\cdot,s))(x)ds \tag{3.3.34}$$

is continuously differentiable with respect to x_i on Ω_T for $i=1,\dots,n$. Furthermore, since $\partial_x^\alpha u_0\in L^\infty(\Omega)\cap C(\Omega)$ for all $|\alpha|\le k$ and $g(\cdot,t)\in L^1(\mathbb{R}^n)\cap C(\mathbb{R}^n)$, for $t\in(0,T]$, we have that

$$(g(\cdot,t)*\partial_x^\alpha u_0(\cdot))(x) \tag{3.3.35}$$

is continuously differentiable with respect to x_i on Ω_T for $i=1,\dots,n$. Therefore, $\partial_x^\beta u$ exists for each $|\beta| = |\alpha|+1$. Moreover, on recalling the approach used to prove Proposition 3.3.6 and (3.3.34) and (3.3.35), it follows that $\partial_x^\beta u\in L^\infty(\overline{\Omega}_T)\cap C(\overline{\Omega}_T)$. Thus it follows from mathematical induction, on recalling the regularity of the initial data, that $\partial_x^\alpha u$ exists on $\overline{\Omega}_T$ and $\partial_x^\alpha u\in L^\infty(\overline{\Omega}_T)\cap C^{|\alpha|}(\overline{\Omega}_T)$ for all $|\alpha| = 3,\dots,k$.

Now, using the differentiation properties for convolution products and utilising (3.3.1), for $|\alpha| = 3, \ldots, k$, we obtain:

$$\begin{aligned}\partial_x^\alpha u(x,t) &= \partial_x^\alpha (g(\cdot,t) * u_0(\cdot))(x)| + \int_0^t \partial_x^\alpha (g(\cdot,t-s) * F(\cdot,s))(x)ds \\ &= (g(\cdot,t) * \partial_x^\alpha u_0(\cdot))(x) + \int_0^t (g(\cdot,t-s) * \partial_x^\alpha f(u(\cdot,s), Ju(\cdot,s)))(x)ds, \end{aligned} \quad (3.3.36)$$

for all $(x,t) \in \Omega_T$. Thus,

$$\begin{aligned}|\partial_x^\alpha u(x,t)| &\leq \|g(\cdot,t)\|_{L^1(\mathbb{R}^n)} \|\partial_x^\alpha u_0\|_{L^\infty(\Omega)} + \int_0^t \|g(\cdot,t-s)\|_{L^1(\mathbb{R}^n)} \|\partial_x^\alpha f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)} ds \\ &= \|\partial_x^\alpha u_0\|_{L^\infty(\Omega)} + \int_0^t \|\partial_x^\alpha f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)} ds \\ &= \|\partial_x^\alpha u_0\|_{L^\infty(\Omega)} + t\|\partial_x^\alpha f(u,Ju)\|_{L^\infty(\overline{\Omega}_T)},\end{aligned}$$

for all $(x,t) \in \Omega_T$. Finally, to obtain (3.3.31) we apply the same steps used to prove Propositions 3.3.6 and 3.3.7 starting from (3.3.36). $\square$

Remark 3.3.14. Assuming the conditions of Proposition 3.3.13 and recalling that $g(x-y, t-s) = G(x,t;y,s)$ is the fundamental solution to the heat equation, it is readily established, on further examination of the proof, that $u \in W_x^{k,\infty}(\overline{\Omega}_T) \cap C^{k,\lfloor \frac{k}{2} \rfloor}(\overline{\Omega}_T)$.

Remark 3.3.15. Note that the bounds in (3.3.30)-(3.3.31) solely depend on

$$\|\partial_x^\alpha u_0\|_{L^\infty(\Omega)},\ T,\ \|u\|_{L^\infty(\overline{\Omega}_T)},\ ||\varphi||_{L^1(\mathbb{R}^n)} \text{ and } \|\partial_x^\alpha f\|_{L^\infty(U)}. \quad (3.3.37)$$

Specifically the bounds in (3.3.30)-(3.3.31) do not depend on the lower derivatives of u and Ju in $\overline{\Omega}_T$, since these are also bounded by terms solely dependent on the terms in (3.3.37). Thus, Proposition 3.3.13 can be used to bound higher order derivatives of solutions to (CP) provided that the relevant quantities in (3.3.37) are bounded.

We compile the results regarding the well-posedness for (CP) in the following theorem.

Theorem 3.3.16. *Suppose that (CP), with φ, f and u_0 prescribed, is a priori bounded on $\overline{\Omega}_T$ for $0 < T < \infty$ and let $\varepsilon > 0$. Then, there exists a unique solution $u : \overline{\Omega}_T \to \mathbb{R}$ to*

(CP). *Moreover, for all initial data* $\tilde{u}_0$ *and integral kernels* $\tilde{\varphi}$ *which are sufficiently close to* u_0 *and* φ *in the* L^∞ *and* L^1 *norms respectively, the unique solution* $\tilde{u} : \overline{\Omega}_T \to \mathbb{R}$ *to* (CP) *with* $\tilde{\varphi}, \tilde{u}_0$ *and* f *exists, and satisfies* $\|u - \tilde{u}\|_{L^\infty(\overline{\Omega}_T)} < \varepsilon$.

Proof. This follows directly from Theorem 3.3.1, Corollary 3.3.4 and Proposition 3.3.5. □

Theorem 3.3.16 is interpreted as a local in time well-posedness result for (CP). To establish global in time well-posedness results for (CP), knowledge of the asymptotic behaviour for solutions to (CP) as $t \to \infty$ is required.

We now provide an illustration of the application of Theorem 3.3.16 on the non-local Fisher-KPP problem considered in [LCS20] and [BN22]. Consider the following Cauchy problem: let $\Omega = \mathbb{R}$ and suppose that:

$$\partial_t u = \partial_{xx} u + u^2(1 - Ju) \qquad \text{on } \Omega_T; \tag{3.3.38}$$

$$u(x, 0) = u_0(x) \qquad \forall x \in \Omega; \tag{3.3.39}$$

$$u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T), \tag{3.3.40}$$

with prescribed non-negative initial data $u_0 \in L^\infty(\Omega) \cap C(\Omega)$. Further assume that the integral kernel φ has positive mass on a closed ball containing 0 and satisfies

$$\varphi \in L^1(\mathbb{R}), \quad \|\varphi\|_{L^1(\mathbb{R})} = 1 \text{ and } \varphi \geq 0 \quad \text{on } \mathbb{R}. \tag{3.3.41}$$

Via [LCS20, Theorem 1.1] it follows that any solution u to (3.3.38)-(3.3.40) is a priori bounded on Ω_T with bound independent of $T > 0$, i.e. $u \in L^\infty(\overline{\Omega}_\infty)$.[1] Thus, via Remark 3.3.2 and Corollary 3.3.4 a solution to (3.3.38)-(3.3.40) exists and is unique. Furthermore, via Proposition 3.3.5 for any finite time T, solutions to (3.3.38)-(3.3.40) depend continuously to initial data (and integral kernels). We note that to consider whether solutions to

[1] For completeness we summarise the techniques the authors of [LCS20] utilise to obtain bounds on solutions of (3.3.38)-(3.3.40). First they obtain L^p-bounds on solutions of (3.3.38)-(3.3.40), that depend on the exponents of the creation and removal terms of (3.3.38) (here 2 and 1 respectively), $\|u_0\|_{L^\infty(\Omega)}$, the spatial dimension, and, the mass of the integral kernel around 0. Subsequently, via an iterative argument over p they manage to obtain L^∞ estimates on the solution that depends on the aforementioned quantities.

(3.3.38)-(3.3.40) depend continuously on the initial data on $\overline{\Omega}_\infty$, knowledge of the large-t structure of solutions is required.

3.4 Conclusion

In this section, motivated by results concerning (HE), we obtained local in time well-posedness for (CP), as well as derivative estimates for the solutions to (CP). We highlight that Theorem 3.3.1 will be used to provide approximating functions for solutions to Cauchy problems associated with non-local reaction-diffusion equations with non-Lipschitz nonlinear terms f. Specifically, see the proof of Theorem 4.2.2.

It should also be noted that the well-posedness results and methods presented in Section 3.3, can be adapted to provide related well-posedness results for systems of non-local reaction-diffusion equations, where the solution is represented by the vector valued function $\mathbf{u} : \overline{\Omega}_T \to \mathbb{R}^m$. For example, in [NB23] the authors consider the predator-prey system

$$\partial_t u = D_u \partial_{xx} u + u(1 - Ju - \alpha v);$$
$$\partial_t v = D_v \partial_{xx} v + v(1 - Jv - \beta u),$$

where the integral kernel φ is a top-hat function and appropriate initial data is considered. Another recent example (of an SIR type model), containing a system of non-local integro-differential equations, that the theory developed here can be applied is seen in [WZ23]. Natural extensions for the scalar case include: generalising the linear differential part of the integro-differential operator, by utilising fundamental solutions for second order linear parabolic partial differential operators, as discussed in in the previous chapter (see for instance [Fri08, LSU68]); considering spatial domains other than $\mathbb{R}^n$; and alternative non-local interaction terms. More specifically, by assuming sufficient regularity on the coefficients in the linear part of the integro-differential operator and nonlinearity, one can use the parametrix method (as described in [Fri08, Ch. 1]) to obtain integral representations of solutions to related Cauchy problems. Utilising these integral representations with a similar methodology to that presented here one can obtain results similar to Theorem 3.3.16. Further discussion is provided in the final chapter.

The higher derivative estimates on solutions to (CP), illustrated in Theorem 3.3.13 can be useful, for example: to bound the truncation error of finite difference schemes used to approximate solutions to (CP), as used in Section 4.4, [NBLM23], and as illustrated in [DMR22]; and in justifying the existence of higher order terms in asymptotic approximations of solutions to u in various space-time limits (see, for example the general approach illustrated in [LN03]).

CHAPTER 4

ON THE CAUCHY PROBLEM FOR THE NON-LOCAL REACTION DIFFUSION EQUATION WITH HÖLDER CONTINUOUS REGULARITY & APPLICATIONS

4.1 Introduction

Let $\Omega = \mathbb{R}^d$. Consider the following Cauchy problem:

$$\begin{cases} \partial_t u = \Delta u + f(u, Ju), & \text{on } \Omega_T; \\ u(x,0) = u_0(x), & \forall x \in \Omega; \\ u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T), \end{cases} \tag{$\tilde{CP}$}$$

with prescribed $u_0 \in L^\infty(\Omega) \cap C(\Omega)$, with $Ju : L^\infty(\Omega_T) \to L^\infty(\Omega_T)$ given by the convolution product (with argument u),

$$Ju(x,t) = \int_{\mathbb{R}^d} \varphi(x-y)u(y,t)dy \quad \forall (x,t) \in \Omega_T,$$

with (prescribed) integral kernel $\varphi \in L^1(\mathbb{R}^d)$, and (prescribed) nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$ which is locally Hölder continuous of degree $\alpha \in (0,1)$, i.e. for any compact subset $\mathcal{K}$ of $\mathbb{R}^2$ there exists a non-negative Hölder constant $C_{\mathcal{K}}$ such that

$$|f(u_1,v_1)-f(u_2,v_2)| \leq C_{\mathcal{K}}(|u_1-u_2|^{\alpha}+|v_1-v_2|^{\alpha}) \quad \forall (u_1,v_1),(u_2,v_2) \in \mathcal{K}. \tag{4.1.1}$$

When (4.1.1) holds, for brevity, we write that $f \in H_{\alpha}$. Henceforth, unless stated otherwise, when we refer to $(\tilde{CP})$, we assume all the conditions mentioned above.

In Section 4.2 we demonstrate that $(\tilde{CP})$ admits solutions up to a time δ, (see Theorem 4.2.2). Moreover, we establish how one can extend such a solution until either $t = T$ or a blow-up occurs (see Corollary 4.2.4). The solutions are constructed by first constructing solutions to sequences of Cauchy problems where the nonlinearity are Lipschitz continuous and tend to the f of $(\tilde{CP})$ (from above or below); to achieve this, we utilise results in Chapter 3. Those methods allow for regularity estimates to be obtained, that in turn can be applied to inductively construct a solution to $(\tilde{CP})$. If moreover we assume that f is non-decreasing in Ju, and that $\varphi \geq 0$ we subsequently illustrate that constructed minimal and maximal solutions to $(\tilde{CP})$ exist up to a time $t = \delta$ (see Theorem 4.2.3). Similarly these can be extended until $t = T$ or until blow-up occurs (see Corollary 4.2.4).

In Section 4.3 we utilise the results presented previously to demonstrate that the scalar non-local analogue to the source problem arising in isothermal autocatalytic chemical kinetics is locally well-posed (for details on the local case, see [MN15b, NK93]). Namely, we consider the Cauchy problem associated with $(\tilde{CP})$ for the case when $\Omega = \mathbb{R}$ and $f(Ju) = (Ju)_+^p$, and note that here f is non-decreasing in Ju. We refer to this Cauchy problem as $(CP)_+$ and establish that it is locally well-posed in time. Existence of global solutions is established via a priori bounds and the results in Section 4.2. To demonstrate the uniqueness of solutions to $(CP)_+$, we establish comparison principles following the overall approach of [AE87] that ultimately allows us to also demonstrate the local in time continuous dependence of $(CP)_+$.

In Section 4.4 we consider a finite difference method to approximate the solution to a problem related to $(CP)_+$. Specifically we establish the conditional convergence of the solution of the finite difference scheme to the solution of $(CP)_+$, as $\delta x, \delta t \to 0$.

In Section 4.5 we establish a formal large-t asymptotic approximation of the solution to $(CP)_+$. This shows that the lower bound of the solution to $(CP)_+$ is asymptotically stable for $p \in (0, 1/3)$ and unstable for $p \in (1/3, 1)$, under specified assumptions on the initial condition and the integral kernel. The formal results presented here are shown to be in excellent agreement with the numerical simulations that are provided. These are used to infer the conditions for which global well-posedness in time holds for $(CP)_+$.

4.2 Local existence

We begin by providing the definition of maximal and minimal solutions.

Definition 4.2.1. Let $f \in H_\alpha$ for some $\alpha \in (0,1)$, $\varphi \in L^1(\mathbb{R}^d)$, and suppose that $u_0 \in L^\infty(\Omega) \cap C^2(\Omega)$. Let

$$\mathcal{S} = \{u : \overline{\Omega}_T \to \mathbb{R} : u \text{ is a solution to } (\tilde{CP}) \text{ on } \overline{\Omega}_T\}. \tag{4.2.1}$$

A function $\overline{u} : \overline{\Omega}_T \to \mathbb{R}$ is said to be a *maximal solution* to the given $(\tilde{CP})$ if $\overline{u} \in \mathcal{S}$ and, for all $u \in \mathcal{S}$ we have

$$\overline{u}(x,t) \geq u(x,t) \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.2.2}$$

Similarly, a function $\underline{u} : \overline{\Omega}_T \to \mathbb{R}$ is said to be a *minimal solution* to the given $(\tilde{CP})$ if $\underline{u} \in \mathcal{S}$ and, for all $u \in \mathcal{S}$ we have

$$\underline{u}(x,t) \leq u(x,t) \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.2.3}$$

It follows that for a given $(\tilde{CP})$, when $\overline{u} = \underline{u}$ on $\overline{\Omega}_T$, then $(\tilde{CP})$ has a unique solution on $\overline{\Omega}_T$. We now state the main results of this section.

Theorem 4.2.2 (Local Hölder existence). *Consider ($\tilde{CP}$) with $f \in H_\alpha$, for some $\alpha \in (0,1)$, $\varphi \in L^1(\mathbb{R}^d)$, and $u_0 \in L^\infty(\Omega) \cap C(\Omega)$. Then, there exists a solution $u : \overline{\Omega}_T \to \mathbb{R}$ to ($\tilde{CP}$) on $\overline{\Omega}_T$, with $T = \delta$ given by*

$$\delta = \min\left\{\frac{m_0 + a'}{c'}, \frac{m_0 - b'}{c'}\right\}, \tag{4.2.4}$$

with

$$m_0 = (1 + \|\varphi\|_{L^1(\mathbb{R}^d)})\|u_0\|_{L^\infty(\Omega)} + 1; \tag{4.2.5}$$

$$a' = \inf_{x\in\mathbb{R}} u_0(x) - 1/2, \;\; b' = \sup_{x\in\mathbb{R}} u_0(x) + 1/2, \tag{4.2.6}$$

and

$$c' = \max\left\{\left|\inf_{(\xi,\eta)\in[-m_0,m_0]^2} f(\xi,\eta) - 1\right|, \left|\sup_{(\xi,\eta)\in[-m_0,m_0]^2} f(\xi,\eta) + 1\right|\right\}. \tag{4.2.7}$$

Theorem 4.2.3 (Local Hölder existence of minimal and maximal solutions). *Consider ($\tilde{CP}$) with $f \in H_\alpha$ for some $\alpha \in (0,1)$ such that f is non-decreasing in Ju, $\varphi \in L^1(\mathbb{R}^d)$ and $\varphi \geq 0$, and, $u_0 \in L^\infty(\Omega) \cap C(\Omega)$. Then, there exist minimal and maximal solutions to ($\tilde{CP}$) on $\overline{\Omega}_T$, with $T = \delta$ as in (4.2.4). In addition, with $\underline{u}, \overline{u} : \overline{\Omega}_\delta \to \mathbb{R}$ being the minimal and maximal solutions of ($\tilde{CP}$) respectively, we have*

$$\max\{\|\underline{u}\|_{L^\infty(\overline{\Omega}_\delta)}, \|\overline{u}\|_{L^\infty(\overline{\Omega}_\delta)}\} \leq m_0. \tag{4.2.8}$$

The proof of Theorems 4.2.2 and 4.2.3 will be illustrated in the remainder of this section.

Global solutions, be they minimal, maximal or other, can be constructed by glueing together solutions of ($\tilde{CP}$), by repeatedly applying Theorem 4.2.2 or 4.2.3, as described in [MN15a, Remark 8.4] and Remark 3.3.2. Heuristically, glueing solutions together is described as follows: suppose, without loss of generality, that $T > 2\delta$ and $u^1 : \overline{\Omega}_\delta \to \mathbb{R}$ is a solution to ($\tilde{CP}$). Consider the analogue to ($\tilde{CP}$) on $\Omega \times [\delta, 2\delta]$, equipped with initial data $u_{0,2}(x) = u^1(x, \delta)$, for all $x \in \Omega$. Via Theorem 4.2.2, this Cauchy problem has a solution $u^2 : \Omega \times [\delta, 2\delta] \to \mathbb{R}$ on $\Omega \times [\delta, 2\delta]$. Moreover, as noted in Remark 4.2.19, $u^2 \in L^\infty(\Omega \times [\delta, 2\delta]) \cap C^{2,1}(\Omega \times [\delta, 2\delta])$ and

$$u^2(x, \delta) = u^1(x, \delta),\ \partial_t u^2(x, \delta) = \partial_t u^1(x, \delta),$$

$$\partial_{x_i} u^2(x, \delta) = \partial_{x_i} u^1(x, \delta),\ \partial_{x_i x_j} u^2(x, \delta) = \partial_{x_i x_j} u^1(x, \delta),$$

for all $x \in \Omega$ (derivatives with respect to t are in fact taken as right and left limits which coincide). Hence, the concatenation of u^1 with u^2 is, by construction, a solution to ($\tilde{CP}$) for all $t \in [0, 2\delta]$. If $\tilde{CP}$ is a-priori bounded on $\overline{\Omega}_T$, we may repeat this argument to construct a solution to ($\tilde{CP}$) on $\overline{\Omega}_T$. Otherwise, we can repeat this argument to construct a solution to ($\tilde{CP}$) on $\overline{\Omega}_T$ for any $T < T_{MAX}$, where as $t \to T_{MAX}^-$ the solution blows-up. For clarity, we have:

Corollary 4.2.4. *Consider* ($\tilde{CP}$) *with $f \in H_\alpha$ for some $\alpha \in (0, 1)$, $\varphi \in L^1(\mathbb{R}^d)$, and suppose $u_0 \in L^\infty(\Omega) \cap C(\Omega)$. Then, the local solution $u : \overline{\Omega}_T \to \mathbb{R}$ to* ($\tilde{CP}$) *in Theorem 4.2.2 can either be extended to a global solution to* ($\tilde{CP}$) *on $\overline{\Omega}_\infty$, or there exists T_{MAX}, such that u cannot be extended onto $\overline{\Omega}_{T^*}$ for any $T^* \geq T_{MAX}$. In the latter case, $\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $T \to T_{MAX}^-$. Moreover, if f is non-decreasing with respect to Ju and $\varphi \geq 0$ the conclusion holds similarly for solutions to* ($\tilde{CP}$) *provided by Theorem 4.2.3.*

To apply known results about ($\tilde{CP}$) where f is Lipschitz continuous, we require the following density result.

Proposition 4.2.5 (Lipschitz density)**.** *Consider $f \in H_\alpha$ with $\alpha \in (0, 1)$. Let $C_\mathcal{K}$ be a Hölder constant for f on $\mathcal{K} \subseteq \mathbb{R}^2$, with $\mathcal{K} = [a, b]^2$ for $a < b$. Then, on $\mathcal{K}$, for any $\varepsilon > 0$, there exists a Lipschitz continuous function $g : \mathcal{K} \to \mathbb{R}$ such that*

$$|f(x, y) - g(x, y)| < \varepsilon, \quad \forall (x, y) \in \mathcal{K}, \tag{4.2.9}$$

where g is also a Hölder continuous function of degree α on $\mathcal{K}$ with Hölder constant $5C_{\mathcal{K}}$. If additionally, f is non-decreasing in y, then g as above, is also non-decreasing in y on $\mathcal{K}$.

Proof. Let $\varepsilon > 0$ and $C_{\mathcal{K}} > 0$ be a Hölder constant for f on $\mathcal{K}$. We set δ to be

$$\delta = \left(\frac{\varepsilon}{4C_{\mathcal{K}}}\right)^{\frac{1}{\alpha}}. \tag{4.2.10}$$

Then, for all $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathcal{K}$ such that $|\xi_1 - \xi_2| < \delta$ and $|\eta_1 - \eta_2| < \delta$, we have

$$|f(\xi_1, \eta_1) - f(\xi_2, \eta_2)| < \frac{\varepsilon}{2}. \tag{4.2.11}$$

Since $\mathcal{K} = [a, b]^2 \subseteq \mathbb{R}^2$, we set $N = \lceil (b-a)/\delta \rceil$ and partition $\mathcal{K}$ as follows:

$$Z_{nm} = [x_{n-1}, x_n] \times [y_{m-1}, y_m], \tag{4.2.12}$$

for all $n, m = 1, 2, \ldots, N$, with

$$x_n = y_n = a + \frac{(b-a)}{N} n, \quad \forall n = 0, 1, \ldots, N. \tag{4.2.13}$$

Then, for any $n, m = 1, 2, \ldots, N$, we define $g_{nm} : Z_{nm} \to \mathbb{R}$ to be the bi-linear interpolant

$$\begin{aligned} g_{nm}(x, y) = \frac{N^2}{(b-a)^2} \Big[& f(x_n, y_m)(x - x_{n-1})(y - y_{m-1}) + f(x_{n-1}, y_m)(x_n - x)(y - y_{m-1}) \\ & + f(x_n, y_{m-1})(x - x_{n-1})(y_m - y) + f(x_{n-1}, y_{m-1})(x_n - x)(y_m - y) \Big], \end{aligned} \tag{4.2.14}$$

for all $(x, y) \in Z_{nm}$ and define $g : \mathcal{K} \to \mathbb{R}$ as

$$g(x, y) = g_{nm}(x, y), \quad \forall (x, y) \in Z_{nm}, \tag{4.2.15}$$

for each $n, m = 1, 2, \ldots, N$. It now follows that

$$\frac{\partial g_{nm}}{\partial x}(x, y) = \frac{N^2}{2(b-a)}\big[f(x_n, y_m)(y - y_{m-1}) - f(x_{n-1}, y_m)(y - y_{m-1}) \\ + f(x_n, y_{m-1})(y_m - y) - f(x_{n-1}, y_{m-1})(y_m - y)\big], \tag{4.2.16}$$

and

$$\frac{\partial g_{nm}}{\partial y}(x, y) = \frac{N^2}{(b-a)}\big[f(x_n, y_m)(x - x_{n-1}) + f(x_{n-1}, y_m)(x_n - x) \\ - f(x_n, y_{m-1})(x - x_{n-1}) - f(x_{n-1}, y_{m-1})(x_n - x)\big], \tag{4.2.17}$$

for all $(x, t) \in Z_{nm}$, and $n, m = 1, 2, \ldots, N$. It follows from (4.2.16) and (4.2.17) that

$$\left|\frac{\partial g_{nm}}{\partial x}(x, y)\right| \leq \frac{N}{(b-a)}\big[\,|f(x_n, y_m) - f(x_{n-1}, y_m)| + |f(x_n, y_{m-1}) - f(x_{n-1}, y_{m-1})|\,\big] \tag{4.2.18}$$

for all $(x, y) \in Z_{nm}$ and, analogously,

$$\left|\frac{\partial g_{nm}}{\partial y}(x, y)\right| \leq \frac{N}{(b-a)}\big[|f(x_n, y_m) - f(x_n, y_{m-1})| + |f(x_{n-1}, y_m) - f(x_{n-1}, y_{m-1})|\big], \tag{4.2.19}$$

for all $(x, y) \in Z_{nm}$ and each $n, m = 1, 2, \ldots, N$. From (4.2.18) and (4.2.19) it follows that g as defined by (4.2.15) and (4.2.14), is Lipschitz continuous with Lipschitz constant given by

$$L_{\mathcal{K}}^{l} = \max_{\substack{1 \leq n \leq N \\ 1 \leq m \leq N}} \left\{ \frac{N}{(b-a)}\big[|f(x_n, y_m) - f(x_{n-1}, y_m)| + |f(x_n, y_{m-1}) - f(x_{n-1}, y_{m-1})| \\ + |f(x_n, y_m) - f(x_n, y_{m-1})| + |f(x_{n-1}, y_m) - f(x_{n-1}, y_{m-1})|\big] \right\}. \tag{4.2.20}$$

Moreover, via (4.2.17), if f is non-decreasing with respect to y, then g is also non-decreasing with respect to y.

We now demonstrate that g, as constructed, is within ε distance of f on $\mathcal{K}$. For each $(x, y) \in \mathcal{K}$, there exists (n, m) such that $\max\{|x - x_n|, |y - y_m|\} < \delta$. Thus, via (4.2.10)-(4.2.11) and (4.2.14), we obtain

$$\begin{aligned} |f(x,y) - g(x,y)| &\leq |f(x,y) - f(x_n, y_m)| + |f(x_n, y_m) - g(x,y)| \\ &\leq |f(x,y) - f(x_n, y_m)| + |g(x_n, y_m) - g(x,y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{4.2.21}$$

To show that g is Hölder continuous, with Hölder constant equal to $5C_{\mathcal{K}}$, independent of $n, m \in \mathbb{N}$, we first consider $(x, y) \in Z_{nm}$, for $1 \leq n, m \leq N$. Recall that $g = f$ at (x_n, y_m) and from (4.2.14) it follows that

$$\begin{aligned} \left| \frac{\partial g}{\partial x}(x,y) \right| &\leq \sup_{y \in [m-1.m]} \left\{ \frac{|f(x_n, y) - f(x_{n-1}, y)|}{(x_n - x_{n-1})} \right\} \\ &\leq \frac{|f(x_n, y_m) - f(x_{n-1}, y_m)| + |f(x_n, y_{m-1}) - f(x_{n-1}, y_{m-1})|}{(x_n - x_{n-1})} \\ &\leq \frac{C_{\mathcal{K}}|x_n - x_{n-1}|^{\alpha} + C_{\mathcal{K}}|x_n - x_{n-1}|^{\alpha}}{(x_n - x_{n-1})} \\ &= 2C_{\mathcal{K}}|x_n - x_{n-1}|^{\alpha-1}, \end{aligned} \tag{4.2.22}$$

and similarly

$$\left| \frac{\partial g}{\partial y}(x,y) \right| \leq 2C_{\mathcal{K}}|y_m - y_{m-1}|^{\alpha-1}.$$

Using the mean value theorem with (4.2.22), we obtain

$$\begin{aligned} |g(x,y) - g(x', y')| &\leq |\nabla g(\theta) \cdot ((x,y) - (x', y'))| \\ &\leq \left| \frac{\partial g}{\partial x}(\theta) \right| |x - x'| + \left| \frac{\partial g}{\partial y}(\theta) \right| |y - y'| \\ &= 2C_{\mathcal{K}} \left(|x - x'|^{\alpha} + |y - y'|^{\alpha} \right), \end{aligned} \tag{4.2.23}$$

for all $(x, y), (x', y') \in Z_{nm}$, for some $\theta \in Z_{nm}$. Now, assume that $(x, y) \in Z_{nm}$ and

$(x', y') \in Z_{kl}$ for $k, l = 1, 2, \ldots, N$, $Z_{nm} \neq Z_{kl}$. If Z_{nm} and Z_{kl} have a common edge we may choose (x^*, y^*) to be the intersection of the line segment connecting (x, y) and (x', y') with this edge and, following the same steps as those used to produce (4.2.23), obtain

$$\begin{aligned} |g(x,y) - g(x',y')| &\leq |g(x,y) - g(x^*,y^*)| + |g(x^*,y^*) - g(x',y')| \\ &\leq 4C_{\mathcal{K}}(|x - x'|^{\alpha} + |y - y'|^{\alpha}). \end{aligned} \tag{4.2.24}$$

If Z_{nm} and Z_{kl} do not have a common edge, then the line connecting (x, y) and (x', y') intersects the nearest edges of Z_{nm} and Z_{kl}. Denote these intersection points in Z_{nm} and Z_{kl} as (x_π, y_π) and (x'_π, y'_π) respectively. Then, using (4.2.23) and $f \in H_\alpha$, it follows that

$$\begin{aligned} |g(x,y) - g(x',y')| &\leq |g(x,y) - g(x_\pi,y_\pi)| + |g(x_\pi,y_\pi) - g(x'_\pi,y_\pi)| \\ &\quad + |g(x'_\pi,y_\pi) - g(x'_\pi,y'_\pi)| + |g(x'_\pi,y'_\pi) - g(x',y')| \\ &\leq 2C_{\mathcal{K}}\left(|x - x_\pi|^{\alpha} + |y - y_\pi|^{\alpha}\right) + 3C_{\mathcal{K}}\left(|x_\pi - x'_\pi|^{\alpha}\right) \\ &\quad + 3C_{\mathcal{K}}\left(|y_\pi - y'_\pi|^{\alpha}\right) + 2C_{\mathcal{K}}\left(|x'_\pi - x'|^{\alpha} + |y'_\pi - y'|^{\alpha}\right) \\ &= 5C_{\mathcal{K}}(|x - x'|^{\alpha} + |y - y'|^{\alpha}). \end{aligned} \tag{4.2.25}$$

Thus g, via (4.2.23)-(4.2.25) is Hölder continuous of degree α in $\mathcal{K}$, with Hölder constant $5C_{\mathcal{K}}$. This completes the proof, as required. $\square$

Note that the Stone–Weierstrass approximation theorem provides a polynomial function $g : [a, b]^2 \to \mathbb{R}$ that is 'ε-close' to f in the L^∞-norm. However, for such g, the Stone-Weierstrass approximation theorem doesn't provide an upper bound on the Hölder constant of g, for fixed Hölder degree equal to that of f. Hence the need for Proposition 4.2.5.

Proposition 4.2.6. *Let $f \in H_\alpha$ for $\alpha \in (0, 1)$, $\mathcal{K} = [a, b]^2$ and $C_{\mathcal{K}}$ be a Hölder constant for f on $\mathcal{K}$. Then, there exist sequences $\{\overline{f}_n\}_{n \in \mathbb{N}}$ and $\{\underline{f}_n\}_{n \in \mathbb{N}}$, such that for every $n \in \mathbb{N}$ the functions $\overline{f}_n, \underline{f}_n : \mathbb{R}^2 \to \mathbb{R}$ satisfy:*

(a) $\overline{f}_n$ and $\underline{f}_n$ are Lipschitz continuous on every compact subset of $\mathbb{R}^2$;

(b) $\overline{f}_n$ *and* $\underline{f}_n$ *are Hölder continuous on every compact subset of* $\mathbb{R}^2$*, with Hölder constant* $5C_{\mathcal{K}}$*, independent of* $n \in \mathbb{N}$*;*

(c) $\overline{f}_n \to f$ *and* $\underline{f}_n \to f$ *as* $n \to \infty$ *uniformly on* $\mathcal{K}$*;*

(d) $\underline{f}_n \leq f \leq \overline{f}_n$ *on* $\mathcal{K}$ *for all* $n \in \mathbb{N}$*;*

(e) $\{\overline{f}_n(\xi, \eta)\}_{n\in\mathbb{N}}$ *is decreasing for all* $(\xi, \eta) \in \mathcal{K}$*, and* $\{\underline{f}_n(\xi, \eta)\}_{n\in\mathbb{N}}$ *is increasing for all* $(\xi, \eta) \in \mathcal{K}$*;*

(f) if $f(\xi, \eta)$ *is non-decreasing with respect to* η*, then* $\underline{f}_n(\xi, \eta)$ *and* $\overline{f}_n(\xi, \eta)$ *are also non-decreasing with respect to* η*;*

(g) $\|\underline{f}_n\|_{L^\infty(\mathbb{R}^2)} = \|\underline{f}_n\|_{L^\infty(\mathcal{K})}$ *and* $\|\overline{f}_n\|_{L^\infty(\mathbb{R}^2)} = \|\overline{f}_n\|_{L^\infty(\mathcal{K})}$*.*

Proof. The proof follows closely that of [MN15a, Proposition 8.9]. In particular, let $\mathcal{K} = [a, b]^2 \subseteq \mathbb{R}^2$ and denote $g_n : \mathbb{R}^2 \to \mathbb{R}$ as g in Proposition 4.2.5 with $\varepsilon = 2^{-n}$, for each $n \in \mathbb{N}$. Then define $\overline{f}_n : \mathbb{R}^2 \to \mathbb{R}$ to be

$$\overline{f}_n(\xi, \eta) = \begin{cases} g_{n+2}(\xi, \eta) + \dfrac{1}{2^n}, & (\xi, \eta) \in \mathcal{K}; \\ g_{n+2}(\xi^*, \eta^*) + \dfrac{1}{2^n}, & (\xi, \eta) \notin \mathcal{K}, \end{cases} \tag{4.2.26}$$

for each $n \in \mathbb{N}$ where (ξ^*, η^*) the nearest point in $\mathcal{K}$ to $(\xi, \eta) \notin \mathcal{K}$, with respect to the Euclidean distance. The functions $\underline{f}_n$ are defined analogously[1]. Now, (a), (b) and (g) follow immediately via (4.2.26). Statement (c) follows after an application of the triangle inequality. After establishing (c), statement (d) follows trivially. Statements (e) and (f) follows via (4.2.26) with Proposition 4.2.5. The details are left as an exercise to the reader. □

For the remainder of this section we suppose that the functions f, φ and u_0 are fixed, and $\{\overline{f}_n\}_{n\in\mathbb{N}}$, $\{\underline{f}_n\}_{n\in\mathbb{N}}$ described in Proposition 4.2.6, are defined using $\mathcal{K} = [-m_0, m_0]^2$ where m_0 is given by (4.2.5).

[1]I.e., with the $+$ signs in (4.2.26) replaced by $-$ signs.

Remark 4.2.7. We will henceforth consider instances of ($\tilde{CP}$) with the reaction functions $\overline{f}_n$ and $\underline{f}_n$, with initial data $u_0+1/(2n) \in L^\infty(\Omega)\cap C(\Omega)$ and $u_0-1/(2n) \in L^\infty(\Omega)\cap C(\Omega)$ respectively and fixed integral kernel φ. These sequences of problems will be referred to as $(\text{CP})^u_n$ and $(\text{CP})^l_n$ respectively, with the superscripts 'u' and 'l' denoting upper and lower.

We further note that, for $G : D_G \to \mathbb{R}$ as given by (3.2.1) the solution $\underline{u}_n : \overline{\Omega}_T \to \mathbb{R}$ to $(\text{CP})^l_n$ is represented by

$$\begin{aligned} \underline{u}_n(x,t) = &\int_{\mathbb{R}^d} G(x,t;y,0)\left(u_0(y)-\frac{1}{2n}\right)dy \\ &+\int_0^t\int_{\mathbb{R}^d} G(x,t;y,s)\underline{f}_n(\underline{u}_n(y,s),J\underline{u}_n(y,s))dyds \end{aligned} \tag{4.2.27}$$

$$\begin{aligned} = &\int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^n}\left(u_0(x+2\sqrt{t}z)-\frac{1}{2n}\right)dz \\ &+\int_0^t\int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d}\underline{f}_n(\underline{u}_n(x+2\sqrt{t-s}z,s),J\underline{u}_n(x+2\sqrt{t-s}z,s))dzds \end{aligned} \tag{4.2.28}$$

for all $(x,t) \in \overline{\Omega}_T$. The solution to $(\text{CP})^u_n$ can be expressed similarly.

Proposition 4.2.8. *For all $n \in \mathbb{N}$, the unique solutions $\overline{u}_n, \underline{u}_n : \overline{\Omega}_T \to \mathbb{R}$ to $(\text{CP})^u_n$ and $(\text{CP})^l_n$ respectively, exist on $\overline{\Omega}_T$ (for any $T>0$). Moreover,*

$$-c't+a' \le \min\{\underline{u}_n(x,t),\overline{u}_n(x,t)\} \le \max\{\underline{u}_n(x,t),\overline{u}_n(x,t)\} \le c't+b' \quad \forall (x,t) \in \overline{\Omega}_T, \tag{4.2.29}$$

for any $T>0$, where a', b' and c' are given by (4.2.5)-(4.2.7). Furthermore, if $f(\xi,\eta)$, is non-decreasing with respect to η, and $\varphi \ge 0$, then

$$\underline{u}_n(x,t) \le \overline{u}_n(x,t) \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.2.30}$$

Proof. Consider $\underline{v}, \overline{v} : \overline{\Omega}_T \to \mathbb{R}$ given by

$$\underline{v} = a' - c't, \qquad \forall (x,t) \in \overline{\Omega}_T; \tag{4.2.31}$$

$$\overline{v} = b' + c't, \qquad \forall (x,t) \in \overline{\Omega}_T. \tag{4.2.32}$$

Let $P_{\pm} : L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T) \to \mathbb{R}(\Omega_T)$ be given by

$$P_{\pm}[v] = \Delta v \pm c' - \partial_t v \quad \forall v \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T). \tag{4.2.33}$$

Then via (4.2.31)-(4.2.33), and (4.2.7), upon recalling $|\underline{f}_n|, |\overline{f}_n| < c'$ on $\mathbb{R}^2$, it follows that

$$\begin{aligned} &P_-[\underline{v}] \geq 0 \geq P_-[\underline{u}_n]; \\ &P_-[\underline{v}] \geq 0 \geq P_-[\overline{u}_n]; \\ &P_+[\underline{u}_n] \geq 0 \geq P_+[\overline{v}]; \\ &P_+[\overline{u}_n] \geq 0 \geq P_+[\overline{v}], \end{aligned} \tag{4.2.34}$$

on Ω_T. Additionally, via (4.2.31)-(4.2.32) and (4.2.6) it follows that

$$\underline{v} \leq \underline{u}_n < \overline{u}_n \leq \overline{v} \qquad \text{on } \partial\Omega_T. \tag{4.2.35}$$

The inequalities of (4.2.29) now follow from the comparison principle for the inhomogeneous heat equation (see for example, [MN14]).

Now suppose that $f(\xi, \eta)$ is non-decreasing with respect to η. Via Proposition 4.2.6, $\overline{f}_n$ is Lipschitz continuous and non-decreasing with respect to Ju. Since

$$0 = \Delta\overline{u}_n + \overline{f}_n(\overline{u}_n, J\overline{u}_n) - \partial_t\overline{u}_n \leq \Delta\underline{u}_n + \overline{f}_n(\underline{u}_n, J\underline{u}_n) - \partial_t\underline{u}_n \qquad \text{on } \Omega_T, \tag{4.2.36}$$

if $\varphi \geq 0$, an application of the comparison principle, provided by Theorem 2.4.3, yields (4.2.30). The existence and uniqueness of $\underline{u}_n$ and $\overline{u}_n$ follow from the a priori bounds in (4.2.29) and Theorem 3.3.16. This completes the proof, as required. □

Corollary 4.2.9. *For $\delta > 0$ given by (4.2.4), it follows that*

$$|\underline{u}_n| \leq m_0 \text{ and } |\overline{u}_n| \leq m_0 \qquad \text{on } \overline{\Omega}_\delta, \tag{4.2.37}$$

for all $n \in \mathbb{N}$. Moreover, if f is non-decreasing in Ju and $\varphi \geq 0$, then

$$-m_0 \leq \underline{u}_n \leq \overline{u}_n \leq m_0 \qquad \text{on } \overline{\Omega}_\delta, \tag{4.2.38}$$

for all $n \in \mathbb{N}$.

Proof. Follows directly upon substitution of (4.2.4) into Proposition 4.2.8. □

Note that that the bounds on $\overline{u}_n$ and $\underline{u}_n$ in Proposition 4.2.8 are independent of $n \in \mathbb{N}$. For the remainder of this section we will only present results for $\underline{u}_n$, noting that to establish the full result in Theorem 4.2.3 that $\overline{u}_n$ can be treated in the same manner.

Proposition 4.2.10 (Derivative estimate I)**.** *Let $\underline{u}_n : \overline{\Omega}_\delta \to \mathbb{R}$ be the (unique) solution to $(\text{CP})^l_n$, for $n \in \mathbb{N}$. Then,*

$$\|\partial_{x_i}\underline{u}_n(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi t}} + \frac{2c'\sqrt{t}}{\sqrt{\pi}}, \tag{4.2.39}$$

for all $t \in (0,\delta]$, $i = 1,2,\ldots,d$ and c' is as in (4.2.7).

Proof. The result follows directly from Proposition 3.3.7. □

Proposition 4.2.11 (Derivative estimate II)**.** *Let $\underline{u}_n : \overline{\Omega}_\delta \to \mathbb{R}$ be the (unique) solution to $(\text{CP})^l_n$, for $n \in \mathbb{N}$. Then,*

$$\|\partial_{x_ix_j}\underline{u}_n(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{t} + K_1 + K_2(c')^\alpha t^\alpha \tag{4.2.40}$$

for all $t \in (0,\delta]$ and $i,j = 1,2,\ldots,d$ where

$$K_1 = \frac{K\|u_0\|^\alpha_{L^\infty(\Omega)}}{\sqrt{\pi}^\alpha}\mathrm{B}\left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right);$$

$$K_2 = \frac{K 2^\alpha}{\sqrt{\pi}^\alpha} \mathrm{B}\left(\frac{\alpha}{2}+1, \frac{\alpha}{2}\right);$$

$$K = I_{ij,\alpha} 2^\alpha 5 C_{\mathcal{K}} (1 + \|\varphi\|^\alpha_{L^1(\mathbb{R}^n)});$$

$$I_{ij,\alpha} = \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left| z_i z_j - \frac{\delta_{ij}}{2} \right| |z|^\alpha dz,$$

with $C_{\mathcal{K}}$ *a Hölder constant for* $f \in H_\alpha$ *on* $\mathcal{K} = [-m_0, m_0]^2$.

Proof. The first term of the right hand side of (4.2.40) is estimated as in (3.2.10); for the second and third terms we proceed as follows. Denote $F : \overline{\Omega}_\delta \to \mathbb{R}$ to be

$$F(x,t) = \underline{f}_n(\underline{u}_n(x,t), J\underline{u}_n(x,t)), \quad \forall (x,t) \in \overline{\Omega}_\delta.$$

By differentiating under the integral sign the second integral of (4.2.27), noting that the improper integrals and their derivatives are well-defined, following a change of variables we obtain:

$$\begin{aligned}
&\left| \partial_{x_i x_j} \left(\int_0^t \int_{\mathbb{R}^n} G(x,t;y,s) F(y,s) dy ds \right) \right| \\
&= \left| \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left(z_i z_j - \frac{\delta_{ij}}{2} \right) \frac{F(x + 2\sqrt{t-s}z, s)}{t-s} dz ds \right| \\
&\leq \left| \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left(z_i z_j - \frac{\delta_{ij}}{2} \right) \frac{F(x + 2\sqrt{t-s}z, s) - F(x,s)}{t-s} dz ds \right| \\
&\quad + \left| \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left(z_i z_j - \frac{\delta_{ij}}{2} \right) \frac{F(x,s)}{t-s} dz ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left(z_i z_j - \frac{\delta_{ij}}{2} \right) \frac{F(x + 2\sqrt{t-s}z, s) - F(x,s)}{t-s} dz ds \right| \\
&\leq \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left| z_i z_j - \frac{\delta_{ij}}{2} \right| \left| \frac{F(x + 2\sqrt{t-s}z, s) - F(x,s)}{t-s} \right| dz ds, \qquad (4.2.41)
\end{aligned}$$

for all $(x,t) \in \overline{\Omega}_\delta$. Let $\xi_{ij} : \mathbb{R}^d \to [0, \infty)$ be given by

$$\xi_{ij}(z) = \frac{e^{-z^2}}{\sqrt{\pi}^d} \left| z_i z_j - \frac{\delta_{ij}}{2} \right| \quad \forall z \in \mathbb{R}^d \text{ and } i,j = 1, \ldots, d.$$

Via Proposition 4.2.6 that $\underline{f}_n$ is locally Hölder continuous, with Hölder constant $5C_{\mathcal{K}}$ on $\mathcal{K}$. Moreover, since J is linear with respect to u, it follows from (4.2.41) that

$$\begin{aligned}
&\left|\partial_{x_ix_j}\left(\int_0^t\int_{\mathbb{R}^n} G(x,t;y,s)F(y,s)dyds\right)\right| \\
&\le \int_0^t\int_{\mathbb{R}^n}\xi_{ij}(z)5C_{\mathcal{K}}\left(\frac{|\underline{u}_n(x+2\sqrt{t-s}z,s)-\underline{u}_n(x,s)|^\alpha}{t-s}\right. \\
&\qquad\qquad\left.+\frac{|J\underline{u}_n(x+2\sqrt{t-s}z,s)-J\underline{u}_n(x,s)|^\alpha}{t-s}\right)dzds,
\end{aligned} \tag{4.2.42}$$

for all $(x,t)\in\overline{\Omega}_\delta$. By applying the Mean Value Theorem and via Proposition 4.2.10 we have[1]

$$|\underline{u}_n(x+2\sqrt{t-s}z,s)-\underline{u}_n(x,s)|\le\|\nabla\underline{u}_n(\cdot,s)\|_\infty 2\sqrt{t-s}|z|, \tag{4.2.43}$$

and, via (4.2.43),

$$\begin{aligned}
|J\underline{u}_n(x+2\sqrt{t-s}z,s)-J\underline{u}_n(x,s)| &\le \int_{\mathbb{R}^d}\varphi(y)\|\nabla\underline{u}_n(\cdot,s)\|_\infty 2\sqrt{t-s}|z|dy \\
&\le \|\varphi\|_{L^1(\mathbb{R}^n)}\|\nabla\underline{u}_n(\cdot,s)\|_\infty 2\sqrt{t-s}|z|,
\end{aligned} \tag{4.2.44}$$

for all $(x,t)\in\overline{\Omega}_\delta$, $z\in\mathbb{R}^d$ and $0<s\le t$. Substituting (4.2.43) and (4.2.44) into (4.2.42), upon recalling (4.2.39) and the definition of K, we obtain

$$\begin{aligned}
&\left|\partial_{x_ix_j}\left(\int_0^t\int_{\mathbb{R}^n} G(x,t;y,s)F(y,s)dyds\right)\right| \\
&\le \int_0^t\int_{\mathbb{R}^d}\xi_{ij}(z)5C_{\mathcal{K}}2^\alpha(1+\|\varphi\|^\alpha_{L^1(\mathbb{R}^n)})|z|^\alpha\|\nabla\underline{u}_n(\cdot,s)\|^\alpha_\infty(t-s)^{\frac{\alpha}{2}-1}dzdt \\
&= I_{ij,\alpha}C_{\mathcal{K}}2^\alpha(1+\|\varphi\|^\alpha_{L^1(\mathbb{R}^n)})\int_0^t\|\nabla\underline{u}_n(\cdot,s)\|^\alpha_\infty(t-s)^{\frac{\alpha}{2}-1}dt \\
&\le K\int_0^t\left|\frac{\|u_0\|_{L^\infty(\Omega)}}{\sqrt{\pi s}}+\frac{2c'\sqrt{s}}{\sqrt{\pi}}\right|^\alpha(t-s)^{\frac{\alpha}{2}-1}dt \\
&\le \frac{K\|u_0\|^\alpha_{L^\infty(\Omega)}}{\sqrt{\pi}^\alpha}\int_0^t s^{-\frac{\alpha}{2}}(t-s)^{\frac{\alpha}{2}-1}ds+\frac{K2^\alpha(c')^\alpha}{\sqrt{\pi}^\alpha}\int_0^t s^{\frac{\alpha}{2}}(t-s)^{\frac{\alpha}{2}-1}ds
\end{aligned}$$

[1] Here, we again utilise the shorthand notation $||\nabla v(\cdot,s)||_\infty$ to mean $\max_{i=1,\dots,n}\{||\partial_{x_i}v(\cdot,s)||_{L^\infty(\Omega)}\}$.

$$= \frac{K\|u_0\|^{\alpha}_{L^\infty(\Omega)}}{\sqrt{\pi}^{\alpha}} \int_0^1 s^{-\frac{\alpha}{2}}(1-s)^{\frac{\alpha}{2}-1} ds + \frac{K2^{\alpha}(c')^{\alpha}}{\sqrt{\pi}^{\alpha}} t^{\alpha} \int_0^1 s^{\frac{\alpha}{2}}(1-s)^{\frac{\alpha}{2}-1} ds$$
$$= \frac{K\|u_0\|^{\alpha}_{L^\infty(\Omega)}}{\sqrt{\pi}^{\alpha}} \mathrm{B}\left(1-\frac{\alpha}{2}, \frac{\alpha}{2}\right) + \frac{K2^{\alpha}}{\sqrt{\pi}^{\alpha}} \mathrm{B}\left(\frac{\alpha}{2}+1, \frac{\alpha}{2}\right)(c')^{\alpha} t^{\alpha},$$

for all $t \in (0, \delta]$, which yields (4.2.40), as required. □

Utilising Proposition 4.2.11 we obtain the following time derivative estimates.

Remark 4.2.12. Let $\underline{u}_n : \overline{\Omega}_\delta \to \mathbb{R}$ be the (unique) solution to $(\mathrm{CP})^l_n$ for $n \in \mathbb{N}$. Then,

$$\|\partial_t \underline{u}_n(\cdot, t)\|_{L^\infty(\Omega)} \leq \sum_{i=1}^{d} \left(\frac{\|u_0\|_{L^\infty(\Omega)}}{t} + K_1 + K_2(c')^{\alpha} t^{\alpha} \right) + c', \tag{4.2.45}$$

for all $t \in (0, \delta]$, with K_1 and K_2 as in Proposition 4.2.11 and c' as in (4.2.7).

Note that the bounds on the derivatives of $\underline{u}_n$ in Propositions 4.2.10 and 4.2.11 as well as in Remark 4.2.12 are independent of $n \in \mathbb{N}$. Now, using the uniform bounds on solutions of $(\mathrm{CP})^l_n$ obtained in Proposition 4.2.8 and the uniform derivative estimates in Proposition 4.2.10 and Remark 4.2.12, we can use the equicontinuity of $\{\underline{u}_n\}_{n\in\mathbb{N}}$ on compact subsets of Ω_δ. Then, we may apply the Arzelá-Ascoli Theorem to obtain

Lemma 4.2.13. *There exists a function $\underline{u} \in L^\infty(\Omega_\delta) \cap C(\Omega_\delta)$ such that for all $M > 0$ and $0 < \delta' < \delta$, the sequence of functions $\{\underline{u}_n\}_{n\in\mathbb{N}}$, has a convergent subsequence $\{\underline{u}_{n_j}\}_{j\in\mathbb{N}}$ that satisfies*

$$\underline{u}_{n_j} \to \underline{u}, \ \text{as } j \to \infty \text{ uniformly on } [-M, M]^d \times [\delta', \delta]. \tag{4.2.46}$$

Proof. Let $M > 0$ and $0 < \delta' < \delta$. Then, via Corollary 4.2.9, Proposition 4.2.10 and Remark 4.2.12, it follows that

$$|\underline{u}_n(x,t)|, |\partial_{x_i}\underline{u}_n(x,t)|, |\partial_t \underline{u}_n(x,t)| \leq K \quad \forall (x,t) \in [-M, M]^d \times [\delta', \delta], \tag{4.2.47}$$

for all $n \in \mathbb{N}$ and $i = 1, \ldots, d$, with K independent of $n \in \mathbb{N}$ and $M > 0$. Thus, $\{\underline{u}_n\}_{n\in\mathbb{N}}$ is uniformly bounded and uniformly equicontinuous on $[-M, M]^d \times [\delta', \delta]$. Therefore, via the

Arzelá-Ascoli compactness theorem (see [RF68, Section 10.1]), there exists a subsequence $\{\underline{u}_{n_i}\}_{i\in\mathbb{N}}$ and a function $\underline{u}^M \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T)$ such that

$$\underline{u}_{n_i} \to \underline{u}^M \quad \text{uniformly on } [-M, M]^d \times [\delta', \delta]. \tag{4.2.48}$$

By repeating this argument, with M replaced by $M+1$, δ' replaced by $\delta'/2$, $\{\underline{u}_n\}$ replaced by $\{\underline{u}_{n_i}\}$, and $\{\underline{u}_{n_i}\}$ replaced by $\{\underline{u}_{n_{i_j}}\}$, an induction argument establishes the existence of $\underline{u} : \Omega_\delta \to \mathbb{R}$ which satisfies (4.2.46), as required. □

Using Lemma 4.2.13, and by setting $\underline{u}(x, 0) = u_0(x)$ for all $x \in \Omega$, we can define $\underline{u}$ on the entirety of $\overline{\Omega}_\delta$. Moreover, for $\underline{u} : \overline{\Omega}_\delta \to \mathbb{R}$, it follows that

$$\|\underline{u}\|_{L^\infty(\overline{\Omega}_\delta)} \le m_0 \quad \text{on } \overline{\Omega}_\delta. \tag{4.2.49}$$

To complete the proof of Theorem 4.2.2 we will construct $\underline{u}$ as a limit of $\underline{u}_n$ using the the (equivalent) integral equation of $(\tilde{CP})$ and $(\text{CP})^l_n$.

Lemma 4.2.14. *The function $\underline{u}$ in Lemma 4.2.13 satisfies*

$$\begin{aligned}\underline{u}(x,t) = &\int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} u_0(x + 2\sqrt{t}z)dz \\ &+ \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} f(\underline{u}(x + 2\sqrt{t-s}z, s), J\underline{u}(x + 2\sqrt{t-s}z, s))dzds,\end{aligned} \tag{4.2.50}$$

for all $(x,t) \in \overline{\Omega}_\delta$. Moreover, $\underline{u}$ is a solution to $(\tilde{CP})$ on $\overline{\Omega}_\delta$.

Proof. For all $n \in \mathbb{N}$, by construction, $\underline{u}_n : \overline{\Omega}_\delta \to \mathbb{R}$ is a solution to $(\text{CP})^l_n$ on $\overline{\Omega}_\delta$. It also follows via Remark 3.3.2 that

$$\begin{aligned}\underline{u}_n(x,t) = &\int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left(u_0(x + 2\sqrt{t}z) - \frac{1}{2n} \right) dz \\ &+ \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \underline{f}_n(\underline{u}_n(x + 2\sqrt{t-s}z, s), J\underline{u}_n(x + 2\sqrt{t-s}z, s))dzds \\ := &\ \underline{v}_n(x,t) + \underline{w}_n(x,t), \quad \forall (x,t) \in \overline{\Omega}_\delta,\end{aligned} \tag{4.2.51}$$

with $\underline{v}_n, \underline{w}_n : \overline{\Omega}_\delta \to \mathbb{R}$. First note that

$$\begin{aligned}\lim_{n\to\infty} \underline{v}_n(x,t) &= \lim_{n\to\infty} \left[\int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} u_0(x + 2\sqrt{t}z) dz - \int_{\mathbb{R}^d} \frac{e^{-z^2}}{2n\sqrt{\pi}^d} dz \right] \\ &= \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}} u_0(x + 2\sqrt{t}z) dz \\ &:= \underline{v}(x,t), \quad \forall (x,t) \in \overline{\Omega}_\delta, \end{aligned} \tag{4.2.52}$$

with $\underline{v} : \overline{\Omega}_\delta \to \mathbb{R}$. Now, let $\underline{w} : \overline{\Omega}_\delta \to \mathbb{R}$ be given by

$$\underline{w}(x,t) = \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} f(\underline{u}(x + 2\sqrt{t-s}z, s), J\underline{u}(x + 2\sqrt{t-s}z, s)) dz ds, \tag{4.2.53}$$

for all $(x,t) \in \overline{\Omega}_\delta$. Fix $M_c > 0$, $\delta_c \in (0,\delta)$ and $\varepsilon > 0$ and set $M > 0$ and $\delta' > 0$ such that

$$e^{-M^2 d} < \frac{\varepsilon}{6\delta c'}, \tag{4.2.54}$$

and

$$\delta' < \min\left\{ \frac{\varepsilon}{6\delta c'}, \delta_c \right\}. \tag{4.2.55}$$

Now, via Lemma 4.2.13, there exists $\{\underline{u}_{n_j}\}_{j\in\mathbb{N}} \in L^\infty([-\tilde{M},\tilde{M}]^d \times [\delta',\delta]) \cap C([-\tilde{M},\tilde{M}]^d \times [\delta',\delta])$ such that $\underline{u}_{n_j} \to \underline{u}$ uniformly on $[-\tilde{M},\tilde{M}]^d \times [\delta',\delta]$ with

$$\tilde{M} = M_c + 2\sqrt{\delta} M. \tag{4.2.56}$$

Additionally, via Lemma 4.2.13 and Proposition 4.2.6 there exists $N \in \mathbb{N}$ such that for all $n_j > N$

$$\left\| f(\underline{u}, J\underline{u}) - \underline{f}_{n_j}(\underline{u}_{n_j}, J\underline{u}_{n_j}) \right\|_{L^\infty([-\tilde{M},\tilde{M}]^d \times [\delta',\delta])} < \frac{\varepsilon}{3\delta}. \tag{4.2.57}$$

Now, for $(x,t) \in [-M_c, M_c]^d \times [\delta_c, \delta]$, it follows that

$$
\begin{aligned}
&\left|(\underline{w} - \underline{w}_{n_j})(x,t)\right| \\
&\leq \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left|(f(\underline{u}, J\underline{u}) - \underline{f}_{n_j}(\underline{u}_{n_j}, J\underline{u}_{n_j}))(x + 2\sqrt{t-s}z, s)\right| dzds \\
&= \int_{\delta'}^t \int_{[-M,M]^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left|(f(\underline{u}, J\underline{u}) - \underline{f}_{n_j}(\underline{u}_{n_j}, J\underline{u}_{n_j}))(x + 2\sqrt{t-s}z, s)\right| dzds \\
&\quad + \int_0^{\delta'} \int_{[-M,M]^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left|(f(\underline{u}, J\underline{u}) - \underline{f}_{n_j}(\underline{u}_{n_j}, J\underline{u}_{n_j}))(x + 2\sqrt{t-s}z, s)\right| dzds \qquad (4.2.58)\\
&\quad + \int_0^t \int_{(\mathbb{R}\setminus[-M,M])^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left|(f(\underline{u}, J\underline{u}) - \underline{f}_{n_j}(\underline{u}_{n_j}, J\underline{u}_{n_j}))(x + 2\sqrt{t-s}z, s)\right| dzds.
\end{aligned}
$$

Via (4.2.57), the first integral on the right hand side of (4.2.58) is bounded above by

$$
\begin{aligned}
&\int_{\delta'}^t \int_{[-M,M]^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} \left\|f(\underline{u}, J\underline{u}) - \underline{f}_{n_j}(\underline{u}_{n_j}, J\underline{u}_{n_j})\right\|_{L^\infty([-\tilde{M},\tilde{M}]^d \times [\delta',\delta])} \\
&< \frac{\varepsilon}{3\delta} \int_0^t \int_{\mathbb{R}^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} dzds \\
&= \frac{\varepsilon}{3}. \qquad (4.2.59)
\end{aligned}
$$

Moreover, via (4.2.55), the second integral on the right hand side of (4.2.58) is bounded above by

$$
\int_0^{\delta'} \int_{[-M,M]^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} 2c' dzds < 2c'\delta' < \frac{\varepsilon}{3}. \qquad (4.2.60)
$$

Furthermore, via (4.2.54), the third integral on the right hand side of (4.2.58) is bounded above by

$$
\begin{aligned}
\int_0^t \int_{(\mathbb{R}\setminus[-M,M])^d} \frac{e^{-z^2}}{\sqrt{\pi}^d} 2c' dzds &= 2^{d+1} c' \delta \left(\int_M^\infty \frac{e^{-\zeta^2}}{\sqrt{\pi}} d\zeta \right)^d \\
&< 2^{d+1} c' \delta e^{-M^2 d} \left(\int_0^\infty \frac{e^{-\zeta^2}}{\sqrt{\pi}} d\zeta \right)^d \\
&< \frac{\varepsilon}{3}. \qquad (4.2.61)
\end{aligned}
$$

Therefore, via (4.2.58)-(4.2.61), it follows that $\underline{w}_{n_j} \to \underline{w}$, as $j \to \infty$, on $[-M_c, M_c]^d \times [\delta_c, \delta]$, and thus, via (4.2.52), (4.2.53) and (4.2.51), it follows that (4.2.50) is satisfied for $(x,t) \in [-M_c, M_c]^d \times [\delta_c, \delta]$. Given that $M_c, \delta_c > 0$ were arbitrary, it follows that (4.2.50) is satisfied on Ω_δ. Further noting that $\underline{w} \to 0$ uniformly for $x \in \mathbb{R}^d$ as $t \to 0^+$, since $u_0 \in L^\infty(\Omega) \cap C(\Omega)$, establishes (4.2.50) on $\overline{\Omega}_\delta$ as well as the continuity of $\underline{u}$ on $\overline{\Omega}_\delta$.

Finally, since $f \in H_\alpha$, it follows, as described at the end of the proof of Theorem 3.3.1, that $\underline{u} \in C^{2,1}(\Omega_\delta) \cap C(\Omega_\delta)$, and moreover is a solution to $(\tilde{CP})$. This completes the proof, as required. $\square$

The proof of Theorem 4.2.2 is now complete.

Remark 4.2.15. Note that the sequence $\{\underline{u}_n\}_{n\in\mathbb{N}}$ of solutions to $(\text{CP})^l_n$, can be replaced by the sequence $\{\overline{u}_n\}_{n\in\mathbb{N}}$ of solutions to $(\text{CP})^u_n$ in each result from Proposition 4.2.10 up to and including Lemma 4.2.14, where $\underline{u}$ should also be replaced by, a potentially distinct, $\overline{u}$.

For the remainder of this section we will further assume that:

$$f(\xi,\eta) \text{ is non-decreasing with respect to } \eta \text{ and } \varphi \geq 0. \tag{4.2.62}$$

We will establish the convergence of $\underline{u}_n$ and $\overline{u}_n$ to respective minimal and maximal solutions of $(\tilde{CP})$.

Proposition 4.2.16. *Let $\underline{u}_n, \underline{u}_{n+1} : \overline{\Omega}_\delta \to \mathbb{R}$ be the (unique) solutions to $(\text{CP})^l_n$ and $(\text{CP})^l_{n+1}$ respectively, and suppose (4.2.62) is satisfied. Then, for all $n \in \mathbb{N}$*

$$\underline{u}_n \leq \underline{u}_{n+1} \quad \text{on } \overline{\Omega}_\delta. \tag{4.2.63}$$

Proof. Let $Q : L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta) \cap C^{2,1}(\Omega_\delta) \to \mathbb{R}(\Omega_\delta)$ be given by

$$Q[v] = \Delta v + \underline{f}_n(v, Jv) - v_t, \tag{4.2.64}$$

for all $v \in L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta) \cap C^{2,1}(\Omega_\delta)$. Observe that via Propositions 4.2.6 and 4.2.8 it follows that

$$Q[\underline{u}_{n+1}] \leq Q[\underline{u}_n] \quad \text{on } \Omega_\delta, \tag{4.2.65}$$

and

$$\underline{u}_{n+1} \geq \underline{u}_n \quad \text{on } \partial\Omega_\delta. \tag{4.2.66}$$

Hence, via (4.2.62), (4.2.65) and (4.2.66), by applying the comparison principle, Theorem 2.4.3, yields (4.2.63), as required. □

Proposition 4.2.17. *Suppose that (4.2.62) is satisfied and let $\underline{u}_n : \overline{\Omega}_\delta \to \mathbb{R}$ be the (unique) solution to $(\text{CP})_n^l$ on $\overline{\Omega}_\delta$. Further assume that $v \in L^\infty(\overline{\Omega}_\delta) \cap C(\overline{\Omega}_\delta) \cap C^{2,1}(\Omega_\delta)$ satisfies*

$$\Delta v + f(v, Jv) - \partial_t v \leq 0 \quad \text{on } \Omega_\delta, \tag{4.2.67}$$

and

$$u_0(x) \leq v(x, 0) \quad \forall x \in \Omega. \tag{4.2.68}$$

Then,

$$\underline{u}_n \leq v \quad \text{on } \overline{\Omega}_\delta. \tag{4.2.69}$$

Proof. The proof follows directly from Theorem 2.4.3. □

Remark 4.2.18. Proposition 4.2.17 implies that, if $u : \overline{\Omega}_\delta \to \mathbb{R}$ is any solution of $(\tilde{CP})$, then $\underline{u}_n \leq u$ on $\overline{\Omega}_\delta$. Thus, if (4.2.62) is satisfied, it follows that $\underline{u}$, as given in Lemma 4.2.14, is a minimal solution to $(\tilde{CP})$, as described in Definition 4.2.1. By considering

$\{\overline{u}_n\}_{n\in\mathbb{N}}$, it is similarly established that there exists a maximal solution to ($\tilde{CP}$) on $\overline{\Omega}_\delta$, denoted by $\overline{u}$.

The proof of Theorem 4.2.3 is now complete.

Remark 4.2.19. Propositions 3.3.6 and 3.3.8 can be directly generalised for the case where the function f is locally Hölder continuous. Therefore, as in the Lipschitz case (illustrated by Remark 3.3.12), if $u_0 \in W^{2,\infty}(\Omega)\cap C^2(\Omega)$, then, for a solution $u : \overline{\Omega}_\delta \to \mathbb{R}$ to ($\tilde{CP}$), it follows that $\partial_{x_i}u$, $\partial_{x_i x_j}u$ and $\partial_t u$ are continuous on $\overline{\Omega}_\delta$, i.e:

$$u \in L^\infty(\overline{\Omega}_\delta)\cap C^{2,1}(\overline{\Omega}_\delta).$$

4.3 The Cauchy problem with $f(Ju) = (Ju)_+^p$ for $p \in (0,1)$

We consider $\Omega = \mathbb{R}$ throughout this section. We introduce $f : \mathbb{R} \to \mathbb{R}$, given by $f(v) = (v)_+^p = (\max\{v, 0\})^p$, for all $v \in \mathbb{R}$. Now consider the Cauchy problem given by

$$\begin{cases} \partial_t u = \Delta u + (Ju)_+^p, & \text{on } \Omega_T; \\ u(x,0) = u_0(x), & \forall x \in \Omega; \\ u \in L^\infty(\overline{\Omega}_T)\cap C(\overline{\Omega}_T)\cap C^{2,1}(\Omega_T), \end{cases} \tag{4.3.1}$$

with prescribed $u_0 \in L^\infty(\Omega)\cap C(\Omega)$, with $Ju : L^\infty(\Omega_T) \to L^\infty(\Omega_T)$ given by

$$Ju(x,t) = \int_{-\infty}^{\infty} \varphi(x-y)u(y,t)dy \quad \forall (x,t)\in\Omega_T, \tag{4.3.2}$$

with (prescribed) $\varphi \in L^1(\mathbb{R})$ such that

$$\varphi \geq \sigma \quad \text{on } [-\delta,\delta] \text{ and } \varphi \geq 0 \quad \text{on } \mathbb{R}, \tag{4.3.3}$$

for constants $\sigma, \delta > 0$ and $p \in (0,1)$, and further assume that

$$u_0 \text{ is a non-identically null, non-negative function on } \Omega. \tag{4.3.4}$$

We denote the Cauchy problem given by (4.3.1)-(4.3.4) as $(CP)_+$. Henceforth, unless stated otherwise, when we refer to $(CP)_+$, we assume all the conditions mentioned above. In the remainder of this section we will demonstrate that $(CP)_+$ is well-posed locally in time.

For $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ as in (4.3.1) that satisfies (4.3.4) we define the following:

$$\mu_0 := \inf_{x\in\Omega} u_0(x) \quad \text{and} \quad M_0 := \sup_{x\in\Omega} u_0(x).$$

Proposition 4.3.1. *Let $u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ be a solution to $(CP)_+$. Then*

$$\mu_0 \leq u(x,t) \leq (M_0^{1-p} + \|\varphi\|^p_{L^1(\Omega)}(1-p)t)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.3.5}$$

Proof. The lower bound on u in (4.3.5) follows from an application of the comparison principle for the heat equation, after noting that $(Jv)^p_+ \geq 0$ on Ω_T.

To obtain the upper bound on u in (4.3.5) we proceed as follows: define $\overline{f} : \mathbb{R} \to \mathbb{R}$ to be

$$\overline{f}(Ju) = \begin{cases} (Ju)^p, & Ju \geq \|\varphi\|_{L^1(\Omega)} M_0; \\ \|\varphi\|^p_{L^1(\Omega)} M_0^p, & Ju < \|\varphi\|_{L^1(\Omega)} M_0, \end{cases} \tag{4.3.6}$$

and note that $\overline{f}$ is locally Lipschitz continuous and non-decreasing on $\mathbb{R}$. Next, let $v : \overline{\Omega}_T \to \mathbb{R}$ be given by

$$v(x,t) = \left(M_0^{1-p} + \|\varphi\|^p_{L^1(\Omega)}(1-p)t\right)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_T, \tag{4.3.7}$$

and define $P : L^\infty(\Omega_T) \cap C^{2,1}(\Omega_T) \to \mathbb{R}(\Omega_T)$ to be

$$P[w] = \partial_{xx} w + \overline{f}(Jw) - \partial_t w, \quad \forall w \in L^\infty(\Omega_T) \cap C^{2,1}(\Omega_T). \tag{4.3.8}$$

It now follows, from (4.3.1), (4.3.6) and (4.3.7), that

$$P[u] = \overline{f}(Ju) - f(Ju) \geq 0 = P[v] \qquad \text{on } \Omega_T. \tag{4.3.9}$$

Moreover, from (4.3.7) we have

$$v \geq u \qquad \text{on } \partial\Omega_T. \tag{4.3.10}$$

Via (4.3.9) and (4.3.10), an application of the comparison principle, Theorem 2.4.3, implies the upper bound in (4.3.5). This completes the proof, as required. □

Thus it has been established, via Proposition 4.3.1 that $(CP)_+$ is a priori bounded on $\overline{\Omega}_T$ for all $T > 0$. We therefore have

Theorem 4.3.2 (Existence)**.** *There exist global minimal and maximal solutions to $(CP)_+$ denoted by $\underline{u}, \overline{u} : \overline{\Omega}_\infty \to \mathbb{R}$, respectively. Moreover,*

$$\mu_0 \leq \underline{u}(x,t) \leq \overline{u}(x,t) \leq \left(M_0^{1-p} + \|\varphi\|^p_{L^1(\Omega)}(1-p)t \right)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.3.11}$$

Proof. Since $f \in H_\alpha$ and $u_0 \in L^\infty(\Omega) \cap C(\Omega)$, the result follows directly from Corollary 4.2.4 with the a priori bounds in Proposition 4.3.1. □

Remark 4.3.3. If $u : \overline{\Omega}_T \to \mathbb{R}$ is a solution to $(CP)_+$, then $v : \overline{\Omega}_T \to \mathbb{R}$ given by

$$v(x,t) = ku(x,t) \quad \forall (x,t) \in \overline{\Omega}_T$$

for any constant $k > 0$, is a solution to $(CP)_+$ with initial data $v_0 = ku_0$, and integral kernel $\varphi_k = k^{(1-p)/p}\varphi$. Therefore, without loss of generality, if required, one can always set $\|\varphi\|_{L^1(\Omega)} = 1$.

In what follows we will utilise the following lemmas:

Lemma 4.3.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be given by*

$$F(x) = \int_{x-\delta}^{x+\delta} e^{-\beta y^2} dy - \delta e^{-\delta^2 \beta} e^{-\beta x^2} \quad \forall x \in \mathbb{R}, \tag{4.3.12}$$

where β and δ are positive constants. Then $F(x) \geq 0$ for all $x \in \mathbb{R}$.

Proof. Since F is an even function of x we restrict our attention to $x \in [0, \infty)$. First note that $F(0) > 0$ since

$$\int_{-\delta}^{\delta} e^{-\beta y^2} dy \geq 2\delta e^{-\beta\delta^2} > \delta e^{-\beta\delta^2} > 0. \tag{4.3.13}$$

Next note that

$$F'(x) = e^{-\beta\delta^2} e^{-\beta x^2} (e^{-2\beta\delta x} - e^{2\beta\delta x} + 2\beta\delta x),$$

for all $x \in [0, \infty)$, and that $F'(x) < 0$ for $x > 0$. Finally, by noting that

$$\lim_{x \to \infty} F(x) = 0$$

it follows that $F \geq 0$, as required. $\square$

In what follows, we use the semigroup notation $S_\tau v(x)$, for $\tau \in (0, \infty)$ to denote

$$S_\tau v(x) = (4\pi\tau)^{-1/2} \int_{-\infty}^{\infty} e^{-|x-y|^2/4\tau} v(y) dy,$$

for each $v \in L^\infty(\Omega) \cap C(\Omega)$ and $(x, \tau) \in \Omega_\infty$. Additionally, throughout this section, we regularly use the following result: for all $u_0, v_0 \in L^\infty(\Omega) \cap C(\Omega)$, if

$$u_0 \geq v_0 \quad \text{on } \Omega$$

then

$$S_\tau u_0(x) \geq S_\tau v_0(x), \quad \forall x \in \Omega.$$

We further note that this notation is consistent with the fundamental solution notation used in previous chapters, and is used here for brevity.

Lemma 4.3.5. *Let $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ and satisfy (4.3.4). Moreover, suppose that $u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ is a non-negative function on $\overline{\Omega}_T$ and*

$$u(x,t) \geq S_t u_0(x) + \int_0^t S_{t-s}(Ju)^p(x,s)ds, \tag{4.3.14}$$

for all $(x,t) \in \overline{\Omega}_T$, with Ju as in (4.3.2)-(4.3.3) and $p \in (0,1)$. Then

$$u(x,t) > ((\sigma\delta)^p(1-p)t)^{\frac{1}{1-p}} \quad \forall (x,t) \in \Omega_T. \tag{4.3.15}$$

Proof. We first consider the case when

$$u_0(x) \geq ce^{-a|x|^2} \quad \forall x \in \mathbb{R}, \tag{4.3.16}$$

for constants $c, a > 0$. Note that the following identity holds

$$S_t\left(e^{a|x|^2}\right) = (1+4at)^{-\frac{1}{2}} e^{\frac{-a|x|^2}{1+4at}} \quad \forall (x,t) \in \Omega_\infty. \tag{4.3.17}$$

Since $u \geq 0$ on $\overline{\Omega}_T$ and $\varphi \geq 0$ on $\mathbb{R}$, via (4.3.14) and (4.3.17) it follows that

$$u(x,t) \geq S_t u_0(x) \geq S_t\left(ce^{-a|x|^2}\right) = c(1+4at)^{-\frac{1}{2}} e^{\frac{-a|x|^2}{1+4at}}, \tag{4.3.18}$$

for all $(x,t) \in \overline{\Omega}_T$. Substituting (4.3.18) for u into (4.3.14), recalling that $u_0 \geq 0$ and φ satisfies (4.3.3), and using Lemma 4.3.4, we additionally obtain

$$u(x,t) \geq \int_0^t S_{t-s}\left(\int_{-\infty}^{\infty} \varphi(x-y)c(1+4as)^{-\frac{1}{2}} e^{\frac{-a|y|^2}{1+4as}} dy\right)^p ds$$

$$
\begin{aligned}
&= \int_0^t S_{t-s}\left(c(1+4as)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\varphi(y)e^{\frac{-a|x-y|^2}{1+4as}}dy\right)^p ds \\
&\geq \int_0^t S_{t-s}\left(c\sigma(1+4as)^{-\frac{1}{2}}\int_{x-\delta}^{x+\delta} e^{\frac{-a|y|^2}{1+4as}}dy\right)^p ds \\
&\geq \int_0^t c^p(\sigma\delta)^p(1+4as)^{\frac{-p}{2}}e^{\frac{-ap\delta^2}{1+4as}}S_{t-s}e^{\frac{-ap|x|^2}{1+4as}}dyds, \qquad (4.3.19)
\end{aligned}
$$

for all $(x,t) \in \overline{\Omega}_T$. Utilising (4.3.17), inequality (4.3.19) can be re-written as

$$
u(x,t) \geq \int_0^t c^p(\sigma\delta)^p(1+4as)^{\frac{1-p}{2}}e^{\frac{-ap\delta^2}{1+4as}}(1+4as+4ap(t-s))^{-\frac{1}{2}}e^{\frac{-ap|x|^2}{1+4as+4ap(t-s)}}ds, \quad (4.3.20)
$$

for all $(x,t) \in \overline{\Omega}_T$. Furthermore, for $p \in (0,1)$, $0 \leq s \leq t < \infty$ and $k \in \mathbb{N}_0$ we also have

$$
1+4ap^{k+1}t \leq 1+4ap^k s+4ap^{k+1}(t-s) \leq 1+4ap^k t. \qquad (4.3.21)
$$

Using (4.3.21), with $k=0$, and (4.3.20) we update the bound of u to be

$$
\begin{aligned}
u(x,t) &\geq c^p(\sigma\delta)^p\int_0^t (1+4as)^{\frac{1-p}{2}}e^{\frac{-ap\delta^2}{1+4as}}(1+4at)^{-\frac{1}{2}}e^{\frac{-ap|x|^2}{1+4apt}}ds \qquad (4.3.22)\\
&= c^p(\sigma\delta)^p(1+4at)^{-\frac{1}{2}}e^{\frac{-ap|x|^2}{1+4apt}}\int_0^t (1+4as)^{\frac{1-p}{2}}e^{\frac{-ap\delta^2}{1+4as}}ds \qquad (4.3.23)\\
&\geq c^p(\sigma\delta)^p(1+4at)^{-\frac{1}{2}}te^{\frac{-ap|x|^2}{1+4apt}} \quad \forall (x,t) \in \overline{\Omega}_T. \qquad (4.3.24)
\end{aligned}
$$

Now, substituting (4.3.24) into (4.3.14), after using Lemma 4.3.4 and (4.3.17), we obtain

$$
\begin{aligned}
u(x,t) &\geq \int_0^t S_{t-s}\left(c^p(\sigma\delta)^p(1+4as)^{-\frac{1}{2}}s\int_{-\infty}^{\infty}\varphi(x-y)e^{\frac{-ap|y|^2}{1+4aps}}dy\right)^p ds \\
&= \int_0^t c^{p^2}(\sigma\delta)^{p^2}(1+4as)^{-\frac{p}{2}}s^pS_{t-s}\left(\int_{-\infty}^{\infty}\varphi(x-y)e^{\frac{-ap|y|^2}{1+4aps}}dy\right)^p ds \\
&\geq \int_0^t c^{p^2}(\sigma\delta)^{p^2+p}(1+4as)^{-\frac{p}{2}}s^pe^{\frac{-ap^2\delta^2}{1+4aps}}S_{t-s}e^{\frac{-ap^2|x|^2}{1+4aps}}ds \\
&= \int_0^t c^{p^2}(\sigma\delta)^{p^2+p}(1+4as)^{-\frac{p}{2}}s^pe^{\frac{-ap^2\delta^2}{1+4aps}}\frac{(1+4aps)^{\frac{1}{2}}e^{\frac{-ap^2|x|^2}{1+4aps+4ap^2(t-s)}}}{(1+4aps+4ap^2(t-s))^{\frac{1}{2}}}ds, \\
&\geq c^{p^2}(\sigma\delta)^{p^2+p}e^{-ap^2\delta^2}\int_0^t\left(\frac{1+4aps}{(1+4as)^p}\right)^{\frac{1}{2}}s^p\frac{e^{\frac{-ap^2|x|^2}{1+4aps+4ap^2(t-s)}}}{(1+4aps+4ap^2(t-s))^{\frac{1}{2}}}ds \qquad (4.3.25)
\end{aligned}
$$

for all $(x,t) \in \overline{\Omega}_T$. By applying (4.3.21), with $k=1$, and the reverse Bernoulli inequality to (4.3.25) we obtain the bound

$$u(x,t) \geq c^{p^2}(\sigma\delta)^{p^2+p}e^{-ap^2\delta^2}(1+4apt)^{-\frac{1}{2}}\frac{1}{p+1}t^{p+1}e^{\frac{-ap^2|x|^2}{1+4ap^2t}} \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.3.26}$$

By iterating this procedure, for every $k \in \mathbb{N}_0$ and $(x,t) \in \overline{\Omega}_T$ we hence obtain that

$$u(x,t) \geq c^{p^{k+1}}(\sigma\delta)^{p^{k+1}+p^k+\cdots+p}c_k(1+4ap^kt)^{-\frac{1}{2}}e^{-kap^{k+1}\delta^2}t^{p^k+p^{k-1}+\cdots+1}e^{\frac{-ap^{k+1}|x|^2}{1+ap^kt}} \tag{4.3.27}$$

where

$$\begin{aligned} c_k &= (1+p+\cdots+p^k)^{-1}(1+p+\cdots+p^{k-1})^{-p^1}\dots(1+p)^{-p^{k-1}} \\ &= \prod_{j=0}^{k-1}\left(\sum_{l=0}^{k-j}p^l\right)^{-p^j} \qquad \forall k \in \mathbb{N}. \end{aligned}$$

To show that c_k is is bounded below uniformly in k we take logarithms and obtain that

$$\log c_k = -\sum_{j=0}^{k}p^j\log\left(\sum_{l=0}^{k-j}p^l\right) \geq \log(1-p)\sum_{j=1}^{k-1}p^j \geq \frac{1}{1-p}\log(1-p),$$

and thus,

$$c_k \geq (1-p)^{\frac{1}{1-p}} \qquad \forall k \in \mathbb{N}. \tag{4.3.28}$$

Using (4.3.28) and allowing k to go to infinity, we obtain (4.3.15) for the case when $u_0(x) \geq ce^{-a|x|^2}$ (the strict inequality follows from the fact that $S_tu_0 > 0$), as specified. If however, $a,c>0$ do not exist such that $u_0 \geq ce^{-a|x|^2}$ for all $x \in \mathbb{R}$, then, since u_0 is continuous and satisfies (4.3.4), it follows from the strong maximum for the heat equation that there exist constants $a_t, c_t > 0$ (dependent on t) such that

$$u(x,t) > c_te^{-a_tx^2}, \quad \forall (x,t) \in \Omega_T. \tag{4.3.29}$$

Using (4.3.29) and the previous argument, the result follows on considering $u_0(x) = u(x,t_0)$ and letting $t_0 \to 0^+$. This completes the proof, as required. $\square$

Corollary 4.3.6. *Suppose that the conditions of Lemma 4.3.5 are satisfied. Then,*

$$u(x,t) \geq \left(\|\varphi\|^p_{L^1(\Omega)}(1-p)t\right)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.3.30}$$

Proof. Let $t_0 \in (0,T)$. Consider $v : \overline{\Omega}_{T-t_0} \to \mathbb{R}$ given by

$$v(x,t) = u(x,t+t_0), \quad \forall (x,t) \in \overline{\Omega}_{T-t_0}, \tag{4.3.31}$$

and define $f_{t_0} : \mathbb{R} \to \mathbb{R}$ to be

$$f_{t_0}(Jv) = \begin{cases} (Jv)^p, & Jv \geq \|\varphi\|_{L^1(\Omega)}((\sigma\delta)^p(1-p)t_0)^{\frac{1}{1-p}}; \\ \|\varphi\|_{L^1(\Omega)}((\sigma\delta)^p(1-p)t_0)^{\frac{1}{1-p}}, & Jv < \|\varphi\|_{L^1(\Omega)}((\sigma\delta)^p(1-p)t_0)^{\frac{1}{1-p}}. \end{cases} \tag{4.3.32}$$

Note that f_{t_0} is Lipschitz continuous and non-decreasing on $\mathbb{R}$. Since u satisfies (4.3.14), via Lemma 4.3.5 and (4.3.32), on denoting $v(x,0) = v_0(x)$, for $x \in \Omega$, it follows that v satisfies

$$\begin{aligned} v(x,t) &\geq S_t v_0(x) + \int_0^t S_{t-s}(Jv(x,s))^p ds \\ &> ((\sigma\delta)^p(1-p)t_0)^{\frac{1}{1-p}} + \int_0^t S_{t-s} f_{t_0}(Jv(x,s))ds, \end{aligned} \tag{4.3.33}$$

for all $(x,t) \in \overline{\Omega}_{T-t_0}$. Now consider $w : \overline{\Omega}_{T-t_0} \to \mathbb{R}$ given by

$$w(x,t) = \left((\sigma\delta)^p(1-p)t_0 + \|\varphi\|^p_{L^1(\Omega)}(1-p)t\right)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_{T-t_0}, \tag{4.3.34}$$

and note that

$$w(x,t) \leq ((\sigma\delta)^p(1-p)t_0)^{\frac{1}{1-p}} + \int_0^t S_{t-s} f_{t_0}(Jw(x,s))ds, \quad \forall (x,t) \in \overline{\Omega}_{T-t_0}. \tag{4.3.35}$$

Additionally, via (4.3.31) and (4.3.34)

$$v \geq w \qquad \text{on } \partial\overline{\Omega}_{T-t_0}. \tag{4.3.36}$$

Via (4.3.33), (4.3.35), (4.3.36), and the comparison principle of Theorem 2.4.3, we have

$$v \geq w \qquad \text{on } \overline{\Omega}_{T-t_0}, \tag{4.3.37}$$

or equivalently

$$u(x, t+t_0) \geq \Big((\sigma\delta)^p(1-p)t_0 + \|\varphi\|^p_{L^1(\Omega)}(1-p)t\Big)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_{T-t_0}. \tag{4.3.38}$$

By letting $t_0 \to 0^+$ in (4.3.38) we obtain (4.3.30), as required. □

To show that the Cauchy problem (4.3.1) admits a unique solution we will employ the following comparison principle.

Proposition 4.3.7. *[Comparison principle] Let $u, v \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ be non-negative and satisfy*

$$u(x,t) \geq S_t u_0(x) + \int_0^t S_{t-s}(Ju(x,s))^p ds; \tag{4.3.39}$$

$$v(x,t) \leq S_t v_0(x) + \int_0^t S_{t-s}(Jv(x,s))^p ds, \tag{4.3.40}$$

for all $(x,t) \in \Omega_T$, with Ju and Jv as in (4.3.2)-(4.3.3) and $p \in (0,1)$, and $u_0, v_0 \in L^\infty(\Omega) \cap C(\Omega)$ satisfy (4.3.4) and

$$u_0 \geq v_0 \quad \text{on } \Omega. \tag{4.3.41}$$

Then $u \geq v$ on $\overline{\Omega}_T$.

Proof. Define $g : \overline{\Omega}_T \to \mathbb{R}$ to be

$$g(x,t) = v(x,t) - u(x,t) \quad \forall (x,t) \in \overline{\Omega}_T. \tag{4.3.42}$$

We will show that $g_+ \equiv 0$ on $\overline{\Omega}_T$. Via (4.3.39)-(4.3.41), it follows that

$$\begin{aligned} g(x,t) &\leq S_t(v_0(x) - u_0(x)) + \int_0^t S_{t-s}(Jv(x,s)^p - Ju(x,s)^p)ds \\ &\leq \int_0^t S_{t-s}(Jv(x,s)^p - Ju(x,s)^p)ds \\ &\leq \int_0^t S_{t-s}(Jv(x,s) - Ju(x,s))_+^p ds \\ &= \int_0^t S_{t-s}(Jg(x,s))_+^p ds, \end{aligned} \tag{4.3.43}$$

for all $(x,t) \in \overline{\Omega}_T$. By taking the positive parts of both sides of (4.3.43) it follows that[1]

$$g_+(x,t) \leq \|\varphi\|_{L^1(\Omega)}^p \int_0^t \|g_+(\cdot,s)\|_{L^\infty(\Omega)}^p ds \quad \forall (x,t) \in \Omega_T, \tag{4.3.44}$$

and hence,

$$\|g_+(\cdot,t)\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^1(\Omega)}^p \int_0^t \|g_+(\cdot,s)\|_{L^\infty(\Omega)}^p ds \quad \forall t \in [0,T]. \tag{4.3.45}$$

Via a non-linear Grönwall type inequality (see [MPF91, Theorem 1, p. 360-361]) it follows that

$$\|g_+(\cdot,t)\|_{L^\infty(\Omega)} \leq (\|\varphi\|_{L^1(\Omega)}^p (1-p)t)^{\frac{1}{1-p}}, \tag{4.3.46}$$

for all $t \in [0,T]$. Additionally, via the mean value theorem, for each $(x,s) \in \Omega_T$, there

[1] Note that the integral in (4.3.44) is well-defined via an application of the monotone convergence theorem (see e.g. [Apo74]).

exists $\eta = \eta_{x,s}$ between $Jv(x,s)$ and $Ju(x,s)$ we have

$$(Jv(x,s))^p - (Ju(x,s))^p = p(Jv(x,s) - Ju(x,s))\eta^{p-1} = pJg(x,s)\eta^{p-1}. \tag{4.3.47}$$

When $u \leq v$, utilising Corollary 4.3.6, we obtain

$$\eta \geq Ju(x,s) \geq \|\varphi\|_{L^1(\Omega)}(\|\varphi\|^p_{L^1(\Omega)}(1-p)s)^{\frac{1}{1-p}} \quad \forall (x,s) \in \Omega_T. \tag{4.3.48}$$

Substituting (4.3.48) into (4.3.47) yields

$$((Jv(x,s))^p - (Ju(x,s))^p)_+ \leq \frac{p(J(v-u)_+(x,s))}{\|\varphi\|_{L^1(\Omega)}(1-p)}s^{-1}, \tag{4.3.49}$$

for all $(x,s) \in \Omega_T$. Now, via (4.3.49) it follows that

$$\int_0^t S_{t-s}((Jv(x,s))^p - (Ju(x,s))^p)_+ ds \leq \int_0^t S_{t-s}\frac{p(J(v-u)_+(x,s))}{\|\varphi\|_{L^1(\Omega)}(1-p)}s^{-1}ds \quad \forall (x,t) \in \Omega_T. \tag{4.3.50}$$

Upon recalling (4.3.43), inequality (4.3.50) implies that

$$g_+(x,t) \leq \int_0^t \frac{ps^{-1}}{\|\varphi\|_{L^1(\Omega)}(1-p)}S_{t-s}Jg_+(x,s)ds, \tag{4.3.51}$$

from which we obtain

$$\|g_+(\cdot,t)\|_{L^\infty(\Omega)} \leq \int_0^t \frac{p}{1-p}s^{-1}\|g_+(\cdot,s)\|_{L^\infty(\Omega)}ds \quad \forall t \in [0,T]. \tag{4.3.52}$$

Now, consider $G : [0,T] \to \mathbb{R}$ to be

$$G(t) = \frac{p}{1-p}\int_0^t s^{-1}\|g_+(\cdot,s)\|_{L^\infty(\Omega)}ds, \quad \forall t \in [0,T], \tag{4.3.53}$$

noting that, via (4.3.44), G is well-defined. Using (4.3.53), (4.3.52) can be re-written as

$$\frac{G'(t)}{G(t)} \leq \frac{p}{1-p} t^{-1}, \quad \forall t \in (0,T]. \tag{4.3.54}$$

For any $\varepsilon \in (0,T)$, it follows from integrating (4.3.54) that

$$G(t) \leq G(\varepsilon)\varepsilon^{-\frac{p}{1-p}} t^{\frac{p}{1-p}}, \tag{4.3.55}$$

for all $t \in [\varepsilon, T]$. Note that, via (4.3.46) and (4.3.53), it follows that

$$G(\varepsilon) \leq \frac{p}{1-p} \int_0^{\varepsilon} s^{-1} \|\varphi\|_{L^1(\Omega)}^{\frac{p}{1-p}} (1-p)^{\frac{1}{1-p}} s^{\frac{1}{1-p}} ds = p\|\varphi\|_{L^1(\Omega)}^{\frac{p}{1-p}} (1-p)^{\frac{1}{1-p}} \varepsilon^{\frac{1}{1-p}}, \tag{4.3.56}$$

for all $\varepsilon > 0$. By substituting (4.3.56) into (4.3.55) we obtain

$$G(t) \leq p\|\varphi\|_{L^1(\Omega)}^{\frac{p}{1-p}} (1-p)^{\frac{1}{1-p}} \varepsilon^{\frac{1}{1-p}} \varepsilon^{-\frac{p}{1-p}} t^{\frac{p}{1-p}} = p\|\varphi\|_{L^1(\Omega)}^{\frac{p}{1-p}} (1-p)^{\frac{1}{1-p}} t^{\frac{p}{1-p}} \varepsilon, \tag{4.3.57}$$

for all $t \in [\varepsilon, T]$. Letting $\varepsilon \to 0$ shows that $G \equiv 0$ on $[0,T]$ which, via (4.3.53) and (4.3.42), yields

$$u(x,t) \geq v(x,t), \quad \forall (x,t) \in \overline{\Omega}_T, \tag{4.3.58}$$

as required. □

Corollary 4.3.8. *Let $u, v \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ be non-negative and satisfy*

$$\partial_t u \geq \partial_{xx} u + (Ju)^p, \quad \text{on } \Omega_T; \tag{4.3.59}$$

$$\partial_t v \leq \partial_{xx} v + (Jv)^p, \quad \text{on } \Omega_T; \tag{4.3.60}$$

$$u(x,0) = u_0(x) \geq v_0(x) = v(x,0), \quad \forall x \in \Omega, \tag{4.3.61}$$

where Ju, Jv are as in (4.3.2)-(4.3.3), $u_0, v_0 \in L^\infty(\Omega) \cap C(\Omega)$ and satisfy (4.3.4) and $p \in (0,1)$. Then, $u \geq v$ on $\overline{\Omega}_T$.

Proof. Via a Duhamel principle, if u and v satisfy (4.3.59)-(4.3.61), then they satisfy the conditions of Proposition 4.3.7, from which the result follows, as required. □

Additionally we have

Corollary 4.3.9. *Let $u : \overline{\Omega}_\infty \to \mathbb{R}$ be a global solution to $(CP)_+$. Then*

$$\left(\mu_0^{1-p} + \|\varphi\|_{L^1(\Omega)}^p (1-p)t\right)^{\frac{1}{1-p}} \leq u(x,t) \leq \left(M_0^{1-p} + \|\varphi\|_{L^1(\Omega)}^p (1-p)t\right)^{\frac{1}{1-p}}, \qquad (4.3.62)$$

for all $t > 0$.

Proof. The upper bound on u follows directly from Proposition 4.3.1 and Corollary 4.3.8. If $\mu_0 > 0$, then, the lower bound on u follows on considering the sub-solution $v : \overline{\Omega}_T \to \mathbb{R}$ given by

$$v(x,t) = \left(\mu_0^{1-p} + \|\varphi\|_{L^1(\Omega)}^p (1-p)t\right)^{\frac{1}{1-p}}, \quad \forall (x,t) \in \overline{\Omega}_T,$$

for any $T > 0$ and by applying the comparison principle of Corollary 4.3.8. Note that if $\mu_0 = 0$, then, the lower bound follows from Corollary 4.3.6. □

Theorem 4.3.10 (Uniqueness)**.** *$(CP)_+$ has a unique global solution.*

Proof. Let $u_1, u_2 : \overline{\Omega}_\infty \to \mathbb{R}$ both be solutions $(CP)_+$ with initial data u_0 and integral kernel φ. Via Corollary 4.3.8 it follows that

$$u_1(x,t) \leq u_2(x,t) \leq u_1(x,t), \quad \forall (x,t) \in \overline{\Omega}_T,$$

for all $T > 0$. Hence $u_1 = u_2$ on $\overline{\Omega}_\infty$, as required. □

We now demonstrate local-in-time continuous dependence for solutions to $(CP)_+$.

Theorem 4.3.11 (Local-in-time continuous dependence)**.** *Let $u : \overline{\Omega}_\infty \to \mathbb{R}$ be the solution to $(CP)_+$. Moreover, let $\tilde{u} : \overline{\Omega}_\infty \to \mathbb{R}$ be the solution to $(CP)_+$ with initial data $\tilde{u}_0 \in$*

$L^\infty(\Omega)\cap C(\Omega)$ and integral kernel $\tilde{\varphi}\in L^1(\Omega)$. Then, for all $\varepsilon>0$ and $T>0$, there exists $\delta>0$ (that is independent of $\tilde{u}$) such that, whenever

$$\|u_0-\tilde{u}_0\|_{L^\infty(\Omega)}<\delta, \quad \textit{and} \quad \|\varphi-\tilde{\varphi}\|_{L^1(\Omega)}<\delta, \tag{4.3.63}$$

then

$$\|u-\tilde{u}\|_{L^\infty(\overline{\Omega}_T)}<\varepsilon. \tag{4.3.64}$$

Proof. Let $\varepsilon>0$. Without loss of generality suppose that $\|\tilde{\varphi}\|_{L^1(\Omega)}\leq\|\varphi\|_{L^1(\Omega)}$. Via (4.3.63), for $\delta<\|\varphi\|_{L^1(\Omega)}/2$, it follows that $\|\tilde{\varphi}\|_{L^1(\Omega)}\geq\|\varphi\|_{L^1(\Omega)}-\delta>0$, and hence $\|\tilde{\varphi}\|_{L^1(\Omega)}$ can be controlled by $\|\varphi\|_{L^1(\Omega)}$. Denote

$$M=\max\{\|u\|_{L^\infty(\overline{\Omega}_T)},\|\tilde{u}\|_{L^\infty(\overline{\Omega}_T)}\}$$

and observe, via (4.3.63) by considering $\delta<1$, that M can be seen to be bounded by a term only dependent on $\|u_0\|_{L^\infty(\Omega)}$, T, $\|\varphi\|_{L^1(\Omega)}$ and p, via Corollary 4.3.9.

Since $f:\mathbb{R}\to\mathbb{R}$ is uniformly Hölder continuous, via Lemma 4.2.14 and (4.3.63) it follows that

$$\begin{aligned}
|u(x,t)-\tilde{u}(x,t)| &< \delta+\int_0^t\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{\pi}}\left|Ju(x+2\sqrt{t-s}z,s)^p-\tilde{J}\tilde{u}(x+2\sqrt{t-s}z,s)^p\right|dzds\\
&\leq \delta+\int_0^t\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{\pi}}\left|Ju(x+2\sqrt{t-s}z,s)^p-\tilde{J}u(x+2\sqrt{t-s}z,s)^p\right|dzds\\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{\pi}}\left|\tilde{J}u(x+2\sqrt{t-s}z,s)^p-\tilde{J}\tilde{u}(x+2\sqrt{t-s}z,s)^p\right|dzds\\
&= \delta+\int_0^t\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{\pi}}\left|(J-\tilde{J})u(x+2\sqrt{t-s}z,s)\right|^p dzds\\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{\pi}}\left|\tilde{J}(u-\tilde{u})(x+2\sqrt{t-s}z,s)\right|^p dzds, \quad \forall(x,t)\in\overline{\Omega}_T,
\end{aligned} \tag{4.3.65}$$

where $\tilde{J}$ denotes J in (4.3.2) with φ replaced with $\tilde{\varphi}$. It follows that

$$\begin{aligned}\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \leq \delta + &\int_0^t \|\varphi-\tilde{\varphi}\|^p_{L^1(\Omega)}\|u(\cdot,s)\|^p_{L^\infty(\Omega)}ds \\ &+\int_0^t \|\tilde{\varphi}\|^p_{L^1(\Omega)}\|u(\cdot,s)-\tilde{u}(\cdot,s)\|^p_{L^\infty(\Omega)}ds,\end{aligned} \tag{4.3.66}$$

for all $t\in[0,T]$. Recalling (4.3.63), (4.3.66) becomes

$$\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \leq \delta + \delta^p M^p T + \|\tilde{\varphi}\|^p_{L^1(\Omega)}\int_0^t \|u(\cdot,s)-\tilde{u}(\cdot,s)\|^p_{L^\infty(\Omega)}ds, \tag{4.3.67}$$

for all $t\in[0,T]$. Applying the non-linear Grönwall's inequality to the inequality in (4.3.67) further yields

$$\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \leq \left((\delta+\delta^pM^pT)^{1-p} + \|\tilde{\varphi}\|^p_{L^1(\Omega)}(1-p)t\right)^{\frac{1}{1-p}}, \tag{4.3.68}$$

for all $t\in[0,T]$. In particular, for all sufficiently small δ we introduce

$$t_0 = \frac{(\delta+\delta^pM^pT)^{1-p}}{\|\tilde{\varphi}\|^p_{L^1(\Omega)}(1-p)} \in (0,T), \tag{4.3.69}$$

for which, we have

$$\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \leq 2^{\frac{1}{1-p}}(\delta+\delta^pM^pT), \quad \forall t\in[0,t_0]. \tag{4.3.70}$$

Furthermore, via the mean value theorem, and utilising the lower bound from Corollary 4.3.9, we have

$$\begin{aligned}&\left|\tilde{J}u(\xi,\tau)^p - \tilde{J}\tilde{u}(\xi,\tau)^p\right| \\ &\leq p\left((\|\tilde{\varphi}\|^p_{L^1(\Omega)}(1-p)\tau)^{\frac{1}{1-p}}\|\tilde{\varphi}\|_{L^1(\Omega)}\right)^{p-1}\|\tilde{\varphi}\|_{L^1(\Omega)}\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \\ &= \frac{p}{(1-p)}\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)}\tau^{-1}, \quad \forall(\xi,\tau)\in\Omega_T.\end{aligned} \tag{4.3.71}$$

We now consider $t \in [t_0, T]$. Substitution of (4.3.69)-(4.3.70) into (4.3.65) yields

$$\begin{aligned}
&\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \\
&\leq \delta + \delta^p M^p T + \int_0^{t_0} \|\tilde{J}(u-\tilde{u})(\cdot,s)\|^p_{L^\infty(\Omega)} ds + \int_{t_0}^t \|(\tilde{J}u^p - \tilde{J}\tilde{u}^p)(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&\leq \delta + \delta^p M^p T + \frac{2^{\frac{p}{1-p}}(\delta + \delta^p M^p T)}{1-p} + \int_{t_0}^t \|(\tilde{J}u^p - \tilde{J}\tilde{u}^p)(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&:= C + \int_{t_0}^t \|(\tilde{J}u^p - \tilde{J}\tilde{u}^p)(\cdot,s)\|_{L^\infty(\Omega)} ds
\end{aligned} \tag{4.3.72}$$

for all $t \in [0, T]$ for $C = C_{\delta,M,T,p}$ given by

$$C = \delta + \delta^p M^p T + \frac{2^{\frac{p}{1-p}}(\delta + \delta^p M^p T)}{1-p}. \tag{4.3.73}$$

Note that, for fixed M, T and p, we have $C = O(\delta^p)$, as $\delta \to 0^+$. Substitution of (4.3.71) into (4.3.72) yields

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty(\Omega)} \leq C + \frac{p}{1-p} \int_{t_0}^t \|u(\cdot,s) - \tilde{u}(\cdot,s)\|_{L^\infty(\Omega)} s^{-1} ds, \tag{4.3.74}$$

for all $t \in [t_0, T]$. Define $g : [t_0, T] \to \mathbb{R}$ to be

$$g(t) = \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty(\Omega)}, \quad \forall t \in [t_0, T]. \tag{4.3.75}$$

After substituting (4.3.75) into (4.3.74), via Grönwall's inequality we have

$$g(t) \leq C t^{\frac{p}{1-p}} t_0^{-\frac{p}{1-p}} \leq C T^{\frac{p}{1-p}} t_0^{-\frac{p}{1-p}}, \quad \forall t \in [t_0, T]. \tag{4.3.76}$$

Upon substitution of (4.3.69) into (4.3.76), it follows that

$$g(t) \leq C(\|\tilde{\varphi}\|^p_{L^1(\Omega)}(1-p)T)^{\frac{p}{1-p}}(\delta + \delta^p M^p T)^{-p}, \tag{4.3.77}$$

for all $t \in [t_0, T]$. Thus, via (4.3.73), it follows from (4.3.77) that

$$g(t) = O(\delta^{p(1-p)}), \quad \text{as } \delta \to 0^+, \tag{4.3.78}$$

uniformly for $t \in [t_0, T]$. By letting $\delta \to 0^+$ in (4.3.69), it follows from (4.3.78), (4.3.75) and (4.3.70) that there exists $\delta > 0$, sufficiently small, so that (4.3.64) holds, as required.

□

Theorem 4.3.12. *$(CP)_+$ is well-posed, locally in time.*

Proof. Follows directly from Theorem 4.3.10 and Theorem 4.3.11. □

4.4 Numerical approximation of $(CP)_+$

For illustrative purposes, we consider the qualitative properties of the solution to $(CP)_+$ with

$$0 < \mu_0 = \inf_{x \in \mathbb{R}} u_0 < \sup_{x \in \mathbb{R}} u_0 = M_0 \quad \text{and} \quad u_0 \in W^{4,\infty}(\mathbb{R}) \cap C^4(\mathbb{R}), \tag{4.4.1}$$

and integral kernel $\varphi = \varphi_\sigma$ given by

$$\varphi_\sigma(x) = \begin{cases} \sigma, & \text{if } |x| \le \frac{1}{2\sigma}; \\ 0, & \text{otherwise,} \end{cases} \tag{4.4.2}$$

and note that $\|\varphi\|_{L^1(\mathbb{R})} = 1$.

By taking any kernel that is sufficiently close to φ_σ, the continuous dependence result Theorem 4.3.11 implies that solutions will be sufficiently close to one another. A similar conclusion is valid for initial data that are sufficiently close to each other.

For a boundary value problem related to $(CP)_+$, we establish that its explicit finite difference approximation scheme converges, and hence, can provide a uniform approximation to the solution to $(CP)_+$, on a truncated domain.

Let $u : \overline{\Omega}_\infty \to \mathbb{R}$ be the solution to $(CP)_+$ and $\overline{u}, \underline{u} : \overline{\Omega}_\infty \to \mathbb{R}$ be the upper and lower bounds of u, respectively, as given in Corollary 4.3.9. Now, define $v : \overline{\Omega}_\infty \to [0,1]$ to be

$$v = \frac{u - \underline{u}}{\overline{u} - \underline{u}} \quad \text{on } \overline{\Omega}_\infty. \tag{4.4.3}$$

It follows immediately that

$$v \in L^\infty(\overline{\Omega}_\infty) \cap C(\overline{\Omega}_\infty) \cap C^{2,1}(\Omega_\infty), \tag{4.4.4}$$

and moreover, that

$$v_t - v_{xx} = \frac{1}{\overline{u} - \underline{u}} \left[(J(v(\overline{u} - \underline{u}) + \underline{u}))^p - \underline{u}^p - v(\overline{u}^p - \underline{u}^p) \right], \quad \text{on } \Omega_\infty. \tag{4.4.5}$$

Moreover, for $v_0 : \mathbb{R} \to [0,1]$, we have $v_0(x) = v(x,0)$, for all $x \in \mathbb{R}$, and assume that

$$\operatorname{supp} v_0 \subseteq [-X_0, X_0] \quad \text{and} \quad v_0 \in C^4(\mathbb{R}), \tag{4.4.6}$$

for some $X_0 > 0$. In this scenario, we have the following proposition.

Proposition 4.4.1. *Let $v : \overline{\Omega}_T \to \mathbb{R}$ satisfy (4.4.4)-(4.4.6). Then,*

$$\lim_{|x| \to \infty} v(x,t) = 0, \tag{4.4.7}$$

uniformly for $t \in [0,T]$.

Proof. Observe that, via the mean value theorem, for each $(x,t) \in \Omega_T$, there exists a $\theta \in (\underline{u}, \underline{u} + (\overline{u} - \underline{u})Jv)$ such that

$$\begin{aligned} &\frac{1}{\overline{u} - \underline{u}} \left[(J(v(\overline{u} - \underline{u}) + \underline{u}))^p - \underline{u}^p - v(\overline{u}^p - \underline{u}^p) \right] \\ &\quad = \frac{1}{\overline{u} - \underline{u}} \left[p\theta^{p-1}(\overline{u} - \underline{u})Jv - v(\overline{u}^p - \underline{u}^p) \right] \\ &\quad \leq p\underline{u}^{p-1}Jv. \end{aligned} \tag{4.4.8}$$

Hence via, (4.4.5) and (4.4.8), it follows that

$$v_t + v_{xx} - p\underline{u}^{p-1}Jv \le 0 \quad \text{on } \Omega_T. \tag{4.4.9}$$

Moreover, consider $\overline{v} : \overline{\Omega}_T \to \mathbb{R}$ given by

$$\overline{v}(x,t) = \frac{ce^{kt}}{1+x^2} \quad \forall (x,t) \in \overline{\Omega}_T, \tag{4.4.10}$$

with

$$k = 8 + \frac{p}{\mu_0^{1-p}}\left(1 + \frac{5}{2\min\{\sigma,\sigma^2\}}\right) \quad \text{and} \quad c = 1 + X_0^2. \tag{4.4.11}$$

We note that $\overline{v} \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ with

$$\overline{v}_t(x,t) = k\overline{v}(x,t); \tag{4.4.12}$$

$$\overline{v}_{xx}(x,t) = \overline{v}(x,t)\left(\frac{8x^2}{(1+x^2)^2} - \frac{2}{1+x^2}\right), \tag{4.4.13}$$

for all $(x,t) \in \Omega_T$. Observe that

$$\begin{aligned} J\overline{v}(x,t) &= \sigma \int_{-\frac{1}{2\sigma}}^{\frac{1}{2\sigma}} \overline{v}(x-y,t)dy \\ &= \sigma c e^{kt} \int_{-\frac{1}{2\sigma}}^{\frac{1}{2\sigma}} \frac{dy}{1+(x-y)^2} \\ &\le \sigma c e^{kt} \int_{-\frac{1}{2\sigma}}^{\frac{1}{2\sigma}} \frac{1}{1+x^2} + \frac{|y||2x-y|}{(1+x^2)(1+(x-y)^2)} dy \\ &= \sigma \overline{v}(x,t) \int_{-\frac{1}{2\sigma}}^{\frac{1}{2\sigma}} 1 + \frac{|y||2x-y|}{1+(x-y)^2} dy, \end{aligned} \tag{4.4.14}$$

for all $(x,t) \in \Omega_T$. For $|x| \le \sigma^{-1}$, the integral on the right hand side of (4.4.14) is bounded above by

$$\int_{-\frac{1}{2\sigma}}^{\frac{1}{2\sigma}} 1 + \frac{|y||2x-y|}{1+(x-y)^2} dy \le \frac{1}{\sigma}\left(1 + \frac{5}{4\sigma^2}\right), \tag{4.4.15}$$

and, for $|x| > \sigma^{-1}$, by noting that $|y| \leq (2\sigma)^{-1} \leq |x|/2$, the integral on the right hand side of (4.4.14) is bounded above by

$$\int_{-\frac{1}{2\sigma}}^{\frac{1}{2\sigma}} 1 + \frac{|y||2x-y|}{1+(x-y)^2} dy \leq \frac{1}{\sigma}\left(1 + \frac{5}{2\sigma}\right). \tag{4.4.16}$$

Via (4.4.15)-(4.4.16), it follows from (4.4.14) that

$$J\overline{v}(x,t) \leq \left(1 + \frac{5}{2\min\{\sigma, \sigma^2\}}\right)\overline{v}(x,t), \tag{4.4.17}$$

for all $(x,t) \in \Omega_T$. Thus, via (4.4.12)-(4.4.13), (4.4.17) and (4.4.11) it follows that

$$\overline{v}_t - \overline{v}_{xx} - p\underline{u}^{p-1}J\overline{v} \geq 0 \quad \text{on } \Omega_T. \tag{4.4.18}$$

Moreover, via (4.4.11) it follows that

$$\overline{v} \geq v \quad \text{on } \partial\Omega_T. \tag{4.4.19}$$

Therefore, via (4.4.9), (4.4.18) and (4.4.19), by applying the comparison principle of Theorem 2.4.3, it follows that

$$v \leq \overline{v} \quad \text{on } \overline{\Omega}_T,$$

and hence (4.4.7) is satisfied, as required. □

We now consider a numerical approximation of (4.4.5)-(4.4.7) by considering the grid

$$\left\{(x_i, t_j) \in \mathbb{R}^2 : \ x_i = -X + (i-1)\delta x, \ t_j = (j-1)\delta t, \ i = 1, \dots, N_x, \ j = 1, \dots, N_t\right\}, \tag{4.4.20}$$

for

$$\delta x = \frac{2X}{N_x - 1} \quad \text{and} \quad \delta t = \frac{T}{N_t - 1}.$$

On the grid we denote the numerical approximation as v_i^j. To impose property (4.4.7) on the approximation, we consider X to be sufficiently large and set

$$v_1^j = v_{N_x}^j = 0 \quad \forall 1 \leq j \leq N_t. \tag{4.4.21}$$

We represent the initial data with compact support on $[-X, X]$ as $v_i^1 \in [0,1]$, for all $1 < i < N_x$. The integro-differential equation (4.4.5) is approximated with

$$\begin{aligned} v_i^{j+1} =& v_i^j + \delta t \left(\frac{v_{i-1}^j - 2v_i^j + v_{i+1}^j}{\delta x^2} \right) \\ &+ \frac{\delta t}{\overline{u}^j - \underline{u}^j} \left([J_i^j v(\overline{u}^j - \underline{u}^j) + \underline{u}^j]^p - \underline{u}^{j^p} - v_i^j (\overline{u}^{j^p} - \underline{u}^{j^p}) \right), \end{aligned} \tag{4.4.22}$$

for all $1 < i < N_x$ and $1 \leq j < N_t$ with: $\overline{u}^j$ and $\underline{u}^j$ the functions $\overline{u}$ and $\underline{u}$ evaluated at t_j, respectively; $J_i^j v$ given by

$$J_i^j v = \sum_{\mathcal{X}_i} \left(\frac{v_i^j + v_{i+1}^j}{2} \right) \delta x \sigma, \tag{4.4.23}$$

with

$$\mathcal{X}_i = \left\{ l \in \mathbb{Z} : \ -\frac{1}{2\sigma} \leq -X + (l-1)\delta x - x_i \leq \frac{1}{2\sigma} \right\},$$

where we denote $v_i^j = 0$ if $i < 1$ or $i > N_x$ and $1 \leq j \leq N_t$; and we ensure that

$$\frac{1}{2\sigma} = M \delta x, \tag{4.4.24}$$

for some $M \in \mathbb{N}$, so that $\|\varphi_\sigma\|_{L^1(\mathbb{R})} = 1$ is precisely represented in the numerical approximation. We now establish bounds on v_i^j.

Proposition 4.4.2. *Let v_i^j be the solution to (4.4.21)-(4.4.24). Then, $v_i^j \in [0,1]$ for all*

$1 \leq i \leq N_x$ *and* $1 \leq j \leq N_t$*, provided that*

$$\delta t \leq \left\{ \frac{\delta x^2}{4}, \frac{\mu_0^{1-p}}{2p} \right\}. \tag{4.4.25}$$

Proof. We establish the result via mathematical induction, namely, we consider the statement $P(j)$ given by

$$0 \leq v_i^j \leq 1 \quad \forall 1 \leq i \leq N_x,$$

for each $1 \leq j \leq N_t$. Via the initial condition on v_i^1 it follows that $P(1)$ is true. Now, suppose that $P(j)$ is true for some $1 \leq j < N_t$. Then, via (4.4.25)

$$\begin{aligned} v_i^{j+1} &\leq v_i^j + \frac{\delta t}{\delta x^2}(v_{i-1}^j - 2v_i^j + v_{i+1}^j) + \frac{\delta t}{\overline{u}^j - \underline{u}^j}(\overline{u}^{j^p} - \underline{u}^{j^p})(1 - v_i^j) \\ &\leq v_i^j + \frac{\delta t}{\delta x^2}(v_{i-1}^j - 2v_i^j + v_{i+1}^j) + \delta t p \underline{u}^{j^{p-1}}(1 - v_i^j) \\ &\leq v_i^j + \frac{\delta t}{\delta x^2}(v_{i-1}^j - 2v_i^j + v_{i+1}^j) + \frac{1}{2}(1 - v_i^j) \\ &= v_i^j \left(\frac{1}{2} - \frac{2\delta t}{\delta x^2} \right) + (v_{i-1}^j + v_{i+1}^j)\frac{\delta t}{\delta x^2} + \frac{1}{2} \end{aligned} \tag{4.4.26}$$

for all $1 < i < N_x$. Via (4.4.25) the coefficients of v_i^j, v_{i+1}^j and v_{i-1}^j on the right hand side of (4.4.26) are non-negative, and hence

$$u_i^{j+1} \leq \frac{1}{2} \max_{1 \leq i \leq N_x} \{v_i^j\} + \frac{1}{2} \leq 1. \tag{4.4.27}$$

Similarly, via (4.4.25)

$$\begin{aligned} v_i^{j+1} &\geq v_i^j + \frac{\delta t}{\delta x^2}(v_{i-1}^j - 2v_i^j + v_{i+1}^j) - \frac{\delta t}{\overline{u}^j - \underline{u}^j} v_i^j (\overline{u}^{j^p} - \underline{u}^{j^p}) \\ &\geq v_i^j + \frac{\delta t}{\delta x^2}(v_{i-1}^j - 2v_i^j + v_{i+1}^j) - \delta t p \underline{u}^{j^{p-1}} v_i^j \\ &= v_i^j + \frac{\delta t}{\delta x^2}(v_{i-1}^j - 2v_i^j + v_{i+1}^j) - \frac{v_i^j}{2} \\ &\geq v_i^j \left(\frac{1}{2} - \frac{2\delta t}{\delta x^2} \right) + (v_{i-1}^j + v_{i+1}^j)\frac{\delta t}{\delta x^2} \end{aligned}$$

$$\geq \frac{1}{2} \min_{1\leq i\leq N_x} \{v_i^j\}$$
$$\geq 0 \tag{4.4.28}$$

It follows from (4.4.27) and (4.4.28) that $P(j+1)$ is true. The result now follows via mathematical induction, as required. $\square$

We note that in the practical implementation of this method we can ensure $v_i^{j+1} \in [0,1]$ provided that

$$\delta t < \min\left\{\frac{\delta x^2}{4}, \frac{\underline{u}^j}{2p}\right\},$$

if solving using using the known values of v_i^j.

Before we establish the conditional convergence of v_i^j as $\delta t, \delta x \to 0$, we note the truncation error of (4.4.22) in the following proposition.

Proposition 4.4.3. *Let $v : \overline{\Omega}_T \to \mathbb{R}$ be the solution to (4.4.4)-(4.4.6), and let $\hat{v}_i^j$ denote $v(x_i, t_i)$ for each $1 \leq i \leq N_x$ and $1 \leq j \leq N_t$. Then*

$$\begin{aligned}\hat{v}_i^{j+1} =& \hat{v}_i^j + \delta t \left(\frac{\hat{v}_{i-1}^j - 2\hat{v}_i^j + \hat{v}_{i+1}^j}{\delta x^2}\right) \\ &+ \frac{\delta t}{\overline{u}^j - \underline{u}^j}\left([J_i^j \hat{v}(\overline{u}^j - \underline{u}^j) + \underline{u}^j]^p - \underline{u}^{j^p} - \hat{v}_i^j(\overline{u}^{j^p} - \underline{u}^{j^p})\right) + \sigma_i^j,\end{aligned} \tag{4.4.29}$$

with the truncation error σ_i^j such that

$$|\sigma_i^j| \leq c \min\{\delta t^2, \delta x^2 \delta t\},$$

with c dependant on $\|\partial_t^2 v\|_{L^\infty(\overline{\Omega}_T)}$, $\|\partial_x^2 v\|_{L^\infty(\overline{\Omega}_T)}$, $\|\partial_x^4 v\|_{L^\infty(\overline{\Omega}_T)}$ and μ_0^{-1}.

Proof. First note that for $f : (0,\infty) \to \mathbb{R}$ given by

$$f(w) = w^p \quad \forall w \in (0,\infty),$$

it follows that $f \in C^\infty((0,\infty))$. Moreover, by recalling (4.4.1), it follows from Proposition

3.3.13 and Corollary 4.3.9 that

$$\partial_t^2 u, \partial_x^3 u, \partial_x^4 u \in L^\infty(\Omega_T) \cap C(\Omega_T),$$

and hence, via (4.4.3), it follows that

$$\partial_t^2 v, \partial_x^3 v, \partial_x^4 v \in L^\infty(\Omega_T) \cap C(\Omega_T). \tag{4.4.30}$$

Using (4.4.30), (4.4.29) follows by applying Taylor's theorem to v, centred at (x_i, t_j), on noting that the truncation error for the trapezium rule in (4.4.23) is bounded by $c\delta x^2$ with constant c depending on $\|\partial_x^2 v\|_{L^\infty(\overline{\Omega}_T)}$ and σ. This completes the proof, as required. □

We can now establish the following convergence result.

Theorem 4.4.4. *Let $\varepsilon > 0$, $v : \overline{\Omega}_T \to \mathbb{R}$ be the unique solution to (4.4.4)-(4.4.7), $\hat{v}_i^j = v(x_i, t_j)$, v_i^j be the solution to (4.4.21)-(4.4.24) and $e_i^j = \hat{v}_i^j - v_i^j$. Then, there exists $X > 0$ such that*

$$|e_i^j| < \varepsilon \quad \forall 1 \le i \le N_x \text{ and } 1 \le j \le N_t,$$

for all sufficiently small δx and

$$\delta t \le \min\left\{\frac{\delta x^2}{4}, \frac{\mu_0^{1-p}}{2p}\right\}. \tag{4.4.31}$$

Proof. Via Proposition 4.4.1, for any $\varepsilon' > 0$, we can choose X sufficiently large so that $|e_1^j|, |e_{N_x}^j| < \varepsilon'/2$, for all $1 \le j \le N_t$. Moreover, we denote

$$w^j = \max_{1 \le i \le N_x} \left\{|e_i^j|, \frac{\varepsilon'}{2}\right\},$$

and note that $w^1 = \varepsilon'/2$, by setting $v_i^1 = \hat{v}_i^1$. Now, via (4.4.22) and Proposition 4.4.3, it follows that

$$\begin{aligned}
e_i^{j+1} &= e_i^j + \frac{\delta t}{\delta x^2}(e_{i-1}^j - 2e_i^j + e_{i+1}^j) \\
&\quad + \frac{\delta t}{\overline{u}^j - \underline{u}^j}[(J_i^j \hat{v}(\overline{u}^j - \underline{u}^j) + \underline{u}^j)^p - (J_i^j v(\overline{u}^j - \underline{u}^j) + \underline{u}^j)^p - e_i^j(\overline{u}^{j^p} - \underline{u}^{j^p})] + \sigma_i^j \\
&= e_i^j \left(1 - \frac{2\delta t}{\delta x^2} - \frac{\delta t}{\overline{u}^j - \underline{u}^j}(\overline{u}^{j^p} - \underline{u}^{j^p})\right) + \frac{\delta t}{\delta x^2}(e_{i-1}^j + e_{i+1}^j) + J_i^j e \delta t p \theta_i^{j^{p-1}} + \sigma_i^j,
\end{aligned} \tag{4.4.32}$$

for $1 < i < N_x$ and $1 \le j < N_t$, with θ_i^j given via the mean value theorem. Via (4.4.31), the coefficients of e_i^j, e_{i-1}^j, e_{i+1}^j and $J_i^j e$, on the right hand side of (4.4.32) are non-negative. Hence, via (4.4.31) and Proposition 4.4.3 we obtain

$$\begin{aligned}
e_i^{j+1} &\ge \min_{1 \le i \le N_x} \{e_i^j\} \left(1 - \delta t \frac{\overline{u}^{j^p} - \underline{u}^{j^p}}{\overline{u}^j - \underline{u}^j} + \delta t p \theta_i^{j^{p-1}}\right) + \sigma_i^j \\
&\ge -w^j \left(1 + \frac{\delta t p}{\mu_0^{1-p}}\right) - c\delta x^2 \delta t,
\end{aligned} \tag{4.4.33}$$

for all $1 < i < N_x$ and $1 \le j < N_t$, for a constant c independent of i and j. Proceeding as in (4.4.33) it also follows that

$$e_i^{j+1} \le w^j \left(1 + \frac{\delta t p}{\mu_0^{1-p}}\right) + c\delta x^2 \delta t, \tag{4.4.34}$$

for all $1 < i < N_x$ and $1 \le j < N_t$. Combining (4.4.33) and (4.4.34) it follows that

$$w^{j+1} \le w^j + \frac{\delta t p}{\mu_0^{1-p}} w^j + c\delta x^2 \delta t, \tag{4.4.35}$$

for all $1 \le j < N_t$. Summing (4.4.35) for $j = 1, \ldots, N-1 < N_t$, it follows that

$$w^N \le w^1 + \sum_{j=1}^{N-1} \left(\frac{\delta t p}{\mu_0^{1-p}} w^j\right) + c\delta x^2 T. \tag{4.4.36}$$

Now, by applying the discrete Grönwall inequality to the inequality in (4.4.36) (see [Hol09,

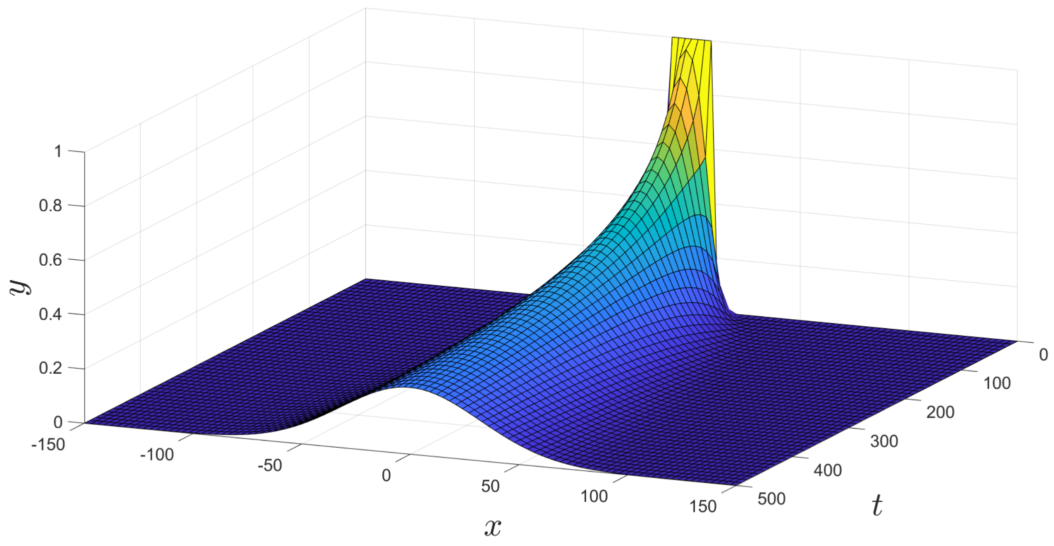

Figure 4.1: An approximation of $y = v(x,t)$. Here $p = 0.1$, $T = 500$, $X = 150$, $N_x = 1501$, $\mu_0 = 1$, $M_0 = 2$, $\sigma = 1/2$ and $v_0(x) = \mathbb{1}_{[-10,10]}(x)$.

Section 5]) we obtain

$$w^N \leq (w^1 + c\delta x^2 T)\exp\left(\sum_{j=1}^{N-1} \frac{\delta t p}{\mu_0^{1-p}}\right) \leq (w^1 + c\delta x^2 T)\exp\left(\frac{pT}{\mu^{1-p}}\right), \tag{4.4.37}$$

for each $1 < N \leq N_t$. By selecting

$$\varepsilon' = \varepsilon \exp\left(-\frac{pT}{\mu^{1-p}}\right) \text{ and } \delta x < \sqrt{\frac{\varepsilon'}{2cT}},$$

it follows, from (4.4.37) that

$$w^N \leq \varepsilon$$

for all $1 \leq N \leq N_t$, which completes the proof, as required. □

Remark 4.4.5. We note that the approach utilised here to establish conditional convergence is standard and illustrated in the local case in [LST94]. Moreover, as in [LST94], Von-Neumann type stability can be considered, albeit, we omit the details here for brevity.

We also note that in all other simulations we have considered, with: initial data of the form $\mathbb{1}_{[-m,m]}$; and various parameters p, μ_0, M_0 and σ in their respective ranges, the

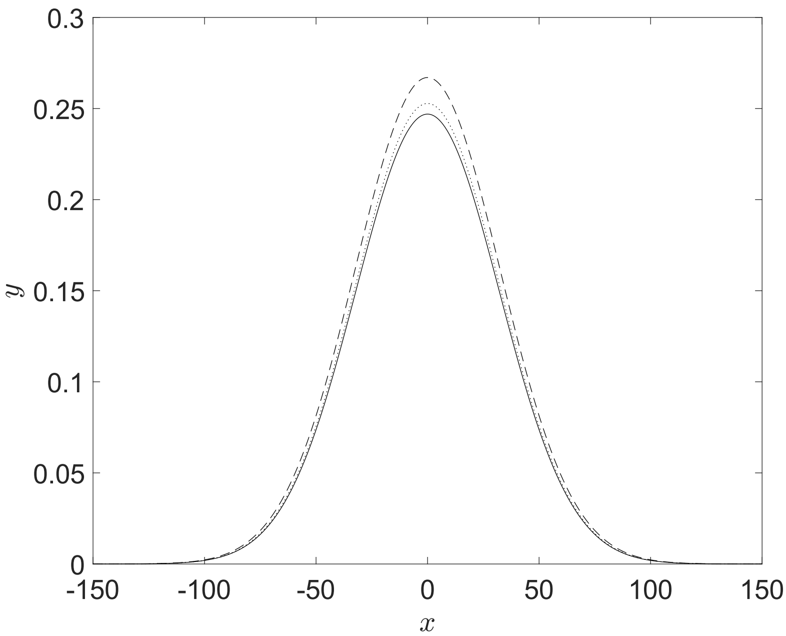

Figure 4.2: A comparison of numerical approximations of $v(x, 500)$ for $p = 0.1$, 0.5 and 0.9 (from lowest to highest respectively). The other parameters are as in Figure 4.1.

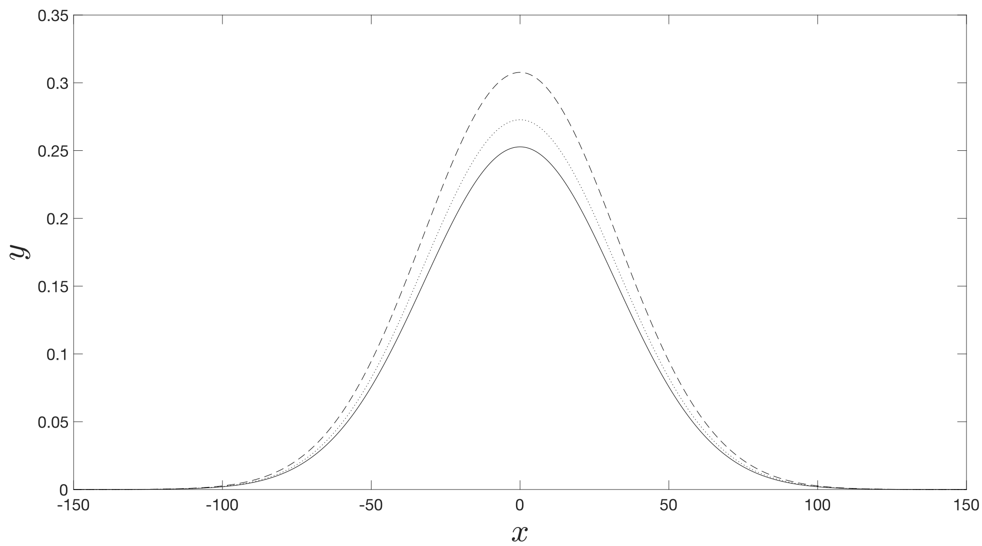

Figure 4.3: A comparison of numerical approximations of $v(x, 500)$ for $M_0 = 2$, 10 and 100 (from lowest to highest respectively). The other parameters are as in Figure 4.1.

numerical simulations produced results similar in appearance to that depicted in Figure 4.1. We note that Figures 4.2 and 4.3 indicate a structural dependence of v upon p, μ_0 and M_0, for large-t, which we explore in the next section.

4.5 Large-t structure of $(CP)_+$

In Section 4.3 we have seen that $\overline{u}$ and $\underline{u}$ are the maximal and minimal solutions to $(CP)_+$. The numerical analysis in Section 4.4 and the theory for the local case of $(CP)_+$, in [MN15a, Chapter 9.2], indicate that the solution $u : \Omega_\infty \to \mathbb{R}$ to $(CP)_+$ converges to $\underline{u}$, as $t \to \infty$, for sufficiently small values of $p \in (0, 1)$ when u_0 and φ are of compact support. In addition to the assumptions on u_0 and φ described by $(CP)_+$, in this section we further assume that

$$u_0 \in L^\infty(\mathbb{R}) \text{ and } \varphi \in L^1(\mathbb{R}) \text{ are even and have compact support,}$$

and begin by formulating the linearised initial value problem. We write,

$$u(x, t) = \underline{u}(x, t) + \varepsilon w(x, t), \quad \forall (x, t) \in \Omega \times [1, \infty), \tag{4.5.1}$$

with $\varepsilon \ll 1$ and, upon recalling $\mu_0 = 0$ and $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$, it follows that

$$\underline{u}(x, t) = ((1 - p)t)^{1/(1-p)}, \quad \forall (x, t) \in \Omega_\infty.$$

On substituting (4.5.1) into the integro-differential equation of $(CP)_+$, by neglecting terms of $O(\varepsilon^2)$, as $\varepsilon \to 0$, we obtain an evolution equation for w given by

$$w_t(x, t) = w_{xx}(x, t) + \frac{p}{(1 - p)t} \int_{\mathbb{R}} \varphi(y) w(x - y, t) dy, \quad \forall (x, t) \in \Omega \times [1, \infty), \tag{4.5.2}$$

with the associated initial condition given by

$$w(x,1) = w_1(x), \quad \forall x \in \Omega, \tag{4.5.3}$$

with $w_1 : \Omega \to \mathbb{R}$ such that $w_1 \in C^2(\Omega) \cap W^{2,\infty}(\Omega)$, and we further suppose that w_1 is an even function and $w_1(x) \to 0$, at least exponentially, as $|x| \to \infty$.

Utilising known results from Fourier analysis, the evolution equation (4.5.2) has a general solution of the form

$$w(x,t) = \int_{\mathbb{R}} F(t,k) e^{ikx} dk, \quad \forall (x,t) \in \Omega \times [1,\infty), \tag{4.5.4}$$

for some $F : [1,\infty) \times \mathbb{R} \to \mathbb{C}$, to be determined. In order to determine F, we substitute the elementary solution

$$w(x,t) = F(t,k) e^{ikx}, \tag{4.5.5}$$

into (4.5.2) and obtain the linear ordinary differential equation

$$F_t(t,k) = \left(\frac{p\hat{\varphi}(k)}{(1-p)t} - k^2 \right) F(t,k), \quad \forall (t,k) \in (1,\infty) \times \mathbb{R}, \tag{4.5.6}$$

with initial condition, obtained by inverting the transformation in (4.5.4) when evaluated at $t = 1$, which is given by

$$F(1,k) = \frac{1}{2\pi} \int_{\mathbb{R}} w_1(\xi) e^{-i\xi k} d\xi = \hat{w}_1(k), \tag{4.5.7}$$

with $\hat{w}_1$ denoting the Fourier transform of w_1, and $\hat{\varphi}(k) = \int_{\mathbb{R}} \varphi(\xi) e^{-ik\xi} d\xi$ denoting the non-unitary Fourier transform of φ. Note that $\hat{w}_1$ and $\hat{\varphi}$ are well-defined since $w_1, \varphi \in L^1(\Omega)$. It now follows that the solution to (4.5.6)-(4.5.7) on $[1,\infty) \times \mathbb{R}$ is given by

$$F(t,k) = \hat{w}_1(k) e^{-\left(k^2 t - \frac{p}{1-p}\hat{\varphi}(k) \log t\right)}, \quad \forall (t,k) \in [1,\infty) \times \mathbb{R}. \tag{4.5.8}$$

Via the conditions on w_1 it follows that $\hat{w}_1 \in C(\mathbb{R})$ and $\hat{w}_1(k) = O(k^{-2})$ as $|k| \to \infty$. Hence, it follows that the solution to (4.5.2)-(4.5.3), obtained by substitution of (4.5.8) into (4.5.4), is given by

$$w(x,t) = \int_{\mathbb{R}} \hat{w}_1(k) e^{-\left(k^2 t - \frac{p}{1-p}\hat{\varphi}(k)\log t\right)} e^{ikx} dk, \quad \forall (x,t) \in \mathbb{R} \times [1,\infty). \tag{4.5.9}$$

Recalling that w_1 and φ are even, from (4.5.9) it follows that

$$|w(x,t)| \leq 2\int_0^\infty |\hat{w}_1(k)| e^{-\left(k^2 t - \frac{p}{1-p}\hat{\varphi}(k)\log t\right)} dk, \quad \forall (x,t) \in \mathbb{R} \times [1,\infty). \tag{4.5.10}$$

Since $\hat{\varphi} \in C^2(\mathbb{R})$ upon recalling that $\hat{\varphi}(0) = \|\varphi\|_{L^1(\mathbb{R})} = 1$, via Taylor's theorem, it follows that

$$\begin{aligned} \hat{\varphi}(k) &= \int_{\mathbb{R}} \varphi(\xi)\cos(k\xi) d\xi \\ &= \hat{\varphi}(0) + \hat{\varphi}'(0)k + \frac{\hat{\varphi}''(\theta)}{2!}k^2 \\ &:= 1 - k^2\gamma(k), \quad \forall k \in \mathbb{R}, \end{aligned} \tag{4.5.11}$$

where θ is between 0 and k, and $\gamma \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Thus, utilising (4.5.11) and following the change of variables $k = \sqrt{s}$, (4.5.10) can be re-written as

$$\begin{aligned} |w(x,t)| &\leq 2t^{\frac{p}{1-p}} \int_0^\infty |\hat{w}_1(k)| e^{-tk^2\left(1+\gamma(k)\frac{p}{1-p}\frac{\log t}{t}\right)} dk \\ &= t^{\frac{p}{1-p}} \int_0^\infty \frac{|\hat{w}_1(\sqrt{s})|}{\sqrt{s}} e^{-ts\left(1+\gamma(\sqrt{s})\frac{p}{1-p}\frac{\log t}{t}\right)} ds, \quad \forall (x,t) \in \mathbb{R} \times [1,\infty). \end{aligned} \tag{4.5.12}$$

Note that $|\hat{w}_1(\sqrt{s})|/\sqrt{s}$ is absolutely integrable on $[0,\infty)$, and

$$\gamma(\sqrt{s})\frac{p}{1-p}\frac{\log t}{t} \to 0, \text{ uniformly on } [0,\infty) \text{ as } t \to \infty. \tag{4.5.13}$$

Thus it follows that for each $\varepsilon \in (0, \frac{1}{2})$, there exists a sufficiently large $T_\varepsilon > 0$ such that

$$\int_0^\infty \frac{|\hat{w}_1(\sqrt{s})|}{\sqrt{s}} e^{-ts\left(1+\gamma(\sqrt{s})\frac{p}{1-p}\frac{\log t}{t}\right)} ds \leq \int_0^\infty \frac{|\hat{w}_1(\sqrt{s})|}{\sqrt{s}} e^{-ts(1-\varepsilon)} ds \quad \forall t \geq T_\varepsilon. \tag{4.5.14}$$

Provided that $\hat{w}_1(0) \neq 0$, it follows from the continuity of $\hat{w}_1$ that $\hat{w}_1$ is one-signed on $[0, X)$, for some $X > 0$. It follows that $|\hat{w}_1(k)|$ is three times continuously differentiable on $[0, X)$ and moreover, we note that $\hat{w}_1'(0) = 0$. Hence via Taylor's theorem, $|\hat{w}_1(\sqrt{s})|$ is once continuously differentiable on $[0, \sqrt{X})$. Thus, by utilising the method used to establish Watson's lemma (see for instance [Mil06, Chapter 2]), it follows that

$$\int_0^\infty \frac{|\hat{w}_1(\sqrt{s})|}{\sqrt{s}} e^{-ts(1-\varepsilon)} ds = |\hat{w}_1(0)| \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{(1-\varepsilon)t}} + O(t^{-3/2}), \quad \text{as } t \to \infty. \tag{4.5.15}$$

By substituting (4.5.15) into (4.5.12), we obtain

$$||w(\cdot, t)||_{L^\infty(\Omega)} \leq \frac{|\hat{w}_1(0)|\Gamma(\frac{1}{2})}{\sqrt{1-\varepsilon}} t^{\frac{p}{1-p}-\frac{1}{2}} + O(t^{\frac{p}{1-p}-\frac{3}{2}}), \quad \text{as } t \to \infty. \tag{4.5.16}$$

By considering $w(0, t)$ in (4.5.9), it follows as in (4.5.10)-(4.5.12) that

$$w(0, t) = t^{\frac{p}{1-p}} \int_0^\infty \frac{\hat{w}_1(\sqrt{s})}{\sqrt{s}} e^{-ts\left(1+\gamma(\sqrt{s})\frac{p}{1-p}\frac{\log t}{t}\right)} ds, \quad \forall t \in [1, \infty). \tag{4.5.17}$$

By treating cases when $\hat{w}_1(0)$ is positive and negative separately, it follows as in (4.5.13)-(4.5.16), albeit bounding the exponential term in (4.5.14) from below rather than from above, that for each $\varepsilon \in (0, \frac{1}{2})$ we have,

$$|w(0, t)| \geq \frac{|\hat{w}_1(0)|\Gamma(\frac{1}{2})}{\sqrt{1+\varepsilon}} t^{\frac{p}{1-p}-\frac{1}{2}} + O(t^{\frac{p}{1-p}-\frac{3}{2}}), \quad \text{as } t \to \infty. \tag{4.5.18}$$

It follows immediately from (4.5.18) that

$$||w(\cdot, t)||_{L^\infty(\Omega)} \geq \frac{|\hat{w}_1(0)|\Gamma(\frac{1}{2})}{\sqrt{1+\varepsilon}} t^{\frac{p}{1-p}-\frac{1}{2}} + O(t^{\frac{p}{1-p}-\frac{3}{2}}), \quad \text{as } t \to \infty. \tag{4.5.19}$$

Thus it follows from (4.5.16) and (4.5.19) that, provided that $\hat{w}_1(0) \neq 0$, we have

$$||w(\cdot,t)||_{L^\infty(\Omega)} \sim |\hat{w}_1(0)|\Gamma(\tfrac{1}{2})t^{\frac{p}{1-p}-\frac{1}{2}}, \quad \text{as } t \to \infty. \tag{4.5.20}$$

The estimate in (4.5.20), when substituted into (4.5.1), indicates that for sufficiently small values of $p \in (0,1)$ we have

$$\|u(\cdot,t) - \underline{u}(t)\|_{L^\infty(\Omega)} = O\left(t^{\frac{3p-1}{2(1-p)}}\right), \quad \text{as } t \to \infty. \tag{4.5.21}$$

To numerically investigate the exponent of t in (4.5.21), we consider local and non-local instances of $(CP)_+$. Specifically, for $p \in (0,1)$ we approximate v, as described in Section 4.4 with $\mu_0 = 0.001$, $M_0 = 1.001$, $v_0(x) = \mathbb{1}_{[-10,10]}(x)$, considering φ_{loc} to be the Dirac δ function for the local case, and

$$\varphi(x) = \begin{cases} \frac{1}{2}, & \text{if } |x| \leq 1; \\ 0, & \text{otherwise,} \end{cases}$$

for the non-local case, on sufficiently large discretised spatial domains, with sufficiently small $\delta x > 0$, for $t \in [0, 2000]$.

From the numerical approximations, we estimated $g(t_k) = \|u(\cdot,t_k)\|_{L^\infty(\Omega)}$ for $t_k = 1000 + 50k$ for $k = 1,\ldots,20$. A Pearson correlation coefficient was used to verify a linear correlation between $\log(t_k)$ and $\log(g(t_k))$. Finally, regression was used to estimate $b = b(p)$ in:

$$\|u(\cdot,t) - \underline{u}(t)\|_{L^\infty(\Omega)} = \|((\overline{u} - \underline{u})v)(\cdot,t)\|_{L^\infty(\Omega)} \sim at^b, \quad \text{as } t \to \infty. \tag{4.5.22}$$

The numerical approximations for $b(p)$ in (4.5.22) are illustrated, along with the asymptotic estimate for $b(p)$ arising from (4.5.21), in Figure 4.4. We observe that the asymptotic and numerical estimates for $b(p)$ in (4.5.22) are in excellent agreement.

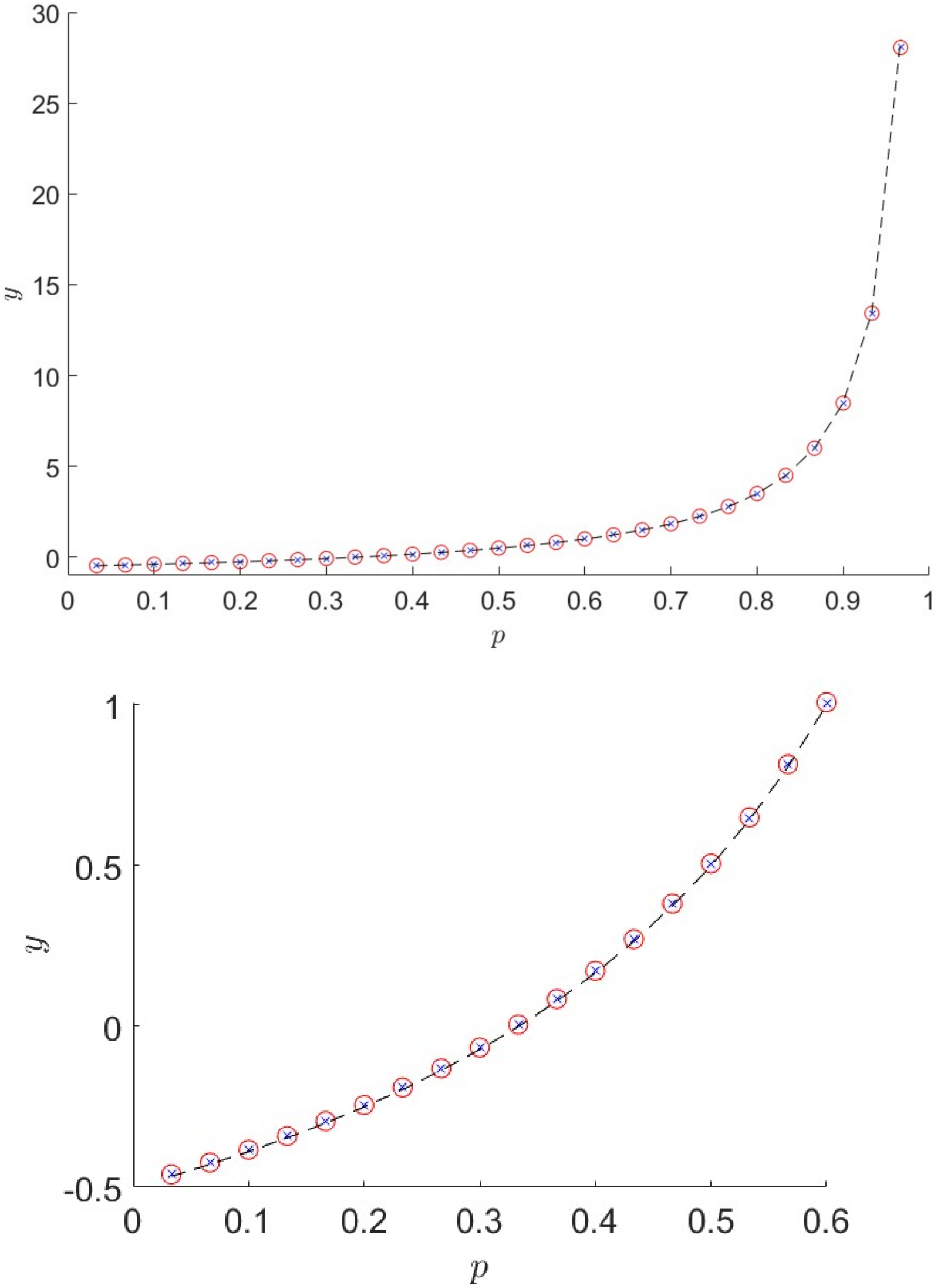

Figure 4.4: Here we illustrate $y(p) = b(p)$ for b as in (4.5.22) with $p \in (0, 1)$ above and $p \in (0, 3/5)$ below. The numerical approximation from the local and non-local numerical solutions to $(CP)_+$ are shown by the blue crosses and red circles respectively. The asymptotic approximation (dashed curve) is given by $y(p) = \frac{3p-1}{2(1-p)}$.

Remark 4.5.1. Let $u : \overline{\Omega}_\infty \to \mathbb{R}$ be the solution to $(CP)_+$ with initial data $u_0 \in L^\infty(\Omega)$ and integral kernel $\varphi \in L^1(\Omega)$. Moreover suppose that u_0 and φ are of compact support and $\|\varphi\|_{L^1(\Omega)} = 1$. Then, for all $p \in (0, 1/3)$

$$u(x,t) = ((1-p)t)^{\frac{1}{1-p}} + O(t^{\frac{3p-1}{2(1-p)}}), \quad \text{as } t \to \infty, \tag{4.5.23}$$

and hence,

$$\left\| u(\cdot,t) - ((1-p)t)^{\frac{1}{1-p}} \right\|_{L^\infty(\Omega)} \to 0, \quad \text{as } t \to \infty.$$

Moreover, for $p \in (1/3, 1)$ it follows that

$$\left\| u(\cdot,t) - ((1-p)t)^{\frac{1}{1-p}} \right\|_{L^\infty(\Omega)} \nrightarrow 0, \quad \text{as } t \to \infty. \tag{4.5.24}$$

Alternatively, from (4.5.23) and (4.5.24), it follows that for $(CP)_+$, the lower bound $((1-p)t)^{\frac{1}{1-p}}$ is asymptotically stable as $t \to \infty$ for $p \in (0, 1/3)$ and unstable for $p \in (1/3, 1)$ as $t \to \infty$. Furthermore, we infer that $(CP)_+$ is globally well-posed in time if $p \in (0, 1/3)$.

4.6 Conclusion

In Section 4.2 we assumed throughout that $u_0 \in L^\infty(\Omega) \cap C(\Omega)$. If $u_0 \in W^{2,\infty}(\Omega) \cap C(\Omega)$, then the derivative estimates in Proposition 4.2.10 and Proposition 4.2.11 are improved, to be bounded on $\overline{\Omega}_\delta$, similarly to those in Proposition 3.3.6 and Proposition 3.3.8 (the latter differing due to the assumption that f is locally Hölder continuous and not locally Lipschitz continuous). One can potentially relax the Hölder condition on f to a Dini condition (as given in [BS23]). To establish this, using the approach herein, one would need to extend the related derivative estimates for f and the Lipschitz density result.

In Section 4.2 we opted for a construction to show the existence of solutions to $(\tilde{CP})$. It should be noted that a solution can also be obtained by the use of the Leray-Schauder fixed-point theorem (assuming one can show that the conditions of the theorem are satisfied at a sufficient convex and closed subset of $L^\infty(\overline{\Omega}_T) \cap C^{2,1}(\overline{\Omega}_T)$ for the equivalent integral equation associated with $(\tilde{CP})$, (see [Fri08, Chapter 7])). However, this method doesn't readily support the construction of maximal and minimal solutions for $(\tilde{CP})$ that the method herein accommodates.

All the results presented in Section 4.3 are presented for $\Omega = \mathbb{R}$. Changing (4.3.3) so that φ has positive mass on a closed ball centred at 0 and then adapting Lemma 4.3.4 (taking the integral over the closed ball and obtaining a similar bound) would allow for all results present in Section 4.3 to extend directly to $\mathbb{R}^d$. Moreover, ideally, the condition that φ has a positive mass centred at 0 can be relaxed to φ having positive mass centered *somewhere* in Ω. This relaxation however, adds technicalities to the proof of the lower bound in Lemma 4.3.5, that require some care to address. To prove global in time well-posedness for $(CP)_+$ for $p \in (0.1/3)$, one could potentially consider compactly supported initial data and then establish an upper bound for the solution to $(CP)_+$, of the form $((1-p)t)^{\frac{1}{1-p}} + O(t^\alpha)$, as $t \to \infty$, for some $\alpha < 0$, similar to that established for the local case in [MN15a, Section 9.2]. More broadly, the approaches illustrated in [GV12] could similarly be used to establish the stability of $((1-p)t)^{\frac{1}{1-p}}$ as $t \to \infty$, which could inform on the well-posedness or ill-posedness results for $(CP)_+$. We note that the numerical approximations for a in (4.5.22) were not observed to be in agreement with $\Gamma(1/2)|\hat{w}_1(0)|$ in (4.5.20). This is likely due to the large times required to ensure the term in (4.5.13) is insignificant when applying Watson's Lemma, but also, for applicable linearization theory for this class of nonlinear evolution equations to take effect. However, for the purpose of inferring stability properties of the linearization of $(CP)_+$, and the stability of $\underline{u}$ for $(CP)_+$, we reiterate that the computations leading to (4.5.20) which dictate the dependence on t, suffice.

A more detailed estimate for $w(x,t)$ in (4.5.9) which incorporates the spatial structure in x would also be of interest. To achieve this, one could use the method of stationary phase to estimate (4.5.9). In the setting described in Section 4.5 one would need to take care to consider the poles of $\hat{\varphi}$, which is possible to address with specific instances of φ, but nontrivial to address with φ as considered therein.

We finally note that the results in Section 4.4 are readily extended for the problem $(CP)_+$ considered with $\Omega = \mathbb{R}^d$, however, in practice the computational complexity grows exponentially with d, making this not ideal for $d > 3$.

CHAPTER 5

CONCLUDING REMARKS

In this short chapter we discuss possible extensions to the results presented in Chapters 2-4, as well as related results which have not been considered herein. We note that even though, the results presented in Chapters 2-4 are not necessarily useful in 'real-world applications' in the wider context of the theory of PDEs they provide insight on how one treats problems of this class of non-local problems. Furthermore, the results highlight the similarities and differences with the local theory established (see [MN15a]) one can expect when working on generalising results from the local setting.

5.1 General remarks and extensions for Chapter 2

Recall the $t-$blow-up, as $t \to 0^+$, of the coefficients of the integro-differential operator discussed in Chapter 2, i.e., for any compact subset $X \subseteq \Omega$, the coefficients are $O(t^{-\beta})$, as $t \to 0^+$ for $x \in X$. It is understood that $\beta = 1$ is, without further spatial constraints on the coefficients, a limiting case. It would be of interest to consider if greater $t-$blow-up would be permitted, provided that further constraints are imposed on solutions to the integro-differential inequalities or the coefficients of the operators considered.

As discussed in the conclusion of Chapter 2, we expect that Proposition 2.3.14 holds in more general (unbounded) domains than $\mathbb{R}^n$. Notably, in Proposition 2.3.14, if $\Omega = \mathbb{R}^n$ is relaxed to $\Omega \subsetneq \mathbb{R}^n$, where Ω is unbounded, with sufficiently smooth boundary (e.g. $\partial\Omega$

is C^2), then provided that (2.5.1) is satisfied accordingly, then one can likely establish a related minimum principle.

Additionally note that the weak minimum principle of Proposition 2.3.14 was established under the regularity condition (2.3.74), restricting the coefficients of a_{ij}, b_i, and their derivatives, to be bounded and Hölder continuous functions as well as having c and d be continuous and bounded functions. These conditions were imposed to define the fundamental solution for adjoint equation, and hence it is worthwhile to inquire:

What are the weakest regularity conditions one can impose on the coefficients of P as in (2.2.4) *and still be able to define fundamental solutions for the adjoint equation?*

Answering this question would allow for a natural generalisation of Theorem 2.4.3. The recently published results in [BS23] provide a starting point to what may be *an optimal* result of this type.

5.2 General remarks and extensions for Chapter 3

A natural extension to the well-posedness theory presented in Chapter 3 is to systems of integro-differential equations. Utilising the ideas presented in Chapter 3 we can obtain a theorem analogous to Theorem 3.3.16, where the unknown function is represented by a vector valued function. The changes required to establish such a result are illustrated here.

Let $\Omega = \mathbb{R}^n$ and suppose that $\mathbf{u} : \overline{\Omega}_T \to \mathbb{R}^m$ is given by

$$\mathbf{u} = (u_1, u_2, \dots, u_m) \quad \text{on } \overline{\Omega}_T, \tag{5.2.1}$$

with $u_i \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ for all $i = 1, 2, \dots, m$. We also denote:

$$\partial_t \mathbf{u} = (\partial_t u_1, \partial_t u_2, \dots, \partial_t u_m) \quad \text{on } \Omega_T; \tag{5.2.2}$$

$$\Delta \mathbf{u} = (\Delta u_1, \Delta u_2, \dots, \Delta u_m) \quad \text{on } \Omega_T. \tag{5.2.3}$$

Furthermore, for $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$, and $\varphi_{ijk} \in L^1(\mathbb{R}^n)$ we define

$$\mathbf{J}\mathbf{u} = (J^1\mathbf{u}, J^2\mathbf{u}, \ldots, J^m\mathbf{u}) \quad \text{on } \Omega_T \tag{5.2.4}$$

with

$$J^i\mathbf{u} = \begin{pmatrix} \varphi_{i11} * u_1 & \varphi_{i12} * u_1 & \ldots & \varphi_{i1p} * u_1 \\ \vdots & \vdots & & \vdots \\ \varphi_{im1} * u_m & \varphi_{im2} * u_m & \ldots & \varphi_{imp} * u_m \end{pmatrix} \quad \text{on } \Omega_T. \tag{5.2.5}$$

Moreover, for $\mathbf{f} : \mathbb{R}^m \times \mathbb{R}^{m \times m \times p} \to \mathbb{R}^m$ we denote

$$\mathbf{f}(\mathbf{u}, \mathbf{J}\mathbf{u}) = (f_1(\mathbf{u}, J^1\mathbf{u}), f_2(\mathbf{u}, J^2\mathbf{u}), \ldots, f_m(\mathbf{u}, J^m\mathbf{u})) \quad \forall (\mathbf{u}, \mathbf{J}\mathbf{u}) \in \mathbb{R}^m \times \mathbb{R}^{m \times m \times p} \tag{5.2.6}$$

where for all $i = 1, 2, \ldots, m$, f_i is assumed to be locally Lipschitz continuous in the variables $(\mathbf{u}, J^i\mathbf{u})$. Now, consider the Cauchy problem,

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} + \mathbf{f}(\mathbf{u}, \mathbf{J}\mathbf{u}), & \text{on } \Omega_T; \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), & \forall x \in \Omega; \\ u_i \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T), \end{cases} \tag{CP*}$$

with prescribed initial data $u_{0i} \in L^\infty(\Omega) \cap C(\Omega)$, for all $i = 1, 2, \ldots, m$. For the i^{th} equation in (CP*) we have

$$\partial_t u_i = \Delta u_i + f_i(\mathbf{u}, J^i\mathbf{u}) \quad \text{on } \Omega_T. \tag{5.2.7}$$

In this setting, and by following the same steps used to obtain Theorem 3.3.16 the following statement holds.

Theorem. *Suppose (CP*), with φ_{ijk}, $\mathbf{f}$ and $\mathbf{u}_0$ prescribed, for all $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$, is a priori bounded on $\overline{\Omega}_T$ for $0 < T < \infty$. Then, there exists a unique* *solution* $\mathbf{u} : \overline{\Omega}_T \to \mathbb{R}^m$ *to* (CP*)*. Moreover, for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for all initial data* $\tilde{\mathbf{u}}_0$ *and kernels* $\tilde{\varphi}_{ijk}$ *which are within* δ *distance of* $\mathbf{u}_0$ *and* φ_{ijk} *in the* L^∞ *and* L^1 *norms respectively (for all* i, j, k*), the unique solution* $\tilde{\mathbf{u}} : \overline{\Omega}_T \to \mathbb{R}$ *to* (CP*) *with* $\tilde{\varphi}_{ijk}, \tilde{\mathbf{u}}_0$ *and* $\mathbf{f}$ *exists, and satisfies* $\|\mathbf{u} - \tilde{\mathbf{u}}\|_{L^\infty(\overline{\Omega}_T)} < \varepsilon$*.*

Alternatively, let $\Omega_T = \mathbb{R}^n \times (0, T]$ and $a_{ij}, b_i, f : \overline{\Omega}_T \to \mathbb{R}$ for $1 \leq i, j \leq n$ be such that: a_{ij} satisfies (2.2.3); and a_{ij}, b_i and f satisfy sufficient regularity and boundedness conditions. Then, using the methods described in [Fri08] or [ZC22] to construct classical solutions to the Cauchy problem

$$\partial_t u = \sum_{i,j=1}^{n} a_{ij} \partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + f \quad \text{on } \Omega_T;$$

$$u(x, 0) = u_0(x) \quad \forall x \in \mathbb{R}^n,$$

one may obtain the (local in time) well-posedness for the following Cauchy problem

$$\begin{cases} \partial_t u = \displaystyle\sum_{i,j=1}^{n} a_{ij} \partial_{x_i x_j} u + \sum_{i=1}^{n} b_i \partial_{x_i} u + f(u, Ju), & \text{on } \Omega_T; \\ u(x, 0) = u_0(x), & \forall x \in \Omega; \\ u \in L^\infty(\overline{\Omega}_T) \cap C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T), \end{cases} \tag{CP**}$$

with prescribed initial data $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ and with $Ju : L^\infty(\Omega_T) \to L^\infty(\Omega_T)$ as in (2.2.1).

5.3 General remarks and extensions for Chapter 4

In Chapter 4 we utilised the comparison theory developed in Chapter 2 to produce a priori bounds for u. Using those bounds, we were able to demonstrate the local existence of solutions to ($\tilde{CP}$). Ideally, criteria on f, which would allow one to ascertain uniform-in-time a priori bounds on solutions to ($\tilde{CP}$), that do not rely on the monotonicity of f with respect to Ju could be established. We recall that such problem-specific uniform-in-time a priori bounds for non-local Fisher-KPP-type equations (see [LCS20, Pen18]) are somewhat technical, and not necessarily readily generalised.

We note that in the local existence result for ($\tilde{CP}$), if f is not non-decreasing with respect to Ju, then the existence of maximal and minimal solutions is not established. We highlight that an example of ($\tilde{CP}$) for which it is demonstrated that maximal and minimal solutions do not exist would be a welcome addition to the literature. An analogous, non-local, problem to that considered in [MN17] would seem to provide a natural starting point to address this.

A more comprehensive treatment with an aim of determining the large$-t$ structure of the solution to $(CP)_+$ can potentially be achieved. A first step in this direction would be making the formal analysis in Section 4.5 rigorous. Then, the influence of the spatial dimension d (for Ω replaced with $\mathbb{R}^d$) and the initial data (see Figures 4.2-4.3), could also be investigated. Following this, a comparable result to Remark 4.5.1 could provide additional insight on whether or not pattern formations can occur in the large$-t$ structure of u.

Concerning the numerical approximation in Section 4.4, we highlighted that the approximation cannot be practically implemented for large spatial dimension d. However, using recently developed methods (see for example [BHJK23]), potentially a numerical approximation can be achieved for higher spatial dimensions d, at the cost of the $||\cdot||_\infty$ error estimates being replaced by $||\cdot||_p$ norm estimates.

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