AUTOMORPHISM GROUP OF A MATSUO ALGEBRA

by

FAARIE ALHARBI

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The University of Birmingham

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CONTENTS

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ABSTRACT

Axial algebras are commutative, not necessarily associative algebras generated by axes, where axes are idempotents whose adjoint actions are semisimple and such that the product of eigenvectors is controlled by a specific fusion law. Matsuo algebras, defined using Fischer spaces and 3-transposition groups, are examples of axial algebras of Jordan type η. In this thesis, we show that Matsuo algebras with $\eta \neq \frac{1}{2}$ almost never contain additional axes, and consequently, their automorphism groups almost always coincide with the automorphism group of the underlying groups of 3-transpositions. Furthermore, we identify all exceptions to this general rule in the irreducible case. Namely, we find three infinite series and an additional sporadic example of Matsuo algebras having extra axes and a larger group of automorphisms. In particular, for specific η, we show that $M = M_\eta(S_n)$ is isomorphic to a quotient of $M' = M_\eta(S_{n+1})$, and similarly $M = M_\eta(O^{\varepsilon}_{2m}(2))$ is isomorphic to a quotient of $M' = M_\eta(Sp_{2m}(2))$. These are examples of what we call aligned pairs of groups of 3-transpositions. We use the information about the spectrum of the diagram of irreducible 3-transposition groups, provided in [HS21], to find all aligned pairs of irreducible 3-transposition groups. We conclude that, apart from the examples above, there is only one additional sporadic aligned pair $(Fi_{23}, \Omega_8(3) : S_3)$.

INTRODUCTION

An *axial algebra* A is a commutative non-associative algebra generated by *axes*. The *axes* are idempotent elements in the algebra. The adjoint action of *axes* decomposes A into a direct sum of eigenspaces. Further, the multiplication of eigenvectors obeys a specific *fusion law*.

Axial algebras are interesting because they are related to groups. If a fusion law is graded, for example by the group C_2, then to each axis c in an axial algebra A there is associated a natural involution τ_c and the group generated by these involutions, the *Miyamoto group*, is a subgroup of the automorphism group of A.

The history of axial algebras is connected to the discovery of the Monster group $\mathbf{M}$, the largest of the 26 sporadic simple groups. This group was initially predicted to exist by Bernd Fischer in his unpublished work around 1973, and it was also conjectured by Griess [Gri76] in 1975. Later, in 1982, Griess [Gri82] constructed it as the automorphism group of the Griess algebra, a commutative non-associative real algebra with a dimension of $196,884$.

The famous observation, made by John McKay in 1978, was the starting point of the research around the Monster group. He observed that $196,883 + 1 = 196,884$, where the left-hand side of this equation is the sum of the first two smallest dimensions of irreducible representations of the Monster group, and the number $196,884$, on the right-hand side, is the coefficient of the linear term of the modular function $j(\tau) = q^{-1} + 744 + 196,884q + 21,493760q^2 + \dots$ where $q = e^{2\pi i\tau}$. This connection between the character degrees of $\mathbf{M}$ and the modular function's coefficient is known as the *Monstrous Moonshine*.

Building on McKay's observation, Thompson in [Tho79], found that, also, the sum of the first three smallest character degrees of $\mathbf{M}$ is exactly the quadratic coefficient of the modular function $j(\tau)$. That is, $1 + 196,883 + 212,968,76 = 21,493760$. Also, he wrote the next three coefficients of $j(\tau)$ as linear combinations of some initial character degrees of $\mathbf{M}$. Further, he subsequently conjectured the existence of a natural infinite-dimensional graded representation $V = \oplus_{i\geqslant -1} V_i$ of $\mathbf{M}$ such that $J(\tau) = q^{-1} + 744 + \sum_{i\geqslant 1} \dim(V_i)q^i$.

Thompson's conjecture was expanded by Conway and Norton [CN79] who have associated McKay-Thompson's series, $T_g(q) = \sum_{n\geqslant -1} H_n(g)q^n$, which is defined to each conjugacy class $[g]$ of $\mathbf{M}$ with $T_g(q)$ having the so-called genus-zero property. Based on their computations on T_g, Conway and Norton conjectured the existence of an infinite-dimensional graded representation of $\mathbf{M}$, where each H_n is a character of $\mathbf{M}$. In 1979, Atkin, Fong, and Smith, proved that such representation exists and had confirmed that H_n are indeed characters of the Monster [Smi85]. This representation was then explicitly constructed by Frenkel, Lepowsky and Meurman [FLM88] in 1988. Namely they constructed the *Moonshine module* $V^\natural := \bigoplus_{i\geqslant 0} V_i^\natural$, an infinite-dimensional graded module with infinitely many bilinear products. The characters of $V^\natural$ are given by the coefficients of $j(\tau) - 744$. The module $V^\natural$ is an example of Vertex Operator Algebras (VOAs) introduced by Borcherds in 1986 [Bor86]. Hence, $V^\natural$ is often referred to as the *Moonshine* VOA. Each subspace

$V_i^\natural$ of $V^\natural$ is called the graded i-part. The graded 2-part, $V_2^\natural$, of $V^\natural = \bigoplus_{i\geqslant 1} V_i^\natural$ is the $196,884$-dimensional Griess algebra. Hence, the Monster group $\mathbf{M}$ can be also obtained as the automorphism group of $V_2^\natural$. Moreover, the vertex operator algebra $V^\natural$ possesses the property of having $\dim V_0^\natural = 1$ and $\dim V_1^\natural = 0$, placing it within the class of vertex operator algebras known as OZ-type VOA. In this VOA class, the graded 2-part, typically denoted V_2, coincides with the Griess algebra.

Focusing on the OZ VOA class, Miyamoto [Miy96], introduced special vectors of V_2, called *Ising vectors*. He proved that every Ising vector induces an automorphism of order two of the VOA. These automorphisms are known as *Miyamoto involutions*. A similar correspondence between involutions and elements of algebra was observed earlier by Conway and Norton [Con85] while studying properties of the Griess algebra. In particular, they found that the idempotents of the Griess algebra, called $2A$-axes, are in bijection with the $2A$-involutions in the Monster group.

Miyamoto initiated the classification of the OZ VOAs that are generated by two distinct Ising vectors. However, the majority of the classification was completed by Sakuma [Sak07], who showed that the graded 2-part in any 2-generated OZ VOA is isomorphic to one of the following algebras

$$2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A$$

These all arise as subalgebras of the Griess algebras and their notation reflects the conjugacy classes of the products of two 2A-involutions, say τ_c, τ_d, in the Monster group, where c, d are distinct $2A$-axes in the Griess algebra. Norton in [Con85] showed that the subalgebra generated by two different 2A-axes, c and d, is uniquely identified by the conjugacy class of the element $\tau_c\tau_d$. Hence, the two-generated subalgebras of the Griess algebra are labeled by the names of these conjugacy classes.

In 2009, Ivanov [Iva09] introduced *Majorana algebras*, a new class of commutative, non-associative algebras defined over $\mathbb{R}$, by axiomatizing the properties of the Griess algebra $V_2^\natural$ that Sakuma used in his proof. Majorana algebra contains distinguished idempotents, called *Majorana axes*, analogous to the Ising vectors of the OZ VOA. That is, every Majorana axis gives an automorphism of order two of the algebra, called the *Majorana involution*. The adjoint action with respect to Majorana axes decompose the algebra as a direct sum of eigenspaces with eigenvalues $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$, and the product of two eigenvectors is controlled by the fusion rules described in Table 1.

$\diamond$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 1: Fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$

This was the first appearance of a fusion law as one of the axioms of a class of algebras. Entries of Table 1 can be interpreted as follows, for example the entry $0 \diamond \frac{1}{32} = \{\frac{1}{32}\}$ means the product of a 0-eigenvector and a $\frac{1}{32}$-eigenvector is a $\frac{1}{32}$-eigenvector. In general, the sets in the image of $\diamond$ would be written without the set notation, so it is $1, 0$ instead of $\{1, 0\}$, and the empty set is represented by a blank entry.

In 2015, Hall, Rehren and Shpectorov [HRS15b; HRS15a] introduced the class of *axial algebras* as a generalisation of Majorana algebras. Specifically, they eliminated the unnecessary axioms from the definition of Majorana algebras and permitted arbitrary choices for the field and the fusion law.

Some important examples of algebras that led to axial algebras include Majorana algebras, *Jordan algebras*, and *Matsuo algebras*. Jordan algebras are commutative non-associative algebras satisfying $(ab)(aa) = a(b(aa))$ for all a and b in the algebra. Matsuo algebras are defined using *Fischer spaces* which are related to 3-*transposition groups*. A 3-transposition group (G, D) is a group G generated by a normal set $D \subseteq G$ of involutions, such that for all $c, d \in D$, $|cd| \leqslant 3$. For a 3-transposition group (G, D), the Matsuo algebra $M_\eta(G, D)$, defined over a field $\mathbb{F}$ of characteristic not 2, with $\eta \in \mathbb{F} \setminus \{0, 1\}$, is the algebra whose basis is D and the multiplication is given by the following rules;

$$x \cdot y = \begin{cases} x, & \text{if } x = y \\ 0, & \text{if } xy \text{ is of order 2} \\ \frac{\eta}{2}(x + y - z), & \text{if } xy \text{ is of order 3 and } z = x^y \end{cases}$$

These algebras were first introduced by Matsuo [Mat03] and later generalized by Hall, Rehren, and Shpectorov in [HRS15a]. Jordan and Matsuo algebras are examples of axial algebras of Jordan type η because they satisfy the fusion law $\mathcal{J}(\eta)$ in Table 2. The primary result in the classification of axial algebras of Jordan type states that all algebras of Jordan type η, when $\eta \neq \frac{1}{2}$, are Matsuo algebras or their factor algebras (see [HRS15a], with a correction in [HSS18a]). Matsuo algebras with $\eta \neq \frac{1}{2}$ is our main area of study in this thesis. In particular, we are interested in studying the automorphism groups of these algebras.

Another important type of axial algebra are those which satisfy the Monster fusion law $\mathcal{M}(\alpha, \beta)$ depicted in Table 3. The Griess algebra over $\mathbb{R}$, which is of dimension 196884, is an example of algebras satisfying the fusion law $\mathcal{M}(\alpha, \beta)$ with $\alpha = \frac{1}{4}$, $\beta = \frac{1}{32}$. In fact, the Griess algebra is an example of Majorana algebra. It is worth

$\diamond$	1	0	η
1	1		η
0		0	η
η	η	η	1,0

Table 2: Jordan fusion law $\mathcal{J}(\eta)$

noting that, by taking α-eigenspace to be trivial and $\beta = \eta$, the class of algebras of Jordan type $\mathcal{J}(\eta)$ can be seen as subclass of algebras of Monster type $\mathcal{M}(\alpha, \beta)$.

$\diamond$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 3: Fusion law $\mathcal{M}(\alpha, \beta)$

By definition, an axial algebra A is generated by a set of axes X. However, the set X is not assumed to be the complete set of all axes in A. That is, some axial algebras can possibly contain additional axes. The problem of finding all axes in A is related to finding the full automorphism group of A. For instance, the set of 2A-axes in the Griess algebra is the complete set of all 2A-axis in the algebra and the Monster group consists of all the automorphisms of the Griess algebra. Similarly, among the complete automorphism groups of Jordan algebras are the classical groups and the group G_2. Previous research on the aforementioned problem includes the work of Castillo-Ramirez [Cas13] who found all idempotents and automorphisms of the Norton-Sakuma algebras. Additionally, in [Cas15], Castillo-Ramirez found all the idempotents of the 6- and 9-dimensional Majorana representations of S_4, denoted as $V_{(2B,3C)}$ and $V_{(2A,3C)}$, respectively. Also, there is a work in [Gor+23], conducted by Gorshkov, McInroy, Shpectorov, and Shumba, in which the authors use a computational method to find automorphism groups of some algebras of Monster type $\{\frac{1}{4}, \frac{1}{32}\}$.

In this PhD thesis, we investigate the full automorphism group of a Matsuo algebra $M := M_\eta(G, D)$ with $\eta \neq \frac{1}{2}$. First, in Theorem 4.4.5, we show that D is (almost always) the complete set of all primitive axes of Jordan type η in M. This is equivalent to saying that the automorphism group of M is (almost always) equal to the automorphism group of (G, D). Then, in Section 4.6, we look for the exceptions to this general situation in the irreducible case. In this direction, we have identified three infinite sequences and an extra sporadic example of Matsuo algebras with additional axes and a larger than expected automorphism groups. In particular, we show that, for specific η, $M = M_\eta(S_n, (1,2)^{S_n})$ is isomorphic to a quotient of $M' = M_\eta(S_{n+1}, (1,2)^{S_{n+1}})$, and $M = M_\eta(O^\varepsilon_{2m}(2), D)$ is isomorphic to a quotient of $M' = M_\eta(Sp_{2m}(2), D')$, where D, D' are the classes of transvections in $O^\pm_{2m}(2)$ and $Sp_{2m}(2)$, respectively. Additionally, for $\eta = -\frac{1}{4}$, $M = M_\eta(\Omega_8(3) : S_3)$ is isomorphic to a quotient of $M' = M_\eta(Fi_{23})$. We note that the pairs of 3-transposition groups in these examples are related to particular pairs of 3-transposition groups, which we called *aligned pairs*. A 3-transposition group (G, D) and its subgroup (H, C) are *aligned* if for a critical value η of G, $\overline{C}$ spans the quotient $\overline{M'}$ of $M' = M_\eta(G, D)$. We have used the paper [HS21], which provides all the information we need about the diagram (D) associated with the group (G, D), to analyse all cases of pairs of irreducible 3-transposition groups. In particular, we considered two main cases; the pairs that are in cross characteristics and the pairs that are in same characteristic. In the cross characteristics case, we used the fact that if H_n is a subgroup of G_m, then it will act non-trivially on the natural module of G_m. Consequently, the lower bound on the dimension of a non-trivial module for H_n is at most m (these lower bounds are provided in [SZ93]). From this observation we get an inequality of the form $f(n) \leqslant g(m)$ for some simple functions f and g, which goes in the opposite direction to a second inequality derived from the definition of aligned pairs. Therefore, by solving these inequalities, we obtain a finite number of pairs that we can examine individually. For the same characteristic case, we used the condition of aligned pairs, which says that if $M' = M_\eta(G, D)$ is aligned with $M = M_\eta(H, C)$, then $\dim M' - \dim R(M')$ is equal $\dim M$ if η is not critical for M, or it is equal to $\dim M - \dim R(M)$ if η is also critical for M. Here, $R(M')$ denote the radical ideal of M', and similarly $R(M)$ denote the radical of M. Using this condition we can exclude the pairs that cannot be aligned, i.e., the cases where the sides of the condition are not both odd or not both even. For the remaining cases, we can find limits on n and m for which the condition holds true, then we study each pair (n, m) separately. Further, we study the case $((G, D), (H, C))$, where (G, D) is not irreducible. After studying all cases, we conclude that aside from the identified examples, no further examples can arise from the irreducible case. This main result is presented in Corollary 7.3.2 of Theorem 7.3.1.

The content of this thesis is as follows: In Chapter 1, we define axial algebras and introduce the main examples. Further, we discuss how axial algebras with C_2-graded fusion laws are linked to groups. In Chapter 2, we prove the claim that the automorphism group of a finite $\mathcal{F}$-axial algebra A, with $1/2 \notin \mathcal{F}$, is finite. In Chapter 3, we define Matsuo algebras using 3-transposition groups and Fischer spaces. Further, we investigate Matsuo algebra with respect to the symplectic group. In Chapter 4, we introduce the concept of aligned pairs and give examples of Matsuo algebras with additional axes. In Chapters 5 and 6, we search for further examples of aligned pairs, considering the cross characteristics case in Chapter 5 and the same characteristic case in Chapter 6. In Chapter 7, we introduce reducible 3-transposition groups and give examples of special type of aligned pairs. Further, we consider the situation $((G, D), (H, C))$, where G is reducible.

CHAPTER 1

AXIAL ALGEBRAS

An *algebra* is a vector space over a field $\mathbb{F}$ equipped with additional $\mathbb{F}$-bilinear product. The multiplication in algebra may or may not be associative. Likewise, we assume that an axial algebra is not necessarily associative.

1.1 Axial Algebras and Fusion Laws

In order to define axial algebra, we start with the definition of a fusion law:

Definition 1.1.1. Let $\mathcal{F}$ be a finite subset of a field $\mathbb{F}$, and let $\diamond : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ be a map from $\mathcal{F} \times \mathcal{F}$ to the set of all subsets of $\mathcal{F}$. Then, a *fusion law* $\mathcal{F}$ over $\mathbb{F}$ is the pair $(\mathcal{F}, \diamond)$.

Let A be a commutative algebra over $\mathbb{F}$. For an element $c \in A$, the map ad_c from A to itself, given by $v \mapsto cv$ for $v \in A$, is called the *adjoint map* of c. For $\lambda \in \mathbb{F}$, the set of eigenvectors $A_\lambda(c) = \{v \in A \mid cv = \lambda v\}$ is the λ-eigenspace of ad_c. Note that, if λ is not an eigenvalue of ad_c, then $A_\lambda(c) = 0$. For $\Lambda \subseteq \mathbb{F}$, let $A_\Lambda(c) := \oplus_{\lambda \in \Lambda} A_\lambda(c)$.

Definition 1.1.2. A non-zero element $c \in A$ is said to be an $\mathcal{F}$-axis if it satisfies the following:

(i) c is an idempotent, i.e, $c^2 = c$.

(ii) ad_c is semisimple with all eigenvalues in $\mathcal{F}$, equivalently $A = A_{\mathcal{F}}(c)$.

(iii) For $\lambda, \mu \in \mathcal{F}$, the product of $A_\lambda(c)$ and $A_\mu(c)$ obeys the fusion law. That is, $A_\lambda(c)A_\mu(c) \subseteq A_{\lambda\Diamond\mu}(c)$.

The first condition $c^2 = c$ implies that 1 is an eigenvalue of ad_c, i.e., $c \in A_1(c)$. Therefore, we will always assume that $\mathcal{F}$ contains 1.

Definition 1.1.3. An axis c is called primitive if $A_1(c) = \langle c \rangle$. That is, if c spans the 1-eigenspace of ad_c.

Remark. We will use $\langle\langle \ldots \rangle\rangle$ for algebra generation.

Definition 1.1.4. Let A be a commutative algebra generated by a set of $\mathcal{F}$-axes denoted by X. Then, A is called an $\mathcal{F}$-*axial algebra*. If all the axes in X are primitive, then we call A a *primitive axial algebra*.

Example 1.1.5. *Let c be an idempotent in an associative algebra A, and suppose that ad_c is semisimple. If $z \in A_\lambda(c)$, then $\lambda z = cz = (cc)z = c(cz) = c(\lambda z) = \lambda cz = \lambda^2 z$. So, $\lambda = 1$, or 0. Thus, A can be decomposed as $A = A_1(c) \oplus A_0(c)$. Let $\mathcal{A} = \{1, 0\}$, we want to check that multiplication of eigenvectors satisfies the fusion rules in Table 1.1. Consider the eigenvectors $a, b \in A_1(c)$ and $w, z \in A_0(c)$. First, $c(ab) = (ca)b = ab$, so $A_1(c)A_1(c) \subseteq A_1(c)$. Next, note that $aw = (ca)w = c(aw) = c(wa) = (cw)a = 0a = 0$, so $A_1(c)A_0(c) = \{0\}$. Finally, $c(wz) = (cw)z = 0z = 0$, so $A_0(c)A_0(c) \subseteq A_0(c)$. Hence, A is an axial algebra satisfying the fusion law $\mathcal{A}$.*

$\diamond$	1	0
1	1	
0		0

Table 1.1: Fusion law $\mathcal{A}$

Furthermore, if A is a primitive $\mathcal{A}$-axial algebra with generating axes X, then A is associative and $A \cong \mathbb{F}X$. To prove this statement, we will use the Seress property of fusion laws, defined as follows.

Definition 1.1.6. A fusion law $\mathcal{F}$ that contains $0, 1$ is called Seress if $1 \diamond \lambda \subseteq \{\lambda\}$ and $0 \diamond \lambda \subseteq \{\lambda\}$ for all $\lambda \in \mathcal{A}$.

If a fusion law is Seress, then by definition, $1 \diamond 0 = \emptyset$. The fusion law in Table 1.1 is Seress since it contains $1, 0$, and satisfies the Seress property. Since $0 \diamond 0 \subseteq \{0\}$, it follows that, for all axis $c \in A$, $A_0(c)$ is a subalgebra.

Lemma 1.1.7 (Seress Lemma). *Let c be an $\mathcal{F}$-axis, where $\mathcal{F}$ is Seress. Then c associates with $A_{\{1,0\}}(c)$.*

Proof. We will show that $c(uv) = (cu)v$ for all $u \in A$, $v \in A_1(c) \oplus A_0(c)$. Note that the target equality is linear in u and it is also linear in v. So v can be taken from $A_1(c)$ or $A_0(c)$ and u can be taken to be an eigenvector. Assume $u \in A_\lambda(c)$. Then $cu = \lambda u$. Furthermore, $(cu)v = \lambda uv$. Since $\mathcal{F}$ is Seress, $uv \subseteq A_\lambda(c)$ for $v \in A_1(c)$ or $v \in A_0(c)$. Hence, $c(uv) = \lambda uv = (cu)v$. □

Proposition 1.1.8. *Let A be a primitive $\mathcal{A}$-axial algebra with a generating set X. Then every primitive idempotent $d \neq c$ lies in $A_0(c)$.*

Proof. Suppose $d \in X - \{c\}$. Then we will deduce that $d \in A_0(c)$. Write $d = \mu c + f$, where $\mu \in \mathbb{F}$ and $f \in A_0(c)$. Since c, d are idempotents and $f \in A_0(c)$ (i.e., $cf = 0$), we have

$$\mu c + f = (\mu c + f)^2 = \mu^2 c^2 + 2\mu c f + f^2 = \mu^2 c + f^2.$$

So $\mu = \mu^2$ and $f = f^2$. Therefore, $\mu = 1$ or 0, and f is an idempotent. If $\mu = 1, f = 0$, we get $d = c$, which is impossible. If $\mu = 1$ and f is not zero, then $d = c + f$ but $cd = c(c+f) = c^2 + cf = c$ implies that $c \in A_1(d)$, which contradicts the primitivity of d. Otherwise, $\mu = 0$, which gives the desired result that $d = f$. Thus, $d \in A_0(c)$. Therefore, $X - \{c\} \subseteq A_0(c)$. □

Proposition 1.1.9. *Let A be a primitive $\mathcal{A}$-axial algebra. Let $X = \{c_1, \ldots, c_n\}$ be the set of generating axes. Then $A \cong \mathbb{F}X$.*

Proof. We proceed by induction on n. If A has dimension one, then $A = \langle\langle c_1 \rangle\rangle = \langle c_1 \rangle \cong \mathbb{F}$. Let $B = \langle\langle X - \{c_1\} \rangle\rangle = \langle\langle c_2, \ldots, c_n \rangle\rangle$. Then by the induction assumption we have $B = \mathbb{F} \oplus \cdots \oplus \mathbb{F}$ ($n-1$ copies of $\mathbb{F}$). From Proposition 1.1.8, we have that $B \subseteq A_0(c_1)$. Note that $\langle c_1 \rangle + B$ is a subalgebra, i.e., it is closed under product. Also it contains X, so $\langle c_1 \rangle + B \supseteq \langle\langle X \rangle\rangle = A$. Thus, $A = \langle c_1 \rangle \oplus B = A_1(c_1) \oplus B = \mathbb{F} \oplus \cdots \oplus \mathbb{F}$ (n copies of $\mathbb{F}$). □

Next, we will introduce some examples of another type of axial algebras called axial algebras of Jordan type η. These are the commutative axial algebras that satisfy the fusion law $\mathcal{J}(\eta)$ given in Table 1.2.

Example 1.1.10. *Let A be a non-associative $\mathbb{F}$-algebra, and let $c \in A$ be a nonzero axis such that $A = \langle\langle c \rangle\rangle$, i.e., A is generated by the single axis c. If A contains another nonzero idempotent, call it d, then for some nonzero $\alpha \in \mathbb{F}$, we have that $\alpha c = d = d^2 = \alpha^2 c$, which implies that $\alpha = 1$. Hence, c is the only nonzero idempotent in A. Namely, c is the identity element of A, and $A \cong \mathbb{F}$. Since* ad_c

has only one eigenvalue, which is 1, $A_1(c) = \langle c \rangle$ *and the eigenspaces* $A_0(c)$ *and* $A_\eta(c)$ *are both trivial. This is an example of primitive axial algebras of Jordan type* η*, denoted as* $1A$*, where* η *can take any value.*

$\diamond$	1	0	η
1	1		η
0		0	η
η	η	η	1,0

Table 1.2: Jordan fusion law $\mathcal{J}(\eta)$

Example 1.1.11. *Let* $3C(\eta)$ *be an* $\mathbb{F}$*-algebra for* $\eta \in \mathbb{F} \setminus \{0, 1\}$*, and let* $B = \{b_0, b_1, b_2\}$ *be a basis for it, where* b_1, b_2*, and* b_3 *are nonzero idempotents. The product of* b_0, b_1*,* b_2 *is described in Table 1.3. Let* $3C(\eta) = A$*. Note that,* $b_i b_i =$

	b_0	b_1	b_2
b_0	b_0	$\frac{\eta}{2}(b_0 + b_1 - b_2)$	$\frac{\eta}{2}(b_0 + b_2 - b_1)$
b_1	$\frac{\eta}{2}(b_1 + b_0 - b_2)$	b_1	$\frac{\eta}{2}(b_1 + b_2 - b_0)$
b_2	$\frac{\eta}{2}(b_2 + b_0 - b_1)$	$\frac{\eta}{2}(b_2 + b_1 - b_0)$	b_2

Table 1.3: Multiplication of $\{b_0, b_1, b_2\}$

$1b_i$*, so* $b_i \in A_1(b_i)$*. Thus,* $\langle b_i \rangle \subseteq A_1(b_i)$*. Let* $\{i, j, k\} = \{0, 1, 2\}$*, note that,* $b_i(\eta b_i - b_j - b_k) = \eta b_i - \frac{\eta}{2}(b_i + b_j - b_k) - \frac{\eta}{2}(b_i + b_k - b_j) = \eta b_i - \frac{\eta}{2} b_i - \frac{\eta}{2} b_j + \frac{\eta}{2} b_k - \frac{\eta}{2} b_i - \frac{\eta}{2} b_k + \frac{\eta}{2} b_j = 0$*. Therefore,* $\langle \eta b_i - b_j - b_k \rangle \subseteq A_0(b_i)$*. Also,* $b_i(b_j - b_k) = \frac{\eta}{2}(b_i + b_j - b_k) - \frac{\eta}{2}(b_i + b_k - b_j) = \eta(b_j - b_k)$*. So,* $\langle b_j - b_k \rangle \subseteq A_\eta(b_i)$*. Notice that the eigenvectors* $b_i, \eta b_i - b_j - b_k$*, and* $b_j - b_k$ *are linearly independent because they correspond to distinct eigenvalues of* ad_{b_i}*. As* $3C(\eta)$ *is 3-dimensional, it follows that* $\dim(A_1(b_i) \oplus A_0(b_i) \oplus A_\eta(b_i)) = 3$*. That is,* ad_{b_i} *is semisimple, i.e.,* $3C(\eta)$ *decomposes as* $A_1(b_i) \oplus A_0(b_i) \oplus A_\eta(b_i)$*. Moreover, in this algebra,* $A_0(b_i)$ *is a subalgebra and the above decomposition satisfies the fusion law* $\mathcal{J}(\eta)$*.*

Example 1.1.12. *Another example of non-associative axial algebras of Jordan type η is Jordan algebras. Let c be an idempotent in a Jordan algebra A, then c satisfies the fusion law $\mathcal{J}(\eta)$ with $\eta = \frac{1}{2}$, as in Table 1.4. If we consider any idempotent in A, then we can decompose A with respect to that idempotent into the direct sum of* 0-, 1-, *and* $\frac{1}{2}$*-eigenspaces. This property is called the* Peirce decomposition.

$\diamond$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0		0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

Table 1.4: Fusion law $\mathcal{J}(\frac{1}{2})$

Example 1.1.13. *Consider an associative $\mathbb{F}$-algebra with a basis of two idempotents $\{c_1, c_2\}$ where $c_1c_2 = 0$. The algebra $2B$ is a primitive axial algebra of Jordan type η spanned by the axes c_1 and c_2. The eigenvalues of ad_{c_i} are only $0, 1$. So the η-eigenspace, $2B_\eta(c_i)$, is trivial, i.e., $2B$ decomposes as $2B_1(c_i) \oplus 2B_0(c_i)$. Here, η can take any value apart from $0, 1$.*

$\diamond$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 1.5: The fusion law $\mathcal{M}(\alpha, \beta)$

The class of axial algebras of Jordan type η is a subclass of a larger class of axial algebras called axial algebras of Monster type (α, β). This class consists of axial algebras that satisfy the Monster fusion law $\mathcal{M}(\alpha, \beta)$ given in Table 1.5. A well-known example of algebras of this type is as follows;

Example 1.1.14. *The Griess algebra, an algebra of dimension* $196,884$ *is an example of axial algebras of Monster type* (α, β)*. It is generated by a specific set of primitive idempotents called* $2A$*-axes. These axes satisfy the fusion law* $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$*, which is the fusion law* $\mathcal{M}(\alpha, \beta)$ *in Table 1.5 with* $\alpha = \frac{1}{4}$ *and* $\beta = \frac{1}{32}$*. Moreover, the automorphism group of the Griess algebra is the Monster group, the largest sporadic simple group.*

1.2 Grading

In the previous examples of fusion laws, one can split the set of eigenvalues into two parts, say $\mathcal{F}^+, \mathcal{F}^-$, such that for all $\lambda, \mu \in \mathcal{F}$ we have that,

$$\lambda \diamond \mu \subseteq \begin{cases} \mathcal{F}^+, & \text{if } \lambda, \mu \text{ are from same part} \\ \mathcal{F}^-, & \text{if } \lambda, \mu \text{ are from different parts} \end{cases}$$

Under this partition, the set $\{\mathcal{F}^+, \mathcal{F}^-\}$ together with $\diamond$ is a group isomorphic to the cyclic group of order two, C_2. Fusion laws with this property are called C_2-graded. For example, the Monster fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$, shown in Table 1.5, is C_2-graded with $\mathcal{M}^+ = \{1, 0, \frac{1}{4}\}$ and $\mathcal{M}^- = \{\frac{1}{32}\}$. Similarly, the Jordan fusion law $\mathcal{J}(\frac{1}{2})$, Table 1.4, is C_2-graded with $\mathcal{J}^+ = \{1, 0\}$ and $\mathcal{J}^- = \{\frac{1}{2}\}$. Note that, grading of $\mathcal{F}$ by a group of order two results in algebra A being 2-graded. In particular, if $\mathcal{F}$ is a C_2-graded fusion law and $c \in A$ is an axis, then the map $\tau_c : A \to A$ defined by

$$u \mapsto \begin{cases} u, & \text{if } u \in A_{\mathcal{F}^+}(c) \\ -u, & \text{if } u \in A_{\mathcal{F}^-}(c) \end{cases}$$

is an automorphism of A of order two, known as the *Miyamoto involution* of c. Because each axis has a different decomposition of the algebra, we obtain different Miyamoto involutions.

Definition 1.2.1. The subgroup of $\mathrm{Aut}(A)$ that generated by all τ_c, $c \in X$ is called the Miyamoto group.

algebra(s)	**Miyamoto groups**
Griess algebra	The Monster group M
Jordan algebras	Classical groups, G_2, and groups of type F_4
Matsuo algebras	3-transposition groups, symmetric groups, Fi_{22},Fi_{23},Fi_{24}

Table 1.6: Miyamoto groups for the corresponding C_2-graded axial algebras

Notice that, by considering these examples of axial algebras in Table 1.6, we obtained all different types of simple groups that appeared as the Miyamoto groups of these algebras. Hence, axial algebras help in studying these types together.

CHAPTER 2

AUTOMORPHISM GROUPS OF AXIAL ALGEBRAS

In this chapter, we aim to show that the automorphism group of a finite-dimensional $\mathcal{F}$-axial algebra A, with $1/2 \notin \mathcal{F}$, is finite. This result is essential for our approach in Chapter 4.

2.1 Algebraic Groups

Suppose that A is a finite-dimensional axial algebra, the goal of this section is to show that $G = \mathrm{Aut}(A)$ is an algebraic group. We start by introducing the basic definitions and examples from algebraic geometry.

2.1.1 Algebraic Geometry

In this chapter, let $\mathbb{F}$ be an algebraically closed field. Let $\mathbb{A}^n(\mathbb{F}) = \{(a_1, \ldots, a_n) \in \mathbb{F}^n\}$ be an affine n-space over $\mathbb{F}$, and let $\mathbb{F}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables. If $S \subseteq \mathbb{F}[x_1, x_2, \ldots, x_n]$ is a set of polynomials, say, $S = \{f_1, \ldots, f_m\}$, with coefficients from $\mathbb{F}$, then the vanishing set of S,

$$V(S) := \{a \in \mathbb{A}^n : f(a) = 0, \text{ for all } f \in S\}$$

is called an affine *algebraic subset* of $\mathbb{A}^n(\mathbb{F})$. An affine algebraic set is also called an *affine variety*. The collection of all algebraic sets in $\mathbb{A}^n(\mathbb{F})$ form the *closed sets* of a topology on $\mathbb{A}^n(\mathbb{F})$ known as the *Zariski topology*. A non-empty algebraic set Z is called *irreducible* if we cannot write Z as $Z = Z_1 \cup Z_2$ for some proper closed subsets $Z_1, Z_2 \subset Z$. Every algebraic set Z can be written uniquely as a union of finitely many irreducible components of Z, i.e., $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_n$ where $Z_1, Z_2, \ldots, Z_n$ are irreducible closed subsets with $Z_i \nsubseteq Z_j$ for all $i \neq j$.

Proposition 2.1.1. *Let S be any set of polynomials in $\mathbb{F}[x_1, \ldots, x_n]$. If $I = \langle S \rangle$ is an ideal in $\mathbb{F}[x_1, \ldots, x_n]$, then $V(S) = V(I)$.*

Proof. As I is the ideal generated by $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$, write

$$I = \left\{ \sum_{i=1}^{n} f_i h_i : f_i \in \mathbb{F}[x_1, \ldots, x_n], h_i \in S \right\}$$

Let $a \in V(S)$, then $h(a) = 0$ for all $h \in S$. Take any $h \in I$, then we have that $h(a) = \sum_{i=1}^{n} f_i(a) h_i(a) = 0$, as $h(a) = 0$ for all $h \in S$. Hence $a \in V(I)$. Therefore, $V(S) \subseteq V(I)$. Conversely, let $b \in V(I)$, so $f(b) = 0$ for all $f \in I$. Since $S \subseteq I$, it follows that $h(b) = 0$ for all $h \in S$. So, $b \in V(S)$, hence $V(I) \subseteq V(S)$. □

Next, we want to give a definition of affine variety that does not rely on embedding in $\mathbb{A}^n$. Let us start with the definition of $\mathbb{F}$-algebra.

The content up to the end of this subsection is compiled based on [Fab21].

Definition 2.1.2. Let B be a ring that is also an $\mathbb{F}$-vector space. Then, B is an $\mathbb{F}$-algebra if $\lambda(b \cdot c) = (\lambda b) \cdot c$ holds for all $\lambda \in \mathbb{F}$ and $b, c \in B$.

Example 2.1.3. *Various examples of* $\mathbb{F}$*-algebra include* $\mathbb{F}$ *itself,* $\mathbb{F}[X]$*, and the collection of* $\mathbb{F}$*-valued functions on a set* V *denoted as* $\mathrm{Map}\,(V, \mathbb{F}) := \{f \mid f : V \to \mathbb{F}\}$*, where* V *represents an arbitrary set.*

All these examples are commutative algebras, and we will below only consider commutative algebras.

Definition 2.1.4. Let B, C be $\mathbb{F}$-algebras. A map $\phi : B \to C$ is a *morphism* of algebras ($\mathbb{F}$-algebra homomorphism) if it is both a ring homomorphism and an $\mathbb{F}$-linear map. Write $\mathrm{Hom}(B, C)$ to denote the set consisting of all $\mathbb{F}$-algebra homomorphisms $\phi : B \to C$.

Definition 2.1.5. Let $C \subseteq B$, where B is an $\mathbb{F}$-algebra. Then, C is a subalgebra of B if it is both a subring and an $\mathbb{F}$-subspace of B.

Definition 2.1.6. Let B be an $\mathbb{F}$-algebra, and $S \subseteq B$ is a set. The subalgebra generated by S, denoted $\mathbb{F}[S]$, is the smallest $\mathbb{F}$-subalgebra of B containing S.

Definition 2.1.7. An algebra B is said to be finitely generated $\mathbb{F}$-algebra if $B = \mathbb{F}[b_1, \ldots, b_n]$ for some finite set $S := \{b_1, \ldots, b_n\} \subseteq B$.

Definition 2.1.8. A ring R is said to be Noetherian if every ideal in R is finitely generated.

Theorem 2.1.9 (Hilbert Basis Theorem)**.** *If* R *is Noetherian, then* $R[x_1, \ldots, x_n]$ *is Noetherian as well.*

Let $B = \mathbb{F}[b_1, \ldots, b_n]$ be a finitely generated $\mathbb{F}$-algebra. As B is finitely generated, we can consider the polynomial algebra in n indeterminates, $\mathbb{F}[T_1, \ldots, T_n]$, and define the map

$$\phi : \mathbb{F}[T_1, \ldots, T_n] \to \mathbb{F}[b_1, \ldots, b_n]$$

by $\phi(T_i) = b_i$. Since ϕ is a surjective algebra homomorphism, it follows from the first isomorphism theorem that $\mathbb{F}[T_1, \ldots, T_n]/\ker\phi \cong B$, where $\ker\phi$ is the kernel of ϕ, i.e., an ideal in $\mathbb{F}[T_1, \ldots, T_n]$. Therefore, every finitely generated $\mathbb{F}$-algebra is a quotient of a polynomial ring in finitely many variables. Also, by Hilbert's basis theorem, $\mathbb{F}[T_1, \ldots, T_n]$ is Noetherian, so we get that B is Noetherian too. That is, every ideal in B is finitely generated.

If $B \subseteq \text{Map}(V, \mathbb{F})$ is a subalgebra, then there exists an $\mathbb{F}$-algebra homomorphism $\text{ev}_x : B \to \mathbb{F}$, $\text{ev}_x(f) \mapsto f(x)$, where ev_x is the evaluation map at x.

Definition 2.1.10. An *affine algebraic variety* is a pair (V, B) where V is a set and $B \subset \text{Map}(V, \mathbb{F})$ is a finitely generated $\mathbb{F}$-algebra of functions from V to $\mathbb{F}$ satisfying the following:

- for any two distinct elements $x, y \in V$ there exists a function $f \in B$ such that $f(x) \neq f(y)$; and
- every $\mathbb{F}$-algebra homomorphism $\mathcal{E} : B \to \mathbb{F}$ is given by the evaluation map ev_y at some $y \in V$; that is, $\mathcal{E}(f) = f(y)$ for all f in B.

Remark. If (V, B) is an affine variety, then $B := \mathbb{F}[V] \subseteq \text{Map}(V, \mathbb{F})$ is called the *coordinate algebra* of V.

Example 2.1.11. (i) *$(V, B) = (\mathbb{A}^n(\mathbb{F}), \mathbb{F}[x_1, \ldots, x_n])$ is an affine variety.*

(ii) *Let $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$, and let I be the ideal generated by S, i.e., $I = (S)$. The affine set $V := V(S) = V(I) \subseteq \mathbb{A}^n(\mathbb{F})$ together with $B := \left\{ f_{|V} : f \in \mathbb{F}[x_1, \ldots, x_n] \right\} \cong \mathbb{F}[x_1, \ldots, x_n]/I(V)$, where $I(V) = \{f \in B :$*

$f(x) = 0$ for all $x \in V\}$, is an affine variety. So, an affine set in $\mathbb{A}^n(\mathbb{F})$ is an affine variety.

Definition 2.1.12. Let $(X, \mathbb{F}[X])$ and $(Y, \mathbb{F}[Y])$ be two affine varieties. A map $\varphi : X \to Y$ is a *morphism* of affine varieties if $(g \circ \varphi : X \to \mathbb{F}) \in \mathbb{F}[X]$ for all $g \in \mathbb{F}[Y]$. The map $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X]$, $g \mapsto g \circ \varphi$ is an algebra homomorphism called the *comorphism* of φ.

Definition 2.1.13. Let $\varphi : X \to Y$ be a morphism of affine varieties. Then, φ is an *isomorphism* if there exists a morphism $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{Id}_X$ and $\varphi \circ \psi = \mathrm{Id}_Y$.

Lemma 2.1.14. *Let $\psi : X \to Y$ be a morphism of affine varieties and suppose that the comorphism $\psi^* : \mathbb{F}[Y] \to \mathbb{F}[X]$ is surjective. Then the image $\psi(X) \subseteq Y$ is closed and $\psi_{|X} : X \to \psi(X)$ is an isomorphism.*

Proposition 2.1.15. *Every affine variety is isomorphic to a variety on an affine set.*

Proof. Let $(X, \mathbb{F}[X])$ be an affine variety and let $f_1, \ldots, f_n$ be algebra generators of $\mathbb{F}[X]$, i.e., $\mathbb{F}[X] = \mathbb{F}[f_1, \ldots, f_n]$. Define a map $\psi : X \to \mathbb{A}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$. Consider the i-th coordinate function $X_i \in \mathbb{F}[\mathbb{A}^n] = \mathbb{F}[X_1, \ldots, X_n]$. Note that, $(X_i \circ \psi)(x) = X_i(\psi(x)) = f_i(x)$. That is, $X_i \circ \psi = f_i \in \mathbb{F}[X]$. Since $X_1, \ldots, X_n$ generate $\mathbb{F}[\mathbb{A}^n]$ as algebra, we have that $g \circ \psi \in \mathbb{F}[X]$ for all $g \in \mathbb{F}[\mathbb{A}^n]$. So, by definition ψ is a morphism. By definition, $f_i = X_i \circ \psi = \psi^*(X_i)$, thus $\mathbb{F}[X] = \mathbb{F}[\psi^*(X_1), \ldots, \psi^*(X_n)]$. Hence, $\psi^* : \mathbb{F}[\mathbb{A}^n] \to \mathbb{F}[X]$ is surjective. So, by Lemma 2.1.14, $\psi(X)$ is closed and $\psi_{|X} : X \to \psi(X)$ is an isomorphism. Hence, X is isomorphic to a closed set in $\mathbb{A}^n$. □

Corollary 2.1.16. *Affine varieties are precisely the closed sets in $\mathbb{A}^n$.*

Definition 2.1.17. Let $\mathbb{F}[X]$ and $\mathbb{F}[Y]$ be finitely generated $\mathbb{F}$-algebras. The *tensor product* of $\mathbb{F}[X]$ and $\mathbb{F}[Y]$ over $\mathbb{F}$ is an $\mathbb{F}$-algebra defined as $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] = \{\sum_{i=1}^{n} \lambda_i(f_i \otimes g_i) \mid f_i \in \mathbb{F}[X], g_i \in \mathbb{F}[Y], \text{ and } \lambda_i \in \mathbb{F}\}$. Moreover, the multiplication structure on $\mathbb{F}[X] \otimes \mathbb{F}[Y]$ is given by $(\sum_i \lambda_i f_i \otimes g_i)(\sum_j \lambda_j f'_j \otimes g'_j) = \sum_{ij} \lambda_i \lambda_j f_i f'_j \otimes g_i g'_j$.

Note that, if $f_1, \ldots, f_n$ are generators of $\mathbb{F}[X]$ and $g_1, \ldots, g_m$ are generators of $\mathbb{F}[Y]$, then $\mathbb{F}[X] \otimes \mathbb{F}[Y]$ is finitely generated by $f_1 \otimes 1, \ldots, f_n \otimes 1, 1 \otimes g_1, \ldots, 1 \otimes g_m$.

If $(X, \mathbb{F}[X])$ and $(Y, \mathbb{F}[Y])$ are two affine varieties, then the direct product $(X \times Y, \mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y])$ is an affine variety. This new variety can be constructed using the map $f \otimes g : X \times Y \to \mathbb{F}$ defined as $f \otimes y(x, y) = f(x) \cdot g(y)$ and expanded by linearity, i.e, $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y]$ naturally embeds in $\mathrm{Map}\,(X \times Y, \mathbb{F})$.

2.1.2 Algebraic Groups and Finite-Dimensional Algebras

In this subsection, after introducing algebraic groups and its related properties, we will show that the automorphism group of a finite-dimensional algebra is an algebraic group.

Definition 2.1.18. An *affine algebraic group* is an affine variety G such that G is a group, and the group operations $m : G \times G \to G, (x, y) \mapsto xy$ and $i : G \to G$, $x \mapsto x^{-1}$ are morphisms of affine varieties.

For an algebraic group G, the identity element e is contained in only one irreducible component of G. This component is denoted by G°. If $G = G^\circ$, then G is called *connected*.

Proposition 2.1.19. *Let G be an algebraic group. Then G° is a normal subgroup of G, the irreducible components of G are exactly the cosets of G°, and $[G : G^\circ]$ is* *finite.*

Proof. Since G° is an irreducible variety, the product $G^\circ \times G^\circ$ is irreducible. The map $m : G^\circ \times G^\circ \to G$ is a morphism of varieties, so it sends irreducible components to irreducible components. Note that, $m(e, e) = e$, so the image of m, $G^\circ G^\circ$, is an irreducible component of G containing e. Since G° is the only irreducible component containing e, we have that $G^\circ = G^\circ G^\circ$. Therefore, G° is closed under multiplication. Now, to show that G° is closed under inverses. As the map $i : G \to G$, $g \mapsto g^{-1}$ is a homomorphism, $i(G^\circ)$ is an irreducible component of G. Also, $i(G^\circ)$ contains e, as $i(e) = e$. Hence $i(G^\circ) = G^\circ$. Thus, G° is closed under inverses, and so it is a subgroup of G. Furthermore, for $g \in G$, the conjugation map $\psi : G \to G$, $x \mapsto gxg^{-1}$, is a homomorphism. So $gG^\circ g^{-1}$ is an irreducible component containing e as $\psi(e) = geg^{-1} = e$. Hence $gG^\circ g^{-1} = G^\circ$. So, G° is a normal subgroup of G. Now, to show that the irreducible components of G are the cosets of G°. Let Y be any irreducible component of G. Take $y \in Y$. The left translation by y^{-1}, $G \to G$, $g \mapsto y^{-1}g$ is a homomorphism. So, $y^{-1}Y$ is an irreducible component of G. It also contains e, as $y \mapsto y^{-1}y = e$. So, $y^{-1}Y = G^\circ$. Hence, $Y = yG^\circ$. Similarly, every coset of G° is an irreducible component being a translate of a component. Since there are only finitely many irreducible components, we deduce that $[G : G^\circ] < \infty$. □

Example 2.1.20. *The additive group $G = \mathbb{F}$ with group operation $m(b_1, b_2) = b_1 + b_2$ and inverse map $i(b) = -b$ is an algebraic group known as G_a.*

Example 2.1.21. *Let $M_n(\mathbb{F})$ be the space of all $n \times n$ matrices with entries from $\mathbb{F}$. The general linear group $GL(n, \mathbb{F}) = \{T \in M_{n\times n}(\mathbb{F}) : \det(T) \neq 0\}$ is an algebraic group. The group operation is matrix multiplication, and the inverse map is matrix inversion. The case of $GL(1, \mathbb{F})$, denoted by $G_m := \mathbb{F}^\times$, is an algebraic group with* *group operation $m(b_1, b_2) = b_1 b_2$ and inverse $i(b) = 1/b$.*

Example 2.1.22. *Any closed subgroup of algebraic group is an algebraic group.*

Proposition 2.1.23. *Suppose that A is a finite-dimensional algebra. Then $\mathrm{Aut}(A)$ is an affine algebraic group.*

Proof. Let $\dim A = n$ with basis $\{b_1, b_2, ..., b_n\}$. Then, $\mathrm{Aut}(A) \leqslant \mathrm{GL}(A) \cong \mathrm{GL}(n, \mathbb{F}) \subseteq M_{n\times n}(\mathbb{F})$. For the linear map $\phi \in \mathrm{Aut}(A)$, let

$$M = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix}$$

be it matrix with respect to the same basis. By definition of matrix representation with respect to a basis, we have that $\phi(b_i) = \sum_{j=1}^{n} x_{ij} b_j$. For $b_i, b_j \in A$, $b_i b_j \in A$. As $\{b_1, b_2, ..., b_n\}$ is a basis of A, $b_i b_j$ can be represented in terms of basis, say $b_i \cdot b_j = \sum_{k=1}^{n} \gamma_{ij}^k b_k$ for some structure constants γ_{ij}^k for $k = 1, 2, \cdots, n$. So,

$$\phi(b_i \cdot b_j) = \phi(\sum_{k=1}^{n} \gamma_{ij}^k b_k) = \sum_{k=1}^{n} \gamma_{ij}^k \phi(b_k) = \sum_{k=1}^{n} \gamma_{ij}^k (\sum_{u=1}^{n} x_{ku} b_u)$$

On the other hand,

$$\phi(b_i) \cdot \phi(b_j) = (\sum_{s=1}^{n} x_{is} b_s) \cdot (\sum_{t=1}^{n} x_{jt} b_t) = \sum_{s,t=1}^{n} x_{is} x_{jt} (b_s \cdot b_t) = \sum_{s,t=1}^{n} x_{is} x_{jt} (\sum_{k=1}^{n} \gamma_{st}^k b_k)$$

As $\phi \in \mathrm{Aut}(A)$, $\phi(b_i \cdot b_j) = \phi(b_i) \cdot \phi(b_j)$. So we have that,

$$\sum_{k=1}^{n} \gamma_{ij}^k \sum_{u=1}^{n} x_{ku} b_u = \sum_{s,t=1}^{n} x_{is} x_{jt} (\sum_{k=1}^{n} \gamma_{st}^k b_k)$$

This gives degree two polynomial equations in the variables x_{ij}, $1 \leqslant i, j \leqslant n$

whose solutions are the elements of $\mathrm{Aut}(A)$. Hence, $\mathrm{Aut}(A)$ is closed in $\mathrm{GL}(n,\mathbb{F})$. Therefore, $\mathrm{Aut}(A)$ is an algebraic group. $\square$

2.2 Lie Algebra of an Algebraic Group

In this section, we will associate a Lie algebra to an algebraic group G. The goal is to show that, when $G = \mathrm{Aut}(A)$, the Lie algebra on *the tangent space of G at the identity* acts by differentiation on the finite-dimensional algebra A.

The content up to the end of this chapter is compiled based on [Hum75].

2.2.1 Tangent Space

Let X be an affine variety with coordinate algebra $\mathbb{F}[X]$. A *derivation* δ of $\mathbb{F}[X]$ is an $\mathbb{F}$-linear map from $\mathbb{F}[X]$ back to itself, such that $\delta(fg) = \delta(f)\cdot g + f\cdot\delta(g)$ for all $f, g \in \mathbb{F}[X]$. The vector space of all derivations of $\mathbb{F}[X]$ is denoted $\mathrm{Der}(\mathbb{F}[X])$. If $\delta : \mathbb{F}[X] \to \mathbb{F}[X]$ is a derivation, then $\overline{\delta} := \mathrm{ev}_x \circ \delta$ is a *point derivation*. For $x \in X$, let $\mathbb{F}_x = \mathbb{F}[X]/I_x$ where $I_x := \ker \mathrm{ev}_x$, the kernel of evaluation at x, is the maximal ideal of $\mathbb{F}[X]$ corresponding to x. Then

$$\mathrm{Der}(\mathbb{F}[X], \mathbb{F}_x) = \left\{\overline{\delta} : \mathbb{F}[X] \to \mathbb{F}_x : \overline{\delta}(fg) = \overline{\delta}(f)\mathrm{ev}_x(g) + \mathrm{ev}_x(f)\overline{\delta}(g)\right\}$$

is the vector space of point derivations at point x.

Definition 2.2.1. Let X be an affine variety, and let $x \in X$. The *tangent space* of X at x, denoted by $T(X)_x$, is defined as the space of all point derivations at the point x. That is, $T(X)_x := \mathrm{Der}(\mathbb{F}[X], \mathbb{F}_x)$.

Definition 2.2.2. Let $\phi : X \to Y$, $x \mapsto y$, be a morphism of affine varieties. The linear map $d\phi_x : T(X)_x \to T(Y)_{\phi(x)}$, $\overline{\delta} \mapsto (d\phi_x)(\overline{\delta})$, where $(d\phi_x)(\overline{\delta})(f) := \overline{\delta}(\phi^* f)$, for $f \in \mathbb{F}[Y]$, is called the *differential* of ϕ at x.

Let E/F be a field extension. A subset $\{e_1, \ldots, e_n\}$ of E is algebraically independent over F if no non-zero polynomial $f(x_1, \cdots, x_n)$ with coefficients from F satisfies that $f(e_1, \ldots, e_n) = 0$. A maximal subset of algebraically independent elements of E over F is called a *transcendence basis* of E/F. The size of transcendence basis is called the *transcendence degree* of E/F.

Definition 2.2.3. Let X be an irreducible variety. The field $\mathbb{F}(X)$ is the field of fractions of $\mathbb{F}[X]$, i.e., elements of $\mathbb{F}(X)$ are f/g, $f, g \in \mathbb{F}[X]$ and $g \neq 0$.

As $\mathbb{F}(X)$ is a finitely generated field extension of $\mathbb{F}$, the transcendence degree of $\mathbb{F}(X)/\mathbb{F}$ is a finite number. This number is defined to be the *dimension* of X, $\dim X$.

Theorem 2.2.4. *Let X be an irreducible affine variety. Then* $\dim T(X)_x \geqslant \dim X$ *for all $x \in X$. Equality holds for all x in some dense open subset of X. In particular, the set of smooth points is nonempty.*

The points $x \in X$, where $\dim T(X)_x = \dim X$, are called *simple points.* If all points of X are *simple*, then X is called *smooth.*

Proposition 2.2.5. *Let G be a connected algebraic group. Then G is smooth, i.e.,* $\dim T(G)_x = \dim G$ *for all $x \in G$. In particular,* $\dim T(G)_e = \dim G$.

Proof. Recall that the group product $m : G \times G \to G$, $m(x, y) \mapsto xy$ is by definition a morphism of varieties. After fixing $y \in G$, define $\phi_y : G \to G$ by $\phi_y(x) = m(x, y) = xy$. That is, $\phi_y(x) \mapsto xy$. As m is a morphism of varieties, ϕ_y

is also a morphism of varieties. Note that for all $x \in G$, $\phi_y \phi_{y^{-1}}(x) = \phi_y(xy^{-1}) = xy^{-1}y = x$. Similarly, $\phi_{y^{-1}}\phi_y(x) = x$. Hence, $\phi_y \circ \phi_{y^{-1}} = \phi_{y^{-1}} \circ \phi_y = \mathrm{Id}$. Therefore, $\phi_y : G \to G$ is an isomorphism of varieties. Now, let $x_0 \in G$ be a simple point and take $y = x_0^{-1}x$. Notice that, $\phi_y(x_0) = x_0 y = x_0 x_0^{-1} x = x$. Take $\phi_y : G \to G$ and its differential map at x_0 and x respectively, $d\phi_y : T(G)_{x_0} \to T(G)_x$. Since ϕ_y is an isomorphism, $d\phi_y$ is also an isomorphism of vector spaces. As x_0 is simple, then by Theorem 2.2.4, $\dim G = \dim T(G)_{x_0}$. But $T(G)_{x_0} \cong T(G)_x$, so $\dim G = \dim T(G)_x$. Hence, x is also simple point. As this holds for arbitrary $x \in G$, each point of G is simple. Therefore, G is smooth. □

2.2.2 Lie Algebra

Definition 2.2.6. A Lie algebra is a vector space V over some field $\mathbb{F}$ together with a bilinear map $[\cdot,\cdot] : V \times V \to V$, called the Lie bracket, satisfying the following axioms:

- (alternativity): $[x, x] = 0$, for all $x \in V$.
- (Jacobi identity): $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in V$.

Note that by expanding $[x + y, x + y]$ using the bilinearity of $[\cdot,\cdot]$ together with the alternativity axiom, we obtain $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$. Hence, we have $[x, y] = -[y, x]$, which implies that $[\cdot,\cdot]$ is anti-symmetric.

Example 2.2.7. *Let V be any associative $\mathbb{F}$-algebra. Define the bracket $[\cdot,\cdot]$ to be the commutator operation, i.e, $[x, y] = xy - yx$. Then V together with $[\cdot,\cdot]$ is a Lie algebra.*

Example 2.2.8. *Let G be the general linear group* $\mathrm{GL}(n, \mathbb{F})$*. The tangent space* $T(G)_e$ *is identified as the general linear Lie algebra* $\mathfrak{gl}(n, \mathbb{F})$*, which is defined as the vector space of all* $n \times n$ *matrices over* $\mathbb{F}$*. The Lie bracket operation in* $\mathfrak{gl}(n, \mathbb{F})$ *is given by the commutator of matrices.*

Proposition 2.2.9. *Let* $\mathcal{B}$ *be an arbitrary algebra over a field* $\mathbb{F}$*. The set of all derivations of* $\mathcal{B}$*, Der*$(\mathcal{B})$*, forms a Lie algebra.*

Proof. Note that $\mathrm{Der}(\mathcal{B})$ consists of derivations which are linear maps from $\mathcal{B}$ to $\mathcal{B}$, so $\mathrm{Der}(\mathcal{B}) \subseteq \mathrm{End}_{\mathbb{F}}(\mathcal{B})$, where this is an associative algebra under composition. Thus, it is enough to show that $\mathrm{Der}(\mathcal{B})$ is closed under the bracket operation. Suppose that $\delta, \psi \in \mathrm{Der}(\mathcal{B})$ and let $x, y \in G$. Then we have that

$$\begin{aligned}
[\delta, \psi](xy) &= \delta\psi(xy) - \psi\delta(xy) = \delta(\psi(xy)) - \psi(\delta(xy)) \\
&= \delta(\psi(x)y + x\psi(y)) - \psi(\delta(x)y + x\delta(y)) \\
&= \delta(\psi(x)y) + \delta(x\psi(y)) - \psi(\delta(x)y) - \psi(x\delta(y)) \\
&= \delta(\psi(x))y + \psi(x)\delta(y) + \delta(x)\psi(y) + x\delta(\psi(y)) - \\
&\psi(\delta(x))y - \delta(x)\psi(y) - \psi(x)\delta(y) - x\psi(\delta(y)) \\
&= \delta(\psi(x))y - \psi(\delta(x))y + x\delta(\psi(y)) - x\psi(\delta(y)) \\
&= [\delta, \psi](x)y + x[\delta, \psi](y).
\end{aligned}$$

Hence, $\mathrm{Der}(\mathcal{B})$ is a Lie algebra. □

Corollary 2.2.10. *Let G be an algebraic group,* $B = \mathbb{F}[G]$*. The set of all derivations of B, Der*(B)*, forms a Lie algebra.*

Let $g \in G$. The map $\lambda_g : \mathbb{F}[G] \to \mathbb{F}[G]$, given by

$$(\lambda_g k)(h) = k(g^{-1}h)$$

is called the *left-translation* by g. A derivation $\delta : \mathbb{F}[G] \to \mathbb{F}[G]$ is *left-invariant* if $\lambda_g \delta = \delta \lambda_g$ for all $g \in G$. Define $L(G)$ to be the $\mathbb{F}$-vector space of all left-invariant derivations. That is, $L(G) := \{\delta \in \text{Der}(\mathbb{F}[G]) \mid \lambda_g \delta = \delta \lambda_g\}$.

Proposition 2.2.11. *The space $L(G)$ is a Lie algebra, a subalgebra of Der$(\mathbb{F}[G])$.*

Proof. It suffices to show that the bracket of two left invariant derivations is again left invariant. Take two derivations $\delta_1, \delta_2 \in L(G)$, and let λ_g be a left translation by $g \in G$. Then, $\lambda_g[\delta_1, \delta_2] = \lambda_g(\delta_1\delta_2 - \delta_2\delta_1) = \lambda_g(\delta_1\delta_2) - \lambda_g(\delta_2\delta_1) = (\delta_1\delta_2)\lambda_g - (\delta_2\delta_1)\lambda_g = [\delta_1, \delta_2]\lambda_g$, as λ_g commutes with δ_1 and δ_2. So, $L(G)$ is a Lie algebra, a subalgebra of Der$(\mathbb{F}[G])$. □

The following lemma allows us to have a Lie algebra structure on the tangent space $T(G)_e$.

Lemma 2.2.12. *The map $L(G) \to T(G)_e$, $D \mapsto \overline{D} := ev_e \circ D$, is a linear isomorphism from the Lie algebra of G, $L(G)$, to the tangent space $T(G)_e$.*

Proof. Consider the inverse linear isomorphism $T(G)_e \to L(G)$, $C \mapsto \tilde{C}$, where $\tilde{C}$ is defined as

$$\tilde{C}(k)(g) := C(\lambda_{g^{-1}}k), \text{ where } k \in \mathbb{F}[G], g \in G, \text{ and } C \in T(G)_e.$$

Let us check that $\tilde{C}$ is a left-invariant derivation. First, we check that it is a derivation. That is, we show that $\tilde{C}(k_1k_2)(g) = [\tilde{C}(k_1)(g)]k_2(g) + k_1(g)[\tilde{C}(k_2)(g)]$.

$$\begin{aligned}
\tilde{C}(k_1k_2)(g) &= C(\lambda_{g^{-1}}(k_1k_2)) \\
&= C((\lambda_{g^{-1}}k_1)(\lambda_{g^{-1}}k_2)) \\
&= C(\lambda_{g^{-1}}k_1)[(\lambda_{g^{-1}}k_2)(e)] + [(\lambda_{g^{-1}}k_1)(e)]C(\lambda_{g^{-1}}k_2) \\
&= C(\lambda_{g^{-1}}k_1)k_2(g) + k_1(g)C(\lambda_{g^{-1}}k_2) \\
&= [\tilde{C}(k_1)(g)]k_2(g) + k_1(g)[\tilde{C}(k_2)(g)]
\end{aligned}$$

Now, we check that $\tilde{C}$ is left-invariant. That is, we show that $\tilde{C}(\lambda_g k)(h) = \lambda_g \tilde{C}(k)(h)$, for $g, h \in G$ and $k \in \mathbb{F}[G]$. Note that, $\tilde{C}(\lambda_g k)(h) = \tilde{C}(k)(g^{-1}h) = C(\lambda_{(g^{-1}h)^{-1}}k) = C(\lambda_{h^{-1}}\lambda_g k) = \tilde{C}(\lambda_g k)(h)$. Thus, $\tilde{C}$ is a left-invariant derivation. Remains to show that the maps $L(G) \to T(G)_e$ and $T(G)_e \to L(G)$ are two sided inverses. That is, to show that $\overline{\tilde{D}} = D$ and $\overline{\tilde{C}} = C$. First, note that, $\tilde{\overline{D}}(k)(g) = \overline{D}(\lambda_{g^{-1}}k) = \mathrm{ev}_e(D(\lambda_{g^{-1}}k)) = \lambda_{g^{-1}}(D(k))(e) = D(k)(g)$. Also, $\overline{\tilde{C}}(k) = \tilde{C}(k)(e) = C(\lambda_{e^{-1}}k) = C(k)$, as desired. Therefore, $L(G) \cong T(G)_e$. □

Using this isomorphism, we can define a Lie bracket on the tangent space $T(G)_e$ by letting $[x, y] = \mathrm{ev}_e \circ [\tilde{x}, \tilde{y}]$. Let $\mathfrak{g} := T(G)_e$. Since G is smooth and $L(G) \cong T(G)_e := \mathfrak{g}$, it follows that $\dim L(G) = \dim \mathfrak{g} = \dim G^\circ$.

2.3 Actions of Algebraic Group and Lie Algebra

Definition 2.3.1. Let G be a group and V be an $\mathbb{F}$-vector space. A *representation* of G on V is a group homomorphism from G to the general linear group of V, denoted $\mathrm{GL}(V)$. In other words, a representation of G is a map $\sigma : G \to \mathrm{GL}(V)$ that satisfies $\sigma(g_1g_2) = \sigma(g_1)\sigma(g_2)$ for all $g_1, g_2 \in G$. Here, $\mathrm{GL}(V)$ refers to the group of invertible linear transformations of V.

Recall that, if $\sigma : G \to G'$ is homomorphism of algebraic groups, then $d\sigma : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras.

Remark. If $\sigma : G \to \mathrm{GL}(V)$ is a representation, we will write $g \cdot v$ instead of $\sigma(g)(v)$, for $g \in G$ and $v \in V$. Similarly, for $\delta \in \mathfrak{g}$, we will write $\delta \cdot v$ instead of $d\sigma(\delta)(v)$.

Definition 2.3.2. Let V be an $\mathbb{F}$-vector space. The *dual space* of V, denoted as V^*, is the set of all linear functionals from V to $\mathbb{F}$. That is, $V^* := \{\beta : V \to \mathbb{F} \mid \beta \text{ is linear}\}$. Here, a linear functional is a linear map that assigns scalars from $\mathbb{F}$ to vectors in V.

Let $\sigma : G \to \mathrm{GL}(V)$ be a representation of G, where G is an algebraic group. Then σ induces an action of G on V given by the rule $g \cdot v := \sigma(g)(v)$, where $g \in G$ and $v \in V$. The *dual representation* σ^* of σ is a representation of G, $\sigma^* : G \to \mathrm{GL}(V^*)$ defined as $(g \cdot f)(v) = f(\sigma(g^{-1})(v)) = f(g^{-1} \cdot v)$, where $g \in G, f \in V^*$, and $v \in V$.

Again, let G be an algebraic group and suppose that $\sigma : G \to \mathrm{GL}(V_1)$ and $\tau : G \to \mathrm{GL}(V_2)$ are representations of G on $\mathbb{F}$-vector spaces V_1 and V_2 respectively. For $g \in G$, the operation, $g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2)$, defines a representation $G \to \mathrm{GL}(V_1 \otimes V_2)$, where $V_1 \otimes V_2$ is the tensor product vector space which has as a basis the set of all $v_1 \otimes v_2$, where v_1 runs through a basis of V_1 and v_2 runs through a basis of V_2.

The following proposition describes the actions of Lie algebra $\mathfrak{g}$ on V_1^* and $V_1 \otimes V_2$.

Proposition 2.3.3. *Let $\sigma : G \to \mathrm{GL}(V_1)$ and $\tau : G \to \mathrm{GL}(V_2)$ be representations of an algebraic group G. Then,*

- $\mathfrak{g}$ *acts on* V_1^* *by the rule* $(\delta \cdot f)(v_1) := -f(\delta \cdot v_1)$*, where* $g \in G, \delta \in \mathfrak{g}, f \in V_1^*, v \in V$.

- $\mathfrak{g}$ *acts on* $V_1 \otimes V_2$ *as* $\delta \cdot (v_1 \otimes v_2) := (\delta \cdot v_1) \otimes v_2 + v_1 \otimes (\delta \cdot v_2)$*, where* $g \in G, \delta \in \mathfrak{g}$*, and* $v_1 \otimes v_2$ *is a typical generator of* $V_1 \otimes V_2$.

There is an identification of the vector space $V_1^* \otimes V_2$ with $\mathrm{Hom}(V_1, V_2)$ (the set of all linear maps from V_1 to V_2) given by $(f \otimes v_2)(v_1) = f(v_1)v_2$.

Definition 2.3.4. Let G be a closed subgroup of $\mathrm{GL}(V)$. A non-zero vector $v \in V$ is called *invariant* if $g \cdot v = v$ for all $g \in G$.

Proposition 2.3.5. *Let* G *be a closed subgroup of* $\mathrm{GL}(V)$*. If* $u \in \mathbb{F}^n$ *is an invariant vector under* G*, then all elements of* $\mathfrak{g}$ *send* u *to* 0.

Proof. Let $u \in \mathbb{F}^n$ be a vector which is fixed by G, i.e., $gu = u$ for all $g \in G$. Let $V =$ span $\{u\}$ be a subspace of $\mathbb{F}^n$. Then, $\mathrm{GL}(V) \cong \mathrm{GL}(1, \mathbb{F}) \cong \mathbb{F}^\times$. That is, define $\mathrm{GL}(V) \to \mathbb{F}^\times$ by $\{u \to cu\} \mapsto c$. Now, consider the action $\sigma : G \to \mathrm{GL}(V)$, given by $g \mapsto \phi_g : u \mapsto g \cdot u$. As G fixes u, $\phi_g = e$. Thus, σ can be restricted to $\sigma : G \to e \subseteq \mathrm{GL}(V)$. Hence this map induces $d\sigma :$ Lie$(G) \to$ Lie$(e) \subseteq$ Lie$(\mathrm{GL}(V))$. Therefore, $d\sigma : \mathfrak{g} \to \{0\} \subseteq \mathfrak{gl}(V)$. Thus, $d\sigma(\delta)u = 0$ for all $\delta \in \mathfrak{g}$. □

Theorem 2.3.6. *Let* $\mathcal{A}$ *be a finite-dimensional* $\mathbb{F}$*-algebra, and let* G *be a closed subgroup of* $\mathrm{GL}(\mathcal{A})$ *consisting of algebra automorphisms. Then,* $\mathfrak{g}$ *consists of derivations of* $\mathcal{A}$.

Proof. To give $\mathcal{A}$ the structure of $\mathbb{F}$-algebra is just to specify a bilinear map $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, or equivalently an $\mathbb{F}$-linear map $\psi' : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Note that, $\psi' \in \mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$. As $\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2$, so $\mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A}) \cong (\mathcal{A} \otimes$

$\mathcal{A})^* \otimes \mathcal{A} \cong \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}$. Let ψ corresponds to $t \in \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}$. For $\sigma \in \mathrm{GL}(\mathcal{A})$ to be an automorphism of $\mathcal{A}$, i.e., $\sigma(u \cdot v) = \sigma(u) \cdot \sigma(v)$ for all $u, v \in \mathcal{A}$, we must have that $\sigma \cdot (\psi(u \otimes v)) = \psi(\sigma(u) \otimes \sigma(v))$. That is, it is necessary and sufficient that $\sigma \cdot \psi = \psi$. Therefore, each $g \in G$ fixes t. Hence, using Proposition 2.3.4 each δ in $\mathfrak{g}$ must send t to 0. Now, the action of $\delta \in \mathfrak{gl}(\mathcal{A})$ on $\mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$, $\mathfrak{gl}(\mathcal{A}) \times \mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A}) \to \mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$ is given by $(\delta, h) \mapsto \delta \cdot h$, where $\delta \cdot h$ is defined as

$$(\delta \cdot h)(v_1 \otimes v_2) = \delta \cdot (h(v_1 \otimes v_2)) - h(\delta \cdot v_1 \otimes v_2 + v_1 \otimes \delta \cdot v_2)$$

for $h \in \mathrm{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A}), v_1, v_2 \in \mathcal{A}$. Let us explain the above relation, note that, $\delta \cdot (h(v_1 \otimes v_2)) = (\delta \cdot h)(v_1 \otimes v_2) + h(\delta \cdot (v_1 \otimes v_2)) = (\delta \cdot h)(v_1 \otimes v_2) + h(\delta \cdot v_1 \otimes v_2 + v_1 \otimes \delta \cdot v_2)$. Hence, $(\delta \cdot h)(v_1 \otimes v_2) = \delta \cdot (h(v_1 \otimes v_2)) - h(\delta \cdot v_1 \otimes v_2 + v_1 \otimes \delta \cdot v_2)$. Now, to say that $\delta \cdot t = 0$, i.e., in this case to say $\delta \cdot h = 0$ gives

$$\delta \cdot (h(v_1 \otimes v_2)) = h(\delta \cdot v_1 \otimes v_2 + v_1 \otimes \delta \cdot v_2) = h(\delta \cdot v_1 \otimes v_2) + h(v_1 \otimes \delta \cdot v_2)$$

Notice that h here is giving only algebra structure. Hence, one can understand $h(v_1 \otimes v_2) := v_1 \cdot v_2$, where $v_1 \cdot v_2$ is just algebra level multiplication. So, we have

$$\delta(v_1 \cdot v_2) = (\delta v_1) \cdot v_2 + v_1 \cdot (\delta v_2)$$

So, $\mathfrak{g}$ acts as derivation. □

Now, we have come to the main result of this chapter.

Corollary 2.3.7 ([Gor+23]). *Suppose that $\mathcal{F}$ is a fusion law that does not contain $\frac{1}{2}$. Then any finite-dimensional $\mathcal{F}$-axial algebra A has a finite automorphism group.*

Proof. Let $G = \mathrm{Aut}(A)$. Then, by Theorem 2.1.23, G is an affine algebraic group. If $\dim G = 0$, then G is finite. Suppose that $\dim G > 0$. Then, we know that $\dim \mathfrak{g} = \dim G$, so $\mathfrak{g}$ is of positive dimension, hence non-zero. Now, by Theorem 2.3.6, we know that $\mathfrak{g}$ acts as derivations on A. That is, let $\delta \in \mathfrak{g}$, then for every axis $a \in A$, $a = a^2$ implies that

$$\delta a = \delta(a)a + a\delta(a) = 2a\delta(a)$$

So, $a\delta(a) = \frac{1}{2}\delta(a)$. Since $\frac{1}{2} \notin \mathcal{F}$, it cannot be an eigenvalue of the adjoint of a. So, we deduce that $\delta(a) = 0$ for all axes. Then $\delta = 0$ which is a contradiction. Therefore, $G = \mathrm{Aut}(A)$ is finite. $\square$

CHAPTER 3

MATSUO ALGEBRAS

In this chapter, we introduce another type of axial algebra called Matsuo algebra, which satisfies a fusion law similar to the one for Jordan algebras.

3.1 3-Transposition Groups

Definition 3.1.1. Let D be a normal subset of involutions of a group G satisfying the following conditions:

(i) G is generated by D; and

(ii) for all $a, b \in D$, $|ab| \leqslant 3$.

Then (G, D) is a group of 3-transpositions.

Remark. If D in the above definition is a single conjugacy class, then (G, D) is called a 3-*transposition group.* This distinction in names is suggested in [HS21].

The following theorem, due to Bernd Fischer, provides a classification for all finite 3-transposition groups.

Theorem 3.1.2 ([CH95]). *Let (G, D) be a finite 3-transposition group which has no non-central solvable normal subgroups. Then, up to a center, we identify D as one of the following classes:*

(a) *The transposition class of a symmetric group S_n;*

(b) *The transvection class of $O^{\pm}_n(2)$, where $O^{\pm}_n(2)$ is the isometry group of a non-degenerate orthogonal space over $\mathbb{F}_2$;*

(c) *The transvection class of $Sp_{2n}(2)$, where $Sp_{2n}(2)$ is the isometry group of a non-degenerate symplectic space over $\mathbb{F}_2$;*

(d) *The reflection class of $O^{\pm}_n(3)$, where $O^{\pm}_n(3)$ denotes the isometry group of a non-degenerate orthogonal space over $\mathbb{F}_3$;*

(e) *The transvection class of $\mathrm{SU}_n(2)$, the isometry group of a non-degenerate unitary space over $\mathbb{F}_4$;*

(f) *A unique class of involutions in one of the five groups $\Omega^+_8(2) : S_3$, $\Omega^+_8(3) : S_3$, Fi_{22}, Fi_{23}, or Fi_{24}.*

3.2 Fischer Spaces and Matsuo Algebras

A pair $(\mathcal{P}, \mathcal{L})$ that consists of a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$, where $\mathcal{L} \subseteq 2^{\mathcal{P}}$ is called a *point-line geometry*. Let x, y be distinct points in $\mathcal{P}$. We write $x \sim y$ if there is a line passing through x and y, that is, x and y are *collinear*. If there is no such line, we write $x \nsim y$. A point-line geometry where every line consists precisely of three points and any two points are contained in at most one line is called a *partial triple system*. In this system, if x, y are collinear, then there exists a unique line through x and y. A pair $(\mathcal{P}', \mathcal{L}')$, where $\mathcal{P}' \subseteq \mathcal{P}$ is a *subspace* of

$(\mathcal{P}, \mathcal{L})$ if every line $l \in \mathcal{L}$ consisting of at least two points from $\mathcal{P}'$ is contained in $\mathcal{P}'$. So, $l \in \mathcal{L}' \subseteq \mathcal{L}$. The subspace generated by three points that do not lie on the same line is a *plane*.

Definition 3.2.1. A *Fischer space* is a partial triple system $(\mathcal{P}, \mathcal{L})$ such that if l_1, l_2 are two intersecting lines then the subspace generated by l_1, l_2 is isomorphic to the dual affine plane of order 2 or isomorphic to the affine plane of order 3.

We can associate with every group of 3-transpositions (G, D), a geometry $\Upsilon = (\mathcal{P}, \mathcal{L})$. This is done by letting $\mathcal{P} = D$, and $\mathcal{L} = \{\{x, y, z\} \subseteq D : |xy| = 3, z = x^y\}$. In fact, this geometry is a Fischer space. Conversely, the 3-transposition group can be recovered up to the center of G from Υ.

Let $\Upsilon = (\mathcal{P}, \mathcal{L})$ be a partial triple system. In the following, we give the general definition of a Matsuo algebra associated to Υ.

Definition 3.2.2. Let $\mathbb{F}$ be a field of characteristic not 2, and $\eta \in \mathbb{F}$ be different from $1, 0$. Then, the associated Matsuo algebra $M = M_\eta(\Upsilon)$, defined over $\mathbb{F}$, is the algebra whose basis is $\mathcal{P}$ and the multiplication is given by:

$$x \cdot y = \begin{cases} x, & \text{if } x = y \\ 0, & \text{if } x \nsim y \\ \frac{\eta}{2}(x + y - z), & \text{if } x \sim y, \{x, y, z\} \text{ is a line.} \end{cases}$$

Depending on the number of points in Υ, we have different names for M, as in Table 3.1. These names were used in [HRS15b].

Proposition 3.2.3 ([Gal+16])**.** *If Υ is disconnected with connected components Υ_i, $i \in I$, then $M_\eta(\Upsilon) = \oplus_{i \in I} M_\eta(\Upsilon_i)$.*

Υ consists of a single axis	$M \cong \mathbb{F}$	$1A$
Υ consists of two non-collinear points	$M \cong \mathbb{F}^2$	$2B$
Υ contains 3 points that form a line	M is 3-dimensional	$3C(\eta)$

Table 3.1: Names of small Matuso algebras

We see that each $x \in \mathcal{P}$ is an idempotent in $M_\eta(\Upsilon)$. In fact, these idempotents are axes, and according to [HRS15a] they satisfy the fusion law $\mathcal{J}(\eta)$ given in Table 3.2. This shows that Matsuo algebras are of Jordan type η. Apart from the case

$\diamond$	1	0	η
1	1		η
0		0	η
η	η	η	1,0

Table 3.2: Fusion law $\mathcal{J}(\eta)$

$\eta = \frac{1}{2}$, Matsuo algebras of Jordan type η have been classified in [HRS15a].

The following theorem is an important result in the classification of algebras of Jordan type $\eta \neq \frac{1}{2}$. It was first introduced in [HRS15a] then corrected and completed in [HSS18a].

Theorem 3.2.4. *Every Jordan algebra $\mathcal{J}(\eta)$ with $\eta \neq \frac{1}{2}$ is either a Matsuo algebra or a quotient of Matsuo algebra.*

In the following definition, we introduce a property that usually appears in the common examples of axial algebras.

Definition 3.2.5. A *Frobenius form* on an axial algebra A is a non-zero bilinear form $(\cdot, \cdot)$ such that (u, vw)=(uv, w) for all $u, v, w \in A$.

Although the Frobenius form is not assumed in case of algebra of Jordan type η, it was shown in [HSS18b] that each algebra of Jordan type η admits a unique

Frobenius form such that every axis has length one. For a Matsuo algebra $M = M_\eta(G, D)$ this unique form is given by:

$$(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \nsim y \\ \frac{\eta}{2}, & \text{if } x \sim y \end{cases}$$

for all $x, y \in D$.

3.3 Symplectic Example

Let V be an m-dimensional vector space over a field $\mathbb{F}$, and $(\cdot,\cdot) : V \times V \to \mathbb{F}$ be a bilinear form. The form $(\cdot,\cdot)$ is said to be *degenerate* if there exists a non-zero element $w \in V$ such that $(w, v) = 0$ for all $v \in V$. In other words, the form is degenerate if $V^\perp = \{w \in V \mid (w, v) = 0 \text{ for all } v \in V\} \neq \{0\}$. Correspondingly, if $V^\perp = \{0\}$ we call the form *non-degenerate*.

Definition 3.3.1. The bilinear form $(\cdot,\cdot)$ is called *symplectic* if $(v, v) = 0$ for all $v \in V$. A vector space that admits a non-degenerate symplectic form is called a *symplectic space*.

Proposition 3.3.2. *Suppose that V is a finite-dimensional vector space with a non-degenerate symplectic form. Then V has an even dimension.*

Definition 3.3.3. Let $(\cdot,\cdot)$ be a symplectic form on V. The *symplectic group*, denoted by $Sp(V)$, is the set of all elements of $GL(V)$ that preserve the form $(\cdot,\cdot)$, i.e.,

$$Sp(V) = \{\alpha \in GL(V) \mid (x^\alpha, y^\alpha) = (x, y) \text{ for all } x, y \in V\}$$

3.3.1 Transvections

Let V be a vector space over $\mathbb{F}$, of dimension m, and $T \leqslant V$ be a hyperplane, i.e., T has dimension $m-1$. A *transvection* on V is defined as:

Definition 3.3.4. A non-identity linear map $r : V \to V$ is called a transvection if

(i) $r(v) = v$ for all $v \in T$,

(ii) $r(v) - v \in T$ for all $v \in V$.

From now on, we assume that V is a symplectic space over a field $\mathbb{F}$ of characteristic two. For a non-zero vector $u \in V$ define the map $r_u : V \to V$ by

$$r_u(v) = v + (v, u)u, \text{ for all } v \in V$$

Proposition 3.3.5. *The map r_u is a linear map.*

Proof. Suppose $x, y \in V$ and $\mu \in \mathbb{F}_2$. Then

$$\begin{aligned} r_u(x + \mu y) &= (x + \mu y) + (x + \mu y, u)u \\ &= x + \mu y + ((x, u) + (\mu y, u))u \\ &= x + \mu y + (x, u)u + \mu(y, u)u \\ &= x + (x, u)u + \mu(y + (y, u)u) \\ &= r_u(x) + \mu r_u(y) \end{aligned}$$

So, r_u is a linear map. □

The following proposition shows that r_u is an involution, therefore it is invertible.

Proposition 3.3.6. *The map r_u is an involution.*

Proof.

$$\begin{aligned} r_u^2(v) &= r_u(v + (v,u)u) \\ &= v + (v,u)u + (v + (v,u)u, u)u \\ &= v + (v,u)u + (v,u)u + ((v,u)u, u)u \\ &= v + 2(v,u)u + (u,v)(u,u)u \\ &= v. \end{aligned}$$

The penultimate equality is due to the fact that $\mathbb{F}$ has characteristic 2, and that $(u, u) = 0$. □

Proposition 3.3.7. *The map r_u preserves the form $(\ ,\ )$, i.e., $(r_u(v), r_u(w)) = (v, w)$ for all v,w in V.*

Proof. For any u,v,w in V, we have

$$\begin{aligned} (r_u(v), r_u(w)) &= (v + (v,u)u, w + (w,u)u) \\ &= (v,w) + ((v,u)u, w) + (v, (w,u)u) + ((v,u)u, (w.u)u) \\ &= (v,w) + (v,u)(u,w) + \overline{(w,u)}(v,u) + 0 \text{ (as } (u,u) = 0 \text{)} \\ &= (v,w) + (v,u)(u,w) + (u,w)(v,u) \\ &= (v,w) + 2(v,u)(u,w) \\ &= (v,w) + 0 \\ &= (v,w). \end{aligned}$$

The penultimate equality is due to the fact that $\mathbb{F}$ has characteristic 2. □

Therefore, we deduce that $r_u \in Sp(V)$.

Proposition 3.3.8. *The map r_u is a transvection.*

Proof. Let T be the hyperplane $u^\perp$. Then, for all $v \in T$, we have that $(v, u) = 0$, so $(v, u)u = 0$. Hence, $r_u(v) = v$ for all $v \in T$. For the second property, note that $r_u(v) - v = (v, u)u$. Since $(u, u) = 0$, so $u \in T$. Also, $(v, u)u \in T$ as $(v, u)u$ is a scalar multiple of u. Therefore, $r_u(v) - v \in T$ for all $v \in V$. Thus, r_u is a transvection on V. $\square$

Note that, in the previous proof, r_u is the identity if and only if u lies in the radical of $(\cdot, \cdot)$. In this case we exclude r_u from being called a transvection.

Next, we will verify that the set of all transvections forms a class of 3-transpositions in $Sp_{2m}(2)$. In the following lemma we use the notation v^{r_u} instead of $r_u(v)$, i.e., $v^{r_u} := r_u(v) = v + (v, u)u$.

Lemma 3.3.9. *Let β be an element of $Sp_{2m}(2)$, then $r_u^\beta = r_{u^\beta}$.*

Proof. Consider an arbitrary vector $v \in V$. Then

$$
\begin{aligned}
v^{r_u^\beta} &= v^{\beta^{-1} r_u \beta} = ((v^{\beta^{-1}})^{r_u})^\beta \\
&= (v^{\beta^{-1}} + (v^{\beta^{-1}}, u)u)^\beta \\
&= v^{\beta^{-1}\beta} + (v^{\beta^{-1}}, u)u^\beta \\
&= v + (v^{\beta^{-1}}, u)u^\beta \\
&= v + (v^{\beta^{-1}\beta}, u^\beta)u^\beta \\
&= v + (v, u^\beta)u^\beta \\
&= v^{r_{u^\beta}}
\end{aligned}
$$

Therefore, $r_u^\beta = r_{u^\beta}$. So, the set of all transvections is closed under conjugation. $\square$

Proposition 3.3.10. *For $u, v \in V$, r_u and r_v commute if $(u, v) = 0$, and $r_v^{r_u} = r_u^{r_v}$ if $(u, v) = 1$.*

Proof. We have two situations. If $(u, v) = 0$, then $r_u(v) = v$. So, $r_v^{r_u} = r_{r_u(v)} = r_v$. Thus, r_u, r_v commute. In other words, $|r_u r_v| = 2$. Suppose $(u, v) = 1$. Then, $r_u(v) = v + (v, u)u = v + u$. So, $r_v^{r_u} = r_{r_u(v)} = r_{u+v} = r_{v+u} = r_{r_v(u)} = r_u^{r_v}$. Moreover, $r_u^{r_v} = r_v^{r_u}$ implies that $r_v^{-1} r_u r_v = r_u^{-1} r_v r_u$, but $r_u^{-1} = r_u$ (as r_u is an invertible involution), so we have that $r_v r_u r_v = r_u r_v r_u$. By multiplying the left side by $(r_v^{-1} r_u^{-1} r_v^{-1})$ we get,

$$1 = r_v^{-1} r_u^{-1} r_v^{-1} r_u r_v r_u = r_v r_u r_v r_u r_v r_u = (r_v r_u)^3$$

Thus, $|r_v r_u| = 3$. $\square$

Corollary 3.3.11. *The set $D := \{r_u \mid u \in V\}$ of all transvections in V, forms a class of 3-transpositions in the symplectic group $Sp_{2m}(2)$.*

The symplectic group $\mathrm{Sp}_{2m}(2)$ is generated by transvections. Since V is defined over $\mathbb{F}_2$, then every one-space in V contains a unique non-zero vector. Therefore, we have a bijection between vectors and one-spaces. Also, the map that sends $v \mapsto r_v$ is a bijection between one-spaces and transvections. Therefore, in the definition of Matsuo algebra we can identify transvections with one-spaces.

Definition 3.3.12. A pair of vectors (x, y) is called *hyperbolic* if $(x, y) = 1$. The subspace $\langle x, y \rangle$ is called a *hyperbolic plane.*

Now, since the symplectic group over $\mathbb{F}_2$ is a 3-transposition group generated by transvections, we can associate a Fischer space $\Upsilon = (\mathcal{P}, \mathcal{L})$ to it. The point set of

Υ is the set of singular one-spaces of V, and two points $\langle x \rangle$ and $\langle y \rangle$ are collinear if (x, y) is a hyperbolic pair. In this case, the triple $\{\langle x \rangle, \langle y \rangle, \langle x + y \rangle\}$ is a line in $\mathcal{L}$, where $\langle x + y \rangle$ corresponds to the transvection $r_x^{r_y} = r_{x+y}$. So, the Matsuo algebra $M = M_\eta(\Upsilon)$, that is associated to the symplectic group $Sp_{2m}(2)$, is defined as follows:

Definition 3.3.13. Let $\eta \in \mathbb{K} \setminus \{0, 1\}$ where $\mathbb{K}$ is a field with characteristic not two. The Matsuo algebra $M = M_\eta(\Upsilon)$ over $\mathbb{K}$ corresponding to Υ, has as basis the set of singular one-spaces of V, and the product on D is given by

$$\langle x \rangle \cdot \langle y \rangle = \begin{cases} \langle x \rangle, & \text{if } \langle x \rangle = \langle y \rangle, \\ 0, & \text{if } (x, y) = 0, \\ \frac{\eta}{2}(\langle x \rangle + \langle y \rangle - \langle x + y \rangle), & \text{if } (x, y) = 1. \end{cases}$$

CHAPTER 4

AUTOMORPHISM GROUP OF A MATSUO ALGEBRA

In this chapter, in Theorem 4.4.5, we show that, in almost all cases, the class D of axes of $M = M_\eta(G, D)$ is the set of all axes of Jordan type η in M. As we will see in Theorem 4.5.7, this is equivalent to show that the full automorphism group of M is, in almost all cases, the same as the automorphism group of the underlying 3-transposition group (G, D). Also, we look for the exceptions to this general situation. That is, we give examples of Matsuo algebras with additional axes and larger groups of automorphisms.

We will only consider the case $\eta \neq \frac{1}{2}$. In this case, according to Theorem 3.2.4, every algebra of Jordan type η is either a Matsuo algebra or a quotient of Matsuo algebra.

4.1 Diagram of 3-Transposition Group

Before defining the diagram, we will introduce the definition of Matsuo algebra with respect to a 3-transposition group (G, D).

Definition 4.1.1. Let $\eta \in \mathbb{F}\backslash\{0,1\}$ where $\mathbb{F}$ is a field of characteristic not 2. Given a 3-transposition group (G, D), the Matsuo algebra $M = M_\eta(G, D)$ associated to (G, D) is the algebra whose basis is D and the multiplication on D is given by,

$$c \cdot d = \begin{cases} c, & \text{if } c = d, \\ 0, & \text{if } cd \text{ is of order } 2, \\ \frac{\eta}{2}(c + d - e), & \text{if } cd \text{ is of order } 3 \text{ and } e = c^d. \end{cases}$$

Recall that for every axis $c \in D$ we have a Miyamoto involution τ_c. These involutions generate the Miyamoto group of M, $\mathrm{Miy}(M)$, which is a subgroup of $\mathrm{Aut}(M)$. Additionally, D is a closed set of axes of Jordan type η, that is, $D^{\tau_d} = D$ for all $\tau_d \in \mathrm{Miy}(M)$.

Recall that a Matsuo algebra admits a Frobenius form which is defined on D as follows:

$$(c, d) = \begin{cases} 1, & \text{if } |cd| = 1, \\ 0, & \text{if } |cd| = 2, \\ \frac{\eta}{2}, & \text{if } |cd| = 3. \end{cases}$$

The *radical* of the Frobenius form on M, denoted by $M^\perp$, is the set of all elements of M which are orthogonal to entire M. In particular,

$$M^\perp = \{v \in M \mid (v, u) = 0 \text{ for all } u \in M\}$$

Proposition 4.1.2. *Let $(.,.)$ be a Frobenius form on M and $d \in M$ be an axis. Then $(M_\lambda(d), M_\mu(d)) = 0$, for all distinct $\lambda, \mu \in \mathcal{F}$.*

Proof. Consider $w \in M_\lambda(d)$ and $z \in M_\mu(d)$. Then,

$$\lambda(w, z) = (\lambda w, z) = (dw, z) = (w, dz) = (w, \mu z) = \mu(w, z)$$

Since $\lambda \neq \mu$, we deduce that $(w, z) = 0$. $\square$

The diagram is a graph related to groups of 3-transpositions, defined as follows:

Definition 4.1.3. Suppose that (G, D) is a 3-transposition group. The *diagram* of (G, D), denoted by (D), is a graph on D with edge set $E = \{\{c, d\} \subseteq D : |cd| = 3\}$.

Definition 4.1.4. The length of element $g \in G$ is the smallest $k \in \mathbb{N}$ such that $g = c_1 c_2 \dots c_k$ for some $c_1, c_2, \dots c_k \in D$.

Proposition 4.1.5. *The distance in the diagram (D) between involutions $c, d \in D$ is equal to the smallest $k \in \mathbb{N}$ such that there exists an element of length k conjugating c to d. If no such element exists then the distance is ∞.*

Proof. If $c, e \in D$ and $c' = c^e$ then $c = c'$ if c, e commute, and c, c' are adjacent in the diagram (D). Suppose that $c^g = d$ and $g = c_1 c_2 \dots c_k$ for some $c_1, c_2, \dots c_k \in D$. Let $d_0 = c, d_1 = c^{c_1}, d_2 = c^{c_1 c_2}, \dots, d_k = c^{c_1 c_2 \dots c_k} = c^g = d$. By the above argument the consecutive involutions d_{i-1}, d_i either equal or adjacent in the diagram (D). So the distance between c and d is less than or equal to k. This shows that the distance at the diagram is no greater than the minimal length of a conjugating element. Conversely, suppose that the distance between c and d is k and $c = d_0, d_1, \dots, d_k = d$ is a shortest path between them. For each $i = 1, 2, \dots, k$, let $c_i = d_{i-1}^{d_i}$. Then, $d_{i-1}^{c_i} = d_i$. Therefore, $c^{c_1 c_2 \dots c_k} = d$. This shows that the shortest length of a conjugating element is at most k and so we have equality, as claimed. $\square$

Corollary 4.1.6. *The 3-transposition group (G, D) is connected if and only if D is a single conjugacy class.*

Remark. We will assume from now on that (G, D) is connected.

In the next section, we will record the main results about ideals in axial algebras.

4.2 Ideals Types

The structure of ideals in axial algebras was developed in [KMS19]. Suppose A is a primitive axial algebra with set of generating axes D. According to [KMS19] there are two distinguish kinds of ideals in A:

1. ideals containing no axes from D, and
2. ideals containing an axis from D.

The first type of ideals is related to the *radical* of the algebra defined as follows:

Definition 4.2.1. Suppose that A is a primitive axial algebra with a generating set D. The *radical*, denoted as $R(A)$, is the unique largest ideal of A not containing any axes from D.

The next theorem provides a useful link between the radical of the algebra and the radical of the Frobenius form.

Theorem 4.2.2 ([KMS19])**.** *Suppose that A is a primitive axial algebra with a Frobenius form $(\cdot, \cdot)$. Then, the radical $R(A, D)$ of A coincides with the radical $A^{\perp}$ of the Frobenius form if and only if $(a, a) \neq 0$ for all $a \in D$.*

Recall that, in case of Matsuo algebra every axis c satisfies $(c, c) = 1$. Therefore, the radical, $R(M)$, of Matsuo algebra coincides with the radical of the Frobenius form, $M^{\perp}$. In general, if $A^{\perp} = 0$, then A contains no ideals from the first type.

Now, to describe the second type of ideals we need the concept of *projection*. For a primitive axis $c \in A$, we can write $A = \oplus_{\lambda \in \mathcal{F}} A_{\lambda}(c)$. Let $v \in A$ be an arbitrary element, then $v = \sum v_{\lambda}$ where $v_{\lambda} \in A_{\lambda}(c)$ for each $\lambda \in \mathcal{F}$. In particular, $v_1 \in A_1(c) = \langle c \rangle$. That is, $v_1 = \beta c$ for some $\beta \in \mathbb{F}$. The *projection* of v onto c is v_1. Suppose A is a primitive axial algebra with generating set D. The *projection graph* of A is defined as follows.

Definition 4.2.3. The projection graph Ω of A is the graph whose vertex set is D, and a directed edge from c to d if the projection of c onto d, $c_1 \neq 0$. Where $c_1 = \beta d$ for some $\beta \in \mathbb{F}$.

Remark. In case of Matsuo algebra the projection graph Ω is exactly the diagram (D).

If A admits a Frobenius form $(.,.)$ with $(.,.) \neq 0$ on D, then Ω is undirected simple graph, and c, d are neighbours in Ω if and only if $(c, d) \neq 0$.

As shown in [KMS19], the connectivity of the projection graph Ω implies that A has no proper ideals from the second type. In particular, it can be shown that if I is an ideal of A that contains an element of D, then it will have to contain all the generating set D, but then I is the entire A. That is, if I is a proper ideal of A, then it will not contain any of the known axes of D.

Definition 4.2.4 ([KMS19])**.** A directed graph Ω is said to be *strongly connected* if every vertex is reachable from every other vertex by a directed path.

Theorem 4.2.5 ([KMS19])**.** *Let Ω be the projection graph of a primitive axial* *algebra A. If Ω is strongly connected, then every proper ideal of A is contained in the radical $R(A)$.*

Therefore, if the graph Ω is strongly connected, then A is simple if and only if $R(A) = 0$.

4.3 Critical Values

Definition 4.3.1. A value of η is said to be critical for Matsuo algebra $M = M_\eta(G, D)$ if $R(M) \neq 0$.

Proposition 4.3.2. *A value η is critical if and only if 0 is an eigenvalue of the Gram matrix $\mathcal{M}$ of the Frobenius form on $M = M_\eta(G, D)$.*

Proof. Suppose that η is a critical value for $M = M_\eta(G, D)$. Then, since the radical of the Frobenius form, $M^\perp$, coincides with the radical $R(M)$ of M, η is critical if and only if $M^\perp \neq 0$, that is, if and only if the Frobenius form on M is degenerate. This means that $|\mathcal{M}| = 0$. Hence, 0 must be an eigenvalue of $\mathcal{M}$. □

The Gram matrix of the Frobenius form on the basis D of M consists of 1's on the diagonal and either 0 or $\frac{\eta}{2}$ off-diagonal (depending on whether the two axes from D commute or not). Therefore, it can be expressed as

$$\mathcal{M} = I + \frac{\eta}{2} C$$

where I is the identity matrix and C is the adjacency matrix of the diagram (D). It follows that we can write the eigenvalues of $\mathcal{M}$ in terms of the eigenvalues of the matrix C. That is, for eigenvalues δ of $\mathcal{M}$ and ζ of C we have

$$\delta = 1 + \frac{\eta}{2}\zeta$$

Since both matrices $\mathcal{M}$ and C are symmetric, they are diagonalizable. So, the dimension of the eigenspace is equal to the multiplicity of the eigenvalue. From the formula above, we see that the multiplicity of δ will be the same as the multiplicity of ζ, and the δ-eigenspace for $\mathcal{M}$ will be the same as the ζ-eignespace for C.

We know from Proposition 4.3.2 that η is critical if and only if 0 is an eigenvalue of the Gram matrix $\mathcal{M}$, which equivalently means that η satisfies the equation $0 = 1 + \frac{\eta}{2}\zeta$. That is, $\eta = -\frac{2}{\zeta}$ for a non-zero eigenvalue ζ of C.

Proposition 4.3.3. *For every 3-transposition group, there exist only finitely many critical values η.*

Proof. Let (G, D) be a 3-transpositions group with diagram adjacency matrix C. Note that η is critical if and only if $\eta = -\frac{2}{\zeta}$ for a non-zero eigenvalue ζ of C. Since C is a matrix of finite size, it has only finitely many eigenvalues. Hence, there are only finitely many values of η for which the radical is nonzero. □

Remark. By the eigenvalues or the eigenspaces of the diagram (D) we mean the eigenvalues and the eigenspaces of the adjacency matrix of the the diagram (D).

Now, in order to find the critical values η for a given 3-transposition group (G, D), we need to know the eigenvalues ζ of the diagram (D). All such eigenvalues, with their multiplicities, were explicitly found in [HS21] for all groups of 3-transpositions (see Table 4.1). In particular, in Table 4.1, the parameters k, r, and s are the

Graph	Extended parameters $\left(n,k,\lambda,\mu;\left\{[r]^f,[s]^g\right\}\right)$
I2	
$(\mathrm{Sym}(m)), m \geqslant 4$	$\left(\begin{pmatrix} m \\ 2 \end{pmatrix}, 2(m-2), m-2, 4; \left\{[m-4]^{m-1}, [-2]^{m(m-3)/2}\right\}\right)$
I3	
$(\mathrm{O}^{\varepsilon}_{2m}(2))$ $\varepsilon = \pm, m \geqslant 2$	$(2^{2m-1} - \varepsilon 2^{m-1}, 2^{2m-2} - \varepsilon 2^{m-1}, 2^{2m-3} - \varepsilon 2^{m-2}, 2^{2m-3} - \varepsilon 2^{m-1};$ $\left\{[\varepsilon 2^{m-1}]^{(2^m - \varepsilon 1)\left(2^{m-1}-\varepsilon 1\right)/3}, [-\varepsilon 2^{m-2}]^{\left(2^{2m}-4\right)/3}\right\}\right)$
I4	
$(\mathrm{Sp}_{2m}(2))$ $m \geqslant 2$	$(2^{2m}-1, 2^{2m-1}, 2^{2m-2}, 2^{2m-2};$ $\left\{[2^{m-1}]^{2^{2m-1}-2^{m-1}-1}, [-2^{m-1}]^{2^{2m-1}+2^{m-1}-1}\right\}\right)$
I5	
$({}^{+}\Omega^{\varepsilon}_{m}(3))$ odd $m \geqslant 5$ $\varepsilon = \pm$	$\left(\left(3^{m-1} - \varepsilon 3^{(m-1)/2}\right)/2, 3^{m-2} - 2\varepsilon 3^{(m-3)/2} - 1, ,\right.$ $2\left(3^{m-3} - \varepsilon 3^{(m-3)/2} - 1\right), 2\left(3^{m-3} - \varepsilon 3^{(m-3)/2}\right);$ $\left.\left\{\left[3^{(m-3)/2} - 1\right]^f, \left[-3^{(m-3)/2} - 1\right]^g\right\}\right)$
with	$f = \left(3^{m-1} - 1 - (\varepsilon - 1)\left(3^{(m-1)/2} - 1\right)\right)/4$ $g = \left(3^{m-1} - 1 - (\varepsilon + 1)\left(3^{(m-1)/2} + 1\right)\right)/4$
$({}^{+}\Omega^{\varepsilon}_{m}(3))$ even $m \geqslant 6$ $\varepsilon = \pm$	$\left(\left(3^{m-1} - \varepsilon 3^{(m-2)/2}\right)/2, 3^{m-2} - 1, ,\right.$ $2\left(3^{m-3} - 1\right), 2\left(3^{m-3} + \varepsilon 3^{(m-4)/2}\right);$ $\left.\left\{\left[-\varepsilon 3^{(m-2)/2} - 1\right]^d, \left[\varepsilon 3^{(m-4)/2} - 1\right]^e\right\}\right)$
with	$d = \left(3^{m/2} - \varepsilon\right)\left(3^{(m-2)/2} - \varepsilon\right)/8$ $e = \left(3^m - 9\right)/8$
I6	
$(\mathrm{SU}_m(2))$ $m \geqslant 4$	$((2^{2m-1} - (-2)^{m-1} - 1)/3, 2^{2m-3},$ $3(2^{2m-5}) + (-2)^{m-3}, 3(2^{2m-5});$ $\{[-(-2)^{m-3}]^d, [-(-2)^{m-2}]^e\})$
with	$d = 8(2^{2m-3} - (-2)^{m-2} - 1)/9$ $e = 4(2^{2m-3} - 7(-2)^{m-3} - 1)/9$
I7	
(Fi_{22})	$(3510, 2816, 2248, 2304; \{[8]^{3080}, [-64]^{429}\})$
(Fi_{23})	$(3167, 28160, 25000, 25344; \{[8]^{30888}, [-352]^{782}\})$
(Fi_{24})	$(306936, 275264, 246832, 247104; \{[80]^{249458}, [-352]^{57477}\})$
$\left(P\Omega^{+}_{8}(2) : S_3\right)$	$(360, 296; \{[8]^{105}, [-4]^{252}, [-64]^{2}\})$
$\left(P\Omega^{+}_{8}(3) : S_3\right)$	$(3240, 2888; \{[8]^{2457}, [-28]^{780}, [352]^{2}\})$

Table 4.1: The spectrum of the diagram (D), quoted from [HS21].

eigenvalues of (D), where k is the valency of the graph and it always has multiplicity one, and the remaining two eigenvalues, r with multiplicity f and s with multiplicity g, are roots of some monic polynomials of degree two, so they are algebraic integers. The parameter n represents the number of points in (G, D). That is, $n = |D| = 1 + f + g$, as (D) is diagonalisable.

Therefore, using Table 4.1, we can find all the critical values η for any given 3-transposition group. In the next example, we will consider finding all the critical values of the Matsuo algebra associated with the 3-transposition group $G = Sp_{2m}(2)$.

Example 4.3.4. *Consider the group $(G, D) = (Sp_{2m}(2), D)$, where D is the class of symplectic transvections in G. Then, using Table 4.1, the eigenvalues ζ of the diagram (D) are:*

(i) $\zeta_1 = 2^{2m-1}$ *with multiplicity* 1,

(ii) $\zeta_2 = 2^{m-1}$ *with multiplicity* $2^{2m-1} - 2^{m-1} - 1$,

(iii) $\zeta_3 = -2^{m-1}$ *with multiplicity* $2^{2m-1} + 2^{m-1} - 1$.

So, the critical values of Matsuo algebra $M = M_\eta(G, D)$ are $\eta_1 = -\frac{2}{\zeta_1} = -\frac{2}{2^{2m-1}} = -2^{2-2m}$, $\eta_2 = \frac{-2}{\zeta_2} = \frac{-2}{2^{m-1}} = -2^{2-m}$, and $\eta_3 = \frac{-2}{\zeta_3} = \frac{-2}{-2^{m-1}} = 2^{2-m}$. Furthermore, since (D) is diagonalisable, the sum of the multiplicities of ζ_1, ζ_2, and ζ_3 equals to the number of points in G. That is,

$$|D| = 1 + (2^{2m-1} - 2^{m-1} - 1) + (2^{2m-1} + 2^{m-1} - 1) = 2^{2m} - 1$$

To summarize, Matsuo algebra $M = M_\eta(G, D)$ is simple if and only if η is not critical. If η is critical, then $R(M)$ coincides with the 0-eigenspace of $\mathcal{M}$, equivalently it coincides with the ζ-eigenspace of the adjacency matrix of the diagram (D). In the previous example, M is simple for all η different from η_1, η_2, η_3, and dimension of $R(M)$ is 1 for η_1, $2^{2m-1} - 2^{m-1} - 1$ for η_2, and $2^{2m-1} + 2^{m-1} - 1$ for η_3.

4.4 Never-critical Values

Let $M = M_\eta(G, D)$, we aim to show that (almost always) D is the complete set of all axes of Jordan type η in M, and that $\mathrm{Aut}(M) = \mathrm{Aut}(G, D)$. To do so, we need η to be of the following type.

Definition 4.4.1. A number $\eta \in \mathbb{F} \setminus \{0, 1\}$ is called *never-critical* if it is not the critical number for any group of 3-transpositions (G, D).

Note that, the eigenvalues ζ, listed in Table 4.1, are always integers, in all cases. Recall that, the formula for critical values is given by $\eta = -\frac{2}{\zeta}$, where ζ is an eigenvalue of the diagram (D). So, η has to be a ratio of two integer numbers. This means that all the critical numbers are in the prime subfield of $\mathbb{F}$. Hence, all numbers outside of the prime subfield of $\mathbb{F}$ are never-critical. In particular, all irrational numbers and rationals $\pm\frac{a}{b}$ with $a > 2$ are never-critical. Also, all numbers outside the interval $[-1, 1]$ except for ± 2, are never-critical. Therefore, a great majority of numbers cannot arise as critical numbers. This leads us to the main theorem of this chapter, which we will discuss after stating the following results from [HRS15a].

Theorem 4.4.2 ([HRS15a])**.** *Let $\mathbb{F}$ be a field of characteristic not 2 with $\eta \in \mathbb{F}$ for $\eta \notin \{0, 1\}$. Let A be a primitive axial algebra of Jordan type η that is generated* *by two axes. Then A is one of the following:*

(i) *A is an algebra $\mathbb{F}$ of type $1A$ over $\mathbb{F}$;*

(ii) *A is an algebra $\mathbb{F} \oplus \mathbb{F}$ of type $2B$ over $\mathbb{F}$;*

(iii) *A is an algebra of type $3C(\eta)$ of dimension 3 over $\mathbb{F}$.*

Proposition 4.4.3 ([HRS15a])**.** *Let A be an axial algebra of Jordan type η. Let c and d be two axes of A with the corresponding Miyamoto involutions $\tau(c)$ and $\tau(d)$. Let W be the subalgebra generated by c and d.*

(i) *If W is of type $1A$, then $c = d$, $\tau(c) = \tau(d)$, $|\tau(c)\tau(d)| = 1$.*

(ii) *If W is of type $2B$, then $\tau(c)\tau(d) = \tau(d)\tau(c)$ and $|\tau(c)\tau(d)| = 2$.*

(iii) *If W is of type $3C(\eta)$, then $\tau(c)^{\tau(d)} = \tau(d)^{\tau(c)}$ and $|\tau(c)\tau(d)| = 3$.*

Proposition 4.4.4 ([HRS15a])**.** *Let A be a primitive axial algebra of Jordan type $\eta \neq \frac{1}{2}$ over a field of characteristic not two that is generated by the set X of axes. Assume that every subalgebra generated by two elements of X has type $1A$, $2B$, or $3C(\eta)$. If $c, d \in X$ with $\tau(c) = \tau(d)$ then $c = d$.*

Theorem 4.4.5. *Suppose that (G, D) is a connected 3-transposition group. Let $\eta \in \mathbb{F} \setminus \{0, 1\}$ where $\mathbb{F}$ is a field of characteristic not 2. If η is never-critical, then D is the complete set of all axes of Jordan type η in M and $\mathrm{Aut}(M) = \mathrm{Aut}(G, D)$.*

Proof. Suppose by contradiction that D is not the complete set of all primitive axes of Jordan type η in M. Then, we have that $D \subsetneq D'$, where D' is the entire set of all primitive axes of Jordan type η in M. Let $\overline{D} := \{\tau_d \mid d \in D\}$ and $\overline{D'} := \{\tau_d \mid d \in D'\}$ be the sets of Miyamoto involutions that correspond to D

and D', respectively. Further, let $\overline{G} = \langle \overline{D} \rangle$ and $\overline{G'} = \langle \overline{D'} \rangle$ be the corresponding Miyamoto groups.

Note that $(\overline{G'}, \overline{D'})$ is a group of 3-transpositions. Indeed, for any Jordan axes $c, d \in D'$. We know from Proposition 4.4.3 that there are two cases for $\eta \neq \frac{1}{2}$. First, if $cd = 0$, then c, d generate the subalgebra $2B$, and $|\tau_c \tau_d| = 2$, as τ_c, τ_d commute. If $cd \neq 0$, then the subalgebra generated by c, d is $3C(\eta)$, with $cd = \frac{\eta}{2}(c + d - e)$, where $c^{\tau_d} = d^{\tau_c} = e \in D'$. Also $|\tau_c \tau_d| = 3$, in this case. This means that $(\overline{G'}, \overline{D'})$ is a group of 3-transpositions.

Also D' is connected. Since (G, D) is connected it is enough to show that any $c \in D' \setminus \{d\}$ is connected in the diagram on D' to some axis from D. Suppose by contradiction that this is not the case. Then $cd = 0$ for all $d \in D$. However, D spans the entire algebra M which means that $cM = 0$. On the other hand, $c \in M$ so $c = c^2 = 0$, a contradiction. Thus, D' is connected and therefore $(\overline{G}', \overline{D}')$ is connected too. So, $(\overline{G}', \overline{D}')$ is a 3-transposition group.

By Proposition 4.4.4, the map $d \mapsto \tau_d$ is a bijection between D' and $\overline{D'}$, so we have that $\overline{D} \subsetneq \overline{D}'$. Suppose that $\overline{D}' \subseteq \overline{G}$. Recall that $\overline{D}$ is normal in $\overline{G}$ and D has a connected diagram. Hence $\overline{D}$ is a single conjugacy class. The same logic applies to $\overline{D}'$, so $\overline{D}'$ is also a single conjugacy class in $\overline{G}$. This is a contradiction, since one conjugacy class cannot be properly contained in another conjugacy class. Thus $\overline{D}'$ is not contained in $\overline{G}$ which means that $\overline{G}$ is a proper subgroup of $\overline{G}'$.

Let $M' = M_\eta(\overline{G}', \overline{D}')$. Consider the linear map $\phi : M' \to M$ sending each basis element $\tau_d \in \overline{D}'$ to $d \in M$ (we use that τ is bijective on D'). We note that this ϕ

is an algebra homomorphism. We just need to verify that for all $c, d \in D'$ we have

$$\phi(\tau_c \circ \tau_d) = c \cdot d \qquad (*)$$

(where $\circ$ is the product in M' and $\cdot$ is the product in M). We have two cases; first if $|\tau_c\tau_d| \leqslant 2$ then we have that $\tau_c \circ \tau_d = 0$ and also $c \cdot d = 0$ in M, so $(*)$ holds in this case. If $|\tau_c\tau_d| = 3$, then since $\tau_c^{\tau_d} = \tau_{c^{\tau_d}}$, we have that $\tau_c\tau_d = \frac{\eta}{2}(\tau_c + \tau_d - \tau_c^{\tau_d}) = \frac{\eta}{2}(\tau_c + \tau_d - \tau_{c^{\tau_d}})$. Therefore, $\phi(\tau_c \circ \tau_d) = \frac{\eta}{2}(\tau_c + \tau_d - c^{\tau_d})$. But the multiplication of cd in M is $\frac{\eta}{2}(c + d - c^{\tau_d})$. So $(*)$ holds in this case too. Hence ϕ is a homomorphism. That is, M must be isomorphic to a quotient of M' for the same value η. As η is never-critical, M' is simple. Therefore, we must have $M \cong M'$, which is a contradiction, since $|D| < |D'|$. □

Note that, for a larger group of automorphisms of M to exist, we need to have a proper quotient of a larger Matsuo algebra M' to be isomorphic to M, but in the never-critical case there are no non-trivial quotients. So, in this case we cannot have a larger group, which means we do not have any extra axes. Since η is (almost always) never-critical, we can say that D is (almost always) the complete set of all axes of Jordan type η in M. However, in Section 4.6, we are going to see that there are some exceptions to this general situation.

4.5 Aligned 3-Transposition Groups

Definition 4.5.1. Suppose (G, D) is a connected 3-transposition group. Then a 3-transposition group (H, C) is a subgroup of (G, D) if H is a subgroup of G and $C = H \cap D$.

Definition 4.5.2. The pair $((G, D), (H, C))$, consisting of a 3-transposition group and its subgroup, is *aligned* if for a critical value η of G we have that $\overline{C}$ is a spanning set of $\overline{M'} = M'/R(M')$, where $M' := M_\eta(G, D)$. Moreover, $M := M_\eta(H, C)$ is a subalgebra of M'.

We call the critical value η from the above definition *aligning*.

Proposition 4.5.3. *Suppose that $((G, D), (H, C))$ is aligned. Then,*

$$\dim M' - \dim R(M') = \begin{cases} \dim M, & \text{if } \eta \text{ is not critical for } H; \\ \dim M - \dim R(M), & \text{if } \eta \text{ is also critical for } H. \end{cases}$$

Where $M' := M_\eta(G, D)$, $M := M_\eta(H, C)$, and η is aligning.

Proof. First, if η is not critical for H, then $R(M) = 0$. That is, M is simple. Consider the algebra homomorphism $\psi : M \to M'/R(M')$. Since $\overline{C}$ spans $\overline{M'}$, ψ is surjective. Also, it is injective as $\ker \psi = M \cap R(M') \subseteq R(M) = 0$. Therefore, $M \cong M'/R(M')$. So, $\dim M = \dim M' - \dim R(M')$. If η is also critical for H, then M has a radical, $R(M)$. Since ψ is surjective, it follows that $M \cap R(M') = R(M)$. So, $M/M \cap R(M') = M/R(M) \cong \overline{M'}$. Therefore, $\dim M - \dim R(M) = \dim \overline{M'} = \dim M' - \dim R(M')$. □

Theorem 4.5.4. *Suppose that $((G, D), (H, C))$ is an aligned pair of 3-transposition groups such that $1 \neq H \neq G$. Then we have the following*

(i) *$N_G(H)$ is a proper subgroup of G.*

(ii) *For a maximal subgroup K of G such that $N_G(H) \leqslant K$, take $C' = K \cap D$ and $H' = \langle C' \rangle$. Then (H', C') is a subgroup of (G, D), the pair $((G, D), (H', C'))$ is aligned, and $K = N_G(H')$.*

Proof. Note that the condition $1 \neq H \neq G$ is equivalent to $\emptyset \neq C \neq D$. For part (i), assume by contradiction that $G = N_G(H)$. Since (G, D) is connected, D is a single conjugacy class of G. Since $D \cap H \neq \emptyset$ and $H \trianglelefteq G$ we have that $D \subseteq H$. This means that $D = C$, but this is a contradiction. Thus H is not normal in G, i.e., $N_G(H)$ is a proper subgroup of G. For part (ii), to show that (H', C') is a subgroup of (G, D), we need to argue that $H' \cap D = C'$. Since $H' \subseteq K$ and $C' = K \cap D$, we have that $H' \cap D \subseteq K \cap D = C'$. Conversely, $C' \subseteq \langle C' \rangle = H'$, and also $C' = K \cap D \subseteq D$, so $C' \subseteq H' \cap D$, therefore $H' \cap D = C'$. Note that C' is a normal subset of K, so H' is a normal subgroup of K. Therefore, $K \leqslant N_G(H')$, and hence $N_G(H') = K$ or G. But $N_G(H') \neq G$ as $D \nsubseteq K$. Lastly, we know that $\overline{C} \subseteq \overline{C'}$ is a spanning set of $\overline{M} = M/R(M)$ where $M = M_\eta(G, D)$, so $\overline{C'}$ is also a spanning set for $\overline{M}$, hence the pair $((G, D), (H', C'))$ is aligned. □

Lemma 4.5.5. *Suppose that A is an algebra isomorphic to a Matsuo algebra $M = M_\eta(G, D)$, or a quotient of M. Then for axes c, d either $\tau_c \neq \tau_d$ or $\tau_c = 1 = \tau_d$.*

Proof. Suppose that d, d' are two distinct axes which give the same Miyamoto involution $\tau = \tau_d = \tau_{d'}$. Suppose that there exists a point c that is collinear to d. Then $|\tau_c \tau| = 3$ and therefore c is also collinear to d'. Furthermore, suppose that e is the third point on the line cd and similarly e' is the third point on the line cd', then we have that $e = c^{\tau_d} = c^{\tau_{d'}} = e'$. Therefore, the lines cd and cd' are the same because they contain the same two points c, e. Hence, we also must have $d = d'$, which is a contradiction. This shows that d cannot be collinear to any other axis. This means that $\langle\langle d, c \rangle\rangle$ is the algebra $2B$ for all axes $c \neq d$. This implies that $\tau_d = 1$. □

Lemma 4.5.6 ([HRS15a])**.** *If ρ is an automorphism of A and c is an axis, then c^ρ is an axis with $\tau(c)^\rho = \tau(c^\rho)$.*

In the next theorem we show that whenever a Matsuo algebra has a larger group of automorphisms then we get an aligned pair. We will prove this theorem using a similar argument as in the proof of Theorem 4.4.5.

Theorem 4.5.7. *Suppose $\eta \neq \frac{1}{2}$. Let $M = M_\eta(G, D)$ be a connected Matsuo algebra such that $D \subsetneq D'$, where D' is the set of all primitive axes of Jordan type η in M. Let $\overline{D} = \{\tau_d \mid d \in D\}$ and similarly $\overline{D'} = \{\tau_d \mid d \in D'\}$. Then $\overline{G} = \langle \overline{D} \rangle$ is a proper subgroup of the connected 3-transposition group $\overline{G'} = \langle \overline{D'} \rangle$. Furthermore, the pair $((\overline{G}, \overline{D}), (\overline{G'}, \overline{D'}))$ is aligned.*

Proof. First, we want show that $(\overline{G}', \overline{D}')$ is a group of 3-transpositions. Take any two Jordan axes $c, d \in D'$. Then by Proposition 4.4.3 there are two cases for $\eta \neq \frac{1}{2}$. First, if $cd = 0$, then c, d generate the subalgebra $2B$ in which case $|\tau_c \tau_d| \leqslant 2$. If $cd \neq 0$, then the subalgebra generated by c, d is $3C(\eta)$, and $|\tau_c \tau_d| = 3$. This implies that $(\overline{G}', \overline{D}')$ is a group of 3-transpositions.

Now, we show that the diagram on D' is connected. Since (G, D) is connected it is suffices to show that any $c \in D' \setminus \{d\}$ is connected in the diagram on D' to some axis from D. Suppose by contradiction that this is not the case. Then $cd = 0$ for all $d \in D$. However, D spans the entire algebra M which means that $cM = 0$. On the other hand, $c \in M$ so $c = c^2 = 0$, a contradiction. Thus, D' is connected and therefore $(\overline{G}', \overline{D}')$ is connected too. So, $(\overline{G}', \overline{D}')$ is a 3-transposition group.

Since the map $d \mapsto \tau_d$ is also a bijection between D' and the set of τ-involutions $\overline{D'}$, we have that $\overline{D} \subsetneq \overline{D}'$. Suppose that $\overline{D}' \subseteq \overline{G}$. Recall that $\overline{D}$ is normal in $\overline{G}$ and D has a connected diagram. Hence $\overline{D}$ is a single conjugacy class. The same logic applies to $\overline{D}'$, so $\overline{D}'$ is also a single conjugacy class in $\overline{G}$. This is a contradiction, since one conjugacy class cannot be properly contained in another conjugacy class.

Thus $\overline{D}'$ is not contained in $\overline{G}$ which means that $\overline{G}'$ properly contains $\overline{G}$.

Let $M' = M_\eta(\overline{G}', \overline{D}')$. Consider the linear map ϕ from M' to M sending each basis element $\tau_d \in \overline{D}'$ to $d \in M$ (we use that τ is bijective on D'). We note that this ϕ is an algebra homomorphism. We just need to verify that for all $c, d \in D'$ we have

$$\phi(\tau_c \circ \tau_d) = c \cdot d \tag{$*$}$$

(where $\circ$ is the product in M' and $\cdot$ is the product in M). We have two cases; first if $|\tau_c\tau_d| \leqslant 2$ then we have that $\tau_c \circ \tau_d = 0$ and also $c \cdot d = 0$ in M, so ($*$) holds in this case. If $|\tau_c\tau_d| = 3$, then since $\tau_c^{\tau_d} = \tau_{c^{\tau_d}}$, we have that $\tau_c\tau_d = \frac{\eta}{2}(\tau_c + \tau_d - \tau_c^{\tau_d}) = \frac{\eta}{2}(\tau_c + \tau_d - \tau_{c^{\tau_d}})$. Therefore, $\phi(\tau_c \circ \tau_d) = \frac{\eta}{2}(\tau_c + \tau_d - c^{\tau_d})$. But the multiplication of cd in M is $\frac{\eta}{2}(c + d - c^{\tau_d})$. So ($*$) holds in this case too. Hence ϕ is a homomorphism.

Now, since $\dim M' > \dim M$ it follows that ϕ is not bijective. So M' has a non-trivial radical, which means that η is a critical value for M'. By the first isomorphism theorem we have that $M \cong M'/\ker(\phi)$, however if we factor out the entire radical we may get a smaller factor of M'. Note that D spans $M'/\ker(\phi)$, so the image of D spans the smaller factor of M' as well. Therefore, the pair $((\overline{G}', \overline{D}'), (\overline{G}, \overline{D}))$ is aligned. □

4.6 Examples

In this section, we give examples which are exceptions to the general situation stating that the full automorphism group of a Matsuo algebra M comes from the automorphisms of the 3-transposition groups (G, D). In particular, we will demonstrate that the situation where the Matsuo algebra has additional axes and larger group of automorphisms exists.

Example 4.6.1. *Consider $H = S_n$ as a subgroup of $G = S_{n+1}$. Let $C = (1,2)^H$ be the class of transpositions in H and $D = (1,2)^G$ be the class of transpositions in G. Then, clearly $C \subsetneq D$, and the Matsuo algebra $M := M_\eta(H,C)$ is a subalgebra of $M' := M_\eta(G,D)$. From Table 4.1, select the eigenvalue $\zeta = n-3$ of the diagram (D). Then, $\eta = -\frac{2}{\zeta} = -\frac{2}{n-3}$ is a critical value for M'. Thus, M' is not simple. So, it has a radical $R(M')$ of dimension n, equals to the multiplicity of ζ.*

Consider the factor algebra $\overline{M'} = M'/R(M')$ (we will adopt the bar notation). Then, $\dim \overline{M'} = \dim M' - \dim R(M') = |D'| - n = \binom{n+1}{2} - n = \frac{n(n+1)-2n}{2} = \frac{n(n-1)}{2} = \binom{n}{2} = \dim M$.

On the other hand, $M := M_\eta(H,C)$ is simple because $\eta = -\frac{2}{n-3}$ is not critical for it. Therefore, $M \cong \overline{M}$, where $\overline{M}$ is the image of M in $\overline{M'}$. Looking at the dimensions we see that $\dim \overline{M} = \dim M = \dim \overline{M'}$, so $\overline{M} \cong \overline{M'}$. That is, $\overline{M'}$ is a copy of M. Therefore, as M is isomorphic to a quotient of M', it has additional axes, which are the images of the vectors from D, and hence it has a larger group of automorphisms.

In the next two examples, we will take the groups $O^+_{2m}(2)$, $O^-_{2m}(2)$ as subgroups inside $Sp_{2m}(2)$. According to Table 4.1, the number of points in $Sp_{2m}(2)$ is $2^{2m}-1$, and the number of points in $O^\varepsilon_{2m}(2)$ is $2^{2m-1} - \varepsilon 2^{m-1}$, where $\varepsilon = \pm 1$. First, we consider the plus case, $O^+_{2m}(2)$.

Example 4.6.2. *Consider $H = O^+_{2m}(2)$ as a subgroup of $G = Sp_{2m}(2)$. Let C be the class of transvections of $H = O^+_{2m}(2)$, and similarly D be the class of transvections of $G = Sp_{2m}(2)$. Then, $C \subset D$, and the Matsuo algebra $M := M_\eta(H,C)$ is a subalgebra of $M' := M_\eta(G,D)$. Now, from Table 4.1, choose the eigenvalue $\zeta = -2^{m-1}$ for the diagram (D). Then, $\eta = -\frac{2}{-2^{m-1}} = 2^{2-m}$ is a* *critical value for* M'. *So* M' *has a radical* $R(M')$ *of dimension* $2^{2m-1} + 2^{m-1} - 1$, *equals to the multiplicity of* ζ. *Note that,* $\dim \overline{M'} = \dim M'/R(M') = \dim M' - \dim R(M') = 2^{2m} - 1 - (2^{2m-1} + 2^{m-1} - 1) = 2^{2m} - 2^{2m-1} - 2^{m-1} = \frac{1}{2}2^{2m} - 2^{m-1} = 2^{2m-1} - 2^{m-1} = \dim M$. *On the other hand,* $M = M_\eta(H, C)$ *is simple because* $\eta = -\frac{2}{-2^{m-1}} = 2^{2-m}$ *is not critical for it. Therefore,* $M \cong \overline{M}$, *where* $\overline{M}$ *is the image of* M *in* $\overline{M'}$. *Note that,* $\dim \overline{M} = \dim M = \dim \overline{M'}$, *so* $\overline{M} \cong \overline{M'}$. *That is,* $\overline{M'}$ *is a copy of* M. *Therefore, as* M *is isomorphic to a quotient of* M', *we deduce that* M *has additional axes, which are the images of the vectors from* D. *Consequently,* M *has a larger group of automorphisms.*

Next, consider $O^-_{2m}(2)$ as a subgroup of $Sp_{2m}(2)$.

Example 4.6.3. *The same argument, as in Example 4.6.2, applies for* $H = O^-_{2m}(2)$ *and* $G = Sp_{2m}(2)$, *except we choose a different eigenvalue for the diagram* (D). *Namely, choose the eigenvalue* $\zeta = 2^{m-1}$. *Then* $\eta = -\frac{2}{2^{m-1}} = -2^{2-m}$ *is a critical value for* $M' := M_\eta(G, D)$, *and* $\dim R(M') = 2^{2m-1} - 2^{m-1} - 1$. *Note that,* $\dim \overline{M'} = \dim M'/R(M') = \dim M' - \dim R(M') = 2^{2m} - 1 - (2^{2m-1} - 2^{m-1} - 1) = 2^{2m} - 2^{2m-1} + 2^{m-1} = \frac{1}{2}2^{2m} + 2^{m-1} = 2^{2m-1} + 2^{m-1} = \dim M$. *So, similar argument as above shows that* $M \cong \overline{M'}$. *Therefore,* M *has additional axes and a larger group of automorphisms.*

Example 4.6.4. *Consider the pair* $(G, H) = (Fi_{23}, \Omega^+_8(3) : S_3)$. *Let* $M' = M_\eta(G)$ *and* $M = M_\eta(H)$. *The number of points in* G *is* 31671, *i.e.,* $\dim M' = |D| = 31671$. *Further, the spectrum of* (D) *is given by*

$$\left(\left([28160]^1, [8]^{30888}, [-352]^{782}\right)\right)$$

So, the critical values of G *are* $\eta_0 = \frac{-2}{28160} = -\frac{1}{14080}$, $\eta_1 = -\frac{2}{8} = -\frac{1}{4}$, *and* $\eta_2 = \frac{2}{352} = \frac{1}{176}$. *Choose the critical value* $\eta_1 = -\frac{1}{4}$. *For this critical value,* $\dim R(M') = 30888$.

So, $\dim \overline{M'} = |D| - \dim R(M') = 31671 - 30888 = 783$. *On the other hand, the group* $H = \Omega_8^+(3) : S_3$ *has number of points* 3240, *i.e.,* $\dim M = |C| = 3240$. *Further the spectrum of* (C) *is*

$$\Big(\big([2888]^1, [8]^{2457}, [-28]^{780}, [-352]^2\big)\Big)$$

Note that, $\eta = -\frac{2}{8} = -\frac{1}{4}$ *is also critical for* H*. So,* H *has a radical,* $R(M)_{-\frac{1}{4}}$*, of dimension* 2457*. Therefore,* $\dim M/R(M)_{-\frac{1}{4}} = |C| - R(M)_{-\frac{1}{4}} = 3240 - 2457 = 783 = \dim M'/R(M')_{-\frac{1}{4}}$. *Since* H *is a maximal subgroup in* G*, the pair* $(G, H) = (Fi_{23}, \Omega_8^+(3) : S_3)$ *is another example of aligned pairs.*

In the remainder of the text, we are going to analyse all cases of aligned pairs of 3-transposition groups in Theorem 3.1.2. In the next chapter we will deal with the pairs that are in cross characteristic, the pairs where the big group and the small group are defined over fields of different characteristics. Next, we will study the pairs that are in same characteristic. Note that, if we assign characteristic 0 to the symmetric group, then it is in cross characteristic with any positive characteristic group in Theorem 3.1.2.

CHAPTER 5

CROSS CHARACTERISTIC CASE

In this chapter, we will search for new examples of aligned pairs of irreducible 3-transposition groups. We will consider the pairs that are in cross characteristic. This is the situation where the big group and the small group are defined over fields of different characteristics. The method that we will use to analyse such pairs is based on the following two observations:

Suppose that (G_m, D) and (H_n, C) are 3-transposition groups in cross characteristic where G_m is defined over $\mathbb{F}_{p^a}$ and H_n is defined over $\mathbb{F}_{q^b}$ for distinct primes p and q.

1. Note that if H_n is a subgroup of G_m then H_n will act non-trivially on the natural module of G_m, the m-dimensional vector space over $\mathbb{F}_{p^a}$. The paper, [SZ93], by Seitz and Zalesskii, provided a lower bound on the dimension of a non-trivial module for H_n over $\mathbb{F}_{p^a}$. This bound must be at most m because H_n does act on the m-dimensional module non-trivially.

2. Let $M := M_\eta(H_n, C)$ and $M' := M_\eta(G_m, D)$ be the corresponding Matsuo

algebras, where η is critical for M'. We know from Proposition 4.5.3 that if (G_m, H_n) is aligned, then

$$\dim M' - \dim R(M') = \begin{cases} \dim M, & \text{if } \eta \text{ is not critical for } M \\ \dim M - \dim R(M), & \text{if } \eta \text{ is also critical for } M \end{cases}$$

That is, $\dim M = |C| \geqslant \dim \overline{M'}$.

Recall that the adjacency matrix of the diagram (D), associated to (G, D), is diagonalisable with multiplicities $1, f$, and g. Let ζ be an eigenvalue in the spectrum of the adjacency matrix of (D). If $\eta = -2/\zeta$ is critical for M', then the radical of M', $R(M')$, coincides with the ζ-eigenspace of the adjacency matrix of (D). So, dimension of $R(M')$ is either 1 or f or g. Therefore, as $\dim M' = |D| = 1 + f + g$, $\dim \overline{M'} = \dim M' - \dim R(M') = 1 + d$ where $d = f$ or g. To make the inequality, $\dim M \geqslant \dim \overline{M'}$, covers both cases of the radical, we should always take d to be the minimum of f and g. Thus, the inequality $\dim M \geqslant \dim \overline{M'}$ can be written as $|C| \geqslant 1 + \min\{f, g\}$.[1]

From the above two observations, we should obtain two opposite inequalities. These inequalities can be used to remove all but a finite number of exceptions, a finite number of pairs (n, m). After that, we can deal with each pair separately.

Remark. Throughout this chapter the 3-transposition groups (G, D) and (H, C) will be denoted as G and H. Moreover, M' and M refer to the Matsuo algebras that corresponds to G and H respectively.

In the following lemma we record some of the remarkable isomorphisms that

[1] The case $\dim R(M') = 1$ can be easily verified without including it in our method, therefore we just apply the method for $\dim R(M') = f, g$.

occur between irreducible 3-transposition groups. These are gathered from [HS21] (Lemma(5.2)) and the Atlas of finite groups [Con+85].

Lemma 5.0.1. (a) *$S_3 \cong O_2^-(2) \cong Sp_2(2) \cong \mathrm{SU}_2(2)$;*

(b) *$S_5 \cong O_4^-(2)$;*

(c) *$S_6 \cong Sp_4(2) \cong \Omega_4^-(3)$;*

(d) *$S_8 \cong O_6^+(2)$;*

(e) *$O_6^-(2) \cong \Omega_5(3) \cong \mathrm{SU}_4(2)$;*

5.1 Case 1: Groups in Char(2) Inside Groups in Char(3)

First, let us consider the case where $G = \Omega_m(3)$ (m is odd) and H is isomorphic to $Sp_{2n}(2), O_{2n}^{\pm}(2)$, or $\mathrm{SU}_n(2)$.[1]

Proposition 5.1.1. *Suppose that $G = \Omega_m(3)$, and $H \cong Sp_{2n}(2)$, $\cong O_{2n}^{\pm}(2)$, or $\cong \mathrm{SU}_n(2)$. Then, (G, H) is not an aligned pair.*

Proof. Consider the group $G = \Omega_m(3)$, where m is odd. According to Table 4.1, the eigenvalues of the diagram (D) of the group $G = \Omega_m^{\varepsilon}(3)$, are $k = 3^{m-2} - 2 \cdot 3^{\frac{m-3}{2}} - 1, r = 3^{\frac{m-3}{2}} - 1$, and $s = -3^{\frac{m-3}{2}} - 1$, where k has multiplicity[2] 1, r has multiplicity $f = (3^{m-1}-1)/4$, and s has multiplicity $g = (3^{m-1}-1-2(3^{\frac{m-1}{2}}+1))/4$. The critical values η of the Matsuo algebra $M' := M_\eta(G)$ are given by the formula $\eta = -\frac{2}{\zeta}$, where $\zeta = k, r$ or s.

[1] In the classical groups, $\mathbb{F} = \mathbb{F}_{q^\alpha}$, where $q = p^n$, p is prime, and $\alpha = 1$ in the cases of the symplectic and orthogonal groups and 2 in the case of the unitary group.

[2] The multiplicity of k represents the number of connected components of (D), so it is 1 in all cases because we assumed that G is connected.

Using observation number 1, the minimal possible degree of a non-trivial module for $H = Sp_{2n}(2)$, $O^{\pm}_{2n}(2)$, or $\mathrm{SU}_n(2)$ is at most m. These lower bounds are provided in [SZ93]. So for $H = Sp_{2n}(2)$ we have that,

$$m \geqslant (2^n - 1)(2^{n-1} - 1)/3 \geqslant 2^{n-1}2^{n-2}/4 \geqslant 2^{2n-5} \tag{5.1}$$

For $H = O^{\pm}_{2n}(2)$, the minimal dimension of a non-trivial module for the plus case, $H = O^{+}_{2n}(2)$, is $2^{n-2}(2^{n-1}-1) = 2^{2n-3}-2^{n-2}$, and for the minus case, $H = O^{-}_{2n}(2)$, it is $(2^{n-1}+1)(2^{n-2}-1) = 2^{2n-3} - 2^{n-2} - 1$. So, in both cases we have that,

$$m \geqslant 2^{2n-3} - 2^{n-2} - 1 \tag{5.2}$$

That is, $m \geqslant 2^{2n-3} - 2^{n-2} - 1 = 2^{n-2}(2^{n-1} - 1) - 1$. So, $m \geqslant 2^{n-2} - 1$.

For $H = \mathrm{SU}_n(2)$, the minimal possible degree for a non-trivial module for H is $(2^n - 1)/3$. Therefore, we have that,

$$m \geqslant \frac{2^n - 1}{3} \tag{5.3}$$

So, $m \geqslant 2^{n-2} - 1$. Therefore, in all cases of H we have that $m \geqslant 2^{n-3}$.

Now, we use observation number 2 to obtain the reverse inequality. Let $M = M_\eta(H)$ be the Matsuo algebra that is associated to H, where $H = Sp_{2n}(2)$, $O^{\pm}_{2n}(2)$, or $\mathrm{SU}_n(2)$. Then, we want to write, $|C| \geqslant 1+\min\{f, g\}$, for each group H. Recall that, the size of C and the multiplicities, f, g of (D), are all given in Table 4.1. In this case, $1 + \min\{f, g\} = 1 + g = 1 + (3^{m-1} - 1 - 2(3^{\frac{m-1}{2}} + 1))/4$. First, for

$H = Sp_{2n}(2)$, $|C| = 2^{2n} - 1$, so we have that,

$$2^{2n} - 1 \geqslant 1 + (3^{m-1} - 1 - 2(3^{\frac{m-1}{2}} + 1))/4 \tag{5.4}$$

For $H = O^{\pm}_{2n}(2)$, the number of points in H is $2^{2n-1} \pm 2^{n-1}$. Thus, we have that,

$$2^{2n-1} \pm 2^{n-1} \geqslant 1 + (3^{m-1} - 1 - 2(3^{\frac{m-1}{2}} + 1))/4 \tag{5.5}$$

Lastly, for $H = \mathrm{SU}_n(2)$, $|C| = (2^{2n-1} - (-2)^{n-1} - 1)/3$, so we have that,

$$(2^{2n-1} - (-2)^{n-1} - 1)/3 \geqslant 1 + (3^{m-1} - 1 - 2(3^{\frac{m-1}{2}} + 1))/4 \tag{5.6}$$

So, in all cases, we have $2^{2n} + 2^n \geqslant 1 + (3^{m-1} - 1 - 2(3^{\frac{m-1}{2}} + 1))/4$. That is, $2^{2n+2} + 2^{n+2} \geqslant 3^{m-1} - 2(3^{m-2}) - 3 = 3^{m-2} - 3$.

Now, from the inequality (5.3), $m \geqslant 2^{n-3}$, so $8m \geqslant 2^n$. Thus, $4(8m)^2 + 4(8m) \geqslant 3^{m-2} - 3$. So, $288m^2 \geqslant 3^{m-2}$. Thus, $m \leqslant 11$. As $2^{n-3} \leqslant m \leqslant 11$, $n - 3 \leqslant 3$. So $n \leqslant 6$. Therefore, the pairs (n, m) that satisfy both inequalities, $m \geqslant 2^{n-3}$ and $2^{2n+2} + 2^{n+2} \geqslant 3^{m-1} - 2(3^{m-2}) - 3 = 3^{m-2} - 3$, are $(2, 3)$, $(2, 5)$, $(3, 3)$, $(3, 5)$, $(3, 7)$, $(4, 3)$, $(4, 5)$, $(4, 7)$, $(5, 5)$, $(5, 7)$, $(5, 9)$, $(6, 9)$. Recall that the critical values of $M' := M_\eta(\Omega_m(3))$ are $\eta = \frac{-2}{\pm 3^{\frac{m-3}{2}} - 1}$. So, for $m = 3$, η is undefined or 1. Therefore, the pairs with $m = 3$ can be discarded. Now, if η is not critical for H, then (G, H) is aligned if $\dim \overline{M'} = \dim M' - \dim R(M') = \dim M$. However, if η is also critical for H, then (G, H) is aligned if $\dim M' - \dim R(M') = \dim M - \dim R(M)$. In Table 5.2, we check these two conditions. First, in Table 5.1, we determine the cases where η is critical for both groups in the pair (G, H). Note that we do not compute the critical values of H in the cases where $\dim M < \dim M' - \dim R(M')$ because in such cases the codimension of the radicals cannot be equal.

	$G = \Omega_m(3)$	$H = Sp_{2n}(2)$	$H = O_{2n}^{\pm}(2)$	$H = \mathrm{SU}_n(2)$
(n, m)	$\eta = \frac{-2}{\pm 3^{\frac{m-3}{2}} - 1}$	$\eta = \frac{-2}{\pm 2^{n-1}}$	$\eta = \frac{-2}{\varepsilon 2^{n-1}}$, $\eta = \frac{-2}{-\varepsilon 2^{n-2}}$	$\eta = \frac{-2}{-(-2)^{n-3}}$, $\eta = \frac{-2}{-(-2)^{n-2}}$
$(3, 5)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\dim M < \dim \overline{M'}$
$(4, 5)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(4, 7)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{5}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(5, 5)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{8}$, $\eta = \frac{1}{8}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{8}, \frac{1}{4})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{8}, -\frac{1}{4})$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{1}{4}$
$(5, 7)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{8}$, $\eta = \frac{1}{8}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{8}, \frac{1}{4})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{8}, -\frac{1}{4})$	$\dim M < \dim \overline{M'}$
$(6, 9)$	$\eta_1 = -\frac{1}{13}$, $\eta_2 = \frac{1}{14}$	$\eta = -\frac{1}{16}$, $\eta = \frac{1}{16}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{16}, \frac{1}{8})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{16}, -\frac{1}{8})$	$\dim M < \dim \overline{M'}$

Table 5.1: Critical values of G and H in Case 1

(n, m)	$\dim \overline{M'}, G = \Omega_m(3)$	$\dim M, H = Sp_{2n}(2)$	$\dim M, H = O_{2n}^{\pm}(2)$	$\dim M, H = \mathrm{SU}_n(2)$
$(2, 5)$	20 or 16	15	6 or 10	3
$(3, 5)$	20 or 16	63 [28]	28 or 36 [16]	9
$(3, 7)$	183 or 169	63	28 or 36	9
$(4, 5)$	20 or 16	255	120 or 136	45 [21]
$(4, 7)$	183 or 169	255 [136]	120 or 136	45
$(5, 5)$	20 or 16	1023	496 or 528	165
$(5, 7)$	183 or 169	1023	496 or 528	165
$(5, 9)$	1641 or 1600	1023	496 or 528	165
$(6, 9)$	1641 or 1600	4095	2016 or 2080	693

Table 5.2: Dimensions of $\overline{M'}$ and M in Case 1

From Table 5.1, we have four cases where η is critical for both groups in the pair (G_m, H_n); $\eta = 1/2$ is critical for $(G, H) = (\Omega_5(3), Sp_6(2))$, $\eta = -1$ is critical for $(G, H) = (\Omega_5(3), O_6^-(2))$, $\eta = -1$ is critical for $(G, H) = (\Omega_5(3), \mathrm{SU}_4(2))$, $\eta = -1/4$ is critical for $(G, H) = (\Omega_7(3), Sp_8(2))$, and $\eta = -1/4$ is critical for $(G, H) = (\Omega_7(3), O_8^+(2))$. First, consider the pair $(G, H) = (\Omega_m(3), Sp_{2n}(2))$ with $(n, m) = (3, 5)$, the group $G = \Omega_5(3)$ has a critical value $\eta = -\frac{-2}{-3^{\frac{m-3}{2}} - 1} = \frac{-2}{-3^{\frac{5-3}{2}} - 1} = \frac{1}{2}$, and also the group $H = Sp_6(2)$ has a critical value $\eta = \frac{-2}{-2^{n-1}} = \frac{-2}{-2^{3-1}} = \frac{1}{2}$. That is, $\eta = \frac{1}{2}$ is critical for both groups. So, in this case we compare the codimension of $R(M')_{\frac{1}{2}}$ in $M' = M_{\frac{1}{2}}(\Omega_5(3))$ with the codimension of $R(M)_{\frac{1}{2}}$ in $M = M_{\frac{1}{2}}(Sp_6(2))$. In this pair, $\dim M = 2^{2n} - 1 = 2^6 - 1 = 63$ and $\dim R(M)_{\frac{1}{2}} = 2^{2n-1} + 2^{n-1} - 1 = 2^5 + 2^2 - 1 = 35$. So, $\dim M/R(M) = 63 - 35 = 28$. But since $\dim \overline{M'} = 20$, the pair is not aligned. Similarly, for the remaining cases, we use Table 5.1 to see if η is also critical for H, if it is critical for H then we compute the codimension of $R(M)_\eta$ in a square brackets next to $\dim M$ in Table 5.2. From Table 5.2, we see that when η is not critical for H, $\dim \overline{M'} \neq \dim M$ in all cases. In the cases where η is critical for both groups, there is only one pair, $(G, H) = (\Omega_5^+(3), O_6^-(2))$, that satisfies the condition of aligned pairs, i.e., $\dim M' - R(M')_{-1} = 16 = \dim M - \dim R(M)_{-1}$. However, as $\Omega_5^+(3) \cong O_6^-(2)$, this pair is not aligned. Therefore, no examples of aligned pairs can arise from the case $G = \Omega_m^+(3)$, and $H \cong Sp_{2n}(2)$, $O_{2n}^{\pm}(2)$, or $\mathrm{SU}_n(2)$. □

Next, let us consider Case 1 with $G = \Omega_m^{\pm}(3)$, m is even. First, consider the plus case, $G = \Omega_m^+(3)$.

Proposition 5.1.2. *Suppose that $G = \Omega_m^+(3)$, where m is even, and $H \cong Sp_{2n}(2)$, $\cong O_{2n}^{\pm}(2)$, or $\cong \mathrm{SU}_n(2)$. Then, (G, H) is not an aligned pair.*

Proof. From Proposition 5.1.1, we know that in all cases of H, $m \geqslant 2^{n-3}$. That is, the minimal degree of a nontrivial module for each H is at most m, and the inequality $m \geqslant 2^{n-3}$ covers all cases of H, similar to Proposition 5.1.1. The opposite inequality is given by $|C| \geqslant 1 + \min\{d, e\}$, where $|C|$ is the number of points in H and e, d are multiplicities of the diagram (D). According to Table 4.1, $\min\{d, e\} = d = (3^{m/2} - 1)(3^{(m-2)/2} - 1)/8$.

First, consider $H = Sp_{2n}(2)$, the number of points in H is $2^{2n} - 1$. So we have that,

$$2^{2n} - 1 \geqslant 1 + (3^{m-1} - 3^{(m-2)/2})/8 \geqslant (3^{m-1} - 3^{m-2})/9 \geqslant 2 \cdot 3^{m-4}$$

For $H = O_{2n}^{\pm}(2)$, the number of points in H is $2^{2n-1} \pm 2^{n-1}$. So we have,

$$2^{2n-1} + 2^{n-1} \geqslant 1 + (3^{m-1} - 3^{(m-2)/2})/8 \geqslant (3^{m-1} - 3^{m-2})/9 \geqslant 2 \cdot 3^{m-4}$$

For $H = \mathrm{SU}_n(2)$, the number of points in H is $(2^{2n-1} - (-2)^{n-1} - 1)/3$. so,

$$(2^{2n-1} - (-2)^{n-1} - 1)/3 \geqslant 1 + (3^{m-1} - 3^{(m-2)/2})/8 \geqslant 2 \cdot 3^{m-4}$$

So, in all cases of H we have that $2^{2n+2} \geqslant 2 \cdot 3^{m-4}$. We know from first inequality that $m \geqslant 2^{n-3}$, so $3^m \geqslant 3^{2^{n-3}} = 3^{2n-6}$. Thus, $3^{m-4} \geqslant 3^{2n-10}$. Now, $2^{2n+1} \geqslant 3^{m-4} \geqslant 3^{2n-10}$. So, $n \leqslant 13$. Therefore, $3^{m-1} \leqslant 2^{2n+1} \leqslant 2^{27}$, so $m \leqslant 19$. The pairs (n, m) that satisfy both inequalities, $m \geqslant 2^{n-3}$ and $2^{2n+1} \geqslant 3^{m-3}$, are $(2, 2)$, $(2, 4)$, $(2, 6)$, $(3, 2)$, $(3, 4)$, $(3, 6)$, $(4, 2)$, $(4, 4)$, $(4, 6)$, $(4, 8)$, $(5, 4)$, $(5, 6)$, $(5, 8)$, $(6, 8)$, $(6, 10)$. Let $M' := M_\eta(G)$ and $M = M_\eta(H)$ be the associated Matsuo algebras to G and H. In Table 5.3 we determine the cases where η is critical for both groups. From Table 5.3, we see that there are four cases where η is also critical

	$G = \Omega_m^+(3)$	$H = Sp_{2n}(2)$	$H = O_{2n}^{\pm}(2)$	$H = \mathrm{SU}_n(2)$
(n, m)	$\eta = \frac{-2}{-3^{\frac{m-2}{2}}-1}$, $\eta = \frac{-2}{3^{\frac{m-4}{2}}-1}$	$\eta = \frac{-2}{\pm 2^{n-1}}$	$\eta = \frac{-2}{\varepsilon 2^{n-1}}$, $\eta = \frac{-2}{-\varepsilon 2^{n-2}}$	$\eta = \frac{-2}{-(-2)^{n-3}}$, $\eta = \frac{-2}{-(-2)^{n-2}}$
$(2, 4)$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{2}{0}$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = 4$, $\eta_2 = 2$
$(3, 4)$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{2}{0}$	$\eta = -\frac{1}{2}$, $\eta = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(4, 4)$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{2}{0}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(4, 6)$	$\eta_1 = \frac{1}{5}$, $\boldsymbol{\eta_2 = -1}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\boldsymbol{\eta_1 = -1}$, $\eta_2 = \frac{1}{2}$
$(5, 6)$	$\eta_1 = \frac{1}{5}$, $\eta_2 = -1$	$\eta = -\frac{1}{8}$, $\eta = \frac{1}{8}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{8}, \frac{1}{4})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{8}, -\frac{1}{4})$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{1}{4}$
$(5, 8)$	$\eta_1 = -\frac{1}{14}$, $\boldsymbol{\eta_2 = -\frac{1}{4}}$	$\eta = -\frac{1}{8}$, $\eta = \frac{1}{8}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{8}, \frac{1}{4})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{8}, \boldsymbol{-\frac{1}{4}})$	$\eta_1 = \frac{1}{2}$, $\boldsymbol{\eta_2 = -\frac{1}{4}}$
$(6, 8)$	$\eta_1 = -\frac{1}{14}$, $\boldsymbol{\eta_2 = -\frac{1}{4}}$	$\eta = -\frac{1}{16}$, $\eta = \frac{1}{16}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{16}, \frac{1}{8})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{16}, -\frac{1}{8})$	$\boldsymbol{\eta_1 = -\frac{1}{4}}$, $\eta_2 = \frac{1}{8}$
$(6, 10)$	$\eta_1 = -\frac{1}{41}$, $\eta_2 = -\frac{1}{13}$	$\eta = -\frac{1}{18}$, $\eta = \frac{1}{18}$	-	-

Table 5.3: Critical values of G and H in Case 1 (m is even)

for H; $\eta = -1$ is critical for both groups in the pair $(G, H) = (\Omega_6^+(3), \mathrm{SU}_4(2))$, $\eta = -\frac{1}{4}$ is critical for both groups in $(G, H) = (\Omega_8^+(3), O_{10}^-(2))$, $\eta = -\frac{1}{4}$ is critical for both groups in the pair $(G, H) = (\Omega_8^+(3), \mathrm{SU}_5(2))$, and $\eta = -\frac{1}{4}$ is critical for both groups in the pair $(G, H) = (\Omega_8^+(3), \mathrm{SU}_6(2))$. For these cases we compute the codimension of $R(M)$ and compare it with that of $R(M')$. The codimensions of $R(M)$ are computed in the square brackets in Table 5.4. However, in all of these cases, the codimension of $R(M)$ does not coincide with codimension of $R(M')$. So, when η is also critical for H, we have no examples of aligned pairs.

Now consider the cases where η is not critical for H. From Table 5.4, the pair $(2, 4)$ which corresponds to $(G, H) = (\Omega_4^+(3), O_4^-(2))$, satisfies the condition of aligned pairs, as $\dim \overline{M'} = 12 - 2 = 10 = \dim M$. However, in this case, the eigenvalue of (D) that has multiplicity $d := \dim R(M')$ is $-3^{\frac{m-2}{2}} - 1$, which implies that $\eta := \frac{-2}{-3^{\frac{4-2}{2}}-1} = 1$ is critical for $M' = M_\eta(\Omega_4^+(3))$, but this is not possible as $\eta \notin \{0, 1\}$. So, $(G, H) = (\Omega_4^+(3), O_4^-(2))$ is not aligned. Similarly, the pair $(2, 4)$ for $(G, H) = (\Omega_4^+(3), \mathrm{SU}_2(2))$ satisfies the condition of aligned pairs, as $\dim \overline{M'} = 12 - 9 = 3 = \dim M$. However, in this case, the radical of G, $R(M')$, is the multiplicity of the eigenvalue $3^{\frac{m-4}{2}} - 1$, thus the critical value η of G is given by $\eta = -\frac{2}{3^{\frac{m-4}{2}}-1}$, which is undefined when $m = 4$. So, the pair $(G, H) = (\Omega_4^+(3), \mathrm{SU}_2(2))$ is not aligned. Since these are the only suspects in Table 5.4, we conclude that no possible aligned pairs can arise from the case $G = \Omega_m^+(3)$, m is even, and $H \cong Sp_{2n}(2)$, $\cong O_{2n}^\pm(2)$, or $\cong \mathrm{SU}_n(2)$.

(n, m)	$\dim \overline{M'}, G = \Omega_m^+(3)$	$\dim M, H = Sp_{2n}(2)$	$\dim M, H = O_{2n}^\pm(2)$	$\dim M, H = \mathrm{SU}_n(2)$
$(2, 4)$	10 or 3	15	6 or 10	3
$(2, 6)$	91 or 27	15	6 or 10	3
$(3, 4)$	10 or 3	63	28 or 36	9
$(3, 6)$	91 or 27	63	28 or 36	9
$(4, 4)$	10 or 3	255	120 or 136	45
$(4, 6)$	91 or 27	255	120 or 136	45 [21]
$(5, 4)$	10 or 3	1023	496 or 528	165
$(5, 6)$	91 or 27	1023	496 or 528	165
$(5, 8)$	820 or 261	1023	496 or 528 [188]	165
$(6, 8)$	820 or 261	4095	2016 or 2080	693 [253]
$(6, 10)$	7381 or 2421	4095	2016 or 2080	693

Table 5.4: Dimensions of $\overline{M'}$ and M, Case 1 with $G = \Omega_m^+(3)$ (m even)

□

Next, consider Case 1 with $G = \Omega_m^-(3)$, m is even.

Proposition 5.1.3. *Suppose that $G = \Omega_m^-(3)$, where m is even, and $H \cong Sp_{2n}(2)$, $\cong O_{2n}^\pm(2)$, or $\cong \mathrm{SU}_n(2)$. Then, (G, H) is not an aligned pair.*

Proof. Similar to Proposition 5.1.1 and Proposition 5.1.2, in all cases of H, we have that,

$$m \geqslant 2^{n-3} \tag{5.7}$$

Now, we write the opposite inequality $|C| \geqslant 1 + \min\{d, e\}$, where C is the set of points in H and $d = (3^{m/2}+1)(3^{(m-2)/2}+1)/8$, $e = (3^m - 9)/8$ are multiplicities in the spectrum of (D). Let $M' = M_\eta(G)$. According to Table 4.1, the number of points in $G = \Omega_m^-(3)$ is $(3^{m-1} + 3^{(m-2)/2})/2$, and dimension of $R(M')$ is d for $\eta = \frac{-2}{3^{\frac{m-2}{2}}-1}$ and e for $\eta = \frac{2}{3^{\frac{m-4}{2}}+1}$. First, consider $H = Sp_{2n}(2)$, the number of points in H is $2^{2n} - 1 = |C|$. So we have that,

$$2^{2n} - 1 \geqslant 1 + (3^{m/2}+1)(3^{(m-2)/2}+1)/8 \geqslant 3^{m/2}3^{(m-2)/2}/9 \geqslant 3^{m-3}$$

For $H = O_{2n}^{\pm}(2)$, the number of points in H is $2^{2n-1} \pm 2^{n-1}$. So we have,

$$2^{2n-1} \pm 2^{n-1} \geqslant 1 + (3^{m/2}+1)(3^{(m-2)/2}+1)/8 \geqslant 3^{m-3}$$

For $H = \mathrm{SU}_n(2)$, $|C| = (2^{2n-1} - (-2)^{n-1} - 1)/3$. So we have that,

$$(2^{2n-1} - (-2)^{n-1} - 1)/3 \geqslant (3^{m/2}+1)(3^{(m-2)/2}+1)/8 \geqslant 3^{m-3}$$

So, in all cases of H we have that,

$$2^{2n+2} \geqslant 1 + (3^{m-1} - 3^{(m-2)/2})/2 \geqslant 3^{m-3} \tag{5.8}$$

We know from inequality (5.7) that $m \geqslant 2^{n-3}$, so $3^m \geqslant 3^{2^{n-3}} = 3^{2n-6}$. So $3^{m-3} \geqslant 3^{2n-9}$. Now, by (5.8), $2^{2n+2} \geqslant 3^{m-2} \geqslant 3^{2n-9}$, so $n \leqslant 13$. By using $n \leqslant 13$ in (5.8) we get $3^{m-3} \leqslant 2^{2n+2} \leqslant 2^{28}$, so $m \leqslant 19$. The pairs (n, m) that satisfy both inequalities, (5.7) and (5.8), are $(2,2)$, $(2,4)$, $(2,6)$, $(3,2)$, $(3,4)$, $(3,6)$, $(4,2)$, $(4,4)$, $(4,6)$, $(4,8)$, $(5,4)$, $(5,6)$, $(5,8)$, $(6,8)$, $(6,10)$. In Table 5.7, we calculate $\dim \overline{M'}$ and $\dim M$ for each (n,m). Additionally, if η is also critical for H, then we compute the codimension of $R(M)$ in a square brackets next to $\dim M$ in the same table. First, in Table 5.6, we determine the pairs (G,H) where η is also critical for H. From Table 5.6, we see that $\eta = -1$ is critical for $(G,H) = (\Omega_4^-(3), Sp_4(2))$. And from Table 5.7, we see that $\dim M' - \dim R(M')_{-1} = 10 = \dim M - \dim R(M)_{-1}$. However, this pair is not aligned because $\Omega_4^-(3) \cong Sp_4(2)$. Next, the pair $(G,H) = (\Omega_4^-(3), O_4^-(2))$, which corresponds to $(n,m) = (2,4)$, satisfies the condition of aligned pairs, as $\dim \overline{M'} = 15 - 5 = 10 = \dim M$. However, since $G = \Omega_4^-(3) \cong Sp_4(2)$, $(G,H) = (\Omega_4^-(3), O_4^-(2)) \cong (Sp_4(2), O_4^-(2))$ is a particular case of Example 4.6.2. Therefore, it is not a new example. Since these pairs are the only suspects in this case, we conclude that no further examples can arise from this Case.

	$G = \Omega_m^-(3)$	$H = Sp_{2n}(2)$	$H = O_{2n}^{\pm}(2)$	$H = \mathrm{SU}_n(2)$
(n,m)	$\eta = \frac{-2}{3^{\frac{m-2}{2}}-1}$, $\eta = \frac{-2}{-3^{\frac{m-4}{2}}-1}$	$\eta = \frac{-2}{\pm 2^{n-1}}$	$\eta = \frac{-2}{\varepsilon 2^{n-1}}$, $\eta = \frac{-2}{-\varepsilon 2^{n-2}}$	$\eta = \frac{-2}{-(-2)^{n-3}}$, $\eta = \frac{-2}{-(-2)^{n-2}}$
$(2,4)$	$\eta_1 = -1$, $\eta_2 = 1$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1,\eta_2) = (-1,2)$, $\varepsilon = -, (\eta_1,\eta_2) = (1,-2)$	$\eta_1 = 4$, $\eta_2 = 2$
$(3,4)$	$\eta_1 = -1$, $\eta_2 = 1$	$\eta = -\frac{1}{2}$, $\eta = \frac{1}{2}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{2},1)$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{2},-1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(4,4)$	$\eta_1 = -1$, $\eta_2 = 1$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{4},\frac{1}{2})$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{4},-\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(5,4)$	$\eta_1 = -1$, $\eta_2 = 1$	$\eta = -\frac{1}{8}$, $\eta = \frac{1}{8}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{8},\frac{1}{4})$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{8},-\frac{1}{4})$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{1}{4}$
$(3,6)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{2}$, $\eta = \frac{1}{2}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{2},1)$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{2},-1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(4,6)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{4},\frac{1}{2})$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{4},-\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(5,8)$	$\eta_1 = -\frac{1}{13}$, $\eta_2 = \frac{1}{5}$	$\eta = -\frac{1}{8}$, $\eta = \frac{1}{8}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{8},\frac{1}{4})$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{8},\boldsymbol{-\frac{1}{4}})$	$\eta_1 = \frac{1}{2}$, $\boldsymbol{\eta_2 = -\frac{1}{4}}$
$(6,8)$	$\eta_1 = -\frac{1}{13}$, $\eta_2 = \frac{1}{5}$	$\eta = -\frac{1}{16}$, $\eta = \frac{1}{16}$	$\varepsilon = +, (\eta_1,\eta_2) = (-\frac{1}{16},\frac{1}{8})$, $\varepsilon = -, (\eta_1,\eta_2) = (\frac{1}{16},-\frac{1}{8})$	$\boldsymbol{\eta_1 = -\frac{1}{4}}$, $\eta_2 = \frac{1}{8}$
$(6,10)$	$\eta_1 = -\frac{1}{40}$, $\eta_2 = \frac{1}{14}$	$\eta = -\frac{1}{18}$, $\eta = \frac{1}{18}$	-	-

Table 5.5: Critical values of G and H, $\varepsilon = -1$, Case 1

From Table 5.5, the pairs where η is also critical for H are listed in Table 5.6

(n,m)	(G,H)	η
$(2,4)$	$(\Omega_4^-(3), Sp_4(2))$	-1
$(2,4)$	$(\Omega_4^-(3), O_4^+(2))$	-1
$(3,4)$	$(\Omega_4^-(3), O_6^-(2))$	-1
$(3,6)$	$(\Omega_6^-(3), O_6^-(2))$	$\frac{1}{2}$
$(4,4)$	$(\Omega_4^-(3), \mathrm{SU}_4(2))$	-1

Table 5.6: (G,H) with same critical value

(n, m)	$\dim \overline{M'}, G = \Omega_m^-(3)$	$\dim M, H = Sp_{2n}(2)$	$\dim M, H = O_{2n}^{\pm}(2)$	$\dim M, H = \mathrm{SU}_n(2)$
$(2, 4)$	10 or 6	15 [10]	$(+, -) = (6[5], 10)$	3
$(2, 6)$	91 or 36	15	$(+, -) = (6, 10)$	3
$(3, 4)$	10 or 6	63	$(+, -) = (28, 36[16])$	9
$(3, 6)$	91 or 36	63	$(+, -) = (28, 36[21])$	9
$(4, 4)$	10 or 6	255	$(+, -) = (120, 136)$	45[21]
$(4, 6)$	91 or 36	255	$(+, -) = (120, 136)$	45
$(5, 4)$	10 or 6	1023	$(+, -) = (496, 528)$	165
$(5, 6)$	91 or 36	1023	$(+, -) = (496, 528)$	165
$(5, 8)$	820 or 288	1023	$(+, -) = (496, 528)$	165
$(6, 8)$	820 or 288	4095	$(+, -) = (2016, 2080)$	693
$(6, 10)$	7381 or 2502	4095	$(+, -) = (2016, 2080)$	693

Table 5.7: Dimensions of $\overline{M'}$ and M, Case 1 with $G = \Omega_m^-(3)$, (m even)

□

5.2 Case 2: Groups in Char(3) Inside Groups in Char(2)

In this section we will consider the opposite of Case 1. That is, we will take the smaller group H to be isomorphic to the orthogonal group $\Omega_n^{\pm}(3)$ and the larger group G to be $Sp_{2m}(2), O_{2m}^{\pm}(2)$, or $\mathrm{SU}_m(2)$. First, consider $H = \Omega_n(3)$, where n is odd.

Proposition 5.2.1. *Suppose that $G = Sp_{2m}(2), O_{2m}^{\pm}(2)$, or $\mathrm{SU}_m(2)$, and $H \cong \Omega_n(3)$ where n is odd. Then, (G, H) is not an aligned pair.*

Proof. According to [SZ93], the minimal possible degree for a non-trivial module for H is $\frac{q^{2n}-1}{q^2-1} - \frac{q^n-1}{q-1}$, $q = 3$. So, for $G = Sp_{2m}(2), O_{2m}^{\pm}(2)$, or $\mathrm{SU}_m(2)$ we have the following inequality,

$$2m \geqslant \frac{3^{2n}-1}{8} - \frac{3^n-1}{2} \tag{5.9}$$

So, $2m \geqslant \frac{3^{2n}-4\cdot 3^n-5}{8} \geqslant \frac{3^{2n}-9\cdot 3^n-9}{9} \geqslant 3^{2n-2} - 3^n - 1 \geqslant 3^n(3^{n-2}-1) \geqslant 3^n$. That is,

$2m \geqslant 3^{n-1}$.

Let $M' = M_\eta(G)$ where $G = Sp_{2m}(2)$, $O^+_{2m}(2)$, or $\mathrm{SU}_m(2)$, and let $M = M_\eta(H) = M_\eta(\Omega^+_m(3))$. Now for each G we want write $|C| \geqslant 1 + \min\{d, e\}$, where $|C| = \frac{3^{n-1}-3^{\frac{n-1}{2}}}{2}$ is the number of points in H and e, d are multiplicities of the diagram (D). First, consider the case $G = Sp_{2m}(2)$. The smaller multiplicity in the spectrum of $G = Sp_{2m}(2)$ is $2^{2m-1} - 2^{m-1} - 1$. So we have that,

$$\frac{3^{n-1} - 3^{\frac{n-1}{2}}}{2} \geqslant 2^{2m-1} - 2^{m-1} \tag{5.10}$$

Thus, $3^{n-1} \geqslant 2^{2m} - 2^m \geqslant 2^m$.

Consider $G = O^\pm_{2m}(2)$. For the plus case, $G = O^+_{2m}(2)$, the smaller multiplicity for $G = O^+_{2m}(2)$ is $\frac{(2^m-1)(2^{m-1}-1)}{3}$. So we have, $\frac{3^{n-1}-3^{\frac{n-1}{2}}}{2} \geqslant 1 + \frac{(2^m-1)(2^{m-1}-1)}{3} \geqslant \frac{2^{m-1}2^{m-2}}{4} \geqslant 2^{2m-5}$, i.e., $3^{n-1} \geqslant 2^{2m-5}$. For $G = O^-_{2m}(2)$, the smaller multiplicity is $(2^{2m} - 4)/3$, so we have that $3^{n-1} \geqslant \frac{3^{n-1}-3^{\frac{n-1}{2}}}{2} \geqslant 1 + \frac{2^{2m}-4}{3} \geqslant 2^{2m-2} - 1$. So for both cases of $G = O^\pm_{2m}(2)$, we have that $3^{n-1} \geqslant 2^{2m-5}$.

Next, consider $G = \mathrm{SU}_m(2)$. The smaller multiplicity in this case is $4(2^{2m-3} - 7(-2)^{m-3} - 1)/9$. Therefore, we have that,

$$\frac{3^{n-1} - 3^{\frac{n-1}{2}}}{2} \geqslant 1 + 4(2^{2m-3} - 7(-2)^{m-3} - 1)/9 \tag{5.11}$$

If m is even, then $-7(-2)^{m-3} > 0$. So, inequality (5.11) simplifies to $3^{n-1} \geqslant 8(2^{2m-3})/9 \geqslant 2^{2m}/16 \geqslant 2^{2m-4}$. If m is odd, then $3^{n-1} \geqslant 4(2^{2m-3} - 8 \cdot 2^{m-3} - 1)/9 \geqslant 4(2^{2m-3} - 2^m - 2)/9 \geqslant (2^{2m-1} - 2^{m+2} - 8)/16 \geqslant 2^{2m-5} - 2^{m-2} - 1 \geqslant 2^{2m-5} - 2^{2m-6} \geqslant 2^{2m-6}$. So in both cases, $3^{n-1} \geqslant 2^{2m-6}$. Therefore, in all cases of G, we have that

$$3^{n-1} \geqslant 2^{2m-6} \tag{5.12}$$

From inequality (5.9), we have that $2m \geqslant 3^{n-1}$, so $2m \geqslant 3^{n-1} \geqslant 2^{2m-6}$. Thus, $m \geqslant 2^{2m-7}$. Therefore, $m \leqslant 4$. So, $3^{n-1} \leqslant 2m \leqslant 8$, thus $n \leqslant 3$. Therefore, the (n,m) pairs that satisfy both inequalities, (5.9) and (5.12), are $(3,1), (3,2), (3,3), (3,4)$. In Table 5.10, we check the condition of aligned pairs. In this case, $\dim M = (3^{n-1} - 3^{\frac{n-1}{2}})/2$, and depending on $\dim R(M')$, $\dim \overline{M'} =$

- $2^{2m} - 2^{2m-1} + 2^{m-1}$ or $2^{2m} - 2^{2m-1} - 2^{m-1}$, for $G = Sp_{2m}(2)$,
- $2^{2m-1} \pm 2^{m-1} - ((2^m \pm 1)(2^{m-1} \pm 1)/3)$ or $2^{2m-1} \pm 2^{m-1} - ((2^{2m} - 4)/3)$, for $G = O^{\pm}_{2m}(2)$, and
- $[(2^{2m-1} - (-2)^{m-1} - 1)]/3 - 8[(2^{2m-3} - (-2)^{m-2} - 1)]/9$, or $[(2^{2m-1} - (-2)^{m-1} - 1)]/3 - 4[(2^{2m-3} - 7(-2)^{m-3} - 1)]/9$, for $G = \mathrm{SU}_m(2)$.

	$H = \Omega_n(3)$	$G = Sp_{2m}(2)$	$G = O^{\pm}_{2m}(2)$	$G = \mathrm{SU}_m(2)$
(n,m)	$\eta = \frac{-2}{\pm 3^{\frac{n-3}{2}} - 1}$	$\eta = \frac{-2}{\pm 2^{m-1}}$	$\eta = \frac{-2}{\varepsilon 2^{m-1}}$, $\eta = \frac{-2}{-\varepsilon 2^{m-2}}$	$\eta = \frac{-2}{-(-2)^{m-3}}$, $\eta = \frac{-2}{-(-2)^{m-2}}$
$(5,2)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = -4$, $\eta_2 = 2$
$(5,3)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = -1$
$(5,4)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(7,2)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{5}$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = -4$, $\eta_2 = 2$
$(7,3)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{5}$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = -1$
$(7,4)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{5}$	$\eta = -\frac{1}{4}$, $\eta = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$

Table 5.8: Critical values of G and H, Case 2 (n odd)

First, in Table 5.8, we compute the critical values for both groups in each case. The pairs where η is also critical for H are listed in Table 5.9. In Table 5.9 we see that $\dim M' - \dim R(M') \neq \dim M - \dim R(M)$ in all cases except for the pair $(G,H) = (O_6^-(2), \Omega_5(3))$. However, since $O_6^-(2) \cong \Omega_5(3)$, this pair is not aligned. For the remaining cases, η is only critical for G. But as listed in Table 5.10, $\dim M \neq \dim \overline{M'} = \dim M' - \dim R(M')$, for all the pairs (n,m). Therefore,

(n,m)	(G,H)	η	$\dim M'/R(M')_\eta$	$\dim M/R(M)_\eta$
$(5,2)$	$(Sp_4(2),\Omega_5(3))$	$\eta=-1$	10	16
$(5,2)$	$(O_4^+(2),\Omega_5(3))$	$\eta=-1$	5	16
$(5,3)$	$(Sp_6(2),\Omega_5(3))$	$\eta=\frac{1}{2}$	28	21
$(5,3)$	$(O_6^-(2),\Omega_5(3))$	$\eta=\frac{1}{2}$	21	21
$(5,3)$	$(\mathrm{SU}_3(2),\Omega_5(3))$	$\eta=-1$	9	16
$(5,4)$	$(O_8^+(2),\Omega_5(3))$	$\eta=\frac{1}{2}$	36	21
$(5,4)$	$(\mathrm{SU}_4(2),\Omega_5(3))$	$\eta=-1$	21	16
$(7,4)$	$(O_8^+(2),\Omega_7(3))$	$\eta=-\frac{1}{4}$	85	169
$(7,4)$	$(Sp_8(2),\Omega_7(3))$	$\eta=-\frac{1}{4}$	8	169

Table 5.9: Pairs (G,H) where η is also critical for H

(n,m)	$\dim(M), H=\Omega_n^+(3)$	$\dim(\overline{M'})$, $G=Sp_{2m}(2)$	$\dim(\overline{M'})$, $G=O_{2m}^{\pm}(2)$	$\dim(\overline{M'})$, $G=\mathrm{SU}_m(2)$
$(3,3)$	3	36 or 28	(21 or 8), (21 or 16)	1 or 9
$(3,4)$	3	136 or 120	(85 or 36), (85 or 52)	21 or 25
$(3,5)$	3	528 or 496	(341 or 165), (341 or 188)	45 or 121
$(5,2)$	36	10 or 6	(5 or 2), (5 or 6)	3 or 1
$(5,3)$	36 [21]	36 or 28	(21 or 8), (21 or 16)	1 or 9
$(5,4)$	36	136 or 120	(85 or 36), (85 or 52)	21 or 25
$(5,5)$	36	528 or 496	(341 or 165), (341 or 188)	45 or 121
$(7,2)$	351	10 or 6	(5 or 2), (5 or 6)	3 or 1
$(7,3)$	351	36 or 28	(21 or 8), (21 or 16)	1 or 9
$(7,4)$	351	136 or 120	(85 or 36), (85 or 52)	21 or 25
$(7,5)$	351	528 or 496	(341 or 165), (341 or 188)	45 or 121

Table 5.10: Dimension of M and $\overline{M'}$, Case 2, n is odd

no further examples can arise from the case $G = Sp_{2m}(2), O^{\pm}_{2m}(2)$, or $\mathrm{SU}_m(2)$, and $H \cong \Omega_n(3)$ (n is odd). □

Next, let us consider Case 2 with $H = \Omega^{\pm}_n(3)$, n is even.

Proposition 5.2.2. *Suppose that $G = Sp_{2m}(2), O^{\pm}_{2m}(2)$, or $\mathrm{SU}_m(2)$ and $H \cong \Omega^{\pm}_n(3)$, where n is even. Then (G,H) is not an aligned pair.*

Proof. For the plus case $H \cong \Omega^+_n(3)$, the minimal possible degree for a non-trivial module for H is $3(3^{2n-2}-1)/8-(3^{n-1}-1)/2-7$. That is, $(3^{2n-1}-1-4(3^{n-1}-1)-56)/8 = (3^{2n-1}-4\cdot 3^{n-1}-53)/8$. This bound must be at most $2m$ as H is supposed to act non-trivially on the natural module of G_m or G_{2m}. So the first inequality in this case is $2m \geqslant (3^{2n-1}-4\cdot 3^{n-1}-53)/8$. So, $2m \geqslant (3^{2n-1}-9\cdot 3^{n-1}-$

$63)/9 \geqslant 3^{2n-3} - 3^{n-1} - 7 \geqslant 3^{n-1}(3^{n-2} - 1) - 7 \geqslant 3^{n-1} - 7$. For the minus case, $H \cong \Omega^-_{2n}(3)$, the lower bound on the dimension of the non-trivial module for H is $3(3^{2n-2}-1)/8 - 3^{n-1} - n + 2$. So we have that, $2m \geqslant (3^{2n-1} - 8 \cdot 3^{n-1} - 8n + 16)/8$. So, $2m \geqslant (3^{2n-1} - 9 \cdot 3^{n-1} - 9n)/9 \geqslant 3^{2n-3} - 3^{n-1} - n \geqslant 3^{n-1}(3^{n-2} - 1) - n \geqslant 3^{n-1} - n$. Therefore, the inequality

$$2m \geqslant 3^{n-1} - n \geqslant 3^{n-1} - 3^{n-2} = 2 \cdot 3^{n-2} \tag{5.13}$$

covers both cases of $H = \Omega^{\pm}_n(3)$. Now, we want to find the opposite inequalities for both cases of $H = \Omega^{\pm}_{2n}(3)$. First, consider $H = \Omega^+_n(3)$. The number of points in $(\Omega^+_{2n}(3))$ is $(3^{n-1} - 3^{\frac{n-2}{2}})/2$. This number has to be at least one plus the smaller multiplicity of the diagram (D). We know from Proposition 5.2.1 that in all cases of G, i.e., $G = Sp_{2m}(2), O^{\pm}_{2m}(2), \mathrm{SU}_m(2)$, we have $(3^{n-1} - 3^{\frac{n-2}{2}})/2 \geqslant 2^{2m-6}$. That is, $3^{n-1} \geqslant 2^{2m-6}$. Similarly for $H = \Omega^-_{2n}(3)$, we have that $(3^{n-1} + 3^{\frac{n-2}{2}})/2 \geqslant 2^{2m-6}$. That is, $3^{n-1} + 3^{n-2} \geqslant 2^{2m-6}$, so $4 \cdot 3^{n-2} \geqslant 2^{2m-6}$. Therefore, for both cases we have that

$$4 \cdot 3^{n-1} \geqslant 2^{2m-6} \tag{5.14}$$

Now, by multiplying (5.13) by 6 and use it in (5.14), we get that $12m \geqslant 4 \cdot 3^{n-1} \geqslant 2^{2m-6}$. That is, $12m \geqslant 2^{2m-6}$, thus $m \leqslant 8$. By substituting $m \leqslant 8$ in (5.13), we get that $3^{n-2} \leqslant m \leqslant 16$, so $n - 2 \leqslant 2$, thus $n \leqslant 4$. In Table 5.13, we see that $\dim M \neq \dim \overline{M'}$ for all the pairs (n, m). Therefore, if η is not critical for $H = \Omega^{\pm}_n(3)$, then we have no examples of aligned pairs. In Tables 5.11 and 5.12, we compute the critical values of G and H, and we list the pairs that have same η in Table 5.14. In Table 5.14, we see that $\dim \overline{M'} \neq \dim M - \dim R(M)$, in all the cases except for $(G, H) = (Sp_4(2), \Omega^-_4(3))$. However, this pair is not aligned because $Sp_4(2) \cong \Omega^-_4(3)$. Therefore, no examples arise from this Case.

	$H = \Omega_n^+(3)$	$G = Sp_{2m}(2)$	$G = O_{2m}^{\pm}(2)$	$G = \mathrm{SU}_m(2)$
(n,m)	$\eta = \frac{-2}{-3^{\frac{n-2}{2}}-1}$, $\eta = \frac{-2}{3^{\frac{n-4}{2}}-1}$	$\eta = \frac{-2}{\pm 2^{m-1}}$	$\eta = \frac{-2}{\varepsilon 2^{m-1}}$, $\eta = \frac{-2}{-\varepsilon 2^{m-2}}$	$\eta = \frac{-2}{-(-2)^{m-3}}$, $\eta = \frac{-2}{-(-2)^{m-2}}$
$(4,2)$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{2}{0}$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = 4$, $\eta_2 = 2$
$(4,3)$	$\eta_1 = \frac{1}{2}$, $\eta_2 = -\frac{2}{0}$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(6,2)$	$\eta_1 = \frac{1}{5}$, $\eta_2 = -\frac{1}{4}$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = 4$, $\eta_2 = 2$
$(6,3)$	$\eta_1 = \frac{1}{5}$, $\eta_2 = -\frac{1}{4}$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(6,4)$	$\eta_1 = \frac{1}{5}$, $\eta_2 = -\frac{1}{4}$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(6,5)$	$\eta_1 = \frac{1}{5}$, $\eta_2 = -\frac{1}{4}$	$\eta_1 = -\frac{1}{8}$, $\eta_2 = \frac{1}{8}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{8}, \frac{1}{4})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{8}, \boldsymbol{-\frac{1}{4}})$	$\eta_1 = \frac{1}{2}$, $\boldsymbol{\eta_2 = -\frac{1}{4}}$

Table 5.11: Critical values of G and H, $H = \Omega_n^+(3)$

	$G = \Omega_n^-(3)$	$H = Sp_{2m}(2)$	$H = O_{2m}^{\pm}(2)$	$H = \mathrm{SU}_m(2)$
(n,m)	$\eta = \frac{-2}{3^{\frac{n-2}{2}}-1}$, $\eta = \frac{-2}{-3^{\frac{n-4}{2}}-1}$	$\eta = \frac{-2}{\pm 2^{m-1}}$	$\eta = \frac{-2}{\varepsilon 2^{m-1}}$, $\eta = \frac{-2}{-\varepsilon 2^{m-2}}$	$\eta = \frac{-2}{-(-2)^{m-3}}$, $\eta = \frac{-2}{-(-2)^{m-2}}$
$(4,2)$	$\eta_1 = -1$, $\eta_2 = 1$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = 4$, $\eta_2 = 2$
$(4,3)$	$\eta_1 = -1$, $\eta_2 = 1$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(6,2)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -1$, $\eta_2 = 1$	$\varepsilon = +, (\eta_1, \eta_2) = (-1, 2)$, $\varepsilon = -, (\eta_1, \eta_2) = (1, -2)$	$\eta_1 = 4$, $\eta_2 = 2$
$(6,3)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{2}, 1)$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{2}, -1)$	$\eta_1 = 2$, $\eta_2 = 2$
$(6,4)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{4}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{4}, \frac{1}{2})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{4}, -\frac{1}{2})$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$
$(6,5)$	$\eta_1 = -\frac{1}{4}$, $\eta_2 = \frac{1}{2}$	$\eta_1 = -\frac{1}{8}$, $\eta_2 = \frac{1}{8}$	$\varepsilon = +, (\eta_1, \eta_2) = (-\frac{1}{8}, \frac{1}{4})$, $\varepsilon = -, (\eta_1, \eta_2) = (\frac{1}{8}, \boldsymbol{-\frac{1}{4}})$	$\eta_1 = \frac{1}{2}$, $\boldsymbol{\eta_2 = -\frac{1}{4}}$

Table 5.12: Critical values of G and H, $H = \Omega_n^-(3)$

(n,m)	$\dim M, H = \Omega_n^{\pm}(3)$	$\dim \overline{M'}$, $G = Sp_{2m}(2)$	$\dim \overline{M'}$, $G = O_{2m}^{\pm}(2)$	$\dim \overline{M'}$, $G = \mathrm{SU}_m(2)$
$(4,2)$	12 and 15	10 or 6	(5 or 2), (5 or 6)	3 or 1
$(4,3)$	12 and 15	36 or 28	(21 or 8), (21 or 16)	1 or 9
$(4,4)$	12 and 15	136 or 120	(85 or 36), (85 or 52)	21 or 25
$(4,5)$	12 and 15	528 or 496	(341 or 165), (341 or 188)	45 or 121
$(6,2)$	117 and 126	10 or 6	(5 or 2), (5 or 6)	3 or 1
$(6,3)$	117 and 126	36 or 28	(21 or 8), (21 or 16)	1 or 9
$(6,4)$	117 and 126	136 or 120	(85 or 36), (85 or 52)	21 or 25
$(6,5)$	117 and 126	528 or 496	(341 or 165), (341 or 188)	45 or 121

Table 5.13: Dimension of M and $\overline{M'}$

(n,m)	(G,H)	η	$\dim M'/R(M')_\eta$	$\dim M/R(M)_\eta$
$(4,3)$	$(Sp_6(2), \Omega_4^+(3))$	$\eta = \frac{1}{2}$	28	10
$(4,3)$	$(O_6^-(2), \Omega_4^+(3))$	$\eta = \frac{1}{2}$	21	10
$(6,4)$	$(Sp_8(2), \Omega_6^+(3))$	$\eta = -\frac{1}{4}$	136	27
$(6,4)$	$(O_8^+(2), \Omega_6^+(3))$	$\eta = -\frac{1}{4}$	85	27
$(6,5)$	$(O_{10}^-(2), \Omega_6^+(3))$	$\eta = -\frac{1}{4}$	188	27
$(4,2)$	$(Sp_4(2), \Omega_4^-(3))$	$\eta = -1$	10	10
$(4,2)$	$(O_4^+(2), \Omega_4^-(3))$	$\eta = -1$	5	10
$(4,3)$	$(O_6^-(2), \Omega_4^-(3))$	$\eta = -1$	16	10
$(6,4)$	$(Sp_8(2), \Omega_6^-(3))$	$\eta = -\frac{1}{4}$	136	91
$(6,4)$	$(O_8^-(2), \Omega_6^-(3))$	$\eta = -\frac{1}{4}$	85	91
$(6,5)$	$(O_{10}^-(2), \Omega_6^-(3))$	$\eta = -\frac{1}{4}$	188	91
$(6,5)$	$(\mathrm{SU}_5(2), \Omega_6^-(3))$	$\eta = -\frac{1}{4}$	45	91

Table 5.14: Pairs (G,H) where η is also critical for H

□

5.3 Case 3: Symmetric Group

If we assign characteristic 0 to the symmetric group S_n, then we have the following cross characteristic cases:

1. $H \cong S_n$ inside $G = \Omega^{\pm}_m(3), O^{\pm}_{2m}(2), Sp_{2m}(2), \mathrm{SU}_m(2)$.
2. $H \cong \Omega^{\pm}_n(3), O^{\pm}_{2n}(2), Sp_{2n}(2), \mathrm{SU}_n(2)$ inside $G = S_m$.

Proposition 5.3.1. *Suppose that $G = \Omega^{\pm}_m(3)$, $O^{\pm}_{2m}(2)$, $Sp_{2m}(2)$, or $\mathrm{SU}_m(2)$ and $H \cong S_n$. Then (G, H) is not aligned.*

Proof. According to [Wag76], the lower bound for the dimension of nontrivial irreducible module for the symmetric group over a field $\mathbb{F}$ of characteristic two is $n-1$ or $n-2$ for all n. So we have that,

$$m \geqslant n - 2 \tag{5.15}$$

The dimension of the Matsuo algebra $M := M_\eta(H)$, for $H = S_n$, is $\binom{n}{2}$. First, consider $G = \Omega^{\varepsilon}_m(3)$ and let $M' := M_\eta(G)$ be its associated Matsuo algebra. Recall that, if (G, H) is aligned, then the set of points in H spans the factor algebra $\overline{M'} = M'/R(M')$. That is, $\dim M \geqslant \dim \overline{M'} = 1 + d$, where d is a multiplicity in the spectrum of the diagram of G. If $G = \Omega_m(3)$, where m is odd, then according to Table 4.1, the smaller multiplicity for the diagram of G is $(3^{m-1}-1-2(3^{\frac{m-1}{2}}+1))/4$. So we have that, $n^2 \geq \binom{n}{2} \geqslant 1 + (3^{m-1} - 1 - 2(3^{\frac{m-1}{2}} + 1))/4$. Thus, $n^2 \geqslant (3^{m-1}-2\cdot 3^{\frac{m-1}{2}}-3)/4 \geqslant (3^{m-1}-3\cdot 3^{\frac{m-1}{2}})/4 = (3^{m-1}-3^{m-2})/4 = 1/2\cdot 3^{m-2} \geqslant 3^{m-3}$. For $G = \Omega^{+}_m(3)$, m is even, the smaller multiplicity is $(3^{m/2} - 1)(3^{(m-2)/2} - 1)/8$. So we have, $n^2 \geqslant (3^{m/2} - 1)(3^{(m-2)/2} - 1)/8 \geqslant (3^{m-1} - 2 \cdot 3^{(m-2)/2})/8 \geqslant 3^{m-1} - 2 \cdot$

$3^{m-2}/9 \geqslant 3^{m-4}$. For the minus case, $G = \Omega_m^-(3)$, the smaller multiplicity is given by $(3^{m/2}+1)(3^{(m-2)/2}+1)/8$. So, in this case $n^2 \geqslant 1+(3^{m/2}+1)(3^{(m-2)/2}+1)/8 \geqslant (3^{m-1}+4\cdot 3^{(m-2)/2}+1)/9 \geqslant 3^{m-3}$. Therefore, in all cases of G we have that,

$$n^2 \geqslant 3^{m-4} \tag{5.16}$$

Next, consider $G = O_{2m}^{\pm}(2)$. For the plus case, $G = O_{2m}^{+}(2)$, the smaller multiplicity for G is $(2^m-1)(2^{m-1}-1)/3$. Thus we have, $n^2 \geqslant \binom{n}{2} \geqslant 1+(2^m-1)(2^{m-1}-1)/3 \geqslant 2^{m-1}2^{m-2}/4 \geqslant 2^{2m-5}$. For the minus case, $G = O_{2m}^{-}(2)$, the smaller multiplicity is $(2^m+1)(2^{m-1}+1)/3$, thus we have $n^2 \geqslant \binom{n}{2} \geqslant 1+(2^m+1)(2^{m-1}+1)/3 \geqslant 2^{2m-1}/4 \geqslant 2^{2m-3}$. Therefore, in both cases of $G = O_{2m}^{\pm}(2)$ we have that,

$$n^2 \geqslant 2^{2m-5} \tag{5.17}$$

For $G = Sp_{2m}(2)$, the smaller multiplicity of G is $2^{2m-1}-2^{m-1}-1$. So we have that,

$$n^2 \geqslant 2^{2m-1}-2^{m-1} \geqslant 2^{m-1}(2^m-1) \geqslant 2^{m-1} \tag{5.18}$$

Lastly, for $G = \mathrm{SU}_m(2)$, the smaller multiplicity of $\mathrm{SU}_m(2)$ is $4(2^{2m-3}-7(-2)^{m-3}-1)/9$. Therefore, we have that,

$$n^2 \geqslant \binom{n}{2} \geqslant 1+4(2^{2m-3}-7(-2)^{m-3}-1)/9 \tag{5.19}$$

We know from the discussion after (5.11) that (5.19) can be simplified into $n^2 \geqslant 2^{m-5}$.

Therefore, in all cases of G, we have that $n^2 \geqslant 2^{m-5}$. Now, from inequality (5.15),

$m+2 \geqslant n$, so $m^2+4m+4 \geqslant n^2$. Therefore, $m^2+4m+4 \geqslant 2^{m-5}$. So, $m \leqslant 12$. Since $n-1 \leqslant m \leqslant 12$, we have that $n \leqslant 13$. Now, the pairs (n, m) that satisfy both inequalities, (5.15) and (5.19), are $(2,3), (3,3)$, $(4,3)$, $(5,3)$, $(2,5)$, $(3,5)$, $(4,5)$, $(5,5)$, $(6,5)$, $(7,5)$, $(2,7)$, $(3,7)$, $(4,7)$, $(5,7)$, $(6,7)$, $(7,7)$, $(8,7)$, $(9,7)$, $(4,9)$, $(5,9)$, $(6,9)$, $(7,9)$, $(8,9)$, $(10,9)$, $(11,9)$, $(8,11)$, $(9,11)$, $(10,11)$, $(11,11)$, $(12,11)$, $(13,11)$. In Table 5.15, we compute $\dim \overline{M'}$ and $\dim M$ for the cases (n, m) with m is odd. Similarly, in Table 5.17 we listed $\dim \overline{M'}$ and $\dim M$ for all (n, m) with m is even. Note that, only in the cases where $\dim M > \dim \overline{M'}$, it is possible that $\dim \overline{M'} = \dim M - \dim R(M)$. So, only for these cases, we compute the critical values of G and H, see Tables 5.16 and 5.18. In the current case, $\dim M = \binom{n}{2}$, and $\dim \overline{M'} =$

- $(3^{m-1}-3^{\frac{m-1}{2}})/2-(3^{m-1}-1-2(3^{\frac{m-1}{2}}+1))/4$, or $(3^{m-1}-3^{\frac{m-1}{2}})/2-(3^{m-1}-1)/4$ for $G = \Omega_m(3)$,
- $(3^{m-1}-\varepsilon 3^{\frac{m-2}{2}})/2-(3^{\frac{m}{2}}-\varepsilon)(3^{\frac{m-2}{2}}-\varepsilon)/8$, or $(3^{m-1}-\varepsilon 3^{\frac{m-2}{2}})/2-(3^m-9)/8$ for $G = \Omega^{\varepsilon}_{2m}(3)$, where $\varepsilon = \pm 1$
- $2^{2m}-2^{2m-1}+2^{m-1}$, or $2^{2m}-2^{2m-1}-2^{m-1}$ for $G = Sp_{2m}(2)$,
- $2^{2m-1} \pm 2^{m-1} - ((2^m \pm 1)(2^{m-1} \pm 1)/3)$, or $2^{2m-1} \pm 2^{m-1} - ((2^{2m}-4)/3)$ for $G = O^{\pm}_{2m}(2)$,
- $[(2^{2m-1}-(-2)^{m-1}-1)]/3-8[(2^{2m-3}-(-2)^{m-2}-1)]/9$, or $[(2^{2m-1}-(-2)^{m-1}-1)]/3-4[(2^{2m-3}-7(-2)^{m-3}-1)]/9$ for $G = \mathrm{SU}_m(2)$.

Let us now check the candidates of Table 5.15 and Table 5.16. First, consider the pair $(n, m) = (7,5)$, this corresponds to $M' = M_\eta(\Omega^+_5(3))$ and $M = M_\eta(S_7)$. Note that $(G, H) = (\Omega^+_5(3), S_7)$ satisfies the condition of aligned pairs, as $\dim(\overline{M'}) =$

(n,m)	$\dim \overline{M'}, G = \Omega_m^+(3)$	$\dim \overline{M'}, G = O_{2m}^{\pm}(2)$(+),(-)	$\dim \overline{M'}, G = Sp_{2m}(2)$	$\dim \overline{M'}, G = \mathrm{SU}_m(2)$	$\dim(M), H = S_n$
$(4,3)$	3 or 1	(21 or 8),(21 or 16)	36 or 28	1 or 9	6
$(5,3)$	3 or 1	(21 or 8),(21 or 16)	36 or 28	1 or 9	10
$(2,5)$	21 or 16	(341 or 156),(341 or 188)	528 or 496	45 or 121	1
$(3,5)$	21 or 16	(341 or 156), (341 or 188)	528 or 496	45 or 121	3
$(4,5)$	21 or 16	(341 or 156), (341 or 188)	528 or 496	45 or 121	6
$(5,5)$	21 or 16	(341 or 156), (341 or 188)	528 or 496	45 or 121	10
$(6,5)$	21 or 16	(341 or 156), (341 or 188)	528 or 496	45 or 121	15
$(7,5)$	21 or 16	(341 or 156), (341 or 188)	528 or 496	45 or 121	21
$(4,9)$	1641 or 1600	(87381 or 43436),(87381 or 43948)	131328 or 130816	14365 or 29241	6
$(5,9)$	1641 or 1600	(87381 or 43436),(87381 or 43948)	131328 or 130816	14365 or 29241	10
$(6,9)$	1641 or 1600	(87381 or 43436),(87381 or 43948)	131328 or 130816	14365 or 29241	15
$(7,9)$	1641 or 1600	(87381 or 43436),(87381 or 43948)	131328 or 130816	14365 or 29241	21
$(8,3)$	3 or 1	(21 or 8),(21 or 16)	36 or 28	1 or 9	28
$(9,3)$	3 or 1	(21 or 8),(21 or 16)	36 or 28	1 or 9	36
$(10,9)$	1641 or 1600	(87381 or 43436),(87381 or 43948)	131328 or 130816	14365 or 29241	45
$(11,9)$	1641 or 1600	(87381 or 43436),(87381 or 43948)	131328 or 130816	14365 or 29241	55
$(8,11)$	14763 or 14641	(1398101 or 698028),(1398101 or 700076)	2098176 or 2096128	232221 or 466489	28
$(9,11)$	14763 or 14641	(1398101 or 698028),(1398101 or 700076)	2098176 or 2096128	232221 or 466489	36
$(10,11)$	14763 or 14641	(1398101 or 698028),(1398101 or 700076)	2098176 or 2096128	232221 or 466489	45
$(11,11)$	14763 or 14641	(1398101 or 698028),(1398101 or 700076)	2098176 or 2096128	232221 or 466489	55
$(12,11)$	14763 or 14641	(1398101 or 698028),(1398101 or 700076)	2098176 or 2096128	232221 or 466489	66
$(13,11)$	14763 or 14641	(1398101 or 698028),(1398101 or 700076)	2098176 or 2096128	232221 or 466489	78

Table 5.15: Dimension of M and $\overline{M'}$, m is odd

(n,m)	(G,H)	Critical values of G	The critical value of H
$(4,3)$	$(\Omega_3(3), S_4)$	η_1 is undefined, $\eta_2 = 1$	undefined
$(4,3)$	$(\mathrm{SU}_3(2), S_4)$	$\eta_1 = 2$, $\eta_2 = -1$	undefined
$(5,3)$	$(\Omega_3(3), S_5)$	η_1 is undefined, $\eta_2 = 1$	$\eta = -2$
$(5,3)$	$(\mathrm{SU}_3(2), S_5)$	$\eta_1 = 2$, $\eta_2 = -1$	$\eta = -2$
$(5,3)$	$(O_6^+(2), S_5)$	$\eta_1 = \frac{1}{2}$, $\eta_2 = 1$	$\eta = -2$
$(7,5)$	$(\Omega_5(3), S_7)$	$\eta_1 = -1$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{2}{3}$
$(8,3)$	$(\Omega_3(3), S_8)$	η_1 is undefined, $\eta_2 = 1$	$-\frac{1}{2}$
$(8,3)$	$(O_6^{\pm}(2), S_8)$	$(+,-) = (-\frac{1}{2}, 1), (\frac{1}{2}, -1)$	$\eta = -\frac{1}{2}$
$(8,3)$	$(Sp_6(2), S_8)$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{1}{2}$
$(8,3)$	$(\mathrm{SU}_3(2), S_8)$	$\eta_1 = 2$, $\eta_2 = -1$	$\eta = -\frac{1}{2}$
$(9,3)$	$(\Omega_3(3), S_9)$	η_1 is undefined, $\eta_2 = 1$	$\eta = -\frac{2}{5}$
$(9,3)$	$(O_6^{\pm}(2), S_9)$	$(+,-) = (-\frac{1}{2}, 1), (\frac{1}{2}, -1)$	$\eta = -\frac{2}{5}$
$(9,3)$	$(Sp_6(2), S_9)$	$\eta_1 = -\frac{1}{2}$, $\eta_2 = \frac{1}{2}$	$\eta = -\frac{2}{5}$
$(9,3)$	$(\mathrm{SU}_3(2), S_9)$	$\eta_1 = 2$, $\eta_2 = -1$	$\eta = -\frac{2}{5}$

Table 5.16: The critical values of (G,H) from Table 5.15

$36 - 15 = 21 = \binom{7}{2} = \dim(M)$. However, in this case, $H = S_7$ cannot be embedded into $G = \Omega_5^+(3)$ because the order of S_7, which is 5040, does not divide the order of $\Omega_5^+(3)$, which is 25920. So $(\Omega_5^+(3), S_7)$ is not an example. Similarly, for $(n,m) = (9,3)$, the pair $(G,H) = (Sp_6(2), S_9)$ satisfies the condition, as $\dim \overline{M'} = 63 - 27 = 36 = \dim M$, however the group S_9 cannot be embedded into $Sp_6(2)$

(n,m)	$\dim \overline{M'}, G = \Omega^{\pm}_{m}(3)$	$\dim \overline{M'}, G = O^{\pm}_{2m}(2)$(+),(-)	$\dim \overline{M'}, G = Sp_{2m}(2)$	$\dim \overline{M'}, G = SU_m(2)$	$\dim(M), H = S_n$
$(2,2)$	1 or 2	5 or 2 or 6	10 or 6	3 or 1	1
$(2,4)$	10 or 3 or 6	85 or 36 or 52	136 or 120	21 or 25	1
$(2,6)$	91 or 27 or 36	1365 or 652 or 716	2080 or 2016	253 or 441	1
$(3,2)$	1 or 2	5 or 2 or 6	10 or 6	3 or 1	3
$(3,4)$	10 or 3 or 6	85 or 36 or 52	136 or 120	21 or 25	3
$(3,6)$	91 or 27 or 36	1365 or 652 or 716	2080 or 2016	253 or 441	3
$(3,8)$	820 or 261 or 288	21845 or 10796 or 11052	32896 or 326460	3741 or 7225	3
$(4,2)$	1 or 2	5 or 2 or 6	10 or 6	3 or 1	6
$(4,4)$	10 or 3 or 6	85 or 36 or 52	136 or 120	21 or 25	6
$(4,6)$	91 or 27 or 36	1365 or 652 or 716	2080 or 2016	253 or 441	6
$(4,8)$	820 or 261 or 288	21845 or 10796 or 11052	32896 or 326460	3741 or 7225	6
$(5,4)$	10 or 3 or 6	85 or 36 or 52	136 or 120	21 or 25	10
$(5,6)$	91 or 27 or 36	1365 or 652 or 716	2080 or 2016	253 or 441	10
$(5,8)$	820 or 261 or 288	21845 or 10796 or 11052	32896 or 326460	3741 or 7225	10
$(6,8)$	820 or 261 or 288	21845 or 10796 or 11052	32896 or 326460	3741 or 7225	15
$(6,10)$	7381 or 2421	349525 or 174252 or 175276	524800 or 523776	58653 or 116281	15
$(7,4)$	10 or 3 or 6	85 or 36 or 52	136 or 120	21 or 25	21
$(9,6)$	91 or 27 or 36	1365 or 652 or 716	2080 or 2016	253 or 441	36
$(9,4)$	10 or 3 or 6	85 or 36 or 52	136 or 120	21 or 25	36

Table 5.17: Dimension of M and $\overline{M'}$, m even

(n,m)	(G,H)	Critical values of G	η of H
$(5,4)$	$(\Omega^{\varepsilon}_4(3), S_5)$	$(\eta_1,\eta_2)^+, (\eta_1,\eta_2)^- = (\frac{1}{2},$ undefined$), (-1,1)$	$\eta = -2$
$(7,4)$	$(\Omega^{\varepsilon}_4(3), S_7)$	$(\eta_1,\eta_2)^+, (\eta_1,\eta_2)^- = (\frac{1}{2},$ undefined$), (-1,1)$	$\eta = -\frac{2}{3}$
$(7,4)$	$(\mathrm{SU}_4(2), S_7)$	$\eta_1 = -1,\ \eta_2 = \frac{1}{2}$	$\eta = -\frac{2}{3}$
$(9,4)$	$(\Omega^{\varepsilon}_4(3), S_9)$	$(\eta_1,\eta_2)^+, (\eta_1,\eta_2)^- = (\frac{1}{2},$ undefined$), (-1,1)$	$\eta = -\frac{2}{5}$
$(9,4)$	$(O^{\pm}_8(2), S_9)$	$(\eta_1,\eta_2)^+, (\eta_1,\eta_2)^- = (-\frac{1}{4},\frac{1}{2}), (\frac{1}{4},-\frac{1}{2})$	$\eta = -\frac{2}{5}$
$(9,4)$	$(\mathrm{SU}_4(2), S_9)$	$\eta_1 = -1,\ \eta_2 = \frac{1}{2}$	$\eta = -\frac{2}{5}$
$(9,6)$	$(\Omega^{\pm}_6(3), S_9)$	$(\eta_1,\eta_2)^+, (\eta_1,\eta_2)^- = (\frac{1}{5},-1), (-\frac{1}{4},\frac{1}{2})$	$\eta = -\frac{2}{5}$

Table 5.18: The critical values of (G,H), from Table 5.17

because according to Atlas [Con+85] S_8 is maximal in $Sp_6(2)$. So, $(Sp_6(2), S_9)$ is not an example. The case $(G,H) = (Sp_6(2), S_8)$ satisfies the condition, as $\dim \overline{M'} = 63 - 35 = 28 = \dim M_\eta(S_8)$. However, it is not a new example because $S_8 \cong O^+_6(2)$, i.e, $(Sp_6(2), S_8) \cong (Sp_6(2), O^+_6(2))$ which is a particular case of Example 4.6.2. From Table 5.16, we see that $\eta = -\frac{1}{2}$ is critical for both groups in the pairs $(G,H) = (O^+_6(2), S_8)$ and $(G,H) = (Sp_6(2), S_8)$, but we have already checked that these are not examples. So, no examples arise from the Tables 5.15 and 5.16. Now, consider the cases that satisfy the condition in Table 5.17. These are listed as follows;

- $(n,m) = (4,2)$, For the cases $(G,H) = (Sp_4(2), S_4)$, $(G,H) = (O_4^-(2), S_4)$, $\dim \overline{M'} = \dim M = 6$

- $(n,m) = (4,4)$, For $(G,H) = (\Omega_4^-(3), S_4)$, $\dim \overline{M'} = \dim M = 6$

- $(n,m) = (5,4)$, For $(G,H) = (\Omega_4^-(3), S_5)$, $\dim \overline{M'} = \dim M = 10$

- $(n,m) = (7,4)$, For $(G,H) = (\mathrm{SU}_4(2), S_7)$, $\dim \overline{M'} = \dim M = 21$

- $(n,m) = (9,4)$, For $(G,H) = (O_8^+(2), S_9)$, $\dim \overline{M'} = \dim M = 36$

- $(n,m) = (9,6)$, For $(G,H) = (\Omega_6^-(3), S_9)$, $\dim \overline{M'} = \dim M = 36$

First, consider the pair $(G,H) = (Sp_4(2), S_4)$, using Atlas [Con+85], we see that S_4 is a maximal subgroup in $Sp_4(2)$, however the corresponding critical value of $M' := M_\eta(Sp_4(2))$ is $\eta := -2/-2^{2-1} = 1$. Thus, $(G,H) = (Sp_4(2), S_4)$ is not an aligned pair. The second case, $(G,H) = (O_4^-(2), S_4)$ is isomorphic to the case (S_5, S_4) because $O_4^-(2) \cong S_5$. So it is a particular case of Example 4.6. The third case, $(G,H) = (\Omega_4^-(3), S_4)$ is similar to the first case $(Sp_4(2), S_4)$ because $\Omega_4^-(3) \cong Sp_4(2)$. The next pair $(G,H) = (\Omega_4^-(3), S_5)$ is isomorphic to (S_6, S_5). So, it is not a new example. The pair $(G,H) = (\mathrm{SU}_4(2), S_7)$ is not an example because $|S_7| = 5040$ does not divide $|\mathrm{SU}_4(2)| = 25,920$. Also, the pair $(G,H) = (\Omega_6^-(3), S_9)$ is not an example because S_9 is not a maximal subgroup of $\Omega_6^-(3)$. Lastly, consider the case $(G,H) = (O_8^+(2), S_9)$. Using Atlas [Con+85], we see that A_9 is maximal in G, so S_9 cannot be embedded into G. Thus, $(G,H) = (O_8^+(2), S_9)$ is not aligned. In Table 5.18, we see that η is only critical for G, in all cases. So, no more candidates other than the ones that we have checked from Table 5.17. Therefore, we deduce that no further examples of aligned pairs can arise from Case 3. □

Next, let us consider the second part of Case 3, the situation where $G = S_m$ and $H \cong \Omega^+_{2n+1}(3), \Omega^\pm_{2n}(3), O^\pm_{2n}(2), Sp_{2n}(2)$, or $\mathrm{SU}_n(2)$.

Proposition 5.3.2. *Suppose that $G = S_m$, the symmetric group of m elements, and $H \cong \Omega^+_{2n+1}(3), \Omega^\pm_{2n}(3), O^\pm_{2n}(2), Sp_{2n}(2), \mathrm{SU}_n(2)$. Then (G, H) is not an aligned pair.*

Proof. Recall that if $H \cong \Omega^\pm_n(3), O^\pm_{2n}(2), Sp_{2n}(2)$, or $\mathrm{SU}_n(2)$ is a subgroup of $G = S_m$. Then H would act non-trivially on the natural module of S_m. So the lower bound of the dimension of a non-trivial irreducible module for each H is at most m. These lower bounds of the nontrivial irreducible modules of H are provided in [SZ93]. First, for $H = \Omega_n(3)$, n is odd, we have that,

$$m \geqslant (3^{2n} - 1)/8 - (3^n - 1)/2 \geqslant (3^{2n} - 9 \cdot 3^n + 3)/9 \geqslant 3^{2n-2} - 3^n \geqslant 3^n \quad (5.20)$$

For $H = \Omega^+_n(3)$, n is even, we have that, $m \geqslant 3(3^{2n-2} - 1)/8 - (3^{n-1} - 1)/2 - 7$. That is,

$$m \geqslant (3^{2n-1} - 4 \cdot 3^{n-1} - 56)/8 \geqslant 3^{2n-3} - 3^{n-1} - 7 \geqslant 3^{n-1} - 7 \quad (5.21)$$

For $H = \Omega^-_n(3)$, n is even, we have that,

$$m \geqslant 3(3^{2n-2} - 1)/8 - 3^{n-1} - n + 2 \geqslant 3^{2n-3} - 3^{n-1} - n \geqslant 3^{n-1} - n, \text{ for } n \geqslant 3 \quad (5.22)$$

For $H = O^+_{2n}(2)$, we have that, $m \geqslant 2(2^{2n-1} - 1)/3 - (2^{n-1} - 1) - 7$. That is,

$$m \geqslant (2^{2n} - 2 - 3 \cdot 2^{n-1} + 3 - 21)/3 \geqslant 2^{2n-2} - 2^{n-1} - 5 \geqslant 2^{n-1} - 5 \quad (5.23)$$

For $H = O^{-}_{2n}(2)$, we have that $m \geqslant (2^{2n-1} - 2)/3 - (2^{n-1} - n + 2)$. That is,

$$m \geqslant (2^{2n-1} - 3 \cdot 2^{n-1} - 3n)/3 \geqslant 2^{2n-3} - 3 \cdot 2^{n-3} - n \tag{5.24}$$

For $H = Sp_{2n}(2)$, we have that,

$$m \geqslant (2^{n} - 1)(2^{n-1} - 1)/3 \geqslant 2^{n-1}2^{n-2}/4 \geqslant 2^{2n-5} \tag{5.25}$$

For $H = \mathrm{SU}_n(2)$, we have that,

$$m \geqslant (2^{n} - 1)/3 \geqslant 2^{n-1} - 1 \tag{5.26}$$

So, in all cases we have that,

$$m \geqslant 2^{n-2} \tag{5.27}$$

Let $M' = M_\eta(G)$ and $M = M_\eta(H)$ be the Matsuo algebras associated to G and H. If (G, H) is aligned, then the set of points in H, where $H = \Omega^{\pm}_{n}(3)$, $O^{\pm}_{2n}(2)$, $Sp_{2n}(2)$, or $\mathrm{SU}_n(2)$, is a spanning set for $\overline{M'} = M'/R(M')$. That is, $\dim M \geqslant \dim \overline{M'} = 1 + d$, where $d = f, g$ is a multiplicity in the spectrum of (D). Recall that, we take d to be the smaller of f, g. In this case, $d = g = m(m-3)/2$. So, we have that,

$$\dim M \geqslant 1 + m(m-3)/2$$

First, consider $H = \Omega_n(3)$, n is odd. Then, the number of points in H is $(3^{n-1} - 3^{\frac{n-1}{2}})/2$. So we have that,

$$(3^{n-1} - 3^{\frac{n-1}{2}})/2 \geqslant (m^2 - 3m + 2)/2, \text{ so } 3^{n-1} \geqslant (m^2 - 3m + 2)/2 \tag{5.28}$$

For $H = \Omega^{+}_{n}(3)$, n is even, the number of points in H is $(3^{n-1} - 3^{\frac{n-2}{2}})/2$, so we have that,

$$(3^{n-1} - 3^{\frac{n-2}{2}})/2 \geqslant (m^2 - 3m + 2)/2, \text{ so } 3^{n-1} \geqslant (m^2 - 3m + 2)/2 \tag{5.29}$$

For $H = \Omega_n^-(3)$ (n even), the number of points in H is $(3^{n-1}+3^{\frac{n-2}{2}})/2$, so we have,

$$(3^{n-1}+3^{\frac{n-2}{2}})/2 \geqslant (m^2-3m+2)/2, i.e, 3^{n-1}+3^{n-2} \geqslant m, \text{ so } 4{\cdot}3^{n-2} \geqslant (m^2-3m+2)/2 \tag{5.30}$$

For $H = O_{2n}^+(2)$, the number of points in H is $2^{2n-1} - 2^{n-1}$. Thus, we have that,

$$2^{2n-1} - 2^{n-1} \geqslant (m^2 - 3m + 2)/2, \text{ so } 2^{2n-1} \geqslant (m^2 - 3m + 2)/2 \tag{5.31}$$

For $H = O_{2n}^-(2)$ the number of points in H is $2^{2n-1} + 2^{n-1}$. Thus, we have that,

$$2^{2n-1} + 2^{n-1} \geqslant (m^2 - 3m + 2)/2 \tag{5.32}$$

For $H = Sp_{2n}(2)$, the number of points in H is $2^{2n} - 1$. So,

$$2^{2n} - 1 \geqslant (m^2 - 3m + 2)/2 \tag{5.33}$$

Finally, when $H = \mathrm{SU}_n(2)$, the number of points in H is $(2^{2n-1} - (-2)^{n-1} - 1)/3$. So we have,

$$(2^{2n-1} - (-2)^{n-1} - 1)/3 \geqslant (m^2 - 3m + 2)/2 \tag{5.34}$$

Consider (5.34). If n is odd, then $(2^{2n-1} - 2^{n-1} - 1)/3 \geqslant (m^2 - 3m + 2)/2$, so $2^{n-2}(2^n - 1) \geqslant (m^2 - 3m + 2)/2$. If n is even, then $(2^{2n-1} + 2^{n-1} - 1)/3 \geqslant (m^2 - 3m + 2)/2$, so, $2^{n-2}(2^n + 1) \geqslant (m^2 - 3m + 2)/2$. Thus, to cover both cases of n, take $(m^2 - 3m + 2)/2 \leqslant 2^{n-2}(2^n + 1)$. Therefore, in all cases we have that

$$m^2 - 3m + 2 \leqslant 2^{2n+1} \tag{5.35}$$

Now, in inequality (5.27) we have $2^{2n-2} \leqslant m$. By using this in (5.35) we get that, $2^{2n+1} \geqslant 2^{4n-4} - 3 \cdot 2^{2n-1} + 2$. This inequality is true for $n \leqslant 3$. Therefore, $m^2 - 3m + 2 \leqslant 2^7 = 128$. So, $m \leqslant 12$. The eigenvalues of the adjacency matrix of the diagram (D) of $G = S_m$ are $k = 2(m-2)$, $\zeta_1 = m-4$, and $\zeta_2 = -2$. Since k has multiplicity 1 and ζ_2 leads to $\eta = 1$, we only consider the eigenvalue $\zeta_1 = m - 4$, which has multiplicity $m - 1$. Therefore, $\dim \overline{M'} = \dim M' - \dim R(M') = \binom{m}{2} - (m-1)$. In Table 5.19, we list all the cases that satisfy the condition, $\dim \overline{M'} = \dim M$, where η is not critical for H. First, the pair $(G, H) = (S_5, O_4^+(2))$ is not aligned because the order of G, which is 120, is not divisible by the order of H, which is 36. Second, the pair $(G, H) = (S_6, O_4^-(2))$ is not a new example because $O_4^-(2) \cong S_5$. Similarly, since $H = \Omega_4^-(3) \cong Sp_4(2) \cong S_6$, the pairs $(S_7, \Omega_4^-(3))$ and $(S_7, Sp_4(2))$ are not new examples. Also, since $H = O_6^+(2) \cong S_8$, the pair $(S_9, O_6^+(2))$ is not a new example. The cases $(G, H) = (S_{10}, \Omega_5(3))$ and $(G, H) = (S_{10}, O_6^-(2))$ are also not examples because using Atlas we see that $H = \Omega_5(3) \cong O_6^-(2)$ is not maximal in S_{10}. Similarly, $(G, H) = (S_{11}, \mathrm{SU}_4(2))$ is not aligned as $H = \mathrm{SU}_4(2)$ is not maximal in S_{11}. Now, we consider the cases where η is also critical for H. In the Tables 5.20 and 5.21, we compute the codimension of the radical of H as well as the corresponding critical values. Note that, in some cases, the codimension of $R(M)_\eta$ in M agrees with the codimension of $R(M')_{\eta'}$ in M', however if $\eta \neq \eta'$, then this is not a possible example. Considering Tables 5.20 and 5.21, there are only two cases such that $\dim M - \dim R(M)_\eta = \dim M' - \dim R(M')_\eta$. These are, $(G, H) = (S_5, O_4^-(2))$ and $(G, H) = (S_6, \Omega_4^-(3))$. However, these cases are not examples, as $O_4^-(2) \cong S_5$ and $\Omega_4^-(3) \cong S_6$.

m	$\dim \overline{M'}, G = S_m$	$\dim M, H = \Omega_n^{\pm}(3)$	$\dim M, H = Sp_{2n}(2)$	$\dim M, H = O_{2n}^{\pm}(2)$	$\dim M, H = \mathrm{SU}_n(2)$
5	[6], $\eta = -2$	-	-	6, $H = O_4^+(2)$	-
6	[10], $\eta = -1$	-	-	10, $H = O_4^-(2)$	-
7	[15], $\eta = -\frac{2}{3}$	15, $H = \Omega_4^-(3)$	15, $H = Sp_4(2)$	-	-
8	[21], $\eta = -\frac{1}{2}$	-	-	-	-
9	[28], $\eta = -\frac{2}{5}$	-	-	28, $H = O_6^+(2)$	-
10	[36], $\eta = -\frac{1}{3}$	36, $H = \Omega_5(3)$	-	36, $H = O_6^-(2)$	-
11	[45], $\eta = -\frac{2}{7}$	-	-	-	45, $H = \mathrm{SU}_4(2)$
12	[55], $\eta = -\frac{1}{4}$	-	-	-	-
13	[66], $\eta = -\frac{2}{9}$	-	-	-	-
14	[78], $\eta = -\frac{1}{5}$	-	-	-	-
15	[91], $\eta = -\frac{2}{11}$	-	-	-	-
16	[105], $\eta = -\frac{1}{6}$	-	-	-	-
17	[120], $\eta = -\frac{2}{13}$	-	-	120, $H = O_8^+(2)$	-
18	[136], $\eta = -\frac{1}{7}$	-	-	136, $H = O_8^-(2)$	-

Table 5.19: $\dim \overline{M'} = \dim M$, η not critical for H

n	$H = Sp_{2n}(2)$	$H = O_{2n}^+(2)$	$H = O_{2n}^-(2)$	$H = \mathrm{SU}_n(2)$
2	[10],[6] $(-1, 1)$	[5],[2] $(-1, 2)$	[5],[6] $(1, -2)$	[3],[1] $(-4, 2)$
3	[36],[28] $(-\frac{1}{2}, \frac{1}{2})$	[21],[8] $(-\frac{1}{2}, 1)$	[21],[16] $(\frac{1}{2}, -1)$	[1],[9] $(2, -1)$
4	[136],[120] $(-\frac{1}{4}, \frac{1}{4})$	[85],[36] $(-\frac{1}{4}, \frac{1}{2})$	[85],[52] $(\frac{1}{4}, -\frac{1}{2})$	[21],[25] $(-1, \frac{1}{2})$
5	[528],[496] $(-\frac{1}{8}, \frac{1}{8})$	[341],[156] $(-\frac{1}{8}, \frac{1}{4})$	[341],[188] $(\frac{1}{8}, -\frac{1}{4})$	[45],[121] $(\frac{1}{2}, -\frac{1}{4})$
6	[2080],[2016] $(-\frac{1}{16}, \frac{1}{16})$	[1365],[652] $(-\frac{1}{16}, \frac{1}{8})$	[1365],[716] $(\frac{1}{16}, -\frac{1}{8})$	[253],[441] $(-\frac{1}{4}, \frac{1}{8})$
7	[8256],[8128] $(-\frac{1}{32}, \frac{1}{32})$	[5461],[2668] $(-\frac{1}{32}, \frac{1}{16})$	[5461],[2796] $(\frac{1}{32}, -\frac{1}{16})$	[861],[1849] $(\frac{1}{8}, -\frac{1}{16})$

Table 5.20: $[\dim M - \dim R(M)_{\eta_1}]$, $[\dim M - \dim R(M)_{\eta_2}]$, (η_1, η_2)

n	$H = \Omega_n(3)$	n	$H = \Omega_n^+(3)$	$H = \Omega_n^-(3)$
5	[16],[21] $(-1, \frac{1}{2})$	4	[10],[3] $(\frac{1}{2}, -)$	[10],[6] $(-1, 1)$
7	[169],[183] $(-\frac{1}{4}, \frac{1}{5})$	6	[91],[27] $(\frac{1}{5}, -1)$	[91],[36] $(-\frac{1}{4}, \frac{1}{2})$
9	[1600],[1641] $(-\frac{1}{13}, \frac{1}{14})$	8	[820],[261] $(\frac{1}{14}, -\frac{1}{4})$	[820],[288] $(-\frac{1}{13}, \frac{1}{5})$
11	[14641],[14763] $(-\frac{1}{40}, \frac{1}{41})$	10	[7381],[2421] $(\frac{1}{41}, -\frac{1}{13})$	[7381],[2502] $(-\frac{1}{40}, \frac{1}{14})$
13	[132496],[132861] $(-\frac{1}{121}, \frac{1}{122})$	12	[66430],[22023] $(\frac{1}{122}, -\frac{1}{40})$	[66430],[22266] $(-\frac{1}{121}, \frac{1}{41})$

Table 5.21: $[\dim M - \dim R(M)_{\eta_1}]$, $[\dim M - \dim R(M)_{\eta_2}]$, (η_1, η_2)

□

5.4 Case 4: Sporadic Groups

In this section, we consider the pairs that involve sporadic groups. First, we take G, the larger group in the pair (G, H), to be one of the sporadic groups $Fi_{22}, Fi_{23}, Fi_{24}$, $\Omega_8^+(2) : S_3$, or $\Omega_8^+(3) : S_3$.

Proposition 5.4.1. *Suppose that $G_m = Fi_{22}, Fi_{23}, Fi_{24}$, $\Omega_8^+(2) : S_3$, or $\Omega_8^+(3) : S_3$, and $H_n \cong S_n, O_{2n}^{\pm}(2), Sp_{2n}(2)$, $\Omega_n^{\pm}(3)$, or $\mathrm{SU}_n(2)$. Then (G, H) is not aligned.*

Proof. Let $M' = M_\eta(G)$ and $M = M_\eta(H)$ be the associated Matsuo algebras to G and H. Recall that, if (G, H) is aligned and M is simple, i.e., η not critical for it, then $\dim \overline{M'} = \dim M' - \dim R(M') = \dim M$. But if (G, H) is aligned and η is also critical for M, then $\dim \overline{M'} = \dim M - \dim R(M)$. We will check this condition for each group G with each group H. First, we want to know the values of n for which $|H_n|$ divides $|G_m|$. The orders of G_m are as follows, $|Fi_{22}| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, $|Fi_{23}| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $|Fi_{24}| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, $|\Omega_8^+(2) : S_3| = 1045094400$, and $|\Omega_8^+(3) : S_3| = 29713078886400$. And the orders of H_n are; $|S_n| = n!$, $|Sp_{2n}(2)| = 2^{n^2} \Pi_{i=1}^n (2^{2i} - 1)$, $|O_{2n}^\varepsilon(q)| = 1/k \cdot q^{n(n-1)} \cdot (q^n - \varepsilon) \Pi_{i=1}^{n-1} (2^{2i} - 1)$ $(q = 2, 3)$ and $k = gcd(4, q^n - \varepsilon)$, $|\Omega_{2n+1}(3)| = 3^{n^2} \Pi_{i=1}^n (3^{2i} - 1)$, and $|\mathrm{SU}_n(2)| = 2^{\frac{1}{2}n(n-1)} \Pi_{i=2}^n (2^i - (-1)^i)$. The bound on n such that $|H_n|$ divides $|G_m|$ are computed in Table 5.22. The dimensions of $\overline{M'}$ with respect to $G = Fi_{22}, Fi_{23}, Fi_{24}, \Omega_8^+(2) :$

H_n	$\lvert H_n\rvert$ divides $\lvert Fi_{22}\rvert$	$\lvert H_n\rvert$ divides $\lvert Fi_{23}\rvert$	$\lvert H_n\rvert$ divides $\lvert Fi_{24}\rvert$	$\lvert H_n\rvert$ divides $\lvert\Omega_8^+(2) : S_3\rvert$	$\lvert H_n\rvert$ divides $\lvert\Omega_8^+(3) : S_3\rvert$
S_n	$n \leqslant 13$	$n \leqslant 13$	$n \leqslant 14$	$n \leqslant 10$	$n \leqslant 10$
$Sp_{2n}(2)$	$n \leqslant 3$	$n \leqslant 4$	$n \leqslant 4$	$n \leqslant 3$	$n \leqslant 3$
$O_{2n}^+(2)$	$n \leqslant 4$	$n \leqslant 4$	$n \leqslant 4$	$n \leqslant 4$	$n \leqslant 4$
$O_{2n}^-(2)$	$n \leqslant 3$	$n \leqslant 4$	$n \leqslant 5$	$n \leqslant 3$	$n \leqslant 3$
$\Omega_n(3)$	$n \leqslant 3$	$n \leqslant 3$	$n \leqslant 3$	$\lvert\Omega_n(3)\rvert \nmid \lvert\Omega_8^+(2) : S_3\rvert$	$n \leqslant 3$
$\Omega_{2n}^+(3)$	$n \leqslant 3$	$n \leqslant 5$	$n \leqslant 5$	$n \leqslant 2$	$n \leqslant 5$
$\Omega_{2n}^-(3)$	$n \leqslant 3$	$n \leqslant 3$	$n \leqslant 3$	$n \leqslant 3$	$n \leqslant 3$
$\mathrm{SU}_n(2)$	$n \leqslant 6$	$n \leqslant 6$	$n \leqslant 6$	$n \leqslant 4$	$n \leqslant 4$

Table 5.22: Bounds of n such that $|H_n|$ divides $|G|$

S_3, and $\Omega_8^+(3) : S_3$, are computed in Table 5.23. Moreover, dimensions of $M = M_\eta(H)$, for $H = S_n, Sp_{2n}(2), O_{2n}^\pm(2), \Omega_{2n+1}(3), \Omega_{2n}^\pm(3)$, or $\mathrm{SU}_n(2)$ are computed in Table 5.24.

G	$\dim M'$	$\dim R(M')$	$\dim \overline{M'}$
Fi_{22}	3510	3080 or 429	430 or 3081
Fi_{23}	31671	30888 or 782	783 or 30889
Fi_{24}	306936	249458 or 57477	249459 or 57478
$\Omega_8^+(2) : S_3$	360	105 or 252 or 2	255 or 108 or 358
$\Omega_8^+(3) : S_3$	3240	2457 or 780 or 2	783 or 2460 or 3238

Table 5.23: Dimension of $\overline{M'}$ for sporadic groups

m	dim $M_\eta(S_n)$	dim $M_\eta(Sp_{2n}(2))$	dim $M_\eta(O^+_{2n}(2))$	dim $M_\eta(O^-_{2n}(2))$	dim $M_\eta(\Omega_{2n+1}(3))$	dim $M_\eta(\Omega^+_{2n}(3))$	dim $M_\eta(\Omega^-_{2n}(3))$	dim $M_\eta(\mathrm{SU}_n(2))$
3	3	63	28	36	3	–	–	9
4	6	255	120	136	–	12	15	45
5	10	1023	496	528	36	–	–	165
6	15	4095	2016	2080	–	117	126	693
7	21	16383	8128	8265	351	–	–	2709
8	28	65535	32640	32896	–	1080	1107	10965
9	36	262143	130816	131328	3240	–	–	43605
10	45	1048575	523776	524800	–	9801	9882	174933
11	55	4194303	2096128	2098176	29403	–	–	698709
12	66	–	–	–	–	88452	88695	2796885
13	78	–	–	–	265356	–	–	–
14	91	–	–	–	–	796797	797526	–

Table 5.24: Dimension of M, $M = M_\eta(H)$

Furthermore, in Table 5.25, we compute the codimensions of $R(M)$ for every group H, where the codimesnion of the radical of M are given by the following formulas:

- $\binom{n}{2} - (n-1) = (n^2 - 3n + 2)/2$, for $H = S_n$.
- $2^{2n} - 2^{2n-1} + 2^{n-1}$, or $2^{2n} - 2^{2n-1} - 2^{n-1}$ for $H = Sp_{2n}(2)$.
- $2^{2n-1} - \varepsilon 2^{n-1} - ((2^n - \varepsilon 1)(2^{n-1} - \varepsilon 1)/3)$, or $2^{2n-1} - \varepsilon 2^{n-1} - ((2^{2n} - 4)/3)$ for $H = O^\varepsilon_{2n}(2)$.
- $(3^{n-1} - 3^{\frac{n-1}{2}})/2 - (3^{n-1} - 1 - 2(3^{\frac{n-1}{2}} + 1))/4$, or $(3^{n-1} - 3^{\frac{n-1}{2}})/2 - (3^{n-1} - 1)/4$ for $H = \Omega_n(3)$, n is odd.
- $(3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2 - (3^{\frac{n}{2}} - \varepsilon)(3^{\frac{n-2}{2}} - \varepsilon)/8$, or $(3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2 - (3^n - 9)/8$ for $H = \Omega_n(3)$, n is even.
- $[(2^{2n-1} - (-2)^{n-1} - 1)]/3 - 8[(2^{2n-3} - (-2)^{n-2} - 1)]/9$, or $[(2^{2n-1} - (-2)^{n-1} - 1)]/3 - 4[(2^{2n-3} - 7(-2)^{n-3} - 1)]/9$ for $H = \mathrm{SU}_n(2)$.

n	codim $R(M), H = S_n$	codim $R(M), H = Sp_{2n}(2)$	codim $R(M), H = O^+_{2n}(2)$	codim $R(M), H = O^-_{2n}(2)$	codim $R(M), H = \Omega_{2n+1}(3)$	codim $R(M), H = \Omega^+_{2n}(3)$	codim $R(M), H = \Omega^-_{2n}(3)$	codim $R(M), H = \mathrm{SU}_n(2)$
3	1	36 or 28	8 or 21	21 or 16	3 or 1	-	-	1 or 9
4	3	136 or 120	36 or 85	85 or 52	-	10 or 3	10 or 6	21 or 25
5	6	528 or 496	156 or 341	341 or 188	21 or 16	-	-	45 or 121
6	10	2080 or 2016	652 or 1365	1365 or 716	-	91 or 27	91 or 36	253 or 441
7	15	8256 or 8128	2668 or 5461	5461 or 2796	183 or 169	-	-	861 or 1849
8	21	32896 or 32640	10796 or 21845	21845 or 11052	-	820 or 261	820 or 288	3741 or 7225
9	28	131328 or 130816	43436 or 87381	87381 or 43948	1600 or 1641	-	-	14365 or 29241
10	36	524800 or 523776	174252 or 349525	349525 or 175276	-	7381 or 2421	7381 or 2502	58653 or 116281
11	45	2098176 or 2096128	698028 or 1398101	1398101 or 700076	14641 or 14763	-	-	232221 or 466489
12	55	-	-	-	-	66430 or 22023	66430 or 22266	933661 or 1863225
13	66	-	-	-	132496 or 132861	-	-	-
14	78	-	-	-	-	597871 or 198927	597871 or 199656	-

Table 5.25: Codimension of $R(M)$ for $M = M_\eta(H)$

Looking at Tables 5.24 and Table 5.25, we see that neither $\dim M$ nor $\dim M - \dim R(M)$ coincide with $\dim M' - \dim R(M')$, from Table 5.23, in all the cases, except for the case $(G, H) = (\Omega_8^+(2) : S_3, Sp_8(2))$. In this case, $\dim M = 255 = 360 - 105 = \dim M' - \dim R(M')$, however this pair cannot be aligned as $|H| = 47377612800$ is bigger than $|G| = 1045094400$. Therefore, we conclude that no examples of aligned pairs can arise from this case. □

Now, let us consider the opposite of the previous case, i.e., we will take the sporadic groups as subgroups inside the other irreducible 3-transposition groups.

Proposition 5.4.2. *Suppose that* $G = S_m$, $Sp_{2m}(2)$, $O_{2m}^{\varepsilon}(2)$, $\Omega_m(3)$, $\Omega_m^{\varepsilon}(3)$, *or* $\mathrm{SU}_m(2)$, *and* $H \cong \Omega_8^+(2) : S_3$, $\Omega_8^+(3) : S_3$, Fi_{22} , Fi_{23}, *or* Fi_{24}. *Then,* (G, H) *is not aligned.*

Proof. First, assume that η is only critical for G. Then, (G, H) is aligned if $|D| - d = |C|$, where d is a multiplicity in the spectrum of (D) and $|C| = 360$, 3240, 3510, or 306936 for $H = \Omega_8^+(2) : S_3$, $\Omega_8^+(3) : S_3$, Fi_{22}, Fi_{23}, or Fi_{24} respectively. First, consider $G = S_m$. Write $|D| - d = |C|$ as,

$$\binom{m}{2} - (m-1) = (m^2 - 3m + 2)/2 = |C| \tag{5.36}$$

Now, if $H = \Omega_8^+(2) : S_3$ i.e., when $|C| = 360$, the above equation has no integer solutions. Therefore, the pair $(G, H) = (S_m, \Omega_8^+(3) : S_3)$ is not aligned. For $H = \Omega_8^+(3) : S_3$, i.e., when $|C| = 3240$, $m = 82$ is a solution to (5.36), thus $(G, H) = (S_{82}, \Omega_8^+(3) : S_3)$ is a possible example. However, since $\Omega_8^+(3) : S_3$ is not maximal in S_{82}, $(G, H) = (S_{82}, \Omega_8^+(3) : S_3)$ is not aligned. For $H = Fi_{22}, Fi_{23}$, i.e., when $|C| = 3510$ or 31671, Equation (5.36) has no integer solutions. So, (S_m, Fi_{22}) and (S_m, Fi_{23}) are not aligned. Lastly, for $H = Fi_{24}$, i.e., when $|C| = 306936$,

$m = 785$ is a solution to (5.36). So, the pair (S_{785}, Fi_{24}) is a possible example. However, as order of Fi_{24} does not divide the order of S_{785}, the pair is not aligned. Next, consider $G = Sp_{2m}(2)$. In this case, $|D| - d_i = |C|$ is given by;

$$2^{2m} - 2^{2m-1} \pm 2^{m-1} = |C| \tag{5.37}$$

This equation has no integer solutions when $|C| = 360$, 3240, 3510, 31671, or 306936. Therefore, $G = Sp_{2m}(2)$ is not aligned with any of the sporadic groups. Next, consider $G = O^{\varepsilon}_{2m}(2)$. In this case the $|D| - d_i = |C|$ are given by

$$2^{2m-1} - \varepsilon 2^{m-1} - ((2^m - \varepsilon 1)(2^{m-1} - \varepsilon 1)/3) = |C| \tag{5.38}$$

$$2^{2m-1} - \varepsilon 2^{m-1} - ((2^{2m} - 4)/3) = |C| \tag{5.39}$$

However, both of the above equations has no integer solutions when $|C| = 360$, 3240, 3510, 31671, or 306936. Therefore, $G = O^{\varepsilon}_{2m}(2)$ is not aligned with any of the sporadic groups. Next, consider $G = \Omega_m(3)$, m is odd. In this case, $|D| - d_i = |C|$ are given by

$$(3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} + 1)/4 = |C| \tag{5.40}$$

and $(3^{m-1} + 3)/4 = |C|$. Similarly, in this case both equations has no integer solutions when $|C| = 360$, 3240, 3510, 31671, or 306936. So, $G = \Omega_m(3)$, (m is odd) is not aligned with any of the sporadic groups. Next, consider $G = \Omega^{\varepsilon}_m(3)$, m is even. In this case, $|D| - d_i = |C|$ are given by $(3^{m-1} - \varepsilon 3^{\frac{m-2}{2}})/2 - (3^{\frac{m}{2}} - \varepsilon)(3^{\frac{m-2}{2}} - \varepsilon)/8 = (3^m - 1)/8 = |C|$, and $(3^{m-1} - \varepsilon 3^{\frac{m-2}{2}})/2 - (3^m - 9)/8 = (4 \cdot 3^{m-1} \pm 4 \cdot 3^{\frac{m-2}{2}} - 3^m + 9)/8 = |C|$. Also in this case both equations has no integer solutions when $|C| = 360$, 3240, 3510, 31671, or 306936. Therefore, $G = \Omega^{\varepsilon}_m(3)$, m is even, is not aligned with any of the sporadic groups. Lastly, consider

$G = \mathrm{SU}_m(2)$. In this case, the equations $|D| - d_1 = |C|$ and $|D| - d_2 = |C|$ are given by $[(2^{2m-1} - (-2)^{m-1} - 1)]/3 - 8[(2^{2m-3} - (-2)^{m-2} - 1)]/9 = |C|$ and $[(2^{2m-1} - (-2)^{n-1} - 1)]/3 - 4[(2^{2m-3} - 7(-2)^{m-3} - 1)]/9 = |C|$. These simplifies to $(2^{2m-1} + 7(-2)^{m-1} + 5)/9 = |C|$ and $(2^{2m} + (-2)^{m+1} + 1)/9 = |C|$, where $|C| = 360, 3240, 3510, 31671$, or 306936. Similarly in this case both equations has no integer solutions. So $G = \mathrm{SU}_m(2)$ is not aligned with any of the sporadic groups. Therefore, if η is not critical for H, then (G, H) is not aligned.

Now, we want to find the cases (G, H) where η is also critical for H. Clearly, a critical value $\eta = -\frac{2}{\zeta}$ of G (ζ is a nonzero eigenvalue of the adjacency matrix of (D)) is critical for H if and only if ζ is an eigenvalue of (C). The eigenvalues of (C) are provided in Table 4.1 and listed as follows:

H	ζ_1	ζ_2	ζ_3
$\Omega_8(2) : S_3$	8	-4	-64
$\Omega_8(2) : S_3$	8	-28	-352
Fi_{22}	8	-64	
Fi_{23}	8	-352	
Fi_{24}	80	-352	

Table 5.26: The eigenvalues of (C)

First, consider $G = S_m$, we want to find the pairs (S_m, H) where η is also critical for H. Consider $(G, H) = (S_m, \Omega_8(2) : S_3)$. The diagram on (D) for S_m has the eigenvalues $k = 2(m-2)$, $r = m - 4$, and $s = -2$. We might ignore the eigevalues k and s as k always has multiplicity one and s leads to $\eta = 1$. So, $\eta = \frac{-2}{m-4}$ is critical for $H = \Omega_8^+(2) : S_3$ if and only if $m - 4 = 8$, i.e., if and only if $m = 12$. So, $\eta = -1/4$ is critical for both groups in the pair $(S_{12}, \Omega_8(2) : S_3)$. Similarly, as 8 is an eigenvalue for $\Omega_8(3) : S_3$, Fi_{22}, and Fi_{23}, $\eta = -1/4$ is critical for both groups in the pairs $(S_{12}, \Omega_8(2) : S_3)$, (S_{12}, Fi_{22}), and (S_{12}, Fi_{23}). For, $H = Fi_{24}$, $\eta = -\frac{2}{m-4}$ is critical for H if and only if $m - 4 = 80$, i.e., if and only if $m = 84$. So, $\eta = -\frac{1}{40}$

is critical for both groups in the pair $(G,H) = (S_{84}, Fi_{24})$. In Table 5.27, we list these cases as well as the codimensions of the radicals in each case. Similarly, we do the same for $G = Sp_{2m}(2)$, $O^{\varepsilon}_{2m}(2)$, $\Omega_m(3)$, $\Omega^{\varepsilon}_m(3)$, and $\mathrm{SU}_m(2)$ in Tables 5.28, 5.29, 5.30, 5.31, 5.32, 5.33, and 5.34, respectively. Looking at these tables we see that $\dim M' - \dim R(M') \neq \dim M - \dim R(M)$ in all the cases. Therefore, since the condition of aligned pairs does not hold when η is critical for H and when η is not critical for H, we conclude that no further examples of aligned pairs can arise from this case.

$(G,H) = (S_m, H)$	η	$\dim M' - \dim R(M')$	$\dim M - \dim R(M)$
$(S_{12}, \Omega_8(2) : S_3)$	$-\frac{1}{4}$	55	255
$(S_{12}, \Omega_8(3) : S_3)$	$-\frac{1}{4}$	55	783
(S_{12}, Fi_{22})	$-\frac{1}{4}$	55	430
(S_{12}, Fi_{23})	$-\frac{1}{4}$	55	783
(S_{84}, Fi_{24})	$-\frac{1}{40}$	3403	57478

Table 5.27: Pairs (S_m, H) with same critical values

$(G,H) = (Sp_{2m}(2), H)$	η	$\dim M' - \dim R(M')$	$\dim M - \dim R(M)$
$(Sp_8(2), \Omega_8(2) : S_3)$	$-\frac{1}{4}$	136	255
$(Sp_6(2), \Omega_8(2) : S_3)$	$\frac{1}{2}$	28	108
$(Sp_{14}(2), \Omega_8(2) : S_3)$	$\frac{1}{32}$	8128	358
$(Sp_8(2), \Omega_8(3) : S_3)$	$-\frac{1}{4}$	136	783
$(Sp_8(2), Fi_{22})$	$-\frac{1}{4}$	136	430
$(Sp_{14}(2), Fi_{22})$	$\frac{1}{32}$	8128	3081
$(Sp_8(2), Fi_{23})$	$-\frac{1}{4}$	136	783

Table 5.28: Pairs $(Sp_{2m}(2), H)$ with same critical values

$(G,H) = (O^+_{2m}(2), H)$	η	$\dim M' - \dim R(M')$	$\dim M - \dim R(M)$
$(O^+_8(2), \Omega_8(2) : S_3)$	$-\frac{1}{4}$	85	255
$(O^+_8(2), \Omega_8(2) : S_3)$	$\frac{1}{2}$	36	108
$(O^+_{16}(2), \Omega_8(2) : S_3)$	$\frac{1}{32}$	10796	358
$(O^+_8(2), \Omega_8(3) : S_3)$	$-\frac{1}{4}$	85	783
$(O^+_8(2), Fi_{22})$	$-\frac{1}{4}$	85	430
$(O^+_{16}(2), Fi_{22})$	$\frac{1}{32}$	10796	3081
$(O^+_8(2), Fi_{23})$	$-\frac{1}{4}$	85	783

Table 5.29: Pairs $(O^+_{2m}(2), H)$ with same critical values

$(G,H)=(O^-_{2m}(2),H)$	η	$\dim M'-\dim R(M')$	$\dim M-\dim R(M)$
$(O^-_6(2),\Omega_8(2):S_3)$	$\frac{1}{2}$	21	108
$(O^-_{14}(2),\Omega_8(2):S_3)$	$\frac{1}{32}$	5461	358
$(O^-_{10}(2),\Omega_8(2):S_3)$	$-\frac{1}{4}$	188	255
$(O^-_{10}(2),\Omega_8(3):S_3)$	$-\frac{1}{4}$	188	783
$(O^-_{14}(2),Fi_{22})$	$\frac{1}{32}$	5461	3081
$(O^-_{10}(2),Fi_{22})$	$-\frac{1}{4}$	188	430
$(O^-_{10}(2),Fi_{23})$	$-\frac{1}{4}$	188	783

Table 5.30: Pairs $(O^-_{2m}(2),H)$ with same critical values

$(G,H)=(\Omega_m(3),H)$	η	$\dim M'-\dim R(M')$	$\dim M-\dim R(M)$
$(\Omega_7(3),\Omega_8(2):S_3)$	$-\frac{1}{4}$	169	255
$(\Omega_5(3),\Omega_8(2):S_3)$	$\frac{1}{2}$	21	108
$(\Omega_7(3),\Omega_8(3):S_3)$	$-\frac{1}{4}$	169	783
$(\Omega_9(3),\Omega_8(3):S_3)$	$\frac{1}{14}$	1641	2460
$(\Omega_7(3),Fi_{22})$	$-\frac{1}{4}$	169	430
$(\Omega_7(3),Fi_{23})$	$-\frac{1}{4}$	169	783
$(\Omega_{11}(3),Fi_{24})$	$-\frac{1}{40}$	14641	57478

Table 5.31: Pairs $(\Omega_m(3),H)$ with same critical values

$(G,H)=(\Omega^+_m(3),H)$	η	$\dim M'-\dim R(M')$	$\dim M-\dim R(M)$
$(\Omega^+_8(3),\Omega_8(2):S_3)$	$-\frac{1}{4}$	261	255
$(\Omega^+_4(3),\Omega_8(2):S_3)$	$\frac{1}{2}$	10	108
$(\Omega^+_8(3),\Omega_8(3):S_3)$	$\frac{1}{14}$	820	2460
$(\Omega^+_8(3),\Omega_8(3):S_3)$	$-\frac{1}{4}$	261	783
$(\Omega^+_8(3),Fi_{22})$	$-\frac{1}{4}$	261	430
$(\Omega^+_8(3),Fi_{23})$	$-\frac{1}{4}$	261	783
$(\Omega^+_{12}(3),Fi_{24})$	$-\frac{1}{40}$	22023	57478

Table 5.32: Pairs $(\Omega^+_m(3),H)$ with same critical values

$(G,H)=(\Omega^-_m(3),H)$	η	$\dim M'-\dim R(M')$	$\dim M-\dim R(M)$
$(\Omega^-_6(3),\Omega_8(2):S_3)$	$-\frac{1}{4}$	91	255
$(\Omega^-_6(3),\Omega_8(2):S_3)$	$\frac{1}{2}$	36	108
$(\Omega^-_6(3),\Omega_8(3):S_3)$	$-\frac{1}{4}$	91	783
$(\Omega^-_{10}(3),\Omega_8(3):S_3)$	$\frac{1}{14}$	2502	2460
$(\Omega^-_6(3),Fi_{22})$	$-\frac{1}{4}$	91	430
$(\Omega^-_6(3),Fi_{23})$	$-\frac{1}{4}$	91	783
$(\Omega^-_{10}(3),Fi_{24})$	$-\frac{1}{40}$	7381	57478

Table 5.33: Pairs $(\Omega^-_m(3),H)$ with same critical values

$(G,H)=(\mathrm{SU}_m(2),H)$	η	$\dim M' - \dim R(M')$	$\dim M - \dim R(M)$
$(\mathrm{SU}_6(2),\Omega_8(2):S_3)$	$-\frac{1}{4}$	253	255
$(\mathrm{SU}_5(2),\Omega_8(2):S_3)$	$\frac{1}{2}$	45	108
$(\mathrm{SU}_9(2),\Omega_8(2):S_3)$	$\frac{1}{32}$	14365	358
$(\mathrm{SU}_5(2),\Omega_8(2):S_3)$	$-\frac{1}{4}$	121	255
$(\mathrm{SU}_4(2),\Omega_8(2):S_3)$	$\frac{1}{2}$	25	108
$(\mathrm{SU}_8(2),\Omega_8(2):S_3)$	$\frac{1}{32}$	7225	358
$(\mathrm{SU}_6(2),\Omega_8(3):S_3)$	$-\frac{1}{4}$	253	783
$(\mathrm{SU}_5(2),\Omega_8(3):S_3)$	$-\frac{1}{4}$	121	783
$(\mathrm{SU}_6(2),Fi_{22})$	$-\frac{1}{4}$	253	430
$(\mathrm{SU}_5(2),Fi_{22})$	$-\frac{1}{4}$	121	430
$(\mathrm{SU}_9(2),Fi_{22})$	$\frac{1}{32}$	14365	2460
$(\mathrm{SU}_8(2),Fi_{22})$	$\frac{1}{32}$	7225	2460
$(\mathrm{SU}_6(2),Fi_{23})$	$-\frac{1}{4}$	253	783
$(\mathrm{SU}_5(2),Fi_{23})$	$-\frac{1}{4}$	121	783

Table 5.34: Pairs $(\mathrm{SU}_m(2),H)$ with same critical values

□

Next, we want to consider the cases of (G,H) where both groups are sporadic groups. First, the group Fi_{24}, has order $|Fi_{24}| = 1255205709190661721292800$ and it is divisible by $|\Omega_8^+(2):S_3| = 1045094400$, $|\Omega_8^+(3):S_3| = 29713078886400$, $|Fi_{22}| = 64561751654400$, and $|Fi_{23}| = 4089470473293004800$. So, we will examine the cases (G,H) where $G = Fi_{24}$ and $H \cong \Omega_8^+(2):S_3, \Omega_8^+(3):S_3$, Fi_{22}, Fi_{23}. Similarly, $|Fi_{23}|$ is divisible by $|Fi_{22}|$, $|\Omega_8^+(3):S_3|$, and $|\Omega_8^+(2):S_3|$. So, we will consider the cases $G = Fi_{23}$ and $H \cong Fi_{22}, \Omega_8^+(3):S_3, \Omega_8^+(2):S_3$. Moreover, as $|Fi_{22}|$ is only divisible by $|\Omega_8^+(2):S_3|$, we will consider the case $(G,H) = (Fi_{22}, \Omega_8^+(2):S_3)$. Finally, $|\Omega_8^+(3):S_3|$ is only divisible by $|\Omega_8^+(2):S_3|$. So, we will consider the pair $(G,H) = (\Omega_8^+(3):S_3, \Omega_8^+(2):S_3)$

Proposition 5.4.3. *Suppose that $G = Fi_{24}$ and $H \cong \Omega_8^+(2):S_3, \Omega_8^+(3):S_3$, Fi_{22}, Fi_{23}. Then (G,H) is not aligned.*

Proof. Let $M' := M_\eta(Fi_{24})$ and $M := M_\eta(H)$ be the corresponding Matsuo algebras. The number of points in G is 306936, i.e, $\dim M' = |D| = 306936$. Moreover, the diagram related to G has the spectrum,

$$\Big(\big([275264]^1, [80]^{249458}, [-352]^{57477}\big)\Big)$$

So, the critical values of G are $\eta_1 = \frac{-2}{275264} = -\frac{1}{137632}$, $\eta_2 = \frac{-2}{80} = -\frac{1}{40}$, and $\eta_3 = \frac{-2}{-352} = \frac{1}{176}$. Further, for η_1

$$\dim \overline{M'} = \dim M' - R(M')_{\eta_1} = 306936 - 249458 = 57478$$

And for η_2,

$$\dim \overline{M'} = \dim M' - R(M')_{\eta_2} = 306936 - 57477 = 249459$$

On the other hand, $\dim M$ is as follows:

- For $H = \Omega_8^+(2) : S_3$, $\dim M = 360$,
- For $H = \Omega_8^+(3) : S_3$, $\dim M = 3240$,
- For $H = Fi_{22}$, $\dim M = 3510$,
- For $H = Fi_{23}$, $\dim M = 31671$.

Clearly, $\dim M = |C| < \dim \overline{M'}$, in all cases. Therefore, no examples of aligned pairs arise from this case. □

Proposition 5.4.4. *Suppose that $G = Fi_{23}$ and $H \cong \Omega_8^+(2) : S_3, \Omega_8^+(3) : S_3$, Fi_{22}. Then (G, H) is not aligned.*

Proof. Let $M' := M_\eta(Fi_{23})$ and $M := M_\eta(H)$ be the corresponding Matsuo algebras. The number of points in $G := Fi_{23}$ is $31671 = |D|$. Moreover, the diagram related to G has the spectrum,

$$\left(\left([28160]^1, [8]^{30888}, [-352]^{782}\right)\right)$$

So, the critical values of G are $\eta_0 = \frac{-2}{28160} = -\frac{1}{14080}$, $\eta_1 = -\frac{2}{8} = -\frac{1}{4}$, and $\eta_2 = \frac{2}{352} = \frac{1}{176}$. For the critical value $\eta_1 = -\frac{1}{4}$, $\dim R(M')_{\eta_1} = 30888$. So, $\dim \overline{M'} = |D| - R(M')_{\eta_1} = 31671 - 30888 = 783$. And, for $\eta_2 = \frac{1}{176}$, $\dim \overline{M'} = |D| - R(M')_{\eta_2} = 31671 - 782 = 30889$. First, if $H = \Omega_8^+(2) : S_3$, then (G, H) cannot be aligned because $|C| = 360 < \dim \overline{M'}$. Next, consider $H = \Omega_8^+(3) : S_3$. In this case, $|C| = 3240$, and the spectrum related to H is

$$\left(\left([2888]^1, [8]^{2457}, [-28]^{780}, [-352]^2\right)\right)$$

Recall that, $(G, H) = (Fi_{23}, \Omega_8^+(3) : S_3)$ is aligned for $\eta = -\frac{2}{8} = -\frac{1}{4}$. So, we will examine the other eigenvalues. For the eigenvalue -352, the critical value $\eta_2 = \frac{1}{176}$ is also critical for H, and in this case $\dim R(M) = 2$. However, $\dim M - \dim R(M) = 3240 - 2 = 3238 \neq \dim(M') - R(M') = 30889$. Since the other critical values of G are not critical for H, and $\dim \overline{M'} \neq \dim M$, we conclude that, $(G, H) = (Fi_{23}, \Omega_8^+(3) : S_3)$ is only aligned when $\eta = -\frac{1}{4}$ as shown in Example 4.6.4. Next, consider the group $H = Fi_{22}$. The number of points in H is 3510 and it has the spectrum,

$$\left(\left([2816]^1, [8]^{3080}, [-64]^{492}\right)\right)$$

Note that, $\eta = -\frac{1}{4}$ is also critical for H. However, $\dim M - \dim R(M)_{\eta=-\frac{1}{4}} = 3510 - 3080 = 430 \neq \dim M' - \dim R(M')_{\eta=-\frac{1}{4}} = 783$. Since the other critical values of G are not critical for H and $\dim \overline{M'} = (783 \text{ or } 30889) \neq |C| = 3510$, we conclude that (G, H) is not aligned. $\square$

Proposition 5.4.5. *The pair* $(G, H) = (Fi_{22}, \Omega_8^+(2) : S_3)$ *is not an aligned pair.*

Proof. Let $M' := M_\eta(Fi_{22})$ and $M := M_\eta(\Omega_8^+(2) : S_3)$ be the corresponding Matsuo algebras. We know from Proposition 5.4.4 that $\overline{M'}$ has dimension equals to 430 or 3081. However the number of points in $(\Omega_8^+(2) : S_3)$ is only 360. Therefore, the pair $(Fi_{22}, \Omega_8^+(2) : S_3)$ cannot be aligned. $\square$

Proposition 5.4.6. *Suppose that* $G = \Omega_8^+(3) : S_3$ *and* $H = \Omega_8^+(2) : S_3$*, then* (G, H) *is not aligned.*

Proof. Let $M' = M_\eta(G)$ and $M = M_\eta(H)$ be the corresponding Matsuo algebras. We know from Table (4.1) that $\dim M' = 3240$ and $\dim R(M')$ is either 2457, 780, or 2, depending on the value of η. So, $\dim \overline{M'} = 3240 - 2457 = 783$, $3240 - 780 = 2460$, or $3240 - 2 = 3238$. On the other hand, $\dim M = 360$. Clearly, $|C| = \dim M = 360 < \dim \overline{M'} = 783, 2460$, or 3238. So, (G, H) cannot be aligned. $\square$

CHAPTER 6

SAME CHARACTERISTIC CASE

In this chapter, we search for aligned pairs of Matsuo algebras considering the case where the underlying 3-transposition groups are defined over fields of same characteristic. Recall that, the examples of aligned Matsuo algebras discovered earlier in Section 4.6, namely $(M', M) = (M_\eta(S_{n+1}), M_\eta(S_n))$ and $(M', M) = (M_\eta(Sp_{2n}(2)), M_\eta(O^{\pm}_{2n}(2)))$, are of this type. So, the remaining same characteristic cases can be listed as follows:

1. $(G, H) = (Sp_{2m}(2), Sp_{2n}(2)), (Sp_{2m}(2), O^{\pm}_{2n}(2))(n \neq m)$, $(Sp_{2m}(2), \mathrm{SU}_n(2))$,
2. $(G, H) = (O^{\pm}_{2m}(2), Sp_{2n}(2)), (O^{\pm}_{2m}(2), O^{\pm}_{2n}(2)), (O^{\pm}_{2m}(2), \mathrm{SU}_n(2))$,
3. $(G, H) = (\mathrm{SU}_m(2), Sp_{2n}(2)), (\mathrm{SU}_m(2), O^{\pm}_{2n}(2)), (\mathrm{SU}_m(2), \mathrm{SU}_n(2))$, and
4. $(G, H) = (\Omega^{\pm}_m(3), \Omega^{\pm}_m(3))$

Remark. Throughout this chapter, G and H refer to the 3-transposition groups (G, D) and (H, C). The corresponding Matsuo algebras for G and H are denoted as $M' := M_\eta(G)$ and $M := M_\eta(H)$, where η is critical for M'. Additionally, we use same notations as in Table 4.1, i.e., $1, f$ and g are the multiplicities related to the adjacency matrix of the diagram (D). Equivalently the dimension of the radical, $R(M')$, is 1 or f or g. Further, $\dim \overline{M'} = \dim M'/R(M') = \dim M' - \dim R(M') = |D| - d$, where $d := 1, f$ or g.

Recall that for G to be aligned with H, we must have that $|D| - d = |C|$. We will use this condition to decide which groups can be possibly aligned with each other. For example, if $|D| - d$ is even, then G can only be aligned with the groups H that have even number of points, and so on.

6.1 Case 1

Let us consider the first case in the above list. We want to show that no possible examples can arise from that case.

Proposition 6.1.1. *Suppose that $G = Sp_{2m}(2)$ and $H = Sp_{2n}(2), \mathrm{SU}_n(2), O^{\pm}_{2n}(2)$ where n of $O^{\pm}_{2n}(2)$ is different from m. Then (G, H) is not aligned.*

Proof. Suppose that $G = Sp_{2m}(2)$. According to Table 4.1, the number of points in $G = Sp_{2m}(2)$ is $2^{2m} - 1 = |D|$, and the multiplicities of the eigenvalues of (D) are $f = 2^{2m-1} + 2^{m-1} - 1$ and $g = 2^{2m-1} - 2^{m-1} - 1$. Note that, $|D| - f = 2^{2m} - 1 - (2^{2m-1} + 2^{m-1} - 1) = 2^{2m-1} - 2^{m-1} = 2^{m-1}(2^m - 1)$ is even. Also, $|D| - g = 2^{2m} - 1 - (2^{2m-1} - 2^{m-1} - 1) = 2^{m-1}(2^m + 1)$, is even. So, G can only be aligned with the 3-transposition groups that have even number of points. If $H = Sp_{2n}(2)$, then (G, H) cannot be aligned because the number of points in H, $2^{2n} - 1 = |C|$, is odd. This also applies to $H = \mathrm{SU}_n(2)$ as the number of points in H, $\frac{1}{3}(2^{2n-1} - (-2)^{n-1} - 1) = |C|$, is also odd. Now, consider $G = Sp_{2m}(2)$ with $H = O^{+}_{2n}(2)$. First, we compare $|C|$ with $|D| - f$, then we compare $|C|$ with $|D| - g$. Note that the number of points in H, $2^{2n-1} - 2^{n-1} = |C|$, coincides with $|D| - f = 2^{2m-1} - 2^{m-1}$. So, $|D| - f = |C|$ only holds when $n = m$, for $\eta = \frac{-2}{-2^{m-1}} = \frac{1}{2^{m-2}}$. Now, let us compare $|D| - g = 2^{2m-1} + 2^{m-1}$ with $|C| = 2^{2n-1} - 2^{n-1}$. We claim that $|D| - g \neq |C|$ for all n, m with $n \neq m$. Suppose that there exists distinct n and m such that $2^{2m-1} + 2^{m-1} = 2^{2n-1} - 2^{n-1}$. We can rewrite this equation as, $2^{m-1}(2^m + 1) = 2^{n-1}(2^n - 1)$. Since $n \neq m$, either $m > n$ or $n > m$. If $m > n$, then $2^{m-1} > 2^{n-1}$, and $(2^m + 1) > (2^n - 1)$. So, $2^{m-1}(2^m + 1)$ is strictly greater than $2^{n-1}(2^n - 1)$, which is a contradiction. Similarly, if $n > m$, then $2^{n-1} > 2^{m-1}$, and $2^n - 1 \geqslant 2^m + 1$. Thus, $2^{n-1}(2^n - 1) > 2^{m-1}(2^m + 1)$. Therefore, $|D| - g \neq |C|$ when $n \neq m$. Since $|D| - d \neq |C|$, for $d = f$ or g, we conclude that $(G, H) = (Sp_{2m}(2), O^+_{2n}(2))$, with $n \neq m$, cannot be aligned. Finally, consider G with $H = O^-_{2n}(2)$, in this case, $|D| - g = 2^{2m-1} + 2^{m-1}$ coincides with the formula of $|C| = 2^{2n-1} + 2^{n-1}$. So, $|D| - g = |C|$ only holds when $n = m$, for $\eta = -\frac{2}{2^{m-1}} = -\frac{1}{2^{m-2}}$. Similar argument as above shows that $|D| - f = 2^{2m-1} - 2^{m-1} \neq 2^{2n-1} + 2^{n-1} = |C|$ for all n, m with $n \neq m$. So, $(Sp_{2m}(2), O^-_{2n}(2))$ cannot be aligned when $n \neq m$. Therefore, we conclude that no further examples of aligned pairs can arise from Case 1. □

6.2 Case 2

In Case 2, we consider $G = O^{\pm}_{2m}(2)$. In this case we have two situations;

1. $|D| - f = 2^{2m-1} \pm 2^{m-1} - (2^m \pm 1)(2^{m-1} \pm 1)/3 = 1/3(2^{2m} - 1)$ is odd, and
2. $|D| - g = 2^{2m-1} \pm 2^{m-1} - (2^{2m} - 4)/3$ is even.

In situation 1, when $|D| - f$ is odd, G can only be aligned with the 3-transposition groups that have odd number of points. In particular, G can be aligned with

$H = Sp_{2n}(2)$ and $H = \mathrm{SU}_n(2)$ only. In the second situation, when $|D| - g$ is even, G can only be aligned with $H = O^{\pm}_{2n}(2)$ because $O^{\pm}_{2n}(2)$ is the only group with even number of points. In the following propositions we aim to show that no further examples of aligned pairs can arise from the two situations.

Proposition 6.2.1. *The pairs* $(G, H) = (O^{\pm}_{2m}(2), Sp_{2n}(2)), (O^{\pm}_{2m}(2), \mathrm{SU}_n(2))$ *are not aligned.*

Proof. First, consider the case $G = O^{\pm}_{2m}(2)$ with $H = Sp_{2n}(2)$. We want to know the values of n and m that satisfy the condition $|D| - f = |C|$. Here, $|D| = 2^{2m-1} - \varepsilon 2^{m-1}, f = (2^m - \varepsilon 1)(2^{m-1} - \varepsilon)/3$, and $|C| = 2^{2n} - 1$. Note that, $|D| - f = 2^{2m-1} - \varepsilon 2^{m-1} - (2^m - \varepsilon)(2^{m-1} - \varepsilon)/3 = (2^{2m} - 1)/3$, for $\varepsilon = \pm 1$. That is, in this case, $|D| - f$ for $G = O^{+}_{2m}(2)$ is the same as $|D| - f$ for $G = O^{-}_{2m}(2)$. So, we can cover both cases of $G = O^{\pm}_{2m}(2)$ by solving $|D| - f = |C|$. Write $(2^{2m} - 1)/3 = 2^{2n} - 1$. So, $2^{2m} - 3 \cdot 2^{2n} = -2$. This equation is only true for $(n, m) = (0, 0)$ which is not a suitable solution. Therefore, the pair $(O^{\pm}_{2m}(2), Sp_{2n}(2))$ cannot be aligned. Now, consider the case, $G = O^{\pm}_{2m}(2)$ with $H = \mathrm{SU}_n(2)$. The number of points in H is $(2^{2n-1} - (-2)^{n-1} - 1)/3$, so write $|D| - f = |C|$ as $(2^{2m} - 1)/3 = (2^{2n-1} - (-2)^{n-1} - 1)/3$. So, we have that $2^{2m} - 2^{2n-1} + (-2)^{n-1} = 0$. However, this equation is only true for $(n, m) = (0, 0), (1, 0)$. Thus, the pair $(O^{\pm}_{2m}(2), \mathrm{SU}_n(2))$ cannot be aligned too. Therefore, no further examples can arise from the case $G = O^{\pm}_{2m}(2)$ and $H = Sp_{2n}(2), \mathrm{SU}_n(2)$. □

Now, let us consider the second situation of Case 2, the situation where $d = g$ and $|D| - g$ is even.

Proposition 6.2.2. *The pair* $(G, H) = (O^{\pm}_{2m}(2), O^{\pm}_{2n}(2))$ *is not aligned.*

Proof. Suppose that $G = O^{\varepsilon}_{2m}(2)$ and $H = O^{\delta}_{2n}(2)$, where $\varepsilon = \pm 1$ and $\delta = \pm 1$. Write the equation $|D| - g = |C|$ as; $2^{2m-1} - \varepsilon 2^{m-1} - (2^{2m} - 4)/3 = 2^{2n-1} - \delta 2^{n-1}$. That is, $3 \cdot 2^{2m-1} - \varepsilon 3 \cdot 2^{m-1} - 2^{2m} + 4 = 3 \cdot 2^{2n-1} - \delta 3 \cdot 2^{n-1}$. So,

$$2^{2m-1} - \varepsilon 3 \cdot 2^{m-1} - 3 \cdot 2^{2n-1} + \delta 3 \cdot 2^{n-1} = -4 \tag{6.1}$$

Now, for $(\varepsilon, \delta) = (+, -)$ which corresponds to $(G, H) = (O^{+}_{2m}(2), O^{-}_{2n}(2))$, the equation (6.1) has solutions $(n, m) = (0, 0)$, $(0, 1)$, $(3, 4)$. For the pair $(G, H) = (O^{+}_{2m}(2), O^{+}_{2n}(2))$, the solutions are $(n, m) = (1, 0), (1, 1)$. For the pair $(G, H) = (O^{-}_{2m}(2), O^{-}_{2n}(2))$, we have only one solution $(n, m) = (1, 1)$. Similarly, for $(G, H) = (O^{-}_{2m}(2), O^{+}_{2n}(2))$, we have only one solution $(n, m) = (2, 2)$. The only suitable solutions are $(n, m) = (3, 4), (2, 2)$. First, the solution $(n, m) = (3, 4)$ corresponds to the groups $G = O^{+}_{8}(2)$ and $H = O^{-}_{6}(2)$. In this case, $|D| - g = 120 - 84 = 36 = 2^5 + 2^2 = |C|$. The eigenvalue of the diagram (D), that corresponds to the multiplicity g, is $-2^{m-2} = -2^{4-2} = -4$, i.e., $\eta = -2/-4 = 1/2$ is critical for G. But $\eta = 1/2$ is also critical for H because the diagram (C) has an eigenvalue $-2^{n-1} = -2^2 = -4$. Therefore, $\dim \overline{M'} = 36 > 36 - \dim R(M)$, and so the pair $(G, H) = (O^{+}_{8}(2), O^{-}_{6}(2))$ is not aligned. Second, the pair $(G, H) = (O^{-}_{4}(2), O^{+}_{4}(2))$ cannot be aligned as $|H| = 36$ does not divide $|G| = 60$. Therefore, we conclude that no possible examples can arise from the pairs $(G, H) = (O^{\pm}_{2m}(2), O^{\pm}_{2n}(2))$. □

6.3 Case 3

Consider $G = \mathrm{SU}_m(2)$. First, if $d = f$, then $|D| - f = \frac{1}{3}(2^{2m-1} - (-2)^{m-1} - 1) - \frac{8}{9}(2^{2m-3} - (-2)^{m-2} - 1) = \frac{1}{9}(3 \cdot 2^{2m-1} - 3(-2)^{m-1} - 3 - 8 \cdot 2^{2m-3} + 8(-2)^{m-2} + 8) = \frac{1}{9}(2^{2m-1} + 14(-2)^{m-2} + 5)$ is odd. Also, if $d = g$, $|D| - g = \frac{1}{3}(2^{2m-1} - (-2)^{m-1} - 1) - \frac{4}{9}(2^{2m-3} - 7(-2)^{m-3} - 1) = \frac{1}{9}(3 \cdot 2^{2m-1} - 3(-2)^{m-1} - 3 - 4 \cdot 2^{2m-3} + 28(-2)^{m-3} + 4) =$

$\frac{1}{9}(2^{2m} + (-2)^{m+1} + 1)$ is odd. Therefore, $G = \mathrm{SU}_m(2)$ can be aligned with the 3-transposition groups that have odd number of points. In particular, it can be aligned with $H = Sp_{2n}(2)$ and $H = \mathrm{SU}_n(2)$.

Proposition 6.3.1. *The pair (G, H) where $G = \mathrm{SU}_m(2)$ and $H = Sp_{2n}(2), \mathrm{SU}_n(2)$ is not an aligned pair.*

Proof. First, consider $H = Sp_{2n}(2)$. The number of points in H is $2^{2n} - 1 = |C|$. Now, we want to find the (n, m) pairs that satisfy the equation $|D| - d = |C|$. From the above discussion we know that $|D| - f = \frac{1}{9}(2^{2m-1} + 14(-2)^{m-2} + 5)$. So, write $|D| - f = |C|$ as $2^{2m-1} - 7(-2)^{m-1} + 5 = 9 \cdot 2^{2n} - 9$. That is,

$$2^{2m-1} - 7(-2)^{m-1} - 9 \cdot 2^{2n} = -14 \tag{6.2}$$

The only integer solutions to (6.2) are $(n, m) = (0, 1), (1, 2)$. Next, write $|D| - g = |C|$ as; $\frac{1}{9}(2^{2m} + (-2)^{m+1} + 1) = 2^{2n} - 1$. So we have that,

$$2^{2m} + (-2)^{m+1} - 9 \cdot 2^{2n} = -10 \tag{6.3}$$

The only solution to (6.3) is $(0, 0)$. As the equations (6.2) and (6.3) have no suitable solutions, we conclude that $(\mathrm{SU}_m(2), Sp_{2n}(2))$ cannot be aligned. Now, consider $G = \mathrm{SU}_m(2)$ with $H = \mathrm{SU}_n(2)$, the number of points in H is $\frac{1}{3}(2^{2n-1} - (-2)^{n-1} - 1) = |C|$. First, consider $|D| - d = |C|$ with $d = f$. Write $|D| - f = |C|$ as $\frac{1}{9}(2^{2m-1} + 14(-2)^{m-2} + 5) = \frac{1}{3}(2^{2n-1} - (-2)^{n-1} - 1)$. That is, $2^{2m-1} - 7(-2)^{m-1} + 5 = 3 \cdot 2^{2n-1} - 3(-2)^{n-1} - 3$. So,

$$2^{2m-1} - 7(-2)^{m-1} - 3 \cdot 2^{2n-1} + 3(-2)^{n-1} = -8 \tag{6.4}$$

The integer solutions to (6.4) are $(n, m) = (0, 1), (1, 1), (2, 2), (4, 5)$. The only suitable solution is $(n, m) = (4, 5)$. This solution corresponds to the pair $G = \mathrm{SU}_5(2)$ and $H = \mathrm{SU}_4(2)$. In this case, $|D| - f = 45 = |C|$. The eigenvalue that corresponds to the multiplicity f is $-(-2)^{m-3} = -(-2)^2 = -4$, so $\eta = -2/-4 = 1/2$ is critical for G. But $\eta = 1/2$ is also critical for H as $\zeta := -(-2)^{n-2} = -4$ is an eigenvalue for the diagram (C) of $\mathrm{SU}_4(2)$. Therefore, $|D| - f = 45 > \dim M - \dim R(M)$, and so the pair $(\mathrm{SU}_5(2), \mathrm{SU}_4(2))$ is not aligned. Now, consider $|D| - d = |C|$, for $d = g$. Write $|D| - g = |C|$ as $\frac{1}{9}(2^{2m} + (-2)^{m+1} + 1) = \frac{1}{3}(2^{2n-1} - (-2)^{n-1} - 1)$. So, $2^{2m} + (-2)^{m+1} + 1 = 3 \cdot 2^{2n-1} - 3(-2)^{n-1} - 3$. So we have that,

$$2^{2m} + (-2)^{m+1} - 3 \cdot 2^{2n-1} + 3(-2)^{n-1} = -4 \tag{6.5}$$

The solutions of (6.5) are $(n, m) = (0, 0), (0, 1), (3, 3)$. Since both equations, (6.4) and (6.5), have no suitable solutions, other than $(n, m) = (4, 5)$, we conclude that $(G, H) = (\mathrm{SU}_m(2), \mathrm{SU}_n(2))$ cannot be aligned. □

6.4 Case 4

As we are considering the same characteristic case, the group $G = \Omega^{\pm}_m(3)$ can only be aligned with $H = \Omega^{\pm}_n(3)$ (the pair $(\Omega^{\pm}_n(3), \Omega_8(3) : S_3)$ was considered in the previous chapter). We start by listing $|D| - f$ and $|D| - g$ for each case of G, i.e., for $G = \Omega_{2m+1}(3), \Omega^{+}_{2m}(3)$, and $\Omega^{-}_{2m}(3)$. Then, we will equate $|D| - f$ and $|D| - g$ with the size of C in $H = \Omega_{2n+1}(3), \Omega^{+}_{2n}(3)$, and $\Omega^{-}_{2n}(3)$.

1. If $G = \Omega_m(3)$, (m is odd);

   (a) $|D| - f = (3^{m-1} - 3^{\frac{m-1}{2}})/2 - (3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} - 3)/4 = (3^{m-1} + 3)/4$.

   (b) $|D| - g = (3^{m-1} - 3^{\frac{m-1}{2}})/2 - (3^{m-1} - 1)/4 = (3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} + 1)/4$.

2. If $G = \Omega_m^+(3)$, (m is even);

   (a) $|D| - f = (3^{m-1} - 3^{\frac{m-2}{2}})/2 - (3^{\frac{m}{2}} - 1)(3^{\frac{m-2}{2}} - 1)/8 = (3^m - 1)/8.$

   (b) $|D| - g = (3^{m-1} - 3^{\frac{m-2}{2}})/2 - (3^m - 9)/8 = (3^{m-1} - 4 \cdot 3^{\frac{m-2}{2}} + 9)/8.$

3. If $G = \Omega_m^-(3)$, (m is even);

   (a) $|D| - f = (3^{m-1} + 3^{\frac{m-2}{2}})/2 - (3^{\frac{m}{2}} + 1)(3^{\frac{m-2}{2}} + 1)/8 = (3^m - 1)/8.$

   (b) $|D| - g = (3^{m-1} + 3^{\frac{m-2}{2}})/2 - (3^m - 9)/8 = (3^{m-1} + 4 \cdot 3^{\frac{m-2}{2}} + 9)/8.$

Remark. In the following propositions, we will start by considering $|D| - f = |C|$, then $|D| - g = |C|$.

Proposition 6.4.1. *Suppose that $G = \Omega_m(3)$, m is odd, and $H = \Omega_n(3), \Omega_n^+(3)$, or $\Omega_n^-(3)$, then (G, H) is not an aligned pair.*

Proof. First, consider G with $H = \Omega_n(3)$. The number of points in H is $(3^{n-1} - 3^{\frac{n-1}{2}})/2$. So, the equation $|D| - f = |C|$ is $(3^{m-1} + 3)/4 = (3^{n-1} - 3^{\frac{n-1}{2}})/2$. That is,

$$3^{m-1} - 2 \cdot 3^{n-1} + 2 \cdot 3^{\frac{n-1}{2}} + 3 = 0 \tag{6.6}$$

The only solution to (6.6) is $(n, m) = (3, 3)$. Next, write $|D| - g = |C|$ as $(3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} + 1)/4 = (3^{n-1} - 3^{\frac{n-1}{2}})/2$. So,

$$3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} - 2 \cdot 3^{n-1} + 2 \cdot 3^{\frac{n-1}{2}} + 1 = 0 \tag{6.7}$$

The only solution to (6.7) is $(n, m) = (1, 1)$. Since the solutions of (6.6) and (6.7) are not suitable, we deduce that $(G, H) = (\Omega_m(3), \Omega_n(3))$ is not aligned. Now, consider $G = \Omega_m(3)$, m is odd, with $H = \Omega_n^\varepsilon(3)$, n is even. The number of points in H is $(3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. So, write $|D| - f = |C|$ as $(3^{m-1} + 3)/4 = (3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$.

That is,

$$3^{m-1} - 2 \cdot 3^{n-1} + 2\varepsilon \cdot 3^{\frac{n-2}{2}} + 3 = 0 \tag{6.8}$$

When $\varepsilon = +1$, equation (6.8) has only one solution, $(n, m) = (2, 1)$, and it has no proper solution when $\varepsilon = -1$. Now, consider $|D| - g = |C|$, write this as, $(3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} + 1)/4 = (3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. That is,

$$3^{m-1} - 2 \cdot 3^{\frac{m-1}{2}} - 2 \cdot 3^{n-1} + 2\varepsilon \cdot 3^{\frac{n-2}{2}} + 1 = 0 \tag{6.9}$$

Solutions to (6.9) are $(n, m) = (2, 3)$ (when $\varepsilon = +1$) and $(n, m) = (1, 3)$ (when $\varepsilon = -1$). Since equations (6.8) and (6.9) have no suitable solutions, we deduce that no examples of aligned pairs arise from $(G, H) = (\Omega_m(3), \Omega_n^{\varepsilon}(3))$. Therefore, no examples arise from the case $G = \Omega_m(3)$, m is odd, and $H = \Omega_n(3), \Omega_n^+(3)$, or $\Omega_n^-(3)$. □

Proposition 6.4.2. *Suppose that $G = \Omega_m^+(3)$, m is even, and $H = \Omega_n^+(3), \Omega_n^-(3)$ or $\Omega_n(3)$, then (G, H) is not an aligned pair.*

Proof. First, consider G with $H = \Omega_n^{\varepsilon}(3)$, n is even. Then number of points in H is $(3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. So, $|D| - f = |C|$ is $(3^m - 1)/8 = (3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. That is,

$$3^m - 4 \cdot 3^{n-1} + 4\varepsilon \cdot 3^{\frac{n-2}{2}} - 1 = 0 \tag{6.10}$$

And $|D| - g = |C|$ is $(3^{m-1} - 4 \cdot 3^{\frac{m-2}{2}} + 9)/8 = (3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. That is,

$$3^{m-1} - 4 \cdot 3^{\frac{m-2}{2}} - 4 \cdot 3^{n-1} + \varepsilon 4 \cdot 3^{\frac{n-2}{2}} + 9 = 0 \tag{6.11}$$

The (n, m) solutions to (6.10) with $\varepsilon = +1$ are $(0, 0), (2, 2)$, and it has no proper solutions when $\varepsilon = -1$. Also, equation (6.11) has solutions $(n, m) = (2, 0), (2, 2)$

when $\varepsilon = +1$, and it has no solutions when $\varepsilon = -1$. Since the solutions of (6.10) and (6.11) are not suitable, the pairs $(G, H) = (\Omega_m^+(3), \Omega_n^+(3)), (\Omega_m^+(3), \Omega_n^-(3))$ cannot be aligned.

Next, consider G with $H = \Omega_n(3)$ (n is odd), the number of points in H is $(3^{n-1} - 3^{\frac{n-1}{2}})/2$. Thus, the equation $|D| - f = |C|$ is $(3^m - 1)/8 = (3^{n-1} - 3^{\frac{n-1}{2}})/2$. That is,

$$3^m - 4 \cdot 3^{n-1} + 4 \cdot 3^{\frac{n-1}{2}} - 1 = 0 \tag{6.12}$$

And $|D| - g = |C|$ is $(3^{m-1} - 4 \cdot 3^{\frac{m-2}{2}} + 9)/8 = (3^{n-1} - 3^{\frac{n-1}{2}})/2$. That is,

$$3^{m-1} - 4 \cdot 3^{\frac{m-2}{2}} - 4 \cdot 3^{n-1} + 4 \cdot 3^{\frac{n-1}{2}} + 9 = 0 \tag{6.13}$$

The only solution to (6.12) is $(n, m) = (1, 0)$, and the only solution to (6.13) is $(n, m) = (3, 4)$. The pair $(n, m) = (3, 4)$ corresponds to the groups $G = \Omega_4^+(3)$ and $H = \Omega_3(3)$. In this case, $|D| - g = 12 - 9 = 3 = |C|$. However, in this case, the eigenvalue of (D), that corresponds to g, is $\zeta := \varepsilon 3^{\frac{m-4}{2}} - 1$, i.e., $\zeta = 0$ when $\varepsilon = +1$ and $m = 4$. This implies that $\eta = \frac{-2}{\zeta}$ is undefined. Hence, $(G, H) = (\Omega_4^+(3), \Omega_3(3))$ is not aligned. Therefore, no examples of aligned pairs can arise from this situation. □

Proposition 6.4.3. *Suppose that $G = \Omega_m^-(3)$, m is even, and $H = \Omega_n^-(3), \Omega_n^+(3)$ or $\Omega_n(3)$, then (G, H) is not an aligned pair.*

Proof. First, consider G with $H = \Omega_n^{\pm}(3)$. The number of points in H is $(3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. So, write $|D| - f = |C|$ as $(3^m - 1)/8 = (3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. That is,

$$3^m - 4 \cdot 3^{n-1} + 4\varepsilon \cdot 3^{\frac{n-2}{2}} - 1 = 0 \tag{6.14}$$

Also, write $|D| - g = |C|$ as $(3^{m-1} + 4 \cdot 3^{\frac{m-2}{2}} + 9)/8 = (3^{n-1} - \varepsilon 3^{\frac{n-2}{2}})/2$. That is,

$$3^{m-1} + 4 \cdot 3^{\frac{m-2}{2}} - 4 \cdot 3^{n-1} + 4\varepsilon \cdot 3^{\frac{n-2}{2}} + 9 = 0 \tag{6.15}$$

The equation (6.14) has solutions $(0,0), (2,2)$ for $\varepsilon = +1$, and it has no solution for $\varepsilon = -1$. The equation (6.15) has no solution for $\varepsilon = +1$, and it has a solution $(2,2)$ for $\varepsilon = -1$. Since these solutions are not suitable for our approach, we conclude that $(G, H) = (\Omega_m^-(3), \Omega_n^{\pm}(3))$ cannot be aligned.

Now, consider G with $H = \Omega_n(3)$, n is odd. The number of points in H is $(3^{n-1} - 3^{\frac{n-1}{2}})/2$. Thus, the equation $|D| - f = |C|$ is $(3^m - 1)/8 = (3^{n-1} - 3^{\frac{n-1}{2}})/2$. That is,

$$3^m + 4 \cdot 3^{n-1} + 4 \cdot 3^{\frac{n-1}{2}} - 1 = 0 \tag{6.16}$$

And, write $|D| - g = |C|$ as $(3^{m-1} + 4 \cdot 3^{\frac{m-2}{2}} + 9)/8 = (3^{n-1} - 3^{\frac{n-1}{2}})/2$. That is,

$$3^{m-1} + 4 \cdot 3^{\frac{m-2}{2}} - 4 \cdot 3^{n-1} + 4 \cdot 3^{\frac{n-1}{2}} + 9 = 0 \tag{6.17}$$

The equation (6.16) has no integer solution, however equation (6.17) has a solution $(n, m) = (5, 6)$. This solution corresponds to the groups $G = \Omega_6^-(3)$ and $H = \Omega_5(3)$. In this case, $|D| - g = 126 - 90 = 36 = (3^4 - 3^2)/2 = |C|$. However, the critical value of G in this case, $\eta = 1/2$, is also critical for H, but $\dim M' - \dim R(M') = 36 \neq 21 = \dim M - \dim R(M)$. Therefore, $(G, H) = (\Omega_6^-(3), \Omega_5(3))$ is not aligned. Hence, no further examples of aligned pairs arise from the case $(G, H) = (\Omega_m^-(3), \Omega_n(3))$. Therefore, no examples arise from the case $G = \Omega_m^-(3)$, m is even, and $H = \Omega_n^-(3), \Omega_n^+(3)$ or $\Omega_n(3)$. $\square$

CHAPTER 7

REDUCIBLE 3-TRANSPOSITION GROUPS

In this chapter, we introduce reducible 3-transposition groups and give examples of special type of aligned pairs.

7.1 The Covers of 3-Transposition Groups

Let (G, D) be a 3-transposition group. According to [CH95], there are two graphs related to the class D, the *commuting graph* and the *diagram*. In the commuting graph the vertex set is D, and two vertices are connected by an edge if and only if they commute. The diagram is just the complement graph of the commuting graph on D.

For each $d \in D$, let $A_d := \{c \in D \mid |cd| = 2\}$ be the set of neighbours of d in the commuting graph D, and $D_d := \{c \in D \mid |cd| = 3\}$ be the set of neighbours of d in the diagram of D. Then, the relations τ and θ are defined as, $d\tau d'$ if and only if $A_d = A_{d'}$ and $d\theta d'$ if and only if $D_d = D_{d'}$. Furthermore, let $\tau(G) = \langle dd' \mid d\tau d' \rangle$ and $\theta(G) = \langle dd' \mid d\theta d' \rangle$.

Theorem 7.1.1 ([CH95])**.** *$\tau(G)$ is a normal 2-group of G, and $\theta(G)$ is a normal 3-group of G.*

We say that (G, D) is *irreducible* if $\tau(G) = 1 = \theta(G)$. Irreducible 3-transposition groups are equivalent to the groups in the Fischer's list (Theorem 3.1.2). Note that, if $\tau(G) \neq 1$, then $\theta(G) = 1$.

Let $h \geqslant 1$. Suppose that $\tau(G) \neq 1$. The group (G, D) is a 2-*cover* if $G = 2^{\bullet h}\overline{G}$, where $\overline{G} = G/\tau(G)$ and $|D| = 2^h|\overline{D}|$. Similarly, suppose that $\theta(G) \neq 1$. The group (G, D) is a 3-*cover* if $G = 3^{\bullet h}\widehat{G}$, where $\widehat{G} = G/\theta(G)$ and $|D| = 3^h|\widehat{D}|$.

The covers of 3-transposition groups are also called *reducible* 3-transposition groups. The following Theorem, from [HS21], lists all the covers of the 3-transposition groups.

Theorem 7.1.2 ([HS21])**.** *Let (G, D) be a finite 3-transposition group. Then, for integers m and h, we identify D, up to a center, as one of the following classes:*

PR1. $3^{\bullet h} : S_2$, *for all* $h \geqslant 1$;

PR2(a). $2^{\bullet h} : S_m$, *for all* $h \geqslant 0$ *and* $m \geqslant 4$;

PR2(b). $3^{\bullet h} : S_m$, *for all* $h \geqslant 1$ *and* $m \geqslant 4$;

PR2(c). $3^{\bullet h} : 2^{\bullet 1} : S_m$, *for all* $h \geqslant 1$ *and* $m \geqslant 4$;

PR2(d). $4^{\bullet h} : 3^{\bullet 1} : S_m$, *for all* $h \geqslant 1$ *and* $m \geqslant 4$;

PR3. $2^{\bullet h} : O^{\varepsilon}_{2m}(2)$, *for all* $h \geqslant 0$ *and all* $m \geqslant 3$, $(m, \varepsilon) \neq (3, +)$;

PR4. $2^{\bullet h} : Sp_{2m}(2)$, *for all* $h \geqslant 0$ *and all* $m \geqslant 3$;

PR5. $3^{\bullet h}\Omega^{\varepsilon}_{m}(3)$, $\varepsilon = \pm$, *for all* $h \geqslant 0$ *and all* $m \geqslant 5$;

PR6. $4^{\bullet h}\mathrm{SU}_m(2)'$, *$h \geqslant 0$ and all $m \geqslant 3$;*

PR7. $Fi_{22}, Fi_{23}, Fi_{24}, P\Omega_8^+(2) : S_3, P\Omega_8^+(3) : S_3$;

PR8. $4^{\bullet h} : (3 \cdot \Omega_6^-(3))$, *all $h \geqslant 1$;*

PR9. $3^{\bullet h} : (2 \times Sp_6(2))$, *all $h \geqslant 1$;*

PR10. $3^{\bullet h} : (2 \cdot O_8^+(2))$, *all $h \geqslant 1$;*

PR11. $3^{\bullet 2h} : (2 \times \mathrm{SU}_5(2))$, *all $h \geqslant 1$;*

PR12. $3^{\bullet 2h} : 4^{\bullet 1} : \mathrm{SU}_3(2)'$, *all $h \geqslant 1$.*

The number of points and the spectrum of the diagram (D), of each group in Theorem 7.1.2, are all calculated in a table in [HS21], which we quoted in page 119. The following Theorem explains how the spectrum of the covers, $G = 2^h\overline{G}$ and $G = 3^h\widehat{G}$, are related to the spectrum of the original 3-transposition groups $\overline{G}$ and $\widehat{G}$.

Theorem 7.1.3 ([HS21]). *Let $h \geqslant 1$. If $(\overline{G}, \overline{D})$, or respectively $(\widehat{G}, \widehat{D})$, has spectrum $(k; \ldots, [\zeta_i]^{m_i}, \ldots)$, then*

(a) *The 2-cover $(G, D) = 2^{\bullet h}(\overline{G}, \overline{D})$ has spectrum $((2^h k; 0, \ldots, [2^h\zeta_i]^{m_i}, 0, \ldots))$, and the multiplicity of each eigenvalue $2^h\zeta_i$ of $2^{\bullet h}\overline{G}$ is the same as the multiplicity of the eigenvalue ζ_i of $\overline{G}$. The number of points in $G = 2^{\bullet h}\overline{G}$ is $|D| = 2^h|\overline{D}|$. The multiplicity of the eigenvalue 0 is equal to $|D| - |\overline{D}| = (2^h - 1)|\overline{D}|$.*

(b) *The 3-cover $(G, D) = 3^{\bullet h}(\widehat{G}, \widehat{D})$ has spectrum $((3^h(k+1)-1; -1, \ldots, [3^h(\zeta_i + 1) - 1]^{m_i}, -1, \ldots))$, and the multiplicity of each eigenvalue $[3^h(\zeta_i + 1) - 1]$ of*

$G = 3^{\bullet h}\widehat{G}$ *is the same as the multiplicity of* ζ_i *of* $\widehat{G}$. *Moreover, the number of points in* G *is* $|D| = 3^h|\widehat{D}|$. *The multiplicity of* -1 *is* $|D|-|\widehat{D}| = (3^h-1)|\widehat{D}|$.

Example 7.1.4 ([HS21])**.** *Let* $(\overline{G}, \overline{D}) = (S_m, (1,2)^{\overline{D}})$, *or respectively* $(\widehat{G}, \widehat{D}) = (S_m, (1,2)^{\widehat{D}})$. *Let us find the number of points and the spectrum of* $(G, D) = 2^{\bullet h}(\overline{G}, \overline{D})$ *and* $(G, D) = 3^{\bullet h}(\widehat{G}, \widehat{D})$. *First, recall that the number of points in the diagram* $(\overline{D})$ *is* $\binom{m}{2}$, *and the spectrum of* $(\overline{G}, \overline{D})$ *is* $(2(m-2); [m-4]^{m-1}, [-2]^{m(m-3)/2})$. *Now, using Theorem 7.1.3, we have that,*

(a) *The number of points in* $G = 2^{\bullet h}\overline{G}$ *is* $|D| = 2^h|\overline{D}| = 2^h\binom{m}{2} = 2^{h-1}m(m-1)$, *and the spectrum of* G *is* $((2^{h+1}(m-2); [2^h(m-4)]^{m-1}, [-2^{h+1}]^{m(m-3)/2}, [0]^t))$, *where* $t = (2^h-1)|\overline{D}| = (2^h-1)\binom{m}{2} = (2^h-1)\frac{m(m-1)}{2}$.

(b) *The number of points in* $G = 3^{\bullet h}\widehat{G}$ *is* $|D| = 3^h|\widehat{D}| = 3^h\binom{m}{2} = 3^h m(m-1)/2$, *and the spectrum of* G *is* $((3^h(2m-3)-1; [3^h(m-3)-1]^{m-1}, [-3^h-1]^{m(m-3)/2}, [-1]^t))$, *where* $t = (3^h-1)|\widehat{D}| = (3^h-1)\frac{m(m-1)}{2}$.

Label	Diagram (G)
Size n	Spectrum $((k;\dots,[\zeta_i]^{m_i},\dots))$
PR1 3^h	$\left(3^{\bullet h}:\mathrm{Sym}(2)\right)$ $\left(\left(3^h-1;[-1]^{-1+3^h}\right)\right)$
PR2(a) $m\geqslant 4: 2^{h-1}m(m-1)$	$\left(2^{\bullet h}:\mathrm{Sym}(m)\right)$ $\left(\left(2^{h+1}(m-2);\left[2^h(m-4)\right]^{m-1},[0]^{\star},\left[-2^{\mathrm{h+1}}\right]^{m(m-3)/2}\right)\right)$
PR2(b) $m\geqslant 4: 3^h m(m-1)/2$	$\left(3^{\bullet h}:\mathrm{Sym}(m)\right)$ $\left(\left(3^h(2m-3)-1;\left[3^h(m-3)-1\right]^{m-1},[-1]^{\star},\left[-3^h-1\right]^{m(m-3)/2}\right)\right)$
PR2(c) $m\geqslant 4: 3^h m(m-1)$	$\left(3^{\bullet h}:2^{\bullet 1}:\mathrm{Sym}(m)\right)$ $\left(\left(3^h(4m-7)-1;\left[3^h(2m-7)-1\right]^{m-1},\left[3^h-1\right]^{m(m-1)/2},[-1]^{\star},\left[-\mathbf{3}^{\mathrm{h+1}}-1\right]^{m(m-3)/2}\right)\right)$
PR2(d) $m\geqslant 4: 3\left(2^{2h-1}\right)m(m-1)$	$\left(4^{\bullet h}:3^{\cdot 1}:\mathrm{Sym}(m)\right)$ $\left(\left(4^h(6m-10);\left[4^h(3m-10)\right]^{m-1},[0]^{\star},\left[-4^h\right]^{m(m-1)},\left[-4^{\mathrm{h+1}}\right]^{m(m-3)/2}\right)\right)$
PR3 $m\geqslant 3,\varepsilon=+: 2^h\left(2^{2m-2}-2^{m-1}\right)$	$\left(2^{\bullet h}:\mathrm{O}_{2m}^{\varepsilon}(2)\right)$ $\left(\left(2^h\left(2^{2m-2}-2^{m-1}\right);\left[2^{h+m-1}\right]^{\left(2^m-1\right)\left(2^{m-1}-1\right)/3},[0]^{\star},\left[-2^{\mathbf{h+m}-2}\right]^{\left(2^{2m}-4\right)/3}\right)\right)$
PR4 $m\geqslant 2, 2^h(2^{2m}-1)$	$\left(2^{\bullet h}:Sp_{2m}(2)\right)$ $\left(\left(2^{2m-1+h};\left[2^{m-1+h}\right]^{2^{2m-1}-2^{m-1}-1},[0]^{\star},\left[-2^{\mathbf{h+m}-1}\right]^{2^{2m-1}+2^{m-1}-1}\right)\right)$
PR5 odd $m\geqslant 5$	$\left(3^{\bullet h+}\Omega_m^{\varepsilon}(3)\right)$ $\left(\left(3^h\left(3^{m-2}-2\varepsilon 3^{(m-3)/2}\right)-1;\left[3^{(m-3)/2+h}-1\right]^f,[-1]^{\star},\left[-\mathbf{3}^{\mathrm{(m-3)/2+h}}-\mathbf{1}\right]^g\right)\right)$
$\varepsilon=+: 3^h\left(3^{m-1}-3^{(m-1)/2}\right)/2$	for $f=\left(3^{m-1}-1\right)/4$ and $g=\left(3^{m-1}-1-2\left(3^{(m-1)/2}+1\right)\right)/4$
$\varepsilon=-: 3^h\left(3^{m-1}+3^{(m-1)/2}\right)/2$	for $f=\left(3^{m-1}-1+2\left(3^{(m-1)/2}-1\right)\right)/4$ and $g=\left(3^{m-1}-1\right)/4$
even $m\geqslant 6,\varepsilon=+:$	$\left(\left(3^{m-2+h}-1;\left[3^{(m-4)/2+h}-1\right]^f,[-1]^{\star},\left[-\mathbf{3}^{\mathrm{(m-2)/2}+h}-1\right]^g\right)\right)$
$3^h\left(3^{m-1}-3^{(m-2)/2}\right)/2$	for $f=\left(3^m-9\right)/8$ and $g=\left(3^{m/2}-1\right)\left(3^{(m-2)/2}-1\right)/8$
even $m\geqslant 6,\varepsilon=-:$	$\left(\left(3^{m-2+h}-1;\left[3^{(m-2)/2+h}-1\right]^f,[-1]^{\star},\left[-3^{\mathrm{(m-4)/2}+h}-1\right]^g\right)\right)$
$3^h\left(3^{m-1}+3^{(m-2)/2}\right)/2$	for $f=\left(3^{m/2}+1\right)\left(3^{(m-2)/2}+1\right)/8$ and $g=\left(3^m-9\right)/8$
PR6 even $m\geqslant 4: 4^h\left(2^{2m-1}-1+2^{m-1}\right)/3$	$\left(4^{\bullet h}\mathrm{SU}_m(2)'\right),\quad m\geqslant 3$ $\left(\left(2^{2h+2m-3};\left[2^{2h+m-3}\right]^f,[0]^{\star},\left[-2^{2\ \mathrm{h+m-2}}\right]^g\right)\right)$
PR8 $h\geqslant 1: 126\left(4^h\right)$	$\left(4^{\bullet h}:\left(3^{\cdot +}\Omega_6^{-}(3)\right)\right)$ $\left(\left(5\left(4^{h+2}\right);\left[2^{2h+3}\right]^{35},[0]^{*},\left[-4^{\mathrm{h+1}}\right]^{90}\right)\right)$
PR9 $h\geqslant 1: 7\left(3^{h+2}\right)$	$\left(3^{\bullet h}:(2\times\mathrm{Sp}_6(2))\right)$ $\left(\left(11\left(3^{h+1}\right)-1;\left[5\left(3^h\right)-1\right]^{27},[-1]^{\star},\left[-3^{h+1}-1\right]^{35}\right)\right)$
PR10 $h\geqslant 1: 40\left(3^{h+1}\right)$	$\left(3^{\bullet h}:\left(2\cdot\mathrm{O}_8^{+}(2)\right)\right)$ $\left(\left(19\left(3^{h+1}\right)-1;\left[3^{h+2}-1\right]^{35},[-1]^{\star},\left[-\mathbf{3}^{\mathrm{h+1}}-1\right]^{84}\right)\right)$
PR11 $h\geqslant 1: 55\left(3^{2h+1}\right)$	$\left(9^{\bullet h}:(2\times\mathrm{SU}_5(2))\right)$ $\left(\left(43\left(3^{2h+1}\right)-1;\left[3^{2h+2}-1\right]^{44},[-1]^{\star},\left[-3^{2h+1}-1\right]^{120}\right)\right)$
PR12 $h\geqslant 1: 4\left(3^{2h+2}\right)$	$(9^{\bullet h}:4^{\bullet 1}:\mathrm{SU}_3(2)')$ $\left(\left(11\left(3^{2h+1}\right)-1;\left[3^{2h}-1\right]^{27},[-1]^{\star},\left[-3^{2h+1}-1\right]^{8}\right)\right)$

In the spectrum of the 2-covers, we have the eigenvalue 0, which can be disregarded as it does not lead to a critical value η. However, the eigenvalue -1, in the spectrum of the 3-covers, leads to $\eta = 2$, and, in fact, it leads to many examples of aligned pairs.

7.2 Examples of Aligned Pairs

It turns out that each 3-cover, $(G, D) = 3^{\bullet h}(\widehat{G}, \widehat{D})$ with $\eta = 2$, forms an aligned pair with its original group $(\widehat{G}, \widehat{D})$. However, in this type of aligned pairs we do not get extra automorphisms as all the axes from $D = 3^h\widehat{D}$ fold into the kernel of the map $\phi : M' \to \overline{M'}$, where $M' = M_\eta(G, D)$. In the following, we give examples of this type.

Remark. We will write $(G, \widehat{G})$ to denote the pair $((G, D), (\widehat{G}, \widehat{D}))$. Furthermore, since the adjacency matrix of the diagram (D) is semisimple, $|D| = 1 + \star + \sum_i m_i$, where $\star$ denotes the multiplicity of the eigenvalue -1 and m_i denotes the multiplicity of the eigenvalue ζ_i in the spectrum of (D).

Example 7.2.1. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet h}S_m, S_m)$*. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$*. The number of points in* M' *is* $3^h m(m-1)/2$*. Furthermore,* (D) *has the spectrum*

$$\left(\left(3^h(2m-3) - 1; \left[3^h(m-3) - 1\right]^{m-1}, [-1]^\star, \left[-3^h - 1\right]^{m(m-3)/2}\right)\right)$$

Choose the eigenvalue -1*. Then,* $\eta = -2/-1 = 2$ *is critical for* G*. The multiplicity of* -1 *is the dimension of the radical,* $R(M')$*. So,* $\dim R(M') = \star = (3^h - 1)|\widehat{D}| = (3^h - 1)\binom{m}{2} = (3^h - 1)m(m-1)/2$*. Note that,* $\dim \overline{M'} = \dim M' - \dim R(M') = (3^h m(m-1) - (3^h - 1)m(m-1))/2 = (3^h m(m-1) -$

$(3^h m(m-1)-m(m-1))/2 = m(m-1)/2 = \binom{m}{2} = \dim M$. *Therefore,* $(3^{\bullet h} S_m, S_m)$ *is an aligned pair.*

Example 7.2.2. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet h} 2^{\bullet 1} S_m, 2^{\bullet 1} S_m)$. *Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$. *Now, the number of points in* G *is* $3^h m(m-1)$, *and the spectrum of* (D) *is*

$$\Big(\big(3^h(4m-7)-1; [3^h(2m-7)-1]^{m-1}, [-1]^\star, [3^h-1]^{m(m-1)/2}, [-3^{h+1}-1]^{m(m-3)/2}\big)\Big)$$

Choose the eigenvalue -1. *Note that,*

$$\dim M' = |D| = 1 + (m-1) + \star + m(m-1)/2 + m(m-3)/2$$

Thus, $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + (m-1) + m(m-1)/2 + m(m-3)/2 = m + (m(m-1) + m(m-3))/2 = (2m^2 - 2m)/2 = m(m-1)$. *Since the number of points in* $\widehat{G} := (2^{\bullet 1} S_m)$ *is* $m(m-1)$, *the pair* $(3^{\bullet h} 2^{\bullet 1} S_m, 2^{\bullet 1} S_m)$ *is aligned.*

Example 7.2.3. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet h} \Omega^\varepsilon_m(3), \Omega^\varepsilon_m(3))$, *where* m *is odd. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$. *Now, the number of points in* $\widehat{G}$ *is* $(3^{m-1} - \varepsilon 3^{(m-1)/2})/2 = \dim M$. *First, consider* $\varepsilon = +$. *Note that,* $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + f + g$, *where* $f = (3^{m-1}-1)/4$, *and* $g = (3^{m-1} - 1 - 2(3^{(m-1)/2}+1))/4$. *Thus,* $\dim \overline{M'} = (4 + 3^{m-1} - 1 + 3^{m-1} - 1 - 2 \cdot 3^{(m-1)/2} - 2)/4 = (2 \cdot 3^{m-1} - 2 \cdot 3^{(m-1)/2})/4 = (3^{m-1} - 3^{(m-1)/2})/2 = \dim M$. *Similarly, for* $\varepsilon = -$, *we have*

$$\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + f + g$$

Where $f = (3^{m-1} - 1 + 2(3^{(m-1)/2} - 1))/4$ *and* $g = (3^{m-1}-1)/4$. *Thus,* $\dim \overline{M'} = (2 \cdot 3^{m-1} + 2 \cdot 3^{(m-1)/2})/4 = (3^{m-1} + 3^{(m-1)/2})/2 = \dim M$. *Therefore,* $(G, \widehat{G})$ *is* *aligned.*

Example 7.2.4. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet h}\Omega_m^{\varepsilon}(3), \Omega_m^{\varepsilon}(3))$*, where* m *is even. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$*. The number of points in* $\widehat{G}$ *is* $(3^{m-1} - \varepsilon 3^{(m-2)/2})/2 = |\widehat{D}| = \dim M$*. First, consider the plus case* $\varepsilon = +$*. Note that,* $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + f + g$*, where* $f = (3^m - 9)/8$ *and* $g = (3^{m/2} - 1)(3^{(m-2)/2} - 1)/8$*. Thus,* $\dim \overline{M'} =$

$$(8 + 3^m - 9 + 3^{m-1} - 3^{m/2} - 3^{(m-2)/2} + 1)/8 = (4 \cdot 3^{m-1} - 4 \cdot 3^{(m-2)/2})/8$$

That is, $\dim \overline{M'} = (3^{m-1} - 3^{(m-2)/2})/2 = \dim M$*. Now, consider the case* $\varepsilon = -1$*. In this case,* $f = (3^{m/2} + 1)(3^{(m-2)/2} + 1)/8$ *and* $g = (3^m - 9)/8$*. So,* $\dim \overline{M'} = (8 + (3^{m/2} + 1)(3^{(m-2)/2} - 1) + 3^m - 9)/8 = (3^{m-1} + 3^m + 3^{(m-2)/2} + 3^{m/2})/8 = (4 \cdot 3^{m-1} - 4 \cdot 3^{(m-2)/2})/8 = (3^{m-1} + 3^{(m-2)/2})/2 = \dim M$*. Therefore,* $(G, \widehat{G}) = (3^{\bullet h}\Omega_m^{\varepsilon}(3), \Omega_m^{\varepsilon}(3))$ *is an aligned pair.*

Example 7.2.5. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet h}2 \times Sp_6(2), Sp_6(2))$*. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$*. Recall that the number of points in* $(Sp_{2m}(2))$ *is given by the formula* $2^{2m} - 1$*. Thus, the number of points in* $Sp_6(2)$ *is* $2^{2(3)} - 1 = 64 - 1 = 63$*. On the other hand,* $G = (3^{\bullet h} : (2 \times Sp_6(2)))$ *has spectrum*

$$\Big(\big(11(3^{h+1}) - 1; [5(3^h) - 1]^{27}, [-1]^{\star}, [-3^{h+1} - 1]^{35}\big)\Big)$$

Thus, $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + 27 + 35 = 63$*. So, the pair* $(G, \widehat{G}) = (3^{\bullet h}2 \times Sp_6(2), Sp_6(2))$ *is aligned.*

Example 7.2.6. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet h} : (2 \cdot O_8^+(2)), O_8^+(2))$*. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$*. Recall that the number of points in* $(O_{2m}^+(2))$ *is given by* $2^{2m-1} - 2^{m-1}$*. So,* $\dim M = |\widehat{D}| = 2^{8-1} - 2^{4-1} = 2^7 - 2^3 = 120$*. On the other* *hand, the spectrum of* $G = (3^{\bullet h}(2 \cdot O_8^+(2)))$ *is given by*

$$\Big(\Big(19(3^{h+1}) - 1; [3^{h+2} - 1]^{35}, [-1]^{\star}, [-3^{h+1} - 1]^{84}\Big)\Big)$$

Hence, $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + 35 + 84 = 120 = \dim M$*. Thus, the pair* $(G, \widehat{G}) = (3^{\bullet h}(2 \cdot O_8^+(2)), O_8^+(2))$ *is aligned.*

Example 7.2.7. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet 2h}(2 \times \mathrm{SU}_5(2)), \mathrm{SU}_5(2))$*. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$*. Recall that the number of point in* $(\mathrm{SU}_m(2))$ *is given by* $(2^{2m-1} - (-2)^{m-1} - 1)/3$*. So, the number of points in* $\mathrm{SU}_5(2)$ *is* $(2^9 - (-2)^4 - 1)/3 = 495/3 = 165$*. That is,* $\dim M = |\widehat{D}| = 165$*. On the other hand, the spectrum of* $G := (3^{\bullet 2h}(2 \times \mathrm{SU}_5(2)))$ *is given by*

$$\Big(\Big(43(3^{2h+1}) - 1; [3^{2h+2} - 1]^{44}, [-1]^{\star}, [-3^{2h+1} - 1]^{120}\Big)\Big)$$

Hence, $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + 44 + 120 = 165 = \dim M$*. Thus,* $(G, \widehat{G}) = (3^{\bullet 2h} : (2 \times \mathrm{SU}_5(2)), \mathrm{SU}_5(2))$ *is an aligned pair.*

Example 7.2.8. *Consider the pair* $(G, \widehat{G}) = (3^{\bullet 2h}4^{\bullet 1}\mathrm{SU}_3(2)', 4^{\bullet 1}\mathrm{SU}_3(2)')$*. Let* $M' = M_\eta(G)$ *and* $M = M_\eta(\widehat{G})$*. Recall that the number of points in* $(\mathrm{SU}_m(2))$ *is given by* $(2^{2m-1} - (-2)^{m-1} - 1)/3$*. So, the number of points in* $(4^{\bullet 1}\mathrm{SU}_3(2)')$ *is* $4 \cdot (2^5 - (-2)^2 - 1)/3 = 4 \cdot (32 - 5)/3 = 36$*. That is,* $\dim M = |\widehat{D}| = 36$*. On the other hand, the spectrum of* $G = (3^{\bullet 2h}4^{\bullet 1}\mathrm{SU}_3(2)')$ *is given by*

$$\Big(\Big(11(3^{2h+1}) - 1; [3^{2h} - 1]^{27}, [-1]^{\star}, [-3^{2h+1} - 1]^{8}\Big)\Big)$$

Hence, $\dim \overline{M'} = \dim M' - \dim R(M') = |D| - \star = 1 + 27 + 8 = 36 = \dim M$*. Thus, the pair* $(G, \widehat{G}) = (3^{\bullet 2h}4^{\bullet 1}\mathrm{SU}_3(2)', 4^{\bullet 1}\mathrm{SU}_3(2)')$ *is an aligned pair.*

7.3 Classification of Aligned Pairs

In this section, we summarize our findings on aligned pairs of 3-transposition groups.

Let (H, C) be an irreducible 3-transposition group. In Chapters 5 and 6, we investigated the case when H is a subgroup of another irreducible 3-transposition group and we concluded that, apart from the examples in Section 4.6, no further examples of aligned pairs can arise. In the following theorem we show that this is also the case when H is a subgroup of a reducible 3-transposition group.

Theorem 7.3.1. *Suppose that $((G, D), (H, C))$ is an aligned pair, where $|C| = m$ and $|D| = n$. Suppose further that H is irreducible and G is reducible. Then,*

(i) *$\theta(G) \neq 1$ and $\zeta = -1$, i.e., $\eta = 2$;*

(ii) *$n = 3^h m$ and $G = \theta(G)H$.*

Proof. We have two cases; the case $\tau(G) \neq 1$ and the case $\theta(G) \neq 1$. We claim that the first case, $\tau(G) \neq 1$, cannot arise at all. Indeed, suppose by contradiction that $\tau(G) \neq 1$. Let $Q = \tau(G) \neq 1$ and $\overline{G} = G/Q$. Then, $n = 2^h n'$ where n' is $|\overline{D}|$. Clearly, $n' \geqslant m$ since C maps into $\overline{D}$ injectively. Then, we have that either $\zeta = 0$ or $\zeta = 2^h \zeta'$ for some eigenvalue of $(\overline{D})$. But the eigenvalue $\zeta = 0$ is not possible as in this case we get $(\eta = -\frac{2}{0})$. For $\zeta \neq 0$, let m_ζ be its multiplicity. Recall that, m_ζ is the same as $m_{\overline{\zeta}}$. Note that, $m \leqslant n' = \frac{n}{2^h} \leqslant (2^h - 1)n' = n - n' < n - m_{\overline{\zeta}}$. So, we cannot have that $m = n - m_{\overline{\zeta}}$.

In the second case, $\tau(G) = 1$ but $\theta(G) \neq 1$. Let $Q = \theta(G)$ and $\widehat{G} = G/Q$. So, $n = 3^h n'$ where $n' = |\widehat{G}|$. In this case, ζ is either -1 or $3^h(\widehat{\zeta}+1)-1$. We rule out the case $\zeta = 3^h(\widehat{\zeta}+1)-1$. Here, $m_\zeta = m_{\overline{\zeta}} < n'$. So, $m \leqslant n' = \frac{n}{3^h} \leqslant (3^h-1)n' = n-n' < n - m_\zeta$. So we conclude that $\zeta = -1$. Now, for $\zeta = -1$, we have $m_{-1} = (3^h - 1)n'$. In this situation, $m \leqslant n' = 3^h n' - (3^h - 1)n' = n - m_{-1}$. So, we have $m = n - m_{-1}$ if and only if $m = n'$. So, C maps onto $\widehat{D}$ bijectively. From here we deduce that $G = QH$ because $D \subseteq QC$ and so $G = \langle D \rangle = \langle QC \rangle = Q\langle C \rangle = QH$. $\square$

In the previous theorem, when $Q = \theta(G) \neq 1$ with $\eta = 2$, $G = QH$ and H form an aligned pair. However, as C maps onto $\widehat{D}$ bijectively, we cannot get extra automorphisms out of this pair. These are the aligned pairs given in the previous section.

Corollary 7.3.2. *Suppose that (H, C) is an irreducible 3-transposition group. Let $\eta \in \mathbb{F} \setminus \{0, 1\}$ where $\mathbb{F}$ is a field of characteristic not 2. Then C is the complete set of all axes of Jordan type η in $M := M_\eta(H, C)$ and $\mathrm{Aut}(M) = \mathrm{Aut}(H, C)$, except for the following cases:*

(i) *$M = M_\eta(S_n)$ for $\eta = -\frac{2}{n-3}$. In this case, (S_{n+1}, S_n) is aligned, and M is isomorphic to a quotient of $M' := M_\eta(S_{n+1})$.*

(ii) *$M = M_\eta(O^+_{2n}(2))$ for $\eta = \frac{1}{2^{n-2}}$. In this case, $(Sp_{2n}(2), O^+_{2n}(2))$ is aligned, and M is isomorphic to a quotient of $M' := M_\eta(Sp_{2n}(2))$.*

(iii) *$M = M_\eta(O^-_{2n}(2))$ for $\eta = -\frac{1}{2^{n-2}}$. In this case, $(Sp_{2n}(2), O^-_{2n}(2))$ is aligned, and M is isomorphic to a quotient of $M' := M_\eta(Sp_{2n}(2))$.*

(iv) *$M = M_\eta(\Omega_8(2) : S_3)$ for $\eta = -\frac{1}{4}$. In this case, $(Fi_{23}, \Omega_8(2) : S_3)$ is aligned, and M is isomorphic to a quotient of $M' := M_\eta(Fi_{23})$.*

CONCLUSION

In this PhD thesis, we have studied the automorphism group of a Matsuo algebra, $M = M_\eta(G, D)$ for $\eta \neq \frac{1}{2}$. We showed that the full automorphism group of a Matsuo algebra is, in almost all cases, the same as the automorphism group of the underlying 3-transposition group (G, D) (and equivalently D is, almost always, the complete set of all axes of Jordan type η in M). Then, we identified all the exceptions to this general result in the irreducible case. In particular, we have found three infinite series and an additional sporadic example of Matsuo algebras having extra axes and a larger group of automorphisms. Namely, for specific η, $M = M_\eta(S_n)$ is isomorphic to a quotient of $M' = M_\eta(S_{n+1})$, and $M = M_\eta(O^{\varepsilon}_{2m}(2))$, $\varepsilon = \pm 1$, is isomorphic to a quotient of $M' = M_\eta(Sp_{2m}(2))$. Moreover, for $\eta = -\frac{1}{4}$, $M = M_\eta(\Omega_8(3) : S_3)$ is isomorphic to a quotient of $M' = M_\eta(Fi_{23})$. These are examples of what we called aligned pairs of groups of 3-transpositions.

Based on this work, some important questions arise within this line of research.

Question 7.3.3. *Are there any other non-irreducible aligned pairs and can they all be classified ?*

Since we have only considered the irreducible case, there might be more examples of aligned pairs arising from the reducible case. Information about the spectrum of the diagram of the reducible 3-transposition groups is also provided in [HS21]. Therefore, with a suitable method, one can examine all the pairs in that case.

Question 7.3.4. *Find the full automorphism groups and the full set of axes of suitable type for the case* $\eta = \frac{1}{2}$.

This situation will be quite different because in some of the cases the algebra contains infinitely many idempotents. Therefore, our current methods do not generally apply in this context. However, it is very important case within the scope of our research.

BIBLIOGRAPHY

[Bor86] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proceedings of the National Academy of Sciences* 83.10 (1986), pp. 3068–3071.

[Cas13] A. Castillo-Ramirez. Idempotents of the Norton-Sakuma algebras. *Journal of Group Theory* 3 (2013), pp. 419–444.

[Cas15] A. Castillo-Ramirez. Associative subalgebras of low-dimensional Majorana algebras. *Journal of Algebra* 421 (2015), pp. 119–135.

[Con85] J. H. Conway. A simple construction for the Fischer-Griess monster group. *Inventiones mathematicae* 79 (1985), pp. 513–540.

[CN79] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bulletin of the London Mathematical Society* 11.3 (1979), pp. 308–339.

[Con+85] J. H. Conway et al. ATLAS of finite groups: maximal subgroups and ordinary characters for simple groups (1985).

[CH95] H. Cuypers and J. I. Hall. The 3-transposition groups with trivial center. *Journal of Algebra* 178 (1995), pp. 149–193.

[Fab21] A. Faber. "MAGIC074 Algebraic geometry". Lecture notes. 2021.

[FLM88] I. Frenkel, J. Lepowsky, and A. Meurman. Vertex operator algebras and the Monster. *Academic Press, Inc.* 134 (1988).

[Gal+16] A. Galt et al. Double axes and subalgebras of Monster type in Matsuo algebras. *Communications in Algebra* 49.10 (2016), pp. 4208–4248.

[Gor+23] I. B. Gorshkov et al. Automorphism groups of axial algebras (2023). arXiv: 2311.18538 [math.RA].

[Gri76] R. L. Griess. "The structure of the Monster simple group". *Proceedings of the conference on finite groups*. Ed. by William R. Scott and Fletcher Gross. Academic Press, 1976, pp. 113–118.

[Gri82] R. L. Griess. The friendly giant. *Inventiones mathematicae* 69 (1982), pp. 1–102.

[HRS15a] J. I. Hall, F. Rehren, and S. Shpectorov. Primitive axial algebras of Jordan type. *Journal of Algebra* 437 (2015), pp. 79–115.

[HRS15b] J. I. Hall, F. Rehren, and S. Shpectorov. Universal axial algebras and a theorem of Sakuma. *Journal of Algebra* 421 (2015), pp. 394–424.

[HSS18a] J. I. Hall, Y. Segev, and S. Shpectorov. Miyamoto involutions in axial algebras of Jordan type half. *Israel Journal of Mathematics* 223 (2018), pp. 261–308.

[HSS18b] J. I. Hall, Y. Segev, and S. Shpectorov. On primitive axial algebras of Jordan type. *Bulletin of the Institute of Mathematics Academia Sinica NEW SERIES* 13.4 (2018), pp. 397–409.

[HS21] J. I. Hall and S. Shpectorov. The spectra of finite 3-transposition groups. *Arabian Journal of Mathematics* 10 (2021), pp. 611–638.

[Hum75] J.E. Humphreys. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer, 1975. ISBN: 9780387901084.

[Iva09] A. A. Ivanov. The Monster group and Majorana involutions. *Cambridge Tracts in Mathematics* 176 (2009), p. 252.

[KMS19] S. M. S. Khasraw, J. McInroy, and S. Shpectorov. On the structure of axial algebras. *Transactions of the American Mathematical Society* 373.3 (2019), pp. 2135–2156.

[Mat03] A. Matsuo. 3-Transposition groups of symplectic type and vertex operator algebras. *Journal of the Mathematical Society of Japan* 57 (2003).

[Miy96] M. Miyamoto. Griess algebras and conformal vectors in vertex operator algebras. *Journal of Algebra* 179 (1996), pp. 523–548.

[Sak07] S. Sakuma. 6-Transposition property of τ-involutions of vertex operator algebras. *International Mathematics Research Notices* 9 (2007).

[SZ93] M. G. Seitz and E. A. Zalesskii. On the minimal degrees of projective representations of the finite Chevalley groups. *Journal of Algebra* 158 (1993), pp. 233–243.

[Smi85] S. D. Smith. On the head characters of the Monster simple group. *Contemp. Math.* 45 (1985), pp. 303–313.

[Tho79] J. G. Thompson. Some numerology between the Fischer-Griess Monster and the elliptic modular function. *Bulletin of the London Mathematical Society* 11.3 (1979), pp. 352–353.

[Wag76] A. Wagner. The faithful linear representation of least degree of S_n and A_n over a field of characteristic 2. *Mathematische Zeitschrift* 151 (1976), pp. 127–138.