ORDINARY AND MODULAR REPRESENTATION THEORY OF TRUNCATED CURRENT LIE ALGEBRAS

by

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Abstract

In this thesis, we study the representation theory of truncated current Lie algebras associated to Lie algebras of reductive groups. After giving the necessary preliminaries, we begin by considering the representation theory of these Lie algebras in characteristic 0 by defining a generalisation of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ for reductive Lie algebras and using this to study the problem of computing composition multiplicities of Verma modules. In particular, we give an inductive procedure to compute these multiplicities in terms of the composition multiplicities of Verma modules for reductive Lie algebras, which are famously given by the Kazhdan–Lusztig polynomials. These results have been published in [7] and [8].

We then move on to consider the representation theory of truncated current Lie algebras in prime characteristic. Here, after proving some elementary structural results such as the classification of semisimple and nilpotent elements, we tackle three main problems. The first is on upper and lower bounds for the dimensions of simple modules; we give an upper bound on the dimensions of simple modules for all p-characters and a lower bound for certain p-characters. Then we classify simple modules for certain p-characters. Finally, we finish by computing the composition multiplicities for projective modules for the restricted enveloping algebras of truncated current Lie algebras, and show they can be given in terms of the composition multiplicities of baby Verma modules for the corresponding reductive Lie algebra. These results are the subject of the preprint [9].

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CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, we study the representation theory of a class of Lie algebras called *truncated current Lie algebras*, sometimes also known as *(generalised) Takiff Lie algebras.* These Lie algebras are of the form $\mathfrak{g}_m := \mathfrak{g} \otimes \Bbbk[t]/(t^{m+1})$ where $\mathfrak{g}$ is the Lie algebra of a reductive group over an algebraically closed field $\Bbbk$. They first appeared in work of Takiff [51] who showed that in the $m = 1$ case they have polynomial rings of symmetric invariants; these results were subsequently generalised to all m by Raïs–Tauvel in [45] and then more recently to the case of truncated multicurrent algebras by Macedo–Savage in [31].

The representation theory of truncated current Lie algebras in characteristic 0 was first studied in detail by Wilson in [56], in which the theory of highest weight modules for $\mathfrak{g}_m$ was developed. Subsequently, in the $\mathfrak{g} = \mathfrak{sl}_2$, $m = 1$ case several generalisations of the BGG category $\mathcal{O}$ were investigated by Mazorchuk–Söderberg in [34]; one of the main objects of study in this thesis is a generalisation of one of their definitions to all choices of $\mathfrak{g}$ and m. The $\mathfrak{g} = \mathfrak{sl}_2$, $m = 1$ case was further explored by Zhu [58] and [59] in which several families of simple modules were described.

On the other hand, the representation theory of truncated current Lie algebras in prime characteristic, and indeed of non-reductive Lie algebras in general, is less explored. There are many results in the reductive case that it is widely believed should hold more generally, for example the Kac–Weisfeiler conjectures made in [55] and subsequently generalised in [28]. Truncated current Lie algebras provide an excellent proving ground for such conjectures since they are sufficiently closely related to reductive Lie algebras that one can often exploit the reductive case to obtain information about the truncated current case.

Truncated current Lie algebras, and in particular the results on symmetric invariants, also have applications to the theory of vertex algebras. More specifically, in the problem of determining the centre of the critical level affine vertex algebra associated to $\mathfrak{g}$, the semiclassical limit of the centre decomposes as a direct limit of the algebras $S(\mathfrak{g}_m)^{\mathfrak{g}_m}$ discussed above (see [15, §3.4] and [37, §6]). There are also relationships to the theory of W-algebras; for example this result on the centre of the affine vertex algebra together with techniques involving affine W-algebras were used by Arakawa–Premet [2] to solve Vinberg's problem for centralisers. Finally, we also mention work of Kamgarpour [29] which discusses connections between truncated current Lie algebras and the geometric Langlands program.

1.1 Main results

We now discuss the main results of the thesis. Some of the notation used here will not be introduced formally until Chapter 2, although we will keep the use of such notation to a minimum. The characteristic 0 results were first published in a special case in a paper of the author [7] and then subsequently in full generality in joint work of the author and Lewis Topley [8]. The prime characteristic results can be found in the preprint [9], also joint work of the author and Lewis Topley.

In characteristic 0, our main focus is on generalising several results on the representation theory of Lie algebras of reductive groups to the truncated current case. In particular, we give a generalisation of the definition of Verma modules to truncated current Lie algebras, and aim to answer the question of their composition multiplicities. Fix $\mathfrak{g}$ the Lie algebra of a reductive algebraic group, $\mathfrak{h} \subseteq \mathfrak{g}$ some maximal torus, and $\mathfrak{b} \subseteq \mathfrak{g}$ a Borel subalgebra containing $\mathfrak{h}$. We consider the representation theory of the truncated current Lie algebra $\mathfrak{g}_m$ defined earlier. We first introduce in §3.1 a generalisation of the BGG category $\mathcal{O}$ to truncated current Lie algebras. We then decompose this category into direct summands denoted $\mathcal{O}^{(\mu)}$, where $\mu = (\mu_1, \dots, \mu_m) \in (\mathfrak{h}^*)^m$. By proving several equivalences between these $\mathcal{O}^{(\mu)}$ we are able to reduce to the problem of computing composition multiplicities of Verma modules to the $\mu_m = 0$ case. The first of these is a parabolic induction style result:

Theorem 1.1.1. *Let $\mu = (\mu_1, \dots, \mu_m) \in (\mathfrak{h}^*)^m$ be such that the coadjoint stabiliser $\mathfrak{g}^{\mu_m} = \mathfrak{l}$ is in standard Levi form. Then the categories $\mathcal{O}^{(\mu)}(\mathfrak{l}_m)$ and $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ are equivalent.*

This is proved in §3.2. The second equivalence is a generalisation of an equivalence via twisting functors for reductive Lie algebras described in §2.3.3, which will be the main focus of §3.3.

Theorem 1.1.2. *Let α be a simple root and $\mu \in \mathfrak{h}^*$ not orthogonal to α. The categories $\mathcal{O}^{(\mu)}$ and $\mathcal{O}^{(s_\alpha(\mu))}$ are equivalent.*

See Theorem 3.3.11 for a precise description of the functors inducing this equivalence of categories.

The equivalences given in Theorems 1.1.1 and 1.1.2 allow us to reduce the problem of computing composition multiplicities of Verma modules to the subcategory $\mathcal{O}^{(0)}$. Since Verma modules for truncated current Lie algebras are in general not finite length modules, we first make a short digression in §3.4.1 to give a satisfactory definition of composition multiplicities for infinite length modules in $\mathcal{O}$. We then compute the composition multiplicities of Verma modules in $\mathcal{O}^{(0)}$ via a direct calculation, which yields the following result:

Theorem 1.1.3. *The composition multiplicities of a given Verma module for $\mathfrak{g}_m$ can be given in terms of the composition multiplicities of certain Verma modules for $\mathfrak{l}_{m-1}$, where $\mathfrak{l} \subseteq \mathfrak{g}$ is some Levi factor of $\mathfrak{g}$.*

A precise formula can be found in Corollary 3.4.8. In particular, applying this result inductively the composition multiplicities of Verma modules for $\mathfrak{g}_m$ can be given in terms of those of Verma modules for some reductive Lie algebra $\mathfrak{l}$, which as we discuss in §2.4.3 are given by the Kazhdan–Lusztig polynomials.

In the prime characteristic chapter of the thesis, we begin in §4.1 by proving some basic results on semisimple and nilpotent elements in truncated current Lie algebras. We then prove various Morita equivalences between certain reduced enveloping algebras in §4.2. These results are vital to proving the results in the remainder of Chapter 4. In §4.3, we investigate the Kac–Weisfeiler conjectures described in §2.4.3 for truncated current Lie algebras. The two main results in this section are as follows:

Theorem 1.1.4. *The first Kac–Weisfeiler conjecture holds for $\mathfrak{g}_m$ where $\mathfrak{g}$ is the Lie algebra of a reductive algebraic group under the standard hypotheses.*

Proposition 1.1.5. *Let $\mathfrak{g}$ be the Lie algebra of a reductive algebraic group under the standard hypotheses. Then for $\mathfrak{g}_m$, we have:*

(i) The second Kac–Weisfeiler conjecture holds for all homogeneous p-characters.

(ii) The second Kac–Weisfeiler conjecture holds for all semisimple p-characters.

We also give a family of examples of Lie algebras which are not the Lie algebras of reductive algebraic groups where the second Kac–Weisfeiler conjecture holds for all p-characters.

Proposition 1.1.6. *The second Kac-Weisfeiler conjecture holds for $(\mathfrak{sl}_2)_m$ in characteristic greater than 2.*

In §4.4, we give a partial classification of the simple modules for truncated current Lie algebras. We deal in particular with the case where the p-character χ is homogeneous and its nilpotent part is in standard Levi form; see §4.4.1 for more detail on these assumptions and why they are necessary. Under these assumptions we obtain a complete classification of simple $U_\chi(\mathfrak{g}_m)$-modules.

Finally, in §4.5 we compute the Cartan invariants for the restricted enveloping algebra $U_0(\mathfrak{g}_m)$; that is, the composition multiplicities of the projective $U_0(\mathfrak{g}_m)$-modules. We obtain the following result (see Theorem 4.5.9 for a precise formula):

Theorem 1.1.7. *The Cartan invariants for $U_0(\mathfrak{g}_m)$ are given in terms of the composition multiplicities of certain $U_0(\mathfrak{g})$-modules, known as (restricted) baby Verma modules.*

CHAPTER 2

PRELIMINARIES

2.1 Notation and conventions

In this section, we introduce some basic definitions and notation that we will use throughout the thesis. Throughout, we work over an algebraically closed field $\Bbbk$. In §2.3 and Chapter 3 our field will have characteristic 0, while in §2.4 and Chapter 4 our field will have characteristic $p > 0$; elsewhere we make no assumptions on the characteristic unless explicitly stated.

We assume familiarity with basic notions from the theory of $\Bbbk$-algebras and modules such as the definition of Noetherian and Artinian modules, see for example [20]. Unless otherwise specified, all algebras in this thesis are unital but not necessarily commutative. In addition, we assume elementary definitions from category theory such as adjoint functors, left and right exact functors, and full subcategories; again these can be found in for example [20]. We will in particular often use the language of abelian categories and definitions such as projective and injective objects in abelian categories. We also assume basic notions from the theory of algebraic groups such as the definition of reductive and semisimple groups, see e.g. [49], and some very basic algebraic geometry.

We now define some module theoretic notation we will use throughout the thesis. Let M be a finite length module over an algebra A, and let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ be a composition series for M. Then for a simple A-module L, the composition multiplicity of L in M, denoted $[M : L]$ is the number of quotients M_i/M_{i-1} in the composition series that are isomorphic to L. Note that by the Jordan–Hölder Theorem $[M : L]$ is independent of the choice of composition series.

Let A be an algebra over a field $\Bbbk$, and let M, N be respectively right and left A-modules. Then the *tensor product* of M and N over A, denoted $M \otimes_A N$, is the vector space spanned by elements of the form $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations:

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$

$$(m \cdot r) \otimes n = m \otimes (r \cdot n).$$

Note that this is just a vector space over $\Bbbk$ with no inherent action of A. However for any $\Bbbk$-algebra B, a left B-action on M induces a left B-action on $M \otimes_A N$ via $b \cdot (m \otimes n) = (b \cdot m) \otimes n$. Similarly a right B-action on N induces a right B-action on $M \otimes_A N$ via $(m \otimes n) \cdot b = m \otimes (n \cdot b)$. Unadorned tensor products are always considered to be over $\Bbbk$.

We now define *Lie algebras*, certain examples of which will be the main objects of study in this thesis.

Definition 2.1.1. *A Lie algebra* $\mathfrak{g}$ *over* $\Bbbk$ *is a* $\Bbbk$*-vector space equipped with a bilinear map*

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the Lie bracket on $\mathfrak{g}$*, such that for all* $x, y, z \in \mathfrak{g}$ *we have:*

$$[x, y] = -[y, x]$$

and

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Example 2.1.2. The main classes of Lie algebras we consider in this thesis are all examples of Lie algebras of linear algebraic groups; these are often referred to as *algebraic* Lie algebras. Let G be a linear algebraic group over $\Bbbk$. Then $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G, can be described in the following way. Let $\mathrm{Der}(\Bbbk[G])$ be the set of derivations on the ring of functions of G. A derivation $D \in \mathrm{Der}(\Bbbk[G])$ is said to be *left-invariant* if $D(g \cdot f) = g \cdot D(f)$ for all $g \in G$ and $f \in \Bbbk[G]$. Here G acts on $\Bbbk[G]$ via left translation, i.e. $(g \cdot f)(y) = f(g^{-1}y)$. If D_1, D_2 are two left-invariant derivations, then the commutator $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is another left-invariant derivation, and the set $\mathfrak{g} = \mathrm{Lie}(G)$ of such derivations carries the structure of a Lie algebra. For example, if $G = \mathrm{GL}_n$ then $\mathrm{Lie}(G) = \mathfrak{gl}_n$, which can be viewed (under a suitable identification) as the Lie algebra consisting of all $n \times n$ matrices with Lie bracket given by $[A, B] = AB - BA$.

The two main classes of Lie algebras that will appear in this thesis are Lie algebras of reductive groups (see §2.2) and their corresponding truncated current Lie algebras (see §2.5). We will be particularly interested in studying modules over these Lie algebras, which are defined in the following way:

Definition 2.1.3. *Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Then a $\mathfrak{g}$-module is a $\Bbbk$-vector space M equipped with an action of $\mathfrak{g}$ such that for all $x, y \in \mathfrak{g}$ and $m \in M$ we have:*

$$x \cdot (y \cdot m) - y \cdot (x \cdot m) = [x, y] \cdot m$$

Just as one can define the group algebra of a group, one can define an associative algebra $U(\mathfrak{g})$ called the *universal enveloping algebra* of $\mathfrak{g}$. This is an associative algebra such that modules for the Lie algebra $\mathfrak{g}$ are equivalent to modules for the associative algebra $U(\mathfrak{g})$. It is defined in the following way:

Definition 2.1.4. *Let $T(\mathfrak{g})$ be the free tensor algebra on $\mathfrak{g}$. Then*

$$U(\mathfrak{g}) := T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] : x, y \in \mathfrak{g})$$

Finally, we note that if $\mathfrak{g}$ is a finite-dimensional Lie algebra (as all Lie algebras considered in this thesis will be) then $U(\mathfrak{g})$ is a Noetherian algebra. In particular, for a finite-dimensional Lie algebra $\mathfrak{g}$, all finitely generated $U(\mathfrak{g})$-modules are Noetherian.

Two important representations of $\mathfrak{g}$ are the *adjoint* and *coadjoint* representations. These are defined as follows:

- The adjoint module has underlying vector space $\mathfrak{g}$, with action given by $x \cdot y = [x, y]$ for all $x, y \in \mathfrak{g}$.
- The coadjoint module has underlying vector space $\mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, with action given by $(x \cdot \chi)(y) = -\chi([x, y])$ for all $x, y \in \mathfrak{g}$ and $\chi \in \mathfrak{g}^*$.

For most classes of the Lie algebras considered in this thesis these modules are isomorphic, although this need not be the case in general. We define the *centraliser* $\mathfrak{g}^x$ of an element $x \in \mathfrak{g}$ to be the Lie subalgebra $\mathfrak{g}^x := \{y \in \mathfrak{g} : y \cdot x = 0\} \subseteq \mathfrak{g}$. Analogously, the *coadjoint stabiliser* $\mathfrak{g}^\chi$ of $\chi \in \mathfrak{g}^*$ is the Lie subalgebra $\mathfrak{g}^\chi = \{y \in \mathfrak{g} : y \cdot \chi = 0\}$. We then define the *index* of $\mathfrak{g}$ to be $\mathrm{ind}(\mathfrak{g}) := \min_{\chi \in \mathfrak{g}^*} \dim \mathfrak{g}^\chi$.

Similarly, if G is an algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$, then G acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ via respectively the adjoint and coadjoint actions as follows, where we view $\mathfrak{g}$ as the set of left-invariant derivations of $\Bbbk[G]$:

- The adjoint module has underlying vector space $\mathfrak{g}$, with action given by $(g \cdot D)(f) = D(g \cdot f)$ for all $g \in G$, $D \in \mathfrak{g} \subseteq \mathrm{Der}(\Bbbk[G])$, and $f \in \Bbbk[G]$.
- The coadjoint module has underlying vector space $\mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, with action given by $(g \cdot \chi)(D) = \chi(g^{-1} \cdot D)$ for all $g \in G$, $y \in \mathfrak{g} \subseteq \mathrm{Der}(\Bbbk[G])$, and $\chi \in \mathfrak{g}^*$; here G acts on $\mathfrak{g}$ via the adjoint action.

The adjoint actions of both G and $\mathfrak{g} = \mathrm{Lie}(G)$ on $\mathfrak{g}$ then extend naturally to actions on both $U(\mathfrak{g})$ and $S(\mathfrak{g})$. We will sometimes refer $U(\mathfrak{g})$ equipped with the adjoint action of $\mathfrak{g}$ as giving $U(\mathfrak{g})$ the structure of an 'ad($\mathfrak{g}$)-module' to distinguish the adjoint action from the action of $\mathfrak{g}$ on $U(\mathfrak{g})$ by left or right multiplication.

2.2 Reductive groups and their Lie algebras

In this thesis, two classes of Lie algebras we will be concerned with are Lie algebras of semisimple groups and Lie algebras of reductive groups; see e.g. [26, Chapter II.1] for a slightly more detailed and general introduction to much of the material of this section. An algebraic group G is said to be *semisimple* if its solvable radical $R(G)$ is trivial and *reductive* if its unipotent radical $R_u(G)$ is trivial. Since $R_u(G) \subseteq R(G)$ all semisimple groups are automatically reductive.

Example 2.2.1. Some examples of reductive groups are as follows:

- Any simple algebraic group is semisimple; this includes for example $\mathrm{SL}_n(\Bbbk)$, $\mathrm{SO}_n(\Bbbk)$, and $\mathrm{Sp}_{2n}(\Bbbk)$ (see [32, Table 9.2] for a full classification of the simple algebraic groups).
- The group GL_n is reductive, but not semisimple.
- The direct product of copies of $\mathbb{G}_m$, the multiplicative group of the field $\Bbbk$, is reductive. If H is an algebraic group isomorphic to such a direct product of copies of $\mathbb{G}_m$ we call H a *torus*.

We now fix some notation which we shall use throughout the thesis. Fix a reductive group G, let $\mathfrak{g} = \mathrm{Lie}(\mathfrak{g})$, and choose some maximal (with respect to inclusion) torus H inside G. Let $\mathfrak{h} = \mathrm{Lie}(H) \subseteq \mathfrak{g}$. Then $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ whose adjoint action on $\mathfrak{g}$ is semisimple; we call such a subalgebra a *toral subalgebra* or just a *torus* of $\mathfrak{g}$ (in positive characteristic we also insist that a torus is closed under the map $(\bullet)^{[p]}$ that we will discuss in §2.4.1). In fact, since H is a maximal torus of G, it follows that $\mathfrak{h}$ is a maximal torus of $\mathfrak{g}$. Under the adjoint action of G all maximal tori in G are conjugate, as are all maximal tori of $\mathfrak{g}$.

We write $X^*(H)$ for the character group of H, the group of homomorphisms of algebraic groups $H \to \mathbb{G}_m$. The Lie algebra $\mathfrak{g}$ is then graded by the action of H as $\mathfrak{g} = \sum_{\alpha \in X^*(H)} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : h \cdot x = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$; here H acts on $\mathfrak{g}$ via the adjoint action. We write $\Phi = \{\alpha \in X^*(T) : \mathfrak{g}_\alpha \neq 0\} \setminus \{0\}$. This root decomposition of $\mathfrak{g}$ has the properties that $\mathfrak{g}_0 = \mathfrak{h}$ and $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Phi$.

Remark 2.2.2. Given an element $\alpha \in X^*(H)$, one can differentiate to obtain an element $d\alpha \in \mathfrak{h}^*$. If $x \in \mathfrak{g}_\alpha$, then one can easily show that x also lies in the $d\alpha$ eigenspace for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

In characteristic 0, the homomorphism of abelian groups $d : X^*(H) \to \mathfrak{h}^*$ is injective, so in Chapter 3 we will usually abuse notation and write α instead of $d\alpha$. We then view Φ as a subset of $\mathfrak{h}^*$ rather than of $X^*(H)$. On the other hand, in prime characteristic the map d is no longer injective, so in Chapter 4 we will always distinguish between elements of $X^*(H)$ and elements of $\mathfrak{h}^*$.

Now we make a choice of positive roots $\Phi^+ \subseteq \Phi$ which gives a corresponding choice of simple roots $\Delta \subseteq \Phi$. In everything that follows we assume that we have made some fixed choice of maximal torus $\mathfrak{h}$, positive roots Φ^+, and simple roots Δ. This choice of positive roots gives rise to a partial order on $\mathfrak{h}^*$ given by $\lambda \leq \mu$ if $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Phi^+$. We now fix a basis for $\mathfrak{g}$ as follows. For each $\alpha \in \Phi^+$, we choose some $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and some $f_\alpha \in \mathfrak{g}_{-\alpha} \setminus \{0\}$; we also sometimes adopt the convention that $e_{-\alpha} = f_\alpha$. We then set $h_\alpha = [e_\alpha, f_\alpha]$ for all $\alpha \in \Phi^+$, and by rescaling e_α and f_α if necessary we may assume that $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, f_\alpha] = -2f_\alpha$. The set $\{e_\alpha : \alpha \in \Phi^+\} \cup \{h_\alpha : \alpha \in \Delta\} \cup \{f_\alpha : \alpha \in \Phi^+\}$ is then a basis for $\mathfrak{g}$ provided G is semisimple. If G is reductive then we extend this to a basis for $\mathfrak{g}$ by adding additional basis elements to the set $\{h_\alpha : \alpha \in \Delta\}$ to extend it to a basis of $\mathfrak{h}$. In characteristic 0 these additional elements can always be chosen to lie in $\mathfrak{z}(\mathfrak{g})$, the centre of $\mathfrak{g}$.

The choice of positive roots yields a *triangular decomposition* of $\mathfrak{g}$ in the following way: let $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and let $\mathfrak{n}^- = \bigoplus_{\alpha \in \Phi \setminus \Phi^+} \mathfrak{g}_\alpha$. Then by the previous paragraph $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. We also write $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$ for the Borel subalgebra of $\mathfrak{g}$ corresponding to this choice of torus and positive roots.

The *Weyl group* W of G with respect to a choice of torus H is defined to be $W :=$

$N(H)/H$ where $N(H)$ denotes the normaliser of H in G. Since H is self-centralising, $w \in W$ acts on H by choosing some lift $\overline{w} \in N(H)$ of w and acting by conjugation. This then induces actions on $X^*(H)$, $\mathfrak{h} = \mathrm{Lie}(H)$, and $\mathfrak{h}^*$ in a natural way.

For many purposes, a more useful choice of action of w on $\mathfrak{h}^*$ is actually a shifted version of this action, denoted $\bullet$, defined as follows. Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then $w \bullet \lambda := w(\lambda + \rho) - \rho$, where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. In the characteristic 0 case, there is also a W-invariant inner product on $\mathfrak{h}^*$ which we denote $(\cdot, \cdot)$; see for example [23, §0.2] or [22, §9]. For $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Phi$ we then define $\langle \lambda, \alpha \rangle = (\lambda, \alpha^\vee)$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. This bracket $\langle \cdot, \cdot \rangle$ plays an important role in the representation theory of $\mathfrak{g}$; see for example [23, Theorem 2.5] and §2.3.3 below.

Example 2.2.3. Let $G = \mathrm{SL}_n$ and assume the characteristic of $\Bbbk$ does not divide n (we will discuss why we make this assumption in more detail in §2.4.2). Then $\mathfrak{g} = \mathfrak{sl}_n$, the Lie algebra of traceless $n \times n$ matrices. The natural choice of torus $H \subseteq \mathrm{SL}_n$ is $H = \{\text{determinant 1 diagonal } n \times n \text{ matrices}\}$, and the corresponding maximal torus of $\mathfrak{g}$ is $\mathfrak{h} = \{\text{traceless diagonal } n \times n \text{ matrices}\}$. The Weyl group W is isomorphic to the symmetric group on n elements. With respect to this choice of torus the root system Φ of $\mathfrak{g}$, when viewed as lying in $\mathfrak{h}^*$, is given by $\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i, j \leq n, i \neq j\}$, where $\epsilon_i \in \mathfrak{h}^*$ is given by

$$\epsilon_i(\mathrm{diag}(x_1, x_2, \ldots, x_n)) = x_i$$

For $\epsilon_i - \epsilon_j \in \Phi$, the corresponding root space $\mathfrak{g}_{\epsilon_i - \epsilon_j}$ is spanned by the elementary matrix E_{ij}. There is a natural choice of positive roots $\Phi^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}$ and simple roots $\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\}$. The triangular decomposition of $\mathfrak{sl}_n$ in this case is given by:

$$\mathfrak{n}^+ = \begin{pmatrix} 0 & * & \cdots & * & * \\ 0 & 0 & & & * \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & * \\ 0 & & \cdots & 0 & 0 \end{pmatrix}, \qquad \mathfrak{h} = \begin{pmatrix} * & 0 & \cdots & 0 & 0 \\ 0 & * & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & * & 0 \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}, \qquad \mathfrak{n}^- = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ * & 0 & & & 0 \\ \vdots & & \ddots & & \vdots \\ * & & & 0 & 0 \\ * & * & \cdots & * & 0 \end{pmatrix}$$

2.3 Category $\mathcal{O}$ for reductive Lie algebras

2.3.1 Definition

We now proceed to give a brief overview of the theory of category $\mathcal{O}$ for Lie algebras of reductive groups, following [23]. Let G be a reductive group over $\mathbb{C}$, let $\mathfrak{g} = \operatorname{Lie} G$, and retain all notation from the previous section. The category $U(\mathfrak{g})$-mod of all $U(\mathfrak{g})$-modules is difficult to study since it does not display certain nice homological properties. On the other hand, if we only consider finite-dimensional representations of $\mathfrak{g}$ the problem becomes much easier. For example, if G is semisimple, then the finite-dimensional representations of $\mathfrak{g}$ are always semisimple and the simple modules are easy to classify (see for example [23, Theorem 1.6]); the general reductive case is not much more difficult. It is therefore natural to ask if there is some category of $U(\mathfrak{g})$-modules which includes some of the infinite-dimensional modules but is still well-behaved. Such a category was introduced by Bernstein–Gelfand–Gelfand in the 1970s. This category, known as the BGG category $\mathcal{O}$, is defined as follows:

Definition 2.3.1. *The category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-mod whose objects are the modules M satisfying the following conditions:*

($\mathcal{O}$1) *M is finitely generated.*

($\mathcal{O}$2) *The Lie subalgebra $\mathfrak{h}$ acts semisimply on M.*

($\mathcal{O}3$) *The Lie subalgebra* $\mathfrak{n}^+$ *acts locally nilpotently on* M.

We now briefly list some basic properties of this category, all of which can be found for example in [23, §1-3].

- The category $\mathcal{O}$ is abelian.
- The category $\mathcal{O}$ is Noetherian, and Artinian, i.e. every object in $\mathcal{O}$ is both Noetherian and Artinian. In particular, all modules in $\mathcal{O}$ have finite length.
- The category $\mathcal{O}$ is closed under taking submodules, quotients, and finite direct sums.
- Any indecomposable module in $\mathcal{O}$ admits a generalised central character χ. Hence there is a direct sum decomposition $\mathcal{O} = \bigoplus_\chi \mathcal{O}^\chi$, where $\mathcal{O}^\chi$ is the full subcategory of $\mathcal{O}$ whose objects are the modules with generalised central character χ.
- Each category $\mathcal{O}^\chi$ contains only finitely many simple modules.
- Each category $\mathcal{O}^\chi$ contains enough projectives and injectives, i.e. every $M \in \mathcal{O}^\chi$ is (isomorphic to) a quotient of a projective object in $\mathcal{O}^\chi$ and a submodule of an injective object in $\mathcal{O}^\chi$.

The simple modules in the BGG category $\mathcal{O}$ can be constructed as follows. First, we define a class of modules called the *Verma modules*. Let $\lambda \in \mathfrak{h}^*$. Then the Verma module of weight λ, denoted $M(\lambda)$, is defined by:

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda$ is the one-dimensional $U(\mathfrak{b})$-module where $h \in \mathfrak{h}$ acts by $\lambda(h)$ and $\mathfrak{n}^+$ acts by 0. Furthermore, we have (see for example [23, Theorem 1.2(f) & Theorem 1.3]) the following theorem:

Theorem 2.3.2. *Each* $M(\lambda)$ *has a unique maximal submodule and hence a unique simple quotient. Write* $L(\lambda)$ *for this unique simple quotient. Then the* $L(\lambda)$ *are pairwise non-isomorphic, and any simple module in* $\mathcal{O}$ *is isomorphic to some* $L(\lambda)$.

2.3.2 The Kazhdan–Lusztig conjecture

We now continue from the previous section to describe an important result in the theory of category $\mathcal{O}$, the famous Kazhdan–Lusztig conjecture. This was conjectured by Kazhdan–Lusztig in 1979 [30] and proved independently by Beilinson–Bernstein [5] and Brylinski–Kashiwara [6] in 1981.

To motivate this result, we first define the notion of the formal character of a module. This will use the fact that by ($\mathcal{O}2$), any module $M \in \mathcal{O}$ is (as a vector space) a direct sum $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ where $M^\lambda = \{v \in M : h \cdot v = \lambda(h) \text{ for all } h \in \mathfrak{h}\}$, the λ weight space of M. These weight spaces have the following properties, see for example [23, §1.1]:

Lemma 2.3.3. *Let $M \in \mathcal{O}$. Then*

(i) Each M^λ is finite-dimensional.

(ii) If M is indecomposable then the set $\{\lambda \in \mathfrak{h}^* : M^\lambda \neq 0\}$ is bounded above with respect to the partial order on $\mathfrak{h}^*$ described earlier.

We can now give the following definition:

Definition 2.3.4. *The formal character of M is the function*

$$\begin{aligned} \operatorname{ch} M : \mathfrak{h}^* &\to \mathbb{Z}_{\geq 0} \\ \lambda &\mapsto \dim(M^\lambda) \end{aligned}$$

These formal characters are related to composition multiplicities via the following lemma:

Lemma 2.3.5. *Let $M \in \mathcal{O}$. Then there exist unique non-negative integers k_μ such that $\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}^*} k_\mu \operatorname{ch} L(\mu)$. Furthermore, $k_\mu = [M : L(\mu)]$ for all $\mu \in \mathfrak{h}^*$.*

Proof. To show existence, observe that for $N \subseteq M \in \mathcal{O}$ we have $\operatorname{ch} M = \operatorname{ch} N + \operatorname{ch}(M/N)$. Applying this to a composition series $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$, we see that

$\operatorname{ch} M = \sum_{i=1}^{k} \operatorname{ch}(M_i/M_{i-1})$ so the character of M is indeed the sum of characters of simple modules. Uniqueness follows from the fact that $L(\mu)^{\lambda} = 0$ unless $\lambda \leq \mu$ and the fact that $\dim(L(\mu)^{\mu}) = 1$. □

One motivation for the Kazhdan–Lusztig conjecture was the problem of computing the formal characters of simple modules in category $\mathcal{O}$. In particular, it is easy to show that $[M(\lambda) : L(\mu)] = 0$ unless $\lambda \geq \mu$ and that $[M(\lambda) : L(\lambda)] = 1$, so by inverting a suitable matrix one obtains an expression for $\operatorname{ch} L(\mu)$ in terms of the $[M(\lambda) : L(\mu)]$ and the $\operatorname{ch} M(\lambda)$. Since the characters of Verma modules are easy to compute, to find the characters of simple modules it suffices to compute the composition multiplicities $[M(\lambda) : L(\mu)]$ of the Verma modules. This motivated the following result, known as the Kazhdan–Lusztig conjecture and proved by Beilinson–Bernstein [5] and Brylinski–Kashiwara [6].

Theorem 2.3.6. *The composition multiplicities $[M(\lambda) : L(\mu)]$ are given by the values of certain polynomials, called Kazhdan–Lusztig polynomials, at 1.*

For a detailed constriction of the Kazhdan–Lusztig polynomials via the Hecke algebra we refer the reader to [23, §8.2]. We also remark that the precise statement only applies to composition multiplicities in the principal block $\mathcal{O}^0$; however there exist various category equivalences that allow us to reduce to this case. For our purposes, the precise definition of the Kazhdan–Lusztig polynomials is not so important; what is important is that these composition multiplicities $[M(\lambda) : L(\mu)]$ are in principle computable and are given by these polynomials.

2.3.3 Twisting functors on category $\mathcal{O}$

There are several types of equivalences between direct summands of the BGG category $\mathcal{O}$. The most relevant such family of equivalences for the purposes of this thesis are the *twisting functors*. These were first defined by Arkipov in [4]; see also work of Andersen–Stroppel [1] and the overview given in [10, §2]. We begin by fixing a simple root α, and define a $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule in the following way. Consider the localisation U_α of $U(\mathfrak{g})$ with respect to the multiplicative subset generated by f_α. This is in particular a $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule, and contains $U(\mathfrak{g})$ as a sub-bimodule in a natural way. We also define an automorphism ϕ_α of $\mathfrak{g}$ by lifting $s_\alpha \in W \cong N(T)/T$ to an element of $N(T) \subseteq G$ and conjugating by this element. The bimodule S_α we are interested in is then given by $S_\alpha := \phi_\alpha(U_\alpha/U(\mathfrak{g}))$, where for a given $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule M we let $\phi_\alpha(M)$ denote the bimodule obtained by twisting the left action (but not the right action) on M by ϕ_α. There is then a right exact functor:

$$\begin{aligned} T_\alpha : \; U(\mathfrak{g})\text{-mod} &\longrightarrow U(\mathfrak{g})\text{-mod} \\ M &\longmapsto S_\alpha \otimes_{U(\mathfrak{g})} M \end{aligned}$$

In fact one can define functors T_w for each $w \in W$ by declaring that for $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ a minimal length expression for w, the functor $T_w := T_{s_{\alpha_1}} \circ T_{s_{\alpha_2}} \circ \cdots \circ T_{s_{\alpha_k}}$. Alternatively, there is a more sophisticated (but equivalent) way to define T_w directly; see for example [1, §2].

Although these functors T_w are defined as functors $U(\mathfrak{g})\text{-mod} \to U(\mathfrak{g})\text{-mod}$, they in fact descend to equivalences between certain direct summands of $\mathcal{O}$. To make this precise, we introduce the notation of a *block* of category $\mathcal{O}$. These are subcategories of the categories $\mathcal{O}^\chi$ defined earlier. Define an equivalence relation $\sim$ on $\mathfrak{h}^*$ by setting $\lambda \sim \mu$ if there exists a non-split extension of $L(\lambda)$ by $L(\mu)$ and extending this to an equivalence relation. The block of $\mathcal{O}$ corresponding to an equivalence class $[\lambda]$ under this relation is the full subcategory of $\mathcal{O}$ containing precisely the modules whose composition factors are all of the form $L(\nu)$ for $\nu \in [\lambda]$.

Definition 2.3.7. *Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ and define*

$$\mathfrak{h}^*_{\mathrm{dom}} := \{\lambda \in \mathfrak{h}^* : \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{<0} \textit{ for all } \alpha \in \Phi^+\},$$

the set of dominant weights in $\mathfrak{h}^*$. Then for $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$, we define $\mathcal{O}_\lambda$ to be the block of $\mathcal{O}$ corresponding to the equivalence class $[\lambda]$; as λ ranges over $\mathfrak{h}^*_{\mathrm{dom}}$ these $\mathcal{O}_\lambda$ form a complete irredundant list of blocks of $\mathcal{O}$.

The category $\mathcal{O}$ decomposes as a direct sum $\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*_{\mathrm{dom}}} \mathcal{O}_\lambda$; see for example [23, §1.13]. Twisting functors then yields equivalences between these blocks by the following result; a proof can be found in [10, Theorem 2.1].

Theorem 2.3.8. *Let $\mu \in \mathfrak{h}^*_{\mathrm{dom}}$ and $\alpha \in \Delta$ be such that $\langle \mu, \alpha^\vee \rangle \notin \mathbb{Z}$. Then $T_{s_\alpha} : \mathcal{O}_\mu \to \mathcal{O}_{s_\alpha \mu}$ is an equivalence of categories.*

2.4 Restricted Lie algebras

2.4.1 Definition

We now move on to discuss some preliminaries for the chapter of the thesis dealing with the representation theory of truncated current Lie algebras over fields of prime characteristic. For §2.4-§2.5 and the entirety of Chapter 4, we fix $\Bbbk$ an algebraically closed field of characteristic $p > 0$; all Lie algebras, associative algebras, vector spaces etc. will be over $\Bbbk$. The first important concept we will need is that of a *restricted Lie algebra*. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Then a p^{th} *power map* on $\mathfrak{g}$ is a map

$$(\bullet)^{[p]} : \mathfrak{g} \to \mathfrak{g}$$

such that the map $\xi : \mathfrak{g} \to U(\mathfrak{g})$ given by $\xi(x) = x^p - x^{[p]}$ satisfies:

$$\xi(x) \in Z(\mathfrak{g})$$
$$\xi(x + y) = \xi(x) + \xi(y)$$
$$\xi(\alpha x) = \alpha^p \xi(x)$$

for any $x, y \in \mathfrak{g}$ and $\alpha \in \Bbbk$. A Lie algebra $\mathfrak{g}$ equipped with such a map is called a restricted Lie algebra. In addition, we write $Z_p(\mathfrak{g}) \subset Z(\mathfrak{g})$ for the image of ξ and refer to it as the *p-centre* of $U(\mathfrak{g})$.

Example 2.4.1. Let $\mathfrak{g}$ be the one-dimensional Lie algebra, and let $0 \neq x \in \mathfrak{g}$. Then for any $\lambda \in \Bbbk$, there is a p^{th} power map on $\mathfrak{g}$ given by $(\alpha x)^{[p]} = \alpha^p \lambda x$.

Example 2.4.2. Let G be an algebraic group over $\Bbbk$. Then there is a natural restricted structure on $\mathfrak{g} = \mathrm{Lie}(G)$ given by viewing $\mathfrak{g}$ as invariant derivations on G (as described in §2.1) and taking $x^{[p]}$ to be the p^{th} power of x as a derivation (which in characteristic p will again be an invariant derivation). In the case $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n$ or Sp_{2n}, the p^{th} power map on $\mathfrak{g}$ obtained in this way is the same as the map given by taking the p^{th} matrix power, where we view $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$ as subalgebras of $\mathfrak{gl}_n$ or $\mathfrak{gl}_{2n}$ in the usual way.

Any p^{th} power map $(\bullet)^{[p]}$ allows us to define a notion of semisimple and nilpotent elements for any restricted Lie algebra in the following way:

Definition 2.4.3. *Let $\mathfrak{g}$ be a restricted Lie algebra. Then an element $x \in \mathfrak{g}$ is:*

- *Semisimple if* $x \in \mathrm{span}\{x^{[p]^i} : i \geq 1\}$
- *Nilpotent if* $x^{[p]^i} = 0$ *for sufficiently large* i.

We can then give a notion of 'Jordan decomposition' in any restricted Lie algebra via the following result.

Lemma 2.4.4. *Let $\mathfrak{g}$ be a restricted Lie algebra and let $x \in \mathfrak{g}$. Then there exist unique $s, n \in \mathfrak{g}$ such that s is semisimple, n is nilpotent, $x = s + n$, and $[s, n] = 0$. Furthermore, $\mathfrak{g}^x = (\mathfrak{g}^s)^n$.*

Proof. Such s and n exist by [50, Theorem 2.3.5]. To see that $\mathfrak{g}^x = (\mathfrak{g}^s)^n$, we use the general fact that there always exists a faithful restricted representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ and by the classical Jordan decomposition we have $\mathfrak{gl}(V)^x = (\mathfrak{gl}(V)^s)^n$. The result then follows by taking the intersection of both sides with $\mathfrak{g}$. $\square$

2.4.2 The standard hypotheses for reductive groups

In small characteristics, many families of Lie algebras of reductive groups exhibit very different behaviour from the characteristic 0 or large characteristic cases. For example, in general $\mathfrak{sl}_n$ is a simple Lie algebra but if p divides n then it has a one-dimensional centre. To exclude these cases, in the prime characteristic portion of this thesis we impose the following conditions on our reductive groups G:

(S1) The derived subgroup of G is simply connected.

(S2) The characteristic p is 'good' for each irreducible component of the root system of G. This is defined in the following way: make a choice of simple roots $\{\alpha_1, \dots, \alpha_n\}$, and write the highest root $\alpha^\#$ as a sum of simple roots $\alpha^\# = \sum k_i \alpha_i$. Then p is good for an irreducible root system if it does not divide any k_i.

(S3) There is a non-degenerate G-invariant bilinear form κ on $\mathfrak{g} = \mathrm{Lie}(G)$.

These conditions are called the *standard hypotheses* and we will refer to a reductive group G satisfying (S1) – (S3) as a *standard reductive group*; we briefly mention that this definition is slightly stronger than the one sometimes used elsewhere in the literature (see the remarks in [36, §4] for a more detailed discussion of the various notions of 'standard reductive groups').

Remark 2.4.5. The condition (S3) ensures the existence of an isomorphism $f : \mathfrak{g} \to \mathfrak{g}^*$ between the adjoint and coadjoint modules given by $(f(x))(y) = \kappa(x, y)$

Example 2.4.6. The 'bad' primes for each irreducible root system are as follows:

- Type A_n: None.
- Types B_n, C_n, D_n: $p = 2$.
- Types E_6, E_7, F_4, G_2: $p = 2, 3$.
- Type E_8: $p = 2, 3, 5$.

Furthermore, if G is almost simple and (S1) and (S2) hold, then in addition (S3) holds unless G is of type A_n for n such that p divides $n+1$.

In this thesis we will primarily be concerned with the Lie algebras of standard reductive groups as opposed to the groups themselves. These Lie algebras can be classified in the following way (see for example [27, §2.9] or [44, §2.1]):

Proposition 2.4.7. *Let G be a standard reductive group and let $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\mathfrak{g}$ is a direct sum of Lie algebras of the following forms:*

- *Simple Lie algebras of types B, C, D, E, F, G in good characteristic.*
- *$\mathfrak{sl}_n$ where p does not divide n.*
- *$\mathfrak{gl}_n$ where p divides n.*
- *Restricted tori, i.e. abelian Lie algebras equipped with a bijective p-th power map.*

2.4.3 Representation theory and the Kac–Weisfeiler conjectures

We now discuss a few relevant features of the representation theory of restricted Lie algebras which will use later in Chapter 4. The first of these is the following result, which first appeared in [13, Theorem 5.1] and illustrates some of the differences between the characteristic 0 and prime characteristic settings.

Proposition 2.4.8. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field of characteristic $p > 0$. Then the simple $U(\mathfrak{g})$-modules are all finite-dimensional, and the dimensions of the simple modules are bounded above.*

Remark 2.4.9. In the case $\mathfrak{g}$ is restricted, this result will follow from the discussion below; however it is true in general even without the restricted hypothesis.

Remark 2.4.10. This result is in contrast to the characteristic 0 case, where any non-abelian Lie algebra has infinite-dimensional simple modules. For example, if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

then for most values of λ the Verma module $M(\lambda)$ described in §2.3.1 is an infinite-dimensional simple module. Even if we restrict our attention to finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules, there is (up to isomorphism) exactly one simple module of any positive integer dimension and hence the dimensions of finite dimensional simple modules are not bounded.

Now, if $\mathfrak{g}$ is a restricted Lie algebra and M is a simple $U(\mathfrak{g})$-module, then for all $x \in \mathfrak{g}$ the element $x^p - x^{[p]}$ acts on M by scalars since it lies in $Z(\mathfrak{g})$. Using the axioms of a p^{th} power map, one can deduce that in fact there exists $\chi \in \mathfrak{g}^*$ such that $x^p - x^{[p]}$ acts on M by $\chi(x)^p$ for all $x \in \mathfrak{g}$; we call this χ the *p-character* of the simple module M. Hence M can also be viewed as a simple module for the algebra $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p)$, which is a finite-dimensional algebra of dimension $p^{\dim(\mathfrak{g})}$ referred to as a *reduced enveloping algebra.*

By Proposition 2.4.8 the simple modules for a restricted Lie algebra are all finite-dimensional of dimension less than some fixed upper bound. It is not hard to show (see e.g. [24, Remark 2.8]) that this upper bound is at most $p^{\frac{1}{2}\dim(\mathfrak{g})}$. On the other hand, finding a precise value for this upper bound is much more difficult. One can also ask more generally what the possible dimensions of simple modules are. These questions led Kac–Weisfeiler to make the follow conjectures in [55].

Conjecture 2.4.11 (KW1)**.** *Let $\mathfrak{g}$ be a restricted Lie algebra. Then the maximal dimension of a simple $U(\mathfrak{g})$-module is $p^{\frac{1}{2}(\dim(\mathfrak{g})-\mathrm{ind}(\mathfrak{g}))}$.*

Theorem 2.4.12 ([40], KW2)**.** *Let G be a standard reductive group, and let $\mathfrak{g} = \mathrm{Lie}(G)$. Then for any p-character χ and any simple $U_\chi(\mathfrak{g})$-module M, we have that $p^{\frac{1}{2}(\dim(\mathfrak{g})-\dim(\mathfrak{g}^\chi))}$ divides* $\dim M$*.*

Remark 2.4.13. The statement of (KW1) still makes sense even if $\mathfrak{g}$ is not restricted; however there exists a counterexample in this case (see [54]).

Remark 2.4.14. Although (KW2) was proved by Premet in [40], Kac later conjectured in a review of Premet's paper [28] that it holds for any algebraic group G under certain mild conditions on the characteristic.

Outside of the case where G is a standard reductive group, very little is known about classes of groups for which (KW2) holds. On the other hand, there are several classes of Lie algebras for which (KW1) is known to hold; these include:

- Any restricted completely solvable Lie algebra [55].
- Any restricted Lie algebra $\mathfrak{g}$ such that there exists $\chi \in \mathfrak{g}^*$ with $\mathfrak{g}^\chi$ a torus [43]. In particular, this includes all Lie algebras of standard reductive groups.
- Any centraliser $(\mathfrak{gl}_n)^e$ of a nilpotent element e in $\mathfrak{gl}_n$ [53, Theorem 4].

In addition, a result of Martin–Stewart–Topley [33, Theorem 1.1] states that for any fixed n, there exists p_0 such that (KW1) holds for any restricted subalgebra of $\mathfrak{gl}_n$ over an algebraically closed field of characteristic greater than p_0 (although their methods do not yield an explicit bound on p_0).

2.5 Truncated current groups and Lie algebras

The main class of Lie algebras we study in this thesis are the *truncated current Lie algebras*, which are defined as follows:

Definition 2.5.1. *Let $\mathfrak{g}$ be a Lie algebra. Then the m^{th} truncated Lie algebra on $\mathfrak{g}$, denoted $\mathfrak{g}_m$, is:*

$$\mathfrak{g}_m := \mathfrak{g} \otimes \mathbb{k}[t]/(t^{m+1})$$

where the Lie bracket on $\mathfrak{g}_m$ is given by

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j}.$$

If G is an algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$, then the following result (see for example [38, Appendix]) shows that $\mathfrak{g}_m$ is also the Lie algebra of an algebraic group G_m called the *group of m-jets on G*. It also provides an alternative characterisation of truncated current Lie algebras via jet schemes; for our purposes we usually only need the existence of the group G_m, so we will not discuss the theory of jet schemes in detail here.

Theorem 2.5.2. *Let G be an algebraic group and let $\mathfrak{g} = \mathrm{Lie}(G)$. Write G_m for the group of m-jets on G. Then $\mathfrak{g}_m = \mathrm{Lie}(G_m)$.*

Remark 2.5.3. In the case $m = 1$, the group G_1 can be viewed as the semidirect product $G \ltimes \mathfrak{g}$, where G acts on $\mathfrak{g}$ via the adjoint representation and we view $\mathfrak{g}$ as a group under addition. Similarly, the Lie algebra $\mathfrak{g}_1$ can be viewed as the semidirect product $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ab}}$ where $\mathfrak{g}_{\mathrm{ab}}$ is the Lie algebra with the same underlying vector space as $\mathfrak{g}$ but the abelian Lie bracket. In the case $m > 1$, one can still view G_m as $G \ltimes \mathfrak{g}^m$ where G acts diagonally on $\mathfrak{g}^m$ via the adjoint action, but here the group structure one endows $\mathfrak{g}^m$ with is more complicated then just addition.

We will be interested in truncated current Lie algebras in the case $\mathfrak{g}$ is the Lie algebra of a reductive group (under the standard hypotheses when we work in prime characteristic). Fix a reductive group G with Lie algebra $\mathfrak{g}$ and recall the notation of §2.2. We now define some notation used throughout the thesis.

The associative algebra $\Bbbk[t]/(t^{m+1})$ has a natural grading given by letting t^i lie in degree i, and this grading induces a grading on the Lie algebra $\mathfrak{g}_m$ by letting $x \otimes t^i$ lie in degree i. We write $\mathfrak{g}_m^{(i)}$ for the i^{th} graded piece of $\mathfrak{g}_m$, and similarly write $\mathfrak{g}_m^{(\geq i)}$ and $\mathfrak{g}_m^{(<i)}$ for the sums of the graded pieces of degrees $\geq i$ and $< i$ respectively.

When writing elements of $\mathfrak{g}_m$ of the form $x \otimes t^i$, we will often omit the tensor product sign and simply write xt^i. Additionally, for $\alpha \in \Phi^+$ and $0 \leq i \leq m$, we set:

$$e_{\alpha,i} := e_\alpha t^i$$

$$h_{\alpha,i} := h_\alpha t^i$$

$$f_{\alpha,i} := f_\alpha t^i$$

The set $\{e_{\alpha,i} : \alpha \in \Phi^+, 0 \leq i \leq m\} \cup \{h_{\alpha,i} : \alpha \in \Delta, 0 \leq i \leq m\} \cup \{f_{\alpha,i} : \alpha \in \Phi^+, 0 \leq i \leq m\}$ is then a basis for $\mathfrak{g}_m$ provided G is semisimple; again if G is reductive then we extend this to a basis by adding additional elements of $\mathfrak{h}$.

At several points we consider linear functions on $\mathfrak{g}_m$ and $\mathfrak{h}_m$. Given $\lambda \in \mathfrak{h}_m^*$, we set:

$$\lambda_i = \lambda|_{\mathfrak{h}_m^{(i)}}$$

$$\lambda_{\geq i} = \lambda|_{\mathfrak{h}_m^{(i)}}$$

We may view λ_i as an element of $\mathfrak{h}^*$ by identifying $\mathfrak{h}_m^{(i)}$ with $\mathfrak{h}$, and we often identify $\mathfrak{h}_m^*$ with $(\mathfrak{h}^*)^{m+1}$ by identifying λ with $(\lambda_0, \ldots, \lambda_m)$. We will also sometimes write elements of $\mathfrak{h}_m^*$ in the form (λ, μ) for $\lambda \in \mathfrak{h}^*$ and $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$, where $\mathfrak{h}^* \oplus (\mathfrak{h}_m^{(\geq 1)})^*$ is identified with $\mathfrak{h}_m^*$ in the natural way. Furthermore, we will also sometimes view $\lambda \in \mathfrak{h}^*$ as an element of $\mathfrak{g}^*$ by declaring that $\lambda(\mathfrak{n}) = \lambda(\mathfrak{n}^-) = 0$, for instance when we wish to consider the coadjoint stabiliser $\mathfrak{g}^\lambda$ of an element $\lambda \in \mathfrak{h}^*$.

If $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}$ is a non-degenerate G-invariant bilinear form on $\mathfrak{g}$ then there is a non-degenerate G_m-invariant bilinear form κ_m on $\mathfrak{g}$ given by:

$$\kappa_m(xt^i, yt^j) = \delta_{i+j,m}\kappa(x, y).$$

Hence if $\mathfrak{g}$ is the Lie algebra of a reductive group in either characteristic 0 or prime characteristic under the standard hypotheses, we have an isomorphism of $\mathfrak{g}_m$-modules $\mathfrak{g}_m \cong \mathfrak{g}_m^*$ by the same argument as in Remark 2.4.5.

2.5.1 Symmetric invariants and the centre of the enveloping algebra

We now describe the structure of the symmetric invariants $S(\mathfrak{g}_m)^{G_m}$ and of the centre $Z(\mathfrak{g}_m)$ of the enveloping algebra $U(\mathfrak{g}_m)$ of a truncated current Lie algebra. We begin by defining a collection of divided power operators $\partial^{(k)}$ on $S(\mathfrak{g}_m)$ as follows. Set $\partial^{(0)} = \mathrm{id}_{S(\mathfrak{g}_m)}$, and then for $1 \leq k \leq m$ we define inductively:

$$\partial^{(k)}(xt^j) = \begin{cases} \binom{j}{k} xt^{j-k} & \text{if } j \geq k, \\ 0 & \text{otherwise}, \end{cases}$$
$$\partial^{(k)}(fg) = \sum_{i+j=k} \partial^{(i)}(f)\partial^{(j)}(g).$$

for any $0 \leq j \leq m$, $x \in \mathfrak{g}$, and f, g homogeneous elements of $S(\mathfrak{g}_m)$.

Now, by the Chevalley Restriction Theorem, $S(\mathfrak{g})^G$ is a graded polynomial algebra generated by $\mathrm{rank}(\mathfrak{g})$ algebraically independent homogeneous elements. Choose such elements $p_1, \ldots, p_r$. There is an inclusion map $S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_m^{(m)})$ given by extending the map $\mathfrak{g} \hookrightarrow \mathfrak{g}_m^{(m)}$ which sends $x \mapsto x \otimes t^m$. By considering the action of G_m on $\mathfrak{g}_m^{(m)}$, this map then descends to an inclusion $S(\mathfrak{g})^G \to S(\mathfrak{g}_m^{(m)})^{G_m}$. Abusing notation, view the p_i as elements of $S(\mathfrak{g}_m)^{G_m}$ via this inclusion.

The following result was proved in characteristic 0 by Raïs–Tauvel [45, §3], and in a slightly different form in prime characteristic by Arakawa–Topley–Villareal [3, Theorem 4.4].

Theorem 2.5.4. *The symmetric invariants $S(\mathfrak{g}_m)^{G_m}$ form a polynomial algebra in $(m+1)\,\mathrm{rank}(\mathfrak{g})$ variables, generated by the algebraically independent elements $\{\partial^{(k)}(p_j) : 1 \leq j \leq r, 0 \leq k \leq m\}$.*

Now, for $0 \leq k \leq m$, we define another map $d^{(k)} : S(\mathfrak{g}_m^{(m)}) \to S(\mathfrak{g}_m)$ by setting

$$d^{(k)}(f) = \sum_x \binom{m}{k} x_{m-k} \frac{df}{dx_m}$$

where the sum is taken over the basis of $\mathfrak{g}$ fixed in §2.2 and we write x_i as shorthand for $x \otimes t^i$. Here by df/dx_m we mean that we view f as a polynomial in our chosen basis elements of $\mathfrak{g} \otimes t^m$ and then differentiate with respect to x_m. For example, if x, y are two basis elements of $\mathfrak{g}$ and $f = x_m^2 y_m$, then $df/dx_m = 2x_m y_m$ and $df/dy_m = x_m^2$, so $d^{(k)}(f) = \binom{m}{k}(2x_{m-k}x_m y_m + y_{m-k}x_m^2)$.

The following corollary with be useful at several points. In characteristic 0 it was proved in [45, Lemma 3.2(ii)], but the same argument is still valid in prime characteristic; in any case we shall only need it in characteristic 0.

Lemma 2.5.5. *For any $1 \leq j \leq r, 0 \leq k \leq m$, there exists $q_j^{(k)} \in S(\mathfrak{g}_m^{(m-k+1)})$ such that $\partial^{(k)} p_j = d^{(k)} p_j + q_j^{(k)}$.*

Although the map $d^{(k)}$ does not preserve G_m invariants, we do have the following result which we shall use later:

Lemma 2.5.6. *The map $d^{(k)}$ preserves $\operatorname{ad}(\mathfrak{h})$ invariants, where $\mathfrak{h}$ is viewed as a subalgebra of $\mathfrak{g}_m$ via the natural inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_m^{(0)}$.*

Proof. The $\operatorname{ad}(\mathfrak{h})$-invariants in $S(\mathfrak{g}_m)$ are just the zero weight space for the action of $\operatorname{ad}(\mathfrak{h})$, but $d^{(k)}$ preserves $\operatorname{ad}(\mathfrak{h})$ weight spaces by definition. □

We now seek to use this description of $S(\mathfrak{g}_m)^{G_m}$ to describe $Z(\mathfrak{g}_m)$, the centre $U(\mathfrak{g}_m)$. For now we assume that we are in characteristic 0, the only case that we require in this thesis, although we will later make some brief remarks on the situation in prime characteristic. By [14, §2.4], there is an isomorphism $\omega : S(\mathfrak{g}_m) \to U(\mathfrak{g}_m)$ of $\operatorname{ad}(\mathfrak{g}_m)$-modules, called the symmetrisation map. Explicitly, this map is given by setting

$$\omega(x_1 \dots x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \dots x_{\sigma(k)}$$

for any elements $x_1, \dots, x_k \in \mathfrak{g}_m$ and extending linearly. We have $Z(\mathfrak{g}_m) = U(\mathfrak{g}_m)^{\mathfrak{g}_m}$ and since we are in characteristic 0 by exponentiating the action of $\mathfrak{g}_m$ to an action of G_m this is equal to $U(\mathfrak{g}_m)^{G_m}$. Hence ω descends to an isomorphism $S(\mathfrak{g}_m)^{G_m} \to Z(\mathfrak{g}_m)$. We then obtain as an immediate corollary of Theorem 2.5.4 the following:

Corollary 2.5.7. *In characteristic* 0, $Z(\mathfrak{g}_m)$ *is a polynomial algebra generated by the* $(m+1)\,\mathrm{rank}(\mathfrak{g})$ *elements* $\{\omega(\partial^{(k)}(p_j)) : 1 \leq j \leq r, 0 \leq k \leq m\}$.

Remark 2.5.8. In prime characteristic the definition of the map ω given above is no longer valid. However, by [16, Theorem 1.2] the existence of an isomorphism of $\mathfrak{g}_m$-modules $U(\mathfrak{g}_m) \xrightarrow{\sim} S(\mathfrak{g}_m)$ is equivalent to the splitting of the inclusion $\mathfrak{g}_m \hookrightarrow U(\mathfrak{g}_m)$ of $\mathfrak{g}_m$-modules. Under the assumptions on $\mathfrak{g}$ given in §2.4.2, this inclusion does indeed split; we will show this later in §2.5.3. A further problem is that it is no longer the case that $Z(\mathfrak{g}_m) = U(\mathfrak{g}_m)^{G_m}$, since the action of the Lie algebra $\mathfrak{g}_m$ can no longer be exponentiated to an action of the group G_m. This can also be seen by observing that $Z(\mathfrak{g}_m)$ contains the p-centre $Z_p(\mathfrak{g}_m)$ described in §2.4.1, which in general is not contained in $U(\mathfrak{g}_m)^{G_m}$. In fact, one can show that $Z(\mathfrak{g}_m)$ is isomorphic to the tensor product of $U(\mathfrak{g}_m)^{G_m}$ and $Z_p(\mathfrak{g}_m)$ over their intersection.

2.5.2 Indices of truncated current Lie algebras

Recall that in §2.1 we defined the index of a Lie algebra $\mathfrak{g}$ to be the minimal dimension of a coadjoint stabiliser in $\mathfrak{g}$. We now discuss these quantities for truncated current Lie algebras. The following result is [45, Theorem 2.8(i)].

Theorem 2.5.9. *Let* $\mathfrak{g}$ *be any Lie algebra over a field of characteristic* 0. *Then* $\mathrm{ind}(\mathfrak{g}_m) = (m+1)\,\mathrm{ind}(\mathfrak{g})$.

Most of the arguments from [45] are still valid in prime characteristic, but a crucial step fails if $m \geq p$. We give a proof in this case when $\mathfrak{g}$ is the Lie algebra of an algebraic group satisfying the hypotheses of §2.4.2.

Theorem 2.5.10. *Let* $\mathfrak{g}$ *be the Lie algebra of a reductive group over an algebraically closed field of characteristic* $p > 0$ *satisfying the standard hypotheses. Then* $\mathrm{ind}(\mathfrak{g}_m) =$

$(m+1)\operatorname{ind}(\mathfrak{g})$.

Proof. Let $\mathfrak{h}^{\text{reg}}$ denote the set of elements of $\mathfrak{h}$ which are regular as elements of $\mathfrak{g}$ (i.e. that have minimal dimensional centraliser), and define $\mathfrak{h}_m^{\text{reg}} := \{\sum x_i t^i \in \mathfrak{h}_m \mid x_0 \in \mathfrak{h}^{\text{reg}}\}$. Observe that if $x_0 \in \mathfrak{h}^{\text{reg}}$ then $\mathfrak{g}^{x_0} = \mathfrak{h}$. For $x \in \mathfrak{h}_m^{\text{reg}}$ it is easy to check that $\mathfrak{g}_m^x = \mathfrak{h}_m$; the inclusion $\subseteq$ follows from the fact $\mathfrak{h}$ is commutative, while the inclusion $\supseteq$ can be shown by considering an arbitrary element $\sum y_i t^i \in \mathfrak{g}_m^x$ and using the fact $x_0 \in \mathfrak{h}^{\text{reg}}$ to inductively show that each $y_i \in \mathfrak{h}$. Furthermore since $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{n}^- \oplus \mathfrak{n}^+$ we have that $[\mathfrak{h}_m, \mathfrak{g}_m] \subseteq \mathfrak{n}_m^- \oplus \mathfrak{n}_m^+$. By considering dimensions, we deduce that $\operatorname{ad}(x)\mathfrak{g}_m = \mathfrak{n}_m^- \oplus \mathfrak{n}_m^+$ for $x \in \mathfrak{h}_m^{\text{reg}}$.

Now consider the morphism

$$\varphi : G_m \times \mathfrak{h}_m \to \mathfrak{g}_m$$

given the restriction of the adjoint action map $G_m \times \mathfrak{g}_m \to \mathfrak{g}_m$. By [52, Lemma 1.6] the differential $d_{(1,x)}\varphi : \mathfrak{g}_m \times \mathfrak{h}_m \to \mathfrak{g}_m$ is given by

$$d_{(1,x)}\varphi(u, v) = [x, u] + v.$$

For $x \in \mathfrak{h}_m^{\text{reg}}$ we see that the image of $d_{(1,x)}\varphi$ is $\mathfrak{n}_m^- \oplus \mathfrak{n}_m^+ \oplus \mathfrak{h}_m = \mathfrak{g}_m$, i.e. $d_{(1,x)}$ is surjective. It follows that $G_m \times \mathfrak{h}_m^{\text{reg}} \to \mathfrak{g}_m$ is a dominant morphism. Hence the conjugates of $\mathfrak{h}_m^{\text{reg}}$ are dense in $\mathfrak{g}_m$. The set of regular elements of $\mathfrak{g}_m$ is open dense and so intersects the conjugates of $\mathfrak{h}_m^{\text{reg}}$ non-trivially. Hence there is a regular element whose centraliser is conjugate to $\mathfrak{h}_m$, which has dimension $(m+1)\operatorname{rank}(\mathfrak{g}) = (m+1)\operatorname{ind}(\mathfrak{g})$. The existence of the isomorphism $\mathfrak{g}_m \cong \mathfrak{g}_m^*$ of $\mathfrak{g}_m$-modules then implies that $\operatorname{ind}(\mathfrak{g}_m) = (m+1)\operatorname{ind}(\mathfrak{g})$, completing the proof. □

2.5.3 Support varieties in the truncated current case

The final preliminary we shall need for the prime characteristic chapter of the thesis is the notion of the support variety of a module. Let M be a $U_\chi(\mathfrak{g})$-module for some restricted Lie algebra $\mathfrak{g}$ and $\chi \in \mathfrak{g}^*$. Then the *support variety* of M, denoted $\mathcal{V}_\mathfrak{g}(M)$, is an algebraic variety which reflects in certain ways the properties of the module. We will not give a definition of $\mathcal{V}_\mathfrak{g}(M)$ here; instead we refer the reader to [17, Definition 6.1] and only mention the properties we require. Before moving on to the main results we shall use, we mention two other interesting properties of support varieties. Firstly, the dimension of $\mathcal{V}_\mathfrak{g}(M)$ is equal to the growth rate of a minimal projective resolution of M in the category $U_\chi(\mathfrak{g})$-mod [17, Remark 6.3], and secondly there is a close relationship between support varieties and rank varieties [17, Theorem 6.4]. Their main use to us however will come from the following result, which is [17, Proposition 6.2 & Proposition 7.1]. This result gives a condition for certain modules to be projective, which will be useful later.

Theorem 2.5.11. *Let M be a $U_\chi(\mathfrak{g})$-module. Then:*

(i) M is projective as a $U_\chi(\mathfrak{g})$-module if and only if $\mathcal{V}_\mathfrak{g}(M) = 0$.

(ii) If $\mathfrak{l} \subseteq \mathfrak{g}$ is a restricted subalgebra, then $\mathcal{V}_\mathfrak{l}(M) = \mathcal{V}_\mathfrak{g}(M) \cap \mathfrak{l}$.

One can also define the support variety of the category of $U_\chi(\mathfrak{g})$-modules, denoted $\mathcal{V}_\mathfrak{g}(\chi)$, by:

$$\mathcal{V}_\mathfrak{g}(\chi) = \mathcal{V}_\mathfrak{g}(M_1 \oplus \cdots \oplus M_k)$$

where $M_1, \ldots, M_k$ form a complete set of irreducible $U_\chi(\mathfrak{g})$-modules. By [41, Theorem 2.1(v), (vii)], we have that $\mathcal{V}_\mathfrak{g}(M) \subseteq \mathcal{V}_\mathfrak{g}(\chi)$ for any $U_\chi(\mathfrak{g})$-module M.

We now state a result on support varieties in the truncated current case which we will need later. To prove this result we first need the following lemma:

Lemma 2.5.12. *Let $\mathfrak{g}$ be the Lie algebra of a standard reductive group in characteristic $p > 0$. Then the inclusion of $\mathrm{ad}(\mathfrak{g}_m)$-modules $\mathfrak{g}_m \hookrightarrow U(\mathfrak{g}_m)$ splits.*

Proof. First consider the case $\mathfrak{g} = \mathfrak{gl}_n$. In this case $(\mathfrak{gl}_n)_m$ can be identified with the Lie algebra of $n \times n$ matrices with coefficients in $\Bbbk[t]/(t^{m+1})$. We then obtain a map $U((\mathfrak{gl}_n)_m) \to (\mathfrak{gl}_n)_m$ which sends the PBW monomial $x_1 \otimes x_2 \otimes \cdots \otimes x_k$ to the product of matrices $x_1 x_2 \ldots x_k$ which gives the required splitting.

Now let $\mathfrak{g}$ be the Lie algebra of a standard reductive group. By [18, Theorem A], $\mathfrak{g}$ satisfies Richardson's property, i.e. there exists a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}_n$ and an $\mathrm{ad}(\mathfrak{g})$-submodule $M \subseteq \mathfrak{gl}_n$ such that $\mathfrak{gl}_n = \mathfrak{g} \oplus M$ as $\mathrm{ad}(\mathfrak{g})$-modules. This then extends to a faithful representation $\rho_m : \mathfrak{g}_m \to (\mathfrak{gl}_n)_m$ such that $(\mathfrak{gl}_n)_m = \mathfrak{g}_m \oplus (M \otimes k[t]/(t^{m+1}))$ as $\mathrm{ad}(\mathfrak{g}_m)$-modules. The desired splitting is then given by the composition $U(\mathfrak{g}_m) \to U((\mathfrak{gl}_n)_m) \to (\mathfrak{gl}_n)_m \to \mathfrak{g}_m$, where the first map is induced by ρ_m, the second is the the map given in the previous paragraph, and the third is projection along the decomposition $(\mathfrak{gl}_n)_m = \mathfrak{g}_m \oplus (M \otimes k[t]/(t^{m+1}))$. □

By [16, Theorem 1.2] this is equivalent to the following corollary:

Corollary 2.5.13. *As* $\mathrm{ad}(\mathfrak{g}_m)$*-modules* $U(\mathfrak{g}_m)$ *and* $S(\mathfrak{g}_m)$ *are isomorphic.*

This enables us to give the following description of the support variety of the category of $U_\chi(\mathfrak{g}_m)$-modules:

Proposition 2.5.14. *Let* $\mathfrak{g}$ *be the Lie algebra of a standard reductive group in characteristic* $p > 0$*. Then for any* $\chi \in \mathfrak{g}_m^*$*, we have* $\mathcal{V}_{\mathfrak{g}_m}(\chi) = \mathcal{N}^{[p]}(\mathfrak{g}_m^\chi)$*, where* $\mathcal{N}^{[p]}(\mathfrak{g}_m^\chi)$ *denotes the set of elements in* $\mathfrak{g}_m^\chi$ *annihilated by the* p^{th}*-power map on* $\mathfrak{g}_m^\chi$*.*

Proof. The argument from [43, Proposition 6.3] applies for any restricted Lie algebra $\mathfrak{l}$ admitting an isomorphism $U(\mathfrak{l}) \cong S(\mathfrak{l})$ of $\mathrm{ad}(\mathfrak{l})$-modules, so the desired result follows from Corollary 2.5.13. □

CHAPTER 3

ORDINARY REPRESENTATION THEORY OF TRUNCATED CURRENT LIE ALGEBRAS

3.1 Category $\mathcal{O}$ for truncated current Lie algebras

3.1.1 Definition and first properties of $\mathcal{O}$

Throughout this chapter, we work over $\mathbb{C}$ (although all our results will be valid over any algebraically closed field of characteristic 0). Recall the notation for reductive groups and their Lie algebras introduced in §2.2 and for truncated current groups and Lie algebras introduced in §2.5. Our overall aim in this chapter is to generalise some of the results in the reductive case described in §2.3 and to state and prove a precise version of Theorem 1.1.3.

We begin by stating our definition of category $\mathcal{O}$ for $\mathfrak{g}_m$, which is a generalisation of the $\mathfrak{g} = \mathfrak{sl}_2$, $m = 1$ case treated in [34]. We note that in the case $m = 0$, the category $\mathcal{O}(\mathfrak{g}_0)$ we obtain is precisely BGG category $\mathcal{O}$ for $\mathfrak{g}_0 = \mathfrak{g}$.

Definition 3.1.1. *The category $\mathcal{O}(\mathfrak{g}_m)$ is the full subcategory of $U(\mathfrak{g}_m)$*-mod *with objects M satisfying the following:*

($\mathcal{O}1$) *M is finitely generated.*

($\mathcal{O}2$) *$\mathfrak{h}_m^{(0)}$ acts semisimply on M.*

($\mathcal{O}3$) $\mathfrak{h}_m^{(\geq 1)} \oplus \mathfrak{n}_m^+$ *acts locally finitely on* M.

Since $U(\mathfrak{g}_m)$ is a Noetherian algebra, by ($\mathcal{O}1$) every $M \in \mathcal{O}(\mathfrak{g}_m)$ is Noetherian. Furthermore, it is not hard to see that $\mathcal{O}(\mathfrak{g}_m)$ is closed under submodules, quotients, and finite direct sums.

Remark 3.1.2. It is perhaps more natural to insist that all of $\mathfrak{h}_m$ acts semisimply on M rather than just $\mathfrak{h}_m^{(0)}$. However, in the category defined by this stronger condition every module can be obtained by taking a module in the BGG category $\mathcal{O}$ for $\mathfrak{g}$ and letting $\mathfrak{g}_m^{(\geq 1)}$ acting trivially; see for example the case $\mathfrak{g} = \mathfrak{sl}_2$, $m = 1$ dealt with in [34, Corollary 5].

Let $M \in \mathcal{O}(\mathfrak{g}_m)$ and $\lambda \in \mathfrak{h}^*$, which we identify with $(\mathfrak{h}_m^{(0)})^*$ in the natural way. Define the λ *weight space of* M to be

$$M^\lambda := \{v \in M \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}_m^{(0)}\}.$$

By ($\mathcal{O}2$) we have $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ as vector spaces, and this is in fact a module grading of M where $U(\mathfrak{g}_m)$ is equipped with the natural grading of the root lattice. The elements of M^λ are called *weight vectors of weight* $\lambda \in \mathfrak{h}^*$. If $v \in M^\lambda$ is a weight vector satisfying $\mathfrak{n}_m^+ \cdot v = 0$ then we say that v is a *maximal vector of weight* λ.

Now let $\mu \in \mathfrak{h}_m^*$ and recall that $\mu_i := \mu|_{\mathfrak{h}_m^{(i)}}$. We say that $v \in M^{\mu_0}$ is a *highest weight vector of weight* μ if v is maximal of weight μ and

$$h \cdot v = \mu(h)v \text{ for all } h \in \mathfrak{h}_m.$$

The following basic properties of weight spaces of $M \in \mathcal{O}(\mathfrak{g}_m)$ can be shown using the same argument as in BGG category $\mathcal{O}$, see for example [23, §1.1]:

Lemma 3.1.3. *For any* $M \in \mathcal{O}(\mathfrak{g}_m)$*, we have* $\dim(M^\lambda) < \infty$ *for all* $\lambda \in \mathfrak{h}^*$*. In addition,* $\{\lambda \in \mathfrak{h}^* : M^\lambda \neq 0\} \subseteq \bigcup_{\lambda \in I}(\lambda - \mathbb{Z}_{\geq 0}\Phi^+)$ *for some finite subset* $I \subseteq \mathfrak{h}^*$.

We also have the following useful corollary, which allows us obtain highest weight vectors from maximal vectors.

Corollary 3.1.4. *Suppose that $M \in \mathcal{O}$ admits a nonzero maximal vector of weight $\lambda \in \mathfrak{h}^*$ in M. Then M admits a nonzero highest weight vector of weight μ for some $\mu \in \mathfrak{h}_m^*$ satisfying $\mu_0 = \lambda$.*

Proof. Let V be the space of maximal vectors of weight λ. This is finite-dimensional since it is a subspace of M^λ, which is finite-dimensional by Lemma 3.1.3. Now, the action of $\mathfrak{h}_m^{(\geq 1)}$ preserves weight spaces (since $\mathfrak{h}_m^{(\geq 1)}$ commutes with $\mathfrak{h}_m^{(0)}$) and maximal vectors (by considering the commutation relations in $U(\mathfrak{g}_m)$) and hence $\mathfrak{h}_m^{(\geq 1)}$ acts on V. But since $\mathfrak{h}_m^{(\geq 1)}$ is commutative there exists some common eigenvector $v \in V$ for this action. Then by definition v is a highest weight vector of weight $(\lambda, \mu_1, \dots, \mu_m) \in \mathfrak{h}_m^*$ for some $\mu_1, \dots, \mu_m \in \mathfrak{h}^*$ where, as mentioned earlier, we identify $\mathfrak{h}_m^*$ with $(\mathfrak{h}^*)^{m+1}$ in the natural manner. □

3.1.2 Highest weight modules

We say M is a *highest weight module of weight* $\lambda \in \mathfrak{h}_m^*$ if M is generated by a highest weight vector of weight λ. The following result on highest weight filtrations is analogous to a result for BGG category $\mathcal{O}$, see for example [23, Corollary 1.2].

Lemma 3.1.5. *Let $M \in \mathcal{O}(\mathfrak{g}_m)$. Then M has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ such that each M_{i+1}/M_i is a highest weight module. We call such a filtration a highest weight filtration of M.*

Proof. By ($\mathcal{O}$1) and ($\mathcal{O}$2), there exists a finite set $\{v_1, v_2, \dots, v_n\}$ of weight vectors which generate M. Let V be the $U(\mathfrak{n}_m \oplus \mathfrak{h}_m^{(\geq 1)})$-module generated by the v_i, which is finite-dimensional by ($\mathcal{O}$3), and proceed by induction on $\dim(V)$, observing that if m is maximal, then so is $(ht^i) \cdot m$ for any $h \in \mathfrak{h}$ and $0 \leq i \leq n$:

If $\dim(V) = 1$, then any non-zero element of V is a highest weight vector and generates M, so M is highest weight. If $\dim(V) > 1$ then pick some λ maximal among the weights of V. Then any vector of weight λ in V is maximal, so by a similar argument to the proof of Corollary 3.1.4 there is some $v \in V^\lambda$ which is a highest weight vector generating a highest weight submodule of M. Let M_1 be the highest weight submodule of M generated by v. Now consider $\overline{M} = M/M_1$, which is generated by the image $\overline{V}$ of V in M/M_1. Since $\dim(\overline{V}) < \dim(V)$ we are done by induction. □

For $\lambda \in \mathfrak{h}_m^*$ we define the *Verma module of weight* λ, denoted $M(\lambda)$, to be

$$M(\lambda) := U(\mathfrak{g}_m) \otimes_{U(\mathfrak{b}_m)} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda$ is the one-dimensional $U(\mathfrak{b}_m)$-module on which $\mathfrak{h}_m$ acts via λ, and $\mathfrak{n}_m$ acts by 0. The Verma modules are universal highest weight modules in the sense that every highest weight module is a quotient of a Verma module. They have the following properties generalising the classical case:

Lemma 3.1.6. *Let $\lambda \in \mathfrak{h}_m^*$. Then:*

(i) $\dim M(\lambda)^{\lambda_0} = 1$, *and hence* $\dim M^{\lambda_0} = 1$ *for every highest weight module M of weight $\lambda \in \mathfrak{h}_m^*$.*

(ii) Every highest weight module M admits a central character which depends only on its highest weight λ. In other words, for any $\lambda \in \mathfrak{h}_m^*$ there is a homomorphism $\chi_\lambda : Z(\mathfrak{g}_m) \to \mathbb{C}$ such that for any highest weight module M of highest weight λ we have $z \cdot v = \chi_\lambda(z)v$ for all $z \in Z(\mathfrak{g}_m)$ and all $v \in M$.

(iii) $M(\lambda)$ admits a unique maximal submodule and a unique simple quotient, which we denote $L(\lambda)$.

(iv) Every simple object in $\mathcal{O}(\mathfrak{g}_m)$ is isomorphic to precisely one of these simple modules. Thus the modules

$$\{L(\lambda) \mid \lambda \in \mathfrak{h}_m^*\}$$

give a complete set of representatives for the isomorphism classes of simple modules in $\mathcal{O}(\mathfrak{g}_m)$*.*

Proof. The proofs of all four parts of this lemma are very similar to the classical case (see for example [23, Theorem 1.2]); we include them for the reader's convenience. For part (i), first observe that $M(\lambda)$ has a basis of the form $\{\prod_{\alpha\in\Phi^+,0\leq j\leq m} f_{\alpha,j}^{i_{\alpha,j}} \otimes 1_\lambda : i_{\alpha,j} \in \mathbb{Z}_{\geq 0}\}$, where this product is taken over some fixed ordering on $\Phi^+ \times \{0, 1, \dots, m\}$. The fact that $\dim M(\lambda)^{\lambda_0} = 1$ then follows from the fact that $\prod_{\alpha\in\Phi^+,0\leq j\leq m} f_{\alpha,j}^{i_{\alpha,j}} \otimes 1_\lambda$ has weight $\lambda_0 - \sum_{\alpha,j} i_{\alpha,j}\alpha$. Now since $M(\lambda)$ lies in $\mathcal{O}$ which is closed under taking submodules, all submodules of $M(\lambda)$ are weight modules for the action of $\mathfrak{h}_m^{(0)}$. Since $1 \otimes 1_\lambda$ generates $M(\lambda)$, for any proper submodule $N \subseteq M(\lambda)$ we have $N^{\lambda_0} = 0$ and hence for any highest weight module $M := (M(\lambda)/N)$ we have $\dim M^{\lambda_0} = 1$ as required.

Part (ii) follows from the fact that $Z(\mathfrak{g}_m) \subseteq U(\mathfrak{g}_m)^{\mathfrak{h}}$ and hence any $z \in Z(\mathfrak{g}_m)$ preserves M^{λ_0} and hence acts on any $v \in M^{\lambda_0}$ by a scalar (using part (a)). But every element of M is of the form $u \cdot v$ for some $v \in M^{\lambda_0}$ and $u \in U(\mathfrak{g}_m)$, so since $z \cdot (u \cdot v) = u \cdot (z \cdot v)$ the result follows.

For part (iii), observe that since any non-zero element of $M(\lambda)^{\lambda_0}$ generates $M(\lambda)$, any proper submodule of $M(\lambda)$ must be weight (as observed in part (i)) and lie in a sum of weight spaces whose weights are strictly less than λ_0. Hence the sum of all proper submodules of $M(\lambda)$ must again lie in a sum of weight spaces whose weights are strictly less than λ_0 and so is still proper, and is therefore the unique maximal submodule of $M(\lambda)$.

Part (iv) follows from Lemma 3.1.5 since it implies that any simple module must be highest weight and hence must be a quotient of some Verma module $M(\lambda)$, but the only such simple quotient is $L(\lambda)$. □

3.1.3 Decomposition of category $\mathcal{O}$

A standard technique in studying the BGG category $\mathcal{O}$ is to consider modules with a fixed central character for $U(\mathfrak{g})$. This refinement is also useful in our more general setting (see Theorem 3.2.2), but it will also be convenient to instead decompose $\mathcal{O}(\mathfrak{g}_m)$ by decomposing modules $M \in \mathcal{O}(\mathfrak{g}_m)$ into generalised eigenspaces for the action of $\mathfrak{h}_m^{(\geq 1)}$.

Fix $M \in \mathcal{O}(\mathfrak{g}_m)$ and $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$. We define the *generalised eigenspace of eigenvalue* μ to be

$$M^{(\mu)} = \{v \in M \mid (h - \mu(h))^k v = 0 \text{ for all } k \gg 0,\ h \in \mathfrak{h}_m^{(\geq 1)}\}.$$

This definition then allows us to state the following lemma:

Lemma 3.1.7. *Each $M \in \mathcal{O}(\mathfrak{g}_m)$ admits a direct sum decomposition:*

$$M = \bigoplus_{\mu \in (\mathfrak{h}_m^{(\geq 1)})^*} M^{(\mu)}$$

and each $M^{(\mu)}$ is a $U(\mathfrak{g}_m)$-submodule of M.

Proof. Since $\mathfrak{h}_m$ preserves the weight spaces of M and each of these weight space is finite-dimensional, it follows that each M^λ decomposes into generalised eigenspaces for $\mathfrak{h}_m^{(\geq 1)}$. Therefore M admits a decomposition of the claimed form and it suffices to show that each summand $M^{(\mu)}$ is a $U(\mathfrak{g}_m)$-submodule of M. In turn, it is then enough to show that $x \cdot v \in M^{(\mu)}$ for all $v \in M^{(\mu)}$ and $x = e_{\alpha,i}, h_{\alpha,i}$, or $f_{\alpha,i}$.

If $x = h_{\alpha,i}$, then $[x, (ht^j - \mu(ht^j))^n] = 0$ for any $h \in \mathfrak{h}$, $1 \leq j \leq m$ and $n \geq 0$, so $(ht^j - \mu(ht^j))^n \cdot (x \cdot v) = x \cdot ((ht^j - \mu(ht^j))^n \cdot v)$. But since $v \in M^{(\mu)}$, this is 0 for sufficiently large n and hence $x \cdot v \in M^{(\mu)}$. We now deal with the case $x = e_{\alpha,i}$; the case $x = f_{\alpha,i}$ is very similar. In the case $i = m$, since $e_{\alpha,m}$ and ht^j commute for $j \geq 1$ the same argument as above shows $e_{\alpha,m} \cdot v \in M^{(\mu)}$ for any $v \in M^{(\mu)}$. Otherwise, observe that:

$$\begin{aligned}[e_{\alpha,i}, (ht^j - \mu(ht^j))^n] &= \sum_{k=0}^{n-1} (ht^j - \mu(ht^j))^k [e_{\alpha,i}, ht^j - \mu(ht^j)] (ht^j - \mu(ht^j))^{n-k-1} \\ &= \sum_{k=0}^{n-1} (ht^j - \mu(ht^j))^k \alpha(h) e_{\alpha,i+j} (ht^j - \mu(ht^j))^{n-k-1}\end{aligned}$$

Hence for $v \in M^{(\mu)}$, we have:

$$\begin{aligned}(ht^j - \mu(ht^j))^n \cdot (e_{\alpha,i} \cdot v) = e_{\alpha,i} \cdot ((ht^j - \mu(ht^j))^n \cdot v) \\ - \sum_{k=0}^{n-1} (ht^j - \mu(ht^j))^k \alpha(h) e_{\alpha,i+j} (ht^j - \mu(ht^j))^{n-k-1} \cdot v\end{aligned}$$

By induction, $e_{\alpha,i+j} \cdot v \in M^{(\mu)}$ for $j \geq 1$, so for sufficiently large n this expression is 0 and hence $e_{\alpha,i} \cdot v \in M^{(\mu)}$. □

Now we define the *Jordan block of* $\mathcal{O}(\mathfrak{g}_m)$ *of weight* $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ to be the full subcategory $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ of $\mathcal{O}(\mathfrak{g}_m)$ whose objects are the modules $M \in \mathcal{O}$ such that $M = M^{(\mu)}$. We remark that these are not blocks in the sense of Definition 2.3.7 but are somewhat larger; however they play a similar role in the structure of category $\mathcal{O}$. The following direct sum decomposition of $\mathcal{O}(\mathfrak{g}_m)$, which we refer to as the *Jordan decomposition* of $\mathcal{O}(\mathfrak{g}_m)$, is an immediate corollary of Lemma 3.1.7.

Corollary 3.1.8.

$$\mathcal{O}(\mathfrak{g}_m) = \bigoplus_{\mu \in (\mathfrak{h}_m^{\geq 1})^*} \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$$

Remark 3.1.9. It is not hard to see that if $\lambda \in \mathfrak{h}_m^*$ and $\mu = \lambda|_{\mathfrak{h}_m^{(\geq 1)}}$ then both $M(\lambda)$ and $L(\lambda)$ lie in $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$. Combining Corollary 3.1.8 with the fact that Verma modules have unique maximal submodules and are therefore indecomposable, it follows that $L(\lambda)$ cannot occur as a subquotient of $M(\nu)$ unless $\lambda_{\geq 1} = \nu_{\geq 1}$

Let $\mathfrak{g}_m \to \mathfrak{g}_{m-1}$ be the natural quotient map with kernel $\mathfrak{g}_m^{(m)}$, and consider the pullback functor

$$p : \mathcal{O}(\mathfrak{g}_{m-1}) \longrightarrow \mathcal{O}(\mathfrak{g}_m)$$

induced by this map.

Lemma 3.1.10. *Let $\lambda \in \mathfrak{h}_{m-1}^*$ and define $\nu \in \mathfrak{h}_m^*$ by $\nu(h_i) = \lambda(h_i)$ for $i = 0, \ldots, m-1$ and $\nu(h_m) = 0$ for all $h \in \mathfrak{h}$. Then $p(L(\lambda)) \cong L(\nu)$ as $\mathfrak{g}_m$-modules.*

Proof. This follows immediately from the observations that $p(L(\lambda))$ is a simple highest weight module of weight ν, and that $L(\nu)$ is the unique such simple module in $\mathcal{O}(\mathfrak{g}_m)$. □

Now we state and prove an easy equivalence between Jordan blocks of $\mathcal{O}(\mathfrak{g}_m)$ which arise by twisting by an automorphism of $U(\mathfrak{g}_m)$. We write $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ for the derived subalgebra of $\mathfrak{g}$. Note that $(\mathfrak{g}_m)' = (\mathfrak{g}')_m$, and so we may write $\mathfrak{g}_m'$ unambiguously.

For $\lambda \in \mathfrak{h}_m^*$ we recall the notation $\lambda_{\geq 1} := \lambda|_{\mathfrak{h}_m^{(\geq 1)}}$. Any such λ can be extended to an element of $\mathfrak{g}_m^*$ by declaring that $\lambda(\mathfrak{n}_m^{\pm}) = 0$, and we may abuse notation by identifying $\mathfrak{h}_m^*$ with a subspace of $\mathfrak{g}_m^*$.

Lemma 3.1.11. *Suppose that $\lambda, \nu \in \mathfrak{h}_m^*$ are such that $\lambda|_{\mathfrak{g}_m'} = \nu|_{\mathfrak{g}_m'}$. Then the categories $\mathcal{O}^{(\lambda_{\geq 1})}(\mathfrak{g}_m)$ and $\mathcal{O}^{(\nu_{\geq 1})}(\mathfrak{g}_m)$ are equivalent.*

Proof. Since $\nu - \lambda$ is 0 on $\mathfrak{g}_m'$ and $\mathfrak{g}_m = \mathfrak{g}_m' \oplus \mathfrak{z}(\mathfrak{g}_m)$, we can define an automorphism ϕ of $U(\mathfrak{g}_m)$ by declaring $\phi(x) = x - (\nu - \lambda)(x)$ for all $x \in \mathfrak{g}$ and then extending to all of $U(\mathfrak{g}_m)$. Twisting by ϕ then defines an autoequivalence of $U(\mathfrak{g}_m)$-mod, and since ϕ preserves $U(\mathfrak{h}_m)$, $U(\mathfrak{h}_m^{(0)})$, and $U(\mathfrak{n}_m^+)$ it also defines an autoequivalence of $\mathcal{O}(\mathfrak{g}_m)$. Finally, suppose $M \in \mathcal{O}^{(\lambda_{\geq 1})}(\mathfrak{g}_m)$, and that $(x - \lambda(x))^n \cdot v = 0$ for some $v \in M$, $x \in \mathfrak{g}$, and $n \geq 0$. Then writing $\cdot$ for the untwisted action and $\cdot_\phi$ for the twisted action, we have

$$(x - \nu(x))^n \cdot_\phi v = (x - \lambda(x))^n \cdot v = 0$$

and hence $\phi(M) \in \mathcal{O}^{(\nu_{\geq 1})}(\mathfrak{g}_m)$. It is then clear that twisting by ϕ defines an equivalence between $\mathcal{O}^{(\lambda_{\geq 1})}(\mathfrak{g}_m)$ and $\mathcal{O}^{(\nu_{\geq 1})}(\mathfrak{g}_m)$ with quasi-inverse given by twisting by ϕ^{-1}. □

3.2 Parabolic induction

In this section we aim to prove the following theorem (a more precise version of Theorem 1.1.1), which allows us to relate the category $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ to a Jordan block of $\mathcal{O}(\mathfrak{l}_m)$ for a suitable Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$. Recall that if $\nu \in \mathfrak{h}^*$ then we extend ν to an element of $\mathfrak{g}^*$ by setting $\nu(\mathfrak{n}^{\pm}) = 0$ and write $\mathfrak{g}^{\nu}$ for the coadjoint stabiliser of ν viewed as an element of $\mathfrak{g}^*$.

Theorem 3.2.1. *Let $\lambda \in \mathfrak{h}_m^*$. Suppose that the stabiliser $\mathfrak{l} = \mathfrak{g}^{\lambda_m}$ is in standard Levi form, let $\mathfrak{p} := \mathfrak{l} + \mathfrak{n}^+ = \mathfrak{l} \oplus \mathfrak{r}$ be the standard parabolic subalgebra with Levi factor $\mathfrak{l}$ containing $\mathfrak{b}$, and write $\mathfrak{r}$ for the nilradical of $\mathfrak{p}$. Set $\mu = \lambda_{\geq 1} \in (\mathfrak{h}_m^{(\geq 1)})^*$. Then the categories $\mathcal{O}^{(\mu)}(\mathfrak{l}_m)$ and $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ are equivalent, and the functors inducing the equivalence are parabolic induction and $\mathfrak{r}_m$-invariants. The first is given by:*

$$U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} (\bullet) : \mathcal{O}^{(\mu)}(\mathfrak{l}_m) \longrightarrow \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$$

sending $M \longmapsto U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M$, where we inflate a $U(\mathfrak{g}_m^{\mu_m})$-module M into a $U(\mathfrak{p}_m)$-module by letting $\mathfrak{r}_m$ act by 0. The second is given by

$$(\bullet)^{\mathfrak{r}_m} : \mathcal{O}^{(\mu)}(\mathfrak{g}_m) \longrightarrow \mathcal{O}^{(\mu)}(\mathfrak{l}_m)$$

sending $M \longmapsto M^{\mathfrak{r}_m}$, where $M^{\mathfrak{r}_m} = \{v \in M : \mathfrak{r}_m \cdot v = 0\}$.

We first observe that $U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} (\bullet)$ is left adjoint to $(\bullet)^{\mathfrak{r}_m}$ since for M, N lying in the appropriate categories we have inverse isomorphisms

$$\mathrm{Hom}_{\mathfrak{l}_m}(M, N^{\mathfrak{r}_m}) \underset{\eta}{\overset{\theta}{\rightleftarrows}} \mathrm{Hom}_{\mathfrak{g}_m}(U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M, N)$$

given by $\theta(f)(u \otimes v) = u \cdot f(v)$ and $\eta(g)(v) = g(1 \otimes v)$ for $u \in U(\mathfrak{g}_m)$ and $v \in M$.

In order to show that the adjoint functors $U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} (\bullet)$ and $(\bullet)^{\mathfrak{r}_m}$ are equivalences we consider the unit and counit of this adjunction. Let $\mathbb{1}_{\mathcal{C}}$ denote the identity endofunctor of a category $\mathcal{C}$. The unit is the natural transformation $\psi : \mathbb{1}_{\mathcal{O}(\mathfrak{g}_m)} \to (U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} (\bullet))^{\mathfrak{r}_m}$ obtained by applying θ to the identity map $N^{\mathfrak{r}_m} \to N^{\mathfrak{r}_m}$, whilst the counit is the natural transformation $\varphi : U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} (\bullet)^{\mathfrak{r}_m} \to \mathbb{1}_{\mathcal{O}(\mathfrak{l}_m)}$ obtained by applying η to the identity map $U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M \to U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M$. In particular, for M, N in the appropriate categories we that have $\psi_M : M \to (U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M)^{\mathfrak{r}_m}$ is given by $v \mapsto 1 \otimes v$ and that $\varphi_N : U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} N^{\mathfrak{r}_m} \to N$ is given by $u \otimes v \mapsto u \cdot v$.

To prove Theorem 3.2.1 it therefore suffices to show that ψ and φ are both natural isomorphisms. The proof is given in Section 3.2.2 and depends heavily on the exactness of the functor $(\bullet)^{\mathfrak{r}_m}$. The proof the exactness of this functor requires a careful study of the central characters of highest weight modules, which we now discuss.

3.2.1 Central characters

The main step in proving the exactness of the functor $(\bullet)^{\mathfrak{r}_m}$ is the following result which leads to a vanishing criterion for extensions. The proof relies on the well-known fact that if two $U(\mathfrak{g}_m)$-modules admit different generalised central characters then there do not exist any non-split extensions between them.

For $\nu \in \mathfrak{h}^*$, we define $\Phi_\nu := \{\alpha \in \Phi : \nu(h_\alpha) = 0\} \subseteq \Phi$, the root system of the stabiliser $\mathfrak{g}^\nu$. We also recall the notation χ_λ for central characters introduced in Lemma 3.1.6(ii). The main theorem we prove in this section is the following:

Theorem 3.2.2. *Let $\lambda, \lambda' \in \mathfrak{h}_m^*$ such that $\lambda_{\geq 1} = \lambda'_{\geq 1}$ and $\mathfrak{g}^{\lambda_m}$ is in standard Levi form. Then $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda_0 - \lambda'_0 \in \mathbb{C}\Phi_{\lambda_m}$.*

Before proceeding to the proof of this theorem, we state the following corollary, which will be a key ingredient in the proof of Theorem 3.2.1.

Corollary 3.2.3. *Let $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ be such that $\mathfrak{g}^{\mu_m}$ is in standard Levi form, and let $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ be indecomposable, Then there is a unique coset $\Xi_M \in \mathfrak{h}^*/\mathbb{C}\Phi_{\mu_m}$ such that if N is a highest weight subquotient of M of weight $\lambda \in \mathfrak{h}_m^*$, then $\lambda_{\geq 1} = \mu$ and $\lambda_0 + \mathbb{C}\Phi_{\mu_m} = \Xi_M$.*

Proof. By Lemma 3.1.7 along with Remark 3.1.9, all subquotients of M must lie in the Jordan block $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ and hence $\lambda_{\geq 1} = \mu$. Since M has finite-dimensional weight spaces by Lemma 3.1.3 and is indecomposable it admits a generalised central character, i.e. there is a unique maximal ideal $\mathfrak{m}$ of $Z(\mathfrak{g}_m)$ such that $\mathfrak{m}^k M = 0$ for $k \gg 0$. Thus all of the highest weight subquotients have the same central character by Lemma 3.1.6(ii). Corollary 3.2.3 now follows from Theorem 3.2.2. □

Remark 3.2.4. Later we shall see that Theorem 3.2.2 and Corollary 3.2.3 hold even without the standard Levi type hypothesis; this will follow from Theorem 3.3.11. However since the the proof of the latter theorem will use Theorem 3.2.2 we retain this hypothesis to keep these dependencies clear.

We now proceed to prove Theorem 3.2.2. We begin by recalling several facts from §2.5.1. First recall that the symmetric invariants $S(\mathfrak{g})^{\mathfrak{g}}$ are generated by $r = \mathrm{rank}(\mathfrak{g})$ algebraically independent homogeneous elements $p_1, \dots, p_r$. Furthermore, we also recall the definitions of the maps $\partial^{(j)}, d^{(j)} : S(\mathfrak{g}_m^{(m)}) \to S(\mathfrak{g}_m)$. As discussed in §2.5.1 there is an embedding $S(\mathfrak{g})^{\mathfrak{g}} \hookrightarrow S(\mathfrak{g}_m)^{\mathfrak{g}_m}$; we will often abuse notation by identifying $p_1, \dots, p_r$ with their images under this embedding. We then have that $S(\mathfrak{g}_m)^{\mathfrak{g}_m}$ is generated by the $r(m+1)$ elements $\partial^{(j)} p_i,\ i = 1, \dots, r,\ j = 0, \dots, m$. Finally, there is an isomorphism $\omega : S(\mathfrak{g}_m)^{\mathfrak{g}_m} \to Z(\mathfrak{g}_m)$. The elements of the centre that we are interested in are:

$$z_i^{(j)} := \omega(\partial^{(j)} p_i) \quad \text{for } i = 1, \dots, r \text{ and } j = 0, \dots, m$$

Now let $U(\mathfrak{g}_m)^{\mathfrak{h}}$ be the invariant subalgebra under the adjoint action of $\mathfrak{h}$, and let $U(\mathfrak{g}_m)\mathfrak{n}_m^+$ be the left ideal of $U(\mathfrak{g}_m)$ generated by $\mathfrak{n}_m^+$. The intersection $U(\mathfrak{g}_m)\mathfrak{n}_m^+ \cap U(\mathfrak{g}_m)^{\mathfrak{h}}$ is an ideal of $U(\mathfrak{g}_m)^{\mathfrak{h}}$, and it is not hard to see that the quotient by this ideal is isomorphic to $U(\mathfrak{h}_m)$. This yields a map $\pi' : U(\mathfrak{g}_m)^{\mathfrak{h}} \to U(\mathfrak{h}_m)$ which can be described in the following way. Let x be a PBW monomial with the respect to the ordered basis of $\mathfrak{g}_m$ defined in §2.5; in particular the order on this basis is compatible with the choice of Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$. Then π' sends x to itself if $x \in U(\mathfrak{h}_m)$ and 0 otherwise. The observations of [23, §1.7] are still valid in our situation, and so we see that χ_λ coincides with the composition

$$U(\mathfrak{g}_m)^{\mathfrak{h}} \xrightarrow{\pi'} U(\mathfrak{h}_m) = \mathbb{C}[\mathfrak{h}_m^*] \xrightarrow{\mathrm{ev}_\lambda} \mathbb{C}$$

where the function ev_λ is the extension of $\lambda \in \mathfrak{h}_m^*$ to a map $\mathbb{C}[\mathfrak{h}_m^*] \cong S(\mathfrak{h}_m) \to \mathbb{C}$. This allows us to view χ_λ as a function not just from $Z(\mathfrak{g}_m)$ to $\mathbb{C}$ but from $U(\mathfrak{g}_m)^{\mathfrak{h}}$ to $\mathbb{C}$.

Now for $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ we define two maps $\xi_\mu, \eta_\mu : \mathfrak{h}^* \to \mathbb{C}^r$. For $\nu \in \mathfrak{h}^*$ we write (ν, μ) for the element of $\mathfrak{h}_m^*$ which restricts to ν on $\mathfrak{h}_m^{(0)} \cong \mathfrak{h}$ and to μ on $\mathfrak{h}_m^{(\geq 1)}$. The first map ξ_μ is given by:

$$\xi_\mu(\nu) := ((\chi_{(\nu,\mu)} \circ \omega \circ d^{(m)})p_1, \ldots, (\chi_{(\nu,\mu)} \circ \omega \circ d^{(m)})p_r) \in \mathbb{C}^r$$

whilst the second map η_μ is given by:

$$\eta_\mu(\nu) := ((\mathrm{ev}_{(\nu,\mu)} \circ d^{(m)} \circ \pi)p_1, \ldots, (\mathrm{ev}_{(\nu,\mu)} \circ d^{(m)} \circ \pi)p_r) \in \mathbb{C}^r.$$

Here π denotes the composition $S(\mathfrak{g}) \to S(\mathfrak{h}) \hookrightarrow S(\mathfrak{g})$, where the map $S(\mathfrak{g}) \to S(\mathfrak{h})$ is given by restriction along the triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$.

Example 3.2.5. Consider the case $\mathfrak{g} = \mathfrak{sl}_2$ and $m = 1$. We let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}_2$ and write x_i for $x \otimes t^i$. The algebra $S(\mathfrak{sl}_2)^{SL_2}$ is a polynomial algebra generated by the element $p = \frac{1}{2}h^2 + 2ef$. Now, we can compute $\xi_\mu(\nu)$ as follows:

$$\begin{aligned}
\xi_\mu(\nu) &= (\mathrm{ev}_{(\nu,\mu)} \circ \pi' \circ \omega \circ d^{(1)})(\frac{1}{2}h^2 + 2ef) \\
&= (\mathrm{ev}_{(\nu,\mu)} \circ \pi' \circ \omega)(2h_0h_1 + 4e_0f_1 + 4f_0e_1) \\
&= (\mathrm{ev}_{(\nu,\mu)} \circ \pi')(2h_0h_1 + 2e_0f_1 + 2f_1e_0 + 2f_0e_1 + 2e_1f_0) \\
&= (\mathrm{ev}_{(\nu,\mu)} \circ \pi')(2h_0h_1 + 4f_0e_1 + 4f_1e_0 + 4h_1) \\
&= \mathrm{ev}_{(\nu,\mu)}(2h_0h_1 + 4h_1) \\
&= 2\nu(h)\mu(h) + 4\mu(h)
\end{aligned}$$

Similarly, we can compute $\eta_\mu(\nu)$:

$$\begin{aligned}\eta_\mu(\nu) &= (\mathrm{ev}_{(\nu,\mu)} \circ d^{(1)} \circ \pi)(\frac{1}{2}h^2 + 2ef)\\ &= (\mathrm{ev}_{(\nu,\mu)} \circ d^{(1)})(\frac{1}{2}h^2)\\ &= \mathrm{ev}_{(\nu,\mu)}(2h_0h_1)\\ &= 2\nu(h)\mu(h)\end{aligned}$$

In particular, two Verma modules $M(\nu,\mu)$ and $M(\nu',\mu)$ lying in the same Jordan block have the same central character if and only if $\xi_\mu(\nu) = \xi_\mu(\nu')$. But by the above calculations, this is true if and only if $\eta_\mu(\nu) = \eta_\mu(\nu')$, and we see this occurs precisely when either $\mu = 0$ or $\nu = \nu'$.

Our approach to determining when Verma modules have the same central character in the general will follow the same approach as the example above. We start with the following lemma:

Lemma 3.2.6. *$\xi_\mu(\nu) - \eta_\mu(\nu)$ depends only on μ_m, and in particular does not depend on ν.*

Proof. It suffices to show that if we take any PBW monomial $x \in S(\mathfrak{g})^{\mathfrak{h}}$ and identify it with an element of $S(\mathfrak{g}_m)^{\mathfrak{h}}$ via the inclusion $S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_m)$ the following holds:

$$d^{(m)}\pi(x) - \pi'\omega d^{(m)}(x) \in U(\mathfrak{h}_m^{(m)}).$$

This is because we can then apply $\mathrm{ev}_{(\nu,\mu)}$ to both terms to see that $\xi_\mu(\nu) - \eta_\mu(\nu)$ depends only on μ_m.

Let $x \in S(\mathfrak{g})^{\mathfrak{h}}$ be a monomial in the standard PBW ordering; we then consider four cases:

Case (1): $x \in S(\mathfrak{h})$. In this case, $\pi(x) = x$ and since $d^{(m)}(x) \in S(\mathfrak{h}_m)$, we have $d^{(m)}\pi(x) = d^{(m)}(x) = \pi'\omega d^{(m)}(x)$ since the restriction of $\pi' \circ \omega$ to $S(\mathfrak{h}_m)$ is the identity map.

Case (2): x is of the form $f_\beta h_{\alpha_1} h_{\alpha_2} \dots h_{\alpha_k} e_\beta$, in which case:

$$\begin{aligned}
d^{(m)}(\pi(x)) &= d^{(m)}(0) = 0 \\
\pi'(\omega d^{(m)}(x)) &= \pi'(\omega(f_{\beta,m} h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} e_{\beta,0} + h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} e_{\beta,m} f_{\beta,0} \\
&+ \sum_j f_{\beta,m} h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_{j-1},m} h_{\alpha_{j+1},m} \dots h_{\alpha_k,m} e_{\beta,m} h_{\alpha_j,0})) \\
&= \frac{1}{2} h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} h_{\beta,m} + \frac{1}{2} h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} h_{\beta,m} + 0 \\
&= h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} h_{\beta,m}.
\end{aligned}$$

The first equality follows from the definition of $d^{(m)}$. To show the second equality, we first observe that π' and ω are both linear. Considering the first term, we see that for any $0 \le i \le j \le k$, we have:

$$\begin{aligned}
&\pi'(h_{\alpha_1,m} \dots h_{\alpha_i,m} e_{\beta,0} h_{\alpha_{i+1},m} \dots h_{\alpha_j,m} f_{\beta,m} h_{\alpha_{j+1},m} \dots h_{\alpha_k,m}) \\
&\quad = \pi'(h_{\alpha_1,m} \dots h_{\alpha_i,m} e_{\beta,0} h_{\alpha_{i+1},m} \dots h_{\alpha_k,m} f_{\beta,m}) \\
&\quad = \pi'(h_{\alpha_1,m} \dots h_{\alpha_j,m} h_{\alpha_{j+1},m} \dots h_{\alpha_k,m} [e_{\beta,0}, f_{\beta,m}] \\
&\quad + \sum_{l \ge i} h_{\alpha_1,m} \cdots h_{\alpha_l,m} [e_{\beta,0}, h_{\alpha_{l+1},m}] h_{\alpha_{l+2},m} \dots h_{\alpha_k,m} f_{\beta,m} \\
&\quad + f_{\beta,m} h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} e_{\beta,0}) \\
&\quad = \pi'(h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} h_{\beta,m} \\
&\quad + \sum_{l \ge i} \beta(h_{\alpha_l}) f_\beta h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} e_{\beta,m} \\
&\quad + f_{\beta,m} h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} e_{\beta,0}) \\
&\quad = h_{\alpha_1,m} h_{\alpha_2,m} \dots h_{\alpha_k,m} h_{\beta,m}
\end{aligned}$$

and

$$
\begin{aligned}
&\pi'(h_{\alpha_1,m}\dots h_{\alpha_i,m}f_{\beta,m}h_{\alpha_{i+1},m}\dots h_{\alpha_j,m}e_{\beta,0}h_{\alpha_{j+1},m}\dots h_{\alpha_k,m})\\
&\quad = \pi'(f_{\beta,m}h_{\alpha_1,m}\dots h_{\alpha_j,m}e_{\beta,0}h_{\alpha_{j+1},m}\dots h_{\alpha_k,m})\\
&\quad = \pi'(\sum_{l\geq j} f_{\beta,m}h_{\alpha_1,m}\dots h_{\alpha_l,m}[e_{\beta,0},h_{\alpha_{l+1},m}]h_{\alpha_{l+2},m}\dots h_{\alpha_k,m}\\
&\quad + f_{\beta,m}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}e_{\beta,0})\\
&\quad = \pi'(\sum_{l\geq j}\beta(h_{\alpha_l})f_{\beta,m}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}e_{\beta,m}\\
&\quad + f_{\beta,m}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}e_{\beta,0})\\
&\quad = 0.
\end{aligned}
$$

Now, $\omega(f_{\beta,m}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}e_{\beta,0})$ is a sum of $(k+2)!$ terms. Half of these terms are of the form

$$\frac{1}{(k+2)!}h_{\alpha_1,m}\dots h_{\alpha_i,m}e_{\beta,0}h_{\alpha_{i+1},m}\dots h_{\alpha_j,m}f_{\beta,m}h_{\alpha_{j+1},m}\dots h_{\alpha_k,m}$$

and half are of the form

$$\frac{1}{(k+2)!}h_{\alpha_1,m}\dots h_{\alpha_i,m}f_{\beta,m}h_{\alpha_{i+1},m}\dots h_{\alpha_j,m}e_{\beta,0}h_{\alpha_{j+1},m}\dots h_{\alpha_k,m}$$

so $\pi'(\omega(f_{\beta,m}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}e_{\beta,0})) = \frac{1}{2}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}h_{\beta,m}$. Similar arguments for the other terms show that

$$\pi'(\omega(h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}e_{\beta,m}f_{\beta,0})) = \frac{1}{2}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_k,m}h_{\beta,m}$$

and

$$\pi'(\omega(f_{\beta,m}h_{\alpha_1,m}h_{\alpha_2,m}\dots h_{\alpha_{j-1},m}h_{\alpha j+1,m}\dots h_{\alpha_k,m}e_{\beta,m}h_{\alpha_j,0})) = 0.$$

Case (3): x is of the form $f_{\alpha_1}\dots f_{\alpha_n}h_{\beta_1}\dots h_{\beta_k}e_\gamma$ where $n>1$ and $\gamma=\sum\alpha_i$ (so in particular $\gamma > \alpha_i$ for all i). In this case:

$$\begin{aligned} d^{(m)}(\pi(x)) &= d^{(m)}(0) = 0 \\ \pi'(\omega d^{(m)}(x)) &= \pi'(\omega(\sum_i f_{\alpha_1,m}\dots f_{\alpha_{i-1},m}f_{\alpha_{i+1},m}\dots f_{\alpha_n,m}h_{\beta_1,m}\dots h_{\beta_k,m}e_{\gamma,m}f_{\alpha_i,0} \\ &+ \sum_j f_{\alpha_1,m}\dots f_{\alpha_n,m}h_{\beta_1,m}\dots h_{\beta_{j-1},m}h_{\beta j+1,m}\dots h_{\beta_k,m}e_{\gamma,m}h_{\beta_j,0} \\ &+ f_{\alpha_1,m}\dots f_{\alpha_n,m}h_{\beta_1,m}\dots h_{\beta_k,m}e_{\gamma,0})) \\ &= 0+0+0 \end{aligned}$$

Case (4): x is of the form $f_{\alpha_1}\dots f_{\alpha_n}h_{\beta_1}\dots h_{\beta_k}e_{\gamma_1}\dots e_{\gamma_p}$, where $n \geq 1, p > 1$ and $\sum \alpha_i = \sum \gamma_k$. In this case:

$$\begin{aligned} d^{(m)}(\pi(x)) =& d^{(m)}(0) = 0 \\ \pi'(\omega d^{(m)}(x)) &= \pi'(\omega(\sum_i f_{\alpha_1,m}\dots f_{\alpha_{i-1},m}f_{\alpha_{i+1},m}\dots f_{\alpha_n,m}h_{\beta_1,m}\dots h_{\beta_k,m}e_{\gamma_1,m}\dots e_{\gamma_p,m}f_{\alpha_i,0} \\ &+ \sum_j f_{\alpha_1,m}\dots f_{\alpha_n,m}h_{\beta_1,m}\dots h_{\beta_{j-1},m}h_{\beta j+1,m}\dots h_{\beta_k,m}e_{\gamma_1,m}\dots e_{\gamma_p,m}h_{\beta_j,0} \\ &+ \sum_l f_{\alpha_1,m}\dots f_{\alpha_n,m}h_{\beta_1,m}\dots h_{\beta_k,m}e_{\gamma_1,m}\dots e_{\gamma_{l-1},m}e_{\gamma_{l+1},m}\dots e_{\gamma_p,m}e_{\gamma_k,0})) \\ &= 0+0+0 \end{aligned}$$ □

Lemma 3.2.7. *For $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ and $\nu, \nu' \in \mathfrak{h}^*$, we have $\chi_{(\nu,\mu)} = \chi_{(\nu',\mu)}$ if and only if $\chi_{(\nu,\mu)}(z_i^{(m)}) = \chi_{(\nu',\mu)}(z_i^{(m)})$ for $i = 1, \dots, r$.*

Proof. Since the centre $Z(\mathfrak{g}_m)$ is generated by the elements $\omega(\partial^{(k)}p_j)$ it follows that $\chi_{(\nu,\mu)} = \chi_{(\nu',\mu)}$ if and only if the characters coincide on these elements. Thus it suffices to show that $\chi_{(\nu,\mu)}(z_i^{(j)}) = \chi_{(\nu',\mu)}(z_i^{(j)})$ for all $j = 0, \dots, m-1$.

We have $p_i \in S(\mathfrak{g}_m^{(m)})$ and it follows from the description of $\partial^{(j)}$ in §2.5.1 that $\partial^{(j)}p_i \in S(\mathfrak{g}_m^{(\geq m-j)})$. Hence we have $z_i^{(j)} \in U(\mathfrak{g}_m^{(\geq m-j)})$ and therefore $\chi_{(\nu,\mu)}(z_i^{(j)})$ only depends on μ. Hence $\chi_{(\nu,\mu)}(z_i^{(j)}) = \chi_{(\nu',\mu)}(z_i^{(j)})$ for all $j < m$, completing the proof. □

We now apply Lemmas 3.2.6 and 3.2.7 to prove Theorem 3.2.2:

Proof of Theorem 3.2.2. Let λ, λ' satisfy the assumptions of the theorem. Thanks to Lemma 3.2.7 we must show that the given condition on λ_0, λ'_0 is equivalent to the condition that $\chi_\lambda(z_i^{(m)}) = \chi_{\lambda'}(z_i^{(m)})$ for all i.

Using Lemma 2.5.5, we have $z_i^{(m)} = \omega(d^{(m)}p_i) + \omega(q_i^{(m)})$. Since $q_i^{(m)} \in S(\mathfrak{g}_m^{\geq 1})$ it follows that $\chi_\lambda(z_i^{(m)}) = \chi_{\lambda'}(z_i^{(m)})$ if and only if $\chi_\lambda \circ \omega \circ d^{(m)}(p_i) = \chi_{\lambda'} \circ \omega \circ d^{(m)}(p_i)$ for all $i = 1, \ldots, r$. We note that this second equality is well-defined because by Lemma 2.5.6 $d^{(m)}$ sends $\mathfrak{h}$-invariants to $\mathfrak{h}$-invariants and ω is $\mathfrak{h}_m$-equivariant.

Setting $\mu := \lambda_{\geq 1} = \lambda'_{\geq 1}$ we can apply Lemma 3.2.6 to see that the central characters χ_λ and $\chi_{\lambda'}$ coincide if and only if $\eta_\mu(\lambda_0) = \eta_\mu(\lambda'_0)$. Let $\pi : \mathfrak{h}^* \to \mathfrak{h}^*/W \cong \mathbb{C}^r$ be the quotient map. It follows from the Chevalley restriction theorem [11, 3.1.37] that we can write this in coordinates as $\pi(\nu) \to (p_1|_{\mathfrak{h}^*}(\nu), \ldots, p_r|_{\mathfrak{h}^*}(\nu))$. Now it is easy to see by a direct comparison of the two definitions that $\eta_\mu(\lambda_0)$ coincides with the differential $d_{\lambda_m}\pi(\lambda_0)$ of the quotient map π.

We now claim that $\ker \eta_\mu = \ker d_{\lambda_m}\pi = \mathbb{C}\Phi_{\lambda_m}$. This implies that $\eta_\mu(\lambda_0) = \eta_\mu(\lambda'_0)$ if and only if $\lambda_0 - \lambda'_0 \in \mathbb{C}\Phi_{\mu_m}$ completing the proof.

To see that $\ker d_{\lambda_m}\pi = \mathbb{C}\Phi_{\lambda_m}$, first observe that it follows from Chevalley's restriction theorem and a result of Richardson [46, Proposition 1.2] that $\dim \ker d_{\lambda_m}\pi = \dim \mathfrak{h} - \dim \mathfrak{z}(\mathfrak{g}^{\lambda_m})$. Note that $\mathbb{C}\Phi_{\lambda_m}$ has a basis consisting of simple roots α such that $h_\alpha \in [\mathfrak{g}^{\lambda_m}, \mathfrak{g}^{\lambda_m}]$, so we have $\dim \mathbb{C}\Phi_{\lambda_m} = \dim \ker d_{\lambda_m}\pi = \dim \ker \eta_\mu$ and so the claim will follow if we can prove the inclusion $\mathbb{C}\Phi_{\lambda_m} \subseteq \ker \eta_\mu$.

Let Δ_{λ_m} be a system of simple roots for Φ_{λ_m} and let $\alpha \in \Delta_{\lambda_m}$. Then it is easily verified that $f_{\alpha,m} \otimes 1_\lambda$ is highest weight of weight (λ_0, μ), and hence $M(\lambda_0, \mu)$ and $M(\lambda_0 + \alpha, \mu)$ have the same central character. Therefore $\alpha \in \ker \eta_\mu$ for any $\alpha \in \Delta_{\lambda_m}$, and hence $\mathbb{C}\Phi_{\lambda_m} \subseteq \ker \eta_\mu$ as required.

□

3.2.2 The parabolic induction functor

Fix a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{b} \subseteq \mathfrak{p}$ with corresponding Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{r}$. In this section we prove that for $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ such that $\mathfrak{g}^{\mu_m} = \mathfrak{l}$ the functor $(\bullet)^{\mathfrak{r}_m}$ is an exact functor from $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ to $\mathcal{O}^{(\mu)}(\mathfrak{l}_m)$, and then use this to prove Theorem 3.2.1. First we need the following lemma.

Proposition 3.2.8. *Let $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ be an indecomposable module and let $\Xi_M \in \mathfrak{h}^*/\mathbb{C}\Phi_{\mu_m}$ be the coset determined by M as in Corollary 3.2.3. If $\mathfrak{l} = \mathfrak{g}^{\mu_m}$ is a standard Levi subalgebra then*

$$M^{\mathfrak{r}_m} = \bigoplus_{\nu \in \Xi_M} M^\nu.$$

Proof. First observe that since M is indecomposable, then the weight λ of any highest weight subquotient of M must satisfy $\lambda_m = \mu_m$. We also note that since $\mathfrak{g}^{\mu_m}$ is the Levi factor of a standard parabolic, we have $\mathfrak{r} = \operatorname{span}\{e_\alpha : \alpha \in \Phi^+ \setminus \Phi_{\mu_m}\}$. By Lemma 3.1.5 we have a finite filtration $0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ such that M_i/M_{i-1} has highest weight $\lambda^{(i)} \in \mathfrak{h}_m^*$. By Lemma 3.1.3, the weights of M lie in the set

$$\bigcup_{i=1}^{k} \{\lambda_0^{(i)} - \mathbb{Z}_{\geq 0}\Phi^+\}.$$

Furthermore, by Corollary 3.2.3 we have $\lambda_0^{(i)} + \mathbb{C}\Phi_{\mu_m} = \Xi_M$ for all i. It follows that the weights of M actually lie in the set $\lambda^{(i)} + \mathbb{C}\Phi_{\mu_m} - \sum_{\beta \in \Phi^+} \mathbb{Z}_{\geq 0}\beta$, for any choice of i.

In particular if $\nu \in \mathfrak{h}^*$ satisfies $\nu \in \lambda_0^{(j)} + \mathbb{C}\Phi_{\mu_m}$ for some j, then for any $\alpha \in \Phi^+ \setminus \Phi_{\mu_m}$, we have that $\nu + \alpha$ does not lie in $\lambda_0^{(i)} - \mathbb{Z}_{\geq 0}\Phi^+$ for any i. Therefore $\mathfrak{r}_m \cdot M^\nu = 0$.

Conversely, suppose $v \in M^{\mathfrak{r}_m}$ is of weight $\nu \in \mathfrak{h}^*$. Since $\mathfrak{h}_m$ acts locally finitely and preserves weight spaces we can find a common eigenvector for $\mathfrak{h}_m$ of weight ν in $U(\mathfrak{h}_m) \cdot v$. Suppose the eigenvalue of this eigenvector is $\lambda' \in \mathfrak{h}_m^*$ (which by assumption satisfies $\lambda'_0 = \nu$). Then a quotient of $M(\lambda')$ occurs as a submodule of M. All highest weight modules are indecomposable since they admit unique maximal submodules, so the generalised central character of M must be $\chi_{\lambda'}$. Now Theorem 3.2.2 implies that

$\nu \in \lambda_0' + \mathbb{C}\Phi_{\mu_m}$. By Corollary 3.2.3 we see that for all i, λ_0' and $\lambda_0^{(i)}$ lie in the same coset of $\mathfrak{h}^*$ modulo $\mathbb{C}\Phi_{\mu_m}$ and so $\nu \in \lambda_0^{(i)} + \mathbb{C}\Phi_{\mu_m} = \Xi_M$. □

Corollary 3.2.9. *Suppose that $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ such that $\mathfrak{g}^{\mu_m} = \mathfrak{l}$ is a standard Levi subalgebra. Then the functor $(\bullet)^{\mathfrak{r}_m} : \mathcal{O}^{(\mu)}(\mathfrak{g}_m) \to \mathcal{O}^{(\mu)}(\mathfrak{g}_m^{\mu_m})$ is exact.*

Proof. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\mathcal{O}^{(\mu)}(\mathfrak{g}_m)$. Since f is injective, the restriction of f to $L^{\mathfrak{r}_m}$ is still injective. If $m \in \ker(g) \cap M^{\mathfrak{r}_m}$, then $m = f(l)$ for some $l \in L$ and for any $r \in \mathfrak{r}_m$, we have $f(r \cdot l) = r \cdot m = 0$, so $r \cdot l \in \ker(f) = 0$. Hence $l \in L^{\mathfrak{r}_m}$, and therefore $0 \to L^{\mathfrak{r}_m} \xrightarrow{f} M^{\mathfrak{r}_m} \xrightarrow{g} N^{\mathfrak{r}_m}$ is exact, i.e. we have left exactness of $(\bullet)^{\mathfrak{r}_m}$.

To complete the proof, we need only to show that $g : M^{\mathfrak{r}_m} \to N^{\mathfrak{r}_m}$ is surjective, and by Theorem 3.2.2 it suffices to consider the case where M and N both have filtrations by highest weight modules of weights (λ_i, μ) where $\lambda_i - \lambda_j \in \mathbb{Z}\Phi_{\mu_m}$ for all i, j. Let $v \in N^{\mathfrak{r}_m}$ have weight λ. By Proposition 3.2.8, we must have $\lambda \in \lambda_i + \mathbb{Z}\Phi_{\mu_m}$ for any λ_i. There then exists some $w \in g^{-1}(v)$ which is also of weight λ, so it is enough to show that any element of M of weight $\lambda \in \lambda_i + \mathbb{Z}\Phi_\mu$ is in $M^{\mathfrak{r}_m}$. But the weight of any element of M lies in $\bigcup(\lambda_i - \mathbb{Z}_{\geq 0}\Phi^+)$. In particular, if $\alpha \in \Phi^+ \backslash \Phi_{\mu_m}$ and $\lambda \in \lambda_i + \mathbb{C}\Phi_{\mu_m}$, then $\lambda + \alpha \nleq \lambda_i$ for any λ_i, so $M^{\lambda+\alpha} = 0$. Hence if $v \in M$ is of weight $\lambda \in \lambda_i + \mathbb{C}\Phi_{\mu_m}$, then for any $e_{\alpha,m} \in \mathfrak{r}_m$ we have $e_{\alpha,m} \cdot v \in M^{\lambda+\alpha} = 0$ so $v \in M^{\mathfrak{r}_m}$ as required. □

Now we have exactness of $(\bullet)^{\mathfrak{r}_m}$, we can finally prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Recall from the discussion following the statement of Theorem 3.2.1 that it suffices to show that the maps $\psi_M : M \to (U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M)^{\mathfrak{r}_m}$ sending $v \mapsto 1 \otimes v$ and $\varphi_N : U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} N^{\mathfrak{r}_m} \to N$ sending $u \otimes v \mapsto u \cdot v$ are always isomorphisms.

The PBW theorem implies that $U(\mathfrak{g}_m)$ is free over $U(\mathfrak{p}_m)$ and hence $U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} (\bullet)$ is an exact functor. Furthermore $(\bullet)^{\mathfrak{r}_m}$ is exact by Corollary 3.2.9. It is therefore enough to check that φ_N and ψ_M are isomorphisms on highest weight modules, since a standard argument using the length of a highest weight filtration from Lemma 3.1.5 will allow us to conclude that φ_N and ψ_M are in fact isomorphisms for all $N \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ and all $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m^{\mu_m})$

The map ψ_M is an isomorphism for highest weight modules M thanks to Proposition 3.2.8. Now suppose that $N \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ is a highest weight module with highest weight generator v. Since $\mathfrak{r}_m \cdot v = 0$ it follows that v lies in the image of φ_N and so φ_N is surjective. To prove injectivity, let $K = \ker(\varphi_N)$ and consider the short exact sequence

$$0 \to K \hookrightarrow U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} N^{\mathfrak{r}_m} \overset{\varphi_N}{\to} N \to 0.$$

By Corollary 3.2.9 we have another short exact sequence

$$0 \to K^{\mathfrak{r}_m} \hookrightarrow (U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} N^{\mathfrak{r}_m})^{\mathfrak{r}_m} \to N^{\mathfrak{r}_m} \to 0.$$

Now set $M = N^{\mathfrak{r}_m}$, which is a highest weight $U(\mathfrak{g}_m^{\mu_m})$-module generated by v. The map $(U(\mathfrak{g}_m) \otimes_{U(\mathfrak{p}_m)} M)^{\mathfrak{r}_m} \to M$ is a $U(\mathfrak{g}_m)$-equivariant map uniquely determined by $1 \otimes m \mapsto m$. Therefore it is the left inverse of ψ_M, which we already know to be bijective. It follows that $K^{\mathfrak{r}_m} = 0$, but since every nonzero object in $\mathcal{O}(\mathfrak{g}_m)$ admits a nonzero highest weight vector, it follows that $K = 0$ and so φ_N is an isomorphism for all highest weight modules N, completing the proof. $\square$

3.3 Twisting Functors

3.3.1 Definition of twisting functors

Now we proceed to state and prove a precise version of Theorem 1.1.2. Throughout this section we fix a simple root $\alpha \in \Delta$ and write $U := U(\mathfrak{g}_m)$ for the sake of brevity.

Recall that a right Ore set S in a non-commutative ring R is a multiplicatively closed subset such that for all $r \in R$ and $s \in S$ there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$.

A left Ore set is defined dually. In order for R to admit a right ring of quotients with respect to S it is necessary that S be a right Ore set. When R has no zero divisors this condition is also sufficient, and if S is both a right and left Ore set the left and right quotients are isomorphic; see [35, §2.1].

Also recall the notation $e_{\alpha,i}, h_{\alpha,i}, f_{\alpha,i}$ from Section 2.5. Let F_α be the multiplicative subset of U generated by $\{f_{\alpha,0}, \ldots, f_{\alpha,m}\}$ and note that these elements are pairwise commutative.

Lemma 3.3.1. *F_α is both a left and right Ore set in U.*

Proof. We show that F_α is a left Ore set; the proof that it is a right Ore set is almost identical. Let $x \in \mathfrak{g}_m \subseteq U$, fix $n = 0, \ldots, m$ and let $j \geq 0$. It is easily verified by induction that

$$f_{\alpha,n}^j x = \sum_{k=0}^{j} \binom{j}{k} (\operatorname{ad} f_{\alpha,n})^k (x) f_{\alpha,n}^{j-k}$$

But $\operatorname{ad}(f_{\alpha,n})$ is nilpotent, so there exists l such that $(\operatorname{ad} f_{\alpha,n})^l = 0$. Hence for sufficiently large j, we have $f_{\alpha,n}^j x = u_j f_{\alpha,n}^{j-l}$ for some $u_j \in U$. Applying this repeatedly, we see that given a PBW monomial $v \in U$ and some $i \geq 0$, we can find $u \in U$ and $j \geq 0$ such that $f_{\alpha,n}^j v = u f_{\alpha,n}^i$. But U is spanned by such monomials v, so we have shown that $\{f_{\alpha,n}^i : i \geq 0\}$ is a left Ore set. Combining these statements for $n = 0, \ldots, m$ it follows that F_α is a left Ore set since F_α is commutative. □

Now let U_α be the localisation of U with respect to F_α. We wish to explicitly describe a basis for U_α. We introduce the following notation: Let

$$a : \Phi^+ \times \{0, 1, \ldots, m\} \to \mathbb{Z}_{\geq 0}$$

$$b : \Delta \times \{0, 1, \ldots, m\} \to \mathbb{Z}_{\geq 0}$$

$$c : \Phi^+ \setminus \{\alpha\} \times \{0, 1, \ldots, m\} \to \mathbb{Z}_{\geq 0}$$

be arbitrary maps of sets, and define

$$v(a,b,c) := \Big(\prod_{i=0}^{n}\prod_{\beta\in\Phi^+} e_{\beta,i}^{a_{\beta,i}}\Big)\Big(\prod_{i=0}^{n}\prod_{\beta\in\Delta} h_{\beta,i}^{b_{\beta,i}}\Big)\Big(\prod_{i=0}^{n}\prod_{\beta\in\Phi^+\setminus\{\alpha\}} f_{\beta,i}^{c_{\beta,i}}\Big) \in U$$

where we take the product with respect to some fixed choice of ordering on the basis of $\mathfrak{g}$. These elements can also be described as precisely the PBW monomials in U which have no factor in F_α.

Lemma 3.3.2. *Let a, b, c, and $v(a,b,c)$ be as above. Then*

(i) A basis for U_α is given by the elements $f_{\alpha,0}^{i_0}\cdots f_{\alpha,m}^{i_m}v(a,b,c)$ where $i_j\in\mathbb{Z}$.

(ii) A basis for U_α is given by the elements $v(a,b,c)f_{\alpha,0}^{i_0}\cdots f_{\alpha,m}^{i_m}$ where $i_j\in\mathbb{Z}$.

Proof. Note that U_α is spanned by unordered monomials in $\mathfrak{g}_m$ and the $f_{\alpha,i}^{-1}$. Using the left Ore condition we can rewrite any such monomial as a span of monomials of the form described in (i). Furthermore, if there is a linear dependence between the latter monomials, we can left multiply by appropriate elements of F_α to obtain a linear dependence between PBW monomials in U, which must be zero. This proves (i), and (ii) follows by a symmetrical argument. □

We now give more precise relations between certain generators of U_α, which will be useful later.

Lemma 3.3.3. *For any $i,j\in\mathbb{Z}$, the following relations hold in U_α, for any $h\in\mathfrak{h}$ and any $\beta\in\Phi^+\setminus\{\alpha\}$:*

(i) $[e_{\alpha,i}, f_{\alpha,j}^{-1}] = -f_{\alpha,j}^{-2}h_{\alpha,i+j} - 2f_{\alpha,j}^{-3}f_{\alpha,i+2j}$.

(ii) $[h_i, f_{\alpha,j}^{-1}] = \alpha(h)f_{\alpha,j}^{-2}f_{\alpha,i+j}$.

(iii) $[e_{\beta,i}, f_{\alpha,j}^{-1}] = r_1 f_{\alpha,j}^{-2}e_{\beta-\alpha,i+j} + r_2 f_{\alpha,j}^{-3}e_{\beta-2\alpha,i+2j} + r_3 f_{\alpha,j}^{-4}e_{\beta-3\alpha,i+3j}$

for some $r_1,r_2,r_3\in\mathbb{C}$. We adopt the convention that $e_{\gamma,i}=0$ if $\gamma\notin\Phi$ or if $i>m$.

Proof. These relations can be verified by multiplying by powers of $f_{\alpha,j}$ to obtain an expression which holds in U. We give an explicit calculation for part (i); the other relations can be proved similarly (in part (iii) we require the general fact that $\beta - 4\alpha$ is never a root for any roots α and β). Using the relations in U we have:

$$f_{\alpha,j}^3 e_{\alpha,i} = f_{\alpha,j}^2 e_{\alpha,i} f_{\alpha,j} - f_{\alpha,j} h_{\alpha,i+j} f_{\alpha,j} - 2 f_{\alpha,i+2j} f_{\alpha,j}.$$

We then multiply on the left by $f_{\alpha,j}^{-3}$ and on the right by $f_{\alpha,j}^{-1}$ to obtain (i). □

Let $V_\alpha \subseteq U_\alpha$ be the span of the monomials appearing in Lemma 3.3.2(i) such that $i_j \geq 0$ for some $j = 0, \ldots, m$. We then have the following lemma:

Lemma 3.3.4. *V_α is a U-U-sub-bimodule of U_α.*

Proof. We first show that V_α is a right U-module. Let i_l, j_l be such that at least one $i_l \geq 0$ and all $j_l \geq 0$, and let $v = v(a,b,c)$ and $v' = v(a',b',c')$ for appropriate functions a, b, c and a', b', c'. Then by the PBW theorem we have $v f_{\alpha,0}^{j_0} \ldots f_{\alpha_m}^{j_m} v' = \sum_p (f_{\alpha,0}^{k_{p_0}} \ldots f_{\alpha,m}^{k_{p_m}} v_p)$ for some $k_{p_l} \geq 0$, and $v_p = v(a_p, b_p, c_p)$ for suitable functions a_p, b_p, c_p. Hence:

$$f_{\alpha,0}^{i_0} \ldots f_{\alpha,m}^{i_m} v \cdot (f_{\alpha,0}^{j_0} \ldots f_{\alpha,m}^{j_m} v') = \sum_p (f_{\alpha,0}^{i_0} \ldots f_{\alpha,m}^{i_m} f_{\alpha,0}^{k_{p_0}} \ldots f_{\alpha,m}^{k_{p_m}} v_p)$$

$$= \sum_p (f_{\alpha_0}^{i_0 + k_{p_0}} \ldots f_{\alpha,m}^{i_m + k_{p_m}} v_p)$$

In particular this expression lies in V_α. But U is spanned by elements of the form $f_{\alpha,0}^{j_0} \ldots f_{\alpha_m}^{j_m} v'$, so V_α is a right U-module.

We now claim that V_α is a span of monomials of the type appearing in 3.3.2(ii) with $i_j \geq 0$ for some j; once we have this, it follows that V_α is a left U-module by a similar argument to above. In general, if $i_0, \ldots, i_m \in \mathbb{Z}$ and $v = v(a,b,c)$ for appropriate a, b, c then using the fact that F_α is an Ore set, in U_α we have:

$$v f_{\alpha,0}^{i_0} \ldots f_{\alpha,m}^{i_m} = \sum_p f_{\alpha,0}^{i_{p_0}} \ldots f_{\alpha,m}^{i_{p_m}} v_p$$

where $i_{p_l} \geq 0$ and the v_p are of the same form as v. In fact, we see that if $i_j \geq 0$, then $i_{p_j} \geq 0$ for all p, so in this case $vf_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m} \in V_\alpha$. Hence

$$V_\alpha \supseteq \mathrm{span}\{v(a,b,c)f_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m} \mid \text{ at least one of the } i_j \geq 0\}.$$

The inclusion

$$V_\alpha \subseteq \mathrm{span}\{v(a,b,c)f_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m} \mid \text{ at least one of the } i_j \geq 0\}$$

follows by a similar argument, completing the proof. □

We now consider the U-U-bimodule $S_\alpha := U_\alpha/V_\alpha$. First we observe the following consequence of Lemma 3.3.2.

Corollary 3.3.5. *S_α has a basis given by*

$$\{f_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m} v(a,b,c) \mid i_j < 0 \textit{ for all } j\}$$

and another basis given by:

$$\{v(a,b,c)f_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m} \mid i_j < 0 \textit{ for all } j\}$$

We will abuse notation slightly and usually denote an element of U_α and its coset in S_α by the same symbol. At several points we will use the fact that in S_α the element $f_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m}$ is zero if any $i_j \geq 0$.

Now we pick a special automorphism of $\mathfrak{g}_m$. Just as in §2.3.3, the simple root α we fixed earlier gives rise to a reflection $s_\alpha \in W = N_G(\mathfrak{h})/\mathfrak{h}$. We lift s_α arbitrarily to an element of $N_G(\mathfrak{h})$, which gives an automorphism of $\mathfrak{g}$ via the adjoint representation. We denote this automorphism by ϕ_α. Note that ϕ_α permutes the root spaces as s_α, i.e. it sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{s_\alpha(\beta)}$ and preserves $\mathfrak{h}$. Furthermore, after rescaling e_α and f_α if necessary, we may assume that $\phi_\alpha(e_\alpha) = f_\alpha$ and $\phi_\alpha(f_\alpha) = e_\alpha$. We extend ϕ_α to an automorphism of $\mathfrak{g}_m$ using the rule $\phi_\alpha(xt^i) = \phi_\alpha(x)t^i$ for all $x \in \mathfrak{g}$; by abuse of notation we also denote this automorphism ϕ_α.

If M is a left U-module, we let $\phi_\alpha(M)$ denote the module obtained by twisting the left action on M by ϕ_α and write $\cdot_\alpha$ for this action. More precisely, if $v \in M$ and $u \in U$ then $u \cdot_\alpha v := \phi_\alpha(u) \cdot v$. Similarly, we can also twist the action on M by ϕ_α^{-1}, and we use the notation $\cdot_{\alpha^{-1}}$ for the twisted action in this case.

Now let $\mathcal{C}$ be the full subcategory of U-mod whose objects are the $\mathfrak{h}$-semisimple modules.

Lemma 3.3.6. *$\mathcal{O}(\mathfrak{g}_m)$ is a Serre subcategory of the category $\mathcal{C}$.*

Proof. By Corollary 3.1.8, it suffices to show that for all $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ and all $M_1, M_2 \in \mathcal{O}^{(\mu)}$, we have that $M_1, M_2 \in \mathcal{C}$ and any $\mathfrak{h}$-semisimple extension M between M_1 and M_2 lies in $\mathcal{O}^{(\mu)}$. By the definition of $\mathcal{O}^{(\mu)}$, if $M_i \in \mathcal{O}^{(\mu)}$ then M_i must be $\mathfrak{h}$-semisimple and hence lie in $\mathcal{C}$. On the other hand, to show $M \in \mathcal{O}^{(\mu)}$, we must show that:

(i) M is finitely generated.

(ii) For all $h \in \mathfrak{h}$ and $1 \leq i \leq m$, $h_i - \mu_i(h)$ acts locally nilpotently on M (where we recall that we use h_i as shorthand for $h \otimes t^i$).

(iii) The subalgebra $\mathfrak{h}_{(\geq 1)}$ acts locally finitely on M.

(iv) The subalgebra $\mathfrak{n}_m \subseteq \mathfrak{g}_m$ acts locally nilpotently on M.

For (i), let X_1 be a finite generating set for M_1 and let X_2 be a finite generating set for M_2. Let X_2' be a set containing a choice of preimage for each element of X_2 under the surjection $M \to M_2$. Then $X_1 \cup X_2'$ is a finite generating set for M. For (ii), let $v \in M$. Then for some $j \geq 0$, $(h_i - \mu_i(h))^j \cdot (v + M_1) = 0 \in M/M_1 \cong M_2$, so $(h_i - \mu_i(h))^j \cdot v =: v' \in M_1$, and for some $k \geq 0$, $(h_i - \mu_i(h))^k \cdot v' = 0$, so we have $(h_i - \mu_i(h))^{j+k} \cdot v = 0$. Finally, we have that $\dim(M^\lambda) = \dim(M_1^\lambda) + \dim(M_2^\lambda)$ for any $\lambda \in \mathfrak{h}^*$, so in particular the weight spaces of M are finite-dimensional and bounded above. Hence (iii) and (iv) hold. □

Let $\mathcal{H} : U\text{-mod} \to \mathcal{C}$ be the functor given by letting $\mathcal{H}(M)$ be the sum of the $\mathfrak{h}$-weight spaces in M, and then define two endofunctors of $\mathcal{C}$ by setting:

$$T_\alpha M := \phi_\alpha(S_\alpha \otimes_U M)$$

$$G_\alpha M := \mathcal{H}(\mathrm{Hom}_U(S_\alpha, \phi_\alpha^{-1}(M))).$$

Note that the action of U on $G_\alpha M$ is given by

$$(u \cdot f)(s) = f(s \cdot u) \text{ for } u \in U,\ f \in G_\alpha M,\ s \in S_\alpha.$$

Also note that if $M, N \in \mathcal{C}$ and $\chi \in \mathrm{Hom}_{\mathcal{C}}(M, N)$, then $T_\alpha(\chi) : T_\alpha M \to T_\alpha N$ and $G_\alpha(\chi) : G_\alpha M \to G_\alpha N$ are given by:

$$T_\alpha(\chi)(s \otimes m) = s \otimes \chi(m) \text{ for } s \in S_\alpha,\ m \in M$$

$$G_\alpha(\chi)(\rho) = \chi \circ \rho \text{ for } \rho \in \mathrm{Hom}_U(S_\alpha, \phi_\alpha^{-1}(M)))$$

.

In order to see these functors are well defined, the only non-trivial conditon to check is that $T_\alpha M \in \mathcal{C}$ for any $M \in \mathcal{C}$. For $M \in \mathcal{C}$, we have that $T_\alpha M$ is spanned by $\{f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes w : i_j > 0, w \in M\}$, and so $T_\alpha M \in \mathcal{C}$ will follow from:

Lemma 3.3.7. *For any $w \in M^\lambda$, we have*

$$f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes w \in (T_\alpha M)^{s_\alpha(\lambda) - (i_0 + \cdots + i_m)\alpha}.$$

Proof. For $w \in M^\lambda$ and $h \in \mathfrak{h}$ we have

$$\begin{aligned} h \cdot_\alpha (f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes w) &= (s_\alpha(h) f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \otimes w \\ &= f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} (s_\alpha(h) - (i_0 + \dots + i_m)\alpha(h)) \otimes w \\ &= f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes (s_\alpha(h) - (i_0 + \dots + i_m)\alpha(h)) w \\ &= f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes (s_\alpha(\lambda) - (i_0 + \dots + i_m)\alpha)(h) w \\ &= (s_\alpha(\lambda) - (i_0 + \dots + i_m)\alpha)(h)(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes w). \end{aligned}$$

and the desired result follows. □

The following result, characterising weight elements of $G_\alpha M$, will be very useful later.

Lemma 3.3.8. *Let $g \in \mathrm{Hom}_U(S_\alpha, \phi_\alpha^{-1}(M))$. Then g has weight $\lambda \in \mathfrak{h}^*$ if and only if $g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m})$ has weight $\lambda + (i_0 + \dots + i_m)\alpha$ in $\phi_\alpha^{-1}(M)$ for all $i_0, \dots, i_m \in \mathbb{Z}_{\geq 0}$, i.e. if and only if this vector has weight $s_\alpha(\lambda) - (i_0 + \dots + i_m)\alpha$ in M.*

Proof. First assume that g has weight λ. Then for any $h \in \mathfrak{h}$ and $i_j \geq 0$:

$$\begin{aligned} \lambda(h) g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) &= (h \cdot g)(f_{\alpha_0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \\ &= g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} h) \\ &= g((h - (i_0 + \dots + i_m)\alpha(h)) f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \\ &= (h - (i_0 + \dots + i_m)\alpha(h)) \cdot g(f_{\alpha,0}^{-i_0} f_{\alpha,m}^{-i_m}) \end{aligned}$$

so $g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m})$ has weight $\lambda + (i_0 + \dots + i_m)\alpha$ in $\phi_\alpha^{-1}(M)$, and hence has weight $s_\alpha(\lambda) - (i_0 + \dots + i_m)\alpha$ in M.

On the other hand, suppose $g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m})$ has weight $s_\alpha(\lambda) - (i_0 + \dots + i_m)\alpha$ in M for all $i_j \geq 0$. Then for any $h \in \mathfrak{h}, i_j \geq 0$, $u = v(a, b, c)$ for functions a, b, c of the appropriate form we have:

$$\begin{aligned}(h\cdot g)(uf_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m}) &= g(uf_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m}h)\\ &= u\cdot g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m}h)\\ &= u\cdot g((h-(i_0+\cdots+i_m)\alpha(h))f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})\\ &= u\cdot (h-(i_0+\cdots+i_m)\alpha(h))\cdot g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})\\ &= u\cdot\lambda(h)g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})\\ &= \lambda(h)g(uf_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})\end{aligned}$$

and hence g has weight λ. □

We then obtain the following corollary:

Corollary 3.3.9. *Let $g\in\mathrm{Hom}_U(S_\alpha,\phi_\alpha^{-1}(M))$, Then g is a weight vector if and only if $g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})$ is a weight vector for all $i_j\geq 0$. In particular, any element of $G_\alpha(M)$ is the direct sum of such weight vectors.*

Proof. Suppose $g\in\mathrm{Hom}_U(S_\alpha,\phi_\alpha^{-1}(M))$ is such that $g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})$ is a weight vector for all $i_j\geq 0$. Since $e_{\alpha,k}\cdot_{\alpha^{-1}} g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m}) = g(f_{\alpha_0}^{-i_0}\dots f_{\alpha,k}^{-i_k+1}\dots f_{\alpha,m}^{-i_m})$, in $\phi_\alpha^{-1}(M)$ we have that $\mathrm{wt}(g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m}))+\alpha=\mathrm{wt}(g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,k}^{-i_k+1}\dots f_{\alpha,m}^{-i_m}))$ for all $0\leq k\leq m$. Hence if in $\phi_\alpha^{-1}(M)$ the weight of $g(f_{\alpha,0}^{-1}\dots f_{\alpha,m}^{-1})$ is λ', we have that the weight of $g(f_{\alpha,0}^{-i_0}\dots f_{\alpha,m}^{-i_m})$ is $\lambda'-2\alpha+(i_0+\cdots+i_m)\alpha$. Applying Lemma 3.3.8, we see that g is indeed a weight vector. □

The following easy lemma is the first step in our ultimate goal of showing that the functors T_α and G_α are equivalences between appropriate Jordan blocks of $\mathcal{O}$.

Lemma 3.3.10. *T_α is right exact and G_α is left exact.*

Proof. For any module $M\in U$-mod, we have

$$\phi_\alpha(S_\alpha\otimes_U M)\cong\phi_\alpha(S_\alpha)\otimes_U M$$

and since $\phi_\alpha(S_\alpha) \otimes_U (\bullet)$ is right exact, T_α is also right exact.

Similarly, G_α is a composition of two left exact functors $\mathrm{Hom}_U(S_\alpha, \phi_\alpha^{-1}(\bullet))$ and $\mathcal{H}$ and so is left exact. □

3.3.2 Twisting functors between blocks of category $\mathcal{O}$

We now observe that the Weyl group action on $\mathfrak{h}^*$ extends naturally to an action on $(\mathfrak{h}_m^{\geq 1})^*$ by letting W act diagonally through the identification $(\mathfrak{h}_m^{\geq 1})^* = (\mathfrak{h}^*)^m$, and this vector space parameterises the Jordan blocks of $\mathcal{O}(\mathfrak{g}_m)$. Retaining the notation of the previous section, we can now state a precise version of Theorem 1.1.2.

Theorem 3.3.11. *Let $\mu \in (\mathfrak{h}_m^{\geq 1})^*$ be such that $\mu(h_{\alpha,m}) \neq 0$. The functors T_α and G_α restrict to functors*

$$\begin{aligned} T_\alpha &: \mathcal{O}^{(\mu)}(\mathfrak{g}_m) \longrightarrow \mathcal{O}^{(s_\alpha(\mu))}(\mathfrak{g}_m) \\ G_\alpha &: \mathcal{O}^{(s_\alpha(\mu))}(\mathfrak{g}_m) \longrightarrow \mathcal{O}^{(\mu)}(\mathfrak{g}_m) \end{aligned}$$

which form a quasi-inverse pair of equivalences.

The proof of Theorem 3.3.11 is broken down into a series of lemmas which we record and prove over the course of this section. The theorem will follow directly from combining Lemmas 3.3.12, 3.3.19, 3.3.22, 3.3.23 and 3.3.24.

For the rest of the section we keep $\alpha \in \Delta$ fixed and let $\mu \in (\mathfrak{h}_m^{\geq 1})^*$ be such that $\mu(h_{\alpha,m}) \neq 0$. The following result is the first step in the proof of Theorem 3.3.11.

Lemma 3.3.12. *T_α restricts to a functor $\mathcal{O}^{(\mu)}(\mathfrak{g}_m) \to \mathcal{O}^{(s_\alpha(\mu))}(\mathfrak{g}_m)$.*

Proof. For $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ let $l(M)$ denote the minimal length of a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k = M$ such that the sections M_i/M_{i-1} are all highest weight modules; such a filtration exists by Lemma 3.1.5. We have an exact sequence:

$$0 \to M_1 \to M \to M/M_1 \to 0$$

and since T_α is right exact, we have another exact sequence

$$T_\alpha M_1 \to T_\alpha M \to T_\alpha(M/M_1) \to 0$$

Using Lemma 3.3.6 and the fact that $T_\alpha M_1$ is a highest weight module, we can reduce the claim that $T_\alpha(M) \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ to the case where $l(M) = 1$, i.e. M is a highest weight module.

Since $T_\alpha M$ is $\mathfrak{h}$-semisimple and a quotient of a highest weight module is still highest weight, by induction and Lemma 3.3.6 it suffices to show that if M is highest weight of weight $(\lambda, \mu_1, \ldots, \mu_m)$, then $T_\alpha M$ is highest weight of weight $(\lambda', s_\alpha(\mu_1), \ldots, s_\alpha(\mu_m))$ for some $\lambda' \in \mathfrak{h}^*$.

For the rest of the proof, fix M highest weight of weight $\lambda \in \mathfrak{h}_m^*$, and let $v \in M$ be a highest weight generator of M. We aim to show that $f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes v$ is highest weight and generates $T_\alpha M$, which will complete the proof of the lemma.

To see that $f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes v$ is maximal, we use Lemma 3.3.7 to see that for any $i \geq 0$, the element $e_{\alpha,i} f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes v \in T_\alpha M$ lies in an $\mathfrak{h}$-eigenspace whose weight is not a weight of $T_\alpha M$. To see that it is a genuine highest weight vector, one can use Lemma 3.3.3(ii) to show that h_i acts via $s_{\alpha_i}(h_i)$.

To see that $f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes v$ generates $T_\alpha M$ we use the fact that $Uv = M$ and the two bases of S_α from Corollary 3.3.5, to check that every element of $T_\alpha M$ lies in the submodule generated by the set $\{f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m} \otimes v \mid i_k > 0\}$. Let L denote the span of this set. Note that it is an $(\mathfrak{sl}_2)_m$-module, where $(\mathfrak{sl}_2)_m$ is the truncated current Lie algebra on $(\mathfrak{sl}_2)_\alpha := \langle e_\alpha, h_\alpha, f_\alpha \rangle$. To complete the proof we show that L is a simple $(\mathfrak{sl}_2)_m$-module.

Let $\mathfrak{t}_m \subseteq (\mathfrak{sl}_2)_m$ denote the span of $h_{\alpha,0}, \ldots, h_{\alpha,m}$ and let $\gamma = \lambda|_{\mathfrak{t}_m} \in \mathfrak{t}_m^*$ give the action of $\mathfrak{t}_m$ on $f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes v$. If $M(\gamma)$ denotes the Verma module of highest weight γ then there is a nonzero homomorphism $M(\gamma) \to L$ and the dimensions of the weight spaces are the same. Therefore it remains to show that the Verma module $M(\gamma)$ is simple. By Lemma 3.3.3(ii) we see that $\gamma(h_{\alpha,m}) = (s_\alpha \lambda)(h_{\alpha,m}) \neq 0$ and so we can apply Theorem 3.2.1 to see that $\mathcal{O}^{(\gamma_{\geq 1})}((\mathfrak{sl}_2)_m)$ is equivalent to $\mathcal{O}^{(\gamma_{\geq 1})}(\mathfrak{t}_m)$. In the latter category, all highest weight modules are simple, and so it follows that $M(\gamma)$ is simple as required. □

Our next steps towards proving Theorem 3.3.11 are to prove the following results on the invariants of a module $M \in \mathcal{O}^{(\mu)}$ with respect to a certain subalgebra of $\mathfrak{g}$, and then prove a result on elements of the tensor product $S_\alpha \otimes_U M$. Throughout, we let $(\mathfrak{sl}_2)_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle \subseteq \mathfrak{g}$.

Lemma 3.3.13. *Let $\mu \in (\mathfrak{h}_m^{\geq 1})^*$ be such that $\mu_m(h_\alpha) \neq 0$, let $M \in \mathcal{O}^{(\mu)}$, and let $\mathfrak{a} = \langle f_{\alpha,0}, \ldots, f_{\alpha,m} \rangle \subseteq \mathfrak{g}_m$. Then M is free as a $U(\mathfrak{a})$-module, and any basis of $M^{\langle e_{\alpha,0},\ldots,e_{\alpha,m}\rangle}$ as a vector space is a free generating set for M.*

Proof. It suffices to consider only the case where M is indecomposable. First consider the case where $\mathfrak{g} = \mathfrak{sl}_2$ and M is a Verma module. In this case, $M^{\langle e_{\alpha,0},\ldots,e_{\alpha,m}\rangle}$ is one-dimensional, spanned by any highest weight vector of M. But by definition M is generated freely by this as a $U(\mathfrak{n}_m^-)$-module, and in this case $\mathfrak{a} = \mathfrak{n}_m^-$ so the lemma holds.

If $\mathfrak{g} = \mathfrak{sl}_2$ and M is indecomposable but not a Verma module, then by Lemma 3.1.5 and [56, Theorem 7.1] M has a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k = M$ such that each quotient is a Verma module. To show any basis of $M^{\langle e_{\alpha,0},\ldots,e_{\alpha,m}\rangle}$ freely generates M as a $U(\mathfrak{a})$-module, it suffices to check one choice.

Choose $\Omega = \Psi \cup \{v\}$, where Ψ is a basis of $M_{k-1}^{\langle e_{\alpha,0},\ldots,e_{\alpha,m}\rangle}$ and $v \in M^{\langle e_{\alpha,0},\ldots,e_{\alpha,m}\rangle} \setminus M_{k-1}$. Let $M' \subseteq M$ be the $U(\mathfrak{a})$-submodule generated by Ω. Then observe that M/M_{k-1} is a Verma module with highest weight generator $v + M_{k-1}$. Hence for any $w \in M$, there is some $w' \in M'$ such that $w - w' \in M_{k-1}$. But $\Psi \subseteq \Omega$, and by induction Ψ generates M_{k-1} as a $U(\mathfrak{a})$-module, so $M_{k-1} \subseteq M'$. Hence $w - w' \in M'$, and so $w \in M'$ for any $w \in M$, i.e. $M = M'$.

Now we show that Ω generates M freely, i.e. that the set $\Omega' = \{f_{\alpha,0}^{i_0} \ldots f_{\alpha,m}^{i_m} v : i_0, \ldots, i_m \geq 0, v \in \Omega\}$ is linearly independent. We already know that Ω' spans M, so it is enough to show that the number of $f_{\alpha,0}^{i_0} \ldots f_{\alpha,m}^{i_m} v$ of weight λ' is equal to the dimension of $M^{\lambda'}$ for any $\lambda' \in \mathfrak{h}^*$. Now, every quotient M_i/M_{i-1} is isomorphic to the same Verma module $M(\lambda, \mu)$ and hence all $v \in \Omega$ have weight λ and also $\dim(M^{\lambda'}) = k \dim(M(\lambda, \mu)^{\lambda'})$. These two facts together imply that the number of $f_{\alpha,0}^{i_0} \dots f_{\alpha,m}^{i_m} v$ of weight λ' is equal to the dimension of $M^{\lambda'}$, so Ω generates M freely and the lemma holds in the case $\mathfrak{g} = \mathfrak{sl}_2$.

Finally, we deal with the case where $\mathfrak{g}$ is any reductive Lie algebra. Let Υ be a basis of $M^{\langle e_{\alpha,0},\dots,e_{\alpha,m}\rangle}$. Let $v \in M$ and let N be the $((\mathfrak{sl}_2)_\alpha)_m$-submodule of M generated by v. It is easy to check that $N \in \mathcal{O}^{(\mu_1(h_\alpha),\dots,\mu_m(h_\alpha))}((\mathfrak{sl}_2)_\alpha)$, since all the axioms except finite generation follow from the fact $M \in \mathcal{O}^\mu$, while by definition N is generated by one element. Hence we can apply the lemma in the $\mathfrak{sl}_2$ case to see that $v = \sum_j f_{\alpha,0}^{i_{j,0}} \dots f_{\alpha,m}^{i_{j,m}} v_k$, for some $i_{j,k} \geq 0$ and $v_k \in N^{\langle e_{\alpha,0},\dots,e_{\alpha,m}\rangle} \subseteq M^{\langle e_{\alpha,0},\dots,e_{\alpha,m}\rangle} = \operatorname{span} \Upsilon$, so v is in the $U(\mathfrak{a})$-module generated by Υ.

Suppose Υ does not generate M freely as a $U(\mathfrak{a})$-module. Then $\sum_j f_{\alpha,0}^{i_{j,0}} \dots f_{\alpha,m}^{i_{j,m}} b_j = 0$ for some $i_{j,k} \geq 0$ and $b_j \in \Upsilon$. But since this sum is finite, we can let N be the $((\mathfrak{sl}_2)_\alpha)_m$-module generated by $\{b_1, \dots, b_k\}$ and obtain a contradiction to the $\mathfrak{sl}_2$ case. Hence the lemma holds. □

Corollary 3.3.14. *Again, let $\mu \in (\mathfrak{h}_m^{\geq 1})^*$ be such that $\mu_m(h_\alpha) \neq 0$, let $M \in \mathcal{O}^{(\mu)}$, and let N_1 be a finitely generated $((\mathfrak{sl}_2)_\alpha)_m$-submodule of M. Then there exists a $U(\mathfrak{a})$-submodule N_2 of M such that $M = N_1 \oplus N_2$, where again $\mathfrak{a} = \langle f_{\alpha,0}, \dots, f_{\alpha,m}\rangle$.*

Proof. Let Ψ be a basis for $N_1^{\langle e_{\alpha,0},\dots,e_{\alpha,m}\rangle}$, which freely generates N_1 as a $U(\mathfrak{a})$-module by the previous lemma. Since $N_1^{\langle e_{\alpha,0},\dots e_{\alpha,m}\rangle} \subseteq M^{\langle e_{\alpha,0},\dots,e_{\alpha,m}\rangle}$, we can extend Ψ to a basis Ω for $M^{\langle e_{\alpha,0},\dots,e_{\alpha,m}\rangle}$, so if we let N_2 be the $U(\mathfrak{a})$-module generated by $\Omega \setminus \Psi$ then by the previous lemma we have $M = N_1 \oplus N_2$. □

We now recall the following general result about tensor products. Let R be a ring, and let M and N be right and left R-modules respectively, and let V be a $\mathbb{C}$-vector space. We say a $\mathbb{C}$-bilinear map $\varphi : M \times N \to V$ is *R-balanced* if for any $v \in M$, $w \in N$, and $r \in R$, we have $\varphi(v \cdot r, w) = \varphi(v, r \cdot w)$. We then have the following standard result on the tensor product $M \otimes_R N$:

Lemma 3.3.15. *The element $v \otimes w$ of $M \otimes_R N$ is zero if and only if for any vector space V and R-balanced map $\varphi : M \times N \to V$, we have that $\varphi(v, w) = 0$.*

We observe that if we set $\mathfrak{a} = \langle f_{\alpha,0}, \dots, f_{\alpha,m} \rangle$ as above and set $A = \text{span}\{f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} : i_0, \dots, i_m > 0\} \subseteq S_\alpha$, which is a $U(\mathfrak{a})$-$U(\mathfrak{a})$-sub-bimodule of S_α, then we have (by considering the two bases of S_α from Corollary 3.3.5) that as U-$U(\mathfrak{a})$-bimodules:

$$S_\alpha \cong U \otimes_{U(\mathfrak{a})} A,$$

and as $U(\mathfrak{a})$-U-bimodules:

$$S_\alpha \cong A \otimes_{U(\mathfrak{a})} U.$$

In particular, for any left U-module M, we have that as left $U(\mathfrak{a})$-modules:

$$S_\alpha \otimes_U M \cong (A \otimes_{U(\mathfrak{a})} U) \otimes_U M \cong A \otimes_{U(\mathfrak{a})} M.$$

The following lemma, which we will prove using Corollary 3.3.14, Lemma 3.3.15 and the above observation, we be very useful in next part of the proof of Theorem 3.3.11:

Lemma 3.3.16. *Suppose $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ is such that $\mu_m(h_\alpha) \neq 0$ and let $M \in \mathcal{O}^{(\mu)}$. Let $v \in M^\lambda \setminus \{0\}$ for some $\lambda \in \mathfrak{h}^*$. Then the following are equivalent:*

(a) In $T_\alpha M$, we have $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v = 0$.

(b) For any vector space V and $U(\mathfrak{a})$-balanced map $\varphi : A \times M \to V$ we have that $\varphi(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}, v) = 0$.

(c) There exist $v_0, \dots v_m \in M$ such that $v = f_{\alpha,0}^{i_0} \cdot v_0 + \dots + f_{\alpha,m}^{i_m} \cdot v_m$.

Proof. We first note that (a) and (b) are equivalent by the above observation and Lemma 3.3.15. Also, if $m = f_{\alpha,0}^{i_0} \cdot v_0 + \dots + f_{\alpha,m}^{i_m} \cdot v_m$ for some $v_0, \dots, v_m \in M$, then $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v = f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes f_\alpha^{i_0} \cdot v_0 + \dots + f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes f_{\alpha,m}^{i_m} \cdot v_m = f_{\alpha,0}^{0} \dots f_{\alpha,m}^{-i_m} \otimes v_0 + \dots + f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{0} \otimes v_m = 0$, so certainly (c) implies (a), and in fact this is true for any $M \in U$-mod (which we will use later). To show (b) implies (c), suppose $v \in M$ cannot be written in the form $f_{\alpha,0}^{i_0} \cdot v_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v_m$. We seek a $U(\mathfrak{a})$-balanced map $\varphi : A \times M \to V$ such that $\varphi(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}, v) \neq 0$. It suffices to consider the case where M is indecomposable.

First consider the case where $\mathfrak{g} = \mathfrak{sl}_2$ and M is the Verma module $M(\lambda, \mu_1, \dots, \mu_m)$ where $\mu_m \neq 0$. Let $v \in M$. Then $v = \sum k_{a_0,\dots,a_m} f_{\alpha,0}^{a_0} \dots f_{\alpha,m}^{a_m} \otimes 1$, where all but finitely many of the $k_{a_0,\dots,a_m}$ are zero. Observe that v is of the form $v = f_{\alpha,0}^{i_0} \cdot v_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v_m$ for some $v_0, \dots, v_m \in M$ if and only if $k_{a_0,\dots,a_m} = 0$ for all $a_k < i_k$.

Now suppose v is not of the form $v = f_{\alpha,0}^{i_0} \cdot v_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v_m$. Then pick some $0 < a_k \leq i_k$ such that $k_{i_0-a_0,\dots,i_m-a_m} \neq 0$. We can define a $U(\mathfrak{a})$-balanced map $\varphi : A \times M \to \mathbb{C}$ by setting:

$$\varphi(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}, f_{\alpha,0}^{j_0} \dots f_{\alpha,m}^{j_m} \otimes 1) = \begin{cases} 1 & \text{if } i_k - j_k = a_k \text{ for all } 0 \leq k \leq m \\ 0 & \text{otherwise} \end{cases}$$

and extending bilinearly. By construction we have that φ is a $U(\mathfrak{a})$-balanced map such that $\varphi(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}, v) \neq 0$.

Now we consider the case where $\mathfrak{g} = \mathfrak{sl}_2$ and M is indecomposable. If M is a Verma module, we are done by the above. If not, let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k = M$ be a filtration of M such that each section is a highest weight module. In fact, by [56, Theorem 7.1], in this case each Verma module is simple, so each section is in fact a Verma module. Let $v \in M$ be such that v cannot be written as $f_{\alpha,0}^{i_0} \cdot v_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v_m$. Consider the quotient M/M_1. We have one of two cases:

(1) If $v + M_1$ cannot be written as $f_{\alpha,0}^{i_0} \cdot v_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v_m + M_1$, then by induction on k we can find a $U(\mathfrak{a})$-balanced map $\varphi : A \times (M/M_1) \to \mathbb{C}$ such that $\varphi(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}, v + M_1) \neq 0$. We can then lift this to a $U(\mathfrak{a})$-balanced map $\overline{\varphi} : A \times M \to \mathbb{C}$ by setting $\overline{\varphi}(u, v') = \varphi(u, v' + M_1)$, which clearly satisfies $\overline{\varphi}(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}, v) \neq 0$.

(2) If $v + M_1 = f_{\alpha,0}^{i_0} \cdot v'_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v'_m + M_1$, then let $w = v - f_{\alpha,0}^{i_0} \cdot v'_0 - \cdots - f_{\alpha,m}^{i_m} \cdot v'_m \in M_1$. Now w cannot be written as $f_{\alpha,0}^{i_0} \cdot w_0 + \cdots + f_{\alpha,m}^{i_m} \cdot w_m$ or else we could take $v_0 = v'_0 + w_0$

and obtain a contradiction. But M_1 is a Verma module, so we have already shown there exists a $U(\mathfrak{a})$-balanced map $\varphi : A \times M_1 \to \mathbb{C}$ such that $\varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, w) \neq 0$. Now by Corollary 3.3.14, there exists a $U(\mathfrak{a})$-submodule M' of M such that $M = M_1 \oplus M'$. Hence

$$(\varphi \oplus 0) : A \times M \cong A \times (M_1 \oplus M') \cong (A \times M_1) \oplus (A \times M') \to \mathbb{C}$$

is a $U(\mathfrak{a})$-balanced map such that:

$$\begin{aligned}(\varphi \oplus 0)(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, v) &= (\varphi \oplus 0)(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, w + f_{\alpha,0}^{i_0} \cdot v_0' + \cdots + f_{\alpha,m}^{i_m} \cdot v_m') \\ &= \varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, w) \neq 0\end{aligned}$$

Finally, let $\mathfrak{g}$ be an arbitrary reductive Lie algebra, let $M \in \mathcal{O}^{(\mu)}$, and let $v \in M$ be such that v cannot be written as $v = f_{\alpha,0}^{i_0} \cdot v_0 + \cdots + f_{\alpha,m}^{i_m} \cdot v_m$. Let M_1 be the $((\mathfrak{sl}_2)_\alpha)_m$-module generated by v, and let M_2 be a $U(\mathfrak{a})$-module such that $M = M_1 \oplus M_2$ as in Corollary 3.3.14. By the $\mathfrak{sl}_2$ case, we have a $U(\mathfrak{a})$-balanced map $\varphi : A \times M_1 \to \mathbb{C}$ such that $\varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, v) \neq 0$. Then

$$(\varphi \oplus 0) : A \times M \cong A \times (N_1 \oplus N_2) \cong (A \times N_1) \oplus (A \times N_2) \to \mathbb{C}$$

is a $U(\mathfrak{a})$-balanced map such that $(\varphi \oplus 0)(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, v) = \varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}, v) \neq 0$ as required. □

In the applications for which we need this lemma, the following corollary will often be sufficient:

Corollary 3.3.17. *Retain the setup of the previous lemma and let $\lambda \in \mathfrak{h}^*$. Then there is some $k \in \mathbb{N}$ such that $M^{\lambda+k\alpha} = 0$. Furthermore for any such k and any $v \in M^\lambda$, we have that $f_{\alpha,0}^{-k} \ldots f_{\alpha,m}^{-k} \otimes v = 0$ if and only if $v = 0$.*

Proof. The existence of $k \in \mathbb{N}$ such that $M^{\lambda+k\alpha} = 0$ follows from Lemma 3.1.3. By

Lemma 3.3.16, we have that $f_{\alpha,0}^{-k} \dots f_{\alpha,m}^{-k} \otimes v = 0$ if and only if there exist $v_0, \dots, v_m \in M$ such that $v = f_{\alpha,0}^{k} \cdot v_0 + \dots + f_{\alpha,m}^{k} \cdot v_m$. Replacing $v_0, \dots, v_m$ with their projections onto the $\lambda + k\alpha$ weight space if necessary, we may assume $v_0, \dots, v_k \in M^{\lambda + k\alpha} = 0$, so $v = 0$. $\square$

Finally, we need one more lemma:

Lemma 3.3.18. *Let $M \in U$-mod. Then any element of $S_\alpha \otimes_{U(\mathfrak{g}_m)} M$ can be written in the form $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v$, for some $i_0, \dots, i_m > 0$ and $v \in M$.*

Proof. Certainly any element of $S_\alpha \otimes M$ can be written as a sum of elements of the form $s \otimes m$ for some $s \in S_\alpha$ and $m \in M$. We have that any $s \in S_\alpha$ is the sum of elements of the form $f_{\alpha,0}^{-i,0} \dots f_{\alpha,m}^{-i_m} u$, where $u = v(a,b,c)$, and $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} u \otimes v = f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes (u \cdot v)$ for any $v \in M$, so any element of $S_\alpha \otimes M$ can be written as the sum of elements of the form $f_{\alpha_0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v$.

Now let $x \in S_\alpha \otimes M$, and write $x = \sum_a f_{\alpha,0}^{-i_{0,a}} \dots f_{\alpha,m}^{-l_{m,a}} \otimes v_a$. Let $j_k = \max_a i_{k,a}$. Then:

$$x = \sum_a (f_{\alpha,0}^{-i_{0,a}} \dots f_{\alpha,m}^{-i_{m,a}} \otimes v_a) = \sum_a (f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m} \otimes f_{\alpha,0}^{j_0 - k_{0,a}} \dots f_{\alpha,m}^{j_m - i_{m,a}} v_a)$$
$$= f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m} \otimes \sum_a (f_\alpha^{j_0 - i_{0,a}} \dots f_\alpha^{j_m - i_{m,a}} v_a)$$

$\square$

We can now complete the next step in the proof of Theorem 3.3.11, which is to show that there is an isomorphism between M and $G_\alpha T_\alpha M$.

Lemma 3.3.19. *For any $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$, the map $\psi_M : M \to G_\alpha T_\alpha M$ defined by $\psi_M(v)(s) = s \otimes v$ is an isomorphism.*

Proof. First observe that ψ_M is certainly a homomorphism since for any $v \in M$, $u \in U$ and $s \in S_\alpha$, we have $(u \cdot \psi_M(v))(s) = \psi_M(v)(s \cdot u) = (s \cdot u) \otimes v = s \otimes (u \cdot v) = \psi_M(u \cdot v)(s)$.

To see that ψ_M is always injective, let $M \in \mathcal{O}^{(\mu)}$ and let $v \in M$ be a non-zero weight vector of weight λ say. Then, applying Corollary 3.3.17, there exists $k \in \mathbb{N}$ such that

$M^{\lambda+k\alpha} = 0$, and $\psi_M(v)(f_{\alpha,0}^{-k} \dots f_{\alpha,m}^{-k}) = f_{\alpha,0}^{-k} \dots f_{\alpha,m}^{-k} \otimes v = 0$ if and only if $v = 0$. Hence $\psi_M(v) = 0$ if and only if $v = 0$. Since M and $G_\alpha T_\alpha M$ are both $\mathfrak{h}$-semisimple, this suffices to show that ψ_M is injective.

We now show that ψ_M is surjective. Let $g \in G_\alpha T_\alpha M$. This g is a map $S_\alpha \to S_\alpha \otimes M$, and we may assume g is a weight element of $G_\alpha T_\alpha M$, of weight λ say. We first use Lemma 3.3.18 to see that, for all $i_0, \dots, i_m > 0$ we can find $j_{i_0,\dots,i_m,l} > 0$, and $v_{i_0,\dots,i_m} \in M$ such that $g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = f_{\alpha,0}^{-j_{i_0,\dots i_m,0}} \dots f_{\alpha,m}^{-j_{i_0,\dots i_m,m}} \otimes v_{i_0,\dots,i_m}$.

If $i_l > j_{i_0,\dots,i_m,l}$, then we can replace $m_{i_0,\dots,i_m}$ with $f_\alpha^{i_l - j_{i_0,\dots,i_m,l}} \cdot m_{i_0,\dots,i_m}$ and set $j_{i_0,\dots,i_m} = i_l$. Hence we may assume that $i_l \leq j_{i_0,\dots,i_m,l}$. On the other hand, suppose $i_l < j_{i_0,\dots,i_m,l}$. Then since g is a $U(\mathfrak{g}_m)$-homomorphism, we have:

$$0 = f_{\alpha,l}^{i_l} \cdot g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = f_{\alpha,0}^{-j_{i_0,\dots,i_m,0}} \dots f_{\alpha,l}^{-j_{i_0,\dots,i_m,l}+i_l} \dots f_{\alpha,m}^{-j_{i_0,\dots,i_m,m}} \otimes v_{i_0,\dots,i_m}$$

By Lemma 3.3.16 we have that $v_{i_0,\dots,i_m} = f_{\alpha,0}^{j_{i_0,\dots,i_m,0}} \cdot v'_0 + \dots + f_{\alpha,l}^{j_{i_0,\dots,i_m,l}-i_l} \cdot v'_l + \dots + f_{\alpha,m}^{j_{i_0,\dots,i_m,m}} \cdot v'_m$ for some $v'_0, \dots, v'_m \in M$. Hence:

$$\begin{aligned} g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = f_{\alpha,0}^{-j_{i_0,\dots,i_m,0}} & \dots f_{\alpha,m}^{-j_{i_0,\dots,i_m,m}} \otimes \\ & (f_{\alpha,0}^{j_{i_0,\dots,i_m,0}} \cdot v'_0 + \dots + f_{\alpha,l}^{j_{i_0,\dots,i_m,l}-i_l} \cdot v'_l + \dots + f_{\alpha,m}^{j_{i_0,\dots,i_m,m}} \cdot v'_m) \\ & = f_{\alpha,0}^{-j_{i_0,\dots,i_m,0}} \dots f_{\alpha,l}^{-i_l} \dots f_{\alpha,m}^{-j_{i_0,\dots,i_m,m}} \otimes v'_l \end{aligned}$$

so, replacing $v_{i_0,\dots,i_m}$ with v'_l, we may assume that $j_{i_0,\dots,i_m,l} = i_l$.

Since g has weight λ, we have that for any $h \in \mathfrak{h}$,

$$\begin{aligned} \lambda(h) g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) &= (h \cdot g)(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} h) \\ &= g((h - (i_0 + \dots + i_m)\alpha(h)) f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \\ &= (h - (i_0 + \dots + i_m)\alpha(h)) \cdot g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \end{aligned}$$

so $g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v_{i_0,\dots i_m}$ has weight $\lambda + (i_0 + \dots + i_m)\alpha$. But by a similar calculation as Lemma 3.3.7 (but using the untwisted action on $S_\alpha \otimes M$ rather than the twisted action used there), if $v \in M$ has weight λ', then $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v$ has weight $\lambda' + (i_0 + \dots + i_m)\alpha$ in $S_\alpha \otimes M$. Hence the $v_{i_0,\dots,i_m}$ all have weight λ in M.

Now we choose k such that $M^{\lambda + k\alpha} = 0$. Then for any $i_0, \dots, i_m \geq k$, we have

$$f_{\alpha,0}^{-k} \cdots f_{\alpha,m}^{-k} \otimes v_{k,\dots,k} = g(f_{\alpha,0}^{-k} \cdots f_{\alpha,m}^{-k}) = f_{\alpha,0}^{i_0 - k} \cdots f_{\alpha,m}^{i_m - k} \cdot g(f_{\alpha,0}^{-i_0} \cdots f_{\alpha,m}^{-i_m})$$

$$= f_{\alpha,0}^{-k} \cdots f_{\alpha,m}^{-k} \otimes v_{i_0,\dots,i_m}$$

so $f_{\alpha,0}^{-k} \cdots f_{\alpha,m}^{-k} \otimes (v_{k,\dots,k} - v_{i_0,\dots,i_m}) = 0$. But, again applying Corollary 3.3.17, we must have $v_{k,\dots,k} - v_{i_0,\dots,i_m} = 0$, i.e. $v_{k,\dots,k} = v_{i_0,\dots,i_m}$. Let $v = v_{k,\dots,k}$. Then we argue that $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v = f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes v_{i_0,\dots,i_m}$ for all $i_0, \dots, i_m > 0$: if $i_0 \dots, i_m \geq k$ we have just seen this, and if not then we can use that fact that

$$f_{\alpha,0}^{-i_0} \cdots f_{\alpha,l}^{-i_l+1} \cdots f_{\alpha,m}^{-i_m} \otimes v_{i_0,\dots,i_m} = f_{\alpha,l} \cdot g(f_{\alpha,0}^{-i_0} \cdots f_{\alpha,m}^{-i_m}) = g(f_{\alpha,0}^{-i_0} \cdots f_{\alpha,l}^{-i_l+1} \cdots f_{\alpha,m}^{-i_m})$$

$$= f_{\alpha,0}^{-i_0} \cdots f_{\alpha,l}^{-i_l+1} \cdots f_{\alpha,m}^{-i_m} \otimes v_{i_0,\dots i_l - 1,\dots i_m}$$

Since S_α is generated by $\{f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} : i_0, \dots, i_m > 0\}$, we conclude that $g = \psi_M(v)$ for this v chosen above, completing the proof that ψ_M is surjective. □

The next result is an easy but useful consequence of Frobenius reciprocity.

Lemma 3.3.20. *Let M be a U-module, and let $A \subseteq S_\alpha$ and $\mathfrak{a} \subseteq \mathfrak{g}_m$ be as in Lemma 3.3.13. Let $\varphi : A \to M$ be a $U(\mathfrak{a})$-homomorphism. Then φ extends uniquely to a U-homomorphism $\varphi : S_\alpha \to M$.*

Proof. Frobenius reciprocity states that for algebras $R \subseteq S$, an R-module N and an S-module M, there is an isomorphism $\mathrm{Hom}_S(S \otimes_R N, M) \cong \mathrm{Hom}_R(N, M)$ given by the restriction map $\varphi \mapsto \varphi|_{1 \otimes R}$. Setting $R = U(\mathfrak{a})$, $S = U$, and $N = A$, the result follows from the earlier observation that $S_\alpha \cong U \otimes_{U(\mathfrak{a})} A$ as left U-modules. □

The next result is a technical result which will be used later to construct certain elements of $G_\alpha M$.

Lemma 3.3.21. *Let $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ and let $\mathcal{I}$ be a subset of $\mathbb{Z}^{m+1}$ satisfying:*

(1) $(i_0, \ldots, i_m) \in \mathcal{I}$ whenever any $i_k \leq 0$.

(2) If $(i_0, \ldots, i_m) \in \mathcal{I}$, then $(i_0, \ldots, i_k - 1, \ldots, i_m) \in \mathcal{I}$ for any $0 \leq k \leq m$.

and let $\{v_{i_0,\ldots,i_m} \in M : (i_0, \ldots, i_m) \in \mathcal{I}\}$ be a collection of elements of M satisfying:

(i) $v_{i_0,\ldots,i_m} = 0$ whenever any $i_k \leq 0$.

(ii) $e_{\alpha,k} \cdot v_{i_0,\ldots,i_m} = v_{i_0,\ldots,i_k-1,\ldots,i_m}$ whenever $(i_0, \ldots, i_m) \in \mathcal{I}$

Then there exists a $U(\mathfrak{a})$-homomorphism $\varphi : A \to \phi_\alpha^{-1}(M)$ such that $\varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}) = v_{i_0,\ldots,i_m}$, which by Lemma 3.3.20 extends to a U-homomorphism $\varphi : S_\alpha \to \phi_\alpha^{-1}(M)$. Moreover, if there exists $\lambda \in \mathfrak{h}^*$ such that the weight of $v_{i_0,\ldots,i_m}$ is $\lambda - (i_0 + \cdots + i_m)\alpha$ for all $(i_0, \ldots, i_m) \in \mathcal{I}$, then we can choose φ to also be weight and hence lie in $G_\alpha M$.

Proof. We construct elements $v_{i_0,\ldots,i_m}$ for $(i_0, \ldots, i_m) \in \mathbb{Z}^{m+1} \setminus \mathcal{I}$ inductively such that the $v_{i_0,\ldots,i_m}$ satisfy conditions (i) and (ii) above for any $(i_0, \ldots, i_m) \in \mathbb{Z}^{m+1}$. Then observe that (i) and (ii) ensure that setting $\varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}) = v_{i_0,\ldots,i_m}$ defines a $U(\mathfrak{a})$-homomorphism $\varphi : A \to \phi_\alpha^{-1}(M)$, since it is enough to check that $v_{i_0,\ldots,i_k-1,\ldots i_m} = f_{\alpha,k} \cdot_{\alpha^{-1}} \varphi(f_{\alpha,0}^{-i_0} \ldots f_{\alpha,m}^{-i_m}) = f_{\alpha,k} \cdot_{\alpha^{-1}} v_{i_0,\ldots,i_m} = e_{\alpha,k} \cdot v_{i_0,\ldots,i_m}$ for any $0 \leq k \leq m$ and $(i_0, \ldots, i_m) \in \mathbb{Z}^{m+1}$.

To construct such $v_{i_0,\ldots,i_m}$, let $(i_0, \ldots, i_m) \in \mathbb{Z}^{m+1} \setminus \mathcal{I}$ be such that $i_0 + \cdots + i_m$ is minimal among elements of $\mathbb{Z}^{m+1} \setminus \mathcal{I}$. Then in particular, $(i_0, \ldots, i_k - 1, \ldots, i_m) \in \mathcal{I}$ for each $0 \leq k \leq m$. Recall that $(\mathfrak{sl}_2)_\alpha := \langle e_\alpha, h_\alpha, f_\alpha \rangle$ and let N be the $((\mathfrak{sl}_2)_\alpha)_m$-module generated by $\{v_{i_0,\ldots,i_k-1,\ldots,i_m} : 0 \leq k \leq m\}$, so $N \in \mathcal{O}^{(\mu)}(((\mathfrak{sl}_2)_\alpha)_m)$. Here we make a slight abuse of notation, identifying μ with its restriction to $\mathfrak{h}_m \cap ((\mathfrak{sl}_2)_\alpha)_m$. Then we use the following claim to construct $v_{i_0,\ldots,i_m} \in N \subseteq M$ such that $e_{\alpha,k} \cdot v_{i_0,\ldots,i_m} = m_{i_0,\ldots,i_k-1,\ldots,i_m}$.

Claim $(*)$**.** Consider the maps:

$$N \xrightarrow{\theta_1} N^{m+1} \xrightarrow{\theta_2} N^{\frac{1}{2}m(m+1)}$$

given by $\theta_1(x) = (e_{\alpha,0} \cdot x, \ldots, e_{\alpha,m} \cdot x)$ and $\theta_2(y_0, \ldots, y_m) = (e_{\alpha,k} \cdot y_l - e_{\alpha,l} \cdot y_k)_{0 \leq k < l \leq m}$. Then $\ker(\theta_2) = \mathrm{im}(\theta_1)$. Furthermore, if $(y_0, \ldots, y_m) \in \ker(\theta_2)$ and $y_k \in N \cap M^\lambda$ for all $0 \leq k \leq m$, then there exists $x \in M^{\lambda-\alpha}$ such that $\theta_1(x) = (y_0, \ldots, y_m)$.

We now proceed to prove this claim. First observe that $\theta_2 \circ \theta_1 = 0$, so certainly $\ker(\theta_2) \supseteq \mathrm{im}(\theta_1)$. Hence we only need to show that $\mathrm{im}(\theta_1) \supseteq \ker(\theta_2)$. Throughout the proof of this claim, we write e_k for $e_{\alpha,k}$ and f_k for $f_{\alpha,k}$.

We first deal with the case where $m = 1$ and $N = M(\gamma)$ is a Verma module. In this case, we consider the restriction of these maps to certain weight spaces in the following way (for any $\lambda \in \mathfrak{h}^*$):

$$N^\lambda \xrightarrow{\theta_1} (N^{\lambda+\alpha})^2 \xrightarrow{\theta_2} N^{\lambda+2\alpha}$$

Now, either $\dim(N^{\lambda+\alpha}) = 0$, in which case $\mathrm{im}(\theta_1) = \ker(\theta_2)$ automatically, or the dimensions of these weight spaces satisfy $\dim(N^\lambda) = m+1$, $\dim(N^{\lambda+\alpha}) = m$, and $\dim(N^{\lambda+2\alpha}) = m-1$. Hence by considering dimensions and the fact that $\ker(\theta_2) \supseteq \mathrm{im}(\theta_1)$, it is enough to show that θ_1 is injective and θ_2 is surjective. We can compute that $e_{\alpha,1}$ acts on the basis vectors $f_{\alpha,1}^i f_{\alpha,0}^j \otimes 1$ by:

$$e_{\alpha,1} \cdot (f_{\alpha,1}^i f_{\alpha,0}^j \otimes 1) = \mu j f_{\alpha,1}^i f_{\alpha,0}^{j-1} \otimes 1 - j(j-1) f_{\alpha,1}^{i+1} f_{\alpha,0}^{j-2} \otimes 1$$

so by considering $\theta_2(x, 0)$, we see θ_2 is surjective using an inductive argument. We also see that $e_{\alpha,1} \cdot v = 0$ if and only if $v = f_{\alpha,1}^i \otimes 1$. Since $\mu \neq 0$, we can show that $e_{\alpha,0} \cdot f_1^i \otimes 1 \neq 0$, so θ_1 is injective as required.

We now deal with the case where $m \geq 1$ and $N = M(\lambda)$ is a Verma module in $\mathcal{O}^{(\mu)}(\mathfrak{sl}_2)_m)$, so $\lambda_{\geq 1} = \mu$. We use the following facts, which can be verified by computing the action of e_m on the basis elements $f_0^{i_0} \ldots f_m^{i_m} \otimes 1$ of $M(\lambda)$ and recalling that $\mu(h_\alpha) \neq 0$:

(a) $e_m \cdot N = N$, which can be shown by an inductive argument using a computation similar the $\mathfrak{g} = \mathfrak{sl}_2$ Verma case.

(b) $e_m \cdot v = 0$ if and only if $v \in \text{span}\{f_1^{i_1} \dots f_m^{i_m} \otimes 1 : i_1, \dots, i_m \geq 0\}$. The latter is isomorphic as a $U(\langle e_0, \dots, e_{m-1}\rangle)$-module to the Verma module $M(\gamma)$ over $\text{span}\{e_{\alpha,i}, h_{\alpha,i}, f_{\alpha,i} \mid i = 0, \dots, m-1\} \cong ((\mathfrak{sl}_2)_\alpha)_{m-1}$ where $\gamma(h_{\alpha,i}) := \mu(h_{\alpha,i+1})$.

(c) Applying (b) repeatedly, we see that for any $k \geq 1$, we have that $e_l \cdot v = 0$ for all $k \leq l \leq m$ if and only if $v \in \text{span}\{f_{m-k+1}^{i_{m-k+1}} \dots f_m^{i_m} \otimes 1 : i_{m-k+1}, \dots, i_m \geq 0\}$.

To ease notation slightly we will write $\mathfrak{a}_{(i)} = \langle e_0, \dots, e_i\rangle$, write $(\mathfrak{sl}_2)_{(i)}$ for the Lie algebra $((\mathfrak{sl}_2)_\alpha)_i$ and $\gamma_{(i)}$ for the character of $\text{span}\{h_{\alpha,0}, \dots, h_{\alpha,i}\}$ given by $\gamma_{(i)}(h_{\alpha,j}) := \mu(h_{\alpha,m-i})$.

Now let $(y_0, \dots, y_m) \in \ker(\theta_2)$, i.e. $e_k \cdot y_l - e_l \cdot y_k = 0$ for all $0 \leq k, l \leq m$. By fact (a), there certainly exists an x_m such that $e_m \cdot x_m = y_m$. We then inductively construct x_k for $2 \leq k \leq m$ such that $e_l \cdot x_k = y_l$ for all $k \leq l \leq m$. The cases $k = 0, 1$ will be dealt with by an additional argument immediately afterwards, and we will then obtain some x_0 such that $\theta_1(x_0) = (y_0, \dots, y_m)$.

Suppose we have constructed x_{k+1} such that $e_l \cdot x_{k+1} = y_l$ for all $k+1 \leq l \leq m$. For any $k+1 \leq l \leq m$, consider $e_l \cdot (e_k \cdot x_{k+1} - y_k) = e_k \cdot (e_l \cdot x_{k+1}) - e_l \cdot y_k = e_k \cdot y_l - e_l \cdot y_k = 0$. Hence by fact (c), we have $e_k \cdot x_{k+1} - y_k \in \text{span}\{f_{m-k}^{i_{m-k}} \dots f_m^{i_m} \otimes 1 : i_{m-k}, \dots, i_m \geq 0\}$. But as a $U(\mathfrak{a}_{(k)})$-module, this is isomorphic to $M(\gamma_{(k)})$, so by fact (a) there exists $x'_k \in \text{span}\{f_{m-k}^{i_{m-k}} \dots f_m^{i_m} \otimes 1 : i_{m-k}, \dots, i_m \geq 0\}$ such that $e_k \cdot x'_k = e_k \cdot x_{k+1} - y_k$, and by fact (c) $e_l \cdot x'_k = 0$ for all $k+1 \leq l \leq m$. Setting $x_k = x_{k+1} - x'_k$, we see that $e_k \cdot x_k = e_k \cdot x_{k+1} - e_k \cdot x'_k = e_k \cdot x_{k+1} - (e_k \cdot x_{k+1} - y_k) = y_k$, and for $k+1 \leq l \leq m$ we have $e_l \cdot x_k = e_l \cdot x_{k+1} - e_l \cdot x'_k = y_l - 0$. Hence we have constructed x_k with the desired properties.

Now consider $y'_1 = e_1 \cdot x_2 - y_1$ and $y'_0 = e_0 \cdot x_2 - y_0$. For $2 \leq l \leq n$, we have $e_l \cdot y'_1 = e_l \cdot (e_1 \cdot x_2 - y_1) = e_1 \cdot y_1 - e_l \cdot y_1 = 0$, and similarly $e_l \cdot y'_0 = 0$, so $y'_0, y'_1 \in \text{span}\{f_{m-1}^{i_{m-1}} f_m^{i_m} \otimes 1 : i_{m-1}, i_m \geq 0\}$ which is isomorphic to $M(\gamma_{(1)})$, the Verma module over $(\mathfrak{sl}_2)_{(1)}$, as a $U(\langle e_0, e_1\rangle)$-module. In addition, since $(y_0, \dots, y_m) \in \ker\theta$ we have

$e_0 \cdot y_1' - e_1 \cdot y_0' = 0$, so we can apply the case where N is a Verma module and $n = 1$, proved earlier, to find x' such that $e_0 \cdot x' = y_0'$, $e_1 \cdot x' = y_1'$, and $e_l \cdot x' = 0$ for $2 \leq l \leq n$. Then setting $x_0 = x_2 - x'$, we have that $\theta_1(x_0) = (y_0, \dots, y_m)$.

If N is not a Verma module, let $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = N$ be a filtration of N such that each quotient is a highest weight module. Since $\mu(h_{\alpha,m}) \neq 0$, all these highest weight modules must in fact be Verma modules by [56, Theorem 7.1]. Then, given $(y_0, \dots, y_m) \in \ker(\theta_2)$, in the quotient N/N_{k-1} we have that there exists $x \in N$ such that $e_l \cdot x + N_{k-1} = y_l + N_{k-1}$ for all $0 \leq l \leq m$. Hence $e_l \cdot x - y_l \in N_{k-1}$ for all $0 \leq l \leq n$. But $e_{l'} \cdot (e_l \cdot x - y_l) = e_l \cdot (e_{l'} \cdot x - y_{l'})$ for all $0 \leq l, l', \leq m$, so by induction on k, there exists $x' \in N_{k-1}$ such that $e_l \cdot x' = e_l \cdot x - y_l$ for all $0 \leq l \leq mn$. Note that in this final argument we have used the fact that $y_l' := e_l \cdot x - y_l$ gives a collection of elements lying in the kernel of θ, which allows us to apply the inductive hypothesis. Hence $e_l \cdot (x - x') = y_l$ for all $0 \leq l \leq m$, so $\mathrm{im}(\theta_1) \supseteq \ker(\theta_2)$ as required.

Finally, if $y_0, \dots, y_m \in M^\lambda \cap N$, then given $x \in N \subseteq M$ such that $\theta_1(x) = (y_0, \dots, y_m)$, by considering weight spaces we may replace x with its component $x^{\lambda-\alpha}$ in the $\lambda - \alpha$ weight space and $\theta_1(x) = \theta_1(x^{\lambda-\alpha})$, proving the final part of Claim $(*)$.

Since $e_{\alpha,k} \cdot v_{i_0,\dots,i_l-1,\dots,i_m} = v_{i_0,\dots,i_k-1,\dots,i_l-1,\dots i_m} = e_{\alpha,l} \cdot m_{i_0,\dots,i_k-1,\dots i_m}$ for all $0 \leq k < l \leq m$, this claim then allows us to pick $v_{i_0,\dots,i_m}$ such that $e_{\alpha,k} \cdot v_{i_0,\dots,i_m} = v_{i_0,\dots,i_k-1,\dots i_m}$ for all $0 \leq k \leq m$, and if $v_{i_0,\dots,i_k-1,\dots,i_m}$ has weight λ for all $0 \leq k \leq m$, then we can choose $v_{i_0,\dots,i_m}$ to have weight $\lambda - \alpha$. Hence applying this inductively, we can construct for all $(i_0, \dots, i_m) \in \mathbb{Z}^{m+1}$ elements $v_{i_0,\dots,i_m} \in M$ satisfying conditions (i) and (ii).

Suppose there exists $\lambda \in \mathfrak{h}^*$ such that $\mathrm{wt}(m_{i_0,\dots,i_m}) = \lambda - (i_0 + \cdots + i_m)\alpha$ for all $(i_0, \dots, i_m) \in \mathcal{I}$. Then by construction $v_{i_0,\dots,i_m}$ is weight for all $(i_0, \dots, i_m) \in \mathbb{Z}^{m+1}$, so by Corollary 3.3.9 the function we have constructed is a weight vector. □

We are now ready to establish the main remaining ingredient needed to finish the proof of Theorem 3.3.11.

Lemma 3.3.22. *For any $N \in \mathcal{O}^{(s_\alpha(\mu))}(\mathfrak{g}_m)$, the map $\epsilon_N : T_\alpha G_\alpha N \to N$ given by $\epsilon_N(s \otimes$*

$g) = g(s)$ *is an isomorphism.*

Proof. First we show ϵ_N is a homomorphism. Let $u \in U$, let $s \in S_\alpha$, and let $g \in G_\alpha N$. Then

$$u \cdot \epsilon_N(s \otimes g) = u \cdot g(s) = \phi_\alpha^{-1}\phi_\alpha(u) \cdot g(s) = g(\phi_\alpha(u) \cdot s) = \epsilon_N((\phi_\alpha(u) \cdot s) \otimes g) = \epsilon_N(u \cdot (s \otimes g))$$

so ϵ_N is certainly a U-homomorphism.

Now let $v \in N$ be a weight vector, and choose $a_0, \dots, a_m$ such that $e_{\alpha,k}^{a_k} \cdot v = 0$ for each k. Let

$$\mathcal{I} = \{(i_0, \dots, i_m) : i_k \leq 0 \text{ for some } k \text{ or } i_k \leq a_k \text{ for all } k\} \subseteq \mathbb{Z}^{m+1}$$
$$v_{i_0,\dots,i_m} = e_{\alpha,0}^{a_0 - i_0} \dots e_{\alpha,m}^{a_m - i_m} \cdot v \text{ if } i_k \leq a_k \text{ for all } k$$
$$v_{i_0,\dots,i_m} = 0 \text{ otherwise.}$$

Then we can use Lemma 3.3.21 to construct $g \in \mathrm{Hom}_U(S_\alpha, \phi_\alpha^{-1}(M))$ which is a weight vector and therefore in $G_\alpha N$ such that $g(f_{\alpha,0}^{-a_0} \dots f_{\alpha,m}^{-a_m}) = v$. Hence $\epsilon_N(f_{\alpha,0}^{-a_0} \dots f_{\alpha,m}^{-a_m} \otimes g) = v$, so ϵ_N is surjective.

We now show ϵ_N is injective. By Lemma 3.3.18 any element of $T_\alpha G_\alpha N$ may be written as $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes g$ for some $g \in G_\alpha N$. Suppose $\epsilon_N(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes g) = g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = 0$. We may assume g is weight (i.e. an $\mathfrak{h}$-eigenvector). If not, we may write g as a sum of $g_\lambda \in (G_\alpha N)^\lambda$, which by considering weight spaces must all satisfy $g_\lambda(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = 0$, and then apply the following argument to each g_λ.

Observe that $g(f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m}) = 0$ if either $j_k \leq 0$ for some $0 \leq k \leq n$ or all $j_k \leq i_k$, and choose

$$\mathcal{I} = \left\{ (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1} : \begin{array}{c} j_k \leq i_k \text{ for all } 0 \leq k \leq m-1 \text{ or } j_m \leq i_m \\ \text{or } j_k \leq 0 \text{ for some } 0 \leq k \leq m \end{array} \right\} \subseteq \mathbb{Z}^{m+1}$$

$v_{j_0,\ldots,j_m} = g(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m})$ if $j_k \leq i_k$ for all $0 \leq k \leq m-1$

$v_{j_0,\ldots,j_m} = 0$ if $j_m \leq i_m$ or $j_k \leq 0$ for some $0 \leq k \leq m$.

Then applying Lemma 3.3.21 to this, we construct $g_m \in \mathrm{Hom}_U(S_\alpha, \phi_\alpha^{-1}(M))$ which is a weight vector (and hence in $G_\alpha N$) such that:

(a) $g_m(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m}) = 0$ if $j_m \leq i_m$

(b) $g_m(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m}) = g(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m})$ if $j_k \leq i_k$ for all $0 \leq k \leq m-1$

Note that these two conditions are not mutually exclusive but they are consistent by our earlier observation.

By (a), we can define $g'_m \in G_\alpha N$ by setting

$$g'_m(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m-1}^{-j_{m-1}} f_{\alpha,m}^{-j_m+i_m}) = g_m(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m})$$

and so $g_m = f_{\alpha,m}^{i_m} \cdot g'_m$. By (b) we have $(g - g_m)(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m}) = 0$ if either some $j_k \leq 0$ or if $j_k \leq i_k$ for all $0 \leq k \leq m-1$.

We now construct $g_s \in G_\alpha N$ inductively by setting

$$\mathcal{I} = \left\{ (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1} : \begin{array}{c} j_k \leq i_k \text{ for all } 0 \leq k \leq s-1 \\ \text{or } j_s \leq i_s \text{ or } j_k \leq 0 \text{ for some } 0 \leq k \leq m \end{array} \right\} \subseteq \mathbb{Z}^{m+1}$$

$v_{j_0,\ldots,j_m} = (g - g_m - \cdots - g_{s+1})(f_{\alpha,0}^{-j_0} \ldots f_{\alpha,m}^{-j_m})$ if $j_k \leq i_k$ for all $0 \leq k \leq s-1$

$v_{j_0,\ldots,j_m} = 0$ if $j_s \leq i_s$ or $j_k \leq 0$ for some $0 \leq k \leq m$

Again, these two conditions are not mutually exclusive, but they are consistent.

These g_s satisfy:

(a) $g_s(f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m}) = 0$ if $j_s \leq i_s$

(b) $g_s(f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m}) = (g - g_m - \dots - g_{s+1})(f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m})$ if $j_k \leq i_k$ for all $0 \leq k \leq s-1$

By (a), we can write this g_s in the form $g_s = f_{\alpha,s}^{i_s} \cdot g'_s$ for some $g'_s \in G_\alpha N$. By (b) we have that $(g - g_m - \dots - g_s)(f_{\alpha,0}^{-j_0} \dots f_{\alpha,m}^{-j_m}) = 0$ if $j_k \leq i_k$ for all $0 \leq k \leq s-1$. Hence we see that $g = f_{\alpha,0}^{-i_0} \cdot g'_0 + \dots + f_{\alpha,m}^{-i_m} \cdot g'_m$ for some $g'_0, \dots, g'_m \in G_\alpha N$, so $f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m} \otimes g = \sum (f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \otimes f_{\alpha,k}^{i_k} g'_k = 0$. Hence ϵ_N is injective. □

The last main step in proving Theorem 3.3.11 is to show that G_α genuinely a functor between appropriate blocks of $\mathcal{O}(\mathfrak{g}_m)$ as claimed.

Lemma 3.3.23. *G_α restricts to a functor $\mathcal{O}^{(s_\alpha(\mu))}(\mathfrak{g}_m) \to \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$.*

Proof. Let $N \in \mathcal{O}^{(s_\alpha(\mu))}$. Let $0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_{k-1} \subseteq N_k = N$ be a filtration of N such that each section is a highest weight module. We have a short exact sequence

$$0 \to N_1 \to N \to (N/N_1) \to 0$$

and hence, since G_α is left exact, an exact sequence

$$0 \to G_\alpha N_1 \to G_\alpha N \to G_\alpha(N/N_1).$$

Hence $G_\alpha N$ is an extension of a submodule of $G_\alpha(N/N_1)$ by $G_\alpha N_1$. Since $G_\alpha N$ is automatically $\mathfrak{h}$-semisimple and $\mathcal{O}^{(s_\alpha(\mu))}$ is closed under taking submodules, we can use induction on k and Lemma 3.3.6 to reduce to the case where N is a highest weight module of weight $(\lambda, s_\alpha(\mu_1), \dots, s_\alpha(\mu_m))$.

Let v be a highest weight generator of N. By Lemma 3.3.21, we can find $g_v \in G_\alpha N$ such that $g_v(f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1}) = v$ and g_v is a weight vector. We aim to show that g_v is a highest weight generator of $G_\alpha N$ of weight $(s_\alpha(\lambda) - (m+1)\alpha, \mu_1, \dots, \mu_m)$. By Lemma 3.3.8, we certainly have that g_v is a weight vector of weight $s_\alpha(\lambda) - (m+1)\alpha$.

Let $\beta \in \Phi^+ \setminus \{\alpha\}$. Then $e_{\beta,k} \cdot g_v$ has weight $s_\alpha(\lambda) - (m-1)\alpha + \beta$, so by Lemma 3.3.8, $(e_{\beta,k} \cdot g_v)(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m})$ has weight $\lambda - (i_0 + \dots + i_m - (m+1))\alpha + s_\alpha(\beta)$. Now, since $s_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}$, $\lambda - (i_0 + \dots + i_m - (m+1))\alpha + s_\alpha(\beta) \nleq \lambda$ and so $N^{\lambda - (i_0 + \dots + i_m - (m+1))\alpha + s_\alpha(\beta)} = 0$. In particular, $(e_{\beta,k} \cdot g_v)(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = 0$ for all $i_0, \dots, i_m > 0$, so $e_{\beta,k} \cdot g_v = 0$ for all $0 \leq k \leq m$.

To see that $e_{\alpha,k} \cdot g_v = 0$, first consider the $((\mathfrak{sl}_2)_\alpha)_m$-module $\bigoplus_{l \in \mathbb{Z}_{\geq 0}} N^{\lambda - l\alpha}$. Since N is highest weight, this is generated as a $((\mathfrak{sl}_2)_\alpha)_m$-module by v, which is highest weight of weight $(\lambda(h_\alpha), s_\alpha(\mu_1)(h_\alpha), \dots, s_\alpha(\mu_m)(h_\alpha))$. Hence since $s_\alpha(\mu_m)(h_\alpha) \neq 0$ and this module is clearly non-zero, by [56, Theorem 7.1] this module is isomorphic to

$$M(\lambda(h_\alpha), s_\alpha(\mu_1)(h_\alpha), \dots, s_\alpha(\mu_m)(h_\alpha))$$

and so in particular the only elements $x \in \bigoplus_{l \in \mathbb{N}} N^{\lambda - l\alpha}$ such that $e_{\alpha,k} \cdot x = 0$ for all $0 \leq k \leq m$ are scalar multiples of v. Now suppose that $e_{\alpha,k} \cdot g_v \neq 0$. Then we have a non-zero element $g \in (G_\alpha N)^{s_\alpha(\lambda) - m\alpha}$. By Lemma 3.3.8, $g(f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1}) = x$ must have weight $\lambda - m\alpha$. But also $e_{\alpha,k} \cdot x = 0$ for all $0 \leq k \leq m$, so by the above observation x is a scalar multiple of v and so has weight λ. Hence $x = 0$, and we can now show inductively by a similar argument that $g(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) = 0$ for all $i_0, \dots, i_m > 0$, so $g = 0$ giving a contradiction.

Finally, let $h \in \mathfrak{h}$ and let $1 \leq k \leq m$. Then $h_k \cdot g_v \in (G_\alpha N)^{s_\alpha(\lambda) - (m+1)\alpha}$. We have:

$$\begin{aligned}
(h_k \cdot g_v)(f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1}) &= g_v(f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1} h_k) \\
&= g_v(h_k f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1} + \alpha(h) \sum_{0 \leq l \leq m-k} f_{\alpha,0}^{-1} \dots f_{\alpha,l}^{-2} \dots f_{\alpha,l+k}^{0} \dots f_{\alpha,m}^{-1}) \\
&= g_v(h_k f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1}) \\
&= h_k \cdot v = \mu_k(h) v
\end{aligned}$$

Now suppose $h_k \cdot g_v \neq \mu_k(h) g_v$. Let $(i_0, \dots, i_m)$ be such that $i_0 + \dots + i_m$ is minimal subject to $x := (h_k \cdot g_v - \mu_k(h) g_v)(f_{\alpha,0}^{-i_0} \dots f_{\alpha,m}^{-i_m}) \neq 0$, and note that $(i_0, \dots, i_m) \neq (1, \dots, 1)$ by the calculation above. Then by the minimality of $i_0 + \cdots + i_m$, we have $e_{\alpha,k} \cdot x = 0$ for each $0 \leq k \leq m$. But by Lemma 3.3.8, we have $x \in N^{\lambda-(i_0+\cdots+i_m-(m+1))\alpha}$, which gives a contradiction by the earlier observation that the only elements $x \in \bigoplus_{l\in\mathbb{N}} N^{\lambda-l\alpha}$ such that $e_{\alpha,k} \cdot x = 0$ for all $0 \leq k \leq m$ are scalar multiples of m. Hence $h_k \cdot g_v = \mu_k(h)g_v$, completing the proof that g_v is highest weight of weight $(s_\alpha(\lambda) - (m+1)\alpha, \mu_1, \ldots, \mu_m)$ as claimed.

Now, by 3.3.22, there is an isomorphism ϵ_N between $T_\alpha G_\alpha N$ and N, and furthermore this isomorphism takes $f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes g_v$ to $g_v(f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1}) = v$. Hence $f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes g_v$ is a highest weight generator of $T_\alpha G_\alpha N$. Now let L be the submodule of $G_\alpha N$ generated by g_v, and consider the inclusion $\iota : L \hookrightarrow G_\alpha N$. We wish to show that ι is an isomorphism, which will complete the proof that N is highest weight and therefore the proof that G_α is a functor from $\mathcal{O}^{s_\alpha(\mu)}$ to $\mathcal{O}^{(\mu)}$. First we compute that for any $l \in L, s \in S_\alpha$:

$$\begin{aligned} ((G_\alpha(\epsilon_N) \circ G_\alpha T_\alpha(\iota) \circ \psi_L)(l))(s) &= \epsilon_N((G_\alpha T_\alpha(\iota) \circ \psi_L)(l)(s)) \\ &= \epsilon_N(T_\alpha(\iota) \circ \psi_L(l)(s)) \\ &= \epsilon_N(T_\alpha(\iota)(s \otimes l)) \\ &= \epsilon_N(s \otimes \iota(l)) \\ &= \iota(l)(s) \end{aligned}$$

The first and second equalities hold since by the definition of G_α if $\chi \in G_\alpha T_\alpha G_\alpha N$ and ρ is a map from $T_\alpha G_\alpha N$ to either N or L, then $G_\alpha(\rho)(\chi) = \rho \circ \chi$. The final three equalities hold by the definitions of ψ_L, $T_\alpha(\iota)$, and ϵ_N respectively.

Hence $\iota = G_\alpha(\epsilon_N) \circ G_\alpha T_\alpha(\iota) \circ \psi_L$. Since L is highest weight, by Lemma 3.3.19 the map ψ_L is certainly an isomorphism. Similarly, by Lemma 3.3.22, the map ϵ_N is an isomorphism, so $G_\alpha(\epsilon_N)$ is also an isomorphism. Hence to show ι is an isomorphism it suffices to show $G_\alpha T_\alpha(\iota)$ is an isomorphism, and to show this it suffices to show $T_\alpha(\iota)$ is an isomorphism. We can also conclude from this calculation that $G_\alpha T_\alpha(\iota)$ is injective.

Now we consider $T_\alpha(\iota) : T_\alpha L \to T_\alpha G_\alpha N$. We have that $(T_\alpha(\iota))(f_{\alpha,0}^{-1} \ldots f_{\alpha,m}^{-1} \otimes g_v) =$

$f_{\alpha,0}^{-1} \dots f_{\alpha,m}^{-1} \otimes g_v$ which as discussed above generates $T_\alpha G_\alpha N$, so $T_\alpha(\iota)$ is certainly surjective. Let $K = \ker(T_\alpha(\iota))$ and observe since that K is a submodule of $T_\alpha L \in \mathcal{O}^{(\mu)}$, we have $K \in \mathcal{O}^{(\mu)}$.

We have a short exact sequence:

$$0 \to K \overset{\varphi}{\to} T_\alpha L \overset{T_\alpha(\iota)}{\to} T_\alpha G_\alpha N \to 0$$

and so since G_α is left exact, we have an exact sequence:

$$0 \to G_\alpha K \overset{G_\alpha(\varphi)}{\to} G_\alpha T_\alpha L \overset{G_\alpha T_\alpha(\iota)}{\to} G_\alpha T_\alpha G_\alpha N$$

But $G_\alpha T_\alpha(\iota)$ is injective, so $\mathrm{im}(G_\alpha(\varphi)) = 0$ and hence $G_\alpha K = 0$. Now, if $K \neq 0$ then since $K \in \mathcal{O}^{(\mu)}$ we can use Lemma 3.3.21 to construct a non-zero element of $G_\alpha K$, so $K = 0$, i.e. $T_\alpha(\iota)$ is injective. Hence $T_\alpha(\iota)$ is an isomorphism, so by the earlier discussion ι is an isomorphism, so $G_\alpha N$ is highest weight as required. $\square$

Finally, to complete the proof of Theorem 3.3.11 we need a small argument to check that ψ and ϵ are both natural transformations:

Lemma 3.3.24. *The transformations*

$$\psi : \mathrm{id}_{\mathcal{O}^{(\mu)}(\mathfrak{g}_m)} \to G_\alpha T_\alpha$$

and

$$\epsilon : T_\alpha G_\alpha \to \mathrm{id}_{\mathcal{O}^{(s_\alpha(\mu))}(\mathfrak{g}_m)}$$

are natural.

Proof. To show ψ is natural, we must show that for any $M, N \in \mathcal{O}^{(\mu)}$ and $f : M \to N$ that $\psi_N \circ f = G_\alpha T_\alpha f \circ \psi_M$. Using that $G_\alpha(\rho)(\chi) = \rho \circ \chi$, we see that for any $s \in S_\alpha$ and

$v \in M$:

$$\begin{aligned}(G_\alpha T_\alpha f \circ \psi_M(v))(s) &= G_\alpha T_\alpha f(s \otimes v)\\ &= T_\alpha f(s \otimes v)\\ &= s \otimes f(v)\\ &= \psi_N(f(v))(s)\\ &= (\psi_N \circ f)(v)(s).\end{aligned}$$

To show ϵ is natural, we must show that for any $M, N \in \mathcal{O}^{(s_\alpha(\mu))}$ and $f : M \to N$ that $f \circ \epsilon_M = \epsilon_N \circ T_\alpha G_\alpha f$. This holds since for any $s \in S_\alpha$ and $g \in G_\alpha M$, we have:

$$\begin{aligned}(\epsilon_N \circ T_\alpha G_\alpha f)(s \otimes g) &= \epsilon_N(s \otimes (G_\alpha f)(g))\\ &= \epsilon_N(s \otimes (f \circ g))\\ &= f(g(s))\\ &= f(\epsilon_M(s \otimes g))\\ &= (f \circ \epsilon_M)(s \otimes g)\end{aligned}$$

□

Theorem 3.3.11 now follows by combining Lemmas 3.3.12, 3.3.19, 3.3.22, 3.3.23, and 3.3.24.

We end this section by remarking that Lemma 3.1.11, Theorem 3.2.1, and Theorem 3.3.11 can be combined (together with a small result on the action of the Weyl group on $\mathfrak{h}^*$; see Proposition 3.4.7) to show that every Jordan block $\mathcal{O}^{(\mu_1,\ldots,\mu_m)}(\mathfrak{g}_m)$ is equivalent to another Jordan block $\mathcal{O}^{(\nu_1,\ldots,\nu_{m-1},0)}(\mathfrak{l}_m)$ where $\mathfrak{l}$ is some Levi factor of $\mathfrak{g}$. We will discuss this in more detail later during our calculation of composition multiplicities in §3.4.2. For now, we illustrate this via the following example:

Example 3.3.25. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $m = 1$, let $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$ be the positive roots of $\mathfrak{sl}_3$ labelled in the usual way, and consider the Jordan blocks $\mathcal{O}^{(\mu)}((\mathfrak{sl}_3)_1)$ as μ ranges over $\mathfrak{h}^*$. The possible stabilisers $(\mathfrak{sl}_3)^\mu$ are:

$$\mathfrak{l}_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \qquad \mathfrak{l}_2 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad \mathfrak{l}_3 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

$$\mathfrak{l}_4 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \qquad \mathfrak{l}_5 = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

We then have the following cases:

- If $(\mathfrak{sl}_3)^\mu = \mathfrak{l}_1$ then $\mu = 0$ so we are already in the Jordan block $\mathcal{O}^{(0)}((\mathfrak{sl}_3)_1)$.

- If $(\mathfrak{sl}_3)^\mu = \mathfrak{l}_2 = \mathfrak{h}$, then $(\mathfrak{sl}_3)^\mu$ is the Levi factor of the standard parabolic $\mathfrak{p} = \mathfrak{b}$. Then by Theorem 3.2.1 and Lemma 3.1.11 we have that $\mathcal{O}^{(\mu)}((\mathfrak{sl}_3)_1) \cong \mathcal{O}^{(0)}(\mathfrak{h}_1)$.

- If $(\mathfrak{sl}_3)^\mu = \mathfrak{l}_3$ or $(\mathfrak{sl}_3)^\mu = \mathfrak{l}_4$ then is it the Levi factor of a standard parabolic $\mathfrak{p}_3$ or $\mathfrak{p}_4$ respectively, where:

$$\mathfrak{p}_3 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \qquad \mathfrak{p}_4 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

  Hence we can again apply Theorem 3.2.1 and Lemma 3.1.11 to show that $\mathcal{O}^{(\mu)}((\mathfrak{sl}_3)_1) \cong \mathcal{O}^{(0)}((\mathfrak{gl}_2)_1)$, since $\mathfrak{l}_3 \cong \mathfrak{gl}_2 \cong \mathfrak{l}_4$.

- If $(\mathfrak{sl}_3)^\mu \cong \mathfrak{l}_5$ then there is no parabolic subalgebra $\mathfrak{p}_5$ containing $\mathfrak{b}$ whose Levi factor is $\mathfrak{l}_5$. Hence we cannot directly apply Theorem 3.2.1. Instead, we first use Theorem 3.3.11 to obtain an equivalence $\mathcal{O}^{(\mu)}((\mathfrak{sl}_3)_1) \cong \mathcal{O}^{(s_\alpha(\mu))}((\mathfrak{sl}_3)_1)$ via the twisting functor T_α. Then one can check that $\mathfrak{g}^{s_\alpha(\mu)} = \mathfrak{l}_4$, so by the previous case we see that $\mathcal{O}^{(\mu)}((\mathfrak{sl}_3)_1) \cong \mathcal{O}^{(0)}((\mathfrak{gl}_2)_1)$.

3.4 Composition multiplicities of Verma modules

3.4.1 Definition of composition multiplicities

Let $M \in \mathcal{O}(\mathfrak{g}_m)$. As in the reductive case covered in §2.3.2, we define the *character* of M to be the function $\mathrm{ch}(M) : \mathfrak{h}^* \to \mathbb{Z}_{\geq 0}$ which sends λ to $\dim(M^\lambda)$, and the *support* of M, denoted $\mathrm{supp}(M)$, to be the set $\mathrm{supp}(M) = \{\lambda \in \mathfrak{h}^* : M^\lambda \neq 0\}$.

Since modules in $\mathcal{O}(\mathfrak{g}_m)$ need not have finite length, we require an alternative way to define composition multiplicities. We do this using the using the following result, generalising the case $\mathfrak{g} = \mathfrak{sl}_2$, $m = 1$ treated in [34, Proposition 8].

Lemma 3.4.1. *Let $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$. For any $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$, there exist unique $\{k_\lambda(M) \in \mathbb{Z}_{\geq 0} : \lambda \in \mathfrak{h}_m^*, \lambda_{\geq 1} = \mu\}$, which we refer to as composition multiplicities, such that*

$$\mathrm{ch}(M) = \sum k_\lambda(M)\, \mathrm{ch}(L(\lambda, \mu))$$

Proof. We first show uniqueness. Suppose we have

$$\mathrm{ch}(M) = \sum_{\lambda \in \mathfrak{h}^*} a_\lambda \, \mathrm{ch}(L(\lambda, \mu)) = \sum_{\lambda \in \mathfrak{h}^*} b_\lambda \, \mathrm{ch}(L(\lambda, \mu))$$

for some $a_\lambda, b_\lambda \in \mathbb{Z}_{\geq 0}$, and $a_\lambda \neq b_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Then we let $X = \{\lambda \in \mathfrak{h}^* : a_\lambda > b_\lambda\}$, $Y = \mathrm{supp}(M) \setminus X$, and

$$\chi = \sum_{\lambda \in X} (a_\lambda - b_\lambda)\, \mathrm{ch}(L(\lambda, \mu)) = \sum_{\lambda \in Y} (b_\lambda - a_\lambda)\, \mathrm{ch}(L(\lambda, \mu))$$

Since $a_\lambda \neq b_\lambda$ for some λ, we have that $\chi \neq 0$. Let $\nu \in \mathfrak{h}^*$ be such that $\chi(\nu) \neq 0$ but $\chi(\nu') = 0$ for all $\nu' \geq \nu$. Now, if $\nu \in X$, then the coefficient of $\mathrm{ch}(L(\nu, \mu))$ in the second sum must be 0, and since $\chi(\nu') = 0$ whenever $\nu' \geq \nu$, so must the coefficients of $L(\nu', \mu)$ for $\nu' \geq \nu$. Hence $\chi(\nu) = 0$ giving a contradiction. If instead $\nu \in Y$ we obtain contradiction by an identical argument.

To show existence of the $k_\lambda(M)$, we use induction on $n = \sum_{\lambda' \geq \lambda} \dim(M^{\lambda'})$, which is finite by Lemma 3.1.3. If $n = 0$, then in particular $M^\lambda = 0$, so we must have $k_\lambda(M) = 0$.

Now, let Γ be the set of non-negative integer sums of positive roots, and choose $\nu \in \lambda + \Gamma$ such that ν is maximal subject to the condition that $M^\nu \neq 0$. Then there must exist a highest weight vector of weight ν in M, generating a highest weight submodule K of M. Consider the quotient map $M(\nu, \mu) \to K$, and let $K' \subseteq K$ be the image of the unique maximal submodule $N(\nu, \mu) \subseteq M(\nu, \mu)$ under this quotient map. Both $\sum_{\lambda' \geq \lambda} \dim(M/K)^{\lambda'}$ and $\sum_{\lambda' \geq \lambda} \dim(K')^{\lambda'}$ are $< n$, so we have already constructed $k_\lambda(M/K)$ and $k_\lambda(K')$. We then set:

$$k_\lambda(M) = k_\lambda(M/K) + k_\lambda(K') \text{ if } \nu \neq \lambda$$
$$k_\lambda(M) = k_\lambda(M/K) + k_\lambda(K') + 1 \text{ if } \nu = \lambda$$

□

We now give another interpretation of these quantities $k_\lambda(M)$ which is closer to the notion of composition multiplicities in Artinian categories.

Lemma 3.4.2. *Let $\mu \in (\mathfrak{h}_n^{\geq 1})^*$ and $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$. Then there exists $\mathcal{I} \subseteq (\mathbb{Z}_{\geq 0})^m$ and a descending filtration of M indexed by $\mathcal{I}$ with the lexicographic ordering, such that:*

(a) If $(i_1, \ldots, i_{k-1}, i_k, \ldots, i_m) \in \mathcal{I}$ with $i_k > 0$ then $(i_1, \ldots, i_{k-1}, i_k - 1, 0, \ldots, 0) \in \mathcal{I}$.

(b) Each quotient $M_{(i_1,\ldots,i_m-1)}/M_{(i_1,\ldots,i_m)}$ is simple.

(c) The intersection of all the $M_{(i_1,\ldots,i_m)}$ is trival.

Furthermore, for any such filtration $L(\lambda)$ appears as a quotient $M_{(i_1,\ldots,i_m-1)}/M_{(i_1,\ldots,i_m)}$ precisely $k_\lambda(M)$ times.

Proof. First observe that by Theorems 3.2.1 and 3.3.11 we may reduce to the case $\mu(\mathfrak{h}_m^{(m)}) = 0$. Suppose $m = 1$. In this case M has a filtration $M \supseteq \mathfrak{g}_1^{(1)} M \supseteq (\mathfrak{g}_1^{(1)})^2 M \supseteq \cdots$.

Each quotient $(\mathfrak{g}_1^{(1)})^i M/(\mathfrak{g}_1^{(1)})^{i+1}M$ lies in BGG category $\mathcal{O}$ for $\mathfrak{g}$ and hence has finite length, so this may be refined to a filtration satisfying (a) and (b).

For $m > 1$ consider the filtration $M \supseteq \mathfrak{g}_m^{(m)}M \supseteq (\mathfrak{g}_m^{(m)})^2 M \supseteq \cdots$. Each quotient $(\mathfrak{g}_m^{(m)})^i M/(\mathfrak{g}_m^{(m)})^{i+1}M$ lies in the category $\mathcal{O}^{(\mu)}(\mathfrak{g}_{m-1})$, where we identify μ with an element of $(\mathfrak{h}_{m-1}^{\geq 1})^*$ in the obvious manner, so we may set $M_{(i,0,\ldots,0)} = (\mathfrak{g}_m^{(m)})^i M$ and argue by induction that this may be refined to a filtration satisfying (a) and (b).

To show this filtration satisfies (c), it is enough to show that whenever $\mu(\mathfrak{h}_m^{(m)}) = 0$ every $M \in \mathcal{O}^{(\mu)}(\mathfrak{g}_m)$ has the property that $\bigcap_{i\geq 0}(\mathfrak{g}_m^{(m)})^i M = 0$. By considering weight spaces, this property is preserved by taking quotients and extensions, so it is enough to verify this in the case where $M = M(\lambda)$ is a Verma module. Since $\lambda_m = \mu_m = 0$ this Verma module is graded: we place a grading on $U(\mathfrak{g}_m)$ by setting, for $x \in \mathfrak{g}$,

$$\deg x_i = \begin{cases} 0 & \text{if } i = 0, \ldots, m-1, \\ 1 & \text{if } i = m. \end{cases}$$

We transfer the induced grading on $U(\mathfrak{n}_m^-)$ to $M(\lambda)$ via the isomorphism of $\mathfrak{n}_m^-$-modules $U(\mathfrak{n}_m^-) \cong M(\lambda)$. Now $M(\lambda)$ is a positively graded $\mathfrak{g}_m$-module and $\bigcap_{i\geq 0}(\mathfrak{g}_m^{(m)})^i M(\lambda)$ is contained in the intersection of all graded components, which is zero.

We claim that $(\mathfrak{g}_m^{(m)})^i M(\lambda, \mu_1, \ldots, \mu_{m-1}, 0) = \{u(\mathbf{v_0}, \ldots, \mathbf{v_m}) \otimes 1 : |\mathbf{v_m}| \geq i\}$, where $u(\mathbf{v_0}, \ldots, \mathbf{v_m}) = \prod f_{\alpha,0}^{v_{0\alpha}} \cdots \prod f_{\alpha,n}^{v_{m\alpha}}$ for $\mathbf{v_0}, \ldots, \mathbf{v_m} \in \mathbb{Z}_{\geq 0}^{\Phi^+}$, which immediately implies that

$$\bigcap_{i\geq 0}(\mathfrak{g}_m^{(m)})^i M(\lambda, \mu_1, \ldots, \mu_{m-1}, 0) = 0.$$

We prove this claim by induction; it clearly holds when $i = 0$. Suppose

$$(\mathfrak{g}_m^{(m)})^i M(\lambda, \mu_1, \ldots, \mu_{m-1}, 0) = \{u(\mathbf{v_0}, \ldots, \mathbf{v_n}) \otimes 1 : |\mathbf{v_m}| \geq i\}.$$

Then $(\mathfrak{g}_m^{(m)})^{i+1} M(\lambda, \mu_1, \ldots, \mu_{m-1}, 0)$ is spanned by elements of the form $x_m \cdot u(\mathbf{v_0}, \ldots, \mathbf{v_n}) \otimes 1$, where $x = e_\alpha, f_\alpha$, or h_α. But $x_m \cdot u(\mathbf{v_0}, \ldots, \mathbf{v_n}) \otimes 1 = f_{\alpha_1,0} x_m u(\mathbf{v_0}', \ldots, \mathbf{v_n}) \otimes 1 +$

$[x, f_{\alpha_1}]_n u(\mathbf{v_0}', \ldots, \mathbf{v_m}) \otimes 1$, where $|\mathbf{v_0}'| = |\mathbf{v_0}| - 1$, so by induction on $|\mathbf{v_0}|$ we may assume that $\mathbf{v_0} = \mathbf{0}$. Now x_m commutes with $u(\mathbf{0}, \mathbf{v_1}, \ldots, \mathbf{v_m})$, so if $x = e_\alpha$ or h_α then $x_m \cdot u(\mathbf{0}, \ldots, \mathbf{v_m}) \otimes 1 = 0$, while if $x = f_\alpha$ then $x_m \cdot u(\mathbf{0}, \ldots, \mathbf{v_m}) \otimes 1$ lies in $\{u(\mathbf{v_0}, \ldots, \mathbf{v_m}) \otimes 1 : |\mathbf{v_m}| \geq i + 1\}$. Hence the claim holds.

Finally, we observe that $\mathrm{ch}(M) = \sum_{(i_0,\ldots,i_m)\in\mathcal{I}, i_m>0} \mathrm{ch}(M_{i_0,\ldots,i_m-1}/M_{i_0,\ldots,i_m})$, so by the uniqueness part of Lemma 3.4.1 the final part of the Lemma holds. $\square$

This result justifies the use of the terminology composition multiplicities, for $k_\lambda(M)$. From now on we use the notation $[M : L(\lambda, \mu)] := k_\lambda(M)$ for $M \in \mathcal{O}^{(\mu)}$.

Corollary 3.4.3. *The parabolic induction and invariants functors and the twisting functors T_α, G_α preserve composition multiplicities.*

Proof. Let $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a filtration of M of the form described in Lemma 3.4.2, and let F be T_α, G_α, or a parabolic induction or invariants functor. Then $F(M) = F(M_0) \supseteq F(M_1) \supseteq F(M_2) \supseteq \cdots$ is also a filtration of this form, and for all i we have $F(M_i)/F(M_{i+1}) \cong F(M_i/M_{i+1})$, so $[M : L(\lambda, \mu)] = [F(M) : F(L(\lambda, \mu))]$. $\square$

3.4.2 Computation of composition multiplicities of Verma modules

We now wish to compute the composition multiplicities $[M(\lambda) : L(\nu)]$. Thanks to Lemma 3.1.7 and Lemma 3.4.2 we know that $[M(\lambda) : L(\nu)] = 0$ unless $\lambda_m = \nu_m$ and so, using a combination of twisting functors and parabolic induction functors, we can reduce to the case $\lambda_m = \nu_m = 0$.

Let $p : \mathbb{Z}_{\geq 0}\Phi^+ \to \mathbb{Z}_{\geq 0}$ be Kostant's partition function. For $\alpha \in \mathbb{Z}\Phi$ we write $\alpha_0 \in \mathfrak{h}_{m-1}^*$ to be the element satisfying $\alpha_0(h_i) = 0$ for $i > 0$ and $\alpha_0(h_0) = \alpha(h)$, where $h \in \mathfrak{h}$. Also, for any $\lambda \in \mathfrak{h}_m^*$ we identify $\lambda_{\leq m-1}$ with an element of $\mathfrak{h}_{m-1}^*$ in the obvious manner.

Lemma 3.4.4. *If $M(\lambda_{\leq m-1} - \alpha_0)(\mathfrak{g}_{m-1})$ is the Verma module of weight $\lambda_{\leq m-1} - \alpha_0$ for*

$\mathfrak{g}_{m-1}$ *then*

$$\operatorname{ch} M(\lambda) = \sum_{\alpha \in \mathbb{Z}_{\geq 1 0}\Phi^+} p(\alpha) \operatorname{ch}(M(\lambda_{\leq m-1} - \alpha_0)(\mathfrak{g}_{m-1})).$$

Proof. Note that $\operatorname{ch} M$ can be defined for any semisimple $\mathfrak{h}$-module M that has finite-dimensional weight spaces. Let $\mathbb{C}_{\lambda_0}$ be the $\mathfrak{h}$-module of weight λ_0. Since $M(\lambda) \cong U(\mathfrak{n}_m^-) \otimes \mathbb{C}_{\lambda_0}$ as $\mathfrak{h}$-modules the lemma follows from the facts:

(i) The character of $U(\mathfrak{n}_m^-)$ is equal to the character of $S(\mathfrak{n}_m^-)$.

(ii) $S(\mathfrak{n}_m^-)$ is a free module over $S(\mathfrak{n}_{m-1}^-)$ and, for $\alpha \in \mathbb{Z}_{\geq 0}\Phi^+$, there are $p(\alpha)$ basis vectors of weight $-\alpha$. □

Corollary 3.4.5. *If $\lambda, \nu \in \mathfrak{h}_m^*$ satisfy $\lambda_m = \nu_m = 0$, and $\lambda_{\geq 1} = \nu_{\geq 1}$, and $L(\nu_{\leq m-1})(\mathfrak{g}_{m-1})$ denotes the simple $\mathfrak{g}_{m-1}$-module of highest weight $\nu_{\leq m-1} \in \mathfrak{h}_m^*$ then*

$$[M(\lambda) : L(\nu)] = \sum_{\alpha \in \mathbb{Z}_{\geq 0}\Phi^+} p(\alpha)[M(\lambda_{\leq m-1} - \alpha_0)(\mathfrak{g}_{m-1}) : L(\nu_{\leq m-1})(\mathfrak{g}_{m-1})].$$

Proof. By Lemma 3.1.10 we have $\operatorname{ch} L(\nu) = \operatorname{ch} L(\nu_{\leq m-1})$ for all $\nu \in \mathfrak{h}_m^*$ satisfying $\nu_m = 0$, and so the claim follows from Lemma 3.4.4. □

We now introduce the notation $\mu_m \in \mathfrak{h}^*$, and write $\mathfrak{g}_{m-1}^{\mu_m} := (\mathfrak{g}^{\mu_m})_{m-1} = (\mathfrak{g}_{m-1})^{\mu_m}$.

Corollary 3.4.6. *Let $\mu \in (\mathfrak{h}_m^{(\geq 1)})^*$ such that $\mathfrak{g}^{\mu_m}$ is the Levi factor of a standard parabolic. Then for any $\lambda, \nu \in \mathfrak{h}_m^*$ such that $\lambda_{\geq 1} = \nu_{\geq 1} = \mu$ we have*

$$[M(\lambda) : L(\nu)] = \sum_{\alpha \in \mathbb{Z}_{\geq 0}\Phi^+} p(\alpha)[M(\lambda_{\leq m-1} - \alpha_0)(\mathfrak{g}_{m-1}^{\mu_m}) : L(\nu_{\leq m-1})(\mathfrak{g}_{m-1}^{\mu_m})].$$

Proof. It can be verified that the functors in Lemma 3.1.11 and Theorem 3.2.1 send highest weight modules to highest weight modules of the corresponding weight. The result then follows from said Lemma and Theorem along with Lemma 3.4.2, and Corollary 3.4.5. □

To complete the computation of composition multiplicities for all Verma modules we must show that for any Jordan block $\mathcal{O}^{(\mu)}$, we can apply twisting functors to obtain a

Jordan block $\mathcal{O}^{(\mu')}$ such that $\mathfrak{g}^{\mu'}$ is the Levi factor of a standard parabolic. This is possible thanks to the following result:

Proposition 3.4.7. *Let $\mu \in \mathfrak{h}^*$. Then there exists $w \in W$ and a Levi factor $\mathfrak{l}$ of a standard parabolic $\mathfrak{p}$ such that $\mathfrak{g}^{w(\mu)} = \mathfrak{l}$. Furthermore, if w is chosen to be of minimal length subject to this condition and $w = s_{\alpha_n} s_{\alpha_{n-1}} \dots s_{\alpha_1}$ is a reduced expression for w, then for each $1 \leq i \leq n$, we have $((s_{\alpha_{i-1}} \dots s_{\alpha_1})\mu)(h_{\alpha_i}) \neq 0$.*

Proof. By [12, Lemma 3.8.1], we can certainly find $w = s_{\alpha_n} s_{\alpha_{n-1}} \dots s_{\alpha_1}$ and $\mathfrak{l}$ the Levi factor of a standard parabolic subalgebra satisfying $\mathfrak{g}^{w(\mu)} = \mathfrak{l}$. Now, suppose that w has minimal length such that $g^{w(\mu)} = \mathfrak{l}$ and that for some i we have $((s_{\alpha_{i-1}} \cdots s_{\alpha_1})\mu)(h_{\alpha_i}) = 0$. In general, if $\mu(h_\alpha) = 0$ for some $\alpha \in \Phi$ then $s_\alpha(\mu) = \mu - \mu(h_\alpha)\alpha = \mu$, so in particular, if $w' := s_{\alpha_n} \dots s_{\alpha_{i+1}} s_{\alpha_{i-1}} \dots s_{\alpha_1}$, then $w(\mu) = w'(\mu)$ and so $\mathfrak{l} = \mathfrak{g}^{w(\mu)} = \mathfrak{g}^{w'(\mu)}$. But w' has shorter length than w, giving a contradiction. □

We now define an action $\bullet_m$ of W on $\mathfrak{h}^*$ by $w \bullet_m \lambda := w(\lambda + m\rho) - m\rho$ where ρ is half the sum of the positive roots. Note this generalises the dot action of W on $\mathfrak{h}^*$, which corresponds to the case $m = 0$, and which controls the central characters of $\mathfrak{g}_0$ (see [23, §1.9]). We remark that this m-dot action first appeared in the work of Geoffriau [19] on the centre of the enveloping algebra of $\mathfrak{g}_m$.

If α is any simple root, then since $s_\alpha(\alpha) = -\alpha$ and s_α permutes the other positive roots we have that $s_\alpha(\rho) = \rho - \alpha$. We then have:

$$\begin{aligned} s_\alpha \bullet_m \lambda &= s_\alpha(\lambda + m\rho) - m\rho \\ &= s_\alpha(\lambda) + m\rho - m\alpha - m\rho \\ &= s_\alpha(\lambda) - m\alpha. \end{aligned}$$

We extend this to a W-action on $\mathfrak{h}_m^*$ by setting

$$(s_\alpha \bullet_m \lambda)(h_i) = \begin{cases} (s_\alpha \bullet_m \lambda_i)(h_i) & \text{for } i = 0 \\ (s_\alpha \lambda_i)(h_i) & \text{for } i \neq 0. \end{cases}$$

By the proof of Lemma 3.3.12, the twisting functors T_α take highest weight modules of weight $\lambda \in \mathfrak{h}_m^*$ to highest weight modules of weight $s_\alpha \bullet_m \lambda$, and hence take $M(\lambda)$ to $M(s_\alpha \bullet_m \lambda)$ and similarly take $L(\lambda)$ to $L(s_\alpha \bullet_m \lambda)$. Applying Corollary 3.4.3 again we obtain the following.

Corollary 3.4.8. *Let $\lambda, \nu \in \mathfrak{h}_m^*$ be such that $\lambda_{\geq 1} = \nu_{\geq 1}$. Let $w \in W$ have a reduced expression $w = s_{\alpha_k} s_{\alpha_{k-1}} \cdots s_{\alpha_1}$ for simple reflections s_{α_i} such that:*

(a) $\mathfrak{g}^{w(\mu_m)}$ is the Levi factor of a standard parabolic.

(b) For each $1 \leq i \leq m$, we have $((s_{\alpha_{i-1}} \cdots s_{\alpha_1})\mu)(h_{\alpha_i}) \neq 0$.

Then we have

$$\begin{aligned} [M(\lambda) : L(\nu)] &= [M(w \bullet_m \lambda) : L(w \bullet_m \nu)] \\ &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}\Phi^+} p(\alpha)[M((w \bullet_m \lambda)_{\leq m-1} - \alpha_0)(\mathfrak{g}_{m-1}^{w(\lambda_m)}) : L((w \bullet_m \nu)_{\leq m-1})(\mathfrak{g}_{m-1}^{w(\mu_m)})]. \end{aligned}$$

CHAPTER 4

MODULAR REPRESENTATION THEORY OF TRUNCATED CURRENT LIE ALGEBRAS

4.1 Structure theory

We now consider truncated current Lie algebras in positive characteristic, starting with some structural results and then moving on to consider their representation theory. Fix an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Recall from §2.4.1 the definition of a restricted Lie algebra, that is, a Lie algebra $\mathfrak{g}$ equipped with a map $(\bullet)^{[p]} : \mathfrak{g} \to \mathfrak{g}$ satisfying certain conditions. If $\mathfrak{g}$ is a restricted Lie algebra, then the corresponding truncated current Lie algebra $\mathfrak{g}_m$ has a natural restricted structure given by:

$$(xt^i)^{[p]} = x^{[p]}t^{ip}$$

for $x \in \mathfrak{g}$ and $0 \leq i \leq m$. If $\mathfrak{g}$ is the Lie algebra of an algebraic group G this restricted structure can be also be obtained from the group G_m described in Theorem 2.5.2. We abuse notation slightly and write $(\bullet)^{[p]}$ for both the p^{th} power map on both $\mathfrak{g}$ and on $\mathfrak{g}_m$. From now on, we fix a standard reductive group G with Lie algebra $\mathfrak{g}$ which carries a natural restricted structure (see §2.4.2).

Recall from §2.5 that there is a non-degenerate G_m-invariant symmetric associative form on $\mathfrak{g}_m$ defined by

$$\begin{aligned} \kappa_m \quad : \quad & \mathfrak{g}_m \times \mathfrak{g}_m \longrightarrow \Bbbk, \\ & xt^i, yt^j \longmapsto \delta_{i+j,m}\kappa(x,y). \end{aligned}$$

where $\kappa : \mathfrak{g}\times\mathfrak{g} \to \Bbbk$ is the nondegenerate form from axiom (H3) of the standard hypotheses (see §2.4.2). Using this non-degenerate pairing we can define an isomorphism of $U(\mathfrak{g}_m)$-modules:

$$\begin{aligned} \hat{\kappa}_m \quad : \quad & \mathfrak{g}_m \longrightarrow \mathfrak{g}_m^* \\ & x \longmapsto \kappa_m(x, \cdot). \end{aligned}$$

Note that this isomorphism preserves the gradings on these modules: it sends homogeneous elements $x \in \mathfrak{g}_m^{(i)}$ to linear functions which vanish on $\bigoplus_{j\neq m-i} \mathfrak{g}_m^{(j)}$. We call such elements of $\mathfrak{g}^*$ *homogeneous* with support in degree $m - i$.

Using Definition 2.4.3, we obtain from the p^{th} power map $(\bullet)^{[p]}$ a notion of semisimple and nilpotent elements. Our first step in studying the representation theory of $\mathfrak{g}_m$ is to classify such elements. We start with the semisimple elements:

Proposition 4.1.1. *Every semisimple element of $\mathfrak{g}_m$ is conjugate under G_m to a semisimple element of $\mathfrak{g}_m^{(0)}$.*

Proof. By [21, Corollary 15.3], since $\mathfrak{g}_m$ is the Lie algebra of the algebraic group G_m, any two Cartan subalgebras of $\mathfrak{g}_m$ are conjugate under the adjoint action of G_m. Furthermore, observe that $\mathfrak{h}_m$ is one such Cartan subalgebra; it is nilpotent, and since $\mathfrak{h}$ is self-normalising in $\mathfrak{g}$ it follows that $\mathfrak{h}_m$ is self-normalising in $\mathfrak{g}_m$. Now, any semisimple element s is contained in some maximal torus, whose centraliser is a Cartan subalgebra (see [50, 2.4.2]) and so s is conjugate to an element $\sum_{i=0}^m h_i t^i$ of $\mathfrak{h}_m$. But since $\mathfrak{h}$ is abelian, we see that the Jordan decomposition of $\sum_{i=0}^m h_i t^i$ must be $h_0 + \sum_{i=1}^m h_i t^i$ (using the fact that $\mathfrak{g}_m^{(\geq 1)}$ consists of nilpotent elements). Since this is conjugate to the semisimple element s, the nilpotent part of this decomposition must be 0 and so we see that in fact we must have $h_i = 0$ for $i > 0$. Hence s is conjugate to h_0 which is a semisimple element of $\mathfrak{g}_m^{(0)}$. □

Proposition 4.1.2. *The nilpotent elements of $\mathfrak{g}_m$ are precisely those elements of the form*

$$x_0 + \sum_{i=1}^{m} x_i t^i$$

where x_0 is a nilpotent element of $\mathfrak{g}$ and $x_1, \ldots, x_m \in \mathfrak{g}$.

Proof. Let $x \in \mathfrak{g}$ be of the form above. Using the axioms of a restricted Lie algebra inductively, along with the fact that $\mathfrak{g}_m^{(\geq 1)}$ is a nilpotent ideal we see that $x^{[p]^i} = 0$ for sufficiently large i, so x is nilpotent.

Now suppose that x is not of the form above and assume for a contradiction that x is nilpotent. Then writing $x = \sum_{i=0}^m x_i t^i$ we see that $x_0 = x_0^s + x_0^n$ is the Jordan decomposition of x_0 in $\mathfrak{g}_m^{(0)} \cong \mathfrak{g}$ with $x_0^s \neq 0$. According to [50, Theorem 2.3.4] there exists a $k > 0$ such that $x^{[p]^k}$ is a semisimple element, and hence we must have $x^{[p]^k} = 0$ since x is nilpotent. However since $\mathfrak{g}_m^{(\geq 1)}$ is a nilpotent ideal which is stable under the p^{th} power map it follows from the axioms for restricted Lie algebras that if we write $x^{[p]^k} = \sum_{i=0}^m x_{i,k} t^i$ then $x_0^{[p]^k} = x_{0,k}$. However $x_0^{[p]^k} \neq 0$ for any $k > 0$ since we assumed that x_0 was not nilpotent, giving a contradiction. □

Remark 4.1.3. The isomorphism $\hat{\kappa}_m : \mathfrak{g}_m \to \mathfrak{g}_m^*$ allows us to transport the notations of semisimple and nilpotent to p-characters $\chi \in \mathfrak{g}_m^*$ and define the Jordan decomposition $\chi = \chi_s + \chi_n$ of χ; we shall often make use without further comment.

4.2 Some Morita equivalences

We now proceed to prove several Morita equivalences between certain reduced enveloping algebras $U_\chi(\mathfrak{g}_m)$. Fix $0 \leq k \leq m$. We say that $\chi \in \mathfrak{g}^*$ *is supported in degree less than or* *equal to* k if

$$\chi\left(\mathfrak{g}_m^{(>k)}\right) = 0.$$

Using the isomorphism $\hat{\kappa}_m$, every $\chi \in \mathfrak{g}^*$ can be expressed uniquely as $\hat{\kappa}_m(x)$ for some $x \in \mathfrak{g}$; χ is then supported in degree less than or equal to k if and only if $x \in \mathfrak{g}_m^{(\geq m-k)}$.

The following result will be useful the classification of simple $U(\mathfrak{g}_m)$-modules.

Proposition 4.2.1. *Let* $\chi \in \mathfrak{g}_m^*$ *be supported in degree less than or equal to* k*, and let* $\psi = \chi|_{\mathfrak{g}_k} \in \mathfrak{g}_k^*$*, where we identify* $\mathfrak{g}_k$ *with* $\mathfrak{g}_m^{(\leq k)} \subseteq \mathfrak{g}_m$ *as vector spaces.*

(i) The full subcategory of $U_\chi(\mathfrak{g}_m)$-mod *whose objects are the modules annihilated by* $\mathfrak{g}_m^{(>k)}$ *is equivalent to* $U_\psi(\mathfrak{g}_k)$-mod*. This subcategory contains all simple* $U_\chi(\mathfrak{g}_m)$*-modules.*

(ii) $\dim \mathfrak{g}_m - \dim \mathfrak{g}_m^\chi = \dim \mathfrak{g}_k - \dim \mathfrak{g}_k^\psi$.

Proof. Let I_k be the ideal of $U_\chi(\mathfrak{g}_m)$ generated by $\mathfrak{g}_m^{(>k)}$. Using the PBW theorem for reduced enveloping algebras, we observe that the map $\mathfrak{g}_m \to U_\chi(\mathfrak{g}_m)/I_k$ factors through $\mathfrak{g}_m \twoheadrightarrow \mathfrak{g}_k$ and induces an isomorphism $U_\psi(\mathfrak{g}_k) \cong U_\chi(\mathfrak{g}_m)/I_k$. Furthermore, I_k is generated by nilpotent elements and so since $U_\chi(\mathfrak{g}_m)$ is Artinian I_k is contained in the Jacobson radical by [35, Theorem 0.1.12]. Hence every simple $U_\chi(\mathfrak{g}_m)$-module is annihilated by I_k, completing the proof of part (i).

Now observe that $\mathfrak{g}_m^{(>k)}$ is an ideal of $\mathfrak{g}_m$ on which χ is 0, so $\mathfrak{g}_m^{(>k)} \subseteq \mathfrak{g}_m^\chi$ and hence $\mathfrak{g}_m^\chi + \mathfrak{g}_m^{(\leq k)} = \mathfrak{g}_m$. Now $\dim(\mathfrak{g}_m) = \dim(\mathfrak{g}_m^\chi + \mathfrak{g}_m^{(\leq k)}) = \dim(\mathfrak{g}_m^\chi) + \dim(\mathfrak{g}_m^{(\leq k)}) - \dim(\mathfrak{g}_m^\chi \cap \mathfrak{g}_m^{(\leq k)})$, so it is enough to show that $\mathfrak{g}_m^\chi \cap \mathfrak{g}_m^{(\leq k)} = \mathfrak{g}_k^\psi$ (where we again identify $\mathfrak{g}_k$ with $\mathfrak{g}_m^{(\leq k)}$). But this follows from the fact $\mathfrak{g}_m/\mathfrak{g}_m^{(>k)} \cong \mathfrak{g}_k$ and the definition of ψ, completing the proof. □

We now prove a parabolic induction theorem for $\mathfrak{g}_m$ analogous to Theorem 3.2.1, which allows us to reduce the problem of classifying $U_\chi(\mathfrak{g}_m)$-modules to the case where χ is nilpotent. It is very similar in spirit and proof to a famous result of Friedlander and Parshall [17, Theorem 3.2] for reductive Lie algebras. We start with the following result, which is easy to prove but vital to our later reductions.

Lemma 4.2.2. *Suppose* $\chi \in \mathfrak{g}_m^*$ *is such that* $\chi([\mathfrak{g}_m, \mathfrak{g}_m]) = 0$. *Then the categories* $U_\chi(\mathfrak{g}_m)$-mod *and* $U_0(\mathfrak{g}_m)$-mod *are equivalent.*

Proof. Let $\lambda \in \mathfrak{g}_m^*$ be the unique linear function satisfying $\chi(x)^p - \chi(x^{[p]}) = \lambda(x)^p$ for all $x \in \mathfrak{g}_m$. This λ certainly exists; for all $x \in \mathfrak{g}_m$, set $\lambda(x)$ to be the unique p-th root of $\chi(x)^p - \chi(x^{[p]})$. Since $[\mathfrak{g}_m, \mathfrak{g}_m]$ is a restricted Lie subalgebra, it follows that $\lambda([\mathfrak{g}_m, \mathfrak{g}_m]) = 0$ and so λ defines a one dimensional $U(\mathfrak{g}_m)$-module $\Bbbk_\lambda$ with p-character χ. The functor $M \mapsto M \otimes_{U(\mathfrak{g}_m)} \Bbbk_\lambda$ is then an equivalence $U_0(\mathfrak{g}_m)$-mod $\to U_\chi(\mathfrak{g}_m)$-mod with quasi-inverse $M \mapsto M \otimes_{U(\mathfrak{g}_m)} \Bbbk_{-\lambda}$. □

Suppose χ is a p-character with Jordan decomposition $\chi = \chi_s + \chi_n$. By Proposition 4.1.1, after conjugating by an element of G_m we may suppose that χ_s vanishes on $\mathfrak{h}_m^{(<m)} \oplus \mathfrak{n}_m^{\pm}$. The stabiliser $\mathfrak{g}^{\chi_s}$ is a Levi subalgebra of $\mathfrak{g}$ and, since χ_s is supported on $\mathfrak{g}_m^{(m)}$, we have that $(\mathfrak{g}_m)^{\chi_s} = (\mathfrak{g}^{\chi_s})_m$ and hence we may write $\mathfrak{g}_m^{\chi_s}$ unambiguously.

Let $\mathfrak{p} = \mathfrak{g}^{\chi_s} \oplus \mathfrak{r}$ be a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{r}$. Note that $\chi(\mathfrak{r}_m) = 0$ by Lemma 2.4.4, and so $U_\chi(\mathfrak{g}_m^{\chi_s})$-modules can be naturally inflated to $U_\chi(\mathfrak{p}_m)$-modules by letting $\mathfrak{r}_m$ act trivially.

Consider the functors

$$\begin{aligned} U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} (\bullet) \quad &: \quad U_\chi(\mathfrak{g}_m^{\chi_s})\text{-mod} \longrightarrow U_\chi(\mathfrak{g}_m)\text{-mod}, \\ &\quad\quad M \longmapsto U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} M. \\ (\bullet)^{\mathfrak{r}_m} \quad &: \quad U_\chi(\mathfrak{g}_m)\text{-mod} \longrightarrow U_\chi(\mathfrak{g}_m^{\chi_s})\text{-mod}, \\ &\quad\quad M \longmapsto M^{\mathfrak{r}_m}. \end{aligned}$$

Theorem 4.2.3. *The functors* $U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} (\bullet)$ *and* $(\bullet)^{\mathfrak{r}_m}$ *are quasi-inverse equivalences of categories.*

Proof. Our approach is very similar to [24, §7.4], in which the $m = 0$ case is proven.

We first show that each $U_\chi(\mathfrak{g})$-module M is free as a $U_0(\mathfrak{r}_m)$-module. By Proposition 2.5.14 we have $\mathcal{V}_{\mathfrak{r}_m}(M) \subseteq \mathcal{N}^{[p]}(\mathfrak{g}_m^\chi) \cap \mathfrak{r}_m = 0$, and hence M is projective as a $U_\chi(\mathfrak{r}_m)$-

module. Furthermore since $\chi([\mathfrak{r}_m, \mathfrak{r}_m]) = \chi(\mathfrak{r}_m^{[p]}) = 0$, the argument from [24, Corollary 7.2] shows that that in fact M is free as a $U_\chi(\mathfrak{r}_m)$-module.

It now follows that both functors are exact. The algebra $U_0(\mathfrak{r}_m)$ is a Frobenius algebra by [17, Proposition 1.2] and hence has a simple socle, so $\dim U_0(\mathfrak{r}_m)^{\mathfrak{r}_m} = 1$. Since $\dim(\mathfrak{r}) = \dim(\mathfrak{g}/\mathfrak{p})$, we have that $\dim M = p^{\dim \mathfrak{r}_m} \dim M^{\mathfrak{r}_m}$ and $\dim(U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} N) = p^{\dim \mathfrak{r}_m} \dim N$. In particular $\dim M = \dim(U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} M^{\mathfrak{r}_m})$ and $\dim N = \dim(U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} N)^{\mathfrak{r}_m}$.

To complete the proof we must show that the unit and counit of this adjunction are natural isomorphisms. In particular, we wish to show that the maps $\varphi_M : U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} M^{\mathfrak{r}_m} \to M$ and $\psi_N : N \to (U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{p}_m)} N)^{\mathfrak{r}_m}$ given by $\varphi_M(u \otimes m) = u \cdot m$ and $\psi_N(n) = 1 \otimes n$ are isomorphisms. We first observe that ψ_N is certainly injective, and so by considering dimensions is an isomorphism. Now we show that φ_M is an isomorphism for M simple: it must be surjective as it has non-zero image, and so considering dimensions once again it is an isomorphism. Since our functors are exact and these categories are all Artinian, a standard argument using the short Five Lemma shows φ_M is an isomorphism for any M. □

Corollary 4.2.4. *If χ is a regular semisimple p-character then*

$$U_\chi(\mathfrak{g}_m) \cong \mathrm{Mat}_{p^{\frac{(m+1)}{2}(\dim(\mathfrak{g}) - \mathrm{ind}(\mathfrak{g}))}} U_0(\mathfrak{h}_m)$$

Proof. Using [42, Theorem 2.3(ii)] and Theorem 4.2.3 we have the isomorphism $U_\chi(\mathfrak{g}_m) \cong \mathrm{Mat}_{p^{\frac{(m+1)}{2}(\dim(\mathfrak{g}) - \mathrm{ind}(\mathfrak{g}))}} U_\chi(\mathfrak{h}_m)$. This requires the notion of a χ-admissible subalgebra (which we discuss later in §4.3), but in the case χ is regular semisimple the conditions for a subalgebra to be χ-admissible reduce to the subalgebra consisting of nilpotent elements. In particular, $\mathfrak{n}_m$ is a χ-admissible subalgebra of $\mathfrak{g}_m$ of dimension $\frac{m+1}{2}(\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g}))$. The equivalence described in Lemma 4.2.2 then induces an algebra isomorphism $U_0(\mathfrak{h}_m) \cong U_\chi(\mathfrak{h}_m)$. □

We now describe the simple and projective modules over $U_0(\mathfrak{h}_m)$. Since $\mathfrak{h}_m$ is commutative we have that the simple modules are the 1-dimensional modules $\Bbbk_\lambda$ for $\lambda \in \mathfrak{h}^*$ such that $\lambda(h)^p = \lambda(h^{[p]})$, where $\mathfrak{h}_m^{(0)}$ acts by λ and $\mathfrak{h}_m^{(\geq 1)}$ acts by 0. Note that there are $p^{\dim(\mathfrak{h})}$ such λ. Each simple module $\Bbbk_\lambda$ then has a unique projective cover in the category $U_0(\mathfrak{h}_m)$-mod, which we denote $Q^{\mathfrak{h}_m}(\lambda)$, and these $Q^{\mathfrak{h}_m}(\lambda)$ form a complete set of irreducible projective modules for $U_0(\mathfrak{h}_m)$.

Proposition 4.2.5. *The only composition factor of* $Q^{\mathfrak{h}_m}(\lambda)$ *is* $\Bbbk_\lambda$*, which occurs with multiplicity* $p^{m\dim(\mathfrak{h})}$.

Proof. Consider the module $P(\lambda) := U_0(\mathfrak{h}_m) \otimes_{U_0(\mathfrak{h}_m^{(0)})} \Bbbk_\lambda$. As a $U_0(\mathfrak{h}_m^{(\geq 1)})$-module this is isomorphic to $U_0(\mathfrak{h}_m^{(\geq 1)})$ and so, since $\mathfrak{h}_m^{(\geq 1)}$ is p-nilpotent, $P(\lambda)$ is indecomposable by [24, Corollary 3.4]. Hence $\dim(Q^{\mathfrak{h}_m}(\lambda)) \geq \dim(P(\lambda)) = p^{m\dim(\mathfrak{h})}$. But $p^{(m+1)\dim(\mathfrak{h})} = \dim(U_0(\mathfrak{h}_m)) \geq \sum_\lambda \dim(Q^{\mathfrak{h}_m}(\lambda)) \geq \sum_\lambda \dim(P(\lambda)) = p^{\dim(\mathfrak{h})}p^{m\dim(\mathfrak{h})}$, so in fact $Q^{\mathfrak{h}_m}(\lambda) \cong P(\lambda)$. Hence $\mathfrak{h}_m^{(0)}$ acts by λ on $Q^{\mathfrak{h}_m}(\lambda)$ so the only composition factor of $Q^{\mathfrak{h}_m}(\lambda)$ is $\Bbbk_\lambda$, and since $\dim(Q^{\mathfrak{h}_m}(\lambda)) = \dim(P(\lambda)) = p^{m\dim(\mathfrak{h})}$ it must occur with multiplicity $p^{m\dim(\mathfrak{h})}$ as required. □

4.3 The Kac–Weisfeiler conjectures in the truncated current case

Recall from §2.4.3 the following conjectures, originally made in [55]:

Conjecture 4.3.1 (KW1)**.** *Let* $\mathfrak{g}$ *be a restricted Lie algebra and let* $M(\mathfrak{g})$ *be the maximal dimension among simple* $U(\mathfrak{g})$*-modules. Then* $M(\mathfrak{g}) = p^{\frac{1}{2}(\dim(\mathfrak{g}) - \mathrm{ind}(\mathfrak{g}))}$.

Theorem 4.3.2 ([40], KW2)**.** *Let* $\mathfrak{g}$ *be the Lie algebra of a reductive group* G *under the standard hypotheses. Then for any p-character* χ *and any simple* $U_\chi(\mathfrak{g})$*-module* M*,* $p^{\frac{1}{2}(\dim\mathfrak{g} - \dim\mathfrak{g}^\chi)}$ *divides* $\dim M$.

Most of the known results on (KW1) mentioned in §2.4.3 do not apply to truncated current Lie algebras. The major exception is the case $\mathfrak{g} = \mathfrak{gl}_n$; here (as observed in

[57]) if e is a nilpotent element in $\mathfrak{gl}_{(m+1)n}$ corresponding to the rectangular partition $(m+1, m+1, \ldots, m+1)$ then $(\mathfrak{gl}_{(m+1)n})^e \cong (\mathfrak{gl}_n)_m$. Hence by [53, Theorem 4] (KW1) holds for $(\mathfrak{gl}_n)_m$. The result [33, Theorem 1.1] also implies that if we fix $\mathfrak{g}$ and m, then in sufficiently large characteristic (KW1) holds for $\mathfrak{g}_m$, without giving an explicit bound. We give a proof of (KW1) for $\mathfrak{g}_m$ which is valid where $\mathfrak{g}$ the Lie algebra of a reductive group G with no assumptions beyond the standard hypotheses.

As mentioned in §2.4.3, Premet proved (KW2) in [40], subsequently giving other proofs in [42, §2.6] and [43, Theorem 5.6]. In [28], Kac conjectured that the statement (KW2) holds for all Lie algebras of algebraic groups, but very little is known about this statement outside the standard reductive case. We show that (KW2) holds for certain classes of p-characters of the truncated current Lie algebra $\mathfrak{g}_m$, and also give a proof for all p-characters of $(\mathfrak{sl}_2)_m$ in characteristic greater than 2.

4.3.1 The first Kac–Weisfeiler conjecture

Recall that earlier in the proof of Theorem 2.5.10 we defined $\mathfrak{h}_m^{\text{reg}} := \{\sum x_i t^i \in \mathfrak{h}_m \mid x_0 \in \mathfrak{h}^{\text{reg}}\}$.

Proposition 4.3.3. *Let $\chi = \hat{\kappa}_m(h)$ for some $h \in \mathfrak{h}_m^{\text{reg}}$. Then the algebra $U_\chi(\mathfrak{g}_m)$ has (up to isomorphism) precisely $p^{\operatorname{rank}(\mathfrak{g})}$ simple modules and the same number of projective indecomposable modules. The simple modules have dimension $p^{\frac{m+1}{2}(\dim(\mathfrak{g})-\operatorname{rank}(\mathfrak{g}))}$ while the projective indecomposable modules have dimension $p^{\frac{m+1}{2}(\dim(\mathfrak{g})+\operatorname{rank}(\mathfrak{g}))-\operatorname{rank}(\mathfrak{g})}$.*

Proof. Let $h = \sum_{x=0}^m x_i t^i$. Observe that the Jordan decomposition of h has semisimple part x_0 and nilpotent part $\sum_{x=1}^m x_i t^i$, so by Theorem 4.2.3 we obtain an equivalence between $U_\chi(\mathfrak{g}_m)$ and $U_\chi(\mathfrak{h}_m)$. Furthermore, by Lemma 4.2.2 together with Proposition 4.2.5 and the preceeding discussion, the simple modules for $U_\chi(\mathfrak{h}_m)$ are 1-dimensional, the projective modules are $p^{m \operatorname{rank}(\mathfrak{g})}$-dimensional, and there are $p^{\operatorname{rank}(\mathfrak{g})}$ of each. Now, from the proof of Theorem 4.2.3, for any $U_\chi(\mathfrak{h}_m)$-module N we have $\dim(U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{b}_m)} N) = p^{\dim \mathfrak{b}_m} \dim N = p^{\frac{m+1}{2}(\dim(\mathfrak{g})-\operatorname{rank}(\mathfrak{g}))} \dim N$ and the desired result follows. □

Corollary 4.3.4. *The first Kac–Weisfeiler conjecture holds for $\mathfrak{g}_m$ where $\mathfrak{g}$ is the Lie algebra of a standard reductive group.*

Proof. By the proof of Theorem 2.5.10, the conjugates of $\mathfrak{h}_m^{\mathrm{reg}}$ are dense in $\mathfrak{g}_m$. Recall that we defined $M(\mathfrak{g}_m)$ to be the maximal dimension of a simple $U(\mathfrak{g}_m)$-module. By [43, Proposition 4.2(1)] the set of p-characters χ such that all simple $U_\chi(\mathfrak{g}_m)$-modules have dimension $M(\mathfrak{g}_m)$ is non-empty and open, and so in particular intersects the conjugates of $\hat{\kappa}_m(\mathfrak{h}_m^{\mathrm{reg}})$ non-trivially. Now by Proposition 4.3.3, if χ is conjugate to an element of $\hat{\kappa}_m(\mathfrak{h}_m^{\mathrm{reg}})$, then all simple $U_\chi(\mathfrak{g}_m)$-modules have dimension $p^{\frac{m+1}{2}(\dim(\mathfrak{g})-\mathrm{rank}(\mathfrak{g}))}$ which is equal to $p^{\frac{1}{2}(\dim(\mathfrak{g}_m)-\mathrm{ind}(\mathfrak{g}_m))}$ by Theorem 2.5.10. Hence $M(\mathfrak{g}_m) = p^{\frac{1}{2}(\dim(\mathfrak{g}_m)-\mathrm{ind}(\mathfrak{g}_m))}$. □

4.3.2 The second Kac–Weisfeiler conjecture

We now investigate the second Kac–Weisfeiler conjecture for the case of truncated currents on the Lie algebra of a general standard reductive group. In particular, although we do not show the statement holds for all p-characters, we give several families for which it does hold. Following [42, §2.3], for $\chi \in (\mathfrak{g}_m)^*$ a nilpotent p-character, we define a χ-*admissible subalgebra* of $\mathfrak{g}_m$ to be a subalgebra $\mathfrak{m}$ satisfying:

(i) $\mathfrak{m}$ consists of nilpotent elements.

(ii) $\chi(\overline{[\mathfrak{m},\mathfrak{m}]}) = 0$, where $\mathfrak{a}$ denotes the $[p]$-closure of a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$.

(iii) $\mathcal{V}_{\mathfrak{g}_m}(\chi) \cap \mathfrak{m} = 0$.

By [42, Theorem 2.3(ii)], if there exists a χ-admissible subalgebra $\mathfrak{m}$ such that $\dim(\mathfrak{m}) = \frac{1}{2}(\dim \mathfrak{g}_m - \dim \mathfrak{g}_m^\chi)$ then (KW2) holds for χ.

Proposition 4.3.5. *Let $\mathfrak{g}$ be the Lie algebra of a standard reductive group. Then for $\mathfrak{g}_m$, we have:*

(i) (KW2) holds for all homogeneous p-characters.

(ii) (KW2) holds for all semisimple p-characters.

Proof. These two claims follow directly from Proposition 4.2.1 and Theorem 4.2.3 respectively, since these results allow to reduce to the $\chi = 0$ case and in this $\mathfrak{m} = 0$ is clearly a χ-admissible subalgebra of the correct dimension. □

Proposition 4.3.6. *The second Kac-Weisfeiler conjecture holds for $(\mathfrak{sl}_2)_m$ provided $p > 2$.*

Proof. First observe that by Theorem 4.2.3 we may reduce to the reductive case. Let $x = \sum_{j=i}^{m} x_j t^j$ where $x_i \neq 0$ be a nilpotent element of $(\mathfrak{sl}_2)_m$, and let $\chi = \hat{\kappa}_m(x)$. Choose a nilpotent $y_0 \in \mathfrak{sl}_2$ such that $[x_i, y_0] \neq 0$; this is always possible since $p > 2$. We claim that $\mathfrak{n}_l = \text{span}\{y_0 t^k : 0 \leq k \leq l\}$ is χ-admissible for any $l \leq m - i$ and that $\dim((\mathfrak{sl}_2)_m)^\chi) \geq 2i + m + 1$, which by our earlier discussion implies that (KW2) holds for $(\mathfrak{sl}_2)_m$. By Proposition 4.1.2 we have that condition (i) holds, and since $\mathfrak{n}$ is commutative condition (ii) also holds. Now, by Proposition 2.5.14, we have $\mathcal{V}_{(\mathfrak{sl}_2)_m}(\chi) = \mathcal{N}^{[p]}(((\mathfrak{sl}_2)_m)^\chi)$ so to show condition (iii) it suffices to show that $\sum_{k=0}^{m-i} \lambda_k y_0 t^k \notin ((\mathfrak{sl}_2)_m)^\chi$ for any scalars $\lambda_k \in \Bbbk$. But $[x, y_0 t^k] = [x_i, y_0] t^{i+k} + \sum_{j=i+1}^{m} [x_j, y_0] t^{j+k} \neq 0$, so by considering the smallest k such that $\lambda_k \neq 0$ we see this holds. Hence $\mathfrak{n}_l$ is indeed χ-admissible for any $l \leq m - i$. Finally, observe that

$$((\mathfrak{sl}_2)_m)^{(\geq m-i+1)} \oplus \text{span}\{\sum_{j=i}^{m} x_j t^{j-i+k} : 0 \leq k \leq m - i\} \subseteq ((\mathfrak{sl}_2)_m)^\chi$$

and so $\dim((\mathfrak{sl}_2)_m)^\chi \geq 2i + m + 1$ as required. □

4.4 Classification of simple modules

4.4.1 Baby Verma modules and their simple quotients

The aim of this section is to classify the simple $U_\chi(\mathfrak{g})$-modules for p-characters χ which are homogeneous and nilpotent of standard Levi type.

In order to justify our restrictions on p-characters we make some basic observations about the role of baby Verma modules. In the case $m = 0$ they are one of the main tools for explicit construction of simple modules. Our approach here is to pick a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$, inflate one dimensional $U_\chi(\mathfrak{h}_m)$-modules to $U_\chi(\mathfrak{b}_m)$-modules and then induce. Note that this is only possible if we can choose a G_m-conjugate of χ which vanishes on $\mathfrak{n}_m$ where $\mathfrak{n} \subseteq \mathfrak{b}$ is the radical. When $m > 0$ this is not always the case; consider, for example, the case where $\mathfrak{g} = \mathfrak{sl}_2$ and $m = 1$, and let $\{e, h, f\} \subseteq \mathfrak{sl}_2$ be the standard basis. Then for $\chi = \hat{\kappa}_m(e + ft)$ there is no choice of $\mathfrak{b}$ such that $\chi(\mathfrak{n}_m) = 0$.

Now assume there exists a choice of Borel subalgebra $\mathfrak{b}$ such that $\chi(\mathfrak{n}_m) = 0$ and fix such a choice. Then we can define the *baby Verma modules*. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{h}$ a maximal torus of $\mathfrak{g}$. Define

$$\Lambda_\chi = \{\lambda \in \mathfrak{h}_m^* : \lambda(ht^i)^p - \lambda((ht^i)^{[p]}) - \chi(ht^i)^p = 0 : h \in \mathfrak{h}, 0 \leq i \leq m\} \subseteq \mathfrak{h}_m^*$$

Then for any $\lambda \in \Lambda_\chi$, the baby Verma module $M_\chi(\lambda)$ for $U_\chi(\mathfrak{g}_m)$ is given by

$$M_\chi(\lambda) := U_\chi(\mathfrak{g}_m) \otimes_{U_\chi(\mathfrak{b}_m)} \mathbb{k}_\lambda$$

where $\mathbb{k}_\lambda$ is the 1-dimensional module on which $\mathfrak{h}_m$ acts by λ and $\mathfrak{n}_m$ acts by 0.

Lemma 4.4.1. *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}$, and let $\chi \in \mathfrak{g}^*$ be such that $\chi(\mathfrak{n}) = 0$. Then every simple $U_\chi(\mathfrak{g}_m)$-module is isomorphic to a quotient of $M_\chi(\lambda)$ for some $\lambda \in \Lambda_\chi$.*

Proof. The proof of the analogous statement [24, Proposition 6.7] is still valid here, replacing $\mathfrak{n}, \mathfrak{b}$ and $\mathfrak{n}^-$ with $\mathfrak{n}_m, \mathfrak{b}_m$ and $\mathfrak{n}_m^-$ respectively. □

We consider the case where $m > 0$ and χ is a nilpotent homogeneous p-character of degree k, i.e. where $\chi(\bigoplus_{i \neq k} \mathfrak{g}(i)) = 0$. In this case, we can always find a Borel $\mathfrak{b}$ such that $\chi(\mathfrak{b}_m) = 0$. In fact by Proposition 4.2.1, without loss of generality we can assume that

$k = m$ so let $\chi = \hat{\kappa}_m(x)$ for some $x \in \mathfrak{g} \cong \mathfrak{g}_m^{(0)}$. Then by Theorem 4.2.3 and Proposition 4.1.2 we can also assume that x is a nilpotent element of $\mathfrak{g}$.

We focus in particular on the case where $\chi = \hat{\kappa}_m(e)$ for some $e \in \mathfrak{g}$ of *standard Levi type*. This means e is a regular nilpotent element of some Levi subalgebra of $\mathfrak{g}$, see [24, §10] for more detail.

For the rest of this section, we fix such an $e \in \mathfrak{g}$ and $\chi = \hat{\kappa}_m(e)$. We can choose a maximal torus $\mathfrak{h} = \mathrm{Lie}(H) \subseteq \mathfrak{g}$, let Φ be the root system with respect to H, and choose a set of simple roots Δ such that the minimal Levi subalgebra containing e has root system generated by a subset of Δ.

Pick root vectors e_α for $\alpha \in \Phi$ such that $e = \sum_{\alpha \in I} e_\alpha$ for some $I \subseteq \Delta$. Let $\mathfrak{b}^+ \subseteq \mathfrak{g}$ be the Borel subalgebra corresponding to this choice of simple roots, $\mathfrak{b}^-$ the opposite Borel corresponding to $-\Delta$, and $\mathfrak{n}^-, \mathfrak{n}$ their respective nilradicals.

Finally, we let $\mathfrak{g}_I$ be the Levi subalgebra $\mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbb{Z}I} \mathfrak{g}_\alpha$ of $\mathfrak{g}$ corresponding to the subset $I \subseteq \Delta$, and let $\mathfrak{r}$ and $\mathfrak{r}^-$ be the nilradicals of the parabolic subalgebras $\mathfrak{g}_I + \mathfrak{b}$ and $\mathfrak{g}_I + \mathfrak{b}^-$ respectively.

Proposition 4.4.2. *Let $\chi = \hat{\kappa}_m(e)$ for some $e \in \mathfrak{g}$ in standard Levi type, and fix a toral basis $\{h_1, \ldots, h_r\}$ of $\mathfrak{h}$. Then $\Lambda_\chi = \{\lambda \in \mathfrak{h}^* : \lambda(\mathfrak{h}_m^{(\geq 1)}) = 0, \lambda(h_j) \in \mathbb{F}_p \subset \Bbbk\}$.*

Proof. Consider first the equation $\lambda(ht^i)^p - \lambda((ht^i)^{[p]}) - \chi(ht^i)^p = 0$ in the case $i = m$. Here there is a unique solution $\lambda(ht^m) = \chi(ht^m) = 0$ since $(ht^i)^{[p]} = 0$. We then see inductively for $i = m-1, m-2, \ldots, 1$ that $\lambda((ht^i)^{[p]}) = 0$ and so again $\lambda(ht^i) = \chi(ht^i) = 0$. Now, if $h \in \mathfrak{h}$ is such that $h^{[p]} = h$ then since $\chi(h) = 0$ we have that $\lambda(h)^p - \lambda(h^{[p]}) = \lambda(h)^p - \lambda(h) = 0$. Hence $\lambda(h_j)$ can be chosen to be any value in $\mathbb{F}_p$ for all $1 \leq j \leq r$. □

In light of this, from here on we regard elements of Λ_χ as elements of $\mathfrak{h}^*$ rather than of $\mathfrak{h}_m^*$. For χ in standard Levi form, we have the following result analogous to [24, Proposition 10.2] which gives a relationship between the baby Verma modules and the simple $U_\chi(\mathfrak{g}_m)$-modules.

Lemma 4.4.3. *Let $\chi \in \mathfrak{g}^*$ be nilpotent and in standard Levi form. Then each $M_\chi(\lambda)$ has a unique maximal submodule and hence a unique simple quotient $L_\chi(\lambda)$.*

Proof. Just as for Lemma 4.4.1, the proof of the analogous statement [24, Proposition 10.2] is still valid here, replacing $\mathfrak{n}, \mathfrak{b}$ and $\mathfrak{n}^-$ with $\mathfrak{n}_m, \mathfrak{b}_m$ and $\mathfrak{n}_m^-$ respectively. □

4.4.2 Simple modules

By Theorem 4.2.3, along with Lemmas 4.4.1 and 4.4.3, to classify the simple $U_\chi(\mathfrak{g}_m)$-modules for χ in standard Levi form supported in degree m it suffices to determine when $L_\chi(\lambda) \cong L_\chi(\mu)$ for $\lambda, \mu \in \Lambda_\chi$. We first consider the case $I = \Delta$, i.e. when e is a regular nilpotent element of $\mathfrak{g}$ and $\mathfrak{g}_I = \mathfrak{g}$.

Proposition 4.4.4. *Let $\chi = \hat{\kappa}_m(e)$ for $e \in \mathfrak{g}$ regular nilpotent. Then $M_\chi(\lambda)$ is simple for all $\lambda \in \Lambda_\chi$.*

Proof. Letting $\mathfrak{m} = \mathfrak{n}_m^-$ and observing that $\mathfrak{g}_m^\chi \cap \mathfrak{n}_m^- = 0$, we can apply Proposition 2.5.14 to see that $\mathfrak{m}$ is a χ-admissible subalgebra (note that χ vanishes on $[\mathfrak{m}, \mathfrak{m}]$ since we have assumed that e is in standard Levi form). It follows by the same argument as used in the second paragraph of the proof of Theorem 4.2.3 that any $U_\chi(\mathfrak{g}_m)$-module is free as a $U_\chi(\mathfrak{m}_m)$-module. Now let N be a $U_\chi(\mathfrak{g}_m)$-submodule of $M_\chi(\lambda)$. By dimensional considerations N is free of rank 0 or 1 over $U_\chi(\mathfrak{m}_m)$, and hence $N = 0$ or $N = M_\chi(\lambda)$, i.e. $M_\chi(\lambda)$ is simple. □

Proposition 4.4.5. *Let $\chi = \hat{\kappa}_m(e)$ for $e \in \mathfrak{g}$ regular nilpotent. Then $L_\chi(\lambda) \cong L_\chi(\mu)$ if and only if $\lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})}$.*

Proof. Fix some $\lambda \in \Lambda_\chi$ and $\alpha \in \Delta$, and consider the element $f_\alpha t^m \otimes 1_\lambda \in M_\chi(\lambda)$ for some $f_\alpha \in \mathfrak{g}_{-\alpha}$. This element is highest weight since if $x \in \mathfrak{n}_m$ then $[x, f_\alpha t^m] \in (\mathfrak{b}_m)^{(m)}$, so $x \cdot (f_\alpha t^m \otimes 1_\lambda) = f_\alpha t^m \otimes (x \cdot 1_\lambda) + 1 \otimes ([x, f_\alpha t^m] \cdot 1_\lambda) = 0$. Furthermore, it generates $M_\chi(\lambda)$ since $(f_\alpha t^m)^{p-1} \cdot (f_\alpha t^m \otimes 1_\lambda) = \chi(f_\alpha t^m)^p \otimes 1_\lambda$ and $\chi(f_\alpha t^m) \neq 0$. Hence we have a surjective homomorphism from $M_\chi(\lambda - d\alpha)$ to $M_\chi(\lambda)$, which by considering dimensions must in fact be an isomorphism. But we also have that $M_\chi(\lambda)$ is simple for all $\lambda \in \Lambda_\chi$ by Corollary 4.4.4, so $L_\chi(\lambda) = M_\chi(\lambda)$. Hence it suffices to show that $\mathrm{span}_{\mathbb{F}_p}\{d\alpha : \alpha \in \Delta\} = \{\lambda \in \Lambda_\chi : \lambda|_{\mathfrak{z}(\mathfrak{g})} = 0\}$, but this can be verified by a case-by-case computation on each irreducible component of the root system Φ.

On the other hand, if $\lambda|_{\mathfrak{z}(\mathfrak{g})} \neq \mu|_{\mathfrak{z}(\mathfrak{g})}$ then $\mathfrak{z}(\mathfrak{g}_m)$ acts on $L_\chi(\lambda)$ and $L_\chi(\mu)$ by different scalars, so they have different central characters and hence must be non-isomorphic. □

The following Lemma, analogous to a result [24, Proposition 10.7] in the reductive case, allows us to extend this classification of simple modules to p-characters $\chi = \hat{\kappa}_m(e)$ for any $e \in \mathfrak{g}$ of standard Levi type.

Lemma 4.4.6. *There is a bijection between simple $U_\chi(\mathfrak{g}_m)$-modules and simple $U_\chi((\mathfrak{g}_I)_m)$-modules sending M to $M^{\mathfrak{r}_m}$. This bijection takes the simple $U_\chi(\mathfrak{g}_m)$-module $L_\chi(\lambda)$ to the simple $U_\chi((\mathfrak{g}_I)_m)$-module $L_\chi(\lambda)$.*

Proof. The first part follows from general results on graded modules, namely Corollary 1.4 and Theorems 1.1 and 1.2 in [48]. To apply these results, we require only the fact that $\mathfrak{r}_m$ and $\mathfrak{r}_m^-$ act nilpotently on the baby Verma modules (and hence on their simple quotients) which follows from the definition of $M_\chi(\lambda)$ and the fact that $\chi(\mathfrak{r}_m^-) = 0$. The second part follows easily once we observe that if we take a highest weight generator of weight λ for the $U_\chi(\mathfrak{g}_m)$ module $L_\chi(\lambda)$, then this element lies in $M^{\mathfrak{r}_m}$ and generates it as a $U_\chi((\mathfrak{g}_I)_m)$-module. □

Theorem 4.4.7. $L_\chi(\lambda) \cong L_\chi(\mu)$ *if and only if* $\lambda|_{\mathfrak{z}(\mathfrak{g}_I)} = \mu|_{\mathfrak{z}(\mathfrak{g}_I)}$.

Proof. By Lemma 4.4.6, $L_\chi(\lambda) \cong L_\chi(\mu)$ if and only if the corresponding simple $U_\chi((\mathfrak{g}_I)_m)$-modules are isomorphic. But by Proposition 4.4.5 this occurs precisely when $\lambda|_{\mathfrak{z}(\mathfrak{g}_I)} = \mu|_{\mathfrak{z}(\mathfrak{g}_I)}$. □

4.4.3 The general linear algebra

Here we explain how the results of the previous section give a complete classification of simple $U(\mathfrak{g}_m)$-modules with homogeneous p-characters when $\mathfrak{g} = \mathfrak{gl}_n$.

Suppose $\chi \in \mathfrak{g}^*$ is homogeneous. Then by Proposition 4.2.1(i) we may assume that χ is supported in degree m. Decomposing χ into semisimple and nilpotent parts as $\chi = \chi_s + \chi_n$, we consider the stabiliser $(\mathfrak{g}_m)^{\chi_s} = (\mathfrak{g}^{\chi_s})_m$. Note that $\mathfrak{g}^{\chi_s}$ is a Levi subalgebra of $\mathfrak{gl}_n$, and therefore it is isomorphic to $\mathfrak{l} = \mathfrak{gl}_{n_1} \times \cdots \times \mathfrak{gl}_{n_k}$ for some $\sum_{i=1}^k n_i$ and $1 \leq k \leq n$. Furthermore $\chi([\mathfrak{l}_m, \mathfrak{l}_m]) = 0$.

The classification of simple $U_\chi(\mathfrak{l})$-modules follows easily from the classification of the simple $U_{\chi|_{\mathfrak{gl}_{n_i}}}(\mathfrak{gl}_{n_i})$ modules for $i = 1, ..., k$ and so we have now reduced the problem to classifying simple $U_\chi(\mathfrak{g}_m)$-modules, where the semisimple part of χ is supported on the centre of $\mathfrak{g}_m$. Using Lemma 4.2.2 we may assume that χ is nilpotent and supported in degree m. But every nilpotent $\chi \in (\mathfrak{gl}_n)^*$ can be placed in standard Levi form, and so the classification of simple $U_\chi(\mathfrak{gl}_n)$-modules can be deduced from our earlier results.

Remark 4.4.8. This classification can even be applied in the case where the semisimple and nilpotent parts are each homogeneous but are supported in different degrees.

Remark 4.4.9. Outside type A not all nilpotent elements have standard Levi type, so the above classification breaks down. If $p > 2$ and $\mathfrak{g}$ is simple of type B, C or D then the nilpotent orbits are still classified by partitions (although in type D certain partitions correspond to two distinct orbits). The partitions corresponding to standard Levi nilpotent orbits are as follows; see for example [27, §1]:

- Type B: All parts of the partition occur with even multiplicity, with the exception of one odd part which can occur with odd multiplicity.

- Type C: All parts occur with even multiplicity, with the possible exception of one even part which can occur with odd multiplicity.

- Type D: There are $2k$ parts which occur with even multiplicity, say $\lambda_{j_1}, ..., \lambda_{j_{2k}}$. The final two parts of the partition have sizes $2n - \sum_{i=1}^{2k} \lambda_{j_i} - 1$ and 1.

If the nilpotent part of a homogeneous p-character χ corresponds to a partition satisfying these properties, we obtain a classification of simple $U_\chi(\mathfrak{g})$-modules from the above results in the same manner as in the type A case.

4.5 Cartan invariants for the restricted enveloping algebras

Recall that for any finite dimensional algebra A with simple modules $L_1, \ldots, L_s$ there is for each L_i a unique (up to isomorphism) projective cover P_i. These P_i form a complete set of indecomposable projective A-modules. The composition multiplicities of the P_i are known as the Cartan invariants of A.

In Corollary 4.2.4 and Proposition 4.2.5 we calculated the Cartan invariants of $U_\chi(\mathfrak{g}_m)$ whene χ is regular semisimple. In this section we deal with the opposite extreme; the case where $\chi = 0$.

Recall from earlier that we have a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h} = \mathrm{Lie}(H)$ for some choice of torus $H \subseteq G_m$. We consider baby Verma modules $Z_0(\lambda)$ for the restricted enveloping algebra $U_0(\mathfrak{g}_m)$. Since $\chi = 0$ is fixed, from now on we omit the subscript in the notation for baby Verma modules and instead write $Z(\lambda)$. We also define $Z^{\mathfrak{g}}(\lambda)$ to be the baby Verma module for $U_0(\mathfrak{g})$:

$$Z^{\mathfrak{g}}(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} \mathbb{k}_\lambda$$

where $\mathbb{k}_\lambda$ is the 1-dimensional $\mathfrak{b}$ module on which $\mathfrak{h}$ acts by λ and $\mathfrak{n}$ acts by 0.

We can inflate this to a $U_0(\mathfrak{g}_m)$-module by letting $\mathfrak{g}_m^{(\geq 1)}$ act by 0; by abuse of notation we also label this module $Z^{\mathfrak{g}}(\lambda)$. Now observe that by Proposition 4.2.1, the simple $U_0(\mathfrak{g}_m)$-modules are just the simple modules for $U_0(\mathfrak{g})$ with $\mathfrak{g}_m^{(\geq 1)}$ acting by 0. As with the baby Verma modules, these modules are labelled by Λ_0, so again abusing notation we write $L(\lambda)$ for both the simple $U_0(\mathfrak{g})$-module of weight λ and the simple $U_0(\mathfrak{g}_m)$-module of weight λ. Here as in the previous section that we may view Λ_0 as a subset of $\mathfrak{h}^*$ using Proposition 4.4.2.

Recall that if N is a $U_0(\mathfrak{g}_m)$-module, then we write $[N : L(\lambda)]$ for the composition multiplicity of $L(\lambda)$ in N. Similarly, if $\{M(\lambda) : \lambda \in \Lambda_0\}$ is a family of $U_0(\mathfrak{g}_m)$-modules and N admits a filtration such that each section is isomorphic to some $M(\lambda)$, then we write $(N : M(\lambda))$ to denote the number of times $M(\lambda)$ occurs as a subquotient in the filtration. Of course, the quantities $(N : M(\lambda))$ may not be well-defined in general; however, whenever we use this notation we shall show that they are.

The following proposition is not difficult to prove and is useful in the computation of composition multiplicities.

Proposition 4.5.1. *If* $(N : M(\mu))$ *exists, then for any* $\lambda \in \Lambda_0$ *we have* $[N : L(\lambda)] = \sum_{\mu\in\Lambda_0}(N : M(\mu))[M(\mu) : L(\lambda)]$.

Proof. Let $0 = N_0 \subset N_1 \subset \cdots \subset N_k = N$ be a filtration of N in which for all $\mu \in \Lambda_0$, $M(\mu)$ appears as a quotient N_i/N_{i-1} precisely $(N : M(\mu))$ times. Then refining this filtration to a composition series we see that $[N : L(\lambda)] = \sum_{i=1}^{k}[N_i/N_{i-1} : L(\lambda)]$ and the desired result follows. □

4.5.1 Graded $U_0(\mathfrak{g}_m)$-modules

We now wish to compute the composition multiplicities of projective $U_0(\mathfrak{g}_m)$-modules. One tool we will use in doing so is a grading on $U_0(\mathfrak{g}_m)$ and a category of graded $U_0(\mathfrak{g}_m)$-modules, generalising a well-known technique in the $m = 0$ case, see for example [24, §11].

We grade $U_0(\mathfrak{g}_m)$ by $X^*(H)$, the character lattice of the maximal torus $H \subseteq G$, in the following way. First define a $X^*(H)$-grading on $U(\mathfrak{g}_m)$ by letting $\mathfrak{g}_\alpha t^i$ lie in grading α, $\mathfrak{h}_m$ lie in grading 0, and extending this to all of $U(\mathfrak{g}_m)$. Then we observe that the kernel of the quotient map $U(\mathfrak{g}_m) \to U_0(\mathfrak{g}_m)$ is generated by homogeneous elements, so this grading descends to an $X^*(H)$ grading on $U_0(\mathfrak{g}_m)$. We consider a category $\mathcal{C}$ of certain $X^*(H)$-graded $U_0(\mathfrak{g}_m)$-modules whose gradings are compatible with the action of action of $\mathfrak{h}$ in the following sense. Let $M = \bigoplus_{\gamma \in X^*(H)} M_\gamma$ be an $X^*(H)$-graded $U_0(\mathfrak{g}_m)$-module. Then the objects of the category $\mathcal{C}$ are all $X^*(H)$-graded $U_0(\mathfrak{g}_m)$-modules such that $\mathfrak{h}$ acts on M_γ by $d\gamma$ for all $\gamma \in X^*(H)$. The observations of [24, §11.4] actually show that this special case allows one to understand the category of all $X^*(H)$ graded $U_0(\mathfrak{g}_m)$-modules.

For $M \in \mathcal{C}$, we write $\operatorname{ch} M$, the character of M, for the function $X^*(H) \to \mathbb{Z}_{\geq 0}$ which sends $\gamma \in X^*(H)$ to $\dim M_\gamma$ (analogously to the definition given in the characteristic 0 case given in §3.4.1). We use the same notation for composition multiplicities in $\mathcal{C}$ as in the ungraded case. We emphasise that graded and ungraded composition multiplicities are not the same, since a shift by $p\gamma$ in the grading of any module in $\mathcal{C}$ yields a non-isomorphic graded module with isomorphic underlying ungraded module.

Let $\gamma \in X^*(H)$. Then we define the *graded baby Verma module* $\widehat{Z}(\gamma)$ to be the module $Z(d\gamma)$ with grading given by letting $\prod (f_\alpha t^i)^{m_{\alpha,i}} \otimes 1_{d\gamma}$ lie in grading $\gamma - \sum \alpha m_{\alpha,i}$. As noted above, if $\beta \in X^*(H)$, then $\widehat{Z}(\gamma)$ and $\widehat{Z}(\gamma + p\beta)$ have isomorphic module structures but different gradings. We also define $\widehat{Z}^{\mathfrak{g}}(\gamma)$ in a similar way.

Lemma 4.5.2. *For all $\gamma \in X^*(H)$, $\widehat{Z}(\gamma) \in \mathcal{C}$ and $\widehat{Z}^{\mathfrak{g}}(\gamma) \in \mathcal{C}$.*

Proof. The first claim follows immediately from the observation in $\widehat{Z}(\gamma)$ that $\mathfrak{h}_m^{(0)}$ acts on $\prod (f_\alpha t^i)^{m_{\alpha,i}} \otimes 1_{d\gamma}$ by $d\gamma - \sum (d\alpha) m_{\alpha,i}$, and a similar argument applies for $\widehat{Z}^{\mathfrak{g}}(\gamma)$. ∎

By Lemma 4.4.3 and standard results on graded modules, we also see that $\widehat{Z}(\gamma)$ has a unique simple quotient $\widehat{L}(\gamma)$ which is isomorphic to $L(d\gamma)$ as a $U_0(\mathfrak{g}_m)$-module.

Proposition 4.5.3. *The $\widehat{L}(\gamma)$ form a complete set of isomorphism classes of simple objects in $\mathcal{C}$, and the $\operatorname{ch} \widehat{L}(\gamma)$ form a basis for the additive group of linear functions $X^*(H) \to \mathbb{Z}$. Hence the graded composition multiplicities of any module $M \in \mathcal{C}$ are determined by $\operatorname{ch} M$.*

Proof. By [25, §1.5], the argument from which still applies here, all simple objects in $\mathcal{C}$ are simple as $U_0(\mathfrak{g}_m)$-modules and by Theorem 4.4.7 we have a classification of simple $U_0(\mathfrak{g}_m)$-modules. Let $M \in \mathcal{C}$ be a simple object isomorphic as a $U_0(\mathfrak{g}_m)$-module to $L(\lambda)$ for some $\lambda \in \Lambda_0$. For any grading on $L(\lambda)$, the element $1 \otimes 1_\lambda$ must be homogeneous and so must lie in grading γ for some $\gamma \in X^*(H)$ such that $d\gamma = \lambda$. Then by considering the $X^*(H)$ grading on $U_0(\mathfrak{g}_m)$ and module structure of $L(\lambda)$, we must have $M \cong \widehat{L}(\gamma)$. The second assertion follows from the observation that $\mathrm{ch}(\widehat{L}(\gamma))(\gamma) = 1$ and $\mathrm{ch}(\widehat{L}(\gamma))(\beta) = 0$ unless $\beta \leq \gamma$. □

4.5.2 Composition multiplicities of restricted baby Verma modules

We now use the graded versions of the simple modules just introduced to compute the composition multiplicities of the baby Verma modules, which is a vital intermediate step towards our goal of computing the Cartan invariants for $U_0(\mathfrak{g}_m)$. The following formula allows us to express these in terms of the composition multiplicities for the baby Verma modules for the original reductive Lie algebra $\mathfrak{g}$.

Theorem 4.5.4. *For any $\lambda, \mu \in \Lambda_0$, we have:*

$$[Z(\lambda) : L(\mu)] = \begin{cases} l_\mu p^{\frac{m}{2}(\dim(\mathfrak{g})-\mathrm{rank}(\mathfrak{g}))-\mathrm{rank}(\mathfrak{g})} & \text{if } \lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})} \\ 0 & \text{otherwise} \end{cases}$$

where $l_\mu = \sum_{\nu \in \Lambda_0} [Z^{\mathfrak{g}}(\nu) : L(\mu)]$. In particular, these composition multiplicities depend only on $\lambda|_{\mathfrak{z}(\mathfrak{g})}$.

Proof. We first prove a relationship between the characters of the families of graded modules $\widehat{Z}(\gamma)$ and $\widehat{Z}^{\mathfrak{g}}(\gamma)$. This will then allow us to deduce a relationship between their composition multiplicities in the category $\mathcal{C}$ of graded modules introduced in the previous section, from which the theorem follows by forgetting the gradings everywhere.

Start by defining a generalisation of Kostant's partition function $p_m : X^*(H) \to \mathbb{Z}_{\geq 0}$ by

$$p_m(\gamma) = |\{(m_{i,\alpha})_{\alpha\in\Phi^+,1\leq i\leq m} : \sum \alpha m_{i,\alpha} = \gamma, 0 \leq m_{i,\alpha} \leq p-1\}|$$

and observe in particular that $p_m(\gamma) = 0$ if $(d\gamma)|_{\mathfrak{z}(\mathfrak{g})} \neq 0$. Let $I = \{((m_{i,\alpha})_{\alpha\in\Phi^+,1\leq i\leq m} : 0 \leq m_{i,\alpha} \leq p-1\}$ and fix $\gamma \in X^*(H)$. For each $\mathbf{m} \in I$, we can define an $X^*(H)$-graded subspace $\widehat{Z}_{\mathbf{m}} = \{\prod f_\alpha^{n_\alpha} \prod (f_\alpha t^i)^{m_{\alpha,i}} \otimes 1_\gamma : 0 \leq n_\alpha \leq p-1\}$ of $\widehat{Z}(\gamma)$, and furthermore we have a decomposition $\widehat{Z}(\gamma) = \bigoplus_{\mathbf{m}\in I} \widehat{Z}_{\mathbf{m}}$. Now observe that $\operatorname{ch} \widehat{Z}_{\mathbf{m}} = \operatorname{ch} \widehat{Z}^{\mathfrak{g}}(\gamma - \sum \alpha m_{i,\alpha})$. Hence we obtain the formula $\operatorname{ch} \widehat{Z}(\gamma) = \sum_{\beta\in X^*(H)} p_m(\gamma-\beta) \operatorname{ch} \widehat{Z}^{\mathfrak{g}}(\beta)$, and can then immediately deduce that $[\widehat{Z}(\gamma) : \widehat{L}(\delta)] = \sum_{\beta\in X^*(H)} p_m(\gamma-\beta)[\widehat{Z}^{\mathfrak{g}}(\beta) : \widehat{L}(\delta)]$. Forgetting the gradings on these modules, we then see that:

$$[Z(d\gamma) : L(d\delta)] = \sum_{\beta\in X^*(H)} p_m(\gamma-\beta)[Z^{\mathfrak{g}}(d\beta) : L(d\delta)].$$

Set $\lambda = d\gamma$, $\mu = d\delta$, and $\nu = d\beta$. If $\lambda|_{\mathfrak{z}(\mathfrak{g})} \neq \mu|_{\mathfrak{z}(\mathfrak{g})}$ then for all $\beta \in X^*(H)$ either $p_m(\gamma-\beta) = 0$ or $[Z^{\mathfrak{g}}(\nu) : L(\mu)] = 0$, so $[Z(\lambda) : L(\mu)] = 0$. On the other hand, if $\lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})}$, then using the fact that $d\beta = d(\beta + p\xi)$ for any $\xi \in X^*(H)$ we have $[Z(\lambda) : L(\mu)] = \sum_{\delta\in X^*(H)} p_m(\gamma - p\delta) \sum_{\nu\in\Lambda_0} [Z^{\mathfrak{g}}(\nu) : L(\mu)]$. Hence it remains only to show that for any $\gamma \in X^*(H)$, we have $\sum_{\delta\in X^*(H)} p_m(\gamma - p\delta) = p^{\frac{m}{2}(\dim(\mathfrak{g})-\operatorname{rank}(\mathfrak{g}))-\operatorname{rank}(\mathfrak{g})}$.

Let $S_\gamma = \bigcup_{\delta\in X^*(H)} \{(m_{i,\alpha})_{\alpha\in\Phi^+,1\leq i\leq m} : \sum \alpha m_{i,\alpha} = \gamma - p\delta, 0 \leq m_{i,\alpha} \leq p-1\}$. Then observe that for any $\beta \in \Phi^+$ there is a bijection $f : S_\gamma \to S_{\gamma+\beta}$ given by

$$f(\mathbf{m})_{i,\alpha} = \begin{cases} m_{i,\alpha} & \text{if } i > 0 \text{ or } \alpha \neq \beta \\ m_{i,\alpha} + 1 \bmod p & \text{if } i = 0 \text{ and } \alpha = \beta \end{cases}$$

Hence $|S_\gamma| = |S_\delta|$ for all $\gamma, \delta \in X^*(H)$. Now, observe that $\sum_{\gamma\in X^*(H)} p_m(\gamma) = p^{\frac{m}{2}(\dim(\mathfrak{g})-\operatorname{rank}(\mathfrak{g}))}$ and we have $p^{\operatorname{rank}(\mathfrak{g})}$ distinct sets S_γ, so for any $\gamma \in X^*(H)$, $|S_\gamma| = \sum_{\delta\in X^*(H)} p_m(\gamma - p\delta) = p^{\frac{m}{2}(\dim(\mathfrak{g})-\operatorname{rank}(\mathfrak{g}))-\operatorname{rank}(\mathfrak{g})}$, completing the proof. □

Corollary 4.5.5. *If $\mathfrak{z}(\mathfrak{g}) = 0$ then the composition multiplicities $[Z(\lambda) : M(\lambda)]$ are inde-*

pendent of λ, so all baby Verma modules have the same composition factors and multiplicities.

The fact the composition multiplicities depend only on $\lambda|_{\mathfrak{z}(\mathfrak{g})}$ is analogous to a result that states that for a the Lie algebra of a reductive group, restricted baby Verma modules with linked weights all have the same composition factors. The difference in this case is that the only condition for our weights to be linked is that they take the same values on $\mathfrak{z}(\mathfrak{g})$. This bears some similarity to Theorem 3.2.2 in the characteristic 0 case. For the following corollary, we recall the definition of a block from §2.3.3.

Corollary 4.5.6. *Let $\lambda, \mu \in \Lambda_0$. If $\lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})}$, then $L(\lambda)$ and $L(\mu)$ lie in the same block of $U_0(\mathfrak{g}_m)$. In particular, if $\mathfrak{z}(\mathfrak{g}) = 0$ then $U_0(\mathfrak{g}_m)$ has only one block.*

Proof. Suppose $\lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})}$. Then by Corollary 4.5.5 we have that $[Z(\lambda) : L(\lambda)]$ and $[Z(\lambda) : L(\mu)]$ are both non-zero, and by Lemma 4.4.3 we have that $Z(\lambda)$ is indecomposable. Hence $L(\lambda)$ and $L(\mu)$ lie in the same block. □

Remark 4.5.7. Theorem 4.5.9 will imply that the condition $\lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})}$ is also necessary for $L(\lambda)$ and $L(\mu)$ to lie in the same block. More generally, it is easy to see that the number of blocks of $U_0(\mathfrak{g}_m)$ is $p^{\dim \mathfrak{z}(\mathfrak{g}_m)} = p^{(m+1)\dim \mathfrak{z}(\mathfrak{g})}$.

4.5.3 Cartan invariants for $U_0(\mathfrak{g}_m)$

Recall the notation $Q^{\mathfrak{h}_m}(\lambda)$ from Proposition 4.2.5. We define the following families of $U_0(\mathfrak{g}_m)$-modules:

$$Z_{proj}(\lambda) = U_0(\mathfrak{g}_m) \otimes_{U_0(\mathfrak{b}_m^+)} Q^{\mathfrak{h}_m}(\lambda)$$

$$DZ(\lambda) = (U_0(\mathfrak{g}_m) \otimes_{U_0(\mathfrak{b}_m^-)} (\Bbbk_\lambda)^*)^*$$

where for $Z_{proj}(\lambda)$ we inflate $Q^{\mathfrak{h}_m}(\lambda)$ to a $U_0(\mathfrak{b}_m^+)$-module by letting $\mathfrak{n}_m^+$ act by 0, and for $DZ(\lambda)$ we inflate $(\Bbbk_\lambda)^*$ to a $U_0(\mathfrak{b}_m^-)$-module by letting $\mathfrak{n}_m^-$ act by 0. Here the module structure on the dual M^* of a $U_0(\mathfrak{l})$-module M for some Lie algebra $\mathfrak{l}$ is given by $(x \cdot f)(m) =$

$-f(x \cdot m)$ for $x \in \mathfrak{l}$, $f \in M^*$, and $m \in M$. Finally, we write $Q(\lambda)$ for the projective cover of $L(\lambda)$.

These definitions allow us to state the following theorem, which is a special case of [39, Theorem 1.3.6].

Theorem 4.5.8. *For all $\lambda, \mu \in \Lambda_0$, we have $(Q(\lambda) : Z_{\text{proj}}(\mu)) = [DZ(\mu) : L(\lambda)]$.*

This theorem is the key ingredient in the proof of the following result, which gives a formula for the Cartan invariants of $U_0(\mathfrak{g}_m)$.

Theorem 4.5.9.

$$[Q(\lambda) : L(\mu)] = \begin{cases} l_\lambda l_\mu p^{m \dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})} \ \textit{if } \lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})} \\ 0 \ \textit{otherwise} \end{cases}$$

where again $l_\lambda = \sum_{\nu \in \Lambda_0} [Z^{\mathfrak{g}}(\nu) : L(\lambda)]$.

Proof. We aim to find formulae for $[DZ(\mu) : L(\lambda)]$ and $(Z_{\text{proj}}(\mu) : Z(\lambda))$; the result will then follow by Theorem 4.5.8 and several applications of Proposition 4.5.1. First, we claim that given $\gamma \in X^*(H)$ such that $\mu = d\gamma$ we can construct an $X^*(H)$-grading on $DZ(\mu)$ such that the graded module $\widehat{DZ}(\gamma) \in \mathcal{C}$ has the same character as $\widehat{Z}(\gamma)$; it is then immediate that $[DZ(\mu) : L(\lambda)] = [Z(\mu) : L(\lambda)]$ for all $\lambda, \mu \in \Lambda_0$. Observe that $(\Bbbk_\mu)^* \cong \Bbbk_{-\mu}$, so we can define an $X^*(H)$-grading on $U_0(\mathfrak{g}_m) \otimes_{U_0(\mathfrak{b}_m^-)} (\Bbbk_\mu)^*$ by letting $\prod (e_\alpha t^i)^{m_{\alpha,i}} \otimes 1_{-\mu}$ lie in grading $-\gamma + \sum \alpha m_{\alpha,i}$.

Now, for any finite-dimensional $M \in \mathcal{C}$ we can choose a basis $\{x_1, \ldots, x_k\}$ such that x_i lies in grading δ_i, and then define a grading on the dual module M^* by choosing a dual basis $\{f_1, \ldots, f_k\}$ and letting f_i lie in grading $-\delta_i$. The resulting graded module M^* also lies in $\mathcal{C}$, and we see that $\operatorname{ch} M^*(\delta) = \operatorname{ch} M(-\delta)$. Applying this to $M = U_0(\mathfrak{g}_m) \otimes_{U_0(\mathfrak{b}_m^-)} (\Bbbk_\mu)^*$ equipped with the grading described earlier, we obtain a graded module $\widehat{DZ}(\gamma)$ such that $\operatorname{ch} \widehat{DZ}(\gamma) = \operatorname{ch} \widehat{Z}(\gamma)$ as required.

Next, we observe that $U_0(\mathfrak{g}_m)$ is free as a $U_0(\mathfrak{b}_m^+)$-module and hence the functor

$U_0(\mathfrak{g}_m) \otimes_{U_0(\mathfrak{b}_m^+)} (-)$ is exact. Hence by Proposition 4.2.5 we see that $(Z_{proj}(\lambda) : Z(\mu)) = \delta_{\lambda,\mu} p^{m\,\mathrm{rank}(\mathfrak{g})}$. We can then apply Proposition 4.5.1 to compute:

$$\begin{aligned}
[Q(\lambda) : L(\mu)] &= \sum_{\nu_1 \in \Lambda_0} (Q(\lambda) : Z_{proj}(\nu_1))[Z_{proj}(\nu_1) : L(\mu)] \\
&= \sum_{\nu_1,\nu_2 \in \Lambda_0} (Q(\lambda) : Z_{proj}(\nu_1))(Z_{proj}(\nu_1) : Z(\nu_2))[Z(\nu_2) : L(\mu)] \\
&= \sum_{\nu_1,\nu_2 \in \Lambda_0} [Z(\nu_1) : L(\lambda)](Z_{proj}(\nu_1) : Z(\nu_2))[Z(\nu_2) : L(\mu)] \\
&= \sum_{\nu \in \Lambda_0} [Z(\nu) : L(\lambda)] p^{m\,\mathrm{rank}(\mathfrak{g})} [Z(\nu) : L(\mu)] \\
&= \begin{cases} l_\lambda l_\mu p^{m\,\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g})} \text{ if } \lambda|_{\mathfrak{z}(\mathfrak{g})} = \mu|_{\mathfrak{z}(\mathfrak{g})} \\ 0 \text{ otherwise} \end{cases}
\end{aligned}$$

□

Remark 4.5.10. It is possible to the use results of [47] along with the theory of translation functors to deduce that for $p > 2h - 1$ these Cartan invariants can actually be calculated in terms of the machinery of p-Kazhdan–Lusztig polynomials.

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