INVARIANT GIBBS MEASURES FOR DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

by

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Abstract

This thesis focuses on the construction of Gibbs measures, their invariance under the flows of Hamiltonian partial differential equations, and their application in understanding macroscopic properties of these partial differential equations. The presentation of these focuses will be carried out through the example of Gibbs measure for the fractional nonlinear Schrödinger equations posed on the torus.

In Chapter 2, we consider the Cauchy problem for the cubic nonlinear fractional Schrödinger equation (FNLS) on the circle, considering initial data distributed via the Gibbs measure. By reformulating the local theory via the random averaging operator theory from Deng-Nahmod-Yue [34, 35], we construct global strong solutions with the flow property for FNLS on the support of the Gibbs measure in the full dispersive range, thus addressing a question proposed by Sun-Tzvetkov [105]. Additionally, we prove the invariance of the Gibbs measure and the sharpness of the result. This chapter mainly comes from [70].

In Chapter 3, we study the Gibbs measures for the *focusing* mass-critical fractional nonlinear Schrödinger equation on the torus $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$. We identify the critical nonlinearity and optimal mass threshold for normalisablity and non-normalisability of the Gibbs measures for the fractional nonlinear Schrödinger equation on the multi-dimensional torus, which extends the works of Lebowitz-Rose-Speer [68], Bourgain [2], and Oh-Sosoe-Tolomeo [85] on the nonlinear Schrödinger equations on one-dimensional torus $\mathbb{T}$. To achieve this purpose, we prove an almost sharp fractional Gagliardo-Nirenberg-Sobolev inequality on the multi-dimensional torus. This chapter is from [69].

Uxori meae Shiting, ob firmum sustentaculum.

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CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

In modern mathematical physics research, Gibbs measures play a significant role. Over the past decades, there have been tremendous advances in the application and mathematical research of Gibbs measures across a variety of fields, including statistical mechanics, constructive quantum field theory, quantum many-body systems, plasma physics, nonlinear optics, and dynamical systems. The problem of constructing and verifying invariant measures for dispersive partial differential equations is one of the most classical problems in the area of nonlinear dispersive equations with random initial data. This dissertation delves into constructing Gibbs measures, their invariance under the evolutions of some Hamiltonian partial differential equations, and using them to analyze the macroscopic behaviors of these partial differential equations.

The dissertation mainly consists of published or pre-published results [69, 70], which are listed below

- Rui Liang, Yuzhao Wang. *Gibbs measure for the focusing fractional NLS on the torus*, SIAM. J. Math. Anal. 54 (2022), no. 6, 6096–6118. [2]

- Rui Liang, Yuzhao Wang. *Gibbs dynamics for the weakly dispersive nonlinear Schrödinger equations*, arXiv:2306.07645 [math.AP].

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1.1 Zakharov's question

At the Sixth I. G. Petrovskii Memorial Meeting of the Moscow Mathematical Society in January 1983, V. E. Zakharov asked, "Numerical experiments demonstrated that some dispersive equations possess the 'returning property', i.e., solutions appear to be very close to the initial state, after some time of rather chaotic evolution. How can this phenomenon be explained?" See [42].

To explain the returning phenomenon, we need to refer to the Poincaré recurrence theorem for measure-preserving transformation. In the following, let us go over the definition of measure-preserving transformation, the Poincaré recurrence theorem and their application to the Hamiltonian systems.

Definition 1 (Measure-preserving transformation)**.** Given a probability space $(X, \mathscr{B}, m)$, a transformation $T : X \to X$ is called *measure-preserving* if T is measurable and $m(T^{-1}[B]) = m(B)$ for all $B \in \mathscr{B}$.

One property shared by all measure-preserving transformations is recurrence. Let us recall the following Poincaré recurrence theorem. See [114, Theorem 1.4].

Theorem 1 (Poincaré recurrence theorem)**.** *Let $T : X \to X$ be a measure-preserving transformation of a probability space $(X, \mathscr{B}, m)$. Let $E \in \mathscr{B}$ with $m(E) > 0$. Then almost all points of E return infinitely often to E under positive iteration by T (i.e., there exists $F \subset E$ with $m(F) = m(E)$ such that for each $x \in F$ there is a sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers with $T^{n_i}(x) \in F$ for each i).*

To answer Zakharov's question, we need to understand the measure-preserving property of a flow of dispersive partial differential equations. Before discussing the dispersive partial differential equation, let us recall the finite-dimensional Hamiltonian system to see how the Poincaré recurrence theorem is used. A finite-dimensional Hamiltonian system is given by the following ordinary differential equations:

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}, \end{cases} \qquad i = 1, 2, \cdots, d, \tag{1.1}$$

where $H(q_1, \cdots, q_d; p_1, \cdots, p_d)$ is called the Hamiltonian. We assume that (1.1) has a unique global-in-time solution $\Phi(t, (q(0), p(0)))$, satisfying $\Phi(0, (q(0), p(0))) = (q(0), p(0))$ and the following *flow property*

$$\Phi(t_2, \Phi(t_1, \cdot)) = \Phi(t_2 + t_1, \cdot) \tag{1.2}$$

for all $t_1, t_2 \in \mathbb{R}$.

Given w in $C^\infty(\mathbb{R})$, from (1.1), we know that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} w(H(t)) &= w'(H(t)) \sum_{i=1}^{d} \Big(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \Big) \\ &= w'(H(t)) \sum_{i=1}^{d} \Big(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \Big) \\ &= 0, \end{aligned} \tag{1.3}$$

where $H(t) := H(q_1(t), \cdots, q_d(t); p_1(t), \cdots, p_d(t))$.

Let us recall the Liouville theorem which claims the invariance of some measures. For a proof of Liouville theorem, we refer the reader to [108, Theorem 1.5].

Theorem 2 (Liouville)**.** *Denote by* $\mathrm{d}x$ *the Lebesgue measure on* $\mathbb{R}^d$*, let* $g \in C^\infty(\mathbb{R}^d; [0, +\infty))$*, and let* $F_i \in C^\infty(\mathbb{R}^d)$*. The solution map* $\Phi(t, \cdot)$ *(assume it exists for all* t *in* $\mathbb{R}$ *with* $\Phi(0, x_0) = x_0$ *for all* x_0 *in* $\mathbb{R}^d$ *and* $\Phi(t_2, \Phi(t_1, \cdot)) = \Phi(t_2 + t_1, \cdot)$ *for all* $t_1, t_2 \in \mathbb{R}$*) of the following ordinary differential equations*

$$\begin{cases} \frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = F_i(x_1(t), \cdots, x_d(t)), \\ x(0) = x_0, \end{cases} \qquad i = 1, \cdots, d,$$

preserves the measure $g\,\mathrm{d}x$ for all $t \in \mathbb{R}$ if and only if

$$\operatorname{div}(gF) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(gF_i) = 0. \tag{1.4}$$

Let us now verify that the finite-dimensional system (1.1) obeys the condition of Theorem 2. From (1.1) and (1.3), we know that

$$\begin{aligned}
&\sum_{i=1}^{d} \Big(\frac{\partial}{\partial q_i}\Big((w \circ H)\frac{\partial H}{\partial p_i}\Big) - \frac{\partial}{\partial p_i}\Big((w \circ H)\frac{\partial H}{\partial q_i}\Big)\Big) \\
&\quad = \sum_{i=1}^{d} \Big(\frac{\partial}{\partial q_i}(w \circ H)\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}(w \circ H)\frac{\partial H}{\partial q_i}\Big) \\
&\quad = \sum_{i=1}^{d} \Big(\frac{\partial}{\partial q_i}(w \circ H)\dot{q}_i + \frac{\partial}{\partial p_i}(w \circ H)\dot{p}_i\Big) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t} w(H(t)) = 0,
\end{aligned} \tag{1.5}$$

where $w \in C^{\infty}(\mathbb{R}; [0, +\infty))$ ($w = 1$ gives the Lebesgue measure $w\mathrm{d}x$, and $w = \mathrm{e}^{-\beta H}$ gives the Gibbs measure $w\mathrm{d}x$). Note that (1.5) means that (1.4) is satisfied for the Hamiltonian system (1.1). Therefore, an application of Theorem 2 proves that the solution map $\Phi(t, \cdot)$ of (1.1) preserves the measure

$$m = (w \circ H)\mathrm{d}x$$

for all $t \in \mathbb{R}$. Namely

$$m(\Phi(t, \cdot)^{-1}[A]) = m(A) \tag{1.6}$$

for all m-measurable set A in $\mathbb{R}^{2d}$.

(1.6) implies that $\Phi(1, \cdot) : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}$ preserves the measure m. Therefore, an application of Theorem 1 yields the following "returning" property.

Theorem 3. *Let $w \in C^{\infty}(\mathbb{R}; [0, +\infty))$. Assume that $H \in C(\mathbb{R}^{2d})$ is such that the ordinary differential equations* (1.1) *has a unique global-in-time solution $\Phi(t, x) = (q(t), p(t))$ for all initial data $x = (q(0), p(0))$. (Here $q(t) := (q_1(t), \cdots, q_d(t))$, $p(t) := (p_1(t), \cdots, p_d(t))$.) We further assume that the solution map $\Phi(t, \cdot)$ satisfies the flow property* (1.2). *Let*

$m = (w \circ H)\mathrm{d}x$. *Then, for any m-measurable set of E with $m(E) > 0$, there exists $F \subset E$ with $m(F) = m(E)$ such that for any $x \in F$ there is a sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers with*

$$\Phi(n_i, x) \in F$$

for all $i \in \mathbb{Z}_+$.

In Theorem 3, the global existence is used for $\Phi(n_i, x)$ to be well-defined. The flow property is used in the following way:

$$\Phi(1, \cdot)^{n_i} = \Phi(n_i, \cdot).$$

From the above example, we see that to apply the Poincaré recurrence theorem, it suffices to verify

- the flow property,
- global existence of the solution m-almost every,
- the invariance of the measure m under the flow.

So far, we have considered the finite-dimensional Hamiltonian system. Now, we move to the dispersive equation, which can be viewed as an infinite-dimensional Hamiltonian system. Consider the nonlinear Schrödinger equation (NLS) given by

$$\mathrm{i}\partial_t u + \partial_{xx} u = |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \tag{1.7}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The associated Hamiltonian to the equation (1.7) is

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}} |u|^4 \mathrm{d}x.$$

Then (1.7) can also be written as

$$\partial_t u = -\mathrm{i}\frac{\delta H}{\delta \overline{u}}. \tag{1.8}$$

Let u_k denote the Fourier coefficients of u, i.e. $u(x) = \sum_k u_k \mathrm{e}^{\mathrm{i}kx}$ and p_k and q_k be given by

$$\begin{cases} q_k = \mathrm{Re}(u_k), \\ p_k = \mathrm{Im}(u_k). \end{cases} \tag{1.9}$$

Denote by

$$H(p,q) = \frac{1}{2}H(u) = \frac{1}{4}\sum_{k\in\mathbb{Z}} |k|^2(|q_k|^2 + |p_k|^2) + \frac{1}{8}\int_{\mathbb{T}} \Big|\sum_{k\in\mathbb{Z}}(q_k + \mathrm{i}p_k)\mathrm{e}^{\mathrm{i}kx}\Big|^4 \mathrm{d}x.$$

Then the equation (1.7), namely (1.8), can be written as the following infinite-dimensional Hamiltonian system:

$$\begin{cases} \dot{q_k} = \frac{\partial H}{\partial p_k}, \\ \dot{p_k} = -\frac{\partial H}{\partial q_k}, \end{cases} \quad k \in \mathbb{Z}.$$

In a seminal work [2], Bourgain constructed the Gibbs measure $\mathrm{d}\rho$ formally written as

$$\mathrm{d}\rho = \mathrm{e}^{-\beta H(u)}\mathrm{d}u,$$

and proved the $\mathrm{d}\rho$-almost everywhere global well-posedness of (1.7), the flow property and the invariance of $\mathrm{d}\rho$ under the flow of (1.7). As a byproduct, the "returning property" of (1.7) follows from Theorem 1, thereby answering Zakharov's question for NLS.

Since Bourgain's work, the subject has attracted substantial attention [110, 31, 24, 67, 90, 96, 106, 62, 87, 34, 10]. See the next Section for further discussion.

1.2 Historical reviews

1.2.1 Dispersive equations with random initial data

One-dimensional case

Friedlander [42] considered the one-dimensional wave equation with cubic nonlinearity

$$u_{tt} - u_{xx} + u^3 = 0, \tag{1.10}$$

and proved the invariance of Gibbs measure. Utilising the invariance of Gibbs measure as a substitute for the invariance of Lebesgue measure established by Liouville's theorem, Friedlander derived the "returning property" by applying the Poincaré recurrence theorem, thereby addressing Zakharov's question mentioned in Section 1.1 for the nonlinear wave flow (1.10). Then Zhidkov [115] extended Friedlander's result to more general nonlinearities.

The invariance of the Gibbs measure for the cubic nonlinear Schrödinger equation was obtained by Bourgain in [2]. Bourgain's impetus [59] was to solve a problem raised by Lebowitz-Rose-Speer's paper on Gibbs measure associated to nonlinear Schrödinger equation on $\mathbb{T}$ with quintic nonlinearity [68]. The problem was finding a flow of a spatially periodic nonlinear Schrödinger equation which is on the support of the Gibbs measure and proving the invariance of the Gibbs measure under the flow. At that time, a significant challenge was the absence of a relevant local-in-time theory on the torus. Bourgain [1] then addressed this by developing almost sure local-in-time dynamics for the equation:

$$\mathrm{i}u_t + u_{xx} + |u|^{p-2}u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{T}$$

with $4 \leq p \leq 6$ by using Fourier restriction norm method and improving Strichartz estimates on the torus by applying analytic number theory results. Later on, Bourgain [2] extended the almost sure local-in-time dynamics globally by using invariance of truncated

Gibbs measure (Bourgain's invariant measure argument) and proved the invariance of the Gibbs measure for the nonlinear Schrödinger equations on one-dimensional torus, which solved the problem raised by Lebowitz-Rose-Speer [68].

Bourgain's probabilistic argument

The equations studied in [42, 115, 2] are all in one spatial dimension. The advantage of the one-dimensional case lies in its deterministic local well-posedness theory. This means that the spatial regularity of the support of the Gibbs measure is above the threshold needed for deterministic well-posedness. Consequently, one can fix the sample point and treat the initial value problem as a deterministic problem. However, in the case of the nonlinear Schrödinger equation on the two-dimensional torus, the initial data in the support of Gibbs measure almost surely falls within $H^{0-}(\mathbb{T}^2) := \bigcap_{\sigma<0} H^{\sigma}(\mathbb{T}^2)$ which is rougher than the deterministic critical regularity $s_c = 0$ and consists of generalised functions rather than just regular functions. A milestone of the subject is [3] where Bourgain considered the following renormalised nonlinear Schrödinger equation with cubic nonlinearity

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u =: |u|^2 u :, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \\ u(0) = u_0 \end{cases} \tag{1.11}$$

where u_0 is in the support of the Gibbs measure and $: |u|^2 u :$ is an appropriate renormalisation of $|u|^2 u$. A renormalisation of $|u|^2 u$ is needed here as a cubic power of a generalised function might not be defined. In (1.11), u_0 is almost surely in $H^{0-}(\mathbb{T}^2)$, which is rougher than the deterministic well-posedness threshold. Hence, one cannot treat the problem as a deterministic problem. Bourgain overcame the difficulties by applying Da Prato-Debussche trick and exploiting the probabilistic smoothing in some multilinear estimates. (1.11) is a Hamiltonian partial differential equation with Hamiltonian

$$H(u) := \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} : |u|^4 : \mathrm{d}x,$$

where $:|u|^4:$ denotes the renormalisation of $|u|^4$. The Gibbs measure associated with the Hamiltonian partial differential equation (1.11) is formally given by

$$\mathrm{e}^{-H(u)}\mathrm{d}u = \exp\left(-\frac{1}{4}\int_{\mathbb{T}^2} :|u|^4: \mathrm{d}x\right)\mathrm{e}^{-\frac{1}{2}\int_{\mathbb{T}^2}|\nabla u|^2\mathrm{d}x}\mathrm{d}u, \tag{1.12}$$

where $\mathrm{d}u$ is an infinite dimensional Lebesgue measure that does not exist. Given that the Hamiltonian is a conserved quantity, the invariance of Gibbs measures becomes a potential replacement for the Liouville theorem (Theorem 2), thereby necessitating the study of invariance of Gibbs measures. The Gibbs measure in (1.12) is called (complex) Φ_2^4 measure in the constructive quantum field theory community. Because the Gibbs measure in (1.12) is absolutely continuous with respect to the Gauss measure $\mathrm{e}^{-\frac{1}{2}\int_{\mathbb{T}^2}|\nabla u|^2\mathrm{d}x}\mathrm{d}u$, choosing the initial data in (1.11) as the following random Fourier series:

$$u_0(x) = u_0^\omega(x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{g_k(\omega)}{|k|}\mathrm{e}^{\mathrm{i}k\cdot x}, \tag{1.13}$$

where $\{g_k\}_{k\in\mathbb{Z}^2\setminus\{0\}}$ is a sequence of standard complex-valued independently identically distributed centred Gaussian random variables, it can be seen that this random initial data falls into the support of the Gibbs measure. Furthermore, a direct computation shows that the random Fourier series in (1.13) belongs to $H^{0-}(\mathbb{T}^2)\setminus L^2(\mathbb{T}^2)$ which contains the support of the Gibbs measure in (1.12). To overcome the difficulty that the random initial data are below the deterministic threshold in terms of regularity, Bourgain proposed decomposing the solution as

$$u(t) = \mathrm{e}^{\mathrm{i}t\Delta}u_0^\omega + v(t). \tag{1.14}$$

(The above decomposition is now known as Bourgain's trick or Da Prato-Debussche trick [37]). Then the first term in (1.14) is in $H^{0-}(\mathbb{T}^2)$ for fixed t, and the second term in (1.14) satisfies the following equation:

$$\mathrm{i}v_t + \Delta v =: |\mathrm{e}^{\mathrm{i}t\Delta}u_0^\omega + v|^2(\mathrm{e}^{\mathrm{i}t\Delta}u_0^\omega + v):, \quad (t,x)\in\mathbb{R}\times\mathbb{T}^2,$$

where $:|e^{it\Delta}u_0^\omega + v|^2(e^{it\Delta}u_0^\omega + v):$ denotes renormalisation which is needed to interpret the nonlinearity. Despite the roughness of $e^{it\Delta}u_0^\omega$, its explicit random structure allows control over the nonlinear interactions between $e^{it\Delta}u_0^\omega$ and $e^{it\Delta}u_0^\omega$, as well as between $e^{it\Delta}u_0^\omega$ and v. Then by utilising some smoothing effects, Bourgain proved that the Duhamel term in the first Picard iterate

$$u^{(1)}(t) = \int_0^t e^{i(t-s)\Delta} : |e^{i(t-s)\Delta}u_0^\omega|^2 e^{i(t-s)\Delta}u_0^\omega : ds \tag{1.15}$$

is in $X^{\frac{1}{2}-,\frac{1}{2}+} \subset C([-T,T];H^{\frac{1}{2}-}(\mathbb{T}^2))$ which allows applying contraction mapping argument for v in $X^{\frac{1}{2}-,\frac{1}{2}+}$.

After Bourgain's introducing this subject to the dispersive PDE community, we have seen a tremendous mass of advances in this area of research over the last decades [110, 31, 24, 67, 90, 96, 106, 62, 87, 34, 10]. See also Bényi-Oh-Pocovnicu's survey on the subject [20].

Random averaging operator argument

There is also a series of important studies on the partial differential equations with stochastic forcing. In the parabolic setting, key contributions are Hairer's theory of regularity structures [53, 54, 55, 56] and the para-controlled calculus developed by Gubinelli-Imkeller-Perkowski [44, 45]. Additionally, an approach utilising Wilsonian renormalisation group analysis was independently introduced by Kupiainen in [60]. For the nonlinear wave equation with stochastic forcing, see [47, 48, 49]. Especially, in [49], the authors introduced the para-controlled operator.

Inspired by the para-controlled calculus [44, 45, 9, 49] and Bringmann's work [7], Deng-Nahmod-Yue [34] developed the theory of random averaging operators. As we have discussed, the initial data in (1.13) is in $H^{0-}(\mathbb{T}^2)\backslash L^2(\mathbb{T}^2)$, which is rougher than the deterministic well-posedness threshold $s_c = 0$. Then, it naturally arises to ask what the threshold for well-posedness is for the random initial data problem. The notion of *probabilistic scaling* s_{pr} was introduced by [34]. In the same paper they introduced the theory of random averaging operators. The probabilistic scaling for the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t u + \Delta u =: |u|^{2r}u :, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d \tag{1.16}$$

is

$$s_{pr} = -\frac{1}{2r}.$$

See [34]. The deterministic well-posedness threshold for (1.16) is

$$s_c = 1 - \frac{1}{r}, \tag{1.17}$$

which is greater than s_{pr}. It is also proven in [3] that $u^{(1)}$ in (1.15) does not belong to $X^{s,\frac{1}{2}+}$ for all $s \geq \frac{1}{2}$ in the two-dimensional setting. However, from (1.17), we know that the space $X^{\frac{1}{2}-,\frac{1}{2}+}$ is still supercritical with respect to deterministic scaling for $d = 2$ and $r \geq 2$. Therefore, a naive Bourgain's trick as in (1.14) is not sufficient for solving (1.16) for $d = 2$ and $r \geq 2$ in $H^{0-}(\mathbb{T}^2)$ which contains the support of the Gibbs measure. Also, a higher-order Picard iterate does not help either, as the remainder will still be in $X^{\frac{1}{2}-,\frac{1}{2}+}$. To move on, we observed that the bad regularity of the first Picard iterate $u^{(1)}$ is only coming from *high-low* interactions:

$$Y = \sum_{N\colon \text{dyadic}} \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta} : \Delta_N u_0^\omega \cdot |\Pi_{\ll N} u|^{2r} : (s)\mathrm{d}s, \tag{1.18}$$

where $\Delta_N := \mathcal{F}^{-1}\left(\mathbf{1}_{|k|\sim N}\mathcal{F}(\cdot)\right)$ and $\Pi_{\ll N} := \mathcal{F}^{-1}\left(\mathbf{1}_{|k|\ll N}\mathcal{F}(\cdot)\right)$ are the standard frequency projectors. A noteworthy observation made by Bringmann [7] is that by exploiting independence between high-low interaction one might be able to apply some stronger probabilistic tools and improve the estimates. Deng-Namhod-Yue replaces $\Pi_{\ll N}u$ in (1.18) by $u_{\ll N}$ which is the solution with initial data $\Pi_{\ll N}u_0^\omega$; and hence the high frequency part and the low frequency part become independent. As a result, large deviation estimates become applicable for these terms, for example.

Later on, inspired by their previous work [33], Deng-Nahmod-Yue considered the following *random averaging operator*:

$$\mathcal{P}^{N,L} : w \longmapsto \sum_{N:\,\text{dyadic}} \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta} : \Delta_N w \cdot |u_{\leq L}|^{2r} : (s)\mathrm{d}s, \tag{1.19}$$

with $L \ll N$ and proved estimates for their operator norms and Hilbert-Schmidt norms by induction. By removing the high-low interaction given by the random averaging operator (1.19), Deng-Nahmod-Yue considered the following ansatz for the solution:

$$u_{\leq N}(t) = \sum_N \mathrm{e}^{\mathrm{i}t\Delta} u_0 + \sum_N \sum_{1\leq L \ll N} \underbrace{\mathfrak{h}^{N,L}(u_0)}_{\text{paralinear term}} + \sum_N v_N(t), \tag{1.20}$$

where the random tensor $\mathfrak{h}^{N,L}$ is given by the random averaging operator $\mathcal{P}^{N,L}$ via (2.36) and (2.38). Although the high-low interaction is poor in terms of regularity, the good news is that it is paralinear. Therefore, if we can estimate the operator norm and Hilbert-Schmidt norm for the random averaging operator (1.19) in the following way:

$$\|\mathcal{P}^{N,L}\|_{\mathrm{OP}} \lesssim L^{-\delta_0}, \quad \|\mathcal{P}^{N,L}\|_{\mathrm{HS}} \lesssim N^{\frac{1}{2}+\delta_1} L^{-\frac{1}{2}},$$

where $0 < \delta_1 \ll \delta_0 \ll 1$, then we can estimate the paralinear term in (1.20), by which Deng-Nahmod-Yue proved that the spatial regularity of Y in (1.18) is similar to $\mathrm{e}^{\mathrm{i}t\Delta} u_0^\omega$ and put the remainder in a favourable space.

Random tensor theory

Deng-Nahmod-Yue then developed a theory of random tensors [35], known as a dispersive counterpart of the theory of regularity structures developed by Hairer [54]. The theory of random tensors is served as a local-in-time theory for subcritical problems above the probabilistic scaling, and aims to propagate the initial data's randomness predominantly in para-multilinear manners. An ansatz for the solution is constructed inductively. Depending on the extent the problem is subcritical, they eliminated many multilinear evolutions of the initial data to make the remainder as smooth as they want, leaving many multilinear evolutions of the initial data treated as random tensors acting on Gaussians in some multilinear ways. Careful dyadic frequency decomposition and precise adjustments are made to ensure the independence of random tensors from the input Gaussian variables. The analysis involves estimating tensor norms that are more refined than the Hilbert-Schmidt norm, to inductively prove the estimates of the random tensors. Furthermore, a selection algorithm is utilised to determine the specific order of random tensors required to prove the merging estimates.

Let us briefly go over some aspects of the theory of random tensors here. In using the random averaging operators, Deng-Nahmod-Yue [34] considered the high-low interaction by (1.19) with $L < N^{1-\delta}$ for some $0 < \delta \ll 1$. However, high-low interaction given by $L < N^{1-\delta}$ is still not very bad, thereby wasting some high-intermediate interaction. In [35], Deng-Nahmod-Yue considered the high-low interaction by (1.19) with $L < N^{\delta}$. Then, they iterate the ansatz (1.20) into the nonlinearity of the equation for a tremendous mass of times, thereby forming some plants (see [35, Definition 3.2] and [72, Definition 2.2] for the definition of plants). Also, instead of describing the high-low interactions in terms of paralinear terms, they describe the high-low interactions in terms of para-multilinear interactions. Precisely, they considered the following ansatz:

$$(u_N)_k = \sum_N \sum_{\text{plant } \mathcal{S}} \sum_{k_{\mathcal{U}}, k_{\mathcal{V}}} \int \mathrm{d}\lambda_{\mathcal{V}} \cdot \underbrace{\mathfrak{h}^{\mathcal{S}}_{kk_{\mathcal{U}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}}) \times \prod_{\mathfrak{l} \in \mathcal{U}} \Delta_{N_{\mathfrak{l}}}(u_0)^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}} \prod_{\mathfrak{f} \in \mathcal{V}} (z_{N_{\mathfrak{f}}})^{\zeta_{\mathfrak{l}}}_{k_{\mathfrak{l}}}(\lambda_{\mathfrak{f}})}_{\text{para-multilinear term}} + \sum_N (z_N)_k, \tag{1.21}$$

where $\mathfrak{h}^{\mathcal{S}}_{kk_{\mathcal{U}}}(k_{\mathcal{V}}, \lambda_{\mathcal{V}})$ are some random tensors. When the size of the plants exceeds δ^{50}, they end the iteration. Hence, by removing a tremendous mass of the poorest high-low interaction from the remainder, the remainder can be made as smooth as they want. In fact, in the remainder, they can accumulate $> \delta^{50}$ many gains from the high-intermediate interactions

$$\|(z_N)_k\|_{\ell_k^2} \leq \prod_{\mathfrak{n}\in\mathcal{S}} N_{\mathfrak{n}}^{-\gamma} \leq ((N^\delta)^{-\gamma})^{|\mathcal{S}|} < N^{-\gamma\delta^{-49}} \quad \text{and} \quad N_{\mathfrak{n}} \geq N^\delta,$$

thereby making the remainder as smooth as they want, and viewing the remainder z_N as a negligible remainder. Through the ansatz (1.21), they propagate the randomness of the initial data predominantly in some para-multilinear manners. δ will be chosen to be closer to zero, corresponding to how close the probabilistically subcritical problem approaches the probabilistic critical threshold, thereby proving the existence of almost sure local-in-time strong solution above the probabilistic scaling.

1.2.2 Fractional nonlinear Schrödinger equation

Fractional nonlinear Schrödinger equation (FNLS)

$$\mathrm{i}\partial_t u - \mathrm{D}^\alpha u = \pm|u|^{p-2}u, \quad (t,x) \in \mathbb{R}\times\mathbb{T} \tag{1.22}$$

has also attracted significant attention. Here, $\mathrm{D}^\alpha = (\sqrt{-\Delta})^{\alpha/2}$ is defined by $\mathcal{F}_x(\mathrm{D}^\alpha f)(k) = |k|^\alpha(\mathcal{F}_x f)(k)$. Plus sign $(+)$ on the right-hand side of (1.22) stands for the defocusing case, whereas minus sign $(-)$ stands for the focusing case. When $\alpha = 2$, the equation (1.22) is the nonlinear Schrödinger equation mentioned previously. FNLS has some backgrounds from fractional quantum mechanics [64], water wave problems [57], and so on. For the one-dimensional case, it can also be viewed as the continuum limit of lattice interaction [61]. In [61], the authors considered the following discrete generalisation of the FNLS with $0 < h \ll 1$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}u_h(t,hm) = h\sum_{n\neq m}\frac{u_h(t,hm)-u_h(t,hn)}{|hm-hn|^{1+s}} \pm |u_h(t,hm)|^2 u_h(t,mh). \tag{1.23}$$

When $1 < s < 2$, (1.23) corresponds to long-range interaction, and in the limit when $h \to 0^+$ it recovers the FNLS equation with parameter $\alpha = s \in (1,2)$. When $s \geq 2$, (1.23) corresponds to short-range interaction, giving classical nonlinear Schrödinger equation (i.e. $\alpha = 2$).

The order of the derivative α also measures the strength of the dispersion. For the one-dimensional case of (1.22), there is no dispersion when $\alpha = 1$ (the half-wave case), weak dispersion when $\alpha \in (1,2)$, and strong dispersion when $\alpha > 2$.

We also review some deterministic well-posedness results for both defocusing and focusing case of (1.22) with cubic nonlinearity (that is $p = 4$). In the case when $\alpha = 2$, Bourgain [2] proved the local well-posedness in $L^2(\mathbb{T})$; then some ill-posedness results below L^2 are obtained [26, 73, 51]. When $\alpha \in (1,2)$, [29] proved local well-posedness result in $H^s(\mathbb{T})$ for $s \geq \frac{2-\alpha}{4}$ and ill-posedness result in $H^s(\mathbb{T})$ for $s < \frac{2-\alpha}{4}$. For the case when $\alpha > 2$, see [75, 95, 18].

We now turn to some probabilistic well-posedness results of (1.22) posed on the one-dimensional torus with $p = 4$. When $\alpha = 2$, Bourgain [2] proved the almost sure global well-posedness in $H^{\frac{1}{2}-}(\mathbb{T})$, and recently Deng-Nahmod-Yue proved the almost sure local well-posedness in $H^{-\frac{1}{2}+}(\mathbb{T})$. For the case when $\alpha > 1$, there are some results answering Zakharov's question mentioned in Section 1.1. Precisely, Demirbas [32] established the flow property for the case $\alpha > \frac{4}{3}$. Then Sun-Tzvetkov [104] improved Demirbas' result by proving flow property for the case $\alpha > \frac{6}{5}$. Subsequently, Sun-Tzvetkov [105] improved their own result to the range $\alpha > \frac{31-\sqrt{233}}{14}$ by applying the theory of random averaging operators, thereby answering Zakharov's question for $\alpha > \frac{31-\sqrt{233}}{14}$. However, despite these advancements, they did not fully address the range $\alpha > 1$ in terms of proving the flow property. Therefore, it remains to prove the flow property for the range $\alpha > 1$, which was an question proposed by [105].

Recently, Liang-Wang [70] addressed this question (see Chapter 2). Also, Liang-Wang-Yue [72] proved the almost sure local well-posedness of nonlinear half-wave equation (case when $\alpha = 1$) for all probabilistically subcritical cases (above its probabilistic scaling).

1.2.3 Gibbs measures

Apart from the problem of invariance of Gibbs measures under Hamiltonian flow, the construction of Gibbs measures has its own interests in constructive quantum field theory. Gibbs measure construction has a far-reaching root from *Euclidean field theory.* See [46]. The most successful models in constructive field theory are the Φ^4_d−models, which are constructed in Bourgain's work [2, 3] for $d = 1, 2$. One way to construct Gibbs measure is constructing Gibbs measure dynamically (stochastic quantisation) [97]. The construction of Φ^4_2 measure via stochastic quantisation was done in [37]. The construction of Φ^4_3 measure via stochastic quantisation had been an open and difficult problem until Martin Hairer's theory on regularity structures [54] was invented. Apart from the theory of the regularity structures, other methods for construting Gibbs measures rely on Gubinelli's theory on paracontrolled distributions [44], and higher order paracontrolled calculus [9], and so forth. In Chapter 3, instead of constructing measures dynamically, we will focus on Barashkov-Gubinelli's approach to constructing Gibbs measures [17].

Let $\mathrm{d}\mu_d$ be the Gaussian measure over distributions on $\mathbb{T}^d$. When $d = 1$, a function in the support of $\mathrm{d}\mu = \mathrm{d}\mu_1$ is in $L^p(\mathbb{T})$ by Sobolev embedding, and consequently the measure $\mathrm{d}\Phi^p_1$ given by

$$\mathrm{d}\Phi^p_1(u) := Z^{-1}\mathrm{e}^{-\frac{1}{p}\|u\|^p_{L^p(\mathbb{T})}}\mathrm{d}\mu(u), \tag{1.24}$$

where Z is a normalisation constant, is absolutely continuous with respect to the Gaussian free field. The measure $\mathrm{d}\Phi^p_1$ is called Φ^p_1 measure.

When $d = 2$, note that the support of $\mathrm{d}\mu_2$ is below L^2 and thus the elements in the support of $\mathrm{d}\mu_2$ are not functions but generalised functions (distributions).

$$\mathrm{d}\Phi^4_2(u) = Z^{-1}\mathrm{e}^{-\frac{1}{4}\|u\|^4_{L^4(\mathbb{T}^2)}}\mathrm{d}\mu_2(u) \tag{1.25}$$

is not a well-defined measure and one needs to introduce renormalisation. In order to resolve this issue, we consider an appropriate renormalisation via frequency truncation in the Fourier side. Precisely, for a given dyadic number N, let σ_N be given by

$$\sigma_N = \left(\int_{H^{0-}(\mathbb{T}^2)}\int_{\mathbb{T}^2} |\Pi_N u(x)|^2 \mathrm{d}x \mathrm{d}\mu_2(u)\right)^{\frac{1}{2}} \sim \log N,$$

where Π_N is defined by $(\Pi_N u)_k = \mathbf{1}_{\langle k\rangle \leq N} u_k$, and define the renormalisation by

$$: |\Pi_N u|^4 := |\Pi_N u|^4 - 4\sigma_N |\Pi_N u|^2 + 2\sigma_N^2. \tag{1.26}$$

Then we consider the following truncated Φ_2^4 measure

$$\mathrm{d}\Phi_{2,N}^4 = Z^{-1} \mathrm{e}^{-\frac{1}{4}\int_{\mathbb{T}^2} :|\Pi_N u|^4 : \mathrm{d}x} \mathrm{d}\mu_2(u).$$

In [92], in the complex-valued setting, the authors apply white noise functional to prove that

$$\left\| \int_{\mathbb{T}^2} : |\Pi_N u|^4 : \mathrm{d}x - \int_{\mathbb{T}^2} : |u|^4 : \mathrm{d}x \right\|_{L^p(\mu_2)} \longrightarrow 0, (N \to \infty) \tag{1.27}$$

where the functional $\int_{\mathbb{T}^2} : |u|^4 : \mathrm{d}x$ is a symbol of the $L^p(\mu_2)$ limit of $\int_{\mathbb{T}^2} : |\Pi_N u|^4 : \mathrm{d}x$ for $1 \leq p < \infty$. The limit in (1.27) also claims the measurability of the limiting functional $G(u) := \frac{1}{4}\int_{\mathbb{T}^2} : |u|^4 : \mathrm{d}x$ with respect to $\mathrm{d}\mu_2$. For the proof of the convergence in (1.27), see also [93, 52, 81].

Then, by utilising Nelson's estimate [76], [92] one can show that

$$\sup_{N\in\mathbb{N}} \left\| \exp\left(-\frac{1}{4}\int_{\mathbb{T}^2} : |u|^4 : \mathrm{d}x\right) \right\|_{L^p(\mathrm{d}\mu_2)} \leq C_p < \infty \tag{1.28}$$

for $1 \leq p < \infty$. Also, let us remark that we can prove (1.28) by using the Boué-Dupuis formula [14, 17, 113] which is the main tool being used in the thesis. With the bound (1.28), one can apply Prokhorov's theorem to obtain weak convergence of the measure and thus we can define the Φ_2^4 measure by the limit of its truncated version $\mathrm{d}\Phi_{2,N}^4$. Also, Φ_2^4 measure can be constructed by Nelson's estimate and Ornstein-Uhlenbeck semi-group. See [46, 92, 98, 102].

Note that the Radon-Nikodym derivatives in (1.24) and (1.25) are bounded, and we call them *defocusing* measures. For the *focusing* case, there are many literatures [68, 2, 27, 21, 84, 4, 6, 5, 85, 99, 12, 89, 69, 66, 71]. For the one-dimensional case, [68] studies the following focusing Gibbs measure.

$$\mathrm{d}\rho_{p,K} = \exp\left(\frac{1}{p}\int_{\mathbb{T}} |u|^p \mathrm{d}x\right) \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\leq K\}} \mathrm{d}\mu(u). \tag{1.29}$$

See also [2, 85]. When the spatial dimension is two, inspired by the Berlin-Kac spheric model, Brydges-Slade proves that the Φ_2^4 measure below is non-integrable [21]:

$$\chi\left(\int_{\mathbb{T}^2} :|u|^2: \mathrm{d}x\right) \exp\left(\lambda\int_{\mathbb{T}^2} :|u|^4: \mathrm{d}x\right) \mathrm{d}\mu_2, \tag{1.30}$$

where χ decays exponentially. As a generalisation of result in [21], Oh-Seong-Tolomeo [84] recently proves that

$$\lim_{L\to\infty} \liminf_{N\to\infty} \int \chi\left(\int_{\mathbb{T}^d} :|u|^2: \mathrm{d}x\right) \mathrm{e}^{\min\left(L, \lambda\int_{\mathbb{T}^d}:|u|^4:\mathrm{d}x\right)} \mathrm{d}\mu_{\frac{d}{2},d}(u) = +\infty \tag{1.31}$$

for any $\lambda > 0$ and any d. In the same paper [84], the authors prove the integrability of the focusing measure in (1.31) with $p = 3$ which is a generalisation of Jaffe's construction of two-dimensional case [6]. For three-dimensional case, [4] studies the focusing Hartree-type Gibbs measure. The method that [84] considers to prove (1.31) is the finite-dimensional Boué-Dupuis formula [14, 17, 113, 84, 81, 89, 91, 69, 101, 38, 71].

It was Lebowitz-Rose-Speer who proposed to study the focusing Gibbs measure (1.29) in [68]. Then Bourgain and Oh-Sosoe-Tolomeo obtained results regarding normalisability or non-normalisability of (1.29) [2, 85]. We summarise Lebowitz-Rose-Speer, Bourgain, and Oh-Sosoe-Tolomeo's results [68, 2, 85] in the Table 1.1.

In Table 1.1, Q is the unique optimiser of Gagliardo-Nirenberg-Sobolev inequality on the real line with $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q'\|_{L^2(\mathbb{R})}^2$.

Oh-Sosoe-Tolomeo's result in the critical case ($p = 6$) with $K = \|Q\|_{L^2(\mathbb{R})}$ is surprising.

	$2 < p < 6$	$p = 6$	$p > 6$
$K < \|Q\|_{L^2(\mathbb{R})}$	normalisable	normalisable	non-normalisable
$K = \|Q\|_{L^2(\mathbb{R})}$	normalisable	normalisable	non-normalisable
$K > \|Q\|_{L^2(\mathbb{R})}$	normalisable	non-normalisable	non-normalisable

Table 1.1: Normalisablility and non-normalisability of Gibbs measures $\mathrm{d}\rho_{p,K}$.

The methods they used to obtain this surprising result are among others coordinate transformations, spectral analysis of second order ordinary differential operators.

Recently, Liang-Wang [69] generalised the results shown in Table 1.1 to fractional nonlinear Schrödinger equations in higher dimensions, except the optimal mass threshold at the critical nonlinearity. Stablishing this result will be the main goal in Chapter 3.

1.3 Preliminaries

In this section, we collect some results that will be used throughout the thesis.

1.3.1 Tensor norms

In this subsection, we define some norms that will be used in the thesis.

We first recall the operator norms and Hilbert-Schmidt norms defined on a Hilbert Space. To simplify the notation, we consider the Hilbert space to be

$$\ell^2(\mathbb{Z}) = \left\{ v = \{z_k\}_{k\in\mathbb{Z}};\ \|v\|_{\ell^2(\mathbb{Z})} = \Big(\sum_{k\in\mathbb{Z}} |z_k|^2 \Big)^{\frac{1}{2}} < \infty \right\},$$

that is the space of square-summable complex-valued sequences. Recall that, for the given bounded operator $H : \ell^2 \to \ell^2$, the operator norm of H is

$$\|H\|_{\mathrm{OP}} = \sup \left\{ \frac{\|Av\|_{\ell^2}}{\|v\|_{\ell^2}} : v \neq 0, v \in \ell^2 \right\}.$$

Denote $\{e_i; i \in \mathbb{Z}\}$ be the canonical orthonormal basis of ℓ^2; see (1.36). Then, we may identify the operator H with its kernel $\{H_{k_1k_2}\}_{k_1,k_2\in\mathbb{Z}}$, where

$$H_{k_1k_2} = \langle He_{k_1}, e_{k_2}\rangle, \quad \text{for} \quad k_1, k_2 \in \mathbb{Z}.$$

With the above notations, we define

$$\|H_{k_1k_2}\|_{k_1\to k_2} := \|H\|_{\mathrm{OP}}.$$

The other important norm in this thesis is the Hilber-Schmidt norm given by

$$\|H\|_{\mathrm{HS}}^2 = \sum_{k\in\mathbb{Z}} \|He_k\|_{\ell^2}^2 = \sum_{k_1,k_2} |H_{k_1k_2}|^2,$$

where $\{H_{k_1k_2}\}_{k_1k_2\in\mathbb{Z}}$ is the kernel of H under the canonical basis of ℓ^2. Similar to the operator norm, we denote $\|H\|_{\mathrm{HS}}$ by $\|H_{k_1k_2}\|_{k_1k_2}$, that is

$$\|H_{k_1k_2}\|_{k_1k_2} := \|H\|_{\mathrm{HS}} = \Big(\sum_{k_1,k_2} |H_{k_1k_2}|^2\Big)^{\frac{1}{2}}$$

to simplify the notation. In what follows, we shall extend the above notations for the operator norm and Hilbert-Schmidt norms to higher dimensions.

Given a positive integer r, a tensor $H = \{H_{k_1\cdots k_r}\}_{k_1,\ldots,k_r\in\mathbb{Z}}$ is a function from $\mathbb{Z}^r$ to $\mathbb{C}$ with variables $(k_1,\ldots,k_r)$. By abusing notation, we also call $H_{k_1\cdots k_r}$ a tensor. For an explanation of how a function from $\mathbb{Z}^r$ to $\mathbb{C}$ is related to the concept of tensor (multilinear map), see Remark 2. In what follows, we assume $H \in \ell^2(\mathbb{Z}^r)$, i.e.

$$\|H\|_{\ell^2(\mathbb{Z}^r)} := \|H_{k_1\cdots k_r}\|_{\ell^2_{k_1\cdots k_r}} = \Big(\sum_{k_1,\ldots,k_r} |H_{k_1\cdots k_r}|^2\Big)^{\frac{1}{2}} < \infty, \tag{1.32}$$

where $\ell^2(\mathbb{Z}^r)$ denotes the set of all complex-valued functions H on $\mathbb{Z}^r$ such that (1.32) holds, and the summation is over $k_1,\ldots,k_r \in \mathbb{Z}$. In the following, the summation over k_j is always assumed to be over $\mathbb{Z}$, unless otherwise stated. For brevity, in the thesis, we will adopt the following notation convention:

$$\|H_{k_1\cdots k_r}\|_{k_1\cdots k_r} := \|H_{k_1\cdots k_r}\|_{\ell^2_{k_1\cdots k_r}}$$

The support of H is the set of $(k_1, \dots, k_r)$ such that $H_{k_1\cdots k_r} \neq 0$. If $1 \le s \le r$, then the tensor norm $\|H_{k_1\cdots k_r}\|_{\ell^2_{k_1\cdots k_s} \to \ell^2_{k_{s+1}\cdots k_r}}$ is given by

$$\|H_{k_1\cdots k_r}\|^2_{\ell^2_{k_1\cdots k_s} \to \ell^2_{k_{s+1}\cdots k_r}} = \sup\left\{ \sum_{k_{s+1},\dots,k_r} \left| \sum_{k_1,\dots,k_s} H_{k_1\cdots k_s k_{s+1}\cdots k_r} z_{k_1\cdots k_r} \right|^2 ; \sum_{k_1,\dots,k_s} |z_{k_1\cdots k_s}|^2 = 1 \right\}. \tag{1.33}$$

For brevity, we also adopt the following notation convention in the thesis:

$$\|H_{k_1\cdots k_r}\|_{k_1\cdots k_s \to k_{s+1}\cdots k_r} := \|H_{k_1\cdots k_r}\|_{\ell^2_{k_1\cdots k_s} \to \ell^2_{k_{s+1}\cdots k_r}}$$

Let us remark that

$$\|H\|_{k_1\cdots k_s \to k_{s+1}\cdots k_r} = \|H\|_{k_{s+1}\cdots k_r \to k_1\cdots k_s} \le \|H\|_{k_1\cdots k_r}. \tag{1.34}$$

Let $f \in \ell^2(\mathbb{Z}^s)$ be a function $f : \mathbb{Z}^s \longrightarrow \mathbb{C}$ such that $\|\{f(k_1, \dots, k_s)\}_{k_1,\dots,k_s \in \mathbb{Z}}\|_{\ell^2_{k_1\cdots k_s}} < \infty$. We may use H to define an operator, and still denoted by $H : \ell^2(\mathbb{Z}^s) \longrightarrow \ell^2(\mathbb{Z}^{r-s})$, given by

$$H[f](k_{s+1}\cdots k_r) = \sum_{k_1,\dots,k_s} H_{k_1\cdots k_s k_{s+1}\cdots k_r} f(k_1, \dots, k_s).$$

Then the tensor norm $\|H\|_{k_1\cdots k_s \to k_{s+1}\cdots k_r}$ defined in (1.33) can be recast as an operator norm.

$$\|H\|^2_{k_1\cdots k_s \to k_{s+1}\cdots k_r} = \|H\|^2_{\ell^2_{k_1\cdots k_s} \to \ell^2_{k_{s+1}\cdots k_r}} := \sup_{\|f(k_1,\cdots,k_s)\|_{\ell^2_{k_1\cdots k_s}} = 1} \|H[f]\|^2_{\ell^2_{k_{s+1}\cdots k_r}}.$$

Remark 1. Given a finite index set $A := \{1, \dots, r\}$, we also adopt the notation convention that

$$k_A := (k_1, \dots, k_r) = k_1 \cdots k_r.$$

A tensor $H_{k_1\cdots k_r}$ can be written as H_{k_A} for brevity. In this notation, we rewrite (1.32) as

$$\|H_{k_A}\|_{k_A} = \Big(\sum_{k_A} |H_{k_A}|^2\Big)^{\frac{1}{2}} < \infty.$$

If (B, C) is a partition of A, namely $B \cap C = \varnothing$ and $B \cup C = A$, the tensor norm defined in (1.33) can be rewritten as $\|H\|_{k_B \to k_C}$ such that

$$\|H\|_{k_B\to k_C}^2 = \sup\Big\{\sum_{k_C}\Big|\sum_{k_B} H_{k_A} z_{k_B}\Big|^2 ; \sum_{k_B} |z_{k_B}|^2 = 1\Big\}.$$

(1.34) can be rewritten as

$$\|H\|_{k_B\to k_C} = \|H\|_{k_C\to k_B} \leq \|H\|_{k_A}.$$

We remark that if $C = \varnothing$, then $B = A$ and we adopt the convention that $\|H\|_{k_B\to k_C} = \|H\|_{k_A}$.

Remark 2. Recall the concept of tensors. By a tensor of rank r, we mean a multilinear map

$$H : \underbrace{\ell^2 \times \cdots \times \ell^2}_{r \text{ copies}} \longrightarrow \mathbb{C}, \tag{1.35}$$

such that

$$\begin{aligned}
H(\alpha u^{(1)} + \beta v^{(1)}, u^{(2)}, \cdots, u^{(r)}) &= \alpha H(u^{(1)}, u^{(2)}, \cdots, u^{(r)}) + \beta H(v^{(1)}, u^{(2)}, \cdots, u^{(r)});\\
H(u^{(1)}, \alpha u^{(2)} + \beta v^{(2)}, \cdots, u^{(r)}) &= \alpha H(u^{(1)}, u^{(2)}, \cdots, u^{(r)}) + \beta H(u^{(1)}, v^{(2)}, \cdots, u^{(r)});\\
&\cdots\\
H(u^{(1)}, u^{(2)}, \cdots, \alpha u^{(r)} + \beta v^{(r)}) &= \alpha H(u^{(1)}, u^{(2)}, \cdots, u^{(r)}) + \beta H(v^{(1)}, u^{(2)}, \cdots, u^{(r)}),
\end{aligned}$$

hold for all $u^{(1)}, u^{(2)}, \cdots, u^{(r)}, v^{(1)}, v^{(2)}, \cdots, v^{(r)} \in \ell^2$ and $\alpha, \beta \in \mathbb{C}$. Here and elsewhere, $\mathbb{C}$ is the vector space consisting of all complex numbers, endowed with the usual addition and scalar multiplication as the linear structure.

Then we consider the following basis of ℓ^2:

$$\{e_j; j \in \mathbb{Z}\}. \tag{1.36}$$

such that $(e_j)_k = 1$ if $i = k$ and $(e_i)_k = 0$ if $i \neq k$, where $(e_j)_k$ is the k-th component of the sequence e_j.

Then, we denote by

$$H_{k_1\cdots k_r} = H(e_{k_1}, \cdots, e_{k_r}),$$

which is the components of tensor H under the basis (1.36). We see that $H_{k_1\cdots k_r}$ is a complex number indexed by r many integers. Therefore, from this point of view, by abusing notation, we can regard a tensor (1.35) as a mapping from $\mathbb{Z}^r$ to $\mathbb{C}$:

$$H : \mathbb{Z}^r \longrightarrow \mathbb{C}.$$

Also, we can write the action of H as

$$H(u^{(1)}, \cdots, u^{(r)}) = \sum_{k_1,\cdots,k_r} H_{k_1\cdots k_r} u^{(1)}_{k_1} \cdots u^{(r)}_{k_r},$$

where $u^{(i)}_{k_i}$ is the k_i-th component of the sequence $u^{(i)} = \{u^{(i)}_{k_i}\}_{k_i \in \mathbb{Z}}$.

1.3.2 Function spaces

In the study of dispersive equations, one of the most important tools is the Fourier restriction norm method introduced by Bourgian [1] (see also [58]). Given $s, b \in \mathbb{R}$, we define the space $X^{s,b}(\mathbb{R} \times \mathbb{T})$ to be the completion of functions that are smooth in space and Schwartz in time equipped with the following norm:

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} = \big\| \langle k \rangle^s \langle \tau + |k| \rangle^b \mathcal{F}_{t,x}(u)(\tau, k) \big\|_{L^2_\tau \ell^2_k(\mathbb{R}\times\mathbb{Z})}, \tag{1.37}$$

where $\mathcal{F}_{t,x}$ is the time-space Fourier transform. We shall also use the following formulation of the Fourier restriction norm. Let

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)\,\mathrm{e}^{\mathrm{i}kx},$$

where $u_k(t) = \mathcal{F}_x(u)(t,k)$ is the spatial Fourier transform of u. For any $b \in \mathbb{R}$, we denote by

$$X^b := L^2(\mathbb{T}; H^b(\mathbb{R})),$$

where $H^b(\mathbb{R}) = \langle\nabla\rangle^{-b}L^2(\mathbb{R})$. We define the Fourier restriction norm of u (in the interaction picture) by

$$\|u\|_{X^b}^2 := \int_{\mathbb{R}} \langle\tau\rangle^{2b}\|\hat{u}_k(\tau)\|_{\ell_k^2}^2 \mathrm{d}\tau = \int_{\mathbb{R}} \langle\tau\rangle^{2b} \sum_k |\hat{u}_k(\tau)|^2 \mathrm{d}\tau.$$

It is easy to see that the above definition coincides with the following Sobolev norm

$$\|u\|_{X^b} = \|\langle\partial_t\rangle^b u\|_{L^2_{t,x}} = \|u\|_{L^2_x H^b_t}.$$

We call $\|\cdot\|_{X^b}$ Fourier restriction norm because it corresponds to the Fourier restriction norm (1.37) in the following sense: let $v := \mathrm{e}^{-\mathrm{i}t\mathrm{D}^\alpha}u$ be the interaction representation of u; see also (2.21) for the motivation, which implies

$$\|u\|_{X^b} = \|\mathrm{e}^{\mathrm{i}t\mathrm{D}^\alpha}v\|_{X^b} = \|\langle\partial_t\rangle^b \mathrm{e}^{\mathrm{i}t\mathrm{D}^\alpha}v\|_{L^2_{t,x}} = \|\langle\tau + |k|^\alpha\rangle^b \widehat{v}_k\|_{L^2_\tau \ell^2_k} = \|v\|_{X^{0,b}}.$$

In what follows, we shall mainly use the norm X^b, which works well with the iteration argument in the main part of our analysis.

For a time-dependent operator $\mathcal{H}(t)$ given by its kernel $\mathcal{H}(t) = \{H_{kk'}(t)\}_{kk'}$, that is

$$\mathcal{H}(t)[f] = \sum_{k\in\mathbb{Z}} \left(\sum_{k'\in\mathbb{Z}} H_{kk'}(t) f_{k'}\right)\mathrm{e}^{\mathrm{i}kx}.$$

Denote by $\mathcal{L}(V_1, V_2)$ the set of all bounded linear maps from normed space V_1 to normed space V_2. We define

$$Y^b := \mathcal{L}(L^2(\mathbb{T}), X^b),$$

and the operator norm $\|\mathcal{H}\|_{Y^b}$ is given by

$$\begin{aligned}\|\mathcal{H}\|_{Y^b} :=& \|\langle\tau\rangle^b \hat{H}_{kk'}(\tau)\|_{\ell^2_k L^2_\tau \to \ell^2_{k'}} \\ =& \sup\left\{\left(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \sum_k \left|\sum_{k'} \hat{H}_{kk'}(\tau) f_{k'}\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}; \sum_{k'} |f_{k'}|^2 = 1\right\} \\ =& \sup_{\|f\|_{L^2}=1} \|\mathcal{H}(t)[f]\|_{X^b}.\end{aligned} \tag{1.38}$$

We also define

$$Z^b := \mathrm{HS}(L^2(\mathbb{T}), X^b) = \{\mathcal{H} \in \mathcal{L}(L^2(\mathbb{T}), X^b); \|\mathcal{H}\|_{Z^b} < \infty\}$$

where the Hilbert-Schmidt norm $\|\mathcal{H}\|_{Z^b}$ is given by

$$\|\mathcal{H}\|_{Z^b} := \left(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \|\hat{H}_{kk'}(\tau)\|^2_{kk'} \mathrm{d}\tau\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \sum_{k,k'} |\hat{H}_{kk'}(\tau)|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}. \tag{1.39}$$

Now we consider an operator $\mathcal{P}$ over space-time functions given by

$$\mathcal{P}[u](t,x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \left(\sum_{k'\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{P_{kk'}}(\tau,\tau') \hat{u}_{k'}(\tau') \mathrm{d}\tau'\right) \mathrm{e}^{\mathrm{i}(t\tau + kx)} \mathrm{d}\tau,$$

where $\widehat{P_{kk'}}(\tau,\tau')$ is the double temporal Fourier transform given by

$$\widehat{P_{kk'}}(\tau,\tau') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} P_{kk'}(t,t') \mathrm{e}^{-\mathrm{i}(t\tau + t'\tau')} \mathrm{d}t \mathrm{d}t'.$$

We define

$$Y^{b,b'} := \mathcal{L}(X^{b'}, X^b)$$

and the operator norm $\|\mathcal{P}\|_{Y^{b,b'}}$ is defined by

$$\begin{aligned}\|\mathcal{P}\|_{Y^{b,b'}} :=& \|\langle\tau\rangle^{b}\langle\tau'\rangle^{-b'}\widehat{P_{kk'}}(\tau,\tau')\|_{\ell^2_k L^2_\tau \to \ell^2_{k'} L^2_{\tau'}} \\ =& \sup_{\|u\|_{X^{b'}}=1} \left(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \sum_k \left|\sum_{k'} \int_{\mathbb{R}} \widehat{P_{kk'}}(\tau,\tau')\hat{u}_{k'}(\tau')\mathrm{d}\tau'\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\ =& \sup_{\|u\|_{X^{b'}}=1} \|\mathcal{P}[u]\|_{X^b}.\end{aligned} \tag{1.40}$$

We also define

$$Z^{b,b'} := \mathrm{HS}(X^{b'}, X^b) = \{\mathcal{P} \in \mathcal{L}(X^{b'}, X^b); \|\mathcal{P}\|_{Z^{b,b'}} < \infty\}$$

where the $\|\mathcal{P}\|_{Z^{b,b'}}$ norm of the operator $\mathcal{P}$

$$\|\mathcal{P}\|^2_{Z^{b,b'}} = \int_{\mathbb{R}^2} \langle\tau\rangle^{2b}\langle\tau'\rangle^{-2b'} \sum_{k,k'} |\widehat{P_{kk'}}(\tau,\tau')|^2 \mathrm{d}\tau\mathrm{d}\tau'. \tag{1.41}$$

For any finite interval I, define the corresponding localised norms

$$\|u\|_{X^b(I)} := \inf\left\{\|v\|_{X^b}; v \in X^b \text{ and } v = u \text{ on } I\right\}$$

and similarly define $Y^{b,b'}(I)$ and $Z^{b,b'}(I)$. By abusing notations, we will call the above v an extension of u, and will denote v by u. We will use X^b to denote $X^b(I)$ for simplicity unless otherwise specified.

1.3.3 Linear estimates

To conclude this section, we record some basic estimates from [33, 34]. Define the original and truncated Duhamel operators

$$\mathcal{I}v(t) = \int_0^t v(t')\mathrm{d}t', \quad \mathcal{I}_\eta v(t) = \eta(t)\int_0^t \eta(t')v(t')\mathrm{d}t',$$

where $\eta(t)$ is a smooth cut-off function, such that $\eta(t) = 1$ for $|t| \leq 1$; and $\eta(t) = 0$ for $|t| \geq 2$. We have the following kernel estimates. See [33, Lemma 3.1] for a proof.

Lemma 1. We have the formula

$$\widehat{\mathcal{I}_\eta v}(\lambda) = \int_{\mathbb{R}} \mathcal{K}(\lambda, \lambda')\hat{v}(\lambda')\mathrm{d}\lambda', \tag{1.42}$$

where the kernel $\mathcal{K}$ satisfies the following pointwise estimates

$$|\mathcal{K}(\lambda, \lambda')| + |\partial_{\lambda,\lambda'}\mathcal{K}(\lambda, \lambda')| \lesssim_\eta \left(\frac{1}{\langle\lambda\rangle^3} + \frac{1}{\langle\lambda - \lambda'\rangle^3}\right)\frac{1}{\langle\lambda'\rangle} \lesssim_\eta \frac{1}{\langle\lambda\rangle\langle\lambda - \lambda'\rangle}. \tag{1.43}$$

We also need the following short-time bound on Fourier restriction norm (see [34, Proposition 2.7]).

Proposition 1 (Short time bounds)**.** Let η be any Schwartz function, and $\eta_T(t) = \eta(T^{-1}t)$ for $T > 0$. Then for any $u \in X^{b_1}$ we have

$$\|\eta_T \cdot u\|_{X^b} \lesssim_\eta T^{b_1 - b}\|u\|_{X^{b_1}},$$

provided either $0 < b \leq b_1 < 1/2$, or $u_k(0) = 0$ for all $k \in \mathbb{Z}$ and $1/2 < b \leq b_1 < 1$.

1.3.4 Bilinear tensor estimates

In this subsection, we recall some bilinear tensor estimates from [35, Proposition 4.11], which serve as "nonlinear" estimates for operator norms of tensors.

Proposition 2. Given finite sets A_1 and A_2. Consider two tensors $h^{(1)}_{k_{A_1}}$ and $h^{(2)}_{k_{A_2}}$, where $A_1 \cap A_2 = C$. Let $A_1 \Delta A_2 = A$[1] and define the semi-product

$$H_{k_A} = \sum_{k_C} h^{(1)}_{k_{A_1}} h^{(2)}_{k_{A_2}}.$$

[1] $A_1 \Delta A_2 := (A_1 \cup A_2) \backslash (A_1 \cap A_2)$.

For any partition (X,Y) of A, let $X \cap A_i =: X_i$, $Y \cap A_i =: Y_i$ for $i = 1,2$. Then, we have

$$\|H\|_{k_X \to k_Y} \leq \|h^{(1)}\|_{k_{X_1 \cup C} \to k_{Y_1}} \cdot \|h^{(2)}\|_{k_{X_2} \to k_{C \cup Y_2}}.$$

For notation convention used in the statement of Proposition 2, we refer the reader to Remark 1.

The following estimate for special tensor semi-products can be proved using Proposition 2. However, we demonstrate a fundamental proof to showcase the idea of proving Proposition 2.

Lemma 2. The following holds.

$$\|h^{(1)}_{k_1 k_1'} h^{(2)}_{k_2 k_1'}\|_{k_1' \to k_1 k_2} \leq \|h^{(1)}_{k_1 k_1'}\|_{k_1' \to k_1} \|h^{(2)}_{k_2 k_1'}\|_{k_1' \to k_2}.$$

Proof. By using the definition of the tensor norm (1.33), we have

$$\begin{aligned}
\|h^{(1)}_{k_1 k_1'} h^{(2)}_{k_2 k_1'}\|_{k_1' \to k_1 k_2} &= \sup_{\|a_{k_1'}\|_{k_1'}=1} \Big\| \sum_{k_1'} h^{(1)}_{k_1 k_1'} h^{(2)}_{k_2 k_1'} a_{k_1'} \Big\|_{k_1 k_2} \\
&\leq \|h^{(1)}_{k_1 k_1'}\|_{k_1' \to k_1} \sup_{\|a_{k_1'}\|_{k_1'}=1} \|h^{(2)}_{k_2 k_1'} a_{k_1'}\|_{k_1' k_2} \\
&= \|h^{(1)}_{k_1 k_1'}\|_{k_1' \to k_1} \|h^{(2)}_{k_2 k_1'}\|_{k_1' \to k_2}.
\end{aligned}$$

□

We conclude this subsection with a weighted estimate. We refer the reader to [34, Proposition 2.5] for its proof. See also [36, Proposition 2.9].

Proposition 3. Suppose that matrices $h = h_{kk''}$, $h^{(1)} = h^{(1)}_{kk'}$ and $h^{(2)} = h^{(2)}_{k'k''}$ satisfy that

$$h_{kk''} = \sum_{k'} h^{(1)}_{kk'} h^{(2)}_{k'k''},$$

and $h^{(1)}_{kk'}$ is supported in $|k - k'| \lesssim L$, then we have

$$\left\| \left(1 + \frac{|k - k''|}{L}\right)^{\kappa} h_{kk''} \right\|_{\ell^2_{kk''}} \lesssim \|h^{(1)}_{kk'}\|_{k \to k'} \cdot \left\| \left(1 + \frac{|k' - k''|}{L}\right)^{\kappa} h^{(2)}_{k'k''} \right\|_{\ell^2_{k'k''}}.$$

1.3.5 Random tensor estimates

In this subsection, we collect some probability results. We start with the Wiener chaos estimate (see [102, Theorem I.22]).

Lemma 3. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard real-valued Gaussian random variables. Given $k \in \mathbb{Z}_{\geq 0}$, let $\{P_j\}_{j\in\mathbb{N}}$ be a sequence of polynomials in $\bar{g} = \{g_n\}_{n\in\mathbb{Z}}$ of degree at most k. Then, for $p \geq 2$, we have

$$\Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^2(\Omega)}.$$

We recall the definition of A-certain.

Definition 2. Given $\varepsilon > 0$. If some statement S for a random variable holds with probability $\mathbb{P}(S) \geq 1 - C_\varepsilon \mathrm{e}^{-A^\varepsilon}$ for some $A > 0$, we say that this statement S is A-certain.

For a complex number z we define z^+ and z^- by $z^+ = z$ and $z^- = \bar{z}$. For a finite index set A, we also use the notation z^{ζ_j} for $\{\zeta_j\}_{j\in A}$ with $\zeta_j \in \{\pm\}$.

Definition 3. Given a finite set A. We say that a pair (k_i, k_j) for $i, j \in A$ is a pairing in k_A, if $k_i = k_j$ and $\zeta_i + \zeta_j = 0$. We say a pairing is over-paired if $k_i = k_j = k_\ell$ for some $\ell \in A \backslash \{i, j\}$.

Definition 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Given a finite set A, a tensor $h_{k_A}(\omega)$ that depends on $\omega \in \Omega$ is called a random tensor.

We also recall the following Large deviation inequality for multilinear Gaussians. It is a special case of [34, Lemma 4.1]. We give a simpler proof for completeness.

Lemma 4. Let $E \subset \mathbb{Z}$ be a finite set, $a = a_{k_1\cdots k_r}(\omega)$ be a random tensor such that the collection $\{a_{k_1\cdots k_r}\}$ is independent with the collection $\{g_k(\omega); k \in E\}$. Let $\zeta_j \in \{\pm\}$ and assume that in the support of $a_{k_1\cdots k_r}$ there is no pairing in $\{k_1, \cdots, k_r\}$ associated with the signs ζ_j. Given a random variable

$$X(\omega) := \sum_{k_1,\cdots,k_r} a_{k_1\cdots k_r} \prod_{j=1}^{r} \eta_{k_j}(\omega)^{\zeta_j},$$

where $\eta_{k_j} \in \{g_{k_j}, |g_{k_j}|^2 - 1\}$. Then, for any $A > |E|$, we have A-certainly that

$$|X(\omega)|^2 \leq A^\theta \sum_{k_1,\cdots,k_r} |a_{k_1\cdots k_r}(\omega)|^2. \tag{1.44}$$

Proof. By using Lemma 3, we have

$$\mathbb{E}[|X|^p] \lesssim p^{\frac{2rp}{2}} (\mathbb{E}[|X|^2])^{\frac{p}{2}}.$$

Due to the no-pairing assumption and independence, we have

$$\begin{aligned}\mathbb{E}[|X|^2] =& \mathbb{E}\left[\left|\sum_{k_1,\cdots,k_r} a_{k_1\cdots k_r} \prod_{j=1}^{r} \eta_{k_j}(\omega)^{\zeta_j}\right|^2\right] \\ =& \sum_{k_1,\cdots,k_r} |a_{k_1\cdots k_r}|^2 \prod_{j=1}^{r} \mathbb{E}[|\eta_{k_j}(\omega)|^2] \\ \lesssim& \sum_{k_1,\cdots,k_r} |a_{k_1\cdots k_r}|^2,\end{aligned}$$

where we used that $\mathbb{E}[|g_k|^2] = 1$ and $\mathbb{E}[|(|g_k|^2 - 1)|^2] = 2$. Thus we have shown that

$$\mathbb{E}[|X|^p] \lesssim p^{\frac{2rp}{2}} \left(\sum_{k_1,\cdots,k_r} |a_{k_1\cdots k_r}|^2\right)^{\frac{p}{2}}.$$

Then, (1.44) follows by a standard Nelson's argument ([112, Lemma 4.5]). □

Now we are ready to state the random tensor estimate, whose proof can be found in [35, Proposition 4.14]. See also [36, footnote 39] and [10] for related discussions.

Proposition 4. Let A be a finite set and $h_{bck_A} = h_{bck_A}(\omega)$ be a tensor, where each k_j and $(b, c) \in \mathbb{Z}^q$ for some integer $q \geq 2$. Given signs $\zeta_j \in \{\pm\}$, we also assume that $\langle b - b_0\rangle, \langle c - c_0\rangle \lesssim M$ and $\langle k_j\rangle \lesssim M$ for some fixed $b_0, c_0 \in \mathbb{Z}$ and all $j \in A$, where M is a dyadic number, and that in support of h_{bck_A} there is no pairing in k_A. Define the tensor

$$H_{bc} = \sum_{k_A} h_{bck_A} \prod_{j\in A} \eta_{k_j}^{\zeta_j}, \tag{1.45}$$

where we restrict $k_j \in E$ in (1.45), E being a finite set such that $\{h_{bck_A}\}$ is independent with $\{\eta_k; k \in E\}$, and $\eta_{k_j} \in \{g_{k_j}, |g_{k_j}|^2 - 1\}$. Then $T^{-1}M$-certainly, we have

$$\|H_{bc}\|_{b\to c} \lesssim T^{-\theta} M^{\theta} \cdot \max_{(B,C)} \|h\|_{bk_B \to ck_C},$$

for some $0 < T \ll 1$, where (B, C) runs over all partitions of A.

For notation convention used in the statement of Proposition 4, we refer the reader to Remark 1.

Remark 3. In what follows, we may ignore the constant $T^{-\theta}M^{\theta}$, as these constants may be absorbed by choosing $0 < \theta \ll 1$ to be small in our analysis. For instance, the factor $T^{-\theta}$ can be absorbed by Proposition 1 by choosing $\theta \ll b_1 - b$. See also [35, 36] for similar arguments.

1.3.6 Gagliardo-Nirenberg-Sobolev inequality

We first recall the definition of the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ defined by the norm:

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 \mathrm{d}\xi. \tag{1.46}$$

The optimiser for the Gagliardo-Nirenberg-Sobolev inequality with the optimal constant:

$$\|u\|^p_{L^p(\mathbb{R}^d)} \le C_{\mathrm{GNS}}(d, p, s) \|u\|^{\frac{(p-2)d}{2s}}_{\dot{H}^s(\mathbb{R}^d)} \|u\|^{2 + \frac{p-2}{2s}(2s-d)}_{L^2(\mathbb{R}^d)} \tag{1.47}$$

plays an important role in the study of the focusing Gibbs measures. We recall the following result.

Theorem 4 (Theorem 2.1, [16]). *Let $d \ge 1$ and let* (i) *$p > 2$ if $d < 2s$, and* (ii)

$2 < p \leq \frac{2d}{d-2s}$ *if* $d \geq 2s$. *Consider the functional*

$$J^{d,p,s}(u) = \frac{\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{(p-2)d}{2s}} \|u\|_{L^2(\mathbb{R}^d)}^{2+\frac{p-2}{2s}(2s-d)}}{\|u\|_{L^p(\mathbb{R}^d)}^p} \tag{1.48}$$

on $H^s(\mathbb{R}^d)$. *Then, the minimum*

$$C_{\mathrm{GNS}}^{-1} = C_{\mathrm{GNS}}(d,p,s)^{-1} := \inf_{\substack{u \in H^s(\mathbb{R}^d) \\ u \neq 0}} J^{d,p,s}(u) \tag{1.49}$$

is attained at a function $Q \in H^s(\mathbb{R}^d)$.

Remark 4. It is easy to see that functions $u(x) := cQ(b(x-a))$ for all $c \in \mathbb{R}\backslash\{0\}$, $b > 0$, and $a \in \mathbb{R}^d$, are minimisers of the functional (1.48). Therefore, we may assume that

$$\begin{aligned} \|Q\|_{L^2(\mathbb{R}^d)} &= \|Q\|_{\dot{H}^s(\mathbb{R}^d)}, \\ \|Q\|_{\dot{H}^s(\mathbb{R}^d)}^2 &= \frac{2}{p}\|Q\|_{L^p(\mathbb{R}^d)}^p. \end{aligned} \tag{1.50}$$

Under this specified scaling, we have $H_{\mathbb{R}^d}(Q) = 0$, where $H_{\mathbb{R}^d}$ is the Hamiltonian functional given by

$$H_{\mathbb{R}^d}(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\mathrm{D}^s u|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^d} |u|^p \mathrm{d}x.$$

Furthermore, this Q solves the following semilinear elliptic equation on $\mathbb{R}^d$:

$$(p-2)d\mathrm{D}^{2s}Q + (4s + (p-2)(2s-d))Q - 4s|Q|^{p-2}Q = 0. \tag{1.51}$$

In the following, we restrict ourselves to (1.50) unless specified otherwise. In particular, we have

$$C_{\mathrm{GNS}} = \frac{p}{2}\|Q\|_{L^2(\mathbb{R}^d)}^{2-p}. \tag{1.52}$$

The uniqueness (in some sense) of this Q for fractional value s is a very challenging problem, which is only proved for some special cases. See [43] for the case $d = 1$ and

$s \in (0, 1]$.

1.3.7 Fernique's theorem

In this subsection, we recall the following simple corollary of Fernique's theorem [41]. See [39, Theorem 2.7] and [85, Lemma 4.2].

Lemma 5. There exists a constant $c > 0$ such that if X is a mean-zero Gaussian process with values in a separable Banach space B with $\mathbb{E}\big[\|X\|_B\big] < \infty$, then

$$\int e^{c\frac{\|X\|_B^2}{(\mathbb{E}[\|X\|_B])^2}} \, d\mathbb{P} < \infty.$$

In particular, for any $t > 1$,

$$\mathbb{P}\big(\|X\|_B \geq t\big) \lesssim \exp\bigg[-\frac{ct^2}{\big(\mathbb{E}\big[\|X\|_B\big]\big)^2}\bigg].$$

1.4 Notation conventions

Throughout the thesis, we will denote by $A \lesssim B$, $A \lesssim_{a_1,\cdots,a_n} B$ and $A \ll B$ any estimate of the form $A \leq CB$, $A \leq C_{a_1\cdots a_n} B$ and $A \leq cB$ respectively, where $C \gg 1$ is an absolute constant, $C_{a_1,\cdots,a_n}$ is a constant that only depends on parameters $a_1, \cdots, a_n$, and $c > 0$ is a small constant. We also denote by $A \sim_{a_1,\cdots,a_n} B$ if $A \lesssim_{a_1,\cdots,a_n} B$ & $B \lesssim_{a_1,\cdots,a_n} A$ holds. We also use $a+$ (and $a-$) to mean $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$. We denote by

$$\begin{cases} \langle k \rangle := \sqrt{1 + |k|^2}; \\ [\![k]\!] := (1 + |k|^\alpha)^{1/\alpha}. \end{cases} \tag{1.53}$$

We denote by $\bar{z}$ the complex conjugate of z.

In this thesis, we adopt the convention of equipping the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ with the normalised Lebesgue measure $dx_{\mathbb{T}^d} = (2\pi)^{-d} dx$. This approach allows us to bypass the need to consider the factor 2π in various calculations. To streamline our notation, we employ $\mathrm{d}x$ to represent both the standard Lebesgue measure on $\mathbb{R}^d$ and the normalised Lebesgue measure on $\mathbb{T}^d$.

We also use $(\mathcal{F}_x f)(k)$ to denote the Fourier coefficients of f, i.e.

$$(\mathcal{F}_x f)(k) = \int_{\mathbb{T}} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

which will be abbreviated as f_k, i.e.

$$f_k := (\mathcal{F}_x f)(k).$$

We may also abuse notations and write $f = (f_k)$. With this notation, the inverse Fourier transform formula rewrites as follows

$$f(x) = \sum_{k\in\mathbb{Z}} f_k \mathrm{e}^{\mathrm{i}kx}.$$

Let $u = u(t,x)$ be a space-time distribution. We denote $u_k(t) := \mathcal{F}_x(u(\cdot,t))(k)$ to be the spatial Fourier coefficient of $u(\cdot,t)$. Given $k \in \mathbb{Z}$, the Fourier coefficient $u_k(t)$ is still a function of time. We use $\widehat{u_k}$ to represent the temporal Fourier transform of u only, i.e.

$$\hat{u}_k(\tau) := \mathcal{F}_t(u_k)(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} u_k(t)\mathrm{e}^{-\mathrm{i}t\tau}\mathrm{d}t.$$

We use t, t' to denote temporal variables and τ, τ', τ_i denote Fourier variables of time. When we work with space-time function spaces, we use short-hand notations such as $C_T H_x^s = C([0,T]; H^s(\mathbb{T}))$.

Let $N_i \geq \frac{1}{2}$ for $i = 1,2,3$ be dyadic numbers. We use $N_{\max}$, N_{med}, and $N_{\min}$ to denote the largest, second largest, and smallest number among N_1, N_2, and N_3, respectively. Denote also by $k_{\max}$ the k_i such that $N_i = N_{\max}$. Similarly, we define k_{med} and $k_{\min}$. We use $a \wedge b$ and $a \vee b$ to denote $\min(a,b)$ and $\max(a,b)$, respectively. We denote by $\#S$ or

$|S|$ the cardinal number of the set S. We also use the indicator function $\mathbf{1}_B$ defined by

$$\mathbf{1}_B = \begin{cases} 1, & x \in B; \\ 0, & \text{otherwise.} \end{cases} \tag{1.54}$$

Let X_i for $i = 1, 2$ be normed space equipped with norm $\| \cdot \|_{X_i}$. Then we define

$$\|x\|_{X_1 \cap X_2} = \|x\|_{X_1} + \|x\|_{X_2}, \quad x \in X_1 \cap X_2$$

and

$$\|x\|_{X_1 \cup X_2} = \min(\|x\|_{X_1}, \|x\|_{X_2}), \quad x \in X_1 \cup X_2.$$

We define analogously norms on the intersection of more than two normed spaces.

For any N, we denote by $\mathcal{B}_{\leq N}$ the σ-algebra generated by the random variable $\{g_k\}$ for $\langle k \rangle \leq N$.

CHAPTER 2

GIBBS DYNAMICS FOR WEAKLY DISPERSIVE NLS

2.1 Introduction

In this chapter, we will consider (1.22) with $p = 4$ and $d = 1$, the fractional cubic nonlinear Schrödinger equation (FNLS) on the one-dimensional torus $\mathbb{T}$:

$$\begin{cases} \mathrm{i}\partial_t u - \mathrm{D}^\alpha u = \mp |u|^2 u, \\ u(0) = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{2.1}$$

where $\mathrm{D}^\alpha = |\partial_x|^\alpha$ is defined the Fourier multiplier $\mathcal{F}_x(\mathrm{D}^\alpha u)(k) = |k|^\alpha u_k$. Recall that we use the symbol $\mp$ to indicate the type of nonlinearity in FNLS (2.1), which can be either defocusing (+) or focusing (−). Recall also that the parameter $\alpha > 0$ measures the dispersion effect in FNLS (2.1). The dispersion becomes stronger as α increases. Recall also the following backgrounds. When $\alpha = 2$, FNLS (2.1) reduces to the well-known cubic nonlinear Schrödinger equation (NLS), which models nonlinear phenomena in optics and plasma physics, see [103] for more details. When $\alpha \neq 2$, FNLS (2.1) arises in fractional quantum mechanics [64], and also in the water wave models [57]. Moreover, it is shown in [61] that FNLS (2.1) describes the continuum limit of lattice interactions when $\alpha \in (1, 2]$. Specifically, the case when $1 < \alpha < 2$ corresponds to the long-range lattice interactions, while the case when $\alpha = 2$ corresponds to the short-range or quick-decaying interactions.

We also note that the cubic nonlinear half-wave equation ($\alpha = 1$) has various physical applications, such as [74, 30] and gravitational collapse [40]. However, when $\alpha = 1$, the equation (2.1) is no longer dispersive. (2.1) also known as Majda-McLaughlin-Tabak (MMT) model [74, 30] in the study of wave turbulence. In this thesis, we focus on FNLS (2.1) for $\alpha \in (1, 2)$, where the dispersion is weaker than that of the cubic nonlinear Schrödinger equation (NLS) (for (2.1) with $\alpha = 2$).

Recall that the equation (2.1) is a Hamiltonian PDE with the conserved Hamiltonian:

$$H(u) = \int_{\mathbb{T}} \left(\frac{1}{2} |\mathrm{D}^{\alpha/2} u|^2 \pm \frac{1}{4} |u|^4 \right) \mathrm{d}x. \tag{2.2}$$

The dynamics of the equation (2.1) also preserves the L^2-norm:

$$M(u) = \frac{1}{2} \int_{\mathbb{T}} |u|^2 \mathrm{d}x. \tag{2.3}$$

The well-posedness of FNLS (2.1) in the low regularity setting has been extensively studied. If $\alpha = 2$, (2.1) is well-posed in $L^2(\mathbb{T})$ in the deterministic sense [1]. However, when $\alpha < 2$, where the equation is less dispersive, the deterministic local well-posedness in $L^2(\mathbb{T})$ is not available. Nevertheless, it has been proved in [32, 29] that FNLS (2.1) with $1 < \alpha < 2$ is locally well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{2-\alpha}{4}$. The main purpose of this thesis is to study FNLS (2.1) from a probabilistic point of view. In particular, we focus on the invariance of the Gibbs measure (2.5) under the flow of (2.1), where the Gibbs measure is a probability measure on distributions on $\mathbb{T}$. See Subsection 2.1.1 for a precise definition.

The existence and invariance of Gibbs measures for nonlinear PDEs are fascinating topics that have attracted a lot of attention in recent years on this topic. We would like to acknowledge the pioneering works of [42, 68, 2, 3, 4, 110, 111, 77, 78, 11, 12, 13, 85, 34, 35, 36, 91, 88, 104, 105, 69, 10]. Here we did not aim to cover all the relevant literature on this topic. We would like to acknowledge that the pioneering works of Bourgain [2, 3] inspired many subsequent studies on the invariant measure for Hamiltonian PDEs, especially those by Tzvetkov [110, 111, 22, 23], that popularised this line of research. One of the main motivations for studying such measures is to understand the long-time behaviour of solutions to nonlinear PDEs, especially those that exhibit chaotic or turbulent phenomena. One such example is the famous question posed by Zakharov mentioned in Section 1.1. The key ingredients to explain "returning" phenomenon are the existence of an invariant Gibbs measure and the flow property of the dynamics, which enable the use of Poincaré recurrence theorem. However, proving the existence and invariance of such a measure is highly nontrivial, as it requires overcoming several analytical and probabilistic challenges. See [42, 68, 2, 3, 10] and references therein for further discussion. We also refer readers to [79] for a nice review of other invariant measures.

For the NLS case, i.e. (2.1) with $\alpha = 2$, Lebowitz-Rose-Speer [68] considered the Gibbs measure of the form

$$\mathrm{d}\rho = Z^{-1}\mathrm{e}^{-\beta H(u)}\mathrm{d}u \tag{2.4}$$

where $H(u)$ is the Hamiltonian given in (2.2), and Z is a normalisation constant. In particular, they showed that the Gibbs measure (2.4) is a well-defined probability measure on $H^{1/2-}(\mathbb{T})$ for the defocusing case; while for the focusing case, as explained in Subsection 2.1.1, their result only holds with the L^2-cut-off $\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\leq K\}}$ for some $K > 0$. Bourgain [2] proved that the Gibbs measure (2.4) is invariant under the flow of NLS and global well-posedness almost surely on the statistical ensemble.

For the fractional NLS (2.1) with $\alpha \in (1, 2)$, the invariance of the Gibbs measure and the "returning" property of the dynamics are not yet fully understood apriori to our recent work [70]. When $\alpha > \frac{4}{3}$, these problems can be settled by using the deterministic theory [32, 29] since the local well-posedness holds on the support of the Gibbs measure. To go beyond the threshold $\frac{4}{3}$, Sun-Tzvetkov [104] exploited a probabilistic argument, where they managed to handle the case $\alpha > \frac{6}{5}$. When $\alpha \in \left(1, \frac{6}{5}\right]$, they also proved the convergence of the Galerkin approximation scheme for the FNLS by using the Bourgain-Bulut approach [11, 12, 13]. However, this argument is insufficient to show the flow property of the limiting dynamics, thus preventing us from applying the Poincaré recurrence theorem. Recently, Sun-Tzvetkov [105] further improved their results in [104] by using the random averaging operator introduced by Deng-Nahmod-Yue [34], and extended the range of α to $\alpha > \frac{31-\sqrt{233}}{14}$. They also conjectured the existence of global strong solutions to (2.1) for all $\alpha > 1$ (see [105, Question 1.1]).

The main goal of the Gibbs dynamics part of the thesis is to affirmatively answer Sun-Tzvetkov's question and conclude this line of research. Specifically, we build strong solutions, satisfying the flow property, to FNLS (2.1) in the full range $\alpha > 1$. The range $\alpha > 1$ is also expected to be optimal. As a matter of fact, we shall show that there is a dramatic change in the regularity of the second Picard iterate between $\alpha > 1$ and $\alpha = 1$. See Theorem 7 for further discussion.

Before proceeding, we recall the notion of Gibbs measure.

2.1.1 Gibbs measures

The Gibbs measure, denoted by $\mathrm{d}\rho$, can be formally written as

$$\mathrm{d}\rho = Z^{-1}\mathrm{e}^{-H(u)-M(u)}\mathrm{d}u, \tag{2.5}$$

where Z is a normalisation constant, $H(u)$ and $M(u)$ are the Hamiltonian and mass of (2.1) given in (2.3) and (2.3), respectively. As both Hamiltonian and mass are conserved along the dynamics of (2.1), it is expected that Gibbs measure (2.5) is invariant under the dynamics of (2.1). By adding the mass term to (2.5), we avoid some technical issues at zero frequency. We may define the Gibbs measure $\mathrm{d}\rho$ as an absolutely continuous measure with respect to the following massive Gaussian free measure:

$$\mathrm{d}\mu = \mathrm{e}^{-\int_{\mathbb{T}}(|\mathrm{D}_x^{\alpha/2}u|^2+|u|^2)\mathrm{d}x}\mathrm{d}u = \prod_{k\in\mathbb{Z}} \mathrm{e}^{-[\![k]\!]^{\alpha}|u_k|^2}\mathrm{d}u_k\mathrm{d}\overline{u}_k, \tag{2.6}$$

where u_k is the Fourier coefficient of u. The above Gaussian measure given in (2.6) is the induced probability measure under the map

$$\omega \longmapsto u_0^\omega(x) = \sum_{k\in\mathbb{Z}} \frac{g_k(\omega)}{[\![k]\!]^{\frac{\alpha}{2}}} \mathrm{e}^{\mathrm{i}kx}. \tag{2.7}$$

Here $[\![k]\!] := (1+|k|^\alpha)^{\frac{1}{\alpha}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Typical elements on the support of μ can be represented as the random Fourier series given in (2.7). An easy computation shows that

$$u_0^\omega \in \bigcap_{s<\frac{\alpha-1}{2}} H^s(\mathbb{T}) \setminus H^{\frac{\alpha-1}{2}}(\mathbb{T}) \tag{2.8}$$

almost surely.

With the above notations, the defocusing Gibbs measure ρ can be recast as the following weighted Gaussian free measure

$$\mathrm{d}\rho = Z^{-1}\mathrm{e}^{-\frac{1}{4}\int_{\mathbb{T}}|u|^4\mathrm{d}x}\mathrm{d}\mu. \tag{2.9}$$

When $\alpha > 1$, we note that $\|u\|_{L^4(\mathbb{T})} < \infty$ almost surely with respect to the Gaussian measure μ. Thus, the defocusing Gibbs measure (2.9) is a well-defined probability measure on $H^s(\mathbb{T})$ for $s < \frac{\alpha-1}{2}$, absolutely continuous with respect to the Gaussian measure μ. When $\alpha \le 1$, one has $\|u\|_{L^4(\mathbb{T})} = \infty$ almost surely, thus a renormalisation is needed. We refer the readers to [104] for further discussion. On the other hand, the focusing Gibbs measure formally given by

$$\mathrm{d}\rho = Z^{-1}\mathrm{e}^{\frac{1}{4}\int_{\mathbb{T}}|u|^4\mathrm{d}x}\mathrm{d}\mu$$

cannot be normalised as a probability measure since

$$\mathbb{E}_\mu\Big[\mathrm{e}^{\frac{1}{2}\int_{\mathbb{T}}|u|^4\mathrm{d}x}\Big] = \infty.$$

Inspired by Lebowitz-Rose-Speer [68] and Oh-Sosoe-Tolomeo [85], In Chapter 3, we considered the following focusing Gibbs measure with a cut-off,

$$\mathrm{d}\rho = Z^{-1}\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\leq K\}}\mathrm{e}^{\frac{1}{4}\int_{\mathbb{T}}|u|^4\mathrm{d}x}\mathrm{d}\mu, \tag{2.10}$$

associated with the focusing FNLS (2.1). In particular, the authors showed that the Gibbs measure (2.10) is a probability measure for any finite cut-off size K, provided $\alpha > 1$. See [69] for further discussion.

2.1.2 Gauge transform

Given a solution $u \in C([-T,T]; L^2(\mathbb{T}))$ to (2.1), we introduce the following invertible gauge transform

$$u(t) \longmapsto \mathcal{G}(u)(t) := \mathrm{e}^{2\mathrm{i}t\int_{\mathbb{T}}|u|^2\mathrm{d}x}u(t), \tag{2.11}$$

A direct computation with the mass conservation shows that the gauged function, which we still denote by u, solves the following renormalised FNLS:

$$\mathrm{i}\partial_t u - \mathrm{D}^\alpha u = \left(|u|^2 - 2\int_{\mathbb{T}}|u|^2\mathrm{d}x\right)u. \tag{2.12}$$

Note that the gauge transform $\mathcal{G}$ in (2.11) is invertible in $C([-T,T]; L^2(\mathbb{T}))$. In particular, we can freely convert solutions to (2.1) into solutions to (2.12) and vice-versa as long as they are in $C([-T,T]; L^2(\mathbb{T}))$. By rewriting (2.12) in the Duhamel formulation, we have

$$u(t) = S(t)u_0^\omega - \mathrm{i}\int_0^t S(t-t')\mathcal{N}(u)(t')\mathrm{d}t', \tag{2.13}$$

where $S(t) = \mathrm{e}^{-\mathrm{i}t\mathrm{D}^\alpha}$ denotes the linear evolution and

$$\mathcal{N}(u) = \left(|u|^2 - 2\int_{\mathbb{T}}|u|^2\mathrm{d}x\right)u.$$

We decompose the nonlinearity $\mathcal{N}(u)$ into non-resonant and resonant parts i.e.

$$\mathcal{N}(u) = \mathcal{Q}(u,u,u) + \mathcal{R}(u,u,u),$$

where the trilinear forms are defined as

$$\begin{aligned} \mathcal{Q}(u,v,w)(t,x) &:= \sum_{k\in\mathbb{Z}} \sum_{\Gamma(k)} u_{k_1}(t)\overline{v_{k_2}(t)} w_{k_3}(t) \mathrm{e}^{\mathrm{i}(k_1-k_2+k_3)x}; \\ \mathcal{R}(u,v,w)(t,x) &:= \sum_{k\in\mathbb{Z}} u_k(t)\overline{v_k(t)} w_k(t) \mathrm{e}^{\mathrm{i}kx}. \end{aligned} \tag{2.14}$$

Here u_k, v_k, w_k denote the spatial Fourier transform of u, v, w respectively, and $\Gamma(k)$ denotes the hyperplane of $\mathbb{Z}^3$,

$$\Gamma(k) := \left\{ (k_1,k_2,k_3) \in \mathbb{Z}^3; k = k_1 - k_2 + k_3 \text{ and } k_1, k_3 \neq k_2 \right\}. \tag{2.15}$$

When all the arguments are the same, we simply denote by $\mathcal{Q}(u) = \mathcal{Q}(u,u,u)$ and $\mathcal{R}(u) = \mathcal{R}(u,u,u)$. The term $\mathcal{Q}(u)$ denotes the non-resonant part of the renormalised nonlinearity $\mathcal{N}(u)$, and $\mathcal{R}(u)$ denotes the resonant part. Then, we have

$$\mathcal{N}(u) = \mathcal{Q}(u) + \mathcal{R}(u).$$

Remark 5. When $\alpha > 1$, the solution to (2.13) we construct will be in $H^s(\mathbb{T}) \backslash H^{\frac{\alpha-1}{2}}(\mathbb{T})$ for any $s < \frac{\alpha-1}{2}$, almost surely. See Theorem 5 below. Thus the solution to (2.13) lies in $L^2(\mathbb{T})$, where the gauge transform $\mathcal{G}$ is invertible, i.e. the renormalised FNLS (2.12) is equivalent to the original FNLS (2.1). The use of gauge transformation (2.11) removes some troublesome resonances, which improves the main counting estimates in Lemma 8 and thus the tensor estimates in Lemma 12.

2.1.3 Main results

In what follows, we consider the Cauchy problem of defocusing FNLS (2.12) with Gaussian random data u_0^ω given in (2.7). Let $N \in 2^{\mathbb{Z}_{\geq 0}} \cup \{1/2\}$ be a dyadic number, define projections Π_N such that

$$(\Pi_N f)_k = \mathbf{1}_{\langle k\rangle \leq N} \cdot f_k, \tag{2.16}$$

and $\Delta_N := \Pi_N - \Pi_{N/2}$. In particular, we note

$$(\Pi_{1/2} f)_k = 0. \tag{2.17}$$

We start with a truncated version of (2.12), with truncated random initial data

$$\begin{cases} \mathrm{i}\partial_t u_N - \mathrm{D}^\alpha u_N = \Pi_N \left[\left(|u_N|^2 - 2 \int_{\mathbb{T}} |u_N|^2 \mathrm{d}x \right) u_N \right], \\ u_N(0) = \Pi_N u_0^\omega = \sum_{\langle k\rangle \leq N} \frac{g_k(\omega)}{[\![k]\!]^{\alpha/2}} \mathrm{e}^{\mathrm{i}kx}, \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{2.18}$$

From (2.17)-(2.18), we also note that

$$u_{1/2} = 0.$$

In this thesis, we call the unique limits of solutions to (2.12) that satisfy the flow property the strong solutions to (2.12). It follows from [104, Theorem 7] (or see [104, Theorem 4]) that (2.12) is globally well-posed almost surely.

Our first goal is to construct local-in-time solutions to (2.12) almost surely with respect to the random initial data (2.7), which is addressed in the following theorem.

Theorem 5. *Given $\alpha > 1$, there exists $\varepsilon > 0$ such that the followings hold. Let u_0 be as in* (2.7) *and u_N be the solution to the truncated system* (2.18). *Then for $0 < T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with $\mathbb{P}(\Omega_T^c) \leq C_\varepsilon \mathrm{e}^{-T^{-\varepsilon}}$ where $\varepsilon > 0$ is independent of T, such that for $\omega \in \Omega_T$, the sequence $\{u_N\}$ converges to a unique limit u in $C([-T,T]; H^{\frac{\alpha-1}{2}-}(\mathbb{T}))$. The limit $u(t)$ solves* (2.12).

Remark 6. In proving Theorem 5, we will assume $\alpha \in (1, 2)$. For the case when $\alpha \geq 2$, [32, 104, 105] have already given a proof. In particular, Theorem 5 has been shown for $\alpha > \frac{31-\sqrt{233}}{14} \approx 1.124$ in [105] by using the theory of random averaging operators developed by Deng-Nahmod-Yue [34], with a precursor [7]. The random averaging operator can be seen as the dispersive version of the well-known paracontrolled distribution method developed by Gubinelli-Imkeller-Perkowski [44]. The key observation is to decompose the solution into a combination of linear, para-linear, and nonlinear parts, such that the para-linear term (or para-controlled term) possesses a randomness structure and the nonlinear part is smoother.

Remark 7. The main difficulty in establishing Theorem 5 comes from the weak dispersion when α is small. Besides the random averaging operator, we need to exploit several new ideas and techniques to overcome the difficulty arising from the weaker dispersion. First of all, we employ the random tensor introduced in [35, 36], which allows us to simplify the multilinear estimates and exploit more refined counting results. See Subsection 2.2.2, Subsection 2.5.1, and Subsection 2.6.1 for more details. Secondly, we improve the counting estimates by exploiting some key properties of the random structure. In particular, we show that the worst-case scenario of bad countings does not happen when considering the interaction between high and low frequencies, which is one of the crucial observations. We make crucial use of the decay fact in the basic counting Lemma 8. This enables us to reduce two dimensions in our key counting estimates; see Lemma 10. This contrasts sharply with the half-wave equation, where at most one dimension can be reduced. See Theorem 7. We also use the Γ-condition to enhance some of the counting arguments. See Subsection 2.2.1 for further discussion. Last but not least, we take advantage of the crucial cancellation that arises from the conservation structure of the random averaging operator, namely the unitary property. See Subsection 2.4 for further discussion. It is important to note that the counting arguments become more complicated in higher dimensions. Specifically, the Γ-condition does not eliminate the impact of high-frequency terms due to the inner product structure. We plan to address these issues in our future work.

Theorem 5 shows that the FNLS (2.1) is almost surely locally well-posed with the random initial data (2.7). To extend the local solution globally, one may adapt Bourgain's invariant measure argument [2, 3]. See also [34, Section 6] for a detailed argument in the random averaging operator setting. More precisely, we use the invariance of the fractional Gibbs measure under the finite-dimensional approximation of the FNLS flow to obtain a uniform control on the solutions and then apply a PDE approximation argument to extend the local solutions to (2.1) obtained from Theorem 5 to global ones. The proof of Theorem 5, using a random averaging operator approach, also implies the stability of the truncated solution map. See [34, Section 5]. As a consequence, we obtain the group property of the flow map and the invariance of the fractional Gibbs measure under the resulting global flow of the FNLS (2.1).

Theorem 6. *Let $\alpha > 1$. Then, the defocusing renormalised FNLS* (2.12) *on $\mathbb{T}$ is globally well-posed almost surely with respect to the Gibbs measure ρ in* (2.9). *Moreover, the flow maps satisfy the usual group (flow) property and keep the Gibbs measure* $\mathrm{d}\rho$ *invariant.*

Once we have Theorem 5, the proof of Theorem 6 is standard by now, and thus we omit the proof. See Deng-Nahmod-Yue [34, Section 6] for a proof of the 2D NLS case; also see Bourgain [2, 3], Deng-Nahmod-Yue [36], and Sun-Tzvetkov [104] for more discussion. The crucial point is that, in the proof of Theorem 5, we established a probabilistic local well-posedness result with a fine structure of the solution of (2.1). See Subsection 2.1.4 for further discussion. In what follows, we shall focus on the proof of Theorem 5.

Remark 8. The above results are for FNLS (2.1) (or (2.12)) with the defocusing nonlinearity $|u|^2u$. The local-in-time theory still holds if we replace the defocusing nonlinearity with the focusing one $-|u|^2u$. In the focusing case, to use Bourgain's invariant measure argument extending local-in-time dynamics globally, one needs to construct the associated focusing Gibbs measure. The authors [69] constructed the focusing Gibbs measure for

(2.1). See also [100] for a new construction of the focusing Gibbs measure from quantum many-body mechanics.

As pointed out by Sun-Tzvetkov [105], the second Picard's iteration enjoys some smoothing effect when $\alpha > 1$. We further observe that the high-high interaction of the second Picard iterate exhibits an even stronger smoothing property, see Theorem 7 (i) and also Subsubsection 2.6.2, which plays an important role in our analysis. However, when $\alpha = 1$, the second Picard iterate fails to gain any smoothing, see Theorem 7 (ii). To illustrate these ideas, we consider the following truncated version of the second Picard iterate of high high interaction. Given $N \in \mathbb{N}$, define the following Picard second iterate

$$Z_N^{(2)}(t) = \int_0^t \mathrm{e}^{\mathrm{i}(t-t')\mathrm{D}^\alpha} \Pi_N \left[\left(|z_N(t')|^2 - 2\int_{\mathbb{T}} |z_N(t')|^2 \mathrm{d}x \right) z_N(t') \right] \mathrm{d}t', \tag{2.19}$$

where

$$z_N(t) = \sum_{N^{1-\delta} < k < N} \frac{g_k(\omega)}{[\![k]\!]^{\alpha/2}} \mathrm{e}^{\mathrm{i}t|k|^\alpha + \mathrm{i}kx}$$

is the truncation of the random linear solution. In (2.19), we only see the high-high interactions, i.e. $N_{\mathrm{med}} \gtrsim N_{\mathrm{max}}^{1-\delta}$. The following theorem shows a sharp contrast of the smoothing property of (2.19) for $\alpha > 1$ and for $\alpha = 1$, respectively. More precisely, we have the following result.

Theorem 7. *With the above notation and $|t| \ll 1$, we have*

(i) *When $\alpha > 1$, we have $\|Z_N^{(2)}(t)\|_{L^2(\mathbb{T})} \lesssim N^{-\frac{1}{2}-}$;*

(ii) *When $\alpha = 1$, we have $\|Z_N^{(2)}(t)\|_{L^2(\mathbb{T})} \gtrsim (\log N)^3$.*

Remark 9. Theorem 7 shows a dramatic change in the regularity of the second Picard iterate between $\alpha > 1$ and $\alpha = 1$. In particular, Theorem 7 quantifies the phase transition of the smoothing effect for Picard's iterates at $\alpha = 1$. As the smoothing property of the second Picard iterate plays a crucial rule in the proof of Theorem 5, Theorem 7 also implies that the strategy in proving Theorem 5 does not work for the half-wave equation, i.e. (2.12) with $\alpha = 1$.

Remark 10. Picard's iterate's lack of smoothing property has been also observed for other weakly dispersive models. Oh [80] considered the Szegö equation and proved that the first nontrivial Picard's iterate does not gain regularity compared to the initial data. See also Camps-Gassot-Ibrahim [28] for a similar observation for the cubic Schrödinger half-wave equation.

2.1.4 Main ideas

We explain the key ideas in proving Theorem 5, i.e. how to construct local-in-time solutions to (2.1) with initial data distributed according to the Gibbs measure. We note that the Gibbs measure $\mathrm{d}\rho$ in (2.9) (or in (2.10)) is absolutely continuous with respect to the Gaussian free measure $\mathrm{d}\mu$ in (2.6). Therefore, to prove Theorem 5, it only suffices to consider (2.1) with initial data distributed according to $\mathrm{d}\mu$, i.e. $u|_{t=0} = u_0^\omega$ given in (2.7). The main difference between the weakly dispersive case $\alpha \in (1, 2)$ and the standard case $\alpha = 2$ comes from the counting estimates. For instance, the values of

$$|k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k|^\alpha$$

may be dense in an interval of size 1, under the constraint of $(k_1, k_2, k_3) \in \Gamma(k)$. This causes the loss of regularity in establishing linear and multilinear estimates regarding solutions to (2.12).

The ansatz

In this subsubsection, we recall the random averaging operator approach. The idea is to include the high-low interactions in the ansatz, and write them as a low-frequency operator applied to the high-frequency. See [34] for further discussion.

To construct the solution to (2.18) locally, we introduce a time cut-off. Let η be any Schwartz function such that $\eta(t) = 1$ for $|t| \le 1$ and $\eta(t) = 0$ for $|t| \ge 2$. Define

$\eta_\delta(t) = \eta(\delta^{-1}t)$ for $\delta \in \mathbb{R}_+$. To simplify the notation, we will denote

$$\chi(t) = \eta_T(t) = \eta\left(\frac{t}{T}\right), \tag{2.20}$$

for some $T \ll 1$ to be determined later. Applying the interaction picture[1] to the unknown $u_N(t)$, still denoted by $u_N(t)$,

$$u_N(t) \leftarrow \mathrm{e}^{\mathrm{i}t\mathrm{D}^\alpha} u_N(t), \tag{2.21}$$

the Duhamel formulation of (2.18), restricted to $|t| \leq T$, becomes

$$u_N(t) = \chi(t)\Pi_N u_0^\omega - \mathrm{i}\chi(t)\int_0^t \Pi_N \mathcal{M}(u_N)(t')\mathrm{d}t' + \mathrm{i}\chi(t)\int_0^t \Pi_N \mathcal{R}(u_N)(t')\mathrm{d}t', \tag{2.22}$$

where $\mathcal{R}$ is as in (2.14) attaching a factor $\chi(t)$, and $\mathcal{M}(u) = \mathcal{M}(u,u,u)$ is the trilinear form defined by

$$\mathcal{M}(u,v,w)_k(t) = \chi(t) \sum_{\substack{k_1-k_2+k_3=k \\ k_2 \notin \{k_1,k_3\}}} \mathrm{e}^{\mathrm{i}t\Phi} \cdot u_{k_1}(t)\overline{v_{k_2}}(t) w_{k_3}(t), \tag{2.23}$$

where

$$\Phi := |k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k|^\alpha. \tag{2.24}$$

In what follows, we focus on the formulation (2.22) in proving Theorem 5. We also note that $u_{1/2} = 0$.

Let y_N be as

$$y_N = u_N - u_{N/2}. \tag{2.25}$$

[1]It is a terminology from quantum mechanics. In quantum mechanics, when dealing with interaction, one usually linearly evolves the observables and evolves the state (wave function) by the remaining perturbation Hamiltonian. It is also called "Dirac picture".

Then from (2.22) we see that y_N in (2.25) satisfies

$$\begin{aligned} y_N(t) =&\chi(t)F_N - \mathrm{i}\sum_{N_{\max}=N}\chi(t)\int_0^t \Pi_N\mathcal{M}(y_{N_1},y_{N_2},y_{N_3})(t')\mathrm{d}t' \\ &- \mathrm{i}\sum_{N_{\max}\leq N/2}\chi(t)\int_0^t \Delta_N\mathcal{M}(y_{N_1},y_{N_2},y_{N_3})(t')\mathrm{d}t' \\ &+ \mathrm{i}\chi(t)\int_0^t(\Pi_N\mathcal{R}(u_N)-\Pi_{N/2}\mathcal{R}(u_{N/2}))(t')\mathrm{d}t', \end{aligned} \tag{2.26}$$

where

$$F_N = \Delta_N u_0^\omega := \Pi_N u_0^\omega - \Pi_{N/2}u_0^\omega, \tag{2.27}$$

with u_0^ω defined in (2.7) and $N_{\max} = \max(N_1, N_2, N_3)$. To construct y_N perturbatively, the main difficulty comes from the low-low-high interactions of the second term in (2.26). To remove this bad interaction, we introduce a random averaging operator. To be more precise, let $0 < \delta \ll 1$ be determined later, and denote

$$L_N = L(N) := \max\{L \in 2^{\mathbb{Z}}; L < N^{1-\delta}\}. \tag{2.28}$$

We define ψ_{N,L_N} to be the solution to the linear equation

$$\psi_{N,L_N}(t) = \chi(t)F_N - 2\mathrm{i}\chi(t)\int_0^t \Pi_N\mathcal{M}(u_{L_N},u_{L_N},\psi_{N,L_N})(t')\mathrm{d}t', \tag{2.29}$$

which is expected to consist of all bad interactions from y_N. We then further decompose y_N into

$$y_N = \psi_{N,L_N} + z_N. \tag{2.30}$$

It is therefore expected that the remainder z_N behaves better than y_N. From (2.26), (2.28), (2.29), and (2.30), we note that the remainder z_N satisfies

$$\begin{aligned}
z_N(t) = &- \mathrm{i} \sum_{N_{\max}=N_{\mathrm{med}}=N} \chi(t) \int_0^t \Pi_N \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})(t')\mathrm{d}t' \\
&- 2\mathrm{i} \sum_{N_1,N_2 \le N/2} \chi(t) \int_0^t \Pi_N \mathcal{M}(y_{N_1}, y_{N_2}, z_N)(t')\mathrm{d}t' \\
&- 2\mathrm{i} \sum_{L_N < N_{\mathrm{med}} \le N/2} \chi(t) \int_0^t \Pi_N \mathcal{M}(y_{N_1}, y_{N_2}, \psi_{N,L_N})(t')\mathrm{d}t' \\
&- \mathrm{i} \sum_{N_1,N_3 \le N/2} \chi(t) \int_0^t \Pi_N \mathcal{M}(y_{N_1}, z_N, y_{N_3})(t')\mathrm{d}t' \\
&- \mathrm{i} \sum_{N_1,N_3 \le N/2} \chi(t) \int_0^t \Pi_N \mathcal{M}(y_{N_1}, \psi_{N,L_N}, y_{N_3})(t')\mathrm{d}t' \\
&- \mathrm{i} \sum_{N_{\max} \le N/2} \chi(t) \int_0^t \Delta_N \mathcal{M}(y_{N_1}, y_{N_2}, y_{N_3})(t')\mathrm{d}t' \\
&+ \mathrm{i}\chi(t) \int_0^t (\Pi_N \mathcal{R}(u_N) - \Pi_{N/2} \mathcal{R}(u_{N/2}))(t')\mathrm{d}t'.
\end{aligned} \tag{2.31}$$

The equation (2.31) will be our main concern when constructing z_N. The main advantage of this formulation is the appearance of the lower bounds of N_{med} in the third term of the right-hand side of (2.31), i.e. $L_N < N_{\mathrm{med}} \le N/2$. Therefore, there is essentially no low-low-high interaction from this formulation. This lower bound of N_{med} comes from the removal of ψ_{N,L_N} from y_N. In what follows, we shall construct z_N in an induction manner. See Section 2.3 for further discussion.

The random averaging operator

Let us have a closer look at ψ_{N,L_N} defined in (2.29). For any $0 \le L \le N/2$ of dyadic $L \in 2^{\mathbb{Z}_{\ge 0}} \cup \{0\}$, consider the linear equation for Ψ:

$$\partial_t \Psi(t) = -2\mathrm{i}\Pi_N \mathcal{M}(u_L, u_L, \Psi)(t), \tag{2.32}$$

where u_L are solutions to (2.22) with N being replaced by L, and $\mathcal{M}$ is the trilinear form given in (2.23). We remark that $u_L(t)$, the solution to (2.18), uniquely exists for $t \in \mathbb{R}$.

If (2.32) has initial data $\Psi(0) = \Psi_0$, then we can rewrite (2.32) as

$$\Psi(t) - \chi(t)\Psi_0 = \mathcal{P}^{N,L}[\Psi](t), \tag{2.33}$$

for $|t| \leq T$, where $\mathcal{P}^{N,L} : X_N^b(J) \longrightarrow X_N^b(J)$, for $J = [-T, T]$, are the linear operators defined by

$$\mathcal{P}^{N,L}[\Psi](t) = -2\mathrm{i}\chi(t) \int_0^t \Pi_N \mathcal{M}(u_L, u_L, \Psi)(t')\mathrm{d}t'. \tag{2.34}$$

Here the space $X_N^b(J)$ is the projection of $X^b(J)$ to its finite Fourier mode, i.e. $\langle k \rangle \leq N$. The operator $\mathcal{P}^{N,L}$ is known as the random averaging operator, which roughly acts as averaging over low-frequency objects. Once the operator $\mathcal{P}^{N,L}$ is properly controlled, as in Definition 5, we can solve (2.33) and get

$$\Psi(t) = \mathcal{H}^{N,L}[\chi(\cdot)\Psi_0](t), \tag{2.35}$$

where

$$\mathcal{H}^{N,L} := (1 - \mathcal{P}^{N,L})^{-1}. \tag{2.36}$$

We denote $H_{kk'}^{N,L}$ the kernel of $\mathcal{H}^{N,L}$ in the following sense, for a function $u : \mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{C}$,

$$\mathcal{F}_t(\mathcal{H}^{N,L}[u])_k(\tau) = \int_{\mathbb{R}} \sum_{k'} \widehat{H_{kk'}^{N,L}}(\tau, \tau')\hat{u}_{k'}(\tau')\mathrm{d}\tau'.$$

Then, the solution (2.35) can be expressed as

$$\begin{aligned}
\Psi(t,x) =& \mathcal{H}^{N,L}[\chi(\cdot)\Psi_0](t,x) \\
=& c\int_{\mathbb{R}} \sum_k \left[\sum_{k'} \left(\int_{\mathbb{R}} \widehat{H_{kk'}^{N,L}}(\tau, \tau')\hat{\chi}(\tau')\mathrm{d}\tau' \right) (\Psi_0)_{k'} \right] \mathrm{e}^{\mathrm{i}(kx+t\tau)}\mathrm{d}\tau.
\end{aligned}$$

In what follows, we set the initial data $\Psi_0 = F_N$ defined in (2.27), and we denote $\psi_{N,L}$ to be the corresponding solution, i.e.

$$\begin{aligned}\psi_{N,L}(t,x) =&\mathcal{H}^{N,L}[\chi(\cdot)F_N](t,x)\\ =&c\int_{\mathbb{R}}\sum_{k}\left[\sum_{\frac{N}{2}<\langle k'\rangle\le N}\left(\int_{\mathbb{R}}\widehat{H_{kk'}^{N,L}}(\tau,\tau')\hat{\chi}(\tau')\mathrm{d}\tau'\right)\frac{g_{k'}}{[\![k']\!]^{\frac{\alpha}{2}}}\right]\mathrm{e}^{\mathrm{i}(kx+t\tau)}\mathrm{d}\tau.\end{aligned} \tag{2.37}$$

From (2.33) and (2.34), we note that $\psi_{N,L}$ defined in (2.37) coincides with that in (2.29) when L is chosen to be L_N in (2.28). We define the operator

$$\mathfrak{h}^{N,L} := \mathcal{H}^{N,L} - \mathcal{H}^{N,L/2}. \tag{2.38}$$

By denoting $\zeta^{N,L} := \psi_{N,L} - \psi_{N,L/2}$, we have

$$\zeta^{N,L} := \psi_{N,L} - \psi_{N,L/2} = \mathfrak{h}^{N,L}[\chi(\cdot)F_N]. \tag{2.39}$$

We also define the flow map version of the operator $\mathcal{H}^{N,L}$, denoted by $\widetilde{\mathcal{H}}^{N,L}(t)$ and defined as

$$\widetilde{\mathcal{H}}^{N,L}(t)[\phi](x) := \mathcal{H}^{N,L}[\chi(\cdot)\phi](t,x) = \sum_{k}\mathrm{e}^{\mathrm{i}kx}\sum_{k'}\tilde{H}_{kk'}^{N,L}(t)\phi_{k'}, \tag{2.40}$$

where $\{\tilde{H}_{kk'}^{N,L}(t)\}_{kk'}$ is the kernel of the operator $\widetilde{\mathcal{H}}^{N,L}$. Then denote

$$\tilde{\mathfrak{h}}^{N,L}(t) := \widetilde{\mathcal{H}}^{N,L}(t) - \widetilde{\mathcal{H}}^{N,L/2}(t), \tag{2.41}$$

whose kernel is given by

$$h_{kk'}^{N,L}(t) = \tilde{H}_{kk'}^{N,L}(t) - \tilde{H}_{kk'}^{N,L/2}(t). \tag{2.42}$$

One of the key observation is that $h_{kk'}^{N,L}$ and $\tilde{H}_{kk'}^{N,L}$ are Borel functions of $\{g_k(\omega)\}_{\langle k\rangle\le L}$, and are thus independent of the Gaussians F_N given by (2.27) and (2.7). Such an independent structure enables one to use probabilistic tools to exploit the high-moment cancellation structure of F_N, see Proposition 4.

The solution structure

We note that, from (2.30), (2.39), and the fact that $\psi_{N,1/2} = F_N$, we have the following formal expansion of y_N on $[-T, T]$.

$$\begin{aligned} y_N(t) =& \chi(t)\psi_{N,L_N}(t) + z_N(t) \\ =& \chi(t)F_N + \sum_{1\le L\le L_N} \zeta^{N,L}(t) + z_N(t) \\ =& \chi(t)F_N + \sum_{1\le L\le L_N} \mathfrak{h}^{N,L}[\chi(\cdot)F_N](t) + z_N(t) \\ =& \chi(t)F_N + \sum_{1\le L\le L_N} \tilde{\mathfrak{h}}^{N,L}[F_N](t) + z_N(t). \end{aligned} \tag{2.43}$$

We will construct z_N and $\mathfrak{h}^{N,L}$, for all $N \ge \frac{1}{2}$ and $1 \le L \le L_N$, by an induction argument. As a matter of fact, the random averaging operator $\mathfrak{h}^{N,L}$ in (2.43) for $L \le L_N$ only depend on $(u_L)_{L\ll N}$, and thus $(y_L)_{L\ll N}$ due to (2.25). Then, from (2.30), we see that $(y_L)_{L\ll N}$ are again determined by $(z_{L'})_{1\le L'\le L}$ and $(\mathfrak{h}^{L',R'})_{L'\le L, R'<(L')^{1-\delta}}$. In all, we see that $\mathfrak{h}^{N,L}$ is determined by $(z_{L'})_{L'\ll N}$ and $(\mathfrak{h}^{L',R'})_{L',R'\ll N}$. On the other hand, from (2.31), we see that z_N depends on $(z_{N'})_{N'\le N}$ and $(\mathfrak{h}^{N',L})_{N'\le N, L<(N')^{1-\delta}}$. See Definition 5 for further details.

Finally, from (2.25), (2.43), and the fact that $u_{1/2} = F_{1/2} = 0$, we arrive at:

$$u(t) = \chi(t)u_0^\omega + \sum_{N\in 2^{\mathbb{Z}_{\ge 0}}} \sum_{1\le L\le L_N} \tilde{\mathfrak{h}}^{N,L}[F_N](t) + \sum_{N\in 2^{\mathbb{Z}_{\ge 0}}} z_N(t), \tag{2.44}$$

provided the summations converge in a proper sense. From (2.44), we see that the expected solution u in Theorem 5 has the following structure

$$u = \text{random linear term } + \text{ random average term } + \text{smoother term}.$$

The random average term bridges the random linear term and the smoother term. On the one hand, it records all quasilinear roughness from u, thus guaranteeing the "smoother term" has higher regularity. On the other hand, it preserves the independent structure of the initial data, such that we may apply probabilistic arguments.

2.2 Preliminary

2.2.1 Counting estimates

Counting estimates are crucial in our analysis. In this subsection, we shall show some almost sharp counting estimates of solutions to some fractional Diophantine equations. To be more precise, given dyadic numbers N, N_1, N_2, $N_3 > 0$ and some fixed number $m \in \mathbb{R}$, define the set

$$\begin{aligned} S =& \{(k, k_1, k_2, k_3) \in \mathbb{Z}^4; k_2 \notin \{k_1, k_3\}, k = k_1 - k_2 + k_3, \\ & |k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k|^\alpha = m + O(1), |k| \leq N, |k_j| \leq N_j \text{ for } j \in \{1, 2, 3\}\}. \end{aligned} \tag{2.45}$$

which is a subset of the hyperplane $k = k_1 - k_2 + k_3$ in $\mathbb{Z}^4$. Denote by S_k the set of $(k_1, k_2, k_3) \in \mathbb{Z}^3$ such that $(k, k_1, k_2, k_3) \in S$ when k is fixed. It is easy to see that

$$S_k \subset \Gamma(k),$$

where $\Gamma(k)$ is the hyperplane of $\mathbb{Z}^3$ defined in (2.15). Similarly, we can define S_{k_1}, S_{kk_1}, $S_{kk_1k_2}$, and etc. For a set A, we use $|A|$ or $\#A$ to denote its cardinality. If A is a finite set, then $|A|$ denotes the number of elements of A. The main purpose of this subsection is to count the cardinality of these sets defined above.

We start with a basic counting estimate from [105, Lemma 2.6].

Lemma 6. Let I, J be two intervals and ϕ be a real-valued C^1 function defined on I. Then, we have

$$|\{k \in I \cap \mathbb{Z}; \phi(k) \in J\}| \lesssim 1 + \frac{|J|}{\inf_{\xi \in I} |\phi'(\xi)|}.$$

In this thesis, we will focus on a special function ϕ. Let

$$\phi_{b,\pm}(x) = |x|^\alpha \pm |x-b|^\alpha, \tag{2.46}$$

where $\alpha \in (1,2)$ and $b \in \mathbb{R}$. Then, we see that $\phi_{b,\pm} \in C^1$ for $\alpha > 1$, and we have

$$\phi'_{b,\pm}(x) = \alpha(\operatorname{sgn}(x)|x|^{\alpha-1} \pm \operatorname{sgn}(x-b)|x-b|^{\alpha-1}). \tag{2.47}$$

Then, we have the following elementary estimate.

Lemma 7. Let $\phi_{b,\pm}$ be as in (2.46).

(i) Let $\alpha \in (1,2)$. Then, we have

$$|\phi'_{b,-}(x)| \gtrsim_\alpha \min(|b||x|^{\alpha-2}, |b|^{\alpha-1}) \tag{2.48}$$

provided $x \neq 0$.

(ii) Let $\alpha \in (1,2)$ and $|b| \geq 1$. Then, we have

$$|\phi'_{b,+}(x)| \gtrsim_\alpha |b|^{\alpha-1} \tag{2.49}$$

for $|2x-b| \gtrsim b$. For $|2x-b| \ll |b|$ if we further assume that $|2x-b| \gtrsim |b|^{1-\frac{\alpha}{2}}$, we have

$$|\phi'_{b,+}(x)| \gtrsim_\alpha |b|^{\frac{\alpha}{2}-1}. \tag{2.50}$$

Proof. (i) We first consider $b < 0$, for which we use (2.47) to get

$$\phi'_{b,-}(x) = \begin{cases} \alpha(-|x|^{\alpha-1} + |x-b|^{\alpha-1}), & x \in (-\infty, b]; \\ \alpha(-|x|^{\alpha-1} - |x-b|^{\alpha-1}), & x \in (b, 0); \\ \alpha(|x|^{\alpha-1} - |x-b|^{\alpha-1}), & x \in (0, +\infty). \end{cases}$$

When $x \in (b, 0)$, it is easy to see that

$$|\phi'_{b,-}(x)| = \alpha(|x|^{\alpha-1} + |x-b|^{\alpha-1}) \gtrsim_\alpha |b|^{\alpha-1},$$

due to the fact that $\alpha > 1$, which is sufficient for (2.48). We turn to the case when $x \in (-\infty, b] \cup (0, +\infty)$, where we have that x and $x - b$ have the same sign. Therefore, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned} |\phi'_{b,-}(x)| &= \alpha(\alpha-1)\left|\int_b^0 |x-b+s|^{\alpha-2}\mathrm{d}s\right| \\ &\gtrsim_\alpha |b| \cdot \min(|x|^{\alpha-2}, |x-b|^{\alpha-2}), \end{aligned} \tag{2.51}$$

where we used the fact that $\alpha - 2 < 0$. If $x \in (-\infty, b]$, then $|x - b| \leq |x|$ and thus $\min(|x|^{\alpha-2}, |x-b|^{\alpha-2}) = |x|^{\alpha-2}$, which, together with (2.51), gives (2.48). In the following, let us assume that $x \in (0, \infty)$. Then we have $|x - b| \geq |x|$. If we further assume that $|x - b| \sim |x|$, then $\min(|x|^{\alpha-2}, |x-b|^{\alpha-2}) \gtrsim |x|^{\alpha-2}$, which is again sufficient for (2.48). It remains to consider the case $|x - b| \gg |x|$, which implies $|x| \ll |b|$ and thus $|x - b| \sim |b|$. Therefore, we have $\min(|x|^{\alpha-2}, |x-b|^{\alpha-2}) = |x-b|^{\alpha-2} \sim |b|^{\alpha-2}$, which, together with (2.51), gives (2.48) again. Thus we finish the proof of (2.48) for $b < 0$. The proof for the case when $b > 0$ is similar and therefore omitted.

(ii) We only consider the case $b < 0$, as the proof for the case $b > 0$ is similar. From (2.47), we have

$$\phi'_{b,+}(x) = \begin{cases} \alpha(-|x|^{\alpha-1} - |x-b|^{\alpha-1}), & x \in (-\infty, b]; \\ \alpha(-|x|^{\alpha-1} + |x-b|^{\alpha-1}), & x \in (b, 0); \\ \alpha(|x|^{\alpha-1} + |x-b|^{\alpha-1}), & x \in (0, +\infty). \end{cases} \tag{2.52}$$

When $x \in (-\infty, b] \cup (0, \infty)$, from (2.52) we have

$$|\phi'_{b,+}(x)| = \alpha||x|^{\alpha-1} + |b-x|^{\alpha-1}| \gtrsim_\alpha |b|^{\alpha-1},$$

since $\alpha > 1$. From now on, we assume $x \in (b, 0]$. From (2.52), we have

$$\begin{aligned} |\phi'_{b,+}(x)| =& \alpha \big| |x|^{\alpha-1} - |x-b|^{\alpha-1} \big| \\ =& \alpha(\alpha-1) \left| \int_x^0 t^{\alpha-2} \mathrm{d}t - \int_0^{x-b} t^{\alpha-2} \mathrm{d}t \right| \\ =& \alpha(\alpha-1) \left| \int_{-x}^{x-b} t^{\alpha-2} \mathrm{d}t \right| \\ \geq& \alpha(\alpha-1) |2x-b| |b|^{\alpha-2}, \end{aligned} \tag{2.53}$$

where we used the facts that $|x|$, $|x-b| \leq |b|$. It is easy to see that (2.53) implies (2.49), provided $|2x-b| \gtrsim |b|$. On the other hand, if $|2x-b| \gtrsim |b|^{1-\frac{\alpha}{2}}$, we have

$$\alpha(\alpha-1)|2x-b||b|^{\alpha-2} \gtrsim |b|^{\frac{\alpha}{2}-1}, \tag{2.54}$$

which, together with (2.53), gives (2.50). Thus we finish the proof of (2.50). □

Now we are ready to show our main counting estimates.

Lemma 8. Let $\alpha \in (1,2)$ and $1 \leq N_1, N_2, N_3 \leq N$. Then we have the following counting estimates

$$\begin{aligned} |S_{kk_1}| \lesssim& (N_2 \wedge N_3)^{2-\alpha} \langle k_1 - k \rangle^{-1} + 1; \\ |S_{kk_3}| \lesssim& (N_1 \wedge N_2)^{2-\alpha} \langle k - k_3 \rangle^{-1} + 1; \\ |S_{k_1k_2}| \lesssim& N_3^{2-\alpha} \langle k_1 - k_2 \rangle^{-1} + 1; \\ |S_{k_2k_3}| \lesssim& N_1^{2-\alpha} \langle k_2 - k_3 \rangle^{-1} + 1, \end{aligned} \tag{2.55}$$

where $\langle k \rangle = (1 + |k|^2)^{1/2}$ is given in (1.53).

Proof. We start with the bound for $|S_{kk_1}|$. We first note that $k_2 \notin \{k_1, k_3\}$ is equivalent to $k_2 \notin \{k_1, k_3\}$ on the hyperplane $k = k_1 - k_2 + k_3$. Moreover, on this hyperplane, for fixed (k, k_1) (such that $|k| \leq N$, $|k_1| \leq N_1$, and $k \neq k_1$), once we further fix k_3, then k_2 is uniquely determined. Therefore, we have

$$
\begin{aligned}
|S_{kk_1}| =&|\{k_3 \in \mathbb{Z}; k_3 \neq k, |k_3| \leq N_3, |k_1+k_3-k| \leq N_2,\\
&\qquad |k_3|^\alpha - |k_1+k_3-k|^\alpha = |k|^\alpha - |k_1|^\alpha - m + O(1)\}|\\
=&|\{k_3 \in \mathbb{Z}; k_3 \neq k, |k_3| \leq N_3, |k_1+k_3-k| \leq N_2,\\
&\qquad \phi_{k-k_1,-}(k_3) = |k|^\alpha - |k_1|^\alpha - m + O(1)\}|,
\end{aligned}
$$

where $\phi_{k-k_1,-}$ is given in (2.46) with $b = k-k_1$. We now apply Lemma 6 and then Lemma 7 to get

$$
\begin{aligned}
|S_{kk_1}| \lesssim& 1 + \frac{1}{\inf_{|k_3|\lesssim N_3} |\phi'_{k-k_1,-}(k_3)|}\\
\lesssim& 1 + (\min(\inf_{|k_3|\leq N_3} |k-k_1||k_3|^{\alpha-2}, |k-k_1|^{\alpha-1}))^{-1} \qquad (2.56)\\
\lesssim& 1 + \langle k-k_1\rangle^{-1} N_3^{2-\alpha},
\end{aligned}
$$

where we use $k \neq k_1$ and $\alpha \in (1,2)$.

By swapping k_2 and k_3 in the above argument, we have

$$
\begin{aligned}
|S_{kk_1}| =&|\{k_2 \in \mathbb{Z}; k_2 \neq k_1, |k_2| \leq N_2, |k_1-k_2-k| \leq N_3,\\
&\qquad |k_2|^\alpha - |k_1-k_2-k|^\alpha = |k_1|^\alpha - |k|^\alpha + m + O(1)\}|\\
=&|\{k_2 \in \mathbb{Z}; k_2 \neq k_1, |k_2| \leq N_2, |k_1-k_2-k| \leq N_3,\\
&\qquad \phi_{k_1-k,-}(k_2) = |k_1|^\alpha - |k|^\alpha + m + O(1)\}|.
\end{aligned}
$$

Then by the same argument as in (2.56), we get

$$
|S_{kk_1}| \lesssim 1 + \langle k-k_1\rangle^{-1} N_2^{2-\alpha}. \qquad (2.57)
$$

Thus we finish the proof of (2.55) for the bound S_{kk_1} by combining (2.56) and (2.57).

The proof for the rest of (2.55) is similar; thus, we omit their details. □

We point out that the decay like $\langle k-k_1\rangle^{-1}$ in (2.55) plays a crucial role in our later analysis. We have the following estimates as a consequence of Lemma 8.

Corollary 1. Let $\alpha \in (1,2)$ and $1 \le N_1, N_2, N_3 \le N$. Then, we have

$$\begin{aligned} |S_k| &\lesssim \min((N_2 \wedge N_3)^{2-\alpha} \log N_1 + N_1, (N_1 \wedge N_2)^{2-\alpha} \log N_3 + N_3); \\ |S_{k_1}| &\lesssim N_3^{2-\alpha} \log N_2 + N_2 \\ |S_{k_2}| &\lesssim \min(N_3^{2-\alpha} \log N_1 + N_1, N_1^{2-\alpha} \log N_3 + N_3); \\ |S_{k_3}| &\lesssim N_1^{2-\alpha} \log N_2 + N_2. \end{aligned} \tag{2.58}$$

Proof. We start with bound for $|S_k|$. We first note that

$$|S_k| \lesssim \min\Bigg(\sum_{|k_1| \lesssim N_1} |S_{kk_1}|, \sum_{|k_3| \le N_3} |S_{kk_3}| \Bigg). \tag{2.59}$$

By using Lemma 8, we have

$$\begin{aligned} \sum_{|k_1| \le N_1} |S_{kk_1}| \lesssim & (N_2 \wedge N_3)^{2-\alpha} \sum_{|k_1| \le N_1} \langle k_1 - k \rangle^{-1} + \sum_{|k_1| \le N_1} 1 \\ \lesssim & (N_2 \wedge N_3)^{2-\alpha} \log N_1 + N_1, \end{aligned} \tag{2.60}$$

and similarly,

$$\begin{aligned} \sum_{|k_3| \le N_3} |S_{kk_3}| \lesssim & (N_1 \wedge N_2)^{2-\alpha} \sum_{|k_3| \le N_3} \langle k - k_3 \rangle^{-1} + \sum_{|k_1| \le N_3} 1 \\ \lesssim & (N_1 \wedge N_2)^{2-\alpha} \log N_3 + N_3. \end{aligned} \tag{2.61}$$

Therefore, the first bound in (2.58) follows from (2.59), (2.60), and (2.61).

By a similar argument, we have

$$\begin{aligned} |S_{k_1}| &\lesssim \sum_{|k_2| \le N_2} |S_{k_1 k_2}| \\ &\lesssim N_3^{2-\alpha} \sum_{|k_2| \le N_2} \langle k_1 - k_2 \rangle^{-1} + \sum_{|k_2| \le N_2} 1 \\ &\lesssim N_3^{2-\alpha} \log N_2 + N_2. \end{aligned}$$

Thus, we finish the proof for the second estimate of (2.58).

The proof for the rest of (2.58) can be handled similarly. Thus, we omit them. □

For the countings of $|S_{k_1k_3}|$ and $|S_{kk_2}|$, the argument in Lemma 8 is not sufficient in dealing with low-high-low interaction. See Subsection 2.6.4. We need the following further decomposition.

$$\begin{cases} S_{k_1k_3}^{\text{bad}} = \{(k,k_2) \in S_{k_1k_3}; |2k-(k_1+k_3)| \ll |k_1+k_3|\}; \\ S_{k_1k_3}^{\text{good}} = \{(k,k_2) \in S_{k_1k_3}; |2k-(k_1+k_3)| \gtrsim |k_1+k_3|\}; \\ S_{kk_2}^{\text{bad}} = \{(k_1,k_3) \in S_{kk_2}; |2k_1-(k+k_2)| \ll |k+k_2|\}; \\ S_{kk_2}^{\text{good}} = \{(k_1,k_3) \in S_{kk_2}; |2k_1-(k+k_2)| \gtrsim |k+k_2|\}. \end{cases} \tag{2.62}$$

For low-high-low interaction that we will encounter in Subsection 2.6.4, k and k_2 must have different signs, which results in the following improved countings estimates.

Lemma 9. Let $\alpha \in (1,2)$. Then we have the following counting estimates

$$\begin{aligned} |S_{kk_2}^{\text{good}}|, |S_{k_1k_2}^{\text{good}}| &\lesssim 1; \\ |S_{kk_2}^{\text{bad}}| &\lesssim |k+k_2|^{1-\frac{\alpha}{2}} + 1; \\ |S_{k_1k_3}^{\text{bad}}| &\lesssim |k_1+k_3|^{1-\frac{\alpha}{2}} + 1. \end{aligned} \tag{2.63}$$

Proof. We start with the estimate of $|S_{kk_2}^{\text{good}}|$ with $k+k_2 \neq 0$. Similar as in the proof of Lemma 8, we may recast the set $S_{kk_2}^{\text{good}}$ as

$$\begin{aligned} |S_{kk_2}^{\text{good}}| =& |S_{kk_2} \cap \{(k_1,k_3) \in S_{kk_2}; k = k_1 - k_2 + k_3, \\ & |2k_1 - (k+k_2)| \gtrsim |k+k_2|\}| \\ =& |\{k_1 \in \mathbb{Z}; k_1 \neq k, |k_1| \leq N_1, |k_1 - k_2 - k| \leq N_3, \\ & \phi_{k+k_2,+}(k_1) = |k|^\alpha + |k_2|^\alpha - m + O(1), \\ & |2k_1 - (k+k_2)| \gtrsim |k+k_2|\}|, \end{aligned}$$

where $\phi_{k+k_2,+}$ is given in (2.46) with $b = k+k_2$. Thus by using Lemma 6 and then (2.49), we have

$$|S^{\text{good}}_{kk_2}| \lesssim 1 + \frac{1}{|k+k_2|^{\alpha-1}} \lesssim 1$$

since $k + k_2 \neq 0$. Thus, we finish the proof of the bound for $|S^{\text{good}}_{kk_2}|$.

Now we turn to $|S^{\text{bad}}_{kk_2}|$. We distinguish two cases. We first consider the case when $|2k_1 - (k+k_2)| \leq |k+k_2|^{1-\frac{\alpha}{2}}$, for which we have the crude estimate

$$\begin{aligned}
&|S^{\text{bad}}_{kk_2} \cap \{(k_1,k_3) \in S_{kk_2}; |2k_1 - (k+k_2)| \leq |k+k_2|^{1-\frac{\alpha}{2}}\}| \\
&\quad \leq |\{k_1 \in \mathbb{Z}; |2k_1 - (k+k_2)| \leq |k+k_2|^{1-\frac{\alpha}{2}}\}| \\
&\quad \lesssim |k+k_2|^{1-\frac{\alpha}{2}} + 1,
\end{aligned} \tag{2.64}$$

which is sufficient for our purpose. From now on, we assume $|2k_1 - (k+k_2)| > |k+k_2|^{1-\frac{\alpha}{2}}$. Similar to the above, we may rewrite

$$\begin{aligned}
&S^{\text{bad}}_{kk_2} \cap \{(k_1,k_3) \in S_{kk_2}; k = k_1 - k_2 + k_3, |2k_1 - (k+k_2)| > |k+k_2|^{1-\frac{\alpha}{2}}\} \\
&\quad = S_{kk_2} \cap \{(k_1,k_3) \in S_{kk_2}; k = k_1 - k_2 + k_3, \\
&\qquad\qquad |k+k_2|^{1-\frac{\alpha}{2}} \leq |2k_1 - (k+k_2)| \ll |k+k_2|\},
\end{aligned}$$

thereby getting

$$\begin{aligned}
&\big|S^{\text{bad}}_{kk_2} \cap \{(k_1,k_3) \in S_{kk_2}; k = k_1 - k_2 + k_3, |2k_1 - (k+k_2)| > |k+k_2|^{1-\frac{\alpha}{2}}\}\big| \\
&\quad = |\{k_1 \in \mathbb{Z}; k_1 \neq k, |k_1| \leq N_1, |k_1 - k_2 - k| \leq N_3, \\
&\qquad\qquad \phi_{k+k_2,+}(k_1) = |k|^\alpha + |k_2|^\alpha - m + O(1), \\
&\qquad\qquad |k+k_2|^{1-\frac{\alpha}{2}} \leq |2k_1 - (k+k_2)| \ll |k+k_2|\}|.
\end{aligned}$$

Then we may apply Lemma 6, together with (2.14), to get

$$\begin{aligned}
&\big|S^{\text{bad}}_{kk_2} \cap \{(k_1,k_3) \in S_{kk_2}; k = k_1 - k_2 + k_3, |2k_1 - (k+k_2)| > |k+k_2|^{1-\frac{\alpha}{2}}\}\big| \\
&\quad \lesssim |k+k_2|^{1-\frac{\alpha}{2}} + 1,
\end{aligned} \tag{2.65}$$

which is again sufficient for our purpose. By collecting (2.64) and (2.65), we proved the estimate for $|S^{\text{bad}}_{kk_2}|$.

The proof for the rest of (2.63) is similar and thus omitted. □

Corollary 2. The following bounds hold.

$$|S^{\text{bad}}_{kk_2}| \lesssim (N_1 \wedge N_3)^{1-\frac{\alpha}{2}}; \quad |S^{\text{bad}}_{k_1k_3}| \lesssim (N \wedge N_2)^{1-\frac{\alpha}{2}},$$

where $S^{\text{bad}}_{kk_2}$ and $S^{\text{bad}}_{k_1k_3}$ are defined in (2.62).

Proof. We only consider the estimate for $|S^{\text{bad}}_{kk_2}|$, as the proof for the bound of $S^{\text{bad}}_{k_1k_3}$ is similar. We first note that for $(k_1, k_3) \in S^{\text{bad}}_{kk_2}$, we have

$$|2k_3 - (k + k_2)| = |2k_1 - (k + k_2)| \ll |k + k_2|,$$

which implies that

$$|k_1| \sim |k + k_2| \sim |k_3|. \tag{2.66}$$

Then recall that $|k_1| \leq N_1$ and $|k_3| \leq N_3$, which together with Lemma 9 and (2.66), implies

$$|S^{\text{bad}}_{kk_2}| \lesssim |k + k_2|^{1-\frac{\alpha}{2}} + 1 \lesssim \min(N_1^{1-\frac{\alpha}{2}}, N_3^{1-\frac{\alpha}{2}}).$$

Thus we finish the proof. □

Remark 11. As a consequence of Lemma 9 and Corollary 2, we have shown that

$$|S_{kk_2}| \lesssim (N_1 \wedge N_3)^{1-\frac{\alpha}{2}}, \quad |S_{k_1k_3}| \lesssim (N \wedge N_2)^{1-\frac{\alpha}{2}}.$$

With this improvement, we may reduce further upper bounds for $|S_{k_1}|$ and $|S_{k_3}|$ as follows

$$|S_{k_1}| \lesssim N_3(N \wedge N_2)^{1-\frac{\alpha}{2}}, \quad |S_{k_3}| \lesssim N_1(N \wedge N_2)^{1-\frac{\alpha}{2}}. \tag{2.67}$$

Finally, we are ready to estimate the size of $|S|$.

Lemma 10. Let S be given in (2.45) with $\alpha \in (1, 2)$ and $1 \leq N_1, N_2, N_3 \leq N$. Then, we have

$$\begin{aligned} |S| \lesssim \min(&N_3^{2-\alpha}(N_1 \wedge N_2)\log(N_1 \vee N_2) + N_1N_2, \\ &N_1^{2-\alpha}(N_2 \wedge N_3)\log(N_2 \vee N_3) + N_2N_3). \end{aligned} \tag{2.68}$$

Proof. We first observe that

$$|S| \lesssim \min\Bigg(\sum_{k_1,k_2} |S_{k_1k_2}|, \sum_{k_2,k_3} |S_{k_2k_3}|\Bigg). \tag{2.69}$$

We shall estimate the right-hand side of (2.69) term by term. From (2.55), we have

$$\begin{aligned} \sum_{k_1,k_2} |S_{k_1k_2}| &\lesssim N_3^{2-\alpha} \sum_{k_1,k_2} \langle k_1 - k_2\rangle^{-1} + \sum_{k_1,k_2} 1 \\ &\lesssim N_3^{2-\alpha} \sum_{|k_1| \leq N_1} \sum_{|k_2| \leq N_2} \langle k_1 - k_2\rangle^{-1} + N_1N_2 \\ &\lesssim N_3^{2-\alpha}(N_1 \wedge N_2)\log(N_1 \vee N_2) + N_1N_2. \end{aligned} \tag{2.70}$$

Similarly, we have

$$\sum_{k_2,k_3} |S_{k_2k_3}| \lesssim N_1^{2-\alpha}(N_2 \wedge N_3)\log(N_2 \vee N_3) + N_2N_3 \tag{2.71}$$

Then (2.68) follows from (2.70) and (2.71). □

Another structure that we shall exploit is the so-called Γ-condition. To be more precise, if there exists a positive number Γ such that

$$|k_{\max}| \leq \Gamma < |k|,$$

where $k_{\max}$ is the frequency corresponding to $N_{\max}$, then we call that S given in (2.45) satisfies the Γ-condition. To simplify the notation, let

$$B_\Gamma = \{(k, k_1, k_2, k_3) \in S; |k_{\max}| \le \Gamma < |k|\}. \tag{2.72}$$

From the fact that $k = k_1 - k_2 + k_3$, we see that

$$\begin{cases} \Gamma \le |k| \le \Gamma + 2N_{\mathrm{med}}; \\ \Gamma - 2N_{\mathrm{med}} \le |k_{\max}| \le \Gamma. \end{cases} \tag{2.73}$$

Thus, both $|k|$ and $|k_{\max}|$ locate in an interval of length $2N_{\mathrm{med}}$.

With the Γ-condition, we can improve the previous counting estimates. Denote by B_Γ^k the set of $(k_1, k_2, k_3) \in \mathbb{Z}^3$ such that $(k, k_1, k_2, k_3) \in B_\Gamma$ when k is fixed. Similarly, we can define $B_\Gamma^{k_1}$, $B_\Gamma^{k_2}$, $B_\Gamma^{k_3}$, and $B_\Gamma^{k_{\min}k_{\mathrm{med}}}$. It is easy to see that

$$B_\Gamma \subset S, \quad B_\Gamma^k \subset S_k, \quad B_\Gamma^{k_i} \subset S_{k_i}, \quad B_\Gamma^{k_{\min}k_{\mathrm{med}}} \subset S_{k_{\min}k_{\mathrm{med}}}. \tag{2.74}$$

Lemma 11. Let S be given in (2.45) with $\alpha \in (1, 2)$, B_Γ be as in (2.72), and $1 \le N_1, N_2, N_3 \le N$. Then, we have

$$|B_\Gamma| \lesssim N_{\min} N_{\mathrm{med}}; \tag{2.75}$$

$$|B_\Gamma^k|, |B_\Gamma^{k_i}| \lesssim N_{\mathrm{med}} \quad \text{for} \quad i = 1, 2, 3; \tag{2.76}$$

$$|B_\Gamma^{k_{\min}k_{\mathrm{med}}}| \lesssim N_{\mathrm{med}}. \tag{2.77}$$

Proof. The key observation is that under the Γ-condition (2.73), both $|k|$ and $|k_{\max}|$ are confined in some intervals of size $2N_{\mathrm{med}}$. We first prove (2.75). Recall from Lemma 8 and

(2.74) that

$$\begin{aligned}|B_\Gamma| &\lesssim \min\Bigg(\sum_{k,k_1}|B_\Gamma^{kk_1}|, \sum_{k,k_3}|B_\Gamma^{kk_3}|, \sum_{k_1,k_2}|B_\Gamma^{k_1k_2}|, \sum_{k_2,k_3}|B_\Gamma^{k_2k_3}|\Bigg)\\ &\lesssim \min\Bigg(\sum_{k,k_1}((N_2\wedge N_3)^{2-\alpha}\langle k_1-k\rangle^{-1}+1), \sum_{k,k_3}((N_1\wedge N_2)^{2-\alpha}\langle k-k_3\rangle^{-1}+1),\\ &\qquad \sum_{k_1,k_2}(N_3^{2-\alpha}\langle k_1-k_2\rangle^{-1}+1), \sum_{k_2,k_3}(N_1^{2-\alpha}\langle k_2-k_3\rangle^{-1}+1)\Bigg),\end{aligned} \tag{2.78}$$

where the summations are under (2.73). We then distinguish three cases. If $N_1 = N_{\min}$, then from (2.73) and (2.78) we have

$$\begin{aligned}|B_\Gamma| &\lesssim \sum_{k,k_1}((N_2\wedge N_3)^{2-\alpha}\langle k_1-k\rangle^{-1}+1)\\ &\lesssim \sum_{k_1}[(N_2\wedge N_3)^{2-\alpha}\log N_{\text{med}}+N_{\text{med}}]\\ &\lesssim N_{\min}N_{\text{med}}^{2-\alpha}\log N_{\text{med}}+N_{\min}N_{\text{med}}\\ &\lesssim N_{\min}N_{\text{med}},\end{aligned} \tag{2.79}$$

where we used the Γ-condition (2.73) in the first step when summing over k. Thus we finish the proof for the case $N_1 = N_{\min}$. If $N_3 = N_{\min}$, the argument is similar and thus omitted. In the following, we assume that $N_2 = N_{\min}$. If we further assume that $N_3 = N_{\text{med}}$, then similar computation as in (2.79) yields

$$\begin{aligned}|B_\Gamma| &\lesssim \sum_{k_1,k_2}(N_3^{2-\alpha}\langle k_1-k_2\rangle^{-1}+1)\\ &\lesssim \sum_{k_2}(N_{\text{med}}^{2-\alpha}\log N_{\text{med}}+N_{\text{med}})\\ &\lesssim N_{\min}N_{\text{med}}^{2-\alpha}\log N_{\text{med}}+N_{\min}N_{\text{med}}\\ &\lesssim N_{\min}N_{\text{med}}.\end{aligned}$$

It remains to consider the case when $N_2 = N_{\min}$ and $N_1 = N_{\text{med}}$. Then, from (2.73) and

(2.78) we have

$$\begin{aligned}
|B_\Gamma| &\lesssim \sum_{k_2,k_3}(N_1^{2-\alpha}\langle k_2-k_3\rangle^{-1}+1)\\
&\lesssim \sum_{k_2} N_{\text{med}}^{2-\alpha}\log N_{\text{med}} + N_{\min}N_{\text{med}}\\
&\lesssim N_{\min}N_{\text{med}}^{2-\alpha}\log N_{\text{med}} + N_{\min}N_{\text{med}}\\
&\lesssim N_{\min}N_{\text{med}}.
\end{aligned}$$

Thus we finish the proof of (2.75).

Now we consider (2.76). We first note that

$$\begin{aligned}
|B_\Gamma^k| &\lesssim \min\left(\sum_{k_1}|B_\Gamma^{kk_1}|, \sum_{k_3}|B_\Gamma^{kk_3}|\right)\\
&\lesssim \min\left(\sum_{k_1}((N_2\wedge N_3)^{2-\alpha}\langle k_1-k\rangle^{-1}+1), \sum_{k_3}((N_1\wedge N_2)^{2-\alpha}\langle k-k_3\rangle^{-1}+1)\right)\\
&\lesssim \sum_{k_{\text{med}}}(N_{\text{med}}^{2-\alpha}\langle k-k_{\text{med}}\rangle^{-1}+1) \lesssim N_{\text{med}}^{2-\alpha+\varepsilon}+N_{\text{med}} \lesssim N_{\text{med}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|B_\Gamma^{k_1}| &\lesssim \sum_{k_2}|B_\Gamma^{k_1k_2}|\\
&\lesssim \sum_{k_2}((N_1\wedge N_3)^{2-\alpha}\langle k_1-k_2\rangle^{-1}+1)\\
&\lesssim \sum_{k_2}(N_{\text{med}}^{2-\alpha}\langle k_1-k_2\rangle^{-1}+1) \lesssim N_{\text{med}},
\end{aligned}$$

where in the last step, we used (2.73), which suggests the summation over k_2 is at most over an interval of size N_{med}. The rest of (2.76) follows similarly.

Finally, we turn to (2.77). From (2.74), we note that

$$\begin{aligned}|B_{\Gamma}^{k_{\min}k_{\text{med}}}| &= |S_{k_{\min}k_{\text{med}}} \cap \{(2.73)\}| \\ &\leq |\{k_{\max} \text{ range in an interval of size at most } 2N_{\text{med}}\}| \\ &\leq 2N_{\text{med}}.\end{aligned}$$

Thus, we finish the proof of (2.77). □

2.2.2 Tensor norm estimates

This subsection will introduce the base tensor operator and then discuss its operator norm estimates. Given $m \in \mathbb{Z}$, $\alpha \in (1,2)$, and dyadic numbers $1 \leq N_1, N_2, N_3 \leq N$, we define the base tensor $\mathrm{T}^{\mathrm{b},m}$ as

$$\mathrm{T}^{\mathrm{b},m} = \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} = \mathbf{1}_S(k, k_1, k_2, k_3), \tag{2.80}$$

where $\mathbf{1}_S$ is the indicator function defined in (1.54), and S is the subset of $\mathbb{Z}^4$ given in (2.45). Here with a slight abuse of notation, we will not distinguish the tensor operator $\mathrm{T}^{\mathrm{b},m}$ and its kernel $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$. From (2.45) we see that

$$\begin{aligned}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} &= \mathbf{1}_{k-k_1+k_2+k_3=0} \cdot \mathbf{1}_{k_2\notin\{k_1,k_3\}} \cdot \mathbf{1}_{\{|k|^\alpha-|k_1|^\alpha+|k_2|^\alpha-|k_3|^\alpha=m+O(1)\}} \\ &\quad \times \mathbf{1}_{|k|\leq N} \cdot \prod_{i=1}^{3} \mathbf{1}_{|k_i|\leq N_i}.\end{aligned}$$

Recall the operator norms defined in (1.33). For example,

$$\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1k_2k_3} = \sum_{k,k_1,k_2,k_3} |\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|^2 = |S|, \tag{2.81}$$

or

$$\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1\to k_2k_3} = \sup\Bigg\{\sum_{k_2,k_3\in\mathbb{Z}}\Bigg|\sum_{k,k_1\in\mathbb{Z}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} z_{kk_1}\Bigg|^2; \sum_{k,k_1\in\mathbb{Z}} |z_{kk_1}|^2 = 1\Bigg\}.$$

Now we are ready to state the estimates for tensor norms, which are the main tools of our later analysis. We start with the Hilbert-Schmidt norm estimate of the base tensor operator $\mathrm{T}^{\mathrm{b},m}$, which is a consequence of Lemma 10.

Lemma 12. Let $\mathrm{T}^{\mathrm{b},m}$ be the base tensor defined in (2.80). Then we have

$$\begin{aligned}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1k_2k_3} \lesssim \min(&N_3^{2-\alpha}(N_1\wedge N_2)\log(N_1\vee N_2)+N_1N_2,\\ &N_1^{2-\alpha}(N_2\wedge N_3)\log(N_2\vee N_3)+N_2N_3).\end{aligned} \tag{2.82}$$

Proof. The estimate (2.82) is a consequence of (2.81) and Lemma 10. □

Lemma 13. Let $\mathrm{T}^{\mathrm{b},m}$ be the base tensor defined in (2.80). Then we have

$$\begin{aligned}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1\to k_2k_3} &\lesssim (N_2\wedge N_3)^{2-\alpha}N_1^{2-\alpha},\\ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_3\to k_1k_2} &\lesssim (N_1\wedge N_2)^{2-\alpha}N_3^{2-\alpha},\\ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_2\to k_1k_3} &\lesssim (N_1\wedge N_3)^{1-\frac{\alpha}{2}}(N\wedge N_2)^{1-\frac{\alpha}{2}}.\end{aligned} \tag{2.83}$$

Proof. We start with the first estimate of (2.83). By the Schur's test, we have

$$\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1\to k_2k_3} \lesssim \left(\sup_{k,k_1\in\mathbb{Z}}\sum_{k_2,k_3\in\mathbb{Z}}|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|\right)\times\left(\sup_{k_2,k_3\in\mathbb{Z}}\sum_{k,k_1\in\mathbb{Z}}|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|\right).$$

which gives the first estimate of (2.83) by using Lemma 8. Same argument as above works for the second estimate of (2.83). For the third estimate, we use Remark 11 instead of Lemma 8. □

Lemma 14. Let $\mathrm{T}^{\mathrm{b},m}$ be the base tensor defined in (2.80). Then we have

$$\begin{aligned}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k\to k_1k_2k_3} &\lesssim \min((N_2\wedge N_3)^{2-\alpha}\log N_1+N_1,(N_1\wedge N_2)^{2-\alpha}\log N_3+N_3);\\ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_1\to kk_2k_3} &\lesssim \min(N_3^{2-\alpha}\log N_2+N_2,N_3(N\wedge N_2)^{1-\frac{\alpha}{2}});\\ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_2\to kk_1k_3} &\lesssim \min(N_3^{2-\alpha}\log N_1+N_1,N_1^{2-\alpha}\log N_3+N_3);\\ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_3\to kk_1k_2} &\lesssim \min(N_1^{2-\alpha}\log N_2+N_2,N_1(N\wedge N_2)^{1-\frac{\alpha}{2}}).\end{aligned} \tag{2.84}$$

Proof. Similar to in the proof of Lemma 13, the estimates of (2.84) are consequences of (2.58). We only show how to get the first estimate, as others follow similarly[1].

By the Schur's test, we have

$$\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k\to k_1k_2k_3} \lesssim \Big(\sup_{k\in\mathbb{Z}} \sum_{k_1,k_2,k_3\in\mathbb{Z}} |\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|\Big) \times \Big(\sup_{k_1,k_2,k_3\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|\Big). \tag{2.85}$$

We note that once k_1, k_2, k_3 are fixed, then k is uniquely determined provided $(k, k_1, k_2, k_3) \in S$, where S is given by (2.45). Therefore, we have

$$\sup_{k_1,k_2,k_3\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}| \leq 1. \tag{2.86}$$

On the other hand, for fixed $k \in \mathbb{Z}$, from (2.80) and (2.58) we note

$$\begin{aligned} \sum_{k_1,k_2,k_3\in\mathbb{Z}} |\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}| &= |S_k| \\ &\lesssim \min((N_2 \wedge N_3)^{2-\alpha} \log N_1 + N_1, (N_1 \wedge N_2)^{2-\alpha} \log N_3 + N_3). \end{aligned} \tag{2.87}$$

Then, the first estimate of (2.84) follows from (2.85), (2.86), and (2.87). □

We will need some estimates of the base tensor $\mathrm{T}^{\mathrm{b},m}$ subject to further restrictions, for which we expect better estimates.

Lemma 15. Let $\mathrm{T}^{\mathrm{b},m}$ be the base tensor defined in (2.80). Then we have

$$\begin{aligned} \|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1k_2k_3} &\lesssim N_1N_3; \\ \|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_2\to k_1k_3} &\lesssim (N_1 \wedge N_3)^{1-\frac{\alpha}{2}}; \\ \|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_1\to kk_2k_3} &\lesssim N_3; \\ \|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_3\to kk_1k_2} &\lesssim N_1. \end{aligned} \tag{2.88}$$

[1]For the second and fourth estimates, we may need to use (2.67) as well.

Proof. We start with the first bound in (2.84). We have

$$\begin{aligned}\|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1k_2k_3} &= \sum_{k_1,k_3}\sum_{k,k_2}|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|^2 \\ &= \sum_{k_1,k_3}|S_{k_1k_3}\cap\{|k_1+k_3|<|k_2|\}|.\end{aligned} \tag{2.89}$$

We observe that under the condition $|k_1+k_3|<|k_2|$, we have

$$|2k-(k_1+k_3)| = |2k_2-(k_1+k_3)| \geq 2|k_2|-|k_1+k_3| > |k_2| > |k_1+k_3|,$$

which, together with (2.62), implies that

$$S_{k_1k_3}\cap\{|k_1+k_3|<|k_2|\}\subset S^{\mathrm{good}}_{k_1k_3}. \tag{2.90}$$

Thus, from (2.85), (2.86), (2.89), (2.90) and Lemma 9 we get

$$\|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1k_2k_3} \leq \sum_{k_1,k_3}|S^{\mathrm{good}}_{k_1k_3}| \lesssim N_1N_3,$$

which finishes the proof of the first estimate of (2.84).

We turn to the estimate of $\|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_2\to k_1k_3}$. By a similar Schur's test argument as in (2.85), we have

$$\begin{aligned}&\|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_2\to k_1k_3} \\ &\quad\lesssim\Big(\sup_{k,k_2\in\mathbb{Z}}\sum_{k_1,k_3\in\mathbb{Z}}|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|\Big)\times\Big(\sup_{k_1,k_3\in\mathbb{Z}}\sum_{k,k_2\in\mathbb{Z}}|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}|\Big) \\ &\quad=\Big(\sup_{k,k_2\in\mathbb{Z}}|S_{kk_2}|\Big)\cdot\Big(\sup_{k_1,k_3\in\mathbb{Z}}|S^{\mathrm{good}}_{k_1k_3}|\Big),\end{aligned}$$

which, together with Lemma 9, gives the desired bound.

For the third estimate $\|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_1\to kk_2k_3}$, we proceed similarly as the above to get

$$\begin{aligned}\|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_1\to kk_2k_3} &\lesssim \sup_{k_1\in\mathbb{Z}}\sum_{k,k_2,k_3\in\mathbb{Z}}|\mathbf{1}_{|k_1+k_3|<|k_2|}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}| \\ &\lesssim \sup_{k_1\in\mathbb{Z}}\sum_{k_3\in\mathbb{Z}}|S^{\mathrm{good}}_{k_1k_3}| \lesssim N_3,\end{aligned} \tag{2.91}$$

where we used (2.90) and Lemma 9.

The last estimate of (2.88) can be handled similarly as in (2.91). Thus, we omit the details. □

We also record the following result, which is a consequence of Lemma 11.

Corollary 3. Let $\mathrm{T}^{\mathrm{b},m}$ be the base tensor defined in (2.80). Then we have

$$\begin{aligned}\|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{kk_1k_2k_3} &\lesssim N_{\min}N_{\mathrm{med}}, \\ \|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_1\to kk_2k_3} &\lesssim N_{\mathrm{med}}, \\ \|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_3\to kk_1k_2} &\lesssim N_{\mathrm{med}}, \\ \|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|^2_{k_2\to kk_1k_3} &\lesssim N_{\mathrm{med}}.\end{aligned}$$

2.3 Induction argument

As explained in Subsubsection 2.1.4, we shall proceed with an induction argument. To be more precise, to construct the solution $u(t)$ to (2.1) with the formal expansion (2.44), we need to (i) construct z_N and $\mathfrak{h}^{N,L}$ with $1 \le L < N^{1-\delta}$ for each dyadic $N \ge 1$; (ii) show the convergence of (2.44) in a proper space. Now we illustrate how to construct z_N and $\mathfrak{h}^{N,L}$ in an induction manner. We first note that $\Pi_{1/2} = 0$[1], which implies that $u_{1/2} = y_{1/2} = F_{1/2} = z_{1/2} - 0$, $\mathcal{H}^{N,1/2} = \Pi_N$, and $\psi_{N,1/2} = \chi(t)F_N$. Then we move to z_1. We note that from (2.30) that

$$y_1 = \psi_{1,1/2} + z_1 = F_1 + z_1. \tag{2.92}$$

[1]Here we used the fact that $\{k \in \mathbb{Z}; \langle k\rangle \le 1/2\} = \varnothing$.

From (2.31) and (2.92), it follows that

$$\begin{aligned} z_1(t) =& y_1(t) - F_1 \\ =& - \mathrm{i}\chi(t) \left(\int_0^t \Pi_1 \mathcal{M}(y_1, y_1, y_1)(t')\mathrm{d}t' + \int_0^t \Pi_1 \mathcal{R}(y_1)(t')\mathrm{d}t' \right) \\ =& - \mathrm{i}\chi(t) \sum_{w_1, w_1', w_1'' \in \{F_1, z_1\}} \left(\int_0^t \Pi_1 \mathcal{M}(w_1, w_1', w_1'')(t')\mathrm{d}t' + \int_0^t \Pi_1 \mathcal{R}(w_1, w_1', w_1'')(t')\mathrm{d}t' \right) \end{aligned} \tag{2.93}$$

which can be solved, provided $T \ll 1$, as Π_1 is the projection to zero frequency. Given z_1 solved from (2.93), we obtain y_1, from (2.30) and $\psi_{1,1/2} = F_1$, and thus u_1 by (2.25). Next, we can use (2.29) to construct $\psi_{N,1}$, which is a linear equation that can be solved locally for small t, where we used u_1 that we obtained earlier. By plugging $\psi_{2,1}$ and y_1 into (2.31), we can solve for z_2[1]. Similarly, we can get y_2 from (2.30) and then u_2 from (2.25). We repeat the above process to construct $\psi_{N,2}$ using (2.29) and u_2. By iterating this procedure, we obtain a sequence of $\{z_L, u_L, \psi_{N,L}\}$ for all dyadic numbers L, N, such that $L < N^{1-\delta}$. To show point (ii), which is the convergence of (2.44) in some space, we need to show that each term in the sequence $\{z_L, u_L, \psi_{N,L}\}$ satisfies some proper bounds, which we elaborated in the following definition.

Definition 5. Given $0 < T \ll 1$, $b > \frac{1}{2}$ sufficiently closed to $\frac{1}{2}$, and let $J = [-T, T]$ and $\chi(t) = \eta_T(t)$. For any dyadic number M, consider the following statements, which we call $\texttt{Local}(M)$:

(i) For the operator $\tilde{\mathfrak{h}}^{N,L}$ defined in (2.41), where $L < \min(M, N^{1-\delta})$, we have

$$\|\tilde{\mathfrak{h}}^{N,L}\|_{Y^b} \leq L^{-\delta_0}, \quad \|\tilde{\mathfrak{h}}^{N,L}\|_{Z^b} \leq N^{\frac{1}{2}+\gamma_0} L^{-\frac{1}{2}}, \tag{2.94}$$

as well as

$$\left\| \left(1 + \frac{|k-k'|}{L} \right)^{\kappa} h_{kk'}^{N,L} \right\|_{Z^b} \leq N. \tag{2.95}$$

[1]We will show this can be done later.

Here the Y^b and Z^b norms are defined in (1.38) and (1.39), respectively.

(ii) For the terms z_N, where $N \leq M$, we have

$$\|z_N\|_{X^b(J)} \leq N^{-\frac{1}{2}-\gamma}. \tag{2.96}$$

(iii) For any dyadic numbers $L_1, L_2 < M \leq N$ the operators $\mathcal{P}^{\pm}$ defined by

$$\begin{aligned}\mathcal{P}^{+}[\psi](t) = \mathcal{P}^{+}_{N,L_1,L_2}[\psi](t) &:= -\mathrm{i}\chi(t)\int_0^t \Pi_N \mathcal{M}(y_{L_1}, y_{L_2}, \psi)(t')\mathrm{d}t' \\ \mathcal{P}^{-}[\psi](t) = \mathcal{P}^{-}_{N,L_1,L_2}[\psi](t) &:= -\mathrm{i}\chi(t)\int_0^t \Pi_N \mathcal{M}(y_{L_1}, \psi, y_{L_2})(t')\mathrm{d}t'\end{aligned} \tag{2.97}$$

whose kernels $P^{\pm}_{kk'}(t,t')$ have Fourier transform $\widehat{P^{\pm}_{kk'}}(\tau,\tau')$, which satisfies $|k-k'| \lesssim L_{\max}$ and

$$\int_{\mathbb{R}^2} \langle\tau\rangle^{2(1-b)}\langle\tau'\rangle^{-2b}\|P^{\pm}_{kk'}(\tau,\tau')\|^2_{k\to k'}\mathrm{d}\tau\mathrm{d}\tau' \lesssim T^{2\theta}L_{\max}^{-6\delta_0}, \tag{2.98}$$

$$\int_{\mathbb{R}^2} \langle\tau\rangle^{2(1-b)}\langle\tau'\rangle^{-2b}\|P^{\pm}_{kk'}(\tau,\tau')\|^2_{kk'}\mathrm{d}\tau\mathrm{d}\tau' \lesssim T^{2\theta}N^{1+\gamma_0}L_{\max}^{-1-4\gamma_0}, \tag{2.99}$$

$$\int_{\mathbb{R}^2} \langle\tau\rangle^{2(1-b)}\langle\tau'\rangle^{-2b}\left\|\left(1+\frac{|k-k'|}{L_{\max}}\right)^{\kappa} P^{\pm}_{kk'}(\tau,\tau')\right\|^2_{kk'}\mathrm{d}\tau\mathrm{d}\tau' \lesssim T^{2\theta}N^2, \tag{2.100}$$

where $L_{\max} = \max(L_1, L_2)$.

In the above definition, we choose the parameters $0 < \varepsilon \ll \delta \ll \gamma_0 \ll \gamma \ll \delta_0 \ll \alpha - 1$ and $\kappa \gg 1$.

Now we are ready to state the main a priori estimate, which is the key ingredient in proving Theorem 5.

Proposition 5. Given $0 < T \ll 1$, the probability that $\texttt{Local}(M/2)$ holds but $\texttt{Local}(M)$ does not hold is less than $C_\theta \mathrm{e}^{-(T^{-1}M)^\theta}$ for some $\theta, C_\theta > 0$.

The proof of Proposition 5 will occupy Section 2.5 and Section 2.6.

2.3.1 Reformulation

As explained at the beginning of this section, the main strategy is to proceed via induction. Recall that $\Pi_{1/2} = 0$, and thus $u_{1/2} = y_{1/2} = F_{1/2} = z_{1/2} = 0$, $\mathcal{H}^{N,1/2} = \Pi_N$, and $\psi_{N,1/2} = \chi(t)F_N$. Therefore, $\texttt{Local}\left(\frac{1}{2}\right)$ holds trivially.

In what follows, let us assume $\texttt{Local}(M')$ holds for all $M' \leq \frac{M}{2}$ for some $M \geq 1$. Then, the aim is to show that $\texttt{Local}(M)$ holds with large probability, i.e. the probability of $\texttt{Local}(M)$ does not hold is less than $C_\theta e^{-(T^{-1}M)^\theta}$ for some $\theta, C_\theta > 0$.

Before proceeding to show $\texttt{Local}(M)$, let us reformulate the estimates given in $\texttt{Local}\left(\frac{M}{2}\right)$. From $\texttt{Local}\left(\frac{M}{2}\right)$, we have (2.94), (2.95), and (2.96) with $N \leq \frac{M}{2}$ and $L < \min(\frac{M}{2}, N^{1-\delta})$. This together with (2.43) implies that

$$y_L(t) = \chi(t)F_L + \sum_{1 \leq R < L^{1-\delta}} \tilde{\mathfrak{h}}^{L,R}[F_L](t) + z_L(t), \tag{2.101}$$

is well-defined for all $L \leq \frac{M}{2}$ and $t \in J = [-T, T]$. To show that $\texttt{Local}(M)$-(ii) holds, that is, (2.96), we substitute y_L for $L \leq \frac{M}{2}$ from (2.101) into (2.31) and then solve for z_M. Likewise, to show that $\texttt{Local}(M)$-(iii) holds, that is, (2.98), (2.99), and (2.100) with $L_1, L_2 < N \leq M$, we substitute y_L from (2.101) into the random averaging operators $\mathcal{P}^\pm$ in (2.97). To simplify the notations, we can assume that y_L is either $\chi(t)F_L$, $\tilde{\mathfrak{h}}^{L,R}[F_L]$, or z_L, depending on the decomposition (2.101). In particular, we assume the input w_L is one of the following three types:

- Type (G):
$$(w_L)_k(t) := \mathbf{1}_{L/2 < \langle k \rangle \leq L} \frac{g_k(\omega)}{[\![k]\!]^{\alpha/2}} \chi(t).$$

- Type (C):
$$(w_L)_k(t) := \sum_{k'} h^{L,R}_{kk'}(t) \frac{g_{k'}(\omega)}{[\![k']\!]^{\alpha/2}}, \tag{2.102}$$

  with $h^{L,R}_{kk'}$ being supported in the set $\{(k,k'); L/2 < \langle k' \rangle \leq L\}$, $\mathcal{B}_{\leq R}$-measurable for some $R < L^{1-\delta}$, and satisfying the bounds

$$\begin{aligned} \|\langle\tau\rangle^b \widehat{h^{L,R}_{kk'}}(\tau)\|_{L^2_\tau(\ell^2_k\to\ell^2_{k'})} &\lesssim R^{-\delta_0}, \\ \|\langle\tau\rangle^b \widehat{h^{L,R}_{kk'}}(\tau)\|_{L^2_\tau \ell^2_{kk'}} &\lesssim L^{\frac{1}{2}+\gamma_0} R^{-\frac{1}{2}}, \end{aligned} \tag{2.103}$$

for some $b > \frac{1}{2}$. Moreover, from (2.95) we may assume that $h^{L,R}_{kk'}$ is supported in $\{|k-k'| \lesssim L^\varepsilon R\}$ for $0 < \varepsilon \ll 1$ that only depends on κ.

- Type (D): $(w_L)_k$ is supported in $\{k \in \mathbb{Z}; \langle k\rangle \le L\}$ and satisfies

$$\|w_L\|_{X^b} \lesssim L^{-\frac{1}{2}-\gamma}, \tag{2.104}$$

for some $b > \frac{1}{2}$, provided $1 \le L \le \frac{M}{2}$.

Remark 12. By letting $h^{L,1/2}_{kk'} := \mathbf{1}_{L/2<\langle k\rangle\le L}\mathbf{1}_{k=k'} \cdot \chi(t)$, we can view type (G) input as a special case of type (C). Therefore, we will only consider type (C) and type (D) in what follows.

2.4 Unitary property

In this section, we prove a crucial cancellation property of $\mathcal{H}^{N,L}$, originated from the L^2 conservation of the linear equation (2.32). We first show that the operator $\widetilde{\mathcal{H}}^{N,L}$ given in (2.36) and (2.40) is well-defined provided

$$\|\mathcal{P}^{N,L}\|_{Y^{b,b}} < 1. \tag{2.105}$$

Here the operator norm $Y^{b,b}$ is given in (1.40). From (2.36), (2.40), and (2.105), we have

$$\begin{aligned}\|\widetilde{\mathcal{H}}^{N,L}[\Psi_0]\|_{X^b} &= \|\mathcal{H}^{N,L}[\chi(\cdot)\Psi_0]\|_{X^b} = \|(1-\mathcal{P}^{N,L})^{-1}[\chi(\cdot)\Psi_0]\|_{X^b} \\ &\leq \|\chi(\cdot)\Psi_0\|_{X^b} + \sum_{n\geq 1} \|(\mathcal{P}^{N,L})^n[\chi(\cdot)\Psi_0]\|_{X^b} \\ &\leq \|\chi(\cdot)\Psi_0\|_{X^b} + \sum_{n\geq 1} \|\mathcal{P}^{N,L}\|^n_{Y^{b,b}} \|\chi(\cdot)\Psi_0\|_{X^b},\end{aligned}$$

which implies that $\widetilde{\mathcal{H}}^{N,L}$ is well-defined under the assumption (2.105). Since $b > \frac{1}{2}$, by the Sobolev embedding, we have that $\widetilde{\mathcal{H}}^{N,L}[\Psi_0] \in C_T L^2_x$. We have the following group property on $[-T,T]$, i.e.

$$\Psi(t_2, \Psi(t_1)) = \Psi(t_1 + t_2, \Psi(0)),$$

where $\Psi(t,W)$ is the solution to (2.33) with initial data $\Psi_0 = W$. Thus $\widetilde{\mathcal{H}}^{N,L}(t)$ is invertible since $\widetilde{\mathcal{H}}^{N,L}(t) \circ \widetilde{\mathcal{H}}^{N,L}(-t) = \mathrm{Id}$. Finally, we note that (2.105) is guaranteed by (2.98). See (2.157). As a consequence, we also have the following unitary property.

Lemma 16. Given $M > \frac{1}{2}$, assume $\texttt{Local}(M)$ holds. For $L < \min(M, N^{1-\delta})$, there exists $T \ll 1$ such that for each $|t| \leq T$, the matrix $\widetilde{\mathcal{H}}^{N,L}(t) = \{\tilde{H}^{N,L}_{kk'}(t)\}_{kk'}$ is unitary, i.e.

$$\sum_k \tilde{H}^{N,L}_{k_1 k}(t) \overline{\tilde{H}^{N,L}_{k_2 k}}(t) = \delta_{k_1}(k_2) \tag{2.106}$$

where $\widetilde{\mathcal{H}}^{N,L}$ is the linear mapping defined in (2.40).

Bourgain [4] first observed this property in the context of Hartree-type NLS. See also [36]. Here we give a proof for reader's convenience.

Proof. From (2.32), we know that

$$\partial_t \sum_k |\Psi_k|^2 = 2\sum_k \mathrm{Re}(\overline{\Psi_k}\partial_t \Psi_k) = 4\,\mathrm{Im}\left(\sum_k \sum_{\substack{k_1-k_2+k_3=k \\ k_2\notin\{k_1,k_3\}}} \mathrm{e}^{\mathrm{i}t'\Phi} \cdot (u_L)_{k_1} \overline{(u_L)_{k_2}} \overline{\Psi_k} \Psi_{k_3}\right).$$

By swapping $(k,k_1,k_2,k_3) \mapsto (k_3,k_2,k_1,k)$ in the summand, we see that we are taking imaginary part of reals, which gives zero. Namely

$$\partial_t \sum_k |\Psi_k|^2 = 0,$$

for $|t| \leq T$. It then follows that

$$\|\Psi(t)\|^2_{L^2(\mathbb{T})} = \|\Psi_0\|^2_{L^2(\mathbb{T})}, \quad \text{for all } |t| \leq T.$$

This shows that $\widetilde{\mathcal{H}}^{N,L}(t)$ is isometric on L^2 for $t \in [-T,T]$. Furthermore, given $t \in [-T,T]$, we know that $\widetilde{\mathcal{H}}^{N,L}(t)$ is invertible and thus surjective. Therefore, for fixed $t \in [-T,T]$, the linear operator $\widetilde{\mathcal{H}}^{N,L}(t)$ is unitary[1], i.e.

$$(\widetilde{\mathcal{H}}^{N,L}(t))^* \widetilde{\mathcal{H}}^{N,L}(t) = \widetilde{\mathcal{H}}^{N,L}(t)(\widetilde{\mathcal{H}}^{N,L}(t))^* = \mathrm{Id},$$

for $|t| \leq T$, which finishes the proof. ☐

We have the following corollary of Lemma 16.

Corollary 4. With the same assumption as in Lemma 16, we have

$$\sum_k \sum_{L_1,L_2 \leq L} h^{N,L_1}_{k_1 k} \overline{h^{N,L_2}_{k_2 k}} \frac{1}{[\![k]\!]^{\alpha/2}} = \sum_k \sum_{L_1,L_2 \leq L} h^{N,L_1}_{k_1 k} \overline{h^{N,L_2}_{k_2 k}} \left(\frac{1}{[\![k]\!]^{\alpha/2}} - \frac{1}{[\![k_1]\!]^{\alpha/2}} \right) \tag{2.107}$$

provided $|t| \leq T$ and $k_1 \neq k_2$.

Proof. From (2.32), we know that

$$\sum_{L_i \leq L} h^{N,L_i}_{k_i k}(t) = \widetilde{\mathcal{H}}^{N,L}_{k_i k}(t), \tag{2.108}$$

for $i = 1,2$. Then it follows from (2.106) and (2.107) that

$$\sum_{L_1,L_2 \leq L} \sum_k h^{N,L_1}_{k_1 k}(t) \overline{h^{N,L_2}_{k_2 k}}(t) = \sum_k \widetilde{\mathcal{H}}^{N,L}_{k_1 k}(t) \widetilde{\mathcal{H}}^{N,L}_{k_2 k}(t) = \delta_{k_1}(k_2)$$

[1]A linear map is unitary if it is surjective and isometric.

for $|t| \leq T$. Thus (2.107) follows. $\square$

Remark 13. Without loss of generality, we may assume that

$$\sum_k h_{k_1 k}^{N,L_1}(t)\overline{h_{k_2 k}^{N,L_2}}(t)\frac{1}{[\![k]\!]^{\alpha/2}} = \sum_k h_{k_1 k}^{N,L_1}(t)\overline{h_{k_2 k}^{N,L_2}}(t)\left(\frac{1}{[\![k]\!]^{\alpha/2}} - \frac{1}{[\![k_1]\!]^{\alpha/2}}\right),$$

in our later analysis. The reason is that from Corollary 4, we can always add

$$0 = \sum_k \sum_{L_1, L_2 \leq L} h_{k_1 k}^{N,L_1}(t)\overline{h_{k_2 k}^{N,L_2}}(t)$$

for $k_1 \neq k_2$ from the beginning of our analysis.

2.5 The random averaging operators

This and the next section will be devoted to the proof of Proposition 5. The plan of the proof is as follows; we first further reduce the problem to a time-independent problem in Subsection 2.5.1; then we prove (2.98) - (2.100) of $\texttt{Local}(M)$ in Subsections 2.5.2 and 2.5.3; in Subsection 2.5.4, we verify (2.94) and (2.95) of $\texttt{Local}(M)$. We defer the proof of (2.96) to the next section.

2.5.1 Further reduction

To make the proof more clear, we can recast the estimates (2.98) - (2.100) in a simpler way. The key point is that the main solution space X^b for $b > \frac{1}{2}$ embeds in C_t in the temporal variable, so we can ignore the t variable in most of our analysis.

Recall the operators $\mathcal{P}^{\pm}$ defined in (2.97). Due to the factor $\chi(t) = \eta(T^{-1}t)$ and $b > 1/2$, by Proposition 1, we can gain a small factor T^{θ}. Recall that $\widehat{P_{kk'}^{\pm}}(\tau, \tau')$ is the temporal Fourier transform of the kernel $P_{kk'}^{\pm}(t, t')$ of $\mathcal{P}^{\pm}$ in (2.97), i.e.,

$$(\widehat{P^{\pm}\psi})_k(\tau) = \sum_{k'} \int_{\mathbb{R}} \widehat{P_{kk'}^{\pm}}(\tau, \tau')\widehat{\psi_{k'}}(\tau')\mathrm{d}\tau'.$$

We know that

$$|k - k'| = |k_1 - k_2| \lesssim L_{\max},$$

where $L_{\max} = \max(L_1, L_2)$. We consider a low modulation cut-off

$$||k|^\alpha - |k'|^\alpha| \lesssim L_{\max}^\kappa. \tag{2.109}$$

For the high modulation part $||k|^\alpha - |k'|^\alpha| \gg L_{\max}^\kappa$, we can gain considerable smoothing. Thus, we merely consider the low modulation part. Note that for the low modulation part, if we further assume that $k \neq k'$, then we have by Taylor expansion that

$$\begin{aligned} ||k'|^\alpha - |k|^\alpha| &= |k'|^\alpha \left| 1 - \left| 1 - \frac{k - k'}{k'} \right|^\alpha \right| \\ &\gtrsim \alpha |k'|^\alpha \frac{|k - k'|}{|k'|} \\ &\geq \alpha |k'|^{\alpha - 1}, \end{aligned}$$

which, together with (2.109), gives

$$|k'| \lesssim L_{\max}^{\frac{\kappa}{\alpha - 1}}. \tag{2.110}$$

Likewise, we can also obtain

$$|k| \lesssim L_{\max}^{\frac{\kappa}{\alpha - 1}}. \tag{2.111}$$

Also, when $k = k'$, the base tensor $h_{kk_1k_2k'}$ does not depends on k and k'. Therefore, we know that when we apply Proposition 4, we only lose a factor $L_{\max}^\theta$, not N^θ.

We start with the operator $\mathcal{P}^\pm$. By using (2.23), (1.42), and (2.97), we may rewrite the temporal Fourier transform of the kernel as

$$
\begin{aligned}
&\widehat{P^{\pm}_{kk'}}(\tau,\tau') \\
&\quad = -\mathrm{i}\sum_{\substack{k_1-k_2=k-k' \\ k_2\notin\{k_1,k'\}}}\int_{\mathbb{R}^2}\mathcal{K}(\tau,\Phi+\tau'+\tau_1-\tau_2)\cdot(\widehat{y_{L_1}})_{k_1}(\tau_1)\cdot(\overline{\widehat{y_{L_2}}})_{k_2}(\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
&\quad = -\mathrm{i}\sum_{m}\int_{\mathbb{R}^2}\mathcal{K}(\tau,m+\Phi-[\Phi]+\tau'+\tau_1-\tau_2) \\
&\qquad\qquad\qquad \times \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\cdot(\widehat{y_{L_1}})_{k_1}(\tau_1)\cdot(\overline{\widehat{y_{L_2}}})_{k_2}(\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2,
\end{aligned}
\tag{2.112}
$$

where Φ is defined in (2.24), $[\Phi]$ is the integer part of Φ, and the base tensor $\mathrm{T}^{\mathrm{b},m}$ is given by (2.80). Then by Minkowski inequality, (2.112), and (1.43), we have

$$
\begin{aligned}
\|\mathcal{P}^{+}\|^2_{Y^{1-b,b'}} &\lesssim \int_{\mathbb{R}^2}\langle\tau\rangle^{2(1-b)}\langle\tau'\rangle^{-2b'}\|\widehat{P^{+}_{kk'}}(\tau,\tau')\|^2_{k\to k'}\mathrm{d}\tau\mathrm{d}\tau' \\
&\lesssim \int_{\mathbb{R}^2}\langle\tau\rangle^{-2b}\langle\tau'\rangle^{-2b'}\Bigg(\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^2}\langle\tau_1\rangle^{-b}\langle\tau_2\rangle^{-b} \\
&\qquad \times \Bigg\|\langle\tau-m+\Phi-[\Phi]-\tau'-\tau_1+\tau_2\rangle^{-1} \\
&\qquad\quad \times \sum_{k_1,k_2}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\cdot(\langle\tau_1\rangle^{b}\widehat{y_{L_1}})_{k_1}(\tau_1)\cdot(\langle\tau_2\rangle^{b}\overline{\widehat{y_{L_2}}})_{k_2}(\tau_2)\Bigg\|_{k\to k'}\mathrm{d}\tau_1\mathrm{d}\tau_2\Bigg)^2\mathrm{d}\tau\mathrm{d}\tau' \\
&\lesssim \int_{\mathbb{R}^2}\langle\tau\rangle^{-2b}\langle\tau'\rangle^{-2b'}\Bigg(\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^2}\langle\tau-m-\tau'-\tau_1+\tau_2\rangle^{-1}\langle\tau_1\rangle^{-b}\langle\tau_2\rangle^{-b} \\
&\qquad \times \Bigg\|\sum_{k_1,k_2}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\cdot(\langle\tau_1\rangle^{b}\widehat{y_{L_1}})_{k_1}(\tau_1)\cdot(\langle\tau_2\rangle^{b}\overline{\widehat{y_{L_2}}})_{k_2}(\tau_2)\Bigg\|_{k\to k'}\mathrm{d}\tau_1\mathrm{d}\tau_2\Bigg)^2\mathrm{d}\tau\mathrm{d}\tau',
\end{aligned}
\tag{2.113}
$$

where $b > b' > \frac{1}{2}$. To deal with the summation over m in (2.113), we need to localise the frequencies k, k_1, k_2, k'. To be more precise, if $m = [\Phi] \in \mathbb{Z}$ ranges over an interval of size R, then from our construction, we have

$$
\sum_{m}\langle\tau-m-\tau'-\tau_1+\tau_2\rangle^{-1} \lesssim \log(1+R). \tag{2.114}
$$

In what follows, we shall use some special cases of (2.114). Recall $\Phi = |k_1|^\alpha - |k_2|^\alpha +$

$|k'|^\alpha - |k|^\alpha$ from (2.24) and the relation $k = k_1 - k_2 + k'$. If we assume $|k_1| \lesssim L_1$, $|k_2| \lesssim L_2$, and (2.109), then we have a refined bound

$$\sum_m \langle \tau - m - \tau' - \tau_1 + \tau_2 \rangle^{-1} \lesssim \log(1 + L_{\max}). \tag{2.115}$$

By using (2.115) and Cauchy-Schwarz inequality in τ_1, τ_2 integrations, we may bound (2.113) by

$$(\log(1 + L_{\max}))^2 \Big\| \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \cdot (\langle \tau_1 \rangle^b \widehat{y_{L_1}})_{k_1}(\tau_1) \cdot (\langle \tau_2 \rangle^b \overline{\widehat{y_{L_2}}})_{k_2}(\tau_2) \Big\|^2_{L^2_{\tau_1\tau_2}(k \to k')} \tag{2.116}$$

where we used that $b, b' > \frac{1}{2}$. From the discussion in Subsection 2.3.1, we proceed with estimating (2.116) by replacing y_{L_j} with an input w_{L_j} of either type (C) or type (D), respectively.

From the definition of type (C), type (D), and (2.116), we may further simplify the estimate (2.98) to a time-independent estimate. To be more precise, to prove (2.98), it suffices to show[1]

$$\|\mathscr{Y}^+_{kk'}\|_{k \to k'} \lesssim L_{\max}^{-4\delta_0}, \tag{2.117}$$

where $\mathscr{Y}^+_{kk'}$ is the time-dependent random matrix given by

$$\mathscr{Y}^+_{kk'} = \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \cdot (w_{L_1})_{k_1} (\overline{w_{L_2}})_{k_2}. \tag{2.118}$$

Here w_{L_j} for $j \in \{1, 2\}$ are of the following two types, with a slight abuse of notation, we still call them type (C) and type (D).

- Type (C), where
$$(w_{L_j})_{k_j} := \sum_{k'_j} h^{L_j,R_j}_{k_jk'_j}(\omega) \frac{g_{k'_j}(\omega)}{[\![k'_j]\!]^{\alpha/2}},$$
  with $h^{L_j,R_j}_{k_jk'_j}$ supported in the set $\{k'_j; L_j/2 < \langle k'_j \rangle \le L_j\}$ and $\mathcal{B}_{\le R_j}$-measurable for

[1]Strictly speaking, to reduce (2.98) to (2.117), one needs to use a standard *meshing argument* used in [34, Lemma 4.2, Claim 5.4], [35, Proposition 6.1], and [36, Subsection 3.4, Subsection 4.1].

some $R_j \le L_j^{1-\delta}$, and satisfying the bounds

$$\begin{aligned} \|h^{L_j,R_j}_{k_j k_j'}\|_{\ell^2_{k_j}\to \ell^2_{k_{j'}}} &\lesssim R_j^{-\delta_0}, \\ \|h^{L_j,R_j}_{k_j k_j'}\|_{\ell^2_{k_j k_{j'}}} &\lesssim L_j^{\frac12+\gamma_0} R_j^{-\frac12}, \end{aligned} \tag{2.119}$$

for $0 < \delta \ll \gamma_0 \ll \gamma \ll \delta_0 \ll \alpha - 1$. Moreover, from (2.95) we may assume that $h^{L_j,R_j}_{k_j k_j'}$ is supported in $\{|k_j - k_j'| \lesssim L_j^{\varepsilon} R_j\}$ for any $\varepsilon > 0$.

- Type (D), where $(w_{L_j})_{k_j}$ is supported in $\{k_j \in \mathbb{Z}; \langle k_j\rangle \le L_j\}$, and satisfies

$$\|(w_{L_j})_{k_j}\|_{\ell^2_{k_j}} \lesssim L_j^{-\frac12-\gamma}. \tag{2.120}$$

A similar argument allows us to reduce (2.99) to

$$\|\mathscr{Y}^+_{kk'}\|_{kk'} \lesssim N^{\frac12+\frac{\gamma_0}{2}} L_{\max}^{-\frac12-3\gamma_0}, \tag{2.121}$$

where $\mathscr{Y}^+_{kk'}$ is as in (2.118) and $0<\varepsilon' \ll 1$.

The same argument as above also works for the estimate (2.98) with the operator $\mathcal{P}^-$, for which we only need to prove the following time-independent random matrix estimates.

$$\|\mathscr{Y}^-_{kk'}\|_{k\to k'} \lesssim L_{\max}^{-4\delta_0}, \tag{2.122}$$

$$\|\mathscr{Y}^-_{kk'}\|_{kk'} \lesssim N^{\frac12+\frac{\gamma_0}{2}} L_{\max}^{-\frac12-3\gamma_0}, \tag{2.123}$$

where the corresponding random tensor $\mathscr{Y}^-_{kk'}$ is given by

$$\mathscr{Y}^-_{kk'} = \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \cdot (w_{L_1})_{k_1} (w_{L_3})_{k_3}. \tag{2.124}$$

Here w_{L_1} and w_{L_3} in (2.124) are of either type (C) satisfying (2.119), or type (D) satisfying (2.120).

The proof of (2.117), (2.121), (2.122), and (2.123) will be detailed in Subsections 2.5.2

and 2.5.3. The above argument yields the following conditional proof of (2.98) – (2.100) in $\texttt{Local}(M)$.

Proof of (2.98) − (2.100) **in** $\texttt{Local}(M)$**.** Given $\texttt{Local}\left(\frac{M}{2}\right)$, we may decompose y_{L_i} into w_{L_i} of type (C) or type (D) for $L_i \le \frac{M}{2}$. From the above argument, we have seen that (2.98) and (2.99) follows from (2.117), (2.122), and (2.123). Finally, the estimate (2.100) follows from (2.99) and the fact $|k-k'| \lesssim L_{\max}$. □

We conclude this subsection by recording a consequence of (2.98) – (2.100).

Corollary 5. Given $\texttt{Local}(M)$ and the above notations, we have

$$\|\mathcal{P}^{\pm}\|_{Y^{b,b}} \lesssim T^{c\theta} L_{\max}^{-2\delta_0}, \tag{2.125}$$

$$\|\mathcal{P}^{\pm}\|_{Z^{b,b}} \lesssim T^{c\theta} N^{\frac{1}{2}+\frac{2\gamma_0}{3}} L_{\max}^{-\frac{1}{2}-\gamma_0}, \tag{2.126}$$

$$\left\| \left(1+\frac{|k-k'|}{L_{\max}}\right)^{\kappa} \widehat{P^{\pm}_{kk'}} \right\|_{Z^{b,b}} \lesssim T^{c\theta} N, \tag{2.127}$$

provided $b > \frac{1}{2}$ and close to $\frac{1}{2}$, where $Y^{b,b}$ and $Z^{b,b}$ are defined in (1.40) and (1.41), respectively.

Proof. From (1.40) and (2.98), we have that

$$\|\mathcal{P}^{\pm}\|_{Y^{1-b,b}} \le \left(\int_{\mathbb{R}^2} \langle\tau\rangle^{2(1-b)}\langle\tau'\rangle^{-2b} \|\widehat{P^{\pm}_{kk'}}(\tau,\tau')\|^2_{k\to k'} \mathrm{d}\tau \mathrm{d}\tau'\right)^{\frac{1}{2}} \lesssim T^{\theta} L_{\max}^{-3\delta_0}, \tag{2.128}$$

for some $b > \frac{1}{2}$. We recall that

$$\mathcal{P}^{\pm}[\psi](t) = -\mathrm{i}\chi(t)\int_0^t \Pi_N \mathcal{M}(y_{L_1}, y_{L_2}, \psi)(t')\mathrm{d}t'.$$

It then follows that

$$\begin{aligned} \|\mathcal{P}^{+}[\psi](t)\|_{X^1} &\sim \|\mathcal{P}^{+}[\psi](t)\|_{L^2_{t,x}([-T,T]\times\mathbb{T})} + \|\partial_t(\mathcal{P}^{+}[\psi](t))\|_{L^2_{t,x}([-T,T]\times\mathbb{T})} \\ &\lesssim L_{\max}^4 \|y_{L_1}\|_{X^b}\|y_{L_2}\|_{X^b}\|\psi\|_{X^0} \lesssim L_{\max}^{12}\|\psi\|_{X^0}, \end{aligned} \tag{2.129}$$

where we used that $y_L = u_L - u_{L/2}$ is Fourier supported in $[-T,T]$, and that $u_L =$

$\sum_{L'\leq L} y_{L'}$ satisfies $\|y_{L'}\|_{X^b} \lesssim L'$ due to the induction argument. In particular, (2.129) implies that

$$\|\mathcal{P}^{\pm}\|_{Y^{1,0}} \lesssim L_{\max}^{12}. \tag{2.130}$$

By using Hölder inequality in τ and τ', we have

$$\begin{aligned}
\|\mathcal{P}^{\pm}\|_{Y^{b,b}} &= \left(\int_{\mathbb{R}^2} \langle\tau\rangle^{2b}\langle\tau'\rangle^{-2b}\|\widehat{P^{\pm}_{kk'}}(\tau,\tau')\|^2_{k\to k'}\mathrm{d}\tau\mathrm{d}\tau'\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^2} \langle\tau\rangle^{2(1-b)}\langle\tau'\rangle^{-2b}\|\widehat{P^{\pm}_{kk'}}(\tau,\tau')\|^2_{k\to k'}\mathrm{d}\tau\mathrm{d}\tau'\right)^{\frac{1}{2b}-\frac{1}{2}} \\
&\quad\times \left(\int_{\mathbb{R}^2} \langle\tau\rangle^{2}\langle\tau'\rangle^{-2b}\|\widehat{P^{\pm}_{kk'}}(\tau,\tau')\|^2_{k\to k'}\mathrm{d}\tau\mathrm{d}\tau'\right)^{1-\frac{1}{2b}} \\
&= \|\mathcal{P}^{\pm}\|^{\frac{1}{b}-1}_{Y^{1-b,b}}\|\mathcal{P}^{\pm}\|^{2-\frac{1}{b}}_{Y^{1,b}}
\end{aligned}$$

which together with (2.128) and (2.130) implies (2.125).

We turn to the proof of (2.126). By using (1.43) and Minkowski inequality, it follows that

$$\begin{aligned}
&\|\langle\tau\rangle\langle\tau'\rangle^{-b}\widehat{P^{\pm}_{kk'}}(\tau,\tau')\|_{L^2_{\tau\tau'}} \\
&\quad\lesssim \sum_{k_1-k_2=k-k'}\int_{\mathbb{R}^2}\left\|\frac{1}{\langle\tau'\rangle\langle\tau-\tau_1+\tau_2-\tau'-\Phi\rangle}\right\|_{L^2_{\tau\tau'}} |(\widehat{y_{L_1}})_{k_1}(\tau_1)(\overline{\widehat{y_{L_2}}})_{k_2}(\tau_2)|\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
&\quad\lesssim \int_{\mathbb{R}^2}\|(\widehat{y_{L_1}})_{k_1}(\tau_1)\|_{\ell^2_{k_1}}\|(\overline{\widehat{y_{L_2}}})_{k_2}(\tau_2)\|_{\ell^2_{k_2}}\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
&\quad\lesssim L_{\max}^{12}.
\end{aligned}$$

Thus by summing over $|k|,|k'|\lesssim N$ we conclude that

$$\|\mathcal{P}^{\pm}\|_{Z^{1,b}} \lesssim N^2 L_{\max}^{12}. \tag{2.131}$$

Then by using Hölder inequality in τ and τ', we have

$$\begin{aligned}
\|\mathcal{P}^\pm\|_{Z^{b,b}} &= \bigg(\int_{\mathbb{R}^2} \langle \tau \rangle^{2b} \langle \tau' \rangle^{-2b} \sum_{k,k'} |\widehat{P_{kk'}}(\tau,\tau')|^2 \mathrm{d}\tau \mathrm{d}\tau' \bigg)^{\frac{1}{2}} \\
&\le \bigg(\int_{\mathbb{R}^2} \langle \tau \rangle^{2(1-b)} \langle \tau' \rangle^{-2b} \sum_{k,k'} |\widehat{P_{kk'}}(\tau,\tau')|^2 \mathrm{d}\tau \mathrm{d}\tau' \bigg)^{\frac{1}{2b}-\frac{1}{2}} \\
&\qquad \times \bigg(\int_{\mathbb{R}^2} \langle \tau \rangle^{2} \langle \tau' \rangle^{-2b} \sum_{k,k'} |\widehat{P_{kk'}}(\tau,\tau')|^2 \mathrm{d}\tau \mathrm{d}\tau' \bigg)^{1-\frac{1}{2b}} \\
&= \|\mathcal{P}^\pm\|_{Z^{1-b,b}}^{\frac{1}{b}-1} \|\mathcal{P}^\pm\|_{Z^{1,b}}^{2-\frac{1}{b}},
\end{aligned}$$

which, together with (2.99) and (2.131), gives (2.126).

The proof for (2.127) is similar; thus, we omit it. □

In the next two subsections, we will establish (2.117) and (2.121) for the low-low-high random matrix $\mathscr{Y}^{+}_{kk'}$ given in (2.118); and (2.122) and (2.123) for the low-high-low random matrix $\mathscr{Y}^{-}_{kk'}$ given in (2.124).

2.5.2 Low-low-high random averaging operators

In this subsection, we focus on the low-low-high random matrix $\mathscr{Y}^{+}_{kk'}$ given in (2.118), and establish (2.117) and (2.121). We consider (2.118) with input (w_{L_1}, w_{L_2}) of types (a) (C, C), (b) (C, D), (c) (D, C), (d) (D, D).

Case (a): $(\mathbf{w_{L_1}}, \mathbf{w_{L_2}})$ of type (C, C)

In this case, the matrix (2.118) can be written as

$$\mathscr{Y}^{+}_{kk'} = \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \cdot \sum_{k_1',k_2'} h^{L_1,R_1}_{k_1k_1'} \overline{h^{L_2,R_2}_{k_2k_2'}} \frac{g_{k_1'} \overline{g_{k_2'}}}{[\![k_1']\!]^{\alpha/2} [\![k_2']\!]^{\alpha/2}}$$

where $\tilde{\mathfrak{h}}^{L_j,R_j} = \{h^{L_j,R_j}_{k_jk_j'}\}$ $(j \in \{1,2\})$ are either identity map over $L_j/2 < \langle k_j \rangle \le L_j$ or satisfy (2.119).

We first consider the non-pairing case, i.e. $k_1' \neq k_2'$, for which we apply Proposition 4 followed by Proposition 2 to get

$$\begin{aligned}\|\mathscr{Y}^{+}_{kk'}\|_{k\to k'} \lesssim &(L_1L_2)^{-\alpha/2}(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1\to k_2k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_2\to k_1k'}) \\ &+ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2\to k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{k\to k_1k_2k'})\prod_{j=1}^{2}\|h^{L_j,R_j}_{k_jk'_j}\|_{k_j\to k'_j},\end{aligned} \tag{2.132}$$

which is enough for our purpose. When $L_1 = L_2$, (2.132) follows from Proposition 4 and then Proposition 2 as $k'_1 \neq k'_2$. In what follows, we only consider (2.132) for the case where $L_1 \neq L_2$. Suppose $L_1 < L_2$ (the case $L_2 < L_1$ is similar). Then the coefficients $h^{L_2,R_2}_{k_2k'_2}$ may not be independent of $(g_{k'_1})_{L_1/2<\langle k'_1\rangle\le L_1}$. Therefore, we cannot use Proposition 4 to k'_1 and k'_2 directly. To overcome this obstacle, we first use Proposition 4 only with respect to k'_2 to obtain a bound for $\sum_{k_2} h^{L_2,R_2}_{k_2k'_2}$. Then we use Proposition 2 to get

$$\begin{aligned}\|\mathscr{Y}^{+}_{kk'}\|_{k\to k'} &\lesssim L_2^{-\alpha/2}\Big\|\sum_{k_1,k_2}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\sum_{k'_1}h^{L_1,R_1}_{k_1k'_1}\overline{h^{L_2,R_2}_{k_2k'_2}}\frac{g_{k'_1}}{[\![k'_1]\!]^{\alpha/2}}\Big\|_{(kk'_2\to k')\cap(k\to k'_2k')} \\ &\lesssim L_2^{-\alpha/2}\Big\|\sum_{k_1}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\sum_{k'_1}h^{L_1,R_1}_{k_1k'_1}\frac{g_{k'_1}}{[\![k'_1]\!]^{\alpha/2}}\Big\|_{(kk_2\to k')\cap(k\to k_2k')}\|h^{L_2,R_2}_{k_2k'_2}\|_{k_2\to k'_2}\end{aligned} \tag{2.133}$$

where $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$. Then, we note that the tensor $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\cdot h^{L_1,R_1}_{k_1k'_1}$ is independent of $(g_{k'_1})_{L_1/2<\langle k'_1\rangle\le L_1}$, which enables us to apply Proposition 4 again with respect to k'_1,

$$\begin{aligned}&\Big\|\sum_{k_1}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\sum_{k'_1}h^{L_1,R_1}_{k_1k'_1}\frac{g_{k'_1}}{[\![k'_1]\!]^{\alpha/2}}\Big\|_{(kk_2\to k')\cap(k\to k'k_2)} \\ &\quad\lesssim L_1^{-\alpha/2}\Big\|\sum_{k_1}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}h^{L_1,R_1}_{k_1k'_1}\Big\|_{(kk_2k'_1\to k')\cap(kk'_1\to k_2k')\cap(kk_2\to k'_1k')\cap(k\to k'_1k_2k')} \\ &\quad\lesssim L_1^{-\alpha/2}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_1k_2\to k')\cap(kk_1\to k_2k')\cap(kk_2\to k_1k')\cap(k\to k_1k_2k')}\|h^{L_1,R_1}_{k_1k'_1}\|_{k_1\to k'_1}.\end{aligned} \tag{2.134}$$

By collecting (2.133) and (2.134), we conclude (2.132) for $L_1 < L_2$. By Lemma 13, Lemma 14, with $N_1 = L_1, N_2 = L_2$, and $N_3 = N$, so that $N_1 \wedge N_2 = \min(N_1, N_2) = L_1$

and $N_1 \wedge N_2 = \min(N_2, N_3) = L_2$, we obtain

$$
\begin{aligned}
&\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_2k_1\to k')\cap(kk_1\to k_2k')\cap(kk_2\to k_1k')\cap(k\to k_1k_2k')} \\
&\quad = \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1\to k_2k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_2\to k_1k'} \\
&\qquad + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2\to k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{k\to k_1k_2k'} \\
&\quad \lesssim L_2^{1-\frac{\alpha}{2}} L_1^{1-\frac{\alpha}{2}} + L_1^{\frac{1}{2}-\frac{\alpha}{4}} L_2^{\frac{1}{2}-\frac{\alpha}{4}} + L_1^{1-\frac{\alpha}{2}}(\log L_2)^{\frac{1}{2}} + L_2^{\frac{1}{2}} + L_2^{1-\frac{\alpha}{2}}(\log L_1)^{\frac{1}{2}} + L_1^{\frac{1}{2}} \\
&\quad \lesssim (L_1L_2)^{1/2}.
\end{aligned}
\tag{2.135}
$$

From (2.132), (2.135), and (2.119), we conclude that

$$
\|\mathscr{Y}^{+}_{kk'}\|_{k\to k'} \lesssim (L_1L_2)^{-\alpha/2} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \lesssim L_{\max}^{-\frac{\alpha-1}{2}},
$$

which is sufficient for (2.117) since $\alpha > 1$ and $\delta_0 \ll \alpha - 1$.

A similar argument as above also gives

$$
\begin{aligned}
\|\mathscr{Y}^{+}_{kk'}\|_{kk'} &\lesssim (L_1L_2)^{-\alpha/2} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2k'} \prod_{j=1}^{2} \|h^{L_j,R_j}_{k_jk'_j}\|_{k_j\to k'_j} \\
&\lesssim (L_1L_2)^{-\alpha/2} (N^{1-\frac{\alpha}{2}} L_{\min}^{\frac{1}{2}} (\log L_{\max})^{\frac{1}{2}} + (L_1L_2)^{\frac{1}{2}}) \\
&\lesssim (L_1L_2)^{-\alpha/2} N^{\frac{1}{2}} L_{\min}^{\frac{1}{2}} L_{\max}^{\varepsilon} \lesssim N^{\frac{1}{2}} L_{\max}^{-\frac{\alpha}{2}+\varepsilon},
\end{aligned}
$$

where $0 < \varepsilon \ll 1$, $L_{\min} = \min(L_1, L_2)$, and $L_{\max} = \max(L_1, L_2)$, which gives (2.121).

Now we consider the pairing case, i.e. $k_1 = k_2$. Note that $L_1 = L_2 = L$. In this case, we have

$$
\begin{aligned}
\mathscr{Y}^{+}_{kk'} &= \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \sum_{k'_1} h^{L_1,R_1}_{k_1k'_1} \overline{h^{L_1,R_2}_{k_2k'_1}} \frac{|g_{k'_1}|^2}{[\![k'_1]\!]^{\alpha}} \\
&= \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \sum_{k'_1} h^{L_1,R_1}_{k_1k'_1} \overline{h^{L_1,R_2}_{k_2k'_1}} \frac{1}{[\![k'_1]\!]^{\alpha}} \\
&\qquad + \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \sum_{k'_1} h^{L_1,R_1}_{k_1k'_1} \overline{h^{L_1,R_2}_{k_2k'_1}} \frac{|g_{k'_1}|^2 - 1}{[\![k'_1]\!]^{\alpha}} \\
&= \mathscr{Y}^{(1)}_{kk'} + \mathscr{Y}^{(2)}_{kk'}.
\end{aligned}
\tag{2.136}
$$

We first consider the term $\mathscr{Y}^{(1)}_{kk'}$. By using Proposition 2, and then Lemma 14, we have

$$
\begin{aligned}
\|\mathscr{Y}^{(1)}_{kk'}\|_{k\to k'} &\lesssim L^{-\alpha}\Big\|\sum_{k_1,k_2,k_1'} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} h^{L_1,R_1}_{k_1k_1'}\overline{h^{L_1,R_2}_{k_2k_1'}}\Big\|_{k\to k'} \\
&\lesssim L^{-\alpha}\min(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2\to k'}, \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{k\to k_1k_2k'})\Big\|\sum_{k_1'} h^{L_1,R_1}_{k_1k_1'}\overline{h^{L_1,R_2}_{k_2k_1'}}\Big\|_{k_1k_2} \\
&\lesssim L^{-\alpha}\min(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2\to k'}, \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{k\to k_1k_2k'})\|h^{L_1,R_1}_{k_1k_1'}\|_{k_1\to k_1'}\|h^{L_1,R_2}_{k_2k_1'}\|_{k_2k_1'} \\
&\lesssim L^{-\alpha}(L^{1-\frac{\alpha}{2}+\varepsilon}+L^{\frac12})L^{\frac12+\gamma_0}\lesssim L^{1-\alpha+\gamma_0},
\end{aligned}
\tag{2.137}
$$

which is sufficient for (2.117) provided $\gamma \ll \alpha - 1$. For the term $\mathscr{Y}^{(2)}_{kk'}$ in (2.136), we apply Proposition 4 with $\eta_{k_1'} = |g_{k_1'}|^2 - 1$, then Lemma 2, and Lemma 14, to get

$$
\begin{aligned}
\|\mathscr{Y}^{(2)}_{kk'}\|_{k\to k'} &\lesssim \Big\|\sum_{k_1,k_2}\sum_{k_1'} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} h^{L_1,R_1}_{k_1k_1'}\overline{h^{L_1,R_2}_{k_2k_1'}}\frac{\eta_{k_1'}}{[\![k_1']\!]^\alpha}\Big\|_{k\to k'} \\
&\lesssim L^{-\alpha}\Big\|\sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} h^{L_1,R_1}_{k_1k_1'}\overline{h^{L_1,R_2}_{k_2k_1'}}\Big\|_{(kk_1'\to k')\cap(k\to k_1'k')} \\
&\lesssim L^{-\alpha}(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2\to k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{k\to k_1k_2k'})\prod_{j=1}^{2}\|h^{L_j,R_j}_{k_jk_j'}\|_{k_j\to k_j'} \\
&\lesssim L^{\frac12-\alpha},
\end{aligned}
\tag{2.138}
$$

which is sufficient for (2.117).

We proceed to consider the estimate (2.121) for the pairing case, where $\mathscr{Y}_{kk'}$ is given by (2.136). We begin with $\mathscr{Y}^{(1)}_{kk'}$ in (2.136). Here we have to use the cancellation that arises from the unitary property of $\tilde{H}^{N,L}_{kk'}$, namely Lemma . Specifically, by Corollary 4 and Remark 13, we can redefine $\mathscr{Y}^{(1)}_{kk'}$, keeping the same notation for $\mathscr{Y}^{(1)}_{kk'}$, as

$$
\mathscr{Y}^{(1)}_{kk'} = \sum_{k_1,k_2}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\sum_{k_1'} h^{L,R_1}_{k_1k_1'}\overline{h^{L,R_2}_{k_2k_1'}}\left(\frac{1}{[\![k_1']\!]^\alpha}-\frac{1}{[\![k_1]\!]^\alpha}\right),
$$

since $k_1 \neq k_2$ and $L_1 = L_2 = L$. From (2.95), we may assume that $|k_1 - k_1'| \lesssim L^\varepsilon R_1$, which implies that

$$\left|\frac{1}{[\![k_1']\!]^\alpha}-\frac{1}{[\![k_1]\!]^\alpha}\right|\lesssim L^{-\alpha-1+\varepsilon}R_1. \tag{2.139}$$

Then, a similar argument as in (2.137) together with (2.139) yields

$$\begin{aligned}\|\mathscr{Y}^{(1)}_{kk'}\|_{kk'}\lesssim&L^{-\alpha-1+\varepsilon}R_1\Big\|\sum_{k_1,k_2,k_1'}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}h^{L_1,R_1}_{k_1k_1'}\overline{h^{L_1,R_2}_{k_2k_1'}}\Big\|_{kk'}\\\lesssim&L^{-\alpha-1+\varepsilon}R_1\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2k'}\|h^{L_1,R_1}_{k_1k_1'}\|_{k_1k_1'}\|h^{L_1,R_2}_{k_2k_1'}\|_{k_2\to k_1'}.\end{aligned} \tag{2.140}$$

By Lemma 12 and (2.119), we get

$$\begin{aligned}\|\mathscr{Y}^{(1)}_{kk'}\|_{kk'}\lesssim&L^{-\alpha-1+\varepsilon}R_1(N^{1-\frac{\alpha}{2}+\varepsilon}L^{\frac12}+L)L^{\frac12+\gamma_0}R_1^{-\frac12}\\\lesssim&N^{1-\frac{\alpha}{2}+\varepsilon}L^{\frac12-\alpha+\varepsilon+\gamma_0}+L^{1-\alpha+\varepsilon+\gamma_0},\end{aligned} \tag{2.141}$$

which is sufficient for the estimate (2.121).

Finally, we turn to $\mathscr{Y}^{(2)}_{kk'}$. Proceeding as in (2.138), we have

$$\begin{aligned}\|\mathscr{Y}^{(2)}_{kk'}\|_{kk'}\lesssim&L^{-\alpha}\Big\|\sum_{k_1,k_2}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}h^{L,R_1}_{k_1k_1'}\overline{h^{L,R_2}_{k_2k_1'}}\Big\|_{(kk'\to k_1')\cap(kk'k_1')}\\\lesssim&L^{-\alpha}\Big(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk'\to k_1k_2}\|h^{L,R_1}_{k_1k_1'}\overline{h^{L,R_2}_{k_2k_1'}}\|_{k_1k_2\to k_1'}\\&\qquad+\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2k'}\|h^{L,R_1}_{k_1k_1'}\overline{h^{L,R_2}_{k_2k_1'}}\|_{k_1'\to k_1k_2}\Big)\\\lesssim&L^{-\alpha}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2k'}\|h^{L,R_1}_{k_1k_1'}\|_{k_1'\to k_1}\|h^{L,R_2}_{k_2k_1'}\|_{k_1'\to k_2}\\\lesssim&L^{-\alpha}(N^{1-\frac{\alpha}{2}+\varepsilon}L^{\frac12}+L)\lesssim N^{1-\frac{\alpha}{2}+\varepsilon}L^{\frac12-\alpha}+L^{1-\alpha},\end{aligned} \tag{2.142}$$

which is again sufficient for (2.121).

Finally, by collecting (2.136), (2.137), (2.138), (2.141), and (2.142), we finish the proof of (2.117) and (2.121) for Case (a).

Case (b) and Case (c): $(\mathbf{w_{L_1}}, \mathbf{w_{L_2}})$ of type (C, D) or (D, C)

Without loss of generality, we only consider that (w_{L_1}, w_{L_2}) are of type (D, C). The random matrix (2.118) can be written as

$$\mathscr{Y}^+_{kk'} = \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} (w_{L_1})_{k_1} \cdot \sum_{k_2'} \overline{h^{L_2,R_2}_{k_2k_2'}} \frac{\overline{g_{k_2'}}}{[\![k_2']\!]^{\alpha/2}},$$

where w_{L_1} satisfies (2.120) and $\tilde{\mathfrak{h}}^{L_2,R_2} = \{h^{L_2,R_2}_{k_2k_2'}\}$ are either identity map over $L_2/2 < \langle k_2 \rangle \leq L_2$ or satisfying (2.119). We apply Proposition 2 to get

$$\|\mathscr{Y}^+_{kk'}\|_{k\to k'} \lesssim \Big\| \sum_{k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \sum_{k_2'} \overline{h^{L_2,R_2}_{k_2k_2'}} \frac{\overline{g_{k_2'}}}{[\![k_2']\!]^{\alpha/2}} \Big\|_{(k\to k_1k')\cup(kk_1\to k')} \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}}, \tag{2.143}$$

where $\|\cdot\|_{X\cup Y} = \min(\|\cdot\|_X, \|\cdot\|_Y)$. Recall that the tensor $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \overline{h^{L_2,R_2}_{k_2k_2'}}$ is independent of F_{L_2}. We can apply Proposition 4 to (2.143), and then apply Proposition 2 to get

$$\begin{aligned}
\|\mathscr{Y}^+_{kk'}\|_{k\to k'} \lesssim & L_2^{-\frac{\alpha}{2}} \min\big(\|\sum_{k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \overline{h^{L_2,R_2}_{k_2k_2'}}\|_{(kk_2'\to k_1k')\cap(k\to k_1k_2'k')}, \\
& \qquad \|\sum_{k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \overline{h^{L_2,R_2}_{k_2k_2'}}\|_{(kk_1k_2'\to k')\cap(kk_1\to k_2'k')}\big) \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \\
\lesssim & L_2^{-\frac{\alpha}{2}} \|h^{L_2,R_2}_{k_2k_2'}\|_{k_2\to k_2'} \min\big(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_2\to k_1k')\cap(k\to k_1k_2k')}, \\
& \qquad \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_1k_2\to k')\cap(kk_1\to k_2k')}\big) \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \\
\lesssim & L_2^{-\frac{\alpha}{2}} \|h^{L_2,R_2}_{k_2k_2'}\|_{k_2\to k_2'} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_1k_2\to k')\cap(kk_1\to k_2k')} \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}}.
\end{aligned} \tag{2.144}$$

By Lemma 13 and Lemma 14, we have

$$\begin{aligned}
\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_1k_2\to k')\cap(kk_1\to k_2k')} &= \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1k_2\to k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1\to k_2k'} \\
&\lesssim L_1^{1-\frac{\alpha}{2}} (\log L_2)^{\frac{1}{2}} + L_2^{\frac{1}{2}} + L_1^{1-\frac{\alpha}{2}} L_2^{1-\frac{\alpha}{2}} \lesssim L_1^{1-\frac{\alpha}{2}} L_2^{\frac{1}{2}},
\end{aligned}$$

which together with (2.144), (2.119), and (2.120), implies that

$$
\begin{aligned}
\|\mathscr{Y}^{+}_{kk'}\|_{k\to k'} &\lesssim L_2^{-\frac{\alpha}{2}} \|h^{L_2,R_2}_{k_2k_2'}\|_{k_2\to k_2'} L_1^{1-\frac{\alpha}{2}} L_2^{\frac{1}{2}} \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \\
&\lesssim L_1^{1-\frac{\alpha}{2}} L_1^{-\frac{1}{2}-\gamma} L_2^{\frac{1}{2}-\frac{\alpha}{2}} \lesssim (L_1L_2)^{\frac{1}{2}-\frac{\alpha}{2}},
\end{aligned}
$$

which proves (2.117) since $\delta_0 \ll \alpha - 1$.

Now we turn to the estimate (2.121). Similar computation as in (2.143) and (2.144) yields

$$
\begin{aligned}
\|\mathscr{Y}^{+}_{kk'}\|_{kk'} &\lesssim \Big\| \sum_{k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \sum_{k_2'} \overline{h^{L_2,R_2}_{k_2k_2'}} \frac{\overline{g_{k_2'}}}{[\![k_2']\!]^{\alpha/2}} \Big\|_{(kk_1k')\cup(kk'\to k_1)} \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \\
&\lesssim L_2^{-\frac{\alpha}{2}} \min\bigg(\Big\| \sum_{k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \overline{h^{L_2,R_2}_{k_2k_2'}} \Big\|_{(kk_1k_2'k')\cap(kk_1k'\to k_2')}, \\
&\qquad \Big\| \sum_{k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \overline{h^{L_2,R_2}_{k_2k_2'}} \Big\|_{(kk_2'k'\to k_1)\cap(kk'\to k_1k_2')} \bigg) \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \qquad (2.145) \\
&\lesssim L_2^{-\frac{\alpha}{2}} \|h^{L_2,R_2}_{k_2k_2'}\|_{k_2\to k_2'} \min(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_1k_2k')\cap(kk_1k'\to k_2)}, \\
&\qquad \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_2k'\to k_1)\cap(kk'\to k_1k_2)}) \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \\
&\lesssim L_2^{-\frac{\alpha}{2}} \|h^{L_2,R_2}_{k_2k_2'}\|_{k_2\to k_2'} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_2k'\to k_1)\cap(kk'\to k_1k_2)} \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}}.
\end{aligned}
$$

By Lemma 13 and Lemma 14, we have

$$
\begin{aligned}
\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{(kk_2k'\to k_1)\cap(kk'\to k_1k_2)} &= \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_2k'\to k_1} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk'\to k_1k_2} \\
&\lesssim N^{1-\frac{\alpha}{2}} (\log L_2)^{\frac{1}{2}} + L_2^{\frac{1}{2}} + L_{\min}^{1-\frac{\alpha}{2}} N^{1-\frac{\alpha}{2}} \qquad (2.146) \\
&\lesssim N^{1-\frac{\alpha}{2}} L_2^{\frac{\alpha-1}{2}} + L_{\min}^{1-\frac{\alpha}{2}} N^{1-\frac{\alpha}{2}}.
\end{aligned}
$$

By collecting (2.144), (2.119), (2.120), and (2.146), we conclude that

$$
\begin{aligned}
\|\mathscr{Y}^{+}_{kk'}\|_{kk'} &\lesssim L_2^{-\frac{\alpha}{2}} \|h^{L_2,R_2}_{k_2k_2'}\|_{k_2\to k_2'} (N^{1-\frac{\alpha}{2}} L_2^{\frac{\alpha-1}{2}} + L_{\min}^{1-\frac{\alpha}{2}} N^{1-\frac{\alpha}{2}}) \|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}} \\
&\lesssim (N^{1-\frac{\alpha}{2}} L_2^{\frac{\alpha-1}{2}} + L_{\min}^{1-\frac{\alpha}{2}} N^{1-\frac{\alpha}{2}}) L_2^{-\frac{\alpha}{2}} L_1^{-\frac{1}{2}-\gamma} \qquad (2.147) \\
&\lesssim N^{1-\frac{\alpha}{2}} L_2^{-\frac{1}{2}} L_1^{-\frac{1}{2}-\gamma} + N^{1-\frac{\alpha}{2}} L_{\min}^{1-\frac{\alpha}{2}} L_2^{-\frac{\alpha}{2}} L_1^{-\frac{1}{2}-\gamma} \\
&\lesssim N^{\frac{1}{2}} L_{\max}^{-\frac{\alpha}{2}},
\end{aligned}
$$

which is sufficient for (2.121).

Therefore, we finish the proof of (2.117) and (2.121) for Case (b): type (C, D). The proof for Case (c): type (D, C) follows similarly.

Case (d): $(\mathbf{w_{L_1}}, \mathbf{w_{L_2}})$ **of type (D, D)**

In this case, the random matrix $\mathscr{Y}^+_{kk'}$ in (2.118) can be written as

$$\mathscr{Y}^+_{kk'} = \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \cdot (w_{L_1})_{k_1} \cdot (\overline{w_{L_2}})_{k_2},$$

where w_{L_1} and w_{L_2} satisfy (2.120). By using Proposition 2, followed by Lemma 13 and (2.120), we obtain

$$\begin{aligned}
\|\mathscr{Y}^+_{kk'}\|_{k\to k'} &\lesssim \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk_1\to k_2k'} \|(w_{L_1})_{k_1}\|_{k_1} \|(\overline{w_{L_2}})_{k_2}\|_{k_2} \\
&\lesssim (L_1L_2)^{1-\frac{\alpha}{2}} (L_1L_2)^{-\frac{1}{2}-\gamma} \\
&\lesssim L^{\frac{1-\alpha}{2}-\gamma} \lesssim L^{-4\delta_0},
\end{aligned}$$

which is sufficient for (2.117), provided $\delta_0 \ll \alpha - 1$.

Similarly, we have

$$\begin{aligned}
\|\mathscr{Y}^+_{kk'}\|_{kk'} &\lesssim \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}\|_{kk'\to k_1k_2} \|(w_{L_1})_{k_1}\|_{k_1} \|(\overline{w_{L_2}})_{k_2}\|_{k_2} \\
&\lesssim N^{1-\frac{\alpha}{2}} L_{\min}^{1-\frac{\alpha}{2}} (L_1L_2)^{-\frac{1}{2}-\gamma} \\
&\lesssim N^{1-\frac{\alpha}{2}} L_{\max}^{-\frac{1}{2}-\gamma} \lesssim N^{\frac{1}{2}} L_{\max}^{-\frac{\alpha}{2}-\gamma},
\end{aligned}$$

which again proves (2.121).

Therefore,we have finished the proof of (2.117) and (2.118) for Case (d).

2.5.3 Low-high-low random averaging operators

In this subsection, we consider the low-high-low random matrix $\mathscr{Y}^-_{kk'}$ defined in (2.124), and prove (2.122) and (2.123). We follow a similar strategy as in the previous subsection, and consider four cases for (2.124) depending on whether (w_{L_1}, w_{L_3}) are (a) (C, C), (b) (C, D), (c) (D, C), (d) (D, D).

Case (a): $(\mathbf{w_{L_1}}, \mathbf{w_{L_3}})$ of type (C, C)

In this case, the matrix (2.124) can be written as

$$\mathscr{Y}^{-}_{kk'} = \sum_{k_1,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3} \sum_{k_1',k_3'} h^{L_1,R_1}_{k_1k_1'} h^{L_3,R_3}_{k_3k_3'} \frac{g_{k_1'} g_{k_3'}}{[\![k_1']\!]^{\alpha/2}[\![k_2']\!]^{\alpha/2}}$$

where $\mathfrak{h}^{L_j,R_j} = \{h^{L_j,R_j}_{k_jk_j'}\}$ $(j \in \{1,3\})$ are either identity map over $L_j/2 < \langle k_j' \rangle \leq L_j$ or satisfying (2.119). Similar argument as in Subsection 2.5.2 yields

$$\begin{aligned} \|\mathscr{Y}^{-}_{kk'}\|_{k\to k'} \lesssim (L_1L_3)^{-\alpha/2}(&\|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1\to k'k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{k\to k_1k'k_3} \\ &+ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1\to k'k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{k\to k_1k'k_3}) \\ &\times \|h^{L_1,R_1}_{k_1k_1'}\|_{k_1\to k_1'} \|h^{L_3,R_3}_{k_3k_3'}\|_{k_3\to k_3'}. \end{aligned} \tag{2.148}$$

By using Lemma 13, Lemma 14, with $N_1 = L_1$, $N_2 \sim N$, and $N_3 = L_3$, we obtain

$$\begin{aligned} &\|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1\to k'k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{k\to k_1k'k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1k_3\to k'} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_3\to k_1k'} \\ &\quad\lesssim (L_1L_3)^{1-\frac{\alpha}{2}} + L_{\max}^{1-\frac{\alpha}{2}}(\log L_{\min})^{\frac{1}{2}} + L_{\min}^{\frac{1}{2}} \\ &\quad\lesssim (L_1L_3)^{\frac{1}{2}}. \end{aligned} \tag{2.149}$$

where $L_{\min} = \min(L_1, L_3)$ and $L_{\max} = \max(L_1, L_3)$. Thus it follows from (2.148), (2.149), and (2.119), that

$$\|\mathscr{Y}^{-}_{kk'}\|_{k\to k'} \lesssim (L_1L_3)^{-\alpha/2} L_1^{1/2} L_2^{1/2} \lesssim L_{\max}^{\frac{1-\alpha}{2}},$$

which is sufficient for (2.122). A similar argument also gives

$$\begin{aligned} \|\mathscr{Y}^{-}_{kk'}\|_{kk'} \lesssim (L_1L_3)^{-\alpha/2}(&\|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk'k_3\to k_1} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1k'\to k_3} \\ &+ \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1k'k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk'\to k_1k_3}) \\ &\times \|h^{L_1,R_1}_{k_1k_1'}\|_{k_1\to k_1'} \|h^{L_3,R_3}_{k_3k_3'}\|_{k_3\to k_3'}. \end{aligned} \tag{2.150}$$

By using Lemma 13, Lemma 14, with $N_1 = L_1$, $N_2 \sim N$, and $N_3 = L_3$, we obtain

$$\begin{aligned}&\|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk'k_3\to k_1} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1k'\to k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk_1k'k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk'\to k_1k_3}\\ &\quad\lesssim L_3^{1-\frac{\alpha}{2}}(\log N)^{\frac{1}{2}} + N^{\frac{1}{2}} + L_1^{1-\frac{\alpha}{2}}(\log N)^{\frac{1}{2}} + L_{\max}^{1-\frac{\alpha}{2}}(L_{\min}\log N)^{\frac{1}{2}} + (L_{\min}N)^{\frac{1}{2}} \\ &\quad\lesssim N^{\frac{1}{2}}L_{\min}^{\frac{1}{2}}.\end{aligned} \tag{2.151}$$

By collecting (2.150) and (4.40), we conclude that

$$\|\mathscr{Y}^{-}_{kk'}\|_{kk'} \lesssim (L_1L_3)^{-\alpha/2}N^{1/2}L_{\min}^{1/2} \lesssim N^{1/2}L_{\max}^{-\alpha/2},$$

from which by choosing $\gamma_0 \ll \alpha - 1$ we finish the proof of (2.123).

Case (b) and Case (c): $(\mathbf{w_{L_1}}, \mathbf{w_{L_3}})$ of type (C, D) or (D, C)

Without loss of generality, we only consider (w_{L_1}, w_{L_3}) is of type (D, C). Thus, the random matrix (2.124) can be written as

$$\mathscr{Y}^{-}_{kk'} = \sum_{k_1,k_3}\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'}(w_{L_1})_{k_1}\sum_{k_3'}h^{L_3,R_3}_{k_3k_3'}\frac{g_{k_3'}}{[\![k_3']\!]^{\alpha/2}},$$

where w_{L_1} satisfies (2.120) and $\mathfrak{h}^{L_3,R_3} = \{h^{L_3,R_3}_{k_3k_3'}\}$ are either identity map over $L_3/2 < \langle k_3'\rangle \le L_3$ or satisfying (2.119). Similar argument as in Subsubsection 2.5.2 yield

$$\begin{aligned}\|\mathscr{Y}^{-}_{kk'}\|_{k\to k'} &\lesssim L_3^{-\alpha/2}\|h^{L_3,R_3}_{k_3k_3'}\|_{k_3\to k_3'}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{(kk_1k_3\to k')\cap(kk_1\to k_3k')}\|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}}\\ &\lesssim L_1^{-\frac{1}{2}-\gamma}L_3^{-\frac{\alpha}{2}}(L_{\max}^{1-\frac{\alpha}{2}}(\log L_{\min})^{\frac{1}{2}} + L_{\min}^{\frac{1}{2}} + (L_1L_3)^{1-\frac{\alpha}{2}})\\ &\lesssim L_1^{-\frac{1}{2}-\gamma}L_3^{-\frac{\alpha}{2}}(L_1L_3)^{1-\frac{\alpha}{2}} \lesssim L_{\max}^{\frac{1}{2}-\frac{\alpha}{2}-\gamma},\end{aligned}$$

which is again sufficient for (2.122) provided $\delta_0 \ll \alpha - 1$.

Similar computation as in (2.145), (2.146), and (2.147) yields

$$\begin{aligned}\|\mathscr{Y}^-_{kk'}\|_{kk'} &\lesssim L_3^{-\alpha/2}\|h^{L_3,R_3}_{k_3k_3'}\|_{k_3\to k_3'}\|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{(kk'k_3\to k_1)\cap(kk'\to k_1k_3)}\|(w_{L_1})_{k_1}\|_{\ell^2_{k_1}}\\ &\lesssim L_1^{-\frac12-\gamma}L_3^{-\frac{\alpha}{2}}(L_3^{1-\frac{\alpha}{2}}(\log N)^{\frac12}+N^{\frac12}+L_{\min}^{\frac12-\frac{\alpha}{4}}N^{\frac12-\frac{\alpha}{4}})\\ &\lesssim N^\varepsilon L_1^{-\gamma-\frac12}L_3^{1-\alpha}+L_1^{-\frac12-\gamma}L_3^{-\frac{\alpha}{2}}N^{\frac12}+N^{\frac12-\frac{\alpha}{4}}L_1^{-\gamma-\frac{\alpha}{4}}L_3^{\frac12-\frac{3\alpha}{4}}\\ &\lesssim N^{\frac12}L_{\max}^{-\frac12-\gamma}\end{aligned}$$

which is sufficient for (2.123) as $\gamma_0 \ll \gamma$. Here we used that $\alpha \in (1,2)$.

Therefore, we finished the proof of (2.122) and (2.123) for Case (b): type (C, D). The proof for Case (c): type (D, C) follows similarly.

Case (d): $(\mathbf{w_{L_1}}, \mathbf{w_{L_3}})$ of type (D, D)

In this case, the matrix (2.124) can be written as

$$\mathscr{Y}^-_{kk'} = \sum_{k_1,k_3}\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\cdot(w_{L_1})_{k_1}\cdot(w_{L_3})_{k_3},$$

where w_{L_1} and w_{L_3} satisfy (2.120). By using Proposition 2, followed by Lemma 13 and (2.120), we obtain

$$\begin{aligned}\|\mathscr{Y}^-_{kk'}\|_{k\to k'} &\lesssim \|\mathrm{T}^{\mathrm{b},m}_{kk_1k'k_3}\|_{kk'\to k_1k_3}\|(w_{L_1})_{k_1}\|_{k_1}\|(w_{L_3})_{k_3}\|_{k_3}\\ &\lesssim (NL_{\min})^{\frac12-\frac{\alpha}{4}}(L_1L_3)^{-\frac12-\gamma}\\ &\lesssim N^{\frac12-\frac{\alpha}{4}}L_{\max}^{-\frac12-\gamma}\lesssim N^{\frac12}L_{\max}^{-\frac12-\frac{\alpha}{4}-\gamma},\end{aligned}$$

where we used that $N \geq L_{\max}$, which proves (2.123).

Therefore, we have finished the proof of (2.122) and (2.124) for Case (d): type (D, D).

2.5.4 Random tensors

The main purpose of this subsection is to prove (2.94) and (2.95) in $\texttt{Local}(M)$ by assuming $\texttt{Local}\left(\frac{M}{2}\right)$. Let

$$\mathfrak{p}^{N,L} = \mathcal{P}^{N,L} - \mathcal{P}^{N,L/2}, \tag{2.152}$$

where $\mathcal{P}^{N,L}$ be the linear operator defined in (2.34). Recall from (2.36) and (2.152) that

$$\mathcal{H}^{N,L} = \Pi_N(1-\mathcal{P}^{N,L})^{-1} = \Pi_N + \sum_{n=1}^{\infty}(\mathcal{P}^{N,L})^n, \tag{2.153}$$

and thus the operator $\mathfrak{h}^{N,L}$ in (2.38) has the following expansion,

$$\mathfrak{h}^{N,L} = \sum_{n=1}^{\infty}(-1)^{n-1}(\mathcal{H}^{N,L}\mathfrak{p}^{N,L})^n\mathcal{H}^{N,L}. \tag{2.154}$$

Recall the operator $\mathcal{P}^+ := \mathcal{P}^+_{N,L_1,L_2}$ from (2.97). We have from (2.34) that

$$\begin{aligned}\mathcal{P}^{N,L}[\psi] &= -2\mathrm{i}\chi(t)\int_0^t \Pi_N\mathcal{M}(u_L,u_L,\psi)(t')\mathrm{d}t' \\ &= -2\mathrm{i}\sum_{L_1,L_2\le L}\chi(t)\int_0^t \Pi_N\mathcal{M}(y_{L_1},y_{L_2},\psi)(t')\mathrm{d}t' \\ &= \mathcal{P}^{N,\frac{L}{2}}\psi + \sum_{L_{\max}=L}\mathcal{P}^+_{N,L_1,L_2}\psi,\end{aligned} \tag{2.155}$$

where $\chi(t)$ is the time localisation given in (2.20). Recall that $\mathcal{P}^+_{N,L_1,L_2}$ satisfies the estimates (2.98)-(2.99), which implies that $\mathfrak{p}^{N,L}$ also satisfies the estimates (2.98) and (2.99) but with a logarithm loss coming from the summation over L_i. This logarithm loss is harmless since it can be absorbed. Namely, from (2.152), (2.155), and Corollary 5, we have

$$\begin{cases}\|\mathfrak{p}^{N,L}\|_{X^b\to X^b} \lesssim T^{c\theta}L^{-2\delta_0}(\log L)^2; \\ \|\mathfrak{p}^{N,L}\|_{Z^{b,b}} \lesssim T^{c\theta}N^{\frac{1}{2}+\frac{2\gamma_0}{3}}L^{-\frac{1}{2}-\gamma_0}(\log L)^2,\end{cases} \tag{2.156}$$

which together with (2.152) implies

$$\begin{cases} \|\mathcal{P}^{N,L}\|_{Y^{b,b}} = \|\mathcal{P}^{N,L}\|_{X^b\to X^b} \lesssim T^{c\theta}, \\ \|\mathcal{P}^{N,L}\|_{Z^{b,b}} \lesssim T^{c\theta} N^{\frac{1}{2}+\frac{2\gamma_0}{3}} \end{cases} \tag{2.157}$$

for some $c > 0$.

From the definition of the operator norm, we note that

$$\|\mathcal{A}\mathcal{B}\|_{Y^{b,b}} = \|\mathcal{A}\mathcal{B}\|_{X^b\to X^b} \le \|\mathcal{A}\|_{X^b\to X^b}\|\mathcal{B}\|_{X^b\to X^b} \tag{2.158}$$

which together with (2.153) and (2.157) implies that

$$\begin{aligned} \|\mathcal{H}^{N,L} - \Pi_N\|_{X^b\to X^b} &= \Big\| \sum_{n=1}^{\infty} (\mathcal{P}^{N,L})^n \Big\|_{X^b\to X^b} \le \sum_{n=1}^{\infty} \|(\mathcal{P}^{N,L})^n\|_{X^b\to X^b} \\ &\le \sum_{n=1}^{\infty} \|\mathcal{P}^{N,L}\|^n_{X^b\to X^b} \le \|\mathcal{P}^{N,L}\|_{X^b\to X^b} \sum_{n\ge 1} (CT^{c\theta})^{n-1} \lesssim T^{c\theta}. \end{aligned} \tag{2.159}$$

Also, note that $\|\Pi_N\|_{X^b(J)\to X^b(J)} \lesssim 1$. In particular, we conclude from (2.159) that

$$\|\mathcal{H}^{N,L}\|_{X^b\to X^b} \lesssim 1,$$

which together with (2.154), (2.158), and (2.156), implies

$$\begin{aligned} \|\mathfrak{h}^{N,L}\|_{X^b\to X^b} &\le \sum_{n=1}^{\infty} \|(\mathcal{H}^{N,L}\mathfrak{p}^{N,L})^n \mathcal{H}^{N,L}\|_{X^b\to X^b} \le \sum_{n=1}^{\infty} \|\mathcal{H}^{N,L}\|^{n+1}_{X^b\to X^b} \|\mathfrak{p}^{N,L}\|^n_{X^b\to X^b} \\ &\le \|\mathfrak{p}^{N,L}\|_{X^b\to X^b} \sum_{n=1}^{\infty} C^{n+1} (T^{c\theta} L^{-\delta_0})^{n-1} \lesssim T^{c\theta} L^{-\delta_0}, \end{aligned}$$

which gives the first bound in (2.94) by choosing $T \ll 1$.

We also note that

$$\|\mathcal{A}\mathcal{B}\|_{Z^{b,b}} \le \min\big(\|\mathcal{A}\|_{X^b\to X^b}\|\mathcal{B}\|_{Z^{b,b}}, \|\mathcal{A}\|_{Z^{b,b}}\|\mathcal{B}\|_{X^b\to X^b}\big),$$

which together with (2.154) and (2.157) implies

$$\begin{aligned}
\|\mathfrak{h}^{N,L}\|_{Z^{b,b}} &\leq \sum_{n=1}^{\infty} \|(\mathcal{H}^{N,L}\mathfrak{p}^{N,L})^n \mathcal{H}^{N,L}\|_{Z^{b,b}} \\
&\leq \sum_{n=1}^{\infty} \|\mathcal{H}^{N,L}\|^{n+1}_{X^\alpha \to X^\alpha} \|\mathfrak{p}^{N,L}\|^{n-1}_{X^\alpha \to X^\alpha} \|\mathfrak{p}^{N,L}\|_{Z^{b,b}} \\
&\leq \|\mathfrak{p}^{N,L}\|_{Z^{b,b}} \sum_{n=1}^{\infty} C^{n+1} (T^{c\theta} L^{-\delta_0})^{n-1} \\
&\leq T^{c\theta} N^{\frac{1}{2}+\gamma_0} L^{-\frac{1}{2}},
\end{aligned}$$

provided $T \ll 1$, which completes the proof of the second bound of (2.94).

Finally, from Proposition 3, we have

$$\left\| \left(1 + \frac{|k-k'|}{L}\right)^{\kappa} (\widehat{\mathcal{A}\mathcal{B}})_{kk'} \right\|_{Z^{b,b}} \leq \|\mathcal{A}\|_{X^b \to X^b} \left\| \left(1 + \frac{|k-k'|}{L}\right)^{\kappa} \widehat{\mathcal{B}}_{kk'} \right\|_{Z^{b,b}},$$

which, together with a similar argument as above, gives

$$\left\| \left(1 + \frac{|k-k'|}{L}\right)^{\kappa} h^{N,L}_{kk'} \right\|_{Z^{b,b}} \leq T^{c\theta} N,$$

provided $0 < T \ll 1$, which proves (2.95).

2.6 The remainder terms

In this section, we continue the proof of Proposition 5. We have shown (2.94), (2.95), (2.98), (2.99), and (2.100) of $\texttt{Local}(M)$ in Section 2.5. It remains to prove (2.96), which will be the main focus of this section. To be more precise, we shall prove the bound (2.96) with $N = M$, i.e.,

$$\|z_M\|_{X^b(J)} \leq M^{-\frac{1}{2}-\gamma} \tag{2.160}$$

by assuming $\texttt{Local}\left(\frac{M}{2}\right)$. The main strategy of this section is to show (2.160) by a continuity argument, with the help of (2.94), (2.95), (2.98), (2.99), and (2.100) of $\texttt{Local}(M)$.

Recall that z_M is given by the equation (2.31), whose right-hand side can be written as the combination of the expressions

$$\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})(t) = \chi(t)\int_0^t \Pi_N \mathcal{M}(w_{N_1}, w_{N_2}, w_{N_3})(s)\mathrm{d}s, \tag{2.161}$$

and

$$\mathscr{B}(w_{N_1}, w_{N_2}, w_{N_3})(t) = \chi(t)\int_0^t \Delta_N \mathcal{M}(w_{N_1}, w_{N_2}, w_{N_3})(s)\mathrm{d}s, \tag{2.162}$$

where w_{N_i} are of either type (C) or type (D) defined in Subsection 2.3.1. In particular, we can grouped the right-hand side of (2.31) as follows

(I) $\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})$ with w_{N_j} for $j \in \{1, 2, 3\}$ of any type and $N_{\text{med}} = N_{\max} = M$.

(II) $\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})$ with w_{N_j} for $j \in \{1, 2\}$ are of any type and $N_j \leq M/2$ for $j \in \{1, 2\}$.

(III) $\mathscr{A}(w_{N_1}, w_{N_2}, \psi_{M,L_M})$ with w_{N_j} for $j \in \{1, 2\}$ are of any type with $L_M < \max(N_1, N_2) \leq M/2$.

(IV) $\mathscr{A}(w_{N_1}, z_M, w_{N_3})$ with w_{N_j} for $j \in \{1, 3\}$ are of any type and $N_j \leq M/2$ for $j \in \{1, 3\}$.

(V) $\mathscr{A}(w_{N_1}, \psi_{M,L_M}, w_{N_3})$ where w_{N_j} for $j \in \{1, 3\}$ are of any type with $\max(N_1, N_3) \leq M/2$.

(VI) $\mathscr{B}(w_{N_1}, w_{N_2}, w_{N_3})$ with w_{N_j} for $j \in \{1, 2, 3\}$ are of any type and $N_{\max} \leq M/2$.

(VII) The term

$$\int_0^t \mathcal{R}(w_N, w_N, w_N)(t')\mathrm{d}t'$$

with w_N of any type and $N \in \{M, M/2\}$.

Our goal in this section is to recover the bound for z_M in (2.160) for each of the terms (I)–(VII) above, i.e.

$$\|\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})\|_{X^b} \lesssim M^{-\frac{1}{2}-\gamma}, \tag{2.163}$$

$$\|\mathscr{B}(w_{N_1}, w_{N_2}, w_{N_3})\|_{X^b} \lesssim M^{-\frac{1}{2}-\gamma}, \tag{2.164}$$

for all possibilities types of $(w_{N_1}, w_{N_2}, w_{N_3})$ as above.

2.6.1 Preparation of the proof of (2.96)

We first see that

$$\|\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})\|_{X^1} \lesssim \|\mathcal{M}(w_{N_1}, w_{N_2}, w_{N_3})\|_{L^2_{t,x}} \lesssim M^2 \prod_{i=1}^{3} \|w_{N_i}\|_{X^b} \lesssim M^{14}.$$

Therefore, to prove (2.163), from interpolation, it suffices to show

$$\|\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})\|_{X^{1-b}} \lesssim M^{-\frac{1}{2}-2\gamma}, \tag{2.165}$$

provided $b - \frac{1}{2} \ll \gamma_0$. By (2.23), (1.42), and (2.161), we have

$$\begin{aligned}
&\mathcal{F}_{t,x}(\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3}))(\tau, k) \\
&\quad = -\mathrm{i} \sum_{\substack{k_1-k_2+k_3=k \\ k_2 \notin \{k_1,k_3\}}} \int_{\mathbb{R}^3} \mathcal{K}(\tau, \Phi + \tau_1 - \tau_2 + \tau_3) \\
&\qquad \times (\widehat{w_{N_1}})_{k_1}(\tau_1) \cdot (\overline{\widehat{w_{N_2}}})_{k_2}(\tau_2) \cdot (\widehat{w_{N_3}})_{k_3}(\tau_3) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3 \\
&\quad = -\mathrm{i} \sum_{m} \int_{\mathbb{R}^3} \sum_{k_1,k_2,k_3} \mathcal{K}(\tau, m + \Phi - [\Phi] + \tau_1 - \tau_2 + \tau_3) \times \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \\
&\qquad \times (\widehat{w_{N_1}})_{k_1}(\tau_1) \cdot (\overline{\widehat{w_{N_2}}})_{k_2}(\tau_2) \cdot (\widehat{w_{N_3}})_{k_3}(\tau_3) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3,
\end{aligned} \tag{2.166}$$

where Φ is defined in (2.24), $[\Phi]$ is the integer part of Φ, and the tensor $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is the base tensor given in (2.80). We note that $\Phi - [\Phi] = O(1)$. Then from (2.166), (1.43), and

Minkowski inequality, we obtain

$$\begin{aligned}
&\|\mathscr{A}(w_{N_1}, w_{N_2}, w_{N_3})\|_{X^{1-b}}^2 \\
&\lesssim \int_{\mathbb{R}^2} \langle\tau\rangle^{-2b}\bigg(\sum_m \int_{\mathbb{R}^3} \bigg\| \langle \tau - m + O(1) - \tau_1 + \tau_2 - \tau_3\rangle^{-1} \\
&\qquad \times \sum_{k_1,k_2,k_3} \langle\tau_1\rangle^{-b}\langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \prod_{i=1}^3 \langle\tau_i\rangle^b ((\widehat{w_{N_i}})^{\zeta_i})_{k_i}(\tau_i)\bigg\|_{\ell^2_k} \mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\bigg)^2 \mathrm{d}\tau \\
&\lesssim \int_{\mathbb{R}^2} \langle\tau\rangle^{-2b}\bigg(\sum_m \int_{\mathbb{R}^3} \langle \tau - m - \tau_1 + \tau_2 - \tau_3\rangle^{-1} \langle\tau_1\rangle^{-b}\langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \\
&\qquad \times \bigg\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \prod_{i=1}^3 \langle\tau_i\rangle^b ((\widehat{w_{N_i}})^{\zeta_i})_{k_i}(\tau_i)\bigg\|_{\ell^2_k} \mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\bigg)^2 \mathrm{d}\tau,
\end{aligned}$$

where $\zeta_1, \zeta_3 = +$ and $\zeta_2 = -$. Due to the fact that $|k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k|^\alpha - O(1) = m$, we see that $|m| \lesssim N^\alpha$, which leads to

$$\sum_{m\in\mathbb{Z}} \langle \tau - m - \tau_1 + \tau_2 - \tau_3\rangle^{-1} \lesssim 1 + \log N.$$

Therefore, similar to the argument used in (2.113)-(2.116), in order to prove (2.165), it suffices to show

$$\bigg\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k'} \prod_{i=1}^3 \langle\tau_i\rangle^b ((\widehat{w_{N_i}})^{\zeta_i})_{k_i}(\tau_i)\bigg\|_{L^2_{\tau_1\tau_2\tau_3}\ell^2_k} \lesssim M^{-\frac{1}{2}-3\gamma}, \tag{2.167}$$

where w_{N_i} are of types (C) or (D). In view of (2.103) and (2.104), we may further reduce (2.167) to the following estimate,[1]

$$\|\mathscr{X}_k\|_{\ell^2_k} \lesssim M^{-\frac{1}{2}-3\gamma}, \tag{2.168}$$

where $\mathscr{X}_k$ is given by

[1]Strictly speaking, to reduce (2.167) to (2.168), one needs to use a standard *meshing argument* used in [34, Lemma 4.2, Claim 5.4], [35, Proposition 6.1], and [36, Subsection 3.4, Subsection 4.1].

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1} \cdot (\overline{w_{N_2}})_{k_2} \cdot (w_{N_3})_{k_3} \tag{2.169}$$

and w_{N_j} are of either of the following two types (with a slightly abuse of notation, we still call them type (C) and (D)).

- Type (C), where
$$(w_{N_j})_{k_j} = \sum_{k'_j} h^{N_j,L_j}_{k_jk'_j} \frac{g_{k'_j}(\omega)}{[\![k'_j]\!]^{\alpha/2}},$$
with $h^{N_j,L_j}_{k_jk'_j}(\omega)$ supported in the set $\left\{\frac{N_j}{2} < \langle k'_j\rangle \leq N_j\right\}$, $\mathscr{B}_{\leq L_j}$ measurable for some $L_j \leq N_j^{1-\delta}$, and satisfying the bounds
$$\begin{aligned} \|h^{N_j,L_j}_{k_jk'_j}\|_{\ell^2_{k_j}\to\ell^2_{k'_j}} &\lesssim L_j^{-\delta_0}, \\ \|h^{N_j,L_j}_{k_jk'_j}\|_{\ell^2_{k_jk'_j}} &\lesssim N_j^{\frac{1}{2}+\gamma_0} L_j^{-\frac{1}{2}}, \end{aligned} \tag{2.170}$$
for $\delta \ll \gamma_0 \ll \gamma \ll \delta_0 \ll \alpha - 1$. Moreover using (2.95) we may assume $h^{N_j,L_j}_{k_jk'_j}$ is supported in $|k_j - k'_j| \lesssim N^{\varepsilon} L_j$.

- Type (D), where $(w_{N_j})_{k_j}$ is supported in $\{|k_j| \lesssim N_j\}$, and satisfies
$$\|(w_{N_j})_{k_j}\|_{\ell^2_{k_j}} \lesssim N_j^{-\frac{1}{2}-\gamma}. \tag{2.171}$$

We will now analyze the terms (I)-(VII) in the expression of $\mathscr{X}_k$ given by (2.169). We will consider eight possible scenarios, based on the possible types of $(w_{N_1}, w_{N_2}, w_{N_3})$ in (2.169), i.e. (a) (C, C, C), (b) (C, C, D), (c) (C, D, C), (d) (D, C, C), (e) (D, D, C), (f) (C, D, D), (g) (D, C, D), and (h) (D, D, D).

2.6.2 High-high interactions

In this subsection, we focus on the scenarios where $N_{\mathrm{med}} \gtrsim N_{\max}^{1-\delta} \gtrsim M^{1-\delta}$ for a small positive constant $0 < \delta \ll 1$. In particular, these scenarios cover terms (I) and (III) from the previous section. As we have seen before, the base tensor $\mathrm{T}^{\mathrm{b},m}$ is defined by (2.80), which is essentially the indicator function of the set (2.45) with the constraints and $N_{\mathrm{med}} \gtrsim N_{\max}^{1-\delta}$.

Case (a): $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, C, C)

First, let us consider the non-pairing cases, where $k_1' \neq k_2'$, $k_2' \neq k_3'$. In this case, we have

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{\substack{k_1',k_2',k_3' \\ k_1'\neq k_2', k_2'\neq k_3'}} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_2'}} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_1'}\overline{g_{k_2'}}g_{k_3'}}{[\![k_1']\!]^{\alpha/2}[\![k_2']\!]^{\alpha/2}[\![k_3']\!]^{\alpha/2}} \tag{2.172}$$

where $h^{N_j,L_j}_{k_jk_j'} = h^{N_j,L_j}_{k_jk_j'}(\omega)$, satisfies (2.170) and $\mathrm{T}^{\mathrm{b},m}$ is defined in (2.80).

Applying Proposition 4, Remark 3, and Proposition (2) yields[1]

$$\begin{aligned} \|\mathscr{X}_k\|_k \lesssim& (N_1N_2N_3)^{-\alpha/2} \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_2'}} h^{N_3,L_3}_{k_3k_3'} \Big\|_{kk_1'k_2'k_3'} \\ \lesssim& (N_1N_2N_3)^{-\alpha/2} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2k_3} \prod_{j=1}^{3} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'}. \end{aligned} \tag{2.173}$$

Then from (2.170) and Lemma 12, we can continue with

$$\begin{aligned} \|\mathscr{X}_k\|_k \lesssim& (N_1N_2N_3)^{-\alpha/2} \min(N_3^{2-\alpha}(N_1\wedge N_2)\log(N_1\vee N_2) + N_1N_2, \\ & \qquad N_1^{2-\alpha}(N_2\wedge N_3)\log(N_2\vee N_3) + N_2N_3) \\ \lesssim& (N_1N_2N_3)^{-\alpha/2} N_{\max}^{1/2} (\log N_{\max})^{1/2} N_{\min}^{1/2} \\ \lesssim& N_{\max}^{\frac{1}{2}-\frac{\alpha}{2}+\varepsilon} N_{\mathrm{med}}^{-\frac{\alpha}{2}}, \end{aligned} \tag{2.174}$$

which is sufficient for (2.168) since $N_{\mathrm{med}} \gtrsim N_{\max}^{1-\delta}$, provided $\varepsilon, \delta, \gamma \ll \alpha - 1$.

Secondly, we will consider the case when $k_1' = k_2'$ (the case when $k_2' = k_3'$ is similar,

[1] Here we only consider the case $N_{\min} = N_{\max}$ for simplicity. When $N_{\min} < N_{\max}$, we need to apply Proposition 4 and Propostion 2 repeatedly as in Subsubsection 2.5.2, as where $h^{N_j,L_j}_{k_jk_j'}$ may depend on $g_{k_i'}$ for $i \neq j$.

and we omit its proof), where we have $N_1 = N_2$. For this case, we have

$$\begin{aligned}\mathscr{X}_k &= \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \sum_{k_1',k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{|g_{k_1'}|^2 g_{k_3'}}{[\![k_1']\!]^{\alpha}[\![k_3']\!]^{\alpha/2}} \\ &= \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \sum_{k_1',k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{(|g_{k_1'}|^2-1) g_{k_3'}}{[\![k_1']\!]^{\alpha}[\![k_3']\!]^{\alpha/2}} \\ &\quad + \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \sum_{k_1',k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_3'}}{[\![k_1']\!]^{\alpha}[\![k_3']\!]^{\alpha/2}} \\ &= \mathscr{X}_k^{(1)} + \mathscr{X}_k^{(2)}.\end{aligned} \tag{2.175}$$

For the term $\mathscr{X}_k^{(1)}$, we can apply Proposition 4 with $\eta_{k_1'} = (|g_{k_1'}|^2-1)$, $\eta_{k_3'} = g_{k_3'}$, or $\eta_{k_1'} = |g_{k_1'}|^2 g_{k_1'}$ for over paired cases, and then similar argument as above, to get

$$\begin{aligned}\|\mathscr{X}_k^{(1)}\|_k &\lesssim N_1^{-\alpha} N_3^{-\frac{\alpha}{2}} \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \Big\|_{kk_1'k_3'} \\ &\lesssim N_1^{-\alpha} N_3^{-\frac{\alpha}{2}} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2k_3} \|h^{N_1,L_1}_{k_1k_1'} h^{N_2,L_2}_{k_2k_1'}\|_{k_1k_2\to k_1'} \|h^{N_3,L_3}_{k_3k_3'}\|_{k_3\to k_3'} \\ &\lesssim N_1^{-\alpha} N_3^{-\frac{\alpha}{2}} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2k_3} \prod_{j=1}^{3} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'},\end{aligned} \tag{2.176}$$

where we used Lemma 2 in the last step. By using the same estimate as in (2.174), we have

$$\|\mathscr{X}_k^{(1)}\|_k \lesssim N_{\max}^{\frac{1}{2}-\frac{\alpha}{2}+\varepsilon} N_{\mathrm{med}}^{-\frac{\alpha}{2}}, \tag{2.177}$$

which is again sufficient for (2.168). Now we turn to the second term in (2.175). By using the unitary property of $\tilde{H}^{N,L_N}_{kk'}$ in Lemma 16 (also see Corollary 4 and Remark 13), we may redefine $\mathscr{X}_k^{(2)}$ as follows, still denoted by $\mathscr{X}_k^{(2)}$,

$$\begin{aligned}\mathscr{X}_k^{(2)} &= \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1',k_3'} \left(\frac{1}{[\![k_1']\!]^{\alpha}} - \frac{1}{[\![k_1]\!]^{\alpha}} \right) h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_3'}}{[\![k_3']\!]^{\alpha/2}} \\ &\quad - \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_1'} \frac{g_{k_3'}}{[\![k_3']\!]^{\alpha/2}} \\ &=: \mathscr{X}_k^{(21)} + \mathscr{X}_k^{(22)}.\end{aligned} \tag{2.178}$$

Recall that the support of $h^{N_1,L_1}_{k_1k_1'}$ is in $|k_1 - k_1'| \lesssim N_1^{\varepsilon} L_1$. Therefore,

$$\left| \frac{1}{[\![k_1']\!]^{\alpha}} - \frac{1}{[\![k_1]\!]^{\alpha}} \right| \lesssim N_1^{-\alpha-1+\varepsilon} L_1. \tag{2.179}$$

Let us consider $\mathscr{X}_k^{(21)}$ and $\mathscr{X}_k^{(22)}$ one-by-one. Let us first consider $\mathscr{X}_k^{(21)}$ first. From (2.178) and (2.179), we apply Proposition 4 and Proposition 2 to get

$$\begin{aligned} \|\mathscr{X}_k^{(21)}\|_k \lesssim & N_1^{-\alpha-1+\varepsilon} L_1 N_3^{-\alpha/2} \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \sum_{k_1'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \Big\|_{kk_3'} \\ \lesssim & N_1^{-\alpha-1+\varepsilon} L_1 N_3^{-\alpha/2} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2k_3} \|h^{N_1,L_1}_{k_1k_1'}\|_{k_1k_1'} \prod_{j=2}^{3} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'}, \end{aligned} \tag{2.180}$$

which together with Lemma 12 and (2.170) implies that

$$\|\mathscr{X}_k^{(21)}\|_k \lesssim N_1^{-\alpha-1+\varepsilon} L_1 N_3^{-\frac{\alpha}{2}} N_{\max}^{\frac{1}{2}+\varepsilon} N_{\min}^{\frac{1}{2}} N_1^{\frac{1}{2}+\gamma_0} L_1^{-\frac{1}{2}} \lesssim N_1^{-\alpha+\varepsilon+\gamma_0} N_3^{-\frac{\alpha}{2}} N_{\max}^{\frac{1}{2}+\varepsilon} L_{\min}^{\frac{1}{2}}, \tag{2.181}$$

which is sufficient for (2.168) again since $N_1, N_3 \gtrsim N_{\max}^{1-\delta}$. For the second term of (2.175), by applying Proposition 4 and Proposition 2 similarly as above, we obtain

$$\begin{aligned} \|\mathscr{X}_k^{(22)}\|_k \lesssim & \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_1'} \frac{g_{k_1'}}{[\![k_1']\!]^{\frac{3\alpha}{2}}} \Big\|_k \\ \lesssim & N_{\max}^{-\frac{3\alpha}{2}} \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_1'} \Big\|_{kk_1'} \\ \lesssim & N_{\max}^{-\frac{3\alpha}{2}} \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2k_3} \prod_{j=1}^{3} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'}, \end{aligned} \tag{2.182}$$

where we used Lemma 2 twice in the last step. By using Lemma 12 and (2.170), from (2.182) we get

$$\|\mathscr{X}_k^{(22)}\|_k \lesssim N_{\max}^{-\frac{3\alpha}{2}} N_{\max}^{1+\varepsilon}, \tag{2.183}$$

where we used $N_1 = N_2 = N_3 = N_{\max}$, which is again sufficient for (2.168).

Finally, by collecting (2.175), (2.177), (2.178), (2.181), and (2.183), we conclude that

$$\|\mathscr{X}_k\|_k \lesssim N_{\max}^{-\frac{1}{2}-4\gamma} \lesssim M^{-\frac{1}{2}-4\gamma}.$$

Thus we finish the proof of (2.168).

2.6.2.2 Case (b): $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, C, D)

First, consider the non-pairing case when $k_1' \neq k_2'$. In this case, we have

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \sum_{k_1'\neq k_2'} h^{N_1,L_1}_{k_1k_1'} h^{N_2,L_2}_{k_2k_2'} \frac{g_{k_1'}\overline{g_{k_2'}}}{[\![k_1']\!]^{\alpha/2}[\![k_2']\!]^{\alpha/2}} \|(w_{N_3})_{k_3}\|_{k_3},$$

where $h^{N_j,L_j}_{k_jk_j'} = h^{N_j,L_j}_{k_jk_j'}(\omega)$ satisfies (2.170), $(w_{N_3})_{k_3}$ satisfies (2.171) and $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is defined in (2.80). From Proposition 2, we have

$$\|\mathscr{X}_k\|_k \lesssim \left\| \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \sum_{k_1'\neq k_2'} h^{N_1,L_1}_{k_1k_1'} h^{N_2,L_2}_{k_2k_2''} \frac{g_{k_1'}\overline{g_{k_2'}}}{[\![k_1']\!]^{\alpha/2}[\![k_2']\!]^{\alpha/2}} \right\|_{k\to k_3} \|(w_{N_3})_{k_3}\|_{k_3}. \tag{2.184}$$

Then we apply Proposition 4 and Proposition 2 repeatedly as in Subsubsection 2.5.2 to the right-hand side of (2.184) to get

$$\begin{aligned} \|\mathscr{X}_k\|_k \lesssim &(N_1N_2)^{-\frac{\alpha}{2}} (\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2\to k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{k\to k_1k_2k_3} \\ &\qquad + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1\to k_2k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_2\to k_1k_3}) \\ &\times \prod_{j=1}^{2} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'} \|(w_{N_3})_{k_3}\|_{k_3}. \end{aligned} \tag{2.185}$$

Then by using Lemma 14 and (2.170), from (2.185) we obtain

$$\begin{aligned} \|\mathscr{X}_k\|_k \lesssim &(N_1N_2)^{-\frac{\alpha}{2}} N_3^{-\frac{1}{2}-\gamma} (N_1^{1-\frac{\alpha}{2}} N_2^{\varepsilon} + N_2^{\frac{1}{2}} + (N_1 \wedge N_2)^{1-\frac{\alpha}{2}} N_3^{\varepsilon} + N_3^{\frac{1}{2}} \\ &\qquad + (N_2 \wedge N_3)^{1-\frac{\alpha}{2}} N_1^{1-\frac{\alpha}{2}} + (N_1 \wedge N_3)^{\frac{1}{2}-\frac{\alpha}{4}} N_2^{\frac{1}{2}-\frac{\alpha}{4}}) \\ \lesssim &(N_1N_2)^{-\frac{\alpha}{2}} N_3^{-\frac{1}{2}-\gamma} N_{\min}^{1-\frac{\alpha}{2}} N_{\max}^{\frac{1}{2}}, \end{aligned}$$

which is sufficient for our purpose since $N_{\text{med}} \gtrsim N_{\max}^{1-\delta}$ and $\varepsilon, \delta, \gamma \ll \alpha - 1$.

Let us turn to the pairing case, i.e. $k_1' = k_2'$ (which implies $N_1 = N_2$). By using Lemma 16, we may consider

$$
\begin{aligned}
\mathscr{X}_k = &\sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'} \left(\frac{1}{[\![k_1']\!]^\alpha} - \frac{1}{[\![k_1]\!]^\alpha} \right) h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_1,L_2}_{k_2k_1'}} (w_{N_3})_{k_3} \\
&+ \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_1,L_2}_{k_2k_1'}} \frac{(|g_{k_1'}|^2 - 1)}{[\![k_1']\!]^\alpha} (w_{N_3})_{k_3} \\
=: &\mathscr{X}_k^{(3)} + \mathscr{X}_k^{(4)}.
\end{aligned}
$$

Similar argument as in (2.180) gives

$$
\begin{aligned}
\| \mathscr{X}_k^{(3)} \|_k \lesssim & N_1^{-\alpha-1+\varepsilon} L_1 \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_1,L_2}_{k_2k_1'}} (w_{N_3})_{k_3} \Big\|_k \\
\lesssim & N_1^{-\alpha-1+\varepsilon} L_1 \| \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \|_{kk_1k_2 \to k_3} \| h^{N_1,L_1}_{k_1k_1'} \|_{k_1k_1'} \| h^{N_2,L_2}_{k_2k_2'} \|_{k_2 \to k_2'} \| (w_{N_3})_{k_3} \|_{k_3} \\
\lesssim & N_1^{-\alpha-1+\varepsilon} L_1 N_1^{\frac{1}{2}} N_1^{\frac{1}{2}+\gamma_0} L_1^{-\frac{1}{2}} N_3^{-\frac{1}{2}-\gamma} \\
\lesssim & N_1^{-\alpha+\frac{1}{2}+\varepsilon+\gamma_0} N_3^{-\frac{1}{2}-\gamma},
\end{aligned}
$$

where we used $N_1 = N_2$, which is sufficient for our purpose since $N_1, N_3 \gtrsim N_{\max}^{1-\delta}$. For term $\mathscr{X}_k^{(4)}$, by Proposition 4 with $\eta_{k_1'} = |g_{k_1'}|^2 - 1$, we have

$$
\begin{aligned}
\| \mathscr{X}_k^{(4)} \|_k = & \Big\| \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_1,L_2}_{k_2k_1'}} \frac{(|g_{k_1'}|^2 - 1)}{[\![k_1']\!]^\alpha} (w_{N_3})_{k_3} \Big\|_k \\
\lesssim & N_1^{-\alpha} \Bigg(\Big\| \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_1,L_2}_{k_2k_1'}} \Big\|_{kk_1' \to k_3} \\
& + \Big\| \sum_{k_1,k_2} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_1,L_2}_{k_2k_1'}} \Big\|_{k \to k_1'k_3} \Bigg) \| (w_{N_3})_{k_3} \|_{k_3}.
\end{aligned}
$$

Then similar argument as in (2.185) together with Lemma 14 and Lemma 2 yields

$$
\begin{aligned}
\|\mathscr{X}_k^{(4)}\|_k =& N_1^{-\alpha}(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2\to k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{k\to k_1k_2k_3}) \\
& \|h^{N_1,L_1}_{k_1k_1'}\overline{h^{N_1,L_2}_{k_2k_1'}}\|_{k_1'\to k_1k_2}\|(w_{N_3})_{k_3}\|_{k_3}. \\
\lesssim & N_1^{-\alpha+\frac{1}{2}}N_3^{-\frac{1}{2}-\gamma},
\end{aligned}
$$

which is again sufficient for our purpose.

The proof for Case (d): (D, C, C) is similar to that of Case (b) (C, C, D); we omit the proof.

Case (c): $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, D, C)

In this case, there is no pairing. Therefore, we have

$$
\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}(\overline{w_{N_2}})_{k_2} \sum_{k_1',k_3'} h^{N_1,L_1}_{k_1k_1'} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_1'}g_{k_3'}}{[\![k_1']\!]^{\alpha/2}[\![k_3']\!]^{\alpha/2}},
$$

where $h^{N_j,L_j}_{k_jk_j'} = h^{N_j,L_j}_{k_jk_j'}(\omega)$ satisfies (2.170), $(w_{N_2})_{k_2}$ satisfies (2.171) and $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is defined in (2.80). Applying Proposition 4 and Propostion 2 similarly as in the previous subsection, we have

$$
\begin{aligned}
\|\mathscr{X}_k\|_k \lesssim \quad & (N_1N_3)^{-\frac{\alpha}{2}}(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_3\to k_2} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1\to k_2k_3} \\
& + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_3\to k_1k_2} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{k\to k_1k_2k_3}) \\
& \times \|h^{N_1,L_1}_{k_1k_1'}\|_{k_1'\to k_1}\|h^{N_3,L_3}_{k_3k_3'}\|_{k_3'\to k_3}\|(w_{N_2})_{k_2}\|_{k_2}.
\end{aligned}
$$

Then by using Lemma 14, (2.170), and (2.171), we get

$$
\|\mathscr{X}_k\|_k \lesssim (N_1N_3)^{-\frac{\alpha}{2}} N_2^{-\frac{1}{2}-\gamma} N_{\min}^{1-\frac{\alpha}{2}} N_{\max}^{\frac{1}{2}},
$$

which is sufficient for our purpose since $N_{\mathrm{med}} \gtrsim N_{\max}^{1-\delta}$ and $\varepsilon, \gamma, \delta \ll \alpha - 1$.

Case (e): $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (D, D, C)

In this case, we have

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1} (\overline{w_{N_2}})_{k_2} \sum_{k_3'} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_3'}}{[\![k_3']\!]^{\alpha/2}},$$

where $h^{N_3,L_3}_{k_3k_3'}$ satisfies (2.170), $(w_{N_1})_{k_1}$ and $(\overline{w_{N_2}})_{k_2}$ satisfies (2.171), and the base $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is defined in (2.80). From Proposition 2.60 and Proposition 2, we have

$$\begin{aligned} \|\mathscr{X}_k\|_k \lesssim \quad & N_3^{-\frac{\alpha}{2}} (\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_3\to k_1k_2} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{k\to k_1k_2k_3}) \\ & \times \|h^{N_3,L_3}_{k_3k_3'}\|_{k_3'\to k_3} \|(w_{N_1})_{k_1}\|_{k_1} \|(w_{N_2})_{k_2}\|_{k_2}. \end{aligned}$$

Then by using Lemma 13, (2.170), and (2.171), we have

$$\|\mathscr{X}_k\|_k \lesssim N_3^{-\frac{\alpha}{2}} (N_1N_2)^{-\frac{1}{2}-\gamma} ((N_{\min}N_{\max})^{1-\frac{\alpha}{2}} + (N_1 \wedge N_2)^{1-\frac{\alpha}{2}} (\log N_3)^{\frac{1}{2}} + N_3^{\frac{1}{2}}),$$

which is again sufficient for our purpose since $N_{\mathrm{med}} \gtrsim N_{\max}^{1-\delta}$.

The proof for Case (f): (C, D, D) is similar to that of Case (e): (D, D, C); we omit the proof.

Case (g): $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (D, C, D)

In this case, we have

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1} \sum_{k_2} \overline{h^{N_2,L_2}_{k_2k_2'}} \frac{\overline{g_{k_2'}}}{[\![k_2']\!]^{\alpha/2}} (w_{N_3})_{k_3},$$

where $h^{N_2,L_2}_{k_2k_2'} = h^{N_2,L_2}_{k_2k_2'}(\omega)$ satisfies (2.170), $(w_{N_1})_{k_1}$ and $(w_{N_3})_{k_3}$ satisfies (2.171), and the base tensor $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is defined in (2.80). From Proposition 4 and Proposition 2, we have

$$\begin{aligned} \|\mathscr{X}_k\|_k \lesssim \quad & N_2^{-\alpha/2} (\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2\to k_3} + \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1\to k_2k_3}) \\ & \|h^{N_2,L_2}_{k_2k_2'}\|_{k_2'\to k_2} \|(w_{N_1})_{k_1}\|_{k_1} \|(w_{N_3})_{k_3}\|_{k_3}. \end{aligned} \tag{2.186}$$

Then by using Lemma 13, Lemma 14, (2.170), and (2.171) to get

$$\|\mathscr{X}_k\|_k \lesssim N_2^{-\frac{\alpha}{2}}(N_1N_3)^{-\frac{1}{2}-\gamma}(N_1^{1-\frac{\alpha}{2}}(\log N_2)^{\frac{1}{2}} + N_2^{\frac{1}{2}} + (N_{\min}N_{\max})^{1-\frac{\alpha}{2}}),$$

which is again sufficient for our purpose since $N_{\text{med}} \gtrsim N_{\max}^{1-\delta}$.

Case (h): $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ **of type (D, D, D)**

In this case, we have

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1}(\overline{w_{N_2}})_{k_2}(w_{N_3})_{k_3},$$

where $(w_{N_j})_{k_j}$ for $j \in \{1,2,3\}$ satisfies (2.171), and the base tensor $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is defined in (2.80). By applying Proposition 2, we obtain

$$\|\mathscr{X}_k\|_k \lesssim \min(\|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_3\to k_1k_2}, \|\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1\to k_2k_3}) \prod_{j=1}^{3} \|(w_{N_j})_{k_j}\|_{k_j}.$$

By using Lemma 13 and (2.171), we get

$$\|\mathscr{X}_k\|_k \lesssim (N_1N_2N_3)^{-\frac{1}{2}-\gamma}(N_{\min}N_{\text{med}})^{1-\frac{\alpha}{2}},$$

which is sufficient for (2.171).

2.6.3 Random averaging operator

In this section, we estimate terms (II) and (IV), which correspond to $\mathcal{P}^+$ and $\mathcal{P}^-$ of (2.97), respectively. We only consider the term (II) since the argument for the term (IV) is similar. To show (2.163) for the term (II), it suffices to show

$$\|\mathcal{P}^+[z_M]\|_{X^b} \lesssim M^{-\frac{1}{2}-\gamma}. \tag{2.187}$$

We see from the definition of terms (II) that $N_{\text{med}} = \max(N_1, N_2)$ since both N_1 and N_2 are smaller than N_3 and N_3 is approximately equal to M. The case where $N_{\text{med}} \gtrsim M^{1-\delta}$ corresponds to the high-high interaction, and we have already established (2.187) for this case in the previous subsection. Therefore, we will only concentrate on the case where $N_{\text{med}} \ll M^{1-\delta}$ in the following discussion. Applying (2.125) and (2.171), we obtain

$$\|\mathcal{P}^{\pm}[z_M]\|_{X^b} \lesssim \|\mathcal{P}^{\pm}\|_{Y^{b,b}} \|z_M\|_{X^b} \lesssim M^{-\frac{1}{2}-\gamma}.$$

This completes the proof of (2.163) for this case.

2.6.4 Low-high-low interations

This subsection shows that the term (V) satisfies the bound (2.165). It suffices to show (2.168). We may assume that $N_1, N_3 \ll N_2^{1-\delta}$; otherwise, it has been dealt with in Subsection 2.6.2. As w_{N_2} is of type C, we only need to consider the cases for the types of $(w_{N_1}, w_{N_2}, w_{N_3})$ are (a) (C, C, C), (b) (C, C, D), (d) (D, C, C), (g) (D, C, D). We set $N_2 = M$ in this subsection.

Before proceeding, let us recall that

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1} \cdot \sum_{k_2'} \overline{h^{N_2,L_2}_{k_2k_2'} \frac{g_{k_2'}(\omega)}{[\![k_2']\!]^{\alpha/2}}} \cdot (w_{N_3})_{k_3} \tag{2.188}$$

where we may take that $N_1, N_3 \ll N_2^{1-\delta}$ with $h^{N_2,L_2}_{k_2k_2'}(\omega)$ supported in the set $\{(k_2, k_2'); \frac{N_2}{2} < \langle k_2'\rangle \le N_2, |k_2 - k_2'| \lesssim N_2^{\varepsilon} L_2\}$, $\mathcal{B}_{\le L_2}$ measurable for some $L_2 \le N_2^{1-\delta}$, and satisfying the bounds (2.170). We note that $|k_2 - k_2'| \lesssim N_2^{\varepsilon} L_2 \lesssim N_2^{\delta/2} N_2^{1-\delta}$, which together with $\langle k_2'\rangle \ge N_2/2$ implies that $|k_2| \ge N_2/4$, provided N_2 is sufficiently large. Furthermore, from $N_1, N_3 \ll N_2^{1-\delta}$, we have

$$|k_1 + k_3| \ll |k_2|. \tag{2.189}$$

From (2.49) and (2.189), we have that

$$|2k-(k_1+k_3)| = |2k_2-(k_1+k_3)| \gtrsim |k_1+k_3|,$$

which, together with (2.62), implies that the summand of (2.188) is non-zero only for those (k, k_1, k_2, k_3) such that

$$(k, k_2) \in S^{\text{good}}_{k_1 k_3}, \tag{2.190}$$

where $S^{\text{good}}_{k_1 k_3}$ is given in (2.62). The above observation will be crucial in our later analysis. In particular, instead of (2.188), we will consider the following

$$\mathscr{X}_k = \sum_{\substack{k_1,k_2,k_3\\ |k_1+k_3|\ll|k_2|}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1} \cdot \sum_{k_2'} \overline{h^{N_2,L_2}_{k_2k_2'}} \frac{\overline{g_{k_2'}(\omega)}}{[\![k_2']\!]^{\alpha/2}} \cdot (w_{N_3})_{k_3}. \tag{2.191}$$

Now, let us consider (2.191) with the restriction (2.189) (or (2.190)) case-by-case.

Case (a) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, C, C)

There is no pairing since $N_1, N_3 \ll N_2 = M$. So we have

$$\mathscr{X}_k = \sum_{\substack{k_1,k_2,k_3\\ |k_1+k_3|\ll|k_2|}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1',k_2',k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_2'}} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_1'}\overline{g_{k_2'}}g_{k_3'}}{[\![k_1']\!]^{\frac{\alpha}{2}}[\![k_2']\!]^{\frac{\alpha}{2}}[\![k_3']\!]^{\frac{\alpha}{2}}}.$$

Same argument as in Subsubsection 2.6.2, followed by Proposition 2 and Lemma 15, yields

$$\begin{aligned}\|\mathscr{X}_k\|_k &\lesssim (N_1N_2N_3)^{-\frac{\alpha}{2}} \Bigg\| \sum_{\substack{k_1,k_2,k_3\\ |k_1+k_3|\ll|k_2|}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_2'}} h^{N_3,L_3}_{k_3k_3'} \Bigg\|_{kk_1'k_2'k_3'} \\ &\lesssim (N_1N_2N_3)^{-\frac{\alpha}{2}} \big\| \mathbf{1}_{|k_1+k_3|\ll|k_2|} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \big\|_{kk_1k_2k_3} \prod_{j=1}^{3} \big\| h^{N_j,L_j}_{k_jk_j'} \big\|_{k_j\to k_j'} \\ &\lesssim (N_1N_2N_3)^{-\frac{\alpha}{2}} (N_1N_3)^{\frac{1}{2}} \lesssim N_2^{-\frac{\alpha}{2}},\end{aligned}$$

which is sufficient for our purpose.

Case (b) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, C, D)

Similar to the above, there is no pairing since $N_1 \ll N_2 = M$. From (2.191), we have

$$\mathscr{X}_k = \sum_{\substack{k_1,k_2,k_3 \\ |k_1+k_3|\ll|k_2|}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1',k_2'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_2'}} \frac{g_{k_1'}\overline{g_{k_2'}}}{[\![k_1']\!]^{\frac{\alpha}{2}}[\![k_2']\!]^{\frac{\alpha}{2}}} (w_{N_3})_{k_3},$$

where $h^{N_j,L_j}_{k_jk_j'}$ for $j = 1, 2$ satisfy (2.170), $(w_{N_3})_{k_3}$ satisfies (2.171) and $\mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}$ is defined in (2.80). By a similar argument as in (2.185) together with (1.34), we have

$$\begin{aligned}
\|\mathscr{X}_k\|_k &\lesssim \Bigg\| \sum_{\substack{k_1,k_2 \\ |k_1+k_3|\ll|k_2|}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1',k_2'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_2'}} \frac{g_{k_1'}\overline{g_{k_2'}}}{[\![k_1']\!]^{\frac{\alpha}{2}}[\![k_2']\!]^{\frac{\alpha}{2}}} \Bigg\|_{k\to k_3} \|(w_{N_3})_{k_3}\|_{k_3} \\
&\lesssim (N_1N_2)^{-\frac{\alpha}{2}} \big\| \mathbf{1}_{|k_1+k_3|<|k_2|} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \big\|_{kk_1k_2k_3} \prod_{j=1}^{2} \big\| h^{N_j,L_j}_{k_jk_j'} \big\|_{k_j\to k_j'} \|(w_{N_3})_{k_3}\|_{k_3}.
\end{aligned}$$

Then we apply Lemma 15, together with (2.170) and (2.171), to get

$$\|\mathscr{X}_k\|_k \lesssim (N_1N_2)^{-\frac{\alpha}{2}} N_3^{-\frac{1}{2}-\gamma} (N_1N_3)^{\frac{1}{2}} \lesssim N_2^{-\frac{\alpha}{2}},$$

which is enough for our purpose.

The proof for Case (d): (D, C, C) is similar to that of Case (b): (C, C, D); we omit the proof.

Case (g) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (D, C, D)

In this case, we consider

$$\mathscr{X}_k = \sum_{\substack{k_1,k_2,k_3 \\ |k_1+k_3|\ll|k_2|}} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_2'} \overline{h^{N_2,L_2}_{k_2k_2'}} \frac{\overline{g_{k_2'}}}{[\![k_2']\!]^{\frac{\alpha}{2}}} (w_{N_1})_{k_1} (w_{N_3})_{k_3}.$$

By a similar argument as in (2.186), we have

$$\|\mathscr{X}_k\|_k \lesssim N_2^{-\frac{\alpha}{2}} \big\|\mathbf{1}_{|k_1+k_2|<|k_3|} \mathrm{T}^{\mathrm{b},m}_{kk_1k_3k_2}\big\|_{kk_1k_2k_3} \big\|h^{N_2,L_2}_{k_2k_2'}\big\|_{k_2'\to k_2} \|(w_{N_1})_{k_1}\|_{k_1} \|(w_{N_3})_{k_3}\|_{k_3}.$$

Then from Lemma 15, (2.170), and (2.171), we have

$$\begin{aligned} \|\mathscr{X}_k\|_k &\lesssim N_2^{-\frac{\alpha}{2}} N_1^{-\frac{1}{2}-\gamma} N_3^{-\frac{1}{2}-\gamma} \big\|\mathbf{1}_{|k_1+k_2|<|k_3|} \mathrm{T}^{\mathrm{b},m}_{kk_1k_3k_2}\big\|_{kk_1k_2k_3} \\ &\lesssim N_1^{-\frac{1}{2}-\gamma} N_2^{-\frac{\alpha}{2}} N_3^{-\frac{1}{2}-\gamma} N_1^{\frac{1}{2}} N_3^{\frac{1}{2}} \lesssim N_2^{-\frac{\alpha}{2}}, \end{aligned}$$

which is again sufficient for (2.168).

2.6.5 The Γ-condition case

In this subsection, we consider (2.164), which shows that the term (VI), defined at the beginning of Section 2.6, satisfies the bound

$$\|\mathscr{B}(w_{N_1}, w_{N_2}, w_{N_3})\|_{X^b} \lesssim M^{-\frac{1}{2}-\gamma}, \tag{2.192}$$

where $\mathscr{B}$ is given in (2.162). Same argument as in Subsection 2.6.1 reduces (2.192) to the the following estimate

$$\|\mathscr{X}_k\|_k \lesssim M^{-\frac{1}{2}-3\gamma}, \tag{2.193}$$

where $\mathscr{X}_k$ is given in (2.169).

To prove (2.193), we make several assumptions. To make the term (VI) non-trivial, we should have $N_{\max} \gtrsim M$. If $N_2 \gtrsim M$, then term (VI) can be handled in the same way as the terms (IV) and (V). Therefore, we only need to consider the cases that N_1 or $N_3 \gtrsim M$. Due to the symmetry between w_{N_1} and w_{N_3}, we may assume $N_3 \gtrsim M$ in what follows. To prove (2.193), it only suffices to show

$$\|\mathscr{X}_k\|_k \lesssim N_3^{-\frac{1}{2}-3\gamma}, \tag{2.194}$$

where $\mathscr{X}_k$ is given in (2.169). We may further assume that $N_1, N_2 \ll N_3^{1-\delta}$ as otherwise it has been dealt with in Subsection 2.6.2. Furthermore, we may assume that $w_{N_3} = \psi_{N_3,L_{N_3}}$, otherwise, it can be handled by using the random averaging operator as in Subsection 2.6.3.

Due to the projections Δ_M ($\frac{M}{2} < [\![k]\!] \leq M$), the requirement $N_{\max} \leq \frac{M}{2}$ in the definition of the term (VI), we observe that

$$\langle k\rangle > \frac{M}{2} \geq \langle k_3\rangle \Longrightarrow |k| \geq \left(\left(\frac{M}{2}\right)^2 - 1\right)^{1/2} \geq |k_3| = |k - k_1 + k_2| \geq |k| - 2N_{\text{med}}$$
$$\Longrightarrow |k_3| + 2N_{\text{med}} \geq |k| \geq \Gamma \geq |k_3|,$$

where $\Gamma := ((M/2)^2 - 1)^{1/2}$. Therefore, to make $\mathscr{X}_k$ is non-trivial, we need $(k, k_1, k_2, k_3) \in B_\Gamma$, where B_Γ is given in (2.72).

With the above argument, we focus on (2.194) with

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathbf{1}_{B_\Gamma}(k, k_1, k_2, k_3) \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (w_{N_1})_{k_1} \cdot (\overline{w_{N_2}})_{k_2} \cdot \sum_{k_3'} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_3'}}{[\![k_3']\!]^{\frac{\alpha}{2}}}, \tag{2.195}$$

where $N_1, N_2 \ll N_3$. In the following, we distinguish several cases for the possible types of (w_{N_1}, w_{N_2}): Case (a) (C, C), Case (c) (C, D), Case (d) (D, C), Case (e) (D, D).

Case (a) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, C, C)

We start with the non-pairing cases. Same argument as in Subsubsection 2.6.2, followed by Corollary 3, yields

$$\|\mathscr{X}_k\|_k \lesssim (N_1N_2N_3)^{-\frac{\alpha}{2}} \|\mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\|_{kk_1k_2k_3} \lesssim (N_1N_2M)^{-\frac{\alpha}{2}} N_{\min}^{\frac{1}{2}} N_{\text{med}}^{\frac{1}{2}} \lesssim N_3^{-\frac{\alpha}{2}},$$

which is sufficient for (2.194).

We turn to the pairing cases. Similar as in Subsubsection 2.6.2, we can write (2.195) (note w_{N_1} and w_{N_2} are both of type (C)) as

$$\begin{aligned}
\mathscr{X}_k &= \sum_{k_1,k_2,k_3} \mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'\neq k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{|g_{k_1'}|^2 g_{k_3'}}{[\![k_1']\!]^{\alpha}[\![k_3']\!]^{\frac{\alpha}{2}}} \\
&= \sum_{k_1,k_2,k_3} \mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'\neq k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{(|g_{k_1'}|^2-1) g_{k_3'}}{[\![k_1']\!]^{\alpha}[\![k_3']\!]^{\frac{\alpha}{2}}} \\
&\qquad + \sum_{k_1,k_2,k_3} \mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot \sum_{k_1'\neq k_3'} h^{N_1,L_1}_{k_1k_1'} \overline{h^{N_2,L_2}_{k_2k_1'}} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_3'}}{[\![k_1']\!]^{\alpha}[\![k_3']\!]^{\frac{\alpha}{2}}} \\
&= \mathscr{X}_k^{(1)} + \mathscr{X}_k^{(2)}.
\end{aligned}$$

For the first term $\mathscr{X}_k^{(1)}$, we follow the computation in (2.176) and use Corollar 3 to get

$$\|\mathscr{X}_k^{(1)}\|_k \lesssim N_1^{-\alpha} N_3^{-\frac{\alpha}{2}} \big\|\mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\big\|_{kk_1k_2k_3} \prod_{j=1}^{3} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'} \lesssim N_1^{-\alpha+1} N_3^{-\frac{\alpha}{2}},$$

which is enough for our purpose. As to the second term $\mathscr{X}_k^{(2)}$, we follow the computation in (2.178), (2.180), and (2.182) to get

$$\begin{aligned}
\|\mathscr{X}_k^{(2)}\|_k \lesssim & N_1^{-\alpha-1+\varepsilon} L_1 N_3^{-\frac{\alpha}{2}} \big\|\mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3}\big\|_{kk_1k_2k_3} \|h^{N_1,L_1}_{k_1k_1'}\|_{k_1k_1'} \prod_{j=2}^{3} \|h^{N_j,L_j}_{k_jk_j'}\|_{k_j\to k_j'} \\
&+ N_3^{-\frac{3\alpha}{2}+1+\varepsilon} \\
\lesssim & N_1^{-\alpha+\varepsilon+\gamma_0} N_3^{-\frac{\alpha}{2}} N_{\mathrm{med}}^{\frac{1}{2}} N_{\mathrm{min}}^{\frac{1}{2}} + N_3^{-\frac{3\alpha}{2}+1+\varepsilon},
\end{aligned}$$

which is again sufficient since $N_{\mathrm{min}} = N_{\mathrm{med}} = N_1$.

Case (b) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (C, D, C) and Case (c) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (D, C, C)

We consider Case (b) only as the argument for Case (c) is similar. In this case, we can write (2.195) as

$$\mathscr{X}_k = \sum_{k_1,k_2,k_3} \mathbf{1}_{B_\Gamma} \mathrm{T}^{\mathrm{b},m}_{kk_1k_2k_3} \cdot (\overline{w_{N_2}})_{k_2} \sum_{k_1',k_3'} h^{N_1,L_1}_{k_1k_1'} h^{N_3,L_3}_{k_3k_3'} \frac{g_{k_1'} g_{k_3'}}{[\![k_1']\!]^{\frac{\alpha}{2}}[\![k_3']\!]^{\frac{\alpha}{2}}}.$$

By a similar argument as in Subsubsection 2.6.2, we obtain

$$\|\mathscr{X}_k\|_k \lesssim (N_1N_3)^{-\frac{\alpha}{2}} \big\|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_3k_2}\big\|_{kk_1k_3k_2} \big\|h^{N_1,L_1}_{k_1k_1'}\big\|_{k_1'\to k_1} \big\|h^{N_3,L_3}_{k_3k_3'}\big\|_{k_3'\to k_3} \|(w_{N_2})_{k_2}\|_{k_2}.$$

Then by using Corollary 3 and (2.170)-(2.171), we get

$$\|\mathscr{X}_k\|_k \lesssim (N_1N_3)^{-\frac{\alpha}{2}} N_2^{-\frac{1}{2}-\gamma}(N_{\min}N_{\mathrm{med}})^{\frac{1}{2}} \lesssim N_3^{-\frac{\alpha}{2}},$$

which is sufficient for our purpose.

Case (d) : $(\mathbf{w_{N_1}}, \mathbf{w_{N_2}}, \mathbf{w_{N_3}})$ of type (D, D, C)

Similar argument as in Subsubsection 2.6.2, from (2.195) we have

$$\begin{aligned}
\|\mathscr{X}_k\|_k &\lesssim N_3^{-\frac{\alpha}{2}} \big\|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_3k_2}\big\|_{kk_1k_2k_3} \big\|h^{N_3,L_3}_{k_3k_3'}\big\|_{k_3'\to k_3} \|(w_{N_1})_{k_1}\|_{k_1} \|(w_{N_2})_{k_2}\|_{k_2} \\
&\lesssim (N_1N_2)^{-\frac{1}{2}-\gamma} N_3^{-\frac{\alpha}{2}} \big\|\mathbf{1}_{B_\Gamma}\mathrm{T}^{\mathrm{b},m}_{kk_1k_3k_2}\big\|_{kk_1k_2k_3} \\
&\lesssim (N_1N_2)^{-\frac{1}{2}-\gamma} N_3^{-\frac{\alpha}{2}} (N_{\min}N_{\mathrm{med}})^{\frac{1}{2}} \le N_3^{-\frac{\alpha}{2}},
\end{aligned}$$

where we applied Proposition 2 in the second step, which is again sufficient for (2.193).

2.6.6 Resonant case

Finally, we estimate term (VII), the resonant case. In this case, we may assume $M = N_1 = N_2 = N_3$. If $(w_{N_1}, w_{N_2}, w_{N_3})$ are of type (D, D, D), then we have

$$\begin{aligned}
\|\chi(\mathrm{VII})\|_{X^b} &\lesssim \|\mathcal{R}(w_{N_1}, w_{N_2}, w_{N_3})(t)\|_{L^2_{t,x}} \\
&= \Big(\int_{\mathbb{R}} \sum_k |(w_{N_1})_k (w_{N_2})_k (w_{N_3})_k|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \\
&\lesssim \prod_{i=1}^{3} \|w_{N_i}\|_{X^{\frac{1}{3}}} \lesssim \prod_{i=1}^{3} \|w_{N_i}\|_{X^b} \lesssim M^{-\frac{3}{2}-3\gamma},
\end{aligned}$$

where we used (2.103), which is sufficient for (2.96).

If $(w_{N_1}, w_{N_2}, w_{N_3})$ are of type (C, C, C), i.e. Case (a). We have

$$\|\chi(t)(\mathrm{VII})\|_{X^b} \lesssim \|\mathcal{R}(w_{N_1}, w_{N_2}, w_{N_3})(t)\|_{L^2_{t,x}},$$

where w_N is given by (2.102). We only consider the non-pairing cases, i.e.

$$\begin{aligned}
&\big(\mathcal{R}(w_{N_1}, w_{N_2}, w_{N_3})(t)\big)_k \\
&= \chi(t) \sum_{k_1,k_2,k_3} \mathrm{T}_{k,k_1,k_2,k_3} \sum_{\substack{k_1',k_2',k_3' \\ k_2' \notin \{k_1',k_3'\}}} h^{N_1,L_1}_{kk_1'}(t) \overline{h^{N_2,L_2}_{kk_2'}}(t) h^{N_3,L_3}_{kk_3'}(t) \frac{g_{k_1'} \overline{g_{k_2'}} g_{k_3'}}{[\![k_1']\!]^{\frac{\alpha}{2}} [\![k_2']\!]^{\frac{\alpha}{2}} [\![k_3']\!]^{\frac{\alpha}{2}}}.
\end{aligned}$$

Then, from Proposition 4 and then Proposition 2, we have

$$\begin{aligned}
&\|\mathcal{R}(w_{N_1}, w_{N_2}, w_{N_3})(t)\|_{L^2_{t,x}} \\
&\lesssim M^{-\frac{3\alpha}{2}} \big\|\chi(t) h^{N_1,L_1}_{kk_1'} \overline{h^{N_2,L_2}_{kk_2'}} h^{N_3,L_3}_{kk_3'}\big\|_{L^2_T(kk_1'k_2'k_3')} \\
&\lesssim T^{\frac{1}{2}} M^{-\frac{3\alpha}{2}} \|h^{N_1,L_1}_{kk_1'}\|_{L^\infty_t(k_1\to k)} \|h^{N_2,L_2}_{kk_2'}\|_{L^\infty_t(k_2\to k)} \|h^{N_3,L_3}_{kk_3'}\|_{L^\infty_t(k_3 k)} \\
&\lesssim T^{\frac{1}{2}} M^{-\frac{3\alpha}{2}} \prod_{j=1}^{2} \big\|\langle\tau\rangle^b \widehat{h^{N_j,L_j}_{k_j k_j'}}(\tau)\big\|_{L^2_\tau(\ell^2_{k_j} \to \ell^2_{k_j'})} \big\|\langle\tau\rangle^b \widehat{h^{N_3,L_3}_{k_3 k_3'}}(\tau)\big\|_{L^2_\tau(\ell^2_{k_3} \ell^2_{k_3'})} \\
&\lesssim T^{\frac{1}{2}} M^{-\frac{3\alpha}{2}} N_3^{\frac{1}{2}+\gamma_0} \lesssim T^{\frac{1}{2}} M^{-\alpha},
\end{aligned}$$

where we used the fact $b > \frac{1}{2}$ and (2.103), which is sufficient for (2.96).

All the intermediate cases, from Case (b) to Case (g), can be controlled similarly.

2.7 Proof of Theorem 5

We are ready to prove Theorem 5 by using Proposition 5.

Proof of Theorem 5. From Proposition 5, it follows that T-certainly, the event

Local(M) holds for any M. Recall the decomposition (2.43) that

$$\begin{aligned} y_N =&\chi(t)F_N + \sum_{1\le L\le L_N} \mathfrak{h}^{N,L}[\chi(\cdot)F_N] + z_N(t) \\ =&\chi(t)F_N + \sum_{1\le L\le L_N} \tilde{\mathfrak{h}}^{N,L}[F_N] + z_N(t) \end{aligned} \tag{2.196}$$

where F_N is given in (2.27), $\mathfrak{h}^{N,L}$ and $\tilde{\mathfrak{h}}^{N,L}$ are the operators given in (2.38) and (2.41), and z_N is the solution to (2.31). From the definition of Local(M), we see that for each N, the decomposition of y_N in the above is well defined, i.e. both $\mathfrak{h}^{N,L}$ and z_N are well-defined. To prove the local well-posedness part of Theorem 5, it suffices to justify the convergence the summation $\sum y_N$ in some proper sense.

From (2.8) and (2.27), we see that

$$u_0^\omega = \sum_{N\ge 1} F_N \quad \text{converges in } H^{\frac{\alpha-1}{2}-}(\mathbb{T}). \tag{2.197}$$

From (2.40) and (2.42), we also have that

$$\tilde{\mathfrak{h}}^{N,L}[F_N] = \sum_k \mathrm{e}^{\mathrm{i}kx} \sum_{k'} h_{kk'}^{N,L}(t) \frac{g_{k'}}{[\![k']\!]^{\alpha/2}}, \tag{2.198}$$

where $h_{kk'}^{N,L}$, the kernel of $\tilde{\mathfrak{h}}^{N,L}$ given in (2.40) and (2.42), is independent of $\{g_{k'}\}_{N/2<\langle k'\rangle\le N}$. Therefore, by using Proposition 4 (as well as Remark 3), we obtain[1]

$$\begin{aligned} \|\tilde{\mathfrak{h}}^{N,L}[F_N]\|_{X^b(J)}^2 =& \int_{\mathbb{R}} \langle\tau\rangle^{2b} \left\| \sum_{k'} \widehat{h_{kk'}^{N,L}}(\tau) \frac{g_{k'}}{[\![k']\!]^{\alpha/2}} \right\|_{\ell_k^2}^2 \mathrm{d}\tau \\ \lesssim& N^{-\alpha} \int_{\mathbb{R}} \langle\tau\rangle^{2b} (\|\widehat{h_{kk'}^{N,L}}(\tau)\|_{k\to k'}^2 + \|\widehat{h_{kk'}^{N,L}}(\tau)\|_{kk'}^2) \\ =& N^{-\alpha}(\|\tilde{\mathfrak{h}}^{N,L}\|_{Y^b(J)}^2 + \|\tilde{\mathfrak{h}}^{N,L}\|_{Z^b(J)}^2) \\ \lesssim& N^{-\alpha} L^{-2\delta_0} + N^{1-\alpha+2\gamma_0} L^{-1} \\ \lesssim& N^{1-\alpha+2\gamma_0} L^{-2\delta_0}, \end{aligned} \tag{2.199}$$

[1] Here, strictly speaking, to use Proposition 4, one needs to use a standard *meshing argument* used in [34, Lemma 4.2, Claim 5.4], [35, Proposition 6.1], and [36, Subsection 3.4, Subsection 4.1].

where we used (2.94) in the second to last step. Then from (2.95), by choosing $\kappa \gg 1$ large enough, we have $|k - k'| \le LN^{\varepsilon}$ for some $0 < \varepsilon \ll 1$. This together with (2.198) implies that the Fourier support of $\tilde{\mathfrak{h}}^{N,L}[F_N]$ is $N/4 < |k| \le 2N$. Therefore, from (2.199) we have that

$$\begin{aligned}\Big\| \sum_{1 \le L \le L_N} \tilde{\mathfrak{h}}^{N,L}[F_N] \Big\|_{C^0_J H_x^{\frac{\alpha-1}{2}-\gamma}} &\lesssim N^{\frac{\alpha-1}{2}-\gamma} \Big\| \sum_{1 \le L \le L_N} \tilde{\mathfrak{h}}^{N,L}[F_N] \Big\|_{X^b(J)} \\ &\lesssim N^{\frac{\alpha-1}{2}-\gamma} \sum_{1 \le L \le L_N} \|\tilde{\mathfrak{h}}^{N,L}[F_N]\|_{X^b(J)} \\ &\lesssim N^{\gamma_0-\gamma},\end{aligned}$$

where we used the fact that $b > \frac{1}{2}$. Also, recalling that $\gamma_0 \ll \gamma$, we thus conclude that

$$\sum_{N \ge 1} \sum_{1 \le L \le L_N} \mathfrak{h}^{N,L}[\chi(\cdot) F_N] \quad \text{converges in } C(J; H^{\frac{\alpha-1}{2}-\gamma}(\mathbb{T})) \subset C(J; L^2(\mathbb{T})) \tag{2.200}$$

by choosing $\gamma \ll 1$.

From (2.96), we see that

$$\sum_{N \ge 1} z_N \quad \text{converges in } X^{\frac{1}{2}+\gamma-,b}(J), \tag{2.201}$$

and thus in $C(J; H^{\frac{1}{2}+\gamma-}(\mathbb{T}))$.

Finally, by collecting (2.196), (2.197), (2.200), and (2.201), we conclude that the sequence

$$u_N = \sum_{M=1}^{N} y_M$$

converges in $C(J; H^{\frac{\alpha-1}{2}-\gamma}(\mathbb{T}))$ which is a subset of $C(J; L^2(\mathbb{T}))$, i.e. there exists a unique $u \in C(J; H^{\frac{\alpha-1}{2}-\gamma}(\mathbb{T}))$. $\square$

2.8 Proof of Theorem 7

This section proves Theorem 7, which claims that the second Picard iterate $Z_N^{(2)}$ given in (2.19) converges in $H^{\frac{1}{2}+}$ almost surely when $\alpha > 1$, while fails to converges in L^2 almost surely when $\alpha = 1$.

We start with Theorem 7 (i). This part actually follows from Subsubsection 2.6.2. To see this, we note that the interaction representation of $Z_N^{(2)}$ is

$$\begin{aligned} S(-t)Z_N^{(2)}(t) &= \chi(t)\int_0^t \chi(t')S(-t')\Pi_N\Big[\Big(|z_N(t')|^2 - 2\fint_{\mathbb{T}}|z_N(t')|^2\mathrm{d}x\Big)z_N(t')\Big]\mathrm{d}t' \\ &= \chi(t)\int_0^t \Pi_N\mathcal{M}(\widetilde{F}_N, \widetilde{F}_N, \widetilde{F}_N)(s)\mathrm{d}s \\ &= \mathscr{A}(\widetilde{F}_N, \widetilde{F}_N, \widetilde{F}_N), \end{aligned} \tag{2.202}$$

where $\mathscr{A}$ is given in (2.161) and $\widetilde{F}_N$ is similar to the F_N given in (2.27) but with large support, i.e.

$$\widetilde{F}_N = \sum_{N^{1-\delta}<k<N} \mathrm{e}^{\mathrm{i}kx}\frac{g_k(\omega)}{[\![k]\!]^{\frac{\alpha}{2}}}.$$

We note that (2.202) is a special case considered in Subsubsection 2.6.2, i.e. the high-high interaction with $N_{\text{med}} \gtrsim N_{\max}^{1-\delta}$ and $w_{N_1}, w_{N_2}, w_{N_3}$ are of type (G). Then from (2.163), it follows that

$$\|Z_N^{(2)}\|_{X^{0,b}} = \|\mathscr{A}(\widetilde{F}_N, \widetilde{F}_N, \widetilde{F}_N)\|_{X^b} \lesssim N^{-\frac{1}{2}-},$$

which finish the proof of Theorem 7 (i) since $b > \frac{1}{2}$.

Now we turn to Theorem 7 (ii), i.e. the case when $\alpha = 1$. We can rewrite

$$\big(Z_N^{(2)}(t)\big)_k = \sum_{\substack{k_1-k_2+k_3=k\\ k_2\notin\{k_1,k_3\}}} \frac{g_{k_1}\overline{g_{k_2}}g_{k_3}}{[\![k_1]\!]^{\frac{1}{2}}[\![k_2]\!]^{\frac{1}{2}}[\![k_3]\!]^{\frac{1}{2}}} \cdot \mathbf{1}_{\langle k\rangle<N}\prod_{i=1}^{3}\mathbf{1}_{N^{1-\delta}<k_j<N}\cdot\Theta(t,\Phi),$$

where

$$\Theta(t,\Phi) = \chi_\delta(t)\mathrm{e}^{-\mathrm{i}t|k|}\int_0^t \chi_\delta(t')\mathrm{e}^{\mathrm{i}t'\Phi(\bar{k})}\mathrm{d}t',$$

and

$$\Phi(\bar{k}) = |k_1| - |k_2| + |k_3| - |k|.$$

We observe that if $k \geq 0$ and $k_i \geq 0$, then $\Phi(\bar{k}) = 0$ on the hyperplane $k = k_1 - k_2 + k_3$. We also note that $|\Theta(t, 0)| = \chi_\delta(t) \int_0^t \chi_\delta(t')dt'$ for $|t| < \delta$. Then it follows that

$$\begin{aligned}
&\Big(\mathbb{E}\big[\|Z_N^{(2)}(t)\|_{L_x^2}\big]\Big)^2 \\
&= \sum_{k\in\mathbb{Z}} \sum_{\substack{k_1-k_2+k_3=k \\ k_2\notin\{k_1,k_3\}}} \frac{|\Theta(t,\Phi)|^2}{[\![k_1]\!][\![k_2]\!][\![k_3]\!]} \mathbf{1}_{\langle k\rangle<N} \prod_{i=1}^{3} \mathbf{1}_{N^{1-\delta}<k_j<N} \\
&\geq \sum_{k} \sum_{\substack{k_1-k_2+k_3=k \\ k_2\notin\{k_1,k_3\},}} \frac{|\Theta(t,0)|^2}{[\![k_1]\!][\![k_2]\!][\![k_3]\!]} \mathbf{1}_{0\leq k<N} \prod_{i=1}^{3} \mathbf{1}_{N^{1-\delta}<k_j<N} \\
&\gtrsim |\Theta(t,0)|^2 \sum_{N^{1-\delta}<N_1,N_2,N_3\leq N/4} (N_1N_2N_3)^{-1} \\
&\qquad \times \Bigg(\sum_{k} \sum_{\substack{k_1-k_2+k_3=k \\ k_2\notin\{k_1,k_3\},}} \mathbf{1}_{0\leq k\leq N} \prod_{i=1}^{3} \mathbf{1}_{N_j/2<k_j\leq N_j}\Bigg),
\end{aligned} \tag{2.203}$$

where N_1, N_2, N_3 are dyadic numbers and S_N is the set given by

$$\begin{aligned}
S_N = \big\{(k, k_1, k_2, k_3) \in \mathbb{Z}^4 ; & k_1 - k_2 + k_3 = k, k_2 \notin \{k_1, k_3\}, \\
& 0 \leq k < N, k_j \in (N_j/2, N_j)\big\},
\end{aligned}$$

such that $N^{1-\delta} < N_1, N_2, N_3 \leq N/4$. Then it is easy to see that

$$|S_N| = \big|\big\{(k_1, k_2, k_3) \in \mathbb{Z}^3; k_2 \notin \{k_1, k_3\}, k_i \in (N_j/2, N_j)\big\}\big| \gtrsim N_1N_2N_3,$$

which together with (2.203) implies

$$\Big(\mathbb{E}\big[\|Z_N^{(2)}(t)\|_{L_x^2}\big]\Big)^2 \gtrsim |\Theta(t,0)|^2 \sum_{N^{1-\delta}<N_1,N_2,N_3\leq N/4} 1 \gtrsim \delta(\log N)^3.$$

This finishes the proof of Theorem 7 (ii).

CHAPTER 3

FOCUSING GIBBS MEASURES

3.1 Introduction

Gibbs measure for the fractional Schrödinger equation posed on the torus $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$, is formally given by

$$\mathrm{d}\rho_{\alpha,d,p} = Z_{\alpha,d,p}^{-1} \exp\left(\pm\frac{1}{p}\int_{\mathbb{T}^d} |u|^p \mathrm{d}x\right) \mathrm{d}\mu_{\alpha,d}(u) \tag{3.1}$$

for $p > 2$, where $Z_{\alpha,d,p}$ is a normalisation constant and $\mu_{\alpha,d}$ is the Gaussian measure formally given by

$$\mathrm{d}\mu_{\alpha,d}(u) = Z_{\alpha,d}^{-1} \mathrm{e}^{-\frac{1}{2}\int_{\mathbb{T}^d} |\mathrm{D}^{\alpha/2}u|^2 \mathrm{d}x} \mathrm{d}u = Z_{\alpha,d}^{-1} \prod_{k\neq 0} \mathrm{e}^{-\frac{1}{2}|k|^{\alpha}|u_k|^2} \mathrm{d}(\mathrm{Re} u_k)\mathrm{d}(\mathrm{Im} u_k), \tag{3.2}$$

where $Z_{\alpha,d}$ is a normalisation constant, $\mathrm{D}^{\alpha/2} = (\sqrt{-\Delta})^{\alpha/2}$ is the spatial fractional derivative defined as the Fourier multiplier $\mathcal{F}_x(\mathrm{D}^{\alpha/2}u)(k) := |k|^{\alpha/2}u_k$, and $u_k := (\mathcal{F}_x u)(k)$ is the k-th Fourier coefficient of u. In (3.1), the plus sign in the exponent stands for the *focusing* case; whereas the minus sign stands for the defocusing case. In this chapter, we merely focus on the focusing case in the measure construction part of this thesis. When $\alpha = 2$, the measure $\mathrm{d}\mu_{\alpha,d}$ represents the massless Gaussian free field on $\mathbb{T}^d$. The Gaussian measure $\mathrm{d}\mu_{\alpha,d}$ given in (3.2) can be defined as the law of the distribution-valued random variable

$$\gamma : \omega \longmapsto u^\omega, \tag{3.3}$$

with u^ω given by the following random Fourier series:

$$u^\omega(x) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{g_k(\omega)}{|k|^{\alpha/2}} \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{3.4}$$

where $\{g_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ denotes a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By the law of the random variable γ given by (3.3)-(3.4), we mean $\mathbb{P}\circ\gamma^{-1}$. Namely,

$$\mathrm{d}\mu_{\alpha,d}\,(A) := \mathbb{P}\circ\gamma^{-1}(A) = \mathbb{P}(\gamma^{-1}[A]),$$

for all subsets A of the set of distributions on $\mathbb{T}^d$. When $d = 1$, the series (3.4) represents the mean-zero Brownian loop when $\alpha = 2$ (namely $u(0) = u(1)$), and the mean-zero fractional Brownian loop when $1 < \alpha < 3$. See [86, Section 5]. It can be proven that for any $1 \le q \le \infty$ and $\sigma < \frac{\alpha-d}{2}$, u in (3.4) belongs to $\dot{W}^{\sigma,q}(\mathbb{T}^d)\setminus\dot{W}^{\frac{\alpha-d}{2},q}(\mathbb{T}^d)$ almost surely. Here a statement holds almost surely means that the statement holds with probability one; and $\dot{W}^{\sigma,q}(\mathbb{T}^d)$ denotes the (complex-valued) homogeneous Sobolev space

$$\dot{W}^{\sigma,q} := \{u; u \text{ is a distribution on } \mathbb{T}^d \text{ such that } \|u\|_{\dot{W}^{\sigma,q}(\mathbb{T}^d)} < \infty\}$$

with

$$\|u\|_{\dot{W}^{\sigma,q}(\mathbb{T}^d)} = \|\mathrm{D}^\sigma u\|_{L^q(\mathbb{T}^d)} = \|\mathcal{F}_x^{-1}(|k|^\sigma u_k)\|_{L^q(\mathbb{T}^d)}.$$

See [19] for details. When $\sigma \ge 0$, $u \in \dot{W}^{\sigma,q}(\mathbb{T}^d)$ is a function, while it is merely a distribution when $\sigma < 0$. Therefore, the series in (3.4) is a function if $\alpha > d$, and is a distribution when $\alpha \le d$. For the latter case, the potential energy $\pm\frac{1}{p}\int_{\mathbb{T}^d}|u|^p\mathrm{d}x$ in (3.1) does not make sense as it stands and hence renormalisation is needed. In this thesis, we focus on the case $\alpha > d$ where the typical elements in the support of $\mathrm{d}\mu_{\alpha,d}$ are functions.

The Hamiltonian (energy functional) associated with the Gibbs measure $\mathrm{d}\rho_{\alpha,d,p}$ in (3.1) is given by

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\mathrm{D}^{\alpha/2}u|^2 \mathrm{d}x \pm \frac{1}{p}\int_{\mathbb{T}^d} |u|^p \mathrm{d}x. \tag{3.5}$$

The construction of the Gibbs measure is interesting not just in the subject of mathematical physics, including areas such as constructive Euclidean quantum field theory, but also holds significance in the study of Hamiltonian PDEs [68, 2, 3, 82, 83, 81, 84, 85]. An important example of Hamiltonian PDEs corresponding to the Hamiltonian (3.5) is the following fractional nonlinear Schrödinger equation:

$$\mathrm{i}\partial_t u - \mathrm{D}^\alpha u = \pm|u|^{p-2}u, \tag{3.6}$$

where $\mathrm{D}^\alpha = (\sqrt{-\Delta})^\alpha$ is the spatial fractional derivative defined as the Fourier multiplier $\mathcal{F}_x(\mathrm{D}^\alpha u)(k) := |k|^\alpha u_k$. The equation (3.6) represents the nonlinear Schrödinger equation (NLS) for $\alpha = 2$ ([2, 3]), the biharmonic NLS for $\alpha = 4$ ([93, 86, 94]), and the nonlinear half-wave equation for $\alpha = 1$ ([50, 70]). In the seminal work [2, 3], Bourgain proved that the local-in-time dynamics of (3.6) can be extended globally in time by employing the Gibbs measure[1] as a replacement of a conservation law. Over the last decade, there has been tremendous progress in the study of this subject. For a survey on the subject, see [20].

The main challenge in constructing the focusing Gibbs measure $\mathrm{d}\rho_{\alpha,d,p}$ in (3.1) arises from the potential energy being unbounded from above. Indeed, given (3.4) and the fact that $\ell^2(\mathbb{Z}^d) \subset \ell^p(\mathbb{Z}^d)$, we see that

$$\mathbb{E}_{\mu_{\alpha,d}}\Big[\mathrm{e}^{\frac{1}{p}\left(\int_{\mathbb{T}^d}|u|^2\mathrm{d}x\right)^{p/2}}\Big] \geq \mathbb{E}_{\mu_{\alpha,d}}\Big[\mathrm{e}^{\frac{1}{p}\|u\|^p_{\mathcal{F}L^p(\mathbb{T}^d)}}\Big] \geq \prod_{k\in\mathbb{Z}^d\setminus\{0\}} \mathbb{E}\Big[\mathrm{e}^{\frac{|g_k|^p}{|k|^{\alpha p/2}}}\Big] = \infty$$

[1]Strictly speaking, we need to modify the massless fractional Gaussian free field $\mathrm{d}\mu_{\alpha,d}$ in (3.2) by the massive one to avoid an issue at the zeroth frequency. See Remark 15.

for $p > 2$, where $\mathcal{F}L^p(\mathbb{T}^d)$ represents the Fourier Lebesgue space defined by the norm

$$\|u\|_{\mathcal{F}L^p(\mathbb{T}^d)} = \bigg(\sum_{k \in \mathbb{Z}^d} |u_k|^p \bigg)^{1/p}.$$

To overcome this difficulty, in a seminal work [68], Lebowitz-Rose-Speer proposed to consider a mass cut-off for the *focusing* Gibbs measure:

$$\mathrm{d}\rho_{2,1,p,K} = Z_{2,1,p,K}^{-1} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})} \le K\}} \exp\bigg(\frac{1}{p} \int_{\mathbb{T}} |u|^p \mathrm{d}x \bigg) \mathrm{d}\mu_{2,1}(u). \tag{3.7}$$

In [68, 2, 85], Lebowitz-Rose-Speer, Bourgain, and Oh-Sosoe-Tolomeo proved that the focusing Gibbs measure (3.7) on the circle $\mathbb{T}$ is normalisable for (i) $2 < p < 6$ and all $K > 0$ and (ii) $p = 6$ and all $0 < K \le \|Q\|_{L^2(\mathbb{R})}$, where Q is the (unique) minimiser of the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ (see Section 1.3.6) with $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q'\|_{L^2(\mathbb{R})}^2$, while it is non-normalisable for (iii) $p > 6$ and (iv) $p = 6$ and $K > \|Q\|_{L^2(\mathbb{R})}$. See also Table 1.1.

The main purpose of the measure construction part of the thesis is to extend the results in [68, 2, 85] to higher dimensions. To this purpose, we gain some smoothing effects by moving to fractional setting and consider the following focusing Gibbs measure (as a generalisation of (3.7)):

$$\mathrm{d}\rho_{\alpha,d,p,K} = Z_{\alpha,d,p,K}^{-1} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \le K\}} \exp\bigg(\frac{1}{p} \int_{\mathbb{T}^d} |u|^p \mathrm{d}x \bigg) \mathrm{d}\mu_{\alpha,d}(u), \tag{3.8}$$

with $\alpha > d$ (such that the random Fourier series in (3.4) defines a function). Our goal is to identify the sharp condition on (α, d, p, K) such that $\mathrm{d}\rho_{\alpha,d,p,K}$ can be normalised as a probability measure for $\alpha > d$. More specifically, for the subcritical case (i.e. $2 < p < \frac{2\alpha}{d} + 2$), we show that the focusing Gibbs measure $\mathrm{d}\rho_{\alpha,d,p,K}$ in (3.8) is normalisable for any $K > 0$, while we show its non-normalisability for any finite $K > 0$ for the supercritical case (i.e. $p > \frac{2\alpha}{d} + 2$). For the critical case (i.e. $p = \frac{2\alpha}{d} + 2$), we then prove the existence of an optimal mass threshold, given by an optimiser for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^d$ (see Subsection 1.3.6). Below this optimal mass threshold, the focusing Gibbs measure $\mathrm{d}\rho_{\alpha,d,p,K}$ is normalisable, while it becomes non-normalisable above it. See Theorem 8. See [84] for the case $\alpha = d$. We refer the reader to [85] and the references cited therein for additional context and further background.

We now state our main result for this chapter.

Theorem 8. *Let $d \geq 1$, $\alpha > d$, and $p > 2$. Given $K > 0$, define the partition function $Z_{\alpha,d,p,K}$ by*

$$Z_{\alpha,d,p,K} = \mathbb{E}_{\mu_{\alpha,d}} \Big[\mathrm{e}^{\frac{1}{p}\int_{\mathbb{T}^d} |u|^p \mathrm{d}x} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \Big], \tag{3.9}$$

where $\mathbb{E}_{\mu_{\alpha,d}}$ denotes an expection with respect to the law $\mu_{\alpha,d}$ of the random Fourier series in (3.4). *Then, the following statements hold:*

(i) (subcritical case) *If $2 < p < \frac{2\alpha}{d} + 2$, then $Z_{\alpha,d,p,K} < \infty$ for any $K > 0$.*

(ii) (critical case) *Let $p = \frac{2\alpha}{d} + 2$. Then, $Z_{\alpha,d,p,K} < \infty$ if $K < \|Q\|_{L^2(\mathbb{R}^d)}$, and $Z_{\alpha,d,p,K} = \infty$ if $K > \|Q\|_{L^2(\mathbb{R}^d)}$. Here, Q is the optimiser for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^d$ such that $\|Q\|^p_{L^p(\mathbb{R}^d)} = \frac{p}{2}\|\mathrm{D}^{\alpha/2}Q\|^2_{L^2(\mathbb{R}^d)}$.*

(iii) (supercritical case) *If $p > \frac{2\alpha}{d} + 2$, then $Z_{\alpha,d,p,K} = \infty$ for any $K > 0$.*

We summarise the result in Theorem 8 in the Table 3.1.

	$2 < p < \frac{2\alpha}{d} + 2$	$p = \frac{2\alpha}{d} + 2$	$p > \frac{2\alpha}{d} + 2$
$K < \|Q\|_{L^2(\mathbb{R}^d)}$	normalisable	normalisable	non-normalisable
$K = \|Q\|_{L^2(\mathbb{R}^d)}$	normalisable	—	non-normalisable
$K > \|Q\|_{L^2(\mathbb{R}^d)}$	normalisable	non-normalisable	non-normalisable

Table 3.1: Normalisablility and non-normalisability of Gibbs measures $\mathrm{d}\rho_{\alpha,d,p,K}$.

Remark 14. The sharpness of $\alpha > d$ comes from a comparision with [84]. In [84], the authors proved that, in the case $\alpha = d$, the corresponding focusing Gibbs measure is non-normalisable even after renormalisation. Also, the condition $\alpha > d$ is needed, as otherwise, the potential function in (3.8) is ill-defined and we need to introduce renormalisation, which changes the Gibbs measure.

As mentioned above, Theorem 8 extends the one-dimensional results in [68, 2, 85] to $d \geq 1$ and $\alpha > d$. We continue to provide a brief summary of the known results in the one-dimensional case. When $\alpha = 2$ and $d = 1$, [68] proved the non-normalisability in Theorem 8 (the non-normalisability of (3.2) for the supercritical case and the critical case with the mass cut-off size $K > \|Q\|_{L^2(\mathbb{R})}$). The argument in [68] was based on a Cameron-Martin type argument and the sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality (See Subsection 1.3.6):

$$\|u\|_{L^p(\mathbb{R})}^p \leq C_{\text{GNS}} \|u\|_{\dot{H}^1(\mathbb{R})}^{\frac{p}{2}-1} \|u\|_{L^2(\mathbb{R})}^{\frac{p}{2}+1}, \tag{3.10}$$

where C_{GNS} is the optimal constant. [68] also proves the normalisability in Theorem 8 for the subcritical case in Theorem 8 *essentially* but with a gap, and then Bourgain [2] gives the correct proof. Bourgain [2] also proves normalisability in Theorem 8 for the critical case with sufficiently small K. In [2], Bourgain's argument to obtain this result is based on dyadic pigeon hole principle and the Sobolev embedding theorem. Recently, Oh-Sosoe-Tolomeo [85] refined Bourgain's argument and used the sharp GNS inequality to prove the normalisability part in Theorem 8 (ii), thus identifying the optimal mass threshold at the critical mass nonlinearity. In the same paper, Oh-Sosoe-Tolomeo also proved the normalisability of the focusing Gibbs measure at the critical mass threshold $K = \|Q\|_{L^2(\mathbb{R})}$ when $\alpha = 2$ and $d = 1$ with $p = 6$. See Remark 16.

The main difficulties in proving Theorem 1.1 stem from the non-local nature of the fractional derivatives $\mathrm{D}^{\alpha/2} = (\sqrt{-\Delta})^{\alpha/2}$ and the non-integer critical exponent $p = \frac{2\alpha}{d} + 2$. In particular, the non-local derivative causes extra difficulty in localising the GNS inequality, initially on $\mathbb{R}^d$, to the torus $\mathbb{T}^d$. Inspired by [19], we utilise a characterisation of the $\dot{H}^{\alpha/2}(\mathbb{R}^d)$-norm based on high order difference operators (3.18). We then proceed to prove an almost sharp GNS inequality on the torus $\mathbb{T}^d$ (with the sharp constant C_{GNS} in (3.10)) by applying this new characterisation. See Proposition 6.

With the almost sharp GNS inequality on the torus $\mathbb{T}^d$, our proof of Theorem 8 is based on the variational approach developed by Barashkov-Gubinelli [17] and is different from those in [68, 2, 85], thereby providing an alternative proof of the results when $d = 1$ and $\alpha = 2$. We first represent the partition function $Z_{\alpha,d,p,K}$ in (3.9) as a stochastic optimisation problem, by applying the Boué-Dupuis variational formula (Lemma 17). We then prove the normalisability part of Theorem 8, by using the almost sharp GNS inequality on the torus $\mathbb{T}^d$. As for the non-normalisability part, the main task is to construct a sequence of drift terms along which the partition function diverges. Our construction of such drift terms is based on a scaling argument, similar to that in [68, 85]. We point out that our proof of Theorem 8, based on the variational approach, is essentially a physical space approach, rather than the Fourier side approach in [2, 85]. It is thus expected that our method is more flexible in geometric settings. See also Appendix B in [84], where the variational approach was used to prove Theorem 8 (i) with $\alpha > d$ and $p = 4$.

Remark 15. The key idea in proving Theorem 8 lies in controlling the potential energy $\frac{1}{p}\|u\|^p_{L^p(\mathbb{T}^d)}$ by the kinetic energy $\frac{1}{2}\|u\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}$ under the constraint $\|u\|_{L^2(\mathbb{T}^d)} \le K$. From Gagliardo-Nirenberg-Sobolev inequality Proposition 6, we see that the subcritical case $2 < p < \frac{2\alpha}{d} + 2$ corresponds to weaker potential energy. The critical exponent $p = \frac{2\alpha}{d} + 2$ leads to the equivalence of potential and kinetic energy, where a restriction on the size K is needed to guarantee the normalisability. However, for the supercritical case $p > \frac{2\alpha}{d} + 2$, the kinetic energy losses control of the potential energy no matter how small the mass is.

Remark 16. As in [85], Theorem 8 also applies when we replace the mean-zero fractional Brownian loop in (3.4) by the fractional Ornstein-Uhlenbeck loop:

$$u^\omega(x) = \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{\langle k\rangle^{\alpha/2}} \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{3.11}$$

where $\langle k\rangle = (1 + |k|^2)^{1/2}$ and $\{g_k\}_{k\in\mathbb{Z}^d}$ is a sequence of independent standard complex-valued Gaussian random variables. See Remark 4.1 in [85]. The law $\mathrm{d}\tilde{\mu}_{\alpha,d}$ of the fractional

Ornstein-Uhlenbeck loop in (3.11) has the formal density

$$d\tilde{\mu}_{\alpha,d} = \tilde{Z}_{\alpha,d}^{-1} e^{-\frac{1}{2}\|u\|^2_{H^{\alpha/2}(\mathbb{T}^d)}} du.$$

As seen in [2], the measure $d\tilde{\mu}_{\alpha,d}$ is a more natural base Gaussian measure to consider for the (fractional) nonlinear Schrödinger equation (3.6) due to the lack of the conservation of the spatial mean under the dynamics.

Note that Theorem 8 also holds in the real-valued setting (i.e. with an extra assumption that $g_{-k} = \overline{g_k}$ in (3.4)). For example, this is relevant to the study of the dispersion generalised KdV equation on $\mathbb{T}$:

$$\partial_t u + D^\alpha \partial_x u = \partial_x(u^{p-1}).$$

Remark 17. We point out that Oh-Sosoe-Tolomeo [85] also showed the normalisability of the Gibbs measure (3.8) at the critical mass threshold when $\alpha = 2$, $d = 1$ and $p = 6$. This result is quite striking in view of the presence of the minimal mass blowup solution (at this critical mass) for the focusing quintic NLS on $\mathbb{T}$. We will not pursue this question for the fractional focusing Gibbs measure (3.8), as their argument is beyond the scope of the framework developed in this thesis.

Remark 18. Since $\alpha > d$, Theorem 8 only considers the non-singular case, namely, the measures $d\mu_{\alpha,d}$ and $d\rho_{\alpha,d,p,K}$ are supported on functions. One of the reasons for only considering the non-singular case is that the bifurcation phenomena at the critical mass (Theorem 8 (ii)) are only possible when $\alpha > d$. As soon as $\alpha \le d$, we need to introduce a proper renormalisation to define the potential energy $\frac{1}{p}\int_{\mathbb{T}^d} |u|^p dx$, which necessitates p to be an integer. When $\alpha = d$, it was shown in [21, 84] that the renormalised focusing Gibbs measure $d\rho_{\frac{d}{2},d,4,K}$ (with $p = 4$, critical), endowed with a (renormalised) mass cut-off, is not normalisable. It was also shown in [84] that with the cubic interaction ($p = 3$, subcritical), the renormalised focusing Gibbs measure $d\rho_{\frac{d}{2},d,3,K}$ endowed with a renormalised mass cut-off is indeed normalisable. When $d = 2$, this normalisability in the case of the cubic interaction was first observed by Bourgain [6]. When $d = 3$, it has recently been shown that the cubic interaction ($p = 3$) exhibits phase transition between weakly and strongly nonlinear regimes. See [89] for more details.

Remark 19. While the construction of the defocusing Gibbs measures has been extensively studied and well understood due to the strong interest in constructive Euclidean quantum field theory (see [102, 46]), the (non-)normalisability issue of the focusing Gibbs measures, going back to the work of Lebowitz-Rose-Speer [68] and Brydges-Slade [21], has not been fully explored. See related works [99, 11, 27, 81, 85, 84, 89, 109] on the non-normalisability (and other issues) for focusing Gibbs measures. In particular, recent works such as [88, 89] employ the variational approach developed in [17] and establish certain phase transition phenomena in the singular setting.

3.2 Gagliardo-Nirenberg-Sobolev inequality on the torus

In proving Theorem 8, we need the almost sharp Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}^d$.

For a function u defined on $\mathbb{T}^d$, we define the $\dot{H}^s(\mathbb{T}^d)$ norm via

$$\|u\|_{\dot{H}^s(\mathbb{T}^d)}^2 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{2s} |u_k|^2. \tag{3.12}$$

Due to the scaling invariance of the minimisation problem (1.48), it is expected that the GNS inequality (1.47) also holds on the finite domains $\mathbb{T}^d$ with the same optimal constant.

Proposition 6. Let $d \geq 1$ and let (i) $p > 2$ if $d < 2s$, and (ii) $2 < p \leq \frac{2d}{d-2s}$ if $d \geq 2s$. Then, for sufficiently small $\delta > 0$, there is a constant $C = C(\delta) > 0$ such that

$$\|u\|_{L^p(\mathbb{T}^d)}^p \leq (C_{\text{GNS}}(d,p,s) + \delta) \|u\|_{\dot{H}^s(\mathbb{T}^d)}^{\frac{(p-2)d}{2s}} \|u\|_{L^2(\mathbb{T}^d)}^{2 + \frac{p-2}{2s}(2s-d)} + C(\delta) \|u\|_{L^2(\mathbb{T}^d)}^p \tag{3.13}$$

for $u \in H^s(\mathbb{T}^d)$, where C_{GNS} is the constant defined in (1.49) and (1.52).

The main difficulty in showing Proposition 6 is due to the non-local nature of the fractional derivatives. To circumvent this difficulty, we will use the characterisation of the $\dot{H}^s(\mathbb{R}^d)$ norm (1.46) based on the L^2-modulus of continuity. We will continue to recall this characterisation. Notice that when $0 < s < 1$, one has

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^d}\left(|\xi|^{-2s}\int_{\mathbb{R}^d}\frac{|\mathrm{e}^{2\pi\mathrm{i}y\cdot\xi}-1|^2}{|y|^{d+2s}}\mathrm{d}y\right)|\xi|^{2s}|\widehat{u}(\xi)|^2\mathrm{d}\xi.\end{aligned} \tag{3.14}$$

Denote the inner integral, a convergent improper integral for $0 < s < 1$, by

$$c_1(d,s) = |\xi|^{-2s}\int_{\mathbb{R}^d}\frac{|\mathrm{e}^{2\pi\mathrm{i}y\cdot\xi}-1|^2}{|y|^{d+2s}}\mathrm{d}y \ \left(=\int_{\mathbb{R}^d}\frac{|\mathrm{e}^{2\pi\mathrm{i}x_1}-1|^2}{|x|^{d+2s}}\mathrm{d}x\right), \tag{3.15}$$

which is a constant, i.e. independent of ξ. From (3.14) and (3.15), we have the following characterisation of the $\dot{H}^s(\mathbb{R}^d)$ norm in (1.46) (see for example [19]),

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = c_1(d,s)^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y. \tag{3.16}$$

We remark that on the torus $\mathbb{T}^d$ the $H^s(\mathbb{T}^d)$ norm defined in (3.12) has a similar equivalent characterisation. See [19, Proposition 1.3]. However, the identity as (3.16) fails for the torus case due to the lack of rotational invariance.

By using high-order difference operators, we may generalise (3.16) to the cases $s \geq 1$. In particular, we have

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = c_k(d,s)^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\Delta_y^k u(x)|^2}{|y|^{d+2s}}\mathrm{d}x\mathrm{d}y, \tag{3.17}$$

where Δ_y^k is the k-th forward difference operator with spacing y defined by

$$\Delta_y^k u(x) = \sum_{j=0}^{k}(-1)^{k-j}C_k^j u(x+jy), \tag{3.18}$$

where C_k^j are binomial coefficients, and

$$c_k(d,s) = \int_{\mathbb{R}^d} \frac{|\mathrm{e}^{2\pi \mathrm{i} x_1} - 1|^{2k}}{|x|^{d+2s}} \mathrm{d}x. \tag{3.19}$$

The proof of (3.17) is similar to that of (3.16). We thus omit the details.

Now we are ready to prove Proposition 6.

Proof of Proposition 6. For the pedagogical purpose, we present the proof for the case $0 < s < 1$ before demonstrating the general case $s > 0$, as the former is less complex in terms of notation.

We first consider $0 < s < 1$. Let $\psi \in C_0^\infty(B(0, \frac{1}{2}))$ be a bump function with $\|\psi\|_{L^1} = 1$ and $\psi_\delta(x) = \delta^{-d}\psi(\frac{x}{\delta})$. Define $\phi_\delta(x) = \mathbf{1}_{[-\frac{1}{2}+2\delta, \frac{1}{2}-2\delta]^d} * \psi_\delta(x)$. Then the following properties hold

(i) $\phi_\delta \in C_0^\infty(\mathbb{T}^d)$,

(ii) $\phi_\delta(x) = 1$ for $x \in [-\frac{1}{2} + 3\delta, \frac{1}{2} - 3\delta]^d$,

(iii) $\phi_\delta(x) = 0$ for $x \in ([-\frac{1}{2} + \delta, \frac{1}{2} - \delta]^d)^c$,

(iv) $|\mathrm{D}^s \phi_\delta(x)| \lesssim \delta^{-s}$ for all $x \in \mathbb{R}^d$.

Let

$$u_\delta(x) = \begin{cases} \phi_\delta(x) u(x), & x \in [-\frac{1}{2}, \frac{1}{2}]^d; \\ 0, & \text{otherwise}. \end{cases} \tag{3.20}$$

First we claim there exists $C(d) > 0$ such that for any $u \in L^p(\mathbb{T}^d)$ there exists $x_0 \in \mathbb{T}^d$ satisfying the following

$$\|u\|_{L^p(\mathbb{T}^d)}^p \le (1 + C(d)\delta) \|\phi_\delta(\cdot) u(\cdot + x_0)\|_{L^p(\mathbb{R}^d)}^p. \tag{3.21}$$

From the definition of ϕ_δ, it suffices to show

$$\|u\|^p_{L^p(\mathbb{T}^d)} \le (1 + C(d)\delta)\|u(\cdot + x_0)\|^p_{L^p([-\frac{1}{2}+3\delta,\frac{1}{2}-3\delta]^d)}. \tag{3.22}$$

We show (3.22) inductively. Recall that $\delta \ll 1$. When $d = 1$, we may split the interval $[-\frac{1}{2}, \frac{1}{2}]$ into $k = [\frac{1}{6\delta}]$ many equal subintervals. Then, from the pigeonhole principle, there must be a subinterval, say the j-th subinterval $[-\frac{1}{2} + \frac{j-1}{k}, -\frac{1}{2} + \frac{j}{k}]$, such that

$$\int_{[-\frac{1}{2}+\frac{j-1}{k},-\frac{1}{2}+\frac{j}{k}]} |u(x)|^p \mathrm{d}x \le \frac{1}{k}\int_{\mathbb{T}} |u(x)|^p \mathrm{d}x,$$

which implies

$$\begin{aligned}\int_{\mathbb{T}} |u(x)|^p \mathrm{d}x &\le (1 + \frac{1}{k})\int_{-\frac{1}{2}+\frac{1}{2k}}^{\frac{1}{2}-\frac{1}{2k}} \Big|u\Big(x + \frac{2j-1}{2k}\Big)\Big|^p \mathrm{d}x \\ &\le (1 + 12\delta)\int_{-\frac{1}{2}+3\delta}^{\frac{1}{2}-3\delta} \Big|u\Big(x + \frac{2j-1}{2k}\Big)\Big|^p \mathrm{d}x,\end{aligned}$$

provided δ is sufficiently small. Thus we conclude (3.22) for $d = 1$. Let us assume (3.22) holds for all $1, 2, \cdots, d-1$ dimensions. Then for $x \in \mathbb{T}^d$, we may write $x = (x', x_d)$ such that $x' \in \mathbb{T}^{d-1}$ and $x_d \in \mathbb{T}$. Then, from our assumption, there exist $x_d^0 \in \mathbb{T}$ and $x_0' \in \mathbb{T}^{d-1}$ such that

$$\begin{aligned}&\int_{\mathbb{T}^d} |u(x)|^p \mathrm{d}x = \int_{\mathbb{T}}\Big(\int_{\mathbb{T}^{d-1}} |u(x', x_d)|^p \mathrm{d}x'\Big)\mathrm{d}x_d \\ &\quad\le (1 + C\delta)\int_{-\frac{1}{2}+3\delta}^{\frac{1}{2}-3\delta}\Big(\int_{\mathbb{T}^{d-1}} |u(x', x_d + x_d^0)|^p \mathrm{d}x'\Big)\mathrm{d}x_d \\ &\quad\le (1 + C\delta)\int_{\mathbb{T}^{d-1}}\Big(\int_{-\frac{1}{2}+3\delta}^{\frac{1}{2}-3\delta} |u(x', x_d + x_d^0)|^p \mathrm{d}x_d\Big)\mathrm{d}x' \\ &\quad\le (1 + C(1)\delta)\big(1 + C(d-1)\delta\big)\int_{[-\frac{1}{2}+3\delta,\frac{1}{2}-3\delta]^{d-1}}\Big(\int_{-\frac{1}{2}+3\delta}^{\frac{1}{2}-3\delta} |u(x' + x_0', x_d + x_d^0)|^p \mathrm{d}x_d\Big)\mathrm{d}x',\end{aligned}$$

where we used the assumption in the second and fourth steps. Thus we finish the proof of (3.22) for d dimension by taking $x_0 = (x_0', x_d^0)$ and $C(d) = 1 + (C(1) + 2C(d-1))$ provided $C(1)\delta < 1$.

From (3.16) and (3.21), for the translated $u(\cdot + x_0)$, still denoting by u[1], we have

$$\begin{aligned}\|u\|^p_{L^p(\mathbb{T}^d)} &\le (1+C\delta)\|u_\delta\|^p_{L^p(\mathbb{R}^d)}\\ &\le (1+C\delta)C_{\mathrm{GNS}}(d,p,s)\|u_\delta\|^{\frac{(p-2)d}{2s}}_{\dot H^s(\mathbb{R}^d)}\|u_\delta\|^{2+\frac{p-2}{2s}(2s-d)}_{L^2(\mathbb{R}^d)}\\ &\le (1+C\delta)C_{\mathrm{GNS}}(d,p,s)\Big(c_1(d,s)^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{(p-2)d}{4s}}\|u\|^{2+\frac{p-2}{2s}(2s-d)}_{L^2(\mathbb{T}^d)}.\end{aligned}$$

To prove (3.13), since $p > 2$, it suffices to show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y \le (1+C\delta)c_1(d,s)\|u\|^2_{H^s(\mathbb{T}^d)} + C(\delta)\|u\|^2_{L^2(\mathbb{T}^d)}. \tag{3.23}$$

Since the integrand $\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{d+2s}}$ in (3.23) is supported on $(x,y) \in (\mathbb{T}^d \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \mathbb{T}^d)$, we have

$$\text{LHS of (3.23)} \le \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y + C\int_{(\mathbb{T}^d)^c}\int_{\mathbb{T}^d}\frac{|u_\delta(x)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y. \tag{3.24}$$

For the second term in (3.24), since $|x-y| > \delta$ in the integrand, we have

$$\int_{(\mathbb{T}^d)^c}\int_{\mathbb{T}^d}\frac{|u_\delta(x)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y \lesssim \Big(\int_{|y|>\delta}\frac{1}{|y|^{d+2s}}\mathrm{d}y\Big)\|u_\delta\|^2_{L^2(\mathbb{T}^d)} \lesssim \delta^{-2s}\|u\|^2_{L^2(\mathbb{T}^d)}, \tag{3.25}$$

which is sufficient for (3.23). Now we turn to the first term in (3.24). We note

$$\begin{aligned}|u_\delta(x)-u_\delta(y)|^2 &= |\phi_\delta(x)(u(x)-u(y)) + (\phi_\delta(x)-\phi_\delta(y))u(y)|^2\\ &= |\phi_\delta(x)(u(x)-u(y))|^2 + |(\phi_\delta(x)-\phi_\delta(y))u(y)|^2\\ &\quad + 2\phi_\delta(x)(\phi_\delta(x)-\phi_\delta(y))(u(x)-u(y))u(y).\end{aligned}$$

[1]We note that (3.13) is invariant under translation.

Thus we have

$$\begin{aligned}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y &\le \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y\\ &+2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|\phi_\delta(x)(\phi_\delta(x)-\phi_\delta(y))(u(x)-u(y))u(y)|}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|(\phi_\delta(x)-\phi_\delta(y))u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y\\ &= A_1+A_2+A_3.\end{aligned}\tag{3.26}$$

For the term A_1, we have

$$\begin{aligned}A_1 &\le \int_{\mathbb{T}^d}\int_{B(0,2)}\frac{|u(x)-u(x+z)|^2}{|z|^{d+2s}}\mathrm{d}z\mathrm{d}x\\ &=\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\left(|n|^{-2s}\int_{B(0,2)}\frac{|\mathrm{e}^{2\pi\mathrm{i}x\cdot n}-1|^2}{|x|^{d+2s}}\mathrm{d}x\right)|n|^{2s}|u_n|^2,\end{aligned}\tag{3.27}$$

where $B(0,2)\subset\mathbb{R}^d$ is the ball centered at 0 with radius 2. It is easy to see that

$$|n|^{-2s}\int_{B(0,2)}\frac{|\mathrm{e}^{2\pi\mathrm{i}x\cdot n}-1|^2}{|x|^{d+2s}}\mathrm{d}x\le|n|^{-2s}\int_{\mathbb{R}^d}\frac{|\mathrm{e}^{2\pi\mathrm{i}x\cdot n}-1|^2}{|x|^{d+2s}}\mathrm{d}x=c_1(d,s),$$

which together with (3.27) shows the contribution from A_1 is bounded by the right-hand side of (3.23). For the term A_3, since $s<1$, we have

$$A_3\lesssim\delta^{-2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u(y)|^2}{|x-y|^{d+2s-2}}\mathrm{d}x\mathrm{d}y\lesssim\delta^{-2}\|u\|^2_{L^2(\mathbb{T}^d)},\tag{3.28}$$

which is sufficient for our purpose. For A_2, by Young's inequality we have

$$A_2\le\delta A_1+\frac{1}{\delta}A_3,\tag{3.29}$$

which is again acceptable. By collecting (3.26), (3.27), (3.29), and (3.28), we finish the proof of (3.23) and thus (3.13) when $0<s<1$.

In the following we consider the case $s\ge1$. Assume $s\in[k-1,k)$ for some $k\in\mathbb{Z}_+$.

Similarly to (3.23) in the case $0 < s < 1$, it only needs to show

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\Delta_y^k u_\delta(x)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \le (1 + C\delta)c_k(d,s)\|u\|^2_{H^s(\mathbb{T}^d)} + C(\delta)\|u\|^2_{L^2(\mathbb{T}^d)}. \tag{3.30}$$

Similarly to (3.25) and (3.27), we may reduce (3.30) to

$$\int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\Delta_y^k u_\delta(x)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \le (1 + C\delta)c_k(d,s)\|u\|^2_{H^s(\mathbb{T}^d)} + C(\delta)\|u\|^2_{L^2(\mathbb{T}^d)}. \tag{3.31}$$

In the following, we prove (3.31). First note that

$$\Delta_y^k u_\delta(x) = \Delta_y^k(\psi_\delta(x)u(x)) = \sum_{j=0}^{k} C_k^j \Delta_y^{k-j}\psi_\delta(x)\Delta_y^j u(x + (k-j)y).$$

Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\Delta_y^k u_\delta(x)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\sum_{j=0}^{k} C_k^j \Delta_y^{k-j}\psi_\delta(x)\Delta_y^j u(x+(k-j)y)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\psi_\delta(x)\Delta_y^k u(x)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\
&+ \sum_{j=0}^{k-1}\int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|C_k^j \Delta_y^{k-j}\psi_\delta(x)\Delta_y^j u(x+(k-j)y)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\
&+ \sum_{j\neq \ell}\int_{\mathbb{T}^d}\int_{B(0,k)} \frac{C_k^j \Delta_y^{k-j}\psi_\delta(x)\Delta_y^j u(x+(k-j)y) C_k^\ell \Delta_y^{k-\ell}\psi_\delta(x)\Delta_y^\ell u(x+(k-\ell)y)}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\
&= B_1 + B_2 + B_3.
\end{aligned} \tag{3.32}$$

For the term B_1 in (3.32), we have

$$\begin{aligned} B_1 &\leq \int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\Delta_y^k u(x)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}^d}\int_{B(0,k)} \frac{|\mathrm{e}^{2\pi \mathrm{i} y\cdot n}-1|^{2k}}{|y|^{d+2s}}\mathrm{d}y|u_n|^2 \\ &\leq c_k(d,s)\sum_{n\in\mathbb{Z}^d}|n|^{2s}|u_n|^2, \end{aligned} \tag{3.33}$$

where $c_k(d,s)$ is defined in (3.19). Thus the contribution of B_1 is bounded by the right-hand side of (3.31). Similarly, we can control B_2 in (3.32) as

$$\begin{aligned} B_2 &\lesssim \sum_{j=0}^{k-1}\int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\Delta_y^{k-j}\psi_\delta(x)\Delta_y^j u(x+(k-j)y)|^2}{|y|^{d+2s}}\mathrm{d}y\mathrm{d}x \\ &\lesssim \sum_{j=0}^{k-1}\delta^{-2(k-j)}\int_{B(0,k)}\int_{\mathbb{T}^d} \frac{|\Delta_y^j u(x+(k-j)y)|^2}{|y|^{d+2s-2(k-j)}}\mathrm{d}x\mathrm{d}y \\ &\lesssim \sum_{j=0}^{k-1}\delta^{-2(k-j)}\int_{\mathbb{T}^d}\int_{B(0,k)} \frac{|\Delta_y^j u(x)|^2}{|y|^{d+2s-2(k-j)}}\mathrm{d}y\mathrm{d}x \\ &\lesssim \sum_{j=0}^{k-1}\delta^{-2(k-j)}\|u\|^2_{\dot{H}^{s-k+j}(\mathbb{T}^d)} \\ &\lesssim \delta\|u\|^2_{\dot{H}^s(\mathbb{T}^d)} + C(\delta)\|u\|^2_{L^2(\mathbb{T}^d)}, \end{aligned} \tag{3.34}$$

where in the last step we used the interpolation between $L^2(\mathbb{T}^d)$ and $\dot{H}^s(\mathbb{T}^d)$. This shows that the contribution of B_2 is acceptable.

Finally, we turn to B_3 in (3.32). When $j<k$ and $\ell<k$, by Hölder's inequality we have

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{C_k^j\Delta_y^{k-j}\psi_\delta(x)\Delta_y^j u(x+(k-j)y)C_k^\ell\Delta_y^{k-\ell}\psi_\delta(x)\Delta_y^\ell u(x+(k-\ell)y)}{|y|^{d+2s}}\mathrm{d}x\mathrm{d}y \lesssim B_2,$$

which is bounded by (3.34). Without loss of generality, we only consider the case $j=k$.

Then we have $\ell < k$. By Young's inequality we have

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\psi_\delta(x)\Delta_y^k u(x) C_k^\ell \Delta_y^{k-\ell}\psi_\delta(x)\Delta_y^\ell u(x+(k-\ell)y)}{|y|^{d+2s}} \mathrm{d}x\mathrm{d}y \lesssim \delta B_1 + C(\delta) B_2,$$

which is again sufficient for our purpose in view of (3.33) and (3.34).

This finishes the proof of (3.31) as desired. □

Remark 20. Let u be a function defined on $\mathbb{R}^d$. With a slight abuse of notation, we also use u to denote its restriction onto $\mathbb{T}^d$. It follows from (3.17) and (3.19) that

$$\begin{aligned} \|u\|_{\dot{H}^s(\mathbb{T}^d)}^2 &= c_k^{-1} \sum_{n\in\mathbb{Z}^d} \bigg(\int_{\mathbb{R}^d} \frac{|\mathrm{e}^{2\pi \mathrm{i} y\cdot n} - 1|^{2k}}{|y|^{d+2s}} \mathrm{d}y \bigg) |u_n|^2 \\ &= c_k^{-1} \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \frac{|\Delta_y^k u(x)|^2}{|y|^{d+2s}} \mathrm{d}y\mathrm{d}x \\ &\le c_k^{-1} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\Delta_y^k u(x)|^2}{|y|^{d+2s}} \mathrm{d}x\mathrm{d}y = \|u\|_{\dot{H}^s(\mathbb{R}^d)}^2, \end{aligned} \tag{3.35}$$

where $k = [s]$ is the largest integer less than s.

3.3 Proof of Theorem 8

In this section, we prove Theorem 8, which provides sharp criteria for the normalisability of the Gibbs measure (1.29) with focusing interaction.

3.3.1 Variational formulation

In order to prove Theorem 8, we recall a variational formula for the partition functional $Z_{\alpha,d,p,K}$ as in [85]. Let $W(t)$ denote a mean zero cylindrical Brownian motion in $L^2(\mathbb{T}^d)$

$$W(t) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} B_n(t)\, \mathrm{e}^{\mathrm{i}n\cdot x}$$

where $\{B_n\}_{n\in\mathbb{Z}^d\setminus\{0\}}$ is a sequence of mutually independent complex-valued Brownian motions. Then define a centered Gaussian process $Y_\alpha(t)$ by

$$Y_\alpha(t) = \mathrm{D}^{-\alpha/2}W(t) = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{B_n(t)}{|n|^{\alpha/2}}\,\mathrm{e}^{\mathrm{i}n\cdot x}. \tag{3.36}$$

We note that $Y_\alpha(t)$ is well-defined and

$$\mathbb{E}\big[|Y_\alpha(1)|^2\big] = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{\mathbb{E}[|B_n(1)|^2]}{|n|^{\alpha/2}} = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{2}{|n|^{\alpha/2}} < \infty,$$ [1]

provided $\alpha > d$. In particular, we have

$$\mathrm{Law}(Y_\alpha(1)) = \mu_{\alpha,d}, \tag{3.37}$$

where $\mu_{\alpha,d}$ is the massless Gaussian free field given in (3.2).

Let $\mathbb{H}_a$ be the space of drifts, which consists of *mean zero* progressively measurable processes belonging to $L^2([0,1];L^2(\mathbb{T}^d))$, $\mathbb{P}$-almost surely. One of the key tools in this thesis is the following Boué-Dupuis variational formula [14, 113, 14]. See also [15] for the infinite dimensional setting.

Lemma 17. Let Y_α be as in (3.36) with $\alpha > d$. Suppose that $F : H^{\frac{\alpha-d}{2}-}(\mathbb{T}^d) \to \mathbb{R}$ is measurable and bounded from above. Then, we have

$$-\log \mathbb{E}\Big[e^{-F(Y_\alpha(1))}\Big] = \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[F\big(Y_\alpha(1) + I_\alpha(\theta)(1)\big) + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x}\mathrm{d}t\bigg], \tag{3.38}$$

where $I_\alpha(\theta)$ is defined by

$$I_\alpha(\theta)(t) = \int_0^t \mathrm{D}^{-\alpha/2}\Pi_{\neq 0}\theta(\tau)\mathrm{d}\tau$$

and the expectation $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ is with respect to the underlying probability measure $\mathbb{P}$.

[1] $\mathbb{E}\big[|Y_\alpha(1)|^2\big]$ is constant in x.

Since we only consider the non-singular case $\alpha > d$, then $Y_\alpha(t)$ and $I_\alpha(\theta)(1)$ enjoy the following pathwise regularity bounds.

Lemma 18. (i) Given any $\alpha > d$ and any finite $p, q \geq 1$, there exists $C_{\alpha,p} > 0$ such that

$$\mathbb{E}\big[\|Y_\alpha(1)\|^p_{L^q(\mathbb{T}^d)}\big] \leq \mathbb{E}\big[\|Y_\alpha(1)\|^p_{L^\infty(\mathbb{T}^d)}\big] \leq C_{\alpha,p} < \infty. \tag{3.39}$$

(ii) For any $\theta \in \mathbb{H}_a$, we have

$$\|I_\alpha(\theta)(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \leq \int_0^1 \|\theta(t)\|^2_{L^2_x} \mathrm{d}t. \tag{3.40}$$

Proof. Part (i) follows from Hölder's inequality, Sobolev embedding, Minkowski's inequality, and Wiener chaos estimate [20, Lemma 2.4] with $k = 1$. As for Part (ii), the estimate (3.40) follows from Minkowski's and Cauchy-Schwarz's inequalities. □

3.3.2 Integrability

In this subsection, we demonstrate the proof of the integrability part of Theorem 8. Namely, we prove the boundedness of $Z_{\alpha,d,p,K}$ (i) for all $K > 0$ when $2 < p < \frac{2\alpha}{d} + 2$ and (ii) for all $K < \|Q\|_{L^2(\mathbb{R}^d)}$ when $p = \frac{2\alpha}{d} + 2$, where Q is the optimiser for the GNS inequality on $\mathbb{R}^d$.

Proof of Theorem 8 - (i) and the first half of (ii). It suffices to show the following bound

$$Z_{\alpha,d,p,K} = \mathbb{E}_{\mu_{\alpha,d}}\Big[\exp(R_p(u)) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}}\Big] < \infty, \tag{3.41}$$

where $R_p(u)$ is the potential energy denoted by

$$R_p(u) := \frac{1}{p}\int_{\mathbb{T}^d} |u|^p \,\mathrm{d}x. \tag{3.42}$$

Observing that

$$\mathbb{E}_{\mu_{\alpha,d}}\Big[\exp(R_p(u))\cdot\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\le K\}}\Big]\le\mathbb{E}_{\mu_{\alpha,d}}\Big[\exp\Big(R_p(u)\cdot\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\le K\}}\Big)\Big],$$

then the bound (3.41) follows once we have

$$\mathbb{E}_{\mu_{\alpha,d}}\Big[\exp\Big(R_p(u)\cdot\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\le K\}}\Big)\Big]<\infty. \tag{3.43}$$

From (3.37) and the Boué-Dupuis variation formula Lemma 17, it follows that

$$\begin{aligned}&-\log\mathbb{E}_{\mu_{\alpha,d}}\Big[\exp\Big(R_p(u)\cdot\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\le K\}}\Big)\Big]\\&=-\log\mathbb{E}\Big[\exp\Big(R_p(Y_\alpha(1))\cdot\mathbf{1}_{\{\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}\le K\}}\Big)\Big]\\&=\inf_{\theta\in\mathbb{H}_a}\mathbb{E}\Big[-R_p\big(Y_\alpha(1)+I_\alpha(\theta)(1)\big)\cdot\mathbf{1}_{\{\|Y_\alpha(1)+I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)}\le K\}}+\frac{1}{2}\int_0^1\|\theta(t)\|_{L^2_x}^2\mathrm{d}t\Big],\end{aligned} \tag{3.44}$$

where $Y(1)$ is given in (3.36). Here, $\mathbb{E}_{\mu_{\alpha,d}}$ and $\mathbb{E}$ denote expectations with respect to the Gaussian field $\mu_{\alpha,d}$ and the underlying probability measure $\mathbb{P}$ respectively. In the following, we show that the right-hand side of (3.44) has a finite lower bound. The key observation is that (i) in the subcritical setting, we view $Y_\alpha(1)$ as a perturbation with finite $L^2(\mathbb{T}^d)$ norm; (ii) in the critical setting, we have "$Y_\alpha(1)=\Pi_{\le N}Y_\alpha(1)+$ a perturbation" for large $N\gg 1$, where the perturbation term is small under $L^2(\mathbb{T}^d)$ norm with large probability. We, therefore, distinguish two cases depending on subcritical/critical interactions.

Case 1: subcritical $p<\frac{2\alpha}{d}+2$. In this case, we prove (3.43) with a mass cut-off of any finite size K. We first recall an elementary inequality, which is a direct consequence of the mean value theorem and the Young's inequality. Given $p>2$ and $\varepsilon>0$, there exists C_ε such that

$$|z_1+z_2|^p\le(1+\varepsilon)|z_1|^p+C_\varepsilon|z_2|^p \tag{3.45}$$

holds uniformly in $z_1, z_2 \in \mathbb{C}$. From (3.42), (3.45), Proposition 6, and the fact

$$\{\|Y_\alpha(1)+I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)} \le K\} \subset \{\|I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)} \le K + \|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}\},$$

we obtain

$$\begin{aligned}
&R_p\big(Y_\alpha(1)+I_\alpha(\theta)(1)\big)\cdot \mathbf{1}_{\{\|Y_\alpha(1)+I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)}\le K\}}\\
&\le (1+\varepsilon)R_p\big(I_\alpha(\theta)(1)\big)\cdot \mathbf{1}_{\{\|I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)}\le K+\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}\}} + C_\varepsilon R_p(Y_\alpha(1))\\
&\le \frac{1+\varepsilon}{p}(C_{\text{GNS}}+\delta)(K+\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)})^{2+\frac{p-2}{\alpha}(\alpha-d)}\|I_\alpha(\theta)(1)\|_{\dot H^{\alpha/2}(\mathbb{T}^d)}^{\frac{(p-2)d}{\alpha}}\\
&\qquad + C_\delta(K+\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)})^p + C_\varepsilon R_p(Y_\alpha(1)).
\end{aligned}$$

Noting that $\frac{(p-2)d}{\alpha} < 2$ in this case, we apply Young's inequality to continue with

$$\le C + C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^{2+\frac{2\alpha(p-2)}{2\alpha-(p-2)d}} + \frac{1}{4}\|I_\alpha(\theta)(1)\|_{\dot H^{\alpha/2}(\mathbb{T}^d)}^2 + C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^p + CR_p(Y_\alpha(1)) \tag{3.46}$$

where C is a constant depending on $\varepsilon, \delta, p, d, s, \|Q\|_{L^2}$, and K. By collecting (3.44), (3.46) and Lemma 18, we arrive at

$$\begin{aligned}
&-\log \mathbb{E}_{\mu_{\alpha,d}}\Big[\exp\Big(R_p(u)\cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\le K\}}\Big)\Big]\\
&\ge \inf_{\theta\in\mathbb{H}_a}\mathbb{E}\Big[-C - C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^{2+\frac{2\alpha(p-2)}{2\alpha-(p-2)d}} - C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^p - CR_p(Y_\alpha(1))\\
&\qquad\qquad - \frac{1}{4}\|I_\alpha(\theta)(1)\|_{\dot H^{\alpha/2}(\mathbb{T}^d)}^2 + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 \mathrm{d}t\Big]\\
&\ge \inf_{\theta\in\mathbb{H}_a}\mathbb{E}\Big[-C - C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^{2+\frac{2\alpha(p-2)}{2\alpha-(p-2)d}} - C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^p - C\|Y_\alpha(1)\|_{L^p(\mathbb{T}^d)}^p\\
&\qquad\qquad + \frac{1}{4}\int_0^1 \|\theta(t)\|_{L^2_x}^2 \mathrm{d}t\Big]\\
&\ge \mathbb{E}\Big[-C - C\|Y_\alpha(1)\|_{L^p(\mathbb{T}^d)}^p - C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^p - C\|Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}^{2+\frac{2\alpha(p-2)}{2\alpha-(p-2)d}}\Big]\\
&\ge -C - 2CC_{\alpha,p} - CC_{\alpha,2+\frac{2\alpha(p-2)}{2\alpha-(p-2)d}} > -\infty,
\end{aligned}$$

where $C_{\alpha,r}$ is defined in Lemma 18 (i). Thus we finish the proof of (3.43) in the subcritical case.

<u>Case 2</u>: *critical interaction* $p = \frac{2\alpha}{d} + 2$. We shall prove (3.43) below the critical mass threshold $K < \|Q\|_{L^2(\mathbb{R}^d)}$. To get the sharp mass threshold, we view $\Pi_{\geq N} Y_\alpha(1)$ as a perturbation instead. It turns out that as N is getting larger, the probability of $\Pi_{\geq N} Y_\alpha(1)$ being large shrinks exponentially to zero. See (3.55).

Since $\alpha > d$, it follows that

$$\lim_{N\to\infty} \|\Pi_{\geq N} Y_\alpha(1)\|_{L^2(\mathbb{T}^d)} = 0,$$

almost surely. Therefore, given small $\varepsilon > 0$, for $\omega \in \Omega$ almost sure, there exists an unique $N_\varepsilon := N_\varepsilon(\omega)$ such that

$$\|\Pi_{\geq \frac{N_\varepsilon}{2}} Y_\alpha(1)\|_{L^2(\mathbb{T}^d)} > \varepsilon \quad \text{and} \quad \|\Pi_{> N_\varepsilon} Y_\alpha(1)\|_{L^2(\mathbb{T}^d)} \leq \varepsilon. \tag{3.47}$$

Similar argument as before with (3.45), Proposition 6, and (3.47), yields that

$$\begin{aligned}
&R_p\big(Y_\alpha(1) + I_\alpha(\theta)(1)\big) \cdot \mathbf{1}_{\{\|Y_\alpha(1)+I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K\}} \\
&\leq (1+\varepsilon) R_p\big(\Pi_{\leq N_\varepsilon} Y_\alpha(1) + I_\alpha(\theta)(1)\big) \cdot \mathbf{1}_{\{\|\Pi_{\leq N_\varepsilon} Y_\alpha(1)+I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K+\varepsilon\}} + C_\varepsilon R_p(Y_\alpha(1)) \\
&\leq \frac{1+\varepsilon}{p} (C_{\text{GNS}} + \delta)(K+\varepsilon)^{p-2} (\|\Pi_{\leq N_\varepsilon} Y_\alpha(1)\|_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} + \|I_\alpha(\theta)(1)\|_{\dot{H}^{\alpha/2}(\mathbb{T}^d)})^2 \\
&\qquad + C_\varepsilon R_p(Y_\alpha(1)) + C_\delta (K+\varepsilon)^p \\
&\leq \frac{(1+\varepsilon)^2}{p} (C_{\text{GNS}} + \delta)(K+\varepsilon)^{p-2} \|I_\alpha(\theta)(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} + C \|\Pi_{\leq N_\varepsilon} Y_\alpha(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \\
&\qquad + C_\varepsilon R_p(Y_\alpha(1)) + C_\delta (K+\varepsilon)^p,
\end{aligned} \tag{3.48}$$

where C is a constant depending on $\varepsilon, \delta, p, d, \alpha, \|Q\|_{L^2}$, and K. Since $C_{\text{GNS}} = \frac{p}{2}\|Q\|_{L^2}^{2-p}$,

$K < \|Q\|_{L^2(\mathbb{R}^d)}$ and $p > 2$, there exist $\eta, \varepsilon, \delta > 0$ such that

$$\frac{(1+\varepsilon)^2}{p}(C_{\text{GNS}} + \delta)(K+\varepsilon)^{p-2} < \frac{1-\eta}{2}. \tag{3.49}$$

By collecting (3.44), (3.48), (3.49), and Lemma 18, we arrive at

$$\begin{aligned}
&-\log \mathbb{E}_{\mu_{\alpha,d}}\Big[\exp\Big(R_p(u)\cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\le K\}}\Big)\Big] \\
&\ge \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\Big[-\frac{1-\eta}{2}\|I_\alpha(\theta)(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} - C_\varepsilon\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} - C_\delta(K+\varepsilon)^p \\
&\qquad\qquad - C_\varepsilon R_p(Y_\alpha(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x}\mathrm{d}t\Big] \\
&\ge \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\Big[-C_\delta(K+\varepsilon)^p - C_\varepsilon R_p(Y_\alpha(1)) - C_\varepsilon\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} + \frac{\eta}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x}\mathrm{d}t\Big] \\
&\ge \mathbb{E}\Big[-C_\delta(K+\varepsilon)^p - C_\varepsilon R_p(Y_\alpha(1)) - C_\varepsilon\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}\Big] \\
&\ge -C_\delta(K+\varepsilon)^p - C_\varepsilon C_{\alpha,p} - C_\varepsilon \mathbb{E}\big[\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}\big],
\end{aligned}$$

where $C_{\alpha,p}$ is given in (3.39). We remark that $Y_\alpha(1) \notin \dot{H}^{\alpha/2}(\mathbb{T}^d)$ almost surely. Therefore, to prove (3.43), it still needs to show that

$$\mathbb{E}\big[\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}\big] < \infty, \tag{3.50}$$

where N_ε is a random variable given by (3.47).

Noting $Y_\alpha(1)$ is a mean-zero random variable, we may decompose Ω (by ignoring a zero-measure set) as

$$\Omega = \bigcup_{N\ge 1} \Omega_N, \tag{3.51}$$

where

$$\Omega_N = \Big\{\omega \in \Omega : N_\varepsilon(\omega) \in \big[\tfrac{N}{2}, N\big)\Big\}. \tag{3.52}$$

By (3.51) and Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\big[\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot H^{\alpha/2}(\mathbb{T}^d)}\big] &\le \sum_{N\ge 1} \mathbb{E}\big[\|\Pi_{\le N} Y_\alpha(1)\|^2_{\dot H^{\alpha/2}(\mathbb{T}^d)} \cdot \mathbf{1}_{\Omega_N}\big] \\
&\le \sum_{N\ge 1} N^\alpha \mathbb{E}\big[\|\Pi_{\le N} Y_\alpha(1)\|^2_{L^2(\mathbb{T}^d)} \cdot \mathbf{1}_{\Omega_N}\big] \\
&\le \sum_{N\ge 1} N^\alpha \Big(\mathbb{E}\big[\|Y_\alpha(1)\|^4_{L^2(\mathbb{T}^d)}\big]\Big)^{\frac{1}{2}} \cdot \mathbb{P}(\Omega_N)^{\frac{1}{2}} \\
&\le C_{\alpha,4}^{\frac{1}{2}} \sum_{N\ge 1} N^\alpha \mathbb{P}(\Omega_N)^{\frac{1}{2}},
\end{aligned} \tag{3.53}$$

where $C_{\alpha,4}$ is given in (3.39). By a direct computation, we have

$$\mathbb{E}\big[\|\Pi_{\ge \frac{N}{4}} Y_\alpha(1)\|^2_{L^2(\mathbb{T}^d)}\big] \sim N^{d-\alpha}. \tag{3.54}$$

It then follows from (3.47), (3.52), Hölder's inequality, Lemma 5, and (3.54), that

$$\begin{aligned}
\mathbb{P}(\Omega_N) &\le \mathbb{P}\big(\big\{\|\Pi_{\ge \frac{N}{4}} Y_\alpha(1)\|_{L^2} > \varepsilon\big\}\big) \\
&\lesssim \exp\bigg\{-c\Big(\frac{\varepsilon}{\mathbb{E}\big(\|\Pi_{\ge \frac{N}{4}} Y_\alpha(1)\|_{L^2(\mathbb{T}^d)}\big)}\Big)^2\bigg\} \\
&\lesssim \exp\bigg\{-\Big(\frac{c\varepsilon^2}{\mathbb{E}\big[\|\Pi_{\ge \frac{N}{4}} Y_\alpha(1)\|^2_{L^2(\mathbb{T}^d)}\big]}\bigg\} \\
&\lesssim \mathrm{e}^{-\tilde c \varepsilon^2 N^{\alpha-d}},
\end{aligned} \tag{3.55}$$

where c and $\tilde c$ are constant. By collecting (3.53) and (3.55), we conclude that

$$\mathbb{E}\big[\|\Pi_{\le N_\varepsilon} Y_\alpha(1)\|^2_{\dot H^{\alpha/2}(\mathbb{T}^d)}\big] \le C_{\alpha,4}^{\frac{1}{2}} \sum_{N\ge 1} N^\alpha \mathrm{e}^{-\frac{\tilde c}{2}\varepsilon^2 N^{\alpha-d}} < \infty,$$

which finishes the proof of (3.50), and thus (3.43) in the critical case.

Therefore, we finish the proof of Theorem 8 -(i) and the first half of (ii). □

3.3.3 Non-integrability

In this subsection, we prove the rest of Theorem 8, i.e. the non-integrability part of (ii) and (iii). In particular, we show that the partition function

$$Z_{\alpha,d,p,K} = \mathbb{E}_{\mu_{\alpha,d}}\Big[\exp(R_p(u))\mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)}\leq K\}}\Big] = \infty \tag{3.56}$$

under either of the following conditions

$$\begin{aligned} &\text{(i) critical nonlinearity: } p = \frac{2\alpha}{d} + 2 \text{ and } K > \|Q\|_{L^2(\mathbb{R}^d)};\\ &\text{(ii) super-critical nonlinearity: } p > \frac{2\alpha}{d} + 2 \text{ and any } K > 0. \end{aligned} \tag{3.57}$$

Here Q is the optimiser of the GNS inequality given in Theorem 4 and Remark 4. To prove (3.56), we construct, within the ball $\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}$, a sequence of drift terms given by perturbed scaled "solitons", along which the variational formula (3.38) diverges. The existence of such a sequence of scaled solitons is guaranteed by the following lemma;

Lemma 19. Assume (3.57) holds. Then, there exist a series of functions $\{W_\rho\}_{\rho>0} \subset H^{\alpha/2}(\mathbb{T}^d) \cap L^p(\mathbb{T}^d)$ such that

$$\begin{aligned} &\text{(i) } H_{\mathbb{T}^d}(W_\rho) \leq -A_1\rho^{-\frac{dp}{2}+d},\\ &\text{(ii) } \|W_\rho\|^p_{L^p(\mathbb{T}^d)} \leq A_2\rho^{-\frac{dp}{2}+d},\\ &\text{(iii) } \|W_\rho\|_{L^2(\mathbb{T}^d)} \leq K-\eta,\\ &\text{(iv) } \Pi_0 W_\rho \lesssim 1, \end{aligned} \tag{3.58}$$

where $H_{\mathbb{T}^d}$ is the Hamiltonian functional given in (3.5) for the focusing case, i.e.

$$H_{\mathbb{T}^d}(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\mathrm{D}^{\alpha/2}u|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{T}^d} |u|^p \mathrm{d}x.$$

and $A_1, A_2, A_3, \eta > 0$ are constant uniformly in sufficiently small $\rho > 0$.

In the next lemma, we construct an approximation Z_M to $Y_\alpha(1)$ in (3.36) through solving a stochastic differential equation. These Z_M act as controllable stochastic perturbations in defining the drift terms. See (3.63) and (3.64) in the following. Similar approximation has appeared in [84].

Lemma 20. Given $\alpha > d$ and a dyadic number $M \sim \rho^{-1} \gg 1$, define the $Z_M(t)$ by its Fourier coefficients $(Z_M)_n(t)$. Let $(Z_M)_n(t)$ for $0 < |n| \leq M$ be as follows:

$$\begin{cases} \mathrm{d}(Z_M)_n(t) = |n|^{-\alpha/2} M^{\frac{d}{2}}((Y_\alpha)_n(t) - (Z_M)_n(t))\mathrm{d}t \\ (Z_M)_n|_{t=0} = 0, \end{cases} \tag{3.59}$$

and $(Z_M)_n(t) = 0$ for $n = 0$ and $|n| > M$. Then the following holds:

$$\mathbb{E}\big[\|Z_M(1) - Y_\alpha(1)\|^p_{L^p(\mathbb{T}^d)}\big] \lesssim \max(M^{\frac{d-\alpha}{2}}, M^{-\frac{d}{2}+})^{\frac{p}{2}}, \text{ for } p \geq 1, \tag{3.60}$$

$$\mathbb{E}\left[\left\|\mathrm{D}^s \frac{\mathrm{d}}{\mathrm{d}t} Z_M(t)\right\|^2_{L^2(\mathbb{T}^d)}\right] \lesssim \max(M^{\frac{3d-\alpha}{2}}, M^{\frac{d}{2}+}), \tag{3.61}$$

for any $M \gg 1$.

The proofs of Lemma 19 and Lemma 20 will be postponed to the next subsection. Now we are ready to prove the rest of Theorem 8.

Proof of Theorem 8 - the second half of (ii) and (iii). We shall prove (3.56) under conditions (3.57). Observing that

$$\mathbb{E}_{\mu_{\alpha,d}}\Big[\exp(R_p(u)) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}}\Big] \geq \mathbb{E}_{\mu_{\alpha,d}}\Big[\exp\Big(R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}}\Big)\Big] - 1,$$

then (3.56) follows from

$$\mathbb{E}_{\mu_{\alpha,d}}\left[\exp\left(R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}}\right)\right] = \infty. \tag{3.62}$$

To apply Lemma 17, we construct the series of drift terms as follows. Let W_ρ be as in

Lemma 19, and

$$\theta(t) \in \Big\{ -\mathrm{D}^s \frac{\mathrm{d}}{\mathrm{d}t} Z_M(t) + \mathrm{D}^s W_\rho \Big\}_{\rho>0}, \tag{3.63}$$

where $\rho \ll 1$ and $M \sim \rho^{-1}$ is a dyadic number. From (3.63), we have

$$\begin{aligned} I_\alpha(\theta)(1) &= \int_0^1 \mathrm{D}^{-\alpha/2} \Pi_{\neq 0} \theta(t) \mathrm{d}t \\ &= \int_0^1 (\Pi_{\neq 0} W_\rho - \frac{\mathrm{d}}{\mathrm{d}t} Z_M(t)) \mathrm{d}t \\ &= \Pi_{\neq 0} W_\rho - Z_M(1). \end{aligned} \tag{3.64}$$

Thus, from Lemma 17, (3.63), and (3.64) we have

$$\begin{aligned} &-\log \mathbb{E}_{\mu_{\alpha,d}} \Big[\exp \Big(R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \Big) \Big] \\ &= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \Big[\Big(-R_p(Y_\alpha(1) + I_\alpha(\theta)(1)) \cdot \mathbf{1}_{\{\|Y_\alpha(1)+I_\alpha(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K\}} + \frac{1}{2} \int_0^1 \|\theta(t)\|^2_{L^2(\mathbb{T}^d)} \mathrm{d}t \Big) \Big] \\ &\leq \inf_{0<\rho\ll 1} \mathbb{E} \Big[\Big(-R_p(Y_\alpha(1) - Z_M(1) + \Pi_{\neq 0} W_\rho) \cdot \mathbf{1}_{\{\|Y_\alpha(1)-Z_M(1)+\Pi_{\neq 0} W_\rho\|_{L^2(\mathbb{T}^d)} \leq K\}} \\ &\qquad + \frac{1}{2} \int_0^1 \Big\| -\frac{\mathrm{d}}{\mathrm{d}t} Z_M(t) + \Pi_{\neq 0} W_\rho \Big\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \Big) \mathrm{d}t \Big] \\ &= \inf_{0<\rho\ll 1} \mathbb{E} \Big[\Big(-R_p(W_\rho) + \frac{1}{2} \|W_\rho\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \Big) \\ &\qquad + \Big(R_p(W_\rho) - R_p(\Pi_{\neq 0} W_\rho) \Big) \\ &\qquad + \Big(R_p(\Pi_{\neq 0} W_\rho) - R_p(Y_\alpha(1) - Z_M(1) + \Pi_{\neq 0} W_\rho) \Big) \cdot \mathbf{1}_{\{\|Y_\alpha(1)-Z_M(1)+\Pi_{\neq 0} W_\rho\|_{L^2(\mathbb{T}^d)} \leq K\}} \\ &\qquad + R_p(\Pi_{\neq 0} W_\rho) \cdot \mathbf{1}_{\{\|Y_\alpha(1)-Z_M(1)+\Pi_{\neq 0} W_\rho\|_{L^2(\mathbb{T}^d)} > K\}} \\ &\qquad + \frac{1}{2} \int_0^1 \Big\| -\frac{\mathrm{d}}{\mathrm{d}t} Z_M(t) \Big\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} - 2 \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} Z_M(t), W_\rho \Big\rangle_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} \Big) \mathrm{d}t \Big] \\ &= \inf_{0<\rho\ll 1} (\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E}). \end{aligned} \tag{3.65}$$

In what follows, we consider these terms one by one for $0 < \rho \ll 1$.

For term (A), from (3.58) - (i), we have

$$\mathrm{A} = -R_p(W_\rho) + \frac{1}{2}\|W_\rho\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)} = H_{\mathbb{T}^d}(W_\rho) \lesssim -\rho^{-\frac{dp}{2}+d}. \tag{3.66}$$

For term (B), from (3.58) - (iv) and the mean value theorem, we have

$$\begin{aligned}\mathrm{B} &= \frac{1}{p}\int_{\mathbb{T}^d}\big(|W_\rho|^p - |W_\rho - \Pi_0 W_\rho|^p\big)\mathrm{d}x\\ &\lesssim \int_{\mathbb{T}^d}\big(|\Pi_0 W_\rho|^p + |\Pi_0 W_\rho||W_\rho|^{p-1}\big)\mathrm{d}x\\ &\lesssim 1 + \|W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)},\end{aligned}$$

Then, by interpolating (3.58) - (ii) and (iii), we obtain

$$\mathrm{B} \lesssim \rho^{-\frac{d(p-1)}{2}+d}. \tag{3.67}$$

For term (C), by using the mean value theorem we see that

$$\begin{aligned}&\int_{\mathbb{T}^d}\big(|\Pi_{\neq 0}W_\rho|^p - |Y_\alpha(1) - Z_M(1) + \Pi_{\neq 0}W_\rho|^p\big)\mathrm{d}x\\ &\qquad\lesssim \int_{\mathbb{T}^d}\big(|Y_\alpha(1) - Z_M(1)|^p + |Y_\alpha(1) - Z_M(1)||\Pi_{\neq 0}W_\rho|^{p-1}\big)\mathrm{d}x,\end{aligned}$$

which together with Lemma 20 and Lemma 19 gives

$$\begin{aligned}\mathrm{C} &= \mathbb{E}\Big[\Big(R_p(\Pi_{\neq 0}W_\rho) - R_p(Y_\alpha(1) - Z_M(1) + \Pi_{\neq 0}W_\rho)\Big)\cdot \mathbf{1}_{\{\|Y_\alpha(1)-Z_M(1)+W_\rho\|_{L^2(\mathbb{T}^d)}\leq K\}}\Big]\\ &\lesssim \int_{\mathbb{T}^d}\big(\mathbb{E}\big[|Y_\alpha(1) - Z_M(1)|^p\big] + \mathbb{E}\big[|Y_\alpha(1) - Z_M(1)|\big]|\Pi_{\neq 0}W_\rho|^{p-1}\big)\mathrm{d}x\\ &\lesssim \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+})^{\frac{p}{2}} + \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+})^{\frac{1}{2}}\|\Pi_{\neq 0}W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)}\\ &\lesssim (\|\Pi_{\neq 0}W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)} - \|W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)}) + \|W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)}\\ &\lesssim \rho^{-\frac{d(p-1)}{2}+d},\end{aligned} \tag{3.68}$$

where in the last step, to bound $(\|\Pi_{\neq 0}W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)} - \|W_\rho\|^{p-1}_{L^{p-1}(\mathbb{T}^d)})$, we used a similar argument as in estimating term (B). Now we turn to term (D), by using Chebyshev's inequality, (3.58) - (iii), (3.60), and (3.67), we have

$$\begin{aligned} \mathrm{D} &= \mathbb{E}\big[R_p(\Pi_{\neq 0}W_\rho)\cdot \mathbf{1}_{\{\|Y_\alpha(1)-Z_M(1)+\Pi_{\neq 0}W_\rho\|_{L^2(\mathbb{T}^d)}>K\}}\big] \\ &\le R_p(\Pi_{\neq 0}W_\rho)\cdot \mathbb{E}\big[\mathbf{1}_{\{\|Y_\alpha(1)-Z_M(1)\|_{L^2(\mathbb{T}^d)}>K-\|W_\rho\|_{L^2(\mathbb{T}^d)}\}}\big] \\ &\le R_p(\Pi_{\neq 0}W_\rho)\frac{\mathbb{E}[\|Y_\alpha(1)-Z_M(1)\|^2_{L^2(\mathbb{T}^d)}]}{(K-\|W_\rho\|_{L^2(\mathbb{T}^d)})^2} \\ &\lesssim \rho^{-\frac{dp}{2}+d}\max(M^{\frac{d-\alpha}{2}}, M^{-\frac{d}{2}+}) \\ &\lesssim \max(\rho^{-\frac{d(p-1)-\alpha}{2}}, \rho^{-\frac{d(p-3)}{2}+}), \end{aligned} \tag{3.69}$$

where in the last step we use the relation $M \sim \rho^{-1}$. For term (E), from (3.59) and (3.61), we have

$$\mathrm{E} = \frac{1}{2}\int_0^t \mathbb{E}\Big[\Big\| -\frac{\mathrm{d}}{\mathrm{d}t}Z_M(t)\Big\|^2_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}\Big]\mathrm{d}t \lesssim \max(\rho^{-\frac{3d-\alpha}{2}}, \rho^{-\frac{d}{2}+}), \tag{3.70}$$

where we used the fact that Z_M, removing the zero frequency, is a mean zero Gaussian random variable. By collecting estimates (3.66), (3.67), (3.68), (3.69), and (3.70), we conclude that

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E} \lesssim -\rho^{-\frac{dp}{2}+d}, \tag{3.71}$$

where we used (3.57) and the assumption $\alpha > d$.

Finally, the desired estimate (3.62) follows from (3.65) and (3.71). We thus finish the proof of Theorem 8. □

3.3.4 Proof of the auxiliary lemmas

It remains to prove Lemmas 19 and 20, which is the main purpose of this subsection. We first present the proof of Lemma 19.

Proof of Lemma 19. Define $W_\rho \in H^s(\mathbb{T}^d)$ by

$$W_\rho(x) := \alpha\rho^{-\frac{d}{2}}\phi_\delta(x)Q(\rho^{-1}x), \tag{3.72}$$

where ϕ_δ is the same as in Proposition 6, $\alpha > 0$ is to be determined later and Q is given in Theorem 4 and Remark 4. Then (iv) follows directly from $\|W_\rho\|_{L^1(\mathbb{T}^d)} \lesssim \|W_\rho\|_{L^2(\mathbb{T}^d)}$. We only consider (i) – (iii) in what follows. We distinguish two cases based on the conditions in (3.57):

Case 1: critical nonlinearity. In this case, we have $p = \frac{4s}{d} + 2 > 2$ and $K > \|Q\|_{L^2(\mathbb{R}^d)}$. Fix $\alpha > 1$ such that

$$\|\alpha Q\|_{L^2(\mathbb{R}^d)} = \alpha\|Q\|_{L^2(\mathbb{R}^d)} = K - \eta, \tag{3.73}$$

where η is given in (3.58). Recall that $H_{\mathbb{R}^d}(Q) = 0$ from Remark 4. We then have

$$H_{\mathbb{R}^d}(\alpha Q) = \frac{\alpha^2}{2}\int_{\mathbb{R}^d} |\mathrm{D}^s Q|^2 \mathrm{d}x - \frac{\alpha^p}{p}\int_{\mathbb{R}^d} |Q|^p \mathrm{d}x < 0.$$

Then, it follows from Remark 20 that

$$\begin{aligned}
H_{\mathbb{T}^d}(W_\rho) &= \frac{\alpha^2}{2}\int_{\mathbb{T}^d} |\mathrm{D}^s(\phi_\delta Q_\rho)|^2 \mathrm{d}x - \frac{\alpha^p}{p}\int_{\mathbb{T}^d} |\phi_\delta(x)Q_\rho(x)|^p \mathrm{d}x \\
&\le \frac{\alpha^2}{2}\|\mathrm{D}^s(\phi_\delta Q_\rho)\|^2_{L^2(\mathbb{R}^d)} - \frac{\alpha^p}{p}\int_{\mathbb{T}^d} |\phi_\delta(x)Q_\rho(x)|^p \mathrm{d}x,
\end{aligned}$$

where $Q_\rho = \rho^{-\frac{d}{2}}Q(\rho^{-1}x)$. By the fractional Leibniz rule [63, 65] and Sobolev embedding, we may continue with

$$\le \frac{\alpha^2 + \varepsilon}{2}\|\mathrm{D}^s Q_\rho\|^2_{L^2(\mathbb{R}^d)} - \frac{\alpha^p}{p}\int_{\mathbb{T}^d} |\phi_\delta(x)Q_\rho(x)|^p \mathrm{d}x + C\|Q_\rho\|^2_{W^{0+,2}},$$

then by interpolation we can continue with

$$\leq \frac{\alpha^2+2\varepsilon}{2}\rho^{-2s}\|\mathrm{D}^s Q\|_{L^2(\mathbb{R}^d)}^2 - \frac{\alpha^p}{p}\rho^{-\frac{dp}{2}+d}\int_{\mathbb{R}^d}|\phi_\delta(\rho x)Q(x)|^p\mathrm{d}x + C\|Q_\rho\|_{L^2(\mathbb{R}^d)}^2,$$

we have $\int_{\mathbb{R}^d}|\phi_\delta(\rho x)Q(x)|^p\mathrm{d}x > \frac{1}{\alpha^\varepsilon}\int_{\mathbb{R}^d}|Q(x)|^p\mathrm{d}x$, provided that ρ is sufficiently small. Thus, combining with (1.50) and the fact $2s = \frac{dp}{2} - d$ from (3.57) - (i), we may continue with

$$\leq \left(\frac{\alpha^2+2\varepsilon}{p} - \frac{\alpha^{p-\varepsilon}}{p}\right)\rho^{-\frac{dp}{2}+d}\|Q\|_{L^p(\mathbb{R}^d)}^p + C\|Q\|_{L^2(\mathbb{R}^d)}^2,$$

which finishes the proof of (3.58) - (i) by choosing ε small enough and setting

$$A_1 := \left(\frac{\alpha^{p-\varepsilon}}{p} - \frac{\alpha^2+2\varepsilon}{p} - \varepsilon\right)\|Q\|_{L^p(\mathbb{R}^d)}^p.$$

As to (3.58) - (ii) and (iii), we note that

$$\|W_\rho\|_{L^p(\mathbb{T}^d)}^p \leq \alpha^p\rho^{-\frac{pd}{2}+d}\|Q\|_{L^p(\mathbb{R}^d)}^p = A_2\rho^{-\theta},$$
$$\|W_\rho\|_{L^2(\mathbb{T}^d)} \leq \alpha\|Q_\rho\|_{L^2(\mathbb{R}^d)} = \alpha\|Q\|_{L^2(\mathbb{R}^d)} = K - \eta,$$

with $A_2 := \alpha^p\|Q\|_{L^p(\mathbb{R}^d)}^p$ and η being the one in (3.73). Thus, we finish the proof.

Case 2: *super-critical nonlinearity.* In what follows, we assume $p > \frac{4s}{d} + 2$. It only needs to prove (3.58) - (i) and (iii), since (ii) follows the same way as that of Case 1. Given $K > 0$, we choose $\alpha \ll 1$ in (3.72) so that

$$\|W_\rho\|_{L^2(\mathbb{T}^d)} \leq \|W_\rho\|_{L^2(\mathbb{R}^d)} = \alpha\|Q\|_{L^2(\mathbb{R}^d)} < K - \eta,$$

which gives (3.58) - (iii). Similar computation as in the previous case, we have

$$\begin{aligned} H_{\mathbb{T}^d}(W_\rho) &= \frac{\alpha^2}{2}\int_{\mathbb{T}^d} |\mathrm{D}^s(\phi_\delta Q_\rho)|^2 \mathrm{d}x - \frac{\alpha^p}{p}\int_{\mathbb{T}^d} |\phi_\delta Q_\rho|^p \mathrm{d}x \\ &\le \frac{\alpha^2+2\varepsilon}{2}\rho^{-2s}\|\mathrm{D}^s Q\|^2_{L^2(\mathbb{R}^d)} - \frac{\alpha^{p-\varepsilon}}{p}\rho^{-\frac{dp}{2}+d}\|Q\|^p_{L^p(\mathbb{R}^d)} + C\|Q_\rho\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

for sufficiently small ρ and δ. Also, note that $p > \frac{4s}{d} + 2$ implies $-2s > -\frac{dp}{2} + d$. Thus, recalling (1.50), we may continue with

$$\begin{aligned} &\le \left(\frac{\alpha^2+2\varepsilon}{p}\rho^{-2s} - \frac{\alpha^{p-\varepsilon}}{2p}\rho^{-\frac{dp}{2}+d}\right)\|Q\|^p_{L^p(\mathbb{R}^d)} + C\|Q\|^2_{L^2(\mathbb{R}^d)} \\ &\le -\widetilde{A}_1\rho^{-\frac{dp}{2}+d}, \end{aligned}$$

for sufficiently small $\rho > 0$ and some constant $\widetilde{A}_1 > 0$. Thus, we obtain (3.58) - (i). We finish the proof of Lemma 19. $\square$

Next, we present the proof of Lemma 20.

Proof of Lemma 20. Let

$$X_n(t) = (Y_\alpha)_n(t) - (Z_M)_n(t), \quad 0 < |n| \le M. \tag{3.74}$$

Then, from (3.36) and (3.59), we see that $X_n(t)$ solves

$$\begin{cases} \mathrm{d}X_n(t) = -|n|^{-s}M^{\frac{d}{2}}X_n(t)\mathrm{d}t + |n|^{-s}\mathrm{d}B_n(t) \\ X_n(0) = 0 \end{cases}$$

for $0 < |n| \le M$. Solving the above stochastic differential equation yields

$$X_n(t) = |n|^{-s}\int_0^t \mathrm{e}^{-|n|^{-s}M^{\frac{d}{2}}(t-t')}\mathrm{d}B_n(t'). \tag{3.75}$$

Then, from (3.74) and (3.75), we have

$$(Z_M)_n(t) = (Y_\alpha)_n(t) - |n|^{-s} \int_0^t \mathrm{e}^{-|n|^{-s} M^{\frac{d}{2}}(t-t')} \mathrm{d}B_n(t'), \tag{3.76}$$

for $0 < |n| \leq M$. In what follows, we show that Z_M approximates to Y_α as $M \sim \rho^{-1}$ tends to infinity. From (3.76), the independence of $\{B_n\}_{n\in\mathbb{Z}^d}$, and Ito's isometry, we have

$$\begin{aligned} \mathbb{E}\big[|Z_M(1) - Y_\alpha(1)|^2\big] &= \sum_{0<|n|\leq M} |n|^{-2s} \int_0^t \mathrm{e}^{-2|n|^{-s} M^{\frac{d}{2}}(t-t')} \mathrm{d}t' + \sum_{|n|>M} |n|^{-2s} \\ &\lesssim \sum_{0<|n|\leq M} |n|^{-s} M^{-\frac{d}{2}} + M^{-2s+d} \\ &\lesssim \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+}), \end{aligned} \tag{3.77}$$

which is sufficient for (3.60) with $p = 2$.

When $p = 1$, (3.60) follows from (3.77) together with Hölder's inequality

$$\mathbb{E}\big[|Z_M(1) - Y_\alpha(1)|\big] \lesssim \Big(\mathbb{E}\big[|Z_M(1) - Y_\alpha(1)|^2\big]\Big)^{\frac{1}{2}}.$$

Then the case for $1 < p < 2$ follows from interpolation. When $p > 2$, we note that $Z_M(1) - Y_\alpha(1) \in \mathcal{H}_1$, homogeneous Wiener chaoses of order 1. Then, by using Wiener chaos estimate [102, Lemma I.22], we obtain

$$\mathbb{E}\big[|Z_M(1) - Y_\alpha(1)|^p\big] \lesssim \mathbb{E}\big[|Z_M(1) - Y_\alpha(1)|^p\big] \lesssim \Big(\mathbb{E}\big[|Z_M(1) - Y_\alpha(1)|^2\big]\Big)^{\frac{p}{2}},$$

which together with (3.77) implies (3.60) for $p > 2$.

Finally, we turn to (3.61). From (3.59) and (3.74), we have

$$\begin{aligned}\mathbb{E}\Big[\Big\|\mathrm{D}^s\frac{\mathrm{d}}{\mathrm{d}t}Z_M(t)\Big\|^2_{L^2(\mathbb{T}^d)}\Big] &= M^d\sum_{0<|n|\le M}\mathbb{E}\big[|X_n(t)|^2\big]\\ &= M^d\sum_{0<|n|\le M}|n|^{-2s}\int_0^t \mathrm{e}^{-2|n|^{-s}M^{\frac{d}{2}}(t-t')}\mathrm{d}t'\\ &\lesssim M^d\sum_{0<|n|\le M}|n|^{-s}M^{-\frac{d}{2}}\\ &\lesssim \max(M^{\frac{3d}{2}-s}, M^{\frac{d}{2}+}).\end{aligned}$$

We finish the proof of (3.61) and thus we conclude this lemma. □

CHAPTER 4

FURTHER RESEARCH

In this chapter, we will discuss further results related to this thesis and explore some questions that arise from the results of this thesis.

4.1 Half-wave equation above the probabilistic scaling

In [72], the author of the thesis, together with Yuzhao Wang and Haitian Yue, studies the nonlinear half-wave equation of the one-dimensional torus:

$$\begin{cases} (\mathrm{i}\partial_t - |\partial_x|)u = |u|^2 u, \\ u(0) = u_0^\omega, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{4.1}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $|\partial_x| = \sqrt{-\partial_x^2}$ is defined by the Fourier multiplier $\mathcal{F}_x(|\partial_x|u)(k) = |k|(\mathcal{F}_x u)(k)$. Here, u_0^ω is random initial data on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\omega \in \Omega$. In the context of almost sure local well-posedness, the random initial data is given by

$$u_0^\omega = \sum_{k\in\mathbb{Z}} \frac{g_k(\omega)}{\langle k\rangle^\alpha} \mathrm{e}^{\mathrm{i}kx}, \tag{4.2}$$

where $\{g_k\}_{k\in\mathbb{Z}}$ is a sequence of independent standard centered normalized (complex) Gaussian random variables, and α is chosen as

$$\alpha = s + \frac{1}{2}, \qquad s > s_{pr} := 0. \tag{4.3}$$

Here, the value s_{pr} is the critical regularity for the *probabilistic scaling*. The random initial data u_0^ω defined by (4.2) is in $H^{s-}(\mathbb{T}) := \cap_{s'<s} H^{s'}(\mathbb{T})$ almost surely. In [72], we prove the almost sure local well-posedness of (4.1) with random initial data (4.2), in the sense that some appropriate canonical smooth approximations converge to a unique limit. The result can be interpreted as almost sure local well-posedness in $H^{s-}(\mathbb{T})$ with respect to the canonical Gaussian measure—the law of u_0^ω defined by (4.2)—for any $s > s_{pr} = 0$, i.e., *in the full probabilistically subcritical regime.*

In proving the almost sure local well-posedness of (4.1) above its probabilistic scaling, we exploit the theory of random tensors developed in [35]. Due to the non-negativity of probabilistic scaling, the pairings can be viewed as perturbations, streamlining the framework of the theory of random tensors to some extent.

4.2 Further applications of the theory of random tensors

Further applications of the theory of random tensors might facilitate further developments within the dispersive community. More precisely, it is likely that this theory can be used to show almost-sure local well-posedness (in the sense that some appropriate canonical smooth approximations converge to a unique limit) for some equations with initial data that are above the probabilistic scaling criticality (or high-high-to-low criticality). Examples of equations where the theory of random tensors may yield almost-surely local well-posedness include (among others) the following models:

(i) The nonlinear Schródinger equation:

$$\begin{cases} (\mathrm{i}\partial_t - \Delta)u = \bar{u}^2, \\ u(0) = u_0^\omega, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^d,$$

where u_0^ω is as defined in (4.2) and α is greater than some criticality.

(ii) The Schrödinger equation with Hartree-type nonlinearity:

$$\begin{cases} (i\partial_t + \Delta)u = (|u|^2 * V)u, \\ u(0) = u_0^\omega, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3,$$

where u_0^ω is as defined in (4.2) with $\alpha = 1$ and $|\widehat{V}(k)| \lesssim \langle k \rangle^{-\beta}$ with β smaller than the β_0 required in (hence extending the result in) [36].

(iii) The nonlinear Dirac equation

$$(i\gamma^\mu \partial_\mu - 1)u = \mathcal{N}(u)$$

with some appropriate random initial data. Here $\{\gamma^\mu\}_\mu$ is the basis for some Clifford algebra and $\mathcal{N}(u)$ denotes the nonlinearity.

The study of the applicability of the theory of random tensors to show almost-surely local well-posedness for the above models will be carried out in the future.

4.3 Other Research Proposals

In [35], the theory of random tensors is developed to study the local-in-time solution (in the sense that some appropriate canonical smooth approximations converge to a unique limit) to the Cauchy problem of the Schrödinger equation with random initial data in the full probabilistically subcritical regime. Due to the roughness of the initial data, the nonlinearity requires renormalisation. However, in [35], Wick renormalisation suffices, rendering some of the more complicated algebraic structures unnecessary. As the theory of random tensors is a dispersive counterpart of parabolic theory, where in particular the theory of regularity structures has been proven to be fruitful in the study of algebraic structures (see [25]), we might also hope that some interesting algebraic structures can be added to the theory of random tensors for some suitable models.

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