NON-LINEAR QUOTIENT MAPPINGS OF THE PLANE AND INSCRIBED EQUILATERAL POLYGONS IN CENTRALLY SYMMETRIC CONVEX BODIES

by

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ABSTRACT

In the present work, we are concerned with two underlying topics, which at first seem inherently disjoint, but are connected in a non-obvious way.

The first topic we study is the behaviour of non-linear quotient mappings whose domain is the plane; we pay particular attention to Lipschitz quotient mappings of the plane. This forms the underlying material in Chapters 2 and 3.

In Chapter 2 we correct a mistake from Johnson et al. [17]. Namely, we give a valid proof of the statement that for a fixed complex polynomial P in one complex variable there exists a homeomorphism of the plane h such that $P \circ h$ is a Lipschitz quotient mapping of the plane. Further, we introduce the notion of strong co-Lipschitzness, and prove the logical equivalence between the long standing conjecture that all Lipschitz quotient mappings from $\mathbb{R}^n$ to itself are discrete and the necessity for every Lipschitz quotient mapping from $\mathbb{R}^n$ to itself, $n \geq 3$, to be strongly co-Lipschitz.

Chapter 3 is dedicated to improving the lower estimates for the ratio of constants L/c for any 2-fold planar Lipschitz quotient mappings in polygonal norms.

Chapters 4 and 5 concern the existence of inscribed equilateral polygons in centrally symmetric convex bodies. We investigate the extremal inscribed equilateral polygons and determine, for a large class of norms, when such polygons are essentially equivalent.

Finally, we consider the level sets of uniformly continuous, co-Lipschitz mappings defined on the plane; this forms Chapter 6. We show for any uniformly continuous, co-Lipschitz mapping $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$, where $\|\cdot\|$ is any norm of the plane, that the maximal number of components $n(f)$ of the level sets of f is intimately related to weak Lipschitz and co-Lipschitz constants of f as well as the maximal possible edge length, in terms of $\|\cdot\|$, over all inscribed equilateral polygons. Further, we obtain a sharp bound for a certain class of norms $\|\cdot\|$ which possess a particular separation property.

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CONTENTS

CHAPTER 1

INTRODUCTION

To begin this chapter we introduce the main objects of study, namely Lipschitz quotient mappings. We provide some motivation as to why the investigation into such mappings is important, and recall many results for such mappings in, typically, finite dimensional spaces.

The remainder of the chapter is devoted to the development of many of the tools that will be used throughout this thesis, paying particular attention to fundamental properties of planar norms.

1.1 Motivation of the problems

The widely known Lipschitz condition was first introduced by Rudolf Lipschitz when investigating the convergence of the Fourier series of periodic functions in [19]. Since this introduction many variants of such mappings have been introduced and studied, see, for instance, [6], [22] and [27]. The body of this work mainly focuses on *Lipschitz quotient mappings* which are Lipschitz mappings that possess a stronger dual property, namely the *co-Lipschitz* condition. Co-Lipschitz, and moreover co-uniform, mappings originate from the study of general uniform spaces, cf. [12], [16], [38], but were first systematically studied in [1] and [17]. Lipschitz quotient mappings were introduced as non-linear analogues for the standard continuous, linear, surjective operators between Banach spaces, otherwise known as linear quotient mappings; the Open Mapping Theorem, see [32, Theorem 2.11], yields linear quotient mappings are inherently open.

Definition 1.1.1. Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces. A map $f : X \to Y$ is called a *Lipschitz mapping* if there exists $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \quad \text{for all } x, y \in X.$$

If such a constant $L \geq 0$ exists we say f is L-Lipschitz. In other words, if f is not a constant mapping, we require a positive constant L such that

$$f(B_r^X(x)) \subseteq B_{Lr}^Y(f(x)) \quad \text{for all } x \in X \text{ and } r > 0.$$

Here $B_s^Z(z)$ denotes the open ball in (Z, d_Z) with radius $s > 0$ centred at $z \in Z$, where $Z = X, Y$. If f is Lipschitz, then the infimum of all such constants L is called the *Lipschitz constant* of f, and is denoted $\mathrm{Lip}(f)$.

Similarly, we say f is a *co-Lipschitz mapping* if it is continuous and there exists a positive constant $c > 0$ such that

$$B_{cr}^Y(f(x)) \subseteq f\left(B_r^X(x)\right) \quad \text{for all } x \in X \text{ and } r > 0.$$

If such a constant $c > 0$ exists we say f is c-co-Lipschitz. If f is co-Lipschitz, then the supremum of all such constants c is called the *co-Lipschitz constant* of f, and is denoted $\text{co-Lip}(f)$.

Finally, if $f : X \to Y$ is both a Lipschitz and co-Lipschitz mapping, we say f is a *Lipschitz quotient mapping*.

Remark 1.1.2. Observe we have included the additional condition that co-Lipschitz mappings need be continuous. This is to ensure the existence of the co-Lipschitz constant since, assuming the axiom of choice, one can show that there exist functions, for example from $\mathbb{R}$ to $\mathbb{R}$, which are surjective to $\mathbb{R}$ on every non-empty open subset, cf. [3].

Example 1.1.3. Let $(X, \|\cdot\|_X)$ be a normed space over $\mathbb{F}$, $a \in \mathbb{F} \setminus \{0\}$ and $b \in X$, where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Consider $f : X \to X$ given by $f(x) = ax + b$. Then f is a Lipschitz quotient mapping. Moreover, $\mathrm{Lip}(f) = \text{co-Lip}(f) = |a|$ since for every $x \in X$ and $r > 0$,

$$f(B_r^X(x)) = aB_r^X(x) + b = B_{|a|r}^X(ax) + b = B_{|a|r}^X(ax + b) = B_{|a|r}^X(f(x)).$$

As stated in [17], and proved in Lemma 1.2.22 of the present thesis, non-trivial examples of

Lipschitz quotient mappings are $f_n : \mathbb{C} \to \mathbb{C}$ defined by $f_n(re^{i\theta}) = re^{in\theta}$, $n \in \mathbb{N}$. Mappings f_n may also be considered as Lipschitz quotient mappings $\mathbb{R}^2 \to \mathbb{R}^2$, via the obvious identification of $\mathbb{R}^2$ with $\mathbb{C}$.

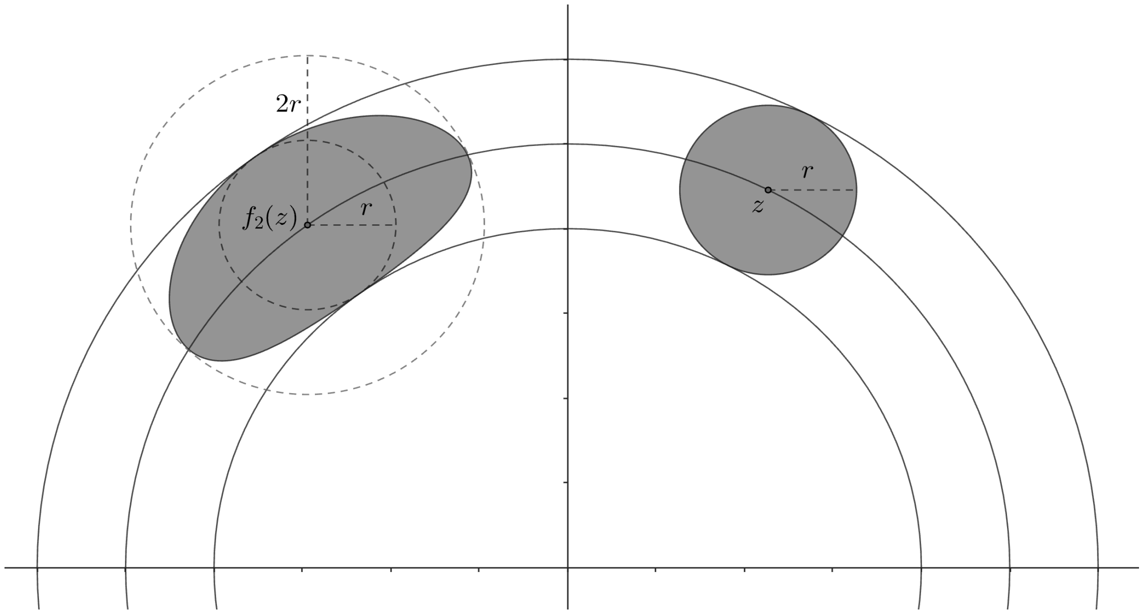

Figure 1.1: The image of a ball $B_r^{|\cdot|}(z)$ under the mapping f_2. This mapping is 2-Lipschitz and 1-co-Lipschitz.

The first systematic study of Lipschitz quotient mappings of the plane is attributed to [17], where the results support the aforementioned notion that Lipschitz quotient mappings are nonlinear analogues for linear quotient mappings between Banach spaces. They also provide the far-reaching result [17, Theorem 2.8 (i)], see also Theorem 1.1.4 of the present thesis, which states that Lipschitz quotient mappings from the plane to itself can be viewed as a reparametrisation of a complex polynomial. As a consequence, the cardinality of each point preimage is bound above by the degree of such polynomial. Note, the boundedness of fibres under planar Lipschitz quotient mappings was first shown in [1], and along with a powerful result of Stoilow [35] proves such a composition exists.

When considering linear quotient mappings $X \to Y$ the point preimages preserve dimension in the following sense: for each $x \in X$, the point preimage of x is an affine subspace of X with dimension $d := \dim(X) - \dim(Y)$. In [21] it is shown for Lipschitz quotient mappings between finite dimensional spaces, if the constants c and L are sufficiently close then point preimages cannot be $(d + 1)$-dimensional.

Such regularity, however, is not guaranteed for Lipschitz quotient mappings between Euclidean spaces of different dimensions with sufficiently small ratio of constants c/L. For example, in [5] an example of a Lipschitz quotient mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ is constructed such that $f^{-1}(0)$ contains a plane. However, in the case of Lipschitz quotient mappings from $\mathbb{R}^2$ to $\mathbb{R}$ [30] provides comprehensive results regarding the structure of the level sets of such mappings. Further, [23] provides estimates for the maximal ratio of constants c/L for such Lipschitz quotient mappings, in the case of the Euclidean norm on the domain; for results in other norms see Chapter 6. Furthermore, little is known concerning the structure of Lipschitz quotient mappings between Euclidean spaces of the same dimension greater than 2. Interestingly, for Lipschitz quotient mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, for $n \geq 3$, it is not even known if $f^{-1}(y)$ is discrete for each $y \in \mathbb{R}^n$, see Conjecture 1.2.39.

It can be shown that Lipschitz quotient mappings between spaces of the same dimension behave almost identically to mappings known as *quasiregular mappings*, as highlighted in [1, pg. 1124]. Quasiregular mappings were introduced as a means to generalise the geometric aspects of the theory of analytic functions in the plane to higher dimensional Euclidean spaces; cf. [31, Chapter 1] for definitions and further results. By a deep theorem of Reshetnyak, which has been presented in [31], quasiregular mappings are *branched covers*, i.e. are both open and discrete. Unfortunately the techniques used to prove the discreteness of the level sets of quasiregular mappings cannot be extended to the class of Lipschitz quotient mappings between spaces of the same finite dimension.

One may conclude further investigation into planar Lipschitz quotient mappings is necessary to highlight the key properties of the underlying geometry which admits the existence of such comprehensive results. This investigation then may provide certain avenues one may consider to try and answer this long-standing conjecture.

The first such avenue considered in the present work concerns questions related to the structure of planar Lipschitz quotient mappings. Namely, we investigate converses of the following theorem, [17, Theorem 2.8 (i)].

Theorem 1.1.4. Let $f : \mathbb{C} \to \mathbb{C}$ be a Lipschitz quotient mapping. Then $f = P \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and P is a complex polynomial of one complex variable.

Note that in Theorem 1.1.4 we did not specify the norms associated to the domain and co-domain. This is justified since passing to an equivalent norm preserves whether a mapping is a Lipschitz quotient; hence in the finite dimensional setting there is no need to specify the norm considered. We also highlight here that the original statement of Theorem 1.1.4 in [17] is given for Lipschitz quotient mappings from $\mathbb{R}^2$ to itself. The restatement in Theorem 1.1.4 follows due to the natural identification of $\mathbb{R}^2$ with $\mathbb{C}$, which is required in the definition of the polynomial P in any case.

This far-reaching result shows that every planar Lipschitz quotient mapping can be viewed as a reparametrisation of a complex polynomial. In Chapter 2 we investigate whether every homeomorphism $h : \mathbb{C} \to \mathbb{C}$ and every complex polynomial $P : \mathbb{C} \to \mathbb{C}$ admits a Lipschitz quotient mapping. In doing so we introduce a stronger notion of co-Lipschitzness and prove its equivalence to the standard pointwise notion under certain conditions.

The next direction in which we investigate the underlying geometry of planar Lipschitz quotient mappings is the extension of the results of [25]. In [25] the ratio of constants c/L for planar Lipschitz quotient mappings were investigated under norms whose unit sphere is isomorphic to a regular equilateral polygon, centred at the origin. We improve the upper bound for the ratio of constants c/L when the point preimages of the planar Lipschitz quotient mapping are at most two.

We then move onto our second titular topic. The work on inscribed equilateral polygons starts by extending the well understood notions of regular equilateral polygons inscribed in a circle. That is, fixing a point on the boundary of this circle, one can find $n-1$ other points, for some $n \geq 3$, such that the Euclidean distance between adjacent pairs of these n points are equal, see Figure 4.1 in Chapter 4. The aim is to extend such a notion to an arbitrary centrally symmetric convex body of the plane, where the distance is measured in the norm induced by this convex body. This has already been considered in [8]. However the statement, and hence proof, of [8, Lemma 2.4] is incorrect. We correct this statement and provide an independent proof of the result in Chapter 4, see Theorem 4.3.5 and Lemma 4.3.6 of the present thesis.

We proceed by investigating the extremal equilateral polygons that can be inscribed in a centrally symmetric convex body; by extremal we mean in terms of the *edge length*, i.e. the distance between the adjacent vertices. We show in Chapter 4 that these extremal equilateral polygons always exist and prove that either the set of possible edge lengths is a single value or a closed interval of positive length, hence is uncountable. The former happens, for instance, in the Euclidean norm. We investigate whether this property is solely observed via the Euclidean norm. In Chapter 5 we provide a countable collection of norms where if the number of vertices is sufficiently large, then every equilateral polygon has the same edge length.

Even though the topics of planar Lipschitz quotient mappings and inscribed equilateral polygons seem disjoint at first, there is an underlying relation between the two. When considering Lipschitz quotient mappings from the plane to the line, we determine an upper bound for the ratio of constants c/L. For a particular class of planar norms, we can use the existence of an equilateral polygon of largest side length to produce a Lipschitz quotient mapping which attains this maximal ratio of constants. This is investigated at the end of Chapter 6.

1.2 Basic properties of Lipschitz quotient mappings

Our main object of study is Lipschitz quotient mappings, as defined in Definition 1.1.1. In this section we introduce many regularity properties of such mappings, paying particular attention to Lipschitz quotient mappings from the plane to itself, known as *planar Lipschitz quotient mappings*. We will first, however, recall standard results regarding Lipschitz quotient mappings; for proofs of the following results see, for example, [37, Section 1.2].

Lemma 1.2.1. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces and $f : X \to Y$ be a Lipschitz quotient mapping. Suppose $|||\cdot|||_X$ and $|||\cdot|||_Y$ are norms on X and Y, respectively, such that $\|\cdot\|_Z$ is equivalent to $|||\cdot|||_Z$ for $Z = X, Y$. Then the mapping f, viewed as a map from $(X, |||\cdot|||_X)$ to $(Y, |||\cdot|||_Y)$, is a Lipschitz quotient mapping.

Proposition 1.2.2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, where X is finite dimensional, and $c > 0$. Then $f : X \to Y$ is c-co-Lipschitz if and only if f satisfies the following condition:

$$\overline{B}^Y_{cr}(f(x)) \subseteq f\left(\overline{B}^X_r(x)\right) \qquad \text{for each } x \in X \text{ and each } r > 0.$$

The following simple lemma concerns transformations of Lipschitz quotient mappings.

Lemma 1.2.3. Suppose $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are normed spaces, $f : X \to Y$, $x_0 \in X$ and $y_0 \in Y$. Let $f_1, f_2 : X \to Y$ be given by $f_1(x) = f(x) + y_0$ and $f_2(x) = f(x + x_0)$. If:

(i) f is L-Lipschitz for some $L > 0$, then both f_1, f_2 are L-Lipschitz, also;

(ii) f is c-co-Lipschitz for some $c > 0$, then both f_1, f_2 are c-co-Lipschitz, also.

In particular, if f is a Lipschitz quotient mapping, then both f_1 and f_2 are Lipschitz quotient mappings, where $\mathrm{Lip}(f) = \mathrm{Lip}(f_1) = \mathrm{Lip}(f_2)$ and $\text{co-Lip}(f) = \text{co-Lip}(f_1) = \text{co-Lip}(f_2)$.

Proof. (i) This follows trivially by the definition of the mappings f_1 and f_2 and by the definition of a Lipschitz mapping.

(ii) Fix $x \in X$ and $r > 0$. To see f_1 is c-co-Lipschitz, note that as f is c-co-Lipschitz,

$$B_{cr}^{Y}(f_1(x)) = B_{cr}^{Y}(f(x)) + y_0 \subseteq f\left(B_r^X(x)\right) + y_0 = f_1\left(B_r^X(x)\right).$$

Similarly, to see f_2 is c-co-Lipschitz, observe that

$$B_{cr}^{Y}(f_2(x)) = B_{cr}^{Y}(f(x+x_0)) \subseteq f\left(B_r^X(x+x_0)\right) = f\left(B_r^X(x)+x_0\right) = f_2\left(B_r^X(x)\right).$$

□

As observed in [21], the maximal cardinality of point preimages under planar Lipschitz quotient mappings proves to be intimately related to how much the co-Lipschitz and Lipschitz constants may differ. We recall the definition for *N-fold mappings* from [25]. First, we introduce some standard notation which will be used throughout.

Notation 1.2.4. Let X be a topological space and $S \subseteq X$. We let ∂S represent the boundary of S and $\mathrm{Int}(S)$ denote the interior of S. Moreover, if S is finite we let $\mathrm{card}(S)$ represent the cardinality of S.

Definition 1.2.5. ([25, Definition 1.2]) A mapping $f : X \to Y$ is called *N-fold*, $N \in \mathbb{N}$, if $\max_{y\in Y} \mathrm{card}\left(f^{-1}(y)\right) = N$, i.e. $\mathrm{card}\left(f^{-1}(y)\right) \leq N$ for each $y \in Y$ and there exists $y_0 \in Y$ such that $\mathrm{card}\left(f^{-1}(y_0)\right) = N$.

Lemma 1.2.6. Suppose X and Y are vector spaces, $x_0 \in X$, $y_0 \in Y$ and $f : X \to Y$ is an N-fold mapping for some $N \in \mathbb{N}$. Let $f_1, f_2 : X \to Y$ be given by $f_1(x) = f(x) + y_0$ and $f_2(x) = f(x + x_0)$. Then f_1 and f_2 are N-fold mappings.

Proof. As f is N-fold, $\mathrm{card}\left(f_1^{-1}(y)\right) = \mathrm{card}\left(f^{-1}(y-y_0)\right) \leq N$ for each $y \in Y$. Moreover, if $y \in Y$ is such that $\mathrm{card}\left(f^{-1}(y)\right) = N$, then $y + y_0 \in Y$ is such that $\mathrm{card}\left(f_1^{-1}(y+y_0)\right) = \mathrm{card}\left(f^{-1}(y)\right) = N$. Hence, f_1 is N-fold. One may argue similarly for f_2. □

The standard planar Lipschitz quotient mappings $f_n(z) = |z|e^{in\arg(z)}$, see Lemma 1.2.22, are centralised in the sense that they map the origin to itself and their associated polynomials, which are obtained via the decomposition in Theorem 1.1.4, are monomials: one may take $h(z) = |z|^{1/n}e^{i\arg(z)}$ and $P(z) = z^n$, so that $f_n(z) = P(h(z))$. This, in some sense, allows such Lipschitz quotient mappings to be as 'efficient' as possible in terms of having the optimal ratio of constants, cf. [21, Theorem 2]. Therefore, one would hope that investigating such centralised mappings would provide some enlightenment about the maximal ratio of constants of general planar Lipschitz quotient mappings. Unfortunately, when considering N-fold Lipschitz quotient mappings where $N \geq 3$, this class of centralised mappings does not reflect the global behaviour of planar Lipschitz quotient mappings. However, when $N = 2$, the below result shows that every 2-fold planar Lipschitz quotient mapping can be associated to a centralised 2-fold planar Lipschitz quotient mapping.

Theorem 1.2.7. Let $f : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, |||\cdot|||)$ be a 2-fold Lipschitz quotient mapping. There exists a homeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $H(0) = 0$ and $F : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, |||\cdot|||)$, given by $F = H^2$, is a Lipschitz quotient mapping which satisfies the following properties:

- there exist $z_0, z_1 \in \mathbb{C}$ such that $F(z) = f(z + z_0) + z_1$ for each $z \in \mathbb{C}$;
- co-Lip(F) = co-Lip(f) and Lip(F) = Lip(f).

Proof. By Theorem 1.1.4 there exists a complex polynomial $P(z) = az^2 + bz + d$, $a \neq 0$, and a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $f = P \circ h$. Define $h_1, P_1 : \mathbb{C} \to \mathbb{C}$ by $h_1(z) = \sqrt{a}h(z)$ and $P_1(z) = z^2 + (b/\sqrt{a})z + d$; here $\sqrt{a}$ represents the principal square root of $a \in \mathbb{C}$. Observe h_1 is a homeomorphism and for each $z \in \mathbb{C}$,

$$(P_1 \circ h_1)(z) = \left(\sqrt{a}h(z)\right)^2 + \frac{b}{\sqrt{a}}\left(\sqrt{a}h(z)\right) + d = (P \circ h)(z) = f(z).$$

Thus $f = P_1 \circ h_1$. Next define $h_2, P_2 : \mathbb{C} \to \mathbb{C}$ by $h_2(z) = h_1(z) + b_1/2$ and $P_2(z) = z^2 + d_1$, where $b_1 = b/\sqrt{a}$ and $d_1 = d - (b_1)^2/4$. Then h_2 is a homeomorphism and for each $z \in \mathbb{C}$,

$$(P_2 \circ h_2)(z) = \left(h_1(z) + \frac{b_1}{2}\right)^2 + d_1 = (P_1 \circ h_1)(z) = f(z).$$

Thus $f = P_2 \circ h_2$ and $h_2 : \mathbb{C} \to \mathbb{C}$ is a homeomorphism.

Define $z_0 := h_2^{-1}(0)$ and $H : \mathbb{C} \to \mathbb{C}$ to be the homeomorphism $H(z) = h_2(z + z_0)$. Then $H(0) = h_2(z_0) = 0$. Further, letting $F = H^2$, note for each $z \in \mathbb{C}$,

$$F(z) = (H(z))^2 = (h_2(z + z_0))^2 = f\,(z + z_0) - d_1.$$

Thus Lemma 1.2.3 yields F is a Lipschitz quotient mapping such that $\mathrm{Lip}(F) = \mathrm{Lip}(f)$ and co-$\mathrm{Lip}(F) =$ co-$\mathrm{Lip}(f)$. □

Such a result motivates the following definition for the aforementioned class of centralised mappings.

Definition 1.2.8. We say a mapping $f : \mathbb{C} \to \mathbb{C}$ is an *N-centred mapping*, $N \in \mathbb{N}$, if there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) = 0$ and $f = h^N$.

Observe that N-centred mappings are naturally N-fold. Moreover, N-centred Lipschitz quotient mappings possess many qualities which one cannot assume about general Lipschitz quotient mappings. Let us first recall the definition of the index (or winding number) of a curve about a point as in [34, Section 7].

Definition 1.2.9. Suppose $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ is a path. We define the *winding number* $w(\gamma, 0)$ of the path γ about the origin to be

$$w\,(\gamma, 0) = \frac{\theta(b) - \theta(a)}{2\pi},$$

where $\theta : [a, b] \to \mathbb{R}$ is a continuous choice of argument along γ.

If $z_0 \in \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C}$ is a path which does not contain z_0, then the winding number of γ about z_0 is $w(\gamma, z_0) = w(\Gamma, 0)$, where $\Gamma = \gamma - z_0$.

Finally, if γ is a closed path it can be shown that the winding number, or index, of γ about z_0 is given by:

$$\mathrm{Ind}_{z_0}\gamma := w(\gamma, z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z - z_0}.$$

With this in mind, we recall [25, Theorem 2.8].

Lemma 1.2.10. ([25, Theorem 2.8]) Let $N \in \mathbb{N}$ and $f : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, |||\cdot|||)$ be an N-fold Lipschitz quotient mapping which preserves orientation. There exist positive constants M and

R such that, for any $\rho > R$,

$$\text{Ind}_0 f\left(\partial B_\rho^{\|\cdot\|}(0)\right) = N \quad \text{and} \quad f\left(\partial B_\rho^{\|\cdot\|}(0)\right) \subseteq \mathbb{C} \setminus B_{c(\rho-M)}^{|||\cdot|||}(0),$$

where $M = \max\{\|z\| : f(z) = 0\}$.

The following improves the above lemma when we consider only centred Lipschitz quotient mappings.

Corollary 1.2.11. Let $N \in \mathbb{N}$, $c > 0$ and $f : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, |||\cdot|||)$ be a continuous, N-centred, c-co-Lipschitz mapping which preserves orientation. Then, for any $\rho > 0$,

$$\text{Ind}_0 f\left(\partial B_\rho^{\|\cdot\|}(0)\right) = N \quad \text{and} \quad f\left(\partial B_\rho^{\|\cdot\|}(0)\right) \subseteq \mathbb{C} \setminus B_{c\rho}^{|||\cdot|||}(0).$$

Proof. For each $\rho > 0$, let $\Gamma_\rho := h\left(\partial B_\rho^{\|\cdot\|}(0)\right)$. Then note $\text{Ind}_0\Gamma_\rho = 1$ since $h(0) = 0$ and h is a homeomorphism. So,

$$\text{Ind}_0 f\left(\partial B_\rho^{\|\cdot\|}(0)\right) = \text{Ind}_0\Gamma_\rho \cdot \text{Ind}_0 z^N = N,$$

since $\text{Ind}_0 z^N = N$.

Now observe $f(z) = 0$ if and only if $z = 0$ since f is N-centred. Moreover, as f is c-co-Lipschitz, it follows by Proposition 1.2.2 that if $z \neq 0$, then

$$0 \in \overline{B}_{|||f(z)|||}^{|||\cdot|||}(f(z)) \subseteq f\left(\overline{B}_{|||f(z)|||/c}^{\|\cdot\|}(z)\right).$$

Therefore there exists $x \in \overline{B}_{|||f(z)|||/c}^{\|\cdot\|}(z)$ such that $f(x) = 0$. Hence $x = 0$ and so $\|z\| \leq |||f(z)|||/c$. □

We turn our attention now to various local notions of Lipschitzness and co-Lipschitzness.

Definition 1.2.12. Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ and $x_0 \in X$. We say that f is *locally Lipschitz at* x_0 if there exist constants $r, L > 0$ such that

$$d_Y(f(y), f(z)) \leq L d_X(y, z) \quad \text{for all } y, z \in B_r^X(x_0). \tag{1.2.1}$$

If such constants exist, we say that f is locally L-Lipschitz at x_0.

Similarly, we say f is *pointwise Lipschitz at* x_0 if the Lipschitz condition (1.2.1) is satisfied for all $y \in B_r(x_0)$ and $z = x_0$. Equivalently, we say f is *pointwise Lipschitz at* x_0 if there exist positive constants L, ρ such that

$$f\left(B_r^X(x_0)\right) \subseteq B_{Lr}^Y\left(f(x_0)\right) \quad \text{for all } r \in (0, \rho).$$

In such a case, we say f is pointwise L-Lipschitz at x_0. We define the *radius of pointwise L-Lipschitz (for f) at* x_0, denoted $r_0^f(x_0)$, to be

$$r_0^f(x_0) := \sup\left\{r > 0 : d_Y\left(f(y), f(x_0)\right) \leq L d_X(y, x_0) \quad \text{for all } y \in B_r^X(x_0)\right\}.$$

Finally, we say f is *pointwise co-Lipschitz at* $x_0 \in X$ if there exist positive constants c, ρ such that

$$B_{cr}^Y\left(f(x_0)\right) \subseteq f\left(B_r^X(x_0)\right) \quad \text{for all } r \in (0, \rho).$$

In such a case, we say f is pointwise c-co-Lipschitz at x_0.

Naturally mappings which are locally L-Lipschitz at x_0 are pointwise L-Lipschitz at x_0, but the reverse implication need not hold in general. For example, for each $d \geq 1$, there exists $f : \mathbb{R}^d \to \mathbb{R}$ such that f is pointwise 1-Lipschitz at the origin, but is not locally Lipschitz; the cases $d \geq 2$ can be seen by extending the case $d = 2$ in [10, Example 2.7].

Such examples become possible since for these mappings the pointwise Lipschitz constant is not uniformly bounded. A standard result about Lipschitz mappings, see Lemma 1.2.17 below, states that a mapping between normed spaces which is pointwise L-Lipschitz on a convex domain, is in fact L-Lipschitz, also.

Unfortunately, without this convexity assumption this may fail to remain true. However, if we impose the condition that the radius of pointwise L-Lipschitzness is uniformly bounded then the notions of being everywhere pointwise and locally Lipschitz are equivalent, even for mappings between metric spaces. If the global lower bound on the radius of pointwise Lipschitzness is removed, however, then the mapping may fail to be locally Lipschitz; to see this consider the mapping in [10, Example 2.7] and the point $x_0 = 0$.

Lemma 1.2.13. Let (X, d_X) and (Y, d_Y) be metric spaces, $L > 0$ and $f : X \to Y$. Suppose f

is pointwise L-Lipschitz at each $x \in X$ and $r_0^f(x)$ be the radius of pointwise L-Lipschitz for f at x for each $x \in X$. If $\inf_{x\in X} r_0^f(x) > 0$, then f is locally L-Lipschitz at every $x \in X$.

Proof. Define $r_0 := \frac{1}{2}\inf_{t\in X} r_0^f(t) > 0$. Fix $x \in X$ and $y, z \in B_{r_0}^X(x)$. Then,

$$d_X(y,z) \leq d_X(x,y) + d_X(x,z) < 2r_0 = \inf_{t\in X} r_0^f(t) \leq r_0^f(y).$$

As $d_X(y,z) < r_0^f(y)$, we note $d_Y(f(y), f(z)) \leq L d_X(y,z)$. □

The rest of Section 1.2 appears in [15, Section 2], with the exception of the proof of Lemma 1.2.22.

Planar mappings which have a structure similar to that of a planar Lipschitz quotient mapping, i.e. a composition of a complex polynomial and a homeomorphism, are locally injective outside of a finite set. This was claimed in [37, Proposition 1.2.9]. However, the proof given in [37] contains a mistake, so we provide a shorter, independent proof below in Proposition 1.2.15.

Definition 1.2.14. Let X, Y be metric spaces. A mapping $f : X \to Y$ is *locally injective* at $x \in X$ if there exists $r = r(x) > 0$ such that $f\big|_{B_r^X(x)}$ is injective.

Proposition 1.2.15. Let $h : \mathbb{C} \to \mathbb{C}$ be a homeomorphism and $P : \mathbb{C} \to \mathbb{C}$ be a non-constant complex polynomial. There exists a finite set $E \subseteq \mathbb{C}$ such that $P \circ h$ is locally injective at each $x \in \mathbb{C} \setminus E$, where $E = h^{-1}(S(P'))$ and $S(P') = \{z \in \mathbb{C} : P'(z) = 0\}$.

Proof. Fix $x_0 \in \mathbb{C} \setminus E$, i.e. $h(x_0) \notin S(P')$. Since $P'(h(x_0)) \neq 0$, by [11, Theorem 7.5], there exists an open neighbourhood $N_{h(x_0)}$ of $h(x_0)$ such that $P\big|_{N_{h(x_0)}}$ is injective. Hence, $P \circ h$ is injective on the open neighbourhood $G = h^{-1}(N_{h(x_0)})$ of x_0.

Thus, $P \circ h$ is locally injective outside of E. Since P' is a non-zero polynomial, $\operatorname{card}(S(P')) \leq \deg(P) - 1$. As h is bijective, $\operatorname{card}(E) = \operatorname{card}(S(P'))$. □

Recall the following standard result regarding Lipschitz mappings.

Lemma 1.2.16. Let X, Y be metric spaces, $A \subseteq X$ dense in X and $L > 0$. If $f : X \to Y$ is a continuous mapping such that $f\big|_A$ is L-Lipschitz, then f is L-Lipschitz.

The following lemma ensures that a mapping which is pointwise Lipschitz everywhere, with a uniform constant, is necessarily Lipschitz, with the same constant. However, for this we need the linear structure induced by normed spaces.

Lemma 1.2.17. Let X, Y be normed spaces, $U \subseteq X$ open and convex and $L > 0$. If $f : X \to Y$ is pointwise L-Lipschitz at each $x \in U$, then $f\big|_U$ is L-Lipschitz.

Recall [5, Section 4] and [25, Lemma 2.3] which introduce a result analogous to Lemma 1.2.17 for co-Lipschitz mappings in the case $U = X = Y = \mathbb{C}$.

Lemma 1.2.18. Let $c > 0$. If $f : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, |\!|\!|\cdot|\!|\!|)$ is continuous and is pointwise c-co-Lipschitz at every $x \in \mathbb{C}$, then f is (globally) c-co-Lipschitz.

Remark 1.2.19. It is claimed in [5, Section 4] that if a mapping is pointwise c-co-Lipschitz at every point of a normed space, then the overall mapping is c-co-Lipschitz. However the proof is not fully correct. It is shown, in [37, Proposition 1.2.7], that if the domain is finite dimensional, then the corresponding statement to Lemma 1.2.18 is correct. The proof heavily relies on the compactness of the unit sphere in the domain. The question whether such a result holds for arbitrary normed spaces is still open.

Homeomorphisms between two metric spaces preserve pointwise co- and Lipschitzness of such mappings and their inverses in the following manner.

Lemma 1.2.20. Let X, Y be metric spaces, $h : X \to Y$ be a homeomorphism, $x_0 \in X$ and $c > 0$. Then h is pointwise c-co-Lipschitz at x_0 if and only if h^{-1} is pointwise $(1/c)$-Lipschitz at $h(x_0)$.

Proof. Since h is pointwise c-co-Lipschitz at x_0 there exists $r_0 > 0$ such that $B^Y_{cr}(h(x_0)) \subseteq h\left(B^X_r(x_0)\right)$ for each $r \in (0, r_0)$. Therefore,

$$h^{-1}\left(B^Y_{cr}\left(h(x_0)\right)\right) \subseteq h^{-1}\left(h\left(B^X_r(x_0)\right)\right) = B^X_r(x_0) = B^X_r\left(h^{-1}\left(h(x_0)\right)\right),$$

for each $r \in (0, r_0)$. Hence, h^{-1} is pointwise $(1/c)$-Lipschitz at $h(x_0)$ on distances smaller than cr_0. The reverse direction follows similarly. □

We now introduce a quick lemma regarding the composition of pointwise co-Lipschitz functions.

Lemma 1.2.21. Let X, Y and Z be metric spaces and $f : X \to Y$, $g : Y \to Z$ be functions. Suppose f is pointwise a-co-Lipschitz at $x \in X$ and g is pointwise b-co-Lipschitz at $f(x) \in Y$ for some constants $a, b > 0$. Then $g \circ f$ is pointwise (ab)-co-Lipschitz at x.

Proof. As f is pointwise a-co-Lipschitz at $x \in X$, there exists $\rho_f > 0$ such that $f\left(B_r^X(x)\right) \supseteq B_{ar}^Y(f(x))$ for each $r \in (0, \rho_f)$. Similarly, there exists $\rho_g > 0$ such that $g\left(B_r^Y(f(x))\right) \supseteq B_{br}^Z(g(f(x)))$ for each $r \in (0, \rho_g)$. Define $\rho := \min(\rho_f, \rho_g/a)$. Then, for each $r \in (0, \rho)$,

$$(g \circ f)\left(B_r^X(x)\right) \supseteq g\left(B_{ar}^Y(f(x))\right) \supseteq B_{abr}^Z((g \circ f)(x)).$$

Hence, $g \circ f$ is pointwise (ab)-co-Lipschitz at x. □

The traditional examples of planar Lipschitz quotient mappings f_n, see Lemma 1.2.22, possess sharp constants, in the sense that the ratios of constants c/L for such mappings are maximal, cf. [21, Theorem 2].

Lemma 1.2.22. For each $n \in \mathbb{N}$, define $f_n : \mathbb{C} \to \mathbb{C}$ to be given by $f_n(z) = |z|e^{in\arg(z)}$. Then f_n is a Lipschitz quotient mapping; namely, f_n is n-Lipschitz and 1-co-Lipschitz with respect to the Euclidean norm.

Proof. To see f_n is n-Lipschitz, we need to show that

$$\left||z|e^{in\arg(z)} - |y|e^{in\arg(y)}\right| \leq n|z-y| \quad \text{for each } y, z \in \mathbb{C}. \tag{1.2.2}$$

Note (1.2.2) is trivially satisfied if $y = 0$ or $z = 0$. Suppose that $y, z \neq 0$. By the homogeneity of (1.2.2), it suffices to prove

$$\left|\rho e^{in\theta} - 1\right| \leq n\left|\rho e^{i\theta} - 1\right| \quad \text{for all } \rho \in (0,1] \text{ and all } \theta \in (-\pi, \pi]. \tag{1.2.3}$$

Fix $\rho \in (0,1]$ and $\theta \in (-\pi, \pi]$. By squaring (1.2.3), we need to show that $\rho^2 - 2\rho\cos(n\theta) + 1 \leq n^2\left(\rho^2 - 2\rho\cos(\theta) + 1\right)$. That is,

$$(\rho-1)^2 + 4\rho\sin^2\left(\frac{n\theta}{2}\right) \leq n^2\left((\rho-1)^2 + 4\rho\sin^2\left(\frac{\theta}{2}\right)\right),$$

which follows since $|\sin(n\theta/2)| \leq n|\sin(\theta/2)|$ for $\theta \in [0, \pi]$; this can be seen by a simple inductive argument. Hence f_n is n-Lipschitz.

Now we shall consider the co-Lipschitzness of f_n. We begin by showing that f_n is pointwise 1-co-Lipschitz at $z = 1$. Indeed, let $r \in (0,1)$ and $y \in B_r(f_n(1)) = B_r(1)$. Define $x :=$

$|y|e^{i\arg(y)/n}$. Observe that $x \neq 0$ and $f_n(x) = y$. So, to conclude f_n is pointwise 1-co-Lipschitz at $z = 1$, it suffices to show $x \in B_r(1)$. Since $r < 1$, note $|\arg(y)| < \pi/2$ and so $\cos(\arg(x)) = \cos(\arg(y)/n) \geq \cos(\arg(y))$. Hence, as $|x| = |y|$,

$$|x-1|^2 = 1 + |x|^2 - 2|x|\cos(\arg(x)) \leq 1 + |y|^2 - 2|y|\cos(\arg(y)) = |y-1|^2 < r^2.$$

So, $x \in B_r(1)$ and thus f_n is pointwise 1-co-Lipschitz at $z = 1$.

Fix $z \in \mathbb{C} \setminus \{0\}$ and let $\rho = \rho(z) = |z|$. Let $r \in (0, \rho)$ and $y \in B_r(f_n(z))$. Since f_n is pointwise 1-co-Lipschitz at $z_0 = 1$,

$$\frac{y}{f_n(z)} = \left|\frac{y}{z}\right| e^{i(\arg(y) - n\arg(z))} \in B_{r/|z|}(1) \subseteq f_n\left(B_{r/|z|}(1)\right),$$

since $r/|z| < 1$. Hence there exists $x' \in B_{r/|z|}(1)$ such that $f_n(x') = |y/z|e^{i(\arg(y)-n\arg(z))}$. Define $x := x'z$ and note both $|x'| = |y/z|$ and $f_n(x) = y$. Moreover, as $x' \in B_{r/|z|}(1)$, then $|x-z| = |z|\cdot|x'-1| < r$. Thus $x \in B_r(z)$ and so $B_r(f_n(z)) \subseteq f_n(B_r(z))$. Hence, f_n is pointwise 1-co-Lipschitz at each $z \in \mathbb{C} \setminus \{0\}$.

Finally as $B_r(f_n(0)) = B_r(0) = f_n(B_r(0))$ for all $r > 0$, we conclude f_n is pointwise 1-co-Lipschitz at all $z \in \mathbb{C}$ and thus, by Lemma 1.2.18, f_n is 1-co-Lipschitz. □

We highlight that in Corollary 1.2.37 below we prove that f_n satisfy properties which are stronger than 1-co-Lipschitzness.

The following lemma concerns the Lipschitz property of variants of the standard Lipschitz quotient mappings f_n introduced in Lemma 1.2.22. Note here and throughout the rest of the thesis, if $k \in \mathbb{N}$ we use $[k]$ to denote the set $\{1, \ldots, k\}$.

Lemma 1.2.23. Let $n \geq 2$ and $k \in [n-1]$. For each $\varepsilon > 0$ there exists $D = D(\varepsilon, k, n) > 0$ such that $g_{k,n} : \mathbb{C} \setminus B_D(0) \to \mathbb{C}$ defined by $g_{k,n}(z) = |z|^{k/n} e^{ik\arg(z)}$ is ε-Lipschitz on $\mathbb{C} \setminus B_D(0)$.

Proof. Fix $\varepsilon > 0$ and consider f_n as in Lemma 1.2.22. Define $h_k(t) = t^{k/n}$ for $t > 0$. Let $T > 0$ be such that h_k is $(\varepsilon/2)$-Lipschitz on $[T, +\infty)$ and let $R > 0$ be such that $(k+1)/R^{1-k/n} < \varepsilon/2$.

Define $D := \min\{T, R\}$ and fix $z_1, z_2 \in \mathbb{C} \setminus B_D(0)$. Then,

$$|g_{k,n}(z_1) - g_{k,n}(z_2)| \leq \left|g_{k,n}(z_1) - |z_2|^{k/n} e^{ik\arg(z_1)}\right| + |z_2|^{k/n}\left|e^{ik\arg(z_1)} - e^{ik\arg(z_2)}\right|. \qquad (1.2.4)$$

As $|z_1|, |z_2| \geq D \geq T$ and as h_k is $(\varepsilon/2)$-Lipschitz on $[T, +\infty)$,

$$\left|g_{k,n}(z_1) - |z_2|^{k/n} e^{ik\arg(z_1)}\right| = |h_k(|z_1|) - h_k(|z_2|)| \leq \frac{\varepsilon}{2}\Big||z_1| - |z_2|\Big| \leq \frac{\varepsilon}{2}|z_1 - z_2|. \tag{1.2.5}$$

Further, since $|z_2| \geq D \geq R$,

$$\begin{aligned}
&|z_2|^{k/n}\left|e^{ik\arg(z_1)} - e^{ik\arg(z_2)}\right| \\
&\quad \leq \left||z_2|^{k/n} - |z_1| \cdot |z_2|^{(k/n)-1}\right| + \left||z_1| \cdot |z_2|^{(k/n)-1} e^{ik\arg(z_1)} - |z_2|^{k/n} e^{ik\arg(z_2)}\right| \\
&\quad = \frac{1}{|z_2|^{1-(k/n)}}\left(\Big||z_1| - |z_2|\Big| + |f_k(z_1) - f_k(z_2)|\right) \\
&\quad \leq \frac{\varepsilon}{2}|z_1 - z_2|,
\end{aligned}$$

where the last inequality follows by our choice of $R > 0$ and Lemma 1.2.22. Substituting this and (1.2.5) into (1.2.4) we obtain

$$|g_{k,n}(z_1) - g_{k,n}(z_2)| \leq \varepsilon|z_1 - z_2|.$$

By the arbitrariness of $z_1, z_2 \in \mathbb{C} \setminus B_D(0)$ we establish the Lipschitzness of $g_{k,n}$. □

The next lemma provides a sufficient property for a mapping between metric spaces to be pointwise co-Lipschitz at a given point. To be able to conveniently refer to this property, we first introduce the following notion.

Definition 1.2.24. Suppose (X, d_X) and (Y, d_Y) are metric spaces and $c > 0$. We say a function $f : X \to Y$ is *strongly c-co-Lipschitz* at $x_0 \in X$ if there exists $\rho > 0$ such that:

(i) $f(x_0) \in \operatorname{Int}\left(f\left(B_\rho^X(x_0)\right)\right)$;

(ii) $d_Y(f(x), f(x_0)) \geq c\, d_X(x, x_0)$ for all $x \in B_\rho^X(x_0)$.

If we do not need to specify c, we shall simply write f is strongly co-Lipschitz at x_0.

Lemma 1.2.25. Let (X, d_X) and (Y, d_Y) be metric spaces and $c > 0$. If $f : X \to Y$ is strongly c-co-Lipschitz at $x_0 \in X$, then f is pointwise c-co-Lipschitz at x_0.

Proof. Let $\rho > 0$ be as in Definition 1.2.24. By property (i) of Definition 1.2.24, there exists $R \in (0, \rho)$ such that

$$B_R^Y(f(x_0)) \subseteq \operatorname{Int}\left(f\left(B_\rho^X(x_0)\right)\right) \subseteq f\left(B_\rho^X(x_0)\right). \tag{1.2.6}$$

Define $r := \frac{R}{2c} > 0$, let $s \in (0, r)$ and fix $y \in B_{cs}^Y(f(x_0))$. Note $cs < cr < R$. Thus (1.2.6) implies $y \in f\left(B_\rho^X(x_0)\right)$. Hence, there exists $x \in B_\rho^X(x_0)$ such that $y = f(x)$. As $x \in B_\rho^X(x_0)$ and $y \in B_{cs}^Y(f(x_0))$, it follows by property (ii) of Definition 1.2.24 that

$$cs > d_Y(y, f(x_0)) = d_Y(f(x), f(x_0)) \geq c d_X(x, x_0).$$

Hence, $x \in B_s^X(x_0)$ and so $y = f(x) \in f\left(B_s^X(x_0)\right)$. Since $y \in B_{cs}^Y(f(x_0))$ was arbitrary, we deduce that $B_{cs}^Y(f(x_0)) \subseteq f\left(B_r^X(x_0)\right)$, and thus f is pointwise c-co-Lipschitz at x_0. □

Corollary 1.2.26. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is an open map, $x_0 \in X$ and there exist positive constants c and r_0 such that $d_Y(f(x), f(x_0)) \geq c d_X(x, x_0)$ for each $x \in B_{r_0}^X(x_0)$. Then, f is pointwise c-co-Lipschitz at x_0.

Remark 1.2.27. When proving pointwise or strong co-Lipschitzness of mappings defined in Chapter 2, we will often consider X to be an open subset of $\mathbb{C}$. In such cases, instead of $B_r^X(x)$, we will consider balls centred at $x \in X$ and open in the Euclidean metric. To be able to use the definition of a co-Lipschitz mapping or Definition 1.2.24 and subsequent results about strongly co-Lipschitz mappings, it is enough to ensure r is sufficiently small so that the Euclidean ball of radius r around x coincides with $B_r^X(x)$.

Remark 1.2.28. Using the notion introduced in Definition 1.2.24, the following implication follows by Lemma 1.2.25:

$$\text{strongly } c\text{-co-Lipschitz at } x_0 \implies \text{pointwise } c\text{-co-Lipschitz at } x_0. \tag{1.2.7}$$

One may naturally ask the question whether a reverse implication holds. In Lemma 1.2.29 below, we show only property (ii) of Definition 1.2.24 needs to be verified for a pointwise co-Lipschitz mapping to be strongly co-Lipschitz.

Lemma 1.2.29. Let (X, d_X), (Y, d_Y) be metric spaces, $f : X \to Y$, $x_0 \in X$ and $c > 0$. Suppose f is pointwise c-co-Lipschitz at x_0. If there exists $\rho_0 > 0$ such that $d_Y(f(x), f(x_0)) \geq c d_X(x, x_0)$ for all $x \in B^X_{\rho_0}(x_0)$, then f is strongly c-co-Lipschitz at x_0.

Proof. It is enough to prove (i) of Definition 1.2.24 is satisfied for some $\rho < \rho_0$. Indeed, as f is pointwise c-co-Lipschitz at x_0, there exists $r_0 > 0$ such that $f\left(B^X_r(x_0)\right) \supseteq B^Y_{cr}(f(x_0))$ for all $r \in (0, r_0)$. Define $\rho := \frac{1}{2}\min(r_0, \rho_0)$. Then,

$$f(x_0) \in B^Y_{c\rho}(f(x_0)) \subseteq f\left(B^X_\rho(x_0)\right).$$

Hence, as $B^Y_{c\rho}(f(x_0))$ is open, we deduce (i) is satisfied. Thus, f is strongly c-co-Lipschitz at x_0. □

The reverse implication of (1.2.7) can easily be seen in the case when the function is locally injective, as we show in the following lemma.

Lemma 1.2.30. Let (X, d_X), (Y, d_Y) be metric spaces, $x_0 \in X$ and $c > 0$. Suppose a mapping $f : X \to Y$ is both pointwise c-co-Lipschitz and locally injective at x_0. Then f is strongly c-co-Lipschitz at x_0.

Proof. Since f is pointwise c-co-Lipschitz at x_0, there exists $r_0 > 0$ such that

$$B^Y_{cr}(f(x_0)) \subseteq f\left(B^X_r(x_0)\right) \quad \text{for all } r \in (0, r_0). \tag{1.2.8}$$

As f is locally injective at x_0, there exists $r_1 > 0$ such that $f\big|_{B^X_{r_1}(x_0)}$ is injective. Define $\rho := \frac{1}{2}\min(r_0, r_1)$. By Lemma 1.2.29, it suffices to show

$$d_Y(f(x), f(x_0)) \geq c d_X(x, x_0) \quad \text{for all } x \in B^X_\rho(x_0). \tag{1.2.9}$$

This is trivially satisfied for $x = x_0$. Suppose, for a contradiction, that (1.2.9) is not satisfied, i.e. there exists $x \in B^X_\rho(x_0) \setminus \{x_0\}$ such that $d_Y(f(x), f(x_0)) < c d_X(x, x_0)$. Define $r := d_X(x, x_0)$, so $0 < r < \rho < r_0$. Hence,

$$f(x) \in B^Y_{cr}(f(x_0)) \subseteq f\left(B^X_r(x_0)\right),$$

where the inclusion follows by (1.2.8). So, there exists $z \in B_r^X(x_0)$ such that $f(z) = f(x)$. Therefore, as $f\big|_{B_\rho^X(x_0)}$ is injective and $r < \rho$, $x = z \in B_\rho^X(x_0)$. It then follows that $r = d_X(x, x_0) < r$, providing contradiction. Hence (1.2.9) is satisfied. □

Corollary 1.2.31. Suppose X and Y are metric spaces, $f : X \to Y$ is a mapping which is locally injective at $x_0 \in X$ and $c > 0$. Then,

$$f \text{ is strongly } c\text{-co-Lipschitz at } x_0 \iff f \text{ is pointwise } c\text{-co-Lipschitz at } x_0.$$

Remark 1.2.32. We highlight the relevance of Corollary 1.2.31 in the context of mappings with the inherent structure of planar Lipschitz quotient mappings. Indeed, Proposition 1.2.15 identifies at which points of the plane a composition $P \circ h$ of a polynomial P and a homeomorphism h is locally injective, hence where the notions of strongly co-Lipschitz and pointwise co-Lipschitz agree. In Corollary 1.2.36 below, we show that these two notions automatically agree for any planar Lipschitz quotient mapping. However, as mentioned in Section 2.1, not all mappings with this underlying structure $P \circ h$ are Lipschitz quotient.

Further, we are able to show the equivalence between the two notions of pointwise co-Lipschitz and strongly co-Lipschitz for discrete co-Lipschitz mappings. To see this we follow the method presented in [20, p. 2091]. Let us first recall the definition of a discrete mapping.

Definition 1.2.33. Let X, Y be topological spaces and $S \subseteq X$. We say:

- S is a *discrete set* if for each $x \in S$ there exists a neighbourhood U of x such that $U \cap S = \{x\}$;
- $f : X \to Y$ is a *discrete mapping* if $f^{-1}(y)$ is a discrete set for each $y \in Y$.

Lemma 1.2.34. Suppose (X, d_X), (Y, d_Y) are metric spaces and $f : X \to Y$ is a discrete c-co-Lipschitz mapping for some $c > 0$. Then f is strongly c-co-Lipschitz at every $x \in X$.

Proof. Fix $x \in X$ and define $\mathcal{A}_x := f^{-1}(f(x))$. Since f is a discrete mapping there exists $r_0 > 0$ such that $B_{2r_0}^X(x) \cap \mathcal{A}_x = \{x\}$. Fix $z \in B_{r_0}^X(x) \setminus \{x\}$ and let $r := d_X(z, x)$. Then $B_r^X(z) \cap \mathcal{A}_x = \emptyset$ and so $f(x) \notin f\left(B_r^X(z)\right)$. Since f is c-co-Lipschitz, $f\left(B_r^X(z)\right) \supseteq B_{cr}^Y(f(z))$. As $f(x) \notin f\left(B_r^X(z)\right)$, this implies $d_Y(f(x), f(z)) \geq cr = c d_X(x, z)$.

Observe that $d_Y(f(x), f(z)) \geq cd_X(x, z)$ is trivially satisfied when $z = x$. Therefore, by Lemma 1.2.29, we conclude f is strongly c-co-Lipschitz at x. □

We highlight that Lemma 1.2.30 and Lemma 1.2.34 are the strongest possible, in the sense that there exist Lipschitz quotient mappings which are 1-co-Lipschitz, but not locally injective, not discrete and not strongly co-Lipschitz at any point. We show this in the following example.

Example 1.2.35. Let $n, k \geq 1$ be integers and $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the standard projection, where both spaces are equipped with the Euclidean norm. Then f is 1-Lipschitz and 1-co-Lipschitz. This trivially follows since $f(B_r(x)) = B_r(f(x))$ for all $r > 0$ and $x \in \mathbb{R}^{n+k}$. Further, it is clear that f is not discrete. Moreover, f is neither locally injective nor strongly c-co-Lipschitz, for any $c > 0$, at any $y \in \mathbb{R}^{n+k}$ as $f^{-1}(y)$ is a k-dimensional hyperplane.

Using Lemma 1.2.34, we deduce the following two corollaries. First we show that planar Lipschitz quotient mappings, or any continuous co-Lipschitz planar mappings, are necessarily strongly co-Lipschitz at every point.

Corollary 1.2.36. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a continuous c-co-Lipschitz mapping for some $c > 0$. Then f is strongly c-co-Lipschitz at each $x \in \mathbb{C}$.

Proof. By [1, Proposition 4.3], or equivalently [17, Proposition 2.1], f is discrete and so Lemma 1.2.34 yields the result. □

Corollary 1.2.37. For every $n \in \mathbb{N}$, let the function $f_n : \mathbb{C} \to \mathbb{C}$ be defined by $f_n(z) = |z|e^{in\arg(z)}$ as in Lemma 1.2.22. Then f_n is strongly 1-co-Lipschitz at every $z \in \mathbb{C}$.

Following Corollary 1.2.36, one may ask the following question.

Question 1.2.38. Suppose $n \geq 3$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz quotient mapping. Is f strongly co-Lipschitz at each $x_0 \in \mathbb{R}^n$?

We note the following logical equivalence between Question 1.2.38 and a long-standing conjecture from [1, p. 1096]. Namely:

Conjecture 1.2.39. Suppose $n \geq 3$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz quotient mapping. Then f is a discrete mapping.

Remark 1.2.40. There are no known Lipschitz quotient mappings $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f^{-1}(y)$ is infinite, for at least one $y \in \mathbb{R}^d$ where $d \geq 3$. However by considering the higher dimensional analogues of the winding maps, namely $f_n^d : \mathbb{C} \times \mathbb{R}^{d-2} \to \mathbb{C} \times \mathbb{R}^{d-2}$ given by $f_n^d = f_n \times \mathrm{id}_{\mathbb{R}^{d-2}}$ where f_n is defined in Lemma 1.2.22, one may note that no global constant $N = N(d)$ exists such that $\mathrm{card}(f^{-1}(y)) \leq N$ for all Lipschitz quotient mappings $f : \mathbb{R}^d \to \mathbb{R}^d$ and all $y \in \mathbb{R}^d$, where $d \geq 3$.

First we note that a positive answer to Conjecture 1.2.39 implies, via an application of Lemma 1.2.34, that every Lipschitz quotient mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ is strongly c-co-Lipschitz everywhere, where $c = \text{co-Lip}(f)$, providing a positive answer to Question 1.2.38.

Conversely, a positive answer to Question 1.2.38, i.e. every Lipschitz quotient mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is strongly co-Lipschitz everywhere, implies Conjecture 1.2.39. This implication is proved in the following simple lemma.

Lemma 1.2.41. Let (X, d_X), (Y, d_Y) be metric spaces and $y \in Y$. If $f : X \to Y$ is strongly co-Lipschitz at every element of $f^{-1}(y)$, then $f^{-1}(y)$ is a discrete set.

In particular, if f is strongly co-Lipschitz at every $x \in X$, then f is a discrete mapping.

Proof. To show $f^{-1}(y)$ is discrete we require to show for each $x \in f^{-1}(y)$ that there exists a neighbourhood U_x of x such that $U_x \cap f^{-1}(y) = \{x\}$.

Fix $x \in f^{-1}(y)$. Since f is strongly co-Lipschitz at x, there exist positive constants c_x and ρ_x such that

$$d_Y\left(f(w), f(x)\right) \geq c_x d_X\left(w, x\right) \qquad \text{for each } w \in B^X_{\rho_x}(x). \tag{1.2.10}$$

Define $U_x := B^X_{\rho_x}(x)$ and let $z \in U_x \cap f^{-1}(y)$. Since $z \in B^X_{\rho_x}(x)$ and $f(z) = y$, by (1.2.10) it follows that

$$0 = d_Y\left(f(z), f(x)\right) \geq c_x d_X(z, x).$$

Thus $z = x$ since $c_x > 0$ and so $U_x \cap f^{-1}(y) = \{x\}$. Since $x \in f^{-1}(y)$ was arbitrary, we conclude $f^{-1}(y)$ is a discrete set. $\square$

1.3 Basic properties of planar curves

First we recall the definition of a simple, closed curve.

Definition 1.3.1. Let X be a normed space and $\Phi \subseteq X$. We say that Φ is a *curve* if there exists a continuous mapping $\phi : [a, b] \subseteq \mathbb{R} \to X$, $a < b$, such that Φ is the image of ϕ. We say that ϕ is a *parametrisation* of Φ.

If in addition ϕ is injective, we say that Φ is a *simple curve.* If however $\phi\big|_{[a,b)}$ is injective and $\phi(a) = \phi(b)$, then we say that Φ is a *simple, closed curve.*

Trivially, one may consider parametrisations of a curve only for $[a, b] = [0, 1]$, as we typically do. Further, note Φ is compact by the continuity of ϕ.

Remark 1.3.2. For ease of notation, if ϕ is a parametrisation of a curve Φ, we may refer to ϕ for both the parametrisation and the curve.

We now recall the definition of the n-dimensional Hausdorff measure.

Definition 1.3.3. Let $(X, \|\cdot\|)$ be a normed space and $E \subseteq X$. The *n-dimensional Hausdorff measure of E* is defined in the following way:

For each $\delta > 0$, define

$$\mathcal{H}^{n,\delta}_{\|\cdot\|}(E) := \inf\left\{\sum_{k=1}^{\infty} (\operatorname{diam}(C_j))^n : E \subseteq \bigcup_{k=1}^{\infty} C_j, \operatorname{diam}(C_j) \leq \delta\right\},$$

where $\operatorname{diam}(C_j)$ denotes the diameter of the set C_j.

Define the n-dimensional Hausdorff measure to be

$$\mathcal{H}^{n}_{\|\cdot\|}(E) = \lim_{\delta \to 0^+} \mathcal{H}^{n,\delta}_{\|\cdot\|}(E).$$

If it is clear from context, we simple write $\mathcal{H}^n(E)$. In this thesis, we will be interested in $\mathcal{H}^1_{\|\cdot\|}$.

Definition 1.3.4. Let $\phi : [a, b] \to \mathbb{R}^d$ be a locally injective parametrisation of a curve Φ and let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$ such that $t_0 = a$, $t_n = b$, $t_j \leq t_{j+1}$ and $\phi\big|_{(t_j,t_{j+1})}$ is injective for each $j \in [n-1] \cup \{0\}$. The *length of ϕ* is

$$\operatorname{length}_{\|\cdot\|}(\phi) = \sum_{j=0}^{n-1} \mathcal{H}^1_{\|\cdot\|}\left(\phi\big|_{(t_j,t_{j+1})}\right).$$

If the norm $\|\cdot\|$ is clear from context we simple write $\operatorname{length}(\phi)$.

Remark 1.3.5. The value $\text{length}_{\|\cdot\|}(\phi)$ is not dependent on the partition chosen; cf. [37, Remark 1.3.8].

Lipschitz mappings behave well when considering the length of curves under their images. Indeed, if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and $f : X \to Y$ is an L-Lipschitz mapping, then for each $E \subseteq X$, note

$$\text{diam}_{\|\cdot\|_Y}(f(E)) \leq L\text{diam}_{\|\cdot\|_X}(E).$$

Hence, $\mathcal{H}^n_{\|\cdot\|_X}(f(E)) \leq L^n \mathcal{H}^n_{\|\cdot\|_X}(E)$ for each $n \in \mathbb{N}$, and so if f is locally injective and ϕ is a parametrisation of a curve, then

$$\text{length}_{\|\cdot\|_Y}(f \circ \phi) \leq L\text{length}_{\|\cdot\|_X}(\phi).$$

We now recall two standard results concerning the length of curves.

Lemma 1.3.6. Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and $x_0, y_0 \in \mathbb{R}^d$ be two distinct points. The length of any locally injective curve ϕ between x_0 and y_0 satisfies $\text{length}_{\|\cdot\|}\phi \geq \|x_0 - y_0\|$.

Lemma 1.3.7. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $\phi : [0,1] \to \mathbb{C}$ be a closed curve. If there exist $R > 0$ and $n \in \mathbb{N}$ such that $\|\phi(t)\| \geq R$ for all $t \in [0,1]$ and $\text{Ind}_0\phi = n$, then $\text{length}_{\|\cdot\|}(\phi) \geq n\mathcal{H}^1_{\|\cdot\|}\left(\partial B^{\|\cdot\|}_R(0)\right)$.

For a given norm $\|\cdot\|$ on the plane, the unit sphere is homotopy equivalent to the Euclidean sphere $\mathbb{S}^1$. As such, it follows that there exists a continuous parametrisation of the unit $\|\cdot\|$-sphere, which has increasing argument; for example, the standard arc-length parametrisation as in [37, Lemma 1.3.3]. Utilising this, it is clear that there exists a parametrisation of the unit sphere whose image has index one about the origin.

Notation 1.3.8. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $z_0 := \partial B^{\|\cdot\|}_1(0) \cap [0, +\infty)$ and $\Gamma_1 : [0,1] \to \partial B^{\|\cdot\|}_1(0)$ be a parametrisation of $\partial B^{\|\cdot\|}_1(0)$ in the anticlockwise direction such that $\Gamma_1\big|_{(0,1)}$ is injective, $\Gamma_1(0) = \Gamma_1(1) = z_0$ and $\text{Ind}_0\Gamma_1 = 1$.

Let $s \in (0,1)$ be such that $\Gamma_1(s) = -z_0$. Define $\theta^{\|\cdot\|} : [0,1] \to \partial B_1^{\|\cdot\|}(0)$ by

$$\theta^{\|\cdot\|}(t) = \begin{cases} \Gamma_1(2st), & \text{if } t \in [0,1/2]; \\ -\Gamma_1(2s(t-1/2)), & \text{if } t \in [1/2,1]. \end{cases}$$

This mapping $\theta^{\|\cdot\|}$ is a parametrisation of the unit sphere whose restriction $\theta^{\|\cdot\|}\big|_{(0,1)}$ is injective.

Lemma 1.3.9. Let $\|\cdot\|$ be a norm on $\mathbb{C}$. Then $\theta^{\|\cdot\|}$ is a parametrisation of $\partial B_1^{\|\cdot\|}(0)$ such that $\theta^{\|\cdot\|}\big|_{(0,1)}$ is injective, $\theta^{\|\cdot\|}(0) = \theta^{\|\cdot\|}(1) = z_0$, $\mathrm{Ind}_0\theta^{\|\cdot\|} = 1$ and $\theta^{\|\cdot\|}(t+1/2) = -\theta^{\|\cdot\|}(t)$ for each $t \in [0,1/2]$.

We now introduce a family of parametrisations of the unit sphere $\partial B_1^{\|\cdot\|}(0)$.

Notation 1.3.10. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $\theta^{\|\cdot\|}$ be a parametrisation of $\partial B_1^{\|\cdot\|}(0)$ as defined in Notation 1.3.8. For each $z \in \partial B_1^{\|\cdot\|}(0)$, let $t_z \in [0,1)$ be the unique value such that $\theta^{\|\cdot\|}(t_z) = z$. Define $\theta_z^{\|\cdot\|} : [0,1] \to \partial B_1^{\|\cdot\|}(0)$ by

$$\theta_z^{\|\cdot\|}(t) := \begin{cases} \theta^{\|\cdot\|}\left(t + t_z\right), & \text{if } t \in [0, 1-t_z]; \\ \theta^{\|\cdot\|}\left(t - (1-t_z)\right), & \text{if } t \in [1-t_z, 1]. \end{cases}$$

If it is clear from context, we suppress the superscript $\|\cdot\|$ from the above notation. Observe that $\theta_{z_0} = \theta$ and in general parametrisations θ_z preserve many of the properties of θ.

Lemma 1.3.11. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $z \in \partial B_1^{\|\cdot\|}(0)$. Then θ_z is a parametrisation of $\partial B_1^{\|\cdot\|}(0)$ in the anticlockwise direction such that $\theta_z\big|_{(0,1)}$ is injective, $\mathrm{Ind}_0\theta_z = 1$, $\theta_z(0) = \theta_z(1) = z$, $\theta_z(1/2) = -z$ and, in general,

$$\theta_z\left((t+1/2) \pmod 1\right) = -\theta_z\left(t\right) \qquad \text{for all } t \in [0,1]. \tag{1.3.1}$$

Remark 1.3.12. Note for each $z \in \partial B_1^{\|\cdot\|}(0)$ that any parametrisation θ_z satisfying (1.3.1) is not unique. In general, this does not affect our considerations. However, when the non-uniqueness becomes an issue we consider a *fixed* parametrisation θ_z, see (4.3.2) of Theorem 4.3.5.

Next we introduce further notation concerning unit vectors in the plane.

Notation 1.3.13. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $x, y \in \partial B_1^{\|\cdot\|}(0)$.

(i) Let $f_x^{\|\cdot\|} : [0,1] \to [0,2]$ be defined by $f_x^{\|\cdot\|}(t) := \left\| \theta_x^{\|\cdot\|}(t) - x \right\|$.

If the norm is clear from context we suppress it in the notation and write f_x.

(ii) If $x \neq y$, let $[x, y]_{\|\cdot\|}$ denote the closed arc of $\partial B_1^{\|\cdot\|}(0)$ formed by x and y, starting at x and traversed in the anticlockwise direction, ending at y.

Analogously, we define $[x, y)_{\|\cdot\|}$, $(x, y]_{\|\cdot\|}$ and $(x, y)_{\|\cdot\|}$.

We now introduce two simple results concerning arcs of spheres in the plane.

Corollary 1.3.14. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $t_1 \in [0,1]$. If $0 \le t_1 < t_2 < t_3 < 1 + t_1$, then

$$\theta_x^{\|\cdot\|}\left(t_2 \pmod 1\right) \in \left(\theta_x\left(t_1 \pmod 1\right), \theta_x\left(t_3 \pmod 1\right)\right)_{\|\cdot\|}.$$

In fact, each parametrisation θ_z, when restricted to $(0,1)$, defines a homeomorphism.

Corollary 1.3.15. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $z \in \partial B_1^{\|\cdot\|}(0)$. Then $\Theta_z := \theta_z\big|_{(0,1)} : (0,1) \to \partial B_1^{\|\cdot\|}(0) \setminus \{z\}$ is a homeomorphism.

Proof. By Lemma 1.3.11, it suffices to verify the openness of Θ_z. Let $U \subseteq (0,1)$ be non-empty and open. Suppose, for a contradiction, that $\Theta_z(U)$ is not open in $\partial B_1^{\|\cdot\|}(0) \setminus \{z\}$. Let $y \in U$ be such that $\Theta_z(U)$ does not contain an open neighbourhood of $\Theta_z(y)$ and let $\varepsilon > 0$ be such that $V := (y - \varepsilon, y + \varepsilon) \subseteq U$. Define $V_1 := (y - \varepsilon, y]$ and $V_2 := [y, y + \varepsilon)$. As V, V_1 and V_2 are connected and Θ_z is continuous, $W := \Theta_z(V)$ and $W_j := \Theta_z(V_j)$, $j = 1, 2$, are connected in $\partial B_1(0) \setminus \{z\} = \Theta_z((0,1))$.

Consider a homeomorphism $\phi : \partial B_1^{\|\cdot\|}(0) \setminus \{z\} \to (0,1)$. Define $Z := \phi(W)$ and $Z_j := \phi(W_j)$ for each $j = 1, 2$. Since $W \subseteq \Theta_z(U)$ does not contain an open neighbourhood of $\Theta_z(y)$, note this implies Z, Z_1 and Z_2 are (connected) intervals in $(0,1)$ which do not contain an open neighbourhood of $a := \phi(\Theta_z(y)) \subseteq (0,1)$. Hence, without loss of generality, $Z = [a, b\rangle$, $Z_1 = [a, b_1\rangle$ and $Z_2 = [a, b_2\rangle$ where $b, b_1, b_2 \in (0,1)$ and $\rangle$ represents either an open or closed end to the intervals. Observe that $Z = Z_1 \cup Z_2$, $a \in Z_1 \cap Z_2$ and $b = \max(b_1, b_2)$. Therefore either $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$. Further, by the injectivity of $\phi \circ \Theta_z$, $Z_j \neq \{a\}$ for each $j = 1, 2$. Thus $Z_1 \cap Z_2 \neq \{a\}$ and so

$$\phi\left(\Theta_z\left((y - \varepsilon, y)\right)\right) \cap \phi\left(\Theta_z\left((y, y + \varepsilon)\right)\right) \neq \emptyset,$$

which contradicts the injectivity of $\phi \circ \Theta_z$ on $(0,1)$.

Therefore W is open in $\partial B_1^{\|\cdot\|}(0) \setminus \{z\}$ and hence Θ_z is an open map, and so defines a homeomorphism. $\square$

We now focus on some simple, yet useful, properties of arcs contained in the unit sphere of a fixed planar norm.

Lemma 1.3.16. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $x, y, v \in \partial B_1^{\|\cdot\|}(0)$ be distinct. Then:

(i) $x \in (v, -v)_{\|\cdot\|}$ if and only if $v \in (-x, x)_{\|\cdot\|}$;

(ii) if $x, y \in [v, -v)_{\|\cdot\|}$ and $y \in (z, x)_{\|\cdot\|}$ for some $z \in (-v, v]_{\|\cdot\|}$ it follows that $x \in (y, -v)_{\|\cdot\|}$;

(iii) $x \in (y, v)_{\|\cdot\|}$ if and only if $v \in (x, y)_{\|\cdot\|}$;

(iv) $x \in (y, v)_{\|\cdot\|}$ if and only if $-x \in (-y, -v)_{\|\cdot\|}$.

Proof. Part (i) of the present lemma follows from Notation 1.3.13 (ii). For (ii), given $\theta_v : [0,1) \to \partial B_1^{\|\cdot\|}(0)$, find $t_w \in (0,1)$ such that $\theta_v(t_w) = w$ for each $w \in \{x, y, -v\}$. Since $x, y \in [v, -v)_{\|\cdot\|}$, $0 \leq t_x, t_y < t_{-v} = 1/2$. As $y \in (z, x)_{\|\cdot\|}$ observe that $t_y < t_x$. Hence, by Corollary 1.3.14,

$$x = \theta_v(t_x) \in (\theta_v(t_y), \theta_v(1/2))_{\|\cdot\|} = (y, -v)_{\|\cdot\|}.$$

For (iii) and (iv), suppose $x \in (y, v)_{\|\cdot\|}$. Given $\theta_y : [0,1) \to \partial B_1^{\|\cdot\|}(0)$, find $t_w \in (0,1)$ such that $\theta_y(t_w) = w$ for $w \in \{v, x\}$. As $x \in (y, v)_{\|\cdot\|}$ and $v \neq y$, note $0 < t_x < t_v < 1$. Hence, $v = \theta_y(t_v) \in (\theta_y(t_x), \theta_y(1))_{\|\cdot\|} = (x, y)_{\|\cdot\|}$. For (iv), $1/2 < t_x + 1/2 < t_v + 1/2 < 3/2$. Therefore, by Lemma 1.3.11 and Corollary 1.3.14,

$$-x = \theta_y\left((t_x + 1/2) \pmod 1\right) \in \left(\theta_y(1/2), \theta_y\left((t_v + 1/2) \pmod 1\right)\right)_{\|\cdot\|} = (-y, -v)_{\|\cdot\|}.$$

The reverse directions of (iii) and (iv) follow by the arbitrariness of x, y, v. $\square$

Lemma 1.3.17. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $x, y, z \in \partial B_1^{\|\cdot\|}(0)$ be distinct vectors such that $y \in (-x, x)_{\|\cdot\|}$ and $z \in (x, -x)_{\|\cdot\|}$. Let $U = (y, z)_{\|\cdot\|} \cap (-z, -y)_{\|\cdot\|}$. Then,

$$U = (y, -y)_{\|\cdot\|} \cap (-z, z)_{\|\cdot\|}. \tag{1.3.2}$$

Moreover,

(i) if $-z \in [y,x)_{\|\cdot\|}$, then $U = (-z,-y)_{\|\cdot\|}$; (ii) if $-z \notin [y,x)_{\|\cdot\|}$, then $U = (y,z)_{\|\cdot\|}$.

Proof. Consider a parametrisation θ_{-x} from Notation 1.3.10. For each $v \in \{y,z,-y,-z\}$, let $t_v \in (0,1)$ be such that $\theta_{-x}(t_v) = v$.

Suppose first that $-z \in [y,x)_{\|\cdot\|}$. Thus, as $y \in (-x,x)_{\|\cdot\|}$, $0 < t_y \le t_{-z} < t_x = 1/2$. Hence,

$$0 < t_y \le t_{-z} < 1/2 \le t_{-y} \le t_z < 1. \tag{1.3.3}$$

Thus, $(-z,-y)_{\|\cdot\|} \subseteq (y,z)_{\|\cdot\|}$ and so $U = (-z,-y)_{\|\cdot\|}$. Further, (1.3.2) follows via (1.3.3).

Suppose now that $-z \notin [y,x)_{\|\cdot\|}$. As $z \neq -x$, $-z \in (x,y)_{\|\cdot\|}$. Now as $z \in (x,-x)_{\|\cdot\|}$, we have $-z \in (-x,x)_{\|\cdot\|}$ by Lemma 1.3.16 (iv), so $0 < t_{-z} < 1/2 < t_z < 1$. Similarly, as $y \in (-x,x)_{\|\cdot\|}$, $0 < t_y < 1/2 < t_{-y} < 1$. As $-z \in (x,y)_{\|\cdot\|}$ and $t_{-z}, t_y \in (0,1/2)$,

$$0 < t_{-z} < t_y < 1/2 < t_z < t_{-y} < 1. \tag{1.3.4}$$

Thus, $(y,z)_{\|\cdot\|} \subseteq (-z,-y)_{\|\cdot\|}$ and so $U = (y,z)_{\|\cdot\|}$. Finally, (1.3.2) follows via (1.3.4). □

1.4 Basic properties of planar norms

We first recall the notion of a strictly convex norm.

Definition 1.4.1. Let $(X, \|\cdot\|)$ be a normed space. We say that $\|\cdot\|$ is a *strictly convex* norm if $\|tx + (1-t)y\| < 1$ for all distinct $x, y \in \partial B_1^{\|\cdot\|}(0)$ and $t \in (0,1)$.

The following result is canonically referred to as the Monotonicity Lemma and is attributed to many different authors, including Grünbaum [13]. However, following [26, Proposition 31], we apply this result in the particular case of unit vectors, and include a 'reverse' application.

Lemma 1.4.2. [Monotonicity Lemma] Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$, $y \in (x,-x]_{\|\cdot\|}$ and $w = (y-x)/\|y-x\|$. If $z \in (x,y)_{\|\cdot\|}$, then:

(i) $\|x - z\| \le \|x - y\|$, with equality if and only if $\partial B_1^{\|\cdot\|}(0)$ contains the straight line segment $[w,z]$;

(ii) $\|y-z\| \leq \|y-x\|$, with equality if and only if $\partial B_1^{\|\cdot\|}(0)$ contains the straight line segment $[-w, z]$.

Proof. We only include a proof of the second statement. Consider first when $y \in (x, -x)_{\|\cdot\|}$, i.e. $y \neq -x$. Let L_1 and L_2 denote the segments given by tz and $tx + (1-t)y$, $t \in [0,1]$, respectively. First note that L_1 and L_2 are not parallel since $y \in (x, -x)_{\|\cdot\|}$ and $z \in (x, y)_{\|\cdot\|}$, so $z \neq (x-y)/\|x-y\|$. Moreover, $L_1 \cap L_2 \neq \emptyset$. Let $p \in \overline{B}_1^{\|\cdot\|}(0)$ be such that $p \in L_1 \cap L_2$.

Now, as $0, z, p \in L_1$ and $x, p, y \in L_2$, note that

$$\begin{aligned} \|z\| + \|y-x\| &= (\|z-p\| + \|p\|) + (\|y-p\| + \|p-x\|) \\ &= (\|p\| + \|x-p\|) + (\|p-y\| + \|z-p\|) \geq \|x\| + \|y-z\|. \end{aligned}$$

Since $\|z\| = \|x\|$, the inequality $\|y-x\| \geq \|y-z\|$ follows.

Let us consider the case of equality. Indeed, by [26, Proposition 1], equality occurs if and only if both the straight line segments

$$\left[\frac{x-p}{\|x-p\|}, \frac{p}{\|p\|}\right], \left[\frac{p-y}{\|p-y\|}, \frac{z-p}{\|z-p\|}\right] \subseteq \partial B_1^{\|\cdot\|}(0).$$

However note that $(x-p)/\|x-p\| = (p-y)/\|p-y\| = (x-y)/\|x-y\|$ and $p/\|p\| = (z-p)/\|z-p\| = z$. Thus, the claim for equality follows.

We now consider the case when $y = -x$. Note in this case that $-w = x/\|x\| = x$. In the case $y = -x$, the inequality $\|y-z\| \leq \|y-x\|$ follows naturally since $\|y-x\| = 2$. Suppose now that $\|y-z\| = \|y-x\| = 2$. Fix an arbitrary $s \in (x, z)_{\|\cdot\|}$. Then, by the previous case as $y \neq -s$, $\|y-s\| \geq \|y-z\| = 2$. Hence, $\|y-s\| = 2$ since $s \in \partial B_1^{\|\cdot\|}(0)$. Therefore, $\|y-s\| = \|y-z\|$ and so the straight line segment $[(s-y)/\|s-y\|, z] \subseteq \partial B_1^{\|\cdot\|}(0)$, for each $s \in (x, z)_{\|\cdot\|}$. Since z is fixed and $\partial B_1^{\|\cdot\|}(0)$ is closed, by letting $s \to x$ we get that $\partial B_1^{\|\cdot\|}(0)$ contains the straight line segment $[-w, z]$.

Suppose now that the straight line segment $[-w, z] = [x, z] \subseteq \partial B_1^{\|\cdot\|}(0)$. Consider now the vector $s := x + (z-x)/2 = (x+z)/2$. Observe that $s \in [x, z] \subseteq \partial B_1^{\|\cdot\|}(0)$. Therefore, as $y = -x$,

$$2 = \|s - (-s)\| = \|x + z\| = \|z - y\|.$$

Hence $\|z-y\|=2$, so the equality $\|y-z\|=\|y-x\|$ holds. $\square$

The following is an application of the Monotonicity Lemma 1.4.2 to four distinct unit vectors.

Lemma 1.4.3. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $w \in (x,-x]_{\|\cdot\|}$. If $y, z \in (x,w)_{\|\cdot\|}$ with $y \in (x,z)_{\|\cdot\|}$, then $\|x-w\| \geq \|y-z\|$. Further, if $\|x-w\| = \|y-z\|$, then $\partial B_1^{\|\cdot\|}(0)$ contains the straight line segments $[x,y], [z,w]$.

Proof. By the Monotonicity Lemma 1.4.2, $\|x-w\| \geq \|x-z\|$. Further, as $z \in (x,-x)_{\|\cdot\|}$, note $x \in (-z,z)_{\|\cdot\|}$ by Lemma 1.3.16 (i). Since $y \in (x,z)_{\|\cdot\|}$, the Monotonicity Lemma 1.4.2 implies $\|z-y\| \leq \|z-x\|$.

Suppose now $\|x-w\| = \|y-z\|$. Then, by the Monotonicity Lemma 1.4.2, $\|z-\eta\| = \|z-x\|$ for every $\eta \in [x,y)_{\|\cdot\|}$ and so $[p,\eta] \subseteq \partial B_1^{\|\cdot\|}(0)$ where $p = (x-z)/\|x-z\|$. Therefore, as p is fixed and $\partial B_1^{\|\cdot\|}(0)$ is closed, $[x,y] \subseteq \partial B_1^{\|\cdot\|}(0)$.

Finally, to see that $[z,w] \subseteq \partial B_1^{\|\cdot\|}(0)$, note that by the Monotonicity Lemma 1.4.2, $\|y-w\| = \|y-\eta\|$ for every $\eta \in (z,w]_{\|\cdot\|}$ and so we can apply the same methodology as above. $\square$

Another immediate consequence of the Monotonicity Lemma 1.4.2 and the continuity of f_z, as defined in Notation 1.3.13, is the following property of strictly convex norms.

Corollary 1.4.4. Let $\|\cdot\|$ be a strictly convex norm on $\mathbb{C}$, $z \in \partial B_1^{\|\cdot\|}(0)$ and $s \in (0,2]$. If:

(i) $s \in (0,2)$, then there exist a unique $w_1 \in (z,-z)_{\|\cdot\|}$ and a unique $w_2 \in (-z,z)_{\|\cdot\|}$ such that $\|z-w_1\| = \|z-w_2\| = s$;

(ii) $s = 2$, then $\|z-w\| = s = 2$ for some $w \in \partial B_1^{\|\cdot\|}(0)$ if and only if $w = -z$.

Recall Notation 1.3.13 (i).

Corollary 1.4.5. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $T_x \in [0,1]$ be such that $f_x(T_x) = 2$. Then f_x is continuous on $[0,1]$ and is increasing on $[0,T_x]$, but is decreasing on $[T_x,1]$.

Further, if $\|\cdot\|$ is strictly convex then $T_x = 1/2$, f_x is strictly increasing on $[0,1/2]$ and strictly decreasing on $[1/2,1]$. Moreover, $g_x := f_x\big|_{[0,1/2]} : [0,1/2] \to [0,2]$ is a homeomorphism.

Proof. The continuity of f_x follows by the continuity of θ_x in Lemma 1.3.11. The monotonicity of f_x on $[0,T_x]$ and $[T_x,1]$ then follow via the Monotonicity Lemma 1.4.2.

Suppose now $\|\cdot\|$ is strictly convex. Then by Notations 1.3.10, 1.3.13 and Corollary 1.4.4, note $T_x = 1/2$ and f_x is strictly increasing/decreasing on $[0,1/2]/[1/2,1]$.

To see that g_x is a homeomorphism, note g_x is surjective as g_x is continuous, $g_x(0) = 0$ and $g_x(1/2) = 2$. Further, g_x is injective as it is strictly increasing. Therefore g_x is bijective. Finally note g_x^{-1} is continuous since every strictly increasing, surjective mapping between non-degenerate intervals is continuous. $\square$

Example 1.4.6. If $\|\cdot\|$ is not a strictly convex norm on $\mathbb{C}$, then g_x is not necessarily strictly increasing. Indeed, let $N : \mathbb{C} \to [0,+\infty)$ be defined by

$$N(s+it) = \begin{cases} |t|, & \text{if } 2|t| > |s|; \\ \frac{1}{3}\left(|s|+|t|\right), & \text{if } |s| \geq 2|t| \text{ and } s \neq 0; \\ 0, & \text{if } s = t = 0. \end{cases}$$

Now let $x = 3$, $y = 2+i$ and $z = 1+i$. Note $N(x) = N(y) = N(z) = 1$ and $y, z \in (x,-x)_N$. Further, $x - y = 1 - i$ and $x - z = 2 - i$. So, $N(x-y) = N(x-z) = 1$. Thus, as $y \in (x,z)_N$, f_x is not strictly increasing.

Corollary 1.4.7. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $x, y, z \in \partial B_1^{\|\cdot\|}(0)$ be such that $a \leq b$ where $a := \|y-x\|$ and $b := \|z-x\|$. Then, for each $c \in [a,b]$ there exists $w \in \Gamma$ such that $\|w-x\| = c$, where $\Gamma = [z,y]_{\|\cdot\|}$ if $x \in [y,z]_{\|\cdot\|}$ and $\Gamma = [y,z]_{\|\cdot\|}$ if $x \in [z,y]_{\|\cdot\|}$.

The following result concerns unit vectors which have maximal distance from a fixed unit vector.

Lemma 1.4.8. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $z \in \partial B_1^{\|\cdot\|}(0)$. Then $f_z^{-1}(2)$ is a non-empty, closed interval contained in $[0,1]$.

Proof. Note, by Lemma 1.3.11, $f_z^{-1}(2) \neq \emptyset$ as $f_z(1/2) = \|\theta_z(1/2) - z\| = \|-z-z\| = 2\|z\| = 2$. Observe $f_z^{-1}(2) \subseteq [0,1]$ is closed, hence compact, by the continuity of f_z. Let $s_1 := \inf\left(f_z^{-1}(2)\right)$ and $s_2 := \sup\left(f_z^{-1}(2)\right)$. Then $f_z^{-1}(2) \subseteq [s_1,s_2] \subseteq [0,1]$.

Fix $t_0 \in [s_1,s_2]$ and consider the following two cases. Suppose that $\theta_z(t_0) \in [\theta_z(s_1), -z]_{\|\cdot\|}$. Then, by the Monotonicity Lemma 1.4.2,

$$2 = \|\theta_z(s_1) - z\| \leq \|\theta_z(t_0) - z\| \leq 2.$$

Hence $f_z(t_0) = 2$ and so $t_0 \in f_z^{-1}(2)$.

Suppose now that $\theta_z(t_0) \in [-z, \theta_z(s_2)]$. Then, by the Monotonicity Lemma 1.4.2.

$$2 = \|\theta_z(s_2) - z\| \leq \|\theta_z(t_0) - z\| \leq 2.$$

Hence $f_z(t_0) = 2$ and so $t_0 \in f_z^{-1}(2)$. Therefore $f_z^{-1}(2) \supseteq [s_1, s_2]$ and thus $f_z^{-1}(s) = [s_1, s_2]$. □

1.5 Basic properties of polygonal norms

This section quotes some results from [25], which we then extend to proceed with our investigation into centred Lipschitz quotient mappings in polygonal norms in Chapter 3. We first introduce the definition of a polygonal norm.

Definition 1.5.1. For each even integer $m \geq 4$, any norm in $\mathbb{C}$ whose unit ball centred at the origin is a regular m-gon is called a *polygonal m-norm.* A *polygonal m-norm* whose unit ball has a vertex at $z = 1$ will be denoted by $\| \cdot \|_m$.

Throughout this section, and the rest of this thesis, when considering the polygonal m-norm, we refer to the arc $(x, y)_{\|\cdot\|_m}$ by $(x, y)_m$, the ball $B_1^{\|\cdot\|_m}$ and the sphere $\partial B_1^{\|\cdot\|_m}(0)$ by $B_1^m(0)$ and $\partial B_1^m(0)$, respectively. Moreover, for any locally injective parametrisation ϕ we denote its length by $\text{length}_m(\phi)$ instead of the usual $\text{length}_{\|\cdot\|_m}(\phi)$.

First we introduce some notation that will be used throughout the rest of this thesis.

Notation 1.5.2. Let $m \geq 4$ be an even integer and $v_1, v_2, \ldots, v_m \in \partial B_1^m(0)$ denote the vertices of the unit sphere of $\| \cdot \|_m$ ordered in the anticlockwise direction, where $v_1 = 1$; that is, $v_j = \cos(2(j-1)\pi/m) + i\sin(2(j-1)\pi/m)$ for each $j \in [m]$. Let $\mathscr{L}_m = m\|v_2 - v_1\|_m$ denote the $\mathcal{H}^1_{\|\cdot\|_m}$-length of the unit sphere $\partial B_1^m(0)$. In what follows, we shall consider all indices modulo m.

For each $j \in [m]$ and integer $N \geq 1$, let $v_{j,k} = v_j + \frac{k}{N}(v_{j+1} - v_j)$, where $k \in [N] \cup \{0\}$. For any $r \geq 0$, let $\mathcal{D}_j^r = [r, +\infty)v_j$ and $\mathbb{D}^r = \bigcup_{j=0}^{n-1} \mathcal{D}_j^r$; for brevity, we write $\mathbb{D}$ instead of $\mathbb{D}^0$. Let $\mathcal{D}_j = \mathbb{R}_+ v_j$ and let $\mathcal{U}_j$ denote the interior of the convex hull of the set $\mathcal{D}_j \cup \mathcal{D}_{j+1}$.

We now recall a few results from [25] which are foundational to the results presented in Section 3.3.

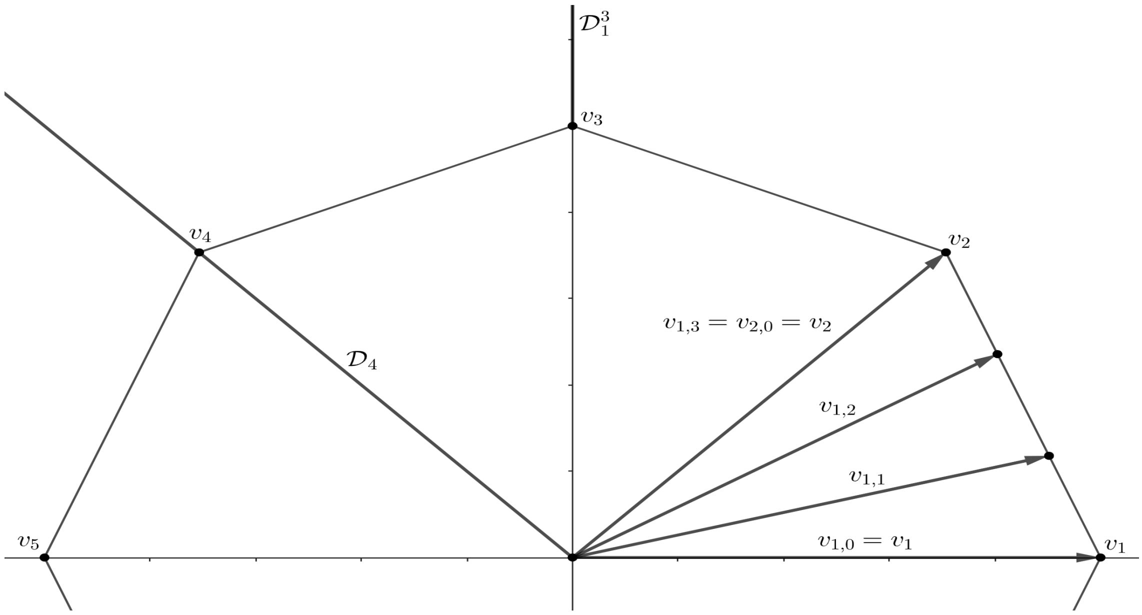

Figure 1.2: Example of Notation 1.5.2 when $m = 8$ and $N = 3$.

Lemma 1.5.3. ([25, Lemma 3.6 (1)]) Let $m \geq 4$ be a multiple of 4 and $r > 0$. Then, $r\|v_{k+1} - v_k\|_m = 2r\tan(\pi/m)$ for each $k \in [m]$.

Lemma 1.5.4. ([25, Lemma 5.1]) Suppose $m \geq 4$ is a multiple of 4, $r > 0$ and $0 < a < r\mathscr{L}_m/m$. Let $P_1 \in [rv_k, rv_{k+1}]$ and $P_2 \in [rv_{k+1}, rv_{k+2}]$ be such that $\|P_1 - rv_{k+1}\|_m = \|P_2 - rv_{k+1}\|_m = a$ for some $k \in [m]$. Then, $\|P_1 - P_2\|_m = 2a\cos^2(\pi/m)$.

Lemma 1.5.5. ([25, Lemma 5.5]) Let $m \geq 4$ be a multiple of 4, $u > 0$, $p \in \mathcal{D}_1$ and $q \in \mathcal{D}_2$. Consider $a_0 = u\cos(2\pi/m)$, $a_1 = u\sec(2\pi/m)$ and $a \in [a_0, a_1]$. Then:

(i) av_2 belongs to the $\left(\frac{m}{4}+1\right)$th side of $D := \partial B^m_{\|uv_1 - av_2\|_m}(uv_1)$, that is the edge of D which is a translation and scaling of the straight line segment $[v_{1+m/4}, v_{2+m/4}]$;

(ii) $\|uv_1 - q\|_m \geq \|uv_1 - av_2\|_m$ if $\|q\|_m \geq a$.

Proposition 1.5.6. ([25, Corollary 5.6]) Let $m \geq 4$ be a multiple of 4 and $r > 0$. If $p \in \mathcal{D}_k^r$ and $q \in \mathcal{D}_{k+1}^r$ for some $k \in [m]$, then $\|p - q\|_m \geq r\mathscr{L}_m/m$.

In a similar manner, we have the following lemma.

Lemma 1.5.7. Let $m \geq 4$ be a multiple of 4. If $\rho \geq R\sec(2\pi/m) > 0$, then we have $\|Rv_k - \rho v_{k+1}\|_m \geq R\tan(2\pi/m)$ for each $k \in [m]$.

Proof. Recalling Lemma 1.5.5 (ii), consider $u = R$, $a = a_1 = R\sec(2\pi/m)$ and $q = \rho v_2$. Since $\rho \geq R\sec(2\pi/m)$, note $\rho \geq u\sec(2\pi/m) = a_1 = a$ and so $\|q\|_m = \|\rho v_2\|_m \geq a$. Therefore, by

Lemma 1.5.5 (ii),

$$\|Rv_1 - \rho v_2\|_m = \|uv_1 - q\|_m \geq \|uv_1 - av_2\|_m = \|Rv_1 - a_1 v_2\|_m.$$

As $a_1 = R\sec(2\pi/m)$, we note $[Rv_1, a_1v_2]$ is a vertical segment, and thus $\|Rv_1 - a_1v_2\|_m = |Rv_1 - a_1v_2|$ since m is divisible by 4. Finally, by considering the right-angled triangle with vertices $0, a_1v_2, Re_1$, one can observe that $|a_1v_2 - Rv_1| = R\tan(2\pi/m)$. □

Before we proceed with recalling the final result from [25], we need to introduce some further notation.

Notation 1.5.8. For $j \in [m]$ and $k \in [N-1] \cup \{0\}$, let $w_{(j-1)N+k} := v_{j,k}$. Moreover, denote the angle between $\mathbb{R}_+ w_l$ and $\mathbb{R}_+ w_{l+1}$ by α_l.

Lemma 1.5.9. ([25, Lemma 5.3, Lemma 5.4]) Let $m \geq 4$ be a multiple of 4 and $N \geq 2$ be an integer. Suppose $r > 0$ and let $s = s(r) > 0$ be such that sw_2 is the intersection between the vertical line through rv_1 and the line $\mathbb{R}_+ w_2$. Then:

i) $s = r\left(1 + \tan(\pi/m)\tan\alpha_0\right)$;

ii) $\|rv_1 - sw_2\|_m = r\tan\alpha_0$ for each $k \in [m]$;

iii) $\tan\alpha_0 = \dfrac{2\tan(\pi/m)}{N + (N-2)\tan^2(\pi/m)}$.

The next lemma allows one to determine the polygonal distance between two points on adjacent rays, and this can be seen to be analogous to the case of $\|\cdot\| = \|\cdot\|_4$; the only caveat is we require some constraints on how far the norms of these two points may vary.

Lemma 1.5.10. Let $m \geq 8$ be a multiple of 4 and $r_0, r_1 > 0$ be such that $\max(r_0, r_1) \leq \min(r_0, r_1)\sec(2\pi/m)$. Then,

$$\|r_1 v_{k+1} - r_0 v_k\|_m = (r_0 + r_1)\tan\left(\frac{\pi}{m}\right) \qquad \text{for each } k \in [m].$$

Proof. Let $d_k := \|r_1 v_{k+1} - r_0 v_k\|_m$. Since rotating $\partial B_d^m(r_0 v_k)$ by an integer multiple of $2\pi/m$ maps it to $\partial B_d^m(r_0 v_1)$, we may assume without loss of generality that $k = 1$. Let $d = d_1$. Now, as $\partial B_1^m(0)$ is symmetric with respect to the ray starting at the origin which contains $(v_1 + v_2)/2$, we may further assume that $r_0 \leq r_1$, that is, $0 < r_0 \leq r_1 \leq r_0\sec(2\pi/m)$.

Consider the sphere $S = \partial B_d^m(r_0 v_1)$ and note $r_1 v_2 \in S$. Let O denote the origin, $A = r_0 v_1$ and B denote the $\left(\frac{m}{4}+1\right)$th vertex of S. Observe by Lemma 1.5.5 that $C = r_1 v_2$ lies on the $\left(\frac{m}{4}+1\right)$th edge of S. Further, let D denote the intersection of the horizontal line through C and the segment $[A, B]$. Similarly, let E denote the intersection between the vertical line through C and the positive real axis; see Figure 1.3 below.

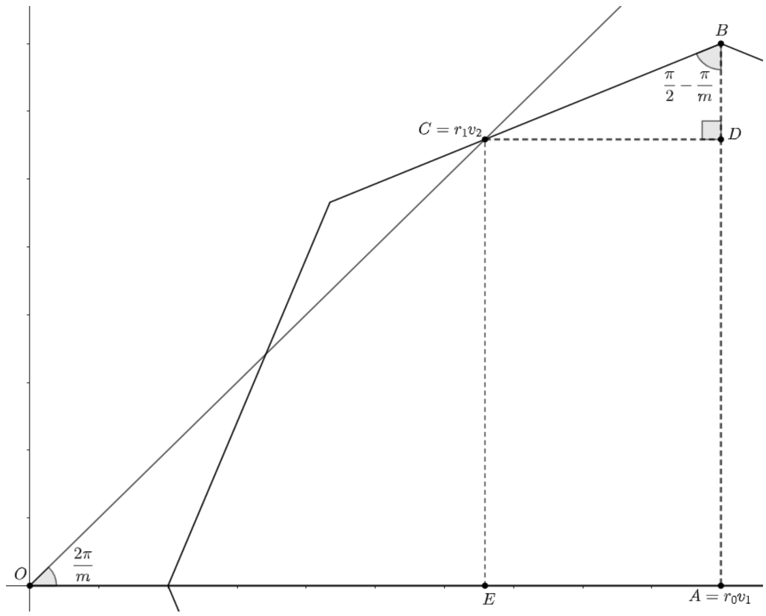

Figure 1.3: The construction used in the proof of Lemma 1.5.10.

Observe that as the acute angle between the segments $[O, C]$ and $[O, E]$ is $2\pi/m$,

$$|A - D| = |E - C| = |C - O| \sin\left(\frac{2\pi}{m}\right) = r_1 \sin\left(\frac{2\pi}{m}\right). \tag{1.5.1}$$

Similarly,

$$|C - D| = |E - A| = |A - O| - |E - O| = r_0 - r_1 \cos\left(\frac{2\pi}{m}\right). \tag{1.5.2}$$

Now, as the angle $\angle CBD = \pi/2 - \pi/m$, $\angle DCB = \pi/m$. Hence, by (1.5.2),

$$\begin{aligned} |B - D| = |C - D| \tan\left(\frac{\pi}{m}\right) &= r_0 \tan\left(\frac{\pi}{m}\right) - r_1 \cos\left(\frac{2\pi}{m}\right) \tan\left(\frac{\pi}{m}\right) \\ &= r_0 \tan\left(\frac{\pi}{m}\right) - r_1 \left(\sin\left(\frac{2\pi}{m}\right) - \tan\left(\frac{\pi}{m}\right)\right) \end{aligned} \tag{1.5.3}$$

Finally, as m is a multiple of 4, note $d = \|r_1 v_2 - r_0 v_1\|_m = |A - B|$. So combining (1.5.1) and

(1.5.3) we obtain

$$d = |A - B| = |A - D| + |B - D| = (r_0 + r_1)\tan\left(\frac{\pi}{m}\right).$$

□

We now provide an explicit formula for the polygonal m-norm of those vectors which have sufficiently small argument.

Lemma 1.5.11. Let $m \geq 4$ be an even integer, $r > 0$ and $z = t + is \in [rv_m, rv_1] \cup [rv_1, rv_2]$. Then,

$$r = \|z\|_m = t + |s|\tan\left(\frac{\pi}{m}\right).$$

Proof. Consider the right-angled triangle with vertices z, $z_1 = tv_1$ and $z_2 = rv_1$. Observe that $\angle z_1 z z_2 = \pi/m$ and so $r - t = |s|\tan(\pi/m)$. □

Remark 1.5.12. If $m \geq 4$ is an even integer, which is not divisible by 4, then for any $k \in [m]$ and $x, y \in [v_k, v_{k+1}]$, it follows that $\|x - y\|_m = |x - y|$ since $x - y$ is parallel to a segment connecting the origin to one of the vertices of $\partial B_1^m(0)$.

Similarly, if m is divisible by 4, then for any $k \in [m]$ and $x, y \in [v_k, v_{k+1}]$, it follows that $\|x - y\|_m = \sec(\pi/m)|x - y|$ since $x - y$ is parallel to an apothem of $\partial B_1^m(0)$ which has Euclidean length $\cos(\pi/m)$, see [25, Lemma 3.6(i)].

Hence, in any case, if in addition $x \in (y, v_{k+1})$ then $\|x - v_{k+1}\|_m < \|y - v_{k+1}\|_m$.

Provided $m \equiv 2 \pmod 4$, any two unit vectors on adjacent edges of the $\|\cdot\|_m$-unit sphere provide equality in the triangle inequality in the following manner.

Lemma 1.5.13. Let $m \geq 6$ be an even integer, which is not divisible by 4, and $x, y \in \partial B_1^m(0)$. If $x \in [v_k, v_{k+1}]$ and $y \in [v_{k+1}, v_{k+2}]$ for some $k \in [m]$, then

$$\|x - y\|_m = \|x - v_{k+1}\|_m + \|y - v_{k+1}\|_m.$$

Proof. First note if $x = y = v_{k+1}$, then this result is vacuously true. Hence, suppose that $x \in [v_k, v_{k+1})$ and $y \in (v_{k+1}, v_{k+2}]$. Since rotating the unit sphere an integer multiple of $2\pi/m$ defines an isometry, we may assume without loss of generality that $[v_k, v_{k+1}]$ denotes the

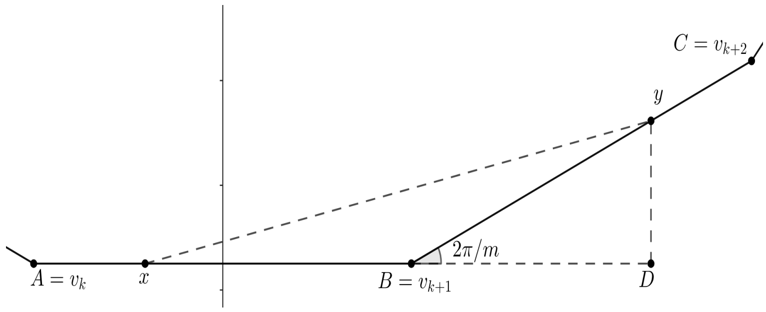

Figure 1.4: Construction in the proof of Lemma 1.5.13

edge of $\partial B_1^m(0)$ that is horizontal and below the real-axis. For ease of notation, let $A := v_k$, $B := v_{k+1}$ and $C := v_{k+2}$. Note $B = \sin(\pi/m) - \cos(\pi/m)i$. Define $\varepsilon := \|x - B\|_m$ and note, by Remark 1.5.12, $\varepsilon = |x - B|$. Hence,

$$x = \left(\sin\left(\frac{\pi}{m}\right) - \varepsilon\right) - \cos\left(\frac{\pi}{m}\right) i.$$

Consider now the vector y. Define D to be the intersection of the straight line containing $[A, B]$ and the vertical line through y; see Figure 1.4. Let $\xi := \|y - B\|_m$ and thus $\xi = |y - B|$. By considering the right-angled triangle with vertices B, D and y, note $\angle DBy = 2\pi/m$. Thus, $|B - D| = \xi\cos(2\pi/m)$ and $|y - D| = \xi\sin(2\pi/m)$. Therefore,

$$y = \left(\sin\left(\frac{\pi}{m}\right) + \xi\cos\left(\frac{2\pi}{m}\right)\right) + i\left(\xi\sin\left(\frac{2\pi}{m}\right) - \cos\left(\frac{\pi}{m}\right)\right).$$

Hence,

$$y - x = \left(\varepsilon + \xi\cos\left(\frac{2\pi}{m}\right)\right) + i\xi\sin\left(\frac{2\pi}{m}\right).$$

Note, as $0 \leq \angle Bxy \leq 2\pi/m$, $(y - x)/\|y - x\|_m \in [v_1, v_2]$. Thus, by Lemma 1.5.11,

$$\begin{aligned}\|y - x\|_m &= \left(\varepsilon + \xi\cos\left(\frac{2\pi}{m}\right)\right) + \xi\sin\left(\frac{2\pi}{m}\right)\tan\left(\frac{\pi}{m}\right) \\ &= \left(\varepsilon + \xi\left(1 - 2\sin^2\left(\frac{\pi}{m}\right)\right)\right) + 2\xi\sin^2\left(\frac{\pi}{m}\right) = \varepsilon + \xi = \|x - B\|_m + \|y - B\|_m.\end{aligned}$$

□

Unfortunately, this does not extend to the case when m is not divisible by four as can be seen in the following example.

Example 1.5.14. Consider $m \geq 4$ to be a multiple of 4. Let $x \in [v_1, v_2)$ and $y \in (v_2, v_3]$ be such that $|x - v_2| = |y - v_2|$. Then, as $x - y$ is parallel to a segment connecting the origin to a vertex of $\partial B_1^m(0)$ note $\|x - y\|_m = |x - y|$. Since $\angle v_1 v_2 v_3 = \pi - (2\pi/m)$, by Remark 1.5.12,

$$\|x - y\|_m = |x - y| = 2|x - v_2| \cos\left(\frac{\pi}{m}\right) = (\|x - v_2\|_m + \|y - v_2\|_m)\cos^2\left(\frac{\pi}{m}\right).$$

Fortunately, we are able to produce a formula similar to that provided in Lemma 1.5.13 for the cases when m is a multiple of four.

Lemma 1.5.15. Let $m \geq 4$ be a multiple of 4 and let $x, y \in \partial B_1^m(0)$. If $x \in [v_k, v_{k+1}]$ and $y \in [v_{k+1}, v_{k+2}]$ for some $k \in [m]$, then

$$\|x - y\|_m = \max(\|x - v_{k+1}\|_m, \|y - v_{k+1}\|_m) + \cos\left(\frac{2\pi}{m}\right)\min(\|x - v_{k+1}\|_m, \|y - v_{k+1}\|_m).$$

Proof. By rotating the unit sphere by an integer multiple of $2\pi/m$, we may assume without loss of generality that $k = 3m/4$. Let $\varepsilon := \|x - v_{k+1}\|_m$ and $\xi := \|y - v_{k+1}\|_m$. Note $x - v_{k+1}$ and $y - v_{k+1}$ are parallel to apothems of $\partial B_1^m(0)$ and thus, by [25, Lemma 3.6 (i)],

$$|x - v_{k+1}| = \varepsilon \cos\left(\frac{\pi}{m}\right) \quad \text{and } |y - v_{k+1}| = \xi \cos\left(\frac{\pi}{m}\right).$$

Therefore,

$$\begin{aligned} x &= v_{k+1} + \varepsilon \cos\left(\frac{\pi}{m}\right)\left(-\cos\left(\frac{\pi}{m}\right) + i \sin\left(\frac{\pi}{m}\right)\right), \\ y &= v_{k+1} + \xi \cos\left(\frac{\pi}{m}\right)\left(\cos\left(\frac{\pi}{m}\right) + i \sin\left(\frac{\pi}{m}\right)\right). \end{aligned}$$

Hence,

$$y - x = (\xi + \varepsilon)\cos^2\left(\frac{\pi}{m}\right) + (\xi - \varepsilon)\sin\left(\frac{\pi}{m}\right)\cos\left(\frac{\pi}{m}\right) i.$$

Observe that $(y - x)/\|y - x\|_m \in [v_m, v_2]_m$ and thus, by Lemma 1.5.11,

$$\begin{aligned} \|y - x\|_m &= (\xi + \varepsilon)\cos^2\left(\frac{\pi}{m}\right) + |\xi - \varepsilon| \sin\left(\frac{\pi}{m}\right)\cos\left(\frac{\pi}{m}\right)\tan\left(\frac{\pi}{m}\right) \\ &= (\xi + \varepsilon)\cos^2\left(\frac{\pi}{m}\right) + |\xi - \varepsilon|\sin^2\left(\frac{\pi}{m}\right) \\ &= \max(\xi, \varepsilon) + \min(\xi, \varepsilon)\cos\left(\frac{2\pi}{m}\right). \end{aligned}$$

□

Below we show, for $m \geq 6$, that $\|\cdot\|_m$ behaves similarly to a strictly convex norm, cf. Corollary 1.4.4.

Lemma 1.5.16. Let $m \geq 6$ be even and $z \in \partial B_1^m(0)$. Then for each $s \in \left[0, \frac{1}{m}\mathcal{H}^1\left(\partial B_1^m(0)\right)\right]$ there exists a unique $w_s \in [z, -z)_m$ such that $\|z - w_s\|_m = s$.

Proof. Without loss of generality, assume $z \in [v_1, v_2)$. Note as $m \geq 6$ that $\cos(2\pi/m) > 0$. Hence by Lemmas 1.5.13 and 1.5.15, note by denoting $\gamma_m = \cos(2\pi/m)$ if m is a multiple of 4, and $\gamma_m = 1$ otherwise, that

$$\|z - v_3\|_m = \|v_2 - v_3\|_m + \gamma_m \|v_2 - z\|_m > \|v_2 - v_3\|_m = \frac{1}{m}\mathcal{H}^1\left(\partial B_1^m(0)\right).$$

Hence, by the Monotonicity Lemma 1.4.2, it suffices to show $g_z : (z, v_3)_m \to [0, 2]$ given by $g_z(w) = \|z - w\|_m$ is strictly increasing, in the sense that if $p_1, p_2 \in (z, v_3)_m$ with $p_1 \in (z, p_2)_m$, then $g_z(p_1) < g_z(p_2)$.

Case 1: m is not divisible by 4. Assume first that $p_1, p_2 \in (z, v_2]$ with $p_1 \in (z, p_2)$. Then, by Remark 1.5.12, $g_z(p_1) < g_z(p_2)$. Now, if $p_1, p_2 \in [v_2, v_3]$ with $p_1 \in (v_2, p_2)$, then $\|v_2 - p_2\|_m > \|v_2 - p_1\|_m$. Hence, by Lemma 1.5.13,

$$g_z(p_2) = \|z - p_2\|_m = \|z - v_2\|_m + \|v_2 - p_2\|_m > \|z - v_2\|_m + \|v_2 - p_1\|_m = \|z - p\|_m = g_z(p_1).$$

Finally, suppose that $p_1 \in (z, v_2)$ and $p_2 \in [v_2, v_3]$. Then, by Remark 1.5.12 and the Monotonicity Lemma 1.4.2,

$$g_z(p_1) = \|z - p_1\|_m < \|z - v_2\|_m \leq g_z(p_2).$$

Therefore g_z is strictly increasing on $(z, v_3]_m$.

Case 2: m is divisible by 4. Assume first that $p_1, p_2 \in (z, v_2]$ with $p_1 \in (z, p_2)$. Then, by Remark 1.5.12, $g_z(p_1) < g_z(p_2)$. Now if $p_1 \in (z, v_2]$ and $p_2 \in (v_2, v_3]$, then by the collinearity of z, p_1, v_2, Lemma 1.5.15 and since both $z \neq v_2$ and $p_2 \neq v_2$,

$$g_z(p_1) \leq \|z - v_2\|_m \leq \max\left(\|z - v_2\|_m, \|p_2 - v_2\|_m\right)$$

$$< \max\left(\|z-v_2\|_m, \|p_2-v_2\|_m\right) + \cos\left(\frac{2\pi}{m}\right)\min\left(\|z-v_2\|_m, \|p_2-v_2\|_m\right)$$

$$= g_z(p_2),$$

as $\cos(2\pi/m)\min\left(\|z-v_2\|_m, \|p_2-v_2\|_m\right) > 0$.

Suppose now $p_1, p_2 \in (v_2, v_3]$ and $p_1 \in (v_2, p_2)$. Hence, $\|p_1 - v_2\|_m < \|p_2 - v_2\|_m$. If $\|p_2 - v_2\|_m \le \|z - v_2\|_m$, then $\|p_1 - v_2\|_m < \|z - v_2\|_m$. Therefore, by Lemma 1.5.15,

$$g_z(p_1) = \|z-v_2\|_m + \cos\left(\frac{2\pi}{m}\right)\|p_1-v_2\|_m < \|z-v_2\|_m + \cos\left(\frac{2\pi}{m}\right)\|p_2-v_2\|_m = g_z(p_2).$$

One can argue similarly if $\|z-v_2\|_m \le \|p_1-v_2\|_m$. Finally, if $\|p_1-v_2\|_m < \|z-v_2\|_m < \|p_2-v_2\|_m$, then

$$g_z(p_1) = \|z-v_2\|_m + \cos\left(\frac{2\pi}{m}\right)\|p_1-v_2\|_m < \|p_2-v_2\|_m + \cos\left(\frac{2\pi}{m}\right)\|z-v_2\|_m = g_z(p_2).$$

Thus, as in all cases $g_z(p_1) < g_z(p_2)$, we conclude g_z is strictly increasing on $(z, v_3)_m$. □

We now determine the exact constants of equivalence between a fixed polygonal norm and the standard Euclidean norm.

Lemma 1.5.17. Let $m \ge 4$ be even. Then, $|z| \le \|z\|_m \le \sec\left(\frac{\pi}{m}\right)|z|$ for each $z \in \mathbb{C}$.

Proof. The result follows trivially if $z = 0$. So suppose $z \in \mathbb{C} \setminus \{0\}$. If $\arg(z) = 2k\pi/m$ for some $k \in [m]$ then $\|z\|_m = |z|$ and so the result follows trivially. Since rotations by an integer multiple of $2\pi/m$ define an isometry in $\|\cdot\|_m$, we may assume without loss of generality that $\arg(z) \in (0, 2\pi/m)$.

Let $\phi_z := \arg(z)$, O be the origin, $A = \|z\|_m v_1$ and $B = z$. Consider the triangle with vertices O, A and B. Note that $\angle OAB = \frac{\pi}{2} - \frac{\pi}{m}$ and so $\angle OBA = \frac{\pi}{2} + \frac{\pi}{m} - \phi_z$. Hence,

$$\begin{aligned}\|z\|_m = |A-O| = \frac{\sin\left(\frac{\pi}{2}+\frac{\pi}{m}-\phi_z\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{m}\right)}|O-B| &= \frac{\cos\left(\frac{\pi}{m}-\phi_z\right)}{\cos\left(\frac{\pi}{m}\right)}|z| \\ &= \left(\cos(\phi_z) + \tan\left(\frac{\pi}{m}\right)\sin(\phi_z)\right)|z|. \qquad (1.5.4)\end{aligned}$$

Let $f : [0, 2\pi/m] \to \mathbb{R}$ be given by $f(\phi) = \cos(\phi) + \tan\left(\frac{\pi}{m}\right)\sin(\phi)$. Note, as $\cos(\phi) \neq 0$ and $m \ge$

4, $f'(\phi) = 0$ if and only if $\phi = \pi/m$. Now, $f(0) = 1 = f(2\pi/m)$ and $f(\pi/m) = \sec(\pi/m) \geq 1$. Therefore, $1 \leq f(\phi) \leq \sec(\pi/m)$ for all $\phi \in [0, 2\pi/m]$. Thus, as $\phi_z \in (0, 2\pi/m)$, the claim of the lemma then follows by (1.5.4). $\square$

CHAPTER 2

THE STRUCTURAL DECOMPOSITION OF PLANAR LIPSCHITZ QUOTIENT MAPPINGS

In this section we focus on converse statements to the groundbreaking result of [17] concerning the structure of planar Lipschitz quotient mappings; namely Theorem 1.1.4. We mainly focus on the question of whether for a fixed non-constant complex polynomial in one variable P does there exist a planar homeomorphism h such that $P \circ h$ is a Lipschitz quotient mapping.

The research presented in this chapter is joint work with O. Maleva. The present author contributed to all results. The work has been accepted for publication, see [15].

2.1 Introduction

This chapter focuses on converses to Theorem 1.1.4, restated below.

Theorem 1.1.4. Let $f : \mathbb{C} \to \mathbb{C}$ be a Lipschitz quotient mapping. Then $f = P \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and P is a complex polynomial of one complex variable.

The authors of [17] pose questions regarding the uniqueness of the homeomorphism h obtained from the decomposition of a planar Lipschitz quotient mapping and whether a converse statement to Theorem 1.1.4 holds also. It is shown that, up to a linear transformation, the homeomorphism obtained via the decomposition of a Lipschitz quotient mapping is unique, see [17, p. 22].

In connection to the structural decomposition of planar Lipschitz quotient mappings, we may ask the following questions concerning converse statements to Theorem 1.1.4.

Question 2.1.1. (a) Can every planar homeomorphism $h : \mathbb{C} \to \mathbb{C}$ be obtained via a decomposition of a Lipschitz quotient mapping? In other words, is it true that for every homeomorphism $h : \mathbb{C} \to \mathbb{C}$ there exists a non-constant complex polynomial P such that $P \circ h$ is a Lipschitz quotient mapping?

(b) Can every non-constant complex polynomial P be obtained via a decomposition of a planar Lipschitz quotient mapping? In other words, is it true that for every non-constant polynomial P there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $P \circ h$ is a Lipschitz quotient mapping?

We begin by considering Question 2.1.1 (a). We provide a planar homeomorphism h such that $P \circ h$ is not Lipschitz quotient for every non-constant complex polynomial P. Indeed, consider the homeomorphism $h : \mathbb{C} \to \mathbb{C}$ given by $h(z) = |z|^2 e^{i \arg(z)}$. Observe that $P \circ h$ is not Lipschitz for every non-constant complex polynomial P. This follows simply as

$$\lim_{R \to +\infty} \frac{|P \circ h(R) - P \circ h(0)|}{R} = +\infty.$$

The main motivation of this chapter is to consider Question 2.1.1 (b), as the authors of [17] do. The authors claim to answer this in [17, Proposition 2.9] in the positive, and provide a sketch proof of the following statement.

Theorem 2.1.2. Let P be a non-constant polynomial in one complex variable with complex coefficients. Then there exists a homeomorphism h of the plane such that $f = P \circ h$ is a Lipschitz quotient mapping.

However, as we show in Section 2.3, the construction of their mapping h is not in fact a homeomorphism of the plane. In this chapter we prove Theorem 2.1.2. To do so we follow the framework provided in [17] but correct oversights in the original sketch. We heavily rely on the new notion of strongly co-Lipschitz defined in Definition 1.2.24. Moreover, with this notion of strongly co-Lipschitz, we consider the following question.

Question 2.1.3. For a fixed homeomorphism $h : \mathbb{C} \to \mathbb{C}$ does there exist a non-constant complex polynomial P such that $P \circ h$ is not a Lipschitz quotient mapping?

We answer Question 2.1.3 in the positive in Theorem 2.2.1.

2.2 Preliminaries

We begin by answering Question 2.1.3 in the positive. Formally, we prove the following.

Theorem 2.2.1. Let $h : \mathbb{C} \to \mathbb{C}$ be a homeomorphism. Then there exists a complex polynomial P in one complex variable such that $P \circ h$ is not Lipschitz quotient.

Naturally Theorem 2.2.1 is a consequence that squaring planar Lipschitz quotient mappings never produces a Lipschitz mapping. We prove this in the following.

Lemma 2.2.2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a Lipschitz quotient mapping. Then $g(z) = (f(z))^2$ is not Lipschitz.

Proof. Suppose f is c_f-co-Lipschitz and L_f-Lipschitz and, for a contradiction, suppose g is L_g-Lipschitz. Let us assume, without loss of generality, that g/f are Lipschitz/ Lipschitz quotient with respect to the Euclidean norm. Now Lemma 1.2.10 provides the existence of a positive constant R such that

$$|f(x)| \geq c_f (|x| - M) \qquad \text{whenever } |x| > R. \tag{2.2.1}$$

Here $M := \max\{|z| : f(z) = 0\}$; note M exists since $\operatorname{card}(f^{-1}(0))$ is at most the degree of the polynomial of P in the decomposition of f obtained via Theorem 1.1.4. Fix $z_0 \in \mathbb{C}$ such that $|z_0| > R + M + L_g/(2c_f^2)$. By Corollary 1.2.36, f is strongly c_f-co-Lipschitz at z_0 and so there exists $r_0 > 0$ such that whenever $w \in B_{r_0}(z_0)$, it follows that $|f(z_0) - f(w)| \geq c_f |z_0 - w|$. As g is L_g-Lipschitz,

$$c_f|z_0 - w| \cdot |f(z_0) + f(w)| \leq \left|(f(z_0))^2 - (f(w))^2\right| = |g(z_0) - g(w)| \leq L_g|z_0 - w|,$$

whenever $w \in B_{r_0}(z_0)$. Hence, for any $w \in B_{r_0}(z_0) \setminus \{z_0\}$, $|f(z_0) + f(w)| \leq L_g/c_f$. Thus, by the continuity of f, $|f(z_0)| \leq L_g/(2c_f)$. However, by our choice of z_0 and (2.2.1), $|f(z_0)| > L_g/(2c_f)$, providing contradiction and so g is not Lipschitz. □

The rest of this section now focuses on the preliminary results needed in the construction in Section 2.3. First we shall introduce some notation.

Notation 2.2.3. For any non-constant complex polynomial P in one complex variable and

$a > 0$ we define the closed set

$$V_a^P = \bigcup_{z_j \in S(P')} \overline{B}_a^{|\cdot|}(z_j), \tag{2.2.2}$$

where P' is the derivative of P and $S(P') = \{z \in \mathbb{C} : P'(z) = 0\}$.

We now state properties of particular functions which are important in the judicious choose of $r > 0$ which we are making in Claim 2.3.5. First, let P be a fixed non-constant complex polynomial of one complex variable, P' be its derivative and $z_j \in S(P')$. Of course if P is non-zero and linear, then $S(P') = \emptyset$. Define the polynomial

$$Q_j(z) := \frac{P(z) - P(z_j)}{(z - z_j)^{m_j}}, \tag{2.2.3}$$

where $m_j \geq 1$ is the multiplicity of z_j as a root of the polynomial $P(z) - P(z_j)$. Note, for future reference, that $P(z) = (z - z_j)^{m_j} Q_j(z) + P(z_j)$. Further, by the maximality of m_j,

$$Q_j(z_j) \neq 0. \tag{2.2.4}$$

We define the expansion of the polynomial Q_j about z_j by

$$Q_j(z) = \sum_{l=0}^{n-m_j} c_{l,j}(z - z_j)^l \tag{2.2.5}$$

where $n = \deg(P)$ and $c_{l,j} \in \mathbb{C}$. Thus (2.2.4) implies $c_{0,j} = Q_j(z_j) \neq 0$ for each j such that $z_j \in S(P')$.

We now define a function which proves useful in the construction of the Lipschitz quotient mapping in Section 2.3. For each $m \geq 1$, let $A_m \subseteq \mathbb{C} \times \mathbb{C}$ be defined by

$$A_m := \left\{(z, w) \in \mathbb{C} \times \mathbb{C} : |z| e^{im \arg(z)} \neq |w| e^{im \arg(w)}\right\} \cup \{(w, w) \in \mathbb{C} \times \mathbb{C} : w \in \mathbb{C} \setminus \{0\}\}.$$

Now, for each $m \geq 1$ and $l \in [m]$ we define the mapping $\Phi_{l,m} : A_m \to \mathbb{C}$ by

$$\Phi_{l,m}(z,w) = \begin{cases} \dfrac{|z|^{\frac{l+m}{m}} e^{i(l+m)\arg(z)} - |w|^{\frac{l+m}{m}} e^{i(l+m)\arg(w)}}{|z|e^{im\arg(z)} - |w|e^{im\arg(w)}}, & \text{if } z \neq w; \\ \\ \dfrac{l+m}{m}|w|^{\frac{l}{m}} e^{il\arg(w)}, & \text{if } z = w. \end{cases} \tag{2.2.6}$$

Lemma 2.2.4. Let $m \geq 1$ and $l \in [m]$. For each $w \in \mathbb{C} \setminus \{0\}$, there exists $\rho > 0$ such that $B_\rho(w) \times \{w\} \subseteq A_m$ and

$$\lim_{\substack{z \to w \\ z \in B_\rho(w)}} \Phi_{l,m}(z,w) = \Phi_{l,m}(w,w).$$

Proof. Note for $w \in \mathbb{C} \setminus \{0\}$ fixed that there exist finitely many points $z \in \mathbb{C}$ such that $(z,w) \notin A_m$; namely this happens exactly when $z \neq w$ but $|z| = |w|$ and $e^{im\arg(z)} = e^{im\arg(w)}$. Hence, there exists $\rho > 0$ such that $B_\rho(w) \times \{w\} \subseteq A_m$.

If $z \in B_\rho(w) \setminus \{w\}$, then $\Phi_{l,m}(z,w) = (g(f(z)) - g(f(w)))/(f(z) - f(w))$ where $f, g : \mathbb{C} \to \mathbb{C}$ are given by $f(z) = |z|e^{im\arg(z)}$ and $g(z) = z^{(l+m)/m}$. As w is fixed, f is continuous at w and g is differentiable at $f(w)$, observe that

$$\lim_{\substack{z \to w \\ z \in B_\rho(w)}} \Phi_{l,m}(z,w) = \lim_{\substack{z \to w \\ z \in B_\rho(w)}} \frac{g(f(z)) - g(f(w))}{f(z) - f(w)} = g'(f(w)) = \Phi_{l,m}(w,w).$$

□

Corollary 2.2.5. Let $m \geq 1$ and $l \in [m]$. For each $w \in \mathbb{C} \setminus \{0\}$ and $\varepsilon > 0$ there exists $\rho > 0$ such that $B_\rho(w) \times \{w\} \subseteq A_m$ and whenever $z \in B_\rho(w)$,

$$|\Phi_{l,m}(z,w)| < \varepsilon + |\Phi_{l,m}(w,w)|. \tag{2.2.7}$$

2.3 Construction of the Lipschitz quotient mapping

Recall the function $h : \mathbb{C} \to \mathbb{C}$ given in [17, Proposition 2.9] (for some large $R > 0$):

$$h(z) = \begin{cases} z, & \text{if } |z| \leq R, \\ \left(\dfrac{2R - |z|}{R}|z| + \dfrac{|z| - R}{R}|z|^{1/n} \right) e^{i\arg(z)}, & \text{if } R \leq |z| \leq 2R, \\ |z|^{1/n} e^{i\arg(z)}, & \text{if } |z| \geq 2R. \end{cases} \tag{2.3.1}$$

The authors of [17] claim first this is a homeomorphism from $\mathbb{C}$ to itself and go on to provide a sketch for a proof of Theorem 2.1.2. However it is clear that h is not injective whenever $n \geq 2$ by observing that for $R > 2^{1/(n-1)}$ the curve $\partial B_{2R}(0)$ is mapped under h inside the open ball $B_R(0)$ where the mapping remains fixed. Further, the authors introduce an amendment to the function h which may further provide points at which h is not injective. They describe how to change the function h defined by (2.3.1) on a finite collection of open balls. However they neglect the fact the prescribed radii of these balls are potentially very small and hence will require a 'scaling' to ensure the function is necessarily injective, as indicated by the $r^{1-(1/m_j)}$ term in (2.3.11). Finally, the authors state the co-Lipschitzness of the function h outside of the union of these balls, but do not verify the co-Lipschitzness on their boundaries, which is intricate.

Below we give a correct construction, for a fixed polynomial P, of a homeomorphism h of the plane to itself such that $P \circ h$ is a Lipschitz quotient mapping. The proof of Theorem 2.1.2 will be split into many claims, which verify the pointwise co- and Lipschitz property of the required functions, and remarks, which utilise earlier lemmata to conclude co- and Lipschitzness on specific regions. To highlight the end of the proof of a claim we use the symbol $\diamondsuit$, whereas the end of the proof of the proposition is highlighted by the usual $\square$.

Proof of Theorem 2.1.2. Fix $n \in \mathbb{N}$. We may assume without loss of generality that P is a monic polynomial of degree n. Indeed if P is not monic, let $a \neq 0$ denote the leading coefficient of P. One can apply the present Proposition to the monic polynomial $Q := P/a$ to find the homeomorphism h such that $f(z) = (Q \circ h)(z)$ is a Lipschitz quotient mapping. Then $(P \circ h)(z) = af(z)$ is a Lipschitz quotient mapping.

Therefore, assume $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. If $n = 1$ define $h(z) := z$ and then $f(z) = (P \circ h)(z) = z + a_0$ is 1-co-Lipschitz and 1-Lipschitz.

Suppose $n \geq 2$. The structure of the proof is as follows: we begin by defining a homeomorphism h_1 of the plane, let $F_1 = P \circ h_1$ and show that F_1 is Lipschitz on $\mathbb{C}$ and pointwise co-Lipschitz on $\mathbb{C}$ with the exception of a small neighbourhood W of finitely many points. Namely, W contains a neighbourhood of the set of roots of the polynomial P', the derivative of P. We use this to show F_1 is strongly co-Lipschitz at each $z \in \mathbb{C} \setminus V$, where $V \supseteq W$. We then proceed by defining an amended homeomorphism h_2 which coincides with h_1 everywhere outside of V, define the new function $F_2 = P \circ h_2$ and prove F_2 is pointwise co- and Lipschitz at the remaining points. Let us introduce some notation which will be important in the construction.

Notation 2.3.1. If $a_k \neq 0$ let $D_k = D\left(1/(2n|a_k|), k, n\right)$ be provided by Lemma 1.2.23, such that $g_{k,n}(z) = |z|^{k/n} e^{ik \arg(z)}$ is $1/(2n|a_k|)$-Lipschitz on $\mathbb{R}^2 \setminus B_{D_k}(0)$; otherwise if $a_k = 0$, let $D_k = 0$.

Let $R > 1$ be such that

(a) the roots of the derivative P' lie inside the open ball of radius R centred at the origin;

(b) $R \geq \max\{D_k : 0 \leq k \leq n-1\}$.

Define $h_1 : \mathbb{C} \to \mathbb{C}$ by

$$h_1(z) = \phi(|z|) e^{i \arg(z)},$$

where

$$\phi(t) = \begin{cases} t^{1/n}, & \text{if } t \geq 2^n R^n; \\ \left(\dfrac{t - R}{2^n R^{n-1} - 1} + R\right), & \text{if } R \leq t \leq 2^n R^n; \\ t, & \text{if } 0 \leq t \leq R. \end{cases}$$

Since $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous, piecewise C^∞ strictly increasing function, h_1 is bijective and continuous. Further we note $h_1^{-1}(z) = \phi^{-1}(|z|) e^{i \arg(z)}$ which is continuous. Hence h_1 is indeed a homeomorphism of $\mathbb{C}$ to itself. Finally, let $U_j := B_{2^n R^n + j}(0)$ for $j = 1, 2$. Define $F_1 = P \circ h_1$.

Claim 2.3.2. F_1 is Lipschitz on U_2.

Proof. We first show that h_1 is Lipschitz on U_2. Note that h_1 is pointwise 1-Lipschitz at each $z_0 \in B_R(0)$, since if $r > 0$ if sufficiently small such that $B_r(z_0) \subseteq B_R(0)$, then $h_1\left(B_r\left(z_0\right)\right) = B_r\left(z_0\right) = B_r\left(h_1(z_0)\right)$.

To see that h_1 is pointwise Lipschitz at each $z_0 \in U_2 \setminus \overline{B}_{R/2}(0)$, first note that ϕ is Lipschitz on $[R/2, 2^n R^n + 2]$. Moreover observe that $e^{i\arg(z)} = z/|z|$ is Lipschitz on $\mathbb{C} \setminus B_{R/2}(0)$, as if $z, w \in \mathbb{C} \setminus B_{R/2}(0)$, then

$$\left|\frac{z}{|z|} - \frac{w}{|w|}\right| \leq \frac{1}{|z| \cdot |w|}\left(|w| \cdot |z - w| + |w| \cdot \Big||w| - |z|\Big|\right) \leq \frac{4}{R}|z - w|.$$

Thus, $h_1(z) = \phi(|z|)e^{i\arg(z)}$ is the product of two bounded Lipschitz functions on the bounded domain $A = \{z \in \mathbb{C} : R/2 \leq |z| \leq 2^n R^n + 2\}$. Therefore, $h_1\big|_A$ is L-Lipschitz for some $L > 0$. In particular, we conclude that h_1 is pointwise L-Lipschitz at each $z \in U_2 \setminus \overline{B}_{R/2}(0)$.

Therefore Lemma 1.2.17 implies h_1 is $\max(1, L)$-Lipschitz on the convex, open set U_2. Now, $F_1 = P \circ h_1$ is the composition of P, a polynomial, which is Lipschitz on the bounded set $h_1(U_2)$ and h_1, which is Lipschitz on U_2. Therefore, F_1 is Lipschitz on U_2. ♢

Claim 2.3.3. F_1 is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$.

Proof. To see F_1 is Lipschitz outside of $\overline{U_1}$ note for $z \notin \overline{U_1}$ that $F_1(z)$ takes the specific form

$$F_1(z) = a_0 + f_n(z) + \sum_{k=1}^{n-1} a_k g_{k,n}(z), \tag{2.3.2}$$

where f_n is defined as in Lemma 1.2.22 and $g_{k,n}$ as in Lemma 1.2.23 for each $k \in [n-1]$.

Hence, as f_n is n-Lipschitz on $\mathbb{C}$ by Lemma 1.2.22, to show F_1 is Lipschitz on $\mathbb{C}\setminus\overline{U_1}$ it suffices to show for each $k \in [n-1]$ that $a_k g_{k,n}$ is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$; this follows by Lemma 1.2.23 and the choice of R and D_k in Notation 2.3.1 (b). Hence F_1 is Lipschitz on $\mathbb{C} \setminus \overline{U_1}$. ♢

Remark 2.3.4. Recall by Claims 2.3.2, 2.3.3 that F_1 is Lipschitz on both $\mathbb{C} \setminus \overline{U_1}$ and U_2. Therefore Lemma 1.2.17 yields that there exists $L_1 > 0$ such that F_1 is L_1-Lipschitz on $\mathbb{C}$.

Claim 2.3.5. Recall (2.2.2)-(2.2.5) from Notation 2.2.3 and the choice of R from Notation 2.3.1. There exists $r \in (0, 1)$ such that:

(i) the balls $\overline{B}_{2r}(z_j)$ around roots $z_j \in S(P')$ of P', are pairwise disjoint;

(ii) $V_{2r}^P \subseteq B_R(0)$;

(iii) $r \leq \min_{j:z_j \in S(P')} \varepsilon_j^{m_j}$, where for each $z_j \in S(P')$ we define $\varepsilon_j > 0$ by

$$\varepsilon_j := \begin{cases} \dfrac{|Q_j(z_j)|}{2(1+n) \sum\limits_{k=1}^{n-m_j} |c_{k,j}|}, & \text{if } n > m_j \text{ and } \sum\limits_{k=1}^{n-m_j} |c_{k,j}| \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

(iv) $|Q_j(z_j)|/2 \leq |Q_j(y)| \leq 2|Q_j(z_j)|$ for each $y \in B_r(z_j)$ such that $z_j \in S(P')$.

Proof. Property (i) is easy to satisfy as there are only finitely many distinct roots in $S(P')$. Next, property (ii) is satisfied for sufficiently small $r > 0$ since $S(P') \subseteq B_R(0)$ and $B_R(0)$ is open. Property (iii) follows naturally by (2.2.4) since each ε_j is positive and there are only finitely many of these terms. Finally, it is possible to satisfy property (iv) since each polynomial Q_j is continuous on $\mathbb{C}$ and $Q_j(z_j) \neq 0$ by (2.2.4). $\diamondsuit$

For the rest of the proof of Theorem 2.1.2, we fix $r \in (0,1)$ provided by Claim 2.3.5. Recall (2.2.2), and define the closed sets W and V to be the following:

$$W = V_{r/2}^P, \quad V = V_r^P. \tag{2.3.3}$$

Claim 2.3.6. There exists $c_0 > 0$ such that F_1 is pointwise c_0-co-Lipschitz at each $z \in U_2 \setminus W$.

Proof. We first show that there exist positive constants L and ξ such that h_1 is pointwise $(1/L)$-co-Lipschitz at each $z \in U_2$ and the polynomial P is pointwise ξ-co-Lipschitz at each $z \in h_1(U_2 \setminus W)$. Then we appeal to Lemma 1.2.21 to conclude that F_1 is pointwise $c_0 := \left(\frac{\xi}{L}\right)$-co-Lipschitz at each $z \in U_2 \setminus W$.

By arguing similarly to the proof of Claim 2.3.2, namely as $h_1^{-1}(z) = \phi^{-1}(|z|)e^{i\arg(z)}$ is the product of two bounded Lipschitz functions, there exists $L > 0$ such that h_1^{-1} is pointwise L-Lipschitz at $h_1(z)$ for each $z \in U_2$. Thus Lemma 1.2.20 and the arbitrariness of $z \in U_2$ implies h_1 is pointwise $(1/L)$-co-Lipschitz at each $z \in U_2$.

Observe by Claim 2.3.5 (ii) that $S(P') \subseteq W \subseteq B_R(0)$. Therefore, as h_1 is the identity on $B_R(0)$ and since $|h_1(z)| \geq R$ for $|z| \geq R$, we conclude that $\overline{h_1(U_2 \setminus W)} \cap S(P') = \emptyset$. As

P' is a polynomial, hence continuous, $|P'|$ assumes its minimal value $2\xi > 0$ on the compact set $\overline{h_1(U_2 \setminus W)}$. In particular for each $z \in h_1(U_2 \setminus W)$ note $P'(z) \neq 0$ and thus, by [11, Theorem 7.5], there exist open neighbourhoods $N_{P(z)} \subseteq F_1(U_2 \setminus W)$ and $N_z \subseteq h_1(U_2 \setminus W)$ of $P(z)$ and z respectively such that $P : N_z \to N_{P(z)}$ is a continuous bijective open mapping, hence a homeomorphism. Further, $(P^{-1})'(P(z)) = 1/P'(z)$. Therefore for each $z \in h_1(U_2 \setminus W)$ it follows that $|(P^{-1})'(P(z))| \leq 1/(2\xi)$. Hence P^{-1} is pointwise $\frac{1}{\xi}$-Lipschitz at $P(z)$. By Lemma 1.2.20 and Remark 1.2.27 we hence conclude P is pointwise ξ-co-Lipschitz at z since $P : N_z \to N_{P(z)}$ is a homeomorphism, N_z and $N_{P(z)}$ are open subsets of $\mathbb{C}$ and $z \in N_z$. We conclude P is pointwise ξ-co-Lipschitz at each $z \in h_1(U_2 \setminus W)$.

Now h_1 is pointwise $\frac{1}{L}$-co-Lipschitz at each $z \in U_2 \setminus W$ and P is pointwise ξ-co-Lipschitz at each $h_1(z) \in h_1(U_2 \setminus W)$. Therefore by Lemma 1.2.21 we conclude F_1 is pointwise c_0-co-Lipschitz at each $z \in U_2 \setminus W$ where $c_0 = \xi/L > 0$. ♢

Remark 2.3.7. Since $(U_2 \setminus W) \cap h_1^{-1}(S(P')) = \emptyset$, by Proposition 1.2.15, F_1 is locally injective at each $z \in U_2 \setminus W$. Further, $U_2 \setminus W$ is open. Therefore Remark 1.2.27, Corollary 1.2.30 and Claim 2.3.6 imply F_1 is strongly c_0-co-Lipschitz at each $z \in U_2 \setminus W$. In particular, for each $z \in U_2 \setminus \mathrm{Int}(V)$ there exists $\rho = \rho(z) > 0$ such that $B_\rho(z) \subseteq U_2 \setminus W$ and

$$|F_1(z) - F_1(x)| \geq c_0 \, |z - x| \qquad \text{for all } x \in B_\rho(z). \tag{2.3.4}$$

Claim 2.3.8. F_1 is $\frac{1}{2}$-pointwise co-Lipschitz at each $z \in \mathbb{C} \setminus \overline{U_1}$.

Proof. Fix any $z_0 \in \mathbb{C} \setminus \overline{U_1}$. Recall $F_1 = P \circ h_1$ where P is a non-constant polynomial of one variable, so is an open map, and h_1 is a homeomorphism. Therefore F_1 is open. By Corollary 1.2.26 and Remark 1.2.27, as $\mathbb{C} \setminus \overline{U_1}$ is open, to check that F_1 is pointwise (1/2)-co-Lipschitz at z_0, it is enough to verify (ii) of Definition 1.2.24 is satisfied; that is, to show that there exists $\rho = \rho(z_0) > 0$ such that

$$|F_1(x) - F_1(z_0)| \geq \frac{|x - z_0|}{2} \quad \text{for each } x \in B_\rho(z_0). \tag{2.3.5}$$

Recall by Corollary 1.2.37 that f_n is strongly 1-co-Lipschitz at z_0. Hence there exists $\rho_1 =$

$\rho_1(z_0) > 0$ such that

$$|f_n(z_0) - f_n(x)| \geq |z_0 - x| \quad \text{for each } x \in B_{\rho_1}(z_0). \tag{2.3.6}$$

Choose $\rho = \rho(z_0) > 0$ sufficiently small such that $\rho < \rho_1$ and $B_\rho(z_0) \subseteq \mathbb{C} \setminus \overline{U_1}$. Let $x \in B_\rho(z_0)$ and put $s = |x - z_0| < \rho$. Recall (2.3.2), that is $F_1 = a_0 + f_n + \sum_{k=1}^{n-1} a_k g_{k,n}$, and so

$$\begin{aligned}
|F_1(x) - F_1(z_0)| &= \left| (f_n(z_0) - f_n(x)) + \sum_{k=1}^{n-1} a_k \left(g_{k,n}(z_0) - g_{k,n}(x)\right) \right| \\
&\geq |f_n(z_0) - f_n(x)| - \sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \qquad (2.3.7) \\
&\geq s - \sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \, , \qquad (2.3.8)
\end{aligned}$$

where the last inequality follows from (2.3.6). We show

$$\sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \leq \frac{s}{2}. \tag{2.3.9}$$

Combining (2.3.9) with (2.3.8) implies (2.3.5) which proves F_1 is pointwise $\frac{1}{2}$-co-Lipschitz at z_0 as claimed.

To see (2.3.9) recall Notation 2.3.1, in particular, recall (b). As $R \geq D_k$, by Lemma 1.2.23, $g_{k,n}$ is $1/(2n|a_k|)$-Lipschitz on $\mathbb{C} \setminus B_R(0)$ for those $k \in [n-1]$ where $a_k \neq 0$. Hence

$$\sum_{k=1}^{n-1} |a_k| \, |g_{k,n}(z_0) - g_{k,n}(x)| \leq \sum_{k=1}^{n-1} \frac{|z_0 - x|}{2n} = \sum_{k=1}^{n-1} \frac{s}{2n} \leq \frac{s}{2}.$$

$\diamondsuit$

Remark 2.3.9. Recall by Claim 2.3.6 that F_1 is pointwise c_0-co-Lipschitz at each $z \in U_2 \setminus W$ and by Claim 2.3.8 that F_1 is pointwise $(1/2)$-co-Lipschitz at each $z \in \mathbb{C} \setminus \overline{U_1}$. Therefore defining $c_1 := \min\left\{c_0, \frac{1}{2}\right\}$ we conclude $c_1 > 0$ and

$$F_1 \text{ is pointwise } c_1\text{-co-Lipschitz at each } z \in \mathbb{C} \setminus W. \tag{2.3.10}$$

We continue by defining the amended homeomorphism $h_2 : \mathbb{C} \to \mathbb{C}$, which coincides with

h_1 on $\mathbb{C} \setminus V$, and prove the pointwise co- and Lipschitz properties of the amended function $F_2 = P \circ h_2$. Indeed, define $h_2 : \mathbb{C} \to \mathbb{C}$ via

$$h_2(z) = \begin{cases} h_1(z), & \text{if } z \notin V; \\ z_j + r^{1-\frac{1}{m_j}} |z - z_j|^{1/m_j} e^{i \arg(z - z_j)}, & \text{if } |z - z_j| \leq r,\ z_j \in S(P'). \end{cases} \tag{2.3.11}$$

See Notation 2.2.3 for the definition of m_j. To check that h_2 is a homeomorphism first note that $h_2\big|_{\mathbb{C}\setminus \mathrm{Int}(V)} = h_1\big|_{\mathbb{C}\setminus \mathrm{Int}(V)}$ and $h_2|_{\overline{B}_r(z_j)}$ is continuous for each $z_j \in S(P')$, thus h_2 is continuous. Further, as $h_2(\overline{B}_r(z_j)) = h_1(\overline{B}_r(z_j)) = \overline{B}_r(z_j)$, both $h_2\big|_{\overline{B}_r(z_j)}$ and $h_2\big|_{\mathbb{C}\setminus \mathrm{Int}(V)}$ are bijective, and $h_2(\mathbb{C} \setminus V) \cap h_2(V) = h_1(\mathbb{C} \setminus V) \cap h_1(V) = \emptyset$, we conclude that $h_2 : \mathbb{C} \to \mathbb{C}$ is bijective. Finally as $h_2^{-1}\big|_{\overline{B}_r(z_j)}$ is continuous for each $z_j \in S(P')$ and $h_2^{-1}\big|_{\mathbb{C}\setminus \mathrm{Int}(V)} = h_1^{-1}\big|_{\mathbb{C}\setminus \mathrm{Int}(V)}$, we conclude h_2 is a homeomorphism of the plane to itself.

Recall that $P(w) = (w - z_j)^{m_j} Q_j(w) + P(z_j)$ and so $F_2(z) = P(h_2(z))$ has the following form:

$$F_2(z) = \begin{cases} F_1(z), & \text{if } z \notin V; \\ P(z_j) + r^{m_j - 1} f_{m_j}(z - z_j) Q_j(h_2(z)), & \text{if } |z - z_j| \leq r,\ z_j \in S(P'), \end{cases} \tag{2.3.12}$$

where f_{m_j} is defined as in Lemma 1.2.22.

Clearly, $F_1(z) = F_2(z)$ for each $z \in \partial V$ as $h_1\big|_{\partial B_r(z_j)} = h_2\big|_{\partial B_r(z_j)}$ for all $z_j \in S(P')$. Moreover, since P is a complex polynomial, hence an open map, and as h_2 is a homeomorphism, we conclude that F_2 is an open map.

Remark 2.3.10. If $m_j = n$ for some $z_j \in S(P')$, then $P(z) = P(z_j) + Q_j(z_j)(z - z_j)^n$ where $Q_j(z_j) \neq 0$. Therefore, $S(P') = \{z_j\}$ and so $F_2(z) = P(z_j) + Q_j(z_j) r^{n-1} f_n(z - z_j)$ for $z \in B_r(z_j)$. Hence, in such a case by Lemma 1.2.22, F_2 is pointwise $(|Q_j(z_j)|\, r^{n-1})$-co-Lipschitz and pointwise $(|Q_j(z_j)|\, n r^{n-1})$-Lipschitz at each $z \in B_r(z_j)$.

Claim 2.3.11. For each $z_j \in S(P')$ there exists $d_j > 0$ such that $F = F_2\big|_{\overline{B}_r(z_j)}$ is d_j-Lipschitz when considered as a function from $\overline{B}_r(z_j)$ to $F_2(\overline{B}_r(z_j))$.

Proof. Fix $z_j \in S(P')$. We shall show that F_2 is pointwise d_j-Lipschitz at each $x \in B_r(z_j)$ for some $d_j > 0$; the claim then follows by applying Lemma 1.2.17 followed by Lemma 1.2.16.

If $m_j = n$, then by Remark 2.3.10 it follows F_2 is pointwise $(|Q_j(z_j)|\, nr^{n-1})$-Lipschitz at each $z \in B_r(z_j)$.

Suppose that $m_j < n$. If $x = z_j$, then for each $y \in B_r(z_j)$, as $F_2(x) = F_2(z_j) = P(z_j)$ and $|f_{m_j}(y - z_j)| = |y - z_j|$,

$$|F_2(x) - F_2(y)| = r^{m_j-1}|Q_j(h_2(y))| \cdot |y - z_j| = r^{m_j-1}|Q_j(h_2(y))| \cdot |x - y|.$$

Since $h_2(B_r(z_j)) = B_r(z_j)$, by Claim 2.3.5 (iv), F_2 is pointwise $(2r^{m_j-1}|Q_j(z_j)|)$-Lipschitz at $x = z_j$.

Suppose now that $x \in B_r(z_j) \setminus \{z_j\}$. Let $\rho_1 > 0$ be such that $B_{\rho_1}(x) \subseteq B_r(z_j)$. Further, for each $l \in [n - m_j]$, let $\rho_{2,l} > 0$ be given by Corollary 2.2.5, where $w = x - z_j \neq 0$, so that for each $z \in B_{\rho_{2,l}}(w)$, $\Phi_{l,m_j}(z, w)$ is well-defined and

$$\left|\Phi_{l,m_j}(z, w)\right| < 1 + \left|\Phi_{l,m_j}(w, w)\right|. \tag{2.3.13}$$

Define $\rho_2 := \min_{l\in[n-m_j]} \rho_{2,l}$ and $\rho := \min(\rho_1, \rho_2)$. Note if $y \in B_\rho(x)$, then $z = y - z_j \in B_\rho(w)$. Considering (2.2.5), (2.3.11), (2.3.12) and Lemma 1.2.22 we deduce that if $y \in B_\rho(x)$, then

$$\begin{aligned} F_2(y) - F_2(x) &= F_2\left(z_j + |y - z_j|e^{i\arg(y-z_j)}\right) - F_2\left(z_j + |x - z_j|e^{i\arg(x-z_j)}\right) \\ &= r^{m_j-1}\left(f_{m_j}(z) - f_{m_j}(w)\right)\left(c_{0,j} + \sum_{l=1}^{n-m_j} r^{\frac{l(m_j-1)}{m_j}} c_{l,j} \cdot \Phi_{l,m_j}(z, w)\right), \end{aligned} \tag{2.3.14}$$

where $z = y - z_j$ and $w = x - z_j$. To see that F_2 is pointwise Lipschitz at x, as f_{m_j} is Lipschitz and $|z - w| = |y - x|$, it suffices to observe that $|\Phi_{l,m_j}(z, w)|$ are uniformly bounded over $z \in B_\rho(w)$ and $|w| = |x - z_j| < r < 1$. Indeed, by (2.3.13) as $l \in [n - m_j]$, observe that

$$\left|\Phi_{l,m_j}(z, w)\right| < 1 + |w|^{l/m_j}\frac{l + m_j}{m_j} \leq 1 + \frac{nr^{1/m_j}}{m_j} \leq 1 + n.$$

Hence, we conclude that there exists $d_j > 0$ such that F_2 is pointwise d_j-Lipschitz at each $x \in B_r(z_j)$, which as explained above, implies the statement of Claim 2.3.11. $\diamondsuit$

Claim 2.3.12. There exists $L > 0$ such that F_2 is L-Lipschitz on $\mathbb{C}$.

Proof. Recall Remark 2.3.4. Since $F_1(z) = F_2(z)$ for $z \in (\mathbb{C} \setminus V) \cup \partial V$ we conclude F_2 is pointwise L_1-Lipschitz at each $z \in \mathbb{C} \setminus V$ and, moreover,

$$|F_2(z) - F_2(w)| \leq L_1 |z - w| \quad \text{for } z \in \partial V \text{ and } w \in \mathbb{C} \setminus V.$$

Therefore, by Claim 2.3.5 (i), Claim 2.3.11 and by defining L to be the maximum of L_1 and $\max_{j:z_j \in S(P')} d_j$, we conclude F_2 is pointwise L-Lipschitz at each $z \in \mathbb{C}$. Hence Lemma 1.2.17 implies that F_2 is L-Lipschitz on $\mathbb{C}$. $\diamondsuit$

We now turn our attention to the co-Lipschitzness of F_2.

Claim 2.3.13. For each $z_j \in S(P')$ and $z \in B_r(z_j)$, the mapping F_2 is pointwise α_j-co-Lipschitz at z, where α_j is defined in (2.3.15).

Proof. Fix $z_j \in S(P')$ and define

$$\alpha_j := \frac{r^{m_j-1} |Q_j(z_j)|}{2}. \tag{2.3.15}$$

If $m_j = n$, then by Remark 2.3.10 it follows that, as $\alpha_j < r^{n-1}|Q_j(z_j)|$, F_2 is pointwise α_j-co-Lipschitz at each $z \in B_r(z_j)$.

Suppose that $m_j < n$. By (2.2.4) we have that $\alpha_j > 0$. To show F_2 is pointwise α_j-co-Lipschitz at each $z \in B_r(z_j)$ we first show for each $z \in \overline{B}_r(z_j)$ that there exists $\rho = \rho(z) > 0$ such that

$$|F_2(z) - F_2(y)| \geq \alpha_j \, |z - y| \tag{2.3.16}$$

for each $y \in B_\rho(z) \cap \overline{B}_r(z_j)$. We emphasize that (2.3.16) holds not only for $z \in B_r(z_j)$ but also for $z \in \partial B_r(z_j)$, and this fact is used later in the proof of Claim 2.3.15.

Consider first when $z = z_j$. Let $\rho = r$ and $y \in B_\rho(z)$. From (2.3.12), we deduce that

$$|F_2(z) - F_2(y)| = r^{m_j-1}|y - z||Q_j(h_2(y))|.$$

Since $h_2(B_r(z_j)) = B_r(z_j)$, by Claim 2.3.5 (iv), we conclude that F_2 satisfies (2.3.16) when $z = z_j$.

Fix $z \in \overline{B}_r(z_j) \setminus \{z_j\}$. Let $\rho_1 = \rho_1(z) > 0$ be defined by

$$\rho_1(z) = \begin{cases} r, & \text{if } z \in \partial B_r(z_j); \\ r - |z - z_j|, & \text{if } z \in B_r(z_j) \setminus \{z_j\}. \end{cases} \tag{2.3.17}$$

By Corollary 1.2.37, since f_{m_j} is strongly 1-co-Lipschitz at $(z - z_j) \in \overline{B}_r(0)$ there exists $\rho_2 = \rho_2(z) > 0$ such that for any $x \in B_{\rho_2}(z - z_j)$ it follows that

$$\left|f_{m_j}(x) - f_{m_j}(z - z_j)\right| \geq |x - (z - z_j)|. \tag{2.3.18}$$

Further by Corollary 2.2.5, for each $l \in [n - m_j]$, let $\rho_{3,l} > 0$ be such that for each $y \in B_{\rho_{3,l}}(z)$, $\Phi_{l,m_j}(y - z_j, z - z_j)$ is well-defined and

$$\left|\Phi_{l,m_j}(y - z_j, z - z_j)\right| < r^{1/m_j} + |z - z_j|^{l/m_j} \frac{l + m_j}{m_j}. \tag{2.3.19}$$

Define $\rho_3 := \min\limits_{l \in [n - m_j]} \rho_{3,l}$ and let $\rho = \rho(z) > 0$ be given by $\rho = \min(\rho_1, \rho_2, \rho_3)$. We claim for $y \in B_\rho(z) \cap \overline{B}_r(z_j)$ that

$$|F_2(y) - F_2(z)| \geq \alpha_j \left|f_{m_j}(y - z_j) - f_{m_j}(z - z_j)\right|. \tag{2.3.20}$$

Fix $y \in B_\rho(z) \cap \overline{B}_r(z_j)$. By using $y \in \overline{B}_r(z_j)$ for $F_2(y)$, $z \neq z_j$ and $y \in B_\rho(z)$ for the well-definedness of $\Phi_{l,m_k}(y - z_j, z - z_j)$ and recalling (2.3.14), it follows that

$$\begin{aligned} &|F_2(y) - F_2(z)| \geq \\ &\quad r^{m_j - 1}\left(|c_{0,j}| - \max_{l \in \{1,\ldots,n-m_j\}} \left|\Phi_{l,m_j}(y - z_j, z - z_j)\right| \cdot \sum_{k=1}^{n-m_j} r^{\frac{k(m_j-1)}{m_j}} |c_{k,j}|\right) \\ &\quad \times \left|f_{m_j}(y - z_j) - f_{m_j}(z - z_j)\right|. \end{aligned}$$

Therefore, since $r < 1$, see Claim 2.3.5, to show (2.3.20) it suffices to prove, as $c_{0,j} = Q_j(z_j)$, that for all $l \in [n - m_j]$,

$$\left|\Phi_{l,m_j}(y - z_j, z - z_j)\right| \sum_{k=1}^{n-m_j} |c_{k,j}| \leq \frac{|Q_j(z_j)|}{2}. \tag{2.3.21}$$

This is trivial when $\sum_{k=1}^{n-m_j} |c_{k,j}| = 0$. Suppose $\sum_{k=1}^{n-m_j} |c_{k,j}| \neq 0$. By property (iii) of Claim 2.3.5, which refers to the inequality (2.2.7) of Corollary 2.2.5, since $|y - z_j| < \rho \leq \rho_3$, $z \in \overline{B}_r(z_j)$, $m_j \geq 1$ and $l \leq n - m_j$, note that

$$\begin{aligned} \left|\Phi_{l,m_j}(y - z_j, z - z_j)\right| &< r^{1/m_j} + |z - z_j|^{l/m_j} \frac{l + m_j}{m_j} && \text{by (2.3.19),} \\ &\leq (1 + n) r^{1/m_j} \\ &\leq \frac{|Q_j(z_j)|}{2 \sum_{k=1}^{n-m_j} |c_{k,j}|} && \text{by Claim 2.3.5 (iii).} \end{aligned}$$

Thus (2.3.21) follows and so (2.3.20) is satisfied, as claimed.

Since $\rho \leq \rho_2$ and $y \in B_\rho(z)$ it follows $(y - z_j) \in B_{\rho_2}(z - z_j)$. Therefore, by (2.3.18),

$$|f_{m_j}(y - z_j) - f_{m_j}(z - z_j)| \geq |(y - z_j) - (z - z_j)| = |y - z|.$$

Hence, combining this with (2.3.20) yields

$$|F_2(y) - F_2(z)| \geq \alpha_j |f_{m_j}(y - z_j) - f_{m_j}(z - z_j)| \geq \alpha_j |y - z|.$$

Thus we deduce that for each $z \in \overline{B}_r(z_j)$ there exists $\rho > 0$ such that (2.3.16) holds for all $y \in B_\rho(z) \cap \overline{B}_r(z_j)$.

If $z \in B_r(z_j)$, by (2.3.17) and since $\rho \leq \rho_1$ we note $B_\rho(z) \subseteq B_r(z_j)$. Hence for each $y \in B_\rho(z)$, (2.3.16) is satisfied. Therefore, since $F_2 = P \circ h_2$ is an open map, by Corollary 1.2.26, Remark 1.2.27 and since $B_r(z_j)$ is open in $\mathbb{C}$, we conclude that F_2 is pointwise α_j-co-Lipschitz at any $z \in B_r(z_j)$. ♢

Remark 2.3.14. Taking $c_2 := \min_{z_j \in S(P')} \alpha_j > 0$ we deduce

$$F_2 \text{ is pointwise } c_2\text{-co-Lipschitz at each } z \in \operatorname{Int}(V). \tag{2.3.22}$$

Claim 2.3.15. *There exists $c_3 > 0$ such that $F_2 : \mathbb{C} \to \mathbb{C}$ is pointwise c_3-co-Lipschitz at each $z \in \partial V$.*

Proof. Let $c_3 := \min(c_0, c_2)$, where $c_0 > 0$ is given by Claim 2.3.6 and $c_2 > 0$ is given by

Remark 2.3.14. Since F_2 is an open map, it suffices by Corollary 1.2.26 to show for each $z \in \partial V$ there exists $\rho = \rho(z) > 0$ such that if $x \in B_\rho(z)$, then

$$|F_2(z) - F_2(x)| \geq c_3 |z - x| . \tag{2.3.23}$$

Fix $z \in \partial V$ and let j be such that $z \in \partial B_r(z_j)$. Let $\rho_1 > 0$ be such that $B_{\rho_1}(z) \subseteq U_2$ and $B_{\rho_1}(z) \cap V \subseteq \overline{B}_r(z_j)$; note such $\rho_1 > 0$ exists by Claim 2.3.5 (i). Since $\partial V \subseteq U_2 \setminus \mathrm{Int}(V)$ and $F_1|_{U_2 \setminus \mathrm{Int}(V)} = F_2|_{U_2 \setminus \mathrm{Int}(V)}$, by (2.3.4) and $c_3 \leq c_0$ there exists $\rho_2 \in (0, \rho_1)$ such that (2.3.23) is satisfied for each $x \in B_{\rho_2}(z) \cap (U_2 \setminus \mathrm{Int}(V)) = B_{\rho_2}(z) \setminus B_r(z_j)$.

Further, by (2.3.16) there exists $\rho \in (0, \rho_2)$ such that (2.3.23) is satisfied for each $x \in B_\rho(z) \cap \overline{B}_r(z_j)$ since $c_3 \leq c_2 \leq \alpha_j$; see Remark 2.3.14.

We then conclude that (2.3.23) is satisfied for each $x \in B_\rho(z)$. As F_2 is an open map, Corollary 1.2.26 implies the statement of Claim 2.3.15. ♢

Claim 2.3.16. There exists $c > 0$ such that F_2 is c-co-Lipschitz on $\mathbb{C}$.

Proof. Let $c := \min(c_1, c_2, c_3)$, where c_1 is given by Remark 2.3.9, c_2 is given by Remark 2.3.14 and c_3 is given by Claim 2.3.15. Recall by (2.3.10) of Remark 2.3.9 that F_1 is pointwise c_1-co-Lipschitz at each $z \in \mathbb{C} \setminus W$. As $F_1(z) = F_2(z)$ for $z \in \mathbb{C} \setminus V$ and $W \subseteq V$, we conclude

$$F_2 \text{ is pointwise } c\text{-co-Lipschitz at each } z \in \mathbb{C} \setminus V. \tag{2.3.24}$$

Also, Remark 2.3.14 implies that

$$F_2 \text{ is pointwise } c\text{-co-Lipschitz at each } z \in \mathrm{Int}(V). \tag{2.3.25}$$

From Claim 2.3.15, (2.3.24) and (2.3.25), we conclude that F_2 is pointwise c-co-Lipschitz at each $z \in \mathbb{C}$. Hence an application of Lemma 1.2.18 implies F_2 is c-co-Lipschitz on $\mathbb{C}$. ♢

Finally, Claims 2.3.12 and 2.3.16 together imply that $f := F_2 = P \circ h_2$ is an L-Lipschitz and c-co-Lipschitz mapping of the plane. □

In this chapter we have shown that for any fixed non-constant complex polynomial P in one complex variable there exists a planar homeomorphism h such that $P \circ h$ is a Lipschitz quotient mapping, proving a converse statement to Theorem 1.1.4. Using the foundational Theorem 1.1.4, Maleva in [21] and [24] was able to introduce a relation between the ratio of constants of a planar Lipschitz quotient mapping and the degree of the polynomial obtained via such a decomposition. In the next section we investigate the sharpness of such a scale, when considering polygonal norms on the plane and improve the estimates obtained in [25].

CHAPTER 3

IMPROVED CONSTANTS OF PLANAR LIPSCHITZ QUOTIENT MAPPINGS IN POLYGONAL NORMS

This chapter focuses on the maximal ratio of constants for planar Lipschitz quotient mappings in polygonal norms. We follow closely the framework in [25] but consider this in the context of centred Lipschitz quotient mappings, see Definition 1.2.8. In doing so, we introduce an improved estimate for such a ratio of constants for 2-fold mappings.

3.1 Introduction

In [21], Maleva discovered a natural relationship between the maximum cardinality of point pre-images of planar Lipschitz quotient mappings in the Euclidean norm and the ratio of its co- and Lipschitz constants. Later, in [22] this ratio was generalised to the setting of an arbitrary planar norm, see Theorem 3.1.1 below; further extensions have been considered in [37, Theorem 2.7] where the domain and co-domain are equipped with distinct norms.

Theorem 3.1.1. ([22, Theorem 1]) Suppose $f : (\mathbb{C}, \|\cdot\|) \to (\mathbb{C}, \|\cdot\|)$ is an N-fold mapping which is L-Lipschitz and c-co-Lipschitz. Then, $c/L \leq 1/N$.

Such a bound invited the question whether there exist mappings for which the maximal ratio of constants is attained. When considering $\|\cdot\| = |\cdot|$, the Euclidean norm, the answer to this question is positive; one may consider the non-trivial winding maps f_n as defined in Lemma 1.2.22.

However, when the norm considered is not the Euclidean norm this question no longer has such an obvious answer. It was claimed in [22] that if the norm was polygonal, then one may construct a similar winding map to that in the Euclidean setting to be able to attain such bounds for the ratio of constants. Unfortunately, this claim is not entirely correct.

The claim is true when $\|\cdot\| = \|\cdot\|_m$ and m is not a multiple of 4, as shown in [25, Corollary 4.4]. However, in [37, Proposition 3.1.3], it is shown that if $\|\cdot\| = \|\cdot\|_4$ then such a 2-winding map is actually 1-co-Lipschitz, but 3-Lipschitz. Upon further investigation, in [25, Theorem 5.12], it is shown that if $\|\cdot\| = \|\cdot\|_m$ and m is divisible by 4, then any N-fold L-Lipschitz and c-co-Lipschitz mapping $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ can never attain such bounds, see Theorem 3.1.2 below.

Theorem 3.1.2. Let $m \geq 4$ be a multiple of 4 and $N \geq 2$ be an integer. If $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ is an N-fold mapping which is L-Lipschitz and c-co-Lipschitz, then $c/L < 1/N$.

In light of such a result, and with the conjecture that the optimal ratio of constants should be attained by considering these winding maps, one may suggest the following.

Conjecture 3.1.3. Suppose $f : (\mathbb{C}, \|\cdot\|_4) \to (\mathbb{C}, \|\cdot\|_4)$ is a 2-fold mapping which is L-Lipschitz and c-co-Lipschitz. Then $c/L \leq 1/3$.

This conjecture forms the motivation for the work presented in this present chapter. A partial progress towards answering this conjecture is provided. We utilise the methodology presented in [25] and apply it to the class of N-centred Lipschitz quotient mappings, to show that for a 2-fold Lipschitz quotient mapping $f : (\mathbb{C}, \|\cdot\|_4) \to (\mathbb{C}, \|\cdot\|_4)$ which is L-Lipschitz and c-co-Lipschitz, then $L/c \geq 2 + (1/38)$. We then generalise this result to N-centred Lipschitz quotient mappings which are equipped with a polygonal norm $\|\cdot\|_m$ where m is divisible by 4.

3.2 Preliminaries

We first introduce a simple result containing the distance to rays in polygonal norms. Recall Notation 1.5.2.

Lemma 3.2.1. Let $m \geq 4$ be a multiple of 4, $c, \rho > 0$ and $z \in \mathbb{C}$ be such that $\mathrm{Re}(z), \mathrm{Im}(z) > 0$. Then, $\mathrm{dist}_m(cz, \mathcal{D}_1^{c\rho}) = c\mathrm{dist}_m(z, \mathcal{D}_1^{\rho})$.

Proof. Suppose first that $\mathrm{Re}(cz) \geq c\rho$. Then, $\mathrm{Re}(z) \geq \rho$ and so, as m is a multiple of 4,

$$\mathrm{dist}_m(cz, \mathcal{D}_1^{c\rho}) = |cz - \mathrm{Re}(cz)| = c\,|z - \mathrm{Re}(z)| = c\mathrm{dist}_m(z, \mathcal{D}_1^{\rho}).$$

Suppose now that $\mathrm{Re}(cz) < c\rho$, so $\mathrm{Re}(z) < \rho$. Observe, for each $s \in (0, \|cz - c\rho v_1\|_m)$, that $\partial B_s^m(cz) \cap \mathcal{D}_1^{c\rho} = \emptyset$. Moreover, $\|cz - t\|_m \geq \|cz - c\rho v_1\|_m$ for each $t \in \mathcal{D}_1^{c\rho}$. Therefore, $\mathrm{dist}_m\left(cz, \mathcal{D}_1^{c\rho}\right) = \|cz - c\rho v_1\|_m$. Since $\mathrm{Re}(z) < \rho$, the argument above shows that $\mathrm{dist}_m\left(z, \mathcal{D}_1^{\rho}\right) = \|z - \rho v_1\|_m$. Thus, as $c > 0$, $\mathrm{dist}_m\left(cz, \mathcal{D}_1^{c\rho}\right) = c\,\mathrm{dist}_m\left(z, \mathcal{D}_1^{\rho}\right)$. □

The following notion is similar to that considered in Notation 1.3.8, but will be used specifically for the unit sphere of polygonal norms. Such a parametrisation exists by considering the standard arc-length parametrisation, for example.

Definition 3.2.2. Let $m \geq 4$ be even and $r > 0$. We define the *standard parametrisation of* $\partial B_r^m(0)$ to be the 1-Lipschitz parametrisation $\gamma_r : [0, r\mathscr{L}_m] \to \partial B_r^m(0)$ such that $\gamma_r\left((k-1)r\mathscr{L}_m/m\right) = rv_k$ for each $k \in [m]$ and so that γ_r is linear on $[(k-1)r\mathscr{L}_m/m, kr\mathscr{L}_m/m]$ for each $k \in [m]$.

Remark 3.2.3. Observe that γ_r is the arc-length parametrisation of $\partial B_r^m(0)$. Moreover, $\gamma_r(0) = rv_1 = r$ and $\mathrm{Ind}_0\gamma_r = 1$.

In a similar way, we define the standard argument parametrisation for the image of the sphere $\partial B_r^m(0)$, similar to the continuous choice of argument as described in Definition 1.2.9.

Definition 3.2.4. Let $m \geq 4$ be even and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be a Lipschitz quotient mapping and $r > 0$ be such that $0 \notin f\left(\partial B_r^m(0)\right)$ and $|\arg(f(\gamma_r(0))| < 2\pi/m$, where γ_r is the standard parametrisation of $\partial B_r^m(0)$. Define $\varphi_r : [0, r\mathscr{L}_m] \to \mathbb{R}$ to be the non-decreasing, continuous function such that $\varphi_r(0) = \arg\left(f(rv_1)\right)$, $\varphi_r(r\mathscr{L}_m) = \varphi_r(0) + 2N\pi$ and, in general,

$$\varphi_r(t) \pmod{2\pi} = \arg\left(f\left(\gamma_r(t)\right)\right) \text{ for each } t \in [0, r\mathscr{L}_m),$$

where $N = \mathrm{Ind}_0 f\left(\partial B_r^m(0)\right)$. We refer to φ_r as the *standard argument parametrisation of* $f(\partial B_r^m(0))$.

Remark 3.2.5. Note that if f is an N-centred planar Lipschitz quotient mapping, then $0 \notin f\left(\partial B_r^m(0)\right)$ for all $r > 0$. Moreover, if f is an N-fold planar Lipschitz quotient mapping, there exists $R_0 > 0$ such that $0 \notin f\left(\partial B_r^m(0)\right)$ for all $r \geq R_0$, cf. Lemma 1.2.10 and Corollary 1.2.11.

We finish this section by introducing a class of points which prove to be prudent in determining an improved upper bound for the ratio of constants for planar Lipschitz quotient mappings in polygonal norms. Recall Definition 3.2.2 and Definition 3.2.4.

Definition 3.2.6. Let $m \geq 4$ be even, $N \geq 1$ be an integer and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be a continuous mapping such that $\mathrm{Ind}_0 f\left(\partial B_r^m(0)\right) = N$ and $0 \notin f\left(\partial B_r^m(0)\right)$ for some $r > 0$.

Let γ_r be the standard parametrisation of $\partial B_r^m(0)$, assume $|\arg(f \circ \gamma_r)(0)| < 2\pi/m$ and let φ_r be the standard argument parametrisation of $f \circ \gamma_r$. For each $j \in [mN] \setminus \{1\}$, define

$$t_j(r) = \sup\left\{t \in [0, r\mathscr{L}_m] : \varphi_r(t) = \frac{2(j-1)\pi}{m}\right\} \quad \text{and} \quad R_j^f(r) = \|f(\gamma_r(t_j(r)))\|_m .$$

If $\arg(f \circ \gamma_r)(0) = \varphi_r(0) \leq 0$, let $t_1(r) = \sup\{t \in [0, r\mathscr{L}_m] : \varphi_r(t) = 0\}$, otherwise let $t_1(r) = \sup\{t \in [0, \mathscr{L}_m] : \varphi_r(t) = 2N\pi\}$. Define $R_1^f(r) = \|f(\gamma_r(t_1(r)))\|_m$.

For each $j \in [mN]$ we refer to $R_j^f(r)v_j$ as an (f, r)-*primitive point*. Finally, we highlight the following relation:

$$f\left(\gamma_r(t_j(r))\right) = R_j^f(r)v_j \quad \text{for each } j \in [mN].$$

Remark 3.2.7. If f is a N-centred mapping and $|\arg(f \circ \gamma_r)(0)| < 2\pi/m$ for some $r > 0$ then the (f, r)-primitive points exist. Moreover if f is a continuous c-co-Lipschitz mapping, then by Corollary 1.2.11, then $R_j^f(r) \geq cr$ for all $r > 0$ and each $j \in [mN]$, where $N = \mathrm{Ind}_0 f\left(\partial B_r^m(0)\right)$.

Remark 3.2.8. Let $m \geq 4$ be a multiple of 4 and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be a Lipschitz quotient mapping. Suppose, for some $r > 0$, that $|\arg(f \circ \gamma_r)(0)| < 2\pi/m$. Let $a > 0$ and $g : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be given by $g = af$. Observe that $|\arg(g \circ \gamma_r)(0)| < 2\pi/m$ and

$$R_j^g(r) = \|g(\gamma_r(t_j(r)))\|_m = a\|f(\gamma_r(t_j(r)))\|_m = aR_j^f(r) \quad \text{for each } j \in [mN].$$

Lemma 3.2.9. Let $m \geq 4$ be even, $N \geq 1$ be an integer, $r > 0$ and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be an N-centred Lipschitz quotient mapping. Let γ_r be the standard parametrisation of $\partial B_r^m(0)$, assume $|\arg(f \circ \gamma_r)(0)| < 2\pi/m$ and let φ_r be the standard argument parametrisation of $f \circ \gamma_r$. Then:

(i) $t_j(r)$ is well-defined for each $j \in [mN]$;

(ii) $\varphi_r(t_j) = 2(j-1)\pi/m < \varphi_r(t)$ for all $t \in (t_j, r\mathscr{L}_m]$ for each $j \in [mN] \setminus \{1\}$. Also, if $\varphi_r(0) \leq 0$, then $\varphi_r(t_1) = 0$ and $\varphi_r(t) > 0$ for all $t \in (t_1, r\mathscr{L}_m]$. Otherwise, $\varphi_r(t_1) = 2N\pi$ and $\varphi_r(t) > 2N\pi$ for all $t \in (t_1, r\mathscr{L}_m]$.

(iii) $0 < t_2 < \cdots < t_{mN} < r\mathscr{L}_m$. Further, if $\varphi_r(0) \leq 0$, then $0 \leq t_1 < t_2$. Otherwise, $t_{mN} < t_1 \leq r\mathscr{L}_m$.

Proof. The proof is a simple, repeated application of an Intermediate Value Theorem type argument, since $f \circ \gamma_r$ is closed and connected. □

3.3 Improved ratio of constants

This section follows closely the framework introduced in [25], but applied to centred Lipschitz quotient mappings. We determine an upper bound for the ratio of the co-Lipschitz and Lipschitz constants for 2-centred Lipschitz quotient mapping. We then apply Theorem 1.2.7 to conclude the same upper bound holds over all 2-fold Lipschitz quotient mappings.

The first result in this section shows that for a centred Lipschitz quotient mapping, the images of all spheres which are centred at the origin behave similar to the images under the standard winding mapping, in the sense that corners of the sphere of $\|\cdot\|_m$ are mapped 'close' to corners of the appropriate sphere, up to some estimate that grows linearly with respect to the radius of this sphere.

Following [25, Remark 2.7], in the rest of this section we assume implicitly that the homeomorphism h from the decomposition of the Lipschitz quotient mapping $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ is orientation preserving. We only mention this comment here and do not repeat this condition in the subsequent results.

Lemma 3.3.1. Let $m \geq 4$ be a multiple of 4. Suppose $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ is an N-centred Lipschitz quotient mapping for some $N \in \mathbb{N}$. Then, for each $R > 0$,

$$\operatorname{dist}_m\left(f\left(Rv_k\right), \mathbb{D}^{cR}\right) \leq \frac{2m(L-cN)R}{\sin(2\pi/m)} \quad \text{for each } k \in [m], \tag{3.3.1}$$

where $c = \text{co-Lip}(f)$ and $L = \text{Lip}(f)$ and $\mathbb{D}^{cR}$ is defined in Notation 1.5.2.

Proof. By Theorem 3.1.2, the right-hand side of (3.3.1) is positive. Since f is N-centred, there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) = 0$ and $f = h^N$, see Definition 1.2.8. By Corollary 1.2.11 note that $\operatorname{Ind}_0 f\left(\partial B^m_{cR}(0)\right) = N$ for each $R > 0$.

Fix $R > 0$ and $k \in [m]$. Define $d_k := \operatorname{dist}_m\left(f\left(Rv_k\right), \mathbb{D}^{cR}\right)$. Observe that if $d_k = 0$, we are done. Suppose $d_k > 0$, i.e. $f(Rv_k) \notin \mathbb{D}^{cR}$. Let $R' := \|f(Rv_k)\|_m$. As $f(Rv_k) \in \mathcal{U}_s$ for some $s \in [m]$, that is $f(Rv_k) \in (R'v_s, R'v_{s+1})$, it follows that

$$0 < d_k \leq \min\left(\|f(Rv_k) - R'v_s\|_m, \|f(Rv_k) - R'v_{s+1}\|_m\right) \leq \frac{1}{2}\|R'v_{s+1} - R'v_s\|_m = \frac{R'}{2}\|v_2 - v_1\|_m.$$

As $f(0) = 0$ and f is L-Lipschitz,

$$R' = \|f(Rv_k)\|_m \leq L\|Rv_k\|_m = LR.$$

Therefore, $0 < d_k \leq (LR/2)\|v_2 - v_1\|_m$.

Define $a_k := d_k/L$. Note $0 < a_k \leq R\|v_2 - v_1\|_m$. Let $P_1 \in [Rv_{k-1}, Rv_k]$ and $P_2 \in [Rv_k, Rv_{k+1}]$ be such that $\|P_1 - Rv_k\|_m = \|P_2 - Rv_k\|_m = a_k$. Let γ be a 1-Lipschitz parametrisation of $\partial B_R^m(0)$ such that $\gamma(0) = P_1$, $\gamma(a_k) = Rv_k$, $\gamma(2a_k) = P_2$ and $\operatorname{Ind}_0\gamma = 1$; for example, consider the arc-length parametrisation of $\partial B_R^m(0)$ starting at P_1.

Then, $\operatorname{Ind}_0(f \circ \gamma) = \operatorname{Ind}_0\left(f\left(\partial B_R^m(0)\right)\right) = N$. Since, by Corollary 1.2.11, $\|f(z)\|_m \geq c\|z\|_m$ for all $z \in \mathbb{C}$, the curve $f \circ \gamma$ does not intersect $B_{cR}^m(0)$. In particular,

$$R' = \|f(Rv_k)\|_m \geq c\|Rv_k\|_m = cR \quad \text{and} \quad \|f(P_j)\|_m \geq c\|P_j\|_m = cR \quad \text{for } j = 1, 2.$$

Let $q_j := f(P_j)$ for $j = 1, 2$. As f is L-Lipschitz, it follows by Lemma 1.5.4, that

$$\|q_1 - q_2\|_m \leq L\|P_1 - P_2\|_m = 2a_k L\cos^2(\pi/m). \tag{3.3.2}$$

Define $\mathcal{U} := \overline{\mathcal{U}_s} \setminus B_{cR}^m(0)$. For any $t \in [0, 2a_k]$, as $f \circ \gamma$ is L-Lipschitz,

$$\|f(\gamma(t)) - f(Rv_k)\|_m = \|f(\gamma(t)) - f(\gamma(a_k))\|_m \leq L|t - a_k| \leq La_k = d_k.$$

Since $\|f(\gamma(t)) - f\left(Rv_k\right)\|_m \leq d_k$ for all $t \in [0, 2a_k]$, $f(Rv_k) \in \mathcal{U}_s$ and by definition of d_k, note $f\left(\gamma(t)\right) \in \overline{\mathcal{U}}_s$ for all $t \in [0, 2a_k]$. Moreover, as f is c-co-Lipschitz and N-centred, $\|f(\gamma(t))\|_m \geq cR$ for all $t \in [0, 2a_k]$. Hence, $(f \circ \gamma)\left([0, 2a_k]\right) \subseteq \mathcal{U}$. By the convexity of $\mathcal{U}$, note $[q_1, q_2] = [(f \circ \gamma)(0), (f \circ \gamma)(2a_k)] \subseteq \mathcal{U}$.

Let $\phi : [0, 2a_k] \to [q_1, q_2]$ be an affine parametrisation, and define

$$\Gamma(t) = \begin{cases} \phi(t), & \text{if } 0 \le t \le 2a_k, \\ (f \circ \gamma)(t), & \text{if } 2a_k \le t \le R\mathscr{L}_m. \end{cases}$$

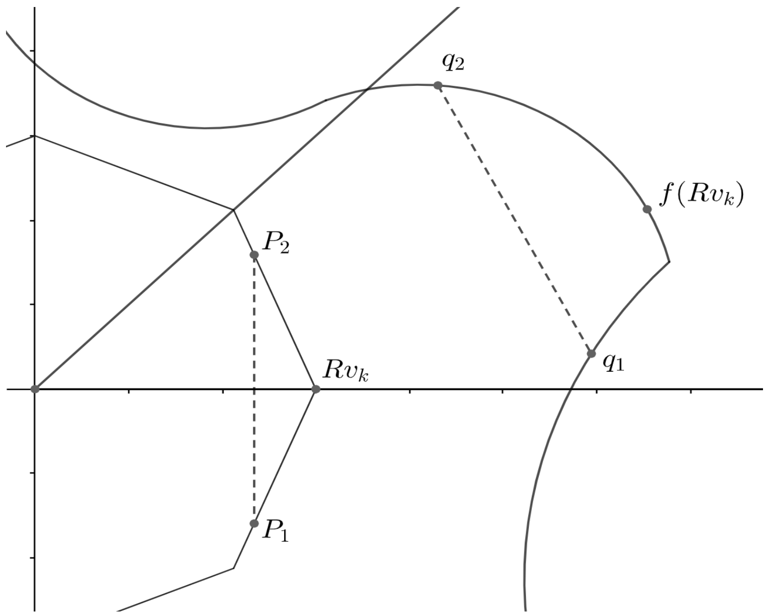

Figure 3.1: The curve of the parametrisation $\Gamma(t)$.

Recall that $\mathrm{Ind}_0(f \circ \gamma) = N$. Since $(f \circ \gamma)([0, 2a_k]) \subseteq \mathcal{U}$ and $\|\Gamma(t)\|_m \ge cR$ for all $t \in [0, R\mathscr{L}_m]$, we infer by Lemma 1.3.7 that

$$\mathrm{length}_m \Gamma = \|q_1 - q_2\|_m + \mathrm{length}_m \left((f \circ \gamma)\big|_{[2a_k, R\mathscr{L}_m]} \right) \ge cNR\mathscr{L}_m.$$

As $f \circ \gamma$ is L-Lipschitz,

$$\mathrm{length}_m \left((f \circ \gamma)\big|_{[2a_k, R\mathscr{L}_m]} \right) \le L \left(R\mathscr{L}_m - 2a_k \right).$$

Combining these two inequalities we obtain, via (3.3.2),

$$\begin{aligned} L(R\mathscr{L}_m - 2a_k) \ge \mathrm{length}_m \left((f \circ \gamma)\big|_{[2a_k, R\mathscr{L}_m]} \right) &\ge cNR\mathscr{L}_m - \|q_1 - q_2\|_m \\ &\ge cNR\mathscr{L}_m - 2a_k L \cos^2(\pi/m). \end{aligned} \tag{3.3.3}$$

As $d_k = a_k L$, rearranging (3.3.3) we obtain

$$(L - cN)R\mathscr{L}_m \geq 2d_k \left(1 - \cos^2(\pi/m)\right) = 2d_k \sin^2(\pi/m).$$

Since $\mathscr{L}_m = 2m\tan(\pi/m)$, see Lemma 1.5.3, we conclude that

$$0 < d_k \leq \frac{(L - cN)R\mathscr{L}_m}{2\sin^2(\pi/m)} = \frac{(L - cN)Rm\tan(\pi/m)}{\sin^2(\pi/m)} = \frac{2m(L - cN)R}{\sin(2\pi/m)}.$$

□

The result below further defends the notion that centred Lipschitz quotient mappings approximately map spheres to spheres. It shows that provided one corner of the sphere is mapped sufficiently close to one of the rays $\mathcal{D}_j$, then the 'midrays' are mapped sufficiently close to the primitive points, similar to how the winding map behaves. Recall Notation 1.5.8.

Lemma 3.3.2. Let $m \geq 4$ be a multiple of 4 and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be an N-centred mapping which is L-Lipschitz and c-co-Lipschitz for some $N \in \mathbb{N}$ and $c, L > 0$. Then for each $T > 0$ there exists a positive constant P_0 such that if

$$\operatorname{dist}_m\left(f(\rho w_1), \mathcal{D}_1^{c\rho}\right) \leq T$$

for some $\rho \geq P_0$, then

$$\left\| f\left(\rho w_k\right) - R_k^f(\rho) v_k \right\|_m \leq T + 2\rho\left(L - cN\right)\tan\left(\frac{\pi}{m}\right)\max\left(\frac{k-1}{N}, m - \frac{k-1}{N}\right),$$

for each $k \in [mN]$.

Proof. By Corollary 1.2.11 note $\operatorname{Ind}_0 f\left(\partial B_r(0)\right) = N$ for all $r > 0$. Consider the N-centred Lipschitz quotient mapping $g = \frac{1}{c}f$; note that g is (L/c)-Lipschitz and 1-co-Lipschitz. For the majority of this proof we will work with g, rather than f.

Let $P_0 > 0$ be given by

$$P_0 = 1 + \frac{T/c}{\cos(\pi/m)\tan(\pi/(2m))}.$$

Fix any $\rho \geq P_0$ and suppose $\operatorname{dist}_m\left(f\left(\rho w_1\right), \mathcal{D}_1^{c\rho}\right) \leq T$. Thus $\operatorname{dist}_m\left(g(\rho w_1), \mathcal{D}_1^{\rho}\right) \leq T/c$ by

Lemma 3.2.1. Let γ_ρ be the standard parametrisation of $\partial B^m_\rho(0)$ with $\gamma_\rho(0) = \gamma_\rho(\rho\mathscr{L}_m) = \rho v_1 = \rho w_1$; see Definition 3.2.2. Consider $\phi := \arg(g \circ \gamma_\rho)(0)$. As g is 1-co-Lipschitz and N-centred, it follows by Corollary 1.2.11 and Lemma 1.5.17 that

$$|g(w_1)| \geq \cos\left(\frac{\pi}{m}\right) \|g(\rho w_1)\|_m \geq \cos\left(\frac{\pi}{m}\right) \|\rho w_1\|_m = \cos\left(\frac{\pi}{m}\right) \rho \geq \cos\left(\frac{\pi}{m}\right) P_0 > \frac{T/c}{\tan(\pi/(2m))}.$$

Hence $\phi = \arg(g(\rho(w_1))) \in (-\pi/(2m), \pi/(2m)) \subseteq (-2\pi/m, 2\pi/m)$, using that the Euclidean distance from $g(\rho w_1)$ to $\mathcal{D}_1$ is less than or equal to $\operatorname{dist}_m(g(\rho w_1), \mathcal{D}_1^\rho) \leq T/c$.

By Corollary 1.2.11, $\operatorname{Ind}_0(g \circ \gamma_\rho) = N$. Let $R^g_k(\rho)v_k$ denote the (g,ρ)-primitive points, as in Definition 3.2.6, such that $R^g_k(\rho) = \|g(\gamma_\rho(t_k(\rho)))\|_m \geq \rho$, by Corollary 1.2.11.

As $\rho > 0$ is fixed, for brevity, in the sequel we simply write R^g_k and t_k. Recall for each $k \in [mN]$, $(g \circ \gamma_\rho)(t_k) = R^g_k v_k \in \mathcal{D}^\rho_k$. Without loss of generality, suppose that $\arg(g \circ \gamma_\rho)(0) \in (-\pi/(2m), 0]$. Hence, t_1 exists and $t_1 \geq 0$. Now, for each $k \in [mN-1]$, we claim that

$$\frac{L}{c}(t_{k+1} - t_k) \geq \operatorname{length}_m\left((g \circ \gamma_\rho)\big|_{[t_k, t_{k+1}]}\right) \geq \frac{\rho\mathscr{L}_m}{m}. \tag{3.3.4}$$

Since $g \circ \gamma_\rho$ is (L/c)-Lipschitz, it suffices to only show the second inequality in (3.3.4) holds.

Fix $k \in [mN-1]$. Since, by Corollary 1.2.11, $\|g(z)\|_m \geq \|z\|_m$ for each $z \in \mathbb{C}$, note that

$$(g \circ \gamma_\rho)(t_k) \in \mathcal{D}^\rho_{k \pmod m} \quad \text{and} \quad (g \circ \gamma_\rho)(t_{k+1}) \in \mathcal{D}^\rho_{(k+1) \pmod m}.$$

The claim then follows via an application of Proposition 1.5.6.

As $\operatorname{dist}_m((g \circ \gamma_\rho)(0), \mathcal{D}^\rho_1) \leq T/c$ it follows by Corollary 1.2.11 and Proposition 1.5.6 that

$$\frac{L}{c}(\rho\mathscr{L}_m - t_{mN}) \geq \operatorname{length}_m\left((g \circ \gamma_\rho)\big|_{[t_{mN}, \rho\mathscr{L}_m]}\right) \geq \frac{\rho\mathscr{L}_m}{m} - \frac{T}{c}. \tag{3.3.5}$$

Let $j \in [mN]$. If $j \geq 2$, by summing (3.3.4) over $k \in [j-1]$, since $t_1 \geq 0$ and as $g \circ \gamma_\rho$ is (L/c)-Lipschitz,

$$\frac{L}{c} t_j \geq \operatorname{length}_m\left((g \circ \gamma_\rho)\big|_{[0, t_j]}\right) \geq (j-1)\frac{\rho\mathscr{L}_m}{m}. \tag{3.3.6}$$

Similarly, summing (3.3.4) over $j \leq k \leq mN - 1$ with (3.3.5),

$$\frac{L}{c}\left(\rho\mathscr{L}_m - t_j\right) \geq \operatorname{length}_m\left((g \circ \gamma_\rho)\big|_{[t_j, \rho\mathscr{L}_m]}\right) \geq (mN - j + 1)\,\frac{\rho\mathscr{L}_m}{m} - \frac{T}{c}. \tag{3.3.7}$$

Rearranging (3.3.6) and (3.3.7), note for each $j \in [mN]$ that if $s_j = (j-1)\rho\mathscr{L}_m/(mN)$, then

$$\begin{aligned} t_j &\geq \frac{c(j-1)\rho\mathscr{L}_m}{mL} = \frac{cN}{L}s_j; \\ t_j &\leq \rho\mathscr{L}_m - \frac{(mN-j+1)c\rho\mathscr{L}_m}{mL} + \frac{T}{L} = \left(1 - \frac{cN}{L}\right)\rho\mathscr{L}_m + \frac{cNs_j}{L} + \frac{T}{L}. \end{aligned}$$

Note by Theorem 3.1.2, $1 - (cN)/L > 0$. Hence,

$$-\left(1 - \frac{cN}{L}\right)s_j \leq t_j - s_j \leq \left(1 - \frac{cN}{L}\right)(\rho\mathscr{L}_m - s_j) + \frac{T}{L}$$

and so, by the definition of s_j,

$$\begin{aligned} |t_j - s_j| &\leq \max\left(s_j\left(1 - \frac{cN}{L}\right), \left(1 - \frac{cN}{L}\right)(\rho\mathscr{L}_m - s_j) + \frac{T}{L}\right) \\ &\leq \frac{T}{L} + \left(1 - \frac{cN}{L}\right)\frac{\rho\mathscr{L}_m}{m}\max\left(\frac{j-1}{N}, m - \frac{j-1}{N}\right). \end{aligned}$$

Note $\gamma_\rho(s_k) = \rho w_k$ for each $k \in [mN]$ by Definition 3.2.2. Since $(g \circ \gamma_\rho)(t_k) = R_k^g v_k \in \mathcal{D}_k^\rho$, $\gamma_\rho(s_k) = \rho w_k$ and $g \circ \gamma_\rho$ is (L/c)-Lipschitz, note

$$\|g(\rho w_k) - R_k^g v_k\|_m \leq \frac{L}{c}|t_k - s_k| \quad \text{for each } k \in [mN],$$

implying that

$$\|g(\rho w_k) - R_k^\rho v_k\|_m \leq \frac{T}{c} + \left(\frac{L}{c} - N\right)\frac{\rho\mathscr{L}_m}{m}\max\left(\frac{j-1}{N}, m - \frac{j-1}{N}\right) \qquad \text{for each } k \in [mN].$$

Finally, as $f = \frac{1}{c}g$, by Remark 3.2.8, note $R_k^f = cR_k^g$ for each $k \in [mN]$. Hence,

$$\|f(\rho w_k) - R_k^f v_k\|_m = c\|g(\rho w_k) - R_k^g v_k\|_m,$$

from which the result follows, since $\mathscr{L}_m = 2m\tan(\pi/m)$ by Lemma 1.5.3. □

Our first improvement follows now, but only in the case when $\|\cdot\| = \|\cdot\|_4$; we present this now.

Theorem 3.3.3. Suppose $N \geq 2$ is an integer and $f : (\mathbb{C}, \|\cdot\|_4) \to (\mathbb{C}, \|\cdot\|_4)$ is an N-centred mapping which is L-Lipschitz and c-co-Lipschitz. Then,

$$\frac{c}{L} \leq \frac{1}{N+\varepsilon_0} \quad \text{where} \quad \varepsilon_0 = \frac{N(N-1)}{24N^2 - 9N - 2}.$$

Proof. Suppose, for a contradiction, that $L/c < N + \varepsilon_0$. Without loss of generality, suppose $c = 1$. Hence, by Theorem 3.1.2, $L = N + \varepsilon$ for some $\varepsilon \in (0, \varepsilon_0)$. By Corollary 1.2.11, as f is 1-co-Lipschitz, $\|f(z)\|_4 \geq \|z\|_4$ for each $z \in \mathbb{C}$. By Lemma 3.3.1,

$$\operatorname{dist}_4 \left(f\left(Rv_1\right), \mathbb{D}^R\right) \leq 8\varepsilon R \quad \text{for each } R > 0.$$

That is, for each $R > 0$ there exists $j = j(R) \in [4]$ such that

$$\operatorname{dist}_4 \left(f(Rv_1), \mathcal{D}^R_{j(R)}\right) \leq 8\varepsilon R. \tag{3.3.8}$$

As rotations by integer multiples of $\pi/2$ are isometries under the polygonal 4-norm, we can assume without loss of generality that $j(1) = 1$. Note the edge length of $\partial B^4_\rho(0)$ is 2ρ and, as $\varepsilon_0 \leq 1/8$, that $2\rho > 2 \cdot 8\varepsilon\rho$ for every $\rho > 0$. Let $g(t) = f(tv_1)$. Since $\mathscr{L}_4 = 8$ and $g\big|_{(0,+\infty)} : (0, +\infty) \to \mathbb{C}$ is continuous, one may conclude by (3.3.8) and Proposition 1.5.6 that $j(\rho) = j(1) = 1$ for all $\rho > 0$, i.e. $\operatorname{dist}_4 (f(\rho v_1), \mathcal{D}^\rho_1) \leq 8\varepsilon\rho$ for each $\rho > 0$.

By Lemma 3.3.2 there exists $P_0 > 0$ such that if $\rho \geq P_0$ and $k \in \{2N, 2N+1\}$, then

$$\left\| f(\rho w_k) - R^f_k(\rho) v_k \right\|_4 \leq 8\varepsilon\rho + 2\varepsilon\rho \left(4 - \frac{k-1}{N}\right) = 16\varepsilon\rho - \frac{2\varepsilon(k-1)\rho}{N}.$$

Fix $r \geq P_0$ and let $s = s(r) > r$ be as in Lemma 1.5.9 i). Then, by Remark 3.2.7 and using the property of $\|\cdot\|_4$ that $\|av_{2N} + bv_{2N+1}\|_4 = |a| + |b|$ for all $a, b \in \mathbb{R}$,

$$\|R^f_{2N+1}(s)v_{2N+1} - R^f_{2N}(r)v_{2N}\|_4 = R^f_{2N+1}(s) + R^f_{2N}(r) \geq s + r.$$

By Lemma 1.5.9 i),iii), we get $s = rN/(N-1)$. Taking $k = 2N$, then $w_k = w_{2N} = v_2$, so since rotation by $\pi/2$ is an isometry of $\|\cdot\|_4$, we may apply Lemma 1.5.9 ii) to get

$$\|rw_k - sw_{k+1}\|_4 = r\tan\alpha_0 = \frac{r}{N-1}. \tag{3.3.9}$$

Hence,

$$\begin{aligned}
&\|f(sw_{k+1}) - f(rv_k)\|_4 \\
&\qquad \geq \|R^f_{k+1}(s)v_{k+1} - R^f_k(r)v_k\|_4 - \|f(rw_k) - R^f_k(r)v_k\|_4 - \|f(sw_{k+1}) - R^f_{k+1}(s)v_{k+1}\|_4 \\
&\qquad \geq (s+r) - \left(16\varepsilon r - \frac{2\varepsilon(2N-1)r}{N}\right) - \left(16\varepsilon s - \frac{2\varepsilon(2N)s}{N}\right) \\
&\qquad = (r+s) - \left(12\varepsilon r + \frac{2\varepsilon r}{N}\right) - 12\varepsilon s \\
&\qquad = (r+s)\,(1-12\varepsilon) - \frac{2\varepsilon r}{N} \\
&\qquad = \frac{r}{N-1}\left((1-12\varepsilon)(2N-1) - 2\varepsilon\left(1 - \frac{1}{N}\right)\right).
\end{aligned} \tag{3.3.10}$$

By the choice of ε_0, note $(1-12\varepsilon)(2N-1) - 2\varepsilon(1-(1/N)) > N+\varepsilon$. Therefore, as f is $(N+\varepsilon)$-Lipschitz, using (3.3.9) and (3.3.10), we get

$$\begin{aligned}
\|f(sw_{2N+1}) - f(rw_{2N})\|_4 &\geq \frac{r}{N-1}\left((1-12\varepsilon)(2N-1) - 2\varepsilon\left(1-\frac{1}{N}\right)\right) \\
&> (N+\varepsilon)\,\frac{r}{N-1} \geq \|f(sw_{2N+1}) - f(rw_{2N})\|_4.
\end{aligned}$$

This contradiction then implies the result. □

Corollary 3.3.4. Suppose $f : (\mathbb{C}, \|\cdot\|_4) \to (\mathbb{C}, \|\cdot\|_4)$ is a 2-fold mapping which is L-Lipschitz and c-co-Lipschitz. Then,

$$\frac{c}{L} \leq \frac{1}{2+(1/38)}.$$

Proof. This follows by Theorem 1.2.7 and Theorem 3.3.3. □

Our aim now is to produce a similar estimate for centred Lipschitz quotient mappings, but in general polygonal norms $\|\cdot\|_m$, rather than simply $\|\cdot\|_4$. The first obstacle occurs from the constraints in Lemma 1.5.10.

Lemma 3.3.5. Suppose m is a multiple of 4, $N \geq 1$ is an integer and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$

is an N-centred mapping which is 1-co-Lipschitz and $(N+\varepsilon)$-Lipschitz, where $\varepsilon \in (0, \varepsilon_1)$ and

$$\varepsilon_1 = \frac{\tan^2(\pi/m)}{m(1-\tan^2(\pi/m))}.$$

Let $r > 0$ be such that $|\arg(f \circ \gamma_r)(0)| < 2\pi/m$. Then, for each $k \in [mN]$, either,

$$R_k(r) \le R_{k+1}(r) \le R_k(r) \sec\left(\frac{2\pi}{m}\right) \qquad \text{or} \qquad R_{k+1}(r) \le R_k(r) \le R_{k+1}(r) \sec\left(\frac{2\pi}{m}\right),$$

where $R_j(r) = R_j^f(r)$ is defined in Definition 3.2.6.

Proof. For brevity, as $r > 0$ is fixed, we simply write R_j instead of $R_j(r) = R_j^f(r)$ for each $j \in [mN]$. Fix $k \in [mN]$ and suppose, without loss of generality, that $R_k \le R_{k+1}$. Suppose, for a contradiction, that $R_{k+1} > R_k \sec(2\pi/m)$. By Corollary 1.2.11, Lemma 1.5.7 and Remark 3.2.7, observe that $\|R_{k+1}v_{k+1} - R_k v_k\|_m \ge R_k \tan(2\pi/m) \ge r \tan(2\pi/m)$.

Also, as $R_j, R_{j+1} \ge r$ for each $j \in [mN]$, it follows by Lemma 1.5.3 and Proposition 1.5.6 that

$$\|R_{j+1}v_{j+1} - R_j v_j\|_m \ge \frac{r}{m}\mathscr{L}_m = 2r\tan\left(\frac{\pi}{m}\right) \qquad \text{for each } j \in [mN].$$

Therefore,

$$\begin{aligned} \text{length}_m\left(f\left(\partial B_r(0)\right)\right) \ge \sum_{j=1}^{mN} \|R_{j+1}v_{j+1} - R_j v_j\|_m &\ge 2r(mN-1)\tan\left(\frac{\pi}{m}\right) + r\tan\left(\frac{2\pi}{m}\right) \\ &= 2r\tan\left(\frac{\pi}{m}\right)\left(mN + \frac{\tan^2(\pi/m)}{1-\tan^2(\pi/m)}\right). \end{aligned}$$

However, as f is $(N+\varepsilon)$-Lipschitz and $\varepsilon < \varepsilon_1$,

$$\begin{aligned} \text{length}_m\left(f\left(\partial B_r(0)\right)\right) \le (N+\varepsilon)r\mathscr{L}_m &= 2(N+\varepsilon)rm\tan\left(\frac{\pi}{m}\right) \\ &< 2r\tan\left(\frac{\pi}{m}\right)\left(mN + \frac{\tan^2(\pi/m)}{1-\tan^2(\pi/m)}\right), \end{aligned}$$

providing the required contradiction. □

Corollary 3.3.6. Suppose m is a multiple of 4, $N \ge 1$ is an integer and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ is an N-centred mapping which is 1-co-Lipschitz and $(N+\varepsilon)$-Lipschitz, where

$\varepsilon \in (0, \varepsilon_1)$ and

$$\varepsilon_1 = \frac{\tan^2(\pi/m)}{m(1-\tan^2(\pi/m))}.$$

Let $r > 0$. If $|\arg(f \circ \gamma_r)(0)| < 2\pi/m$, then

$$\|R_{j+1}(r)v_{j+1} - R_j(r)v_j\|_m = (R_j(r) + R_{j+1}(r)) \tan\left(\frac{\pi}{m}\right) \qquad \text{for all } j \in [mN],$$

where $R_j(r) = R_j^f(r)$ is defined in Definition 3.2.6.

Proof. This follows simply by Lemma 1.5.10 and Lemma 3.3.5. □

Following the framework of the proof of Theorem 3.3.3, we now need to be able to find a lower bound for $\|R_{k+1}(s)v_{k+1} - R_k(r)v_k\|_m$, in particular, to obtain a better estimate for this than the trivial estimate we obtain via Proposition 1.5.6. To obtain such a bound, we first estimate the size of the norm of primitive points of a centred Lipschitz quotient mapping.

Lemma 3.3.7. Let m be a multiple of 4, $N \geq 1$ be an integer and suppose $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ is an N-centred mapping which is 1-co-Lipschitz and $(N+\varepsilon)$-Lipschitz, where $\varepsilon \in (0, \varepsilon_1)$ and

$$\varepsilon_1 = \frac{\tan^2(\pi/m)}{m(1-\tan^2(\pi/m))}.$$

Let $\rho > 0$ and γ_ρ be the standard parametrisation of $\partial B_\rho^m(0)$. Suppose $|\arg(f \circ \gamma_\rho)(0)| < 2\pi/m$. Then,

$$\rho \leq R_k(\rho) \leq (1 + m\varepsilon)\rho \qquad \text{for each } k \in [mN],$$

where $R_k(\rho) = R_k^f(\rho)$ is defined in Definition 3.2.6.

Proof. For ease of notation, and since $\rho > 0$ is fixed, we shall simply write t_k and R_k instead of $t_k(\rho)$ and $R_k^f(\rho)$, respectively. As f is N-centred and 1-co-Lipschitz, note by Remark 3.2.7 that $R_k = R_k(\rho) \geq \rho$ for each $k \in [mN]$. By Corollary 3.3.6, for each $k \in [mN-1]$,

$$\operatorname{length}_m \left((f \circ \gamma_\rho)\big|_{[t_k, t_{k+1}]} \right) \geq \|R_{k+1}v_{k+1} - R_k v_k\|_m = (R_k + R_{k+1}) \tan\left(\frac{\pi}{m}\right). \tag{3.3.11}$$

Summing (3.3.11) over $0 \leq k \leq mN-1$,

$$\operatorname{length}_m (f \circ \gamma_\rho) \geq \sum_{k=1}^{mN-1} \operatorname{length}_m \left((f \circ \gamma_\rho)\big|_{[t_k, t_{k+1}]} \right) + \|R_{mN}v_{mN} - R_1 v_1\|_m$$

$$
\begin{aligned}
&\geq \sum_{k=1}^{mN} (R_k + R_{k+1}) \tan\left(\frac{\pi}{m}\right) \\
&= 2\tan\left(\frac{\pi}{m}\right) \sum_{k=1}^{mN} R_k.
\end{aligned}
$$

As f is $(N+\varepsilon)$-Lipschitz and since $\mathscr{L}_m = 2m\tan(\pi/m)$, see Lemma 1.5.3, observe that

$$
\text{length}_m\,(f \circ \gamma_\rho) \leq (N+\varepsilon)\text{length}_m\left(\partial B_\rho^m(0)\right) = (N+\varepsilon)\cdot 2m\rho\tan\left(\frac{\pi}{m}\right).
$$

Hence, $\sum_{k=1}^{mN} R_k \leq (N+\varepsilon)m\rho$. Let $K \in [mN]$ be such that $R_k \leq R_K$ for every $k \in [mN]$. Then, as $R_k \geq \rho$,

$$
\sum_{k=1}^{mN} R_k = R_K + \sum_{k \neq K} R_k \geq (mN-1)\rho + R_K.
$$

Therefore, $R_k \leq R_K \leq (N+\varepsilon)m\rho - (mN-1)\rho = (1+m\varepsilon)\rho$ for each $k \in [mN]$. □

We are now able to obtain an estimate for $\|R_{k+1}(s)v_{k+1} - R_k(r)v_k\|_m$. However, we can only produce such an estimate when the ratio between the Lipschitz and co-Lipschitz constants is sufficiently close to N.

Lemma 3.3.8. Suppose m is a multiple of 4, $N \geq 2$ is an integer and $r > 0$. Let $s > 0$ be as in Lemma 1.5.9 i), and let γ_r and γ_s denote the standard parametrisation of $\partial B_r^m(0)$ and $\partial B_s^m(0)$, respectively. Suppose $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ is an N-centred mapping which is 1-co-Lipschitz and $(N+\varepsilon_2)$-Lipschitz, where

$$
\varepsilon_2 = \frac{2\tan^2(\pi/m)}{m\left(N + (N-2)\tan^2(\pi/m)\right)}.
$$

Suppose $|\arg(f\circ\gamma_r)(0)|, |\arg(f\circ\gamma_s)(0)| < 2\pi/m$. Then, for each $k \in [mN]$,

$$
\begin{aligned}
\|R_{k+1}(s)v_{k+1} - R_k(r)v_k\|_m &= (R_k(r) + R_{k+1}(s))\tan\left(\frac{\pi}{m}\right) \\
&\geq r\left(2 + \frac{2\tan^2(\pi/m)}{N + (N-2)\tan^2(\pi/m)}\right)\tan\left(\frac{\pi}{m}\right).
\end{aligned}
$$

Proof. Suppose $\mathrm{Lip}(f) = N + \varepsilon$ for some $\varepsilon \in (0, \varepsilon_2)$. We first show that $R_k(r) \leq R_{k+1}(s)$.

Recall, by Lemma 1.5.9 (i),(iii), that

$$s = r\left(1 + \tan\left(\frac{\pi}{m}\right)\tan\alpha_0\right) = r\left(1 + \frac{2\tan^2(\pi/m)}{N + (N-2)\tan^2(\pi/m)}\right).$$

Since $N \geq 2$ and $\tan^2(\pi/m) > 0$, note $2(1-\tan^2(\pi/m)) \leq 2 \leq N + (N-2)\tan^2(\pi/m)$. Hence,

$$\varepsilon < \frac{2\tan^2(\pi/m)}{m\,(N + (N-2)\tan^2(\pi/m))} \leq \frac{\tan^2(\pi/m)}{m(1-\tan^2(\pi/m))}.$$

Therefore the condition $\varepsilon < \varepsilon_1$ of Lemma 3.3.7 is satisfied. Hence,

$$r \leq R_k(r) \leq r(1+m\varepsilon) \quad \text{and} \quad s \leq R_{k+1}(s) \leq s(1+m\varepsilon).$$

Thus,

$$R_k(r) \leq (1+m\varepsilon)r \leq \left(1 + \frac{2\tan^2(\pi/m)}{N + (N-2)\tan^2(\pi/m)}\right) r = s \leq R_{k+1}(s).$$

We next show that $R_{k+1}(s) \leq R_k(r)\sec(2\pi/m)$. Indeed, since $\varepsilon < \varepsilon_2$ and $N \geq 2$,

$$\varepsilon \leq \frac{2\tan^2(\pi/m)}{m\,(N + (N-2)\tan^2(\pi/m))} \leq \frac{\tan^2(\pi/m)}{m}.$$

Observe that $\cos^4(\pi/m) > 2\cos^2(\pi/m) - 1 = \cos(2\pi/m)$, and so

$$\frac{\sec(2\pi/m)}{\sec^2(\pi/m)} > \sec^2(\pi/m) = \tan^2(\pi/m) + 1.$$

Hence, $\varepsilon \leq \frac{1}{m}\left(-1 + \sec(2\pi/m)/\sec^2(\pi/m)\right)$. Therefore,

$$(1+m\varepsilon)\left(1+\tan^2\left(\frac{\pi}{m}\right)\right) = (1+m\varepsilon)\sec^2\left(\frac{\pi}{m}\right) \leq \sec\left(\frac{2\pi}{m}\right).$$

So, as $N \geq 2$,

$$\begin{aligned} s(1+m\varepsilon) = r(1+m\varepsilon)\left(1 + \frac{2\tan^2(\pi/m)}{N + (N-2)\tan^2(\pi/m)}\right) &\leq r(1+m\varepsilon)\left(1+\tan^2\left(\frac{\pi}{m}\right)\right) \\ &\leq r\sec\left(\frac{2\pi}{m}\right). \end{aligned}$$

Finally,

$$R_{k+1}(s) \leq s(1 + m\varepsilon) \leq r \sec\left(\frac{2\pi}{m}\right) \leq R_k(r) \sec\left(\frac{2\pi}{m}\right).$$

By Lemma 1.5.10, as $R_k(r) \leq R_{k+1}(s) \leq R_k(r)\sec(2\pi/m)$, it follows

$$\|R_{k+1}(s)v_{k+1} - R_k(r)v_k\|_m = (R_k(r) + R_{k+1}(s)) \tan\left(\frac{\pi}{m}\right) \qquad \text{for each } k \in [mN].$$

Now, as $R_k(r) \geq r$ and $R_{k+1}(s) \geq s$, we obtain the result via Lemma 1.5.9. □

With these estimates now in place, we are able to follow the framework of the proof of Theorem 3.3.3 to obtain an improved estimate for the upper bound of the ratio of constants of Lipschitz quotient mappings in polygonal norms.

Theorem 3.3.9. Let $m \geq 8$ be a multiple of 4, $N \geq 2$ be an integer and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be an N-centred mapping which is c-co-Lipschitz and L-Lipschitz. Then,

$$\frac{c}{L} \leq \frac{1}{N + \eta},$$

where $\eta = \min(\varepsilon_2, \varepsilon_3)$ and

- $\varepsilon_2 = \dfrac{2\tan^2(\pi/m)}{m(N + (N-2)\tan^2(\pi/m))}$,
- $\varepsilon_3 = \dfrac{N(N-1)\tan^2(\pi/m)}{2N + mN(N + (N-1)\tan^2(\pi/m))\,(1 + \operatorname{cosec}^2(\pi/m)) + (N-2)\tan^2(\pi/m)}$.

Proof. Suppose, for a contradiction, that $L/c < N + \eta$ and $c = 1$, i.e. by Theorem 3.1.2, that $L = N + \varepsilon$ for some $\varepsilon \in (0, \eta)$.

By Lemma 3.3.1, for each $R > 0$ there exists $j = j(R) \in [m]$ such that

$$\operatorname{dist}_m\left(f(Rv_1), \mathcal{D}^R_{j(R)}\right) \leq \frac{2m\varepsilon R}{\sin(2\pi/m)}. \tag{3.3.12}$$

As rotations by integer multiples of $2\pi/m$ are isometries in the polygonal m-norm, we may assume without loss of generality that $j(1) = 1$.

To proceed, we first show that $\varepsilon < \frac{1}{m}\sin^2(\pi/m)$. Consider first when $N \geq 3$. Note as $\varepsilon \leq \varepsilon_2$ that it suffices to verify that $\varepsilon_2 < \frac{1}{m}\sin^2(\pi/m)$ when $N \geq 3$. This is equivalent to showing that $2 < N - 2\sin^2(\pi/m)$, which is satisfied since $m \geq 8$. Suppose now $N =$

2. We claim $\varepsilon_3 < \frac{1}{m}\sin^2(\pi/m)$ in such a case. This is equivalent to showing that $m < 2\cos^2(\pi/m) + m\left(\cos^2(\pi/m) + 2/\sin^2(\pi/m)\right)$, which is satisfied since $m \geq 8$. Therefore, in either case $\varepsilon < \frac{1}{m}\sin^2(\pi/m)$.

Note by Lemma 1.5.3 that the edge length of $\partial B_\rho^m(0)$ is $2\rho\tan(\pi/m)$. So, as $\varepsilon < \frac{1}{m}\sin^2(\pi/m)$, observe that $2\rho\tan(\pi/m) > 4m\varepsilon\rho/\sin(2\pi/m)$ for all $\rho > 0$. Since $g\big|_{(0,+\infty)} : (0,+\infty) \to \mathbb{C}$ given by $g(\rho) = f(\rho v_1)$ is continuous, we conclude by (3.3.12) that $j(\rho) = j(1) = 1$ for all $\rho > 0$, i.e.

$$\mathrm{dist}_m\left(f(\rho v_1), \mathcal{D}_1^\rho\right) \leq \frac{2m\varepsilon\rho}{\sin(2\pi/m)} \qquad \text{for all } \rho > 0. \tag{3.3.13}$$

Now, by Lemma 3.3.2 there exists $P_0 > 0$ such that if $\rho \geq P_0$ and $k \in \left\{\frac{mN}{2}, 1 + \frac{mN}{2}\right\}$, then

$$\|f(\rho w_k) - R_k^f(\rho) v_k\|_m \leq \frac{2m\varepsilon\rho}{\sin(2\pi/m)} + 2\varepsilon\rho\tan\left(\frac{\pi}{m}\right)\left(m - \frac{k-1}{N}\right).$$

We claim that $|\arg(f\circ\gamma_\rho)(0)| < 2\pi/m$ for all $\rho \geq P_0$. Indeed, note that as $N \geq 2$,

$$\varepsilon < \varepsilon_2 \leq \frac{\tan^2(\pi/m)}{m} \leq \frac{\tan(2\pi/m)\sin(2\pi/m)\cos(\pi/m)}{2m}, \tag{3.3.14}$$

where the final inequality follows by considering the function $h(y) = 4y^5 - 4y^2 + 2$ where $y = \cos(\pi/m) \in [0,1]$, and showing that $h(y) \geq h((2/5)^{1/3}) > 0$ for all $y \in [0,1]$.

Therefore, as $c = 1$ note by Corollary 1.2.11 and Lemma 1.5.17 that

$$|(f\circ\gamma_\rho)(0)| \geq \cos\left(\frac{\pi}{m}\right)\|(f\circ\gamma_\rho)(0)\|_m \geq \rho\cos\left(\frac{\pi}{m}\right).$$

So, if $\rho \geq P_0$ it follows by (3.3.13), (3.3.14) and since the Euclidean distance from $f(\rho v_1)$ to $\mathcal{D}_1$ is at most $\mathrm{dist}_m(f(\rho v_1), \mathcal{D}_1^\rho)$, that

$$|\tan(\arg(f\circ\gamma_\rho)(0))| \leq \frac{1}{\rho\cos(\pi/m)}\cdot\frac{2m\varepsilon\rho}{\sin(2\pi/m)} = \frac{2m\varepsilon}{\cos(\pi/m)\sin(2\pi/m)} < \tan\left(\frac{2\pi}{m}\right).$$

Therefore $|\arg(f\circ\gamma_\rho)(0)| < 2\pi/m$ provided $\rho \geq P_0$.

Fix $r > P_0$ and $s > 0$ as in Lemma 1.5.9. Then, as $\varepsilon < \varepsilon_2$ and as f is 1-co-Lipschitz we conclude by Lemma 1.5.9 and Lemma 3.3.8 that

$$\|R^f_{1+(mN/2)}(s)v_{1+(mN/2)} - R^f_{mN/2}(r)v_{mN/2}\|_m \geq r\left(2 + \frac{2\tan^2(\pi/m)}{N + (N-2)\tan^2(\pi/m)}\right)\tan\left(\frac{\pi}{m}\right).$$

Let t_m, s_m and c_m represent $\tan(\pi/m), \sin(\pi/m)$ and $\cos(\pi/m)$, respectively. Let $k = mN/2$, then

$$\begin{aligned}
&\|f(sw_{1+k}) - f(rw_k)\|_m \\
&\geq \|R^f_{1+k}(s)v_{1+k} - R^f_k(r)v_k\|_m - \|f(rw_k) - R^f_k(r)v_k\|_m - \|f(sw_{1+k}) - R^f_{1+k}(s)v_{1+k}\|_m \\
&\geq 2rt_m + \frac{2rt_m^3}{N + (N-2)t_m^2} - \frac{m\varepsilon r}{s_m c_m} - 2\varepsilon r t_m\left(\frac{m}{2} + \frac{1}{N}\right) - \frac{m\varepsilon s}{s_m c_m} - m\varepsilon s t_m \\
&= \frac{2rt_m(N + (N-1)t_m^2)}{N + (N-2)t_m^2} - m\varepsilon\,(r+s)\left(t_m + \frac{1}{s_m c_m}\right) - \frac{2\varepsilon r t_m}{N} \\
&= \frac{2rt_m}{N + (N-2)t_m^2}\left(\left(N + (N-1)t_m^2\right)\left(1 - m\varepsilon\left(1 + \frac{1}{s_m^2}\right)\right) - \frac{\varepsilon(N + (N-2)t_m^2)}{N}\right), \quad (3.3.15)
\end{aligned}$$

using for the last equality Lemma 1.5.9 i), iii) from which we deduce that

$$(r+s)\left(t_m + \frac{1}{2s_m c_m}\right) = \frac{2rt_m}{N + (N-2)t_m^2}\left(1 + \frac{1}{2s_m^2}\right).$$

Note as k is an integer multiple of N, that $w_k = v_{m/2}$. As rotation by $((m/2)-1)2\pi/m$, mapping v_1 to $v_{m/2}$, is an isometry of $\|\cdot\|_m$, we may use Lemma 1.5.9 ii) to get

$$\|rw_k - sw_{k+1}\|_m = r\tan\alpha_0 = \frac{2rt_m}{N + (N-2)t_m^2}.$$

As f is $(N+\varepsilon)$-Lipschitz, it follows that

$$\|f(sw_{1+k}) - f(rw_k)\|_m \leq (N+\varepsilon)\|sw_{k+1} - rv_k\|_m = (N+\varepsilon)\frac{2rt_m}{N + (N-2)t_m^2}.$$

However, as $\varepsilon < \varepsilon_3$, this contradicts (3.3.15). □

Corollary 3.3.10. Let m be a multiple of 4 and $f : (\mathbb{C}, \|\cdot\|_m) \to (\mathbb{C}, \|\cdot\|_m)$ be a 2-fold mapping which is c-co-Lipschitz and L-Lipschitz. Then,

$$\frac{c}{L} \leq \frac{1}{2+\varepsilon_3}, \qquad \text{where} \qquad \varepsilon_3 = \frac{2\tan^2(\pi/m)}{4 + 2m(2 + \tan^2(\pi/m))\,(1 + \operatorname{cosec}^2(\pi/m))}.$$

Proof. This follows by Theorem 1.2.7 and Theorem 3.3.9. □

In this section we improved the best known estimate for the ratio of constants for planar Lipschitz quotient mappings in polygonal norms, when the number of edges is a multiple of four. The next section concerns a different topic altogether, that of inscribing equilateral polygons in centrally symmetric convex bodies in the plane. This topic is then related back to the context of Lipschitz quotient mappings in Chapter 6.

CHAPTER 4

EQUILATERAL POLYGONS IN CENTRALLY SYMMETRIC CONVEX BODIES

In this section we move away from the study of planar Lipschitz quotient mappings and concern ourselves with the existence of equilateral polygons in centrally symmetric, convex bodies. We show that in the boundary of any norm in $\mathbb{C}$ one may inscribe equilateral polygons, i.e. those whose sides all have equal length, and get results about sets of all possible side lengths of such polygons.

4.1 Introduction

Let X be a normed space and $K \subseteq X$. We say K is a *convex body* if it is a compact, convex set with non-empty interior. The *Minkowski functional of* K is the function $p_K : X \to [0, +\infty) \cup \{+\infty\}$ where

$$p_K(x) := \inf \{r > 0 : x \in rK\} ;$$

here we use the convention that $\inf \emptyset = +\infty$. Note that for every norm a closed ball of unit radius centred at the origin is a centrally symmetric, convex body. Moreover, if K is a centrally symmetric, convex body then $K = \{x \in X : p_K(x) \leq 1\}$ and p_K defines a norm on X; cf. [9, Chapter 5]. Therefore there is a one-to-one correspondence between the collection of centrally symmetric, convex bodies in X and the collection of norms on X.

It may seem natural to extend certain geometric notions from the Euclidean setting, where we consider the standard ball as the centrally symmetric, convex body, to a wider class of bodies where one uses the induced norm for any questions on distance. A simple first example to consider, in the Euclidean setting, is the following question: for each unit vector and each $n \geq 3$, can one find a sequence of $n-1$ other unit vectors such that the distance between adjacent vectors is constant? Indeed, if $z \in \partial B_1^{|\cdot|}(0)$ and $n \geq 3$, one can consider the tuple $P = (z_1, \ldots, z_n)$, where $z_j = e^{\varphi_0 + 2(j-1)\pi/n}$ and $\varphi_0 = \arg(z)$; see Figure 4.1 below.

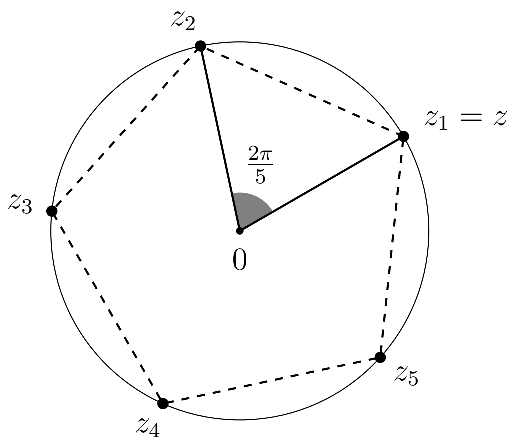

Figure 4.1: Example of P when $n = 5$ and $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$.

The rest of this section focuses on the question of finding similar tuples for boundaries of different centrally symmetric convex bodies in the plane. We first proceed by formally defining what we mean by polygons inscribed in a centrally symmetric convex body in the plane.

Definition 4.1.1. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $n \geq 2$ and $P = (x_1, \ldots, x_n) \in \mathbb{C}^n$ be an n-tuple of distinct unit vectors, in $\|\cdot\|$, ordered in the anticlockwise direction on $\partial B_1^{\|\cdot\|}(0)$. For each $j \in [n]$ we say that x_j is a *vertex* of P and the pair $[x_j, x_{j+1}]$ is an *edge* of P, where we identify x_{n+1} with x_1, and has *edge length* $\|x_{j+1} - x_j\|$. We say $(P, \|\cdot\|)$, or simply P, is an *inscribed polygon in* $\|\cdot\|$.

Somewhat abusing notation we sometimes treat P as an ordered set of vertices; if x_j is a vertex of P we write $x_j \in P$ and we consider polygons with vertices $P \setminus \{x_j\}$, for example.

If P is an inscribed polygon such that $|P| = n$, i.e. P has n distinct vertices, then P is said to be an *(inscribed) n-gon.* If $n = 3$ or $n = 4$, we say that P is a *triangle* or *quadrilateral*, respectively.

We say a polygon $Q = (w_1, \ldots, w_n)$ is equivalent to the polygon $P = (z_1, \ldots, z_n)$, denoted $P \sim_{\|\cdot\|} Q$, if the set of vertices of P and Q coincide, i.e. there exists $k \in [n]$ such that $w_j = z_{(j+k) \pmod n}$ for each $j \in [n]$.

Define

$$\mathcal{F}_n^{\|\cdot\|} := \{P \in \mathbb{C}^n : P \text{ is an inscribed } n\text{-gon in } \|\cdot\|\}$$

to be the collection of all inscribed n-gons for a fixed norm $\|\cdot\|$. We say that $P \in \mathcal{F}_n^{\|\cdot\|}$ is an $\|\cdot\|$-*equilateral polygon* if every edge of P has the same edge length, i.e. $\|x_{j+1}-x_j\| = \|x_2-x_1\|$ for all $j \in [n]$. Define

$$E_n^{\|\cdot\|} := \left\{P \in \mathcal{F}_n^{\|\cdot\|} : P \text{ is an } \|\cdot\|\text{-equilateral polygon}\right\}.$$

For each $P \in E_n^{\|\cdot\|}$, let $e(P, \|\cdot\|)$ denote the edge length of $(P, \|\cdot\|)$.

We will often consider equilateral polygons inscribed in the unit sphere of a polygonal norm. For the set of these polygons we will simply write E_n^m instead of $E_n^{\|\cdot\|_m}$ and E_n^∞ instead of $E_n^{\|\cdot\|_\infty}$.

If the norm is clear from context, we simply write n-gon, equilateral n-gon, $\mathcal{F}_n$, E_n and $e(P)$.

Remark 4.1.2. More restrictive structures have already been considered in [4], [28]. They define a *regular m-gon* to be a cyclically ordered set $p_1, \ldots, p_m$ in $\mathbb{R}^n$ such that $\|p_j - p_k\|_\infty = \|p_{j+l} - p_{k+l}\|_\infty$ for all $j, k, l \in [m]$, where indices are considered modulo m. Questions regarding the existence of such structures are then investigated.

Observe $\sim_{\|\cdot\|}$ is an equivalence relation. Therefore, when we consider the uniqueness of equilateral n-gons in $\|\cdot\|$, we consider uniqueness up to the equivalence classes as determined by $\sim_{\|\cdot\|}$.

When considering inscribed 2-gons in a norm $\|\cdot\|$ on $\mathbb{C}$, observe that any such polygon is automatically equilateral. Further, any two vertices contained in the unit disc form an equilateral 2-gon. The interesting cases only arise when we consider $n \geq 3$. Thus throughout the rest of this thesis, we generally do not concern ourselves with the trivial $n = 2$ case.

In the standard Euclidean geometry, for each $n \geq 3$ and every $z \in \partial B_1^{|\cdot|}(0)$ there exists a unique, up to equivalence, $P \in E_n$ such that $z \in P$; for this P we have $e(P, |\cdot|) = 2\sin(\pi/n)$. The aim of this chapter is to understand when such a framework may exist for a general norm $\|\cdot\|$ on $\mathbb{C}$ and is motivated by the following observation. Suppose $\|\cdot\| = \|\cdot\|_\infty$, $P = (1, i, -1, -i)$ and $Q = (1+i, -1+i, -1-i, 1-i)$. Then $P, Q \in E_4^\infty$, but $e(P) = 1$ and $e(Q) = 2$; see Figure 4.2 below. Therefore the situation as described by the Euclidean norm is not universal over all norms on $\mathbb{C}$.

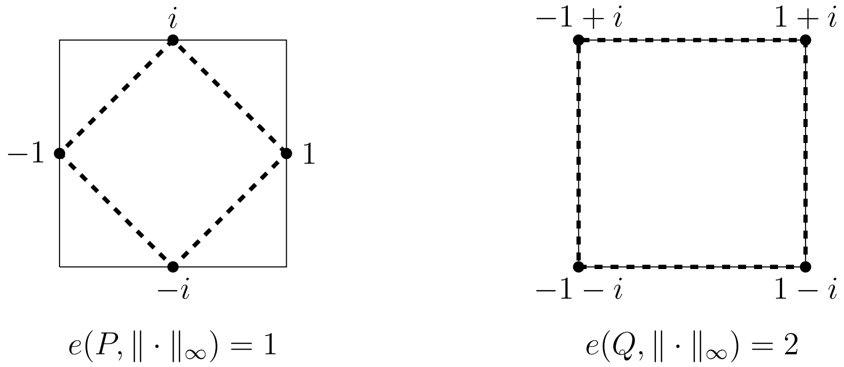

Figure 4.2: Two equilateral 4-gons inscribed in $\|\cdot\|_\infty$ which have different edge lengths.

4.2 General properties of inscribed polygons

As a consequence of the Monotonicity Lemma 1.4.2, we can show that if a vertex and its neighbouring two vertices in an inscribed polygon have maximal edge length, then all vertices have maximal possible length from the initial vertex, also.

Corollary 4.2.1. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $n \geq 3$ and $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$. If $\|x_1 - x_2\| = \|x_1 - x_n\| = 2$, then $\|x_1 - x_j\| = 2$ for all $2 \leq j \leq n$.

In particular, if $P \in E_n^{\|\cdot\|}$ and $e(P) = 2$, then $\|x_j - x_k\| = 2$ for each $j \neq k$.

Proof. If $n = 3$ the result is vacuously true, hence suppose $n \geq 4$. If $j = 2$ or $j = n$ the result follows by the hypothesis, so suppose $3 \leq j \leq n - 1$. Note, if $x_j \in (x_1, -x_1]_{\|\cdot\|}$ then by the Monotonicity Lemma 1.4.2 and since $\|x - y\| \leq 2$ for all $x, y \in \partial B_1^{\|\cdot\|}(0)$,

$$2 \geq \|x_j - x_1\| \geq \|x_2 - x_1\| = 2,$$

so $\|x_j - x_1\| = 2$. Similarly, if $x_j \in [-x_1, x_1)_{\|\cdot\|}$, then $2 \geq \|x_j - x_1\| \geq \|x_n - x_1\| = 2$ and so $\|x_j - x_1\| = 2$. □

Next we prove the intuitive notion that the minimum distance between any two vertices of an inscribed polygon will be attained by two adjacent vectors, which can be readily verified in the Euclidean case.

Corollary 4.2.2. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$, $n \geq 2$. Then there exists $j_0 \in [n]$ such that

$$\min_{j \neq k} \|x_j - x_k\| = \|x_{j_0+1} - x_{j_0}\|.$$

Proof. Note for each fixed $j \in [n]$ that $\min_{\substack{k\in[n]\\ k\neq j}} \|x_j - x_k\| = \min\{\|x_{j+1} - x_j\|, \|x_j - x_{j-1}\|\}$ by the Monotonicity Lemma 1.4.2. □

The following result shows that the collection of equilateral polygons is preserved when considering norms which differ by a linear transformation.

Proposition 4.2.3. Let $\|\cdot\|$ and $|||\cdot|||$ be two norms on $\mathbb{C}$, $n \geq 3$ and $T : \mathbb{C} \to \mathbb{C}$ be a linear transformation such that $T\left(\partial B_1^{\|\cdot\|}(0)\right) = \partial B_1^{|||\cdot|||}(0)$. Then there exists a bijection $h = h_T : E_n^{\|\cdot\|} \to E_n^{|||\cdot|||}$. Moreover, $e(h(P), |||\cdot|||) = e(P, \|\cdot\|)$ for every $P \in E_n^{\|\cdot\|}$.

Proof. Since $0 \notin T\left(\partial B_1^{\|\cdot\|}(0)\right)$, $\ker(T) = \{0\}$. Moreover, as T is surjective, T is invertible and so $T^{-1}\left(\partial B_1^{|||\cdot|||}(0)\right) = \partial B_1^{\|\cdot\|}(0)$. Without loss of generality, suppose T is orientation preserving. Define $h_T : E_n^{\|\cdot\|} \to E_n^{|||\cdot|||}$ by $h_T((x_1, \ldots, x_n)) = (T(x_1), \ldots, T(x_n))$. To see $h = h_T$ is well-defined, let $P = (x_1, \ldots, x_n) \in E_n^{\|\cdot\|}$. Note as $T\left(\partial B_1^{\|\cdot\|}(0)\right) = \partial B_1^{|||\cdot|||}(0)$ that $h(P) \in \mathcal{F}_n^{|||\cdot|||}$. Further, by the homogeneity of T, $|||T(x_j) - T(x_k)||| = |||T(x_j - x_k)||| = \|x_j - x_k\|$ for each $j, k \in [n]$. Thus, $h(P) \in E_n^{|||\cdot|||}$ and $e(h(P), |||\cdot|||) = e(P, \|\cdot\|)$.

The injectivity of h follows by the injectivity of T acting component-wise on $P \in E_n^{\|\cdot\|}$. To see h is surjective note for each $Q \in E_n^{|||\cdot|||}$ we can consider $P = h_{T^{-1}}(Q)$, then $P \in E_n^{\|\cdot\|}$ and $h_T(P) = Q$. □

Remark 4.2.4. In the above proof of Proposition 4.2.3 we assume $E_n^{\|\cdot\|} \neq \emptyset$. In fact, in Theorem 4.3.5, we show that $E_n^{\|\cdot\|} \neq \emptyset$ for all norms $\|\cdot\|$ and $n \geq 3$. However, even without this fact, it is clear that if $E_n^{\|\cdot\|} = \emptyset$, then $E_n^{|||\cdot|||} = \emptyset$, also.

An important, albeit problematic, class of norms which we shall consider is the collection of *rectilinear norms.*

Definition 4.2.5. A norm $\|\cdot\|$ on $\mathbb{C}$ is rectilinear if the unit sphere $\partial B_1^{\|\cdot\|}(0)$ is a parallelogram, i.e. there exists a linear transformation $T : \mathbb{C} \to \mathbb{C}$ such that $\partial B_1^{\|\cdot\|}(0) = T\left(\partial B_1^{\|\cdot\|_\infty}(0)\right)$.

The following results provide sufficient conditions for a planar norm to be rectilinear.

Lemma 4.2.6. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $x, v \in \partial B_1^{\|\cdot\|}(0)$. If both $[x, v], [x, -v] \subseteq \partial B_1^{\|\cdot\|}(0)$, then $\|\cdot\|$ is rectilinear and $[x, v]$, $[x, -v]$ are non-parallel segments of $\partial B_1^{\|\cdot\|}(0)$.

Proof. First observe that $x \neq v$, as otherwise $0 \in [v, -v] \subseteq \partial B_1^{\|\cdot\|}(0)$. Similarly, $x \neq -v$. Now observe that $[-x, -v], [-x, v] \subseteq \partial B_1^{\|\cdot\|}(0)$. Therefore $\partial B_1^{\|\cdot\|}(0)$ is a parallelogram. □

Corollary 4.2.7. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $P = (A, B, C) \in E_3^{\|\cdot\|}$. If there exists $v \in \partial B_1^{\|\cdot\|}(0)$ such that P is a subset of the closed arc $[-v, v]_{\|\cdot\|}$, then $\|\cdot\|$ is rectilinear. In particular, if $e(P) < 1$ then $\|\cdot\|$ is rectilinear.

Proof. Since A, B and C are distinct and $\|A - B\| = \|A - C\|$ it follows, by the Monotonicity Lemma 1.4.2, that the straight line segment $[V, B] \subseteq \partial B_1^{\|\cdot\|}(0)$ where $V = (C - A)/\|C - A\|$. Similarly, since $\|C - B\| = \|C - A\|$, $[-V, B] \subseteq \partial B_1^{\|\cdot\|}(0)$. Therefore, taking $x = B$ and $v = V$ in Lemma 4.2.6, we conclude $\|\cdot\|$ is rectilinear.

If now $e(P) < 1$, observe there exists $v \in \{\pm A, \pm B, \pm C\}$ such that $P \subseteq [v, -v]_{\|\cdot\|}$. Suppose, for a contradiction, this is false. Then, $P \not\subseteq [A, -A]_{\|\cdot\|}$. Hence, $C \in (-A, A)_{\|\cdot\|}$ and so, by Lemma 1.3.16 (iv), $-C \in (A, -A)_{\|\cdot\|}$. Now, if $B \in (-A, A)_{\|\cdot\|}$, then $P \subseteq [-A, A]_{\|\cdot\|}$, a contradiction. Therefore $B \in (A, -A]_{\|\cdot\|}$. Hence, $B \notin (A, -C]_{\|\cdot\|}$ as otherwise $P \subseteq [C, -C]_{\|\cdot\|}$. So, $B \in (-C, -A]_{\|\cdot\|}$. But then, by the Monotonicity Lemma 1.4.2, $\|A+C\| \leq \|A-B\| = e(P)$. Therefore,

$$2 = 2\|A\| \leq \|A + C\| + \|A - C\| \leq 2e(P) < 2,$$

providing contradiction. □

Next we introduce the notion of an *n-rectilinear pair*. Such pairs will play a fundamental role in our considerations that follow; in particular 3- and 4-rectilinear pairs.

Definition 4.2.8. We say a pair $(\|\cdot\|, n)$ is an n-rectilinear pair if $n \geq 3$ and $\|\cdot\|$ is a rectilinear norm.

The next results determine which equilateral polygons may have maximal edge length.

Lemma 4.2.9. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 4$. Then there exists $P \in E_n^{\|\cdot\|}$ such that $e(P) = 2$ if and only if $(\|\cdot\|, n)$ is a 4-rectilinear pair.

Proof. If $\|\cdot\|$ is rectilinear and $n = 4$, then the four vertices of the unit sphere form a polygon $P \in E_4$ such that $e(P) = 2$. Now suppose there exists $P \in E_n^{\|\cdot\|}$ for some norm $\|\cdot\|$ on $\mathbb{C}$ such that $e(P) = 2$ and $n \geq 4$. Then by a classical result of Petty, see [29, Theorem 4], as $\frac{1}{2}P$ is an equilateral set it then follows that $n = 4$ and $\|\cdot\|$ is rectilinear. □

Example 4.2.10. Observe Lemma 4.2.9 fails if $n = 3$, in the sense that there exist non-rectilinear norms $\|\cdot\|$ and equilateral triangles $P \in E_3^{\|\cdot\|}$ such that $e(P, \|\cdot\|) = 2$. Indeed, consider $\|\cdot\| = \|\cdot\|_6$, the polygonal 6-norm, and the equilateral (Euclidean and $\|\cdot\|_6$) triangle to be given by three vertices v_1, v_3, v_5, see Notation 1.5.2, of the sphere $\partial B_1^{\|\cdot\|_6}(0)$.

Lemma 4.2.11. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $n \geq 3$ and $P = (x_1, \ldots, x_n) \in E_n^{\|\cdot\|}$. If $x_2 = -x_1$ or $x_n = -x_1$, then $(\|\cdot\|, n)$ is a 3-rectilinear pair.

Proof. We shall prove $(\|\cdot\|, n)$ is a 3-rectilinear pair when $x_2 = -x_1$; the proof when $x_n = -x_1$ follows similarly. Note as $x_2 = -x_1$ that $e(P) = 2$. We claim this implies $\|x_3 - x_1\| = 2$. Indeed, as $x_3, \ldots, x_n \in (-x_1, x_1)_{\|\cdot\|}$, it follows by the Monotonicity Lemma 1.4.2 that

$$2 = e(P) = \|x_n - x_1\| \leq \|x_3 - x_1\| \leq \|x_1 - (-x_1)\| = 2.$$

Thus $\|x_3 - x_1\| = 2$ and so $Q = (x_1, x_2, x_3) \in E_3$ with $e(Q) = 2$. Hence, by Corollary 4.2.7, $\|\cdot\|$ is rectilinear.

Further, as $P \in E_n$, $x_2 = -x_1$, $e(P) = 2$ and $\|\cdot\|$ is rectilinear, observe $n = 3$ and P is formed of three vertices of $\partial B_1(0)$. □

We now introduce some simple notation about the minimal and maximal edge length of an inscribed polygon.

Notation 4.2.12. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $n \geq 2$ and $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$. Define:

$$\text{i) } d^+(P) = \max_{j \in [n]} \|x_{j+1} - x_j\|; \qquad \text{ii) } d^-(P) = \min_{j \in [n]} \|x_{j+1} - x_j\|.$$

The next result determines the position of vertices of two inscribed polygons which share a common vertex.

Lemma 4.2.13. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. Suppose $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$ and $Q = (y_1, \ldots, y_n) \in \mathcal{F}_n^{\|\cdot\|}$ share a common vertex $x_1 = y_1$. If:

a) $x_{k+1} \in (x_k, -x_k]_{\|\cdot\|}$ and $y_{k+1} \in (y_k, -y_k]_{\|\cdot\|}$ for each $k \in [n-1]$; and

b) $d^+(P) < \min\left(\|y_2 - y_1\|, \|y_3 - y_2\|, \ldots, \|y_n - y_{n-1}\|\right)$,

then $x_k \in (x_1, y_k)_{\|\cdot\|}$ for each $k = 2, \ldots, n$.

Proof. We shall argue recursively; first we show that $x_2 \in (x_1, y_2)_{\|\cdot\|}$. Indeed, by a) and as $x_1 = y_1$, $x_2, y_2 \in (x_1, -x_1]_{\|\cdot\|}$. Furthermore, by b),

$$\|x_2 - x_1\| \leq d^+(P) < \|y_2 - y_1\| = \|y_2 - x_1\|.$$

Hence $x_2 \in (x_1, y_2)_{\|\cdot\|}$, as otherwise this contradicts the Monotonicity Lemma 1.4.2.

Suppose $x_k \in (x_1, y_k)_{\|\cdot\|}$ for some $k \in \{2, \ldots, n-1\}$. For a contradiction, suppose that $x_{k+1} \in [y_{k+1}, x_1)_{\|\cdot\|}$. Consider a parametrisation $\theta_{x_1} : [0, 1) \to \partial B_1^{\|\cdot\|}(0)$ and find $t_z \in (0, 1)$ such that $\theta_{x_1}(t_z) = z$ for each $z \in \{x_k, y_k, x_{k+1}, y_{k+1}, -x_k\}$. Note as $x_k \in (x_1, y_k)_{\|\cdot\|}$ that $0 < t_{x_k} < t_{y_k}$. Similarly, as $x_{k+1} \in [y_{k+1}, x_1)_{\|\cdot\|}$, $t_{y_{k+1}} \leq t_{x_{k+1}} < 1$. Further, by a), $y_{k+1} \in (y_k, -y_k]_{\|\cdot\|}$ and $x_{k+1} \in (x_k, -x_k]_{\|\cdot\|}$. Moreover, $y_{k+1} \in (y_k, y_1)_{\|\cdot\|} = (y_k, x_1)_{\|\cdot\|}$ and $x_{k+1} \in (x_k, x_1)_{\|\cdot\|}$. Hence, $t_{y_k} < t_{y_{k+1}}$. Therefore, $0 < t_{x_k} < t_{y_k} < t_{y_{k+1}} \leq t_{x_{k+1}} < 1$. Thus $y_k, y_{k+1} \in (x_k, x_{k+1}]_{\|\cdot\|} \subseteq (x_k, -x_k]_{\|\cdot\|}$.

If $x_{k+1} \neq y_{k+1}$, then by Lemma 1.4.3 for $x = x_k$, $y = y_k$, $z = y_{k+1}$, $w = x_{k+1}$, and b),

$$\|y_{k+1} - y_k\| \leq \|x_{k+1} - x_k\| \leq d^+(P) < \|y_{k+1} - y_k\|,$$

a contradiction. Similarly if $x_{k+1} = y_{k+1}$, then by the Monotonicity Lemma 1.4.2 and b),

$$d^+(P) < \|y_{k+1} - y_k\| = \|x_{k+1} - y_k\| \leq \|x_{k+1} - x_k\| \leq d^+(P),$$

a contradiction. As in either case we obtain a contradiction, we conclude that $x_{k+1} \notin [y_{k+1}, x_1)_{\|\cdot\|}$, i.e. $x_{k+1} \in (x_1, y_{k+1})_{\|\cdot\|}$. □

It is natural to consider the maximal possible edge length of an equilateral polygon. Here we provide a trivial upper bound for such.

Proposition 4.2.14. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 2$. If $P \in E_n^{\|\cdot\|}$, then $e(P) \leq \frac{1}{n}\mathcal{H}^1\left(\partial B_1^{\|\cdot\|}(0)\right)$.

Proof. Consider $P = (x_1, \ldots, x_n) \in E_n$. Note $\text{length}_{\|\cdot\|}\left((x_k, x_{k+1})_{\|\cdot\|}\right) \geq \|x_{k+1} - x_k\| = e(P)$ for each $k \in [n]$ by Lemma 1.3.6. Hence,

$$\mathcal{H}^1\left(\partial B_1^{\|\cdot\|}(0)\right) = \sum_{k=1}^{n} \text{length}_{\|\cdot\|}\left((x_k, x_{k+1})_{\|\cdot\|}\right) \geq ne(P),$$

from which the result follows. $\square$

Remark 4.2.15. This bound is not sharp in general, for example consider $\|\cdot\| = |\cdot|$ the Euclidean norm. However, if $\|\cdot\|$ is a polygonal m-norm where $m \equiv 2 \pmod 4$ and n is a multiple of m, then it is readily verified that the (Euclidean) equilateral n-gon which contains all vertices of the sphere attains such a bound.

We now introduce the notion of *strict acute visibility*, where an inscribed equilateral polygon possesses this property if each vertex 'sees' the next vertex in the same half of the unit sphere as itself. To motivate the nomenclature, recall that for an equilateral polygon $P \in E_n^{|\cdot|}$ with $n \geq 5$ each side of P is subtended by an arc smaller than a quarter-circle, so angles subtended by these arcs are acute.

Definition 4.2.16. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $n \geq 3$ and $P \in \mathcal{F}_n^{\|\cdot\|}$. We say P has *acute visibility* if $P \cap [-v, v)_{\|\cdot\|} \neq \emptyset$ for every $v \in \partial B_1^{\|\cdot\|}(0)$. Further, we say P has *strict acute visibility* if $P \cap (-v, v)_{\|\cdot\|} \neq \emptyset$ for every $v \in \partial B_1^{\|\cdot\|}(0)$.

Provided $(\|\cdot\|, n)$ is not a 3-rectilinear pair, we show that any equilateral n-gon inscribed in $\|\cdot\|$ has strict acute visibility.

Lemma 4.2.17. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$ be such that $(\|\cdot\|, n)$ is not a 3-rectilinear pair. If $P = (x_1, \ldots, x_n) \in E_n^{\|\cdot\|}$, then P has strict acute visibility. Moreover, for each $k \in [n]$,

$$x_{k+1} \in (v_0, -v_0)_{\|\cdot\|} \quad \text{for each } v_0 \in [x_k, x_{k+1})_{\|\cdot\|}. \tag{4.2.1}$$

Proof. Part 1: We begin by showing that P has acute visibility. For a contradiction suppose there exists $v_0 \in \partial B_1^{\|\cdot\|}(0)$ such that $P \cap [-v_0, v_0)_{\|\cdot\|} = \emptyset$, i.e. $P \subseteq [v_0, -v_0)_{\|\cdot\|}$. Reordering if necessary, let us assume that the first vertex which lies closest to v_0 is x_1. Hence, by the Monotonicity Lemma 1.4.2,

$$e(P) = \|x_2 - x_1\| \leq \|x_3 - x_1\| \leq \cdots \leq \|x_n - x_1\| = e(P).$$

Hence, $\|x_k - x_1\| = e(P)$ for each $k = 2, \ldots, n$. In particular, this implies $Q = (x_1, x_2, x_3) \in E_3$ and $Q \subseteq [v_0, -v_0)_{\|\cdot\|}$. By Corollary 4.2.7 this implies $\|\cdot\|$ is rectilinear.

Since $(\|\cdot\|, n)$ is not a 3-rectilinear pair, we conclude that $n \geq 4$. From above, for each $k = 2, \ldots, n-1$, $\|x_1 - x_k\| = \|x_k - x_{k+1}\|$. Therefore, by the Monotonicity Lemma 1.4.2, for each $k = 2, \ldots, n-1$, the straight line segments $[p_k, x_k], [-p_k, x_k] \subseteq \partial B_1^{\|\cdot\|}(0)$ where $p_k = (x_{k+1} - x_1)/\|x_{k+1} - x_1\|$.

Let V denote the collection of the four vertices of the unit sphere $\partial B_1^{\|\cdot\|}(0)$; recall $\|\cdot\|$ is rectilinear. Note that as x_k lies in the intersection of two non-parallel straight line segments $[p_k, x_k]$ and $[-p_k, x_k]$ which are both contained in $\partial B_1^{\|\cdot\|}(0)$, then $x_k \in V$ for each $k = 2, \ldots, n-1$. Since $n \geq 4$, in particular it follows that $x_2, x_3 \in V$. Thus, as $\|\cdot\|$ is rectilinear, note $e(P) = \|x_3 - x_2\| = 2$. So, by Lemma 4.2.9, $(\|\cdot\|, n)$ is a 4-rectilinear pair. But then as $n = 4$ and $x_2 \in V$ note P is formed by the vectors in V. In particular, note that P then satisfies $P \not\subseteq [v, -v)_{\|\cdot\|}$ for all $v \in \partial B_1^{\|\cdot\|}(0)$, contradicting the existence of v_0. Hence, P has acute visibility.

Part 2: We now prove P has strict acute visibility, by proving (4.2.1). Without loss of generality suppose $k = 1$. We shall consider two cases: $v_0 = x_1$ and $v_0 \in (x_1, x_2)_{\|\cdot\|}$.

First suppose $v_0 = x_1$ and suppose, for a contradiction, that $x_2 \notin (x_1, -x_1)_{\|\cdot\|}$, i.e. $x_2 \in [-x_1, x_1)_{\|\cdot\|}$. Note by Lemma 4.2.11 that $x_2 \neq -x_1$. Therefore $x_2, \ldots, x_n \in (-x_1, x_1)_{\|\cdot\|}$. Let $w \in (-x_1, x_2)_{\|\cdot\|}$ be arbitrary. Then, $w \in (-x_1, x_k)_{\|\cdot\|} \subseteq (-x_1, x_1)_{\|\cdot\|}$ for each $k = 2, \ldots, n$. So, by Lemma 1.3.16 (ii) with $x = x_k$, $y = w$ and $v = -x_1 = z$, note $x_2, \ldots, x_n \in (w, x_1)_{\|\cdot\|}$. Further, as $w \in (-x_1, x_2)_{\|\cdot\|} \subseteq (-x_1, x_1)_{\|\cdot\|}$, note $x_1 \in (w, -w)_{\|\cdot\|}$ by Lemma 1.3.16 (i). Therefore, $P \subseteq [w, -w)_{\|\cdot\|}$, which contradicts Part 1 of the present lemma. So (4.2.1) holds for $v_0 = x_1$.

Consider now when $v_0 \in (x_1, x_2)_{\|\cdot\|}$ and recall a parametrisation θ_{x_1} of $\partial B_1(0)$ from Notation 1.3.10. For each $z \in \{v_0, x_2, -v_0, -x_2\}$, let $t_z \in [0, 1]$ be such that $\theta_{x_1}(t_z) = z$. As $v_0 \in (x_1, x_2)_{\|\cdot\|}$ note $0 < t_{v_0} < t_{x_2}$. Further, by the case $v_0 = x_1$, note $x_2 \in (x_1, -x_1)_{\|\cdot\|}$. So $t_{x_2} < 1/2$. For $z \in \{v_0, x_2\}$, recall $t_{-z} = t_z + 1/2$ and thus

$$0 < t_{v_0} < t_{x_2} < \frac{1}{2} < t_{-v_0} < t_{-x_2} < 1.$$

Hence, $x_2 = \theta_{x_1}(t_{x_2}) \in (\theta_{x_1}(t_{v_0}), \theta_{x_1}(t_{-v_0}))_{\|\cdot\|} = (v_0, -v_0)_{\|\cdot\|}$. □

Remark 4.2.18. Observe the above result fails to hold when $n = 3$ and $\|\cdot\|$ is rectilinear. In particular, suppose $\|\cdot\| = \|\cdot\|_\infty$ and $P_c = (1 + (1-c)i, 1 + i, (1-c) + i)$ for $c \in (0, 1]$. Then $P_c \in E_3^{\|\cdot\|_\infty}$ with $e(P_c) = c$, but $P_c \subseteq [1, -1)_{\|\cdot\|}$.

Note both Lemma 4.2.17 and the present remark show that having strict acute visibility for every $P \in E_n^{\|\cdot\|}$ is equivalent to $(\|\cdot\|, n)$ not being a 3-rectilinear pair. Later in Lemma 4.3.6, we correct [8, Lemma 2.4] and prove the uniqueness of the edge length over all equilateral polygons containing a fixed vector, and we rely heavily on the strict acute visibility of equilateral polygons.

We continue by introducing a result of a similar nature. This result states that if a polygon has acute visibility, then adjacent vertices need to lie in the corresponding halves of the unit sphere.

Lemma 4.2.19. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. If $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$ has acute visibility, then $x_{k+1} \in (x_k, -x_k]_{\|\cdot\|}$ for each $k \in [n]$.

Proof. Fix $k \in [n]$ and suppose $x_{k+1} \in (-x_k, x_k)_{\|\cdot\|}$. Let $v \in (-x_k, x_{k+1})_{\|\cdot\|}$ be arbitrary. Then, by Lemma 1.3.16 (iv), $-v \in (x_k, -x_{k+1})_{\|\cdot\|}$. As $P \cap [-v, v)_{\|\cdot\|} \neq \emptyset$, note there exists $j \in [n]$ such that $x_j \in [-v, v)_{\|\cdot\|}$. Therefore, as $-v \in (x_k, -x_{k+1})_{\|\cdot\|}$ and as $v \in (-x_k, x_{k+1})_{\|\cdot\|}$,

$$x_j \in (x_k, v)_{\|\cdot\|} \subseteq (x_k, x_{k+1})_{\|\cdot\|},$$

a contradiction. □

Recall Notation 4.2.12 for the minimum and maximum edge length of an inscribed polygon.

Lemma 4.2.20. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. If $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$ and $Q = (y_1, \ldots, y_n) \in \mathcal{F}_n^{\|\cdot\|}$ both have acute visibility and $x_1 = y_1$, then

$$d^-(Q) \leq d^+(P) \quad \text{and} \quad d^-(P) \leq d^+(Q).$$

Proof. It suffices to prove that $d^-(Q) \leq d^+(P)$. Suppose, for a contradiction, $d^-(Q) > d^+(P)$. By Lemma 4.2.13, $x_k \in (x_1, y_k)_{\|\cdot\|} = (y_1, y_k)_{\|\cdot\|}$ for each $k \in [n]$. In particular, $x_n \in (y_1, y_n)_{\|\cdot\|}$. Also, since P and Q have acute visibility, $x_1 \in (x_n, -x_n]_{\|\cdot\|} \cap (y_n, -y_n]_{\|\cdot\|}$. Thus, by Lemma 1.3.16 (i), $x_n, y_n \in [-y_1, y_1)_{\|\cdot\|}$. Moreover, as $x_n \in (y_1, y_n)_{\|\cdot\|}$ it follows by Lemma 1.3.16 (iii) that $y_n \in (x_n, y_1)_{\|\cdot\|}$. Hence $y_n \in (x_n, y_1)_{\|\cdot\|} \cap [-y_1, y_1)_{\|\cdot\|}$. By the Monotonicity Lemma 1.4.2, $\|y_1 - y_n\| \leq \|y_1 - x_n\|$ and so

$$d^+(P) < d^-(Q) \leq \|y_n - y_1\| \leq \|x_n - y_1\| = \|x_n - x_1\| \leq d^+(P),$$

providing contradiction. $\square$

Corollary 4.2.21. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. Suppose $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$ has acute visibility. Then, for each $Q \in E_n^{\|\cdot\|}$ where $P \cap Q \neq \emptyset$,

$$d^-(P) \leq e(Q) \leq d^+(P).$$

Proof. Suppose $Q = (y_1, \ldots, y_n) \in E_n$ is such that $P \cap Q \neq \emptyset$. Then, by Lemma 4.2.20, $e(Q) = d^+(Q) \geq d^-(P)$ and $e(Q) = d^-(Q) \leq d^+(P)$. $\square$

We finish this section by identifying a region in which one may perturb a vertex of an equilateral polygon to ensure the new polygon maintains the acute visibility property, which the original polygon has due to Lemma 4.2.17. We also determine for most norms $\|\cdot\|$ on $\mathbb{C}$ and $n \geq 3$ when this radius is strictly positive, provided the quantity $e(P)$ is constant over all $P \in E_n^{\|\cdot\|}$ containing a fixed unit vector.

The condition $e(P)$ is constant over all equilateral polygons P containing a fixed vector is proven to be true for all norms $\|\cdot\|$ on $\mathbb{C}$ and $n \geq 3$, provided $(\|\cdot\|, n)$ is not a 3-rectilinear pair. This is shown in Lemma 4.3.6 without any dependency on the results proven in this section using this condition as an assumption.

Lemma 4.2.22. Suppose $\|\cdot\|$ is a norm on $\mathbb{C}$ and $n \geq 3$ is such that $(\|\cdot\|, n)$ is not a 3-rectilinear pair. Let $P = (x_1, \ldots, x_n) \in E_n^{\|\cdot\|}$ and $r_0 := \min(1, e(P), \|x_2 + x_1\|, \|x_n + x_1\|)$. Then $r_0 > 0$ and $Q_y := (y, x_2, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$ has acute visibility for each $y \in B_{r_0}^{\|\cdot\|}(x_1) \cap \partial B_1^{\|\cdot\|}(0)$.

Proof. Note by Lemma 4.2.11 that $r_0 > 0$. Let $U := B_{r_0}^{\|\cdot\|}(x_1) \cap \partial B_1^{\|\cdot\|}(0)$. We shall first prove that

$$U \subseteq (x_n, -x_n)_{\|\cdot\|} \cap (-x_2, x_2)_{\|\cdot\|}. \tag{4.2.2}$$

Indeed, let $w \in U$ be arbitrary. Observe that $\|w - x_1\| < r_0 \leq e(P)$. By Lemma 4.2.17 and Lemma 1.3.16 (i), note $x_2 \in (x_1, -x_1)_{\|\cdot\|}$ and $x_n \in (-x_1, x_1)_{\|\cdot\|}$. Hence $w \in (x_n, x_2)_{\|\cdot\|}$ as otherwise by the Monotonicity Lemma 1.4.2 one can conclude that $\|w - x_1\| \geq e(P)$. Similarly, as $\|w - x_1\| < r_0 \leq \min(\|x_2 + x_1\|, \|x_n + x_1\|)$, note $w \in (-x_2, -x_n)_{\|\cdot\|}$ since $-x_2 \in (-x_1, x_1)_{\|\cdot\|}$ and $-x_n \in (x_1, -x_1)_{\|\cdot\|}$. Thus, by Lemma 1.3.17,

$$w \in (x_n, x_2)_{\|\cdot\|} \cap (-x_2, -x_n)_{\|\cdot\|} = (x_n, -x_n)_{\|\cdot\|} \cap (-x_2, x_2)_{\|\cdot\|}.$$

The arbitrariness of $w \in U$ proves (4.2.2).

Fix $y \in U$. To see Q_y has acute visibility, it suffices to show that

$$Q_y \cap [v, -v)_{\|\cdot\|} \neq \emptyset \quad \text{for all } v \in \partial B_1^{\|\cdot\|}(0). \tag{4.2.3}$$

The proof is split into four cases; namely, if $v \in [x_j, x_{j+1})_{\|\cdot\|}$ for some $j \in [n-1]$, $v = x_n$, $v \in (x_n, x_1)_{\|\cdot\|} \cap U$ or if $v \in (x_n, x_1)_{\|\cdot\|} \setminus U$.

Case 1: Suppose $v \in [x_j, x_{j+1})_{\|\cdot\|}$ for some $j \in [n-1]$. By Lemma 4.2.17, $x_{j+1} \in (v, -v)_{\|\cdot\|}$ and so $x_{j+1} \in Q_y \cap (v, -v)_{\|\cdot\|}$. Hence (4.2.3) is satisfied in such a case.

Case 2: Suppose $v = x_n$. Then, by (4.2.2), $y \in U \subseteq (x_n, -x_n)_{\|\cdot\|} = (v, -v)_{\|\cdot\|}$ and thus (4.2.3) is satisfied in such a case.

Case 3: Suppose $v \in (x_n, x_1)_{\|\cdot\|} \cap U$. Then, by (4.2.2), $v \in U \subseteq (-x_2, x_2)_{\|\cdot\|}$. So, by Lemma 1.3.16 (i), $x_2 \in (v, -v)_{\|\cdot\|}$. Therefore, as $x_2 \in Q_y$, (4.2.3) is satisfied in such a case.

Case 4: Suppose $v \in (x_n, x_1)_{\|\cdot\|} \setminus U$. Observe U is an open, connected subset of $\partial B_1^{\|\cdot\|}(0)$ and $x_1 \in U$. Thus, there exist $a, b \in \partial B_1^{\|\cdot\|}(0)$ such that $U = (a, b)_{\|\cdot\|}$ where $a \in (-x_1, x_1)_{\|\cdot\|}$ and $b \in (x_1, -x_1)_{\|\cdot\|}$; note $a, b \neq -x_1$ since $r_0 \leq 1$. Now, as $v \in (x_n, x_1)_{\|\cdot\|} \setminus U$ note $x_n \neq a$. So, by (4.2.2), $a \in (x_n, x_1)_{\|\cdot\|}$ and thus

$$v \in (x_n, x_1)_{\|\cdot\|} \setminus U = (x_n, x_1)_{\|\cdot\|} \setminus (a, b)_{\|\cdot\|} = (x_n, a]_{\|\cdot\|}.$$

Let θ_{-x_1} be a parametrisation of $\partial B_1^{\|\cdot\|}(0)$ defined in Notation 1.3.10. Let $t_z \in (0, 1)$ be such that $\theta_{-x_1}(t_z) = z$ for each $z \in \{x_n, v, a, x_1, b, -x_n, -v\}$. As $x_n \in (-x_1, x_1)_{\|\cdot\|}$, note $0 < t_{x_n}$. Further, as $v \in (x_n, a]_{\|\cdot\|}$ and $a \in (-x_1, x_1)_{\|\cdot\|}$,

$$0 < t_{x_n} < t_v \leq t_a < t_{x_1} = \frac{1}{2}. \tag{4.2.4}$$

Since $\|x_1 - (-x_n)\| \geq r_0$, $b \in (x_1, -x_n)_{\|\cdot\|}$ as otherwise this contradicts the Monotonicity Lemma 1.4.2. Thus, as $v \in (x_n, x_1)_{\|\cdot\|}$, it follows by Lemma 1.3.16 (iv) that $-v \in (-x_n, -x_1)_{\|\cdot\|}$. Therefore,

$$\frac{1}{2} = t_{x_1} < t_b < t_{-x_n} < t_{-v} < 1. \tag{4.2.5}$$

Combining (4.2.4) and (4.2.5), we obtain

$$0 < t_{x_n} < t_v \leq t_a < t_{x_1} < t_b < t_{-x_n} < t_{-v} < 1.$$

Hence,

$$U = (a, b)_{\|\cdot\|} = (\theta_{-x_1}(t_a), \theta_{-x_1}(t_b))_{\|\cdot\|} \subseteq [\theta_{-x_1}(t_v), \theta_{-x_1}(t_{-v}))_{\|\cdot\|} = [v, -v)_{\|\cdot\|} .$$

Thus, $y \in U \subseteq [v, -v)_{\|\cdot\|}$ and so (4.2.3) is satisfied in such a case. □

Lemma 4.2.23. Let $\|\cdot\|$ be a rectilinear norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $A_x = \{Q \in E_4^{\|\cdot\|} : x \in Q\}$. Suppose $A_x \neq \emptyset$ and $e(Q)$ is constant over all $Q \in A_x$. Then $P_1 \sim_{\|\cdot\|} P_2$ for each $P_1, P_2 \in A_x$. Moreover, for each $P \in A_x$, we have $e(P) \in [1, 2]$ and $e(P) = 2$ if and only if P is formed by the four vertices of $\partial B_1^{\|\cdot\|}(0)$.

Proof. Without loss of generality, by Proposition 4.2.3, $\|\cdot\| = \|\cdot\|_\infty$. Since $\partial B_1^{\|\cdot\|_\infty}(0)$ is invariant under rotations by $\pi/2$ and under reflections in the axes, we may further assume that $x = -\eta + i$ for some $\eta \in [0, 1]$.

Define $P_\eta := (-\eta + i, -1 - \eta i, \eta - i, 1 + \eta i) =: (x_1, x_2, x_3, x_4)$. Observe that $P_\eta \in A_x$ and $e(P_\eta) = 1 + \eta \in [1, 2]$. We claim that any $Q \in A_x$ satisfies $Q \sim P_\eta$. To proceed, consider $Q = (y_1, y_2, y_3, y_4) \in A_x$ where $y_1 = x = x_1$. Then, $e(Q) = e(P_\eta) = 1 + \eta$.

Consider first the case when $\eta = 1$, i.e. $y_1 = x_1 = -1 + i$. By Lemma 4.2.17, $y_2 \in (-1 + i, 1 - i)_{\|\cdot\|}$. Hence, as $\|y_2 - y_1\| = e(Q) = e(P_1) = 2$, note $y_2 = \xi - i$ for some $\xi \in [-1, 1)$. Arguing similarly and utilising Lemma 1.3.16 (i), we obtain $y_4 = 1 + \epsilon i$ for some $\epsilon \in (-1, 1]$. Now as $e(Q) = 2$, the Monotonicity Lemma 1.4.2 yields $\|y_j - y_k\| = 2$ for each $j \neq k$. In particular, $\|y_4 - y_2\| = 2$. Therefore,

$$2 = \|y_4 - y_2\| = \max\{1 + \epsilon, 1 - \xi\}.$$

So, $1 \in \{\epsilon, -\xi\}$. Without loss of generality, assume $\xi = -1$ and so $y_2 = -1 - i$, as otherwise one may consider the reflection of the equilateral polygon in the line containing the segment $[-1 + i, 1 - i]$. By Lemma 4.2.17, $y_3 \in (-1 - i, 1 + i)_{\|\cdot\|}$. As $\|y_3 - y_2\| = e(Q) = 2$, $y_3 = 1 + \gamma i$ for some $\gamma \in [-1, 1)$. Since $y_3 \in (y_2, y_4)_{\|\cdot\|}$ we conclude $\gamma < \epsilon$. But then, $2 = \|y_4 - y_3\| = \epsilon - \gamma$

and so $\epsilon = 1$ and $\gamma = -1$. Thus, $y_3 = 1 - i$ and $y_4 = 1 + i$. Hence, $Q = P_1$.

The case when $\eta = 0$ follows by a similar case analysis. Now let us consider when $\eta \in (0, 1)$. Observe, in such a case,

$$\partial B_{1+\eta}^{\|\cdot\|_\infty}(y_1) \cap (-\eta + i, \eta - i)_{\|\cdot\|} = \{-1 - \eta i\},$$

hence by Lemma 4.2.17 it follows that $y_2 = -1 - \eta i$. Similarly, as

$$\partial B_{1+\eta}^{\|\cdot\|_\infty}(y_2) \cap (-1 - \eta i, 1 + \eta i)_{\|\cdot\|} = \{\eta - i\},$$

it follows via Lemma 4.2.17 that $y_3 = \eta - i$. Finally, as

$$\partial B_{1+\eta}^{\|\cdot\|_\infty}(y_3) \cap (\eta - i, -\eta + i)_{\|\cdot\|} = \{1 + \eta i\}$$

we conclude by Lemma 4.2.17 that $y_4 = 1 + \eta i$ and hence $Q = P_\eta$. □

We end this section by determining a uniform lower bound for the radius in which one may perturb an equilateral polygon to ensure it maintains its acute visibility.

Lemma 4.2.24. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$ be such that $(\|\cdot\|, n)$ is not a 3-rectilinear pair. Suppose $\inf_{Q \in E_n^{\|\cdot\|}} e(Q)$ is finite and positive. Then there exists $c = c(n, \|\cdot\|) > 0$ such that for any $P = (x_1, \ldots, x_n) \in E_n^{\|\cdot\|}$ it follows that

$$\min \{e(P), \|x_2 + x_1\|, \|x_n + x_1\|\} \geq c.$$

Proof. First note for any $c \in (0, \inf_{Q \in E_n} e(Q)]$ that if $P \in E_n$, then

$$e(P) \geq \inf_{Q \in E_n} e(Q) \geq c > 0.$$

Therefore it suffices to find $c \in (0, \inf_{Q \in E_n} e(Q)]$ such that for any $P = (x_1, \ldots, x_n) \in E_n$, we have

$$\min \{\|x_2 + x_1\|, \|x_n + x_1\|\} \geq c.$$

Suppose no such constant $c > 0$ exists. Then there exists a sequence of vectors $\{x_1^m\}_{m=1}^\infty \subseteq$

$\partial B_1^{\|\cdot\|}(0)$ and a sequence of corresponding polygons $P_m = (x_1^m, \dots, x_n^m) \in E_n$ such that

$$\min\{\|x_2^m + x_1^m\|, \|x_n^m + x_1^m\|\} \to 0 \quad \text{as } m \to +\infty.$$

Via a compactness argument, there exists a subsequence $P_{m_k} = (x_1^{m_k}, \dots, x_n^{m_k}) \in E_n$ such that both $\min\{\|x_2^{m_k} + x_1^{m_k}\|, \|x_n^{m_k} + x_1^{m_k}\|\} \to 0$ and $P_{m_k} \to P = (x_1, \dots, x_n) \in E_n$ as $k \to \infty$.

Passing to a further subsequence, if necessary, we may assume without loss of generality that $\|x_2^{m_k} + x_1^{m_k}\| \to 0$ as $k \to \infty$. Hence,

$$\|x_2 + x_1\| = \lim_{k\to\infty} \|x_2^{m_k} + x_1^{m_k}\| = 0.$$

Therefore $x_2 = -x_1$ and so Lemma 4.2.11 implies $(\|\cdot\|, n)$ is a 3-rectilinear pair, providing contradiction. □

Remark 4.2.25. In Lemma 4.4.4 we show that $\inf_{Q\in E_n^{\|\cdot\|}} e(Q) > 0$ whenever $(n, \|\cdot\|)$ is not a 3-rectilinear pair.

4.3 Existence of equilateral polygons

In this section we show for each norm $\|\cdot\|$ on $\mathbb{C}$, $n \geq 3$ and $x \in \partial B_1^{\|\cdot\|}(0)$ there exists $P \in E_n^{\|\cdot\|}$ such that $x \in P$. This result was already claimed in [8, Lemma 2.4], but the statement in [8] is not entirely correct. It is first claimed that for every norm $\|\cdot\|$ on $\mathbb{C}$, $n \geq 3$ and every unit vector x in $\|\cdot\|$ that there exists $P \in E_n^{\|\cdot\|}$ such that $x \in P$. We verify this is indeed correct in Theorem 4.3.5 but provide an independent proof. They claim further that for any two equilateral polygons $P, Q \in E_n^{\|\cdot\|}$ that if $P \cap Q \neq \emptyset$, then $e(P, \|\cdot\|) = e(Q, \|\cdot\|)$. This is false in general. Indeed, consider $P_c := (1+(1-c)i, 1+i, (1-c)+i) \in E_3^{\|\cdot\|_\infty}$ for each $c \in (0, 2]$, as in Remark 4.2.18. Observe that $1 + i \in P_c$ for every $c \in (0, 2]$, but $e(P_c, \|\cdot\|_\infty) = c$, see Figure 4.3. In Lemma 4.3.6 we prove that such phenomena may only occur in the particular cases when $(\|\cdot\|, n)$ is a 3-rectilinear pair.

First, we introduce the following notation.

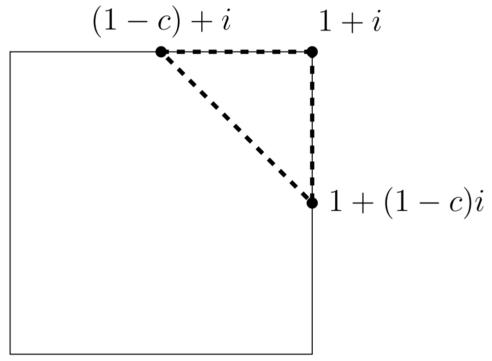

Figure 4.3: Equilateral triangles P_c inscribed in $\|\cdot\|_\infty$ with distinct edge lengths, but share a fixed vertex.

Notation 4.3.1. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $n \geq 2$. Define

$$\alpha(x, n, \|\cdot\|) = \sup \left\{ d^-(P) : x \in P \text{ and } P \in \mathcal{F}_n^{\|\cdot\|} \right\}.$$

Via a standard compactness argument, it can be readily shown there exists an inscribed polygon for which this quantity is attained.

Proposition 4.3.2. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $n \geq 2$. There exists $P \in \mathcal{F}_n^{\|\cdot\|}$ such that $x \in P$ and $d^-(P) = \alpha(x, n, \|\cdot\|)$.

Proof. For brevity, we write α instead of $\alpha(x, n, \|\cdot\|)$. For each $k \in \mathbb{N}$, let $P_k = (x_1^k, \ldots, x_n^k) \in \mathcal{F}_n^{\|\cdot\|}$ be such that $x_1^k = x$ and $d^-(P_k) > \alpha - 1/k$.

Since $\partial B_1^{\|\cdot\|}(0)$ is compact and $\left\{x_2^k\right\}_{k=1}^\infty \subseteq \partial B_1^{\|\cdot\|}(0)$ there exists a convergent subsequence $\{x_2^{k_{2,j}}\}_{k_{2,j}}$ such that $x_2^{k_{2,j}} \to x_2 \in \partial B_1^{\|\cdot\|}(0)$ as $k_{2,j} \to \infty$. Similarly, there exists a convergent subsequence $\{x_3^{k_{3,j}}\}_{k_{3,j}}$ of $\{x_3^{k_{2,j}}\}_{k_{2,j}}$ such that $x_3^{k_{3,j}} \to x_3 \in \partial B_1^{\|\cdot\|}(0)$ as $k_{3,j} \to \infty$. Continuing recursively, we obtain $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$ such that $x_1 = x$.

We claim that $d^-(P) \geq \alpha$. Indeed, fix $l \in [n]$ and $\varepsilon > 0$. Let $K_0 \in \mathbb{N}$ be such that

$$\min\left(\left\|x_l^{k_{n,j}} - x_l\right\|, \left\|x_{l+1}^{k_{n,j}} - x_{l+1}\right\|\right) < \frac{\varepsilon}{3} \quad \text{whenever } k_{n,j} \geq K_0.$$

Take $K > \max(K_0, 3/\varepsilon)$ and note for each $k_{n,l} \geq K$ that

$$\|x_{l+1} - x_l\| \geq \left\|x_{l+1}^{k_{n,j}} - x_l^{k_{n,j}}\right\| - \left\|x_{l+1}^{k_{n,j}} - x_{l+1}\right\| - \left\|x_l^{k_{n,j}} - x_l\right\| > d^-\left(P_{k_{n,j}}\right) - \frac{2\varepsilon}{3} > \alpha - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, note $\|x_{l+1} - x_l\| \geq \alpha$ for each $l \in [n]$. Hence $d^-(P) \geq \alpha$, and thus by the definition of α, $d^-(P) = \alpha$. □

Moreover, we may assume this minimum is attained on the first edge of the inscribed polygon, and is necessarily attained on an edge which contains the fixed vertex x that is of interest.

Proposition 4.3.3. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $n \geq 2$. There exists $P \in \mathcal{F}_n^{\|\cdot\|}$ such that $x_1 = x$ and $d^-(P) = \|x_2 - x_1\| = \alpha(x, n, \|\cdot\|)$.

Proof. For brevity, we write α instead of $\alpha(x, n, \|\cdot\|)$. By Proposition 4.3.2 there exists $Q = (y_1, \ldots, y_n) \in \mathcal{F}_n^{\|\cdot\|}$ such that $y_1 = x$ and $d^-(Q) = \alpha$. If $\|y_2 - y_1\| = \alpha$, then let $P = Q$.

Suppose $\|y_2 - y_1\| > \alpha$, hence $n \geq 3$. Let $Y \in (y_1, y_2)_{\|\cdot\|}$ be such that $\|y_1 - Y\| = \alpha$; note such Y exists by Corollary 1.4.7. Consider $P = (y_1, Y, y_3, \ldots, y_n) \in \mathcal{F}_n^{\|\cdot\|}$. To see that $d^-(P) = \alpha$, note it suffices to verify $\|y_3 - Y\| \geq \alpha$. Indeed, if $y_3 \in (Y, -Y]_{\|\cdot\|}$, then by the Monotonicity Lemma 1.4.2, as $y_2 \in (Y, y_3)_{\|\cdot\|}$,

$$\|y_3 - Y\| \geq \|y_3 - y_2\| \geq \alpha.$$

Suppose now $y_3 \in (-Y, Y)_{\|\cdot\|}$. As $Y \in (y_1, y_2)_{\|\cdot\|} \subseteq (y_1, y_3)_{\|\cdot\|}$ note by Lemma 1.3.16 (iii) that $y_1 \in (y_3, Y)_{\|\cdot\|} \subseteq (-Y, Y)_{\|\cdot\|}$. Hence, by the Monotonicity Lemma 1.4.2,

$$\|y_3 - Y\| \geq \|y_1 - Y\| = \alpha.$$

Thus $\|y_3 - Y\| \geq \alpha$ and so $d^-(P) = \|Y - y_1\| = \alpha$. □

Below we prove the existence of a 'near equilateral' inscribed n-gon. We show that for each unit vector x one can construct an inscribed n-gon for which $n - 1$ edges have edge length precisely $\alpha(x, n, \|\cdot\|)$.

Lemma 4.3.4. Suppose $\|\cdot\|$ is a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}(0)$ and $n \geq 3$. Let

$$\mathcal{P}_{x,n} = \left\{P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|} : x_1 = x, d^-(P) = \|x_{k+1} - x_k\| = \alpha \text{ for all } k \in [n-1]\right\}, \tag{4.3.1}$$

where $\alpha = \alpha(x, n, \|\cdot\|)$. Then, $\mathcal{P}_{x,n} \neq \emptyset$.

Proof. For brevity, we write α instead of $\alpha(x, n, \|\cdot\|)$. By Proposition 4.3.3, there exists $P = (x_1, \dots, x_n) \in \mathcal{F}_n$ such that $x_1 = x$ and $d^-(P) = \|x_2 - x_1\| = \alpha$. If $\alpha = 2$ then, by Corollary 4.2.1, $P \in E_n$ and $e(P) = \alpha$. So, $P \in \mathcal{P}_{x,n}$.

Suppose $\alpha < 2$ and recall Notation 1.3.10. Let $y_1 := x_1$ and $P_1 := P$. Arguing recursively, for each $k \in [n-1]$, let $y_{k+1} = \theta_{y_k}(\tau_k)$, where

$$\tau_k = \min\{t \in [0, 1/2] : f_{y_k}(t) = \alpha\}.$$

If $k < n-1$, let $P_{k+1} := (x_1, y_2, \dots, y_{k+1}, x_{k+2}, \dots, x_n)$ and, if $k = n-1$, let $P_n = (x_1, y_2, \dots, y_n)$. By construction, $\|y_{k+1} - y_k\| = \alpha$ for each $k \in [n-1]$.

To conclude $P_n \in \mathcal{P}_{x,n}$, we need only show that $\|y_n - x_1\| \geq \alpha$. Indeed first note, as $\alpha < 2$, that $y_{k+1} \in (y_k, -y_k)_{\|\cdot\|}$ for each $k \in [n-1]$. We claim that $y_k \in (x_1, x_k]_{\|\cdot\|}$ for each $k \in [n]$. Indeed, if $k = 2$ note $y_2 \in (x_1, x_2]_{\|\cdot\|}$ by the Monotonicity Lemma 1.4.2, since $\|x_2 - x_1\| = \alpha$. Fix $k \in [n-1]$ and suppose $y_k \in (x_1, x_k]_{\|\cdot\|}$. If $x_{k+1} \in [-y_k, y_k]_{\|\cdot\|} \cap (x_k, x_1)_{\|\cdot\|}$, then $y_{k+1} \in (x_1, x_{k+1}]_{\|\cdot\|}$ as $y_{k+1} \in (y_k, -y_k)_{\|\cdot\|}$. So, suppose $x_{k+1} \in (y_k, -y_k)_{\|\cdot\|} \cap (x_k, x_1)_{\|\cdot\|}$. If $y_{k+1} \notin (x_1, x_{k+1}]_{\|\cdot\|}$, then $y_{k+1} \in (x_{k+1}, x_1)_{\|\cdot\|} \cap (y_k, -y_k)_{\|\cdot\|}$. As $x_{k+1} \in (y_k, -y_k)_{\|\cdot\|}$ and $d^-(P) = \alpha$, it follows by the Monotonicity Lemma 1.4.2, that

$$\alpha \leq \|x_{k+1} - x_k\| \leq \|y_k - x_{k+1}\| \leq \|y_{k+1} - y_k\| = \alpha.$$

Thus, $\|y_k - x_{k+1}\| = \alpha$. But, $x_{k+1} \in (y_k, y_{k+1})_{\|\cdot\|}$ which contradicts the definition of y_{k+1}. Therefore, $y_{k+1} \in (x_1, x_{k+1}]_{\|\cdot\|}$.

So, $y_k \in (x_1, x_k]_{\|\cdot\|}$ for each $k \in [n]$ and in particular $y_n \in (x_1, x_n]_{\|\cdot\|}$. Now, if $y_n \in (x_1, -x_1]_{\|\cdot\|}$ note by the Monotonicity Lemma 1.4.2 that $\|y_n - x_1\| \geq \|y_2 - x_1\| = \alpha$. Alternatively, if $y_n \in (-x_1, x_1)_{\|\cdot\|}$ then as $y_n \in (x_1, x_n]_{\|\cdot\|}$, by the Monotonicity Lemma 1.4.2,

$$\|y_n - x_1\| \geq \|x_n - x_1\| \geq d^-(P) = \alpha.$$

So, in either case, $\|y_n - x_1\| \geq \alpha$ and so $P_n \in \mathcal{P}_{x,n}$. □

We are now able to prove the existence of an equilateral polygon containing a fixed unit vector.

Theorem 4.3.5. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $x \in \partial B_1^{\|\cdot\|}$ and $n \geq 3$. There exists $P \in E_n^{\|\cdot\|}$ such that $x \in P$ and $e(P) = \alpha(x, n, \|\cdot\|)$. In particular, $E_n^{\|\cdot\|} \neq \emptyset$.

Proof. For brevity, we write α instead of $\alpha(x, n, \|\cdot\|)$. If $\alpha = 2$, then the result follows by Corollary 4.2.1 and Proposition 4.3.3. Suppose $\alpha < 2$. Let us fix a parametrisation $\theta = \theta_x : [0,1] \to \partial B_1^{\|\cdot\|}(0)$, as in Notation 1.3.10, see also Remark 1.3.12. By Lemma 4.3.4 there exists $P = (x_1, \ldots, x_n) \in \mathcal{P}_{x,n}$, defined in (4.3.1), such that $x_1 = x$ and $x_n = \theta(t^*)$, where

$$t^* = \sup \{t \in [0,1) : \text{ there exists } Q \in \mathcal{P}_{x,n} \text{ such that } \theta(t) \in Q\}. \tag{4.3.2}$$

To see such a polygon exists and $t^* \in (0,1)$, recall the definition (4.3.1) of $\mathcal{P}_{x,n} \neq \emptyset$ and then the existence of P as above follows via a compactness argument.

We claim that $P \in E_n$. To see this, we need to show that $\|x_n - x_1\| = \alpha$. Note it suffices to verify that $\|x_n - x_1\| \leq \alpha$ since $P \in \mathcal{P}_{x,n}$ and so $d^-(P) = \alpha$. For a contradiction, suppose that $\|x_n - x_1\| > \alpha$. If $x_n \in (x_1, -x_1)_{\|\cdot\|}$, let $z_n = -x_1$, otherwise find $z_n \in (x_n, x_1)_{\|\cdot\|} \cap (x_n, -x_n)_{\|\cdot\|}$ such that $\|z_n - x_n\| < \alpha < \|z_n - x_1\|$; such z_n exists by Corollary 1.4.7.

Let $S_n := \{t \in (0, 1/2) : f_{z_n}(1-t) = \alpha\}$ and observe, since $\alpha < 2$, that $S_n \neq \emptyset$ by the monotonicity and continuity of $f_{z_n}(1-t)$. Define $z_{n-1} := \theta_{z_n}(1 - \sup S_n)$.

We claim that $z_{n-1} \in (x_{n-1}, z_n)_{\|\cdot\|}$. Observe this is true if $x_{n-1} \in (z_n, -z_n]_{\|\cdot\|}$. So suppose $x_{n-1} \in (-z_n, z_n)_{\|\cdot\|}$. If $z_{n-1} \notin (x_{n-1}, z_n)_{\|\cdot\|}$ it follows by Lemma 1.4.3 that $\|z_n - x_{n-1}\| = \alpha$ since $\|x_n - x_{n-1}\| = \|z_n - z_{n-1}\| = \alpha$. Thus, taking $Q_1 := (x_1, \ldots, x_{n-1}, z_n)$ we observe $Q_1 \in \mathcal{P}_{x,n}$ and $z_n \in (x_n, x_1)_{\|\cdot\|}$, which contradicts our choice of $P \in \mathcal{P}_{x,n}$ since $\theta^{-1}(z_n) > \theta^{-1}(x_n)$. Hence, $z_{n-1} \in (x_{n-1}, x_n)_{\|\cdot\|} \cap (-z_n, z_n)_{\|\cdot\|}$.

Arguing recursively, suppose $z_k \in (x_k, z_{k+1})_{\|\cdot\|} \cap (-z_{k+1}, z_{k+1})_{\|\cdot\|}$ has been defined for some $k \in \{3, \ldots, n\}$. Define $S_k := \{t \in (0, 1/2) : f_{z_k}(1-t) = \alpha\}$; observe $S_k \neq \emptyset$ by the monotonicity and continuity of f_{z_k}. Let $z_{k-1} := \theta_{z_k}(1 - \sup S_k)$.

We claim $z_{k-1} \in (x_{k-1}, z_k)_{\|\cdot\|} \cap (-z_k, z_k)_{\|\cdot\|}$. Observe this is true if $x_{k-1} \in (z_k, -z_k]_{\|\cdot\|}$. So suppose $x_{k-1} \in (-z_k, z_k)_{\|\cdot\|}$. In such a case, if $z_{k-1} \notin (x_{k-1}, z_k)_{\|\cdot\|}$ it follows by Lemma 1.4.3 that $\|z_k - x_{k-1}\| = \alpha$. Hence, define $Q_{(n+1)-k} := (x_1, \ldots, x_{k-1}, z_k, \ldots, z_n)$ and observe $Q_{(n+1)-k} \in \mathcal{P}_{x,n}$ and $z_n \in (x_n, x_1)_{\|\cdot\|}$. This contradicts our choice of $P \in \mathcal{P}_{x,n}$ as $z_n \in Q_{(n+1)-k}$ and $\theta^{-1}(z_n) > \theta^{-1}(x_n)$. Hence, $z_{k-1} \in (x_{k-1}, z_k)_{\|\cdot\|} \cap (-z_k, z_k)_{\|\cdot\|}$.

This implies $\|z_2 - x_1\| \geq \alpha$. Indeed, if $z_2 \in (x_1, -x_1)_{\|\cdot\|}$ then as $z_2 \in (x_2, z_3)_{\|\cdot\|}$ and

$x_2 \in (x_1, -x_1)_{\|\cdot\|}$, this implies by the Monotonicity Lemma 1.4.2 that $\|z_2 - x_1\| \geq \|x_2 - x_1\| = \alpha$. Conversely, if $z_2 \in (-x_1, x_1)_{\|\cdot\|}$, then $x_n \in (-x_1, x_1)_{\|\cdot\|}$ and thus as $z_2 \in (x_1, z_{n-1})_{\|\cdot\|} \subseteq (x_1, x_n)_{\|\cdot\|}$ it follows by the Monotonicity Lemma 1.4.2 that $\|z_2 - x_1\| \geq \|x_n - x_1\| = \alpha$.

Moreover, $\|z_2 - x_1\| > \alpha$ as if $\|z_2 - z_1\| = \alpha$ one may define $Q_n := (x_1, z_2, \ldots, z_n)$ and note $Q \in \mathcal{P}_{x,n}$, but $z_n \in (x_n, x_1)_{\|\cdot\|}$ which contradicts our choice of $P \in \mathcal{P}_{x,n}$.

Since $\|z_2 - z_1\| > \alpha$, find $w_2 \in (x_2, z_2)_{\|\cdot\|} \cap (-z_3, z_3)_{\|\cdot\|}$ such that $\|w_2 - x_1\| > \alpha$; see Corollary 1.4.7. This implies $\|w_2 - z_3\| > \|z_2 - z_3\| = \alpha$ by the definition of z_2. Arguing recursively, if $n \geq 4$, suppose that $w_k \in (x_k, z_k)_{\|\cdot\|} \cap (-z_{k+1}, z_{k+1})_{\|\cdot\|}$ satisfies $\|w_k - z_{k+1}\| > \alpha$ for some $k = 2, \ldots, n-2$. Find $w_{k+1} \in (x_{k+1}, z_{k+1})_{\|\cdot\|} \cap (-z_{k+2}, z_{k+2})_{\|\cdot\|}$ such that $\|w_{k+1} - w_k\| > \alpha$ by Corollary 1.4.7 since $\|w_k - z_{k+1}\| > \alpha$.

Define $P' := (x_1, w_2, \ldots, w_{n-1}, z_n)$. To see that $d^-(P') > \alpha$, note it suffices to verify $\|w_{n-1} - z_n\| > \alpha$. This follows as $w_{n-1} \in (x_{n-1}, z_{n-1})_{\|\cdot\|}$ and so the Monotonicity Lemma 1.4.2 implies, via the definition of z_{n-1}, that $\|w_{n-1} - z_n\| > \|z_{n-1} - z_n\| = \alpha$. Thus, $d^-(P') > \alpha$, $x = x_1 \in P'$ and $|P'| = n$, which contradicts the definition of α.

As in either case we obtain a contradiction, we conclude that $\|x_n - x_1\| = \alpha$ and hence $P \in E_n$. □

In the above theorem we have shown, independently of [8], that there exists an equilateral polygon containing a fixed unit vector. However, in [8], it is also claimed that the edge length of an equilateral polygon containing a fixed unit vector is constant over all such equilateral polygons. We have already provided, at the beginning of the present section, examples where this is not true. Below we show that this is true whenever $(\|\cdot\|, n)$ is not a 3-rectilinear pair.

Lemma 4.3.6. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$ be such that $(\|\cdot\|, n)$ is not a 3-rectilinear pair, and let $x \in \partial B_1^{\|\cdot\|}(0)$. Then $e(P)$ is constant over all $P \in E_n^{\|\cdot\|}$ such that $x \in P$.

Moreover, if $\|\cdot\|$ is strictly convex there exists a unique equilateral polygon, up to equivalence, $P \in E_n^{\|\cdot\|}$ such that $x \in P$.

Proof. Suppose there exist two polygons $P = (x_1, \ldots, x_n) \in E_n$ and $Q = (y_1, \ldots, y_n) \in E_n$ such that $x_1 = y_1$, but $e(P) < e(Q)$. Since $P, Q \in E_n$, note $e(P) = d^+(P)$ and $e(Q) = d^-(Q)$. So, by Lemma 4.2.13 and Lemma 4.2.17, $x_n \in (x_1, y_n)_{\|\cdot\|}$. Further, by Lemma 4.2.17,

$$x_1 = y_1 \in (x_n, -x_n)_{\|\cdot\|} \cap (y_n, -y_n)_{\|\cdot\|}.$$

Thus, by Lemma 1.3.16 (i), $x_n, y_n \in (-x_1, x_1)_{\|\cdot\|}$. So, by Lemma 1.3.16 (iii), $y_n \in (x_n, x_1)_{\|\cdot\|} \cap (-x_1, x_1)_{\|\cdot\|}$. Hence, by the Monotonicity Lemma 1.4.2,

$$e(P) = \|x_n - x_1\| \geq \|y_n - x_1\| = \|y_n - y_1\| = e(Q) > e(P),$$

a contradiction.

If $\|\cdot\|$ is strictly convex, then by the first part of the present lemma and Corollary 1.4.4 the uniqueness follows. □

4.4 Extremal equilateral polygons

In Theorem 4.3.5 we have shown for any norm $\|\cdot\|$ on $\mathbb{C}$ and $n \geq 2$ that $E_n^{\|\cdot\|} \neq \emptyset$. This section concerns to what extent the edge length of such equilateral polygons may vary.

To begin, we observe the following equality.

Lemma 4.4.1. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. Then,

$$\sup\left\{\alpha(x, n, \|\cdot\|) : x \in \partial B_1^{\|\cdot\|}(0)\right\} = \sup\left\{e(P, \|\cdot\|) : P \in E_n^{\|\cdot\|}\right\}.$$

Proof. Let $A := \{\alpha(x, n, \|\cdot\|) : x \in \partial B_1^{\|\cdot\|}(0)\}$ and $B := \{e(P) : P \in E_n\}$. Note if $(\|\cdot\|, n)$ is not a 3-rectilinear pair then by Theorem 4.3.5 and Lemma 4.3.6 it follows that $A = B$.

Suppose now $\|\cdot\|$ is rectilinear and $n = 3$. Then observe that $\alpha(x, 3, \|\cdot\|) = 2$ for every $x \in \partial B_1^{\|\cdot\|}(0)$; one needs to consider the inscribed (equilateral) triangle formed from x and the vertices of $\partial B_1^{\|\cdot\|}(0)$ which lie on the opposite edge to x. Hence $\sup(A) = 2$. Similarly, by considering any equilateral polygon formed by three vertices of the sphere $\partial B_1^{\|\cdot\|}(0)$ we deduce that $\sup(B) = 2$, also. □

Remark 4.4.2. Following from Lemma 4.4.1, let $A = \{\alpha(x, n, \|\cdot\|) : x \in \partial B_1^{\|\cdot\|}(0)\}$ and $B = \{e(P) : P \in E_n^{\|\cdot\|}\}$. Observe that $\inf A \neq \inf B$, in general. Indeed, if $(\|\cdot\|, n)$ is a 3-rectilinear pair then $\inf A = 2$ since $\alpha(x, 3, \|\cdot\|) = 2$ for each $x \in \partial B_1^{\|\cdot\|}(0)$. However, if for example $\|\cdot\| = \|\cdot\|_\infty$, by considering $P_c = (1 + (1-c)i, 1 + i, (1-c) + i) \in E_3^{\|\cdot\|_\infty}$ we deduce that $\inf B = 0$. This is only possible when $(\|\cdot\|, n)$ is a 3-rectilinear pair, however; one may conclude via Theorem 4.3.5 and Lemma 4.3.6 that $A = B$ whenever $(\|\cdot\|, n)$ is not 3-rectilinear.

To avoid the technicalities as described in Remark 4.4.2, we shall introduce the following definition for the smallest and largest possible edge lengths for equilateral polygons.

Definition 4.4.3. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 2$. Define the *upper* $\|\cdot\|$*-regularity constant* to be

$$\alpha(n, \|\cdot\|) = \sup\left\{\alpha(x, n, \|\cdot\|) : x \in \partial B_1^{\|\cdot\|}(0)\right\} = \sup\left\{e(P, \|\cdot\|) : P \in E_n^{\|\cdot\|}\right\},$$

and the *lower* $\|\cdot\|$*-regularity constant* to be

$$\beta(n, \|\cdot\|) = \inf\left\{e(P, \|\cdot\|) : P \in E_n^{\|\cdot\|}\right\}.$$

We have the following characterisation of when the lower regularity constant is minimal.

Lemma 4.4.4. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. Then $\alpha(n, \|\cdot\|) > 0$ and

$$\beta(n, \|\cdot\|) = 0 \quad \text{if and only if} \quad (\|\cdot\|, n) \text{ is a 3-rectilinear pair.}$$

Proof. We first show that $\beta(3, \|\cdot\|) = 0$ if and only if $\|\cdot\|$ is rectilinear. Indeed, if $\|\cdot\|$ is rectilinear we may assume, by Proposition 4.2.3, that $\|\cdot\| = \|\cdot\|_\infty$. Let $P_c = (1+(1-1/c)i, 1+i, (1-1/c)+i) \in E_3^\infty$ for each $c \in (0, 1]$. Then $e(P_c) = c$ and so $\beta(3, \|\cdot\|) = 0$.

Suppose now that $\beta(3, \|\cdot\|) = 0$. Find a sequence $P_m \in E_3$ such that $e(P_m) \to 0$ as $m \to +\infty$. Hence there exists $M \in \mathbb{N}$ such that $e(P_M) < 1$. Therefore, by Corollary 4.2.7 applied to P_M, we conclude that $\|\cdot\|$ is rectilinear.

We now show that $\beta(n, \|\cdot\|) > 0$ whenever $n \geq 4$. Indeed, for a contradiction, suppose that there exists a norm $\|\cdot\|$ on $\mathbb{C}$ and $n \geq 4$ such that $\beta(n, \|\cdot\|) = 0$. For each $k \in \mathbb{N}$ find $P_k = (x_1^k, \ldots, x_n^k) \in E_n$ such that $e(P_k) < 1/k$. Then, by Lemma 4.2.17 as $n \geq 4$, P_k has acute visibility. However, for each $k \in \mathbb{N}$ and $j \in [n]$,

$$\left\|x_j^k - x_1^k\right\| \leq \sum_{l=1}^{j-1} \left\|x_{l+1}^k - x_l^k\right\| = (j-1)e(P_k) < \frac{n}{k},$$

thus $P \subseteq B_{n/k}^{\|\cdot\|}(x_1^k)$. Hence taking k sufficiently large we obtain a contradiction with the acute visibility of P_k.

Thus $\beta(n, \|\cdot\|) = 0$ if and only if $(\|\cdot\|, n)$ is a 3-rectilinear pair. Therefore, if $(\|\cdot\|, n)$ is not a 3-rectilinear pair, $\alpha(n, \|\cdot\|) \geq \beta(n, \|\cdot\|) > 0$. Now, if $(\|\cdot\|, n)$ is a 3-rectilinear pair, consider $P = (v_1, v_2, v_3) \in E_3$, where v_j are the vertices of $\partial B_1^{\|\cdot\|}(0)$. Then $e(P) = 2$ and so $\alpha(3, \|\cdot\|) = 2 > 0$. □

Remark 4.4.5. Note, via the compactness of $\partial B_1^{\|\cdot\|}(0)$, there exists $P \in E_n^{\|\cdot\|}$ such that $e(P) = \alpha(n, \|\cdot\|)$. Therefore,

$$\alpha(n, \|\cdot\|) = \max\left\{e(P) : P \in E_n^{\|\cdot\|}\right\}.$$

Similarly, if $(\|\cdot\|, n)$ is not a 3-rectilinear pair, there exists $Q \in E_n^{\|\cdot\|}$ such that $e(Q) = \beta(n, \|\cdot\|)$ and so

$$\beta(n, \|\cdot\|) = \min\left\{e(P) : P \in E_n^{\|\cdot\|}\right\}.$$

Let us introduce the following notation.

Notation 4.4.6. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$ be such that $(\|\cdot\|, n)$ is not a 3-rectilinear pair. Define $l_n^{\|\cdot\|} : \partial B_1^{\|\cdot\|} \to (0, 2]$ by $l_n^{\|\cdot\|}(x) = e\left(P_x\right)$, where $P_x \in E_n^{\|\cdot\|}$ is such that $x \in P_x$.

Observe by Theorem 4.3.5, Lemma 4.3.6 and Lemma 4.4.4 that l_n is well-defined. We now show that l_n is 1-Lipschitz continuous.

Theorem 4.4.7. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$ be such that $(\|\cdot\|, n)$ is not a 3-rectilinear pair. Then $l_n^{\|\cdot\|}$ is 1-Lipschitz.

Proof. We split the proof into two parts: we first show that $l_n = l_n^{\|\cdot\|}$ is pointwise 1-Lipschitz at each $x \in \partial B_1^{\|\cdot\|}(0)$ and then utilise Lemma 1.2.13 to conclude the Lipschitzness of l_n.

Part 1: Fix $x \in \partial B_1^{\|\cdot\|}(0)$ and find $P = (x_1, \ldots, x_n) \in E_n$ such that $x_1 = x$ and $e(P) = l_n(x)$. Define

$$r_0 := r_0(x) = \min\left\{1, e(P), \|x_2 + x_1\|, \|x_n + x_1\|\right\}.$$

Note by Lemma 4.2.22 that $r_0 > 0$. Moreover, by (4.2.2), recall

$$x \in B_{r_0}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0) \subseteq (x_n, -x_n)_{\|\cdot\|} \cap (-x_2, x_2)_{\|\cdot\|} =: G.$$

Therefore $G \neq \emptyset$. Next, as $P \in E_n$ and $(\|\cdot\|, n)$ is not a 3-rectilinear pair, by Lemma 4.2.22, the inscribed n-gon $Q_y := (y, x_2, \ldots, x_n) \in \mathcal{F}_n$ has acute visibility for each $y \in B_{r_0}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0)$.

In particular, note for each $y \in B_{r_0}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0)$,

$$d^+(Q_y) \leq \max \{\|y - x_2\|, \|y - x_n\|, e(P)\} \leq e(P) + \|x - y\| = l_n(x) + \|x - y\|.$$

Hence Corollary 4.2.21 applied to Q_y and an equilateral polygon Q containing x implies

$$l_n(y) = e(Q) \leq d^+(Q_y) \leq l_n(x) + \|x - y\| \quad \text{for all } y \in B_{r_0}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0). \tag{4.4.1}$$

Arguing similarly yields $d^-(Q_y) \geq l_n(x) - \|x - y\|$. Hence, by Corollary 4.2.21,

$$l_n(y) = e(Q) \geq d^-(Q_y) \geq l_n(x) - \|x - y\| \quad \text{for all } y \in B_{r_0}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0). \tag{4.4.2}$$

Combining (4.4.1) and (4.4.2) yields $|l_n(x) - l_n(y)| \leq \|x - y\|$ for all $y \in B_{r_0}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0)$ and hence l_n is pointwise 1-Lipschitz at $x \in \partial B_1^{\|\cdot\|}(0)$.

Part 2: For each $x \in \partial B_1^{\|\cdot\|}(0)$, let $P_x = (x_1, \ldots, x_n) \in E_n$ be such that $x_1 = x$. Define

$$R_x := \min \{\|x_2 + x_1\|, \|x_n + x_1\|, e(P_x)\}.$$

By Part 1 of the present theorem, $|l_n(x) - l_n(y)| \leq \|x - y\|$ whenever $y \in B_{r_0(x)}^{\|\cdot\|}(x) \cap \partial B_1^{\|\cdot\|}(0)$, where $r_0(x) = \min\{1, R_x\}$. However, by Lemma 4.2.24, there exists $c > 0$ such that $R_x \geq c$. Therefore,

$$\inf_{x \in \partial B_1^{\|\cdot\|}(0)} r_0(x) = \inf_{x \in \partial B_1^{\|\cdot\|}(0)} \min \{1, R_x\} \geq \min (1, c) > 0.$$

Let $X = \partial B_1^{\|\cdot\|}(0)$ and $Y = [0, 1]$, where X is equipped with the metric induced by the norm $\|\cdot\|$ and Y is equipped with the metric induced by the Euclidean norm. Then, by Lemma 1.2.13, l_n is locally 1-Lipschitz at each $x \in X$. Recall a function which is everywhere locally 1-Lipschitz on a compact metric space is 1-Lipschitz; see, for example, [33, Theorem 2.1]. Hence l_n is 1-Lipschitz. □

Finally, as a consequence of the continuity of l_n, we can deduce that generally the set of possible edge lengths over all equilateral polygons inscribed in a fixed planar norm forms a closed, possibly degenerate, interval.

Corollary 4.4.8. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $n \geq 3$. For each $d \in (\beta(n, \|\cdot\|), \alpha(n, \|\cdot\|)]$

there exists $P \in E_n^{\|\cdot\|}$ such that $e(P) = d$. Moreover, if $(\|\cdot\|, n)$ is not a 3-rectilinear pair, there exists $Q \in E_n^{\|\cdot\|}$ such that $e(Q) = \beta(n, \|\cdot\|)$.

Proof. This follows immediately by Remark 4.4.5 and Theorem 4.4.7. □

Remark 4.4.9. It is not true that when $\|\cdot\|$ is strictly convex, then $\alpha(n, \|\cdot\|) = \beta(n, \|\cdot\|)$ for each $n \geq 3$, as in the Euclidean case. Indeed, consider the norm l_p^2 on $\mathbb{C}$ given by $l_p^2(x + iy) = (|x|^p + |y|^p)^{1/p}$ for some $p > 2$. Then let $P = (1, i, -1, -1)$ and Q denote the quadrilateral formed by the intersection of $\partial B_1^{l_p^2}(0)$ and the lines $y = \pm x$. Then, $P, Q \in E_4^{l_p^2}$ and

$$\beta(4, l_p^2) \leq e(P) = 2^{1/p} < 2^{1-(1/p)} = e(Q) \leq \alpha(4, l_p^2).$$

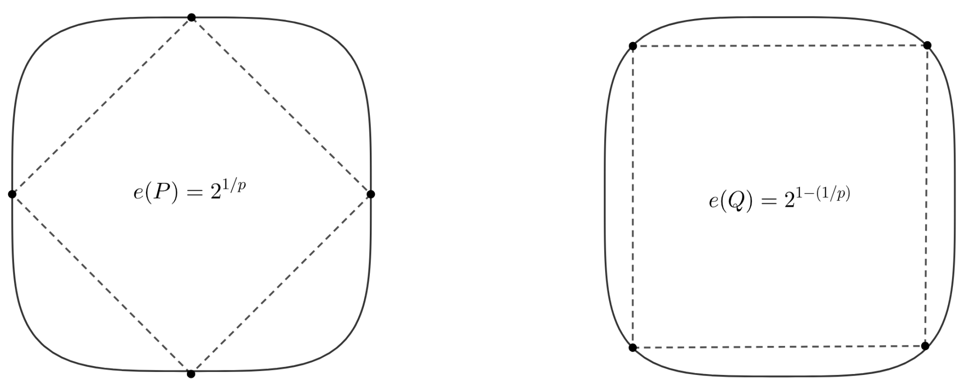

Figure 4.4: Existence of two equilateral polygons in a strictly convex norm with distinct edge lengths.

This section was focused on the generalised concept of equilateral polygons in centrally symmetric convex bodies in the plane, and aimed to correct a statement of [8] and provide an independent proof. We have shown in Corollary 4.4.8 that if the upper and lower norm regularity constants differ then there are uncountable many equivalence classes of equilateral polygons. The next section is devoted to determining when the upper and lower regularity constants for polygonal norms are different.

CHAPTER 5

NORM REGULARITY CONSTANTS IN POLYGONAL NORMS

This section is devoted to the investigation of when the upper and lower norm regularity constants for polygonal norms, see Definition 1.5.1 and Definition 4.4.3, are different. In doing so, we formalise the notion of 'rotating' equilateral polygons in a fixed polygonal norm and show that for polygonal norms where the number of edges is not a multiple of four, then such norms behave in a manner that is similar to the Euclidean norm.

For ease of notation, when considering two equivalent equilateral polygons $P, Q \in E_n^m$ in some polygonal norm $\| \cdot \|_m$ we will simply write $P \sim Q$.

Recall Notation 1.5.2. Throughout this chapter $v_1, \ldots, v_m$ will always denote the vertices of $\partial B_1^m(0)$ ordered in the anticlockwise direction, where $v_1 = 1$.

5.1 Rotating equilateral polygons in polygonal norms

Before we consider the construction of such aforementioned rotated equilateral polygons, we first show that equilateral polygons in polygonal norms behave similarly to those inscribed in a strictly convex norm, in the sense of uniqueness of polygons; see Theorem 4.3.5.

Lemma 5.1.1. Let $m \geq 6$ be even, $n \geq m$ and $P, Q \in E_n^m$. If $P \cap Q \neq \emptyset$, then $P \sim Q$.

Proof. Let $P = (x_1, \ldots, x_n) \in E_n^m$ and $Q = (y_1, \ldots, y_n) \in E_n^m$ be such that $x_1 = y_1$. Then, by Lemma 1.5.16, Proposition 4.2.14 and Lemma 4.3.6, as $e(P) = e(Q) \leq \frac{1}{m}\mathcal{H}^1\left(\partial B_1^m(0)\right)$, there exists a unique $z \in (x_1, -x_1)_m$ such that $\|x_1 - z\|_m = e(P)$. Therefore, as $\|x_2 - x_1\|_m = \|y_2 - x_1\|_m = e(P)$ and $x_2, y_2 \in (x_1, -x_1)_m$ by Lemma 4.2.17, we conclude that $x_2 = y_2$.

Continuing recursively, $x_j = y_j$ for each $j \in [n]$ and so $P = Q$. $\square$

Remark 5.1.2. The above lemma fails when $m = 4$. Indeed consider, for each $r \in [0, 1]$ the equilateral polygon $P(r) \in E_5^4$ given by

$$P(r) = \left(\frac{1-i}{2}, \frac{1+i}{2}, \frac{-1+i}{2}, \left(-1 + \frac{r}{2}\right) - \frac{r}{2}i, \left(\frac{-1}{2} + \frac{r}{2}\right) - \left(\frac{1}{2} + \frac{r}{2}i\right) \right);$$

see Figure 5.1. Then $P(r) \cap P(s) \neq \emptyset$ for all $r, s \in [0, 1]$, but $P(r) \not\sim P(s)$ whenever $r \neq s$.

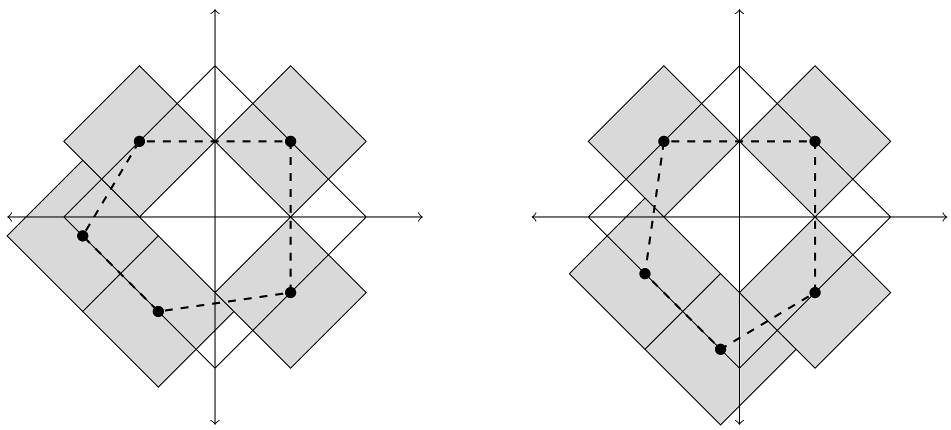

Figure 5.1: Polygons $P(1/4)$ and $P(3/4)$ which both contain $x = (1-i)/2$ but are not equivalent.

The following lemma improves on the inherent strict acute visibility of any equilateral polygon $P \in E_n^m$ where $n \geq m$.

Lemma 5.1.3. Let $m \geq 4$ be even, $n \geq m$ and $P \in E_n^m$. Then $P \cap [v_j, v_{j+1}) \neq \emptyset$ for each $j \in [m]$.

Proof. We show the following sufficient statement: for each $j \in [m]$, if $P \cap [v_j, v_{j+1}) \neq \emptyset$, then $P \cap [v_{j+1}, v_{j+2}) \neq \emptyset$. Without loss of generality, suppose $j = 1$ and $P \cap [v_1, v_2) \neq \emptyset$. Define $x \in P \cap [v_1, v_2)$ to be such that

$$\|x - v_2\|_m = \min_{z \in P \cap [v_1, v_2)} \|z - v_2\|_m.$$

Consider first the case when $m = 4$. For a contradiction, suppose that $P \cap [v_2, v_3) = \emptyset$. Then by Lemma 4.2.17 and either the Monotonicity Lemma 1.4.2 if $y = v_3$ or Lemma 1.4.3 if $y \neq v_3$, one can conclude for each $y \in P \cap (x, -x)_m$ that $\|x - y\|_m \geq \|v_2 - v_3\|_m = 2$. Hence $e(P) = 2$.

However, by Lemma 4.2.9 and Lemma 4.2.23, $P \sim_{\|\cdot\|_4} (v_1, v_2, v_3, v_4)$ and thus $v_2 \in P$, providing a contradiction.

Now, if $m \geq 6$ and $P \cap [v_2, v_3) = \emptyset$, by Proposition 4.2.14, Lemmata 1.5.13, 1.5.15, 4.2.17 and since $n \geq m$, there exists $y \in P \cap (x, -x)_m$ such that, if $H = \mathcal{H}^1\left(\partial B_1^m(0)\right)$,

$$\frac{H}{m} = \|v_3 - v_2\|_m < \|x - y\|_m = e(P) \leq \alpha(n, \|\cdot\|_m) \leq \frac{H}{n} \leq \frac{H}{m}.$$

providing contradiction. □

Now we introduce some new notions which allow one to utilise the underlying symmetry of polygonal norms.

Notation 5.1.4. Let $m \geq 6$ be even and $n \geq m$ be fixed. For every $t \in \left[0, \frac{1}{m}\mathcal{H}^1\left(\partial B_1^m(0)\right)\right]$ and $x \in [v_1, v_2]$ such that $\|x - v_1\|_m = t$, let $P_t = (x_1(t), \ldots, x_n(t)) \in E_n^m$ be the (unique) equilateral n-gon such that $x_1(t) := x \in P_t$.

The following notation of z_j, L_j, X_j, Y_j and c_j is introduced for the important special case of $t = 0$, i.e. $x_1(0) = x = v_1$. Let $P_0 = (z_1, \ldots, z_n)$ where $z_1 = v_1$. For each $j \in [m]$ let $L_j := \{k \in [n] : z_k \in [v_j, v_{j+1}) \cap P_0\}$. Further, let $X_j, Y_j \in [v_j, v_{j+1}) \cap P_0$ be defined by $X_j = z_{\min L_j}$ and $Y_j = z_{\max L_j}$. Finally let, for each $j \in [m]$,

$$c_j := \begin{cases} \cos(2\pi/m), & \text{if } \|X_{j+1} - v_{j+1}\|_m \geq \|Y_j - v_{j+1}\|_m; \\ \sec(2\pi/m), & \text{if } \|Y_j - v_{j+1}\|_m > \|X_{j+1} - v_{j+1}\|_m. \end{cases} \tag{5.1.1}$$

We identify $c_m = c_0$, $c_{m+1} = c_1$, $c_{m+2} = c_2$, etc.

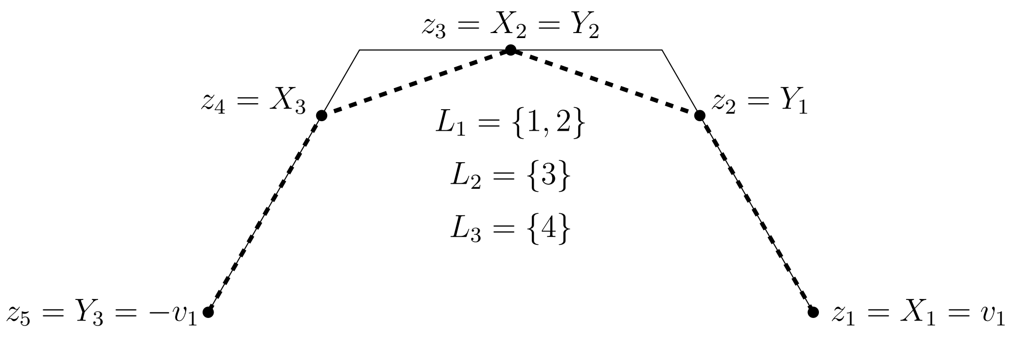

Figure 5.2: Example of Notation 5.1.4 for $P_0 \in E_8^6$.

Remark 5.1.5. The coefficients c_j are used when we 'rotate' equilateral polygons $P_0 \in E_n^m$ in Theorem 5.2.5.

Note that $L_j \neq \emptyset$ for every $j \in [m]$ by Lemma 5.1.3. Moreover, $Y_j \neq v_{j+1}$ for every $j \in [m]$.

Definition 5.1.6. For each even $m \geq 4$ let $\mathcal{L}_m$ denote the dihedral group of isometries of $\partial B_1^m(0)$. For each $n \geq m$, define the positive constant

$$\rho(n,m) := \sup \{\rho > 0 : P_s \cap I(P_0) = \emptyset \text{ for each } s \in (0,\rho) \text{ and each } I \in \mathcal{L}_m\}.$$

Notation 5.1.7. Let $m \geq 4$ be even. For each $j \in [m]$, let $R_j \in \mathcal{L}_m$ denote the rotation of the sphere $\partial B_1^m(0)$ such that $R_j(v_1) = v_j$ and let $S_j \in \mathcal{L}_m$ denote the reflection of the sphere $\partial B_1^m(0)$ such that $S_j(v_1) = v_j$.

Remark 5.1.8. To clarify, by R_1 we mean the identity isometry, id, and by S_1 we mean complex conjugation, i.e. $S_1(z) = \overline{z}$ for each $z \in \mathbb{C}$.

Observe, for each even $m \geq 4$, $\mathcal{L}_m = \{R_j\}_{j=1}^m \cup \{S_j\}_{j=1}^m$. Hence $\operatorname{card}(\mathcal{L}_m) = 2m$. Moreover, the vertex set $\{v_1, \ldots, v_m\}$ is invariant under each $I \in \mathcal{L}_m$, i.e. for each $I \in \mathcal{L}_m$ and each $j \in [m]$ there exists $k \in [m]$ such that $I(v_j) = v_k$. Next observe for each $I \in \mathcal{L}_m$ and each $P \in E_n^m$ that $I(P) \in E_n^m$ with $e(I(P)) = e(P)$. Furthermore, for each $I \in \mathcal{L}_m$ there exists $I^{-1} \in \mathcal{L}_m$ such that $I^{-1} \circ I = I \circ I^{-1} = \text{id}$.

Note $\rho(n,m) > 0$ is well-defined since $\operatorname{card}(\{z \in \partial B_1^m(0) : z \in I(P_0) \text{ for some } I \in \mathcal{L}_m\}) \leq 2nm$ as $\operatorname{card}(I(P_0)) = n$ for each $I \in \mathcal{L}_m$ and $\operatorname{card}(\mathcal{L}_m) = 2m$. Thus there exists $\rho \in \left(0, \frac{1}{m}\mathcal{H}^1(\partial B_1^m(0))\right)$ such that $P_s \cap I(P_0) = \emptyset$ for each $s \in (0,\rho)$ and each $I \in \mathcal{L}_m$.

Finally note that $\rho(n,m) = \sup\{\rho > 0 : I(P_s) \cap P_0 = \emptyset \text{ for each } s \in (0,\rho) \text{ and each } I \in \mathcal{L}_m\}$. This follows since $P_s \cap I(P_0) = \emptyset$ for some $s \in (0,\rho)$ and $I \in \mathcal{L}_m$ if and only if $I^{-1}(P_s) \cap P_0 = \emptyset$ and $I^{-1} \in \mathcal{L}_m$, also.

Remark 5.1.9. Observe that if $v_j \in P_0$ for some $j \in [m]$, then $R_j(P_0) \sim P_0$ and $S_j(P_0) \sim P_0$. Moreover, these three conditions are equivalent. First to see $R_j(P_0) \sim P_0$ and $S_j(P_0) \sim P_0$ note, by definition $R_j(v_1) = S_j(v_1) = v_j$, and so $v_j \in P_0 \cap R_j(P_0) \cap S_j(P_0)$. Hence Lemma 5.1.1 yields the equivalence. To see these three notions are in fact equivalent it suffices to show that either $S_j(P_0) \sim P_0$ or $R_j(P_0) \sim P_0$ implies $v_j \in P_0$; we only show the former since the latter follows almost identically. Suppose that $S_j(P_0) \sim P_0$. Then $v_j = S_j(v_1) \in S_j(P_0) \sim P_0$ and so

$v_j \in P_0$.

The following lemma concerns how the collections $\{X_j\}$ and $\{Y_j\}$ are invariant under the isometries R_j.

Lemma 5.1.10. Let $m \geq 6$ be even and $n \geq m$. If $v_j \in P_0$ then for each integer $a \geq 0$ and each $s \in [m]$,

$$\left(R_j^{-1}\right)^a (v_s) = v_{s-a(j-1)}, \quad \left(R_j^{-1}\right)^a (X_s) = X_{s-a(j-1)} \quad \text{and} \quad \left(R_j^{-1}\right)^a (Y_s) = Y_{s-a(j-1)}.$$

In particular, $R_j^a(P_0) \sim P_0$.

Proof. The first equality follows via the definition of R_j. We only prove the condition concerning X_s since the equivalent condition for Y_s follows similarly. Fix $a \in \mathbb{N}$ and $s \in [m]$. Observe that it suffices to verify that

$$(R_j)^a \left(X_{s-a(j-1)}\right) = X_s. \tag{5.1.2}$$

Indeed, as $X_{s-a(j-1)} \in \left[v_{s-a(j-1)}, v_{s+1-a(j-1)}\right)$ then $R_j^a \left(X_{s-a(j-1)}\right) \in [v_s, v_{s+1})$. Observe that $P_0 \sim R_j(P_0)$ since $R_j(v_1) = v_j \in P_0$. Hence $P_0 \sim R_j^a(P_0)$. Suppose there exists $x \in P_0 \sim R_j^a(P_0)$ such that $\left\|R_j^a(x) - v_s\right\| < \left\|R_j^a \left(X_{s-a(j-1)}\right) - v_s\right\|$. Then, as $R_j^a \in \mathcal{L}_m$, this implies that,

$$\left\|x - v_{s-a(j-1)}\right\| = \left\|x - \left(R_j^{-1}\right)^a (v_s)\right\| < \left\|X_{s-a(j-1)} - \left(R_j^{-1}\right)^a (v_s)\right\| = \left\|X_{s-a(j-1)} - v_{s-a(j-1)}\right\|,$$

which contradicts the definition of $X_{s-a(j-1)}$. Therefore, (5.1.2) is satisfied. □

Lemma 5.1.11. Let $m \geq 6$ be even, $n \geq m$ and $P_0 \in E_n^m$ be such that $v_1 \in P_0$. Suppose $A := \{j : 1 < j \leq m \text{ and } v_j \in P_0\} \neq \emptyset$ and $k_0 := \min A$. Then m is a multiple of $k_0 - 1$.

Moreover, $v_j \in P_0$ for some $j \in [m]$ if and only if $j \equiv 1 \pmod{k_0 - 1}$.

Proof. Let $t := \lfloor m/(k_0 - 1) \rfloor$ and $J := (m + 1) - t(k_0 - 1)$. If m is not a multiple of $(k_0 - 1)$, then $J > 1$. Hence, $1 < J \leq k_0 - 1$. So, $v_J \in (v_1, v_{k_0})_m$. However, by Lemma 5.1.1 and Lemma 5.1.10, $v_J = R_{k_0}^{-t}(v_{m+1}) = R_{k_0}^{-t}(v_1) \in R_{k_0}^{-t}(P_0) \sim P_0$, contradicting the definition of k_0, as $1 < J < k_0$.

Since $R_{k_0}^a(P_0) \sim P_0$ for each $a \geq 0$, if $j \equiv 1 \pmod{k_0 - 1}$ then $v_j \in P_0$. Suppose now $j \in [m]$ is such that $v_j \in P_0$. Find integers $a \geq 0$ and $b \in [k_0 - 1]$ such that $j - 1 = a(k_0 - 1) + b$.

Then, by Lemma 5.1.10 followed by Lemma 5.1.1, $v_{b+1} = R_{k_0}^{-a}(v_j) \in R_{k_0}^{-a}(P_0) \sim P_0$. Hence, as $1 < b+1 \leq k_0$, it follows by the definition of k_0 that $b = k_0 - 1$. Therefore, $j - 1 = a(k_0-1) + (k_0-1)$. So, $j \equiv 1 \pmod{k_0 - 1}$. □

We now show that each of the functions x_k, defined in Notation 5.1.4, which prescribe the path followed by the vertices of each equilateral n-gon, are in fact continuous. First, we introduce a simple proposition concerning convergence in compact metric spaces.

Proposition 5.1.12. Let X be a compact metric space, $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $x \in X$. If every convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x, then $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} x_n = x$.

Proof. Let $\varepsilon > 0$ be fixed and for a contradiction suppose that $\lim_{n\to\infty} x_n \neq x$. Then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $d(x_{n_k}, x) \geq \varepsilon$ whenever $k \geq 1$. But then, as X is compact, there exists a convergent subsequence $\{x_{n_{k_m}}\}_{m=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$, hence of $\{x_n\}_{n=1}^{\infty}$, such that $\lim_{m\to\infty} d\left(x_{n_{k_m}}, x\right) = 0$, providing contradiction. □

Lemma 5.1.13. Let $m \geq 6$ be even and $n \geq m$. For each $k \in [m]$, $x_k : [0, \|v_2 - v_1\|_m] \to \partial B_1^m(0)$ is continuous.

Proof. Let $d := \|v_2 - v_1\|_m$. We first show that x_1 is continuous at each $t \in [0, d]$. Indeed, fix $t \in [0, d]$ and let $\varepsilon > 0$ be given. If $s \in [0, d]$ is such that $|s - t| < \varepsilon$, then by the collinearity of $x_1(t), x_1(s)$ and v_1,

$$\|x_1(s) - x_1(t)\|_m = \Big| \|x_1(t) - v_1\|_m - \|x_1(s) - v_1\|_m \Big| = |s - t| < \varepsilon.$$

Hence the function x_1 is continuous at t.

Suppose now that x_k is continuous at t for some $k \in [n-1]$. We claim that x_{k+1} is continuous at t, also. Indeed, let $\varepsilon > 0$ be given and let $(s_l) \subseteq [0, d]$ be such that $s_l \to t$. Recall Theorem 4.4.7 and let $N_1 \in \mathbb{N}$ be such that $|l_n(x_k(s_l)) - l_n(x_k(t))| < \varepsilon/2$ for each $l \geq N_1$. Take $N_2 \in \mathbb{N}$ to be such that $\|x_k(s_l) - x_k(t)\|_m < \varepsilon/2$ whenever $l \geq N_2$. Define $N := \max(N_1, N_2)$ and note for each $j \geq N$ that

$$\|x_{k+1}(s_l) - x_k(t)\|_m \leq \|x_{k+1}(s_l) - x_k(s_l)\|_m + \|x_k(s_l) - x_k(t)\|_m < l_n(x_k(t)) + \varepsilon.$$

Similarly $\|x_{k+1}(s_l) - x_k(t)\|_m > l_n(x_k(t)) - \varepsilon$ whenever $l \geq N$. Hence

$$\lim_{l\to\infty} \|x_{k+1}(s_l) - x_k(t)\|_m = l_n(x_k(t)).$$

Consider any convergent subsequence $x_{k+1}(s_{l_j})$; such a sequence exists by the compactness of $\partial B_1^m(0)$. Thus, by the continuity of $\|\cdot\|_m$, $\|\lim_{j\to+\infty} x_{k+1}(s_{l_j}) - x_k(t)\|_m = l_n(x_k(t))$. Moreover as $x_{k+1}(s_{l_j}) \in (x_k(s_{l_j}), -x_k(s_{l_j}))_{\|\cdot\|}$, by Lemma 4.2.19, and since $x_k(s_{l_j}) \to x_k(t)$ we observe that $\lim_{j\to+\infty} x_{k+1}(s_{l_j}) \in [x_k(t), -x_k(t)]_m$. By [36, Theorem 4.3.6] note that $\mathcal{H}^1\left(\partial B_1^m(0)\right) \leq 8$. Therefore, by Proposition 4.2.14 and as $n \geq m \geq 6$, it follows $l_n(x_k(t)) \leq \frac{1}{m}\mathcal{H}^1\left(\partial B_1^m(0)\right) \leq 8/6 < 2$. Therefore $\lim_{j\to+\infty} x_{k+1}(s_{l_j}) \neq -x_k(t)$, so by Lemma 5.1.1, $\lim_{j\to+\infty} x_{k+1}(s_{l_j}) = x_{k+1}(t)$. Hence by the compactness of $\partial B_1^m(0)$ and Proposition 5.1.12 it follows $\lim\limits_{l\to+\infty} x_{k+1}(s_l) = x_{k+1}(t)$. Therefore x_{k+1} is continuous at t. □

We provide an alternate definition for the constant $\rho(n,m)$, see Definition 5.1.6; this allows one to conclude the existence of an isometry $I \in \mathcal{L}_m$ which maps $P_{\rho(n,m)}$ to the fixed polygon P_0.

Proposition 5.1.14. Let $m \geq 6$ be even and $n \geq m$. Then $\rho(n,m) = \rho^*$, where

$$\rho^* = \min\left\{\|I(z) - v_1\|_m : z \in P_0,\ I \in \mathcal{L}_m \text{ and } I(z) \in (v_1, v_2]\right\}. \tag{5.1.3}$$

Moreover, there exists $j \in [m]$ such that $R_j(P_0) \sim P_{\rho(n,m)}$.

Proof. For brevity, we write ρ instead of $\rho(n,m)$. To begin observe that $P_0 \cap (v_1, v_2] \neq \emptyset$, by Lemma 1.5.13, Lemma 1.5.15 and Proposition 4.2.14, since $e(P_0) \leq \alpha(n, \|\cdot\|_m) \leq \|v_1 - v_2\|_m$ and as $n \geq m$. Next, we note that $I_0(P_0) \cap P_{\rho^*} \neq \emptyset$ for some $I_0 \in \mathcal{L}_m$, since by definition of ρ^* there exist $I_0 \in \mathcal{L}_m$ and $z \in P_0$ such that $I_0(z) \in P_{\rho^*}$. Moreover, this implies $\rho^* \geq \rho$ by Definition 5.1.6. To see $\rho \geq \rho^*$ it suffices to verify for each $s \in (0, \rho^*)$ and each $I \in \mathcal{L}_m$ that $I(P_0) \cap P_s = \emptyset$. Suppose, for a contradiction, that there exist $s \in (0, \rho^*)$ and $I \in \mathcal{L}_m$ such that $I(P_0) \cap P_s \neq \emptyset$. By Lemma 5.1.1 as $n \geq m \geq 6$ this implies $I(P_0) \sim P_s$. Hence there exists $z \in P_0$ such that $I(z) \in [v_1, v_2]$ with $\|I(z) - v_1\|_m = s \in (0, \rho^*)$. This contradicts (5.1.3). Therefore $\rho \geq \rho^*$ and with $\rho^* \geq \rho$ this implies $\rho^* = \rho$.

Hence there exists $I_0 \in \mathcal{L}_m$ such that $I_0(P_0) \sim P_{\rho(n,m)}$. Observe that either $I_0 = R_j$ or

$I_0 = S_j$ for some $j \in [m]$. Suppose the latter holds. Then, $v_j = S_j(v_1) = I_0(v_1) \in P_{\rho(n,m)}$. Thus, as $v_j \in R_j(P_0) \cap P_{\rho(n,m)}$, we conclude by Lemma 5.1.1 that $R_j(P_0) \sim P_{\rho(n,m)}$. $\square$

Next we show that, in fact, every equilateral polygon inscribed in the unit sphere of a polygonal norm contains a vertex which is within $\rho(n,m)$ of a vertex of the sphere.

Lemma 5.1.15. Let $m \geq 6$ be even and $n \geq m$. Then for each $P \in E_n^m$ there exists $j \in [m]$ and $x \in P$ such that $\|x - v_j\|_m < \rho(n,m)$.

Proof. Recall Notation 5.1.4 for $x_k(t)$, $k \in [n]$ and $t \in [0, \|v_2 - v_1\|_m]$. For each $k \in [n]$ let $g_k(t) := \min_{j \in [m]} \|x_k(t) - v_j\|_m$ and let $f(t) := \min_{k \in [n]} g_k(t)$. By the continuity of x_k and of the norm $\|\cdot\|_m$ we conclude that g_k is continuous for each $k \in [n]$, hence f is continuous. Observe that $f(0) = f(\rho(n,m)) = 0$ since $v_1 \in P_0$ and since $P_{\rho(n,m)} \cap \{v_1, \ldots, v_m\} \neq \emptyset$ by Proposition 5.1.14. Furthermore observe that

$$f(t) \leq g_1(t) \leq \|x_1(t) - v_1\|_m = t$$

for each $t \in [0, \|v_2 - v_1\|_m]$. We show that in fact $0 \leq f(t) < \rho(n,m)$ for each $t \in [0, \|v_2 - v_1\|_m]$. Suppose, for a contradiction, that there exists $t \in (0, \|v_2 - v_1\|_m]$ such that $f(t) \geq \rho(n,m)$. Define

$$t_0 := \inf \{t > 0 : f(t) \geq \rho(n,m)\}.$$

By the continuity of f, observe that

$$\rho(n,m) \leq f(t_0) \leq t_0, \tag{5.1.4}$$

so in particular $t_0 > 0$. We claim that $f(t_0) = \rho(n,m)$. Indeed, if $t_0 = \rho(n,m)$ this follows via (5.1.4). Suppose that $t_0 > \rho(n,m)$ and consider an increasing sequence $\{t_k\} \subseteq (\rho(n,m), t_0)$ such that $t_k \nearrow t_0$. Observe, as $t_k < t_0$, that $f(t_k) < \rho(n,m)$ for each k. Thus, by the continuity of f, we conclude that

$$f(t_0) = \lim_{k \to \infty} f(t_k) \leq \rho(n,m).$$

Hence combining this with (5.1.4) we conclude that $f(t_0) = \rho(n,m)$.

As $f(t_0) = \rho(n,m)$ note there exist $k \in [n]$ and $j \in [m]$ such that $\|x_k(t_0) - v_j\|_m = \rho(n,m)$. So, for one of $I = R_j$ or $I = S_j$, it follows that $I(x_k(t_0)) = x_1(\rho(n,m))$. Hence, by Lemma 5.1.1,

$P_{\rho(n,m)} \sim I(P_{t_0})$. Let $J \in [m]$ be such that $v_J \in P_{\rho(n,m)}$; the existence of such J follows via Proposition 5.1.14. Therefore $v_J \in I(P_{t_0})$ and so $I^{-1}(v_J) \in P_{t_0}$. Hence, as $\{v_1, \ldots, v_m\}$ is invariant under isometries of the sphere, we conclude that $0 = f(t_0) = \rho(n,m)$ which contradicts the definition of $\rho(n,m)$.

Hence $f(t) < \rho(n,m)$ for each $t \in [0, \|v_2 - v_1\|_m]$. □

Corollary 5.1.16. Let $m \geq 6$ and $n \geq m$. If $P \in E_n^m$, then there exists $I \in \mathcal{L}_m$ such that $P \sim I(P_\rho)$ for some $\rho \in [0, \rho(n,m))$.

We proceed by introducing a result which determines the regularity of equilateral polygons inscribed in any polygonal norm, except for $\|\cdot\|_4$; this exception follows simply by the non-uniqueness of equilateral polygons as shown in Remark 5.1.2.

Proposition 5.1.17. Let $m \geq 6$ be even, $n \geq m$ and $P = (x_1, \ldots, x_n) \in E_n^m$. If $\text{Im}(x_1) = 0$, then for each $k \in [n]$,

$$\text{Re}(x_k) = \text{Re}\left(x_{(n+2)-k}\right) \quad \text{and} \quad \text{Im}(x_k) = -\text{Im}\left(x_{(n+2)-k}\right).$$

Moreover, if n is even, then

$$\text{Re}\,(x_k) = -\text{Re}\left(x_{\left(\frac{n}{2}+2\right)-k}\right) \quad \text{and} \quad \text{Im}\,(x_k) = \text{Im}\left(x_{\left(\frac{n}{2}+2\right)-k}\right).$$

Proof. Consider the isometry $I = S_1 \in \mathcal{L}_m$. Then $Q := (I(x_1), I(x_n), \ldots, I(x_2)) \in E_n^m$. As $I(x_1) = x_1$, note $x_1 \in P \cap Q$. Therefore, by Lemma 5.1.1, $P = Q$. In particular, for each $k \in [n]$,

$$x_k = I\left(x_{(n+2)-k}\right) = \overline{x_{(n+2)-k}}.$$

For the second part, we first show that if n is even, then $-x_1 \in P$. Indeed, by the previous part, for each $k \in [n]$ there exists $j \in [n]$ such that $x_k = \overline{x_j}$. Hence, as n is even and since $x_1 = \overline{x_1}$, counting corresponding vertices of P in both $(x_1, -x_1)_m$ and $(-x_1, x_1)_m$, this implies $x_{(n/2)+1} = \overline{x_{(n/2)+1}}$. Hence $\text{Im}(x_{(n/2)+1}) = 0$ and as $x_{(n/2)+1} \neq x_1$, this implies $-x_1 = x_{(n/2)+1} \in P$.

Now let us consider the isometry $J \in \mathcal{L}_m$ given by $J(z) = -\overline{z}$. Consider the equilateral polygon $R \in E_n^m$ formed of the vertices $J(x_k)$, $k \in [n]$. Then as $J(x_1) = -x_1 \in P \cap R$, by

Lemma 5.1.1, $P = R$. In particular, for each $k \in [n]$,

$$x_k = J\left(x_{\left(\frac{n}{2}+1\right)-k}\right) = -\overline{x_{\left(\frac{n}{2}+2\right)-k}}.$$

$\square$

Recall Notation 5.1.4. We introduce a further notion of symmetry observed by the equilateral polygons P_0.

Definition 5.1.18. Let $m \geq 6$ be even, $n \geq m$ and $j \in [m]$. We say that $P_0 \in E_n^m$ is (j, n, m)-*vertex symmetric* if $\|X_j - v_j\|_m = \|Y_{j-1} - v_j\|_m$, where v_j is a vertex of $\partial B_1^m(0)$.

We write P is (n, m)-vertex symmetric, if the value of j does not matter for our considerations.

Example 5.1.19. Observe if $m = 8$ and $n = 12$ then P_0 is $(j, 12, 8)$-vertex symmetric for each $j = 2, 4, 6, 8$, see Figure 5.3.

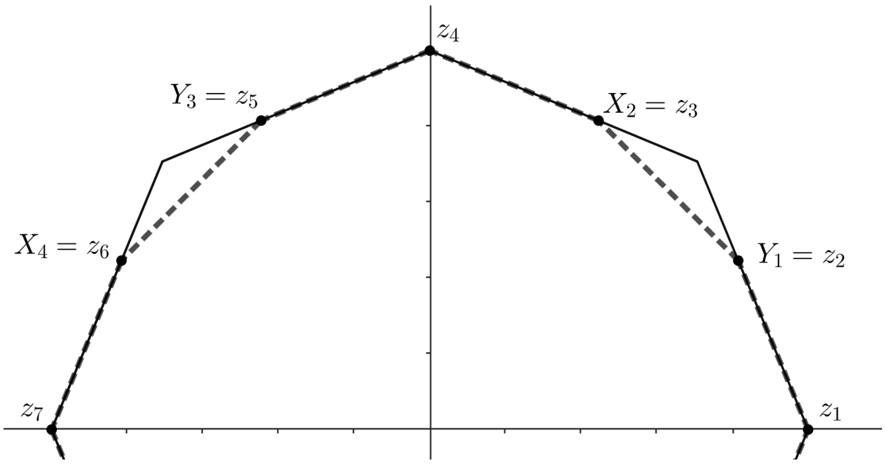

Figure 5.3: The equilateral polygon $P_0(12, 8)$ is vertex symmetric.

As a corollary to Proposition 5.1.17 we can deduce in most cases when the fixed equilateral polygon P_0 is vertex symmetric.

Corollary 5.1.20. Let $m \geq 6$ be even and $n \geq m$. If either:

i) n is odd, or ii) $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 4$,

then P_0 is vertex symmetric.

Proof. Let $P_0 = (x_1, \dots, x_n)$. Suppose first that n is odd. Then, by Proposition 5.1.17, observe that $x_{2+((n-1)/2)} = S_1(x_{1+((n-1)/2)})$. Further, by Lemma 5.1.3 note $x_{1+((n-1)/2)} \in [v_{m/2}, v_{1+(m/2)})$ and $x_{2+((n-1)/2)} \in [v_{1+(m/2)}, v_{2+(m/2)})$. Thus, as $P_0 \sim S_1(P_0)$ by Lemma 5.1.1 we conclude that $Y_{m/2} = x_{1+((n-1)/2)}$, $X_{1+(m/2)} = x_{2+((n-1)/2)}$ and $\|x_{1+((n-1)/2)} - v_{1+(m/2)}\|_m = \|x_{2+((n-1)/2)} - v_{1+(m/2)}\|_m$. Therefore P_0 is $(1 + (m/2), n, m)$-vertex symmetric.

If now $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, one can argue similarly to deduce that P_0 is $(1 + (m/4), n, m)$-vertex symmetric. □

Lemma 5.1.21. Let $m \geq 6$ be even, $n \geq m$ and $j \in [m]$. If P_0 is (j, n, m)-vertex symmetric, then $v_j \notin P_0$ and $v_{2j-1} \in P_0$. Moreover, if $j_0 = \min\{j \in [m] : P_0 \text{ is } (j, n, m)\text{-vertex symmetric}\}$, then $2j_0 - 1 \leq m + 1$ and $v_k \notin P_0$ for each $1 < k < 2j_0 - 1$.

Proof. If $v_j \in P_0$ then $X_j = v_j$. But then, by Remark 5.1.5, $\|X_j - v_j\|_m = 0 < \|Y_{j-1} - v_j\|_m$, thus P_0 is not (j, n, m)-vertex symmetric. Hence $v_j \notin P_0$ and $\|X_j - v_j\|_m = \|Y_{j-1} - v_j\|_m$. Thus $S_{2j-1}(X_j) = Y_{j-1}$ and so $P_0 \sim S_{2j-1}(P_0)$ by Lemma 5.1.1. Hence, $v_{2j-1} = S_{2j-1}(v_1) \in P_0$.

Consider now j_0 as defined in the present lemma. Note that $j_0 \geq 2$ since $v_1 \in P_0$. Also, by the first part of the present lemma, $v_{j_0} \notin P_0$. First note that $j_0 \leq 1 + (m/2)$ since $P_0 \sim S_1(P_0)$ as $v_1 = z_1 = S_1(z_1)$. Therefore $2j_0 - 1 \leq 1 + m$. Suppose now, for a contradiction, there exists $k \in [2j_0 - 2] \setminus \{1\}$ such that $v_k \in P_0$. Without loss of generality, we may assume that $1 < k < j_0$ since $k \neq j_0$ and if $k > j_0$ as $P_0 \sim S_{2j_0-1}(P_0)$ then it suffices to consider the vertex given by $S_{2j_0-1}(v_k) \in P_0$. Note as $k < j_0$ that $\|X_l - v_l\|_m \neq \|Y_{l-1} - v_l\|_m$ for each $l \in [k]$.

Find integer $a \geq 0$ and $b \in [k-1]$ such that $j_0 - 1 = a(k-1) + b$. Then observe, by Lemma 5.1.10, that $X_{j_0} = (R_k)^a(X_{b+1})$, $Y_{j_0-1} = (R_k)^a(Y_b)$ and $v_{j_0} = (R_k)^a(v_{b+1})$. Then, as $R_k \in \mathcal{L}_m$ and $b + 1 \in [k]$, note

$$\|X_{j_0} - v_{j_0}\|_m = \|X_{b+1} - v_{b+1}\|_m \neq \|Y_b - v_{b+1}\|_m = \|Y_{j_0-1} - v_{j_0}\|_m,$$

contradicting our choice of j_0. □

Proposition 5.1.22. Let $m \geq 8$ be divisible by 4 and $n \geq m$. If there exists an even $J \in [m]$ such that $v_J \in P_0$, then P_0 is not vertex symmetric.

Proof. Let $k_0 := \min\{k : 2 \leq k \leq m,\ k \text{ is even and } v_k \in P_0\}$. First note that if $k_0 = 2$, then $v_j \in P_0$ for all $j \in [m]$, so P_0 is not vertex symmetric. So suppose that $k_0 \geq 4$ and, for a contradiction, that P_0 is (j,n,m)-vertex symmetric for some $j \in [m]$. Since $P_0 \sim R_{k_0}^{-a}(P_0)$ for any $a \geq 0$, we may assume without loss of generality that $j \in [k_0]$. Note $j \neq 1$ and $j \neq k_0$, so $2 \leq j \leq k_0 - 1$.

As P_0 is (j,n,m)-vertex symmetric, observe by Lemma 5.1.1 that $P_0 \sim S_{2j-1}(P_0)$ as $S_{2j-1}(Y_{j-1}) = X_j$. If $j \leq k_0/2$, then $2j-1 < k_0$ and $v_{2j-1} = S_{2j-1}(v_1) \in P_0$. Let $l_0 := \min\{l : 2 \leq l \leq m,\ l \text{ is odd and } v_l \in P_0\}$. Note l_0 is well-defined since $v_{2j-1} \in P_0$, $2j-1$ is odd and $1 < 2j-1 < k_0 \leq m$. Since $l_0 \leq 2j-1 < k_0$, we also get $l_0 = \min\{2 \leq s \leq m : v_s \in P_0\}$. Therefore, by Lemma 5.1.11, as $v_{k_0} \in P_0$, we conclude that $k_0 \equiv 1 \pmod{l_0 - 1}$. However, as l_0 is odd, this implies that k_0 is odd also, contradicting that k_0 is even.

As $k_0/2$ is an integer, if $j \leq k_0/2$ does not hold, then $j \geq 1 + (k_0/2)$; this implies that $2j - k_0 \geq 2$. Note that $2j - k_0 < k_0$ in any case, as $j \leq k_0 - 1$. Moreover, as $v_{k_0} \in P_0$, $v_{2j-k_0} = S_{2j-1}^{-1}(v_{k_0}) \in S_{2j-1}^{-1}(P_0) \sim P_0$. Using that $2 \leq 2j - k_0 < k_0$ and since $2j - k_0$ is even, this contradicts the definition of k_0. □

We have the following characterisation of when the equilateral polygon P_0 is necessarily vertex symmetric.

Lemma 5.1.23. Let $m \geq 6$ be even, $n \geq m$ and $P_0 \in E_n^m$ be such that $v_1 \in P_0$. Suppose $k_0 := \min\{j : 1 < j \leq m \text{ and } v_j \in P_0\}$ is well-defined. Then, P_0 is (j,n,m)-vertex symmetric for some $j \in [k_0]$ if and only if k_0 is odd and $j = (k_0+1)/2$.

Proof. By Proposition 5.1.22, we observe that P_0 is not vertex symmetric whenever k_0 is even. So, let us suppose now $k_0 \geq 3$ is odd. We first show that P_0 is $((k_0-1)/2, n, m)$-vertex symmetric. Indeed, as $v_{k_0} \in P_0$, note by Lemma 5.1.1 that $P_0 \sim S_{k_0}(P_0)$. As $v_j \notin P_0$ for all $2 \leq j \leq k_0 - 1$, this then implies that $\|X_{(k_0+1)/2} - v_{(k_0+1)/2}\|_m = \|Y_{(k_0-1)/2} - v_{(k_0+1)/2}\|_m$. Hence P_0 is $((k_0+1)/2, n, m)$-vertex symmetric.

Finally, for a contradiction, suppose there exists $j \in [k_0] \setminus \{(k_0+1)/2\}$ such that P_0 is (j,n,m)-vertex symmetric. Note $2 \leq j \leq k_0 - 1$. We may assume without loss of generality that $j < (k_0+1)/2$ since $S_{k_0}(P_0) \sim P_0$ and $S_{k_0}(v_{(k_0+1)/2}) = v_{(k_0+1)/2}$. But then, as $S_{2j-1}(P_0) \sim P_0$ this implies that $v_{2j-1} = S_{2j-1}(v_1) \in P_0$, contradicting the definition of k_0 as $2j - 1 < k_0$. So, P_0 is not (j,n,m)-vertex symmetric for any $j \in [k_0]$ where $j \neq (k_0+1)/2$. □

We now prove a relation between the coefficients c_k, as defined in (5.1.1), which utilises the inherent symmetry of the unit sphere $\partial B_1^m(0)$. Recall Notation 5.1.4.

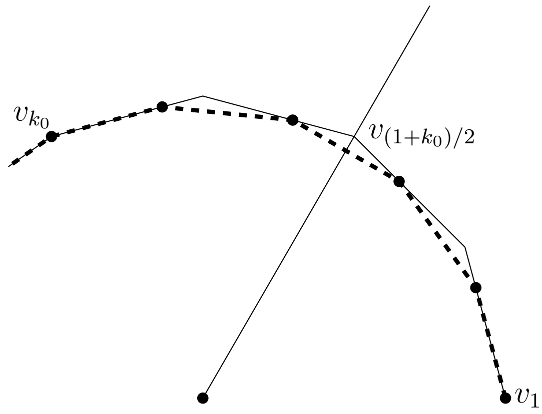

Figure 5.4: Example of the equilateral 15-gon $P_0 \in E_{15}^{12}$ where $k_0 = 5$ is defined as in Lemma 5.1.23.

Lemma 5.1.24. Let $m \geq 6$ be even and $n \geq m$. If $v_k \in P_0$ for some $k \in [m] \setminus \{1\}$, then $c_{k-1} = \sec(2\pi/m)$. Moreover, if $3 \leq k \leq m$ and $v_j \notin P_0$ for each $2 \leq j \leq k-1$, then:

(a) if k is even, $c_j c_{k-1-j} = 1$ for each $j \in [k-2]$;

(b) if k is odd, $c_j c_{k-1-j} = 1$ for each $j \in [k-2] \setminus \left\{\dfrac{k-1}{2}\right\}$.

Proof. From Notation 5.1.4, it follows that if $v_k \in P_0$ then $X_k = v_k$. So, by Remark 5.1.5, $\|X_k - v_k\|_m = 0 < \|Y_{k-1} - v_k\|_m$. Hence $c_{k-1} = \sec(2\pi/m)$.

Suppose now that $v_j \notin P_0$ for each $j \in [k-1] \setminus \{1\}$. By Lemma 5.1.1 $P_0 \sim S_k(P_0)$ as $v_k = S_k(v_1) \in P_0$. Note that for each $j \in [m]$, $S_k(v_j) = v_{k+1-j}$. So, as $X_j, Y_j \in [v_j, v_{j+1})$, note

$$S_k(X_j), S_k(Y_j) \in (v_{k-j}, v_{k+1-j}] \qquad \text{for all } j \in [m].$$

Hence, as $X_j, Y_j \in (v_j, v_{j+1})$ for each $j \in [k-1] \setminus \{1\}$, it follows that $S_k(X_j) = Y_{k-j}$ and $S_k(Y_j) = X_{k-j}$. Note by Lemma 5.1.23 that $\|X_j - v_j\|_m = \|Y_{j-1} - v_j\|_m$ if and only if k is odd and $j = (k+1)/2$. So, by Notation 5.1.4 it follows that $c_j c_{k-1-j} = 1$ whenever k is even, or whenever k is odd and $j \neq (k-1)/2$. ☐

We provide an interesting property for vertex symmetric equilateral polygons.

Lemma 5.1.25. Let $m \geq 6$ be even and $n \geq m$. Then

$$P_0(n, m) \in E_n^m \text{ is } (n, m)\text{-vertex symmetric if and only if } c_1 \dots c_{m-1} = \cos(2\pi/m).$$

Proof. Suppose first that P_0 is vertex symmetric and recall Notation 5.1.4. Let $j_0 \in \{2, \dots m\}$ be the minimal index such that P_0 is (j_0, n, m)-vertex symmetric, i.e.

$$\|X_{j_0} - v_{j_0}\|_m = \|Y_{j_0-1} - v_{j_0}\|_m. \tag{5.1.5}$$

Consider first when $j_0 = 1 + (m/2)$. Hence, $\|X_{j+1} - v_{j+1}\|_m \neq \|Y_j - v_{j+1}\|_m$ for each $j \in [(m/2) - 1]$ and $\|X_{1+(m/2)} - v_{1+(m/2)}\|_m = \|Y_{m/2} - v_{1+(m/2)}\|_m$. Moreover, as $v_1 \in P_0$ and $S_1(v_1) = v_1$ note, by Lemma 5.1.1, that $P_0 \sim S_1(P_0)$. Hence, $\|X_{j+1} - v_{j+1}\|_m \neq \|Y_j - v_{j+1}\|_m$ for each $j \neq m/2$. Thus, $c_j c_{m-j} = 1$ for each $j \neq m/2$. Hence, $c_1 \dots c_{m-1} = c_{m/2} = \cos(2\pi/m)$, by (5.1.1) of Notation 5.1.4, since P_0 is $(1 + (m/2), n, m)$-vertex symmetric.

Suppose now $j_0 \neq 1 + (m/2)$. Hence $v_{2j_0-1} \in P_0$ and $2j_0 - 1 \leq m + 1$, by Lemma 5.1.21. Moreover, as $2j_0 - 1$ is odd and m is even, we have $2j_0 - 1 \leq m - 1$. As $j_0 \neq 1$, we get $1 < 2j_0 - 1 < m$. Define $q := 2j_0 - 2$ and let $k_0 := \min\{1 < j \leq m : v_j \in P_0\}$. Since $v_{q+1} \in P_0$, k_0 is well-defined and $k_0 \leq q + 1$. We show that in fact

$$k_0 = q + 1 = 2j_0 - 1. \tag{5.1.6}$$

Suppose that $k_0 \leq q = 2j_0 - 2$. We claim this implies that $k_0 \leq j_0 - 1$. First note that $k_0 \neq j_0$ since $v_{j_0} \notin P_0$ by Lemma 5.1.21. Moreover, observe that as $v_{k_0} \in P_0$ and since $P_0 \sim_{\|\cdot\|} S_{2j_0-1}(P_0)$ it follows that that $v_d := S_{2j_0-1}(v_{k_0}) \in P_0$ and $1 < d < 2j_0 - 1$. From the minimality of k_0 we have $k_0 = \min(k_0, d) < j_0$. Our choice of j_0 implies, by Lemma 5.1.21,

$$\|X_j - v_j\|_m \neq \|Y_{j-1} - v_j\|_m \text{ for each } j \in [j_0 - 1]. \tag{5.1.7}$$

Find integer $a \geq 0$ and $b \in [k_0 - 1]$ such that $j_0 - 1 = a(k_0 - 1) + b$. Hence observe by Lemma 5.1.10,

$$X_{j_0} = (R_{k_0})^a (X_{b+1}), \quad Y_{j_0-1} = (R_{k_0})^a (Y_b) \quad \text{and} \quad v_{j_0} = (R_{k_0})^a (v_{b+1}).$$

Now as $R_{k_0} \in \mathcal{L}_m$ is an isometry of the unit sphere and since $b + 1 \leq k_0 \leq j_0 - 1$, this implies by (5.1.7),

$$\|X_{j_0} - v_{j_0}\|_m = \|X_{b+1} - v_{b+1}\|_m \neq \|Y_b - v_{b+1}\|_m = \|Y_{j_0-1} - v_{j_0}\|_m,$$

a contradiction. Therefore (5.1.6) is satisfied.

As $v_1, v_{k_0} \in P_0$ we conclude $P_0 \cap R_{k_0}(P_0) \neq \emptyset$, by Lemma 5.1.1. So $P_0 \sim R_{k_0}(P_0)$; note that the minimality of k_0 and Lemma 5.1.11 imply that m is a multiple of $k_0 - 1$, which is equal to q by (5.1.6). Moreover note, by Lemma 5.1.24, as $v_{kq+1} = (R_{k_0})^k(v_1) \in P_0$ that $c_{kq} = \sec(2\pi/m)$ for each $k \geq 0$. Furthermore, for each $k \geq 0$, $c_{kq+1} \ldots c_{(k+1)q-1} = c_1 \ldots c_{q-1}$. Therefore, to see $c_1 \ldots c_{m-1} = \cos(2\pi/m)$ observe that it suffices to verify that $c_1 \ldots c_{q-1} = \cos(2\pi/m)$.

Indeed, if $j_0 = 2$ then $q = 2$ and thus $c_1 \ldots c_{q-1} = c_1 = \cos(2\pi/m)$ by (5.1.1) since P_0 is $(2, n, m)$-vertex symmetric and so $\|X_2 - v_2\|_m = \|Y_1 - v_2\|_m$. If now $j_0 \geq 3$, since $v_{2j_0-1} \in P_0$ and $v_j \notin P_0$ for each $j \in [2j_0 - 2] \setminus \{1\}$ observe that, by Lemma 5.1.24 (b), $c_j = 1/c_{2j_0-j-2}$ for each $j \in [j_0 - 2] = [((k_0-1)/2) - 1]$ and thus $c_1 \ldots c_{q-1} = c_{(k_0-1)/2} = c_{j_0-1} = \cos(2\pi/m)$, where the latter follows from (5.1.1) and (5.1.5).

Suppose now that $P_0 = (z_1, \ldots, z_n)$ is not vertex symmetric. We shall show that $c_1 \ldots c_{m-1} \neq \cos(2\pi/m)$. First note that if n were odd then, by Corollary 5.1.20, P_0 is vertex symmetric. Hence n is even. Thus, by Proposition 5.1.17, observe that $-v_1 = z_{(n/2)+1} \in P_0$. Define again $k_0 := \min\{1 < j \leq m : v_j \in P_0\}$; note k_0 is well-defined as $-v_1 = v_{(m/2)+1} \in P_0$. If $k_0 = 2$, then $P_0 \sim_{\|\cdot\|} R_2(P_0)$, that is, $v_j \in P_0$, and thus $c_j = \sec(2\pi/m)$, for each $j \in [m]$. Hence $c_1 \ldots c_{m-1} = \sec^{m-1}(2\pi/m) > 1 > \cos(2\pi/m)$. Therefore suppose that $k_0 > 2$.

Then k_0 is even by Lemma 5.1.23, so $k_0 \geq 4$. This implies that $c_1 \ldots c_{k_0-2} = 1$ by Lemma 5.1.24 (a). Therefore as $v_{k_0} \in P_0$ we conclude that $c_1 \ldots c_{k_0-1} = c_{k_0-1} = \sec(2\pi/m)$, by Lemma 5.1.23.

Moreover as k_0, such that $v_{k_0} \in P_0$, is minimal, note by Lemma 5.1.11 that m is a multiple of $k_0 - 1$ and $B := \{j \in [m] : v_j \in P_0\} = \{1, k_0, 2k_0 - 1, \ldots, m - k_0 + 2\}$. In particular, this implies $c_j \ldots c_{j+k_0-2} = c_1 \ldots c_{k_0-1}$ for each $j \in B$ and so $c_1 \ldots c_{m-1} = (c_1 \ldots c_{k_0-1})^{m/(k_0-1)} = \sec^{m/(k_0-1)}(2\pi/m) > 1 > \cos(2\pi/m)$.

Therefore if P_0 is not vertex symmetric then $c_1 \ldots c_{m-1} \neq \cos(2\pi/m)$. □

We finish this section by introducing a rather trivial condition for an equilateral polygon inscribed in a polygonal norm to be, under an isometry, equivalent to P_0 provided the latter is vertex symmetric.

Lemma 5.1.26. Let $m \geq 8$ be a multiple of 4 and $n \geq m$ be such that P_0 is vertex symmetric and $P = (x_1, \dots, x_n) \in E_n^m$. Suppose there exists $j \in [n]$ and $k \in [m]$ such that $\|x_{j+1} - x_j\|_m = e(P_0)$ and $\|x_j - v_k\|_m = \|x_{j+1} - v_k\|_m$. Then there exists $I \in \mathcal{L}_m$ such that $P \cap I(P_0) \neq \emptyset$.

Proof. Suppose $P_0 = (z_1, \dots, z_n)$ where $z_1 = v_1$. As P_0 is vertex symmetric note, by Lemma 1.5.15, there exists $l_0 \in [m]$ such that $\|X_{l_0} - v_{l_0}\|_m = e(P_0)/(1+c)$ where $c = \cos(2\pi/m)$. Similarly, as $\|x_j - v_k\|_m = \|x_{j+1} - v_k\|_m$ and since $\|x_{j+1} - x_j\|_m = e(P_0)$ it follows $\|x_j - v_k\|_m = e(P_0)/(1+c)$. Let $I \in \mathcal{L}_m$ be given by $I = R_k \circ R_{l_0}^{-1}$. Then note $I(v_{l_0}) = v_k$ and $I(X_{l_0}) = x_j$. Hence $x_j \in P \cap I(P_0)$. □

5.2 When do the polygonal norm regularity constants differ?

We now determine for a fixed polygonal norm $\| \cdot \|_m$ and fixed $n \geq m$ under what conditions $\alpha(n, \| \cdot \|_m) = \beta(n, \| \cdot \|_m)$. We shall consider the cases when m is a multiple of four or not separately.

Polygonal m-norms where m is not divisible by 4

Interestingly, in polygonal norms where the number of edges is not divisible by four it follows that for every inscribed polygon, provided it is sufficiently separated, then the sum of the edge lengths is equal to the length of the unit sphere.

Corollary 5.2.1. Let $m \geq 4$ be an even integer which is not divisible by 4 and $n \geq m$. If $P = (x_1, \dots, x_n) \in \mathcal{F}_n^{\|\cdot\|_m}$ is such that $P \cap [v_k, v_{k+1}] \neq \emptyset$ for each $k \in [m]$, then

$$\sum_{j=1}^{n} \|x_{j+1} - x_j\|_m = \mathcal{H}^1\left(\partial B_1^m(0)\right).$$

In particular, if $R \in E_n^m$, then $e(R) = \frac{1}{n}\mathcal{H}^1\left(\partial B_1^m(0)\right)$.

Proof. Let $Q \in \mathcal{F}_{n'}$ be the polygon formed by the vertices $P \cup \{v_j : 1 \leq j \leq m\} =: \{y_k : 1 \leq$

$k \leq n'\}$. Then,

$$\sum_{j=1}^{n} \|x_{j+1} - x_j\|_m = \sum_{k=1}^{n'} \|y_{k+1} - y_k\|_m = \sum_{j=1}^{m} \|v_{j+1} - v_j\|_m = \mathcal{H}^1\left(\partial B_1^m(0)\right),$$

where the first equality follows by Lemma 1.5.13.

If $R \in E_n^m$ then Lemma 5.1.3 implies, via the first part of the present corollary, the claim. □

Corollary 5.2.2. Let $m \geq 4$ be an even integer which is not divisible by 4. Then

$$\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m) = \frac{2m\sin(\pi/n)}{n} \quad \text{for each } n \geq m.$$

Proof. This follows by Corollary 5.2.1 and [25, Lemma 3.6 (2)]. □

Polygonal m-norms where m is divisible by 4

We begin our analysis of regularity constants in the case when the number of edges in $\|\cdot\|_m$ is a multiple of four by proving the strict inequality of the norm regularity constants whenever the number of vertices n of the equilateral polygon is a multiple of m.

Theorem 5.2.3. Let $m \geq 4$ be divisible by 4. Then $\beta(n, \|\cdot\|_m) < \alpha(n, \|\cdot\|_m)$ for each $n \geq m$ which is a multiple of m.

Proof. We prove there exists a positive constant δ such that if $w_1, w_2 \in [v_1, v_2]$, $\|w_j - v_1\|_m < \delta$ $(j = 1, 2)$ and $w_1 \neq w_2$ then for equilateral n-gons P_{w_1},$P_{w_2} \in E_n^m$ such that $w_1 \in P_{w_1}$, $w_2 \in P_{w_2}$ one has $e(P_{w_1}) \neq e(P_{w_2})$. By Definition 4.4.3 this implies the statement.

Let $d := \|v_2 - v_1\|_m$, $c = \cos(2\pi/m)$ and

$$\delta := \frac{m}{n} \cdot \frac{d}{\left(\left(1 + \frac{m}{n}\right) + \left(1 - \frac{m}{n}\right) c\right)} = \frac{m}{n} \cdot \frac{d}{(1+c) + \frac{m}{n}(1-c)} = \frac{m}{n} \cdot \frac{d}{2 - (1-c)\left(1 - \frac{m}{n}\right)}. \tag{5.2.1}$$

Further, let $x \in [v_1, v_2]$ be such that $\|x - v_1\|_m \leq \delta$ and

$$e = e(x) := \frac{m}{n}\left(d - (1-c)\|x - v_1\|_m\right) \tag{5.2.2}$$

For each $k \in \left[\frac{n}{m}\right]$, define

$$x_k := v_1 + (\|x - v_1\|_m + (k-1)e) \cdot \frac{v_2 - v_1}{d}.$$

We claim that $x_k \in [v_1, v_2]$ for each $k \in \left[\frac{n}{m}\right]$; to see this it suffices to show $\|x_{n/m} - v_1\|_m \leq d$. Observe that

$$\begin{aligned}\|x_{n/m} - v_1\|_m &= \|x - v_1\|_m + \left(\frac{n}{m} - 1\right) \cdot \frac{m}{n} \cdot (d - (1-c)\|x - v_1\|_m) \\ &= d\left(1 - \frac{m}{n}\right) + \|x - v_1\|_m \left(c + \frac{m}{n}(1-c)\right). \end{aligned} \tag{5.2.3}$$

However, by (5.2.1), $\|x - v_1\|_m \leq \frac{m}{n} \cdot d/\left(c + \frac{m}{n}(1-c)\right)$, and thus by (5.2.3), $\|x_{n/m} - v_1\|_m \leq d$. Now,

$$\left\|x_{n/m} - v_2\right\|_m = \|v_1 - v_2\|_m - \|x_{n/m} - v_1\|_m = d - \|x - v_1\|_m - \left(\frac{n}{m} - 1\right)e. \tag{5.2.4}$$

Next for each $k \in [n/m]$ and each $j \in [m]$, let $x_{(j-1)\frac{n}{m}+k} = R_2^{(j-1)}(x_k)$. Define $P := (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|_m}$. Notice that $x_1 = x$ and further to see that $P_x \in E_n^m$ it suffices to verify that $\left\|x_{(n/m)+1} - x_{n/m}\right\|_m = \|x_2 - x_1\|_m = e$. First we show that $\|x_{(n/m)+1} - v_2\|_m \leq \|x_{n/m} - v_2\|_m$. Indeed, note that by (5.2.1), (5.2.2), (5.2.3) and (5.2.4),

$$\begin{aligned}\left\|x_{\frac{n}{m}} - v_2\right\|_m - \left\|x_{\frac{n}{m}+1} - v_2\right\|_m &= \left\|x_{\frac{n}{m}} - v_2\right\|_m - \|x - v_1\|_m \\ &= d - 2\|x - v_1\|_m - \left(1 - \frac{m}{n}\right)(d - (1-c)\|x - v_1\|_m) \\ &= \frac{m}{n}d - \|x - v_1\|_m \left((1+c) + \frac{m}{n}(1-c)\right) \\ &\geq \frac{m}{n}d - \delta\left((1+c) + \frac{m}{n}(1-c)\right) = 0.\end{aligned}$$

Therefore, by Lemma 1.5.15, (5.2.2), (5.2.3) and (5.2.4),

$$\begin{aligned}\|x_{(n/m)+1} - x_{n/m}\|_m &= \left\|x_{\frac{n}{m}} - v_2\right\|_m + c\left\|x_{\frac{n}{m}+1} - v_2\right\|_m \\ &= \|x_{(n/m)} - v_2\|_m + c\|x - v_1\|_m \\ &= d - (1-c)\|x - v_1\|_m - \left(\frac{n}{m} - 1\right)e \\ &= d - (1-c)\|x - v_1\|_m - \left(1 - \frac{m}{n}\right)(d - (1-c)\|x - v_1\|_m)\end{aligned}$$

$$= \frac{m}{n} d - \frac{m}{n} (1-c) \|x - v_1\|_m = e.$$

Thus, $P \in E_n^m$ and $l_n^{\|\cdot\|_m}(x) = e(P) = e(x)$. It is clear from (5.2.2) that $e(w_1) \neq e(w_2)$ for distinct $w_j \in [v_1, v_2]$ with $\|w_j - v_1\|_m < \delta$, $j = 1, 2$. □

Below we complete our analysis in the particular case of the rectilinear norms by providing a classification of when the regularity constants are equal. Later in Theorem 5.2.10 we extend this to determining when the norm regularity constants of any polygonal m-norm are equal for any $n \geq m$.

Theorem 5.2.4. Let $n \geq 4$. Then, $\alpha(n, \|\cdot\|_4) = \beta(n, \|\cdot\|_4)$ if and only if n is not a multiple of 4. Moreover, if n is not a multiple of 4, $\alpha(n, \|\cdot\|_4) = \beta(n, \|\cdot\|_4) = 2/(\lceil n/4 \rceil)$.

Proof. By Theorem 5.2.3 note if $n = 4k$ then $\beta(n, \|\cdot\|_4) < \alpha(n, \|\cdot\|_4)$. Suppose now n is not a multiple of 4. For this we show for each $x \in [v_1, v_2]$ the existence of an equilateral n-gon P such that $x \in P$ and $e(P) = 2/\lceil n/4 \rceil$. We claim that it suffices to provide the construction in the specific cases $n = 4k+3$ for some $k \in \mathbb{N}$. This follows since, for a given $x \in [v_1, v_2]$, we shall construct an equilateral $(4k+3)$-gon, $P_{4k+3} = (x_1, \ldots, x_{4k+3})$ such that $x_1 = x$, $x_{2k+2} = v_3$, $x_{3k+3} = v_4$ and such that $\|x_{2k+3} - x_{2k+1}\|_4 = \|x_{3k+4} - x_{3k+2}\|_4 = 2/(k+1)$. Therefore we may define $P_{4k+2} := P_{4k+3} \setminus \{v_3\}$ and $P_{4k+1} := P_{4k+2} \setminus \{v_4\}$, see Figure 5.5. Thus $P_{4k+1} \in E_{4k+1}^4$, $P_{4k+2} \in E_{4k+2}^4$, $x \in P_{4k+1} \cap P_{4k+2}$ and $e(P_{4k+1}) = e(P_{4k+2}) = 2/(k+1)$.

Hence suppose $n = 4k+3$ for some $k \in \mathbb{N}$ and $x \in [v_1, v_2]$. Suppose first that $\|x - v_1\|_4 \leq 2/(k+1)$. Now, for each $j \in [k+1]$, let $x_j = x + (j-1) \cdot \frac{2}{k+1} \cdot \frac{v_2 - v_1}{2}$. Observe that $x_{k+1} \in [v_1, v_2]$ and $\|x_{k+1} - v_2\|_4 \leq 2/(k+1)$. Next, for each $j \in \{k+2, \ldots, 2k+2\}$ let $x_j := v_2 + (j - (k+1)) \cdot \frac{2}{k+1} \cdot \frac{v_3 - v_2}{2}$. For each $j \in \{2k+3, \ldots, 3k+3\}$ let $x_j := v_3 + (j - (2k+2)) \frac{2}{k+1} \cdot \frac{v_4 - v_3}{2}$. Finally, for each $j \in \{3k+4, \ldots, 4k+3\}$ let $x_j := v_4 + (j - (3k+3)) \cdot \frac{2}{k+1} \cdot \frac{v_1 - v_4}{2}$. Now, to see that $P = (x_1, \ldots, x_{4k+3}) \in E_{4k+3}^4$ with $e(P) = 2/(k+1)$ it suffices to note $\|x_{k+2} - x_{k+1}\|_4 = \|x_{4k+3} - x_1\|_4 = 2/(k+1)$. This follows since both $\|x_1 - v_1\|_4, \|x_{k+1} - v_2\|_4 \leq 2/(k+1)$ and both $\|x_{k+2} - v_2\|_4 = \|x_{4k+3} - v_1\|_4 = 2/(k+1)$. Hence $P \in E_{4k+3}^4$ with $v_2, v_3 \in P$ and $e(P) = 2/(k+1)$.

If $\|x - v_1\| > 2/(k+1)$ consider the largest positive integer $J \in \mathbb{N}$ such that

$$x - \frac{2J}{k+1} \cdot \frac{v_2 - v_1}{\|v_2 - v_1\|_m} \in (v_1, v_2].$$

Let $y := x - J(v_2 - v_1)/(k+1) \in (v_1, v_2]$ and observe that $\|y - v_1\|_4 \leq 2/(k+1)$ as otherwise this contradicts the maximality of the index J. Using the previous part of the present theorem, construct an equilateral polygon $P \in E_{4k+3}$ such that $y \in P$ and $e(P) = 2/(k+1)$. Observe that as $x \in (v_1, v_2)$, $\|y - x\|_4 = 2J/(k+1)$ and $e(P) = 2/(k+1)$ then $x \in P$.

Therefore for each $x \in \partial B_1^4(0)$ there exists $P \in E_{4k+3}$ such that $x \in P$ and $e(P) = 2/(k+1)$. $\square$

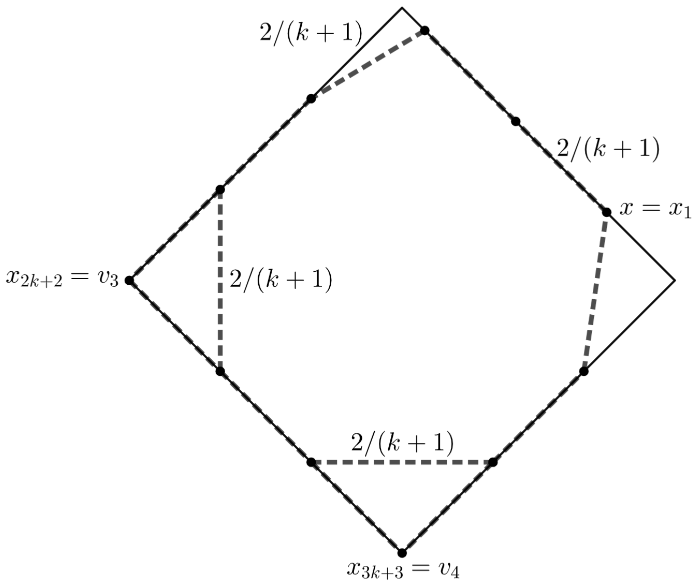

Figure 5.5: The construction of P_{4k+3} in the proof of Theorem 5.2.4 when $k = 2$. The polygons P_{4k+2} and P_{4k+1} are then given by $P_{4k+2} = P_{4k+3} \setminus \{v_4\}$ and $P_{4k+1} = P_{4k+2} \setminus \{v_3\}$.

Recall Notation 5.1.4. For a fixed $m \geq 8$ even and $n \geq m$, suppose $P_0 = (z_1, \ldots, z_n) \in E_n^m$ is such that $z_1 = v_1$. Now for each $\rho \in [0, \rho(n,m)]$, $j \in [m]$ and each $k \in L_j$, define

$$\tilde{x}_k(\rho) := z_k + \rho \prod_{l=1}^{j-1} c_l \frac{v_{j+1} - v_j}{\|v_{j+1} - v_j\|_m}. \tag{5.2.5}$$

Finally, let $Q_{n,m}(\rho) := (\tilde{x}_1(\rho), \ldots, \tilde{x}_n(\rho))$.

The following theorem provides a sufficient condition for $Q_{n,m}(\rho)$ to define an equilateral n-gon inscribed in the unit sphere of $\|\cdot\|_m$. Later, in Lemma 5.2.6, we show for each $\rho \in [0, \rho(n,m)]$ that $\tilde{x}_k(\rho) = x_k(\rho)$ for every $k \in [n]$ whenever the norm regularity constants are equal, see Notation 5.1.4.

Theorem 5.2.5. Let $m \geq 8$ be a multiple of 4 and $n \geq m$. There exists $\delta_0 > 0$ such that for each $\rho \in (0, \delta_0]$:

$$\text{(i) } Q_{n,m}(\rho) \in \mathcal{F}_n^m; \qquad \text{(ii) } \|\tilde{x}_{k+1}(\rho) - \tilde{x}_k(\rho)\|_m = e\left(P_0\right) \quad \text{for each } k \in [n-1].$$

Moreover, if $P_0(n,m)$ is (n,m)-vertex symmetric then δ_0 can be chosen such that $\rho(n,m) \le \delta_0$ and for each $\rho \in (0, \delta_0]$:

$$\text{(a) } Q_{n,m}(\rho) \in E_n^m; \qquad \text{(b) } e(Q_{n,m}(\rho)) = e(P_0); \qquad \text{(c) } Q_{n,m}(\rho) = P_\rho.$$

Proof. Recall Notation 5.1.4. Let $P_0 = (z_1, \ldots, z_n)$ and $d := \|v_2 - v_1\|_m$. For each $j \in [m]$, let

$$A_j := \{k \in L_j : k+1 \in L_{j+1} \text{ and } \|z_k - v_{j+1}\|_m > \|z_{k+1} - v_{j+1}\|_m\}.$$

Observe that $n \in A_m$ since $z_n \in L_m$ and $\|z_n - v_{m+1}\|_m = \|z_n - v_1\|_m > 0 = \|z_1 - v_1\|_m$. Moreover note for each $j \in [m]$ that either $A_j = \emptyset$ or there exists $k \in [n]$ such that $A_j = \{k\}$ and $z_k = Y_j$. Let

$$\delta_1 := \min_{\substack{j \in [m] \\ k \in L_j}} \frac{d - \|z_k - v_j\|_m}{\prod_{l=1}^{j-1} c_l} \quad \text{and} \quad \delta_2 := \min_{j \in [m]} \left\{ \frac{\|z_k - v_{j+1}\|_m - \|z_{k+1} - v_{j+1}\|_m}{\prod_{l=1}^{j-1} c_l + \prod_{l=1}^{j} c_l} : k \in A_j \right\}. \tag{5.2.6}$$

Observe that $\delta_1 > 0$ since $\|z_k - v_j\|_m < d$ for each $j \in [m]$ and $k \in L_j$. Further, $\delta_2 > 0$ by the definition of A_j. Finally, define the positive constant $\delta_0 := \min(\delta_1, \delta_2)$.

Fix $\rho \in (0, \delta_0]$. To show (i) we shall show $\|\tilde{x}_k(\rho)\|_m = 1$ for each $k \in [n]$. Indeed, fix $k \in [n]$ and let $j \in [m]$ be such that $k \in L_j$. As $\rho \le \delta_1$, note

$$\|\tilde{x}_k(\rho) - v_j\|_m = \|z_k - v_j\|_m + \rho \prod_{l=1}^{j-1} c_l \le d, \tag{5.2.7}$$

which, by (5.2.5), implies $\tilde{x}_k(\rho) \in [v_j, v_{j+1}]$. Hence $Q_{n,m}(\rho) \in \mathcal{F}_n^m$.

To see (ii) note we need to show for each $k \in [n-1]$ that

$$\|\tilde{x}_{k+1}(\rho) - \tilde{x}_k(\rho)\|_m = \|z_{k+1} - z_k\|_m. \tag{5.2.8}$$

Note if $k \in [n-1]$ is such that both $k, k+1 \in L_j$ for some $j \in [m]$ then (5.2.8) is satisfied by (5.2.5). Hence it suffices to consider $k \in [n-1]$ such that $k \in L_j$, but $k+1 \in L_{j+1}$. Fix $k \in [n-1]$ such that $k \in L_j$ and $k+1 \in L_{j+1}$ for some $j \in [m]$. We shall consider two cases:

$$\text{(A) } \|X_{j+1} - v_{j+1}\|_m \ge \|Y_j - v_{j+1}\|_m; \qquad \text{(B) } \|Y_j - v_{j+1}\|_m > \|X_{j+1} - v_{j+1}\|_m.$$

Note that $Y_j = z_k$ and $X_{j+1} = z_{k+1}$. Consider (A). In such a case, note

$$\|\tilde{x}_{k+1}(\rho) - v_{j+1}\|_m \geq \|z_{k+1} - v_{j+1}\|_m \geq \|z_k - v_{j+1}\|_m \geq \|\tilde{x}_k(\rho) - v_{j+1}\|_m.$$

Moreover, by (5.1.1), $c_j = \cos(2\pi/m)$. Thus, by Lemma 1.5.15,

$$\begin{aligned}\|\tilde{x}_{k+1}(\rho) - \tilde{x}_k(\rho)\|_m &= \|\tilde{x}_{k+1}(\rho) - v_{j+1}\|_m + \cos\left(\frac{2\pi}{m}\right)\|\tilde{x}_k(\rho) - v_{j+1}\|_m \\ &= \left(\|z_{k+1} - v_{j+1}\|_m + \rho\prod_{l=1}^{j} c_l\right) + \cos\left(\frac{2\pi}{m}\right)\left(\|z_k - v_{j+1}\|_m - \rho\prod_{l=1}^{j-1} c_l\right) \\ &= \|z_{k+1} - v_{j+1}\|_m + \cos\left(\frac{2\pi}{m}\right)\|z_k - v_{j+1}\|_m = \|z_{k+1} - z_k\|_m.\end{aligned}$$

Hence (5.2.8) is satisfied.

Consider (B), i.e. $A_j \neq \emptyset$. Note as $\rho \leq \delta_2$ that

$$\|\tilde{x}_k(\rho) - v_{j+1}\|_m = \|z_k - v_{j+1}\|_m - \rho\prod_{l=1}^{j-1} c_l \geq \|z_{k+1} - v_{j+1}\|_m + \rho\prod_{l=1}^{j} c_l = \|\tilde{x}_{k+1}(\rho) - v_{j+1}\|_m. \quad (5.2.9)$$

Moreover, by (5.1.1), $c_j = \sec(2\pi/m)$. Thus, by Lemma 1.5.15,

$$\begin{aligned}\|\tilde{x}_{k+1}(\rho) - \tilde{x}_k(\rho)\|_m &= \|\tilde{x}_k(\rho) - v_{j+1}\|_m + \cos\left(\frac{2\pi}{m}\right)\|\tilde{x}_{k+1}(\rho) - v_{j+1}\|_m \\ &= \left(\|z_k - v_{j+1}\|_m - \rho\prod_{l=1}^{j-1} c_l\right) + \cos\left(\frac{2\pi}{m}\right)\left(\|z_{k+1} - v_{j+1}\|_m + \rho\prod_{l=1}^{j} c_l\right) \\ &= \|z_k - v_{j+1}\|_m + \cos\left(\frac{2\pi}{m}\right)\|z_{k+1} - v_{j+1}\|_m = \|z_{k+1} - z_k\|_m.\end{aligned}$$

Hence, in either case, (5.2.8) is satisfied. Thus $\|\tilde{x}_{k+1}(\rho) - \tilde{x}_k(\rho)\|_m = e(P_0)$ for each $k \in [n-1]$ proving (ii).

Suppose now that P_0 is vertex symmetric. To prove (a), note by (i) and (ii), it suffices to verify $\|\tilde{x}_n(\rho) - \tilde{x}_1(\rho)\|_m = \|z_n - z_1\|_m$. In such a case, we first show that $\|\tilde{x}_n(\rho) - v_1\|_m \geq \|\tilde{x}_1(\rho) - v_1\|_m$. Indeed, by Lemma 5.1.25, the definition of δ_2 in (5.2.6) and since both $\rho \leq \delta_0 \leq \delta_2$ and $z_1 = v_1$,

$$\|\tilde{x}_n(\rho) - v_1\|_m - \|\tilde{x}_1(\rho) - v_1\|_m = \|z_n - v_1\|_m - \rho\left(\prod_{l=1}^{m-1} c_l + 1\right)$$

$$\geq \|z_n - v_1\|_m - \delta_2 \left(\prod_{l=1}^{m-1} c_l + \prod_{l=1}^{m} c_l \right) \geq 0. \tag{5.2.10}$$

Hence, by Lemma 1.5.15 and Lemma 5.1.25, it follows, using again $z_1 = v_1$,

$$\begin{aligned} \|\tilde{x}_n(\rho) - \tilde{x}_1(\rho)\|_m &= \|\tilde{x}_n(\rho) - v_1\|_m + \cos\left(\frac{2\pi}{m}\right) \|\tilde{x}_1(\rho) - v_1\|_m \\ &= (\|z_n - z_1\|_m - \rho c_1 \dots c_{m-1}) + \cos\left(\frac{2\pi}{m}\right) \rho \\ &= \|z_n - z_1\|_m. \end{aligned} \tag{5.2.11}$$

Thus (5.2.8) is satisfied in such a case and thus (a) is proven. Moreover by (ii) and since $\|\tilde{x}_n(\rho) - \tilde{x}_1(\rho)\|_m = e(P_0)$ this implies (b), also.

To see (c), note $Q_{n,m}(\rho) \in E_n^m$ and $\|\tilde{x}_1(\rho) - v_1\|_m = \rho$. Thus, by Lemma 5.1.1, $Q_{n,m}(\rho) = P_\rho$.

Finally to see $\rho(n,m) \leq \delta_0$, since $Q_{n,m}(\delta_0) = P_{\delta_0}$ by (iii), by Definition 5.1.6 it would be enough to show:

$$\text{There exists } I \in \mathcal{L}_m \text{ such that } I\left(Q_{n,m}(\delta_0)\right) \cap P_0 \neq \emptyset. \tag{5.2.12}$$

Suppose first that $\delta_1 \leq \delta_2$, i.e. $\delta_0 = \delta_1$. Let $j \in [m]$ and $k \in L_j$ be such that the minimum for δ_1 in (5.2.6) is attained. Then observe that in (5.2.7) equality is attained, i.e. $\|\tilde{x}_k(\delta_0) - v_j\|_m = d$. Hence $\tilde{x}_k(\delta_0) = v_{j+1}$. Now observe that, as $v_{j+1} = R_{j+1}(v_1)$ if $I_1 := R_{j+1}^{-1}$, then $I_1(\tilde{x}_k(\delta_0)) = v_1$ and thus $I_1(Q_{n,m}(\delta_0)) \cap P_0 \neq \emptyset$.

Suppose now that $\delta_1 > \delta_2$, i.e. $\delta_0 = \delta_2$. Let $j \in [m]$ and $k \in A_j$ be such that the minimum for δ_2 in (5.2.6) is attained. Then observe that in (5.2.9) equality is attained, i.e. $\|\tilde{x}_k(\delta_0) - v_{j+1}\|_m = \|\tilde{x}_{k+1}(\delta_0) - v_{j+1}\|_m$. Since P_0 is vertex symmetric, and since $\|\tilde{x}_{k+1}(\delta_0) - \tilde{x}_k(\delta_0)\|_m = e(P_0)$ this implies, by Lemma 5.1.26, there exists $I_2 \in \mathcal{L}_m$ such that $I_2(Q_{n,m}(\delta_0)) \cap P_0 \neq \emptyset$. □

We now show that the equilateral polygons P_ρ and $Q_{n,m}(\rho)$ are in fact equivalent whenever ρ is sufficiently small and the norm regularity constants $\alpha(n, \|\cdot\|_n)$ and $\beta(n, \|\cdot\|_m)$ are equal, even if P_0 is not vertex symmetric.

Lemma 5.2.6. Let $m \geq 8$ be a multiple of 4 and $n \geq m$ be such that $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$.

Then there exists $\rho_0 > 0$ such that for any $\rho \in [0, \rho_0]$,

$$\tilde{x}_k(\rho) = x_k(\rho) \quad \text{for each } k \in [n],$$

where x_k and $\tilde{x_k}$ are defined in Notation 5.1.4 and (5.2.5), respectively.

Proof. Since $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$ note that $e(P) = e(P_0)$ for all $P \in E_n^m$. Let $\rho_1 := \min(\delta_0, \|v_2 - v_1\|_m)$ where δ_0 is given by Theorem 5.2.5. Consider $\rho_0 \in (0, \rho_1)$ such that $\|\tilde{x}_n(\rho_0) - v_1\|_m \geq \|\tilde{x}_1(\rho_0) - v_1\|_m$; note such ρ_0 exists since, by (5.2.5),

$$\lim_{t \to 0^+} \|\tilde{x}_n(t) - v_1\|_m = e(P_0) > 0 = \lim_{t \to 0^+} \|\tilde{x}_1(t) - v_1\|_m.$$

Fix $\rho \in [0, \rho_0]$ and consider $P_\rho = (x_1(\rho), \ldots, x_n(\rho)) \in E_n^m$ as in Notation 5.1.4. Observe, by definition, that $\tilde{x}_1(\rho) = x_1(\rho)$. To see that $x_k(\rho) = \tilde{x}_k(\rho)$ for every $k \in [n]$, we shall argue by induction. Suppose $\tilde{x}_k(\rho) = x_k(\rho)$ for some $k \in [n-1]$. By Lemma 5.1.1 there exists a unique $w \in (x_k(\rho), -x_k(\rho))_m$ such that $\|x_k(\rho) - w\|_m = e(P_0)$. However $\tilde{x}_{k+1}(\rho) \in (x_k(\rho), -x_k(\rho))_m$ too, since $\rho \leq \rho_0 < \|v_2 - v_1\|_m$, and by Theorem 5.2.5,

$$\|x_k(\rho) - \tilde{x}_{k+1}(\rho)\|_m = \|\tilde{x}_k(\rho) - \tilde{x}_{k+1}(\rho)\|_m = e(P_0) = \|x_k(\rho) - x_{k+1}(\rho)\|_m.$$

Thus, $x_{k+1}(\rho) = \tilde{x}_{k+1}(\rho)$. □

We now can show that the norm regularity constants for polygonal m-norms where m is a multiple of four are equal if and only if the initial equilateral polygon P_0 is vertex symmetric.

Corollary 5.2.7. Let $m \geq 8$ be divisible by 4 and $n \geq m$. Then:

$$\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m) \text{ if and only if } P_0(n, m) \text{ is } (n, m)\text{-vertex symmetric.}$$

Proof. Suppose first that P_0 is vertex symmetric. To see $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$, we require to show that

$$e(P) = e(P_0) \text{ for each } P \in E_n^m. \tag{5.2.13}$$

Let $P \in E_n^m$ be any equilateral n-gon. By Lemma 5.1.1 and Lemma 5.1.15 there exist $\rho \in [0, \rho(n, m))$ and $I \in \mathcal{L}_m$ such that $P \sim I(P_\rho)$. We therefore conclude by Lemma 4.3.6 that

$e(P) = e\left(I(P_\rho)\right) = e\left(P_\rho\right) = e(P_0)$, where for the last equality we used Theorem 5.2.5 (b),(c).

Suppose now that $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$. Then note for each $P \in E_n^m$ it holds that $e(P) = e(P_0)$. By Lemma 5.2.6 there exists $\rho_0 > 0$ such that $x_k(\rho_0) = \tilde{x}_k(\rho)$ for all $k \in [n]$. Let $\rho_1 \in (0, \rho_0)$ be such that $\|\tilde{x}_n(\rho_1) - v_1\|_m \geq \|\tilde{x}_1(\rho_1) - v_1\|_m$; note such ρ_1 exists since, by (5.2.5),

$$\lim_{t \to 0^+} \|\tilde{x}_n(t) - v_1\|_m = e(P_0) > 0 = \lim_{t \to 0^+} \|\tilde{x}_1(t) - v_1\|_m.$$

As $e(P) = e(P_0)$ note

$$\|\tilde{x}_n(\rho_0) - \tilde{x}_1(\rho_0)\|_m = \|x_n(\rho_0) - x_1(\rho_0)\|_m = e(P) = e(P_0).$$

However as $\|\tilde{x}_n(\rho_0) - v_1\|_m \geq \|\tilde{x}_1(\rho_0) - v_1\|_m$, by Lemma 1.5.15 and recalling both Notation 5.1.4 and (5.2.5), it follows that

$$\|\tilde{x}_n(\rho_0) - \tilde{x}_1(\rho_0)\|_m = \|z_n - z_1\|_m + \rho_0 \left(\cos\left(\frac{2\pi}{m}\right) - c_1 \dots c_{m-1} \right).$$

Hence $\|\tilde{x}_n(\rho_0) - \tilde{x}_1(\rho_0)\|_m = e(P_0) = \|z_n - z_1\|_m$ if and only if $c_1 \dots c_{m-1} = \cos(2\pi/m)$. This, by Lemma 5.1.25, in turn implies P_0 is vertex symmetric. □

Before we can conclude for which pairs (n, m) the initial equilateral polygon $P_0 \in E_n^m$ is vertex symmetric, we introduce the following notation and lemma.

Notation 5.2.8. For each $j \in \mathbb{N}$, let $\nu_2(j) \in \mathbb{N} \cup \{0\}$ denote the power of the prime 2 in the prime factorisation of j, i.e. the largest positive integer such that $j/2^{\nu_2}(j) \in \mathbb{N}$. We will also denote s_n, s_m to be the integers such that $n = 2^{\nu(n)}(1 + 2s_n)$ and $m = 2^{\nu(m)}(1 + 2s_m)$.

Lemma 5.2.9. Let $m \geq 8$ and $n \geq m$ be such that $M := \min(\nu_2(n), \nu_2(m)) \geq 1$. Then $v_{1+(m/2^k)} \in P_0$ for each $k \in [M]$.

Proof. Let $P_0 = (x_1, \dots, x_n) \in E_n^m$ be such that $x_1 = v_1$. As $\nu_2(n) \geq 1$, hence n is even, Proposition 5.1.17 implies that

$$x_{1+(n/2)} = -x_1 = -v_1 = v_{1+(m/2)}.$$

Hence, $v_{1+(m/2)} \in P_0$. Suppose, for some $k \in [M-1]$, that $v_{1+(m/2^k)} \in P_0$. Then, as $v_1 \in P_0$, note $P_0 \sim S_{1+(m/2^k)}(P_0)$. Hence either $v_{1+(m/2^{k+1})} \in P_0$ or P_0 is $\big(1+(m/2^{k+1}), n, m\big)$-vertex symmetric. Suppose, for a contradiction, the latter holds. Let $I_1 = \big(v_1, v_{1+(m/2^{k+1})}\big)_m$ and $I_2 = \big(v_{1+(m/2^{k+1})}, v_{1+(m/2^k)}\big)_m$. Then as P_0 is $\big(1+(m/2^{k+1}), n, m\big)$-vertex symmetric, observe that $v_{1+(m/2^{k+1})} \notin P_0$. Using $P_0 \sim S_{1+(m/2^k)}(P_0)$, we conclude that $\mathrm{card}(P_0 \cap I_1) = \mathrm{card}(P_0 \cap I_2)$. Therefore,

$$\mathrm{card}\left(\left[v_1, v_{1+\frac{m}{2^k}}\right)_m \cap P_0\right) = 1 + \mathrm{card}\left(\left(I_1 \cup I_2 \cup \{v_{1+m/2^{k+1}}\}\right) \cap P_0\right) = 1 + 2\mathrm{card}\left(I_1 \cap P_0\right).$$

Moreover, as $P_0 \sim R_{1+(m/2^k)}(P_0)$, this implies

$$2^{\nu_2(n)}(1+2s_n) = n = \mathrm{card}(P_0) = 2^k \mathrm{card}\left(\left[v_1, v_{(m/2^k)+1}\right)\right) = 2^k\left(1 + 2\mathrm{card}\left(I_1 \cap P_0\right)\right). \quad (5.2.14)$$

As $k < \nu_2(n)$, (5.2.14) yields a contradiction. Therefore $v_{1+(m/2^{k+1})} \in P_0$. □

We are now ready to prove the statement which extends Theorem 5.2.4.

Theorem 5.2.10. Let $m \geq 4$ be even and $n \geq m$. Then $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$ if and only if exactly one of the following is satisfied:

(a) $\min\left(\nu_2(m), \nu_2(n)\right) \leq 1$; (b) $\nu_2(m) > \nu_2(n) \geq 2$.

Proof. The case when $m = 4$ follows by Theorem 5.2.4. Suppose now that $m \geq 6$ and $m \equiv 2 \pmod 4$. Then by Corollary 5.2.2 we obtain that $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$ for each $n \geq m$.

For the remainder of the proof suppose $m \geq 8$ and $m \equiv 0 \pmod 4$, i.e. $\nu_2(m) \geq 2$. We shall show that each of the conditions (a) and (b) yields $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$. Then we show that whenever $\nu_2(n) \geq \nu_2(m) \geq 2$ that $\alpha(n, \|\cdot\|_m) \neq \beta(n, \|\cdot\|_m)$. Let $P_0 = (z_1, \ldots, z_n)$ where $z_1 = v_1$. We show that if $\nu_2(n) \leq 1$ or if $2 \leq \nu_2(n) < \nu_2(m)$, then P_0 is vertex symmetric. By Corollary 5.2.7 this would imply that $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$. Indeed if $\nu_2(n) = 0$, i.e. n is odd, note by Corollary 5.1.20 that P_0 is vertex symmetric. Similarly, if $\nu_2(n) = 1$, i.e. $n \equiv 2 \pmod 4$, then by Corollary 5.1.20 we conclude that P_0 is vertex symmetric.

For ease of notation, let $k_n = \nu_2(n)$ and $k_m = \nu_2(m)$. Assume now that $2 \leq k_n < k_m$ are such that $n = 2^{k_n}(2s_n+1)$ and $m = 2^{k_m}(2s_m+1)$. Observe by Lemma 5.2.9 that $v_{1+(m/2^{k_n})} \in P_0$ and $1 + (m/2^{k_n})$ is odd. Hence $P_0 \sim S_{1+(m/2^{k_n})}(P_0)$ and $P_0 \sim R_{1+(m/2^{k_n})}(P_0)$ by Lemma 5.1.1. Moreover, $\mathrm{card}(P_0 \cap [v_1, v_{1+(m/2^{k_n})})_m) = 1 + 2s_n$.

Since $P_0 \sim S_{1+(m/2^{k_n})}(P_0)$ note either P_0 is $(1+(m/2^{k_n+1}), n, m)$-vertex symmetric or that $v_{1+(m/2^{k_n+1})} \in P_0$. We claim the former holds. Indeed, for a contradiction, suppose that $v_{1+(m/2^{k_n+1})} \in P_0$ and let $I_1 = \left(v_1, v_{1+(m/2^{k_n+1})}\right)_m$ and $I_2 = \left(v_{1+(m/2^{k_n+1})}, v_{1+(m/2^{k_n})}\right)_m$. As $P_0 \sim S_{1+(m/2^{k_n})}(P_0)$, note that $\text{card}(P_0 \cap I_1) = \text{card}(P_0 \cap I_2)$. Hence,

$$\begin{aligned}\text{card}\left(P_0 \cap \left[v_1, v_{1+(m/2^{k_n})}\right)_m\right) &= \text{card}\left(P_0 \cap \left(I_1 \cup I_2 \cup \{v_1\} \cup \{v_{1+(m/2^{k_n+1})}\}\right)\right) \\ &= 2\left(1 + \text{card}\left(P_0 \cap I_1\right)\right),\end{aligned}$$

which contradicts the fact that $\text{card}\left(P_0 \cap \left[v_1, v_{1+m/2^{k_n}}\right)_m\right) = 1+2s_n$ is odd. Thus $v_{1+(m/2^{k_n+1})} \notin P_0$ and so P_0 is $(1+(m/2^{k_n+1}), n, m)$-vertex symmetric.

Finally, let us consider the case when $\nu_2(n) \geq \nu_2(m) \geq 2$. To conclude that $\alpha(n, \|\cdot\|_m) > \beta(n, \|\cdot\|_m)$ by Proposition 5.1.22 and Corollary 5.2.7 it suffices to show there exists $j_0 \in [m]$ such that j_0 is even and $v_{j_0} \in P_0$. Note by Lemma 5.2.9 that $1 + m/2^{k_m} = 2s_m + 2$, so $v_{2s_m+2} \in P_0$. Thus P_0 is not vertex symmetric, and hence $\alpha(n, \|\cdot\|_m) > \beta(n, \|\cdot\|_m)$. □

We have now shown that the polygonal norms $\|\cdot\|_m$ where m is not a multiple of four behave similarly to that of the Euclidean norms, in the sense that $\alpha(n, \|\cdot\|_m) = \beta(n, \|\cdot\|_m)$ when $n \geq m$; the same is not true when m is a multiple of four. A similar discrepancy in the behaviour of polygonal norms can be observed when considering the optimal ratio of constants for planar Lipschitz quotient mappings in polygonal norms, as in Chapter 3.

Further investigation is required into the cases when $n < m$. It is a simple exercise to determine that $\alpha(3, \|\cdot\|_6) = 2$ but $\beta(3, \|\cdot\|_6) \leq 3/2$, see Figure 5.6, and thus it is not true in general that the regularity constants are always equal for polygonal norms whose number of edges is not a multiple of four.

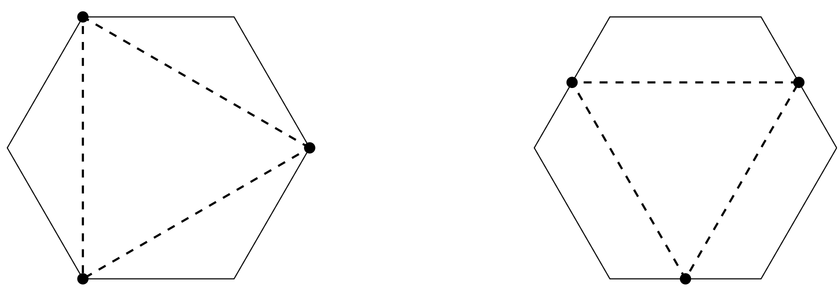

Figure 5.6: Two $\|\cdot\|_6$-equilateral triangles, with distinct edge lengths.

In the next section we return to the study of uniformly continuous, co-Lipschitz mappings. We obtain an upper bound for the ratio of constants for such mappings from $(\mathbb{C}, \|\cdot\|)$ to $\mathbb{R}$, in terms of the upper $\|\cdot\|$-regularity constants. We investigate when such a bound is optimal.

CHAPTER 6

UNIFORMLY CONTINUOUS, CO-LIPSCHITZ MAPPINGS FROM THE PLANE TO THE LINE

This chapter extends and generalises [23] to the case of an arbitrary norm $\|\cdot\|$ on $\mathbb{C}$. Namely, we obtain an upper bound for the ratio of constants of uniformly, continuous co-Lipschitz mappings from $(\mathbb{C}, \|\cdot\|)$ to $\mathbb{R}$, in terms of constants $\alpha(n, \|\cdot\|)$ which depends solely on the norm and n, the maximum number of components over all fibres. Further, we prove that, provided the norm $\|\cdot\|$ satisfies some 'separation property', then this upper bound is in fact sharp. Historically, this study of the uniformly continuous co-Lipschitz mappings $\mathbb{C} \to \mathbb{R}$ motivated our investigation of equilateral polygons inscribed in $\partial B_1^{\|\cdot\|}(0)$ presented in Chapters 4 and 5.

6.1 Introduction

We first introduce the definition of co-uniformly and uniformly continuous mappings as well as non-linear quotient mappings as in [1].

Definition 6.1.1. Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces. A mapping $f : X \to Y$ is said to be uniformly continuous if there exists a continuous, subadditive, monotone function $\Omega_f : [0, +\infty) \to [0, +\infty)$ such that $\Omega_f(r) \to 0$ as $r \to 0$ and

$$f\left(B_r^X(x)\right) \subseteq B^Y_{\Omega_f(r)}\left(f(x)\right) \quad \text{for each } x \in X \text{ and all } r > 0.$$

Similarly, a mapping $f : X \to Y$ is co-uniformly continuous if there exists a continuous, monotone function $\omega_f : (0, +\infty) \to (0, +\infty)$ such that $\omega_f(r) > 0$ for all $r > 0$ and

$$B^Y_{\omega_f(r)}(f(x)) \subseteq f\left(B^X_r(x)\right) \quad \text{for each } x \in X \text{ and all } r > 0.$$

A surjective mapping $f : X \to Y$ is said to be a uniform quotient mapping if it is both uniformly and co-uniformly continuous.

Remark 6.1.2. The above definition of uniformly continuous mappings is equivalent to the standard definition when considering normed spaces; see [2, pg. 11].

As one may observe, Lipschitz mappings may be considered as those uniformly continuous mappings f such that $\Omega_f(r) \leq Lr$ for some $L > 0$ and all $r > 0$. In an analogous manner, one may define co-Lipschitz mappings as a subclass of continuous co-uniformly continuous mappings.

Moreover, any uniformly continuous mapping between two normed spaces $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is 'Lipschitz for large distances' in the sense that for each $\varepsilon > 0$, there exists $r > 0$ such that

$$\|f(x) - f(y)\|_Y \leq \frac{\Omega_f(r)}{r}(1 + \varepsilon)\|x - y\|_X \quad \text{if } \|x - y\|_X \geq r/\varepsilon;$$

cf. [2, Proposition 1.11]. Further, for each $r_0 > 0$, the set $\{\Omega_f(r)/r : r \geq r_0\}$ is bounded, since Ω_f is subadditive. Moreover, since Ω_f is continuous and sub-additive, we may define the *weak Lipschitz constant* of a uniformly continuous mapping f to be

$$L^*_f := \lim_{r \to \infty} \frac{\Omega_f(r)}{r}.$$

Remark 6.1.3. Note the limit L^*_f exists due to Fekete's lemma, see for example [14, Theorem 6.6.1], since Ω_f is continuous and sub-additive. However, for our purpose, one may define $L^*_f = \limsup_{r \to +\infty} \Omega_f(r)/r$, which exists due to the boundedness of the set $\{\Omega_f(r)/r : r \geq r_0\}$ for all $r_0 > 0$. The proofs only require the existence of the limit superior.

This section concerns uniformly continuous, co-Lipschitz mappings from $\mathbb{C}$ to $\mathbb{R}$ and investigates the bound for the ratio of constants of L^*_f and c, similar to that considered in [23].

The work in [23] is motivated by the comprehensive results of [30] which provides answers to two questions posed in [17] concerning the structure of level sets of uniform quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}$. In particular, [30, Theorem 4.11, Theorem 5.1] states for a given uniformly continuous co-Lipschitz mapping $f : \mathbb{C} \to \mathbb{R}$ the following holds:

(i) for every $t \in \mathbb{R}$ every connected component K of $f^{-1}(t)$ has a representation of the following form:

$$K = K_0 \cup \bigcup_{j=1}^{\#e\left(f^{-1}(t)\right)} K_j;$$

where K_0 is a compact tree with $\#e\left(f^{-1}(t)\right)$ ends, each K_j is a closed, unbounded set that is homeomorphic to $[0, +\infty)$, the collection $\{K_j\}_{j=1}^{\#e\left(f^{-1}(t)\right)}$ is pairwise disjoint and each K_j intersects K_0 in exactly one point which is an end point of K_0 and is the end point of K_j;

(ii) there exists $n \in \mathbb{N}$ such that for each $t \in \mathbb{R}$,

$$\#e\left(f^{-1}(t)\right) = 2n;$$

that is, for each $t \in \mathbb{R}$ there exists $R_0 > 0$ such that for every $R > R_0$, $f^{-1}(t) \setminus B(0, R)$ has exactly $2n$ unbounded components;

(iii) the maximum number of connected components of $f^{-1}(t)$, over all $t \in \mathbb{R}$, is equal to n.

Using these comprehensive results Maleva, in [23, Theorem 1], provides a scale of values $0 < \cdots < \rho_{2,1}^{(m)} < \cdots < \rho_{2,1}^{(1)} < 1$ such that if $f : \mathbb{C} \to \mathbb{R}$ is a uniformly continuous, co-Lipschitz mapping, where the domain is equipped with the Euclidean norm, and the ratio $c/L_f^* > \rho_{2,1}^{(m)}$ then $\frac{1}{2}\#e\left(f^{-1}(t)\right) = n(f) \le m$.

This scale is, in fact, given by $\rho_{2,1}^{(m)} = \sin(\pi/(2m+2))$ and Maleva provides examples of Lipschitz quotient mappings $\xi_m : \mathbb{C} \to \mathbb{R}$ such that

$$c_{\xi_m}/L_{\xi_m}^* = \rho_{2,1}^{(m-1)} \quad \text{and} \quad n(\xi_m) = m.$$

The aim of this chapter is to determine such a scale for uniform quotient mappings $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ where $\|\cdot\|$ is any norm on the plane, and investigate the sharpness of such a scale. In doing so, we provide a class of planar norms $\|\cdot\|$ where one can construct an explicit Lipschitz quotient mapping $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ such that this scale is sharp. We show that this is intimately tied to the existence of inscribed equilateral polygons.

6.2 Preliminaries

This short preliminary section recalls and extends some foundational results in the context of convex, planar geometry. Such results will be implemented in the construction of the family of Lipschitz quotient mappings in Section 6.4.

Notation 6.2.1. Let $(X, \|\cdot\|)$ be a normed space, $A, B \subseteq X$, $x, w \in X$ and $a \in A$. We define:

(i) $\mathrm{dist}_{\|\cdot\|}(x, A) = \inf\{\|x - y\| : y \in A\}$;

(ii) $N_{\|\cdot\|}(a, A) = \{z \in X : \|a - z\| = \mathrm{dist}(z, A)\}$;

(iii) $D_{\|\cdot\|}(A, B) = \big\{y \in X : \mathrm{dist}_{\|\cdot\|}(y, A) = \mathrm{dist}_{\|\cdot\|}(y, B)\big\}$;

(iv) $[w, x\rangle = \{z \in X : z = w + r(x - w) \text{ for some } r \geq 0\} = w + \mathbb{R}^+(x - w)$, provided $w \neq x$.

If the context is clear, we may drop the sub- or superscript $\|\cdot\|$, i.e. write $\mathrm{dist}(x, A)$, $N(a, A)$ or $D(A, B)$ instead.

Lemma 6.2.2. Let $(X, \|\cdot\|)$ be a normed space and $A, B \subseteq X$ be non-empty. Then $D_{\|\cdot\|}(A, B)$ is a non-empty, closed subset of X.

Proof. Let $x \in A$ and $y \in B$. If $\mathrm{dist}(x, B) = 0$ or $\mathrm{dist}(y, A) = 0$, then $D(A, B) \neq \emptyset$. Suppose that $\mathrm{dist}(x, B), \mathrm{dist}(y, A) > 0$. Define $f : X \to \mathbb{R}$ by $f(z) = \mathrm{dist}(z, A) - \mathrm{dist}(z, B)$. Then f is Lipschitz continuous and

$$f(x) = -\mathrm{dist}(x, B) < 0 < \mathrm{dist}(y, A) = f(y).$$

Hence, restricting f to $[x, y]$ implies the existence of $z \in [x, y]$ such that $f(z) = 0$, that is $z \in D(A, B)$. To finally conclude that $D(A, B)$ is closed, one need only observe that $D(A, B) = f^{-1}(\{0\})$. □

We now recall a foundational result of Klee, which concerns points where the distance to a closed, convex set is attained. The proof presented in [18] only considers the Euclidean norm. However, it may be trivially extended to a general norm defined on $\mathbb{R}^n$.

Lemma 6.2.3. ([18, p. 248]) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, $L \subseteq \mathbb{R}^n$, $p \in \mathbb{R}^n$ and $q \in L$ be such that $\|p-q\| = \inf\{\|p-x\| : x \in L\} = \mathrm{dist}_{\|\cdot\|}(p, L)$. Then,

$$\|z-q\| = \mathrm{dist}_{\|\cdot\|}(z, L) \quad \text{for each } z \in [p, q]. \tag{6.2.1}$$

If L is convex, then (6.2.1) is satisfied for all $z \in [q, p\rangle$.

Using this result of Klee, we show the attainment result of Lemma 6.2.5. First, we introduce the following notation.

Notation 6.2.4. Let L_1, L_2 be two distinct half rays in $\mathbb{C}$ which have their end-point as the origin. Define R_{L_1,L_2} to be the open region in $\mathbb{C}$ enclosed by the half rays L_1 and L_2, starting at L_1 and traversing in the anticlockwise direction.

Lemma 6.2.5. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $L = [0, t\rangle$ for some $t \in \mathbb{C} \setminus \{0\}$. Suppose $L_1 = [0, t_1\rangle$ and $L_2 = [0, t_2\rangle$ for some distinct $t_1, t_2 \in \mathbb{C} \setminus \{0\}$ such that $L \setminus \{0\} \subseteq R_{L_1,L_2}$. Then for each $z \in L \setminus \{0\}$ there exists $\rho_0(z) > 0$ such that for every $r \in (0, \rho_0(z)]$ there exist $\gamma_1 \in R_{L_1,L} \cap B_r^{\|\cdot\|}(z)$ and $\gamma_2 \in R_{L,L_2} \cap B_r^{\|\cdot\|}(z)$ such that

$$\mathrm{dist}_{\|\cdot\|}(\gamma_j, L) = \|\gamma_j - z\| \quad \text{for } j = 1, 2.$$

Proof. Fix $z \in L \setminus \{0\}$. Let $\rho_0 = \rho_0(z) > 0$ be such that $B_{2\rho_0}^{\|\cdot\|}(z) \subseteq R_{L_1,L_2}$. Fix $r \in (0, \rho_0]$, $w_1 \in R_{L_1,L} \cap B_r^{\|\cdot\|}(z)$ and $w_2 \in R_{L,L_2} \cap B_r^{\|\cdot\|}(z)$. Let $v_1, v_2 \in L$ be such that

$$\mathrm{dist}_{\|\cdot\|}(w_j, L) = \|w_j - v_j\| \quad \text{for } j = 1, 2.$$

Define $\gamma_j := w_j + z - v_j$ for $j = 1, 2$. Note $\gamma_j \in B_r^{\|\cdot\|}(z)$ since $z, v_j \in L$ and so

$$\|\gamma_j - z\| = \|w_j - v_j\| \leq \|w_j - z\| < r. \tag{6.2.2}$$

Further, observe that $\gamma_j \in B_r^{\|\cdot\|}(z)$ for each $j = 1, 2$ and γ_j is a translation of w_j in a direction parallel to L. Therefore, $\gamma_1 \in R_{L_1,L}$ and $\gamma_2 \in R_{L,L_2}$.

For each $j = 1, 2$, let $s_j \in L$ be such that $\mathrm{dist}(\gamma_j, L) = \|\gamma_j - s_j\|$. Let $a_j := s_j + w_j - \gamma_j$.

Note, as $z \in L$, by (6.2.2)

$$\|a_j - z\| \le \|w_j - a_j\| + \|w_j - z\| = \|\gamma_j - s_j\| + \|w_j - z\| = \text{dist}(\gamma_j, L) + \|w_j - z\|$$
$$\le \|\gamma_j - z\| + \|w_j - z\| < 2r.$$

Therefore, as $a_j = s_j - z + v_j$, we conclude that $a_j \in L$. Hence,

$$\text{dist}_{\|\cdot\|}(\gamma_j, L) = \|\gamma_j - s_j\| = \|w_j - a_j\| \ge \text{dist}_{\|\cdot\|}(w_j, L) = \|w_j - v_j\| = \|\gamma_j - z\|.$$

So, as $z \in L$, we conclude that $\text{dist}(\gamma_j, L) = \|z - \gamma_j\|$. □

Recall Notation 6.2.1.

Lemma 6.2.6. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and L_1, L_2 be two rays such that $L_1 \cap L_2 = \{0\}$ and $z \in D_{\|\cdot\|}(L_1, L_2) \setminus \{0\}$. Then,

$$\text{dist}(y, L_j) = \frac{\|y\|}{\|z\|}\text{dist}(z, L_j) \quad \text{for each } y \in [0, z\rangle \text{ and } j = 1, 2.$$

Moreover, if $\gamma_j \in L_j$ is such that $\text{dist}(z, L_j) = \|z - \gamma_j\|$, then $\text{dist}(y, L_j) = \|y - (\|y\|/\|z\|)\gamma_j\|$.

Proof. The result is trivial if $y = 0$. Therefore assume $y \in [0, z\rangle \setminus \{0\}$. For $j = 1, 2$, let $w_j \in L_j$ be such that $\text{dist}(z, L_j) = \|z - w_j\|$. Define $w_j' := \|y\|w_j/\|z\|$. Note as $y, z \in [0, z\rangle \setminus \{0\} \subseteq \mathbb{C}$ that $y = \|y\|z/\|z\|$ and so

$$\|y - w_j'\| = \left\| \frac{\|y\|}{\|z\|}z - \frac{\|y\|}{\|z\|}w_j \right\| = \frac{\|y\|}{\|z\|}\|z - w_j\| = \frac{\|y\|}{\|z\|}\text{dist}(z, L_j).$$

Since $w_j' \in L_j$, note $\text{dist}(y, L_j) \le (\|y\|/\|z\|)\text{dist}(z, L_j)$.

Next, let $\gamma_j' \in L_j$ be such that $\text{dist}(y, L_j) = \|y - \gamma_j'\|$. Let $\gamma_j = \|z\|\gamma_j/\|y\|$ and note $\gamma_j \in L_j$. Moreover,

$$\|z - \gamma_j\| = \left\| \frac{\|z\|}{\|y\|}y - \frac{\|z\|}{\|y\|}\gamma_j \right\| = \frac{\|z\|}{\|y\|}\text{dist}\,(y, L_j)\,.$$

Thus $\text{dist}(z, L_j) \le (\|z\|/\|y\|)\text{dist}(y, L_j)$. □

Corollary 6.2.7. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and L_1, L_2 be two rays such that $L_1 \cap L_2 = \{0\}$. Then $D_{\|\cdot\|}(L_1, L_2) \cap R_{L_1,L_2}$ is a non-trivial cone with vertex at 0.

Remark 6.2.8. Note when $\|\cdot\|$ is a strictly convex norm, then $D_{L_1,L_2} \cap R_{L_1,L_2}$ is exactly one ray which passes through the origin. It is unknown if this extends to the cases when $\|\cdot\|$ is not strictly convex.

6.3 Generalised upper bound for the ratio of constants

We begin by recalling the notation used in [23] which we shall adopt throughout this section.

Notation 6.3.1. Suppose $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ is a uniformly continuous, co-Lipschitz mapping and $t \in \mathbb{R}$. Then:

i) Let $\mathcal{D}_t = \left\{d > 0 : f^{-1}(t) \setminus B_d^{\|\cdot\|}(0) \text{ has exactly } \#e\left(f^{-1}(t)\right) = 2n \text{ ends}\right\}$ and $\Theta\left(f^{-1}(t)\right) := \inf\left(\mathcal{D}_t\right) + 1$.

ii) Denote the unbounded components of $f^{-1}(t) \setminus B^{\|\cdot\|}_{\Theta(f^{-1}(t))}(0)$ by $C_1(f^{-1}(t)), \ldots, C_{2n}\left(f^{-1}(t)\right)$ so that $\arg(z) < \arg(w)$ for any $z \in C_j\left(f^{-1}(t)\right) \cap \overline{B}^{\|\cdot\|}_{\Theta(f^{-1}(t))}(0)$ and $w \in C_{j+1}\left(f^{-1}(t)\right) \cap \overline{B}^{\|\cdot\|}_{\Theta(f^{-1}(t))}(0)$ for each $j \in [2n-1]$.

Remark 6.3.2. For Notation 6.3.1 ii), observe that $C_j\left(f^{-1}(t)\right) \cap \overline{B}^{\|\cdot\|}_{\Omega(f^{-1}(t))}(0)$ need not be a singleton, as in the Euclidean case. However, by [30, Theorem 4.11, Theorem 5.1], it is guaranteed that $C_j\left(f^{-1}(t)\right) \cap C_{j+1}\left(f^{-1}(t)\right) = \emptyset$ for each $j \in [2n-1]$.

Corollary 6.3.3. ([23, Corollary 2]) Let $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ be a uniformly continuous, co-Lipschitz mapping, K be an unbounded component of $\left(\mathbb{C} \setminus B_d^{\|\cdot\|}(0)\right) \setminus f^{-1}(t)$ for some $t \in \mathbb{R}$ and $d > \Theta\left(f^{-1}(t)\right)$. Then for any $\tilde{c} < c$ there exists $R(\tilde{c}) > 0$ such that if $r > R(\tilde{c})$, then there exists $y \in K$ where $\|y\| = r$ and $|f(y) - t| > \tilde{c}\|y\|$.

Theorem 6.3.5 below generalises [23, Lemma 4], which establishes the Euclidean scale of values $\rho_{2,1}^{(m-1)} = \sin(\pi/(2m+2))$, to the case of uniformly continuous co-Lipschitz mappings $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$, where $\|\cdot\|$ is any norm on $\mathbb{C}$. For this we need to recall the definition for the upper norm regularity constants, as in Definition 4.4.3. First, we need this quick lemma concerning the lack of disjointness of closed balls with sufficiently large radii.

Lemma 6.3.4. Let $\|\cdot\|$ be a norm on $\mathbb{C}$, $n \geq 2$ and $P = (x_1, \ldots, x_n) \in \mathcal{F}_n^{\|\cdot\|}$. For each $j \in [n]$, let $\overline{D}_j := \{z \in \mathbb{C} : \|x_j - z\| \leq \alpha(n, \|\cdot\|)/2\}$. Then there exists $j \in [n]$ such that $\overline{D}_j \cap \overline{D}_{j+1} \neq \emptyset$.

Proof. Suppose, for a contradiction, that $\overline{D}_j \cap \overline{D}_{j+1} = \emptyset$ for all $j \in [n]$. Then observe that $\|x_{j+1} - x_j\| > \alpha(n, \|\cdot\|)$ for each $j \in [n]$. Therefore, recalling Notation 4.2.12, Notation 4.3.1 and Definition 4.4.3,

$$\alpha(n, \|\cdot\|) \geq \alpha(x_1, n, \|\cdot\|) \geq d^-(P) = \min_{1 \leq j \leq n} \|x_{j+1} - x_j\| > \alpha(n, \|\cdot\|),$$

providing contradiction. □

Theorem 6.3.5. Let $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ be a uniformly continuous, co-Lipschitz mapping such that $L_f^* < 1$, $n = n(f)$ denotes the maximum number of components of any fibre and $\alpha(2n, \|\cdot\|) \neq 2$. Then for each $t \in \mathbb{R}$ there exists $R_0 = R_0(t) > 0$ such that

$$\min_{j \in [2n]} \max_{x \in \Gamma_j(r)} |f(x) - t| \leq \frac{\alpha(2n, \|\cdot\|) r}{2} \quad \text{for all } r \geq R_0.$$

Here $\Gamma_j(r)$ are the arcs of $\partial B_r^{\|\cdot\|}(0)$ defined in the following way. For each $r > \Theta\left(f^{-1}(t)\right)$ we fix $\#e\left(f^{-1}(t)\right) = 2n$ points $A_j(r) \in C_j\left(f^{-1}(t)\right) \cap \partial B_r^{\|\cdot\|}(0)$ and by $\Gamma_j(r)$, $j \in [2n]$, we denote the closed arc going counter clockwise along $\partial B_r^{\|\cdot\|}(0)$ going from $A_j(r)$ to $A_{j+1}(r)$, considering indices modulo $2n$.

Proof. Since $L_f^* < 1$ there exists $R > 0$ such that $\Omega_f(r) < r$ for each $r \geq R$. For brevity, we write α instead of $\alpha(2n, \|\cdot\|)$. Define

$$R_0 := \max\left(\frac{2R}{\alpha}, \frac{2\Theta\left(f^{-1}(t)\right)}{2 - \alpha}\right).$$

Suppose, for a contradiction, that there exists $r > R_0$ such that for each $j \in [2n]$ it follows that $\max_{x \in \Gamma_j(r)} |f(x) - t| > \alpha r/2$. Since $\alpha r/2 > R$ note $\alpha r/2 > \Omega_f(\alpha r/2)$ and hence as $|f(x) - t| \leq \Omega_f\left((\text{dist}(x, f^{-1}(t))\right)$ for all $x \in \mathbb{C}$, it follows for each $j \in [2n]$ that

$$\max_{x \in \Gamma_j(r)} \Omega_f\left(\text{dist}\left(x, f^{-1}(t)\right)\right) > \Omega_f\left(\frac{\alpha r}{2}\right).$$

Therefore, as Ω_f is an increasing function, $\max_{x \in \Gamma_j(r)} \text{dist}(x, f^{-1}(t)) > \alpha r/2$ for each $j \in [2n]$. Thus, for each $j \in [2n]$, there exists $x_j \in \Gamma_j(r)$ such that $\overline{D}_j \cap f^{-1}(t) = \emptyset$ where $\overline{D}_j = \{y \in \mathbb{C} : \|y - x_j\| \leq \alpha r/2\}$. Hence $\overline{D}_j$ is in the same component of $\mathbb{C} \setminus f^{-1}(t)$ as x_j.

Now, as $r \geq R_0$ and $\alpha \neq 2$, note $r - \alpha r/2 \geq \Theta(f^{-1}(t))$ and so $\overline{D}_j$ is contained in the same component of $\left(\mathbb{C} \setminus B^{\|\cdot\|}_{\Theta(f^{-1}(t))}(0)\right) \setminus f^{-1}(t)$ as x_j. In particular, $\overline{D}_j \cap \overline{D}_k = \emptyset$ for any $j \neq k$. This contradicts Lemma 6.3.4. □

We are now in a position to provide a result analogous to [23, Theorem 1].

Corollary 6.3.6. Let $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ be a c-co-Lipschitz, uniformly continuous mapping and let $n = n(f)$ denote the maximal number of components of $f^{-1}(t)$. Then,

$$\frac{c}{L_f^*} \leq \frac{\alpha(2n, \|\cdot\|)}{2}.$$

Proof. Let $\alpha = \alpha(2n, \|\cdot\|)$. If $\alpha = 2$, the estimate follows trivially as $c \leq L_f^*$. Hence, suppose $\alpha \neq 2$, that is $\alpha \in (0, 2)$. For a contradiction, suppose that $c/L_f^* > \alpha/2$. Without loss of generality, we may assume that $L_f^* < 1$ and $c > \alpha/2$.

Consider $\tilde{c} := \alpha/2$. Then $c > \tilde{c}$. Thus, by Corollary 6.3.3, for each unbounded component K_j, $j \in [2n]$, of $\left(\mathbb{C} \setminus B_{\Theta(f^{-1}(t))+1}(0)\right) \setminus f^{-1}(0)$ consider $R_j(\tilde{c}) > 0$. Then, for each $r > \max_{j \in [2n]} R_j(\tilde{c})$ there exists $y_j \in K_j$, $j \in [2n]$, such that $\|y_j\| = r$ and $|f(y_j)| > \tilde{c}\|y_j\| = \tilde{c}r$. Therefore, for each $r > \max_{j \in [2n]} R_j(\tilde{c})$,

$$\min_{j \in [2n]} \max_{x \in \Gamma_j(r)} |f(x)| > \tilde{c}r = \frac{\alpha r}{2},$$

which contradicts Theorem 6.3.5. □

6.4 A uniformly continuous co-Lipschitz mapping which attains the optimal ratio of constants

We recall the scale $\alpha(m, |\cdot|) = 2\sin(\pi/m)$ is shown to be sharp in [23, Section 2]. This section is devoted to the question whether there exist uniformly continuous co-Lipschitz mappings which prove the scale $\alpha(m, \|\cdot\|)$ is sharp, where $\|\cdot\|$ is any norm on the plane. We show this in the positive for a particular class of norms of the plane. First, we introduce a family of mappings analogous to those defined in [23].

Notation 6.4.1. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $m \in 2\mathbb{N}$. Define $\xi_m : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ in the following way:

Let $P = (x_1, \ldots, x_m) \in E_m^{\|\cdot\|}$ be such that $e(P) = \alpha(m, \|\cdot\|)$ and for each $j \in [m]$ let $L_j = [0, x_j\rangle$ and let $R_j = R_{L_j, L_{j+1}}$ denote the open region enclosed by the half rays L_j and L_{j+1}, starting at L_j and traversed in the anticlockwise direction.

Define $\xi_m : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$ by

$$\xi_m(z) = \begin{cases} \min\left(\operatorname{dist}(z, L_j), \operatorname{dist}(z, L_{j+1})\right), & \text{if } z \in R_j \text{ and } j \text{ is odd;} \\ -\min\left(\operatorname{dist}(z, L_j), \operatorname{dist}(z, L_{j+1})\right), & \text{if } z \in R_j \text{ and } j \text{ is even;} \\ 0, & \text{if } z \in L_j \text{ for some } j \in [m]. \end{cases}$$

Remark 6.4.2. Observe that ξ_m is 1-Lipschitz. Moreover, to see that $\operatorname{Lip}(\xi_m) = 1$, consider $z \in R_j$ for some $j \in [m]$ and let $\gamma \in L_j \cup L_{j+1}$ be such that $|\xi_m(z)| = \|z - \gamma\|$. Then, as $\xi_m(\gamma) = 0$,

$$|\xi_m(z) - \xi_m(\gamma)| = |\xi_m(z)| = \|z - \gamma\|.$$

Let us now introduce a particular class of planar norms.

Definition 6.4.3. Let $m \geq 2$ be even. We say a norm $\|\cdot\|$ on $\mathbb{C}$ is *m-separated* if for every $j \in [m]$,

$$\operatorname{dist}(z, L_j) \geq \|z\| \frac{\alpha(m, \|\cdot\|)}{2} \qquad \text{for all } z \in D_{\|\cdot\|}(L_j, L_{j+1}).$$

The rest of this section is devoted to the following result, which then shows the scale $\alpha(m, \|\cdot\|)/2$ is sharp provided the planar norm is m-separated. In particular, we prove the following theorem.

Theorem 6.4.4. Let $m \geq 2$ be even and $\|\cdot\|$ be an m-separated norm on $\mathbb{C}$. Then ξ_m is a Lipschitz quotient mapping with $\operatorname{Lip}(\xi_m) = 1$ and $\operatorname{co-Lip}(\xi_m) = \alpha(m, \|\cdot\|)/2$.

We shall first show that ξ_m is pointwise 1-co-Lipschitz at each $z \in \mathbb{C} \setminus \bigcup_{j=1}^m D(L_j, L_{j+1})$, regardless of whether the norm is m-separated or not.

Lemma 6.4.5. Let $\|\cdot\|$ be a norm on $\mathbb{C}$ and $m \geq 2$ be even. Then ξ_m is pointwise 1-co-Lipschitz at each $z \in \mathbb{C} \setminus \bigcup_{j=1}^m D_{\|\cdot\|}(L_j, L_{j+1})$.

Proof. Consider first $z \in R_j \setminus D(L_j, L_{j+1})$ for some $j \in [m]$. Without loss of generality, suppose that $\xi_m(z) = \operatorname{dist}(z, L_j)$, then $\xi_m(z) > 0$. Since $R_j \setminus D(L_j, L_{j+1})$ is open in $\mathbb{C}$, there exists

$r_0 > 0$ such that

$$B_{r_0}^{\|\cdot\|}(z) \subseteq R_j \setminus D(L_j, L_{j+1}), \quad r_0 \leq \xi_m(z) \quad \text{and} \quad \xi_m(y) = \operatorname{dist}(y, L_j) \text{ for all } y \in B_{r_0}^{\|\cdot\|}(z).$$

Fix $r \in (0, r_0)$ and $t \in B_r^{|\cdot|}(\xi_m(z)) = (\xi_m(z) - r, \xi_m(z) + r)$. To see that ξ_m is pointwise 1-co-Lipschitz at z it suffices to show there exists $\beta \in B_r^{\|\cdot\|}(z)$ such that $\xi_m(\beta) = t$.

Since $L_j \subseteq \mathbb{C}$ is closed, there exists $\gamma \in L_j$ such that $\xi_m(z) = \operatorname{dist}(z, L_j) = \|z-\gamma\|$. Consider $N(\gamma, L_j) = \{w \in \mathbb{C} : \operatorname{dist}(w, L_j) = \|w - \gamma\|\}$. By Lemma 6.2.3, $R = [\gamma, z\rangle \subseteq N(\gamma, L_j)$.

Fix $\tilde{r} \in (0, r)$ such that $t \in [\xi_m(z) - \tilde{r}, \xi_m(z) + \tilde{r}]$. Note by the Jordan Curve Theorem that $\operatorname{Int}\left(\partial B_{\tilde{r}}^{\|\cdot\|}(z)\right)$ is bounded. Hence any ray which has its end point at z and which non-trivially intersects $\mathbb{C} \setminus \overline{B}_{\tilde{r}}^{\|\cdot\|}(z)$ must necessarily intersect $\partial B_{\tilde{r}}^{\|\cdot\|}(z)$ at least once. Thus, as $\gamma \notin \overline{B}_{\tilde{r}}^{\|\cdot\|}(z)$, the straight line $R = [\gamma, z\rangle$ necessarily intersects $\partial B_{\tilde{r}}^{\|\cdot\|}(z)$ at least twice.

Let $\delta_1, \delta_2 \in R \cap \partial B_{\tilde{r}}^{\|\cdot\|}(z)$ be such that $\|\delta_1 - \gamma\| < \|\delta_2 - \gamma\|$. Then, by the collinearity of z, γ and δ_k, $k = 1, 2$,

$$\xi_m(\delta_k) = \operatorname{dist}(\delta_k, L_j) = \|\delta_k - \gamma\| = \|z - \gamma\| + (-1)^k \|z - \delta_k\| = \|z - \gamma\| + (-1)^k \tilde{r}.$$

Now, as $\xi_m\big|_{[\delta_1, \delta_2]}$ is continuous and $\xi_m(\delta_k) = \xi_m(z) + (-1)^k \tilde{r}$, $k = 1, 2$, there exists $\beta \in [\delta_1, \delta_2] \subseteq \overline{B}_{\tilde{r}}^{\|\cdot\|}(z) \subseteq B_r^{\|\cdot\|}(z)$ such that $\xi_m(\beta) = t$. Therefore, ξ_m is pointwise 1-co-Lipschitz at each $z \in R_j \setminus D(L_j, L_{j+1})$, $j \in [m]$.

Let us now consider $z \in L_j \setminus \{0\}$ for some $j \in [m]$. Let $r_0 > 0$ be such that $r_0 < \min(\|z\|, \rho_0(z))$, where $\rho_0(z)$ is given by Lemma 6.2.5 where we take $L_1 = L_{j-1}$, $L = L_j$ and $L_2 = L_{j+1}$. Recall Notation 6.4.1. Let $r_1 \in (0, r_0)$ be such that that $|\xi_m(w)| = \operatorname{dist}(w, L_j)$ for all $w \in B_{r_1}^{\|\cdot\|}(z)$; note $\mathbb{C} \setminus (D(L_{j-1}, L_j) \cup D(L_j, L_{j+1}))$ is open and so such $r_1 > 0$ exists. Let $|t| < r_1$ and $r \in (0, r_1)$ be such that $t \in [\xi_m(z) - r, \xi_m(z) + r] = [-r, r]$. Suppose, without loss of generality, that $\xi_m(w) = \operatorname{dist}(w, L_j)$ if $w \in \overline{B}_r^{\|\cdot\|}(z) \cap R_j$ and $\xi_m(w) = -\operatorname{dist}(w, L_j)$ if $w \in \overline{B}_r^{\|\cdot\|}(z) \cap R_{j-1}$.

By Lemma 6.2.5 there exists $z_1 \in \partial B_r^{\|\cdot\|}(z) \cap R_j$ such that $\xi_m(z_1) > 0$ and $r = \|z_1 - z\| = \operatorname{dist}(z_1, L_j) = \xi_m(z_1)$. Similarly, there exists $z_2 \in \partial B_r^{\|\cdot\|}(z) \cap R_{j-1}$ such that $\xi_m(z_2) < 0$ and $-r = -\|z_2 - z\| = -\operatorname{dist}(z_2, L_j) = \xi_m(z_2)$. Hence, as $\xi_m\big|_{[z_1, z_2]}$ is continuous, there exists $\beta \in [z_1, z_2] \subseteq B_r^{\|\cdot\|}(z)$ such that $\xi_m(\beta) = t$. Therefore, ξ_m is pointwise 1-co-Lipschitz at

$z \in L_j \setminus \{0\}$, $j \in [m]$. □

If we now assume the norm is m-separated, which is exhibited by the Euclidean norm, for example, we can then show the mapping ξ_m is pointwise $(\alpha(m, \|\cdot\|)/2)$-co-Lipschitz at the remaining points $z \in D(L_j, L_{j+1})$.

Lemma 6.4.6. Let $m \geq 2$ be even and $\|\cdot\|$ be an m-separated norm on $\mathbb{C}$. Then, ξ_m is pointwise $(\alpha(m, \|\cdot\|)/2)$-co-Lipschitz at each $z \in \bigcup_{j=1}^{m} D_{\|\cdot\|}(L_j, L_{j+1})$.

Proof. For brevity, we write α instead of $\alpha(m, \|\cdot\|)$ and recall $\alpha \in (0, 2]$. Fix $j \in [m]$; without loss of generality, let us assume that $j = 1$. Fix $z \in D(L_1, L_2) \setminus \{0\}$ and observe that $\xi_m(z) > 0$. Let $r_0 > 0$ be such that $B_{r_0}^{\|\cdot\|} \subseteq R_1$ and $r_0 < \xi_m(z)$. Observe that $r_0 < 2\xi_m(z)/\alpha$. Fix $r \in (0, r_0)$ and $t \in [\xi_m(z) - (\alpha/2)r, \xi_m(z) + (\alpha/2)r]$. Let $z_1, z_2 \in \partial B_r^{\|\cdot\|}(z) \cap [0, z\rangle$ be such that $\|z_1\| < \|z_2\|$; one may argue via the Jordan Curve Theorem and the connectedness of $\partial B_r^{\|\cdot\|}(z)$ to conclude the existence of such $z_1, z_2 \in \partial B_r^{\|\cdot\|}(z)$. Then, by Lemma 6.2.6, $z_1, z_2 \in D(L_1, L_2)$ and for each $k = 1, 2$, using $\|z_k\| = \|z\| + (-1)^k r$,

$$\xi_m(z_k) = \operatorname{dist}(z_k, L_1) = \operatorname{dist}(z_k, L_2) = \frac{\|z_k\|}{\|z\|}\operatorname{dist}(z, L_2) = \left(1 + (-1)^k \frac{r}{\|z\|}\right) \xi_m(z).$$

Hence, as $\|\cdot\|$ is m-separated, $\xi_m(z_1) \leq \xi_m(z) - (\alpha/2)r$ and $\xi_m(z_2) \geq \xi_m(z) + (\alpha/2)r$. Therefore, by the continuity of $\xi_m\big|_{[z_1, z_2]}$, there exists $\beta \in [z_1, z_2] \subseteq \overline{B}_r^{\|\cdot\|}(z)$ such that $\xi_m(\beta) = t$.

Consider now $z = 0$, $r > 0$ and $t \in [-\alpha r/2, \alpha r/2]$. Let $z_1 \in D(L_m, L_1) \cap \partial B_r^{\|\cdot\|}(0)$ and $z_2 \in D(L_1, L_2) \cap \partial B_r^{\|\cdot\|}(0)$. Then, as $\|\cdot\|$ is m-separated, $\xi_m(z_1) \leq -\alpha r/2$ and $\xi_m(z_2) \geq \alpha r/2$. Hence, as $\xi_m\big|_{[z_1, z_2]}$ is continuous, there exists $\beta \in [z_1, z_2]$ such that $\xi_m(\beta) = t$. □

We are now in a position to prove Theorem 6.4.4.

Proof of Theorem 6.4.4. We have already verified that $\operatorname{Lip}(\xi_m) = 1$ in Remark 6.4.2. Combining Lemma 1.2.18, Lemma 6.4.5 and Lemma 6.4.6 we conclude that ξ_m is $(\alpha(m, \|\cdot\|)/2)$-co-Lipschitz on $\mathbb{C}$, since $\alpha(m, \|\cdot\|) \leq 2$. Hence, co-$\operatorname{Lip}(\xi_m) \geq \alpha(m, \|\cdot\|)/2$. Now, by Corollary 6.3.6 as $\operatorname{Lip}(\xi_m) = 1$, it follows that co-$\operatorname{Lip}(\xi_m) \leq \alpha(m, \|\cdot\|)/2$. □

The question is now which planar norms $\|\cdot\|$ are m-separated. We observe the following sufficient property.

Let $P = (x_1, \dots, x_m) \in E_m^{\|\cdot\|}$ be the polygon from which ξ_m is defined, that is such that $e(P) = \alpha(m, \|\cdot\|)$. If for every $j \in [m]$ and every $z \in D(L_j, L_{j+1})$ it follows that

$$\operatorname{dist}(z, L_j) = \Big\|z - \|z\|x_j\Big\| = \Big\|z - \|z\|x_{j+1}\Big\|, \tag{6.4.1}$$

then $\|\cdot\|$ is m-separated. Indeed, if (6.4.1) is satisfied, then for each $z \in D(L_j, L_{j+1})$,

$$\operatorname{dist}(z, L_j) = \Big\|z - \|z\|x_j\Big\| \geq \|z\| \cdot \|x_{j+1} - x_j\| - \Big\|z - \|z\|x_{j+1}\Big\| = \alpha(m, \|\cdot\|)\|z\| - \operatorname{dist}(z, L_j).$$

Therefore, any norm $\|\cdot\|$ on $\mathbb{C}$ which satisfies (6.4.1) for each $j \in [m]$ and every $z \in D(L_j, L_{j+1})$ is necessarily m-separated.

In particular, if $p \equiv 2 \pmod 4$ and $m \geq p$ is a multiple of p, it can be readily verified that $\|\cdot\|_p$ is m-separated. However, it is not satisfied in general. Indeed, consider $\|\cdot\| = \|\cdot\|_4$ and $m = 4$. Then $\alpha(4, \|\cdot\|_4) = 2$, which is attained at $P = (1, i, -1, -i) \in E_4^{\|\cdot\|_4}$. Let $x_1 = 1$, $x_2 = i$ and $x = (1+i)/2$. Then $x \in D(L_1, L_2)$, but

$$\operatorname{dist}(x, L_1) = \frac{1}{2} < 1 = \frac{\alpha(4, \|\cdot\|_4)}{2}.$$

One may conjecture that, in general, the bound obtained in Corollary 6.3.6 is not optimal. The main issue with the above example where $\|\cdot\| = \|\cdot\|_4$ is that the distance to the rays is not attained at the scaled equilateral polygon, but rather outside of the closed sphere centred at the origin. Potentially, one may be able to define a new constant γ in Theorem 6.3.5 for which $\max_{x \in \Gamma_j(r)} \operatorname{dist}(x, f^{-1}(t)) > \gamma r$ still provides a valid contradiction, without having to revert to the context of equilateral polygons.

CHAPTER 7

FINAL COMMENTS AND FURTHER WORK

In Chapter 2 we answered a couple of converse questions to a groundbreaking result of [17], see Theorem 1.1.4 of the present thesis. In particular, we showed that for each planar complex polynomial $P : \mathbb{C} \to \mathbb{C}$ in one complex variable there exists a planar homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $P \circ h$ is a Lipschitz quotient mapping. In doing so, we introduced the notion of strongly co-Lipschitz mappings and noted the correspondence between the discreteness of Lipschitz quotient mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ and the existence of a Lipschitz quotient mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ which is not strongly co-Lipschitz at a some $x_0 \in \mathbb{R}^n$. Further investigation into continuous, strongly co-Lipschitz mappings between spaces of the same finite dimension could lead to some progress into the long-standing conjecture of [17], see also Conjecture 1.2.39 of the present thesis.

We then turned our attention to questions concerning the optimal ratio of constants for a planar Lipschitz quotient mapping in polygonal norms. This was presented in Chapter 3. In Corollary 3.3.10 such an estimate is provided, and in particular we show that if $f : (\mathbb{C}, \|\cdot\|_4) \to (\mathbb{C}, \|\cdot\|_4)$ is a 2-fold L-Lipschitz, c-co-Lipschitz mapping, then $L/c \geq 2 + (1/38)$. This is a partial progress towards answering Conjecture 3.1.3.

It is unlikely that the improved estimate in Corollary 3.3.10 is optimal. As discussed in [37, Chapter 5] there exist large classes of Lipschitz quotient mapping $f : (\mathbb{C}, \|\cdot\|_\infty) \to (\mathbb{C}, \|\cdot\|_\infty)$ for which $L/c \geq 3$.

Another avenue one may utilise to approach fully answering Conjecture 3.1.3 in the positive, is to further investigate the relations of N-fold Lipschitz quotient mappings to the standard N-winding maps. For each $m \geq 4$ let us denote $f_{m,2}$ to be the 2-fold winding map in $\|\cdot\|_m$, as defined in [25, Notation 3.12]. It can be shown that $f_{m,2}$ is necessarily 2-centred and that any

2-centred Lipschitz quotient mapping $f : \mathbb{C} \to \mathbb{C}$ can be viewed as a reparametrisation of the standard 2-winding map, in that there exists an appropriate homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $f = f_{m,2} \circ h$. Supposing the conjecture that the optimal ratio of constants of a planar Lipschitz quotient mapping in polygonal norms will be attained by the associated winding map, investigating such a decomposition of centred mappings may prove to be useful.

One may note that for any two non-zero, distinct vectors $z, w \in \mathbb{C}$ such that the straight line segment containing both z, w is parallel to either the real or imaginary axis, then it follows that $\|f_{4,2}(z) - f_{4,2}(w)\|_4 = 3\|z - w\|_4$, exhibiting the maximum possible 'growth' as shown by $f_{4,2}$. Therefore, one avenue that may be worth exploring is to consider preimages of the spheres $\partial B_r^{\|\cdot\|_\infty}(0)$, $r > 0$, under centred Lipschitz quotient mappings, or even under the homeomorphism h as above.

The framework in Chapter 3 does not readily extend to the question of the optimal ratio of constants for general N-fold planar Lipschitz quotient mappings in polygonal norms. Fortunately, it is possible to show that for every N-fold Lipschitz quotient mapping f there exists a corresponding N-centred mapping f_1, not necessarily Lipschitz quotient, that does not vary too much from the initial mapping:

$$(1 - \varepsilon) \|f_1(z)\| < \|f(z)\| < (1 + \varepsilon) \|f_1(z)\| \qquad \text{whenever } \|z\| \text{ is sufficiently large.}$$

This suggests that there is potentially some underlying relationship between the class of centred N-fold Lipschitz quotient mappings and the standard N-fold Lipschitz quotient mappings, whenever $N \geq 3$. Further investigation into this is required, and hopefully such an avenue will allow one to apply Theorem 3.3.9 to be able to improve the current known estimates for the ratio of co- and Lipschitz constants for general planar Lipschitz quotient mappings in polygonal norms.

Chapters 4 and 5 concerned the existence and certain structural properties of equilateral polygons inscribed in planar norms, in particular in polygonal norms. In Theorem 4.3.5 and Lemma 4.3.6 we correct a statement of [8] concerning the existence of such equilateral polygons and the uniqueness of the edge of equilateral polygons containing a fixed vector. This was shown to not be true in general, due to the exceptional class of rectilinear norms; in particular, due to the existence of 3-rectilinear pairs.

We proceeded by investigating the extremal edge lengths exhibited by equilateral n-gons inscribed in a fixed planar norm. We show in Corollary 4.4.8 that the collection of possible edge lengths is either a single value, or fills a closed interval of positive length, hence is uncountable. The former is satisfied by the Euclidean norm for all $n \geq 3$ and the latter is satisfied by any rectilinear norm whenever $n \geq 4$ is a multiple of 4, for example.

In Chapter 5 we investigated to what extent the regularity constants, see Definition 4.4.3, of polygonal norms $\|\cdot\|_m$ behave similarly to the regularity constants of the Euclidean norm. In Theorem 5.2.10 a classification is provided for when the regularity constants of polygonal norms are equal. Further investigation is required to determine the behaviour of the regularity constants $\alpha(n, \|\cdot\|_m)$ and $\beta(n, \|\cdot\|_m)$, whenever $n < m$. As mentioned in Chapter 5, when $m = 6$ it is possible to show that $\beta(n, \|\cdot\|_6) < \alpha(n, \|\cdot\|_6)$ for $n = 3, 4, 5$. We conjecture that whenever $m \geq 4$ is even and $n < m$, where $n \neq 6$, then $\alpha(n, \|\cdot\|_m) > \beta(n, \|\cdot\|_m)$.

At first, the exclusion of the case $n = 6$ may appear to be unexpected. However, there is a standard result regarding the existence of equilateral hexagons in Minkowski planes, cf. [26, Proposition 34], which states that $\alpha(6, \|\cdot\|) = \beta(6, \|\cdot\|) = 1$ for any norm $\|\cdot\|$ on $\mathbb{C}$.

Ongoing joint work of the author with S. Dewar investigates the collection of planar norms for which their regularity constants coincide. We were able to show that there exists a comeagre collection of planar norm $\mathcal{D} \subseteq \mathcal{K}_2$ so that for every $\|\cdot\| \in \mathcal{D}$ and every $n \geq 3$, where $n \not\equiv 2 \pmod 4$, we have $\alpha(n, \|\cdot\|) > \beta(n, \|\cdot\|)$. Here $\mathcal{K}_2$ denotes the collection of all planar norms. The topology we assign to $\mathcal{K}_2$ is the metric topology stemming from the Hausdorff metric applied to the centrally symmetric, convex bodies provided by the unit spheres centred at the origin, cf. [7, Section 4]. The techniques used rely heavily on Rigidity Theory, and follow closely the exposition in [7, Chapter 6]. Unfortunately, the current techniques cannot be extended to the cases when $n \not\equiv 2 \pmod 4$. Nonetheless, we conjecture that:

i) There exists a comeagre collection of planar norms $\mathcal{D} \subseteq \mathcal{K}_2$ so that for every $\|\cdot\| \in \mathcal{D}$ and every $n \geq 3$, where $n \neq 6$, we have $\alpha(n, \|\cdot\|) > \beta(n, \|\cdot\|)$,

ii) For $\|\cdot\| \in \mathcal{K}_2$, $\alpha(n, \|\cdot\|) = \beta(n, \|\cdot\|)$ for all $n \geq 3$ if and only if there exists a linear transformation $T : \mathbb{C} \to \mathbb{C}$ such that $T\left(\partial B_1^{\|\cdot\|}(0)\right) = \partial B_1^{|\cdot|}(0)$, where $|\cdot|$ is the standard Euclidean norm.

Our final topic of discussion, in Chapter 6, was that of uniformly continuous, co-Lipschitz mappings $f : (\mathbb{C}, \|\cdot\|) \to \mathbb{R}$, where $\|\cdot\|$ is any norm on $\mathbb{C}$. We provide an upper bound, see Corollary 6.3.6, for the ratio of the co-Lipschitz and weak-Lipschitz constants for such mappings. This, surprisingly, is related to the upper norm regularity constant of the norm $\|\cdot\|$. We then proceeded by, for a particular class of planar norms, providing an example of a family of Lipschitz quotient mappings which attains this maximal ratio. Unfortunately, this was not applicable to all planar norms. Further investigation into an optimal ratio of constants is required. One may notice that the reason why the construction provided does not attain the maximal ratio of constants in all norms stems from Theorem 6.3.5. If one is able to identify a new constant, say $\gamma(2n, \|\cdot\|)$, such that $2\gamma(2n, \|\cdot\|) \leq \alpha(2n, \|\cdot\|)$ and for any $j \in [2n]$ it follows that $\max_{x \in \Gamma_j(r)} \operatorname{dist}(x, f^{-1}(t)) \leq r\gamma(2n, \|\cdot\|)$ holds true for all mappings f, then the same methodology would be applicable to conclude that $c/L_f^* \leq \gamma(2n, \|\cdot\|)$. This is still ongoing work.

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