FEASIBILITY ALGORITHMS FOR DISTANCE-BIREGULAR GRAPHS

by

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Synopsis.

A distance-biregular graph is a finite, undirected bipartite graph where any two vertices in the same part of the bipartition have the same intersection array. In this thesis we find necessary conditions for a pair of arrays to correspond to a distance-biregular graph and use these to construct an algorithm for generating all pairs of feasible arrays corresponding to possible graphs of girth four and smallest valency $b_0' < 20$. The feasible arrays with $b_0' < 10$ are analysed in Chapters 5 and 6; those with $10 \leqslant b_0' < 20$ are listed in Appendix II. Our results raised a number of interesting questions which are listed at the end of Appendix II.

Dedication.

To my wife Judith.

Acknowledgements.

I would like to thank everyone who has encouraged me during the work for this thesis. In particular I would like to thank my supervisor Dr.A.D.Gardiner for all of his help and guidance. I would also like to thank the Science and Engineering Research Council for their financial support.

CONTENTS.

Chapter 1

Introduction

We describe the types of graph we are considering in this thesis and list all known families.

1.1 Graphs

Firstly we need to define exactly what type of graph we will be considering. A graph Γ consists of a set of vertices $V\Gamma$ and a set of edges $E\Gamma \subset V\Gamma \times V\Gamma$. We consider undirected graphs without loops or multiple edges, so $E\Gamma$ is a symmetric, irreflexive relation. A finite (resp. infinite) graph is thus one with a finite (resp. infinite) vertex set. For $u,v \in V\Gamma$ we write $u \sim v$ if $(u,v) \in E\Gamma$ and say that u and v are adjacent, and v is a neighbour of u. A clique in a graph is a subset of $V\Gamma$ whose members are all pairwise adjacent.

We define a path (or walk) of length k from v_0 to v_k to be a sequence $v_0, v_1, \ldots, v_k$ of vertices in a graph, where $v_{i-1} \sim v_i$ for each $i=1,\ldots,k$. We say the path is closed if $v_0 = v_k$. We denote by $d_\Gamma(u, v)$ the length of the shortest path from u to v in Γ. If it is clear which Γ we are considering we use $d(u,v)$. (If no path exists we write $d(u,v)=\infty$.) We may also say that the distance from u to v is $d(u,v)$. This means that a graph Γ is connected if $d(u,v) < \infty$ for all $u,v \in V\Gamma$. The diameter of a connected graph is defined to be the supremum of d on Γ and is denoted by $\mathrm{diam}(\Gamma)$.

We call a connected graph bipartite (resp. n-partite), if the vertex set of Γ can be partitioned into two (resp. n) non-empty subsets of Γ such that if $u \sim v$, then u and v are in different subsets. The complement of a graph Γ is a graph Γ^c

with $V\Gamma^c = V\Gamma$ and $u \sim v$ in Γ^c if and only if $u \not\sim v$ in Γ .

We associate with a labelled graph Γ on n vertices (numbered 1,2,...,n) the nxn adjacency matrix $A(\Gamma)$ defined by

$$A(\Gamma)_{ij} = \begin{cases} 1; & \text{if } i \sim j, \\ 0; & \text{otherwise.} \end{cases}$$

The eigenvalues of the graph Γ are the eigenvalues of the adjacency matrix of Γ . Relabelling the vertices gives rise to a different adjacency matrix $P^{-1}A\,P$ for some permutation matrix P, but does not change the eigenvalues, which are thus an invariant of the graph Γ itself.

The degree (or valency) of a vertex v is the number of neighbours of v. A graph is k-regular if each vertex has valency k. A graph is biregular if it is bipartite and if any two vertices in the same part of the bipartition have the same degree.

For a graph Γ , the graph $\Gamma^{(i)}$ has vertex set $V\Gamma^{(i)} = V\Gamma$ with vertices u,v being adjacent in $\Gamma^{(i)}$ if and only if they are at distance i in Γ . For a (connected) bipartite graph Γ , the graph $\Gamma^{(2)}$ is the disjoint union of two connected graphs; we call each component of $\Gamma^{(2)}$ a derived graph of Γ .

A cycle of length k ($\geqslant 3$) in a graph Γ is a path $v_0, \ldots, v_k$, for which $v_{i-1} \neq v_{i+1}$, i = 1,...,k-1 and $v_0 = v_k$. The girth of a graph Γ is the length of the shortest cycle in Γ .

The line graph of a graph Γ is the graph $L(\Gamma)$ with vertex set $E\Gamma$, two edges e,f in Γ being adjacent in $L(\Gamma)$ if and only if they have a common vertex in Γ . The subdivision graph $S(\Gamma)$ of a graph Γ is the graph obtained from Γ by subdividing each edge with a new vertex; formally $VS(\Gamma) = V\Gamma \cup E\Gamma$ with $x \sim y$ in $S(\Gamma)$ if and only if $x \in V\Gamma$, $y \in E\Gamma$ (or vice versa) and x, y are incident in Γ .

1.2 Incidence Structures

An incidence structure I consists of a pair (P,B), where the set P is the set of points of I and B is a collection of subsets of I called the blocks of I. If two points x and y of an incidence structure determine a unique block l containing them both then we often refer to l not as a block, but as the line xy.

The incidence graph $\Gamma = \Gamma(I)$ of an incidence structure I = (P,B) has vertex set $V\Gamma = P \cup B$ and adjacency defined by pairs (p,b), $p \in P$ and $b \in B$, where $p \sim b$ if and only if $p \in b$. The incidence graph of an incidence structure is clearly a bipartite graph. We say an incidence structure is regular (resp. biregular) if its incidence graph is regular (resp. biregular). The block graph Γ_B of an incidence structure I is that derived graph with vertex set B the blocks of I; the point graph Γ_P is the derived graph with vertex set P.

An incidence structure I = (P,B) with v points , each block having exactly k points, and such that each t-subset of P occurs in exactly λ blocks is called a t-(v,k,λ) block design. Fisher's inequality guarantees that if $t \geqslant 2$ then $|B| \geqslant |P|$.

A 2-(v,k,λ) design is symmetric if and only if $|B| = |P|$. Equivalently a 2-(v,k,λ) design is symmetric if and only if each pair of blocks intersects in a given number of points μ, in which case we necessarily have $\lambda = \mu$.

A quasisymmetric block design with intersection numbers μ_1, μ_2 is a 2-(v,k,λ) design for which any two blocks intersect in either μ_1 or μ_2 points.

Let I = (P,B) be an incidence structure with each block of size k. Then I is a 2-(kl,k,λ) transversal design if and only if the point set can be partitioned into k 'parts' P_i, i=1,...,k, each of size l such that each block contains exactly one point from each P_i and any two points from distinct parts P_i lie in exactly λ blocks.

A generalised n-gon is an incidence structure whose incidence graph satisfies

(i) it is biregular with valencies (s+1) and (t+1);

(ii) the distance between any two vertices is at most n;

(iii) if the distance between two vertices is less than n, there is a unique shortest path joining them;

(iv) for any vertex there is at least one vertex at distance n from it.

1.3 Distance-regularity - local and global.

Let Γ be a connected graph. By $\Gamma_i(u)$ we mean the set of vertices of Γ at distance i from the vertex u, and by $k_i(u)$ the size of $\Gamma_i(u)$. We often write $\Gamma_1(u)$ as $\Gamma(u)$.

Let $u, v \in V\Gamma$ with $d(u,v) = i$. Then

$$c(u,v) = |\Gamma_{i-1}(u) \cap \Gamma_1(v)|,$$

$$a(u,v) = |\Gamma_i(u) \cap \Gamma_1(v)|,$$

and $$b(u,v) = |\Gamma_{i+1}(u) \cap \Gamma_1(v)|.$$

A vertex $u \in V\Gamma$ is distance-regular if for each i, such that $i = 1,\ldots,\mathrm{diam}(\Gamma)$, the numbers c(u,v), a(u,v), b(u,v) are independent of the choice of v in $\Gamma_i(u)$; we then write $c_i(u)$, $a_i(u)$ and $b_i(u)$ in place of c(u,v), a(u,v) and b(u,v) (where v is any vertex in $\Gamma_i(u)$). If u is a distance-regular vertex of a graph Γ, then the array

$$\iota(u) = \begin{bmatrix} * & c_1(u) & \ldots & c_d(u) \\ 0 & a_1(u) & \ldots & a_d(u) \\ b_0(u) & b_1(u) & \ldots & * \end{bmatrix}$$

is the intersection array of u, where $d = \mathrm{diam}(\Gamma)$. The matrix

$$I(u) = \begin{bmatrix} 0 & c_1(u) & 0 & \ldots & & 0 \\ b_0(u) & a_1(u) & c_2(u) & \ldots & & 0 \\ 0 & b_1(u) & a_2(u) & \ldots & & 0 \\ \ldots & . & . & . & & \\ \ldots & & . & . & . & \\ \ldots & & & . & . & . \\ 0 & \ldots & & b_{d-2}(u) & a_{d-1}(u) & c_d(u) \\ 0 & \ldots & & 0 & b_{d-1}(u) & a_d(u) \end{bmatrix}$$

is the intersection matrix for u.

A graph is locally distance-regular if each vertex of Γ is distance-regular. If every vertex of a locally distance-regular graph Γ has the same intersection array then Γ is globally distance-regular. A globally distance-regular graph is usually called a distance-regular graph. A bipartite locally distance-regular graph which is not globally distance-regular is distance-biregular if any two vertices in the same part of the bipartition have the same intersection array. Shawe-Taylor in [5] shows that a locally distance-regular graph is either globally distance-regular or distance-biregular.

The intersection array $\iota(\Gamma)$ of a distance-regular graph Γ is the intersection array of each of its vertices. The standard notation for this is

$$\iota(\Gamma) = \begin{bmatrix} * & c_1 & c_2 & \cdots & c_d \\ 0 & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & b_2 & \cdots & * \end{bmatrix}$$

where d=diam(Γ).

Let Γ be a distance-biregular graph. The two parts of the bipartition of the vertex set $V\Gamma$ are denoted by P and B. The diameter of Γ is d. We denote a typical vertex in P by u and its intersection array by

$$\iota(P) = \begin{bmatrix} * & c_1 & c_2 & \cdots & c_d \\ 0 & 0 & 0 & \cdots & 0 \\ b_0 & b_1 & b_2 & \cdots & * \end{bmatrix}$$

We usually omit the row of zeros and write

$$\begin{bmatrix} * & c_1 & c_2 & \cdots & c_d \\ b_0 & b_1 & b_2 & \cdots & * \end{bmatrix}$$

We denote a typical vertex in B by v and its intersection array by

$$\iota(B) = \begin{bmatrix} * & c_1' & c_2' & \cdots & c_d' \\ b_0' & b_1' & b_2' & \cdots & * \end{bmatrix}$$

The corresponding intersection matrices are denoted by I(P) and I(B) respectively. We let k_i be the numbers $k_i(u)$ for $u \in P$, $i = 0,\ldots,d$ and k_i' be the numbers $k_i(v)$ for $v \in B$, $i = 0,\ldots,d$. We note that $k_{d-1}' \neq 0$ and $k_{d-1} \neq 0$ but that one of the numbers k_d' and k_d may be zero. If $c_d = 0$ then we define d_P to be d-1, otherwise we define d_P to be d. We define d_B similarly.

Known families of distance-biregular graphs.

(i) Complete bipartite graphs.

The complete bipartite graph $K_{b_0,b_0'}$ has intersection arrays:

$$\begin{bmatrix} * & 1 & b_0 \\ b_0 & (b_0'-1) & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & b_0' \\ b_0' & (b_0-1) & * \end{bmatrix}$$

These are the only distance-biregular graphs of diameter two. ie: $d_P = d_B = 2$.

(ii) Quasisymmetric 2-designs.

Let I = (P, B) be a quasisymmetric 2-(v, b_0', c_2) design with block intersection numbers $\mu_1 = c_2'$ and $\mu_2 = 0$. Then, the incidence graph is a distance-biregular graph with $d_P = 3$, $d_B = 4$ and intersection arrays as below.

$$\begin{bmatrix} * & 1 & c_2 & b_0' \\ b_0 & b_0'-1 & b_0-c_2 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & c_2' & c_3' & b_0' \\ b_0' & b_0-1 & b_0'-c_2' & b_0-c_3' & * \end{bmatrix}$$

The value of b_0 is determined by the usual design condition $(v-1)\lambda = b_0(k-1)$; thus $b_0 = (v-1)c_2/(b_0'-1)$. The value of c_3' is determined by taking a non-incident point-block pair u',v' and counting flags (u'',v'') where u'' lies in v' and v'' contains u'; thus $b_0'c_2 = c_2'c_3'$. Hence the arrays are completely determined by v, b_0', c_2 and c_2' . Conversely, any distance-biregular graph with $d_P = 3$ and $d_B = 4$ is the incidence graph of a quasisymmetric 2-design with $\mu_1 = c_2'$ and $\mu_2 = 0$.

$$|P| := 1 + k_2 + k_4 + \ldots + k_{d'} = k_1' + k_3' + \ldots + k_{d''}',$$
$$|B| := k_1 + k_3 + \ldots + k_{d''} = 1 + k_2' + k_4' + \ldots + k_{d'}',$$

where d' is the largest even integer less than or equal to d, and d'' is the largest odd integer less than or equal to d.

We also have $|P| .b_0 = |B| .b_0'$.

Proof

Let u,v be arbitrary vertices in P, B respectively.

(i) The first two expressions follow by counting edges between $\Gamma_i(u)$ and $\Gamma_{i+1}(u)$. The second two expressions follow by counting edges between $\Gamma_i(v)$ and $\Gamma_{i+1}(v)$.

(ii) A vertex in $\Gamma_i(u)$ (respectively $\Gamma_i(v)$) has degree b_0 (respectively b_0') if i is even and b_0' (respectively b_0) if i is odd.

(iii) Let u be adjacent to v. Choose a vertex $x \in \Gamma_i(u) \cap \Gamma_{i-1}(v)$. Then the b_i neighbours of x in $\Gamma_{i+1}(u)$ lie in $\Gamma_i(v)$, so $b_{i-1}' \geqslant b_i$. By symmetry $b_{i-1} \geqslant b_i'$.

(iv) With u and v as in (iii) we can choose a vertex $x \in \Gamma_i(u) \cap \Gamma_{i+1}(v)$. The c_i neighbours of x in $\Gamma_{i-1}(u)$ then lie in $\Gamma_i(v)$, so $c_{i+1}' \geqslant c_i$. By symmetry $c_{i+1} \geqslant c_i'$.

(v) Clearly $P = \{u\} \cup \Gamma_2(u) \cup \Gamma_4(u) \cup \ldots \cup \Gamma_{d'}(u)$

$$= \Gamma_1(u) \cup \Gamma_3(u) \cup \ldots \cup \Gamma_{d''}(u)$$

and $B = \Gamma_1(v) \cup \Gamma_3(v) \cup \ldots \cup \Gamma_{d''}(v)$

$$= \{v\} \cup \Gamma_2(v) \cup \Gamma_4(v) \cup \ldots \cup \Gamma_{d'}(v).$$

Now count the number of edges joining P to B in two ways to get

(v) <u>Generalised n-gon.</u>

A generalised n-gon is a distance-biregular graph, and not a distance-regular graph, if the number of points on each line $s + 1 = b_0'$ differs from the number of lines through each point $t + 1 = b_0$. The intersection array for a point vertex is as below.

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \cdots & 1 & t+1 \\ t+1 & s & t & s & t & \cdots & s & * \end{bmatrix}$$

(vi) <u>The Johnson Biregular Graphs JB(k, n).</u>

Consider the set $\{1, \ldots, n\}$.

Let $P = \{k\text{-subsets}\}$ and $B = \{(k+1)\text{-subsets}\}$ for k a positive integer less than n. If we consider the graph Γ with vertex set $V\Gamma = P \cup B$ and adjacency defined in the usual way (ie: if $u \in P$ and $v \in B$ then $u \sim v$ if $u \subset v$) then we have a distance-biregular graph. The intersection array for a vertex in P is:

$$\left[\begin{matrix} * & 1 & 1 & 2 & \cdots & i & i & \cdots \\ n-k & k & (n-k-1) & (k-1) & \cdots & (k-i+1) & (n-k-i) & \cdots \end{matrix}\right.$$

$$\left.\begin{matrix} \cdots & (n-k-1) & (n-k) & (n-k) \\ \cdots & 1 & (2k-n+1) & * \end{matrix}\right] \quad \text{if } k \geq n-k$$

$$\text{or} \quad \left.\begin{matrix} \cdots & k & k & (k+1) \\ \cdots & 1 & (n-2k) & * \end{matrix}\right] \quad \text{if } k < n-k.$$

(vii) <u>The q-anologue Johnson Biregular graphs JB (k, n).</u>

Consider an n-dimensional vector space over GF(q), where q is the power of a prime.

Let P = {k-subspaces} and B = {(k + 1)-subspaces}. If we consider the graph Γ with vertex set $V\Gamma = P \cup B$ and adjacency defined in the usual way (ie: if $u \in P$ and $v \in B$ then $u \sim v$ if $u \subset v$), then we have a distance-biregular graph. The intersection array for a vertex in P is:

$$\left[\begin{array}{cccccccc} * & 1 & 1 & \cdots & \frac{q^i-1}{q-1} & \frac{q^i-1}{q-1} & \cdots \\ \frac{q^{n-k}-1}{q-1} & \frac{q^{k+1}-q}{q-1} & \frac{q^{n-k}-q}{q-1} & \cdots & \frac{q^{k+1}-q^i}{q-1} & \frac{q^{n-k}-q^i}{q-1} & \cdots \end{array}\right.$$

$$\left.\begin{array}{cccc} \cdots & \frac{q^{n-k-1}-1}{q-1} & \frac{q^{n-k}-1}{q-1} & \frac{q^{n-k}-1}{q-1} \\ \cdots & q^{n-k-1} & \frac{q^{k+1}-q^{n-k}}{q-1} & * \end{array}\right] \quad \text{if } k \geqslant n-k$$

or

$$\left.\begin{array}{cccc} \cdots & \frac{q^{k}-1}{q-1} & \frac{q^{k}-1}{q-1} & \frac{q^{k+1}-1}{q-1} \\ \cdots & q^{k} & \frac{q^{n-k}-q^{k}}{q-1} & * \end{array}\right] \quad \text{if } k < n-k.$$

(viii) Partial Geometries.

A finite partial geometry is an incidence structure $I = (P, B)$ with a symmetric incidence relation satisfying the following axioms

(a) each point is incident with $t + 1$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;

(b) each line is incident with $s + 1$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.

(c) if x is a point and L is a line not incident with x, then there are exactly α ($\alpha > 1$) points $x_1, x_2, \ldots, x_\alpha$ and α lines $L_1, L_2, \ldots, L_\alpha$ such that xIL_iIx_iIL, $i = 1, 2, \ldots, \alpha$.

The intersection arrays, for $\alpha < s + 1$ and $\alpha < t + 1$, are:

$$\begin{bmatrix} * & 1 & 1 & \alpha & t+1 \\ t+1 & s & t & s-\alpha & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 1 & \alpha & s+1 \\ s+1 & t & s & t-\alpha & * \end{bmatrix}$$

(ix). The following infinite family of distance-biregular graphs with $d_P = d_B = 4$ is given in [1].

Consider AG(3, q). We will define the vertices of one part of our bipartition (B) as the q^3 points of AG(3, q) and the vertices of the other part of our bipartition (P) as the affine planes described below.

Take a "spread" in the projective plane at infinity, ie: a set of projective lines in the projective plane with the property that each projective point lies in exactly 0 or d of these projective lines. Let the number of lines in the spread be s. In the affine space each of these projective lines corresponds to q affine planes. Let these qs affine planes be the vertices of P.

The intersection arrays are:

$$\begin{bmatrix} * & 1 & q & s-1 & q^2 \\ q^2 & s-1 & q^2-q & 1 & * \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} * & 1 & d & q(s-1)/d & s \\ s & q^2-1 & s-d & q^2-q(s-1)/d & * \end{bmatrix}$$

If our projective plane is $PG(2, 2^r)$ we can find a spread by considering an oval and looking at the lines that miss it. In $PG(2, 2^r)$ we have $(2^r + 2)$ points in an oval so we have $(2^r)^2 + 2^r + 1 - \binom{2^r + 2}{2} = 2^{r-1}(2^r - 1)$ lines missing our oval. Also, each point in our plane, but not in our chosen oval, has $2^r + 1$ lines through it and of these $(2^r + 2)/2 = 2^{r-1} + 1$ cut the oval. Therefore we can conclude that each point in $PG(2, 2^r)$ lies in exactly 0 or $d = 2^{r-1}$ lines of our spread.

Chapter 2.

In this chapter we will describe known necessary conditions which have to hold for a pair of arrays to correspond to a possible distance-biregular graph. Where possible we have given alternative proofs to those given in [5].

Definition

Let Γ be a distance-biregular graph such that $V\Gamma = P \cup B$. Let u denote a vertex in P and v denote a vertex in B. Suppose that $w_1 \in V\Gamma$ such that $d(u, w_1) = q$, where q is any non-negative integer less than, or equal to, the diameter d.

We define $\alpha_p{}^q{}_t(u, w_1)$ as follows, $0 \leqslant p, t \leqslant d$.

$$\alpha_p{}^q{}_t(u, w_1) := |\Gamma_p(u) \cap \Gamma_t(w_1)| .$$

Suppose that $w_2 \in V\Gamma$ such that $d(v, w_2) = q$, where q is any non-negative integer less than, or equal to, d. Then we define $\beta_p{}^q{}_t(v, w_2)$ as follows, $0 \leqslant p, t \leqslant d$.

$$\beta_p{}^q{}_t(v, w_2) := |\Gamma_p(v) \cap \Gamma_t(w_2)| .$$

Preliminary observations

Firstly we will consider $\alpha_p{}^q{}_t(u, w_1)$.

1A. $\alpha_0{}^q{}_t(u, w_1) = \alpha_t{}^q{}_0(u, w_1) = \delta_{tq}$.

2A. $\alpha_1{}^q{}_t(u, w_1) = 0$ if $t \neq q-1$ or $q+1$.

$$\alpha_1{}^q{}_{q-1}(u, w_1) = \begin{cases} \alpha_{q-1}{}^q{}_1(w_1, u) = c_q & \text{if } q \text{ is even} \\ \beta_{q-1}{}^q{}_1(w_1, u) = c_q' & \text{if } q \text{ is odd} \end{cases}$$

$$\alpha_1{}^q{}_{q+1}(u, w_1) = \begin{cases} \alpha_{q+1}{}^q{}_1(w_1, u) = b_q & \text{if } q \text{ is even} \\ \beta_{q+1}{}^q{}_1(w_1, u) = b_q' & \text{if } q \text{ is odd.} \end{cases}$$

3A. $\alpha_p{}^0{}_q(u, w_1) = \delta_{pq} k_q$.

Secondly we will consider $\beta_p{}^q{}_t(v, w_2)$.

1B. $\beta_0{}^q{}_t(v, w_2) = \beta_t{}^q{}_0(v, w_2) = \delta_{tq}$.

2B. $\beta_1{}^q{}_t(v, w_2) = 0$ if $t \neq q-1$ or $q+1$.

$$\beta_1{}^{q}{}_{q-1}(v, w_2) = \begin{cases} \beta_{q-1}{}^{q}{}_{1}(w_2, v) = c_q' & \text{if } q \text{ is even} \\ \alpha_{q-1}{}^{q}{}_{1}(w_2, v) = c_q & \text{if } q \text{ is odd} \end{cases}$$

$$\beta_1{}^{q}{}_{q+1}(v, w_2) = \begin{cases} \beta_{q+1}{}^{q}{}_{1}(w_2, v) = b_q' & \text{if } q \text{ is even} \\ \alpha_{q+1}{}^{q}{}_{1}(w_2, v) = b_q & \text{if } q \text{ is odd.} \end{cases}$$

3B. $\beta_p{}^{o}{}_q(v, w_2) = \delta_{pq} k_q'$.

Notice that in 1A - 3B above $\alpha_p{}^q{}_t(u, w_1)$ does not depend on our choice of u and w_1 , and $\beta_p{}^q{}_t(v, w_2)$ does not depend on our choice of v and w_2 . In these cases we write $\alpha_p{}^q{}_t$ for $\alpha_p{}^q{}_t(u, w_1)$ and $\beta_p{}^q{}_t$ for $\beta_p{}^q{}_t(v, w_2)$.

We will now show that we can always write $\alpha_p{}^q{}_t$ for $\alpha_p{}^q{}_t(u, w_1)$ and $\beta_p{}^q{}_t$ for $\beta_p{}^q{}_t(v, w_2)$.

ie: $\alpha_p{}^q{}_t(u, w_1)$ depends on q but is independent of our choice of u and w_1 and

$\beta_p{}^q{}_t(v, w_2)$ depends on q but is independent of our choice of v and w_2 .

Theorem 2.1.

Let Γ be a distance-biregular graph such that $V\Gamma = P \cup B$. Choose a vertex $u \in P$ and a vertex $w \in V\Gamma$ such that $d(u,w) = q$. Then $\alpha_p{}^q{}_t(u,w)$ is independent of u and w, $0 \leqslant p,q,t \leqslant d$.

Proof

Let S_p represent the statement:

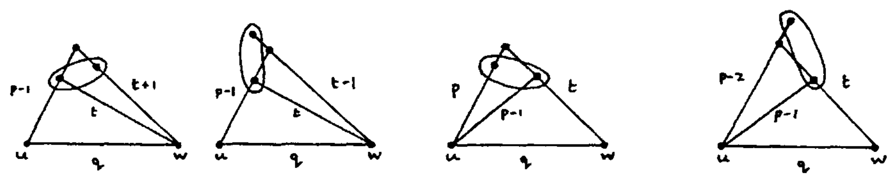
"$\alpha_p{}^q{}_t(u,w)$ is independent of the choice of u and w, for all q and t".

(i) S_0 and S_1 are clearly true since we have mentioned earlier that $\alpha_0{}^q{}_t(u,w)$ and $\alpha_1{}^q{}_t(u,w)$ may be written as $\alpha_0{}^q{}_t$ and $\alpha_1{}^q{}_t$ respectively, $0 \leqslant q,t \leqslant d$.

(ii) Suppose S_n is true for all $n \leqslant p-1$. We will form a recurrence relation to show that this implies S_p is true. We have two cases to consider and in both cases we count 'quadrilaterals' in a certain way.

(I) q even. In this case $w \in P$.

$$c_{t+1}\alpha_{p-1}{}^q{}_{t+1} + b_{t-1}\alpha_{p-1}{}^q{}_{t-1} = c_p \alpha_p{}^q{}_t(u,w) + b_{p-2}\alpha_{p-2}{}^q{}_t$$



Therefore $\alpha_p{}^q{}_t(u,w)$ may be written as

$$(c_{t+1}\alpha_{p-1}{}^q{}_{t+1} + b_{t-1}\alpha_{p-1}{}^q{}_{t-1} - b_{p-2}\alpha_{p-2}{}^q{}_t)/c_p$$

ie: $\alpha_p{}^q{}_t(u,w)$ is independent of our choice of u and w and we may write $\alpha_p{}^q{}_t(u,w) = \alpha_p{}^q{}_t$.

(II) q odd. In this case $w \in B$.

$$c'_{t+1}\alpha_{p-1}{}^q{}_{t+1} + b'_{t-1}\alpha_{p-1}{}^q{}_{t-1} = c_p \alpha_p{}^q{}_t(u,w) + b_{p-2}\alpha_{p-2}{}^q{}_t$$

Therefore $\alpha_p{}^q{}_t(u,w)$ may be written as

$$(c'_{t+1}\alpha_{p-1}{}^q{}_{t+1} + b'_{t-1}\alpha_{p-1}{}^q{}_{t-1} - b_{p-2}\alpha_{p-2}{}^q{}_t)/c_p \quad .$$

ie: $\alpha_p{}^q{}_t(u,w)$ is independent of our choice of u and w and we may write

$$\alpha_p{}^q{}_t(u,w) = \alpha_p{}^q{}_t \quad .$$

ie: In both cases S_p is true. Hence, by induction, we may always write $\alpha_p{}^q{}_t$ in place of $\alpha_p{}^q{}_t(u,w)$.

Theorem 2.2

Let Γ be a distance-biregular graph such that $V\Gamma = P \cup B$. Choose a vertex $v \in B$ and a vertex $w \in V\Gamma$ such that $d(v,w) = q$. Then $\beta_p{}^q{}_t(v,w)$ is independent of v and w, $0 \leqslant p,q,t \leqslant d$.

Proof

Similar to the previous method.

Next we give several simple necessary conditions on the intersection arrays of a distance-biregular graph.

Proposition 2.3[5]

We have the following relationships for a distance-biregular graph:

(i) $k_0 = 1$, and for each $i > 0$, $k_{i+1} = k_i b_i / c_i$;

$k_0' = 1$, and for each $i > 0$, $k_{i+1}' = k_i' b_i' / c_i'$.

The k_i and k_i' are whole numbers.

(ii) $c_i + b_i = \begin{cases} b_0, & \text{if } i \text{ is even,} \\ b_0', & \text{if } i \text{ is odd.} \end{cases}$

$c_i' + b_i' = \begin{cases} b_0, & \text{if } i \text{ is odd,} \\ b_0', & \text{if } i \text{ is even.} \end{cases}$

(iii) $b_{i-1}' \geqslant b_i$ and $b_{i-1} \geqslant b_i'$, $i = 1, \ldots, d-1$.

(iv) $c_{i+1}' \geqslant c_i$ and $c_{i+1} \geqslant c_i'$, $i = 1, \ldots, d-1$.

(v) The following inequalities hold:

$$|P| := 1 + k_2 + k_4 + \ldots + k_{d'} = k_1' + k_3' + \ldots + k_{d''}',$$
$$|B| := k_1 + k_3 + \ldots + k_{d''} = 1 + k_2' + k_4' + \ldots + k_{d'}',$$

where d' is the largest even integer less than or equal to d, and d'' is the largest odd integer less than or equal to d.

We also have $|P| . b_0 = |B| . b_0'$.

Proof

Let u,v be arbitrary vertices in P, B respectively.

(i) The first two expressions follow by counting edges between $\Gamma_i(u)$ and $\Gamma_{i+1}(u)$. The second two expressions follow by counting edges between $\Gamma_i(v)$ and $\Gamma_{i+1}(v)$.

(ii) A vertex in $\Gamma_i(u)$ (respectively $\Gamma_i(v)$) has degree b_0 (respectively b_0') if i is even and b_0' (respectively b_0) if i is odd.

(iii) Let u be adjacent to v. Choose a vertex $x \in \Gamma_i(u) \cap \Gamma_{i-1}(v)$. Then the b_i neighbours of x in $\Gamma_{i+1}(u)$ lie in $\Gamma_{i-1}(v)$, so $b_{i-1}' \geqslant b_i$. By symmetry $b_{i-1} \geqslant b_i'$.

(iv) With u and v as in (iii) we can choose a vertex $x \in \Gamma_i(u) \cap \Gamma_{i+1}(v)$. The c_i neighbours of x in $\Gamma_{i-1}(u)$ then lie in $\Gamma_i(v)$, so $c_{i+1}' \geqslant c_i$. By symmetry $c_{i+1} \geqslant c_i'$.

(v) Clearly
$$P = \{u\} \cup \Gamma_2(u) \cup \Gamma_4(u) \cup \ldots \cup \Gamma_{d'}(u)$$
$$= \Gamma_1(u) \cup \Gamma_3(u) \cup \ldots \cup \Gamma_{d''}(u)$$
and
$$B = \Gamma_1(v) \cup \Gamma_3(v) \cup \ldots \cup \Gamma_{d''}(v)$$
$$= \{v\} \cup \Gamma_2(v) \cup \Gamma_4(v) \cup \ldots \cup \Gamma_{d'}(v).$$

Now count the number of edges joining P to B in two ways to get

$$|P|\ b_0 = |B|\ b_0' .$$

We now turn our attention to the derived (or halved) graphs of a distance-biregular graph.

Proposition 2.4. [5]

Let Γ be a distance-biregular graph. Then the derived graphs of Γ are distance regular.

Proof

Let the derived graph on the vertex set of P be denoted by P. Let $u \in P$ and consider $P_j(u) = \Gamma_{2j}(u)$. Take $w \in P_j(u)$ and let

$$a_j^* := |P_j(u) \cap P_1(w)|$$

$$c_j^* := |P_{j-1}(u) \cap P_1(w)|$$

$$\text{and} \quad b_j^* := |P_{j+1}(u) \cap P_1(w)| .$$

Then $a_j^* = \alpha_{2j}{}^{2j}{}_2$, $c_j^* = \alpha_{2j-2}{}^{2j}{}_2$, and $b_j^* = \alpha_{2j+2}{}^{2j}{}_2$.

ie: The derived graph on the vertex set of P is distance-regular.

Similarly for the derived graph on the vertex set of B.

Lemma 2.5[5]

(a) $c_{2i}\ c_{2i+1} = c_{2i}'\ c_{2i+1}'$, $i \geqslant 1$.

(b) $b_{2i-1}\ b_{2i} = b_{2i-1}'\ b_{2i}'$, $i \geqslant 1$.

Proof

(a) Let us consider a distance-biregular graph Γ . We will count $\beta_2{}^{2i+1}{}_{2i-1}$ in two ways

(i) $\beta_2{}^{2i+1}{}_{2i-1} = \alpha_{2i-1}{}^{2i+1}{}_2 = c_{2i+1}\, c_{2i} / c_2'$.

$$= \frac{c_{2i+1}' \; c_{2i}' \; \ldots \; c_3'}{c_{2i-1}' \; \ldots \; c_2'} = \frac{c_{2i+1}' \; c_{2i}'}{c_2'}$$

Therefore $c_{2i}\, c_{2i+1} = c_{2i}'\; c_{2i+1}'$.

(b) Let the number of vertices in P and B be n and m respectively. The number of edges in Γ is given by:

$$n.b_0 = m.b_0'.$$

We now proceed by induction.

(i) The number of pairs of vertices at distance 3 is

$$\frac{n.b_0.b_1.b_2}{c_2.c_3} \quad \text{(for paths starting in P)}$$

$$= \frac{m.b_0'.b_1'.b_2'}{c_2'.c_3'} \quad \text{(for paths starting in B)}$$

Therefore $b_1 b_2 = b_1' b_2'$.

(ii) Assume the result for pairs of vertices at distance up to 2i+1. Counting paths joining pairs of vertices at distance 2i+3 we have:

$$\frac{n.b_0.b_1 \; \ldots \; b_{2i+1}.b_{2i+2}}{c_2.c_3 \; \ldots \; c_{2i+2}.c_{2i+3}} \quad \text{(for paths starting in P)}$$

$$= \frac{m.b_0'.b_1' \; \ldots \; b_{2i+1}'.b_{2i+2}'}{c_2'.c_3' \; \ldots \; c_{2i+2}'.c_{2i+3}'} \quad \text{(for paths starting in B)}$$

Hence by (a) and the inductive hypothesis

$$b_{2i+1} b_{2i+2} = b_{2i+1}' \; b_{2i+2}' \; .$$

Lemma 2.6.[5]

A distance-biregular graph cannot be regular.

Proof.

We suppose $b_0 = b_0'$ and show that the two arrays $\iota(P)$, $\iota(B)$

would have to be identical.

Let Γ be a regular distance-biregular graph. Then $b_0 = b'_0$ and the first two columns in each array are the same. Suppose that the arrays are equal up to the (2j - 1)-th. column for some $j \geqslant 1$. Now $b_{2i}\, b_{2i-1} = b'_{2i}\, b'_{2i-1}$ for every i. Thus $b_{2j-1} = b'_{2j-1}$ implies $b_{2j} = b'_{2j}$ and hence (as $c_{2j} + b_{2j} = b_0 = b'_0 = c'_{2j} + b'_{2j}$) $c_{2j} = c'_{2j}$. We also know that $c_{2i+1} c_{2i} = c'_{2i+1} c'_{2i}$ for every i, so $c_{2j} = c'_{2j}$ implies $c_{2j+1} = c'_{2j+1}$ and (since $b_0 = b'_0$) we also have $b_{2j+1} = b'_{2j+1}$. Hence the two arrays are identical and the graph is distance-regular.

We will assume from now on that we have a (non-regular) distance-biregular graph with $b_0 > b'_0$.

Chapter 3

This chapter is divided into two parts. The first part deals with the general case of when a pair of arrays are feasible for a distance-biregular graph Γ. The second part then deals mainly with the case when Γ is a distance-biregular graph of girth 4. In [5] it is shown, by considering eigenvalues, that the diameter of a non-regular distance-biregular graph is even so we will use this fact in the later parts of this chapter.

In this chapter we will suppose that Γ is a distance-biregular graph with the same notation as used in the previous chapters. We will continue to find necessary conditions for a pair of arrays to correspond to a distance-biregular graph.

Lemma 3.1.

We have the following necessary integrality conditions:

(a) (i) c_2 divides $c_{2i-1}c_{2i}$ and $c_{2i}\, c_{2i+1}$.

(ii) c_2' divides $c_{2i-1}'\, c_{2i}'$ and $c_{2i}'\, c_{2i+1}'$.

(b) (i) c_2 divides $b_{2i-1}b_{2i}$ and $b_{2i}\, b_{2i+1}$.

(ii) c_2' divides $b_{2i-1}'\, b_{2i}'$ and $b_{2i}'\, b_{2i+1}'$.

Proof

(a) Both (i) and (ii) follow from Proposition 2.4 and Lemma 2.5. For example for (i) we consider $c_i^* = c_{2i-1}c_{2i}/c_2$ and

$$\alpha_{2\;2i-1}^{2i+1} = \frac{c_{2i+1}c_{2i}\;\cdots\;c_3}{c_{2i-1}\;\cdots\;c_1} = \frac{c_{2i+1}c_{2i}}{c_2}\;.$$

(b) (i) This follows from $b_i^* = b_{2i}\, b_{2i+1}/c_2$ and

$$\beta_{2i+1\;2}^{2i-1} = b_{2i-1}'\; b_{2i}'\;/c_2 = b_{2i}\, b_{2i-1}/c_2\;.$$

(ii) is similar.

The first part of the following lemma will be proved in two ways. The first method uses Lemma 2.5 and Proposition 2.3 and the second shows the use of an intersection diagram.

Lemma 3.2

The following conditions must be satisfied.

(a) (i) $b_{2i-1}(c_{2i} - c_{2i-1}') = b_{2i-1}'\,(c_{2i}' - c_{2i-1});$

(ii) $b_{2i}(c_{2i}' - c_{2i-1}) = b_{2i}'(c_{2i} - c_{2i-1}');$

(b) (i) $c_{2i+1}(b_{2i} - b_{2i+1}') = c_{2i+1}'(b_{2i}' - b_{2i+1});$

(ii) $c_{2i}(b_{2i}' - b_{2i+1}) = c_{2i}'(b_{2i} - b_{2i+1}').$

<u>Proof</u>

(a)
$$\begin{aligned} b_{2i-1}(c_{2i} - c_{2i-1}') &= b_{2i-1}(b_0 - b_{2i} - c_{2i-1}') \\ &= b_0 b_{2i-1} - b_{2i} b_{2i-1} - b_{2i-1} c_{2i-1}' \\ &= b_0 b_{2i-1} - b_{2i}' b_{2i-1}' - b_{2i-1} c_{2i-1}' \\ &= (b_0 - c_{2i-1}') b_{2i-1} - b_{2i}' b_{2i-1}' \\ &= b_{2i-1}'(b_{2i-1} - b_{2i}') \\ &= b_{2i-1}'(b_0' - c_{2i-1} - b_0' + c_{2i}') \\ &= b_{2i-1}'(c_{2i}' - c_{2i-1}). \end{aligned}$$

Hence (i) holds.

If $c_{2i} - c_{2i-1}' = 0$ we must have $c_{2i}' - c_{2i-1} = 0$ so (ii) clearly holds.

We may therefore suppose that $c_{2i} \neq c_{2i-1}'$.

We know that $b_{2i} b_{2i-1} = b_{2i}' b_{2i-1}'$ so we have

$$\frac{b_{2i-1}}{b_{2i-1}'} = \frac{c_{2i}' - c_{2i-1}}{c_{2i} - c_{2i-1}'} = \frac{b_{2i}'}{b_{2i}}$$

Whence (ii) follows.

(b) Similarly

$$\begin{aligned} c_{2i}(b_{2i}' - b_{2i+1}) &= c_{2i}(b_{2i}' - b_0' + c_{2i+1}) \\ &= -c_{2i} c_{2i}' + c_{2i}' c_{2i+1}' \\ &= c_{2i}'(c_{2i+1}' - c_{2i}) \\ &= c_{2i}'(b_0 - b_{2i+1}' - c_{2i}) \\ &= c_{2i}'(b_{2i} - b_{2i+1}'). \end{aligned}$$

Hence (ii) holds.

If $b_{2i}' - b_{2i+1} = 0$ we must have $b_{2i} - b_{2i+1}' = 0$ so (i) clearly holds. We may therefore suppose that $b_{2i}' \neq b_{2i+1}$. We know that $c_{2i}\, c_{2i+1} = c_{2i}'\, c_{2i+1}'$ so we have

$$\frac{c_{2i}}{c_{2i}'} = \frac{b_{2i} - b_{2i+1}'}{b_{2i}' - b_{2i+1}} = \frac{c_{2i+1}'}{c_{2i+1}}.$$

Whence (i) follows.

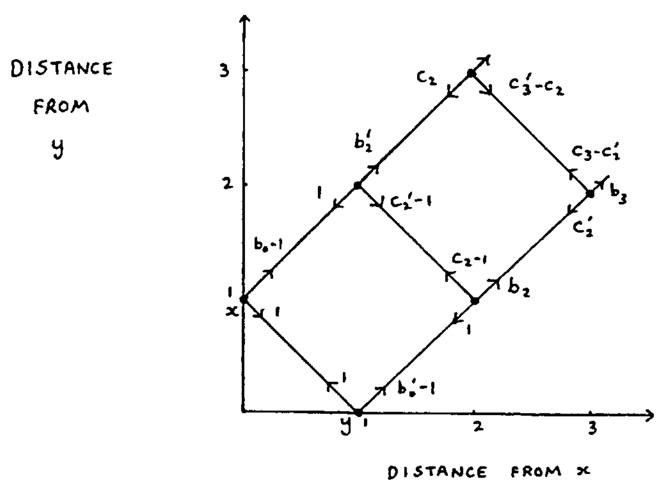
We now give an alternative proof of the first part of the previous lemma to show the use of an intersection diagram.

Let $x \in P$, $y \in B$ and write

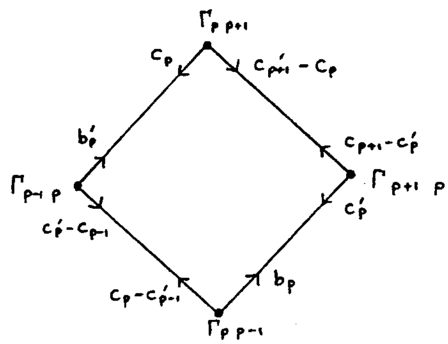
$$\Gamma_{pq} := \Gamma_p(x) \cap \Gamma_q(y), \qquad \text{where } \Gamma_p(z) := \{w : d(z,w) = p\}.$$

In the following intersection diagrams the sets Γ_{pq} are denoted by the black dots. Two dots are joined by a labelled edge in the intersection diagram if there are E edges (respectively F edges) from each vertex in Γ_{pq} to $\Gamma_{p'q'}$ ($\Gamma_{p'q'}$ to Γ_{pq} respectively).

The following intersection diagram summarises the graph structure relative to the edge x,y.



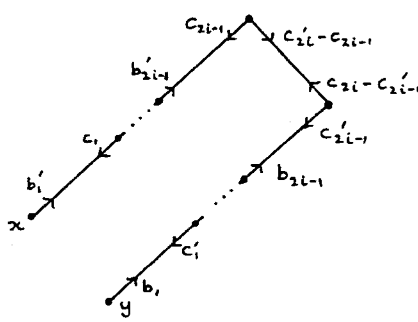
ie: For a general 'square'.



Alternative proof of Lemma 3.2 (a).

(a) If $c_{2i} = c'_{2i-1}$ then $b_0 - c_{2i} = b_0 - c'_{2i-1}$, ie: $b_{2i} = b'_{2i-1}$, and since $b_{2i-1}b_{2i} = b'_{2i-1}b'_{2i}$ we must have $b_{2i-1} = b'_{2i}$ and this gives us $c_{2i-1} = c'_{2i}$ so the results are trivially true. Let us suppose $c_{2i} \neq c'_{2i-1}$.

We count paths of length $4i - 1$ from x to y via $\Gamma_{2i-1\,2i}$ and $\Gamma_{2i\,2i-1}$ in two ways, first starting at x and then starting at y.



From the diagram we obtain

$$b_1'b_2' \dots b_{2i-1}'(c_{2i}' - c_{2i-1})c_{2i-1}'c_{2i-2}' \dots c_3'c_2'$$

$$= b_1b_2 \dots b_{2i-1}(c_{2i} - c_{2i-1}')c_{2i-1}c_{2i-2} \dots c_3c_2$$

ie: $b_{2i-1}'(c_{2i}' - c_{2i-1}) = b_{2i-1}(c_{2i} - c_{2i-1}')$.

We also have $b_{2i-1}'b_{2i}' = b_{2i-1}b_{2i}$ so $b_{2i}(c_{2i}' - c_{2i-1}) = b_{2i}'(c_{2i} - c_{2i-1}')$

Shawe-Taylor in [5] p.34 proves the following feasibility condition.

" Suppose the intersection matrix I(A) corresponds to a distance-biregular graph Γ. Let $\underline{x}$ be the right eigenvector of I(A) corresponding to the non-zero eigenvalue λ and satisfying $x_0 = 1$. Then the coordinates of $\underline{x}$ must satisfy

$$\sum_{i \text{ even}} x_i^2 / k_i = \sum_{i \text{ odd}} x_i^2 / k_i. \quad "$$

We show in the following that this condition automatically holds whenever a matrix has the form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 \\ 0 & m_{32} & 0 & m_{34} \\ 0 & 0 & m_{43} & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 & 0 \\ 0 & m_{32} & 0 & m_{34} & 0 \\ 0 & 0 & m_{43} & 0 & m_{45} \\ 0 & 0 & 0 & m_{54} & 0 \end{bmatrix}$$

The k_i are defined in the following Theorem.

This means that when we consider distance-biregular graphs where $d \leqslant 4$ we need not check that this result holds as the condition is automatically satified.

Theorem 3.3.

Let

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 \\ 0 & m_{32} & 0 & m_{34} \\ 0 & 0 & m_{43} & 0 \end{bmatrix}, \text{ where } m_{21}, m_{23}, m_{32}, m_{34}, m_{43} \neq 0.$$

We define

$$k_0 = 1; \; k_1 = m_{21}; \; k_2 = m_{21} m_{32}/m_{23}$$

$$\text{and } k_3 = (m_{21} m_{32} m_{43})/(m_{23} m_{34}).$$

If we let $\underline{x} = (1 \;\; x_1 \;\; x_2 \;\; x_3)^T$ be the right eigenvector of M associated with the eigenvalue $\lambda\ (\neq 0)$ then we have:

$$\frac{x_0^2}{k_0} + \frac{x_2^2}{k_2} = \frac{x_1^2}{k_1} + \frac{x_3^2}{k_3}$$

ie: $$\sum_{i \text{ even}} x_i^2/k_i = \sum_{i \text{ odd}} x_i^2/k_i.$$

Proof.

Consider the entries in $\underline{x}$. Since $\underline{x}$ is the right eigenvector of M with associated eigenvalue λ we have:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 \\ 0 & m_{32} & 0 & m_{34} \\ 0 & 0 & m_{43} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This gives us: $x_1 = \lambda$; $\quad x_2 = \dfrac{(\lambda x_1 - m_{21})}{m_{23}} = \dfrac{(\lambda^2 - m_{21})}{m_{23}}$

$$\text{and } x_3 = \frac{m_{43}\, x_2}{\lambda} = \frac{m_{43}(\lambda^2 - m_{21})}{m_{23}\lambda}$$

$$\text{or } x_3 = \frac{\lambda x_2 - m_{32}\, x_1}{m_{34}} = \frac{\lambda(\lambda^2 - m_{21}) - m_{32}\, m_{23}\lambda}{m_{23}\, m_{34}} .$$

Equating the two expressions for x_3 means that the eigenvalues of A are precisely the roots of the following quartic.

$$\lambda^4 - (m_{34}\, m_{43} + m_{21} + m_{23} m_{32})\lambda^2 + m_{21}\, m_{34}\, m_{43} = 0. \qquad (*)$$

Consider the values of λ needed for

$$x_0^2/k_0 + x_2^2/k_2 = x_1^2/k_1 + x_3^2/k_3 \qquad (**)$$

We need $$1 + \frac{(\lambda^2 - m_{21})^2}{m_{21}\, m_{23}\, m_{32}} = \frac{\lambda^2}{m_{21}} + \frac{m_{43}(\lambda^2 - m_{21})^2 m_{34}}{\lambda^2 m_{21}\, m_{23}\, m_{32}}$$

ie: $$(\lambda^2 - m_{21}) = \frac{(\lambda^2 - m_{21})^2}{m_{32}\, m_{23}} - \frac{m_{43}(\lambda^2 - m_{21})^2 m_{34}}{\lambda^2 m_{23}\, m_{32}}$$

Now, if $\lambda^2 - m_{21} = 0$ then $x_2 = x_3 = 0$. The second expression for x_3 then gives us $m_{23}\, m_{32}\lambda = 0$ which is not possible since m_{23}, m_{32} and $\lambda \neq 0$.

Therefore, since $\lambda^2 \neq m_{21}$, we have:

$$\lambda^2 m_{23} m_{32} = \lambda^2(\lambda^2 - m_{21}) - m_{34} m_{43}(\lambda^2 - m_{21})$$

$$\text{ie: } \lambda^4 - (m_{34} m_{43} + m_{21} + m_{23} m_{32})\lambda^2 + m_{21} m_{34} m_{43} = 0.$$

Therefore the eigenvalues of M are precisely the values of λ for which (**) holds.

Theorem 3.4.

Let

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 & 0 \\ 0 & m_{32} & 0 & m_{34} & 0 \\ 0 & 0 & m_{43} & 0 & m_{45} \\ 0 & 0 & 0 & m_{54} & 0 \end{bmatrix}$$ where $m_{21}, m_{23}, m_{32}, m_{34}, m_{43}, m_{45}, m_{54} \neq 0$.

We define $k_0 = 1$; $k_1 = m_{21}$; $k_2 = m_{21} m_{32}/m_{23}$;

$k_3 = (m_{21} m_{32} m_{43})/m_{23} m_{34}$ and $k_4 = (m_{21} m_{32} m_{43} m_{54})/(m_{23} m_{34} m_{45})$.

If we let $\underline{x} = (1 \quad x_1 \quad x_2 \quad x_3 \quad x_4)^T$ be the right eigenvalue of M associated with the eigenvalue $\lambda(\neq 0)$ then we have:

$$\frac{x_0^2}{k_0} + \frac{x_2^2}{k_2} + \frac{x_4^2}{k_4} = \frac{x_1^2}{k_1} + \frac{x_3^2}{k_3}$$

$$\text{ie: } \sum_{i \text{ even}} x_i^2/k_i = \sum_{i \text{ odd}} x_i^2/k_i .$$

Proof.

Consider the entries in $\underline{x}$. Since $\underline{x}$ is the right eigenvector of M associated with the eigenvalue λ we have:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ m_{21} & 0 & m_{23} & 0 & 0 \\ 0 & m_{32} & 0 & m_{34} & 0 \\ 0 & 0 & m_{43} & 0 & m_{45} \\ 0 & 0 & 0 & m_{54} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

This gives us: $x_1 = \lambda$; $x_2 = \dfrac{(\lambda x_1 - m_{21})}{m_{23}} = \dfrac{(\lambda^2 - m_{21})}{m_{23}}$;

$$x_3 = \frac{\lambda x_2 - m_{32} x_1}{m_{34}} = \frac{\lambda(\lambda^2 - m_{21}) - m_{32} m_{23} \lambda}{m_{23} m_{34}}$$

$$= \frac{\lambda(\lambda^2 - (m_{21} + m_{23} m_{32}))}{m_{23} m_{34}}$$

$$x_4 = \frac{m_{54} x_3}{\lambda} = \frac{m_{54}(\lambda^2 - (m_{21} + m_{23} m_{32}))}{m_{23} m_{34}}$$

or $$x_4 = \frac{\lambda x_3 - m_{43} x_2}{m_{45}} = \frac{\lambda^2(\lambda^2 - (m_{21} + m_{23} m_{32})) - m_{34} m_{43} (\lambda^2 - m_{21})}{m_{23} m_{34} m_{45}}$$

Equating the two expressions for x_4 means that the eigenvalues of M are precisely the roots of the following quartic.

$$\lambda^4 - (m_{21} + m_{23} m_{32} + m_{34} m_{43} + m_{45} m_{54})\lambda^2$$
$$+ (m_{21} m_{34} m_{43} + m_{21} m_{45} m_{54} + m_{23} m_{32} m_{45} m_{54}) = 0 \qquad (*)$$

Consider the values of λ needed for

$$\frac{x_0^2}{k_0} + \frac{x_2^2}{k_2} + \frac{x_4^2}{k_4} = \frac{x_1^2}{k_1} + \frac{x_3^2}{k_3} \qquad (**)$$

ie: $$1 + \frac{(\lambda^2 - m_{21})^2}{m_{21} m_{23} m_{32}} + \frac{m_{45} m_{54} (\lambda^2 - (m_{21} + m_{23} m_{32}))^2}{m_{21} m_{23} m_{22} m_{34} m_{43}}$$

$$= \frac{\lambda^2}{m_{21}} + \frac{\lambda^2(\lambda^2 - (m_{21} + m_{23} m_{32}))^2}{m_{21} m_{23} m_{32} m_{34} m_{43}}$$

$$\Rightarrow m_{21} m_{23} m_{32} m_{34} m_{43} + m_{34} m_{43} (\lambda^2 - m_{21})^2 + m_{45} m_{54} (\lambda^2 - (m_{21} + m_{23} m_{32}))^2$$
$$= \lambda^2 m_{23} m_{32} m_{34} m_{43} + \lambda^2(\lambda^2 - (m_{21} + m_{23} m_{32}))^2$$

$$\Rightarrow (\lambda^2 - (m_{21} + m_{23} m_{32}))^2(\lambda^2 - m_{45} m_{54})$$
$$= m_{34} m_{43} ((\lambda^2 - m_{21})^2 - \lambda^2 m_{23} m_{32} + m_{21} m_{23} m_{32})$$
$$= m_{34} m_{43} (\lambda^4 - (2m_{21} + m_{23} m_{32})\lambda^2 + (m_{21}^2 + m_{21} m_{23} m_{32}))$$
$$= m_{34} m_{43} (\lambda^2 - (m_{21} + m_{23} m_{32}))(\lambda^2 - m_{21}).$$

Now, if $\lambda^2 = m_{21} + m_{23} m_{22}$ then $x_4 = x_3 = 0$ and, since we also have $m_{43} x_2 + m_{45} x_4 = \lambda x_3$, $x_2 = 0$. Therefore $\lambda^2 = m_{21}$ and $m_{23} m_{32} = 0$

which is a contradiction since $m_{23}m_{32} \neq 0$.

Therefore, since $\lambda^2 \neq m_{21} + m_{23}m_{32}$, we have

$$(\lambda^2 - (m_{21} + m_{23}m_{32}))(\lambda^2 - m_{45}m_{54}) = m_{34}m_{43}(\lambda^2 - m_{21})$$

$$\Rightarrow \lambda^4 - (m_{21} + m_{23}m_{32} + m_{34}m_{43} + m_{45}m_{54})\lambda^2 + (m_{21}m_{34}m_{43} + m_{21}m_{45}m_{54} + m_{23}m_{32}m_{45}m_{54}) = 0.$$

Therefore the eigenvalues of M are precisely the values of λ for which (**) holds.

We will now find some inequalities involving the size of the blocks $\Gamma_i(u)$, where $u \in P$.

Lemma 3.5.

Let Γ be a distance-biregular graph of diameter d. Then

$$\frac{c_{2i}(b_{2i-1} - 1) + b_{2i}(c_{2i+1} - 1)}{c_2} \leqslant k_{2i} - 1$$

for $2 \leqslant 2i < d$

$$\frac{c_{2i-1}(b_{2i-2} - 1) + b_{2i-1}(c_{2i} - 1)}{c_2'} \leqslant k_{2i-1} - 1$$

for $3 \leqslant 2i - 1 < d$.

Proof.

Take two vertices $u, u' \in P$ such that $d(u, u') = 2i$. Since $u' \in \Gamma_{2i}(u)$ there are c_{2i} vertices in $\Gamma_{2i-1}(u)$ each of which is connected to u' and $(b_{2i-1} - 1)$ other vertices in $\Gamma_{2i}(u)$. There are also b_{2i} vertices in $\Gamma_{2i+1}(u)$, each of which is connected to u' and $(c_{2i+1} - 1)$ other vertices in $\Gamma_{2i}(u)$. The vertices other than u' are each counted c_2 times so by considering the number of vertices in $\Gamma_{2i}(u)$ we have our first result. The second inequality follows similarly.

Note that if 2i = d we have

$$\frac{c_{2i}\,(b_{2i-1}-1)}{c_2} \leqslant k_{2i}-1$$

and if d = 2i - 1 we have

$$\frac{c_{2i-1}(b_{2i-2}-1)}{c_2'} \leqslant k_{2i-1}-1 .$$

By the same method as the above we also have a similar result if we consider vertices (v, v') in B.

The following was proved independently in the preprint [1] which was never published.

Lemma 3.6.

Let Γ be a distance-biregular graph with the usual notation for its intersection arrays. Then the following inequalities hold.

$$c'_{2i} > \frac{b'_0}{b_0}\, c_{2i} \qquad \text{and} \qquad c_{2i-1} > \frac{b'_0}{b_0}\, c'_{2i-1}$$

$$b_{2i} > \frac{b_0}{b'_0}\, b'_{2i} \qquad\qquad b'_{2i-1} > \frac{b_0}{b'_0}\, b_{2i-1}$$

$$i = 1,\dots,d/2-1 \qquad\qquad i = 1,\dots,d/2$$

Proof.

We will actually show by induction that we have

$$\frac{b'_{2i}}{b_{2i}} < \frac{b'_0}{b_0} < \frac{c'_{2i}}{c_{2i}} \qquad \text{and} \qquad , \frac{b_{2i-1}}{b'_{2i-1}} < \frac{b'_0}{b_0} < \frac{c_{2i-1}}{c'_{2i-1}}$$

(i) Since $c_1 = c'_1 = 1$ and $b_1 b_2 = b'_1\, b'_2$, from Lemma 2.5, we have

$$\frac{b_1}{b_1'} = \frac{(b_0' - 1)}{(b_0 - 1)} < \frac{b_0'}{b_0} < \frac{1}{1} = \frac{c_1}{c_1'}, \quad \text{and} \quad \frac{b_1}{b_1'} = \frac{b_2'}{b_2} < \frac{b_0'}{b_0}.$$

$$\therefore b_2' = b_0' - c_2' < \frac{b_0'}{b_0}(b_0 - c_2) = b_0' - c_2 \frac{b_0'}{b_0} \Longrightarrow \frac{b_0'}{b_0} < \frac{c_2'}{c_2}$$

$$\text{ie: } \frac{b_2'}{b_2} < \frac{b_0'}{b_0} < \frac{c_2'}{c_2}.$$

(ii) Suppose the results are true for all terms less than or equal to 2i.

Then, by Lemma 2.5, $c_{2i}' c_{2i+1}' = c_{2i} c_{2i+1}$, so by our induction hypothesis $\frac{c_{2i}'}{c_{2i}} = \frac{c_{2i+1}}{c_{2i+1}'} > \frac{b_0'}{b_0}$

This means that $b_0' - b_{2i+1} = c_{2i+1} > \frac{b_0'}{b_0} c_{2i+1}' = \frac{b_0'}{b_0}(b_0 - b_{2i+1}')$

and this gives us $\frac{b_{2i+1}}{b_{2i+1}'} < \frac{b_0'}{b_0} < \frac{c_{2i+1}}{c_{2i+1}'}$

Now, since $b_{2i+1} b_{2i+2} = b_{2i+1}' b_{2i+2}'$ we have $\frac{b_{2i+1}}{b_{2i+1}'} = \frac{b_{2i+2}'}{b_{2i+2}} < \frac{b_0'}{b_0}$

Therefore $b_0' - c_{2i+2}' < \frac{b_0'}{b_0}(b_0 - c_{2i+2})$ so $\frac{b_{2i+2}'}{b_{2i+2}} < \frac{b_0'}{b_0} < \frac{c_{2i+2}'}{c_{2i+2}}$

This gives us the required result.

As a consequence of this we see that d_B is always at least as big as d_P , and since d is even, d_B is even.

<u>Corollary 3.7.</u>

$d_P = 2i \Longrightarrow d_B = 2i$.

<u>Proof.</u>

Since the diameter of the graph is even we know that $d_B = 2i$

or 2i-1. Suppose $d_B = 2i-1$.

Then $c_{2i-1} > \frac{b_0'}{b_0} c_{2i-1}' = \frac{b_0'}{b_0} b_0 = b_0'$. #

Therefore $d_p = 2i \Longrightarrow d_B = 2i$.

Note that this means that $d_B = d$.

The fact that $k_1 = b_0 > b_0' = k_1'$ is part of our basic hypothesis. The following shows that this is the simplest case of a more general result.

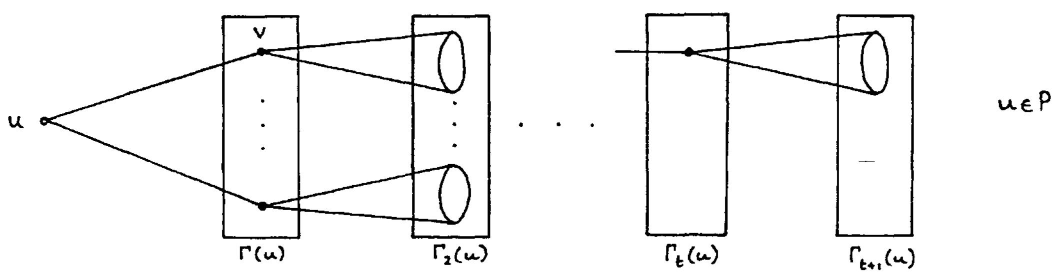
Lemma 3.8.

Let Γ be a distance-biregular graph. Then

$$k_1 + k_2 + \dots + k_{2i+1} > k_1' + k_2' + \dots + k_{2i+1}'$$
$$0 \leqslant i \leqslant d/2 - 1.$$

Proof.

Consider the following diagram.



We will consider the sum $k_1 + k_2 + \dots + k_t$. By Proposition 2.3 we know that for $i \geqslant 2$

$$k_i = \frac{b_0 b_1 \dots b_{i-1}}{c_2 \dots c_i} \quad \text{so}$$

$$k_1 + k_2 + \dots + k_t = b_0 + \frac{b_0 b_1}{c_2} + \dots + \frac{b_0 b_1 \dots b_{t-1}}{c_2 \dots c_t}.$$

Now consider a vertex $v \in B \cap \Gamma(u)$ and the sum $k'_1 + \ldots + k'_t$. This sum represents the number of vertices at distance at most t from v. Since d(u, v) = 1 all the vertices at distance (t - 1) or less from u have to be at distance at most t from v. Also, for a vertex to be at most distance t from v it must be at most distance (t + 1) from u. This leads us to

$$k'_1 + \ldots + k'_t = k_1 + \ldots + k_{t-1} + \frac{b_1 b_2 \ldots b_{t-1}}{c'_2 \ldots c'_{t-1}} + \frac{b_1 b_2 \ldots b_t}{c'_2 \ldots c'_t}$$

Therefore

$$\sum_{\alpha=1}^{t} k_\alpha - \sum_{\alpha=1}^{t} k'_\alpha = k_t - \left(\frac{b_1 b_2 \ldots b_{t-1}}{c'_2 \ldots c'_{t-1}} + \frac{b_1 b_2 \ldots b_t}{c'_2 \ldots c'_t} \right)$$

$$= \frac{b_0 b_1 \ldots b_{t-1}}{c_2 \ldots c_t} - \frac{b_1 b_2 \ldots b_{t-1}}{c'_2 \ldots c'_{t-1}} \left(1 + \frac{b_t}{c'_t}\right)$$

Let t = 2i + 1. This gives us

$$\sum_{\alpha=1}^{t} k_\alpha - \sum_{\alpha=1}^{t} k'_\alpha = \frac{b_0 b_1 \ldots b_{t-1}}{c_2 \ldots c_t} - \frac{b_1 b_2 \ldots b_{t-1}(c'_t + b_t)}{c'_2 \ldots c'_{t-1} c'_t}$$

$$= \frac{b_1 \ldots b_{t-1}}{c_2 \ldots c_t} (b_0 - c'_t - b_t)$$

(by Lemma 2.5)

$$= \frac{b_1 \ldots b_{t-1}}{c_2 \ldots c_t} (b'_t - b_t) > 0$$

(since $b'_{2i+1} > \frac{b_1 b_{2i+1}}{b_0} > b_{2i+1}$).

The same proof shows that $k'_1 + \ldots + k'_{2i} > k_1 + \ldots + k_{2i}$ provided $c'_{2i} < c_{2i}$. We will show in Lemma 3.15 that this holds for all i if Γ is of girth 4.

We know from Lemma 3.6 that $c_2' > b_0' c_2 / b_0$ and we will now find another inequality which will restrict c_2' and c_2 further.

Lemma 3.9.

If Γ is a distance-biregular graph then $c_2 \geqslant c_2'$ with $c_2 = c_2'$ if and only if $c_2 = c_2' = 1$ (ie: Γ is not of girth 4).

Proof.

$b_1 b_2 = b_1' b_2'$ from Lemma 2.5 so $(b_0' - 1)b_2 = (b_0 - 1)b_2'$.

Therefore $c_2 = b_0 - b_2 = b_0 - \dfrac{(b_0 - 1)b_2'}{(b_0' - 1)}$

This gives us $c_2 = (b_0 - b_0') + b_0' - \dfrac{(b_0 - 1)b_2'}{(b_0' - 1)}$

Since $b_2' \leqslant b_0' - 1$ we have

$$c_2 \geqslant \frac{(b_0 - b_0')b_2'}{(b_0' - 1)} + b_0' - \frac{(b_0 - 1)b_2'}{(b_0' - 1)}$$

$$= b_0' + \frac{b_2'(b_0 - b_0' - b_0 + 1)}{(b_0' - 1)}$$

$$= b_0' + \frac{b_2'(-b_0' + 1)}{(b_0' - 1)}$$

$$= b_0' - b_2' = c_2'.$$

Therefore $c_2 \geqslant c_2'$ with $c_2 = c_2'$ if and only if $b_2' = b_0' - 1$

ie: $c_2 = c_2' = 1$.

Combining our restrictions on c_2 and c_2' gives us

$$c_2 \geqslant c_2' > \frac{b_0'}{b_0} c_2$$

The following Lemma is a generalization of a result for distance-regular graphs. It will be used later to enable us to find a bound on the diameter of a distance-biregular graph of girth four which is easily seen to be an improvement on the bound $d \leqslant b_0' - c_2' + 2$ given in [5].

Lemma 3.10.

Let Γ be a distance-biregular graph with the usual intersection arrays. Then the following results hold.

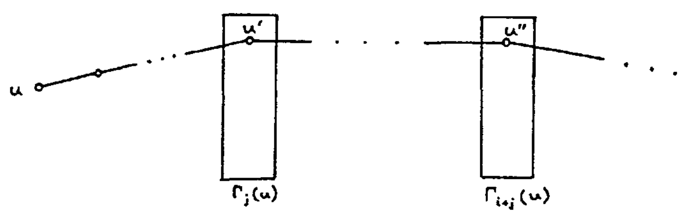
(a) $d_P \geqslant i + j \Longrightarrow \begin{cases} c_i \leqslant b_j & \text{if } i + j \text{ is even} \\ c_i' \leqslant b_j & \text{if } i + j \text{ is odd.} \end{cases}$

(b) $d_B \geqslant i + j \Longrightarrow \begin{cases} c_i' \leqslant b_j' & \text{if } i + j \text{ is even} \\ c_i \leqslant b_j' & \text{if } i + j \text{ is odd.} \end{cases}$

Proof.

(a) Suppose $i + j$ is even and $u \in P$. Take $u' \in \Gamma_j(u)$ and $u'' \in \Gamma_{i+j}(u) \cap \Gamma_i(u')$.

ie:



Then $c_i = |\Gamma(u') \cap \Gamma_{i-1}(u'')| \leqslant |\Gamma(u') \cap \Gamma_{j+1}(u)| = b_j$.

Now suppose that $i + j$ is odd and $u \in P$.

Then $c_i' = |\Gamma(u') \cap \Gamma_{i-1}(u'')| \leqslant |\Gamma(u') \cap \Gamma_{j+1}(u)| = b_j$.

Therefore $c_i \leqslant b_j$ if $i + j$ is even

and $c_i' \leqslant b_j$ if $i + j$ is odd.

(b) This follows similarly, just take $v \in B$.

As a simple consequence of this we have the following.

Corollary 3.11.

Let Γ be a distance-biregular graph and suppose that $c_2' > b_2'$. Then $d_B = 2$ and Γ is $K_{b_0,b_0'}$.

Proof.

Suppose that $c_2' > b_2'$. By Lemma 3.10 we have $d_B < 2 + 2 = 4$. Now d_B is even, by Corollary 3.7, so therefore $d_B = 2$ and hence the intersection arrays are

$$\begin{bmatrix} * & 1 & b_0' \\ b_0' & (b_0 - 1) & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & b_0 \\ b_0 & (b_0' - 1) & * \end{bmatrix}$$

This means that for any distance-biregular graph, other than $K_{b_0,b_0'}$, $c_2' \leqslant b_2'$.

ie: $c_2' \leqslant b_0' - c_2'$ so $c_2' \leqslant b_0'/2$

Since the only distance-biregular graphs of diameter two are $K_{b_0,b_0'}$ we will restrict ourselves to graphs of diameter greater than two.

Corollary 3.12.

Let Γ be a distance-biregular graph (other than $K_{b_0,b_0'}$). Then

$$c_2' \leqslant b_0'/2 \quad \text{and} \quad c_2 < b_0/2$$

Proof.

From above $c_2' \leqslant b_0'/2$ and since, by Lemma 3.6, $c_2' > b_0' c_2/b_0$ we have

$$\frac{b_0'}{2} \geqslant c_2' > \frac{b_0'}{b_0} c_2 \text{ . Therefore } c_2 < \frac{b_0}{2}.$$

From now on we will restrict ourselves to the case where the girth, g, is four.

The first result we consider is a stronger version of Proposition 2.3 (iv).

Proposition 3.13. [1]

Let Γ be a distance-biregular graph of girth 4. Then

$$c'_{i+1} > c_i \quad \text{and} \quad c_{i+1} > c'_i \qquad \text{for } 1 \leqslant i < d_P.$$

Proof.

We have girth 4 so we know from Lemma 3.9 that $c_2 > c'_2 \geqslant 2$.

Let $x \in P$ and $\{x_1, \ldots, x_{b_0}\}$ be its neighbours. Then any pair (x_i, x_j) have to have c'_2 common neighbours and be included in $(c'_2 - 1)$ 4-gons (with x). Let $y \in B$ and $\{y_1, \ldots, y_{b'_0}\}$ be its neighbours. Then any pair (y_i, y_j) will have to have c_2 common neighbours and be included in $(c_2 - 1)$ 4-gons (with y).

Let i be even and $u \in P$. Take $p \in B$ such that $p \in \Gamma_{i+1}(u)$ and let the c_{i+1} neighbours of p in $\Gamma_i(u)$ be $\{q_1, \ldots, q_{c_{i+1}}\}$. Also, let the c_i neighbours of q_1 in $\Gamma_{i-1}(u)$ be $\{r_1, \ldots, r_{c_i}\}$. Then the $c_i(c'_2 - 1)$ cycles of length 4 containing pq_1 and one of the edges q_1r_j are amongst the $(c_{i+1} - 1)(c_2 - 1)$ cycles of length 4 containing pq_1 and one of the edges pq_k. Hence

$$(c_{i+1} - 1)(c_2 - 1) \geqslant c_i(c'_2 - 1).$$

In the same way for a vertex $v \in B$ we have

$$(c'_{i+1} - 1)(c'_2 - 1) \geqslant c'_i(c_2 - 1).$$

Now, since $c_2 > c'_2 \geqslant 2$ we have

$$\frac{c_{i+1} - 1}{c_i} \geqslant \frac{c_2' - 1}{c_2 - 1} \geqslant \frac{c_i'}{c_{i+1}' - 1}$$

$$\text{ie: } (c_{i+1} - 1)(c_{i+1}' - 1) \geqslant c_i c_i' .$$

The case for i odd follows similarly.

We know from Proposition 2.3 (iv) that $c_{i+1}' \geqslant c_i$ and $c_{i+1} \geqslant c_i'$ but we will now show that we cannot have both of these inequalities as equalities.

Suppose $c_{i+1}' = c_i$ and $c_{i+1} = c_i'$. Then

$$c_i c_i' = c_{i+1}' c_{i+1} > (c_{i+1}' - 1)(c_{i+1} - 1) \geqslant c_i c_i' \quad \#$$

Suppose $c_{i+1} > c_i'$.

Consider two adjacent vertices u and v with $u \in P$ and take a point p such that $p \in \Gamma_{i+1}(u) \cap \Gamma_i(v)$. Then, since $c_{i+1} > c_i'$, there is a neighbour q of p such that $q \in \Gamma_i(u) \cap \Gamma_{i+1}(v)$. This means the neighbours of q in $\Gamma_i(v)$ cannot all be in $\Gamma_{i-1}(u)$.

Hence $c_{i+1}' > c_i$.

Similarly if $c_{i+1}' > c_i$ we have $c_{i+1} > c_i'$. Therefore we have our result.

As a simple consequence we have $b_{i-1}' > b_i$ and $b_{i-1} > b_i'$.

We will now generalise the result $c_2 > c_2'$ which we obtained from Lemma 3.9.

Lemma 3.14.

Let Γ be a distance-biregular graph of girth 4. Then

$$c_{2i} > c_{2i}' \quad 2 \leqslant 2i \leqslant d_p \; ; \quad c_{2i+1}' > c_{2i+1} \quad 2 \leqslant 2i \leqslant d_p - 1.$$

Proof.

We know from Lemma 3.2 that $b_{2i-1}(c_{2i} - c'_{2i-1}) = b'_{2i-1}(c'_{2i} - c_{2i-1})$ and from Lemma 3.6 $b'_{2i-1} > b_0 b_{2i-1}/b'_0$. Proposition 3.13 gives us the results $c_{2i} > c'_{2i-1}$ and $c'_{2i} > c_{2i-1}$ so we have

$$\frac{b_{2i-1}}{b'_{2i-1}} = \frac{c'_{2i} - c_{2i-1}}{c_{2i} - c'_{2i-1}} < \frac{b'_0}{b_0} < 1 .$$

Therefore $c_{2i} > c'_{2i} + (c'_{2i-1} - c_{2i-1})$. (*)

We now proceed by induction.

(i) Let $i = 1$. We know from Lemma 3.9 that $c_2 > c'_2$ so since $c_2 c_3 = c'_2 c'_3$ we also have $c'_3 > c_3$.

(ii) Suppose the result is true for all pairs up to $c_{2i-2} > c'_{2i-2}$ and $c'_{2i-1} > c_{2i-1}$. Then $c_{2i} > c'_{2i}$ by (*) and since $c_{2i} c_{2i+1} = c'_{2i} c'_{2i+1}$, $c'_{2i+1} > c_{2i+1}$.

Hence we have our result.

We are now in a position to prove the result stated after Lemma 3.8.

Lemma 3.15.

Let Γ be a distance-biregular graph of girth four. Then

$$k'_1 + k'_2 + \ldots + k'_{2i} > k_1 + k_2 + \ldots + k_{2i}$$

$$1 \leqslant i \leqslant d_p/2.$$

Proof.

We know from Lemma 3.8 that

$$\sum_{\alpha=1}^{t} k_\alpha - \sum_{\alpha=1}^{t} k'_\alpha = \frac{b_0 b_1 \ldots b_{t-1}}{c_2 \ldots c_t} - \frac{b_1 \ldots b_{t-1}}{c'_2 \ldots c'_{t-1}}\left(1 + \frac{b_t}{c'_t}\right)$$

If $t = 2i$, $c_2 \ldots c_{t-1} = c_2' \ldots c_{t-1}'$ from repeated application of Lemma 2.5 (a).

$$\text{Therefore} \quad \sum_{\alpha=1}^{t} k_\alpha - \sum_{\alpha=1}^{t} k_\alpha' = \frac{b_1 \ldots b_{t-1}}{c_2 \ldots c_{t-1}}\left(\frac{b_0}{c_t} - 1 - \frac{b_t}{c_t'}\right)$$

$$= \frac{b_1 \ldots b_{t-1}}{c_2 \ldots c_{t-1}} \frac{(b_0 c_t' - c_t c_t' - b_t c_t)}{c_t c_t'}$$

$$= \frac{b_1 \ldots b_{t-1}}{c_2 \ldots c_{t-1}} \frac{((b_0 - c_t)c_t' - b_t c_t)}{c_t c_t'}$$

$$= \frac{b_1 \ldots b_{t-1}}{c_2 \ldots c_{t-1}} \frac{(b_t c_t' - b_t c_t)}{c_t c_t'}$$

$$= \frac{b_1 \ldots b_{t-1} b_t}{c_2 \ldots c_{t-1} c_t} \frac{(c_t' - c_t)}{c_t'}$$

< 0 from Lemma 3.14.

Hence we have our result.

The following result gives us another bound for c_2, but this time we consider a design within a distance-biregular graph to obtain our inequality.

Lemma 3.16.

Let Γ be a distance-biregular graph of girth 4, and diameter greater than two. Then $c_2 \leqslant b_0' - 1$.

Proof.

Let $u \in P$. Consider $\Gamma(u)$ and $\Gamma_2(u)$ as the points and blocks of a 2-$(b_0, c_2, c_2' - 1)$ design. Then by Fisher's inequality the number of blocks is at least as large as the number of points.

$$\text{ie:} \quad \frac{b_0(b_0' - 1)}{c_2} \geqslant b_0 .$$

Therefore $c_2 \leqslant b_0' - 1$.

We will now investigate what happens when c_2 takes certain values.

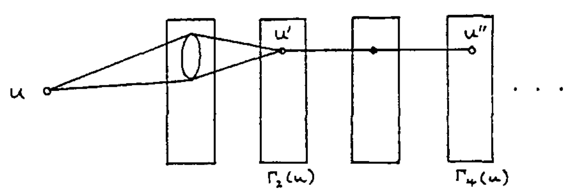
<u>Case 1.</u>

Suppose $c_2 = b_0' - 1$ and let $u \in P$.

This gives us $|\Gamma(u)| = k_1 = k_2 = |\Gamma_2(u)|$ and therefore $\Gamma(u)$ and $\Gamma_2(u)$ form a symmetric $2-(b_0, c_2, c_2' - 1)$ design. We will show that this means that our distance-biregular graph Γ is the incidence graph of $3-(b_0 + 1, b_0', c_2' - 1)$ design.

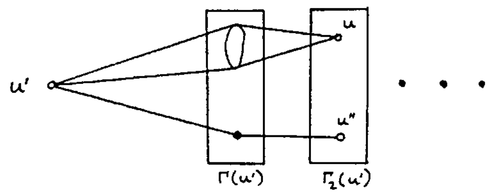
We will refer to the vertices of P as points and the vertices of B as blocks. Firstly we note that since $\Gamma(u)$ and $\Gamma_2(u)$ form a symmetric $2-(b_0, c_2, c_2' - 1)$ design any two points in $\Gamma_2(u)$ have $(c_2' - 1)$ blocks in $\Gamma(u)$ in common, so any two distinct points in $\Gamma_2(u)$ are at distance two in Γ. This means that u and any two points in $\Gamma_2(u)$ have $(c_2' - 1)$ blocks in common. If $d_p = 3$ this clearly means that Γ could be represented as a 3-design as stated above as any three points would have $(c_2' - 1)$ blocks in common. Therefore let us suppose that $d_p \geqslant 4$ and use this to get a contradiction.

Consider the diagram below.



ie: $u' \in \Gamma_2(u)$ and $u'' \in \Gamma_4(u) \cap \Gamma_2(u')$.

Let us turn our attention to u'.



Since $\Gamma(u')$ and $\Gamma_2(u')$ also form a symmetric 2-design any two points in $\Gamma_2(u')$ have $(c_2' - 1)$ blocks in common. However, u and u'' are at distance four in Γ and not distance two. $\#$

Therefore if $c_2 = (b_0' - 1)$ Γ is the incidence graph of a 3-design.

The intersection arrays for Γ are below.

$$\begin{bmatrix} * & 1 & b_0'-1 & b_0' \\ b_0 & b_0'-1 & b_0-(b_0'-1) & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & c_2' & c_3' & b_0' \\ b_0' & b_0-1 & b_2' & b_3' & * \end{bmatrix}$$

b_2', c_2', b_3' and c_3' are found by using the conditions $b_1 b_2 = b_1' b_2'$ $b_2' + c_2' = b_0'$, $c_2 c_3 = c_2' c_3'$ and $b_3' + c_3' = b_0$.

Theorem 3.17 below will show us that in the special case where we also have $k_2 = k_3$, the design is a Hadamard design.

Theorem 3.17. [5]

The existence of a Hadamard matrix of order 4n is equivalent to the existence of a distance-biregular graph with intersection arrays as below.

$$\begin{bmatrix} * & 1 & 2n-1 & 2n \\ 4n-1 & 2n-1 & 2n & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & n & 4n-2 & 2n \\ 2n & 4n-2 & n & 1 & * \end{bmatrix} \quad (*)$$

Proof.

Firstly let us suppose that we have a Hadamard matrix of order 4n. By multiplying certain columns and rows of our original matrix by -1 we can form a new Hadamard matrix, H, where the first row and first column only contain +1's.

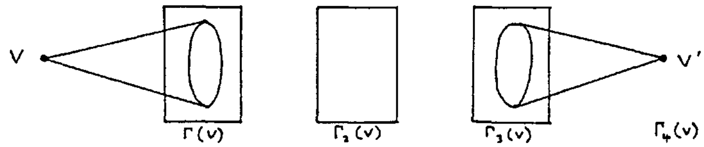
ie: H has the form

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \hdashline 1 & \cdot & & \\ \cdot & & \cdot & \\ 1 & & & \cdot \end{bmatrix}}_{4n} \begin{matrix} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} 4n-1 \end{matrix}$$

Since $HH^T = 4nI$ each row (and column) must have 2n +1's and 2n -1's, where any two rows (or columns) 'overlap' , or intersect, in n +1's and n -1's.

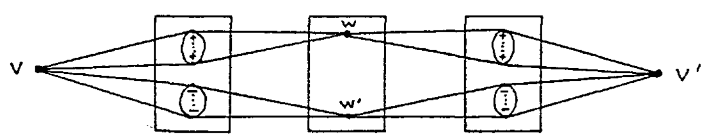
Let H' be the (4n-1)x4n matrix obtained from H by deleting the top row. We will now show how to form a distance-biregular graph from H'. Let our points be the 4n columns of H'. Our blocks are formed from the (4n-1) rows of H'. Each row gives us two blocks. The first is defined as the set of points formed by considering where the 4n columns intersect the row in +1's and the second by considering where the columns intersect in -1's. This gives us our 2(4n-1) = 8n-2 blocks each of which contain 2n points. If we take any two distinct points they intersect in (2n-1) blocks (since any two columns of H overlap in 2n rows and we have removed one row where they overlap). If we take any two distinct blocks they overlap in n points or 0 points since the rows of H overlap in n +1's and n -1's. From these conditions we can form a distance-biregular graph with arrays as shown (*).

Now let us suppose that we have a distance-biregular graph with arrays (*). Let us consider the following.



$v, v' \in B$

Take any vertex $v \in B$ and pair it with the unique vertex $v' \in \Gamma_4(v)$. By doing this for all vertices in B we obtain (4n-1) distinct non-ordered pairs and we will use these to give us the rows of a (4n-1)x4n matrix H'. Label the 4n vertices in P by 1, 2, ... ,4n. Take a pair {v, v'} as described above, $v, v' \in B$. If v is connected to vertex $i \in P$ let the ith. entry in the row be +1 and if v' is connected to i let the entry be -1. (Note that the choice of which of our pair of vertices is v and which is v' is arbitrary.) If we do this for each of the (4n-1) pairs of vertices we form (4n-1) rows each of which has 2n +1's and 2n -1's. By considering another such pair {w, w'} we have the situation below.



This means that any two rows intersect in n +1's and n -1's. If we now form a 4nx4n matrix H from H' by adding a first row of +1's we see that we have a Hadamard matrix of order 4n with $HH^T = 4nI$.

Case 2.

Suppose $c_2 = b'_0 - 2$. (So $b'_0 \geqslant 4$ since $g = 4$.) We will show that this means that $c'_2 = 2$ and that we only have two possibilities for c_3, namely $(b'_0 - 1)$ and b'_0.

We know that $b_0 b_1$ is divisible by c_2 so since $(b'_0 - 1)$ and $(b'_0 - 2)$ are co-prime we can deduce that $(b'_0 - 2)$ divides b_0.

ie: $b_0 = x(b'_0 - 2)$ for some integer x.

Consider $b_1 b_2 = (b'_0 - 1)(b_0 - c_2) = (b'_0 - 1)(x(b'_0 - 2) - (b'_0 - 2))$

$$= (b'_0 - 1)(x - 1)(b'_0 - 2)$$

and $b'_1 b'_2 = (b_0 - 1)b'_2 = (x(b'_0 - 2) - 1)b'_2$

$$= x(b'_0 - 2)b'_2 - b'_2 .$$

Since $b_1 b_2 = b'_1 b'_2$ these two expressions are equal.

$$\text{ie: } (b'_0 - 1)(x - 1)(b'_0 - 2) = x(b'_0 - 2)b'_2 - b'_2$$

By dividing both sides by $(b'_0 - 2)$ we see that $(b'_0 - 2)$ divides b'_2. Since we have girth four we know that $c'_2 \geqslant 2$ and $b'_2 + c'_2 = b'_0$ so we also have $b'_2 \leqslant b'_0 - 2$. Therefore $b'_2 = b'_0 - 2$ and $c'_2 = 2$.

$$\text{ie: } (b'_0 - 1)(x - 1) = x(b'_0 - 2) - 1$$

$$x(b'_0 - 1) - (b'_0 - 1) = x(b'_0 - 2) - 1$$

Therefore $x = (b'_0 - 2)$ and $b_0 = (b'_0 - 2)^2$.

Our arrays start as

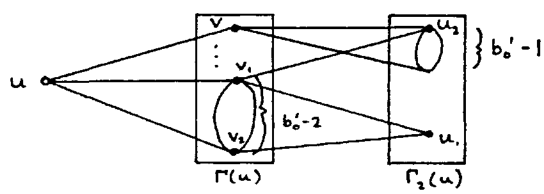
$$\begin{bmatrix} * & 1 & (b'_0 - 2) & \cdots \\ (b'_0 - 2)^2 & (b'_0 - 1) & (b'_0 - 2)(b'_0 - 3) & \end{bmatrix}$$

and $$\begin{bmatrix} * & 1 & 2 & \cdots \\ b_0' & (b_0'-1)(b_0'-3) & (b_0'-2) & \end{bmatrix}.$$

Let us consider a vertex $u \in P$ and the two sets $\Gamma(u)$ and $\Gamma_2(u)$. If $v_1, v_2 \in \Gamma(u)$ then, since $c_2' = 2$, $|\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma_2(u)| = 1$. We will therefore refer to the vertices of $\Gamma(u)$ as points and the vertices of $\Gamma_2(u)$ as lines. (So we have any two points are on exactly one line.)

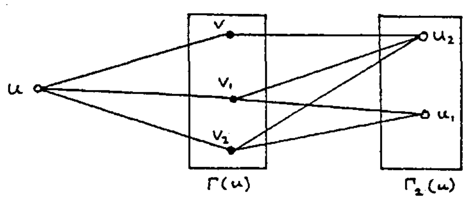
$|\Gamma(u)| = (b_0'-2)^2$ and $\Gamma_2(u) = (b_0'-1)(b_0'-2) = (b_0'-2)^2 + (b_0'-2)$. This suggests we should be looking at $\Gamma(u) \cup \Gamma_2(u)$ as the incidence graph of an affine plane of order $(b_0'-2)$, so we will investigate further.

Suppose $u_1 \in \Gamma_2(u)$ is a line and $v \in \Gamma(u)$ is a point not on u_1.



We will show that there is exactly one line through v which has no point in common with u_1 (ie: there is exactly one vertex, u' say, in $\Gamma_2(u)$ such that $u' \sim v$ but u' is not adjacent to any of the $(b_0'-2)$ neighbours of u_1 in $\Gamma(u)$).

Suppose $v_1, v_2 \in \Gamma(u) \cap \Gamma(u_1)$. Then $v_1 \sim u_1$ and v, v_1 have one common neighbour in $\Gamma_2(u)$, u_2 say. So, each of the $(b_0'-2)$ vertices in $\Gamma(u) \cap \Gamma(u_1)$ has one common neighbour, with v, in $\Gamma_2(u)$. Suppose that v_1, v_2 have the same common neighbour, u_2 say, with v in $\Gamma_2(u)$.



Then, from considering v_1 and v_2, $c_2' \geqslant 3$. #

Therefore there is just one line through v missing u_1 (ie: there is just one vertex, u'' say, in $\Gamma_2(u)$ such that u_1 and u'' have no common neighbours in $\Gamma(u)$). Hence $\Gamma(u) \cup \Gamma_2(u)$ is the incidence graph of an affine plane of order $(b_0' - 2)$.

This means that a necessary condition for a pair of arrays with $c_2' = 2$ and $c_2 = b_0' - 2$ to correspond to a distance-biregular graph is that there exists an affine plane of order $(b_0' - 2)$.

Let us consider affine planes of small order. It is well known that there is a unique affine plane of order n for n = 2, 3, 4, 5, 7, and 8, there is no affine plane of order 6 and there are several affine planes of order 9. It is also shown in [3] that there is no affine plane of order 10.

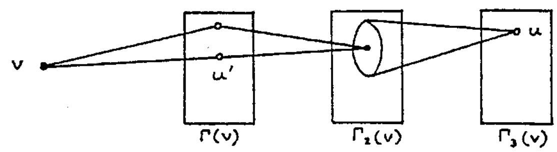
When $c_2' = 2$ we have $c_3 \geqslant c_2 + 1$ as the following Lemma shows. In this Lemma we will consider P as a set of points and B as a set of blocks.

Lemma 3.18.

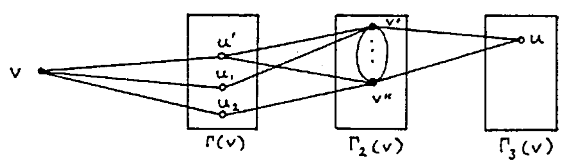
If Γ is a distance-biregular graph with $c_2' = 2$ then $c_3 \geqslant c_2 + 1$.

Proof.

Let $v \in B$ and $u \in P$ such that $d(u, v) = 3$.



u is incident with c_3' blocks in $\Gamma_2(v)$ and each of these blocks have c_2' (= 2) points in common with v. Take any one of these points and label it u'. Now, $d(u, u') = 2$ so u and u' must have c_2 blocks in common and these blocks must all lie in $\Gamma_2(v)$. Let Ω_B be this set of blocks in $\Gamma_2(v)$ and let Ω_P be the set of points in $\Gamma(v)$ at distance two from u. The size of Ω_P is given by $|\Omega_P| = c_2'c_3'/c_2$ and since $c_2c_3 = c_2'c_3'$, $|\Omega_P| = c_3$.



Let $v' \in \Omega_B$. Then, since $c_2' = 2$, v' is adjacent to u' and one other point, u_1, in $\Gamma(v)$. Any other block $v'' \in \Omega_B$ must also be adjacent to u' and one other point, u_2, in $\Gamma(v)$. Since $c_2' = 2$ and v' and v'' are both adjacent to u and u' we see that $u_1 \neq u_2$. Since $|\Omega_B| = c_2$ we must have $|\Omega_P| \geqslant c_2 + 1$. Therefore, combining this with the result above, we have $c_3 \geqslant c_2 + 1$.

(Note that since $c_3' > c_2$ and $c_2 > c_2'$ we also have $c_3' \geqslant c_2' + 2$.)

Let us now return to the case where $b_0 = (b_0' - 2)^2$, $c_2 = b_0' - 2$ and $c_2' = 2$. We know from Lemma 3.18 that $c_3 \geqslant c_2 + 1 = b_0' - 1$ so we have two possible values of c_3 , ie: $c_3 = b_0' - 1$ or b_0'.

(i) Let $c_3 = b_0'$. Our arrays are of the form :

$$\begin{bmatrix} * & 1 & (b_0' - 2) & b_0' \\ (b_0' - 2)^2 & (b_0' - 1) & (b_0' - 2)(b_0' - 3) & * \end{bmatrix}$$

and
$$\begin{bmatrix} * & 1 & 2 & b_0'(b_0' - 2)/2 & b_0' \\ b_0' & (b_0' - 1)(b_0' - 3) & (b_0' - 2) & (b_0' - 2)(b_0' - 4)/2 & * \end{bmatrix}$$

Consider $k_3 = \dfrac{(b_0' - 2)^2(b_0' - 1)(b_0' - 2)(b_0' - 3)}{b_0'(b_0' - 2)}$

$$= \frac{(b_0' - 2)^2(b_0' - 1)(b_0' - 3)}{b_0'}$$

Since this is an integer, and b_0' and $(b_0' - 1)$ are co-prime, we know that $(b_0' - 2)^2(b_0' - 3)$ is divisible by b_0'.

ie: b_0' divides $(b_0')^3 - 7(b_0')^2 + 16b_0' - 12$. This leads us to conclude that b_0' divides 12 and since $b_0 > b_0'$ we only have two possible values for b_0', namely 6 and 12. This gives us the two pairs of arrays below.

(Note that $b_0' \geqslant 4$ and $b_0' = 4$ gives us $b_0 = (b_0' - 2)^2 = 4$. $\#$)

1. $\begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix}$

This pair of arrays is discussed in [5] and it is shown, by considering an extension of PG(2, 4), that there exists a corresponding distance-biregular graph. We show in Chapter 6 that the arrays are realised uniquely.

2. $\begin{bmatrix} * & 1 & 10 & 12 \\ 100 & 11 & 90 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 60 & 12 \\ 12 & 99 & 10 & 40 & * \end{bmatrix}$

Since $c_2' = 2$ and $c_2 = b_0' - 2 = 10$ $\Gamma(u) \cup \Gamma_2(u)$ would form the incidence graph of an affine plane of order 10. However, by [3] this is not possible so these arrays are not feasible.

(ii) If $c_3 = b_0' - 1$ then $b_3 = 1$ and, since $c_2 c_3 = c_2' c_3'$, we have $(b_0' - 2)(b_0' - 1) = 2c_3'$ ie: $c_3' = (b_0' - 2)(b_0' - 1)/2$.

$b_3' + c_3' = (b_0' - 2)^2$ so $b_3' = (b_0' - 2)^2 - (b_0' - 2)(b_0' - 1)/2$

$= (b_0' - 2)(b_0' - 3)/2$

From Proposition 3.13 $c_4' > c_3 = b_0' - 1$ but $c_4' \leqslant b_0'$ so $d_P = d_B = 4$ and we have the general case below.

$$\begin{bmatrix} * & 1 & (b_0' - 2) & (b_0' - 1) & (b_0' - 2)^2 \\ (b_0' - 2)^2 & (b_0' - 1) & (b_0' - 2)(b_0' - 3) & 1 & * \end{bmatrix}$$

and $$\begin{bmatrix} * & 1 & 2 & (b_0' - 1)(b_0' - 2)/2 & b_0' \\ b_0' & (b_0' - 1)(b_0' - 3) & (b_0' - 2) & (b_0' - 2)(b_0' - 3)/2 & * \end{bmatrix}$$

We will now find some more necessary conditions for a pair of arrays to correspond to a distance- biregular graph.

We know from Proposition 3.13 that $c_3 > c_2'$ and that $c_3' > c_2$ but we will now improve on this further. Lemma 3.18 tells us that if $c_2' = 2$ then $c_3 \geqslant c_2 + 1$ so, since $c_2 > c_2'$, $c_3 > c_2' + 1$. We will now show that we always have $c_3 > c_2' + 1$.

<u>Lemma 3.19.</u> [5]

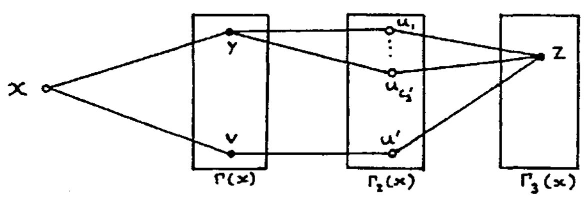
If Γ is a distance-biregular graph of girth 4 we have the following inequalities.

$$c_3 > c_2' + 1 \quad \text{and} \quad c_3' > c_2 + 2.$$

<u>Proof.</u>

Let $x \in P$ and choose any $y \in \Gamma(x)$, $z \in \Gamma_3(x) \cap \Gamma_2(y)$.

In Proposition 3.13 we proved that $c_3 > c_2'$. Let us now assume that $c_3 = c_2' + 1$. Since $z \in \Gamma_2(y)$ and $y \in B$ z and y must have c_2' common neighbours in $\Gamma_2(x)$. Let these be labelled $u_1, u_2, \ldots u_{c_2'}$ Now, z has $c_3 = c_2' + 1$ neighbours in $\Gamma_2(x)$ so let the remaining neighbour be u'. ie: $u' \in \Gamma_2(x) \cap \Gamma(z)$ with $u' \not\sim y$.



Now, $d(y, u') = 3$ so there are c_3 vertices at distance 2 from u' and 1 from y. One of these is x and the other $c_3 - 1 = c_2'$ of them have to be precisely the common neighbours of y and z.

Now consider a vertex $v \in \Gamma(x) \cap \Gamma(u')$. v must be adjacent to exactly $c_2' - 1$ of the vertices $u_1, \ldots, u_{c_2'}$ in order to have c_2' common neighbours with z. Each such vertex, v, must be adjacent to a different set of $c_2' - 1$ vertices, otherwise two vertices, v and v' say, adjacent to the same set would have $c_2' + 1$ common neighbours. Therefore,

$$|\Gamma(x) \cap \Gamma(u')| \leqslant \binom{c_2'}{c_2' - 1} = c_2' . \qquad \text{ie: } c_2 \leqslant c_2'.$$

However, we know that $c_2 > c_2'$ so we have a contradiction.

Therefore $c_3 > c_2' + 1$.

We also know $c_2 c_3 = c_2' c_3'$ so this gives us

$$c_3' = \frac{c_2 c_3}{c_2'} \geqslant \frac{(c_2' + 2)c_2}{c_2'} = c_2 + \frac{2c_2}{c_2'} \geqslant c_2 + 3.$$

$$\text{ie: } c_3' > c_2 + 2.$$

We are now in a position to prove the Proposition below giving us a bound on the diameter of a distance-biregular graph Γ (of girth 4).

<u>Proposition 3.20.</u>

Let Γ be a distance-biregular graph of girth 4. Then we can bound the diameter, d, of Γ as follows. Either $d = 4$ or we have $d > 4$ and one of two cases

(1) $d_P = d_B$ and $d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 2\}$

or (2) $d_P = d_B - 1$ and $d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 3\}$

Proof.

Suppose $d > 4$. ie: $d \geqslant 6$ since d is even by Corollary 3.7. Then $d = d_B = 2j + 2$ for some integer j and this means that $b_2' \geqslant c_{2j}'$ from Lemma 3.10.

We also have $c_{2j}' > c_{2j-1} > c_{2j-2}'$ from Proposition 3.13 so

$$c_{2j}' \geqslant c_{2j-2}' + 2.$$

$$\therefore b_0' - c_2' \geqslant c_{2j}' \geqslant c_{2j-2}' + 2 \geqslant \ldots \geqslant c_4' + 2j - 4 \geqslant c_3 + 2j - 3.$$

We know from Lemma 3.19 that $c_3 \geqslant c_2' + 2$ and this leads us to conclude

$$b_0' - c_2' \geqslant c_2' + 2j - 1 = c_2' + d - 3$$

$$\text{ie: } d \leqslant b_0' - 2c_2' + 3.$$

(1) Suppose $d_p = d_B = 2j + 2$. From Lemma 3.10 we have $b_2 \geqslant c_{2j}$.

Also $c_{2j} > c_{2j-1}' > c_{2j-2}$ so $c_{2j} \geqslant c_{2j-2} + 2$, and $c_3' \geqslant c_2 + 3$.

$$\therefore b_2 = b_0 - c_2 \geqslant c_{2j} \geqslant c_{2j-2} + 2 \geqslant \ldots \geqslant c_4 + 2j - 4 \geqslant c_3' + 2j - 3 \geqslant c_2 + 2j$$

$$\text{ie: } d_p \leqslant b_0 - 2c_2 + 2.$$

Therefore $d_p = d_B \Longrightarrow d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 2\}$

(2) Suppose $d_p = d_B - 1 = 2j + 1$. Then we have $b_2 \geqslant c_{2j-1}'$ so

$$b_2 = b_0 - c_2 \geqslant c_{2j-1}' \geqslant c_{2j-3}' + 2 \geqslant \ldots \geqslant c_3' + 2j - 4 \geqslant c_2 + 2j - 1$$

$$\text{ie: } d_p \leqslant b_0 - 2c_2 + 2.$$

Therefore $d_p = d_B - 1 \Longrightarrow d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 3\}$

In the remainder of this chapter we will consider other bounds on the diameter, d, of a distance-biregular graph Γ. Firstly we will use Lemma 3.1 to find another restriction on c_2 and c_2'.

Lemma 3.21.

(a) If $(b_0, c_2) = 1$ then $d_p = 2j + 1$ for some integer j, and c_2 divides b_{2i+1} and c_{2i+2} for $0 \leqslant i \leqslant j - 1$.

(b) If $d_p = 2j + 2$ for some integer j, then $(b_0, c_2) \neq 1$.

(c) $(b_0', c_2') \neq 1$.

Proof

(a) Suppose that $(b_0, c_2) = 1$.

(1) The result is certainly true for i = 0 since c_2 divides c_2 and by considering the valency of the left-hand derived graph, ie: $\dfrac{b_0 b_1}{c_2} = \dfrac{b_0(b_0' - 1)}{c_2}$, we see that c_2 divides b_1.

(2) Suppose the result is true for all terms up to 2i (so in particular c_2 divides b_{2i-1} and c_{2i}). By again turning our attention to the left-hand derived graph we see that $b_{2i} b_{2i+1}$ is divisible by c_2.

ie: $\dfrac{(b_0 - c_{2i})b_{2i+1}}{c_2} = \dfrac{b_0 b_{2i+1}}{c_2} - \dfrac{c_{2i} b_{2i+1}}{c_2}$ is an integer.

Therefore, since $(b_0, c_2) = 1$ and c_2 divides c_{2i} we have c_2 divides b_{2i+1}.

We also have $c_{2i+1} c_{2i+2}$ is divisible by c_2.

ie: $\dfrac{(b_0' - b_{2i+1})c_{2i+2}'}{c_2} = \dfrac{b_0' c_{2i+2}}{c_2} - \dfrac{b_{2i+1} c_{2i+2}}{c_2}$ is an integer.

We know c_2 divides b_{2i+1} and we also know that c_2 divides $b_1 = (b_0' - 1)$. Therefore $(b_0', c_2) = 1$ and we must have c_2 divides c_{2i+2}.

(b) We now consider what happens if $d_P = 2j + 2$. This would imply that c_2 divides $c_{2j+2} = b_0$ and hence $(b_0, c_2) \neq 1$.

(c) Since d_B is even we must always have $(b_0', c_2') \neq 1$ by considering B in place of P in the above.

The following Lemma uses Lemma 3.21 to give us another bound on d in the special case when $(b_0, c_2) = 1$.

<u>Lemma 3.22.</u>

Let Γ be a distance-biregular graph of girth four. Suppose that $(b_0, c_2) = 1$. Then we have the following bound on the diameter, d, of Γ.

$$d < \frac{2b_0}{c_2} + 2 .$$

<u>Proof.</u>

Let $d = 2j + 2$ for some integer j. Then, from Lemma 3.21 (a) $d_P = 2j + 1$. We know from Lemma 3.21 that c_2 divides c_{2i+2} for $0 \leq i < j$ and Proposition 3.13 gives us $c_{2i+2} > c'_{2i+1} > c_{2i}$.
Therefore the largest diameter we could possibly have would come from an array with entries as below.

$$\begin{bmatrix} * & 1 & c_2 & . & 2c_2 & . & 3c_2 & . & \ldots & (d-2)c_2/2 & c_{d-1} \\ b_0 & b_0' - 1 & b_2 & . & b_4 & . & b_6 & . & \ldots & b_{d-2} & * \end{bmatrix}$$

$$\text{Therefore}\quad \frac{(d-2)c_1}{2} < b_0$$

$$\text{ie:}\quad d < \frac{2b_0}{c_1} + 2\ .$$

We will now find another bound on the diameter of a distance-biregular graph and compare this bound with the bound obtained in Proposition 3.20.

Lemma 3.23.

Let Γ be a distance-biregular graph of girth 4. Then the following inequalities hold.

$$c_{2i} \geq c_2 + (i-1)\left[\frac{b_0}{b_0'}\right] + 2i \qquad 2 \leqslant i \leqslant \frac{d_p}{2}\ .$$

$$c'_{2i+1} \geq c_2 + (i-1)\left[\frac{b_0}{b_0'}\right] + (2i+1) \qquad 1 \leqslant i < \frac{d_p}{2}\ .$$

Proof.

From Lemma 3.2 we know that

$$b'_{2i-1}(c'_{2i} - c_{2i-1}) = b_{2i-1}(c_{2i} - c'_{2i-1})$$

This gives $\quad \dfrac{b'_{2i-1}}{b_{2i-1}}(c'_{2i} - c_{2i-1}) = c_{2i} - c'_{2i-1}\ .$

We also know from Lemma 3.6 that $\dfrac{b'_{2i-1}}{b_{2i-1}} \geq \dfrac{b_0}{b_0'}$ so we have

$$c_{2i} - c'_{2i-1} \geq \frac{b_0}{b_0'}(c'_{2i} - c_{2i-1})$$

Rearranging gives

$$c_{2i} - c'_{2i} \geq c'_{2i-1} + \left(\frac{b_0}{b_0'} - 1\right) c'_{2i} - \frac{b_0}{b_0'}\, c_{2i-1}$$

$$\Longrightarrow c_{2i} - c'_{2i} > (c'_{2i-1} - c_{2i-1}) + \left(\frac{b_0}{b'_0} - 1\right)(c'_{2i} - c_{2i-1})$$

$$\Longrightarrow c_{2i} - c'_{2i} > (c'_{2i-1} - c_{2i-1}) + \frac{b_0}{b'_0} - 1 \qquad (*)$$

Since $c'_{2i} > c_{2i-1}$ from Proposition 3.13.

Now, $c_{2i}\,c_{2i+1} = c'_{2i}\,c'_{2i+1}$ so multiplying (*) by c_{2i+1} gives

$$c'_{2i}\,c'_{2i+1} - c'_{2i}\,c_{2i+1} = c_{2i}\,c_{2i+1} - c'_{2i}\,c_{2i+1} > c_{2i+1}\left((c'_{2i-1} - c_{2i-1}) + \frac{b_0}{b'_0} - 1\right)$$

$$\Longrightarrow c'_{2i+1} - c_{2i+1} > \frac{c_{2i+1}}{c'_{2i}}\left((c'_{2i-1} - c_{2i-1}) + \frac{b_0}{b'_0} - 1\right)$$

Proposition 3.13 tells us that $c_{2i+1} > c'_{2i}$ so we have

$$c'_{2i+1} - c_{2i+1} > (c'_{2i-1} - c_{2i-1}) + \frac{b_0}{b'_0} - 1 \qquad (**)$$

Let us return to (*). We will use the fact that $[b_0/b'_0]$ is the least integer greater than $b_0/b'_0 - 1$.

$$\begin{aligned}
c_{2i} &\geqslant c'_{2i} + (c'_{2i-1} - c_{2i-1}) + [b_0/b'_0] \\
&\geqslant c'_{2i} + (c'_{2i-3} - c_{2i-3}) + 2[b_0/b'_0] \qquad \text{(from (**))} \\
&\geqslant \cdots \\
&\geqslant c'_{2i} + (c'_3 - c_3) + (i-1)[b_0/b'_0] \\
&\geqslant c'_{2i} + (c_2 - c_3) + (i-1)[b_0/b'_0] + 3 \qquad (c'_3 \geqslant c_2 + 3) \\
&\geqslant c'_{2i-2} + (c_2 - c_3) + (i-1)[b_0/b'_0] + 5 \qquad (c'_{2i} > c_{2i-1} > c'_{2i-2}) \\
&\geqslant c'_4 + (c_2 - c_3) + (i-1)[b_0/b'_0] + 3 + (2i-4) \\
&\geqslant c_3 + (c_2 - c_3) + (i-1)[b_0/b'_0] + 3 + (2i-4) + 1
\end{aligned}$$

$$\text{ie: } c_{2i} > c_2 + (i-1)[b_0/b'_0] + 2i.$$

We will now prove the second inequality. From (**) we have

$$c'_{2i+1} \geq (c'_{2i-1} + c_{2i-1}) + c_{2i+1} + [b_0/b'_0]$$

$$\geq \ldots$$

$$\geq c_{2i+1} + (c'_3 - c_3) + (i-1)[b_0/b'_0]$$

$$\geq \ldots$$

$$\geq c_3 + (c'_3 - c_3) + (i-1)[b_0/b'_0] + (2i-2)$$

$$\geq c'_3 + (i-1)[b_0/b'_0] + (2i-2)$$

$$\geq c_2 + (i-1)[b_0/b'_0] + (2i+1) \qquad (c'_3 \geq c + 3)$$

ie: $c'_{2i+1} \geq c_2 + (i-1)[b_0/b'_0] + (2i+1)$.

Thus we have our two inequalities.

We will now use the results of the last lemma to give us a bound on the diameter of Γ .

Proposition 3.24.

Let d be the diameter of a distance-biregular graph of girth 4. Then we have the following bound on d.

$$d \leq \begin{cases} \left\lceil 2\,\dfrac{(b_1 + [b_0/b'_0])}{(2 + [b_0/b'_0])} \right\rceil & d_p = 2i \\ \text{or:} & \\ \left\lceil 2\,\dfrac{(b_1 + 2[b_0/b'_0])}{(2 + [b_0/b'_0])} \right\rceil & d_p = 2i - 1. \end{cases}$$

Proof.

We have two cases to consider.

(a) Firstly we consider the case where $d_p = d_B = d = 2j$ for some positive integer j. This means that $c_d = b_0$ so by Lemma 3.23

$$b_0 = c_d \geq c_2 + (d/2 - 1)[b_0/b'_0] + d$$

$$\Longrightarrow b_0 - c_2 \geqslant d(1 + [b_0/b_0']/2) - [b_0/b_0']$$

$$\therefore d \leqslant \frac{b_2 + [b_0/b_0']}{(1 + [b_0/b_0']/2)} = 2\left(\frac{b_2 + [b_0/b_0']}{2 + [b_0/b_0']}\right)$$

$$\therefore d \leqslant \left[2\,\frac{(b_2 + [b_0/b_0'])}{2 + [b_0/b_0']}\right]$$

(b) Now we consider the case where $d_p = 2j - 1 = d_B - 1 = d - 1$. Using the fact that $c'_{d-1} \leqslant b_0 - 1$ and Lemma 3.23 we have

$$b_0 - 1 \geqslant c'_{2i-1} = c'_{d-1} \geqslant c_2 + (d/2 - 2)[b_0/b_0'] + (d - 1)$$

$$\Rightarrow b_0 - c_2 = b_2 \geqslant d([b_0/b_0']/2 + 1) - 2[b_0/b_0']$$

$$\therefore d \leqslant \frac{b_2 + 2[b_0/b_0']}{([b_0/b_0']/2 + 1)}$$

ie: $$d \leqslant \left[\frac{2(b_2 + 2[b_0/b_0'])}{(2 + [b_0/b_0'])}\right]$$

It is not immediately clear that this bound is ever any better than the one found earlier in Proposition 3.20 so we will now give an example of when it is.

Suppose d_p is odd (and hence that $d_p = d_B - 1$). Then we have :

(i) $d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 3\}$

and (ii) $d \leqslant \left[2\,\frac{(b_2 + 2[b_0/b_0'])}{(2 + [b_0/b_0'])}\right]$

To investigate when (ii) $\leqslant$ (i) we will in fact consider two cases.

(I) When is (ii) $\leqslant b_0' - 2c_2' + 3$? Certainly whenever we have:

$$\frac{2(b_0 - c_2 + 2[b_0/b_0'])}{2 + [b_0/b_0']} \leqslant b_0' - 2c_2' + 3$$

$$\Longrightarrow 2b_0 - 2c_2 + 4[b_0/b_0'] \leqslant (2 + [b_0/b_0'])(b_0' - 2c_2' + 3)$$

$$\Longrightarrow \quad b_0'(2 + [b_0/b_0']) \geqslant 2b_0 - 2c_2 + 2c_2'(2 + [b_0/b_0']) + [b_0/b_0'] - 6$$

eg: If $[b_0/b_0'] = 1$ and $c_2' = 2$ then (ii) is better than (i) when

$$b_0' \geqslant 2(b_0 - c_2 + 7/2)/\,3.$$

(II) When is (ii) $\leqslant b_0 - 2c_2 + 3$? Certainly whenever we have:

$$\frac{2(b_0 - c_2 + 2[b_0/b_0'])}{2 + [b_0/b_0']} \leqslant b_0 - 2c_2 + 3\ .$$

$$\Longrightarrow 2b_0 - 2c_2 + 4[b_0/b_0'] \leqslant (b_0 - 2c_2 + 3)(2 + [b_0/b_0'])$$

$$\Longrightarrow b_0[b_0/b_0'] \geqslant 2c_2(1 + [b_0/b_0']) + [b_0/b_0'] - 6.$$

eg: If $[b_0/b_0'] = 1$ then (ii) is better than (i) when

$$b_0 \geqslant 4c_2 - 5.$$

ie: If b_0 and b_0' are 'close' ($[b_0/b_0'] = 1$) then it would appear that the bound in Proposition 3.24 is better than the bound in Proposition 3.20.

Before the last lemma in this chapter we need to introduce some new notation. For any real number x let $\{x\}$ denote the least integer greater than or equal to x.

Lemma 3.25.

If Γ is a distance-biregular graph of girth 4 then we have the following inequality.

$$b_0'b_1' \geqslant c_2'c_3' \left\{ \frac{b_0 - 1}{\binom{b_0' - 1}{c_2' - 1}} \right\}$$

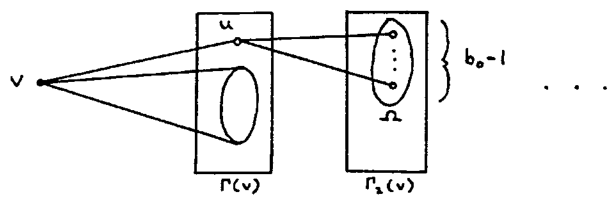
Note that if $\binom{b_0'-1}{c_2'-1} \,\Big|\, (b_0 - 1)$ we have $b_0' \binom{b_0'-1}{c_2'-1} \geqslant c_2' c_3'$.

Proof.

We will think of P as a set of points and B as a set of blocks. Let us consider any vertex, v, in B. Then the intersection array for v is

$$\begin{bmatrix} * & 1 & c_2' & c_3' & \cdots \\ b_0' & b_0 - 1 & b_2' & b_3' & \cdots \end{bmatrix}$$

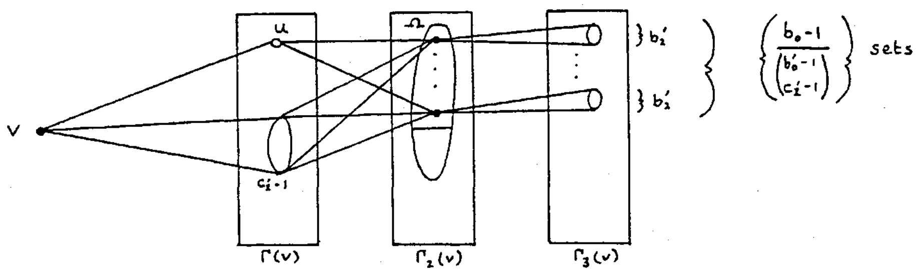
Let $u \in P$ and $\Gamma(v)$. The graph described by this array starts as



Let us consider the set Ω consisting of the $(b_0 - 1)$ neighbours of u in $\Gamma_2(v)$. Each block in Ω is connected to u and $(c_2' - 1)$ other points in $\Gamma(v)$. The number of choices for these $(c_2' - 1)$ point sets is $\binom{b_0'-1}{c_2'-1}$.

This means that at least $\left\{ \dfrac{b_0 - 1}{\binom{b_0'-1}{c_2'-1}} \right\}$ blocks in Ω share the same $(c_2' - 1)$ point sets in $\Gamma(v)$. Now, each pair of blocks in Ω are at distance two in Γ so they must have c_2' common neighbours. This means that if we consider any pair of the $\left\{ \dfrac{b_0 - 1}{\binom{b_0'-1}{c_2'-1}} \right\}$ blocks in Ω described earlier the common neighbours all lie in

$\Gamma(v)$. This means that we must have the situation shown below



ie: $$k_3' = \frac{b_0'(b_0 - 1)b_2'}{c_2'c_3'} \geqslant \binom{b_0 - 1}{\binom{b_0' - 1}{c_2' - 1}} b_2'$$

ie: $$b_0'(b_0 - 1) \geqslant c_2'c_3' \binom{b_0 - 1}{\binom{b_0' - 1}{c_2' - 1}}$$

ie: $$b_0'b_1' \geqslant c_2'c_3' \binom{b_0 - 1}{\binom{b_0' - 1}{c_2' - 1}} .$$

We will now present a useful bound on b_0 in terms of b_0' and c_2' .

<u>Theorem 3.26.</u>

In a distance-biregular graph Γ of girth four the larger valency, b_0, is bounded by

$$b_0 \leqslant \frac{(b_0' - 1)(b_0' - 2)}{(c_2' - 1)} + 1 .$$

<u>Proof.</u>

We have $c_2 > c_2' \geqslant 2$. From Lemma 3.16 $c_2 \leqslant b_0' - 1$ and from Lemma 3.2 (a) (i) we know that $b_1(c_2 - 1) = b_1'(c_2' - 1)$.

$$b_0' - 1 \geqslant c_2 = \frac{b_1'(c_2' - 1)}{b_1} + 1 = \frac{(b_0 - 1)(c_2' - 1)}{(b_0' - 1)} + 1$$

$$\Longrightarrow b_0 \leqslant \frac{(b_0' - 1)(b_0' - 2)}{(c_2' - 1)} + 1$$

Chapter 4.

In this chapter a general method for finding all possible pairs of arrays for distance-biregular graphs of girth four is constructed.

When trying to find combinatorially feasible pairs of arrays, for distance-biregular graphs, a depth-first search is often useful. In this we start with the two initial segments of a pair of arrays and then build up the full arrays using local feasibility conditions. We will suppose that we know b_0' and c_2' and that we are looking for possible values of b_0 and c_2.

The bound on b_0 in Theorem 3.26 is very useful. But, rather than try all possible values of $b_0 \leqslant \frac{(b_0'-1)(b_0'-2)}{(c_2'-1)} + 1$ we shall now restrict the possible values of b_0 and c_2 even further. We will then be in a position to systematically examine the cases for small b_0' and find all possible feasible pairs of arrays in these cases.

We are concerned with arrays where $c_2 > c_2' \geqslant 2$ so suppose that our arrays start as below.

$$\begin{bmatrix} * & 1 & c_2 & \ldots \\ b_0 & b_0'-1 & b_0-c_2 & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & c_2' & \ldots \\ b_0' & b_0-1 & b_0'-c_2' & \end{bmatrix}$$

We are trying to restrict the possible values of b_0 within the range

$$b_0'+1 \leqslant b_0 \leqslant \frac{(b_0'-1)(b_0'-2)}{(c_2'-1)} + 1 .$$

We will start by finding an integrality condition.

We know $b_1 b_2 = b_1' b_2'$. It follows that

$$b_0 - c_2 = \frac{(b_0 - 1)(b_0' - c_2')}{(b_0' - 1)} .$$

$$\therefore\ c_2 = b_0 - \frac{(b_0 - 1)(b_0' - c_2')}{(b_0' - 1)}$$

$$= \frac{b_0 b_0' - b_0 - b_0 b_0' + b_0' + b_0 c_2' - c_2'}{(b_0' - 1)}$$

$$= \frac{b_0 (c_2' - 1) + (b_0' - c_2')}{(b_0' - 1)}$$

$$= \frac{b_0 (c_2' - 1) + (b_0' - 1) - (c_2' - 1)}{(b_0' - 1)}$$

$$= \frac{(b_0 - 1)(c_2' - 1)}{(b_0' - 1)} + 1 \qquad (*)$$

Let $((b_0' - 1),(c_2' - 1)) = p$. This p only depends on b_0' and c_2' and when we use this test we know b_0' and c_2' and hence we know p. This means that $\frac{(b_0 - 1)(c_2' - 1)/\ p}{(b_0' - 1)/\ p} \in \mathbb{Z}$ so $\frac{(b_0' - 1)}{p} \,\Big|\, (b_0 - 1)$

since $((c_2' - 1)/p,\ (b_0' - 1)/p) = 1$.

ie: $b_0 = \frac{k(b_0' - 1)}{p} + 1$ (**) for some positive integer k.

We also know that $b_0 b_1 / c_2$ is an integer (it is in fact the valency of the left-hand derived graph). Therefore $b_0 (b_0' - 1)$ is divisible by c_2. This means that the following must be an integer.

$$\frac{b_0 (b_0' - 1)}{c_2} = \frac{b_0 (b_0' - 1)}{(b_0 - 1)(c_2' - 1)/(b_0' - 1) + 1} \qquad \text{from } (*)$$

$$= \frac{b_0 (b_0' - 1)^2}{(b_0 - 1)(c_2' - 1) + (b_0' - 1)}$$

$$= \frac{b_0(b_0' - 1)^2}{b_0(c_2' - 1) + b_2'} \qquad (***)$$

We will now describe b_0 more fully. Let $(b_0, b_2') = J$. This means that we can express b_0 as qJ for some positive integer q where $(q, b_2') = 1$.

Therefore from (***), $\dfrac{(b_0' - 1)^2 J}{b_0(c_2' - 1) + (b_0' - c_2')}$ is an integer.

Now, since $b_0 = \dfrac{k(b_0' - 1)}{p} + 1$, we have:

$$\frac{(b_0' - 1)^2 J}{(k(b_0' - 1)/p + 1)(c_2' - 1) + (b_0' - c_2')}$$

$$= \frac{p(b_0' - 1)^2 J}{(k(b_0' - 1) + p)(c_2' - 1) + (b_0' - c_2')p}$$

$$= \frac{p(b_0' - 1)^2 J}{k(b_0' - 1)(c_2' - 1) + p(c_2' - 1) + p(b_0' - c_2')}$$

$$= \frac{p(b_0' - 1)^2 J}{k(b_0' - 1)(c_2' - 1) + p(b_0' - 1)}$$

$$= \frac{p(b_0' - 1) J}{k(c_2' - 1) + p}$$

This last expression is a positive integer so let this integer be M.

$$\text{ie:} \quad M = \frac{p(b_0' - 1) J}{k(c_2' - 1) + p}.$$

Re-arranging for k gives:

$$k = \left(\frac{p(b_0' - 1) J}{M} - p\right)\frac{1}{(c_2' - 1)}$$

Substituting this expression for k in (**) leads to

$$b_0 = \left(\frac{(b_0' - 1) J}{M} - 1\right)\frac{(b_0' - 1)}{(c_2' - 1)} + 1$$

$$\therefore\ b_0 = \frac{1}{(c_2' - 1)} \left(\frac{(b_0' - 1)^2 J}{M} - (b_0' - 1) + c_2' - 1 \right)$$

$$= \frac{1}{(c_2' - 1)} \left(\frac{(b_0' - 1)^2 J}{M} - (b_0' - c_2') \right) .$$

$$= \frac{1}{(c_2' - 1)} \left(\frac{(b_0' - 1)^2 J}{M} - b_2' \right)$$

Although this appears to be a complicated expression it is in fact very useful as will be demonstrated in the next chapter.

So to recap we have:

Let $p = (c_2' - 1, b_0' - 1)$, $k = \dfrac{p(b_0 - 1)}{(b_0' - 1)}$,

$J = (b_0, b_2')$ and $M = \dfrac{p(b_0' - 1)\, J}{k(c_2' - 1) + p}$

Then $$b_0 = \frac{1}{(c_2' - 1)} \left(\frac{J\,(b_0' - 1)^2}{M} - b_2' \right) .$$

Chapter 5.

In this chapter we will show that the method developed in Chapter 4 can actually be used by hand to give us all possible pairs of arrays for a distance-biregular graph of girth four when $b_0 > b_0' = 3$, 4, 5, 6, 7, 8 or 9. We will not be discussing the case where $b_0' = 2$ since this is done in detail in [5] where it is shown that the only possibilities are K_{2,b_0} and the subdivision graph of a (k, g)-graph. From the discussion following Lemma 3.16 we know the possibilities when $c_2 = b_0' - 1$ or $b_0' - 2$. We will start the chapter by considering these cases and then consider the cases where $c_2 \leqslant b_0' - 3$.

Lemma 3.21 (c) tells us that $(b_0', c_2') \neq 1$ so if b_0' is any prime number the only possibility for c_2' is b_0' with the only possible distance-biregular graph being $K_{b_0, b_0'}$. Since we are not considering these possibilities here there are no possible cases for $b_0' = 3$, 5 or 7.

(1) Suppose that $c_2 = b_0' - 1$.

We know from Chapter 3 that this means that $k_1 = k_2$, $d_p = 3$, $d_B = 4$ and we have the incidence graph of a $3-(b_0 + 1, b_0', c_2' - 1)$ design with intersection arrays as below.

$$\begin{bmatrix} * & 1 & (b_0' - 1) & b_0' \\ b_0 & (b_0' - 1) & b_0 - (b_0' - 1) & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & c_2' & c_3' & b_0' \\ b_0' & (b_0 - 1) & b_2' & b_3' & * \end{bmatrix}$$

(i) $b_0' = 4$. Using Lemma 3.21 (c) gives us one possible value for c_2', namely $c_2' = 2$. Since $c_2 = b_0' - 1$, $c_2 = 3$ and our arrays start as

$$\begin{bmatrix} * & 1 & 3 & \ldots \\ b_0 & 3 & (b_0 - 3) & \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & \ldots \\ 4 & (b_0 - 1) & 2 & \end{bmatrix}$$

Therefore, since $b_1 b_2 = b_1' b_2'$, we have $3b_0 - 9 = 2b_0 - 2$.

ie: $b_0 = 7$. Now, since $c_2 c_3 = c_2' c_3'$ we have $c_3' = 6$.

Our arrays are $\begin{bmatrix} * & 1 & 3 & 4 \\ 7 & 3 & 4 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 6 & 4 \\ 4 & 6 & 2 & 1 & * \end{bmatrix}$

(ii) $b_0' = 6$. We know that $c_2 = 5$ and by Lemma 3.21 (c) we have $c_2' = 2$ or 3.

(a) $c_2' = 2$. $b_1 b_2 = b_1' b_2' \Longrightarrow 5(b_0 - 5) = (b_0 - 1)4 \Longrightarrow b_0 = 21$.

Our arrays are: $\begin{bmatrix} * & 1 & 5 & 6 \\ 21 & 5 & 16 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 15 & 6 \\ 6 & 20 & 4 & 6 & * \end{bmatrix}$

(b) $c_2' = 3$. $b_1 b_2 = b_1' b_2' \Longrightarrow 5(b_0 - 5) = (b_0 - 1)3 \Longrightarrow b_0 = 11$.

Our arrays are: $\begin{bmatrix} * & 1 & 5 & 6 \\ 11 & 5 & 6 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 3 & 10 & 6 \\ 6 & 10 & 3 & 1 & * \end{bmatrix}$

(iii) $b_0' = 8$. $c_2 = 7$ and by Corollary 3.12 $c_2' \leqslant b_0'/2$ ie: $c_2' \leqslant 4$.
So, by Lemma 3.21 (c), $c_2' = 2$ or 4.

(a) $c_2' = 2$. $b_1 b_2 = b_1' b_2' \Longrightarrow 7(b_0 - 7) = (b_0 - 1)6 \Longrightarrow b_0 = 43$.

If this were a possibility our arrays would be:

$$\begin{bmatrix} * & 1 & 7 & 8 \\ 43 & 7 & 36 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 28 & 8 \\ 8 & 42 & 6 & 15 & * \end{bmatrix}.$$

However, $k_4' = 67.5$ which contradicts the fact that k_4' is an integer. Therefore $c_2' = 2$ is not a possibility.

(b) $c_2' = 4$. By using the same method as above we have one possibility with arrays as below.

$$\begin{bmatrix} * & 1 & 7 & 8 \\ 15 & 7 & 8 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 4 & 14 & 8 \\ 8 & 14 & 4 & 1 & * \end{bmatrix}$$

(iv) $b_0' = 9$. By Corollary 3.12 $c_2' \leqslant b_0'/2$ so $c_2' \leqslant 4.5$. By Lemma 3.21 (c) we therefore have $c_2' = 3$ as our only possibility.
Proceeding as above gives us $b_0 = 29$. This would give us the following array for any vertex in P. $\begin{bmatrix} * & 1 & 8 & 9 \\ 29 & 8 & 21 & * \end{bmatrix}$

However, k_3 is not an integer so we have a contradiction.

(2) Suppose that $c_2 = b_0' - 2$.

(i) $b_0' = 4$. No cases.

(ii) $b_0' = 6$. We have the special case below where $c_3 = b_0'$.

$$\begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix}$$

We also have the general case discussed in Chapter 3.

$$\begin{bmatrix} * & 1 & 4 & 5 & 16 \\ 16 & 5 & 12 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 10 & 6 \\ 6 & 15 & 4 & 6 & * \end{bmatrix}$$

(iii) $b_0' = 8$. Since there is no affine plane of order 6 we have no possible cases for $b_0' = 8$ and $c_2 = 6$.

(iv) $b_0' = 9$. If $c_2 = b_0' - 2 = 7$ then $b_2 = 42$. Therefore, since $b_1 b_2 = b_1' b_2'$, $b_2' = 7$ so $c_2' = 2$. Therefore $(b_0' , c_2') = 1$ which is not possible. Hence there are no possible cases for $b_0' = 9$ and $c_2 = 7$.

We will now suppose that $c_2 \leqslant b_0' - 3$. Lemma 3.2 (a) (i) gives us $b_1(c_2 - 1) = b_1'(c_2' - 1)$ so

$$c_2 = \frac{b_1'(c_2' - 1)}{b_1} + 1 \leqslant b_0' - 3$$

$$\Longrightarrow b_0 \leqslant \frac{(b_0' - 1)(b_0' - 4)}{(c_2' - 1)} + 1 \qquad (*)$$

(3) $b_0' = 4$. In this case $b_0' - 3 = 1$ so there are no further pairs of arrays.

(4) $b_0' = 6$. As described earlier we have two possible values of c_2' , namely $c_2' = 2$ or 3.

(i) $c_2' = 2 \Longrightarrow b_0 \leqslant 10 + 1 = 11$ from (*).

Since $c_2 \leqslant b_0' - 3 = 3$ and $c_2 > c_2' = 2$ we must have $c_2 = 3$.

Therefore $b_1 b_2 = b_1' b_2' \Longrightarrow 5(b_0 - 3) = 4(b_0 - 1) \Longrightarrow b_0 = 11$.

However, this implies that k_2 is not an integer. $\#$

(ii) $c_2' = 3$ again leads to no further cases.

(5) $b_0' = 8$. We know that $c_2' = 2$ or 4.

(i) $c_2' = 2 \Longrightarrow b_0 \leqslant 28 + 1 = 29$ from (*).

We will now use the information we obtained in Chapter 4.

$$p = (c_2' - 1,\ b_0' - 1) = (1,\ 7) = 1.$$

Therefore $b_0 = 7k + 1$ for some positive integer k from Chapter 4 (**). ie: $b_0 = 15$, 22 or 29.

We also know $J = (b_0\ ,\ b_2') = (7k + 1,\ 6) = 1, 2, 3$ or 6. However, with the possible values of b_0 we have, the only possible values for J are 1 (for $b_0 = 22$ and 29) or 3 (for $b_0 = 15$).

(a) $J = 1 \Longrightarrow M = \dfrac{7}{k + 1} \Longrightarrow k = 6$. ie: $b_0 = 43$. $\#$

(b) $J = 3 \Longrightarrow M = \dfrac{21}{k + 1} \Longrightarrow k = 2, 6$ or 20 .

$k = 6$ or 20 would mean that b_0 is too large so the only value of b_0 we need to consider is when $k = 3$ ie: $b_0 = 15$.

We will consider the entries in our pair of possible arrays.

$b_0 = 15$, $b_0' = 8$, $b_1 = 7$, $c_1 = 1$, $b_1' = 14$, $c_1' = 1$, $b_2' = 6$ and $c_2' = 2$.

Since $b_1 b_2 = b_1' b_2'$ we know that $b_2 = 12$ and hence $c_2 = 3$.

We will now try to construct a pair of feasible arrays.

$$c_2 c_3 = c_2' c_3' \Longrightarrow 3c_3 = 2c_3'$$

Lemma 3.19 tells us that $c_3 > c_2' + 1 = 3$, so by the above $c_3 = 4$, 6 or 8.

Lemma 3.19 also tells us that $c_3' > c_2 + 2 = 5$, so $c_3' = 6$, 9 or 12. ($c_3' = 15$ is not a possibility since it would mean that d_8 was odd.)

Let us suppose that $c_3 = 4$. Then $c_3' = 6$, $b_3 = 4$ and $b_3' = 8$.

Now, by Proposition 3.13, $c_4' > c_3 = 4$. $c_4' = 5$ or 6 would mean that k_4' is not an integer so the only possibilities are $c_4' = 7$ or 8. If $c_4' = 7$ then $b_4' = 1$ and, since $b_3 b_4 = b_3' b_4'$, $b_4 = 2$ and $c_4 = 13$. This would imply that k_4 is not an integer so is not possible. Therefore $c_4' = 8 = b_0'$ and, since $d = d_8$, $c_4 = 15 = b_0$.

Our arrays are:

$$\begin{bmatrix} * & 1 & 3 & 4 & 15 \\ 15 & 7 & 12 & 4 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 6 & 8 \\ 8 & 14 & 6 & 9 & * \end{bmatrix}$$

We note that if $c_3 = 6$ or 8 our integrality conditions on the k_i and k_i' lead us to contradictions.

(ii) $c_2' = 4 \Longrightarrow b_0 \leqslant 10$ from (*)

We know that $p = (c_2' - 1, b_0' - 1) = (3, 7) = 1$.

Therefore $b_0 = 7k + 1$, for some positive integer k, from Chapter 4 (**). However, since $b_0 \leqslant 10$ and $b_0 \neq b_0'$, this means that for $b_0' = 8$, $c_2' = 4$ we have no possible pairs of arrays.

ie: For $b_0' = 8$ and $c_2 \leqslant b_0' - 3$ we have one pair of possible arrays

(6) $b_0' = 9$. We know that $c_2' = 3$ is the only possible value of c_2', so $b_0 \leqslant 16 + 1 = 17$ from (*).

We also know that $p = (c_2' - 1, b_0' - 1) = (2, 8) = 2$.

Hence, $b_0 = 4k + 1$ for some positive integer k.

Combining these results gives us just two possible values of b_0 namely, $b_0 = 13$ or 17.

Substituting these values of b_0 in $J = (b_0, b_2')$ tells us that the only possible value of J is 1.

Therefore $$M = \frac{16}{k+2}$$ so k = 3 or 7.

k = 7 gives too large a value of b_0 so the only value of b_0 we need consider is when k = 3 ie: $b_0 = 13$.

We will consider the entries in our pair of arrays.

$b_0 = 13$, $b_0' = 9$, $b_1 = 8$, $c_1 = 1$, $b_1' = 12$, $c_1' = 1$, $b_2' = 6$ and $c_2' = 3$.

Since $b_1 b_2 = b_1' b_2'$ we know $b_2 = 9$ and hence $c_2 = 4$.

Recall that Lemma 3.21 (a) tells us that if $(b_0, c_2) = 1$ then d_p is odd. We will use this, together with results on c_2, c_2', c_3 and c_3', to construct our full arrays.

$$c_2 c_3 = c_2' c_3' \Longrightarrow 4c_3 = 3c_3'$$

Lemma 3.19 tells us $c_3 > c_2' + 1 = 4$, so by the above $c_3 = 6$ or 9.

Lemma 3.19 also tells us that $c_3' > c_2 + 2$ so $c_3' = 8$ or 12.

Let us suppose that $c_3 = 6$. Then, $c_3' = 8$, $b_3 = 3$ and $b_3' = 5$.

We know that d_p is odd so $d_p \geqslant 5$ (since we are assuming that $c_3 \neq b_0'$) but Lemma 3.21 (a) then tells us that c_2 (= 4) divides b_3 (= 3) which is not true. Therefore $c_3 = 9$, $c_3' = 12$ and $d_p = 3$ is our only possibility and we have one pair of feasible arrays for a distance-biregular graph, namely

$$\begin{bmatrix} * & 1 & 4 & 9 \\ 13 & 8 & 9 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 12 & 9 \\ 9 & 12 & 6 & 1 & * \end{bmatrix}$$

So, to summarize the previous results in a table, for $b_0' < 10$ we have the following possibilities.

b_0'	Possible values of b_0.	Number of feasible arrays.
3	None	None
4	7	1
5	None	None
6	11	1
	16	2
	21	1
7	None	None
8	15	2
9	13	1

Once we have $b_0' \geqslant 10$ the methods described here yield a greater number of possible values of b_0 .

Remark 5.1.

Note that for $b_0' < 10$ we have made no use of the global feasibility condition that the multiplicities of the eigenvalues should be integers.

Chapter 6.

We will start this chapter with a summary of results from previous chapters. We will then analyse all of the pairs of arrays obtained in Chapter 5 and attempt to find the distance-biregular graphs to which they correspond.

Summarizing the previous chapters gives us our definition for a pair of feasible arrays for a distance-biregular graph.

We say two intersection arrays are a pair of combinatorially feasible arrays for a distance-biregular graph if the following conditions are satisfied.

1. $$c_i + b_i = \begin{cases} b_0 & \text{if } i \text{ is even} \\ b'_0 & \text{if } i \text{ is odd.} \end{cases}$$

   $$c'_i + b'_i = \begin{cases} b_0 & \text{if } i \text{ is odd} \\ b'_0 & \text{if } i \text{ is even.} \end{cases}$$

2. $b'_{i-1} \geqslant b_i \geqslant b'_{i+1}$ for $1 \leqslant i \leqslant d - 2$.

   $c'_{i+1} \geqslant c_i \geqslant c'_{i-1}$ for $2 \leqslant i \leqslant d - 1$.

3. The numbers defined by the relationships below are positive integers.

   $k_0 = 1$, $k_{i+1} = k_i b_i \,/\, c_{i+1}$ $0 \leqslant i \leqslant d_P - 1$

   $k'_0 = 1$, $k'_{i+1} = k'_i b'_i \,/\, c'_{i+1}$ $0 \leqslant i \leqslant d_B - 1$.

(Since $|d_P - d_B| \leqslant 1$ we have the convention that if $d_B = d_P + 1$ $k_d = 0$.)

4. The following equations hold.

   $n := 1 + k_2 + \ldots + k_{d'} = k'_1 + k'_3 + \ldots + k'_{d''}$.

   and $m := k_1 + k_3 + \ldots + k_{d''} = 1 + k'_2 + \ldots + k'_{d'}$,

   where d' is the largest even integer less than or equal to d and d'' is the largest such odd integer.

   We also have $nb_0 = mb'_0$.

5. The $\alpha_p{}^q{}_t$ and $\beta_p{}^q{}_t$ defined in Chapter 2 are positive integers.

6. The diameter, d, is even.

7. $c_{2i}c_{2i+1} = c'_{2i}\,c'_{2i+1} \qquad 1 \leqslant i \leqslant d/2 - 1$

$b_{2i-1}b_{2i} = b'_{2i-1}b'_{2i} \qquad 1 \leqslant i \leqslant d/2 - 1.$

Now suppose that $b_0 > b'_0$.

8. c_2 divides $c_{2i-1}c_{2i}$, $c_{2i}c_{2i+1}$, $b_{2i-1}b_{2i}$ and $b_{2i}b_{2i+1}$.

c'_2 divides $c'_{2i-1}c'_{2i}$, $c'_{2i}c'_{2i+1}$, $b'_{2i-1}b'_{2i}$ and $b'_{2i}b'_{2i+1}$.

(For $1 \leqslant i \leqslant d/2 - 1$)

9. $b_{2i-1}(c_{2i} - c'_{2i-1}) = b'_{2i-1}(c'_{2i} - c_{2i-1})$

$b_{2i}(c'_{2i} - c_{2i-1}) = b'_{2i}(c_{2i} - c'_{2i-1})$

$c_{2i+1}(b_{2i} - b'_{2i+1}) = c'_{2i+1}(b'_{2i} - b_{2i+1})$

$c_{2i}(b'_{2i} - b_{2i+1}) = c'_{2i}(b_{2i} - b'_{2i+1})$

10. $\dfrac{c_{2i}(b_{2i-1} - 1) + b_{2i}(c_{2i+1} - 1)}{c_2} \leqslant k_{2i} - 1.$

$\dfrac{c_{2i-1}(b_{2i-2} - 1) + b_{2i-1}(c_{2i} - 1)}{c'_2} \leqslant k_{2i-1} - 1.$

11. $c'_{2i} > \dfrac{b'_0}{b_0}c_{2i}$; $b_{2i} > \dfrac{b_0}{b'_0}b'_{2i} \qquad 1 \leqslant i \leqslant d/2 - 1.$

$c_{2i-1} > \dfrac{b'_0}{b_0}c'_{2i-1}$; $b'_{2i-1} > \dfrac{b_0}{b'_0}b_{2i-1} \qquad 1 \leqslant i \leqslant d/2.$

12. If d_P is even then d_B is equal to d_P .

If d_P is odd then d_B is equal to $d_P + 1$.

13. $k_1 + k_2 + \ldots + k_{2i+1} > k'_1 + k'_2 + \ldots + k'_{2i+1}$.

14. $c_2 \geqslant c'_2$ with $c_2 = c'_2$ if and only if $c_2 = c'_2 = 1$.

15. $d_P \geqslant i + j \Longrightarrow \begin{cases} c_i \leqslant b_j & \text{if } i + j \text{ is even} \\ c'_i \leqslant b_j & \text{if } i + j \text{ is odd.} \end{cases}$

$d_B \geqslant i + j \Longrightarrow \begin{cases} c'_i \leqslant b'_j & \text{if } i + j \text{ is even} \\ c_i \leqslant b'_j & \text{if } i + j \text{ is odd.} \end{cases}$

16. If $d > 2$ then $c_2' \leqslant b_0'/2$ and $c_2 < b_0/2$.

From now on we will be considering arrays where $c_2' > 1$. (So the girth, g, is four.)

17. $c'_{i+1} > c_i$ and $c_{i+1} > c'_i$.

$b'_{i-1} > b_i$ and $b_{i-1} > b'_i$.

18. $c_{2i} > c'_{2i}$ and $c'_{2i+1} > c_{2i+1}$.

19. $k'_1 + k'_2 + \ldots + k'_{2i} > k_1 + k_2 + \ldots + k_{2i}$.

20. $c_2 \leqslant (b_0' - 1)$.

21. If $c_2' = 2$ then $c_3 \geqslant c_2 + 1$.

22. $c_3 > c_2' + 1$ and $c_3' > c_2 + 2$.

23. Either $d = 4$ or $d > 4$ and we have one of two cases.

(i) $d_p = d_B$ and $d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 2\}$

(ii) $d_p = d_B - 1$ and $d \leqslant \min\{b_0' - 2c_2' + 3,\ b_0 - 2c_2 + 3\}$

24. If $(b_0, c_2) = 1$ then d_p is odd and c_2 divides b_{2i+1} and c_{2i+2}.

If d_p is even then $(b_0, c_2) \neq 1$.

25. $(b_0', c_2') \neq 1$.

26. $d < 2b_0/c_2 + 2$.

27. $c_{2i} \geqslant c_2 + (i - 1)[b_0/b_0'] + 2i$

$c'_{2i+1} \geqslant c_2 + (i - 1)[b_0/b_0'] + (2i + 1)$

28. If d_p is even $d \leqslant \left[2 \cdot \frac{(b_2 + [b_0/b_0'])}{(2 + [b_0/b_0'])} \right]$

If d_p is odd $d \leqslant \left[2 \cdot \frac{(b_2 + 2[b_0/b_0'])}{(2 + [b_0/b_0'])} \right]$

29. $b_0'(b_0 - 1) \geqslant c_2' c_3' \left(\frac{b_0 - 1}{\binom{b_0' - 1}{c_2' - 1}} \right)$

For girth four we have the following pairs of combinatorially feasible arrays for distance-biregular graphs when $3 \leqslant b_0' \leqslant 9$ and the diameter, d, is greater than two.

1. $\begin{bmatrix} * & 1 & 3 & 4 \\ 7 & 3 & 4 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 6 & 4 \\ 4 & 6 & 2 & 1 & * \end{bmatrix}$

2. $\begin{bmatrix} * & 1 & 5 & 6 \\ 11 & 5 & 6 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 3 & 10 & 6 \\ 6 & 10 & 3 & 1 & * \end{bmatrix}$

3. $\begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix}$

4. $\begin{bmatrix} * & 1 & 4 & 5 & 16 \\ 16 & 5 & 12 & 1 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 10 & 6 \\ 6 & 15 & 4 & 6 & * \end{bmatrix}$

5. $\begin{bmatrix} * & 1 & 5 & 6 \\ 21 & 5 & 16 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 15 & 6 \\ 6 & 20 & 4 & 6 & * \end{bmatrix}$

6. $\begin{bmatrix} * & 1 & 7 & 8 \\ 15 & 7 & 8 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 4 & 14 & 8 \\ 8 & 14 & 4 & 1 & * \end{bmatrix}$

7. $\begin{bmatrix} * & 1 & 3 & 4 & 15 \\ 15 & 7 & 12 & 4 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 6 & 8 \\ 8 & 14 & 6 & 9 & * \end{bmatrix}$

8. $\begin{bmatrix} * & 1 & 4 & 9 \\ 13 & 8 & 9 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 3 & 12 & 9 \\ 9 & 12 & 6 & 1 & * \end{bmatrix}$

It turns out that all of these pairs of combinatorially feasible arrays are realisable and in the rest of this chapter we examine each of them. This suggests that combinatorial feasibility conditions are remarkably strong.

We note that we have excluded the pair of arrays

$$\begin{bmatrix} * & 1 & 6 & 7 & 36 \\ 36 & 7 & 30 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 21 & 8 \\ 8 & 35 & 6 & 15 & * \end{bmatrix}$$

from our list since there is no affine plane of order 6. This pair of arrays passes all of our other feasibility conditions.

In Appendix I we have a listing of a computer program, written by the author, which constructs and tests pairs of arrays by using combinatorial and algebraic feasibility conditions.

We will now consider our feasible arrays. In each case we will try to fully describe any corresponding distance-biregular graphs. We will refer to any possible distance-biregular graph as Γ and take u as any vertex in P.

1. $\begin{bmatrix} * & 1 & 3 & 4 \\ 7 & 3 & 4 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 6 & 4 \\ 4 & 6 & 2 & 1 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 7$, $k_2 = 7$, $k_3 = 7$. $|P| = 8$ and $|B| = 14$.

Firstly we notice that $\Gamma(u)$ and $\Gamma_2(u)$ form a 2-(7, 3, 1) design and Γ is the incidence graph of a quasisymmetric 2-(8, 4, 3) design with $\mu_1 = 2$ and $\mu_2 = 0$. From the section following Lemma 3.16 we know that since $k_1 = k_2 = k_3$ Γ is the incidence graph of a 3-(8, 4, 1) design associated with a Hadamard matrix of order 8. This is in fact the incidence structure formed by considering the 8 points and 14 planes of the 3-dimensional affine space over GF(2). (This can be thought of in terms of a cube and the planes associated with it.)

2. $\begin{bmatrix} * & 1 & 5 & 6 \\ 11 & 5 & 6 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 3 & 10 & 6 \\ 6 & 10 & 3 & 1 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 11$, $k_2 = 11$, $k_3 = 11$. $|P| = 12$ and $|B| = 22$.

Our pair of arrays certainly give us a 2-(12, 6, 5) design and, since $k_1 = k_2 = k_3$, our pair of arrays also give us a 3-(12, 6, 2) design associated with a Hadamard matrix of order 12.

Now suppose that we have a 3-(12, 6, 2) design. We will show that the incidence graph has arrays as above.

Firstly we ask the question ' given any pair of blocks what can they intersect in '? (We would like to show that they can only intersect in 0 or 3.)

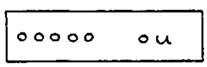
Take any block. Any three points lie in precisely two blocks. ie: Any three points determine another block.

How many ways are there of choosing three points?

$\binom{6}{3} = \frac{6!}{3!3!} = 20.$ ie: We have $1 + 20 = 21$ blocks.

What can we say about the remaining block and its intersection with our original one?

If it intersected our original block in a point, u say, how many blocks would u be contained in?



From our construction above, u would lie in $\binom{5}{2} + 1 = 11$ blocks that intersected with our original block in three points and also this extra block. ie: u would lie in 12 blocks.

We know from Design theory that if we have a t-(v, k, λ) design and we let λ_i denote the number of blocks containing a given set of i points, $0 \leqslant i \leqslant t$, then: $\lambda_i \binom{k-i}{t-i} = \binom{v-i}{t-i}\lambda$.

Therefore, for our 3-(12, 6, 2) design each point of our original block is in $\lambda_1 = 11$ blocks and u cannot lie in 12 blocks.

ie: each block intersects in 0 or 3 points and, since $\lambda_2 = 5$, we have a pair of arrays as shown.

3. $\begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 16$, $k_2 = 20$, $k_3 = 40$. $|P| = 21$ and $|B| = 56$.

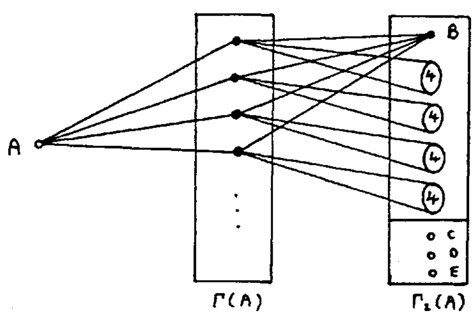
This pair of arrays was discussed in the section following Lemma 3.16 as a special case of when $c_2 = b_0' - 2$.

We know that we have a 2-(21, 6, 4) design and we will now try to find a projective plane (PG(2, 4)) within this design.

We certainly have 21 points but what about the lines?

Each line has 4 + 1 = 5 points on it so we need to try and find our sets of five points first.

Consider the following:



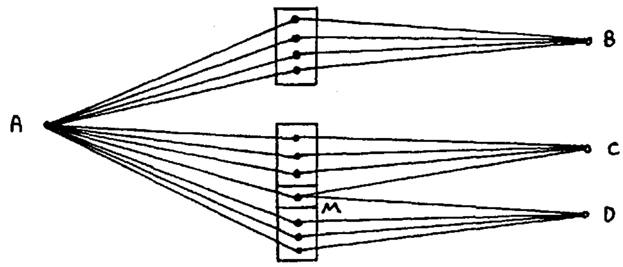
These need to be separate since $c_2' = 2$.

So, given any two points {A, B} there are exactly three other points {C, D, E} which have no common neighbour with the pair {A, B}. (Note that {A, B} = {B, A} and both give us {C, D, E}.)

Therefore we have the set {A, B, C, D, E}. Can we say " choose any pair {x, y} from {A, B, C, D, E} and consider all points which have no common neighbour with {x, y}. The set of points we obtain is {A, B, C, D, E} " ?

Suppose that {C, D} have a common neighbour with A.

ie:



(Note that {C, D} cannot have two common neighbours with A since this would contradict $c_2' = 2$.)

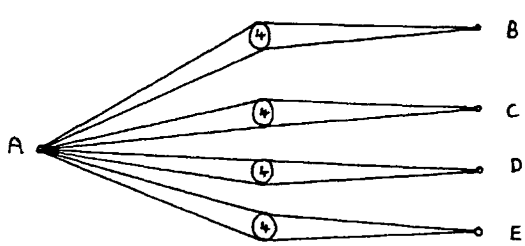
M is connected to three more points $\{x_1, x_2, x_3\}$ in $\Gamma_2(A)$ (since $b_1 = 5$). Also, x_i, $1 \leqslant i \leqslant 3$, is connected to four vertices in $\Gamma(A)$. Since x_i is not connected to any more of C and D's neighbours in $\Gamma(A)$ (as $c_2' = 2$) and is only connected to at most one neighbour of B in $\Gamma(A)$ (again, since $c_2' = 2$) each x_i, $1 \leqslant i \leqslant 3$, must determine two new vertices in $\Gamma(A)$. (These new vertices cannot be connected to more than one x_i otherwise $c_2' \neq 2$.)

ie: The number of vertices in $\Gamma(A) \geqslant 4 + 7 + 6 = 17$.

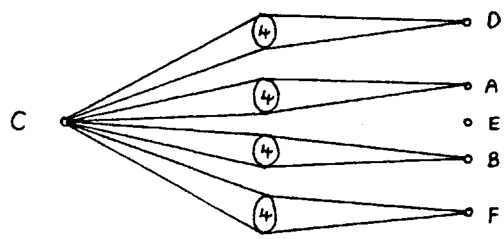
However, $|\Gamma(A)| = 16$.

Therefore {C, D} have no point in common with A.

Now {C, D} were chosen arbitrarily from {C, D, E} and we could have used B instead of A (since {A, B} = {B, A} and both give us {C, D, E}), so we have:



The only case left to consider is whether or not {C, D} have a point in common with E. If so we have a new point F which has no common neighbour with {C, D}. Consider the arrangenent:

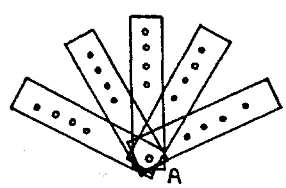


Now, $c'_2 = 2$ and $|\Gamma(C)| = 16$ so E is connected to a neighbour of A in $\Gamma(C)$ and a neighbour of B in $\Gamma(C)$. This means that C and E have a neighbour with A.#

Therefore whichever pair we choose from {A, B, C, D, E}, when we consider the points with no common neighbours with our pair we obtain {A, B, C, D, E} again.

This means that our lines intersect in at most one point but can they intersect in none?

Each point, A, lies in five lines ($|\Gamma_2(A)| = 20 = 4.5$) so we have:



Any other line has to be made up from five of these points. Suppose we have two lines l_1, l_2 which do not intersect. Take l_1 and the lines through any of its points, A say.

Now, l_2 has to be made up of five of these points. If it intersected any of these lines in two points it would have to be one of lines and would thus intersect l_1 in A. $\#$

Therefore l_2 has to intersect l_1 in a point other than A.

ie: Given any two lines they must intersect in a point. Also the number of lines is given by:

$$\frac{\text{(Number of possible points)x(Number of lines a point lies in)}}{\text{Number of points in a line}}$$

This equals 21 so we have found a copy of PG(2, 4) in our 2-(21, 6, 4) design.

4. $\begin{bmatrix} * & 1 & 4 & 5 & 16 \\ 16 & 5 & 12 & 1 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 10 & 6 \\ 6 & 15 & 4 & 6 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 16$, $k_2 = 20$, $k_3 = 48$, $k_4 = 3$. $|P| = 24$, $|B| = 64$.

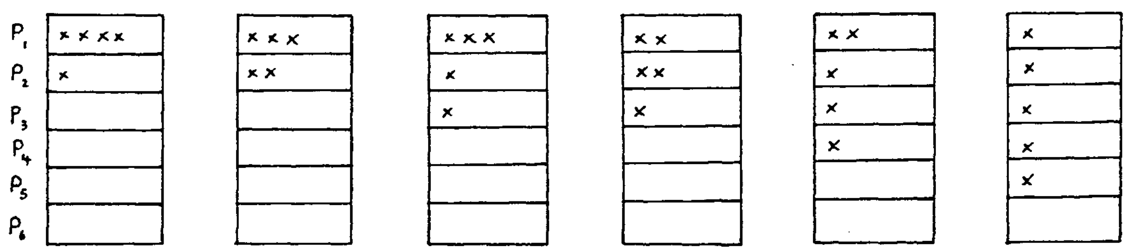
Γ is the incidence graph of a 2-(4.6, 6, 4) transversal design which is in fact a 3-(4.6, 6, 1) transversal design. We show that this design arises from the unique Steiner system S(5, 8, 24). The points are the 24 points of P. The vertices of B correspond to 6-sets in P. We use these 6-sets to define three kinds of 8-sets (octads).

(i) P is partitioned into 6 antipodal parts $P_1, P_2, P_3, P_4, P_5, P_6$ each of size 4. Any two such parts form a natural octad. (There are 15 such octads.)

(ii) Each block of the transversal design meets each antipodal part in exactly one point. If b is a block then the symmetric difference of b and P forms a natural octad. (There are 24.16 = 64.6 such octads.)

(iii) Two 6-sets which meet have just 2 points in common. Their symmetric difference forms a natural octad. Suppose the common points of the two 6-sets lie in P_5 and P_6 . It is not hard to show that for each partition of {1, 2, 3, 4} into two sets of size 2 we get the same octad in another way. Thus each such octad arises in at least 4 ways. (There are $\leqslant \frac{24.20}{2}\binom{4}{2}\Big/4 = 360$ such octads.)

If we now consider the possible configurations for 5-sets



one can show that each lies in at least one octad. It follows that each line lies in exactly one octad (since there are $\leqslant$ 759 octads).

5. $\begin{bmatrix} * & 1 & 5 & 6 \\ 21 & 5 & 16 & * \end{bmatrix}$ amd $\begin{bmatrix} * & 1 & 2 & 15 & 6 \\ 6 & 20 & 4 & 6 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 21$, $k_2 = 21$, $k_3 = 6$. $|P| = 22$ and $|B| = 77$.

Since $k_1 = k_2$ we know that $\Gamma(u)$ and $\Gamma_2(u)$ form a symmetric 2-(21, 5, 1) design (PG(2, 4)) and that Γ is the incidence graph of a 3-(22, 6, 1) design (a one point extension of PG(2,4)). The graph is unique [2] Theorem 8.18.

6. $\begin{bmatrix} * & 1 & 7 & 8 \\ 15 & 7 & 8 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 4 & 14 & 8 \\ 8 & 14 & 4 & 1 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 15$, $k_2 = 15$, $k_3 = 15$. $|P| = 16$ and $|B| = 30$.

We know that Γ is the incidence graph of a quasisymmetric 2-(16, 8, 7) design with $\mu_1 = 4$ and $\mu_2 = 0$ and, since $k_1 = k_2 = k_3$ Γ is the incidence graph of a Hadamard 3-(16, 8, 3) design - for example, the graph formed by the points and hyperplanes of AG(4, 2).

7. $\begin{bmatrix} * & 1 & 3 & 4 & 15 \\ 15 & 7 & 12 & 4 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 2 & 6 & 8 \\ 8 & 14 & 6 & 9 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 15$, $k_2 = 35$, $k_3 = 105$, $k_4 = 28$. $|P| = 64$ and $|B| = 120$.

We believe these arrays are realised (uniquely) by the natural intersection graph on cosets of two subgroups $H \cong A_8$ and $K \cong (\mathbb{Z}_2)^3.((\mathbb{Z}_2)^3.SL(3, 2))$ in a group $G \cong (\mathbb{Z}_2)^6.A_8$.

8. $\begin{bmatrix} * & 1 & 4 & 9 \\ 13 & 8 & 9 & * \end{bmatrix}$ and $\begin{bmatrix} * & 1 & 3 & 12 & 9 \\ 9 & 12 & 6 & 1 & * \end{bmatrix}$

$k_0 = 1$, $k_1 = 13$, $k_2 = 26$, $k_3 = 26$. $|P| = 27$ and $|B| = 39$.

The above pair of arrays corresponds to the incidence graph of a quasisymmetric 2-(27, 9, 4) design with $\mu_1 = 3$ and $\mu_2 = 0$ - for example, the incidence graph of the points and hyperplanes in AG(3, 3).

Appendix I.

In this appendix we give a listing of a computer program written by the author to help in the classification of distance-biregular graphs.

The program is written in PASCAL for the Honeywell Multics system at the University of Birmingham. To find the eigenvalues of a given matrix the program calls a FORTRAN program from the NAG library. The program uses the tests described in this thesis together with tests described in [4] for the distance-regular derived graphs.

```
PROGRAM magic (input, output) ;
{***************************************************************}
{* GUIDE TO THE PROGRAM :                                      *}
{*                                                             *}
{*intmat:=the intersection matrix for the left hand derived *}
{*        graph                                                *}
{*inma  :=the intersection matrix for the right hand derived*}
{*        graph                                                *}
{*entry :=an intersection array used in a subroutine          *}
{*        (gammatest) and which represents both left and      *}
{*        right hand derived graphs, whichever is              *}
{*        appropriate                                          *}
{*sm    :=a matrix whose entries come from alphas (or betas)*}
{*        and whose eigenvalues are found to help us in an    *}
{*        integrality test                                     *}
{*sum   :=this helps us to identify the entries in sm          *}
{*g     :=either an alpha or a beta, whichever is              *}
{*        appropriate                                          *}
{*intger:=this helps us to use the external program f02aff    *}
{*        from the NAG library                                 *}
{*rr    :=this helps us with eigenvalue checks                 *}
{*ri    :=this also helps us with eigenvalue checks            *}
{*no[i] :=the number of vertices at distance i from a vertex*}
{*        of valency r                                         *}
```

```
{*lo[i] :=the number of vertices at distance i from a vertex*}
{*        of valency s                                      *}
{*c[i]  :=the usual entry c_i                               *}
{*b[i]  :=the usual entry b_i                               *}
{*f[i]  :=the usual entry c_i'                              *}
{*e[i]  :=the usual entry b_i'                              *}
{*nod[i]:=the number of vertices at distance i from a given *}
{*        vertex                                            *}
{*add[i]:=this helps us to calculate our alphas and betas   *}
{***********************************************************}

{***********************************************************}
{* f02aff is an external subroutine used to find eigenvalues*}
{***********************************************************}

      $IMPORT
         'f02aff(fortran)': f02aff
         $

      TYPE
         smat = ARRAY [1..21, 1..21] OF real;
         oner = ARRAY [1..21] OF real;
         onei = ARRAY [1..21] OF integer;
         matr = ARRAY [0..20, 0..20, 0..20] OF integer;
```

```
    mat = ARRAY [1..3, 0..20] OF integer;
    sing = ARRAY [0..20] OF integer;

VAR
    intmat, inma : mat;
    sm : smat;
    g : matr;
    intger : onei;
    sum, rr, ri : oner;
    no, lo, c, b, f, nod, add, e : sing;
    i, r, s, d, wrong, sumnoe, sumnoo, sumloe, sumloo,
    pass, stop, j, p, ks, k, sa, ta, ra, n, ifail, lod,
    diameter, ne, fail, js, counter, loc, sip, din, bin,
    jb, r1, r2, r3 : integer;
    error, nr : real;


{***************************************************************}
{* The next part is used to test the eigenvalues of the two    *}
{* derived graphs. It is a FORTRAN subroutine which is called*}
{* from the NAG library                                         *}
{***************************************************************}


PROCEDURE fo2aff (VAR a: smat ; VAR ia, n : integer ;
```

```
VAR rr, ri : oner ; VAR intger : onei ; VAR ifail :
integer) ; EXTERNAL ;

{**************************************************************}
{* The next part simply helps us to set up the initial status*}
{**************************************************************}

PROCEDURE setup;

   BEGIN
      wrong := 0;
      no[0] := 1;
      no[1] := r;
      {r = valency of left hand array}
      i := 1 ;
      REPEAT
         i := i + 1 ;
         IF ((no[i-1]*b[i-1]) MOD c[i] = 0) THEN no[i] :=
         (no[i-1]*b[i-1]) DIV c[i]
         ELSE pass := 0;
      UNTIL i = j ;
   END;
```

```
{***********************************************************}
{* This gives the entries in the right hand array knowing the*}
{* entries in the left hand array                           *}
{***********************************************************}

      PROCEDURE construct ;

         BEGIN
            IF pass = 1 THEN
               BEGIN
                  lo[0] := 1;
                  lo[1] := s;
                  {s = valency of right hand array}
                  IF e[j-1] <> 0 THEN
                     BEGIN
                        IF j MOD 2 = 0 THEN
                           BEGIN
                              IF (b[j]*b[j-1]) MOD e[j-1] = 0
                              THEN
                                 BEGIN
                                    e[j] := (b[j]*b[j-1]) DIV
                                    e[j-1];
                                    f[j] := e[0] - e[j];
                                    IF f[j] < 0 THEN pass := 0
```

```
                END
              ELSE pass := 0;
            END

      ELSE

            BEGIN
              IF (c[j]*c[j-1]) MOD f[j-1] = 0
              THEN
                BEGIN
                  f[j] := (c[j]*c[j-1]) DIV
                  f[j-1];
                  e[j] := b[0] - f[j];
                  IF e[j]<0 THEN pass := 0
                END
              ELSE pass := 0;
            END
        END
      ELSE
        BEGIN
          e[j] := 0;
          f[j] := 0;
        END;
      IF (e[j] = 0) AND (b[j] <> 0) THEN pass := 0;
```

```
                IF (pass = 1) AND (f[j] <> 0) THEN
                   BEGIN
                      IF ((lo[j-1]*e[j-1]) MOD f[j] = 0)
                      THEN lo[j] := (lo[j-1]*e[j-1]) DIV f[j]
                      ELSE pass := 0;
                   END;
                IF f[j] = 0 THEN lo[j] := 0;
                IF (pass = 1) AND (c[j] = e[0]) AND (j MOD 2
                = 1) THEN
                   BEGIN
                      d := j;
                      f[d + 1] := s;
                      e[d + 1] := 0;
                      lo[d + 1] := (lo[d]*e[d]) DIV f[d + 1];
                      IF e[d] = 0 THEN pass := 0
                   END
             END
       END;


{***************************************************************}
{*              Here we calculate our k's and l's              *}
{***************************************************************}
```

```
PROCEDURE circles;

    BEGIN
        IF (pass = 1) THEN
            BEGIN
                sumnoe := 0;
                sumnoo := 0;
                sumloe := 0;
                sumloo := 0;
                i := -1;
                REPEAT
                    i := i + 1;
                    sumnoe := sumnoe + no[2*i];
                    sumloo := sumloo + lo[2*i + 1];
                    sumnoo := sumnoo + no[2*i + 1];
                    sumloe := sumloe + lo[2*i];
                UNTIL i = (d DIV 2) - 1;
                sumnoe := sumnoe + no[2*(d DIV 2)];
                sumloe := sumloe + lo[2*(d DIV 2)];
                IF (d MOD 2 <> 0) THEN
                    BEGIN
                        sumnoo := sumnoo + no[d];
                        sumloe := sumloe + lo[d + 1];
                        sumloo := sumloo + lo[d]
```

```
                    END;
                  IF (sumnoo <> sumloe) OR (sumnoe <> sumloo)
                  THEN pass := 0
                END
            END;

{**************************************************************}
{*               This just tests obvious inequalities         *}
{**************************************************************}

     PROCEDURE inequalities;

        BEGIN
          IF (pass = 1) AND (d > 3) THEN
            BEGIN
              IF e[j - 2] < e[j] THEN pass := 0;
              IF f[j - 2] > f[j] THEN pass := 0
            END
        END;

{**************************************************************}
{*  This constructs our two distance-regular derived graphs   *}
{**************************************************************}
```

```
PROCEDURE derive;

 BEGIN
  IF (pass = 1) AND (j > 3) THEN
    BEGIN
       intmat[1, 0] := 0;
       intmat[2, 0] := 0;
       intmat[3, 0] := (b[0]*b[1]) DIV c[2];
       inma[1, 0] := 0;
       inma[2, 0] := 0;
       inma[3, 0] := (e[0]*e[1]) DIV f[2];
       i := 0;
       REPEAT
          i := i + 1;
          IF (b[2*i]*b[2*i + 1]) MOD c[2] = 0 THEN
             intmat[3, i] := (b[2*i]*b[2*i + 1]) DIV c[2]
          ELSE pass := 0;
          IF (pass = 1) AND ((c[2*i - 1]*c[2*i]) MOD c[2]
             = 0) THEN
             BEGIN
                intmat[1, i] := (c[2*i - 1]*c[2*i]) DIV c[2];
                intmat[2, i] := intmat[3, 0] - intmat[1, i]
                                - intmat[3, i]
```

```
      END
   ELSE pass := 0;
   IF (pass = 1) AND ((e[2*i]*e[2i + 1]) MOD f[2] = 0
   THEN inma[3, i] := (e[2*i]*e[2*i + 1]) DIV f[2]
   ELSE pass := 0;
   IF (pass = 1) AND ((f[2*i - 1]*f[2*i]) MOD f[2] = 0
   THEN
      BEGIN
         inma[1, i] := (f[2*i - 1]*f[2*i]) DIV f[2];
         inma[2, i] := inma[3, 0] - inma[1, i]
                       - inma[3, i]
      END
   ELSE pass := 0;
UNTIL i = j DIV 2 - 1;
IF (pass = 1) AND (j MOD 2 = 0) AND (b[j] = 0) THEN
   BEGIN
      IF (c[j - 1]*c[j]) MOD c[2] = 0 THEN
         BEGIN
            intmat[1, j DIV 2] := (c[j - 1]*c[j])
                                  DIV c[2];
            intmat[2, j DIV 2] := intmat[3, 0]
                                 - intmat[1, j DIV 2];
            intmat[3, j DIV 2] := 0
         END
```

```
            ELSE pass := 0;
            IF ((f[j - 1]*f[j]) MOD f[2] = 0) AND (pass = 1)
            THEN
               BEGIN
                  inma[1, j DIV 2] := (f[j - 1]*f[j]) DIV
                                      f[2];
                  inma[2, j DIV 2] := inma[3, 0]
                                    - inma[1, j DIV 2];
                  inma[3, j DIV 2] := 0
               END
            ELSE pass := 0
         END;
      IF (pass = 1) AND (j MOD 2 = 1) AND (b[j] = 0) THEN
         BEGIN
            IF (c[j - 2] * c[j - 1]) MOD c[2] = 0 THEN
               BEGIN
                  intmat[1, j DIV 2] := (c[j - 2]*c[j - 1])
                                        DIV c[2];
                  intmat[2, j DIV 2] := intmat[3, 0]
                                      - intmat[1, j DIV 2];
                  intmat[3, j DIV 2] := 0
               END
            ELSE pass := 0;
```

```
            IF (pass = 1) AND ((f[j - 2]*f[j - 1]) MOD f[2]
               = 0) THEN
               BEGIN
                  inma[1, j DIV 2] := (f[j - 2]*f[j - 1])
                                      DIV f[2];
                  inma[3, j DIV 2] := (e[j]*e[j - 1])
                                      DIV f[2];
                  inma[2, j DIV 2] := inma[3, 0] -
                       inma[1, j DIV 2] - inma[3, j DIV 2];
                  IF inma[3, j DIV 2] = 0 THEN pass := 0
               END
         END;
      IF (pass = 1) AND (j MOD 2 = 1) AND (b[j] = 0) THEN
         BEGIN
            inma[1, j DIV 2 + 1] := (f[j + 1]*f[j]) DIV
                                    f[2];
            inma[2, j DIV 2 + 1] := inma[3,0] -
                                    inma[1, j DIV 2 + 1];
            inma[3, j DIV 2 + 1] := 0
         END
   END
END;
```

```
{*************************************************************}
{* This sets up the matrix whose eigenvalues we wish to      *}
{* consider.                                                 *}
{*************************************************************}

 PROCEDURE gammatest (VAR entry : mat);

    VAR
       p, ra, sa, ta, x, y, q, i, ia, u : integer;

    BEGIN
       IF (entry[3, js] = 0) AND (pass = 1) AND (js > 1) THEN
          BEGIN
             FOR sa := 0 TO 20 DO
                BEGIN
                   FOR ra := 0 TO 20 DO
                      BEGIN
                         FOR ta := 0 TO 20 DO
                            BEGIN
                               g[sa, ta, ra] := 0
                            END
                      END
                END;
             FOR p := 0 TO js DO
```

```
    BEGIN
        g[0, p, p] := 1;
        g[p, 0, p] := 1;
        g[1, p, p] := entry[2, p];
        g[p, 1, p] := g[1, p, p]
    END;
FOR p := 0 TO js - 1 DO
    BEGIN
        g[1, p + 1, p] := entry[3, p];
        g[p + 1, 1, p] := g[1, p + 1, p]
    END;
FOR p := 1 TO js DO
    BEGIN
        g[1, p - 1, p] := entry[1, p];
        g[p - 1, 1, p] := g[1, p - 1, p]
    END;
IF (pass = 1) THEN
    BEGIN
        d := js;
        nod[0] := 1;
        lod := 0;
        REPEAT
            lod := lod + 1;
            IF lod = 1 THEN ks := entry[3, 0];
```

```
   IF lod <> 1 THEN ks := ks*entry[3, lod - 1]
                                 DIV entry[1, lod];
   nod[lod] := ks;
UNTIL lod = d;
FOR sa := 0 TO d DO
   BEGIN
      FOR ra := 0 TO d DO
         BEGIN
            IF sa = ra THEN g[sa, ra, 0] :=
                                nod[sa];
         END
   END;
sa := 1;
REPEAT
   sa := sa + 1;
   ra := 0;
   REPEAT
      ra := ra + 1;
      ta := sa - 1;
      REPEAT
         ta := ta + 1;
         IF ta <> d THEN
            BEGIN
```

```
                                        g[sa, ta, ra] :=
(entry[3, ta - 1]*g[sa - 1, ta - 1, ra] + (entry[2, ta] -
 entry[2, sa - 1])*g[sa - 1, ta, ra] + entry[1, ta + 1]*
 g[sa - 1, ta + 1, ra] - entry[3, sa - 2] *
 g[sa - 2, ta, ra]);
                                        IF g[sa, ta, ra] MOD
 entry[1, sa] <> 0 THEN pass := 0
                                        ELSE g[sa, ta, ra] :=
 g[sa, ta, ra] DIV entry[1,sa];
                                        g[ta, sa, ra] := g[sa, ta, ra]
                                     END;
                                  IF ta = d THEN
                                     BEGIN
                                        add[0] := g[sa, 0, ra];
                                        FOR p := 1 TO (d - 1) DO
                                            BEGIN
                                               add[p] := add[p - 1]
                                                       + g[sa, p, ra]
                                            END;
                                        g[sa, ta, ra] := nod[sa] -
                                                        add[d - 1];
                                        g[ta, sa, ra] := g[sa, ta, ra]
                                     END;
                                  IF g[sa, ta, ra] < 0 THEN pass := 0;
```

```
            IF pass = 0 THEN
               BEGIN
                  ta := d;
                  ra := d;
                  sa := d
               END;
         UNTIL ta = d;
      UNTIL ra = d;
   UNTIL sa := d;
   error := 0.000001;
   k := entry[3, 0];
   IF pass = 1 THEN
      BEGIN
         FOR p := 1 TO d + 1 DO
            BEGIN
               rr[p] := 0;
               ri[p] := 0
            END;
         FOR p := 1 TO 21 DO
            BEGIN
               FOR q := 1 TO 21 DO
                  BEGIN
                     sm[p, q] := 0.0
                  END
```

```
   END;
ne := 0;
FOR p := 0 TO d DO
   BEGIN
      ne := ne + nod[p]
   END;
FOR p := 1 TO d + 1 DO
   BEGIN
      FOR q := 1 TO d + 1 DO
         BEGIN
            FOR x := 1 TO d + 1 DO
               BEGIN
                  sum[x] := 0
               END;
            FOR i := 1 TO d + 1 DO
               BEGIN
                  FOR y := 1 TO d + 1 DO
                     BEGIN
                        sum[i] := sum[i]
                 + g[i - 1, y - 1, y - 1]
                     END;
                  sum[i] := sum[i]/
                            nod[i - 1];
```

```
                        sum[i] := sum[i]*
                        g[i - 1, p - 1, q - 1];
                        sm[p, q] := sum[i] +
                              sm[p, q]
                    END
                END
            END;
FOR p := 1 TO d + 1 DO
    BEGIN
        intger [p] := 0
    END;
ifail := 0;
ia := 21;
n := d + 1;
f02aff(sm, ia, n, rr, ri, intger, ifail);
IF ifail = 0 THEN
    BEGIN
        nr := ne;
        p := 0;
        REPEAT
            p := p + 1;
            IF abs(round(ri[p])) > error
            THEN pass := 0;
            IF rr[p] <> 0.0 THEN
```

```
                                  BEGIN
                                     IF abs(round(nr/rr[p]) -
                              nr/rr[p]) > error THEN pass := 0
                                  END;
                                IF pass = 0 THEN p := d + 1;
                             UNTIL p = d + 1;
                          END
                    END
              END
        END
  END;
{***************************************************************}
{*             Here we simply print out the results            *}
{***************************************************************}
 PROCEDURE conclusion;
    BEGIN
       IF (pass = 1) AND (b[j] = 0) AND (fail = 0) AND (c[2] <>
                                           1) AND (j > 2) THEN
          BEGIN
             write('*' : 3);
             p := 0;
             REPEAT
                p := p + 1;
                write(c[p] : 3);
```

```
UNTIL p = j;
writeln;
p := -1;
REPEAT;
   p := p + 1;
   write(b[p] : 3);
UNTIL p = j - 1;
write('*' : 3);
writeln;
writeln;
p := -1;
REPEAT
   p := p + 1;
   writeln(' k', p : 2, '=', no[p] : 5);
UNTIL p = j;
writeln;
writeln;
IF j > 3 THEN
   BEGIN
      writeln(' Derived graph :');
      writeln;
      write('*' : 4);
      p := 0;
      REPEAT
```

```
            p := p + 1;
            write(intmat[1, p] : 4);
        UNTIL intmat[3, p] = 0;
        writeln;
        p := -1;
        REPEAT
            p := p + 1;
            writel(intmat[2, p] : 4);
        UNTIL intmat[3, p] = 0;
        writeln;
        p := -1;
        REPEAT
            p := p + 1;
            write(intmat[3, p] : 4);
        UNTIL intmat[3, p + 1] = 0;
        write('*' : 4);
        writeln;
        writeln
    END;
write('*' : 3);
p := 0;
REPEAT
    p := p + 1;
    write(f[p] : 3);
```

```
UNTIL p = j;
IF (b[j] = 0) AND (j MOD 2 = 1) THEN write
                                   (f[j + 1] : 3);
writeln;
p := -1;
REPEAT
   p := p + 1;
   write(e[p] : 3);
UNTIL p = j - 1;
IF (b[j] = 0) AND (j MOD 2 = 1) THEN write
                                      (e[j] : 3);
write('*' : 3);
writeln;
writeln;
p := -1;
REPEAT
   p := p + 1;
   write('  1', p : 2, '=', lo[p] : 5);
UNTIL p = j;
IF (j MOD 2 = 1) AND (b[j] = 0) THEN write
   ('  1', j + 1 : 2, '=', lo[j + 1] : 5);
writeln;
writeln;
IF j > 3 THEN
```

```
BEGIN
   writeln(' Derived graph :');
   writeln;
   write('*' : 4);
   p := 0;
   REPEAT
      p := p + 1;
      write(inma[1, p] : 4);
   UNTIL inma[3, p] = 0;
   writeln;
   p := -1;
   REPEAT
      p := p + 1;
      write (inma[2, p] : 4);
   UNTIL inma[3, p] = 0;
   writeln;
   p := -1;
   REPEAT
      p := p + 1;
      write(inma[3, p] : 4);
   UNTIL inma[3, p + 1] = 0;
   write('*' : 4);
   writeln;
   writeln
```

```
            END;
        writeln
      END
   END;

{***********************************************************}
{*          Now we have the main program 'magic'           *}
{***********************************************************}

BEGIN                                                {magic}

writeln('****************************************************');
writeln('*This program aims to help in the classification of *');
writeln('*distance-biregular graphs.                         *');
writeln('*It does this by making use of several feasibility  *');
writeln('*conditions to form an algorithm.                   *');
writeln('*The program takes two values of s, a value of r and*');
writeln('*a diameter bound. It considers all possible values *');
writeln('*of s in between the two given values, and all      *');
writeln('*possible values of r up to, and including, the     *');
writeln('*chosen bound. It then gives all possible, feasible *');
writeln('*pairs of arrays with these valencies which have a  *');
writeln('*diameter not greater than the one given.           *');
writeln('****************************************************');
```

```
writeln('You are asked to input four things:');
writeln('1.The valency of the lower right-hand array (s1).');
writeln('2.The valency of the higher right-hand array (s2).');
writeln('3.The bound you wish to have on the left-hand valency
                                                        (r)');
writeln('4.The bound on the diameter of graphs you wish to
                                                  consider.');
writeln('The program then outputs various pairs of arrays
                                                   together');
writeln('with the ki, li and derived graphs.');
writeln;
writeln('Please input the right-hand valencies and a bound on
                                                        r.');
writeln('Firstly s1:');
read(r1);
writeln('Secondly s2 (remember that s2 > s1):');
read(r2);
writeln('Thirdly r:');
read(r3);
writeln('Now please input your diameter bound');
read(diameter);
writeln;
writeln('The possible pairs of arrays are as follows:');
writeln;
```

```
FOR s := r1 TO r2 DO
   BEGIN
      FOR r := (s + 1) TO r3 DO
         BEGIN
            counter := 0;
            loc := 0;
            c[0] := 0;
            b[0] := r;
            c[1] := 1;
            b[1] := s - 1;
            f[0] := 0;
            e[0] := s;
            f[1] := 1;
            e[1] := r - 1;
            FOR sa := 0 TO 20 DO
               BEGIN
                  FOR ra := 0 TO 20 DO
                     BEGIN
                        FOR ta := 0 TO 20 DO
                           BEGIN
                              g[sa, ta, ra] := 0
                           END
                     END
               END;
```

```
j := 1;
REPEAT
   stop := 0;
   j := j + 1;
   IF j MOD 2 = 0 THEN c[j] := b[0] + 1
   ELSE c[j] := e[0] + 1;
   REPEAT
      FOR sa := 1 TO 3 DO
         BEGIN
            FOR ra := 0 TO 20 DO
               BEGIN
                  intmat[sa, ra] := 1;
                  inma[sa, ra] := 1
               END
         END;
      pass := 1;
      fail := 0;
      d := j;
      c[j] := c[j] - 1;
      IF (c[2] > e[0] - 1) THEN pass := 0;
      IF pass = 1 THEN
         BEGIN
            IF j MOD 2 = 0 THEN b[j] := b[0] - c[j]
            ELSE b[j] := e[0] - c[j];
```

```
loc := loc + 1;
setup;
construct;
IF (j > 2) AND (c[2] > 1) AND
   (f[3] < c[2] + 3) THEN pass := 0;
IF (b[j] <> 0) AND (j > 2) AND
   pass = 1 THEN
   BEGIN
      IF (c[j] <= f[j - 1]) OR
      (f[j] <= c[j - 1]) THEN pass := 0
   END;
IF b[j] = 0 THEN
   BEGIN
      circles;
   END;
inequalities;
{***************************************************************}
{*          Here we want to test the array so far             *}
{***************************************************************}
IF (j > 2) AND (pass = 1) THEN
   BEGIN
      IF c[3] < f[2] + 2 THEN pass := 0;
      IF f[3] < c[2] + 3 THEN pass := 0;
```

```
        IF (f[2] = 2) AND
         (c[3] < c[2] + 1) THEN pass := 0;
        IF (c[j - 1]*c[j]) MOD c[2] <> 0
                                THEN pass := 0;
        IF (f[j - 1]*f[j]) MOD f[2] <> 0
                                THEN pass := 0;
        IF (b[j - 1]*b[j]) MOD c[2] <> 0
                                THEN pass := 0;
        IF (e[j - 1]*e[j]) MOD f[2] <> 0
                                THEN pass := 0;
        IF (b[j] = 0) AND (j MOD 2 = 1)
                                THEN
           BEGIN
              IF (f[j]*f[j + 1]) MOD f[2]
                           <> 0 THEN pass := 0;
              IF (e[j]*e[j + 1]) MOD f[2]
                           <> 0 THEN pass := 0;
           END;
     END;
  IF (j > 2) AND (pass = 1) THEN
     BEGIN
        IF (b[j] = 0) AND (j MOD 2 = 1)
                                THEN
           BEGIN
```

```
                IF ((c[j]*(b[j - 1] - 1))
        DIV f[2] > no[j] - 1) THEN pass := 0;
                IF ((f[j+1]*(e[j] - 1))
    DIV f[2] > lo[j + 1] - 1) THEN pass := 0;
             END;
          IF (b[j] = 0) AND (j MOD 2 = 0)
                              THEN
             BEGIN
                IF ((c[j]*(b[j - 1] - 1))
        DIV c[2] > no[j] - 1) THEN pass := 0;
                IF ((f[j]*(e[j - 1] - 1))
        DIV f[2] > lo[j] - 1) THEN pass := 0;
             END
       END;
    IF (j > 1) AND (c[2] > r DIV 2) THEN
                                   pass := 0;
    IF (j > 1) AND (f[2] > s DIV 2) THEN
                                   pass := 0;
    IF (j > 2) AND (pass = 1) THEN
       BEGIN
          din := 1;
          bin := 1;
          FOR jb := 1 TO (f[2] - 1) DO
             BEGIN
```

```
                    bin := bin*(s - jb);
                    din := din*jb;
                 END;
              bin := bin DIV din;
              IF (r - 1) MOD (bin) = 0 THEN
                          bin := (r - 1) DIV bin
              ELSE bin := (r - 1) DIV bin + 1;
              bin := f[2]*f[3]*bin;
              IF (s*(r - 1)) < bin THEN
                                         pass := 0;
           END;
        IF (b[j] = 0) AND (pass = 1) THEN
           BEGIN
              counter := counter + 1;
              derive;
              js := j DIV 2;
              gammatest(intmat);
              IF j MOD 2 = 0 THEN js := j DIV 2;
              gammatest(inma);
              IF j MOD 2 = 1 THEN js :=
                                       j DIV 2 + 1;
              gammatest(inma);
           END;
        END;
     conclusion;
```

```
IF (c[j] = c[j - 2]) AND (pass = 0) THEN
   BEGIN
      IF (j = 3) AND (c[2] = 1) THEN
                                 stop := 1;
      ELSE
         BEGIN
            i := 0;
            REPEAT
               i := i + 1;
            UNTIL c[j - 1] <>
                  c[j - i - 2];
            IF (j = 2 + i) AND
              (c[2] = 1) THEN stop := 1;
            ELSE j := j - i;
         END
   END;
IF (j = diameter) AND (stop = 0) AND
((c[j - 1] <> 1) OR (c[j - 2] <> 1) THEN
   BEGIN
      i := 0;
      REPEAT
         pass := 0;
         i := i + 1;
      UNTIL c[j - i] <> c[j - i - 2];
```

```
                    IF (j = 2 + i) AND (c[2] = 1)
                                   THEN stop := 1
                    ELSE j := j - i
                  END;
                IF b[j] = 0 THEN pass := 0;
                IF c[j] < c[j - 2] THEN stop := 1;
                IF b[j] < 0 THEN stop := 1;
                IF c[2] < 1 THEN stop := 1;
                IF stop = 1 THEN pass := 1;
              UNTIL pass = 1;
              IF c[2] = 1 THEN stop := 1;
              IF stop = 1 THEN j := diameter;
            UNTIL j = diameter;
          END;
       END;
    END.                                                {magic}
```

Appendix II.

In this appendix we give a list of pairs of feasible arrays for $9 < b_0' < 20$ and $c_2 > 1$. This list was obtained by using the program in Appendix I. We do not analyse any of the pairs of arrays (some can be excluded by other combinatorial reasons) but include the list to demonstrate the efficiency of the tests described in this thesis and to provide work for further research.

b_0'	Possible values of b_0.	Number of feasible arrays.
10	19	1
	28	2
	46	1
	64	1
11	None	None
12	23	1
	45	5
	56	1
	100	2
	111	1
13	None	None
14	27	3
	40	2
	66	2
	144	1

b'_0	Possible values of b_0.	Number of feasible arrays.
15	22	1
	36	6
16	21	1
	31	1
	91	2
	196	1
17	None	None
18	35	2
	52	2
	120	4
	256	1
19	None	None

$b_0' = 10.$

$$\begin{bmatrix} * & 1 & 9 & 10 \\ 19 & 9 & 10 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 5 & 18 & 10 \\ 10 & 18 & 5 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 4 & 9 & 28 \\ 28 & 9 & 24 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 18 & 10 \\ 10 & 27 & 8 & 10 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 4 & 6 & 28 \\ 28 & 9 & 24 & 4 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 12 & 10 \\ 10 & 27 & 8 & 16 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 10 \\ 46 & 9 & 40 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 30 & 10 \\ 10 & 45 & 8 & 16 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 8 & 9 & 64 \\ 64 & 9 & 56 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 36 & 10 \\ 10 & 63 & 8 & 28 & * \end{bmatrix}$$

$b_0' = 11.$

None.

$b_0' = 12.$

$$\begin{bmatrix} * & 1 & 11 & 12 \\ 23 & 11 & 12 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 6 & 22 & 12 \\ 12 & 22 & 6 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 9 & 12 \\ 45 & 11 & 36 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 36 & 12 \\ 12 & 44 & 9 & 9 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 9 & 11 & 45 \\ 45 & 11 & 36 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 33 & 12 \\ 12 & 44 & 9 & 12 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 9 & 6 & 45 \\ 45 & 11 & 36 & 6 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 18 & 12 \\ 12 & 44 & 9 & 27 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 5 & 12 \\ 45 & 11 & 40 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 30 & 12 \\ 12 & 44 & 10 & 15 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 5 & 8 & 45 \\ 45 & 11 & 40 & 4 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 20 & 12 \\ 12 & 44 & 10 & 25 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 11 & 12 \\ 56 & 11 & 45 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 44 & 12 \\ 12 & 55 & 9 & 12 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 10 & 12 \\ 100 & 11 & 90 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 60 & 12 \\ 12 & 99 & 10 & 40 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 10 & 11 & 100 \\ 100 & 11 & 90 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 55 & 12 \\ 12 & 99 & 10 & 45 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 11 & 12 \\ 111 & 11 & 100 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 66 & 12 \\ 12 & 110 & 10 & 45 & * \end{bmatrix}$$

<u>$b_0' = 13.$</u>

None.

$b_0' = 14.$

$$\begin{bmatrix} * & 1 & 13 & 14 \\ 27 & 26 & 14 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 7 & 26 & 14 \\ 14 & 26 & 7 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 3 & 8 & 27 \\ 27 & 13 & 24 & 6 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 12 & 14 \\ 14 & 26 & 12 & 15 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 3 & 4 & 27 \\ 27 & 13 & 24 & 10 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 6 & 14 \\ 14 & 26 & 12 & 21 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 4 & 13 & 40 \\ 40 & 13 & 36 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 26 & 14 \\ 14 & 39 & 12 & 14 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 4 & 9 & 40 \\ 40 & 13 & 36 & 5 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 18 & 14 \\ 14 & 39 & 12 & 22 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 13 & 66 \\ 66 & 13 & 60 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 39 & 14 \\ 14 & 65 & 12 & 27 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 10 & 66 \\ 66 & 13 & 60 & 4 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 30 & 14 \\ 14 & 65 & 12 & 36 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 12 & 13 & 144 \\ 144 & 13 & 132 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 78 & 14 \\ 14 & 143 & 12 & 66 & * \end{bmatrix}$$

$b_0' = 15.$

$$\begin{bmatrix} * & 1 & 7 & 15 \\ 22 & 14 & 15 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 5 & 21 & 15 \\ 15 & 21 & 10 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 15 \\ 36 & 14 & 30 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 30 & 15 \\ 15 & 35 & 12 & 6 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 14 & 36 \\ 36 & 14 & 30 & 1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 28 & 15 \\ 15 & 35 & 12 & 8 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 10 & 36 \\ 36 & 14 & 30 & 5 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 20 & 15 \\ 15 & 35 & 12 & 16 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 6 & 36 \\ 36 & 14 & 30 & 9 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 12 & 15 \\ 15 & 35 & 12 & 24 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 6 & 28 & 15 \\ 36 & 14 & 30 & 9 & 8 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 12 & 12 & 35 & 15 \\ 15 & 35 & 12 & 24 & 3 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 6 & 5 & 36 \\ 36 & 14 & 30 & 10 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 3 & 10 & 15 \\ 15 & 35 & 12 & 26 & * \end{bmatrix}$$

$b_0' = 16.$

$$\begin{bmatrix} * & 1 & 5 & 16 \\ 21 & 15 & 16 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 4 & 20 & 16 \\ 16 & 20 & 12 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 15 & 16 \\ 31 & 15 & 16 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 8 & 30 & 16 \\ 16 & 30 & 8 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 7 & 12 & 91 \\ 91 & 15 & 84 & 4 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 2 & 42 & 16 \\ 16 & 90 & 14 & 49 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 7 & 10 & 91 \\ 91 & 15 & 84 & 6 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 35 & 16 \\ 16 & 90 & 14 & 56 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 14 & 15 & 196 \\ 196 & 15 & 182 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 105 & 16 \\ 16 & 195 & 14 & 91 & * \end{bmatrix}$$

<u>$b_0' = 17.$</u>

None.

<u>$b_0' = 18.$</u>

$$\begin{bmatrix} * & 1 & 17 & 18 \\ 35 & 17 & 18 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 9 & 34 & 18 \\ 18 & 34 & 9 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 5 & 6 & 35 \\ 35 & 17 & 30 & 12 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 10 & 18 \\ 18 & 34 & 15 & 25 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 4 & 17 & 52 \\ 52 & 17 & 48 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 34 & 18 \\ 18 & 51 & 16 & 18 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 4 & 12 & 52 \\ 52 & 17 & 48 & 6 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 24 & 18 \\ 18 & 51 & 16 & 28 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 15 & 17 & 120 \\ 120 & 17 & 105 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 85 & 18 \\ 18 & 119 & 15 & 35 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 15 & 7 & 120 \\ 120 & 17 & 105 & 11 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 35 & 18 \\ 18 & 119 & 15 & 85 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 8 & 17 & 120 \\ 120 & 17 & 112 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 68 & 18 \\ 18 & 119 & 16 & 52 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 8 & 14 & 120 \\ 120 & 17 & 112 & 4 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 56 & 18 \\ 18 & 119 & 16 & 64 & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 16 & 17 & 256 \\ 256 & 17 & 240 & 1 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 136 & 18 \\ 18 & 255 & 16 & 120 & * \end{bmatrix}$$

<u>$b_0' = 19$.</u>

None.

Questions.

Q1. (cf. Theorem 3.3 and Theorem 3.4.)

Let M be a matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & & \bigcirc \\ m_{21} & 0 & m_{23} & & \\ & m_{32} & \ddots & \ddots & \\ \bigcirc & & \ddots & \ddots & \\ & & & & 0 \end{bmatrix}$$

with all $m_{ij} \neq 0$ and let $\underline{x} = (1 \quad x_1 \quad x_2 \quad \ldots \quad x_d)^T$ be a right eigenvector of M with eigenvalue λ ($\neq 0$). Let $k_0 = 1$, $k_1 = m_{21}$, $k_{i+1} = \dfrac{k_i \; m_{i+2\, i+1}}{m_{i+1\, i+2}}$ $(i \geqslant 1)$.

(Under what additional conditions) is it automatically true that $\sum_{i\ \mathrm{even}} x_i^2/k_i = \sum_{i\ \mathrm{odd}} x_i^2/k_i$?

Q2. (cf. Lemma 3.8 and Lemma 3.15.)

When exactly is $c'_{2i} < c_{2i}$, or equivalently

$$k'_1 + k'_2 + \ldots + k'_{2i} > k_1 + k_2 + \ldots + k_{2i} \; ?$$

Q3. (cf. Proposition 3.20 and Proposition 3.24.)

Is there a common improvement on the bounds

$$d_p = d \Rightarrow d \leqslant \min\{b'_0 - 2c'_2 + 3,\ b_0 - 2c_2 + 2\} \qquad d \leqslant \left[\frac{2(b_0 + [b_0/b'_0])}{(2 + [b_0/b'_0])}\right]$$

$$d_p = d\text{-}1 \Rightarrow d \leqslant \min\{b'_0 - 2c'_2 + 3,\ b_0 - 2c_2 + 3\} \qquad d \leqslant \left[\frac{2(b_0 + 2[b_0/b'_0])}{(2 + [b_0/b'_0])}\right] ?$$

Q4. (cf. Remark 5.1.)

What is the connection between the local feasibility conditions on partial arrays (such as $c_{2i}c_{2i+1} = c'_{2i}c'_{2i+1}$) and global feasibility conditions (such as the integrality of multiplicities of eigenvalues) ? When does local feasibility imply global feasibility, and why ?

Q5. (a) Only one feasible pair of arrays in this thesis has diameter > 4. Is this pair ($b_0' = 15$) realisable ?

(b) Is there some much stronger bound for d in terms of b_0' ? Is there a bound for d independent of b_0' ?

References.

[1]. C.DELORME, Regularite Metrique Forte, (Unpublished).

[2]. D.R.HUGHES AND F.C.PIPER, Design Theory, Cambridge University Press, (1985).

[3]. C.W.H.LAM, J.McKAY, S.SWIERCZ, T.THIEL, announcement, (1988).

[4]. P.R.SECKER, Feasibility Algorithms For Distance-transitive Graphs, M.Sc.(Qual.) Thesis, University of Birmingham, (1986).

[5]. J.S. SHAWE-TAYLOR, Regularity and Transitivity in graphs, Ph.D. Thesis, Royal Holloway and Bedford New College, (1986).