ON $\mathfrak{sl}_2$-TRIPLES IN LIE ALGEBRAS OF REDUCTIVE ALGEBRAIC GROUPS IN POSITIVE CHARACTERISTIC

by

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Abstract

In this thesis we consider $\mathfrak{sl}_2$-triples in $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of a connected reductive algebraic group over a field of positive characteristic $p > 2$.

We focus on good primes for G which are smaller than the coxeter number of G, and determine to what extent the theorems of Jacobson–Morozov and Kostant hold in this setting. To do so, we determine the maximal G-stable closed subvariety $\mathcal{V}$ of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ such that the G-orbits in $\mathcal{V}$ are in bijection with the G-orbits of $\mathfrak{sl}_2$-triples (e, h, f) with $e, f \in \mathcal{V}$.

We also determine the maximal G-stable closed subvariety $\mathcal{V}$ of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ such that any subalgebra $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\mathbb{k})$ with $e, f \in \mathcal{V}$ is G-completely reducible.

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CHAPTER 1

INTRODUCTION

The main focus of this thesis is to extend what is known about $\mathfrak{sl}_2$-triples in the Lie algebras of both the classical and exceptional groups.

The theory of Lie algebras, and understanding their subalgebra structure, has been a key area of research over the last 100 years. As an important component of this, since the 1950s substantial attention has been devoted to understanding $\mathfrak{sl}_2$-triples. For G a simple reductive algebraic group over an algebraically closed field $\Bbbk$, an $\mathfrak{sl}_2$-triple in $\mathfrak{g} = \operatorname{Lie} G$ is defined to be a triple of elements (e, h, f) that generate a subalgebra of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2(\Bbbk)$, see Definition 2.1.5. This simple definition underpins much of the modern theory of Lie algebras and their representations.

The theory of $\mathfrak{sl}_2$-triples has been a key tool in the study of nilpotent orbits in the Lie algebra $\mathfrak{g}$ of a reductive algebraic group G over $\mathbb{C}$. The seminal result in $\mathfrak{sl}_2(\Bbbk)$-theory is the Jacobson–Morozov Theorem, which states that given any $\mathfrak{g} = \operatorname{Lie}(G)$, a semisimple Lie algebra over $\Bbbk = \mathbb{C}$, for each nilpotent element $e \in \mathfrak{g}$ there exists some $\mathfrak{sl}_2(\Bbbk)$-triple (e, h, f). This theorem was claimed by Morozov in

[Mor42] and a complete proof was provided by Jacobson in [Jac51]. This was later extended by Kostant, who proved in [Kos59] that there exists a bijective map

$$\{G\text{-orbits of } \mathfrak{sl}_2\text{-triples}\} \longleftrightarrow \{\text{nilpotent orbits in } \mathfrak{g}\}. \tag{1.0.1}$$

The significance of this result led researchers worldwide to investigate to what extent this result could be extended to fields of characteristic $p > 2$.

We now let $\Bbbk$ be an algebraically closed field of characteristic $p > 2$. Take G to be a connected, reductive algebraic group over $\Bbbk$, and set $\mathfrak{g} = \mathrm{Lie}(G)$.

Under the restriction that p is a good prime for G, a result of Pommerening tells us that for any nilpotent $e \in \mathfrak{g}$, there exists some $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ containing e, see [Pom80]. Note that the restriction to $p > 2$ is enough to ensure that the characteristic is good for the classical algebraic groups, however for G of type G_2, F_4, E_6 or E_7 we require $p > 3$ and for G of type E_8 we require $p > 5$. The result of Pommerening was extended by Stewart–Thomas to $p \geq 3$ for all reductive algebraic groups of exceptional type except for one nilpotent orbit for $G = G_2$, when $p = 3$, see [ST18, Theorem 1.7]. In general the $\mathfrak{sl}_2$-triples found by Pommerening and Stewart–Thomas are not unique up to conjugacy, and so subsequently there was a great deal of interest in extending the theorems of Jacobson–Morozov and Kostant to the setting of a reductive algebraic group G over an algebraically closed field $\Bbbk$ of characteristic $p > 2$. The restriction on p required for there to be a unique $\mathfrak{sl}_2$-triple up to conjugacy by G for each $e \in \mathcal{N}$ was determined in the work of Stewart–Thomas. To state this result we let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathcal{N}$ denote the nilpotent cone of $\mathfrak{g}$. In [ST18, Theorem 1.1] it is shown that there is a bijection

$$\{G\text{-orbits of } \mathfrak{sl}_2\text{-triples in } \mathfrak{g}\} \longrightarrow \{G\text{-orbits in } \mathcal{N}\} \tag{1.0.2}$$

sending the G-orbit of an $\mathfrak{sl}_2$-triple (e, h, f) to the G-orbit of e if and only if $p > h(G)$, where $h(G)$ is the Coxeter number of G.

This is the culmination of a series of earlier bounds on p for there to be a bijection as in (1.0.2). These previous bounds were: $p > 4h - 3$ given by Springer–Steinberg in [SS70, p. III.4.11]; and $p > 3h - 3$, given by Carter, using an argument of Spaltenstein, in [Car93, Section 5.5].

In this thesis we consider restricted Lie algebras, that is Lie algebras such that there exists a map $x \mapsto x^{[p]}$, called the p-th power map, which is equivariant under the adjoint action of G, see Definition 2.1.40 for more detail. In [PS19, Section 2.4] Premet–Stewart define for each nilpotent element $e \in \mathfrak{g}$ the standard $\mathfrak{sl}_2$-triple containing e in a canonical way when the characteristic is good; and we discuss the construction of standard $\mathfrak{sl}_2$-triples in detail in §4.1. In particular we note that standard $\mathfrak{sl}_2$-triples (e, h, f) satisfy $f^{[p]} = 0$, and if $e^{[p]} = 0$, then e and f are conjugate by an element of $\mathrm{Ad}(G)$.

It is thus natural to consider $\mathfrak{sl}_2$-triples $(e, h, f) \in \mathfrak{g}$ in which e is conjugate to f. If $G = \mathrm{GL}_n(\Bbbk)$, then we say that (e, h, f) in $\mathfrak{gl}_n(\Bbbk)$ is a strong $\mathfrak{sl}_2$-triple if e and f are conjugate by $\mathrm{GL}_n(\Bbbk)$. For $G = \mathrm{GL}_n(\Bbbk)$ the $[p]$-power map is the map taking the p-th power of the matrix, hence it follows from the results in [PS19], that if $e \in \mathfrak{gl}_n(\Bbbk)$ is such that $e^p = 0$, then the standard $\mathfrak{sl}_2$-triple containing e is strong. In Chapter 3 we consider if there exist strong $\mathfrak{sl}_2$-triples when $e^p \neq 0$, and prove the following theorem.

Theorem 1. *There exists a surjective map*

$$\left\{ \begin{array}{l} \mathrm{GL}_n(\Bbbk)\text{-}orbits\ of\ strong \\ \mathfrak{sl}_2\text{-}triples\ in\ \mathfrak{gl}_n(\Bbbk) \end{array} \right\} \longrightarrow \{\mathrm{GL}_n(\Bbbk)\text{-}orbits\ in\ \mathcal{N}\}. \tag{1.0.3}$$

That is, given any nilpotent $e \in \mathfrak{gl}_n(\Bbbk)$, there exists some strong $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{gl}_n(\Bbbk)$.

We explain in Proposition 3.3.4 that there is no bijective map between these sets, and note that the existence of such a surjective map remains an open question for the remaining classical Lie algebras.

Returning to the setting of G being any reductive algebraic group, following the results of Stewart– Thomas, it is a natural question to consider to what extent the map (1.0.1) fails to be a bijection in the case where $p \leq h(G)$. A key problem is to determine the maximal G-stable closed subvarieties $\mathcal{V}$ of $\mathcal{N}$ such that the restriction of this map to

$$\{G\text{-orbits of } \mathfrak{sl}_2\text{-triples } (e, h, f) \text{ with } e, f \in \mathcal{V}\} \longrightarrow \{G\text{-orbits in } \mathcal{V}\} \tag{1.0.4}$$

is a bijection. In this thesis we solve this problem for all reductive G. We determine a maximal subvariety $\mathcal{V}$ of $\mathcal{N}$ such that the map in (1.0.4) is a bijection, and prove that $\mathcal{V}$ is the unique maximal such subvariety.

For any $x \in \mathcal{N}$ with associated cocharacter λ_x we consider $\mathrm{ht}(x) = \max\{j \mid \mathfrak{g}(j; \lambda_x) \neq 0\}$ as discussed in §4.1. This definition allows us to state our first main result, given below.

Theorem 2. *Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 0$.*

Let G be a connected, reductive algebraic group over $\Bbbk$, with p a good prime for G. Define $\mathcal{V} \subseteq \mathcal{N}$ to be

$$\mathcal{V} = \{x \in \mathcal{N} \mid \mathrm{ht}(x) \leq 2p - 3\}.$$

Then the map

$$\{G\text{-orbits of } \mathfrak{sl}_2\text{-triples } (e, h, f) \text{ with } e, f \in \mathcal{V}\} \longrightarrow \{G\text{-orbits in } \mathcal{V}\} \tag{1.0.5}$$

given by sending the G-orbit of an $\mathfrak{sl}_2$-triple (e, h, f) to the G-orbit of e is a bijection. Moreover, $\mathcal{V}$ is the unique maximal G-stable closed subvariety of $\mathcal{N}$ that satisfies this property.

As we frequently consider G-stable closed subvarieties $\mathcal{V}$ of $\mathcal{N}$ such that the map in (1.0.5) is a bijection, we use a shorthand for such varieties, and say that such a variety satisfies the $\mathfrak{sl}_2$*-property*. With this terminology, we have that Theorem 2 determines the unique maximal G-stable closed subvariety $\mathcal{V}$ of $\mathcal{N}$ that satisfies the $\mathfrak{sl}_2$-property.

Theorem 2, restricted to G an algebraic group of classical type, is the key result of a joint paper published by the author and Simon Goodwin, for which both authors contributed equally, [GP22, Theorem 1.1].

Within this thesis we also consider G-completely reducible $\mathfrak{sl}_2$-subalgebras. A concept initially defined for reductive algebraic groups by Serre, G-complete reducibility extends the notion of a representation being completely reducible. We say that a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is G-completely reducible if for every parabolic $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, there exists some Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{h}$, this is discussed in more detail in §2.1.5.

In the 2018 paper by Stewart–Thomas, they investigate G-completely reducible subalgebras and assert that for G a connected reductive algebraic group over $\Bbbk$, any semisimple subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$, and hence any $\mathfrak{sl}_2(\Bbbk)$-subalgebra of $\mathfrak{g}$, is G-completely reducible if the characteristic of the field is larger than $h(G)$, [ST18, Theorem 1.3].

In Chapters 5 and 6 we investigate this further, and consider $\mathfrak{sl}_2$-subalgebras $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ generated by some $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with e and f contained in the same nilpotent subvariety. This leads to our second main result, stated below.

Theorem 3. *Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 0$. Let G be a connected, reductive algebraic group over $\Bbbk$, with p a good prime for G. Define $\mathcal{V} \subseteq \mathcal{N}$ to be*

$$\mathcal{V} = \{x \in \mathcal{N} \mid \mathrm{ht}(x) \leq 2p - 3\}.$$

Then any $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ with $e, f \in \mathcal{V}$ is G-completely reducible. Moreover, $\mathcal{V}$ is the unique maximal G-stable closed subvariety of $\mathcal{N}$ that satisfies this property.

Observe that this variety is equal to the variety given in Theorem 2. These results therefore indicate a deeper connection between the nilpotent varieties that satisfy the $\mathfrak{sl}_2$-property, and those such that any $\mathfrak{sl}_2$-algebra generated by an $\mathfrak{sl}_2$-triple with e, f contained in the variety are G-complete reducible.

Structure of this thesis

Throughout this thesis we take $\Bbbk$ to be an algebraically closed field of characteristic $p > 2$, with G a reductive algebraic group over $\Bbbk$ and $\mathfrak{g} = \mathrm{Lie}(G)$. We now give an overview of the structure of this thesis.

In Chapter 2 we begin by giving a summary of Lie algebras, algebraic groups, their structure, and defining the notation that we will use throughout. In particular, in §2.3 we recall some required representation theory of $\mathfrak{sl}_2(\Bbbk)$. We conclude in §2.4 with a discussion on the nilpotent orbits of Lie algebras. In particular we define the Jordan type of nilpotent elements in the classical cases, and recall the closure order on their orbits in §2.4.2.

Chapter 3 is concerned with strong $\mathfrak{sl}_2$-triples. In order to prove Theorem 1, in §3.2 we introduce a method of constructing diagrams to represent the action of nilpotent matrices in $\mathfrak{gl}_n(\Bbbk)$. We then give a series of lemmas on how these diagrams can be used to determine the Jordan type of nilpotent elements in certain cases, see Lemmas 3.2.6 and 3.2.7. In §3.3 we prove Theorem 1 by constructing an infinite family of strong $\mathfrak{sl}_2$-triples for each nilpotent $e \in \mathfrak{gl}_n(\Bbbk)$. We then prove that these triples are not conjugate under $\mathrm{GL}_n(\Bbbk)$, thus showing that the map (1.0.3) cannot be injective.

In Chapter 4 we give preliminary results on $\mathfrak{sl}_2$-triples that we will need in Chapters 5 and 6. In §4.1 we define the standard $\mathfrak{sl}_2$-triples as introduced by Premet–Stewart in [PS19], and note that such an $\mathfrak{sl}_2$-triple can be defined for each nilpotent $e \in \mathfrak{g}$. Using results on standard $\mathfrak{sl}_2$-triples and the properties of $\mathfrak{sl}_2$-triples in $\mathfrak{sl}_p(\Bbbk)$ given in §4.2, we determine conditions $\mathcal{V}$ must satisfy in order to satisfy the $\mathfrak{sl}_2$-property. We conclude in §4.3 with a discussion on the G-completely reducible $\mathfrak{sl}_2(\Bbbk)$-subalgebras of $\mathfrak{g}$, where we determine a set of conditions on a nilpotent subvariety $\mathcal{V} \subseteq \mathcal{N}$ needed to ensure that all $\mathfrak{sl}_2(\Bbbk)$-subalgebras $\langle e, h, f\rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible. We give these results in preparation to apply them in the proof of Theorems 2 and 3 in Chapters 5 and 6.

Chapter 5 sees us turn our attention to proving Theorems 2 and 3 for G one of $\mathrm{GL}_n(\Bbbk), \mathrm{SL}_n(\Bbbk), \mathrm{Sp}_n(\Bbbk), \mathrm{O}_n(\Bbbk), \mathrm{SO}_n(\Bbbk)$. The work in this chapter forms the basis of a paper published by the author and Simon Goodwin [GP22]. We begin by stating Theorems 5.0.1 and 5.0.2, which are equivalent to Theorems 2 and 3 with the variety $\mathcal{V}$ stated in terms of the Jordan types of the nilpotent orbits. In §5.1 we take G to be either $\mathrm{GL}_n(\Bbbk)$ or $\mathrm{SL}_n(\Bbbk)$. We prove the algebra $A := U(\mathfrak{sl}_2(\Bbbk))/\langle e^{p-1}, f^{p-1}\rangle$ is semisimple, and use this to prove that $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property in Corollary 5.1.9. The semisimplicity of A is given by a theorem of Jacobson, [Jac58, Theorem 1], see also [Car93, Theorem 5.4.8], though we provide an alternative proof of this. We then complete the proof of Theorem 5.0.1(a) by noting that Proposition 4.2.2 implies the maximality of $\mathcal{V}$ satisfying the $\mathfrak{sl}_2$-property. In §5.1.4 we explain that the case $G = \mathrm{SL}_n(\Bbbk)$ in Theorem 5.0.1(b) follows quickly.

In Section 5.2 we consider the cases where G is one of $\mathrm{Sp}_n(\Bbbk)$, $\mathrm{O}_n(\Bbbk)$ or $\mathrm{SO}_n(\Bbbk)$. Using Lemma 5.2.1 and Theorem 5.0.1(a) we are able to quickly show that for $G = \mathrm{Sp}_n(\Bbbk)$ or $\mathrm{O}_n(\Bbbk)$, we have that $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property. Then for $G = \mathrm{Sp}_n(\Bbbk)$ we complete the proof of Theorem 5.0.1(c) by using Propositions 4.1.6 and 4.2.2 to deduce maximality. We move on to deal with the case $G = \mathrm{O}_n(\Bbbk)$ in §5.2.2, where we complete a detailed analysis of certain $\mathfrak{sl}_2(\Bbbk)$-modules in the proof of Proposition 5.2.5. This proposition shows that $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property. We then deduce maximality of $\mathcal{V}$ similarly to the previous cases to complete the proof of Theorem 5.0.1(d). We are left to deduce Theorem 5.0.1(e) for $G = \mathrm{SO}_n(\Bbbk)$ which is done in Proposition 5.2.6.

To finish Chapter 5, in §5.3 we deduce Theorem 5.0.2. To do so, we use Lemma 2.1.31 to explain how this follows from Theorem 5.0.1.

In Chapter 6 we take G to be of exceptional type, and prove Theorem 6.0.1. We first explain that this is enough to prove Theorems 2 and 3 for G of exceptional type. In §6.1 we show that $\mathcal{V}$, as defined in Theorem 6.0.1, is maximal with respect to both the $\mathfrak{sl}_2$-property and the G-complete reducibility property. This follows from a combination of results given in Chapter 4.

In §6.2 we give an overview of the approach we take to prove Theorem 6.0.1. In §6.2.1 we use induction on the semisimple rank of G to reduce our $\mathfrak{sl}_2$-triples to those of the form $(e, h, f) = (\overline{e} + e', \overline{h} + h', \overline{f} + f')$ where $(\overline{e}, \overline{h}, \overline{f})$ is the standard $\mathfrak{sl}_2$-triple for $\overline{e}$ contained in the Levi factor $\mathfrak{l}$ of the maximal parabolic $\mathfrak{p}$ containing (e, h, f). In §6.2.2 we then use this reduction and [ST18, Statement (4), pg. 13] to deduce the form of any non-G-completely reducible $\mathfrak{sl}_2$-triples. In §6.2.3 we then summarise four techniques that are used to prove that there are no non-G-completely reducible $\mathfrak{sl}_2(\mathbb{k})$-subalgebras $\mathfrak{h} = \langle e, h, f \rangle \subseteq \mathfrak{g}$ with $e, f \in \mathcal{V}$. In §6.3 we take G to be of type G_2 and demonstrate how this method can be applied by hand. In §6.4 we complete a case-by-case analysis of the remaining exceptional type algebraic groups, using computational techniques in MAGMA.

Finally, in Section 6.5 we prove that this is enough to show that $\mathcal{V}$ also satisfies the $\mathfrak{sl}_2$-property.

CHAPTER 2

PRELIMINARIES

Unless stated otherwise, throughout this thesis we take $\Bbbk$ to be an algebraically closed field of characteristic $p > 2$. All algebraic groups and Lie algebras we work with are over $\Bbbk$. The prime subfield of $\Bbbk$ is denoted by $\mathbb{F}_p$.

2.1 Lie algebras, algebraic groups and their representations

2.1.1 Lie algebras

We begin introducing Lie algebras by defining the fundamental concepts that we use throughout. For a more detailed introduction, refer to [EW06].

Definition 2.1.1. Let $\mathfrak{g}$ be a vector space over $\Bbbk$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ that satisfies:

(a) $[x, x] = 0$ for all $x \in \mathfrak{g}$; and

(b) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z \in \mathfrak{g}$.

Then $\mathfrak{g}$ is a *Lie algebra* and the bilinear map $[\cdot,\cdot]$ is called the *Lie bracket*.

Condition (b) is referred to as the *Jacobi identity*.

We define homomorphisms of Lie algebras, Lie subalgebras, and ideals in the natural way.

Definition 2.1.2. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be Lie algebras over $\mathbb{k}$, we say that $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *homomorphism of Lie algebras* if φ is a linear map, and φ satisfies

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \text{ for all } x, y \in \mathfrak{g}_1.$$

A homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an *isomorphism of Lie algebras* if it is bijective map.

Definition 2.1.3. Let $\mathfrak{g}$ be a Lie algebra.

A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if

$$[x,y] \in \mathfrak{h} \text{ for all } x, y \in \mathfrak{h}.$$

A subspace $I \subseteq \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if

$$[x,y] \in I \text{ for all } x \in \mathfrak{g}, y \in I.$$

We now define the *classical Lie algebras*, $\mathfrak{sl}_n(\mathbb{k})$, $\mathfrak{so}_n(\mathbb{k})$ and $\mathfrak{sp}_{2n}(\mathbb{k})$, which we consider as subsets of matrices. Over $\mathbb{C}$, the classical Lie algebras have the property that every finite-dimensional simple Lie algebra is isomorphic to one of these Lie algebras or one of 5 exceptional Lie algebras [EW06, Theorem 4.12]. Throughout the rest of this chapter, the classical Lie algebras form the basis for most examples.

Example 2.1.4. (i) Let V be an n-dimensional vector space over $\mathbb{k}$. We define the *general linear algebra*, $\mathfrak{gl}(V)$, to be the set of all linear maps $V \to V$, where the Lie bracket is defined to be

$$[x, y] = xy - yx, \quad \text{for all } x, y \in \mathfrak{gl}(V)$$

where here xy represents composition of maps.

We similarly define $\mathfrak{gl}_n(\mathbb{k})$ to be the space of all $n \times n$ matrices over $\mathbb{k}$ where our Lie bracket is defined as above, where in this case xy represents matrix multiplication.

Note that if we have some basis $\{v_1, v_2, \ldots, v_n\}$ of V, then we can define a bijective map $\mathfrak{gl}(V) \to \mathfrak{gl}_n(\mathbb{k})$ that maps a linear map $\alpha : V \to V$ to the matrix (a_{ij}) such that

$$\alpha(v_k) = \sum_{l=1}^{n} a_{lk} v_l.$$

This map is an example of an isomorphism of Lie algebras.

(ii) We define the *special linear Lie algebra*, $\mathfrak{sl}_n(\mathbb{k})$, to be the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{k})$ containing matrices with trace 0.

(iii) We define the *orthogonal Lie algebra*, $\mathfrak{so}_n(\mathbb{k})$, to be the subalgebra of $\mathfrak{gl}_n(\mathbb{k})$ with

$$\mathfrak{so}_n(\mathbb{k}) = \{x \in \mathfrak{gl}_n : x^t J_n = -J_n x\} \text{ where } J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Elements of $\mathfrak{so}_n(\mathbb{k})$ are reflected negatively in the antidiagonal, that is $x = (x_{ij}) \in \mathfrak{gl}_n(\mathbb{k})$ is in $\mathfrak{so}_n(\mathbb{k})$ if $x_{ij} = -x_{n-j+1,n-i+1}$ for all i, j.

For example

$$\mathfrak{so}_3(\mathbb{k}) = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} : a, b, c \in \mathbb{k} \right\}.$$

(iv) Similarly we define the *symplectic Lie algebra*, $\mathfrak{sp}_{2n}(\mathbb{k})$, to be the subalgebra of $\mathfrak{gl}_{2n}(\mathbb{k})$ with

$$\mathfrak{sp}_{2n}(\mathbb{k}) = \left\{ x \in \mathfrak{gl}_{2n} : x^t J = -Jx \right\} \text{ where } J = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}.$$

We have that $x = (x_{ij}) \in \mathfrak{gl}_n(\mathbb{k})$ is in $\mathfrak{sp}_{2n}(\mathbb{k})$ if

$$x_{ij} = \begin{cases} -x_{2n-j+1,2n-i+1} & \text{if either } i, j < n \text{ or } i, j > n, \\ x_{2n-j+1,2n-i+1} & \text{otherwise.} \end{cases}$$

For example

$$\mathfrak{sp}_4(\mathbb{k}) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{1,3} \\ x_{3,1} & x_{3,2} & -x_{2,2} & -x_{1,2} \\ x_{4,1} & x_{3,1} & -x_{2,1} & -x_{1,1} \end{pmatrix} : x_{ij} \in \mathbb{k} \right\}.$$

In particular, we are interested in Lie subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{k})$. The standard basis for $\mathfrak{sl}_2(\Bbbk)$ is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where the corresponding Lie brackets are

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Definition 2.1.5. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Then we say that $e, h, f \in \mathfrak{g}$ is an $\mathfrak{sl}_2$-*triple* if (e, h, f) generate a subalgebra that is a homomorphic image of $\mathfrak{sl}_2(\Bbbk)$. That is, they satisfy the following equations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{2.1.1}$$

Note that if $\mathrm{char}(\Bbbk) \neq 2$, we have that (e, h, f) generate a Lie subalgebra which is either 0 or isomorphic to $\mathfrak{sl}_2(\Bbbk)$.

We see the standard basis of $\mathfrak{sl}_2(\Bbbk)$ given above satisfies these equations, and thus is an $\mathfrak{sl}_2$-triple.

Example 2.1.6. Suppose we have some $\mathfrak{sl}_2$-triple (e, h, f) in a Lie algebra $\mathfrak{g}$. Then $(f, -h, e)$ is also an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.

2.1.2 Structure of algebraic groups

Let G be a linear algebraic group. We write $\mathbb{G}_a = (\Bbbk, +)$, and $\mathbb{G}_m = (\Bbbk^\times, \times)$ to represent the additive and multiplicative groups of the field $\Bbbk$ respectively.

We now define certain subgroups of the algebraic groups, which we require to define root systems in §2.1.3. For further discussion, refer to [MT11, Chapter 6].

Definition 2.1.7. A subgroup $T \leq G$ is a *torus* if T is isomorphic to a direct product of copies of $\mathbb{G}_m$, that is $T \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

We say T is a *maximal torus* if it is maximal satisfying this property with respect to inclusion.

We have that all maximal tori are conjugate in G.

Definition 2.1.8. The *rank* of a linear algebraic group G is the dimension of a maximal torus of G.

To illustrate this, we consider the following example.

Example 2.1.9. (i) Let $G = \mathrm{GL}_n(\mathbb{k})$, then it is clear that D_n is a maximal torus for G where we define D_n to be the invertible diagonal matrices. Thus we have that the rank of $\mathrm{GL}_n(\mathbb{k})$ is n.

(ii) Similarly, we have that $D_n \cap \mathrm{SL}_n(\mathbb{k})$ is a maximal torus of $\mathrm{SL}_n(\mathbb{k})$. Then $\mathrm{SL}_n(\mathbb{k})$ has rank $n-1$.

Definition 2.1.10. Let G be a linear algebraic group. We define a *character* of G to be a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$. We denote the set of characters by

$$X(G) := \{\chi : G \to \mathbb{G}_m\}.$$

Similarly, we define a *cocharacter* of G to be a morphism of algebraic groups $\gamma : \mathbb{G}_m \to G$, with the set of cocharacters denoted by

$$Y(G) := \{\gamma : \mathbb{G}_m \to G\}.$$

Let T be a maximal torus of G. Observe that for $\chi \in X(T), \gamma \in Y(T)$ we have $\chi \circ \gamma \in \operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m)$. Hence for any $\lambda \in \mathbb{G}_m$ we have $\chi \circ \gamma(\lambda) = \lambda^n$ for some $n \in \mathbb{Z}$. We define the map $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$ by $\langle \chi, \gamma \rangle = n$.

We can equivalently define characters and cocharacters of Lie algebras.

We give an example of a character which we will refer back to later.

Example 2.1.11. Let $T = D_n \cap \mathrm{SL}_n(\Bbbk) \subseteq \mathrm{SL}_n(\Bbbk)$ be as in Example 2.1.9, then for each $i \neq j$ we define the character $\chi_{ij} : T \to \mathbb{G}_m$ to be given by

$$\chi_{ij} : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mapsto a_i a_j^{-1}.$$

Definition 2.1.12. Let G be a linear algebraic group, then we say that a subgroup $B \subseteq G$ is a *Borel subgroup* of G if it is a closed, connected, solvable subgroup of G, and it is maximal with respect to these properties.

We remark that all Borel subgroups are conjugate, and thus up to conjugacy, we can take any choice of Borel subgroup.

Example 2.1.13. Let $G = \mathrm{GL}_n(\Bbbk)$, then the subgroup of invertible upper triangular matrices is a Borel subgroup.

Definition 2.1.14. The maximal closed connected solvable normal subgroup of G is referred to as the *radical* of G, and is denoted by $R(G)$.

The maximal closed connected normal unipotent subgroup of G is referred to as the *unipotent radical* of G, and is denoted by $R_u(G)$.

We say that a linear algebraic group is *reductive* if $R_u(G) = 1$, and *semisimple* if G is connected and reductive.

2.1.3 Root systems

Throughout this thesis, we will want to discuss both the classical and exceptional Lie algebras. In order to define the exceptional Lie algebras we first need to understand their construction using root systems. We give a brief overview of this, for more detail refer to [Car93, Chapter 1] and [MT11, Chapter 8].

Within this section take G to be a reductive algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$. We continue notation from §2.1.2, and we take $T \leq G$ to be a maximal torus of G.

Define $W(T)$, the *Weyl group*, to be the quotient of the normaliser of T in G by the centraliser of T in G. Note that this is uniquely determined up to isomorphism, as all maximal tori are conjugate.

Take B to be a Borel subgroup of G containing T, then B has a semidirect product decomposition

$$B = R_u(B) \rtimes T$$

where $R_u(B)$ is the unipotent radical of B.

There exists a unique Borel subgroup B^- of G containing T such that $B \cap B^- = T$. As above we can find a semidirect product decomposition

$$B^- = R_u(B^-) \rtimes T.$$

We consider the minimal proper subgroups of $R_u(B)$ and $R_u(B^-)$ that are normalised by T. These are all connected unipotent subgroups of dimension 1, and each determines an element of X.

Definition 2.1.15. The non-zero elements of X arising in this way are called *roots*. The roots in X form a finite subset which we denote Φ.

For each root $\alpha \in \Phi$, the 1-dimensional unipotent subgroup that determines α is denoted X_α. These are known as the *root subgroups* of G.

We write Φ^+ for the set of *positive roots* arising from the root subgroups of $R_u(B)$, and Φ^- for the set of *negative roots* arising from the root subgroups of $R_u(B^-)$

We write Δ for the set of positive roots that cannot be expressed as a sum of two positive roots. We refer to the elements of Δ as *simple roots*.

A key consequence of these results is that $G = \langle T, X_\alpha : \alpha \in \Phi \rangle$.

The roots arising from the root subgroups in $R_u(B^-)$ are the negatives of the roots arising from $R_u(B)$, so we may refer to $\alpha, -\alpha$ as opposite roots. Then we have that $\langle X_\alpha, X_{-\alpha} \rangle$ is isomorphic to either $\mathrm{SL}_2(\Bbbk)$ or $\mathrm{PGL}_2(\Bbbk)$.

Definition 2.1.16. Let $\alpha^\vee \in Y$ be such that $\alpha^\vee : \mathbb{G}_m \to \langle X_\alpha, X_{-\alpha} \rangle$ and $\langle \alpha, \alpha^\vee \rangle = 2$. Then $\alpha^\vee$ is uniquely determined, and we define $\alpha^\vee$ to be a *coroot* of Y. The set of coroots of Y is denoted $\Phi^\vee$.

Note that each element of W permutes both the roots Φ in X and the coroots $\Phi^\vee$ in Y. Let $\alpha \in \Phi$ and define $s_\alpha \in W$ to be the element of W such that for all $\chi \in X, \gamma \in Y$ we have

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \, \alpha$$

$$s_\alpha(\gamma) = \gamma - \langle \alpha, \gamma \rangle \, \alpha^\vee.$$

We have that $s_\alpha = s_{-\alpha}$ and $s_\alpha^2 = 1$, and that W is generated by the set of s_α for all $\alpha \in \Delta$.

Definition 2.1.17. For each connected reducive group G we refer to $(X, \Phi, Y, \Phi^\vee)$ as the *root datum* of G.

Given any root datum $(X, \Phi, Y, \Phi^\vee)$ there is a connected reductive group G over $\Bbbk$ that has the given root datum, and this group is unique up to isomorphism. For further detail on this classification refer to [MT11, Chapter 9].

In order to better understand these defintions, we continue Examples 2.1.9 and 2.1.11.

Example 2.1.18. Take $G = \mathrm{GL}_n(\Bbbk)$, and recall that $T = D_n$ is a maximal torus for G. Take B to be the subgroup of upper triangular matrices. Then $R_u(B)$ is the subgroup of upper-unitriangular matrices.

Symmetrically we have B^- is the subgroup of lower triangular matrices.

Define the character $\chi_{ij} : T \to G_m$ as in Example 2.1.11.

We have that $\chi_{i,j} = \chi_{i,i+1} + \chi_{i+1,i+2} + \cdots + \chi_{j-1,j}$ for $i < j$, so we see that

$$\Phi(G) = \{\chi_{ij} : i \neq j\}, \quad \Delta = \{\chi_{i,i+1} : 1 \leq i \leq n-1\}.$$

and

$$\Phi^+ = \{\chi_{ij} : i < j\}.$$

We can draw the Dynkin diagrams using these root systems and their bases.

Definition 2.1.19. Let G be a connected, reductive group with root datum $(X, \Phi, Y, \Phi^\vee)$. Label a vertex for each of the simple roots of Δ, and between

$\alpha, \beta \in \Delta$ draw $d_{\alpha\beta}$ lines where

$$d_{\alpha\beta} := \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}.$$

If $d_{\alpha\beta} > 1$, then one of $\langle \alpha, \beta^\vee \rangle$ or $\langle \beta, \alpha^\vee \rangle$ is equal to -1. Draw an arrow pointing from α to β if $d_{\alpha\beta} > 1$ and $\langle \beta, \alpha^\vee \rangle = -1$.

This diagram is known as the *Dynkin diagram* of G.

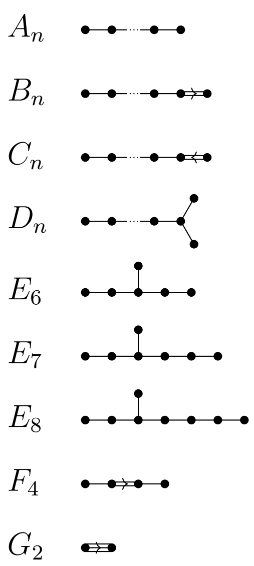
The Dynkin diagram is uniquely determined by G, and is independent of the choice of maximal torus T and Borel subgroup B containing T.

Theorem 2.1.20. *Let G be a simple algebraic group with root datum $(X, \Phi, Y, \Phi^\vee)$. Then the Dynkin diagram for G is one of the following types*

$$A_n (n \geq 1), B_n (n \geq 1), C_n (n \geq 3), D_n (n \geq 4), E_6, E_7, E_8, F_4, G_2.$$

The Dynkin diagrams for these types are shown below.

Definition 2.1.21. We refer to Lie algebras with Dynkin diagrams of type $A-D$ as *classical* Lie algebras and those with Dynkin diagrams of type E, F, G as *exceptional* Lie algebras.



Example 2.1.22. Recall the examples of Lie algebras given in Example 2.1.4, then we have that $\mathfrak{sl}_{n+1}(\Bbbk)$ has type A_n, $\mathfrak{so}_{2n+1}(\Bbbk)$ has type B_n, $\mathfrak{sp}_{2n}(\Bbbk)$ has type C_n and $\mathfrak{so}_{2n}(\Bbbk)$ has type D_n for $n \in \mathbb{Z}_{>0}$. To see a construction of the root systems in characteristic 0, see [EW06, Chapter 12].

2.1.4 Parabolic and Levi subgroups and subalgebras

For any connected, reductive algebraic group G we can define the parabolic and Levi subgroups of G, for statements and proofs of the results given here refer to [MT11, Chapter 12].

We maintain notation from §2.1.2 and §2.1.3. We take G to be a connected reductive algebraic group, $T \leq G$ a maximal torus contained in a Borel subgroup B of G. Let Φ be the root system of G, and Δ the set of simple roots with respect to $T \leq B$, and $S = \{s_\alpha : \alpha \in \Delta\}$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ For a subset $I \subseteq S$ we write

$$\Delta_I := \{\alpha \in \Delta : s_\alpha \in I\} \quad \text{and} \quad \Phi_I := \Phi \cap \sum_{\alpha\Delta_I} \mathbb{Z}\alpha.$$

Before we can define the parabolic and Levi subgroups we must first give some further notation. We state this notation in combination with the results of [MT11, Theorem 8.17].

Let $S = \{s_\alpha : \alpha \in \Delta\}$ be the set of generating relations. Then for each $I \subseteq S$ we define

$$U_I := \left\langle U_\alpha : \alpha \in \Phi^+ \setminus \Phi_I \right\rangle .$$

Definition 2.1.23. For a subset $I \subseteq S$, define $P_I := \langle \alpha \in \Phi^+ \cup \Phi_I \rangle$. Then we say that P_I is a *standard parabolic* of G.

We then define a *parabolic subgroup* to be any subgroup of G conjugate to a standard parabolic of G. We could equivalently define a parabolic subgroup to be any subgroup containing a Borel subgroup. Note that $R_U(P_I) = U_I$.

Define

$$L_I := \langle T, U_\alpha : \alpha \in \Phi_I \rangle \leq P_I$$

then this is a complement to U_I in the parabolic subgroup P_I, so $P_I = U_I \rtimes L_I$. This decomposition is called the *Levi decomposition* of the parabolic subgroup P_I, and L_I is the *Levi complement* of P_I.

The conjugates of the standard Levi complements are called *Levi subgroups* of G.

We now discuss the parabolic subgroups of $G \subseteq \mathrm{GL}_n(\Bbbk)$ in more detail.

Example 2.1.24. The parabolic subgroups of $\mathrm{GL}_n(\Bbbk)$ are the stabilisers of flags of subspaces of $V = \Bbbk^n$. That is for each parabolic subgroup P there exists some flag $\mathfrak{F} : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = \Bbbk^n$ such that

$$P = Stab_{\mathrm{GL}_n(\Bbbk)}(\mathfrak{F}) = \{g \in \mathrm{GL}(V) : g(V_j) = V_j \text{ for all } j = 1, \ldots, r\}.$$

For $G = \mathrm{Sp}_{2n}(\Bbbk)$ or $\mathrm{O}_n(\Bbbk)$ take the bilinear form given by the matrices J and J_n respectively from Example 2.1.4, then the parabolic subgroups of G are $P = Stab_G(\mathfrak{F})$ for some flag $\mathfrak{F} : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_r \subset V$ where each V_i is a totally isotropic subspace of V for $1 \leq i \leq r$, where by totally isotropic we mean a subspace on which the bilinear form vanishes.

Example 2.1.25. Suppose $G = \mathrm{GL}_n(\Bbbk)$ and let $\{e_1, \ldots, e_n\}$ be the standard basis of $\Bbbk^n$. Consider the flag $\mathfrak{F} : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = \Bbbk^n$ where $V_i = \mathrm{span}\{e_1, \ldots, e_{n_1+\cdots+n_i}\}$ and hence $\dim(V_i/V_{i-1}) = n_i$. Then the corresponding parabolic subgroup is

$$P = \left\{ \begin{pmatrix} A_1 & * & * & * \\ & A_2 & * & * \\ & & \ddots & * \\ & & & A_r \end{pmatrix} : A_1 \in \mathrm{GL}_{n_1}(\Bbbk), A_2 \in \mathrm{GL}_{n_2}(\Bbbk), \ldots, A_r \in \mathrm{GL}_{n_r}(\Bbbk) \right\}.$$

Lemma 2.1.26. *Suppose P is a parabolic subgroup of G given as the stabiliser of some flag $\mathfrak{F} : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = k^n$ with $\dim(V_i/V_{i-1}) = n_i$ which can be written as a direct sum decomposition as in Example 2.1.25. A Levi subgroup associated to P is the stabiliser of this direct sum decomposition. All of the conjugates of this stabiliser are also Levi subgroups.*

Example 2.1.27. A Levi subgroup associated to P in Example 2.1.25 is

$$L = \left\{ \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix} : A_1 \in \mathrm{GL}_{n_1}(\Bbbk), A_2 \in \mathrm{GL}_{n_2}(\Bbbk), \ldots, A_r \in \mathrm{GL}_{n_r}(\Bbbk) \right\}.$$

Definition 2.1.28. Let G be a linear algebraic group and consider $\mathfrak{g} = \mathrm{Lie}(G)$. We define a *parabolic subalgebra* of $\mathfrak{g}$ to be some $\mathfrak{p} = \mathrm{Lie}(P)$ for a parabolic subgroup P of G. We define a *Levi subalgebra* of $\mathfrak{g}$ associated to $\mathfrak{p}$ to be $\mathfrak{l} = \mathrm{Lie}(L)$ where L is a Levi subgroup associated to P.

Given a Lie algebra $\mathfrak{g}$ with a parabolic subalgebra $\mathfrak{p}$, then we say that a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is a *Levi factor*.

We now give some further examples for the classical groups.

Example 2.1.29. (i) The Levi subgroups of $\mathrm{O}_n(\Bbbk)$ are of the form

$$L \cong \mathrm{GL}_{n_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{n_{r-1}}(\Bbbk) \times \mathrm{O}_{n-2(n_1+\cdots+n_{r-1})}(\Bbbk)$$

for some $n_1, \ldots, n_{r-1}$.

The Levi subalgebras of $\mathfrak{so}_n(\Bbbk)$ are of the form

$$\mathfrak{l} \cong \mathfrak{gl}_{n_1}(\Bbbk) \times \cdots \times \mathfrak{gl}_{n_{r-1}}(\Bbbk) \times \mathfrak{so}_{n-2(n_1+\cdots+n_{r-1})}(\Bbbk)$$

for some $n_1, \ldots, n_{r-1}$, that is

$$\mathfrak{l} \cong \left\{ \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_{r-1} & & & & \\ & & & B & & & \\ & & & & \tilde{A}_{r-1} & & \\ & & & & & \ddots & \\ & & & & & & \tilde{A}_1 \end{pmatrix} : A_i \in \mathfrak{gl}_{n_i}(\Bbbk), B \in \mathfrak{so}_m(\Bbbk) \right\}$$

where $m = n - 2(n_1 + \cdots + n_{r-1})$ and $(\tilde{A}_k)_{i,j} = -(A_k)_{n_k-j+1, n_k-i+1}$ for each

k.

(ii) Similarly, the Levi subgroups of $\mathrm{Sp}_{2n}(\Bbbk)$ are of the form

$$L \cong \mathrm{GL}_{n_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{n_{r-1}}(\Bbbk) \times \mathrm{Sp}_{2n-2(n_1+\cdots+n_{r-1})}(\Bbbk)$$

for some $n_1, \ldots, n_{r-1}$.

The Levi subalgebras of $\mathfrak{sp}_{2n}(\Bbbk)$ are of the form

$$\mathfrak{l} \cong \mathfrak{gl}_{n_1}(\Bbbk) \times \cdots \times \mathfrak{gl}_{n_{r-1}}(\Bbbk) \times \mathfrak{sp}_{2n-2(n_1+\cdots+n_{r-1})}(\Bbbk)$$

for some $n_1, \ldots, n_{r-1}$, that is

$$\mathfrak{l} \cong \left\{ \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_{r-1} & & & & \\ & & & B & & & \\ & & & & \overline{A_{r-1}} & & \\ & & & & & \ddots & \\ & & & & & & \overline{A_1} \end{pmatrix} : A_i \in \mathfrak{gl}_{n_i}(\Bbbk), B \in \mathfrak{sp}_m(\Bbbk) \right\}$$

where $m = 2n - 2(n_1 + \cdots + n_{r-1})$ and $(\overline{A_k})_{i,j} = -(A_k)_{n_k-j+1, n_k-i+1}$.

2.1.5 G-complete reducibility

We now define what it means for a subalgebra of $\mathfrak{g}$ to be G-completely reducibile. The concept of G-complete reducibility for groups is due to Serre [Ser05], and the natural corresponding notion for subalgebras was introduced by McNinch [McN07].

Definition 2.1.30. We say a subgroup H of G is *G-completely reducible* if whenever H is contained in a parabolic subgroup of G it is contained in a Levi subgroup of G.

Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we say that $\mathfrak{h}$ is *G-completely reducible* if for every parabolic subalgebra $\mathfrak{p}$ such that $\mathfrak{h} \subseteq \mathfrak{p}$ we have that there exists some Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ with $\mathfrak{h} \subseteq \mathfrak{l}$.

We state the following preliminary results which are due to Stewart–Thomas and McNinch.

Lemma 2.1.31. *[ST18, Lemma 3.2] Let G be a simple, simply connected algebraic group of classical type, and $\mathfrak{g} = \mathrm{Lie}(G)$. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is G-completely reducible if and only if it acts completely reducibly on the natural module for $\mathfrak{g}$.*

Lemma 2.1.32. *[McN07, Lemma 4] Let G be reductive and let L be a Levi factor of a parabolic subgroup of G. Suppose that we have a Lie subalgebra $\mathfrak{h} \subset \mathfrak{l} = \mathrm{Lie}(L)$. Then $\mathfrak{h}$ is G-completely reducible if and only if $\mathfrak{h}$ is L-completely reducible.*

2.1.6 Representations and modules

Next we discuss modules and representations of Lie algebras.

Definition 2.1.33. Suppose $\mathfrak{g}$ is a Lie algebra over $\Bbbk$.

- Let V be an n-dimensional vector space over $\Bbbk$. A *representation* of $\mathfrak{g}$ on V is a Lie algebra homomorphism

$$\varphi : \mathfrak{g} \to \mathfrak{gl}(V).$$

- A $\mathfrak{g}$-*module* is a vector space V over $\Bbbk$ together with a map

$$\mathfrak{g} \times V \to V, \quad (x, v) \mapsto x \cdot v$$

that satisfies the following conditions for all $\lambda, \mu \in \Bbbk$, $x, y \in \mathfrak{g}$, and $u, v \in V$:

(a) $(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v)$;

(b) $x \cdot (\lambda u + \mu v) = \lambda(x \cdot u) + \mu(x \cdot v)$; and

(c) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

For $\mathfrak{g}$-modules V_1, V_2 we define the direct sum $V_1 \oplus V_2$ to be a $\mathfrak{g}$-module in the natural way. That is, for $(v_1, v_2) \in V_1 \oplus V_2$, we set $x \cdot (v_1, v_1) = (x \cdot v_1, x \cdot v_2)$ for all $x \in \mathfrak{g}$.

Remark 2.1.34. Given some representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ we can make V into a $\mathfrak{g}$-module by setting

$$x \cdot v = \varphi(x)(v) \text{ for all } x \in \mathfrak{g}, v \in V.$$

Similarly, if V is an $\mathfrak{g}$-module, then we can view V as a representation of $\mathfrak{g}$ by setting $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, to be the homomorphism with

$$\varphi(x)(v) = x \cdot v \text{ for all } x \in \mathfrak{g}, v \in V.$$

Thus after this point we will not differentiate between modules and representations.

Definition 2.1.35. Suppose V is a module for some Lie algebra $\mathfrak{g}$. Then a subspace $W \subseteq V$ is a *submodule* if for all $x \in \mathfrak{g}$, and for all $w \in W$ we have $x \cdot w \in W$.

Definition 2.1.36. We say that a module V for some Lie algebra $\mathfrak{g}$ is *simple* if the only submodules of V are 0 and V.

We say that a $\mathfrak{g}$-module is *completely reducible* if it can be written as a direct sum of simple $\mathfrak{g}$-modules.

2.1.7 Universal enveloping algebras

In Section 2.3.2 we will find the irreducible modules of $\mathfrak{sl}_2(\Bbbk)$, in order to do this we consider universal enveloping algebras and their representations. The following definitions can be found in Chapter 15 in [EW06]. For convenience, we introduce the notation $\mathfrak{s} = \mathfrak{sl}_2(\Bbbk)$.

Definition 2.1.37. Suppose that $\mathfrak{g}$ is some finite-dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, x_2, \ldots, x_n\}$, then we define the *structure constants* to be the $a_{ij}^k \in \Bbbk$ such that

$$[x_i, x_j] = \sum_k a_{ij}^k x_k \text{ for } 1 \leq i, j \leq n.$$

Definition 2.1.38. Given some finite-dimensional Lie algebra $\mathfrak{g}$ over $\Bbbk$, that has basis $\{x_1, x_2, \ldots, x_n\}$ and corresponding structure constants $a_{ij}^k \in \Bbbk$. We define the *universal enveloping algebra* $U(\mathfrak{g})$ to be the algebra generated by $\{X_1, X_2, \ldots, X_n\}$ subject to the relations

$$X_i X_j - X_j X_i = \sum_{k=1}^{n} a_{ij}^k X_k \quad \text{for all } 1 \leq i, j \leq n$$

with no other relations.

The universal enveloping algebra $U(\mathfrak{g})$ has a basis formed by all monomials in the elements $\{X_1, X_2, \ldots, X_n\}$, as proved in [Bir37, Lemma 1], that is it has PBW basis

$$\{X_1^{a_1} X_2^{a_2} \dots X_n^{a_n} : a_i \in \mathbb{Z}_{\geq 0}\}.$$

The elements $\{X_1, X_2, \dots, X_n\}$ are linearly independent, and hence $U(\mathfrak{g})$ is infinite dimensional if $\mathfrak{g} \neq 0$. We mention that in fact $U(\mathfrak{g})$ is independent of the choice of basis, and that $\mathfrak{g}$ is a subspace of $U(\mathfrak{g})$ in degree 1. As a copy of $\mathfrak{g}$ is contained in $U(\mathfrak{g})$, from this point, we will just refer to elements of $U(\mathfrak{g})$ as polynomials in elements of $\mathfrak{g}$.

Example 2.1.39. We saw in Example 2.1.4 that $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{k})$ has standard basis $\{e, h, f\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfy $[h, e] = 2e,\; [h, f] = -2f$ and $[e, f] = h$.

The universal enveloping algebra $U(\mathfrak{s})$ of $\mathfrak{s}$ has PBW basis $\{f^a h^b e^c : a, b, c \in \mathbb{Z}_{\geq 0}\}$. The only relations in $U(\mathfrak{s})$ are

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h.$$

Note that, for example, this means that in $U(\mathfrak{s})$, $e^2 \neq 0$ even though this relation is satisfied by the elements of $\mathfrak{sl}_2(\mathbb{k})$. We can write every element in $U(\mathfrak{s})$ as a sum of elements of the form $f^a h^b e^c$ for some $a, b, c \in \mathbb{k}$, see for example

$$ehf = (he - 2e)f = (h - 2)(ef) = (h - 2)(h + fe)$$

$$= h^2 - 2h + f(he - 2e) - 2fe = h^2 - 2h + fhe - 4fe.$$

The following notation and definitions are taken from [Jan98, Chapter 2]. Throughout this thesis we will want to consider restricted Lie algebras, we define this notion now.

Definition 2.1.40. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Then $\mathfrak{g}$ is *restricted* if there exists a map $\mathfrak{g} \to \mathfrak{g}$ sending $x \mapsto x^{[p]}$ such that

(a) $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p$ for all $x \in \mathfrak{g}$;

(b) $(tx)^{[p]} = t^p x^{[p]}$ for all $x \in \mathfrak{g}, t \in \Bbbk$; and

(c) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$ for all $x, y \in \mathfrak{g}$, where $s_i(x,y)$ is the coefficient of t^{i-1} in the formal expression of $\mathrm{ad}(tx+y)^{p-1}(x)$.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be Lie subalgebra of $\mathfrak{g}$. Then we say that $\mathfrak{h}$ is a *p-subalgebra* if $x^{[p]} \in \mathfrak{h}$ for all $x \in \mathfrak{h}$.

Example 2.1.41. In the familiar example of the classical Lie algebras, where $\mathfrak{g} \subseteq \mathfrak{gl}_n(\Bbbk)$, for $x \in \mathfrak{g}$ we write $x^{[p]}$ to be the p-th power of x as a matrix in $\mathfrak{gl}_n(\Bbbk)$, and we write x^p to be the p-th power of x in $U(\mathfrak{g})$.

Any Lie algebra of an algebraic group is restricted, and hence any Lie algebra we consider in this thesis is restricted.

Definition 2.1.42. Let V be a $\mathfrak{g}$-module. Then V has *p-character* $\chi \in \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \Bbbk)$ if for all $x \in \mathfrak{g}$

$$(x^p - x^{[p]} - \chi(x)^p) \cdot V = 0.$$

Note that any simple $\mathfrak{g}$-module has a p-character.

Definition 2.1.43. Let $\chi \in \mathfrak{g}^*$, then we define the *reduced enveloping algebra* of $\mathfrak{g}$ associated to χ to be

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g}) / \left\langle x^p - x^{[p]} - \chi(x)^p : x \in \mathfrak{g} \right\rangle .$$

There is no distinction in notation between elements in $\mathfrak{g}, U(\mathfrak{g})$ and $U_\chi(\mathfrak{g})$, however it should be clear from context where we are taking our elements from.

Reduced enveloping algebras are finite dimensional. In particular, if the basis of $\mathfrak{g}$ is $\{x_1, x_2, \ldots, x_n\}$ then $U_\chi(\mathfrak{g})$ has basis $\{x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} : 0 \leq a_i < p\}$, see this for example by Proposition 2.8 in [Jan98].

Lemma 2.1.44. *[Jan98, Section 2.7] There is a bijection between $\{\mathfrak{g}$-modules$\}$ and $\{U(\mathfrak{g})$-modules$\}$, that induces for each χ a bijection between $\{\mathfrak{g}$-modules with p-character $\chi\}$ and $\{U_\chi(\mathfrak{g})$-modules$\}$.*

2.2 Wedderburn's Theorem

Let $\Bbbk$ be an algebraically closed field. We now recall some of the representation theory of semisimple Lie algebras. We take these results from the Introduction chapter of [CR90].

Let A be a finite dimensional algebra, with simple modules $\{L_i\}_{i \in I}$ that are defined up to isomorphism for some indexing set I.

Definition 2.2.1. We say an A-module M is *completely reducible* if it can be written as a direct sum of simple A-modules.

We say that A is *semisimple* if every A-module is completely reducible.

Let M be a completely reducible A-module. We define the *homogeneous components* of M to be $\{M_i\}_{i \in I}$ such that

$$M_i = \sum_{V \subseteq M, V \cong L_i} V.$$

Note that $M = \oplus_{i \in I} M_i$, and $M_i = L_i^{\oplus t}$ for some t.

The following theorem is a corollary of Wedderburn's Theorem, stated for example in [CR90, Theorem 3.22, Wedderburn's Theorem], and Proposition 3.33 and Theorem 3.34 of [CR90].

Theorem 2.2.2. *Let A be a finite dimensional semisimple algebra over $\Bbbk$. Label the simple modules of A as $\{L_1, \ldots, L_k\}$. Consider A as a module over itself, then there are a finite number of homogeneous components of A, that we label $\{A_1, \ldots, A_k\}$, and*

$$A = A_1 \oplus \cdots \oplus A_k.$$

Moreover each A_i is a matrix algebra with deg$(A_i) = \dim(L_i)$.

This result only holds when A is a semisimple algebra, so we now give the definition of the Jacobson radical to enable us to apply these ideas for algebras that are not semisimple.

Definition 2.2.3. We define the *Jacobson radical* of A, denoted $\operatorname{rad} A$, to be the intersection of the maximal left ideals of A.

Results in [CR90, §5] tell us that for any algebra A, $\operatorname{rad} A$ is a two sided ideal and if we take the quotient of A by its Jacobson radical, then $A/\operatorname{rad} A$ is semisimple. Thus we can apply Wedderburn's theorem to $A/\operatorname{rad} A$ when A is not semisimple.

2.3 Representations of $\mathfrak{sl}_2(\Bbbk)$

Recall that $\Bbbk$ is an algebraically closed field of characteristic p, and we write $\mathfrak{s} = \mathfrak{sl}_2(\Bbbk)$.

We give some notation for $\mathfrak{s}$-modules and their related $\mathfrak{sl}_2$-triples that is used throughout. Given an $\mathfrak{s}$-module M and $x \in \mathfrak{s}$, we write $x_M \in \mathfrak{gl}(M)$ to denote the matrix representation of the action of x on M. Then we have that (e_M, h_M, f_M) is an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(M)$, and in fact lies in $\mathfrak{sl}(M)$ as $\mathfrak{s}$ is equal to its derived subalgebra and $\mathfrak{sl}(M)$ is the derived subalgebra of $\mathfrak{gl}(M)$.

We now consider some important modules of $\mathfrak{s}$.

Example 2.3.1. We first define the $(n+1)$-dimensional module $V(n)$ of $\mathfrak{sl}_2(\mathbb{C})$. These modules are first defined over $\mathbb{C}$, however we will also explain how to define $V(n)$ over a field of characteristic $p > 0$. We consider $V(n)$ in two ways:

(i) Let us first consider the vector space $\mathbb{C}[X, Y]$ of polynomials in commuting indeterminates X, Y. Let $n \in \mathbb{Z}_{\geq 0}$, and set $V(n)$ to be the subspace of $\mathbb{C}[X, Y]$ containing homogeneous polynomials of degree n.

Then $V(0)$ is the vector space of constant polynomials. For $n \geq 1$, we can take the following set to be a basis of $V(n)$

$$\{X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n\}.$$

We make $V(n)$ into an $\mathfrak{s}$-module by setting

$$e \cdot v = X\frac{\delta v}{\delta Y}, \quad h \cdot v = X\frac{\delta v}{\delta X} - Y\frac{\delta v}{\delta Y}, \quad f \cdot v = Y\frac{\delta v}{\delta X}.$$

Then using the above basis $e_{V(n)}, h_{V(n)}, f_{V(n)}$ are represented by the following matrices respectively

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & n \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} n & & & & \\ & n-2 & & & \\ & & \ddots & & \\ & & & 2-n & \\ & & & & -n \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ n & 0 & & & \\ & \ddots & \ddots & & \\ & & 2 & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

where all other entries in these matrices are zero.

Note that $V(n)$ can be defined for any $n \in \mathbb{Z}_{\geq 0}$, and has dimension $n+1$.

(ii) Typically when working with $\mathfrak{sl}_2$-triples we will want to work with e both nilpotent, and in Jordan normal form. Hence we conjugate the above matrices (i.e. find another basis) to find a form that is more convenient. Conjugating the above matrices by the diagonal matrix

$$\mathrm{diag}(0!, 1!, 2!, \ldots, n!) \in \mathrm{GL}_n(\mathbb{C})$$

gives us the following module of $\mathfrak{sl}_2(\mathbb{C})$, where $e_{V(n)}, h_{V(n)}, f_{V(n)}$ are represented by the following matrices respectively

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} n & & & & \\ & n-2 & & & \\ & & n-4 & & \\ & & & \ddots & \\ & & & & -n \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ a_1 & 0 & & & \\ & a_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & a_n & 0 \end{pmatrix}$$

where in the matrix for $f_{V(n)}$ we have that $a_j = a_{j-1} + n - 2(j-1) =$

$j((n+1)-j)$, and all other entries in these matrices are zero.

We will frequently refer to the $\mathfrak{sl}_2$-triple $(e_{V(n)}, h_{V(n)}, f_{V(n)})$ resulting from this module reduced modulo p. We therefore introduce the shorthand $h(e) := h_{V(n)}, f(e) := f_{V(n)}$.

Now let $\Bbbk$ be a field of characteristic $p > 0$, then reducing the above matrices modulo p makes $V(n)$ a module for $\mathfrak{s}$. To conjugate to the basis in part (b) we needed that $n! \neq 0$, which holds over $\mathbb{C}$ but not necessarily in an arbitrary field of prime characteristic. Thus the two forms on $V(n)$ can reduce differently modulo p, we define the $\mathfrak{s}$-module $V(n)$ in $\Bbbk$ to be formed by reducing the matrices in part (b).

2.3.1 Representations of $\mathfrak{sl}_2$ over $\mathbb{C}$

It is shown in Chapter 8 of [EW06] that the $\mathfrak{sl}_2(\mathbb{C})$-modules $V(n)$ are simple and moreover they are the only simple modules of $\mathfrak{sl}_2(\mathbb{C})$.

Weyl's Theorem, which can be found for example in [EW06, Theorem 8.7], states that any finite-dimensional module for $\mathfrak{sl}_2(\mathbb{C})$ is completely reducible, hence every $\mathfrak{sl}_2(\mathbb{C})$-module can be written as a direct sum of these $V(n)$. Over fields of characteristic $p > 0$, this is no longer true.

2.3.2 Representations of $\mathfrak{sl}_2$ over a field of characteristic p

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 2$.

We next recall some aspects of the representation theory of $\mathfrak{s} = \mathfrak{sl}_2(\Bbbk)$ that we require later. We will explain the classification of simple $\mathfrak{s}$-modules on which e and f act nilpotently, and some information about extensions between these simple modules.

Recall the reduced enveloping algebras as defined in Definition 2.1.43. We consider Section 5 of [Jan98] which finds all the simple modules for the reduced enveloping algebras $U_\chi(\mathfrak{s})$ for any p-character χ. It can be deduced from this that $\mathfrak{s}$-modules with dimension $n < p$ are a direct sum of the simple modules $V(n)$, however there exist modules of dimension greater than or equal to p that are not a sum of simple modules. Most notably, the baby Verma modules of dimension p that are introduced below.

Proposition 2.3.2. *[Jan98, Section 5.2] The only simple $U_0(\mathfrak{s})$-modules are $V(d)$ for $0 \leq d < p$.*

Proposition 2.3.3. *[Jan98, Proposition 5.3] If $\chi \neq 0$ then every simple $U_\chi(\mathfrak{s})$-module has dimension p.*

Definition 2.3.4. For χ a p-character and $\lambda \in \mathbb{k}$, let $\mathfrak{b} = \mathbb{k}h + \mathbb{k}e$ and define m_+ to be an element so that $h \cdot m_+ = \lambda m_+$. We define the *baby Verma module associated to χ and λ* to be

$$Z_\chi(\lambda) = U_\chi(\mathfrak{s}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{k}m_+.$$

A basis for $Z_\chi(\lambda)$ is $\{v_i = f^i \otimes m_+ : i \in \{0, \dots, p-1\}\}$. The standard basis of $\mathfrak{sl}_2(\mathbb{k})$ act on this module as

$$h \cdot v_i = (\lambda - 2i)v_i$$

$$e \cdot v_i = \begin{cases} 0 & \text{if } i = 0, \\ i(\lambda - i + 1)v_{i-1} & \text{if } i > 0. \end{cases}$$

$$f \cdot v_i = \begin{cases} v_{i+1} & \text{if } i < p-1, \\ \chi(f)^p v_0 & \text{if } i = p-1. \end{cases}$$

We consider the submodules of $Z_0(\lambda)$ for $\lambda \in \Bbbk$. We suppose that $\chi = 0$, and hence $0 = \chi(h)^p = \lambda^p - \lambda$, from which we deduce that $\lambda \in \mathbb{F}_p$ and there is no such module for $\lambda \notin \mathbb{F}_p$. We know that e^p acts as zero, so we note that any submodule M of $Z_0(\lambda)$ must contain some $v \neq 0$ so that $e \cdot v = 0$. The only basis vectors that are mapped to 0 by e are v_0 and $v_{\lambda+1}$, hence either $v_0 \in M$ or $v_{\lambda+1} \in M$. If $v_0 \in M$, we use the action of f to see that all of the basis vectors are contained in the submodule, so $M = Z_\chi(\lambda)$. Therefore to find a non-trivial submodule we assume that $v_0 \notin M$. Set $M = \operatorname{span}\{v_{\lambda+1}, \dots, v_{p-1}\}$ and note that this is closed under the action of e, h and f and hence M is a submodule with $p - \lambda - 1$ basis elements. The module M is irreducible as the only basis vector that is mapped to 0 by e is $v_{\lambda+1}$. Note that for each $n \in \mathbb{F}_p$ we have $V(n)$ is isomorphic to the irreducible submodule, M, of $Z_0(p-n-2)$. Equivalently, $V(n)$ is the irreducible quotient of $Z_0(n)$.

We can use the same method of considering possible submodules to see that the baby Verma modules $Z_\chi(\lambda)$ are simple when $\chi \neq 0$.

Remark 2.3.5. When $\Bbbk$ is a field of positive characteristic p, the $\mathfrak{sl}_2(\Bbbk)$-module $V(n)$ is simple if and only if $n < p$.

For $c, d \in \{0, 1, \dots, p-2\}$, it is known that

$$\operatorname{Ext}^1_{\mathfrak{s}}(V(c), V(d)) = 0 \text{ except in the case } d = p - c - 2, \tag{2.3.1}$$

see for example [ST18, Lemma 2.7]. In order to see this, consider the action of the

Casimir element, $c := ef + fe + \frac{1}{2}h^2$, which lies in the centre of $U(\mathfrak{s})$. We have that c acts on $V(d)$ as the scalar $\frac{1}{2}d(d+2)$. Given an $\mathfrak{sl}_2(\mathbb{k})$-module M, we can express M as a direct sum of the generalised eigenspaces for c, and so we have that $\mathrm{Ext}^1_{\mathfrak{s}}(V(d), V(d')) = 0$ for $d' \neq d, p-d-2$. By [Jan03, Proposition 12.9] we have that $\mathrm{Ext}^1_{\mathfrak{s}}(V(d), V(d)) = 0$ for $d \in \{0, 1, \dots, p-2\}$.

Remark 2.3.6. Over $\mathbb{C}$, consider the $\mathfrak{sl}_2(\mathbb{C})$-module $V(n)$ with basis as given in Example 2.1.4(b). Under this basis, $f_{V(n)}$ is represented by the matrix with elements $a_j = j(n-j)$ on the -1-th diagonal, where $a_j \neq 0$ for any j. Hence we can see that $e_{V(n)} \sim (n+1)$ and $f_{V(n)} \sim (n+1)$ and hence they have the same Jordan normal form and thus are conjugate by an element of $\mathrm{GL}_{n+1}(\mathbb{C})$.

Similarly, if we are considering a field $\mathbb{k}$ of prime characteristic $p > 0$, we consider the $\mathfrak{sl}_2$-triple formed by the matrices $(e_{V(n)}, h_{V(n)}, f_{V(n)})$ defined by their action on the $\mathfrak{s}$-module $V(n)$. If $n < p$, we have that $a_j = j(n-j) \neq 0$ for any j and hence $e_{V(n)}$ and $f_{V(n)}$ are conjugate by an element of $\mathrm{GL}_{n+1}(\mathbb{k})$. However, this does not hold for $n \geq p$, as $a_p = p((n+1)-p) = 0 \mod p$.

In Chapter 3 we show that for any nilpotent $e \in \mathfrak{gl}_n(\mathbb{k})$ there exists some $h, f \in \mathfrak{gl}_n(\mathbb{k})$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple, and e is conjugate to f. This leads us to the following definition.

Definition 2.3.7. We say that an $\mathfrak{sl}_2$-triple $(e, h, f) \in \mathfrak{gl}_n(\mathbb{k})$ is a *strong* $\mathfrak{sl}_2$-triple if e is conjugate to f by an element of $\mathrm{GL}_n(\mathbb{k})$.

2.4 Nilpotent orbits for classical Lie algebras

Let G be one of $\mathrm{GL}_n(\mathbb{k})$, $\mathrm{SL}_n(\mathbb{k})$, $\mathrm{Sp}_n(\mathbb{k})$, $\mathrm{O}_n(\mathbb{k})$ or $\mathrm{SO}_n(\mathbb{k})$ (where we assume n is even in the $\mathrm{Sp}_n(\mathbb{k})$ case), and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathcal{N}$ represent the nilpotent cone of $\mathfrak{g}$.

2.4.1 Parametrisation of orbits

We give an overview of the well-known parametrisation of G-orbits in $\mathcal{N}$ in terms of Jordan types. We begin with a discussion on the nilpotent orbits in $\mathcal{N}$, for more details we refer the reader to [Jan04, Section 1].

Throughout, by a partition we mean a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers λ_i such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \ldots, m-1$; we have the convention that $\lambda_i = 0$ for $i > m$. We say that λ is partition of $\lambda_1 + \lambda_2 + \cdots + \lambda_m$. We sometimes use superscripts to denote multiplicities in partitions, so for example may write $(3^2, 2, 1^3)$ as a shorthand for $(3, 3, 2, 1, 1, 1)$. For a partition λ and $i \in \mathbb{Z}_{>0}$, we define $m_i(\lambda)$ to be the multiplicity of i in λ. Given partitions λ and μ we define $\lambda|\mu$ to be the partition with $m_i(\lambda|\mu) = m_i(\lambda) + m_i(\mu)$ for all $i \in \mathbb{Z}_{>0}$

We briefly recap the definition of Jordan Normal Form, as later we will want to take matrices in this form for ease.

Definition 2.4.1. Suppose $a \in \Bbbk$, then the *Jordan block* $J_h(a)$ of size $h \times h$ is the matrix

$$J_h(a) = \begin{pmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \ddots & \ddots & \\ & & & a & 1 \\ & & & & a \end{pmatrix}.$$

A *Jordan matrix* is a matrix of the following block form

$$\begin{pmatrix} J_{h_1}(a_1) & & & \\ & J_{h_2}(a_2) & & \\ & & \ddots & \\ & & & J_{h_k}(a_m) \end{pmatrix}$$

for some $h_i \geq 1, a_i \in \Bbbk$.

We say that a matrix is in *Jordan normal form* if it is a Jordan matrix.

We have taken $\Bbbk$ to be an algebraically closed field, so if we suppose V is a finite dimensional vector space and $f : V \to V$ is a linear operator, then there exists a basis of V so that the matrix of f is in Jordan normal form. Up to the reordering of the Jordan blocks, this Jordan normal form is unique, see for example Theorem 11.23 in [HH83]. Hence we have that every matrix in $\mathfrak{gl}_n(\Bbbk)$ is conjugate to some matrix in Jordan normal form.

Suppose that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n(\Bbbk)$. Let us consider a nilpotent element x in $\mathfrak{g}$, that is $x^n = 0$. Up to conjugacy by $\mathrm{GL}_n(\Bbbk)$ we can assume that x is in Jordan normal form, and since x is nilpotent it only has eigenvalues equal to 0.

Definition 2.4.2. We say that the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ is the *Jordan type* of x if the Jordan normal form of x has Jordan blocks $J_{\lambda_i}(0) = J_{\lambda_i}$.

We write $x \sim \lambda$ to express x having Jordan type λ, and use the notation $\lambda(x)$ to denote the Jordan type of x.

We see that in the cases $G = \mathrm{GL}_n(\Bbbk)$ or $G = \mathrm{SL}_n(\Bbbk)$, the G-orbits in $\mathcal{N}$ are parameterised by their Jordan type. Also in the cases $G = \mathrm{Sp}_n(\Bbbk)$ or $G = \mathrm{O}_n(\Bbbk)$

the G-orbits in $\mathcal{N}$ are parameterised by their Jordan types, and the Jordan types that can occur are known explicitly, as is stated in [Jan04, Theorem 1.6], and restated below.

Proposition 2.4.3. *For a partition λ of n, there is a nilpotent element $x \in \mathfrak{sp}_n(\mathbb{k})$ (respectively $\mathfrak{so}_n(\mathbb{k})$) with Jordan type λ if and only if $m_i(\lambda)$ is even for all odd i (respectively $m_i(\lambda)$ is even for all even i).*

Remark 2.4.4. To describe the parametrisation in the case $G = \mathrm{SO}_n(\mathbb{k})$, we note that the $\mathrm{O}_n(\mathbb{k})$-orbit of $x \in \mathcal{N}$ is either a single $\mathrm{SO}_n(\mathbb{k})$-orbit, or splits into two $\mathrm{SO}_n(\mathbb{k})$-orbits. The former case occurs if the centraliser of e in $\mathrm{O}_n(\mathbb{k})$ contains an element of $\mathrm{O}_n(\mathbb{k}) \setminus \mathrm{SO}_n(\mathbb{k})$ whilst the latter occurs if the centraliser of e in $\mathrm{O}_n(\mathbb{k})$ is contained in $\mathrm{SO}_n(\mathbb{k})$. The centraliser of e in $\mathrm{O}_n(\mathbb{k})$ is contained in $\mathrm{SO}_n(\mathbb{k})$ precisely when all parts of λ are even; such partitions are referred to as *very even* as all parts are even and have even multiplicity.

2.4.2 Closure order on nilpotent orbits

Definition 2.4.5. We introduce a partial ordering, which we refer to as the *dominance order*, on the partitions of n. Given two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, $\mu = (\mu_1 \geq \mu_2 \geq \dots)$, we say that $\mu \preceq \lambda$ if we have that

$$\sum_{i=1}^{r} \mu_i \leq \sum_{i=1}^{r} \lambda_i \text{ for all } r \in \mathbb{Z}_{>0}.$$

The dominance order on partitions induces an order on the nilpotent orbits in $\mathfrak{g}$.

Suppose we have some $\mathfrak{X}$, a closed G-invariant subvariety of the variety $\mathcal{N}$ of nilpotent elements of $\mathfrak{g}$. Then we have that $\mathfrak{X}$ is a union of G-orbits in $\mathcal{N}$. We write this as $\mathfrak{X} = \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ where $\mathcal{O}_\lambda = G \cdot e_\lambda$ for some subset of partitions Λ, and

e_λ is some nilpotent element with Jordan type λ.

We next state a theorem essentially due to Spaltenstein, which shows that the closure order on nilpotent orbits is determined by the dominance order on partitions. In the statement we use the notation $\mathcal{O}_\lambda$ for the G-orbit in $\mathcal{N}$ of elements with Jordan type λ.

Theorem 2.4.6. *Let G be one of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$ or $\mathrm{O}_n(\Bbbk)$. Let λ and μ be partitions of n that parameterise a G-orbit in $\mathcal{N}$. Then $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$ if and only if $\mu \preceq \lambda$.*

To explain why this theorem holds, we first note that there is a Springer isomorphism from the variety $\mathcal{U}$ of unipotent elements in G to $\mathcal{N}$; that is a G-invariant isomorphism of varieties $\mathcal{U} \xrightarrow{\sim} \mathcal{N}$ that maps each element of $\mathcal{U}$ to an element of $\mathcal{N}$ with the same Jordan type. The existence of such a homeomorphism was proved by Springer in [Spr69, Theorem 3.1], Bardsley and Richardson extended this to prove existence of an isomorphism in [BR85]. We refer for example to [Hum95, §6.20] for explicit examples of Springer isomorphisms for $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$ and $\mathrm{O}_n(\Bbbk)$. A result of Spaltenstein, [Spa82, Théorème II.8.2], establishes that the dominance order on partitions determines the closure order for the unipotent classes; we refer also to [Car93, Section 13.4], where this result of Spaltenstein is covered. Thus Theorem 2.4.6 can be deduced using a Springer isomorphism.

We note that the closure order on the nilpotent orbits for the case $G = \mathrm{SO}_n(\Bbbk)$ is also covered in the result of Spaltenstein. Here we have that if λ and μ are distinct partitions of n that parameterise G-orbits $\mathcal{O}_\mu$ and $\mathcal{O}_\lambda$ in $\mathcal{N}$, then $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$ if and only if $\mu \preceq \lambda$. Note that in the case where λ is a very even partition, the two nilpotent orbits corresponding to λ are incomparable. However, if λ and μ are distinct then the relation on their orbits can quickly be deduced from the $G = \mathrm{O}_n(\mathbb{k})$ case. If either λ or μ are not very even, then we can use the $\mathrm{O}_n(\mathbb{k})$ case directly. In the case λ and μ are both very even, then there exists some not very even partition κ parameterising a nilpotent $\mathrm{O}_n(\mathbb{k})$-orbit such that $\mu \preceq \kappa \preceq \lambda$. To see this explicitly, let $\lambda = (\lambda_1, \lambda_1, \ldots, \lambda_k, \lambda_k)$, $\mu = (\mu_1, \mu_1, \ldots, \mu_r, \mu_r)$ be very even partitions. Suppose $\mu \preceq \lambda$ so there must exist some minimal i so that $\lambda_i > \mu_i$, and as these are both even we have $\lambda_i - 1 > \mu_i$. We take

$$\kappa = (\lambda_1, \lambda_1, \ldots, \lambda_i - 1, \lambda_i - 1, \ldots, \lambda_k, \lambda_k)$$

and hence we find $\mu \preceq \kappa \preceq \lambda$. Hence we can use the $\mathrm{O}_n(\mathbb{k})$ case, via $\mathcal{O}_\kappa$, to see that $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$.

2.4.3 Regular and distinguished orbits

We now discuss regular and distinguished nilpotent orbits. We give an overview of some known results.

Definition 2.4.7. We say that a nilpotent element $x \in \mathfrak{g}$ is *regular* if the centraliser of x in G has dimensional equal to the rank of G.

We say that a nilpotent element $x \in \mathfrak{g}$ is *distinguished* if the only Levi subalgebra of $\mathfrak{g}$ that contains x is $\mathfrak{g}$ itself.

Example 2.4.8. Let $\mathfrak{g}$ be one of $\mathfrak{sl}_n(\mathbb{k}), \mathfrak{sp}_{2n}(\mathbb{k}), \mathfrak{so}_n(\mathbb{k})$, and u a regular nilpotent element of G. The Jordan type of u is given by

- $u \sim (n)$ if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$;

- $u \sim (2n)$ if $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{k})$;
- $u \sim (2m+1)$ if $\mathfrak{g} = \mathfrak{so}_{2m+1}(\mathbb{k})$;
- $u \sim (2m-1, 1)$ if $\mathfrak{g} = \mathfrak{so}_{2m}(\mathbb{k})$.

The following well-known result is covered in [Spr66, Theorem 5.9].

Theorem 2.4.9. *The regular nilpotent elements of* $\mathfrak{g}$ *are conjugate under the action of* $\mathrm{Ad}(G)$.

CHAPTER 3

STRONG $\mathfrak{sl}_2$-TRIPLES

Within this chapter, we fix an algebraically closed field $\mathbb{k}$ of characteristic $p > 2$. Let G be $\mathrm{GL}_n(\mathbb{k})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{k})$. As before we write $\mathfrak{s} := \mathfrak{sl}_2(\mathbb{k})$.

Recall that a strong $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{gl}_n(\mathbb{k})$ is defined to be an $\mathfrak{sl}_2$-triple such that $e \sim f$. The main result in this chapter is Theorem 1 which is restated below.

Theorem 1. *There exists a surjective map*

$$\left\{ \begin{array}{l} \mathrm{GL}_n(\mathbb{k})\text{-}\mathit{orbits\ of\ strong} \\ \mathfrak{sl}_2\text{-}\mathit{triples\ in}\ \mathfrak{gl}_n(\mathbb{k}) \end{array} \right\} \longrightarrow \{\mathrm{GL}_n(\mathbb{k})\text{-}\mathit{orbits\ in}\ \mathcal{N}\}. \tag{1.0.3}$$

That is, given any nilpotent $e \in \mathfrak{gl}_n(\mathbb{k})$, there exists some strong $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{gl}_n(\mathbb{k})$.

In order to prove this we define a family of strong $\mathfrak{sl}_2$-triples for each nilpotent $e \in \mathfrak{gl}_n(\mathbb{k})$. In this chapter we also show that there is no bijective map between these sets when $p \leq n$, we do this in Proposition 3.3.4 by constructing an infinite family of non-conjugate $\mathfrak{sl}_2$-triples for each nilpotent orbit.

3.1 Properties of $\mathfrak{sl}_2$-triples in $\mathfrak{gl}_n(\Bbbk)$

Recall the $\mathfrak{s}$-modules $V(n-1)$ as defined in Example 2.3.1, and that given $e \in \mathfrak{gl}_n(\Bbbk)$ a single Jordan block, we use the notation $h(e) := h_{V(n-1)}, f(e) := f_{V(n-1)}$ to represent the action of the standard basis of $\mathfrak{sl}_2(\Bbbk)$ on $V(n-1)$.

We will often require the matrix representation of $\mathfrak{sl}_2$-triples (e, h, f) in $\mathfrak{gl}_n(\Bbbk)$ so in the following we write $e = (e_{ij})$, $h = (h_{ij})$ and $f = (f_{ij})$.

Definition 3.1.1. Let $\mathfrak{g}$ be any Lie algebra, the *centraliser* of $x \in \mathfrak{g}$ is defined to be the set

$$\mathfrak{g}^x = \{y \in \mathfrak{g} : [x, y] = 0\}.$$

Lemma 3.1.2. *Let $e \in \mathfrak{gl}_n(\Bbbk)$ be a single nilpotent Jordan block. That is, it is of the form $J_n(0)$ as in Definition 2.4.1. Then the centraliser of e is*

$$\mathfrak{g}^e = \left\{ \begin{pmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ 0 & x_1 & x_2 & \dots & x_{n-1} \\ 0 & 0 & x_1 & \dots & x_{n-2} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & x_1 \end{pmatrix} : x_1, x_2, \dots, x_n \in \Bbbk \right\}.$$

Proof. Suppose $x = (x_{ij}) \in \mathfrak{gl}_n(\Bbbk)$. We have that

$$(xe)_{ij} = \begin{cases} x_{i,j-1} & \text{if } j > 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (ex)_{ij} = \begin{cases} x_{i+1,j} & \text{if } i < n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $x \in \mathfrak{g}^e$ if and only if both

$$x_{i,1} = x_{n,j} = 0$$

for $1 < i$ and $j < n$, and

$$x_{i,j-1} = x_{i+1,j}$$

for $1 < j$ and $i < n$. This leads us to the general form above. □

Lemma 3.1.3. *Let $h(e) \in \mathfrak{gl}_n(\mathbb{k})$ be of the form given in Example 2.3.1. Then*

$$\mathfrak{g}^{h(e)} = \{(x_{ij}) : x_{ij} = 0 \text{ if } j \neq i + kp \text{ for some } k \in \mathbb{Z}\}.$$

Proof. Suppose $x = (x_{ij}) \in \mathfrak{gl}_n(\mathbb{k})$. We have

$$h(e)_{ij} = \begin{cases} (n-1) - 2(i-1) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and so

$$(xh(e))_{ij} = (n - 1 - 2(j-1))x_{ij} \quad \text{and} \quad (h(e)x)_{ij} = (n - 1 - 2(i-1))x_{ij}.$$

Hence $x \in \mathfrak{g}^{h(e)}$ if and only if $(n - 1 - 2(j-1))x_{ij} = (n - 1 - 2(i-1))x_{ij}$ for all i, j. Thus for all i, j either $x_{ij} = 0$ or $i = j \mod p$. □

Lemma 3.1.4. *Let $e \in \mathfrak{gl}_n(\mathbb{k})$ be a single Jordan block and $x_k \in \mathbb{k}^\times$. Take $f' \in \mathfrak{gl}_n(\mathbb{k})$ to be of the form*

$$f'_{ij} = \begin{cases} x_k & \text{if } j = i + kp - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[h(e), f'] = -2f'$, *and* $(e, h(e), f(e) + f')$ *is an* $\mathfrak{sl}_2$*-triple.*

Proof. Recall that $h(e) = (h(e)_{ij})$ and $f' = (f'_{ij})$ where

$$h(e)_{ij} = \begin{cases} (n-1) - 2(i-1) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}, \qquad f'_{ij} = \begin{cases} x_k & \text{if } j = i + kp - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

$$(h(e)f')_{ij} = \begin{cases} ((n-1) - 2(i-1))x_k & \text{if } j = i + kp - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} (f'h(e))_{ij} &= \begin{cases} ((n-1) - 2(j-1))x_k & \text{if } j = i + kp - 1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} ((n-1) - 2(i + kp - 2))x_k & \text{if } j = i + kp - 1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} ((n-1) - 2(i-2))x_k & \text{if } j = i + kp - 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $(h(e)f' - f'h(e))_{ij} = -2f'_{ij}$ and so $[h(e), f'] = -2f'$.

By Example 2.3.1 we have that $(e, h(e), f(e))$ is an $\mathfrak{sl}_2$-triple and hence $[h(e), e] = 2e$. It follows from Lemma 3.1.2 that

$$[e, f(e) + f'] = [e, f(e)] + [e, f'] = h(e).$$

Thus we see that $(e, h(e), f(e) + f')$ is an $\mathfrak{sl}_2$-triple. □

3.2 Matrix diagrams

In order to further examine the $\mathfrak{sl}_2$-triples created by the action of $\mathfrak{s}$-modules we first give a method of creating diagrams to represent endomorphisms of vector spaces. These will be used to represent the action of e, h, f on $\mathfrak{s}$-modules. These diagrams can be used both to determine when an endomorphism is nilpotent, and further to find the Jordan normal form of nilpotent endomorphisms in certain cases.

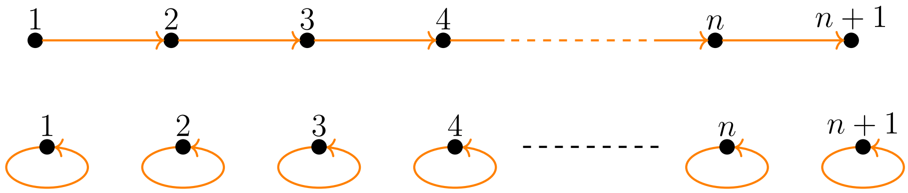
Definition 3.2.1. Let V be a vector space with basis $\{v_1, \ldots, v_n\}$, and $x : V \to V$ an endomorphism of V such that $x : v_i \mapsto \sum_j a_{ij} v_j$. The diagram $D(x)$ is constructed from x by adding a node, labelled i, for each basis element v_i and adding an arrow from i to j if $a_{ij} \neq 0$.

Note that if $D(x)$ has no arrows coming from i, then x maps v_i to 0.

We emphasise that $D(x)$ cannot be used to uniquely define a linear transformation. However, in some cases $D(x)$ can be used to determine the Jordan type of x.

For an $\mathfrak{s}$-module V we consider $e_V, h_V, f_V \in \mathfrak{gl}(V)$ to be the matrices that represent the action of e, h, f on V.

Example 3.2.2. We draw these diagrams for the $\mathfrak{s}$-module $V(n)$ as defined in Example 2.3.1. In this example, we take $\Bbbk = \mathbb{C}$ and consider $V(n)$ as a module for $\mathfrak{sl}_2(\mathbb{C})$. The following are the diagrams for $D(e)$, $D(f)$ and $D(h)$ respectively.



The construction of $D(f)$ for $\Bbbk$ of characteristic $p > 2$ can be seen in the proof of Proposition 3.3.2.

Definition 3.2.3. Let V be a vector space with basis $\{v_1, \ldots, v_n\}$ and $x : V \to V$ an endomorphism of V. Consider the diagram $D(x)$, then a basis element which has arrows coming out, but no arrows coming in, is a *source*. Conversely, a basis element with arrows coming in, but no arrows coming out, is a *sink*.

We define a *chain* of length j starting at v_{i_1} to be some sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ where $v_{i_m} \neq v_{i_l}$ for $l \neq m$, x maps v_{i_j} to 0, and for each $m < j$ there is an arrow from v_{i_m} to $v_{i_{m+1}}$.

We say x contains a *cycle* if there exists some sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ where there is an arrow from v_{i_m} to $v_{i_{m+1}}$ for each $m < j$ and we have $v_{i_1} = v_{i_j}$.

Similarly x creates a *non-oriented cycle* if there exists some sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ where for each $m < j$ there is either an arrow from v_{i_m} to $v_{i_{m+1}}$ or an arrow from $v_{i_{m+1}}$ to v_{i_m} and $v_{i_1} = v_{i_j}$.

Example 3.2.4. Continuing Example 3.2.2, we see that $D(e)$ has a source at v_{n+1}, a sink at v_1, there are no cycles, and for each i the longest chain starting from v_i is of length i.

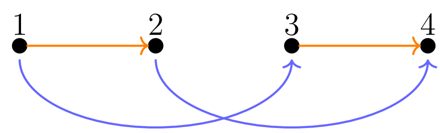
Now consider $D(h)$, then each basis vector is an eigenvector, and hence has an arrow to itself in $D(h)$. Thus there is a cycle at each vector, however there are no other arrows, so there are no sources or sinks.

In certain cases, these diagrams can be used to find the Jordan normal form of nilpotent elements. To see this we consider what information can be determined from the chains and cycles in $D(x)$. First suppose that x acts on basis vector v_i in such a way that $x^j(v_i) = 0$, $x^{j-1}(v_i) \neq 0$ for some j. Then there must exist some chain of length j in $D(x)$ beginning at v_i. Additionally if the diagram has no non-oriented cycles then there do not exist any chains of greater length starting from i. If there are non-oriented cycles we might be able to find a chain of greater length.

Example 3.2.5. Suppose V has basis $\{v_1, v_2, v_3, v_4\}$ and x acts on V as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

then $D(x)$ is the following diagram.



Observe that $x^2(v_1) = x(v_2 + v_3) = x(v_2) + x(v_3) = v_4 - v_4 = 0$, however there is a non-oriented cycle v_1, v_2, v_4, v_3, v_1, and there is a chain v_1, v_2, v_4 of length 3.

Lemma 3.2.6. *Let V be an n-dimensional vector space with basis $\{v_1, \ldots, v_n\}$. Suppose that x is a matrix representing an endomorphism of V. If $D(x)$ has no cycles, then x is nilpotent.*

Proof. First suppose that there is a chain of length $m > n$ starting at v_i. There are only n basis vectors, so this chain must visit some basis vector v_k more than once, and therefore this chain contains a cycle. This is a contradiction, hence all chains in $D(x)$ must have length $\leq n$.

Next suppose for a contradiction that there exists some $m \geq n$ such that $x^m \neq 0$. Let v_i be such that $x^m(v_i) \neq 0$, then there exists some basis vector v_k such that $x^m(v_i) = a_k v_k +$ other terms where $a_k \neq 0$. Thus we can find a chain of length $m+1$ starting at v_i and ending at v_k. This contradicts the above, and thus we conclude that $x^n = 0$ and hence that x is nilpotent. □

Note that a nilpotent matrix $e \in \mathfrak{gl}_n(\Bbbk)$ is conjugate to a single Jordan block if and only if $e^{n-1} \neq 0$ and $e^n = 0$. This inspires the following lemma.

Lemma 3.2.7. *Let $x \in \mathfrak{gl}_n(\Bbbk)$. Suppose that $D(x)$ has no cycles, and there is some v such that $x^{n-1}(v) \neq 0$, so that $D(x)$ has a chain of length n starting at v. Then x has Jordan type (n).*

Proof. By Lemma 3.2.6 it follows that x is nilpotent and so we find the Jordan type of x. We have $x^{n-1} = 0$, and so if $\mathcal{B}_V = \{v_1, \dots v_n\}$ is a basis of V there must be some v_i so that $x^k(v_i) \neq 0$ for all $k < n$.

We define another basis of V on which x acts as the single Jordan block. Set $\mathcal{B}_W = \{w_1, \dots, w_n\}$ where

$$w_j = x^{n-j}(v_i).$$

Suppose for contradiction that the elements of $\mathcal{B}_W$ are not linearly independent. Thus there exist some scalars a_k not all zero so that $\sum a_k w_k = 0$. Take this relation to have the minimal number of non-zero a_k. Let j be the smallest such that $a_j \neq 0$, then up to multiplication by a non-zero scalar we can rearrange this sum into the following form

$$w_j = \sum_{k>j} a_k w_k. \tag{3.2.1}$$

That is

$$x^{n-j}(v_i) = \sum_{k>j} a_k x^{n-k}(v_i).$$

Then x^j acts on this relation, and we find

$$0 = x^n(v_i) = \sum_{k>j} a_k x^{n-k+j}(v_i) = \sum_{k>j} a_k w_{k-j}.$$

This contradicts that (3.2.1) had a minimal number of non-zero coefficients, thus we have that $\mathcal{B}_W$ contains n linearly independent elements, and hence is a basis.

Then we have that

$$x(w_j) = x(x^{n-j}(v_i)) = x^{n-j+1}(v_i) = \begin{cases} w_{j-1} & \text{if } j < n-1, \\ 0 & \text{if } j = n-1. \end{cases}$$

So under this basis x acts as the single Jordan block $J_n(0)$. □

We use the above lemma to consider $\mathfrak{sl}_2$-triples (e, h, f) in $\mathfrak{gl}_n(\Bbbk)$ with e the single Jordan block. If $D(f)$ has no cycles, and a chain of length n beginning at some v such that $f^{n-1}(v) \neq 0$ then f is conjugate to e. Also note that if there is such a chain, it must start from a source, and end in a sink, or there exists a chain of length greater than n.

3.3 A family of strong $\mathfrak{sl}_2$-triples

In this section we construct a strong $\mathfrak{sl}_2$-triple containing e for any regular nilpotent element $e \in \mathfrak{gl}_n(\Bbbk)$.

Example 3.3.1. Recall the n-dimensional $\mathfrak{sl}_2$-module $V(n-1)$ defined in Example 2.3.1. We have that e acts on $V(n-1)$ as a single Jordan block of size n. If $n \leq p$, then f acts as the matrix with $a_j = j(n-j) \neq 0$ for any j. Thus $D(f)$ has no cycles and $f^{n-1}(v_n) = v_1 \neq 0$, and hence f is conjugate to the single Jordan block by Lemma 3.2.7. Thus if $n \leq p$, we see that f is conjugate to e in $V(n-1)$.

We now consider what happens when $n > p$, we split this into two cases, considering when n is a multiple of p separately.

Let $e \in \mathfrak{gl}_n(\Bbbk)$ be regular nilpotent and take $(e, h(e), f(e))$ to be the $\mathfrak{sl}_2$-triple defined in Example 2.3.1(ii).

Proposition 3.3.2. *Suppose $ap < n < (a+1)p$ for some $a \in \mathbb{Z}_{>0}$. There exists some $F \in \mathfrak{gl}_n(\Bbbk)$ such that $(e, h(e), F)$ is a strong $\mathfrak{sl}_2$-triple. Moreover $F = f(e) + f'$ for some $f' \in \mathfrak{g}^e$.*

Proof. To simplify the diagrams, we first consider the case when $p < n < 2p$.

We will define an element $F \in \mathfrak{gl}_n(\Bbbk)$ such that $e \sim F$. We first note that by Lemma 3.2.7, if $D(F)$ has no cycles and there exists some basis vector v such that $F^{n-1}(v) \neq 0$ then $e \backsim F$.

First recall the matrix $f(e)$ with elements a_j as given in Example 2.3.1(ii). Then $a_j = 0$ if and only if $j = p$ or $j = n-p$ so $f(e)$ has the following diagram $D(f(e))$.

It is clear that e is not conjugate to $f(e)$, as $D(f(e))$ has no chain of length n, and hence there is no v_i such that $f(e)^{n-1}(v_i) \neq 0$.

Let $x_1 \in \mathbb{k}$ be a non-zero scalar, and define $f' \in \mathfrak{gl}_n(\mathbb{k})$ by

$$f'_{ij} = \begin{cases} x_1 & \text{if } j = i + p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f' \in \mathfrak{g}^e$ by Lemma 3.1.2, and $[h, f'] = -2f'$ by Lemma 3.1.4.

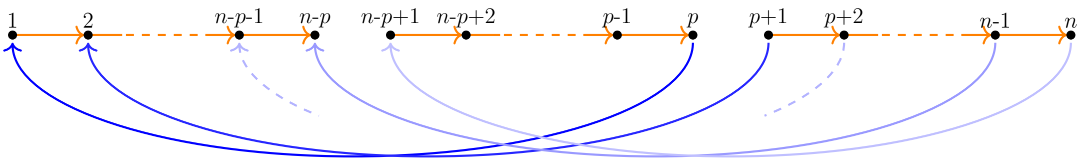
Set $F = f(e) + f'$, so

$$[e, F] = [e, f + f'] = [e, f] + [e, f'] = h + 0 = h, \text{ and}$$

$$[h, F] = [h, f + f'] = [h, f] + [h, f'] = -2f - 2f' = -2F.$$

Given that $[h, e] = 2e$, we deduce that (e, h, F) is an $\mathfrak{sl}_2$-triple. All that is left to show is that $e \backsim F$.

We can draw the diagram $D(F)$ as



which is the diagram $D(f(e))$ along with additional arrows from v_i to v_{i-p+1} for all $i \geq p$. There are no cycles in $D(F)$, and hence $F^n = 0$.

We show that $F^{n-1}(v_{p+1}) \neq 0$. Observe first that

$$F^{n-1}(v_{p+1}) = F^{n-2}(a_{p-1}v_{p+2} + x_p v_2) = F^{n-2}(a_{p-1}v_{p+2}) + F^{n-2}(x_p v_2).$$

Then note that $F^{n-p-i+1}(v_i) = 0$ for all $1 \leq i \leq n-p$, so we have

$$F^{n-1}(v_{p+1}) = F^{n-2}(a_{p-1}v_{p+2}).$$

Similarly, we can continue these steps to find

$$F^{n-1}(v_{p+1}) = F^{p}(a_{p+1} \dots a_{n-1}v_n + x_p a_{p+1} \dots a_{n-2}v_{n-p})$$

$$= F^{p-1}(x_p a_{p+1} \dots a_{n-1} v_{n-p+1}) = \dots = F^{n-p+1}(x_p a_{n-p+1} \dots a_{p-1} a_{p+1} \dots a_{n-1} v_p)$$

$$= F^{n-p}(x_p^2 a_{n-p+1} \dots a_{p-1} a_{p+1} \dots a_{n-1} v_1) = \dots$$

$$= x_p^2 a_1 \dots a_{n-p-1} a_{n-p+1} \dots a_{p-1} a_{p+1} \dots a_{n-1} v_{n-p}.$$

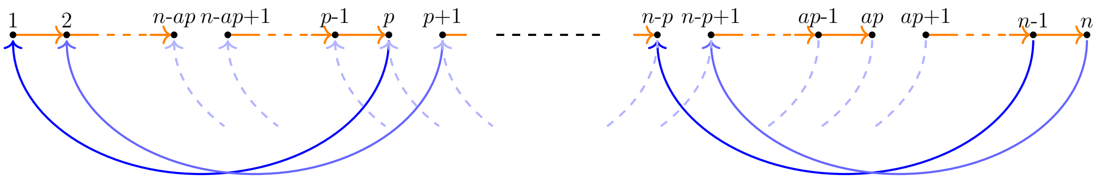
Then as $x_p \neq 0$, and $a_j = 0$ if and only if $j = p$ or $j = n-p$, we have $F^{n-1}(v_{p+1}) \neq 0$.

Therefore by Lemma 3.2.7 it follows that $F \sim (n)$ and hence is conjugate to e.

Thus we are done when $a = 1$, however we claim we can find such an $\mathfrak{sl}_2$-triple for any $ap < n < (a+1)p$ with $a \geq 1$. The proof for these cases is almost identical, we set $F = f(e) + f'$ where

$$f'_{ij} = \begin{cases} x_1 & \text{if } j = i + p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $D(F)$ is shown below.



The only source in this diagram is v_{ap+1}, from which we can find a chain of length n. Once again, $D(F)$ has no cycles and $F^{n-1}(v_{ap+1}) \neq 0$ by a similar agument to above, and so by Lemma 3.2.7 F is conjugate to e. $\square$

Thus, if $e \in \mathfrak{gl}_n(\Bbbk)$ is a single Jordan block, then we can find a strong $\mathfrak{sl}_2$-triple containing e when n is not a multiple of p. When n is a multiple of p we must amend our method slightly, as explained in the following proposition.

Proposition 3.3.3. *Suppose $n = ap$ for some $a \in \mathbb{Z}$, $a > 1$. There exists some $F \in \mathfrak{gl}_n(\Bbbk)$ such that $(e, h(e), F)$ is a strong $\mathfrak{sl}_2$-triple. Moreover $F = f(e) + f'$ for some $f' \in \mathfrak{g}^e$.*

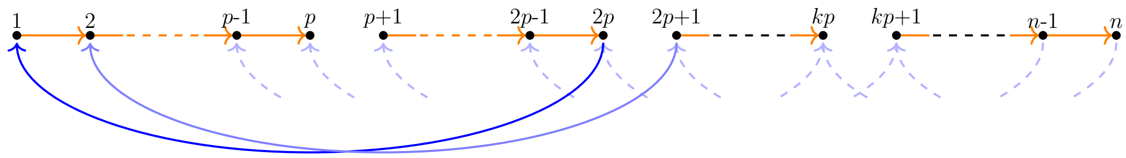
Proof. This proof follows the same structure as the proof of Proposition 3.3.2. We have that $n = ap$ for some $a > 1$, thus in the matrix of $f(e)$ we have that $a_j = j(n-j)$ is equal to 0 if and only if j is a multiple of p. Hence the diagram $D(f(e))$ in this case is

Let $x_2 \in \mathbb{k}$ be a non-zero scalar and define $f' \in \mathfrak{gl}_n(\mathbb{k})$ by

$$f'_{ij} = \begin{cases} x_2 & \text{if } j = i + 2p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $F := f(e) + f'$, then $(e, h(e), F)$ is an $\mathfrak{sl}_2$-triple by Lemmas 3.1.2 and 3.1.4. The proof that $e \sim F$ uses the same method as in the proof of Proposition 3.3.2.

Note that $D(F)$ is the diagram $D(f(e))$ with additional arrows from v_i to v_{i-2p+1} if $i \geq 2p$, as depicted below.



We see that there are no cycles in $D(F)$, and there is a source $v_{n-p+1} = v_{(a-1)p+1}$ with $F^{n-1}(v_{n-p+1}) = x_{2p}^{a-1}(\prod_{j \neq kp} a_j)v_p \neq 0$. Hence, again, using Lemma 3.2.7 we have that $F \sim (n)$ and thus $F \backsim e$. $\square$

Combining Propositions 3.3.2 and 3.3.2 shows that for any single Jordan block $e \in \mathfrak{gl}_n(\mathbb{k})$ there exists some strong $\mathfrak{sl}_2$-triple (e, h, F). Before we prove that a strong $\mathfrak{sl}_2$-triple exists for any nilpotent $e \in \mathfrak{gl}_n(\mathbb{k})$ we first give results on the properties of the strong $\mathfrak{sl}_2$-triples that we have found.

Proposition 3.3.4. *Let $p \leq n$. There is no bijection between the sets given in Theorem 1.*

Proof. We proceed by constructing an infinite family of non-conjugate $\mathfrak{sl}_2$-triples containing $e \in \mathfrak{gl}_n(\Bbbk)$ a single nilpotent Jordan block.

Recall the $\mathfrak{sl}_2$-triple $(e, h(e), F)$ defined in Proposition 3.3.2 (Proposition 3.3.3 respectively if n a multiple of p). We have that $F = f(e) + f'$ where $f'_{ij} = x_1$ for $j = i + p - 1$ ($f'_{ij} = x_2$ for $j = i + 2p - 1$ respectively) for non-zero scalars $x_1, x_2 \in \Bbbk$. Thus we have not just defined one strong $\mathfrak{sl}_2$-triple for e, we have defined a family of strong $\mathfrak{sl}_2$-triples that is parametrised by $\Bbbk^\times$. We now prove that the $\mathfrak{sl}_2$-triples in these families are pairwise non-conjugate.

Let $ap < n < (a + 1)p$. Consider for $\lambda \in \Bbbk$ the n-dimensional $\mathfrak{s}$-module V_λ. The actions of e, h, f are given by $(e_{V_\lambda}, h_{V_\lambda}, f_{V_\lambda}) := (e, h(e), F_\lambda)$ where F_λ is equal to $f(e) + f'$ with $x_1 = \lambda$. That is

$$(F_\lambda)_{ij} = \begin{cases} \lambda & \text{if } j = i + p - 1, \\ a_j & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3.3.2, (e, h, F_λ) and (e, h, F_μ) are strong $\mathfrak{sl}_2$-triples for $\lambda, \mu \in \Bbbk^\times$. We claim that they are not conjugate as $\mathfrak{sl}_2$-triples when $\lambda \neq \mu$.

Suppose, looking for a contradiction, that (e, h, F_λ) is conjugate to (e, h, F_μ) and hence there exists $g \in \mathrm{GL}_n(\Bbbk)$ such that

$$geg^{-1} = e, \quad ghg^{-1} = h, \quad \text{and } gF_\lambda g^{-1} = F_\mu.$$

Hence we require that g centralises h and e, that is $g \in G^h \cap G^e$, and so there exists some $x_k \in \Bbbk$ so that g is of the form

$$g_{ij} = \begin{cases} x_k & \text{if } j = i + kp \text{ for some } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $gF_\lambda = F_\mu g$ and so we calculate

$$(gF_\lambda)_{ij} = \begin{cases} a_j x_0 & \text{if } j = i - 1, \\ \lambda x_{k-1} + a_j x_k & \text{if } j = i + kp - 1 \text{ for some } k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(F_\mu g)_{ij} = \begin{cases} a_{i-1} x_0 & \text{if } j = i - 1, \\ \mu x_{k-1} + a_{i-1} x_k & \text{if } j = i + kp - 1 \text{ for some } k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $j = i + kp - 1$ for some $k > 0$, we have $\lambda x_{k-1} + a_j x_k = \mu x_{k-1} + a_{i-1} x_k$. Note that

$$a_{i-1} = a_{j-kp} = (j - kp)(n - (j - kp)) = j(n - j) = a_j.$$

Therefore $\lambda x_{k-1} = \mu x_{k-1}$ for each $k > 0$, hence as $\mu \neq \lambda$ it follows that $x_{k-1} = 0$. In particular, for $k = 1$ we have $x_0 = 0$, and hence g is not invertible as it has first column zero. We conclude that there does not exist any $g \in \mathrm{GL}_n(\Bbbk)$ that conjugates (e, h, F_λ) to (e, h, F_μ).

Now let n be a multiple of p, and take $\lambda \neq \mu$. The proof that there does not exist any $g \in \mathrm{GL}_n(\Bbbk)$ that conjugates (e, h, F_λ) to (e, h, F_μ) follows in the same way.

We remark that following from Lemmas 3.1.2 and 3.1.4 there is no reason that we cannot also add a scalar on the $(tp-1)$-th diagonal for any $t > 1$ (or $t > 2$ if n is a multiple of p). That is if $(a-1)p < n < ap$, then $(e, h, F_{\lambda_1,\lambda_2,\dots,\lambda_{(a-1)}})$ is a strong $\mathfrak{sl}_2$-triple when $\lambda_1 \neq 0$ and

$$F_{\lambda_1,\lambda_2,\dots,\lambda_{(a-1)}} = \begin{cases} \lambda_t & \text{if } j = i + tp - 1 \text{ for } t \geq 1, \\ a_j & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $e \sim F_{\lambda_1,\lambda_2,\dots,\lambda_{(a-1)}}$.

Adapting the above argument gives us that $(e, h, F_{\lambda_1,\lambda_2,\dots,\lambda_{(a-1)}})$ and $(e, h, F_{\mu_1,\mu_2,\dots,\mu_{(a-1)}})$ are not conjugate as $\mathfrak{sl}_2$-triples if $\lambda_t \neq \mu_t$ for any $1 \leq t \leq (a-1)$. □

A natural question to ask is whether up to conjugacy these families contain all strong $\mathfrak{sl}_2$-triples. Let e be the regular nilpotent element in $\mathfrak{gl}_n(\mathbb{k})$ and $h(e)$ the diagonal matrix as in Example 2.3.1. Suppose $F \in \mathfrak{gl}_n(\mathbb{k})$ is such that $(e, h(e), F)$ is an $\mathfrak{sl}_2$-triple and $e \sim F$. Then F is of the form $f(e) + f'$ where $f' \in \mathfrak{g}^e$, and $[h(e), f'] = -2f'$. Hence F equals $F_{\lambda_1,\lambda_2,\dots,\lambda_{(a-1)}}$ for some $\lambda_1 \neq 0$ and so $(e, h(e), F)$ is of the form found in Proposition 3.3.4. If h is not diagonal then this remains unknown.

Open Question. *Let e be the regular nilpotent element in $\mathfrak{gl}_n(\mathbb{k})$. Up to conjugacy, can we find all strong $\mathfrak{sl}_2$-triples containing e defined by a finite set of coefficients?*

Let $n \in \mathbb{Z}_{>0}$, and take (e, h, F) with $e \sim F$ to be as defined in Proposition 3.3.2.

Let V be an n-dimensional $\mathfrak{s}$-module such that $(e_V, h_V, f_V) = (e, h, F)$.

Suppose that $ap < n < (a+1)p$ for some $a \geq 1$. We use $D(F)$ as in Proposition 3.3.2 to determine structural information about V. We divide the basis of V into $2a+1$ segments $A_1, \dots, A_{2a+1}$ where we define

$$A_{2j} = \{v_{n-(a-j+1)p+1}, \dots, v_{jp}\}$$

$$A_{2j+1} = \{v_{jp+1}, \dots, v_{n-(a-j)p}\}$$

for $j \in \{0, \dots, a\}$. Observe that in $D(F)$ there are no arrows from A_i to A_j if $i < j$.

Set $M_{2a+1} = M$, and let M_j be the subspace generated by $A_1 \cup A_2 \cup \dots \cup A_j$ for $j \in \{0, \dots, a\}$. Each M_j is an $\mathfrak{s}$-submodule as it closed under the action of e, h and f. Set $M_0 = 0$, then

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_j \subseteq M_{j+1} \subseteq \dots \subseteq M_{2a+1} = M.$$

We have that M_{2j+1}/M_{2j} is generated by the image of A_{2j+1} under the natural quotient which has $n - ap$ elements, and similarly, M_{2j}/M_{2j-1} is generated by A_{2j}, which has $(a+1)p - n$ elements.

Lemma 3.3.5. *The series*

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_j \subseteq M_{j+1} \subseteq \dots \subseteq M_{2a+1} = M$$

is a composition series for V, and

$$M_{2j+1}/M_{2j} \cong V(n-ap-1), \qquad M_{2j}/M_{2j-1} \cong V((a+1)p-n-1).$$

Proof. Recall that the modules $V(n)$ are simple for $n < p$ as observed in Remark 2.3.5.

It is clear that e acts on these subquotients as a single Jordan block, so we must check the action of h and F. Consider $N_{2j+1} = M_{2j+1}/M_{2j}$, and note that the M_{2j}/M_{2j-1} case follows in the same way. The $\mathfrak{s}$-module $V(n-ap-1)$ has a basis $\{w_1, \dots, w_{n-ap}\}$ where e, h and f have matrix forms as given in Example 2.3.1. That is, h acts on $V(n-ap-1)$ as

$$\begin{pmatrix} n-1 & & & \\ & n-3 & & \\ & & \ddots & \\ & & & 1-n \end{pmatrix},$$

whereas N_{2j+1} has basis $A_{2j+1} = \{v_{jp+1}, \dots, v_{n-(a-j)p}\}$ where

$$h(v_{jp+i}) = (n-1-2(jp+i-1))v_{jp+i} = (n-1-2(i-1))v_{jp+i}.$$

Hence the action of h on V and h on $V(n-ap-1)$ are represented by the same matrix in the given bases.

Next, we consider consider how F acts on V', recalling that f acts on $V(n-ap-1)$

as

$$\begin{pmatrix} 0 & & & & \\ a_1 & 0 & & & \\ & a_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & a_{n-ap-1} & 0 \end{pmatrix}$$

where $a_k = k((n-ap)-k) = k(n-k)$. We have that $F(v_{jp+i}) = a_{jp+i}v_{jp+i+1} + x_p v_{(j-1)p+i+1}$. However, $v_{(j-1)p+i+1}$ is contained in M_{2j} and $a_{jp+i} = (jp+i)(n-jp-i) = i(n-i)$ and therefore $F(v_{jp+i}) = i(n-i)v_{jp+i+1}$.

Hence under the given bases e, h and f act on V' and $V(n-ap-1)$ identically, and thus these modules are isomorphic. Thus we have proved that the modules M_j form a composition series for V. □

A composition series for the case where n is a multiple of p can be found using the same method.

We now prove Theorem 1 by showing that for any nilpotent $e \in \mathfrak{gl}_n(\Bbbk)$, there exists some strong $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{gl}_n(\Bbbk)$. We use Propositions 3.3.2 and 3.3.3 to prove the general case.

Proof of Theorem 1. Take any nilpotent $e \in \mathfrak{gl}_n(\Bbbk)$, then up to conjugacy by $\mathrm{GL}_n(\Bbbk)$ we can assume that e is in Jordan normal form, suppose that $e \sim (n_1, n_2, \ldots, n_m)$. We can embed e into

$$\mathfrak{gl}_{n_1}(\Bbbk) \oplus \mathfrak{gl}_{n_2}(\Bbbk) \oplus \cdots \oplus \mathfrak{gl}_{n_m}(\Bbbk) \subseteq \mathfrak{gl}_n(\Bbbk).$$

We consider each of these components separately. Recalling notation from

Definition 2.4.1 we have that e restricted to $\mathfrak{gl}_{n_k}(\mathbb{k})$ is $J_{n_k}(0) = J_{n_k}$, that is, in block form

$$e = \begin{pmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_m} \end{pmatrix}.$$

For each n_k we find a strong $\mathfrak{sl}_2$-triple $(J_{n_k}, h_{n_k}, f_{n_k})$ in $\mathfrak{gl}_{n_k}(\mathbb{k})$, if $n_k \leq p$ then take the strong $\mathfrak{sl}_2$-triple given by $V(n-1)$ in Example 2.3.1 and if $n_k > p$ take the strong $\mathfrak{sl}_2$-triple given in either Proposition 3.3.2 or 3.3.3. For each k we have J_{n_k} is conjugate to f_{n_k}, so $f_{n_k} \sim (n_k)$.

We set

$$h = \begin{pmatrix} h_{n_1} & & \\ & \ddots & \\ & & h_{n_m} \end{pmatrix} \text{ and } f = \begin{pmatrix} f_{n_1} & & \\ & \ddots & \\ & & f_{n_m} \end{pmatrix}$$

so that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n(\mathbb{k})$. We have $f \sim (n_1, \ldots, n_m) \sim e$. Hence we have found a strong $\mathfrak{sl}_2$-triple containing e in $\mathfrak{gl}_n(\mathbb{k})$. □

The strong $\mathfrak{sl}_2$-triples that we found in the proof of Theorem 1 also satisfy an additional property, which we discuss in the remark below.

Remark 3.3.6. Let (e, h, f) be a strong $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n(\mathbb{k})$ of the form found in the proof of Theorem 1, and suppose that there is some Levi subalgebra $\mathfrak{l}$ of $\mathfrak{gl}_n(\mathbb{k})$ such that $e, h, f \in \mathfrak{l}$. Suppose that L is a Levi subgroup of $\mathrm{GL}_n(\mathbb{k})$ such that $\mathfrak{l} = \mathrm{Lie}(L)$. Then (e, h, f) is a strong $\mathfrak{sl}_2$-triple in $\mathfrak{l}$, i.e. e is conjugate to f by an element of L.

In general the property of being conjugate by an element of a Levi subgroup is not equivalent to being conjugate by an element of G. For instance, consider any

$\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{gl}_n(\Bbbk)$ where $e \not\sim f$, then we take the following $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_{2n}(\Bbbk)$

$$e' = \begin{pmatrix} e & \\ & f \end{pmatrix}, h' = \begin{pmatrix} h & \\ & -h \end{pmatrix}, f' = \begin{pmatrix} f & \\ & e \end{pmatrix}.$$

It is clear that $e' \sim f'$, so (e',h',f') is a strong $\mathfrak{sl}_2$-triple. Note that e',h',f' are in the Levi subalgebra of $\mathfrak{gl}_{2n}(\Bbbk)$ that is isomorphic to $\mathfrak{gl}_n(\Bbbk) \oplus \mathfrak{gl}_n(\Bbbk) = \mathrm{Lie}(\mathrm{GL}_n(\Bbbk) \times \mathrm{GL}_n(\Bbbk))$, however it is not a strong $\mathfrak{sl}_2$-triple under conjugation by $\mathrm{GL}_n(\Bbbk) \times \mathrm{GL}_n(\Bbbk)$.

In Theorem 1 we showed that for any nilpotent $e \in \mathfrak{gl}_n(\Bbbk)$ there exists a strong $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{gl}_n(\Bbbk)$. However, we cannot use this method to extend this result in full generality to other classical Lie algebras. We give a counterexample to show that Theorem 1 can not be extended to $\mathfrak{g} = \mathfrak{sp}_4(\Bbbk)$.

Example 3.3.7. Suppose that $\Bbbk$ has characteristic 3.

Consider the regular nilpotent element $e \in \mathfrak{sp}_4(\Bbbk)$. Then calculations show that any $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{sp}_4(\Bbbk)$ is of the following form, up to conjugacy, for some $x \in \Bbbk$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we restrict to the case where h is diagonal, we must have that $x = 0$, and hence f is not conjugate to e. We see that if $x \neq 0$, then f is conjugate to e using Lemma 3.2.7, however we currently do not have any way to generalise this.

In general, this problem remains open for the classical Lie algebras $\mathfrak{g} \subseteq \mathfrak{gl}_n(\Bbbk)$.

CHAPTER 4

PROPERTIES OF $\mathfrak{sl}_2$-TRIPLES

Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 2$. In this short chapter we give results on $\mathfrak{sl}_2$-triples for $\mathrm{SL}_p(\Bbbk)$ and G-completely reducible $\mathfrak{sl}_2$-subalgebras for connected, reductive algebraic groups G. We use these results to prove that the varieties given in Theorems 2 and 3 are maximal with respect to the given properties.

4.1 Standard $\mathfrak{sl}_2$-triples

Let G be either of exceptional type or one of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$, $\mathrm{O}_n(\Bbbk)$ or $\mathrm{SO}_n(\Bbbk)$ (where we assume n is even in the $\mathrm{Sp}_n(\Bbbk)$ case). Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. We recall the $[p]$-power map on $\mathfrak{g}$ is given in Definition 2.1.40.

We discuss standard $\mathfrak{sl}_2$-triples as introduced by Premet–Stewart in [PS19, §2.4]. This theory of standard $\mathfrak{sl}_2$-triples is based on the theory of optimal cocharacters associated to nilpotent elements developed by Premet in [Pre03, Section 2]. We note that the material in [PS19, §2.4] is stated only for the case G is a simple group of exceptional type, and that some of [Pre03, Section 2] works under the assumption that the derived subgroup of G is simply connected and there is a non-degenerate G-invariant symmetric bilinear from on G. However, the results that we cover go through for all the groups in our setting, see for instance the arguments given in [Pre03, §2.3].

We recap the construction of standard $\mathfrak{sl}_2$-triples given in [PS19, §2.4]. For $x \in \mathcal{N}$, and cocharacter $\tau : \Bbbk^\times \to G$, we define the *weight spaces* $\mathfrak{g}(j;\tau)$ of τ to be

$$\mathfrak{g}(j;\tau) := \{x \in \mathfrak{g} : \tau(t) \cdot x = t^j x \ \forall t \in \Bbbk^\times\}.$$

We then define associated cocharacters as in [Jan04, §5.3].

Definition 4.1.1. For $x \in \mathcal{N}$ we say a cocharacter $\tau : \Bbbk^\times \to G$ is an *associated cocharacter* for x if both $x \in \mathfrak{g}(2;\tau)$, and there exists a Levi subgroup L in G such that x is distinguished nilpotent in $\mathrm{Lie}(L)$ and $\tau(\Bbbk^\times) \subseteq L'$.

An associated cocharacter gives us a $\mathbb{Z}$-grading of $\mathfrak{g}$ by

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j;\tau).$$

Definition 4.1.2. For any $x \in \mathcal{N}$ with associated cocharacter λ_x we define the *height* of x to be

$$\mathrm{ht}(x) = \max\{j : \mathfrak{g}(j;\lambda_x) \neq 0\}.$$

For any $x \in \mathcal{N}$ there exists some optimal cocharacter for x, denoted $\tau : \Bbbk^\times \to G$, as in the Kempf-Rousseau theory as explained in [Pre03, §2.2]. Let $e \in \mathcal{N}$, then we can choose an optimal cocharacter $\tau : \Bbbk^\times \to G$ such that $e \in \mathfrak{g}(2;\tau)$ by [Pre03, Theorem A]. Moreover, such optimal cocharacters form a single conjugacy class under the adjoint action of G^e. It follows from [Pre03, Proposition 2.5] that such optimal cocharacters for x coincide with the associated cocharacters for x. Thus from this point we may take τ to be an associated cocharacter for x.

Then $\tau(t) \cdot e = t^2 e$ and the centraliser, $\mathfrak{g}^e$, of e, in $\mathfrak{g}$ is contained in the sum of positive weight spaces of τ. Let $h_\tau := \mathrm{d}\tau(1) \in \mathfrak{g}$, then we have $[h_\tau, e] = 2e$. Let $C_G(\tau)$ be the centraliser of τ in G, and let $C^e := G^e \cap C_G(\tau)$. Let T^e be a maximal torus of C^e and let $L := C_G(T^e)$. Then L is a Levi subgroup of G such that e is a distinguished nilpotent element in the Lie algebra $\mathfrak{l}'$ of the derived subgroup L' of L. That is to say that the only Levi subalgebra of $\mathfrak{l}'$ containing e is $\mathfrak{l}'$ itself.

As explained in [PS19, §2.4] the map $\mathfrak{l}'(-2;\tau) \to \mathfrak{l}'(0;\tau)$ is bijective, and hence there is a unique $f \in \mathfrak{l}'(-2;\tau)$ such that (e, h_τ, f) is an $\mathfrak{sl}_2$-triple.

Definition 4.1.3. An $\mathfrak{sl}_2$-triple of the form (e, h_τ, f) is called a *standard* $\mathfrak{sl}_2$*-triple.*

Example 4.1.4. Let $p > 3$ and $\mathfrak{g} = \mathfrak{sl}_4(\Bbbk)$. Take $e \sim (4)$ to be the single Jordan block. We can take $\tau : \Bbbk^\times \to G$ to be the associated cocharacter for e defined by

$$\tau(t) = \begin{pmatrix} t^3 & & & \\ & t^1 & & \\ & & t^{-1} & \\ & & & t^{-3} \end{pmatrix}, \quad \text{and} \quad h_\tau = d\tau(1) = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}.$$

The unique $f \in \mathfrak{g}_0(-2;\tau)$ such that (e, h_τ, f) is an $\mathfrak{sl}_2$-triple is given by

$$f = \begin{pmatrix} 0 & & & \\ 3 & 0 & & \\ & 4 & 0 & \\ & & 3 & 0 \end{pmatrix}.$$

Observe that the $\mathfrak{sl}_2$-triple (e, h_τ, f) is equal to $(e_{V(3)}, h_{V(3)}, f_{V(3)})$.

We now consider some properties of standard $\mathfrak{sl}_2$-triples. We state a result of Premet–Stewart and use this to prove that any G-stable closed subvariety $\mathcal{V} \subseteq \mathcal{N}$ that satisfies the $\mathfrak{sl}_2$-property must satisfy $\mathcal{V} \subseteq \mathcal{N}^{[p]}$.

Lemma 4.1.5. *[PS19, §2.4] Let (e, h_τ, f) be a standard $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Then:*

(a) $f^{[p]} = 0$; and

(b) if $e^{[p]} = 0$, then e is conjugate to f by G.

Since $f^{[p]} = 0$ we can consider $\exp(sf) \in G$ for $s \in \Bbbk$, see for example the start of the proof of [PS19, Proposition 2.7]. Let $\mathcal{N}(e,h,f) := \{ae + bh + cf : a, b, c \in \Bbbk,\ b^2 = -ac\}$ denote the image of the nilpotent cone of $\mathfrak{s} = \mathfrak{sl}_2(\Bbbk)$ in $\mathfrak{g}$. Standard calculations show that by conjugating e by $\tau(t)$ for $t \in \Bbbk^\times$ and then by $\exp(sf) \in G$ for $s \in \Bbbk$, we obtain that

$$\mathcal{N}(e,h,f) \setminus \Bbbk f = \{t^2(e - sh - s^2 f) : t \in \Bbbk^\times, s \in \Bbbk\} \subseteq (\operatorname{Ad} G)e. \tag{4.1.1}$$

It thus follows that $f \in \overline{(\operatorname{Ad} G)e}$ and so $(\operatorname{Ad} G)f \subseteq \overline{(\operatorname{Ad} G)e}$.

Now suppose that $e^{[p]} \neq 0$. Given that $f \in \mathcal{N}$ we can take τ' to be an associated cocharacter for f, and construct a standard $\mathfrak{sl}_2$-triple $(f, h_{\tau'}, e')$ for f. Since $f^{[p]} = 0$, by Lemma 4.1.5(b) we have that e' is conjugate to f. Then $(e')^{[p]} = 0$, and thus e' is not conjugate to e by G. Hence, the $\mathfrak{sl}_2$-triples $(f, h_{\tau'}, e')$ and $(f, -h, e)$ are not conjugate by G.

Suppose now that $\mathcal{V} \subseteq \mathcal{N}$ is a G-stable closed subvariety that contains e, and thus also contains f and e', because $(\operatorname{Ad} G)e' = (\operatorname{Ad} G)f \subseteq \overline{(\operatorname{Ad} G)e}$. Then the map (1.0.5) given by sending the G-orbit of an $\mathfrak{sl}_2$-triple (e, h, f) to the G-orbit of e is not injective. We see this by considering the G-orbits of $(f, -h', e')$ and $(f, -h, e)$, which are distinct, and both map to the G-orbit of f. This argument implies the following proposition, where in the statement we use the notation $\mathcal{N}^{[p]} := \{x \in \mathfrak{g} : x^{[p]} = 0\}$.

Proposition 4.1.6. *Let $\mathcal{V} \subseteq \mathcal{N}$ be a G-stable closed subvariety that satisfies the $\mathfrak{sl}_2$-property. Then $\mathcal{V} \subseteq \mathcal{N}^{[p]}$.*

Corollary 4.1.7. *There is a unique maximal G-stable closed variety $\mathcal{V}$ of $\mathcal{N}$ that satisfies the $\mathfrak{sl}_2$-property.*

Proof. Suppose that $\mathcal{V}$ and $\mathcal{V}'$ are two such maximal G-stable closed subvarieties of $\mathcal{N}$. We consider $\mathcal{V} \cup \mathcal{V}'$, which is a G-stable closed subvariety of $\mathcal{N}$, and it suffices to show that it satisfies the $\mathfrak{sl}_2$-property. Let (e, h, f) and (e, h', f') be $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ with $e, f, f' \in \mathcal{V} \cup \mathcal{V}'$. Without loss of generality we assume that $e \in \mathcal{V}$. By Proposition 4.1.6, we have that $\mathcal{V} \cup \mathcal{V}' \subseteq \mathcal{N}^{[p]}$, and thus $f \in \mathcal{N}^{[p]}$ so that $f^{[p]} = 0$. Thus we can apply the exponentiation argument above to obtain (4.1.1), and deduce that $f \in \mathcal{V}$. Similarly we can deduce that $f' \in \mathcal{V}$. Hence, as $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property, we have that (e, h, f) and (e, h', f') are G-conjugate. Therefore, we have that $\mathcal{V} \cup \mathcal{V}'$ satisfies the $\mathfrak{sl}_2$-property, as required. □

4.2 $\mathfrak{sl}_2$-triples for $\mathrm{SL}_p(\mathbb{k})$

We consider $\mathfrak{sl}_2$-triples for $\mathrm{SL}_p(\mathbb{k})$ and recap the known result that for the case $e \sim (p)$ there are multiple $\mathfrak{sl}_2$-triples (e, h, f) in $\mathfrak{sl}_p(\mathbb{k})$ up to conjugacy by $\mathrm{SL}_p(\mathbb{k})$. We present just two non-conjugate such $\mathfrak{sl}_2$-triples, but note that by using the baby Verma modules as defined in Definition 2.3.4, it can be shown that there is an infinite family of non-conjugate such $\mathfrak{sl}_2$-triples. In Proposition 4.2.1 we explain how this restricts the possible subvarieties $\mathcal{V}$ of $\mathcal{N}$ that satisfy the $\mathfrak{sl}_2$-property, where we recall that we say a subvariety $\mathcal{V}$ of $\mathcal{N}$ satisfies the $\mathfrak{sl}_2$-property if the map (1.0.5) is a bijection. In Corollary 4.2.2 we apply this to the classical Lie algebras. We use the notation from §2.3.2 throughout this section.

Proposition 4.2.1. *Let G be a connected reductive algebraic group. Let L be a Levi subgroup of G whose derived subgroup L' is of type A_{p-1}. Let $\mathfrak{X} \subseteq \mathcal{N}$ be a G-invariant closed subvariety containing a regular nilpotent element of $\mathfrak{l} = \mathrm{Lie}(L)$. Then $\mathfrak{X}$ does not satisfy the $\mathfrak{sl}_2$-property.*

Proof. We first consider $(e_0, h_0, f) := (e_{Z_0(0)}, h_{Z_0(0)}, f_{Z_0(0)})$, the $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(Z_0(0))$ determined by the baby Verma module $Z_0(0)$. We view (e_0, h_0, f) as an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_p(\mathbb{k})$ using the basis of $Z_0(0)$ given in Definition 2.3.4. Similarly there is an $\mathfrak{sl}_2$-triple (e_{p-1}, h_{p-1}, f) in $\mathfrak{sl}_p(\mathbb{k})$ determined by the baby Verma module $Z_0(p-1)$ and the basis of $Z_0(p-1)$. We note that the f in these $\mathfrak{sl}_2$-triples is the same, and that $e_0 \sim (p-1, 1)$ and $e_{p-1} \sim (p)$. Therefore, the $\mathfrak{sl}_2$-triples (e_0, h_0, f) and (e_{p-1}, h_{p-1}, f) are not conjugate by $\mathrm{SL}_p(\mathbb{k})$, and thus the $\mathfrak{sl}_2$-triples $(f, -h_0, e_0)$ and $(f, -h_{p-1}, e_{p-1})$ are not conjugate by $\mathrm{SL}_p(\mathbb{k})$. We note here that this implies that $\mathcal{V} = \mathcal{N}$ does not satisfy the $\mathfrak{sl}_2$-property for the case $G = \mathrm{SL}_p(\mathbb{k})$, as the $\mathrm{SL}_p(\mathbb{k})$-orbits of the $\mathfrak{sl}_2$-triples $(f, -h_0, e_0)$ and $(f, -h_{p-1}, e_{p-1})$ are distinct, and map to the same $\mathrm{SL}_p(\mathbb{k})$-orbit under the map given in (1.0.5).

Note that we can also consider the embedding of the $\mathfrak{sl}_2$-triples $(f, -h_0, e_0)$ and $(f, -h_{p-1}, e_{p-1})$ into $\mathfrak{pgl}_p(\mathbb{k})$, and see that these cannot be conjugate under $\mathrm{PGL}_p(\mathbb{k})$ as they are not conjugate under the action of $\mathrm{SL}_p(\mathbb{k})$.

Suppose that G has a Levi subgroup L whose derived subgroup L' is isomorphic to $\mathrm{SL}_p(\mathbb{k})$ or $\mathrm{PGL}_p(\mathbb{k})$. By identifying the Lie algebra $\mathfrak{l}'$ of L' with $\mathfrak{sl}_p(\mathbb{k})$ or $\mathfrak{pgl}_p(\mathbb{k})$ respectively, we may consider the non-conjugate $\mathfrak{sl}_2$-triples $(f, -h_0, e_0)$ and $(f, -h_{p-1}, e_{p-1})$ inside $\mathfrak{g}$. □

An immediate consequence of this proposition is the following corollary which gives us a restriction on the Jordan types of nilpotent orbits in subvarieties that satisfy the $\mathfrak{sl}_2$-property.

Corollary 4.2.2. *Let G be one of $\mathrm{GL}_n(\mathbb{k})$, $\mathrm{SL}_n(\mathbb{k})$, $\mathrm{Sp}_n(\mathbb{k})$, $\mathrm{O}_n(\mathbb{k})$ or $\mathrm{SO}_n(\mathbb{k})$, and let $\mathcal{V}$ be a G-stable closed subvariety of $\mathcal{N}$. Suppose that $\mathcal{V}$ contains an element of Jordan type $(p, 1^{n-p})$ (respectively $(p^2, 1^{n-2p})$) if G is one of $\mathrm{GL}_n(\mathbb{k})$ or $\mathrm{SL}_n(\mathbb{k})$ (respectively $\mathrm{Sp}_n(\mathbb{k})$, $\mathrm{O}_n(\mathbb{k})$ or $\mathrm{SO}_n(\mathbb{k})$). Then $\mathcal{V}$ does not satisfy the $\mathfrak{sl}_2$-property.*

4.3 G-completely reducible $\mathfrak{sl}_2$-triples

We now give an analogue of Proposition 4.2.1, considering G-completely reducible $\mathfrak{sl}_2$-subalgebras, rather than the $\mathfrak{sl}_2$-property. We use the preliminary results given in §2.1.5 to prove the following proposition.

Proposition 4.3.1. *Let L be a Levi subgroup of G whose derived subgroup L' is of type A_{p-1}. Let $\mathfrak{X} \subseteq \mathcal{N}$ be a G-invariant closed subvariety containing a regular* *nilpotent element in* $\mathfrak{l} = \mathrm{Lie}(L)$. *Then there exists some non-G-completely reducible* $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ *with* $e, f \in \mathfrak{X}$.

Proof. We have that L' must be isomorphic to either $\mathrm{SL}_p(\Bbbk)$ or $\mathrm{PGL}_p(\Bbbk)$.

We first note that there are non-$\mathrm{SL}_p(\Bbbk)$-completely reducible $\mathfrak{sl}_2$-triples in $\mathfrak{sl}_p(\Bbbk)$ containing a regular nilpotent element. We see this by considering the baby Verma modules $Z_0(0)$ with highest weight $0 \in \Bbbk$ which are embedded in $\mathfrak{sl}_p(\Bbbk)$ as discussed in §4.2. As in the proof of Proposition 4.2.1, we write (e_0, h_0, f) to represent the action of $\mathfrak{sl}_2(\Bbbk)$ on $\mathbb{Z}_0(0)$, then we have that f has Jordan type (p). The baby Verma module $Z_0(0)$ has irreducible submodule isomorphic to $V(p-2)$, however $Z_0(0)$ is not completely reducible as the submodule does not have a complement. Hence by Lemma 2.1.31, as $\mathfrak{sl}_2(\Bbbk)$ is not completely reducible on the natural module of $\mathfrak{sl}_p(\Bbbk)$, the subalgebra $\langle e_0, h_0, f \rangle$ is non-$\mathrm{SL}_p(\Bbbk)$-completely reducible.

We also find a non-$\mathrm{PGL}_p(\Bbbk)$-completely reducible $\mathfrak{sl}_2$-triple in $\mathfrak{pgl}_p(\Bbbk) = \mathfrak{gl}_p(\Bbbk)/\mathfrak{z}(\mathfrak{gl}_p(\Bbbk))$ containing a regular nilpotent element. We consider the embedding of $\mathfrak{h} = \langle e_0, h_0, f \rangle$ into $\mathfrak{pgl}_p(\Bbbk)$, which we denote $\overline{\mathfrak{h}} = \langle \overline{e_0}, \overline{h_0}, \overline{f} \rangle$. We see $\overline{\mathfrak{h}} = \langle \overline{e_0}, \overline{h_0}, \overline{f} \rangle$ is contained in $\mathfrak{psl}_p(\Bbbk)$. Then we see that $\mathfrak{h}$ is contained in the parabolic

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & \\ * & b \end{pmatrix} : A \in \mathfrak{gl}_{p-1}(\Bbbk), b \in \Bbbk^{\times} \right\},$$

and hence $\overline{\mathfrak{h}}$ is contained in the parabolic $\overline{\mathfrak{p}}$ which is found by taking the quotient of elements in $\mathfrak{p}$ by the centre. Suppose for contradiction that there exists some Levi $\mathfrak{l}$ of $\mathfrak{p}$ such that the embedding $\overline{\mathfrak{l}}$ has $\overline{\mathfrak{h}} \subseteq \overline{\mathfrak{l}}$. Then we note that $\mathfrak{z}(\mathfrak{gl}_p(\Bbbk)) \subseteq \mathfrak{l}$ and hence $\mathfrak{h} \subseteq \mathfrak{l}$. This is a contradiction, and hence there is no such Levi $\mathfrak{l}$. Thus $\overline{\mathfrak{h}}$ is $\mathrm{PGL}_p(\Bbbk)$-completely reducible.

Thus we use Lemma 2.1.32 to deduce that we have some non-G-completely reducible $\mathfrak{h} = \langle e, h, f \rangle$ isomorphic to its copy in $\mathfrak{l}$, and hence with $e, f \in \mathfrak{X}$. ☐

The next proposition gives an analogue to Proposition 4.1.6 in the context of G-complete reducibility. Let $\mathcal{V} \subseteq \mathcal{N}$ and take $H(\mathcal{V})$ to be the set of $\mathfrak{h} \subseteq \mathfrak{g}$ with $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ for $e, f \in \mathcal{V}$.

Proposition 4.3.2. *Let $\mathcal{V} \subseteq \mathcal{N}$ be a nilpotent subvariety such that all $\mathfrak{h} \in H(\mathcal{V})$ are G-completely reducible. Then any $\mathfrak{h} \in H(\mathcal{V})$ is a p-subalgebra. That is*

$$e^{[p]} = f^{[p]} = 0, \;\; h^{[p]} = h.$$

Hence we deduce that for all $x \in \mathcal{V}$ we have $x^{[p]} = 0$.

Proof. We first note that from [ST18, Lemma 4.3] we have that either $\mathfrak{h}$ is a p-subalgebra or $\mathfrak{h}$ is L-irreducible in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ of $\mathfrak{g}$ with a factor of type A_{p-1}.

Suppose for contradiction that $\mathfrak{h}$ is a G-completely reducible $\mathfrak{sl}_2$-subalgebra that is not a p-subalgebra and hence is L-irreducible in a Levi subalgebra with a factor of type A_{p-1}. We consider the irreducible $\mathfrak{sl}_2(\Bbbk)$-modules of dimension p. Any such module must correspond to some irreducible baby Verma module, $Z_\chi(\lambda)$ as described in §2.3.2.

Suppose that $L \cong \mathrm{SL}_p(\Bbbk)$. Then we describe the action of $\mathfrak{sl}_2(\Bbbk)$ on $Z_\chi(\lambda)$ as elements of $\mathfrak{sl}_p(\Bbbk)$ in §2.3.2. In this case we see that $e^{[p]} = f^{[p]} = 0$. Note that $h^{[p]} - h$ does not have to act as zero on this module, however in any baby Verma module we find that either e or f acts as a single Jordan block of size p, hence is regular nilpotent in $\mathfrak{sl}_p(\Bbbk)$.

Now we suppose that $L \cong \mathrm{PGL}_p(\Bbbk)$. Then we consider the embedding of $Z_\chi(\lambda)$ as elements of $\mathfrak{pgl}_p(\Bbbk)$, by taking the quotient of the matrices described in §2.3.2 by elements in the centre of $\mathfrak{gl}_p(\Bbbk)$. We see that, once again, either e or f is regular nilpotent in $\mathfrak{pgl}_p(\Bbbk)$.

Hence there exists a regular nilpotent element in $\mathfrak{l}$ of type A_{p-1} and thus using Proposition 4.3.1 we deduce that we can find a non-G-completely reducible $\mathfrak{sl}_2$-triple (e, h', f') with $e, f' \in \mathcal{V}$. We have assumed that no such $\mathfrak{sl}_2$-triple exists, and so we must have that every $\mathfrak{h}$ is a p-subalgebra. $\square$

CHAPTER 5

$\mathfrak{sl}_2$-TRIPLES FOR THE CLASSICAL ALGEBRAIC GROUPS

The work presented in this chapter is based on: "On $\mathfrak{sl}_2$-triples for classical algebraic groups in positive characteristic", published joint by the author and Simon Goodwin, [GP22]. All work was contributed equally by both authors.

Let G be one of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$, $\mathrm{O}_n(\Bbbk)$ or $\mathrm{SO}_n(\Bbbk)$, where we assume n is even in the $\mathrm{Sp}_n(\Bbbk)$ case. Let $\mathfrak{g} = \operatorname{Lie} G$ and recall that $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$. As before we write $\mathfrak{s} := \mathfrak{sl}_2(\Bbbk)$.

We refer the reader to §2.4.1 for more detail on the parametrisation of G-orbits in $\mathcal{N}$, but we recollect here that the Jordan normal form of any element in $\mathcal{N}$ corresponds to a partition λ of n, and this uniquely determines a G-orbit in $\mathcal{N}$, except in the case where $G = \mathrm{SO}_n(\Bbbk)$ and λ is a very even partition, for which there are two G-orbits.

Recall that we write $x \sim \lambda$ to denote that the partition of n given by the Jordan normal form of x is λ. In this chapter we prove Theorems 2 and 3 for G as above.

In order to do this we give the variety $\mathcal{V}$ in terms of the Jordan type of nilpotent elements. The subvarieties of $\mathcal{N}$ required for the statement of these theorems are

$$\mathcal{N}^{p-1} := \{x \in \mathcal{N} : x^{p-1} = 0\}, \tag{5.0.1}$$

and

$$^{1}\mathcal{N}^{p} := \{x \in \mathcal{N} : \; x \sim (\lambda_1, \lambda_2, \ldots, \lambda_m), \; \lambda_1 \leq p, \; \lambda_2 < p\}. \tag{5.0.2}$$

By this we mean that the nilpotent elements in $\mathcal{N}^{p-1}$ have Jordan blocks of at most size $p-1$, and elements in $^{1}\mathcal{N}^{p}$ have at most one block of size p, and all remaining blocks are smaller than size p.

Given these definitions, we are now able to state the first main result of this chapter.

Theorem 5.0.1. *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 2$. Let $(G, \mathfrak{g}, \mathcal{V})$ be one of the following:*

(a) $G = \mathrm{GL}_n(\Bbbk)$, $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$, $\mathcal{V} = \mathcal{N}^{p-1}$;

(b) $G = \mathrm{SL}_n(\Bbbk)$, $\mathfrak{g} = \mathfrak{sl}_n(\Bbbk)$, $\mathcal{V} = \mathcal{N}^{p-1}$;

(c) $G = \mathrm{Sp}_n(\Bbbk)$, $\mathfrak{g} = \mathfrak{sp}_n(\Bbbk)$, $\mathcal{V} = \mathcal{N}^{p-1}$;

(d) $G = \mathrm{O}_n(\Bbbk)$, $\mathfrak{g} = \mathfrak{so}_n(\Bbbk)$, $\mathcal{V} = {}^{1}\mathcal{N}^{p}$; *or*

(e) $G = \mathrm{SO}_n(\Bbbk)$, $\mathfrak{g} = \mathfrak{so}_n(\Bbbk)$, $\mathcal{V} = {}^{1}\mathcal{N}^{p}$.

Then the map

$$\{G\text{-orbits of } \mathfrak{sl}_2\text{-triples } (e, h, f) \text{ with } e, f \in \mathcal{V}\} \longrightarrow \{G\text{-orbits in } \mathcal{V}\} \tag{1.0.5}$$

given by sending the G-orbit of an $\mathfrak{sl}_2$-triple (e, h, f) to the G-orbit of e is a bijection. Moreover, $\mathcal{V}$ is the unique maximal G-stable closed subvariety of $\mathcal{N}$ that satisfies this property.

Recall that as we often consider G-stable closed subvarieties $\mathcal{V}$ of $\mathcal{N}$ such that the map in (1.0.5) is a bijection, we use a shorthand for such varieties, and say that such a variety satisfies the $\mathfrak{sl}_2$-*property*. Then Theorem 5.0.1 determines the unique maximal G-stable closed subvariety $\mathcal{V}$ of $\mathcal{N}$ that satisfies the $\mathfrak{sl}_2$-property. Or in other words it states that for $e \in \mathcal{V}$, there exists a unique $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ with $f \in \mathcal{V}$ up to conjugacy by the centralizer of e in G, and moreover, $\mathcal{V}$ is maximal with respect to this property.

Of note here is that when $p > h(G)$ we have that $\mathcal{V} = \mathcal{N}$, as given by [ST18, Theorem 1.1].

In order to prove Theorem 5.0.1 we first consider $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{k})$ in §5.1, we can then immediately apply this result to $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$. In §5.2 we then consider $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{k})$ and $\mathfrak{g} = \mathfrak{so}_n(\mathbb{k})$ cases. We determine that $\mathfrak{sl}_2$-triples in $\mathfrak{sp}_n(\mathbb{k})$ are conjugate by $\mathrm{Sp}_n(\mathbb{k})$ if and only if they are conjugate by $\mathrm{GL}_n(\mathbb{k})$, and hence this case follows from the $\mathfrak{gl}_n(\mathbb{k})$ result. In order to prove the $\mathfrak{so}_n(\mathbb{k})$ case we do a detailed analysis of the Jordan block structure of $\mathfrak{sl}_2$-modules in $\mathfrak{so}_n(\mathbb{k})$.

Our second main result concerns the G-completely reducible $\mathfrak{sl}_2$-subalgebras of the classical Lie algebras.

Theorem 5.0.2. *Let $\mathbb{k}$ be an algebraically closed field with prime characteristic p for some $p > 2$. Let $(G, \mathfrak{g}, \mathcal{V})$ be as in Theorem 5.0.1. Then any $\mathfrak{h} = \langle e, h, f\rangle \cong \mathfrak{sl}_2(\mathbb{k})$ with $e, f \in \mathcal{V}$ is G-completely reducible. Moreover, $\mathcal{V}$ is the unique maximal G-stable closed subvariety of $\mathcal{N}$ that satisfies this property.*

Observe that the variety of nilpotent elements given in Theorems 5.0.1 and 5.0.2 is equal to the variety given in Theorems 2 and 3 when restricted to the classical algebraic groups. Hence to prove Theorems 2 and 3 when G is one of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$, $\mathrm{O}_n(\Bbbk)$ or $\mathrm{SO}_n(\Bbbk)$ it is enough to prove Theorems 5.0.1 and 5.0.2.

5.1 General and special linear groups

For the main part of this section we consider the case $G = \mathrm{GL}_n(\Bbbk)$ and work towards proving Theorem 5.0.1(a). Then in §5.1.4 we consider the case $G = \mathrm{SL}_n(\Bbbk)$ and deduce Theorem 5.0.1(b).

To prove Theorem 5.0.1(a) we work with the algebra

$$A := U(\mathfrak{s})/\langle e^{p-1}, f^{p-1}\rangle.$$

In Corollary 5.1.7 we see that A is semisimple. This follows from [Jac58, Theorem 1], although we give an alternative proof.

We write elements of A as linear combinations of monomials in e, h and f, so there is a possibility of a conflict of notation with elements of $U(\mathfrak{s})$. However, when considering elements, of $U(\mathfrak{s})$ or A, we ensure it is clear from the context which algebra they are contained in.

5.1.1 Simple A-modules and a lower bound for the dimension of A

In the following lemma we give a set of pairwise non-isomorphic simple A-modules. The simple $\mathfrak{s}$-modules $V(d)$ in the statement of the lemma are as given in

Example 2.3.1.

Lemma 5.1.1. *The* $\mathfrak{s}$*-modules* $V(0), V(1), \ldots, V(p-2)$ *are simple* A*-modules, and moreover they are pairwise non-isomorphic.*

Proof. For $0 \leq d \leq p-2$, we have that e^{d+1} and f^{d+1} act as zero on $V(d)$. Hence e^{p-1} and f^{p-1} act as zero on $V(d)$, thus $V(d)$ is an A-module, and it is simple as an A-module as it is simple as an $\mathfrak{s}$-module. For $c \neq d$ we have that $V(c)$ and $V(d)$ have different dimensions, so are not isomorphic. □

In the following corollary we establish a lower bound for $\dim(A/\operatorname{rad} A)$, where $\operatorname{rad} A$ denotes the Jacobson radical of A. We achieve this by applying Wedderburn's theorem to the semisimple algebra $A/\operatorname{rad} A$. For justification of this, and background on the representation theory used here refer to §2.2.

Corollary 5.1.2. *The dimension of* $A/\operatorname{rad} A$ *is greater than or equal to* $\sum_{i=1}^{p-1} i^2$.

Proof. We have that $A/\operatorname{rad} A$ is semisimple and using Lemma 5.1.1 we have that $V(0), V(1), \ldots, V(p-2)$ are distinct simple non-isomorphic modules for $A/\operatorname{rad} A$. From Wedderburn's theorem, Theorem 2.2.2, we deduce that

$$\dim(A/\operatorname{rad} A) \geq \dim(V(0))^2 + \dim(V(1))^2 + \cdots + \dim(V(p-2))^2 = \sum_{i=1}^{p-1} i^2. \ \square$$

5.1.2 A spanning set for A and an upper bound for the dimension of A

We define the subsets

$$S_k := \{f^a h^k e^c : 0 \leq a, c < p-1-k\}$$

of A for each $k < p-1$. The following proposition is proved at the end of this subsection.

Proposition 5.1.3. *The union $S := \bigcup_{k=0}^{p-2} S_k$ is a spanning set for A.*

We note that $|S_k| = (p-1-k)^2$, thus $|S| = \sum_{i=1}^{p-1} i^2$, which is equal to the lower bound of $A/\operatorname{rad} A$ given in Corollary 5.1.2. Therefore, by combining Corollary 5.1.2 and Proposition 5.1.3, we are able to deduce that S is a basis of A so $\dim(A) = \sum_{i=1}^{p-1} i^2$. Further, we have that $\operatorname{rad} A = 0$, so that A is semisimple. Hence we have that $\{V(0), V(1), \ldots, V(p-2)\}$ is a complete set of inequivalent simple A-modules. This is all stated in Corollary 5.1.7, and is then used to prove that $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property in Corollary 5.1.9.

In order to show that S is a spanning set for A, we start with a lemma which gives some relations in $U(\mathfrak{s})$.

Lemma 5.1.4. *Within $U(\mathfrak{s})$, for any $k \in \mathbb{Z}_{>0}$, we have*

(a) $[e^k, h] = -2ke^k$;

(b) $[e^k, f] = khe^{k-1} - k(k-1)e^{k-1}$; *and*

(c) $[h^k, f] \in \operatorname{span}\{fh^i : i = 0, \ldots, k-1\}$.

Proof. (a) We note that e is an eigenvector of $\operatorname{ad} h$ with eigenvalue 2, so e^k is an eigenvector of $\operatorname{ad} h$ with eigenvalue $2k$. Thus $[e^k, h] = -2ke^k$.

(b) We use a simple induction on k. For $k = 1$, we have $[e, f] = h$. Suppose that $[e^k, f] = khe^{k-1} - k(k-1)e^{k-1}$. Then

$$[e^{k+1}, f] = [e^k, f]e + e^k[e, f] = khe^k - k(k-1)e^k + e^k h.$$

From (a) we have that $e^k h = he^k - 2ke^k$, hence $[e^{k+1}, f]$ is equal to

$$khe^k - k(k-1)e^k + he^k - 2ke^k = (k+1)he^k - (k^2+k)e^k = (k+1)he^k - k(k+1)e^k.$$

(c) We show this using induction on k. When $k = 1$, by the definition of $U(\mathfrak{s})$, we have $[h, f] = -2f$. Suppose that $[h^k, f] = \sum_{i=0}^{k-1} a_i f h^i$ for some constants $a_i \in \Bbbk$, then

$$[h^{k+1}, f] = [h, f]h^k + h[h^k, f] = -2fh^k + \sum_{i=0}^{k-1} a_i h f h^i.$$

Recall that $hf = fh - 2f$, and so this is equal to

$$-2fh^k + \sum_{i=0}^{k-1} a_i(fh - 2f)h^i = -2fh^k + \sum_{i=0}^{k-1} a_i f h^{i+1} + \sum_{i=0}^{k-1}(-2a_i) f h^i = \sum_{i=0}^{k} b_i f h^i$$

where

$$b_i = \begin{cases} -2a_0 & \text{if } i = 0, \\ -2 + a_{k-1} & \text{if } i = k, \\ -2a_i + a_{i-1} & \text{otherwise.} \end{cases}$$

Thus the result holds for all $k \in \mathbb{Z}_{>0}$. □

We now prove a lemma giving spanning properties of the sets S_k. Before stating and proving this lemma we explain, in the following remark, how we use an antiautomorphism of $U(\mathfrak{s})$ to reduce the amount of work required.

For $\mathfrak{g}$ some Lie algebra, an antiautomorphism of $U(\mathfrak{g})$ is a map of vector spaces $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ such that $\tau(xy) = \tau(y)\tau(x)$ for all $x, y \in U(\mathfrak{g})$. We note that given such an antiautomorphism, for any $x, y \in U(\mathfrak{g})$,

$$\tau([x,y]) = \tau(xy - yx) = \tau(y)\tau(x) - \tau(x)\tau(y) = [\tau(y), \tau(x)].$$

Remark 5.1.5. Consider the antiautomorphism $\tau : U(\mathfrak{s}) \to U(\mathfrak{s})$ determined by

$$\tau(e) = f, \ \tau(h) = h, \ \tau(f) = e.$$

So for any $a, b, c \in \mathbb{Z}_{\geq 0}$ we have

$$\tau(f^a h^b e^c) = \tau(e)^c \tau(h)^b \tau(f)^a = f^c h^b e^a.$$

As τ stabilises $\langle e^{p-1}, f^{p-1} \rangle$ it gives an antiautomorphism of A, which we also denote by τ. Using τ, given any relation in A we can find an equivalent relation where the powers of e and f are swapped. More precisely, if we have some $r_{a,b,c} \in \Bbbk$ such that $\sum_{a,b,c} r_{a,b,c} f^a h^b e^c = 0$, then

$$0 = \tau \left(\sum_{a,b,c} r_{a,b,c} f^a h^b e^c \right) = \sum_{a,b,c} r_{a,b,c} f^c h^b e^a.$$

Using this, we note that for any relation on elements of A written in the form of a linear combination of monomials $f^a h^b e^c$, there is a another relation determined by swapping the powers of e and f. We also observe here that S_k is stable under

τ for all k.

Lemma 5.1.6. *Let $k \in \mathbb{Z}_{\geq 0}$. Within $A = U(\mathfrak{s})/\langle e^{p-1}, f^{p-1}\rangle$, we have*

(a) if $k < p$ and either $a \geq p-1-k$ or $c \geq p-1-k$, then $f^a h^k e^c \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1})$; and

(b) if $k \geq p$, then $f^a h^k e^c \in \operatorname{span}(S_0 \cup \cdots \cup S_{p-2})$ for any $a, c \geq 0$.

Proof. We work by induction to show that for $k \in \mathbb{Z}_{\geq 0}$ with $k < p$, if $a \geq p-1-k$ or $c \geq p-1-k$, then $f^a h^k e^c \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1})$.

Note that this is clear for $k = 0$, as $e^{p-1} = 0 = f^{p-1}$ in A.

To demonstrate the argument we also cover the case $k = 1$. We need to show that $f^a h e^{p-2} \in \operatorname{span}(S_0)$ for any $a < p-1$ as then we have the analogue for $f^{p-2} h e^c$ using Remark 5.1.5. We have $e^{p-1} = 0$, therefore using Lemma 5.1.4(b) we see

$$0 = [f^a e^{p-1}, f] = f^a [e^{p-1}, f] = (p-1) f^a h e^{p-2} - (p-1)(p-2) f^a e^{p-2}$$

hence we have

$$f^a h e^{p-2} = -2 f^a e^{p-2} \in \operatorname{span}(S_0),$$

and we are done.

Now let $k \in \mathbb{Z}_{\geq 0}$ with $k < p$. For our inductive hypothesis, we suppose that for all $i < k$, if $a \geq p-1-i$ or $c \geq p-1-i$, then $f^a h^i e^c \in \operatorname{span}(S_0 \cup \cdots \cup S_{i-1})$. We first show that

$$f^a h^k e^{p-1-k} \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1}) \ \text{ for any } a < p-1. \tag{5.1.1}$$

In order to show this, we first consider some arbitrary $x \in A$ and show that, if there is some $j < k-1$ such that $x \in \operatorname{span}(S_0 \cup \cdots \cup S_j)$, then $[x, f] \in \operatorname{span}(S_0 \cup \cdots \cup S_{j+1})$. It is enough to show that $[f^a h^j e^c, f] \in \operatorname{span}(S_0 \cup \cdots \cup S_{j+1})$ for any $j < k-1$, $a, c < p-1-j$.

Using Lemma 5.1.4(b) and (c), there exists some $a_i \in \Bbbk$ such that

$$\begin{aligned}
[f^a h^j e^c, f] &= f^a([h^j, f]e^c + h^j[e^c, f]) \\
&= f^a\left(\left(\sum_{i=0}^{j-1} a_i f h^i\right) e^c + c h^{j+1} e^{c-1} - c(c-1) h^j e^{c-1}\right) \\
&= \sum_{i=0}^{j-1} a_i f^{a+1} h^i e^c + c f^a h^{j+1} e^{c-1} - c(c-1) f^a h^j e^{c-1}. \qquad (5.1.2)
\end{aligned}$$

For $i < j < k-1$ we have that $f^{a+1} h^i e^c \in \operatorname{span}(S_0 \cup \cdots \cup S_{j-1})$ using our inductive hypothesis if needed. As a and $c-1 < p-1-j$ we have $f^a h^j e^{c-1} \in S_j$, and as $j+1 < k$, then using the inductive hypothesis if necessary we have $f^a h^j e^{c-1} \in \operatorname{span}(S_0 \cup \cdots \cup S_{j+1})$. Hence we can conclude that each of the terms in (5.1.2) is in $\operatorname{span}(S_0 \cup \cdots \cup S_{j+1})$ and hence

$$[f^a h^j e^c, f] \in \operatorname{span}(S_0 \cup \cdots \cup S_{j+1}). \qquad (5.1.3)$$

We now move on to prove (5.1.1). By our inductive hypothesis, we have that $f^a h^{k-1} e^{p-k} \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-2})$, hence we can use (5.1.3) as $k-2 < k-1$, so we have that $[f^a h^{k-1} e^{p-k}, f] \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1})$. Thus

$$[f^a h^{k-1} e^{p-k}, f] = f^a [h^{k-1}, f] e^{p-k} + f^a h^{k-1} [e^{p-k}, f] \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1}). \quad (5.1.4)$$

We show that the first term on the right hand side of (5.1.4) is in $\operatorname{span}(S_0 \cup \cdots \cup$

$S_{k-2})$. This is done by noting that if $i < k-1$ then using the inductive hypothesis if needed we see $f^{a+1}h^i e^{p-k} \in \text{span}(S_0 \cup \cdots \cup S_i)$ and hence in $\text{span}(S_0 \cup \cdots \cup S_{k-2})$, and thus, using Lemma 5.1.4(c) we see that

$$f^a[h^{k-1}, f]e^{p-k} \in \text{span}(S_0 \cup \cdots \cup S_{k-2}).$$

By rearranging (5.1.4) we obtain that $f^a h^{k-1}[e^{p-k}, f] \in \text{span}(S_0 \cup \cdots \cup S_{k-1})$, we then use Lemma 5.1.4(b) to see

$$f^a h^{k-1}[e^{p-k}, f] = -kf^a h^k e^{p-k-1} - k(k+1)f^a h^{k-1}e^{p-k-1}. \tag{5.1.5}$$

Note that $f^a h^{k-1}e^{p-k-1} \in \text{span}(S_0 \cup \cdots \cup S_{k-1})$ by the induction hypothesis. Thus, we can rearrange (5.1.5) to see

$$kf^a h^k e^{p-k-1} \in \text{span}(S_0 \cup \cdots \cup S_{k-1}).$$

As we have assumed that $0 < k < p$, we have that $k \neq 0$ in $\Bbbk$, and so we deduce (5.1.1)

We next show that (5.1.1) can be used to prove that if $c \geq p-k-1$ we have $f^a h^k e^c \in \text{span}(S_0 \cup \cdots \cup S_{k-1})$. We know by (5.1.1) that we can find some scalars $r_{i,b,j} \in \Bbbk$ so that

$$f^a h^k e^{p-1-k} = \sum_{\substack{0 \leq b \leq k-1 \\ i,j < p-1-b}} r_{i,b,j} f^i h^b e^j.$$

We consider $f^a h^k e^l$ for $l > p-1-k$, and have that

$$f^a h^k e^l = \sum_{\substack{0 \leq b \leq k-1 \\ i,j < p-1-b}} r_{i,b,j} f^i h^b e^{j+(l-p-1-k)}.$$

Using the induction hypothesis $f^i h^b e^{j+(l-p-1-k)} \in \operatorname{span}(S_0 \cup \cdots \cup S_b)$, and hence $f^a h^k e^l \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1})$.

We can use the antiautomorphism τ from Remark 5.1.5 to show that $f^a h^k e^c \in \operatorname{span}(S_0 \cup \cdots \cup S_{k-1})$ when $a \geq p-1-k$, and so we have completed the proof of (a).

In fact we have proved that for any $k \in \mathbb{Z}_{\geq 0}$ with $k < p$ and any $a, b \in \mathbb{Z}_{\geq 0}$ that we have $f^a h^k e^b \in \operatorname{span}(S_0 \cup \cdots \cup S_{p-2})$. As a particular case, we have that $h^{p-1} \in \operatorname{span}(S_0 \cup \cdots \cup S_{p-2})$. Now given $f^a h^k e^b$ with $k \geq p$, we can repeatedly substitute h^{p-1} as an expression in $\operatorname{span}(S_0 \cup \cdots \cup S_{p-2})$ and obtain $f^a h^k e^b$ as a linear combination of terms $f^i h^b e^j$ with $b \leq p-1$. From this we can deduce that $f^a h^k e^b \in \operatorname{span}(S_0 \cup \cdots \cup S_{p-2})$ using part (a) of the lemma. Thus we have proved part (b) of the lemma. □

Using Lemma 5.1.6 we are now able to show that S is a spanning set for A, and hence prove Proposition 5.1.3.

Proof of Proposition 5.1.3. We have that $\{f^a h^b e^c : a, b, c \in \mathbb{Z}_{\geq 0}\}$ is a basis for $U(\mathfrak{s})$, hence as A is obtained from $U(\mathfrak{s})$ by taking the quotient by $\langle e^{p-1}, f^{p-1} \rangle$ we see that

$$\{f^a h^b e^c : a, c < p-1,\ b \in \mathbb{Z}_{\geq 0}\}$$

spans A. By Lemma 5.1.6, every element in this set is contained in the span of S. Hence, S is a spanning set for A. □

5.1.3 Proof of Theorem 5.0.1(a)

Let $G = \mathrm{GL}_n(\Bbbk)$ and $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$. We recall that $\mathcal{N}^{p-1}$ is defined in (5.0.1). In Corollary 5.1.9 it is stated that $\mathcal{N}^{p-1}$ has the $\mathfrak{sl}_2$-property. To prove this corollary we use the fact that A is semisimple. The semisimplicity of A is stated as part of the following corollary, which is proved as explained after the statement of Proposition 5.1.3.

Corollary 5.1.7. *We have that S is a basis of A, so the dimension of A is equal to $\sum_{i=1}^{p-1} i^2$. Further, we have that A is semisimple, and the simple modules of A are $V(0), V(1), \ldots, V(p-2)$.*

Remark 5.1.8. We note that further results can be proved using the arguments for the proof of Corollary 5.1.7 (or deduced from its statement). For any $m < p$, it can be shown that $U(\mathfrak{s})/\langle e^m, f^m \rangle$ is a semisimple algebra with simple modules $V(0), V(1), \ldots, V(m-1)$; also this statement can be proved for the case of $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C})$ for any $m \in \mathbb{Z}_{>0}$. These results are also covered in [Jac58, Theorem 2].

We now explain how A-modules relate to $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. Any A-module M can be considered as an $\mathfrak{s}$-module, and thus we obtain an $\mathfrak{sl}_2$-triple (e_M, h_M, f_M) in $\mathfrak{gl}(M)$, as explained in §2.3. Moreover, we have $e_M^{p-1} = 0 = f_M^{p-1}$, as M is an A-module and $e^{p-1} = 0 = f^{p-1}$ in A. Suppose that $\dim M = n$ and choose an identification $M \cong \Bbbk^n$ as a vector space. We view (e_M, h_M, f_M) as an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Further, given two A-modules M and N, both of dimension n, we have that $M \cong N$ if and only if the $\mathfrak{sl}_2$-triples (e_M, h_M, f_M) and (e_N, h_N, f_N) are conjugate by an element of G.

Hence, there is a one-to-one correspondence between the set of n-dimensional A-modules up to isomorphism and the $\mathfrak{sl}_2$-triples (e, h, f) in $\mathfrak{g}$ with $e, f \in \mathcal{N}^{p-1}$ up to conjugacy by elements of G. Thus proving that $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property is equivalent to proving that for each partition λ of n such that $m_i(\lambda) = 0$ for all $i \geq p$, there is an n-dimensional A-module M_λ on which e acts with Jordan type λ, and this module is unique up to isomorphism.

By Corollary 5.1.7, each A-module is semisimple and hence a direct sum of the simple modules $V(0), \ldots, V(p-2)$. So any n-dimensional A-module satisfies $M \cong \bigoplus_{d=0}^{p-2} V(d)^{\oplus s_d}$ for some $s_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{d=0}^{p-2}(d+1)s_d = n$. We have that e acts on M with Jordan type λ_M, where $m_d(\lambda_M) = s_{d-1}$ for each d.

Thus we see the desired module is $M_\lambda := \bigoplus_{d=0}^{p-2} V(d)^{\oplus m_{d+1}(\lambda)}$. Hence, we have proved the following corollary.

Corollary 5.1.9. *Let $G = \mathrm{GL}_n(\Bbbk)$. Then $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property.*

We now explain that $\mathcal{N}^{p-1}$ is maximal satisfying the $\mathfrak{sl}_2$-property to complete the proof of Theorem 5.0.1(a). Suppose that $\mathcal{V}$ is a G-stable closed subvariety of $\mathcal{N}$ such that $\mathcal{V} \not\subseteq \mathcal{N}^{p-1}$. Then there must exist some $e' \in \mathcal{V}$ with Jordan type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_1 \geq p$. Hence, by Theorem 2.4.6, there exists $e \in \mathcal{V}$ with Jordan type $(p, 1^{n-p})$. Thus, using Corollary 4.2.2, we deduce that $\mathcal{V}$ does not satisfy the $\mathfrak{sl}_2$-property.

5.1.4 Deduction of Theorem 5.0.1(b)

In this short subsection we deal with the case $G = \mathrm{SL}_n(\Bbbk)$ and $\mathfrak{g} = \mathfrak{sl}_n(\Bbbk)$ and explain that Theorem 5.0.1(b) follows quickly from Theorem 5.0.1(a).

We note that the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$ is the same as the nilpotent cone of $\mathfrak{gl}_n(\Bbbk)$, and that two elements in $\mathcal{N}$ are conjugate by $\mathrm{GL}_n(\Bbbk)$ if and only they are conjugate by G, because $\mathrm{GL}_n(\mathbb{k})$ is generated by $\mathrm{SL}_n(\mathbb{k})$ and $Z(\mathrm{GL}_n(\mathbb{k}))$. Thus for any G-stable closed subvariety $\mathcal{V}$ of $\mathcal{N}$, we have that the set of G-orbits in $\mathcal{V}$ is equal to the set of $\mathrm{GL}_n(\mathbb{k})$-orbits in $\mathcal{V}$.

We also note that any $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n(\mathbb{k})$ must lie in $\mathfrak{g}$, as any $\mathfrak{sl}_2$-triple must be contained in the derived subalgebra of $\mathfrak{gl}_n(\mathbb{k})$, which is equal to $\mathfrak{sl}_n(\mathbb{k})$. Hence the set of G-orbits of $\mathfrak{sl}_2$-triples (e, h, f) with $e, f \in \mathcal{V}$ is equal to the set of $\mathrm{GL}_n(\mathbb{k})$-orbits of $\mathfrak{sl}_2$-triples (e, h, f) with $e, f \in \mathcal{V}$. It is now clear that Theorem 5.0.1(b) follows from Theorem 5.0.1(a).

5.2 Symplectic and orthogonal groups

In this section we deal with the cases where G is one of $\mathrm{Sp}_n(\mathbb{k})$, $\mathrm{O}_n(\mathbb{k})$ or $\mathrm{SO}_n(\mathbb{k})$ and prove parts (c), (d) and (e) of Theorem 5.0.1.

5.2.1 Proof that $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property for $\mathrm{Sp}_n(\mathbb{k})$ and $\mathrm{O}_n(\mathbb{k})$, and deduction of Theorem 5.0.1(c)

Let G be one of $\mathrm{Sp}_n(\mathbb{k})$ or $\mathrm{O}_n(\mathbb{k})$ and $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{k})$ or $\mathfrak{so}_n(\mathbb{k})$ respectively. In Proposition 5.2.2 we show that $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property; we recall that $\mathcal{N}^{p-1}$ is defined in (5.0.1). To prove this we want to relate G-conjugacy of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ with $\mathrm{GL}_n(\mathbb{k})$-conjugacy of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$, so that we can apply Theorem 5.0.1(a). This link is given in Lemma 5.2.1 and is based on [Jan04, Theorem 1.4], which states that two elements of $\mathfrak{g}$ are conjugate by G if and only if they are conjugate by $\mathrm{GL}_n(\mathbb{k})$. With minor modifications the proof of [Jan04, Theorem 1.4] goes through to prove the lemma below.

Lemma 5.2.1. *Let G be one of $\mathrm{Sp}_n(\mathbb{k})$ or $\mathrm{O}_n(\mathbb{k})$, and let (e, h, f) and (e', h', f')*

be $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. Then (e,h,f) and (e',h',f') are in the same G-orbit if and only if they are in the same $\mathrm{GL}_n(\Bbbk)$-orbit.

We move on to prove the main result in this subsection.

Proposition 5.2.2. *Let G be one of $\mathrm{Sp}_n(\Bbbk)$ or $\mathrm{O}_n(\Bbbk)$. Then $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property.*

Proof. By the result of Pommerening in [Pom80, §2.1], or the theory of standard $\mathfrak{sl}_2$-triples recapped in §4.1, we have that the map in (1.0.5) for $\mathcal{V} = \mathcal{N}^{p-1}$ is surjective.

Let (e,h,f), (e,h',f') be $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ with $e,f,f' \in \mathcal{N}^{p-1}$. By Corollary 5.1.9, these $\mathfrak{sl}_2$-triples are conjugate by $\mathrm{GL}_n(\Bbbk)$, and thus by Lemma 5.2.1 are conjugate by G. This implies that the map in (1.0.5) for $\mathcal{V} = \mathcal{N}^{p-1}$ is injective. □

Proof of Theorem 5.0.1(c). In this case we take $G = \mathrm{Sp}_n(\Bbbk)$. By Proposition 5.2.2, we have that $\mathcal{N}^{p-1}$ satisfies the $\mathfrak{sl}_2$-property. We complete the proof Theorem 5.0.1(c) by explaining that $\mathcal{N}^{p-1}$ is the maximal G-stable closed subvariety of $\mathcal{N}$ satisfying the $\mathfrak{sl}_2$-property. Let $\mathcal{V}$ be a G-stable closed subvariety of $\mathcal{N}$ such that $\mathcal{V} \nsubseteq \mathcal{N}^{p-1}$. Then there is an element in $\mathcal{V}$ which has Jordan type $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, where either $\lambda_1 > p$, or $\lambda_1 = \lambda_2 = p$. Using Theorem 2.4.6, we deduce that there is an element in $\mathcal{V}$ with Jordan type $(p+1, 1, \dots, 1)$ or $(p, p, 1, \dots, 1)$. For the first possibility we can apply Proposition 4.1.6 to deduce that $\mathcal{V}$ does not satisfy the $\mathfrak{sl}_2$-property, whilst in the second case we can apply Corollary 4.2.2 to deduce that $\mathcal{V}$ does not satisfy the $\mathfrak{sl}_2$-property. □

5.2.2 Proof of Theorem 5.0.1(d)

Let $G = \mathrm{O}_n(\Bbbk)$ and $\mathfrak{g} = \mathfrak{so}_n(\Bbbk)$ and recall that ${}^1\mathcal{N}^p$ is defined in (5.0.2). In Proposition 5.2.5, we prove that ${}^1\mathcal{N}^p$ satisfies the $\mathfrak{sl}_2$-property. This proof requires some analysis of underlying $\mathfrak{s}$-modules, where we recall that $\mathfrak{s} = \mathfrak{sl}_2(\Bbbk)$, and we note that the ideas in the proof have some similarities with those in the proof of [ST18, Lemma 6.2].

In Lemma 5.2.3 we state some well-known general results on module extensions, which we use in the proof of Proposition 5.2.5. We only state this lemma for $\mathfrak{s}$-modules, though it is of course applicable more generally.

Before the statement of Lemma 5.2.3 we introduce some notation. We use the notation $M \cong A|B$ for $\mathfrak{s}$-modules M, A and B, to mean there is short exact sequence $0 \to B \to M \to A \to 0$. When using this notation, we identify B with a fixed submodule of M and A as the corresponding quotient. We also use the notation to cover three (or more) modules, so consider $\mathfrak{s}$-modules of the form $A|B|C$, where A, B and C are $\mathfrak{s}$-modules, and note there is no need to include brackets in the notation $A|B|C$.

In part (a) of the statement of Lemma 5.2.3 we should really define the module $A|C$ occurring there. This can be defined as the quotient of $A|B|C$ by the submodule B given by the splitting $B|C \cong B \oplus C$; or equivalently as the submodule of $A|B|C$ corresponding to the submodule A of $A|B$ given by the splitting $A|B \cong A \oplus B$. The modules $A|B$, $A|C$ and $B|C$ in parts (b) and (c) are defined similarly.

Lemma 5.2.3. *Let M, A, B, and C be $\mathfrak{s}$-modules.*

(a) Suppose that $M \cong A|B|C$, and that $A|B \cong A \oplus B$ and $B|C \cong B \oplus C$. Then

$M \cong (A|C) \oplus B$.

(b) *Suppose that $M \cong (A \oplus B)|C$ and that $B|C \cong B \oplus C$. Then $M \cong (A|C) \oplus B$.*

(c) *Suppose that $M \cong A|(B \oplus C)$ and that $A|B \cong A \oplus B$. Then $M \cong (A|C) \oplus B$.*

We note that (a) can be proved by using splitting maps $B \to B|C$ and $B|C \to C$ for the short exact sequence $C \to B|C \to B$ to construct a short exact sequence $B \to A|B|C \to A|C$. Then a splitting map $A|B \to B$ for the short exact sequence $B \to A|B \to A$ can be used to construct a splitting map $A|B|C \to B$ for the short exact sequence $B \to A|B|C \to A|C$. We have that (b) and (c) are immediate consequences of (a).

We also require an elementary lemma about the action of nilpotent elements in $\mathfrak{s}$-modules, which is stated in Lemma 5.2.4. We only state this lemma for $\mathfrak{s}$-modules, though it is of course applicable more generally. In the statement we use the notation given in §2.3 and §2.4.1.

Lemma 5.2.4. *Let M, A and B be $\mathfrak{s}$-modules and let $x \in \mathfrak{s}$. Suppose that $M \cong A|B$ and that x_A and x_B are nilpotent. Then x_M is nilpotent and $\lambda(x_A)|\lambda(x_B) \preceq \lambda(x_M)$.*

We give an outline of how this lemma can be proved. First we identify M and $A \oplus B$ as vector spaces. We then note that $x_A + x_B$ is in the closure of the $\mathrm{GL}(M)$-orbit of x_M; we see this by observing that $x_A + x_B$ lies in the closure of $\{(\mathrm{Ad}\,\tau(t))x_M : t \in \Bbbk^\times\}$, where $\tau : \Bbbk^\times \to \mathrm{GL}(M)$ is the cocharacter such that $\tau(t)a = a$ for all $a \in A$ and $\tau(t)b = tb$ for all $b \in B$. The proof concludes by noting that $\lambda(x_A + x_B) = \lambda(x_A)|\lambda(x_B)$ and then applying Theorem 2.4.6.

We are now ready to state and prove our main result in this subsection.

Proposition 5.2.5. *Let $G = \mathrm{O}_n(\Bbbk)$. Then ${}^1\mathcal{N}^p$ satisfies the $\mathfrak{sl}_2$-property.*

Proof. By the result of Pommerening in [Pom80, §2.1], or the theory of standard $\mathfrak{sl}_2$-triples recapped in §4.1, the map in (1.0.5) for $\mathcal{V} = {}^1\mathcal{N}^p$ is surjective. The rest of the proof is devoted to proving that this map is in fact injective.

Let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g} = \mathfrak{so}_n(\Bbbk)$ with $e, f \in {}^1\mathcal{N}^p$. Let $V \cong \Bbbk^n$ be the natural module for $G = \mathrm{O}_n(\Bbbk)$, and consider V as a an $\mathfrak{s}$-module by restriction to the subalgebra of $\mathfrak{g}$ spanned by $\{e, h, f\}$. Write $(\cdot\,,\cdot)$ for the G-invariant non-degenerate symmetric bilinear form on V.

The idea of the rest of the proof is to determine the structure of the $\mathfrak{s}$-module V, and observe that it is determined uniquely up to isomorphism by the Jordan type of e. Then at the end of the proof we use this and Lemma 5.2.1 to deduce that the map in (1.0.5) is injective.

Let $M \leq V$ be a maximal isotropic $\mathfrak{s}$-submodule of V, and consider $M^\perp := \{v \in V : (v, m) = 0 \text{ for all } m \in M\} \leq V$, which is an $\mathfrak{s}$-submodule of V. As M is isotropic we have the sequence of submodules

$$0 \leq M \leq M^\perp \leq V. \tag{5.2.1}$$

We have an $\mathfrak{s}$-module homomorphism $\phi : V \to M^*$ defined by $\phi(v)(m) = (m, v)$, where M^* denotes the dual module of M. This induces an isomorphism $V/M^\perp \cong M^*$, and so by an abuse of notation we write M^* for $V/M^\perp$. Also we write N for the $\mathfrak{s}$-module $M^\perp/M$, and note that $(\cdot\,,\cdot)$ induces an $\mathfrak{s}$-invariant non-degenerate symmetric bilinear form on N, which we also denote by $(\cdot\,,\cdot)$. Thus the quotients in the sequence in (5.2.1) are M, N and M^*, or in other words $V \cong M^*|N|M$.

We first consider the $\mathfrak{s}$-module M. Suppose that $e_M^{p-1} \neq 0$. Then $\lambda(e_M)$ contains a part of size p or greater. We note that $\lambda(e_M) = \lambda(e_{M^*})$, so that $\lambda(e_{M^*})$ also contains a part of size p or greater. Using Lemma 5.2.4, we deduce that $\lambda(e_V)$ must have first and second parts greater than or equal to p, but this is contradicts that $e = e_V \in {}^1\mathcal{N}^p$. Thus we have that $e_M^{p-1} = 0$. Similarly, we have $f_M^{p-1} = 0$.

It now follows from Corollary 5.1.7 that we have a direct sum decomposition of the $\mathfrak{s}$-module

$$M = M_1 \oplus \cdots \oplus M_r$$

where each M_i is simple, and $M_i \cong V(d_i)$ for some $d_i \in \{0, 1, \ldots, p-2\}$. We have a corresponding direct sum decomposition

$$M^* = M_1^* \oplus \cdots \oplus M_r^*$$

of M^*, where $M_i^* \cong V(d_i)$ and is dual to M_i via $(\cdot\,,\cdot)$ for each i.

Next we consider the $\mathfrak{s}$-module N, which we recall has a non-degenerate symmetric invariant bilinear form. Let A be a simple submodule of N, and consider $A^\perp \leq N$, which is also a submodule of N. Thus $A \cap A^\perp$ is a submodule of N, and as A is simple we have $A \cap A^\perp$ is equal to 0 or to A. Suppose that $A \cap A^\perp = A$, so that A is an isotropic subspace of N. Let $\overline{A}$ be the submodule of $M^\perp$ corresponding to $A \leq M^\perp/M = N$. Then $\overline{A}$ is isotropic and this contradicts that M is a maximal isotropic subspace of V. Therefore, $A \cap A^\perp = 0$, so that A is non-degenerate, and thus $N = A \oplus A^\perp$.

Hence, N is a semisimple $\mathfrak{s}$-module and in fact we have an orthogonal direct sum decomposition

$$N = N_1 \oplus \cdots \oplus N_s, \tag{5.2.2}$$

where each N_i is a simple $\mathfrak{s}$-module and is a non-degenerate subspace for $(\cdot\,,\cdot)$.

Since $e, f \in {}^1\mathcal{N}^p$, using Lemma 5.2.4, we have that $e_N^p = 0 = f_N^p$, and that $\lambda(e_N)$ and $\lambda(f_N)$ have at most one part of size p. It follows that for each i we have $N_i \cong V(c_i)$ for some $c_i \in \{0, 1, \dots, p-2\}$ with the possible exception of one j for which $N_j = V(c_j)$ where $c_j \in \Bbbk \setminus \{0, 1, \dots, p-2\}$.

We note that for every i such that $N_i \cong V(c_i)$ for some $c_i \in \{0, 1, \dots, p-2\}$, we must have that c_i is even, because $e_{N_i} \in \mathfrak{so}(N_i)$ and $\lambda(e_{N_i}) = (c_i + 1)$, so $c_i + 1$ must be odd as explained in §2.4.1.

If there exists a j for which $N_j = V(c_j)$ where $c_j \in \Bbbk \setminus \{0, 1, \dots, p-2\}$, then we show that $c_j = p-1$. To see this we consider $h_{N_j} \in \mathfrak{so}(N_j)$, which is a semisimple element of $\mathfrak{so}(N_j)$ with eigenvalues $c_j, c_j - 2, \dots, c_j - 2p + 2$. The eigenvalues of a semisimple element of $\mathfrak{so}(N_j)$ must include 0 (and also the multiplicity of an eigenvalue a must be equal to the multiplicity of the eigenvalue $-a$). It follows that we must have $c_j = p - 1$.

Next we show that $c_i \neq c_j$ for $i \neq j$. Suppose that we did have $N_i \cong N_j$ for some $i \neq j$. We denote $N_{i,j} = N_i \oplus N_j$ and consider the $\mathfrak{s}$-module $N'_{i,j} = N'_i \oplus N'_j$, where $N'_i = N_i$ and $N'_j = N_j$ as $\mathfrak{s}$-modules, but we give $N'_i \oplus N'_j$ a non-degenerate $\mathfrak{s}$-invariant symmetric bilinear form so that N_i and N_j are isotropic spaces that are dual to each other. We fix an isomorphism $N'_{i,j} \cong N_{i,j}$ as vector spaces with non-degenerate $\mathfrak{s}$-invariant symmetric bilinear forms. This can be used to view $x_{N'_{i,j}}$ as an element of $\mathfrak{so}(N_{i,j})$ for any $x \in \mathfrak{s}$. By definition we have that $N_i \oplus N_j \cong N'_i \oplus N'_j$ as $\mathfrak{s}$-modules, which implies that the $\mathfrak{sl}_2$-triples $(e_{N_{i,j}}, h_{N_{i,j}}, f_{N_{i,j}})$ and $(e_{N'_{i,j}}, h_{N'_{i,j}}, f_{N'_{i,j}})$ both viewed inside $\mathfrak{so}(N_{i,j})$ are conjugate by $\mathrm{GL}(N_{i,j})$. Now we can apply Lemma 5.2.1 to deduce that $(e_{N_{i,j}}, h_{N_{i,j}}, f_{N_{i,j}})$ and $(e_{N'_{i,j}}, h_{N'_{i,j}}, f_{N'_{i,j}})$ are conjugate by $\mathrm{O}(N_{i,j})$. Under the identification $N'_{i,j} \cong N_{i,j} \leq N$, we have that N'_i is an isotropic $\mathfrak{s}$-submodule of N. However, then the corresponding submodule $\overline{N'_i}$ of $M^\perp$ is isotropic, and this contradicts the maximality of M as an isotropic submodule of V.

To summarise our findings about N, we have that the orthogonal direct sum decomposition in (5.2.2), satisfies that $N_i \cong V(c_i)$ for some $c_i \in \{0, 2, \dots, p-1\}$ for each i and that $c_i \neq c_j$ for $i \neq j$.

Our next goal is to prove that

$$M^\perp \cong M \oplus N \cong M_1 \oplus \cdots \oplus M_r \oplus N_1 \oplus \cdots \oplus N_s. \tag{5.2.3}$$

For $j \in \{1, \dots, r\}$ we define $A_j = \bigoplus_{i \neq j} M_i \leq M$. We consider $M^\perp/A_j$ and aim to show that

$$M^\perp/A_j \cong M_j \oplus N. \tag{5.2.4}$$

Noting that $M^\perp/A_j \cong N|M_j$, we see that by repeated application of Lemma 5.2.3(c) we can deduce (5.2.3) from (5.2.4). Thus we only need to establish (5.2.4).

Using (2.3.1) and the fact that the summands in (5.2.2) are pairwise non-isomorphic, there is at most one i for which $\mathrm{Ext}_{\mathfrak{s}}(M_j, N_i)$ is non-zero.

If $\mathrm{Ext}_{\mathfrak{s}}(M_j, N_i) = 0$ for all i, then we have $N_i|M_j \cong N_i \oplus M_j$ for all i, and thus we obtain (5.2.4) by repeated applications of Lemma 5.2.3(b).

If $\mathrm{Ext}_{\mathfrak{s}}(M_j, N_i) \neq 0$ for some i, i.e. $c_i = p - d_j - 2$, then without loss of generality, we may assume that $i = 1$. Using Lemma 5.2.3(b) we can deduce that

$$M^{\perp}/A_j \cong (N_1|M_j) \oplus N_2 \oplus \cdots \oplus N_s. \tag{5.2.5}$$

We may assume that $N_1|M_j$ is a non-split extension of N_1 by M_j otherwise we obtain (5.2.4). We have that $\dim(N_1|M_j) = p$ and $e_{N_1|M_j}$ or $f_{N_1|M_j}$ has Jordan type (p). Without loss of generality we assume that $e_{N_1|M_j}$ has Jordan type (p). We next consider the $\mathfrak{s}$-module $A_j^{\perp}/A_j$ on which $(\cdot\,,\cdot)$ induces a non-degenerate form. There is an isomorphism $A_j^{\perp}/M \cong (M^{\perp}/A_j)^*$ via $(\cdot\,,\cdot)$, and also an isomorphism $N \cong N^*$ as $(\cdot\,,\cdot)$ is non-degenerate on N. Thus we have that

$$A_j^{\perp}/M \cong (M_j^*|N_1) \oplus N_2 \oplus \cdots \oplus N_s. \tag{5.2.6}$$

Using (5.2.5) and (5.2.6) along with repeated applications of Lemma 5.2.3(b) and (c) we deduce that

$$A_j^{\perp}/A_j \cong (M_j^*|N_1|M_j) \oplus N_2 \oplus \cdots \oplus N_s.$$

From the isomorphism $A_j^{\perp}/M \cong (M^{\perp}/A_j)^*$ we obtain an isomorphism $M_j^*|N_1 \cong (N_1|M_j)^*$. Thus we deduce that $e_{M_j*|N_1}$ has Jordan type (p).

Next we consider $e_{M_j^*|N_1|M_j}$. We can choose a basis for $N_1|M_j$ containing a basis of M_j and such that the matrix of $e_{N_1|M_j}$ with respect to this basis is a single Jordan block J_p of size p; we denote this matrix by $[e_{N_1|M_j}]$, and use similar notation for other matrices considered here. We can pick a basis of M_j^* such that the matrix $[e_{M_j^*}]$ of $e_{M_j^*}$ is a single Jordan block J_{d_j+1} of size $d_j + 1$. By choosing a lift of the basis of M_j^* to $M_j^*|N_1|M_j$ and combining with the basis of $N_1|M_j$ we obtain a basis of $M_j^*|N_1|M_j$ for which the matrix of $e_{M_j^*|N_1|M_j}$ has block form

$$[e_{M_j^*|N_1|M_j}] = \begin{pmatrix} J_p & X \\ & J_{d_j+1} \end{pmatrix},$$

where X is some $p \times (d_j + 1)$ matrix. We can consider the matrix $e_{M_j*|N_1}$ with respect to the basis obtained by projecting our basis of $M_j^*|N_1|M_j$ to $M_j^*|N_1$, and we have

$$[e_{M_j*|N_1}] = \begin{pmatrix} J_{c_1+1} & X' \\ & J_{d_j+1} \end{pmatrix},$$

where X' consists of the bottom $c_1 + 1$ rows of X. The Jordan type of $e_{M_j*|N_1}$ is (p), so by considering $[e_{M_j^*|N_1}]$ we see that the bottom left entry of X' must be non-zero. This follows easily using the concepts from Chapter 3, in particular considering Lemma 3.2.7, we see that $D(e_{M_j*|N_1})$ must have a chain of length p. Given the block structure of $e_{M_j*|N_1}$ we can only find such a chain if there is a non-zero entry in the bottom left of X'.

Thus the bottom left entry of X is non-zero. By considering $[e_{M_j^*|N_1|M_j}]$, we deduce that the Jordan type of $e_{M_j^*|N_1|M_j}$ is $(p+d_j+1)$. Now using Lemma 5.2.4, we deduce that the first part of $\lambda(e_V)$ has size greater than p, which is a contradiction because $e_V = e \in {}^1\mathcal{N}^p$. From this contradiction we deduce that $N_1|M_j$ is in fact a split extension, and so we obtain (5.2.4) as desired.

We have now proved (5.2.3) holds. Also note we have an isomorphism $V/M \cong (M^\perp)^*$ via $(\cdot\,,\cdot)$, and an isomorphism $N \cong N^*$ since $(\cdot\,,\cdot)$ is non-degenerate on N. Thus from (5.2.3) we obtain

$$V/M \cong M^* \oplus N.$$

Hence, by applying Lemma 5.2.3(a) we obtain that

$$V \cong M^*|M \oplus N.$$

Our next step is to prove that $M^*|M \cong M^* \oplus M$. Let us suppose that this is not the case, then, using Lemma 5.2.3(b) and (c) we can find i and j such that the subquotient $M_i^*|M_j$ of $M^*|M$ is a non-split extension. Using (2.3.1) and the fact that $M_i \cong M_i^*$ we have that $i \neq j$. Without loss of generality we can assume that $i = 1$ and $j = 2$, and then we have that $d_1 = p - d_2 - 2$. We consider the subquotient $M_{1,2} = (M_1^* \oplus M_2^*)|(M_1 \oplus M_2)$ of $M^*|M$. By using that $\mathrm{Ext}^1_{\mathfrak{s}}(M_1, M_1^*) = 0 = \mathrm{Ext}^1_{\mathfrak{s}}(M_2, M_2^*)$ along with Lemma 5.2.3(b) and (c), we obtain that $M_{1,2} \cong (M_1^*|M_2) \oplus (M_2^*|M_1)$. We have that $(\cdot\,,\cdot)$ induces a non-degenerate bilinear form on $M_{1,2}$ and that $(M_1^*|M_2)$ and $(M_2^*|M_1)$ are isotropic subspaces of $M_{1,2}$, which are dual via $(\cdot\,,\cdot)$. By assumption we have that $M_1^*|M_2$ is a non-split extension, and it has dimension p. Then by Corollary 5.1.7 we have that either $e_{M_1^*|M_2}$ or $f_{M_1^*|M_2}$ has Jordan type (p). Without loss of generality we assume that $e_{M_1^*|M_2}$ has Jordan type (p). Since $(M_2^*|M_1) \cong (M_1^*|M_2)^*$, we also have that $e_{M_2^*|M_1}$ has Jordan type (p). By using Lemma 5.2.4, we deduce that $\lambda(e_V)$ must have first and second parts greater than or equal to p, but this is not possible as $e = e_V \in {}^1\mathcal{N}^p$. This contradiction implies that $M^*|M \cong M^* \oplus M$ as desired.

We have thus far proved that the $\mathfrak{s}$-module V is semisimple and has the direct sum decomposition

$$V = (M_1^* \oplus \cdots \oplus M_r^*) \oplus (N_1 \oplus \cdots \oplus N_s) \oplus (M_1 \oplus \cdots \oplus M_r) \tag{5.2.7}$$

where $M_i \cong V(d_i) \cong M_i^*$ for each i and $N_j \cong V(c_j)$ for each j. Hence, we see that the isomorphism type of V is uniquely determined by the Jordan type of e. Let (e, h', f') be an $\mathfrak{sl}_2$-triple in $\mathfrak{so}(N)$ with $f' \in {}^1\mathcal{N}^p$. Then writing V' for the $\mathfrak{s}$-module given by $\operatorname{span}\{e, h', f'\}$, we have that V' is isomorphic to V. From this we deduce that (e, h', f') is conjugate to (e, h, f) via $\mathrm{GL}(V) = \mathrm{GL}_n(\Bbbk)$, and thus by Lemma 5.2.1 is conjugate via $\mathrm{O}(V) = \mathrm{O}_n(\Bbbk)$. This gives the desired injectivity of the map in (1.0.5), and completes this proof. □

All that is left to do to prove Theorem 5.0.1(d) is to prove that ${}^1\mathcal{N}^p$ is the unique maximal G-stable subvariety of $\mathcal{N}$ satisfying the $\mathfrak{sl}_2$-property. To show this let $\mathcal{V}$ be a G-stable closed subvariety of $\mathcal{N}$ such that $\mathcal{V} \not\subseteq \mathcal{N}^{p-1}$. Then there is an element in $\mathcal{V}$ which has Jordan type $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, where either $\lambda_1 > p$, or $\lambda_1 = \lambda_2 = p$. Using Theorem 2.4.6, we deduce that there is an element in $\mathcal{V}$ with Jordan type $(p+2, 1, \dots, 1)$ or $(p, p, 1, \dots, 1)$. For the first possibility we can apply Proposition 4.1.6 to deduce that $\mathcal{V}$ does not satisfy the $\mathfrak{sl}_2$-property, whilst in the second case we can apply Corollary 4.2.2 to deduce that $\mathcal{V}$ does not satisfy the $\mathfrak{sl}_2$-property.

5.2.3 Deduction of Theorem 5.0.1(e)

We are left to deal with the case $G = \mathrm{SO}_n(\Bbbk)$, and we prove that ${}^1\mathcal{N}^p$ satisfies the $\mathfrak{sl}_2$-property in Proposition 5.2.6. This is deduced from Proposition 5.2.5 along with considerations of how $\mathrm{O}_n(\Bbbk)$-orbits of $\mathfrak{sl}_2$-triples (e, h, f) in $\mathfrak{g} = \mathfrak{so}_n(\Bbbk)$ with $e, f \in \mathcal{N}$ split into $\mathrm{SO}_n(\Bbbk)$-orbits.

Proposition 5.2.6. *Let $G = \mathrm{SO}_n(\Bbbk)$. Then ${}^1\mathcal{N}^p$ satisfies the $\mathfrak{sl}_2$-property.*

Proof. We know that the map in (1.0.5) for $\mathcal{V} = {}^1\mathcal{N}^p$ is surjective, for the same reasons as the corresponding statement in Proposition 5.2.5.

As explained in Remark 2.4.4 the $\mathrm{O}_n(\Bbbk)$-orbit of $x \in \mathfrak{so}_n(\Bbbk)$ is either a single $\mathrm{SO}_n(\Bbbk)$-orbit, or splits into two $\mathrm{SO}_n(\Bbbk)$-orbits, with the former case occurring precisely when there exists $g \in \mathrm{O}_n(\Bbbk)$ with $\det g = -1$ such that $gx = xg$. The underlying argument can also be applied to $\mathfrak{sl}_2$-triples in $\mathfrak{so}_n(\Bbbk)$. Thus we have that for an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{so}_n(\Bbbk)$ the $\mathrm{O}_n(\Bbbk)$-orbit of (e,h,f) is either a single $\mathrm{SO}_n(\Bbbk)$-orbit or splits into two $\mathrm{SO}_n(\Bbbk)$-orbits. Moreover, we have that the $\mathrm{SO}_n(\Bbbk)$-orbit of (e,h,f) is equal to the $\mathrm{O}_n(\Bbbk)$-orbit if and only if there exists some $g \in \mathrm{O}_n(\Bbbk)$ with $\det g = -1$, $ge = eg$, $gh = hg$ and $gf = fg$.

Let λ be the Jordan type of a nilpotent element in ${}^1\mathcal{N}^p$. Using λ we construct a specific realization of some $e \in {}^1\mathcal{N}^p$ with $e \sim \lambda$ and an $\mathfrak{sl}_2$-triple (e,h,f).

Let $V = \Bbbk^n$ be the natural module for $\mathrm{O}_n(\Bbbk)$. Let $m_i'(\lambda) = \frac{m_i(\lambda)}{2}$ for even i. We can form an orthogonal direct sum decomposition of V of the form

$$V = \bigoplus_{i \text{ odd}} \bigoplus_{j=1}^{m_i(\lambda)} V_{i,j} \oplus \bigoplus_{i \text{ even}} \bigoplus_{j=1}^{m_i'(\lambda)} (U_{i,j} \oplus U_{i,j}'), \tag{5.2.8}$$

where for odd i each $V_{i,j}$ is a non-degenerate subspace of dimension i, and for even i the pair $U_{i,j}$ and $U_{i,j}'$ are isotropic subspaces of dimension i, which are in duality under the symmetric bilinear form on V. Corresponding to this decomposition of V we have a subgroup

$$H \cong \prod_{i \text{ odd}} \mathrm{O}_i(\Bbbk)^{m_i(\lambda)} \times \prod_{i \text{ even}} \mathrm{GL}_i(\Bbbk)^{m_i'(\lambda)}$$

of $\mathrm{O}_n(\Bbbk)$. The Lie algebra of H is

$$\mathfrak{h} \cong \bigoplus_{i \text{ odd}} \mathfrak{so}_i(\Bbbk)^{\oplus m_i(\lambda)} \oplus \bigoplus_{i \text{ even}} \mathfrak{gl}_i(\Bbbk)^{\oplus m_i'(\lambda)}, \tag{5.2.9}$$

and is a subalgebra of $\mathfrak{so}_n(\Bbbk)$.

We choose $e \in \mathfrak{h}$ to be regular nilpotent in each of the summands in (5.2.9). Then by construction we see that $e \sim \lambda$. We can find an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{h}$, for example this now follows from Theorem 5.0.1(a) and (d). By Proposition 5.2.5 we know that (e, h, f) lies in the unique $\mathrm{O}_n(\Bbbk)$-orbit of $\mathfrak{sl}_2$-triples in $\mathfrak{so}_n(\Bbbk)$ with $e \sim \lambda$.

Suppose that λ has an odd part, and let $i \in \mathbb{Z}_{>0}$ be odd and such that $m_i(\lambda) > 0$. Then we can define $g \in \mathrm{O}_n(\Bbbk)$ by declaring that g acts on $V_{i,1}$ by -1 and on all other summands in (5.2.8) by 1. We see that $\det g = -1$, and g lies in the centre of H so that $ge = eg$, $gh = hg$ and $gf = fg$. Hence, the $\mathrm{SO}_n(\Bbbk)$-orbit of (e, h, f) is equal to the $\mathrm{O}_n(\Bbbk)$-orbit, and hence is the unique $\mathrm{SO}_n(\Bbbk)$-orbit of $\mathfrak{sl}_2$-triples with $e \sim \lambda$.

Now suppose that λ is very even. Then we know that the $\mathrm{O}_n(\Bbbk)$-orbit of e splits into two $\mathrm{SO}_n(\Bbbk)$-orbits, and we let $e' \in \mathfrak{so}_n(\Bbbk)$ be a representative of the other $\mathrm{SO}_n(\Bbbk)$-orbit in $(\operatorname{Ad} \mathrm{O}_n(\Bbbk))e$. There is an $\mathfrak{sl}_2$-triple (e', h', f'), which lies in the $\mathrm{O}_n(\Bbbk)$-orbit of (e, h, f). Also (e', h', f') is not in the $\mathrm{SO}_n(\Bbbk)$-orbit of (e, h, f), as e is not conjugate to e' via $\mathrm{SO}_n(\Bbbk)$. It follows that the $\mathrm{O}_n(\Bbbk)$-orbit of (e, h, f) splits into two $\mathrm{SO}_n(\Bbbk)$-orbits, and these are the $\mathrm{SO}_n(\Bbbk)$-orbits of (e, h, f) and (e', h', f'). Now using Proposition 5.2.5 we deduce that the $\mathrm{SO}_n(\Bbbk)$-orbit of (e, h, f) is the only orbit mapping to the $\mathrm{SO}_n(\Bbbk)$-orbit of e by the map in (1.0.5); and that the $\mathrm{SO}_n(\Bbbk)$-orbit of (e', h', f') is the only orbit mapping to the $\mathrm{SO}_n(\Bbbk)$-orbit of e' by the map in (1.0.5).

We have shown that for each $e \in {}^1\mathcal{N}^p$, there is a unique $\mathrm{SO}_n(\Bbbk)$-orbit of $\mathfrak{sl}_2$-triples (e, h, f) with $e, f \in {}^1\mathcal{N}^p$ which maps to the $\mathrm{SO}_n(\Bbbk)$-orbit of e under the map in

(1.0.5). This shows that the map in (1.0.5) is injective for $\mathcal{V} = {}^1\mathcal{N}^p$, and hence that ${}^1\mathcal{N}^p$ satisfies the $\mathfrak{sl}_2$-property. □

To complete the proof of Theorem 5.0.1(e), we are just left to show the maximality of $\mathcal{V} = {}^1\mathcal{N}^p$ subject to satisfying the $\mathfrak{sl}_2$-property, but this can be done using the arguments at the end of §5.2.2.

Hence, we have completed the proof of all parts of Theorem 5.0.1, and therefore have completed the proof of Theorem 2 for G any of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$, $\mathrm{O}_n(\Bbbk)$ or $\mathrm{SO}_n(\Bbbk)$.

5.3 G-completely reducible $\mathfrak{sl}_2$-subalgebras

We now consider Theorem 5.0.2 and find that the proof follows quickly from the proof of Theorem 5.0.1 and the results in §4.3. Recall Lemma 2.1.31 which states that for $\mathfrak{g}$ a classical Lie algebra, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is G-completely reducible if and only if it acts completely reducibly on the natural module for $\mathfrak{g}$.

5.3.1 General linear, special linear and symplectic groups

First consider $\mathfrak{g}$ to be one of $\mathfrak{gl}_n(\Bbbk), \mathfrak{sl}_n(\Bbbk)$ or $\mathfrak{sp}_n(\Bbbk)$. Then in each of these cases we have $\mathcal{V} = \mathcal{N}^{p-1}$.

Let $\mathfrak{h} = \langle e, h, f\rangle \cong \mathfrak{sl}_2(\Bbbk)$ be a subalgebra of $\mathfrak{g}$ with $e, f \in \mathcal{V}$, we show that $\mathfrak{h}$ must be G-completely reducible. Consider the action of $\mathfrak{h}$ on the natural module, V, for $\mathfrak{g}$. We identify $\mathfrak{h}$ with $\mathfrak{s}$, and write $A := U(\mathfrak{h})/\langle e^{p-1}, f^{p-1}\rangle$. We have that $e^{p-1} = 0 = f^{p-1}$, and hence we can consider V as a module for A. By Corollary 5.1.7 we have that A is semisimple, and hence $\mathfrak{h}$ acts completely reducibly. We can then use Lemma 2.1.31 to deduce that $\mathfrak{h}$ is G-completely reducible.

Hence all that is left to do in this case is to show that $\mathcal{V}$ is maximal with respect to this property. We mirror the techniques used to prove maximality for Theorem 5.0.1.

First, let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{k})$ or $\mathfrak{sl}_n(\mathbb{k})$. Suppose that $\mathfrak{X}$ is a G-stable closed subvariety of $\mathcal{N}$ such that $\mathfrak{X} \not\subseteq \mathcal{N}^{p-1}$. We saw in §5.1.3 that there must exist some $e \in \mathfrak{X}$ with Jordan type $(p, 1^{n-p})$. Thus by Proposition 4.3.1 there exists some non-G-completely reducible $\mathfrak{sl}_2$-subalgebra $\mathfrak{h} = \langle e, h, f\rangle$ with $e, f \in \mathcal{V}$.

Now let $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{k})$. Again, suppose that $\mathfrak{X}$ is a G-stable closed subvariety of $\mathcal{N}$ such that $\mathfrak{X} \not\subseteq \mathcal{N}^{p-1}$. In §5.2 we saw that there must exist some $e \in \mathfrak{X}$ with Jordan type either $(p+1, 1, \dots, 1)$ or $(p, p, 1, \dots, 1)$. In the first case we use Proposition 4.3.2 to show that there exists some non-G-completely reducible $\mathfrak{sl}_2$-subalgebra $\mathfrak{h} = \langle e, h, f\rangle$ with $e, f \in \mathfrak{X}$. We can similarly use Proposition 4.3.1 in the second case.

5.3.2 Orthogonal groups

Let $\mathfrak{g} = \mathfrak{so}_n(\mathbb{k})$, so that $\mathcal{V} = {}^1\mathcal{N}^p$. We first note that it is shown in the proof of Proposition 5.2.5, see (5.2.7), that any $\mathfrak{sl}_2$-subalgebra $\mathfrak{h} = \langle e, h, f\rangle$ of $\mathfrak{g}$ with $e, f \in {}^1\mathcal{N}^p$ acts completely reducibly on the natural module for $\mathfrak{g}$. Hence, by Lemma 2.1.31, $\mathfrak{h}$ is G-completely reducible.

All that is left is to prove that ${}^1\mathcal{N}^p$ is maximal with respect to this property. Suppose that $\mathfrak{X}$ is a G-stable closed subvariety of $\mathcal{N}$ such that $\mathfrak{X} \not\subseteq {}^1\mathcal{N}^p$. Then we demonstrate at the end of §5.2.2 that there must exist some $e \in \mathfrak{X}$ with Jordan type either $(p+2, 1, \dots, 1)$ or $(p, p, 1, \dots, 1)$. We can then use either Proposition 4.3.2

or Proposition 4.3.1 respectively to show that there exists some non-G-completely reducible $\mathfrak{sl}_2$-subalgebra $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathfrak{X}$.

This concludes the proof of Theorem 5.0.2, and therefore the proof of Theorem 3 for G any of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SL}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$, $\mathrm{O}_n(\Bbbk)$ or $\mathrm{SO}_n(\Bbbk)$.

CHAPTER 6

$\mathfrak{sl}_2$-TRIPLES FOR THE EXCEPTIONAL ALGEBRAIC GROUPS

Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 0$, and G a simple algebraic group of type G_2, F_4, E_6, E_7, or E_8, such that p is good for G. Recall that under this restriction we have $p > 3$ for G of type G_2, F_4, E_6 and E_7, and $p > 5$ for G of type E_8. Let $\mathfrak{g} = \operatorname{Lie} G$ and recall that $\mathcal{N}$ denotes the nilpotent cone of $\mathfrak{g}$. As before we write $\mathfrak{s} := \mathfrak{sl}_2(\Bbbk)$. We adopt notation from [Ste16] to express the nilpotent orbits of $\mathfrak{g}$.

In this chapter we prove Theorem 6.0.1, which is enough to prove Theorems 2 and 3 for G a group of exceptional type. In order to define the variety $\mathcal{V} \subseteq \mathcal{N}$ in Theorem 6.0.1 we introduce some shorthand. We say that an element $x \in \mathcal{N}$ satisfies the *A_{p-1}-property* if x is not regular nilpotent in any $\mathfrak{l} = \operatorname{Lie}(L)$ where $L \subseteq G$ is a Levi subgroup and $L' \cong \mathrm{SL}_p(\Bbbk)$ or $\mathrm{PGL}_p(\Bbbk)$.

Theorem 6.0.1. *Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 0$. Let G be a simple algebraic group of exceptional type and suppose that p is* *good for G. Define $\mathcal{V} \subseteq \mathcal{N}$ to be*

$$\mathcal{V} := \left\{ x \in \mathcal{N} \;\middle|\; \begin{array}{l} x^{[p]} = 0, \text{ and} \\ x \text{ satisfies the } A_{p-1}\text{-property.} \end{array} \right\}.$$

Then $\mathcal{V}$ satisfies that

(a) *any $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ with $e, f \in \mathcal{V}$ is G-completely reducible; and*

(b) *the map*

$$\{G\text{-orbits of } \mathfrak{sl}_2\text{-triples } (e, h, f) \text{ with } e, f \in \mathcal{V}\} \longrightarrow \{G\text{-orbits in } \mathcal{V}\}$$

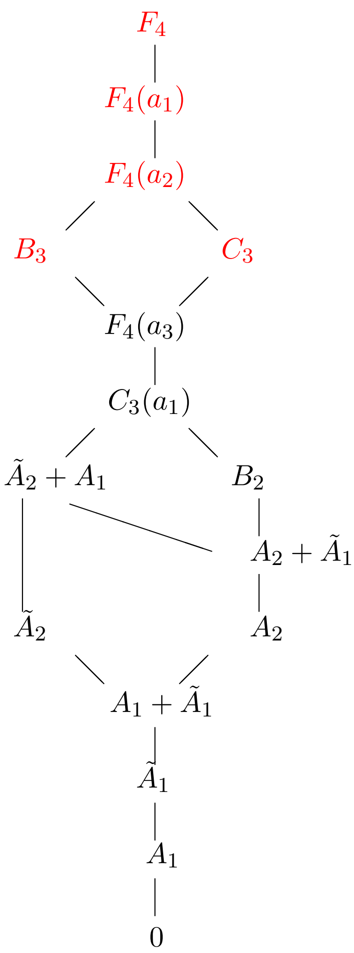
given by sending the G-orbit of an $\mathfrak{sl}_2$-triple (e, h, f) to the G-orbit of e is a bijection.

Moreover, $\mathcal{V}$ is the unique maximal G-stable closed subvariety of $\mathcal{N}$ that satisfies each of these properties.

For G a simple algebraic group of exceptional type, we have checked that the set of nilpotent orbits that satisfy the conditions of $\mathcal{V}$ given in Theorem 6.0.1 is equal to set of nilpotent orbits contained in the variety given in Theorems 2 and 3. This can be seen by comparing the lists of nilpotent orbits that satisfy the conditions for each variety, and noting that the lists are the same. Hence Theorem 6.0.1 is enough to prove Theorems 2 and 3 for the exceptional groups. We refer the reader to §6.6 for a list of nilpotent orbits not contained in $\mathcal{V}$ for each simple algebraic group of exceptional type and prime p which is good for G.

It is not immediately clear that $\mathcal{V}$ is closed. In order to show this we consider the *Hasse diagrams* for the nilpotent orbits of $\mathfrak{g}$. We define a partial ordering on the nilpotent orbits of $\mathfrak{g}$. For X, Y nilpotent orbits of $\mathfrak{g}$ we write $X \preceq Y$ if X is contained in the closure of Y. We then construct the Hasse diagram of this partial ordering. The Hasse diagrams for the unipotent classes of each of the exceptional groups are given in [Car93, §13.4]. We note that these are identical to the Hasse diagrams of the nilpotent classes by the use of Springer isomorphisms, as discussed in §2.4.2.

The Hasse diagram for F_4 is given as an example below. In addition, we take $p = 5$ and show the orbits contained in $\mathcal{V}$ in black, and those not contained in $\mathcal{V}$ in red.



Hasse diagram of nilpotent orbits in F_4

It is clear from the Hasse diagram that $\mathcal{V}$ is a closed subvariety of $\mathcal{N}$ for $p = 5$ and G of type F_4. By considering the Hasse diagram for each group of exceptional type, we can similarly determine that $\mathcal{V}$ is closed for each good prime for G.

We now explain the structure of the proof of Theorem 6.0.1. In §6.1 we show that if $\mathcal{V}$ satisfies the properties given in Theorem 6.0.1 then it is maximal with respect to these properties. We then show that $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property and that any $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ with $e, f \in \mathcal{V}$ is G-completely reducible. To show this we proceed by induction on the semisimple rank of the group. In contrast to the approach taken for the classical cases in Chapter 5, we start by proving that $\mathcal{V}$ as defined in Theorem 6.0.1 satisfies that any $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ such that $e, f \in \mathcal{V}$ is G-completely reducible for each of the groups of exceptional type. This is then used to show that $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property in §6.5. Note that if $h(G)$ is the Coxeter number of G then if $p > h(G)$ all $\mathfrak{sl}_2$-triples are necessarily G-completely reducible by [ST18, Theorem 1.3]. Hence we can assume that $p < h(G)$.

6.1 Showing that $\mathcal{V}$ is maximal

Let $\mathcal{V} \subseteq \mathcal{N}$ be the nilpotent variety as defined in Theorem 6.0.1 and suppose that $\mathcal{V}$ satisfies properties (a) and (b). We now show that $\mathcal{V}$ is maximal with respect to these properties.

We first suppose that $\mathcal{V}$ satisfies property (a) of Theorem 6.0.1.

Lemma 6.1.1. *Suppose that any $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ with $e, f \in \mathcal{V}$ is G-completely reducible. Then $\mathcal{V}$ is maximal with respect to this property.*

Proof. Let $\mathfrak{X}$ be a G-stable closed subvariety of $\mathcal{N}$ such that $\mathfrak{X} \not\subseteq \mathcal{V}$. Then there must exist some $x \in \mathfrak{X}$ such that either $x^{[p]} \neq 0$ or x does not satisfy the A_{p-1}-property. We show that in either of these cases there exists some non-G-completely reducible $\mathfrak{sl}_2(\Bbbk)$-subalgebra $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathfrak{X}$.

First suppose that $x^{[p]} \neq 0$, then it follows immediately from Proposition 4.3.2 that any $\mathfrak{h} \cong \langle e, h, f \rangle$ is not G-completely reducible.

Now suppose that x does not satisfy the A_{p-1}-property, that is x is regular nilpotent in some $\mathfrak{l} = \mathrm{Lie}(L)$ where $L \subseteq G$ is a Levi subgroup and $L' \cong \mathrm{SL}_p(\Bbbk)$ or $\mathrm{PGL}_p(\Bbbk)$. Then by Proposition 4.3.1, there must exist some non-G-completely reducible $\mathfrak{sl}_2$-subalgebra $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathfrak{X}$.

Hence $\mathcal{V}$ is maximal with respect the condition that any $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ with $e, f \in \mathcal{V}$ is G-completely reducible. □

We now suppose that $\mathcal{V}$ satisfies property (b) of Theorem 6.0.1.

Lemma 6.1.2. *Suppose that $\mathcal{V}$ satisfies the $\mathfrak{sl}_2$-property, then it is maximal with respect to satisfying the $\mathfrak{sl}_2$-property.*

Proof. Suppose that $\mathfrak{X}$ is a G-stable closed subvariety of $\mathcal{N}$ such that $\mathfrak{X} \not\subseteq \mathcal{V}$. Then there must exist some $x \in \mathfrak{X}$ such that either $x^{[p]} \neq 0$ or x does not satisfy the A_{p-1}-property.

First suppose that $x^{[p]} \neq 0$, then by Proposition 4.1.6 we have that $\mathfrak{X}$ does not satisfy the $\mathfrak{sl}_2$-property.

Now suppose that x does not satisfy the A_{p-1}-property, that is x is regular nilpotent in some $\mathfrak{l} = \mathrm{Lie}(L)$ where $L \subseteq G$ is a Levi subgroup and $L' \cong \mathrm{SL}_p(\Bbbk)$ or $\mathrm{PGL}_p(\Bbbk)$. Then by Proposition 4.2.1 we have that $\mathfrak{X}$ does not satisfy the $\mathfrak{sl}_2$-property. □

6.2 General strategy to prove Theorem 6.0.1

We prove Theorem 6.0.1 for each of the simple algebraic groups of exceptional type using an induction on the semisimple rank of the Levi subgroups, we follow the same approach for each group of exceptional type, so we first give a summary and justification of the techniques used.

6.2.1 Reducing to a Levi factor

Suppose that we have some $\mathfrak{sl}_2(\Bbbk)$-subalgebra, $\mathfrak{h} = \langle e, h, f \rangle \subseteq \mathfrak{g} = \mathrm{Lie}(G)$ with $e, f \in \mathcal{V}$. Take P to be a parabolic subgroup of G such that $\mathfrak{p} = \mathrm{Lie}(P)$ is a minimal parabolic subalgebra of $\mathfrak{g}$ subject to containing $\mathfrak{h}$. Note that we can assume that $P \neq G$, else $\mathfrak{h}$ is G-irreducible, and hence G-completely reducible.

Consider the Jordan decomposition of $h = h_s + h_n \in \mathfrak{p}$, where h_s is semisimple, h_n is nilpotent and h_s, h_n commute. We choose T to be a maximal torus of P such that $h_s \in \mathfrak{t} = \mathrm{Lie}(T)$, and choose $\mathfrak{l}$ to be a Levi factor of $\mathfrak{p}$ that contains $\mathfrak{t}$. We take $\overline{\mathfrak{h}} = \langle \overline{e}, \overline{h}, \overline{f} \rangle \subseteq \mathfrak{l}$ to be the image of $\mathfrak{h}$ under the projection $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \to \mathfrak{l}$ where we write $\mathfrak{u} = \mathrm{Lie}(U)$ for the unipotent radical U of P. We have that $\overline{\mathfrak{h}}$ cannot be contained in a parabolic $\mathfrak{p}'$ of $\mathfrak{l}$, as $\mathfrak{p}' \oplus \mathfrak{u} \subset \mathfrak{p}$ is a parabolic of $\mathfrak{g}$ which contradicts the minimality of $\mathfrak{p}$. Hence $\overline{\mathfrak{h}}$ is irreducible in $\mathfrak{l}$.

Note that $\overline{e}$ is in the closure of $G \cdot e$, and hence as $\mathcal{V}$ is closed we have that $\overline{e} \in \mathcal{V}$. Consider $\mathcal{V}(\mathfrak{l}) \subseteq \mathcal{N}(\mathfrak{l})$, defined to be the nilpotent subvariety containing all $x \in \mathcal{N}(\mathfrak{l})$ such that $x^{[p]} = 0$, and x is not regular nilpotent in any $l_1 = \mathrm{Lie}(L_1)$ where $L_1 \subseteq L$ is a Levi subgroup of L and $L_1' \cong \mathrm{SL}_p(\Bbbk)$ or $\mathrm{PGL}_p(\Bbbk)$. We claim that $\overline{e} \in \mathcal{V}(\mathfrak{l})$. It follows immediately from $\overline{e} \in \mathcal{V}$ that $\overline{e}^{[p]} = 0$. Suppose that $\overline{e}$

is regular nilpotent in some $\mathfrak{l}_1$ with $L_1 \subseteq L$ is a Levi subgroup and $L_1' \cong \mathrm{SL}_p(\Bbbk)$ or $\mathrm{PGL}_p(\Bbbk)$. Then $L_1 \subseteq G$ a Levi subgroup of G, and hence this contradicts that $\overline{e} \in \mathcal{V}$. The same argument holds for $\overline{f}$, and hence we conclude that $\overline{e}, \overline{f} \in \mathcal{V}(\mathfrak{l})$.

Recall the standard $\mathfrak{sl}_2$-triples as discussed in §4.1. Then we claim that $(\overline{e}, \overline{h}, \overline{f}) \in \mathfrak{l}$ is the standard $\mathfrak{sl}_2$-triple in $\mathfrak{l}$ with $\overline{e} \in \mathfrak{l}$ distinguished; we have that $\overline{\mathfrak{h}}$ is L-completely reducible, and hence it follows that $\overline{\mathfrak{h}}$ is G-completely reducible by Lemma 2.1.32.

We prove this inductively on the semisimple rank of L, that is, the rank of L'. Consider $L' = [L, L]$, and suppose that the rank of L' is equal to n. If $\mathfrak{l}' = \mathrm{Lie}(L')$ is simple, then we use either Theorem 5.0.1 for the groups of classical type, or Theorem 6.0.1 for the lower rank groups of exceptional type to see that $(\overline{e}, \overline{h}, \overline{f})$ is uniquely determined, and hence is the standard $\mathfrak{sl}_2$-triple for $\overline{e}$.

Otherwise, if $\mathfrak{l}$ is not simple, then we consider the simple components of $\mathfrak{l}$, each of rank strictly less than n. We embed $(\overline{e}, \overline{h}, \overline{f})$ into each of the simple components and note that by the inductive hypothesis the $\mathfrak{sl}_2$-triple contained in each component is uniquely defined up to conjugacy. Hence $(\overline{e}, \overline{h}, \overline{f})$ is uniquely determined, and is the standard $\mathfrak{sl}_2$-triple for $\overline{e}$ in $\mathfrak{l}$.

We claim that $\overline{e} \in \mathfrak{l}$ is distinguished, so suppose for contradiction that $\overline{e}$ is not distinguished in $\mathfrak{l}$. Then $\overline{e}$ is contained in a proper Levi subalgebra $\mathfrak{l}_1$ of $\mathfrak{l}$ in which $\overline{e}$ is distinguished. Construct a standard $\mathfrak{sl}_2$-triple in $\mathfrak{l}_1$, which we denote $(\overline{e}, h_1, f_1)$. Recall that the standard $\mathfrak{sl}_2$-triples are uniquely determined up to the choice of maximal torus, and so $(\overline{e}, h_1, f_1)$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and we can assume that $(\overline{e}, \overline{h}, \overline{f}) \in \mathfrak{l}_1$. Therefore $\mathfrak{h} \subseteq \mathfrak{p}_1 = \mathfrak{l}_1 \oplus \mathfrak{u} \subset \mathfrak{p}$, where $\mathfrak{p}_1$ is a parabolic of $\mathfrak{g}$. This contradicts the minimality of $\mathfrak{p}$, and hence $\overline{e} \in \mathfrak{l}$ is distinguished.

Up to conjugacy we can determine $(\overline{e}, \overline{h}, \overline{f})$. Take the Jordan decomposition of $\overline{h}$, written $\overline{h} = \overline{h}_s + \overline{h}_n$. We have that $\overline{h}$ is semisimple, and hence $\overline{h}_n = 0$. We also have $\overline{h}_s = h_s$ because $h_s \in \mathfrak{t} \subseteq \mathfrak{l}$, and so we deduce that $\overline{h} = h_s$.

Recall that as $(\overline{e}, \overline{h}, \overline{f})$ is a standard $\mathfrak{sl}_2$-triple, $\overline{h}$ is defined to be $\mathrm{d}\lambda(1)$ for some associated cocharacter $\lambda : k^* \to T$. We can lift $\overline{h}$ to this associated cocharacter λ, and so the $\overline{\mathfrak{h}}$-module structure on $\mathfrak{u}$ lifts to a compatible $(\lambda(k^*), \overline{\mathfrak{h}})$-module. By this we mean that for $x \in \mathfrak{u}$ we have that if $\lambda(t) \cdot x = t^m x$ then $[\overline{h}, x] = mx$. Note that this is a powerful condition, as we have $\mathfrak{u} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{u}(j; \lambda)$.

6.2.2 Cohomology condition

We continue notation as in §6.2.1. That is, we take $\mathfrak{h} = \langle e, h, f \rangle$ to be an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ with $e, f \in \mathcal{V}$. We choose P to be a parabolic subgroup of G such that $\mathfrak{p} = \mathrm{Lie}(P)$ is a minimal subalgebra of $\mathfrak{g}$ subject to containing $\mathfrak{h}$. We choose T to be a maximal torus of P such that $h_s \in \mathfrak{t} = \mathrm{Lie}(T)$, and take $\mathfrak{l}$ to be a Levi factor of $\mathfrak{p}$ that contains $\mathfrak{t}$. Take $\overline{\mathfrak{h}}$ to be the image of $\mathfrak{h}$ under the projection $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \to \mathfrak{p}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$. We state the following lemmas which give necessary conditions for $\mathfrak{h}$ to be G-completely reducible.

Lemma 6.2.1. *[ST18, Statement (4), pg. 13] Suppose $\mathfrak{h}$ is non-G-completely reducible. There exists an $\overline{\mathfrak{h}}$ composition factor of $\mathfrak{u}$ isomorphic to $L(p-2)$.*

Lemma 6.2.2. *Suppose $\mathfrak{h}$ is non-G-completely reducible. Let T be the maximal torus of P chosen so that $\overline{h} \in \mathrm{Lie}(T)$. Then there exists some set of positive roots $\alpha_1, \dots, \alpha_t$ of G with respect to T such that the submodule generated by e_{α_i} has head isomorphic to $L(p-2)$, the λ-eigenvalues of α_i are equal to $2 - k_i p$ for some $k_i \in \mathbb{Z}$, and $e' = s_1 e_{\alpha_1} + \cdots + s_t e_{\alpha_t}$ for some $s_1, \dots, s_t \in \mathbb{F}_p$.*

Proof. Suppose that we have some non-G-completely reducible $\mathfrak{sl}_2$-subalgebra, $\mathfrak{h} = \langle e, h, f \rangle \subseteq \mathfrak{g} = \mathrm{Lie}(G)$. By Lemma 6.2.1 there exists an $\overline{\mathfrak{h}}$ composition factor of $\mathfrak{u}$ isomorphic to $L(p-2)$, the irreducible $\mathfrak{sl}_2$-module of highest weight $p-2$.

For each positive root α of G the value of $\langle \alpha, \lambda \rangle \in \mathbb{Z}$, the λ-eigenvalue of α, can be deduced from the cocharacters given in [LT11]. The reduction of this modulo p corresponds to the $\mathrm{ad}\,\overline{h}$-eigenvalue of e_α. Note that there exists a weight of a composition factor isomorphic to $L(p-2)$ that is equal to $2-p$.

It follows from the proof of Statement (4) of Stewart–Thomas in [ST18] that e' is formed from sums of e_{α_i} such that α_i is in the $(2-kp)$ λ-eigenspace in $L(p-2)$ as $(\overline{e}+e', \overline{h}+h', \overline{f}+f')$ is non-G-completely reducible. Thus $e' = s_1 e_{\alpha_1} + \cdots + s_t e_{\alpha_t}$ as stated. □

It follows from Lemma 6.2.2 and the analogous result for f' that any non-G-completely reducible $\mathfrak{sl}_2$-triple is of the form $(\overline{e}+e', \overline{h}, \overline{f}+f')$ where:

- $e' = \sum s_j e_{\alpha_j}$ for some α_j in the $(2-kp)$ λ-eigenspace in some $L(p-2)$ composition factor of $\mathfrak{u}$ for some $k \in \mathbb{Z}_{>0}$, and $s_j \in \mathbb{F}_p$;
- $f' = \sum q_j f_{\beta_j}$ for some β_j in the $(kp-2)$ λ-eigenspace in some $L(p-2)$ composition factor of $\mathfrak{u}$ for some $k \in \mathbb{Z}_{>0}$, and $q_j \in \mathbb{F}_p$.

We note that the 0 λ-eigenspace in any $L(p-2)$ composition factor of $\mathfrak{u}$ is 0, so $h' = 0$.

We proceed by showing that there does not exist any such $e', f' \neq 0$ so that $\overline{e}+e', \overline{f}+f' \in \mathcal{V}$ and $\mathfrak{h}$ is non-G-completely reducible.

6.2.3 Summary of techniques used

To now complete the proof of part (a) of Theorem 6.0.1 we show that there do not exist any non-G-completely reducible $\mathfrak{sl}_2$-subalgebras $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$, where $\mathcal{V}$ is the nilpotent variety as defined in Theorem 6.0.1. We describe the method taken for each group of exceptional type below, but we recommend the reader simultaneously consult §6.3 for an example done by hand for G of type G_2.

As described in §6.2 we can find some minimal parabolic $\mathfrak{p}$ containing $\mathfrak{h}$. We consider each parabolic $\mathfrak{p}$ of $\mathfrak{g}$ up to conjugacy, and show that in each case if $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ is such that $\mathfrak{p}$ is the minimal parabolic subject to containing $\mathfrak{h}$ then $\mathfrak{h}$ much be G-completely reducible. As above, we take T to be the torus chosen so that $\overline{h} \in \mathrm{Lie}(T)$.

We consider each $\overline{e}$ which is distinguished in $\mathfrak{l}$ in turn. We then let e be some sum $e = \overline{e} + s_1 e_{\alpha_1} + \cdots + s_t e_{\alpha_t}$, for roots $\alpha_1, \ldots, \alpha_t$ and $s_1, \ldots, s_t \in \mathbb{F}_p$, where e_{α_i} has λ-eigenvalue equal to $2 - k_i p$ for some $k_i \in \mathbb{Z}$. Finally, using one of the following four tools, we show that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible or $e \notin \mathcal{V}$.

Tool number 1: Take the cocharacter λ for $\overline{e}$ as in [LT11]. For each positive root α of G with respect to T in the nilradical of $\mathfrak{p}$, use MAGMA to calculate the λ-eigenvalue of e_α. If there are no α with λ-eigenvalue $(2 - kp)$ for some $k \geq 1$ then we may immediately deduce that $e = \overline{e}$, and $\mathfrak{h}$ is G-completely reducible by Lemma 6.2.2.

Tool number 2: Suppose there exist some positive roots $\alpha_1, \ldots, \alpha_t$ with λ-eigenvalue equal to $(2 - k_i p)$ for some $s_i \in k_i \geq 1$. Consider the nilpotent elements $e := \overline{e} + e'$ where $e' = s_1 e_{\alpha_1} + \cdots + s_t e_{\alpha_t}$ for some $\mathbb{F}_p$. In these cases we calculate the nilpotent orbit of e. If e is not contained in $\mathcal{V}$, then we have a contradiction and can rule out this case.

In order to calculate the nilpotent orbit of e we compute basic properties of e using MAGMA and then apply the following lemma.

Lemma 6.2.3. *The nilpotent orbit of e can be uniquely determined by the following three properties: the dimension of the centraliser of e; the dimension of the centre of the centraliser of e; and the dimensions of terms in the derived series of the nilradical of the centraliser.*

There are a finite number of nilpotent orbits in each of the Lie algebras of the simple algebraic groups of type G_2, F_4, E_6, E_7 and E_8, and hence we can calculate this list of properties for each orbit and determine that this list of invariants is enough to classify each orbit.

We remark that [KST22, Theorem 1.2] yields a similar classification of nilpotent classes of the exceptional algebraic groups. Under our restrictions on p, this result tells us that we can determine the nilpotent orbit of e using information on Jordan block sizes.

If e is contained in $\mathcal{V}$ then there is more to consider. At this point we analyse the module structure of the nilradical to determine for each α_i if there is a factor isomorphic to $L(p-2)$ containing α_i. This analysis is done on a case-by-case basis for each relevant parabolic. If there is no such factor then we deduce by Lemma 6.2.1 that $\mathfrak{h}$ is G-completely reducible. We approach this in one of the following two ways.

Tool number 3: For each α_i, determine the $\mathfrak{u}$-composition factor containing e_{α_i}.

If this is not $L(p-2)$ for each i, then we conclude by Lemma 6.2.1 that $\mathfrak{h}$ is G-completely reducible.

Tool number 4: In the case where $\overline{e}$ and e are in the same nilpotent orbit we show that there exists an element of G that conjugates $\overline{e}$ to e, but commutes with $\overline{h}$ and $\overline{f}$. Recall from §6.2.1 that $\overline{\mathfrak{h}}$ is G-completely reducible, and hence $\mathfrak{h}$ is G-completely reducible.

In order to give such an element of G, we give a certain root α of G, and take $x_\alpha(t) := \exp(t \operatorname{ad} e_\alpha)$. It follows from [Car89, §4.4] that $x_\alpha(t)$ is an element of G, and we then use the formulas given in [Car89, §4.4] to show that $x_\alpha(1)$ conjugates $(\overline{e}, \overline{h}, \overline{f})$ to $(\overline{e}+e', \overline{h}, \overline{f})$.

We give the following lemma to reduce the amount of work needed in each case.

Lemma 6.2.4. *Let $\overline{e} \in \mathcal{N}$ be a distinguished nilpotent element in $\mathfrak{l}$, where $\mathfrak{l} = \operatorname{Lie}(L)$ for some Levi subgroup of G. Take $(\overline{e}, \overline{h}, \overline{f})$ to be a standard $\mathfrak{sl}_2$-triple, where $\overline{h}$ is defined to be $\mathrm{d}\lambda(1)$ for some associated cocharacter $\lambda : k^* \to T$. Let α be a positive simple root of $\mathfrak{g}$, and consider $x_\alpha(1)$. Then*

(a) if $\langle \alpha, \lambda \rangle = 0$, then $x_\alpha(1) \cdot \overline{h} = \overline{h}$; and

(b) if $\overline{f}$ is the sum of negative root vectors in $\mathfrak{l}$, and $e_\alpha \notin \mathfrak{l}$, then $x_\alpha(1) \cdot \overline{f} = \overline{f}$.

Proof. First suppose that $\langle \alpha, \lambda \rangle = 0$, then by the formulas given in [Car89, §4.4] we have

$$x_\alpha(1) \cdot \overline{h} = \overline{h} - [e_\alpha, \overline{h}] = \overline{h}.$$

Next suppose that $\overline{f}$ is the sum of negative root vectors in $\mathfrak{l} \subseteq \mathfrak{g}$. It is clear that no roots can be obtained by adding $\alpha \notin \mathfrak{l}$ to a negative root in $\mathfrak{l}$, and hence $x_\alpha(1)$

commutes with $\overline{f}$. □

We now explain why these four tools are enough to prove that for all (e, h, f) with $e, f \in \mathcal{V}$ we have that $\mathfrak{h} = \langle e, h, f \rangle$ is G-completely reducible. Suppose for a contradiction that there exist (e, h, f) with $e, f \in \mathcal{V}$ so that $\mathfrak{h} = \langle e, h, f \rangle \cong \mathfrak{sl}_2(\Bbbk)$ is non-G-completely reducible. Recall that $\overline{e}$ is distinguished in some proper Levi of $\mathfrak{g}$. In §6.2.2 we explain that (e, h, f) can be written as $(\overline{e} + e', \overline{h}, \overline{f} + f')$ where $(\overline{e}, \overline{h}, \overline{f})$ is a standard $\mathfrak{sl}_2$-triple for $\overline{e}$, and

- $e' = \sum s_j e_{\alpha_j}$ for some α_j in the $(2 - kp)$ λ-eigenspace, e_{α_j} in some $L(p-2)$ composition factor of $\mathfrak{u}$, and $s_j \in \mathbb{F}_p$; and
- $f' = \sum q_j f_{\beta_j}$ for some β_j in the $(kp - 2)$ λ-eigenspace, f_{β_j} in some $L(p-2)$ composition factor of $\mathfrak{u}$, and $q_j \in \mathbb{F}_p$.

Hence we consider all such elements $\overline{e} + e'$ and show using tools number 1-4 that one of the following holds:

(a) no α exist so that α is in the $(2 - kp)$ λ-eigenspace in some $L(p-2)$ composition factor of $\mathfrak{u}$;

(b) $\overline{e} + e' \notin \mathcal{V}$; or

(c) there exists some $g \in G$ such that $g \cdot \overline{e} = \overline{e} + e'$, and g commutes with both $\overline{h}$ and $\overline{f}$.

To conclude we note that in case (a) we have $\overline{e} + e' = \overline{e}$; case (b) cannot occur as we have assumed that $\overline{e} + e' \in \mathcal{V}$; and in case (c) we have that (e, h, f) is conjugate to $(\overline{e}, \overline{h}, \overline{f} + f'')$ where f'' is equal to $g^{-1} \cdot f'$.

Recall that $(\overline{f}+f'', -\overline{h}, \overline{e})$ is also an $\mathfrak{sl}_2$-triple, for which we can use the same tools to conclude that $\overline{f}+f''$ is either $\overline{f}$ or conjugate to $\overline{f}$ without changing $\overline{h}$ or $\overline{e}$.

Hence we conclude that $\overline{\mathfrak{h}} = \langle e, h, f\rangle \cong \mathfrak{sl}_2(\mathbb{k})$ is G-completely reducible when $e, f \in \mathcal{V}$.

6.3 Worked example for G of type G_2

Let G be of type G_2. Recall that $h(G_2) = 6$ and so we are left to consider $p = 5$. Observe from Table 6.3 that $\mathcal{V}$ contains the classes $G_2(a_1), \tilde{A}_1, A_1$.

There are 4 distinct non-conjugate parabolic subalgebras of $\mathfrak{g}$, however we may immediately rule out G itself, and the Borel. Thus we are left with two cases to consider: $\mathfrak{p}_1$ the parabolic subalgebra with distinguished element e_{10}, and $\mathfrak{p}_2$ the parabolic subalgebra with distinguished element e_{01}.

First suppose that $\mathfrak{h} = \langle e, h, f\rangle$ with $e, f \in \mathcal{V}$ is such that $\mathfrak{p}_1$ is the minimal parabolic containing $\mathfrak{h}$. Here we take $\overline{e} = e_{10}$.

We calculate the λ-eigenvalue of e_α for each positive root α of G with respect to T in the nilradical of $\mathfrak{p}_1$. It follows from [LT11, pg 73] that the λ-eigenvalue of $e_{a_1a_2} \in \mathfrak{u}$ is given by $2a_1 - 3a_2$. We output these eigenvalues in Table 6.1.

(a_1, a_2)	$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(3,2)$
λ-eigenvalue	-3	-1	1	3	0

Table 6.1: λ-eigenvalues of $e_{a_1a_2} \in \mathfrak{u}$ for $\mathfrak{p}_1$

We see that e_{01} is the only $e_\alpha \in \mathfrak{u}$ with λ-eigenvalue of the form $2 - kp$ for some $k \in \mathbb{Z}$.

Tool number 2: We calculate the nilpotent orbit of $e = e_{01} + e_{10}$. It is clear that

e is regular in G_2, and hence $e \notin \mathcal{V}$. Thus we deduce that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ and $\overline{e} = e_{10}$ are G-completely reducible.

Now suppose that $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ is such that $\mathfrak{p}_2$ is the minimal parabolic subject to containing $\mathfrak{h}$. Here we take $\overline{e} = e_{01}$.

It follows from [LT11, pg 72] that the λ-eigenvalue of $e_{a_1 a_2} \in \mathfrak{u}$ is given by $a_1 - 2a_2$. We output these eigenvalues in Table 6.2.

(a_1, a_2)	$(1,0)$	$(1,1)$	$(2,1)$	$(3,1)$	$(3,2)$
λ-eigenvalue	1	-1	0	1	-1

Table 6.2: λ-eigenvalues of $e_{a_1 a_2} \in \mathfrak{u}$ for $\mathfrak{p}_2$

Tool number 1: There are no positive roots α with λ-eigenvalue equal to $2 - kp$ for any $k \in \mathbb{Z}$. Hence we deduce that $\mathfrak{h} = \langle e, h, f \rangle$ is G-completely reducible.

Thus we deduce that Theorem 6.0.1(a) holds for G of type G_2.

6.4 Case-by-case analysis

For the remaining algebraic groups of exceptional type computations are completed using MAGMA. Throughout this section we continue using the notation established in §6.2. In each case, we follow the steps detailed in §6.2.3, applying tools number 1 and 2 immediately, and providing a more detailed analysis when we use tool numbers 3 and 4.

6.4.1 G of type F_4

Let G be of type F_4, and note that $h(F_4) = 12$ and so we consider $p \in \{5, 7, 11\}$. We refer the reader to Table 6.5 for a list of classes not contained in $\mathcal{V}$ for each p.

For $p = 11$ tools number 1 and 2 given in §6.2.3 are sufficient to deduce that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible.

Let $p = 7$, following the use of tools number 1 and 2 from §6.2.3 we are left to consider only the parabolic containing distinguished element $\bar{e} = e_{0100}+e_{0010}+e_{0001}$.

The nilpotent element $\bar{e} = e_{0100} + e_{0010} + e_{0001}$ represents the C_3 orbit in F_4. For any root α we have $\langle \alpha, \lambda \rangle$ is given by $-9a_1 + 2a_2 + 2a_3 + 2a_4$ by [LT11, pg. 78]. Hence $\beta = 1110$ satisfies $\langle \beta, \lambda \rangle = -5$.

We now use tool number 3. By further examination of the roots we see that 1000 has eigenvalue -9 with respect to λ, and this is the lowest possible λ-eigenvalue. In characteristic 7 we see that there is no simple module of highest weight 9. We see that β is the only simple root with λ-eigenvalue equal to -5, so we have that e_β is in a $\mathfrak{u}$ composition factor that is isomorphic to $L(2)$. Hence there is no composition factor isomorphic to $L(5)$ and hence there is no non-G-completely reducible $\mathfrak{sl}_2$-triple with nilpotent element $e = \bar{e} + se_{1110}$ for any $s \in \mathbb{F}_p$.

We conclude that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible.

Finally, let $p = 5$, then tools number 1 and 2 given in §6.2.3 are sufficient to deduce that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible in all but two cases.

(a) Let $\bar{e} = e_{1000} + e_{0010} + e_{0001}$ be the representative of the $\tilde{A}_2 + A_1$ orbit in F_4. Then for any root α we have $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 5a_2 + 2a_3 + 2a_4$ by [LT11, pg. 76]. We see that $\beta_1 = 1100$ and $\beta_2 = 0110$ are the only roots in $\mathfrak{u}$ that satisfy $\langle \beta_i, \lambda \rangle = -3$, and so we consider the sums

$$e = \bar{e} + \sum_{i=1}^{2} s_i \beta_i \quad \text{where } s_i \in \mathbb{F}_p.$$

Using calculations in MAGMA and Lemma 6.2.3 we see that the only such sums that are contained in $\mathcal{V}$ are of the form $\overline{e}+se_{1100}-se_{0110}$, each of which is contained in the $\tilde{A}_2+A_1$ orbit.

We proceed with tool number 4. Let $e_2 = e_{0100}$ and set $x_2(t) = \exp(t \operatorname{ad} e_2)$. It follows from the formulas in [Car89, §4.4] that

$$x_2(s) \cdot \overline{e} = \overline{e} + se_{1100} - se_{0110}.$$

By Lemma 6.2.4 we have that $x_2(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(\overline{e} + se_{1100} - se_{0110}, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \left\langle \overline{e} + se_{1100} - se_{0110}, \overline{h}, \overline{f} \right\rangle$ is G-completely reducible.

(b) Take the nilpotent element $\overline{e} = e_{0100} + e_{0001} + e_{0120}$ to be the representative of the $C_3(a_1)$ orbit. Then for any root α we have $\langle \alpha, \lambda \rangle$ is given by $-5a_1 + 2a_2 + 2a_4$ by [LT11, pg. 77]. We note that

$$\beta_1 = 1100, \beta_2 = 1110, \beta_3 = 1120$$

satisfy $\langle \beta_i, \lambda \rangle = -3$. Hence we consider the sums

$$e = \overline{e} + \sum_{i=1}^{3} s_i \beta_i \quad \text{where } s_i \in \mathbb{F}_p.$$

Using calculations in MAGMA and Lemma 6.2.3 we observe that the only such sums that are contained in $\mathcal{V}$ are of the form $\overline{e}+se_{1100}+se_{1120}$, each of these is contained in the $C_3(a_1)$ orbit.

We proceed with tool number 4. Let $e_1 = e_{1000}$ and set $x_1(t) = \exp(t \operatorname{ad} e_1)$.

It follows from the formulas in [Car89, §4.4] that

$$x_1(s) \cdot \overline{e} = \overline{e} + se_{1100} + se_{1120}.$$

By Lemma 6.2.4 we have that $x_1(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(\overline{e} + se_{1100} + se_{1120}, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle \overline{e} + se_{1100} + se_{1120}, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

6.4.2 G of type E_6

Let G be of type E_6. Recall that $h(E_6) = 12$ and so we are left to consider $p \in \{5, 7, 11\}$. We refer the reader to Table 6.5 for a list of classes not contained in $\mathcal{V}$ for each p.

For $p \in \{5, 11\}$, tools number 1 and 2 given in §6.2.3 are sufficient to deduce that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible.

Hence we consider $p = 7$, for which there are 4 remaining cases to consider in more detail following the application of tools number 1 and 2.

(a) First take $\overline{e} = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$ to represent the A_5 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 9a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6$.

Observe that $\beta_1 = {}^{01100}_{1}$ and $\beta_2 = {}^{00110}_{1}$ have λ-eigenvalue -5. Consider the sums $e = \overline{e} + s_1 e_{\beta_1} + s_2 e_{\beta_2}$, for $s_1, s_2 \in \mathbb{F}_p$. Then using calculations in MAGMA and Lemma 6.2.3 we observe that $e \in \mathcal{V}$ if and only if $s_2 = -s_1$, and each element of this form is contained in the A_5 orbit.

We proceed with tool number 4 and consider $e_8 = e_{\substack{00100\\1}}$ and set $x_8(t) = \exp(t \operatorname{ad} e_8)$. It follows from the formulas in [Car89, §4.4] that

$$x_8(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

By Lemma 6.2.4 we have that $x_8(s)$ centralises $\overline{h}$ and $\overline{f}$.

Hence we deduce that $x_8(s)$ conjugates $(\overline{e}, \overline{h}, \overline{f})$ to $(\overline{e} + se_{\beta_1} - se_{\beta_2}, \overline{h}, \overline{f})$ and $\mathfrak{h} = \langle \overline{e} + se_{\beta_1} - se_{\beta_2}, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(b) Now let $\overline{e} = e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$ represent an $A_4 + A_1$ orbit. Then we have $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 - 7a_3 + 2a_4 + 2a_5 + 2a_6$.

Observe that $\beta_1 = {}^{11000}_{\;\;0}$ and $\beta_2 = {}^{01100}_{\;\;0}$ have λ-eigenvalue -5. Consider $e = \overline{e} + s_1 e_{\beta_1} + s_2 e_{\beta_2}$, for $s_1, s_2 \in \mathbb{F}_p$. Then using calculations in MAGMA and Lemma 6.2.3 we observe that $e \in \mathcal{V}$ if and only if $s_2 = -s_1$, in which case e is contained in the $A_4 + A_1$ orbit.

We proceed with tool number 4 and consider $e_3 = e_{\substack{01000\\0}}$ and set $x_3(t) = \exp(t \operatorname{ad} e_3)$. It follows from the formulas in [Car89, §4.4] that

$$x_3(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

By Lemma 6.2.4 we have that $x_3(s)$ centralises $\overline{h}$ and $\overline{f}$.

We deduce that $x_3(s)$ conjugates $(\overline{e}, \overline{h}, \overline{f})$ to $(\overline{e} + se_{\beta_1} - se_{\beta_2}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \langle \overline{e} + se_{\beta_1} - se_{\beta_2}, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(c) Now let $e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$ represent an $A_4 + A_1$ orbit. Then we have $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 - 7a_5 + 2a_6$.

Observe that $\beta_1 = {}^{00110}_{\;\;0}$ and $\beta_2 = {}^{00011}_{\;\;0}$ have λ-eigenvalue -5. Consider

$e = \overline{e} + s_1 e_{\beta_1} + s_2 e_{\beta_2}$, for $s_1, s_2 \in \mathbb{F}_p$. Then $e \in \mathcal{V}$ if and only if $s_2 = -s_1$, and these are each contained in the $A_4 + A_1$ orbit.

We proceed with tool number 4 and consider $e_5 = e_{\substack{00010\\0}}$ and set $x_5(t) = \exp(t \operatorname{ad} e_5)$. It follows from the formulas in [Car89, §4.4] that

$$x_5(s) \cdot \overline{e} = \overline{e} + s e_{\beta_1} - s e_{\beta_2}.$$

By Lemma 6.2.4 we have that $x_5(s)$ centralises $\overline{h}$ and $\overline{f}$.

We deduce that $x_5(s)$ conjugates $(\overline{e}, \overline{h}, \overline{f})$ to $(\overline{e} + s e_{\beta_1} - s e_{\beta_2}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \langle \overline{e} + s e_{\beta_1} - s e_{\beta_2}, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(d) Finally let $e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}}$ be a representative of the $D_5(a_1)$ orbit. Then we have $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_5 - 7a_6$. Observe that $\beta_1 = \substack{00011\\0}$ and $\beta_2 = \substack{00111\\0}$ have λ-eigenvalue -5. Consider $e = \overline{e} + s_1 e_{\beta_1} + s_2 e_{\beta_2}$, for $s_1, s_2 \in \mathbb{F}_p$. Then using calculations in MAGMA and Lemma 6.2.3 we observe that $e \in \mathcal{V}$ if and only if $s_2 = s_1$, in which case e is contained in the $D_5(a_1)$ orbit.

We proceed with tool number 4 and consider $e_6 = e_{\substack{00001\\0}}$ and set $x_6(t) = \exp(t \operatorname{ad} e_6)$. It follows from the formulas in [Car89, §4.4] that

$$x_6(s) \cdot \overline{e} = \overline{e} + s e_{\beta_1} + s e_{\beta_2}.$$

By Lemma 6.2.4 we have that $x_6(s)$ centralises $\overline{h}$ and $\overline{f}$.

We deduce that $x_6(s)$ conjugates $(\overline{e}, \overline{h}, \overline{f})$ to $(\overline{e} + s e_{\beta_1} + s e_{\beta_2}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \langle \overline{e} + s e_{\beta_1} + s e_{\beta_2}, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

6.4.3 G of type E_7

Let G be of type E_7. Recall that $h(E_7) = 18$ and so we are left to consider $p \in \{5, 7, 11, 13, 17\}$. We refer the reader to Table 6.6 for a list of classes not in $\mathcal{V}$ for each p.

Let $p \in \{11, 17\}$, then tools number 1 and 2 given in §6.2.3 are sufficient to deduce that all $\mathfrak{h} = \langle e, h, f\rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible.

Let $p = 13$, then tools number 1 and 2 given in §6.2.3 leave one remaining case for further analysis.

Take $\bar{e} = e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$ to be the representative of the D_6 orbit. We have $\langle \alpha, \lambda\rangle$ is given by $-15a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7$.

Observe that $\beta = {}^{\substack{111000\\0}}$ has λ-eigenvalue -11. We use tool number 3, and consider the module structure of the nilradical to see that ${}^{\substack{100000\\0}}$ has λ-eigenvalue equal to -15, and β is the only root with λ-eigenvalue equal to -11. Therefore e_β is contained in a $\mathfrak{u}$ composition factor that is isomorphic to $L(2)$. Hence there is no composition factor of $\mathfrak{u}$ that is isomorphic to $L(13)$, and so by Lemma 6.2.1 we have $\mathfrak{h}$ is G-completely reducible.

Let $p = 7$. Following the application tools number 1 and 2 given in §6.2.3 we are left to consider the following nine cases.

(a) Take $\bar{e} = e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$ which represents the $A_5 + A_1$ orbit. We have $\langle \alpha, \lambda\rangle$ is given by $2a_1 + 2a_2 - 9a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7$, and hence $\beta = {}^{\substack{111000\\0}}$ yields an eigenvalue of -5.

We use tool number 2 and are left to consider $e = \bar{e} + se_\beta$ for $s \in \mathbb{F}_p$. Using calculations in MAGMA and Lemma 6.2.3 we observe that e is in the A_5+A_1 orbit for each $s \in \mathbb{F}_p$.

We proceed using tool number 4 and consider $e_8 = e_{\substack{110000\\0}}$ and set $x_8(t) = \exp(t \operatorname{ad} e_8)$. It follows from the formulas in [Car89, §4.4] that

$$x_8(s) \cdot \overline{e} = \overline{e} + s e_\beta.$$

By Lemma 6.2.4, $x_8(s)$ centralises both $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(\overline{e} + se_\beta, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle \overline{e} + se_\beta, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(b) Let $\overline{e} = e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$, represent the D_6 orbit, then $\langle \alpha, \lambda \rangle$ is given by $-15a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7$. We see that

$$\beta_1 = \begin{smallmatrix}112100\\1\end{smallmatrix}, \ \beta_2 = \begin{smallmatrix}111110\\1\end{smallmatrix}, \ \beta_3 = \begin{smallmatrix}111111\\0\end{smallmatrix}$$

have λ-eigenvalue equal to -5.

We use tool number 2 and are left to consider $e = \overline{e} + re_{\beta_1} + se_{\beta_2} - (r+s)e_{\beta_3}$ which is in the D_6 orbit for any $r, s \in \mathbb{F}_p$. We use tool number 4 to show that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible. Let $e_{26} = e_{\substack{111100\\1}}$ and $e_{27} = e_{\substack{111110\\0}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$. Then by the formulas in [Car89, §4.4] we have that

$$x_{26}(r) \cdot (x_{27}(r+s) \cdot \overline{e}) = \overline{e} + re_{\beta_1} + se_{\beta_2} - (r+s)e_{\beta_3}.$$

By Lemma 6.2.4 we have that $x_{26}(s)$ and $x_{27}(r+s)$ commute with $\overline{h}$ and $\overline{f}$ and hence we have that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$ and thus $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(c) Let $\overline{e} = e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$ be a representative of the $D_5 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 - 11a_6 + 2a_7$. We see that

$$\beta_1 = {}^{001110}_{1}, \ \beta_2 = {}^{011110}_{0}, \ \beta_3 = {}^{001111}_{0}$$

have λ-eigenvalue equal to -5.

We use tool number 2 and are left to consider $e = \overline{e} + se_{\beta_1} + se_{\beta_2} + re_{\beta_3}$ which is in the $D_5 + A_1$ orbit for $r, s \in \mathbb{F}$. Then we use tool number 4 to show that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible. Let $e_{18} = e_{\substack{001110\\0}}$ and $e_{19} = e_{\substack{000111\\0}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$. Then by the formulas in [Car89, §4.4] we have that

$$x_{18}(s) \cdot (x_{19}(r - s) \cdot \overline{e}) = \overline{e} + se_{\beta_1} + se_{\beta_2} + re_{\beta_3}.$$

By Lemma 6.2.4 we have that $x_{18}(s)$ and $x_{19}(r - s)$ commute with $\overline{h}$ and $\overline{f}$ and hence we have that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$ and thus $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(d) Let $\overline{e} = e_{\substack{100000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$ represent the $A_4 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 4a_2 - 5a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7$. We see that

$$\beta_1 = {}^{010000}_{0}, \ \beta_2 = {}^{111000}_{1}, \ \beta_3 = {}^{011100}_{1}$$

all have λ-eigenvalue equal to -5.

We use tool number 2, and are left to consider $e = \overline{e} + se_{\beta_2} - se_{\beta_3}$ which is in the $A_4 + A_1$ orbit for $s \in \mathbb{F}_p$. Then we use tool number 4 to show that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible. Let $e_{15} = e_{\substack{011000\\1}}$ and set $x_{15}(t) =$

$\exp(t \operatorname{ad} e_{15})$. Then by the formulas in [Car89, §4.4] we have that

$$x_{15}(s) \cdot \overline{e} = \overline{e} + s e_{\beta_2} - s e_{\beta_3}.$$

By Lemma 6.2.4 we have that $x_{15}(s)$ centralises $\overline{h}$ and $\overline{f}$ and hence we have that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$ and thus $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(e) Let $\overline{e} = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}}$ represent an $A_4 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 6a_2 + 2a_3 + 2a_4 + 2a_5 - 5a_6 + 2a_7$. We see that

$$\beta_1 = {\substack{000010\\0}}, \;\; \beta_2 = {\substack{011110\\1}}, \;\; \beta_3 = {\substack{001111\\1}}$$

have λ-eigenvalue equal to -5.

Following the application of tool number 2, we are left to consider $e = \overline{e} + s e_{\beta_2} - s e_{\beta_3}$ which is in the $A_4 + A_1$ orbit for $s \in \mathbb{F}_p$. Then we use tool number 4 to show that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible. Let $e_{21} = e_{\substack{001110\\1}}$ and set $x_{21}(t) = \exp(t \operatorname{ad} e_{21})$. Then by the formulas in [Car89, §4.4] we have that

$$x_{21}(s) \cdot \overline{e} = \overline{e} + s e_{\beta_2} - s e_{\beta_3}.$$

By Lemma 6.2.4 we have that $x_{21}(s)$ centralises $\overline{h}$ and $\overline{f}$ and hence we have that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$ and thus $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(f) Let $\overline{e} = e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$ represent the $(A_5)'$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $-5a_1 - 8a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7$. There are four positive simple roots with λ-eigenvalue equal to -5. Following the use of tool number 2, we consider

$$\beta_1 = \begin{smallmatrix}112100\\1\end{smallmatrix}, \ \beta_2 = \begin{smallmatrix}111110\\1\end{smallmatrix}.$$

We have that $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ is in the $(A_5)'$ orbit for $s \in \mathbb{F}_p$. Then we use tool number 4 to show that $(e, \overline{h}, \overline{f})$ is G-completely reducible. Let $e_{26} = e_{\begin{smallmatrix}111100\\1\end{smallmatrix}}$ and set $x_{26}(t) = \exp(t \operatorname{ad} e_{26})$. Then by the formulas in [Car89, §4.4] we have that

$$x_{26}(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

By Lemma 6.2.4 we have that $x_{26}(s)$ centralises $\overline{h}$ and $\overline{f}$ and hence we have that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$ and thus $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(g) Let $\overline{e} = e_{\begin{smallmatrix}000000\\1\end{smallmatrix}} + e_{\begin{smallmatrix}010000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}001000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}000100\\0\end{smallmatrix}} + e_{\begin{smallmatrix}000001\\0\end{smallmatrix}}$ represent the $D_4 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $-6a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 - 7a_6 + 2a_7$. There are four positive simple roots with λ-eigenvalue equal to -5, these are given by

$$\beta_1 = \begin{smallmatrix}000110\\0\end{smallmatrix}, \ \beta_2 = \begin{smallmatrix}000011\\0\end{smallmatrix}, \ \beta_3 = \begin{smallmatrix}111110\\1\end{smallmatrix}, \ \beta_4 = \begin{smallmatrix}111111\\0\end{smallmatrix}.$$

Following the application of tool number 2, we are left to consider $e = \overline{e} + se_{\beta_1} - se_{\beta_2} + re_{\beta_3} - re_{\beta_4}$ which is in the $D_4 + A_1$ orbit for $r, s \in \mathbb{F}_p$. Then we use tool number 4 to show that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible. Let $e_6 = e_{\begin{smallmatrix}000010\\0\end{smallmatrix}}$ and $e_{27} = e_{\begin{smallmatrix}111110\\0\end{smallmatrix}}$ set $x_i(t) = \exp(t \operatorname{ad} e_i)$. Then by the formulas in [Car89, §4.4] we have that

$$x_6(s) \cdot (x_{27}(r) \cdot \overline{e}) = \overline{e} + se_{\beta_1} - se_{\beta_2} + re_{\beta_3} - re_{\beta_4}.$$

By Lemma 6.2.4 we have that $x_6(s)$ and $x_{27}(r)$ commute with $\overline{h}$ and $\overline{f}$ and hence we have that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$ and thus $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(h) Take $\overline{e} = e_{\substack{100000\\0}} + e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}}$ to be the representative of the $D_5(a_1)$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_5 - 7a_6$, and there are four simple roots with a λ-eigenvalue of -5, namely

$$\beta_1 = {\substack{000110\\0}}, \beta_2 = {\substack{001110\\0}}, \beta_3 = {\substack{000111\\0}},\ \beta_4 = {\substack{001111\\0}}.$$

Following the application of tool number 2, we consider $e = \overline{e} + se_{\beta_1} + se_{\beta_2} + re_{\beta_3} + re_{\beta_4}$ which is in the $D_5(a_1)$ orbit for any $r, s \in \mathbb{F}_p$. We show that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$.

Consider $e_6 = e_{\substack{000010\\0}}$ and $e_{13} = e_{\substack{000011\\0}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

By the formulas in [Car89, §4.4] we have

$$x_6(s) \cdot (x_{13}(r) \cdot \overline{e}) = \overline{e} + se_{\beta_1} + se_{\beta_2} + re_{\beta_3} + re_{\beta_4}.$$

By Lemma 6.2.4 we have that $x_6(s)$ and $x_{13}(r)$ commute with $\overline{h}$ and $\overline{f}$.

Hence we have shown that in each case $(e, \overline{h}, \overline{f})$ is conjugate to the standard $\mathfrak{sl}_2$-triple and hence $\mathfrak{h}$ is G-completely reducible.

(i) Take $\overline{e} = e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{000001\\0}} + e_{\substack{001000\\1}} - e_{\substack{011000\\0}} + e_{\substack{001100\\0}} + e_{\substack{000110\\0}}$ to be a representative of the orbit $D_6(a_2)$. Then $\langle \alpha, \lambda \rangle$ is given by $-9a_1 + 2a_2 + 2a_3 + 2_5 + 2a_7$, and there are three simple roots α with -5 eigenvalue, these are

$$\beta_1 = {\substack{111000\\1}},\ \beta_2 = {\substack{111100\\0}},\ \beta_3 = {\substack{111110\\0}}$$

Following the application of tool number 2, we consider elements of the form $e = \overline{e} + (s-r)e_{\beta_1} + se_{\beta_2} + re_{\beta_3}$ which are in the $D_6(a_2)$ orbit for $r, s \in \mathbb{F}_p$, and we use tool number 4 to see that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$.

Consider $e_8 = e_{\substack{110000\\0}}$ and $e_{14} = e_{\substack{111000\\0}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

Then by the formulas in [Car89, §4.4] we have that

$$x_8(s) \cdot (x_{16}(r) \cdot \overline{e}) = \overline{e} + (s-r)e_{\beta_1} + se_{\beta_2} + re_{\beta_3}.$$

It follows from Lemma 6.2.4 that both $x_8(s)$ and $x_{16}(r)$ commute with $\overline{h}$ and $\overline{f}$. Thus we have completed the required steps for tool number 4 and see that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

Finally let $p = 5$, then tools number 1 and 2 given in §6.2.3 are sufficient to deduce that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible in all but two cases.

(a) Take $\overline{e} = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}} + e_{\substack{000001\\0}}$ to represent the $A_3 + A_2$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_3 - 5a_4 + 2a_5 + 2a_6 + 2a_7$. There are four simple roots α with -3 eigenvalue, these are

$$\beta_1 = {\substack{011000\\0}}, \ \beta_2 = {\substack{001100\\0}}, \ \beta_3 = {\substack{011100\\1}}, \ \beta_4 = {\substack{001100\\1}}.$$

Following the application of tool number 2, we consider the elements of the form $e = \overline{e} + re_{\beta_1} - re_{\beta_2} + se_{\beta_3} - se_{\beta_4}$ which are in the $A_3 + A_2$ orbit for $r, s \in \mathbb{F}_p$. We use tool number 4 to see that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$.

Consider $e_4 = e_{\substack{001000\\0}}$ and $e_9 = e_{\substack{001000\\1}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

Then by the formulas in [Car89, §4.4] we have that

$$x_4(r) \cdot (x_9(s) \cdot \overline{e}) = \overline{e} + re_{\beta_1} - re_{\beta_2} + se_{\beta_3} - se_{\beta_4}.$$

It follows from Lemma 6.2.4 that both $x_4(r)$ and $x_9(s)$ commute with $\overline{h}$ and $\overline{f}$. Thus we have completed the required steps for tool number 4 and see that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(b) Take $\overline{e} = e_{\substack{000000\\1}} + e_{\substack{010000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000001\\0}} + e_{\substack{001000\\1}} + e_{\substack{001100\\0}}$ to represent the $D_4(a_1)+A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $-4a_1+2a_2+2a_3+2a_5-5a_6+2a_7$. There are six simple roots α with -3 eigenvalue, these are

$$\beta_1 = \substack{000110\\0}, \; \beta_2 = \substack{000011\\0}, \; \beta_3 = \substack{001110\\0},$$

$$\beta_4 = \substack{111110\\1}, \; \beta_5 = \substack{111111\\0}, \; \beta_6 = \substack{112110\\1}.$$

Following the application of tool number 2 we consider the elements of the form $e = \overline{e}+re_{\beta_1}-re_{\beta_2}+re_{\beta_3}+se_{\beta_4}-se_{\beta_5}-se_{\beta_6}$ which are in the $D_4(a_1)+A_1$ orbit for $r, s \in \mathbb{F}_p$, and we use tool number 4 to see that $(\overline{e}, \overline{h}, \overline{f})$ is conjugate to $(e, \overline{h}, \overline{f})$.

Consider $e_6 = e_{\substack{000010\\0}}$ and $e_{27} = e_{\substack{111110\\0}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

Then by the formulas in [Car89, §4.4] we have that

$$x_6(r) \cdot (x_{27}(s) \cdot \overline{e}) = \overline{e} + re_{\beta_1} - re_{\beta_2} + re_{\beta_3} + se_{\beta_4} - se_{\beta_5} - se_{\beta_6}.$$

It follows from Lemma 6.2.4 that both $x_6(r)$ and $x_{27}(s)$ commute with $\overline{h}$ and $\overline{f}$. Thus we have completed the required steps for tool number 4 and see that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

6.4.4 G of type E_8

Let G be of type E_8. Recall that $h(E_8) = 30$ and so we are left to consider $p \in \{7, 11, 13, 17, 19, 23, 29\}$. Note we no longer have to consider $p = 5$, as 5 is not a good prime for E_8. We refer the reader to Table 6.7 for a list of classes not contained in $\mathcal{V}$ for each p.

Let $p = 29$, then tools number 1 and 2 given in §6.2.3 are sufficient to deduce that all $\mathfrak{h} = \langle e, h, f \rangle$ with $e, f \in \mathcal{V}$ are G-completely reducible.

Next let $p = 23$. The application of tools number 1 and 2 given in §6.2.3 leave one remaining case for further analysis, which we approach with tool number 3.

Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$ to represent the E_7 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 - 27a_8$. Hence $\beta = {\substack{0001111\\0}}$ has an λ-eigenvalue of -21. Observe that ${\substack{0000001\\0}}$ has λ-eigenvalue equal to -27. We have that β is the only root with λ-eigenvalue equal to -21 and hence we determine that e_β is contained in an $L(4)$ composition factor of $\mathfrak{u}$ and $\overline{e} + se_\beta$ cannot be part of a non-G-completely reducible $\mathfrak{sl}_2$-triple for any $s \in \mathbb{F}_p$.

We next consider $p = 19$, then following the application of tools number 1 and 2 there are three remaining cases to look at.

(a) First take $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ to represent the D_7 orbit, then $\langle \alpha, \lambda \rangle$ is given by $-21a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. Hence $\beta = {\substack{1110000\\0}}$ has λ-eigenvalue equal to -17.

We use tool number 3 here, and observe that $\substack{1000000\\0}$ has λ-eigenvalue -21. Given that β is the only root with λ-eigenvalue equal to -17, the composition factor of $\mathfrak{u}$ containing e_β is isomorphic to $L(2)$.

Hence there is no composition factor isomorphic to $L(17)$, and so by Lemma 6.2.1 we have $\mathfrak{h}$ is G-completely reducible.

(b) Now let $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$ represent the $E_7(a_1)$ orbit, then we have $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + 2a_7 - 21a_8$. Then $\beta = \substack{0000111\\0}$ is the only root with λ-eigenvalue -17.

We use tool number 3 and see that $\substack{0000001\\0}$ has eigenvalue -21, and so the composition factor of $\mathfrak{u}$ containing e_β is isomorphic to $L(2)$.

Hence there is no composition factor isomorphic to $L(17)$, and so by Lemma 6.2.1 we have $\mathfrak{h}$ is G-completely reducible.

(c) Now take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$ to be the representative of the D_7 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 - 27a_8$.

We see that $\beta_1 = \substack{0011111\\1}$ and $\beta_2 = \substack{0111111\\0}$ have λ-eigenvalue -17.

Following the application of tool number 2, we are left to consider $e = \overline{e} + se_{\beta_1} + se_{\beta_2}$ for $s \in \mathbb{F}_p$, then e is in the D_7 nilpotent orbit. Consider $e_{36} = e_{\substack{0011111\\0}}$ and set $x_{36}(t) = \exp(t \operatorname{ad} e_{36})$. Using [Car89, §4.4] we have that

$$x_{36}(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} + se_{\beta_2}.$$

Using Lemma 6.2.4 we note that $x_{22}(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that

$(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

Let $p = 17$, then following the application of tools number 1 and 2, there are four remaining cases to consider.

(a) Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$ to be a representative of the $E_7(a_1)$ orbit, then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + 2a_7 - 21a_8$.

We see that $\beta = {}^{0001111}_{\;\;\;0}$ has λ-eigenvalue -15.

Following the application of tool number 2, we are left to consider $e = \overline{e} + se_\beta$ which is in the $E_7(a_1)$ orbit for all $s \in \mathbb{F}_p$. Consider $e_{22} = e_{\substack{0000111\\0}}$ and set $x_{22}(t) = \exp(t \operatorname{ad} e_{22})$. By the formulas in [Car89, §4.4] we have that

$$x_{22}(s) \cdot \overline{e} = \overline{e} + se_\beta.$$

By Lemma 6.2.4 we have that $x_{22}(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(b) Next suppose $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0000110\\0}}$, a representative of the $E_7(a_2)$ orbit. In this case we find that $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_7 - 17a_8$. There are two roots with λ-eigenvalue equal to -15. Following the application of tool number 2 we are left to consider $\beta = {}^{0000111}_{\;\;\;0}$ for which $e = \overline{e} + se_\beta$ is in the $E_7(a_2)$ nilpotent orbit for all $s \in \mathbb{F}_p$. We consider $e_8 = e_{\substack{0000001\\0}}$ and set $x_8(t) = \exp(t \operatorname{ad} e_8)$.

Using the formulas in [Car89, §4.4] we have that

$$x_8(s) \cdot \overline{e} = \overline{e} + se_\beta.$$

Using Lemma 6.2.4 we observe that $x_8(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(c) Let $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$, a representative of the D_7 orbit. In this case we find that $\langle \alpha, \lambda \rangle$ is given by $-21a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. Thus we see $\beta_1 = {}^{1110000}_{\quad 1}$ and $\beta_2 = {}^{1111000}_{\quad 0}$ have λ-eigenvalue equal to -15.

Then following the application of tool number 2 we are left to consider $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ which are in the D_7 nilpotent orbit for all $s \in \mathbb{F}_p$. We consider $e_{16} = e_{\substack{1110000\\0}}$ and set $x_{16}(t) = \exp(t \operatorname{ad} e_{16})$.

Using the formulas in [Car89, §4.4] we have that

$$x_{16}(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

Using Lemma 6.2.4 we observe that $x_{16}(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(d) Let $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$, be a representative of the $E_6 + A_1$ orbit. In this case we find that $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 - 17a_7 + 2a_8$. Thus we see $\beta_1 = {}^{0000110}_{\quad 0}$ and $\beta_2 = {}^{0000011}_{\quad 0}$ have λ-eigenvalue equal to -15.

Then following the application of tool number 2 we are left to consider $e = \bar{e} + se_{\beta_1} - se_{\beta_2}$ which is in the $E_6 + A_1$ nilpotent orbit for all $s \in \mathbb{F}_p$. We consider $e_7 = e_{\substack{0000010\\0}}$ and set $x_7(t) = \exp(t \operatorname{ad} e_7)$.

Using the formulas in [Car89, §4.4] we have that

$$x_7(s) \cdot \bar{e} = \bar{e} + se_{\beta_1} - se_{\beta_2}.$$

Using Lemma 6.2.4 we observe that $x_7(s)$ centralises $\bar{h}$ and $\bar{f}$.

Thus we have completed tool number 4, and see that $(\bar{e}, \bar{h}, \bar{f})$ and $(e, \bar{h}, \bar{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \bar{h}, \bar{f} \rangle$ is G-completely reducible.

Let $p = 13$, we apply tools number 1 and 2, and are left with the following seven cases to consider.

(a) First take $\bar{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$ to represent the $E_6 + A_1$ orbit, then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 - 17a_7 + 2a_8$. Thus we see $\beta_1 = {\substack{0011110\\0}}$ and $\beta_2 = {\substack{0001111\\0}}$ have λ-eigenvalue -11. We use tool number 3 and consider the module structure of the nilradical and see that since this contains a root with λ-eigenvalue -17, and two with λ-eigenvalue -15, hence there must be factors isomorphic to $L(2)$ and $L(4)$ in the composition series of $\mathfrak{u}$ in characteristic 13. These two factors account for both e_{β_1} and e_{β_2}. Hence there is no composition factor isomorphic to $L(11)$, and so by Lemma 6.2.1 we have $\mathfrak{h}$ is G-completely reducible.

(b) Next, take $\bar{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$ to represent the D_6 orbit, in this case we have that $\langle \alpha, \lambda \rangle$ is given by $-15a_1 +$

$2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 - 10a_8$. Thus we see $\beta_1 = \begin{smallmatrix}1110000\\0\end{smallmatrix}$ and $\beta_2 = \begin{smallmatrix}1121111\\1\end{smallmatrix}$ have eigenvalue -11.

We use tool number 3 and observe that $\begin{smallmatrix}1000000\\0\end{smallmatrix}$ and $\begin{smallmatrix}1111111\\0\end{smallmatrix}$ have λ-eigenvalue -15, and this is the lowest possible eigenvalue. Hence, given that β_1 and β_2 are the only roots with λ-eigenvalue equal to -11, the composition factor of $\mathfrak{u}$ containing each e_{β_i} is isomorphic to $L(2)$.

Hence there is no composition factor isomorphic to $L(17)$, and so by Lemma 6.2.1 we have $\mathfrak{h}$ is G-completely reducible.

(c) Consider $\bar{e} = e_{\begin{smallmatrix}1000000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0000000\\1\end{smallmatrix}} + e_{\begin{smallmatrix}0100000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0010000\\1\end{smallmatrix}} + e_{\begin{smallmatrix}0011000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0001100\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0000110\\0\end{smallmatrix}}$, then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_7 - 17a_8$. Thus we see $\beta_1 = \begin{smallmatrix}0011111\\1\end{smallmatrix}$ and $\beta_2 = \begin{smallmatrix}0111111\\0\end{smallmatrix}$ have λ-eigenvalue -11.

We use tool number 3 and consider the module structure of the nilradical, we see that $\begin{smallmatrix}0000001\\0\end{smallmatrix}$ has λ-eigenvalue equal to -17, and this is the minimal value, $\begin{smallmatrix}0000011\\0\end{smallmatrix}$ and $\begin{smallmatrix}0000111\\0\end{smallmatrix}$ have λ-eigenvalue equal to -15 and β_1, β_2 are the only roots with λ-eigenvalue equal to -11. Hence each of e_{β_1}, e_{β_2} must be part of either an $L(2)$ or an $L(4)$ composition factor, hence neither is contained in a factor isomorphic to $L(11)$. Therefore there is no $\mathfrak{u}$ composition factor isomorphic to $L(11)$, and so by Lemma 6.2.1 we have $\mathfrak{h}$ is G-completely reducible.

(d) Take $\bar{e} = e_{\begin{smallmatrix}1000000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0010000\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0000100\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0000010\\0\end{smallmatrix}} + e_{\begin{smallmatrix}0110000\\1\end{smallmatrix}} + e_{\begin{smallmatrix}0011000\\1\end{smallmatrix}} + e_{\begin{smallmatrix}0111000\\0\end{smallmatrix}}$ to be a representative of the $E_7(a_2)$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_4 + 2a_6 + 2a_7 - 15a_8$. There are two roots with λ-eigenvalue equal to -11, following the application of tool number 2, we consider $\beta = \begin{smallmatrix}0000111\\0\end{smallmatrix}$ for which $e = \bar{e} + se_\beta$ is in the $E_7(a_2)$ nilpotent orbit for each $s \in \mathbb{F}_p$. Consider $e_{13} = e_{\begin{smallmatrix}0000011\\0\end{smallmatrix}}$ and set

$x_{13}(t) = \exp(t \operatorname{ad} e_{13})$. By the formulas given in [Car89, §4.4] we have that

$$x_{13}(s) \cdot \overline{e} = \overline{e} + s e_{\beta}.$$

It follows from Lemma 6.2.4 that $x_{13}(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(e) Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the A_7 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 15a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. Then $\beta_1 = \substack{0110000\\1}$ and $\beta_2 = \substack{0011000\\1}$ have eigenvalue equal to -11.

Following the application of tool number 2, we are left to consider $e = \overline{e} + s e_{\beta_1} - s e_{\beta_2}$ which are in the A_7 nilpotent orbit for each $s \in \mathbb{F}_p$. Consider $e_9 = e_{\substack{0010000\\1}}$ and set $x_9(t) = \exp(t \operatorname{ad} e_9)$. By the formulas given in [Car89, §4.4] we have that

$$x_9(s) \cdot \overline{e} = \overline{e} + s e_{\beta_1} - s e_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_9(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(f) Take $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the D_7 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $-21a_1 + 2a_2 +$

$2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. Then

$$\beta_1 = \substack{1121000\\1},\ \beta_2 = \substack{1111100\\1},\ \beta_3 = \substack{1111110\\0}$$

have eigenvalue equal to -11.

Following the application of tool number 2, we are left to consider $e = \overline{e} + re_{\beta_1} - (r+s)e_{\beta_2} + se_{\beta_3}$ which are in the D_7 nilpotent orbit for each $r, s \in \mathbb{F}_p$. Consider $e_{30} = e_{\substack{1111000\\1}}$ and $e_{31} = e_{\substack{1111100\\0}}$ and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{30, 31\}$. By the formulas given in [Car89, §4.4] we have that

$$x_{30}(r) \cdot (x_{31}(s) \cdot \overline{e}) = \overline{e} + re_{\beta_1} - (r+s)e_{\beta_2} + se_{\beta_3}.$$

It follows from Lemma 6.2.4 that $x_{30}(r)$ and $x_{31}(s)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(g) Let $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} - e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}}$ represent the $D_7(a_2)$ orbit, then $\langle \alpha, \lambda \rangle$ is given by $-13a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_7 + 2a_8$.

The positive simple roots with λ-eigenvalue equal to -11 are $\beta_1 = \substack{1100000\\0}$, $\beta_2 = \substack{1110000\\0}$. Following the application of tool number 2 we are left to consider $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ which is contained in the $D_7(a_2)$ orbit for each $s \in \mathbb{F}_p$.

Consider $e_1 = e_{\substack{1000000\\0}}$. Then set $x_1(t) = \exp(t \operatorname{ad} e_1)$.

Then using the formulas in [Car89, §4.4] we have that

$$x_1(s) \cdot \bar{e} = \bar{e} + s e_{\beta_1} - s e_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_1(s)$ centralises $\bar{h}$ and $\bar{f}$ for all $s \in \mathbb{F}_p$.

Thus we have completed the steps required for tool number 4 and we have shown that $(e, \bar{h}, \bar{f})$ is conjugate to $(\bar{e}, \bar{h}, \bar{f})$ and hence $\mathfrak{h} = \langle e, \bar{h}, \bar{f} \rangle$ is G-completely reducible.

Next consider $p = 11$, following the application of tools number 1 and 2 we consider the following eleven cases.

(a) We first take $\bar{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$, the representative for the D_6 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $-15a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 - 10a_8$ and $\beta_1 = {}^{1221111}_{\;\;\;1}$ and $\beta_2 = {}^{1122111}_{\;\;\;1}$ have λ-eigenvalue -9.

Following the application of tool number 2, we are left to consider $e = \bar{e} + s e_{\beta_1} + s e_{\beta_2}$ is in the D_6 nilpotent orbit for each $s \in \mathbb{F}_p$. Consider $e_{60} = e_{\substack{1121111\\1}}$ and set $x_{60}(t) = \exp(t \operatorname{ad} e_{60})$. Using [Car89, §4.4] we have that

$$x_{60}(s) \cdot \bar{e} = \bar{e} + s e_{\beta_1} + s e_{\beta_2}.$$

By Lemma 6.2.4 we see that $x_{60}(s)$ centralises $\bar{h}$ and $\bar{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\bar{e}, \bar{h}, \bar{f})$ and $(e, \bar{h}, \bar{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \bar{h}, \bar{f} \rangle$ is G-completely reducible.

(b) Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the $A_6 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_+2a_2 - 11a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. Then $\beta_1 = {}^{1100000}_{\ \ 0}$ and $\beta_2 = {}^{0110000}_{\ \ 0}$ have λ-eigenvalue equal to -9.

Following the application of tool number 2, we are left to consider $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ which is in the $A_6 + A_1$ nilpotent orbit for each $s \in \mathbb{F}_p$. Consider $e_3 = e_{\substack{0100000\\0}}$ and set $x_3(t) = \exp(t \operatorname{ad} e_3)$. By the formulas given in [Car89, §4.4] we have that

$$x_3(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_3(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(c) Take $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the $D_5 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $-10a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 - 9a_7 + 2a_8$. There are three positive simple roots with λ-eigenvalue equal to -9, however following the application of tool number 2 we are left to consider

$$\beta_1 = {}^{1111110}_{\ \ 1}, \ \beta_2 = {}^{1111111}_{\ \ 0}$$

for which $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ is in the $D_5 + A_1$ nilpotent orbit for each $s \in \mathbb{F}_p$. Consider $e_{39} = e_{\substack{1111110\\0}}$ and set $x_{39}(t) = \exp(t \operatorname{ad} e_{39})$. By the formulas given in [Car89, §4.4] we have that

$$x_{39}(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_{39}(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(d) Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the A_7 orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 15a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. There are three positive simple roots with λ-eigenvalue equal to -9, these are given by

$$\beta_1 = {}^{1110000}_{\quad 1}, \; \beta_2 = {}^{0111000}_{\quad 1}, \; \beta_3 = {}^{0011100}_{\quad 1}.$$

Following the application of tool number 2, we are left to consider $e = \overline{e} + re_{\beta_1} - (r+s)e_{\beta_2} + se_{\beta_3}$ which is in the A_7 nilpotent orbit for each $r, s \in \mathbb{F}_p$. Consider $e_{17} = e_{\substack{0110000\\1}}$ and $e_{18} = e_{\substack{0011000\\1}}$ set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{17, 18\}$. By the formulas given in [Car89, §4.4] we have that

$$x_{17}(r) \cdot (x_{18}(s) \cdot \overline{e}) = \overline{e} + re_{\beta_1} - (r+s)e_{\beta_2} + se_{\beta_3}.$$

It follows from Lemma 6.2.4 that $x_{17}(r)$ and $x_{18}(s)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(e) Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the $D_5 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 - 10a_6 - a_7 + 2a_8$. There are three positive simple roots with λ-eigenvalue equal to -9, however following the application of tool number 2 we are left to consider

$$\beta_1 = {\begin{smallmatrix}0001110\\0\end{smallmatrix}}, \ \beta_2 = {\begin{smallmatrix}0000111\\0\end{smallmatrix}}$$

for which $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ is in the $D_5 + A_1$ nilpotent orbit for each $s \in \mathbb{F}_p$. Consider $e_{14} = e_{\substack{0000110\\0}}$ and set $x_{14}(t) = \exp(t \operatorname{ad} e_{14})$. By the formulas given in [Car89, §4.4] we have that

$$x_{14}(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_{14}(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(f) Take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000001\\0}}$ to be a representative of the $D_5 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 - 11a_6 + 2a_7 - a_8$. There are three positive simple roots with λ-eigenvalue equal to -9, however following the application of tool number 2 we are left to consider

$$\beta_1 = {\begin{smallmatrix}0001100\\0\end{smallmatrix}}, \ \beta_2 = {\begin{smallmatrix}0000110\\0\end{smallmatrix}}$$

for which $e = \overline{e} + se_{\beta_1} - se_{\beta_2}$ is in the $D_5 + A_1$ nilpotent orbit for each $s \in \mathbb{F}_p$.

Consider $e_6 = e_{\substack{0000100\\0}}$ and set $x_6(t) = \exp(t \operatorname{ad} e_6)$. By the formulas given in [Car89, §4.4] we have that

$$x_6(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} - se_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_6(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(g) Now take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0001100\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}} + e_{\substack{0011100\\1}}$ to be the representative for the $E_7(a_4)$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_4 + 2a_7 - 11a_8$.

There are three simple roots with λ-eigenvalue equal to -9 however following the application of tool number 2 we are left to consider $\beta_1 = {\substack{0000011\\0}}$ and $\beta_2 = {\substack{0000111\\0}}$ for which $e = \overline{e} + se_{\beta_1} + se_{\beta_2}$ is in the $E_7(a_4)$ nilpotent orbit for each $s \in \mathbb{F}_p$. We consider $e_8 = e_{\substack{0000001\\0}}$ and set $x_8(t) = \exp(t \operatorname{ad} e_8)$.

Using [Car89, §4.4] we have that

$$x_8(s) \cdot \overline{e} = \overline{e} + se_{\beta_1} + se_{\beta_2}.$$

It follows from Lemma 6.2.4 that $x_8(s)$ centralises $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(h) Let $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0110000\\0}} +$

$e_{\substack{0011000\\0}}$ be a representative of the $E_6(a_1)+A_1$ orbit, then $\langle\alpha,\lambda\rangle$ is given by $2a_1+2a_2+2a_3+2a_5+2a_6-13a_7+2a_8$. Observe that $\beta_1 = {}^{0001110}_{\;\;\;0}$, $\beta_2 = {}^{0000111}_{\;\;\;0}$ and $\beta_3 = {}^{0011110}_{\;\;\;0}$ have λ-eigenvalue equal to -9.

Following the application of tool number 2, we are left to consider $e = \overline{e} + se_{\beta_1} + re_{\beta_2} + se_{\beta_3}$ which is in the $E_6(a_1)+A_1$ orbit for all $r, s \in \mathbb{F}_p$.

Let $e_{14} = e_{\substack{0000110\\0}}$ and $e_{15} = e_{\substack{0000011\\0}}$. Then for $i \in \{14, 15\}$ set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

Using the formulas in [Car89, §4.4] we see that

$$x_{15}(r) \cdot (x_{14}^s(s) \cdot \overline{e}) = \overline{e} + se_{\beta_1} + re_{\beta_2} + se_{\beta_3}.$$

By Lemma 6.2.4 we have that $x_{15}(r)$ and $x_{14}(s)$ centralises $\overline{h}$ and $\overline{f}$ for all $r, s \in \mathbb{F}_p$.

We have completed the steps for tool number 4 and can conclude that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f}\rangle$ is G-completely reducible.

(i) Let $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}}$ represent the $E_7(a_3)$ orbit, then $\langle\alpha,\lambda\rangle$ is given by $2a_1+2a_4+2a_6+2a_7-15a_8$. Observe that

$$\beta_1 = {}^{0011111}_{\;\;\;0},\ \beta_2 = {}^{0011111}_{\;\;\;1},\ \beta_3 = {}^{0111111}_{\;\;\;0},\ \beta_4 = {}^{0111111}_{\;\;\;1}$$

have λ-eigenvalue equal to -9.

Following the application of tool number 2, we are left to consider $e = \overline{e} + se_{\beta_1} + re_{\beta_2} + re_{\beta_3} + se_{\beta_4}$ which is in the $E_7(a_3)$ orbit for each $r, s \in \mathbb{F}_p$.

Let $e_{22} = e_{\substack{0000111\\0}}$ and $e_{29} = e_{\substack{0001111\\0}}$. Then for $i \in \{22, 29\}$ set $x_i(t) =$

$\exp(t\,\mathrm{ad}\,e_i)$.

Using the formulas in [Car89, §4.4] we see that

$$x_{29}(s)\cdot(x_{22}(r)\cdot\overline{e}) = \overline{e} + se_{\beta_1} + re_{\beta_2} + re_{\beta_3} + se_{\beta_4}.$$

By Lemma 6.2.4 we have that $x_{29}(s)$ and $x_{22}(r)$ commute with $\overline{h}$ and $\overline{f}$ for each $r, s \in \mathbb{F}_p$.

We have completed the steps for tool number 4 and can conclude that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \left\langle e, \overline{h}, \overline{f} \right\rangle$ is G-completely reducible.

(j) Let $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}}$ represent the $D_6(a_1)$ orbit, then $\langle \alpha, \lambda \rangle$ is given by $-11a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + 2a_7 - 8a_8$. Observe that

$$\beta_1 = {\substack{1100000\\0}},\ \beta_2 = {\substack{1110000\\0}},\ \beta_3 = {\substack{1111111\\1}},\ \beta_4 = {\substack{1121111\\1}}$$

have λ-eigenvalue equal to -9.

Following the application of tool number 2, we are left to consider $e = \overline{e} + re_{\beta_1} - re_{\beta_2} + se_{\beta_3} - se_{\beta_4}$ which is in the $D_6(a_1)$ orbit for each $r, s \in \mathbb{F}_p$.

Let $e_1 = e_{\substack{1000000\\0}}$ and $e_{47} = e_{\substack{1111111\\0}}$. Then for $i \in \{1, 47\}$ set $x_i(t) = \exp(t\,\mathrm{ad}\,e_i)$.

Using the formulas in [Car89, §4.4] we see that

$$x_1(r)\cdot(x_{47}(s)\cdot\overline{e}) = \overline{e} + re_{\beta_1} - re_{\beta_2} + se_{\beta_3} - se_{\beta_4}.$$

By Lemma 6.2.4 we have that $x_1(r)$ and $x_{47}(s)$ commute with $\overline{h}$ and $\overline{f}$ for each $r, s \in \mathbb{F}_p$.

We have completed the steps for tool number 4 and can conclude that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \left\langle e, \overline{h}, \overline{f} \right\rangle$ is G-completely reducible.

(k) Let $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0110000\\0}} - e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}}$ represent the $D_7(a_2)$ orbit, then $\langle \alpha, \lambda \rangle$ is given by $-13a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_7 + 2a_8$.

There are three simple roots α with λ-eigenvalue -9, these are

$$\beta_1 = \substack{1110000\\1}, \ \beta_2 = \substack{1111000\\0}, \ \beta_3 = \substack{1111100\\0}.$$

Following the application of tool number 2 we are left to consider $e = \overline{e} + re_{\beta_1} + (r+s)e_{\beta_2} + se_{\beta_3}$ which is contained in the $D_7(a_2)$ orbit for each $r, s \in \mathbb{F}_p$.

Consider $e_9 = e_{\substack{1100100\\0}}$ and $e_{16} = e_{\substack{1110000\\0}}$. Then for $i \in \{9, 16\}$ set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

Then using the formulas in [Car89, §4.4] we have that

$$x_{16}(s) \cdot (x_9(r+s) \cdot \overline{e}) = \overline{e} + re_{\beta_1} + (r+s)e_{\beta_2} + se_{\beta_3}.$$

It follows from Lemma 6.2.4 that $x_{16}(s)$ and $x_9(r+s)$ commute with $\overline{h}$ and $\overline{f}$ for all $r, s \in \mathbb{F}_p$.

Thus we have completed the steps required for tool number 4 and we have shown that $(e, \overline{h}, \overline{f})$ is conjugate to $(\overline{e}, \overline{h}, \overline{f})$ and hence $\mathfrak{h} = \left\langle e, \overline{h}, \overline{f} \right\rangle$ is G-completely reducible.

Finally, let $p = 7$, for which following the application of tools number 1 and 2 we are left with the remaining seven cases.

(a) First we take $\bar{e} = e_{\substack{1000000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ to represent the $A_5 + A_1$ orbit, then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 5a_2 - 6a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8$. There are seven positive roots α with λ-eigenvalue equal to -5, however following the application of tool number 2, we only consider

$$\beta_1 = {}^{1111000}_{\ \ 1}, \ \beta_2 = {}^{0121000}_{\ \ 1}, \ \beta_3 = {}^{0111100}_{\ \ 1},$$

$$\beta_4 = {}^{1221110}_{\ \ 1}, \ \beta_5 = {}^{1222100}_{\ \ 1}$$

for which we have that

$$e = \bar{e} + qe_{\beta_1} + re_{\beta_2} - re_{\beta_2} + se_{\beta_4} - e_{\beta_5}$$

is in the $A_5 + A_1$ nilpotent orbit for any $q, r, s \in \mathbb{F}_p$. We consider

$$e_{23} = e_{\substack{1110000\\1}}, \ e_{25} = e_{\substack{0111000\\1}}, \ e_{51} = e_{\substack{1221100\\1}}$$

and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for each $i \in \{23, 25, 51\}$.

Using [Car89, §4.4] we see that that

$$x_{23}(q-r) \cdot (x_{25}(r) \cdot (x_{51}(s) \cdot \bar{e})) = \bar{e} + qe_{\beta_1} + re_{\beta_2} - re_{\beta_2} + se_{\beta_4} - e_{\beta_5}.$$

We have from Lemma 6.2.4 that $x_{23}(q-r)$, $x_{25}(r)$ and $x_{51}(s)$ commute with $\bar{h}$ and $\bar{f}$.

Thus we have completed tool number 4, and see that $(\bar{e}, \bar{h}, \bar{f})$ and $(e, \bar{h}, \bar{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \bar{h}, \bar{f} \rangle$ is G-completely reducible.

(b) Take $\bar{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}}$ to represent an $A_5 + A_1$ orbit, then we find that $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 - 9a_3 + 2a_4 +$

$2a_5+2a_6+2a_7-5a_8$. There are seven positive roots α with λ-eigenvalue equal to -5, following the application of tool number 2, we reduce to considering

$$\beta_1 = \substack{1110000\\0},\ \beta_2 = \substack{1232111\\1},\ \beta_3 = \substack{1222211\\1},$$

$$\beta_4 = \substack{1232111\\1},\ \beta_5 = \substack{1222211\\1}$$

We have that

$$e = \overline{e} + qe_{\beta_1} + re_{\beta_2} - re_{\beta_3} + se_{\beta_4} + se_{\beta_5}$$

is in the $A_5 + A_1$ orbit for any $r, s \in \mathbb{F}_p$.

Take

$$e_9 = e_{\substack{1100000\\0}},\ e_{11} = e_{\substack{0110000\\0}},\ e_{72} = e_{\substack{1222111\\1}}$$

. Then for $i \in \{9, 11, 72\}$ set $x_i(t) = \exp(t \operatorname{ad} e_i)$.

Using the formulas in [Car89, §4.4] we see that

$$x_{72}(s) \cdot (x_{11}(r) \cdot (x_9(q-r) \cdot \overline{e})) = \overline{e} + qe_{\beta_1} + re_{\beta_2} - re_{\beta_3} + se_{\beta_4} + se_{\beta_5}.$$

We show by Lemma 6.2.4 that $x_{72}(s)$, $x_9(q-r)$ and $x_{11}(r)$ commute with $\overline{h}$ and $\overline{f}$.

We have completed the steps for tool number 4 and can conclude that $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ are all conjugate $\mathfrak{sl}_2$-triples, and hence are all G-completely reducible.

(c) Next take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}}$ which represents an $A_5 + A_1$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 - 9a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 - 6a_7 + 2a_8$. There are seven positive roots α with λ-eigenvalue equal to -5, however following the application of tool number 2 we restrict to considering the roots

$$\beta_1 = \begin{smallmatrix}0110000\\1\end{smallmatrix},\ \beta_2 = \begin{smallmatrix}0011000\\1\end{smallmatrix},$$

$$\beta_3 = \begin{smallmatrix}1111110\\1\end{smallmatrix},\ \beta_4 = \begin{smallmatrix}0121110\\1\end{smallmatrix},\ \beta_5 = \begin{smallmatrix}0111111\\1\end{smallmatrix}$$

which are such that

$$e = \overline{e} + qe_{\beta_1} - qe_{\beta_2} + re_{\beta_3} + re_{\beta_4} + se_{\beta_5}$$

are in the $A_5 + A_1$ orbit for each $q, r, s \in \mathbb{F}_p$.

Consider

$$e_{10} = e_{\substack{0010000\\1}},\ e_{41} = e_{\substack{0111110\\1}},\ e_{42} = e_{\substack{0011111\\1}}$$

and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{10, 41, 42\}$. By [Car89, §4.4]

$$x_{10}(q) \cdot (x_{42}(s) \cdot (x_{41}(r) \cdot \overline{e})) = \overline{e} + qe_{\beta_1} - qe_{\beta_2} + re_{\beta_3} + re_{\beta_4} + se_{\beta_5}.$$

By Lemma 6.2.4 we have that $x_{10}(q)$, $x_{42}(s)$ and $x_{41}(r)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and conclude that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(d) Next take $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}}$ which represents the $A_4 + A_3$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_2 + 2a_3 + 2a_4 - 9a_5 + 2a_6 + 2a_7 + 2a_8$. There are six positive roots α with λ-eigenvalue equal to -5, these are given by

$$\beta_1 = {}^{0011000}_{\;\;1},\ \beta_2 = {}^{0111000}_{\;\;0},\ \beta_3 = {}^{0011100}_{\;\;0},\ \beta_4 = {}^{0000111}_{\;\;0}$$

$$\beta_5 = {}^{1233210}_{\;\;2},\ \beta_6 = {}^{1233211}_{\;\;1}$$

Following the application of tool number 2 we restrict to the cases

$$e = \overline{e} + re_{\beta_1} + re_{\beta_2} - (r+s)e_{\beta_3} + se_{\beta_4} + qe_{\beta_5} - qe_{\beta_6}$$

are in the $A_4 + A_3$ orbit for each $q, r, s \in \mathbb{F}_p$.

Consider

$$e_{12} = e_{{}^{0011000}_{\;\;0}},\ e_{13} = e_{{}^{0001100}_{\;\;0}},\ e_{82} = e_{{}^{1233210}_{\;\;1}}$$

and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{12, 13, 82\}$. By [Car89, §4.4]

$$x_{12}(r) \cdot (x_{13}(s) \cdot (x_{82}(q) \cdot \overline{e})) = \overline{e} + re_{\beta_1} + re_{\beta_2} - (r+s)e_{\beta_3} + se_{\beta_4} + qe_{\beta_5} - qe_{\beta_6}.$$

By Lemma 6.2.4 we have that $x_{12}(r)$, $x_{13}(s)$ and $x_{82}(q)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the steps required for tool number 4, and conclude that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(e) Let $\overline{e} = e_{{}^{0010000}_{\;\;0}} + e_{{}^{0000010}_{\;\;0}} + e_{{}^{1110000}_{\;\;1}} + e_{{}^{1111000}_{\;\;0}} + e_{{}^{0111000}_{\;\;1}} + e_{{}^{0011100}_{\;\;1}} + e_{{}^{0111100}_{\;\;0}}$ be a representative of the $E_7(a_5)$ orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_4 + 2a_7 - 9a_8$ and the following roots have λ-eigenvalue equal to -5,

$$\beta_1 = {}^{0011111}_{\;\;0},\ \beta_2 = {}^{0011111}_{\;\;1},\ \beta_3 = {}^{0111111}_{\;\;0},$$

$$\beta_4 = {}^{1111111}_{0} \ \beta_5 = {}^{0111111}_{1} \ \beta_6 = {}^{1111111}_{1}.$$

Following the application of tool number 2, we are left to consider the following nilpotent elements, which are in the $E_7(a_5)$ orbit for any $q, r, s \in \mathbb{F}_p$

$$\overline{e} + qe_{\beta_1} + re_{\beta_2} + re_{\beta_3} + se_{\beta_4} + se_{\beta_5} + qe_{\beta_6}.$$

Consider $e_{22} = e_{\substack{0000111\\0}}$, $e_{15} = e_{\substack{0000011\\0}}$ and $e_{29} = e_{\substack{0001111\\0}}$ and set for each $i \in \{15, 22, 29\}$, $x_i(t) = \exp(t \operatorname{ad} e_i)$.

We use the formulas in [Car89, §4.4] to see that

$$x_{15}(r) \cdot (x_{29}(q) \cdot (x_{22}(s) \cdot \overline{e})) = \overline{e} + qe_{\beta_1} + re_{\beta_2} + re_{\beta_3} + se_{\beta_4} + se_{\beta_5} + qe_{\beta_6}.$$

Hence we are left to show that $x_{15}(r), x_{29}(q)$ and $x_{22}(s)$ commute with $\overline{h}$ and $\overline{f}$. This follows by Lemma 6.2.4.

We have completed the steps required for tool number 4 and can conclude that $(\overline{e}, \overline{h}, \overline{f})$, is conjugate to $(e, \overline{h}, \overline{f})$ in each of the above cases, and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(f) Let $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0010000\\0}} + e_{\substack{0000100\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0110000\\1}} + e_{\substack{0011000\\1}} + e_{\substack{0111000\\0}}$ be a representative of the $E_6(a_3) + A_1$, orbit. Then $\langle \alpha, \lambda \rangle$ is given by $2a_1 + 2a_4 + 2a_6 - 9a_7 + 2a_8$. There are six positive roots α with λ-eigenvalue equal to -5, these are given by

$$\beta_1 = {}^{0000111}_{0}, \ \beta_2 = {}^{0011110}_{0}, \ \beta_3 = {}^{0001111}_{0},$$

$$\beta_4 = {}^{0011110}_{1}, \ \beta_5 = {}^{0111110}_{0}, \ \beta_6 = {}^{0111110}_{1}.$$

Following the application of tool number 2, we are left to consider

$$e = \overline{e} + se_{\beta_1} + qe_{\beta_2} - qe_{\beta_3} + re_{\beta_4} + re_{\beta_5} + qe_{\beta_6}$$

with $q, r, s \in \mathbb{F}_p$ we have that e is in the $E_6(a_3) + A_1$ orbit.

Consider

$$e_{14} = e_{\substack{0000110\\0}},\ e_{15} = e_{\substack{0000011\\0}},\ e_{21} = e_{\substack{0001110\\0}}$$

and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{14, 15, 21\}$.

By the formulas in [Car89, §4.4]

$$x_{21}(q) \cdot (x_{14}(s) \cdot (x_{15}(s - r) \cdot \overline{e})) = \overline{e} + se_{\beta_1} + qe_{\beta_2} - qe_{\beta_3} + re_{\beta_4} + re_{\beta_5} + qe_{\beta_6}.$$

It follows from Lemma 6.2.4 that $x_{21}(q)$, $x_{14}(s)$ and $x_{15}(s-r)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the requirements for tool number 4 and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(g) Let $\overline{e} = e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0010000\\1}} - e_{\substack{0110000\\0}} + e_{\substack{0011000\\0}} + e_{\substack{0001100\\0}}$ be the representative of the $D_6(a_2)$ orbit. Then $\langle \alpha, \lambda \rangle = -9a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_7 - 6a_8$.

There are six simple roots α with λ-eigenvalue -5, these are

$$\beta_1 = {\substack{1110000\\1}},\ \beta_2 = {\substack{1111000\\0}},\ \beta_3 = {\substack{1111100\\0}}$$

$$\beta_4 = {\substack{1222111\\1}},\ \beta_5 = {\substack{1122111\\1}},\ \beta_6 = {\substack{1122211\\1}}.$$

Then for any $a, b, r, s \in \mathbb{F}_p$ we have that $e = \overline{e} + ae_{\beta_1} + (a+b)e_{\beta_2} + be_{\beta_3} + re_{\beta_4} + (r+s)e_{\beta_5} + se_{\beta_6}$ is in the $D_6(a_2)$ orbit. Following the application of tool number 2, these are the only cases that we need to consider further.

Take

$$e_9 = e_{\substack{1100000\\0}}, \; e_{16} = e_{\substack{1110000\\0}}, \; e_{54} = e_{\substack{1111111\\1}}, \; e_{60} = e_{\substack{1121111\\1}}$$

and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{9, 16, 54, 60\}$.

We find that $x_{16}(b) \cdot (x_9(a+b) \cdot (x_{60}(s) \cdot (x_{54}(r+s) \cdot \overline{e})))$ is equal to

$$\overline{e} + ae_{\beta_1} + (a+b)e_{\beta_2} + be_{\beta_3} + re_{\beta_4} + (r+s)e_{\beta_5} + se_{\beta_6}.$$

It follows from Lemma 6.2.4 that $x_{16}(b), x_9(a+b), x_{60}(s)$ and $x_{54}(r+s)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the requirements for tool number 4 and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples and hence $\mathfrak{h} = \langle e, \overline{h}, \overline{f} \rangle$ is G-completely reducible.

(h) Let $\overline{e} = e_{\substack{1000000\\0}} + e_{\substack{0000000\\1}} + e_{\substack{0100000\\0}} + e_{\substack{0001000\\0}} + e_{\substack{0000010\\0}} + e_{\substack{0000001\\0}} + e_{\substack{0010000\\1}} + e_{\substack{0011000\\0}}$ be the representative of the $D_5(a_1) + A_2$ orbit. Then $\langle \alpha, \lambda \rangle = 2a_1 + 2a_2 + 2a_3 + 2a_5 - 9a_6 + 2a_7 + 2a_8$.

There are seven simple roots α with λ-eigenvalue -5, these are

$$\beta_1 = {\substack{0001110\\0}}, \; \beta_2 = {\substack{0000111\\0}}, \; \beta_3 = {\substack{0011100\\1}}, \; \beta_4 = {\substack{0111100\\0}},$$

$$\beta_5 = {\substack{0011110\\0}}, \; \beta_6 = {\substack{1233321\\2}}, \; \beta_7 = {\substack{1243321\\2}}.$$

Then for any $a, q, r, s \in \mathbb{F}_p$ we have that

$$e = \overline{e} + (r+s)e_{\beta_1} - se_{\beta_2} + (q-r)e_{\beta_3} + qe_{\beta_4} + (s-q)e_{\beta_5} + ae_{\beta_6} - ae_{\beta_7}$$

is in the $D_5(a_1)+A_2$ orbit. Following the application of tool number 2, these are the only cases that we need to consider further.

Take

$$e_{13} = e_{\substack{0001100\\0}},\ e_{14} = e_{\substack{0000110\\0}},\ e_{20} = e_{\substack{0011100\\0}},\ e_{96} = e_{\substack{1233321\\1}}$$

and set $x_i(t) = \exp(t \operatorname{ad} e_i)$ for $i \in \{13, 14, 20, 96\}$.

We find that

$x_{96}(a) \cdot (x_{20}(q) \cdot (x_{13}(r) \cdot (x_{14}(s) \cdot \overline{e})))$ is equal to

$$\overline{e} + (r+s)e_{\beta_1} - se_{\beta_2} + (q-r)e_{\beta_3} + qe_{\beta_4} + (s-q)e_{\beta_5} + ae_{\beta_6} - ae_{\beta_7}.$$

It follows from Lemma 6.2.4 that $x_{96}(a), x_{20}(q), x_{13}(r)$ and $x_{14}(s)$ commute with $\overline{h}$ and $\overline{f}$.

Thus we have completed the requirements for tool number 4 and see that $(\overline{e}, \overline{h}, \overline{f})$ and $(e, \overline{h}, \overline{f})$ are conjugate $\mathfrak{sl}_2$-triples and hence $\mathfrak{h} = \left\langle e, \overline{h}, \overline{f} \right\rangle$ is G-completely reducible.

Hence we have completed the proof of Theorem 6.0.1(a) for each group of exceptional type.

6.5 Unique $\mathfrak{sl}_2$-triples

In this section we complete the proof of Theorem 6.0.1(b) by showing that for each $e \in \mathcal{V}$ there is a unique $\mathfrak{sl}_2$-triple $(e, h, f) \in \mathfrak{g}$ such that $f \in \mathcal{V}$ up to conjugation by G.

Proof of Theorem 6.0.1. Let $\mathcal{V}$ be as in the statement of the theorem. Let $e \in \mathcal{V}$ and let (e, h_0, f_0) be a standard $\mathfrak{sl}_2$-triple taken as in §4.1. Note that $e^{[p]} = 0$ so f_0 is conjugate to e as shown in [PS19, §2.4], and hence f_0 is in $\mathcal{V}$.

Take (e, h, f) to be an $\mathfrak{sl}_2$-triple with $e, f \in \mathcal{V}$. We have shown in §6.4 that $\mathfrak{h} = \langle e, h, f\rangle$ is G-completely reducible and we use this to show that (e, h, f) is G-conjugate to (e, h_0, f_0).

We first recall that $\mathfrak{h}$ is a p-subalgebra by Proposition 4.3.2. Therefore $h^{[p]} = h$, so h is toral, and hence semisimple.

We give a similar argument to the one given in [PS19, §2.4] to deduce that h is fixed up to conjugacy by the centraliser of e.

Let τ be the associated cocharacter for e and $h_0 = d\tau(1)$. Note that all eigenvalues of h_0 belong to $\mathbb{F}_p$, and so we can write $\mathfrak{g}(h_0; i)$ for the eigenspace of $\operatorname{ad}(h_0)$ corresponding to the eigenvalue $i \in \mathbb{F}_p$. Then we have the grading

$$\mathfrak{g}(i; h_0) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(i + jp; \tau).$$

Since e commutes with $h - h_0$, we see that h is a semisimple element of the restricted Lie algebra $\Bbbk h_0 \oplus \mathfrak{g}^e$. Since $\mathfrak{g}^e = \operatorname{Lie}(G^e)$, the second component coincides with the Lie algebra of the normaliser $N_G(\Bbbk e) = \tau(\Bbbk^\times) \cdot G^e$.

Note that the Lie algebra $\mathrm{Lie}(\tau(\Bbbk^\times))$ is a 1-dimensional torus of $\mathfrak{g}$ spanned by the element h_0.

The centraliser $C_G(\tau)$ of $\tau(\Bbbk^\times)$ is a Levi subalgebra of G and $\mathrm{Lie}(C_G(\tau)) = \mathfrak{g}(0;\tau)$. Set $\mathfrak{g}^e(i) = \mathfrak{g}^e \cap \mathfrak{g}(i;\tau)$. The group $C^e := G^e \cap Z_G(\tau)$ is reductive and $\mathfrak{c}^e := \mathrm{Lie}(C^e) = \mathfrak{g}^e(0)$.

As $\tau(\Bbbk^\times) \cdot T^e$ is a maximal torus of $N_G(\Bbbk e)$ contained in $\tau(\Bbbk^\times) \cdot \mathfrak{l}^e$, it follows that since h is semisimple it is conjugate under the adjoint action of $N_G(\Bbbk e)$ to an element of $\Bbbk h_0 \oplus \mathfrak{g}^e(0)$. Hence we can assume that $h \in \Bbbk h_0 + \mathfrak{g}^e(0)$. Then

$$h - h_0 \in \mathfrak{g}^e(0) \cap [e, \mathfrak{g}].$$

If $h \neq h_0$ then the linear map

$$(\mathrm{ad}\, e)^2 : \mathfrak{g}(-2;\tau) \to \mathfrak{g}(2;\tau)$$

is not injective. The computations in [Pre03] then imply that e is regular nilpotent in $\mathrm{Lie}(L)$ for some L, a Levi subgroup of type $A_{p-1}A_r$ in G for some r. This contradicts the definition of $\mathcal{V}$. Hence we have that $h = h_0$.

All that is left to show is that $f = f_0$.

Let T^e be a maximal torus of C^e and let $L = C_G(T^e)$ be a Levi subgroup of G. The Lie algebra $\mathfrak{l}' = \mathrm{Lie}(L^e)$ is stable under the action of $\tau(\Bbbk^\times)$ and contains h_0.

Moreover, e is distinguished in $\mathfrak{l}'$, that is, $e \in \mathfrak{l}'(\tau, 2)$ and $\dim \mathfrak{l}'(0;\tau) = \dim \mathfrak{l}'(2;\tau)$. Since $\mathfrak{g}^e \subseteq \mathrm{Lie}(P^e)$, $\dim \mathfrak{l}'(-2;\tau) = \dim \mathfrak{l}'(2;\tau)$ and the map $\mathrm{ad}\, e : \mathfrak{l}'(-2;\tau) \to$

$\mathfrak{l}'(0;\tau)$ is injective, we must have that $\operatorname{ad} e$ is bijective. As $\mathfrak{h}_0 \in \mathfrak{l}'(0;\tau)$ there is a unique element $f \in \mathfrak{l}'(-2;\tau)$ such that $[e,f] = h_0$.

Thus we have that (e,h,f) is conjugate to (e,h_0,f_0) and we are done. □

6.6 Classes in $\mathcal{V}$

Let $\mathcal{V}$ be the variety given in Theorems 3 and 6.0.1. In this section we explicitly list the classes not contained in $\mathcal{V}$ for each prime which is good for G. In order to determine the elements $x \in \mathcal{N}$ for which $x^{[p]} \neq 0$ we refer to Tables 1,3,5,7 and 9 in [Law95]. We remark that there is an error in the tables for $G = E_8, p = 3$, however this does not coincide with the cases we consider.

p	Classes not in $\mathcal{V}$ for p
$p > 5$	
5	G_2

Table 6.3: Nilpotent classes not in $\mathcal{V}$ for G of type G_2

p	Classes not in $\mathcal{V}$ for p
$p > 11$	
11	F_4
7	as for $p = 11$ and $F_4(a_1)$
5	as for $p = 7$ and $F_4(a_2), C_3, B_3$

Table 6.4: Nilpotent classes not in $\mathcal{V}$ for G of type F_4

p	Classes not in $\mathcal{V}$ for p
$p > 11$	
11	E_6
7	as for $p = 11$ and $E_6(a_1), D_5$
5	as for $p = 7$ and $E_6(a_3), D_5(a_1), A_5, A_4 + A_1, D_4, A_4$

Table 6.5: Nilpotent classes not in $\mathcal{V}$ for G of type E_6

p	Classes not in $\mathcal{V}$ for p
$p > 17$	
17	E_7
13	as for $p = 17$ and $E_7(a_1)$
11	as for $p = 13$ and $E_7(a_2), E_6$
7	as for $p = 11$ and $E_7(a_3), E_6(a_1), D_6, E_7(a_4), D_6(a_1), D_5{+}A_1, A_6, D_5$
5	as for $p = 7$ and $E_7(a_5), E_6(a_3), D_6(a_2), D_5(a_1) + A_1, A_5 + A_1, (A_5)', A_4 + A_2, D_5(a_1), A_4 + A_1, D_4 + A_1, (A_5)'', A_4, D_4$

Table 6.6: Nilpotent classes not in $\mathcal{V}$ for G of type E_7

p	Classes not in $\mathcal{V}$ for p
$p > 29$	
29	E_8
23	as for $p = 29$ and $E_8(a_1)$
19	as for $p = 23$ and $E_8(a_2)$
17	as for $p = 19$ and $E_8(a_3), E_7$
13	as for $p = 17$ and $E_8(a_4), E_8(b_4), E_7(a_1)$
11	as for $p = 13$ and $E_8(a_5), E_8(b_5), D_7, E_7(a_2), E_6 + A_1, E_6$
7	as for $p = 11$ and $E_8(a_6), D_7(a_1), E_8(b_6), E_7(a_3), E_6(a_1) + A_1, A_7, D_7(a_2), D_6, D_5{+}A_2, E_6(a_1), E_7(a_4), A_6{+}A_1, D_6(a_1), A_6, D_5{+}A_1, D_5$

Table 6.7: Nilpotent classes not in $\mathcal{V}$ for G of type E_8

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