Studying categorical aspects of the Landau-Ginzburg B-model using variations of geometric invariant theory

by

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Abstract

In this thesis we use variations of geometric invariant theory to study the derived categories of coherent sheaves associated to complete intersections in toric varieties. In the context of mirror symmetry, a given Calabi-Yau variety may not have a unique mirror associated to it. Finding relations between derived categories associated to distinct mirror constructions leads to unification results under Homological Mirror Symmetry. In particular, in this thesis we prove the equivalence of two constructions to a complete intersection of cubics in $\mathbb{P}^5$, one due to Batyrev and Borisov, the other due to Libgober and Teitelbaum. The proof relies on methods of partial compactifications and variations of geometric invariant theory, and is the first of its kind to relate derived categories for complete intersections and not hypersurfaces.

Singular complete intersections present an obstacle when applying these methods to a wider context, and we do not obtain equivalences of derived categories in general. Partial compactifications and variations of geometric invariant theory however remain a strong tool in studying the derived categories of singular complete intersections. In this thesis, we give a framework in which we can use these methods to obtain crepant categorical resolutions. We illustrate this framework by giving a family of examples which directly generalises the mirror construction by Libgober and Teitelbaum, then categorically resolving the derived categories of coherent sheaves by the derived categories of coherent sheaves associated to a family of Batyrev-Borisov mirrors.

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CONTENTS

CHAPTER 1

INTRODUCTION

Mirror symmetry is a phenomenon originating in string theory postulating that for every Calabi-Yau manifold X, there should be a "dual" Calabi-Yau manifold Y, such that certain properties of the manifold X are "mirrored" by other properties of the manifold Y. Candelas, Lynker and Schimmrigk [13] for example compiled a list of Calabi-Yau threefolds and noted that they pair up in the sense that for most Calabi-Yau threefolds X on the list, there was a mirror partner Y with $H^{p,q}(X) \cong H^{3-p,q}(Y)$. On the level of Euler characteristic, this implies that $\chi(X) = -\chi(Y)$. Generally, mirror symmetry predicts that for a mirror pair X, Y, the complex algebraic structure on the Calabi-Yau variety X is mirrored by the symplectic structure on the Calabi-Yau variety Y.

While the phenomenon of mirror symmetry was, mathematically, first observed on a topological level, there are a few major approaches of how mirror symmetry should be articulated. Most of this thesis focuses on Homological Mirror Symmetry, as introduced by Kontsevich [33]. An early (and since generalised) formulation of Kontsevich's Homological Mirror Symmetry Conjecture states that if two Calabi-Yau manifolds X and Y are mirror to each other, there are two equivalences of categories on them,

$$\mathrm{D^b}(\operatorname{coh} X) \cong \operatorname{Fuk}(Y) \text{ and } \mathrm{D^b}(\operatorname{coh} Y) \cong \operatorname{Fuk}(X). \tag{1.1}$$

The categories involved here are the bounded derived category of coherent sheaves, $\mathrm{D^b}(\operatorname{coh} \circ)$ and the Fukaya category, $\operatorname{Fuk}(\circ)$, which we will introduce in Chapter 2. Roughly,

$\mathrm{D^b}(\operatorname{coh} X)$ encodes algebro-geometric information of X, whilst $\mathrm{Fuk}(X)$ encodes symplectic information.

One of the leading problems in the study of mirror symmetry is figuring out how to construct a mirror partner Y to a given Calabi-Yau manifold X. There are several approaches on how to solve this problem. Given a Calabi-Yau X, the different "recipes" how to obtain a candidate mirror partner are known as mirror constructions. Different articulations of mirror symmetry give different evidence of when a Calabi-Yau Y is considered a mirror partner to X. For example, early formulations of mirror symmetry focused on topological evidence (via the cohomology and Euler characteristics) whereas Homological Mirror Symmetry considers the categorical information in (1.1). Even for those Calabi-Yau manifolds where we know their mirror partner, we may not have a unique one. One famous example of this phenomenon is due to Rødland [43]. This naturally leads to the question whether there is a unifying articulation of mirror symmetry, under which different mirror partners are equivalent. Consider two varieties Y, Y' obtained as mirror to a given Calabi-Yau X via two different mirror constructions. Working towards a unification of mirror constructions, it would be helpful to know if both Y and Y' fit into the context of Homological Mirror Symmetry, that is if $\mathrm{D^b}(\operatorname{coh} Y) \cong \mathrm{D^b}(\operatorname{coh} Y')$. In this thesis, we work towards answering this question by exploring how variations of geometric invariant theory can be used to establish such equivalences of categories. An example we have chosen to exhibit these methods is a connection between mirror constructions by Batyrev-Borisov and Libgober-Teitelbaum.

Libgober and Teitelbaum [35] proposed a mirror to a Calabi-Yau complete intersection V_λ of two cubics in $\mathbb{P}^5$ defined as the zero locus of the two polynomials

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \quad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2.$$

Their proposed mirror $W_{LT,\lambda}$ is a (minimal) resolution of singularities of the variety $V_{LT,\lambda}$ with defining equations $Q_{1,\lambda}, Q_{2,\lambda}$ but in the quotient space $\mathbb{P}^5 / G_{81}$, where G_{81} is a specified order 81 subgroup of $PGL(5,\mathbb{C})$. They showed topological evidence that V_λ and $W_{LT,\lambda}$ are a mirror pair, proving on the level of Euler characteristics that $\chi(V_\lambda) = -\chi(W_{LT,\lambda})$. In [21], Filipazzi and Rota verify a state space isomorphism between the two Calabi-Yau varieties by providing an explicit mirror map.

Batyrev and Borisov in [7] introduced a mirror construction for Calabi-Yau intersections in Fano toric varieties using polytopes, showing mirror duality for $(1,q)$-Hodge numbers. This mirror construction agrees with constructions by Greene-Plesser [24] and Berglund-Hübsch [9] for Fermat hypersurfaces. However, the Batyrev-Borisov mirror to two cubics in $\mathbb{P}^5$ differs from the one given above by Libgober and Teitelbaum. In chapter 3, we work towards a unification of mirror constructions by proving that the bounded derived category of coherent sheaves on the Libgober-Teitelbaum mirror is equivalent to that of a complete intersection $Z \subseteq X_\nabla$ in the Batyrev-Borisov mirror family. On the level of stacks, we will prove the following result.

Theorem 1.0.1 (Theorem 1.1 in [36]). *Let $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 0, 1$. Consider the two polynomials*

$$p_{1,\lambda} = x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8,$$

$$p_{2,\lambda} = x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11}.$$

Let $\mathcal{Z}_\lambda = Z(p_{1,\lambda}, p_{2,\lambda}) \subseteq \mathcal{X}_\nabla$ and $\mathcal{V}_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq [\mathbb{P}^5 / G_{81}]$. Then

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{V}_{LT,\lambda}) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_\lambda).$$

Since [21] relates Libgober-Teitelbaums construction to BHK mirror symmetry, this gives a direct link between the constructions of Batyrev-Borisov and Berglund-Hübsch-Krawitz. Since previously, the constructions by Batyrev-Borisov and Berglund-Hübsch-Krawitz were only linked for hypersurfaces, the results of this thesis provide a first example of a link for the case of complete intersections. We hope this provides a first step towards the unification of the two constructions for complete intersections.

This work is also, to our knowledge, the first application of partial compactifications in variations of geometric invariant theory (VGIT) to prove the equivalence of derived categories for complete intersections, and not hypersurfaces, in Calabi-Yau varieties.

In the case of singular mirror candidates, the methods employed in the proof of Theorem 1.0.1 may fail to apply. This is because the methods rely on studying the category of singularities associated to a Landau-Ginzburg model. For the Batyrev-Borisov mirrors, by construction, the singularities are well-behaved, originating from the quotient singularities of the ambient toric variety. Thus, we cannot expect to obtain an equivalence between the categories of singularities of a singular mirror candidate and a Batyrev-Borisov mirror. VGIT remains a strong tool in this case, as demonstrated by Favero and Kelly [19]. Using their methods, we efficiently can obtain categorical resolutions. Generalising the construction by Libgober and Teitelbaum gives a family of singular complete intersections $Z_n \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/(\mathbb{C}^* \times G_n)\right]$, where G_n is a specified order n^{2n-2} subgroup of $PGL(2n-1, \mathbb{C})$. We show that this family can be categorically resolved by Batyrev-Borisov mirrors to complete intersections of two degree n polynomials in $\mathbb{P}^{2n-1}$. Specifically, we prove the following theorem.

Theorem 1.0.2 (=Theorem 4.2.1)**.** *Let $n \geq 2$, and $\lambda^{2n} \neq 0, n^{2n}$. Consider the complete intersection in $\mathbb{P}^{2n-1}$ given by the vanishing set of the two polynomials*

$$Q_{1,n,\lambda} = x_1^n + x_2^n + \cdots + x_n^n - \lambda x_{n+1} x_{n+2} \ldots x_{2n},$$

$$Q_{2,n,\lambda} = x_{n+1}^n + x_{n+2}^n + \cdots + x_{2n}^n - \lambda x_1 x_2 \ldots x_n.$$

Then $G_n \cong (\mathbb{Z}/n\mathbb{Z})^{2(n-2)} \times (\mathbb{Z}/n^2\mathbb{Z})$ and the hypersurfaces $Q_1 = 0$ and $Q_2 = 0$ are preserved under the action of G_n on $\mathbb{P}^{2n-1}$. Let $Z_n = Z(Q_{1,n,\lambda}, Q_{2,n,\lambda}) \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/(\mathbb{C}^* \times G_n)\right]$, and let Y_n be a Batyrev-Borisov mirror to $Z(Q_{1,n,\lambda}, Q_{2,n,\lambda}) \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/\mathbb{C}^*\right]$.

Then there is a categorical resolution $\mathrm{D^b}(\mathrm{coh}\, Y_n) \to \mathrm{D^b}(\mathrm{coh}\, Z_n)$.

The proof of Theorem 1.0.2 uses methods that apply in greater generality, the extent of which is an interesting area of study. The following Theorem 4.3.7 demonstrates how strong these methods can be to find categorical resolutions, which we expect to be a big step towards a unification result in mirror symmetry.

Theorem 1.0.3 (=Theorem 4.3.7)**.** *Let X_Σ be a projective toric variety with vector bundle $\mathcal{V} = \oplus_{i=1}^r \mathcal{O}(D_i)$ so that there is a section $s_i \in \Gamma(X_\Sigma, \mathcal{O}(D_i))$ for all i and $Z = Z(s_1, \ldots, s_r) \subseteq X_\Sigma$ and $D_1 + \cdots + D_r = -K_{X_\Sigma}$. Let*

$$A = \{m' \in M_{\mathbb{R}} \times \mathbb{R}^r \mid x^m \text{ is nontrivial summand of some } u_i s_i\},$$

and let $S = A \cup \{e_i \mid i = 1, \ldots, r\}$. If $\sigma_S := \mathrm{Cone}(\mathrm{Conv}(S)) \subseteq \sigma_{\mathcal{V}}^\vee$ has the property $(\mathbf{r} - \mathbf{P})$ (see Definition 4.3.1), then there is a categorical resolution of $\mathrm{D^b}(\mathrm{coh}\, Z)$ by a derived category $\mathrm{D^b}(\mathrm{coh}\, Z_{res})$ associated to a complete intersection in a toric stack $[X_{\Sigma'}]$.

We start by giving some background on the mathematical tools used in this thesis in chapter 2. This includes a short introduction to toric geometry, Homological Mirror Symmetry as well as to the different mirror constructions involved.

In chapter 3, we study how methods of partial compactifications and VGIT can be used to provide derived equivalences between complete intersections. This will be illustrated by proving Theorem 1.0.1.

Then, in chapter 4, we explore how methods of VGIT can be used to establish categorical resolutions. We then use these methods to prove Theorem 1.0.2, which gives a family of complete intersections generalising the Libgober-Teitelbaum construction, and Theorem 1.0.3.

Finally, in chapter 5, we explore some further ideas relating to the work of the previous chapters.

CHAPTER 2

BACKGROUND

In this chapter, we will give a gentle introduction to the mathematics lying at the heart of the results presented in this thesis. We start by treating some toric geometry, which allows to effectively study the algebraic geometry of toric varieties by looking at some fundamentally combinatorial objects. We then give a brief introduction to Homological Mirror Symmetry, setting up the categorical terminology used throughout the thesis. Finally, we exhibit the mirror constructions that appear in this thesis.

2.1 Toric Geometry

A lot of the mathematics presented in this thesis is based on Toric geometry. Thus we give a small introduction to it here. A good exposition is the book "Toric Varieties" by Cox, Little and Schenck [15], and this section will be mostly based on it as well.

2.1.1 Toric varieties, Cones and Fans

We begin by introducing the objects at the core of toric geometry: toric varieties.

Definition 2.1.1. A toric variety is an irreducible variety X containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X.

Many varieties appearing in the daily lives of algebraic geometers are toric. For example, affine and projective spaces as well as many of their subvarieties are toric. An important aspect of this large class of varieties is that toric varieties allow for a combinatorial description. This correspondence between toric varieties and certain combinatorial data allows to study the varieties in more detail by identifying combinatorial properties.

A character of a torus T is a group homomorphism $\chi : T \to \mathbb{C}^*$. Looking at the torus $(\mathbb{C}^*)^n$ for example, we have the characters $\mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*) = \mathbb{Z}^n$. Indeed, all characters can be obtained via associating to $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ the character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$, defined by

$$\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \cdot \cdots \cdot t_n^{a_n}.$$

Thus, the characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$. For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T and by identification with $\mathbb{Z}^n$ we associate a character χ^m to an element $m \in M$ as above.

Fix a lattice M of rank d (the character lattice of some toric variety V of dimension d) and its dual lattice N, with the pairing

$$\langle -, - \rangle : M \times N \to \mathbb{Z}.$$

We extend the pairing $\mathbb{R}$-linearly to $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

The combinatorial description of V is obtained by considering appropriate sets in $M_{\mathbb{R}}$ (for polytopes) and $N_{\mathbb{R}}$ (for fans).

One way to think about building toric varieties is by gluing together open subsets of affine toric varieties. We organise this using fans, which are a collection of cones. These cones will correspond to the affine toric subvarieties and the structure of the fan describes how to glue them.

Definition 2.1.2. A convex polyhedral cone in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}\,(S) = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}},$$

where S is a finite subset of $N_{\mathbb{R}}$. We say that σ is generated by S. Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its dual cone is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Furthermore, a cone $\sigma = \operatorname{Cone}(S)$ is called rational if the finite set S can be chosen to be a subset of N.

Definition 2.1.3. Given $m \neq 0$ in $M_{\mathbb{R}}$, we define the hyperplane

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}}.$$

A face of a cone σ is a subcone τ of the form $\tau = H_m \cap \sigma$, for some $m \in \sigma^{\vee}$. We denote this by $\tau \preceq \sigma$. A proper face is a face $\tau \neq \sigma$, written $\tau \prec \sigma$. Cones of dimension 1 are called rays, codimension 1 faces of a cone σ are called facets. A rational polyhedral cone σ is called strongly convex if it is convex and $\sigma \cap (-\sigma) = \{0\}$.

A strongly convex rational polyhedral cone σ is usually given by listing the generators of its edges (since the edges are 1 dimensional cones, they have unique primitive generators in N). Lemma 1.2.15 in [15] shows that such a cone σ is generated by the ray generators of its edges.

Definition 2.1.4. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.

A. σ is smooth or regular if its minimal generators can be extended to a $\mathbb{Z}$-basis of N.

B. σ is simplicial if its minimal generators are linearly indepedent over $\mathbb{R}$.

As mentioned previously, we can associate affine varieties to cones. This is done via monoids and their associated algebras[1]. Given a rational polyhedral cone σ, the lattice points $S_{\sigma} = \sigma^{\vee} \cap M \subseteq M$ form a monoid.

[1]Note that monoids are referred to as semigroups in [15]. As this is not the definition of semigroup we otherwise know, we will use the word monoid.

Lemma 2.1.5 (Gordan's Lemma, Proposition 1.2.27 in [15]). *S_σ is finitely generated and hence an affine monoid.*

The following theorem associates varieties to cones.

Theorem 2.1.6 (Theorem 1.2.18 in [15]). *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational polyhedral cone with monoid $S_\sigma = \sigma^\vee \cap M$. Then*

$$U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

is an affine toric variety. Furthermore,

$$\dim U_\sigma = n \Longleftrightarrow U_\sigma \text{ has torus } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \Longleftrightarrow \sigma \text{ is strongly convex.}$$

Before advancing to fans, we will need to glue affine varieties together. The following result will help us in that aspect.

Proposition 2.1.7 (Proposition 1.3.16 in [15]). *Let σ be a strongly convex rational polyhedral cone and let $\tau \preceq \sigma$ be a face. Write τ in the form $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee \cap M$. Then we have the following equality of monoid algebras*

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}.$$

For the affine sets U_τ, U_σ, we thus have the following isomorphism of affine varieties

$$U_\tau = \operatorname{Spec}(\mathbb{C}[S_\tau]) = \operatorname{Spec}(\mathbb{C}[S_\sigma]_{\chi^m}) = \operatorname{Spec}(\mathbb{C}[S_\sigma])_{\chi^m} = (U_\sigma)_{\chi^m} \subseteq U_\sigma.$$

In particular, if two cones σ, σ' intersect in a common face $\tau = \sigma \cap \sigma'$, we get $U_\sigma \supseteq U_\tau \subseteq U_{\sigma'}$.

We now define the main object of interest in toric geometry.

Definition 2.1.8. A fan Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

A. Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

B. For all $\sigma \in \Sigma$, each face of σ is also in Σ.

C. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (and hence also in Σ).

Furthermore, if Σ is a fan, then the support of Σ is defined to be $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_{\mathbb{R}}$ and $\Sigma(r)$ denotes the set of r-dimensional cones of Σ.

First, we will explain how to use fans to construct toric varieties in general, and then give an example of this. We first require the following two facts from §1 of [15]. Recall that, for a face τ of a cone σ, we can write $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee \cap M$. The proof of Lemma 2.1.5 in [15] then asserts that $S_\tau = S_\sigma + \mathbb{Z}(-m)$. Secondly, if $\tau = \sigma_1 \cap \sigma_2$, then we have $\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m$ for some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$. This implies

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}. \tag{2.1}$$

The construction of a variety X_Σ from a fan Σ can now be described. Consider the collection of affine toric varieties $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$, where σ runs over all cones in Σ. Take σ_1, σ_2 to be any of these cones and consider their intersection face $\tau = \sigma_1 \cap \sigma_2$.

The equalities in (2.1) give an isomorphism $g_{\sigma_2,\sigma_1} : (U_{\sigma_1})_{\chi^m} \simeq (U_{\sigma_2})_{\chi^{-m}}$, acting as the identity on U_τ. Gluing the affine varieties along the subvarieties $(U_\sigma)_{\chi^m}$ fulfills the cocycle conditions, hence giving a variety X_Σ. The authors of [15] go on to prove the following theorem.

Theorem 2.1.9 (Theorem 3.1.5 in [15])**.** *Let Σ be a fan in $N_{\mathbb{R}}$. The variety X_Σ is a normal separated toric variety.*

Example 2.1.10. A standard example of producing varieties from fans is $\mathbb{P}^2$, as found in [26]. Consider the fan $\Sigma \subseteq N_{\mathbb{R}} = \mathbb{R}^2$ in Figure 2.1. It has 7 cones:

$$(0,0),\ \rho_0,\ \rho_1,\ \rho_2,\ \sigma_{0,1},\ \sigma_{2,0},\ \text{and } \sigma_{1,2}.$$

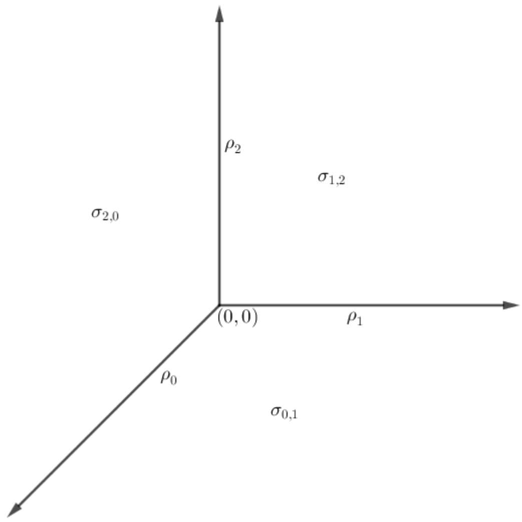

Figure 2.1: A fan for $\mathbb{P}^2$

The toric variety X_Σ associated to this fan is covered by the three affine open sets (corresponding to the 2-dimensional cones):

$$U_{1,2} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_{1,2}}]) \simeq \mathrm{Spec}(\mathbb{C}[x, y])$$
$$U_{2,0} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_{2,0}}]) \simeq \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y])$$
$$U_{0,1} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_{0,1}}]) \simeq \mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]).$$

By above discussion, we see that the gluing data on the coordinate rings is given by:

$$g_{10}^* : \mathbb{C}[x, y]_x \simeq \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}} \tag{2.2}$$
$$g_{21}^* : \mathbb{C}[x, y]_y \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}} \tag{2.3}$$
$$g_{02}^* : \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}} \tag{2.4}$$

Let $(x_0 : x_1 : x_2)$ be the usual homogeneous coordinates on $\mathbb{P}^2$ with standard affine opens U_i, defined by $x_i \neq 0$ for $i = 0, 1, 2$. To show X_Σ is $\mathbb{P}^2$, we define maps $U_{\sigma_{i,i+1}} \to \mathbb{P}^2$

identifying $U_{\sigma_{i,i+1}}$ with U_{i+2} (indices taken modulo 3). The maps are

$$\begin{aligned}
&\mathbb{C}[xy^{-1}, y^{-1}] \to \mathbb{C}\left[\frac{x_0}{x_2}, \frac{x_1}{x_2}\right], \text{ via } y^{-1} \mapsto \frac{x_0}{x_2},\ xy^{-1} \mapsto \frac{x_1}{x_2} \\
&\mathbb{C}[x, y] \to \mathbb{C}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right], \text{ via } x \mapsto \frac{x_1}{x_0},\ y \mapsto \frac{x_2}{x_0} \\
&\mathbb{C}[x^{-1}, x^{-1}y] \to \mathbb{C}\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}\right], \text{ via } x^{-1} \mapsto \frac{x_0}{x_1},\ x^{-1}y \mapsto \frac{x_2}{x_1}
\end{aligned}$$

These maps on the coordinate rings are compatible with the gluing maps g_{ij}^*, giving the inclusions of the varieties corresponding to the rays into the $U_{i,j}$. This shows that X_Σ is indeed isomorphic to $\mathbb{P}^2$.

Gluing affines together like this can quickly become unwieldy. There is a more efficient method of associating the variety X_Σ to a fan Σ; the Cox construction. This will be elaborated upon in section §2.1.4.

Definition 2.1.11. Let $\Sigma \subseteq N_\mathbb{R}$ be a fan.

A. Σ is called smooth or regular if every cone $\sigma \in \Sigma$ is smooth.

B. Σ is called simplicial if every cone $\sigma \in \Sigma$ is simplicial.

C. The fan Σ is complete if $|\Sigma| = N_\mathbb{R}$.

These properties nicely translate to X_Σ, illustrating the usefulness of the interplay of combinatorics and algebraic geometry that toric geometry offers.

Theorem 2.1.12 (Theorem 3.1.19 in [15])**.** *Let X_Σ be the toric variety defined by a fan $\Sigma \subseteq N_\mathbb{R}$. Then:*

A. X_Σ is a smooth variety if and only if Σ is a smooth fan.

B. X_Σ is an orbifold if and only if the fan Σ is simplicial.

C. X_Σ is compact in the classical topology on $\mathbb{C}^n$ if and only if Σ is complete.

It is important to note that every toric variety corresponds to a fan. This is due to the Orbit-Cone corrrespondence. It states that there is a bijective correspondence between cones in Σ and T_N-orbits in X_Σ. In other words, given a toric variety X, one can deduce a fan for it by finding all the orbits of the action on X by its torus T_N. A treatment on this subject can be found in §3.2 of [15]; we will restrict ourselves to the statement of the correspondence.

Theorem 2.1.13 (Theorem 3.2.6 in [15])**.** *Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R}$. Then:*

A. There is a bijective correspondence

$$\begin{aligned} \{Cones\ \sigma\ in\ \Sigma\} &\leftrightarrow \{T_N - orbits\ in\ X_\Sigma\} \\ \sigma &\leftrightarrow O(\sigma). \end{aligned}$$

B. Let $n = \dim N_\mathbb{R}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.

C. The affine open subset U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

D. $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski topologies.

In the later section on f-duality, §2.3.3, we use fan matrices.

Definition 2.1.14. Consider a fan Σ. Each ray $\rho \in \Sigma(1)$ has a primitive generator v_ρ in the underlying lattice. Enumerating these, we get a set $(v_i)_{i=1}^{|\Sigma(1)|}$, written out as column vectors. These are used to form the fan matrix $V = \begin{pmatrix} v_1 \dots v_{|\Sigma(1)|} \end{pmatrix}$ of the variety X_Σ.

Example 2.1.15. Consider the fan for $\mathbb{P}^2$ given above in Figure 2.1. The primitive generators for the rays are $u_{\rho_0} = (-1,-1)$, $u_{\rho_1} = (1,0)$ and $u_{\rho_2} = (0,1)$. This gives a fan matrix $V = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

2.1.2 Divisors on toric varieties

One advantage of describing toric varieties by their fans becomes apparent when studying their divisors. A good introduction to this is §4 in [15], to be followed in this section.

By the Orbit-Cone correspondence, k-dimensional cones σ of a fan Σ correspond to $(n-k)$-dimensional T_N-orbits in X_Σ. So a ray $\rho \in \Sigma(1)$ gives a codimension 1 orbit $O(\rho)$, whose closure is a T_N-invariant prime divisor on X_Σ. To emphasize this, we denote said divisor by D_ρ. Denote by $u_\rho \in \rho \cap N$ the minimal generator of the ray ρ. When $m \in M$, the character $\chi^m : T_N \to \mathbb{C}^*$ is a rational function in $\mathbb{C}(X_\Sigma)^*$, as the torus is open in X_Σ. Since D_ρ is a prime divisor, it defines a valuation $\nu_\rho : \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}$.

Proposition 2.1.16 (Proposition 4.1.1 in [15])**.** *Let X_Σ be the toric variety of a fan Σ. If the ray $\rho \in \Sigma(1)$ has minimal generator u_ρ and χ^m is the character corresponding to $m \in M$, then*

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Together with the Orbit-Cone Correspondence, this proposition implies the following.

Proposition 2.1.17 (Proposition 4.1.2 in [15])**.** *For $m \in M$, the character χ^m is a rational function on X_Σ, and its principal divisor is given by*

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

Definition 2.1.18. Define the group $\mathrm{Div}_{T_N}(X_\Sigma) \subseteq \mathrm{Div}(X_\Sigma)$ to be the group of Weil divisors on X_Σ that are invariant under the torus action.

By Exercise 4.1.1 in [15], the group $\mathrm{Div}_{T_N}(X_\Sigma)$ consists exactly of the divisors

$\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$, including, by Proposition 2.1.17, the principal divisors of the characters of X_Σ. We can write

$$\mathrm{Div}_{T_N}(X_\Sigma) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\, D_\rho \subseteq \mathrm{Div}(X_\Sigma).$$

Using this, we can construct a short exact sequence to calculate the class group of X_Σ.

Theorem 2.1.19 (Theorem 4.1.3 in [15])**.** *We have the exact sequence*

$$M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0, \tag{2.5}$$

where the first map sends m to $\mathrm{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\mathrm{Cl}(X_\Sigma)$. Furthermore, we have a short exact sequence

$$0 \to M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

Remark 2.1.20. A useful way to think about this and calculate the class groups of varieties is via matrices. Assume $|\Sigma(1)| = r$. Pick a basis $e_1, \ldots, e_n$ of M, so that $M \simeq \mathbb{Z}^n$ and thus, as dual lattice, $N \simeq \mathbb{Z}^n$. Then the pairing $\langle -, - \rangle$ is simply the dot product and u_i can be thought of as column vector $(\langle e_1, u_i\rangle, \ldots, \langle e_n, u_i\rangle)^T$. The map $M \to \mathrm{Div}_{T_N}(X_\Sigma)$ becomes the map $A : \mathbb{Z}^n \to \mathbb{Z}^r$, represented by the matrix with rows the ray generators $u_1, \ldots, u_r$. Theorem 2.1.19 implies that $\mathrm{Cl}(X_\Sigma)$ is the cokernel of this map. In particular, the torsion of the group can be found by computing the Smith normal form, which will become useful later on when we work on generalising our own results in Chapter 4.

While one can also classify Cartier divisors by means of toric geometry, this is not relevant for us and we refer the reader to §4.2 of [15].

2.1.3 Polytopes

Definition 2.1.21. A polytope Δ in $M_{\mathbb{R}}$ is a convex hull of a finite set S of points in $M_{\mathbb{R}}$. If this finite set can be chosen to consist of only points of M, we call Δ a lattice polytope. Choosing a minimal generating set S for a polytope Δ, we call the elements of S vertices of the polytope. A polyhedron is the intersection of finitely many closed half-spaces.

Remark 2.1.22. Polygons in $\mathbb{R}^2$ and bounded polyhedra in $\mathbb{R}^3$ are polytopes; however not all polyhedra need to be bounded. A polytope $\Delta \subseteq M_{\mathbb{R}}$ can also be written as finite intersection of closed halfspaces. As such, every polytope is also a polyhedron but not vice versa, as for example the intersection of the closed halfspaces $x \leq 0, y \leq 0$ in $\mathbb{R}^2$ is a polyhedron but not a polytope.

An important class of polytopes are simplices. A polytope Δ of dimension d is called a simplex if it has $d+1$ vertices.

Writing polytopes as intersection of half-spaces is particularly useful when Δ is full dimensional, as then each facet F has a unique supporting hyperplane. In this case, write the supporting hyperplane to the facet F as

$$H_F = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle = -a_F\}.$$

Here, u_F is called an inward-pointing facet normal of the facet F. This allows us to write Δ as

$$\Delta = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \preceq \Delta\}.$$

Definition 2.1.23. Let Δ be a lattice polytope in $M_{\mathbb{R}}$. We define its dual polytope $\Delta^\vee$ to be

$$\Delta^\vee := \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq -1 \ \forall m \in \Delta\}.$$

We call Δ reflexive if the dual polytope $\Delta^\vee$ is also a lattice polytope. If there is an interior point $m \in \Delta$ so that $(\Delta - m)^\vee$ is a lattice polytope, we call Δ reflexive with respect to m.

It is well known that the unique interior lattice point of a reflexive polytope Δ is 0 (see, for example, Exercise 2.3.5 in [15]).

Given two polytopes $P_1 = \text{Conv}(S_1)$ and $P_2 = \text{Conv}(S_2)$, their Minkowski sum is $P_1 + P_2 = \text{Conv}(S_1 + S_2)$, where $S_1 + S_2 := \{m_1 + m_2 \mid m_1 \in S_1, m_2 \in S_2\}$.

We now define some properties of cones and polytopes which appear frequently in the study of mirror symmetry. The following definitions and results can be found in [6, 38, 51].

Definition 2.1.24. Let Δ be a full-dimensional lattice polytope in $M_{\mathbb{R}}$. Then Δ is called a Gorenstein polytope of index r ($r \in \mathbb{Z}_{>0}$) if $r\Delta$ contains a unique interior lattice point m and if $r\Delta - m$ is a reflexive polytope.

Definition 2.1.25. A Gorenstein cone σ is a cone in $M_{\mathbb{R}}$ with generators $v_1, \ldots, v_k \in M$ such that $\langle v_i, n_\sigma \rangle = 1$ for some element $n_\sigma \in N$ which we call the Gorenstein element of σ. The support Δ_σ of σ is the polytope $\text{Conv}(\{v_1, \ldots, v_k\})$ in the hyperplane $\langle x, n_\sigma \rangle = 1$ in $M_{\mathbb{R}}$.

We note that n_σ lies in the interior of $\sigma^\vee$. Define the k^{th} slice of σ as the lattice polytope

$$\sigma_{(k)} := \sigma \cap \{m \in M_{\mathbb{R}} \mid \langle m, n_\sigma \rangle = k\}.$$

Note that the support of σ is the 1^{st} slice, $\sigma_{(1)}$. There is a partition of the monoid $\sigma \cap M$ into the slices.

$$\sigma \cap M = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \sigma_{(k)} \cap M.$$

Definition 2.1.26. A reflexive Gorenstein cone σ is a Gorenstein cone σ whose dual $\sigma^\vee$ is also Gorenstein. Denoting its Gorenstein element with $m_{\sigma^\vee}$, the index of σ is the integer $r = \langle m_{\sigma^\vee}, n_\sigma \rangle$.

The following proposition links the notions of reflexive Gorenstein cones and Gorenstein polytopes.

Proposition 2.1.27 (Proposition 2.7 in [38], Proposition 2.11 in [6])**.** *Let σ be a Gorenstein cone in $M_{\mathbb{R}}$. Then the following are equivalent:*

A. σ is reflexive of index r;

B. $\sigma_{(r)}$ is a reflexive polytope with $m_{\sigma^\vee}$ as unique interior point;

C. The support polytope $\sigma_{(1)}$ of σ is a Gorenstein polytope of index r.

Definition 2.1.28. Let $\Delta_1, \ldots, \Delta_t$ be t lattice polytopes of positive dimension in $M_{\mathbb{R}}$. Define the Cayley polytope $\Delta_1 * \cdots * \Delta_t$ to be

$$\Delta_1 * \cdots * \Delta_t := \mathrm{Conv}((\Delta_1, e_1), \ldots, (\Delta_t, e_t)) \subseteq M_{\mathbb{R}} \times \mathbb{R}^t,$$

where e_i are the standard basis vectors for $\mathbb{R}^t$.

Definition 2.1.29. Let $\Delta_1, \ldots, \Delta_t$ be t lattice polytopes. We say a cone σ is a Cayley cone associated to t lattice polytopes if it can be written as $\sigma = \mathrm{Cone}(\Delta_1 * \cdots * \Delta_t)$.

Definition 2.1.30. Let σ be a reflexive Gorenstein cone of index r. If σ is also a Cayley cone associated to r lattice polytopes, then we say that σ is completely split.

Given a lattice polytope Δ, we can associate the fan of a toric variety to it in two ways: its normal fan or its face fan.

The first fan we discuss is the normal fan. Let $\Delta \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Faces, facets and vertices of Δ will be denoted by Q, F and v respectively.

A vertex $v \in \Delta$ corresponds to the (maximal) cones

$$C_v = \mathrm{Cone}\,(\Delta \cap M - v) \subseteq M_{\mathbb{R}} \text{ and } \sigma_v = C_v^\vee \subseteq N_{\mathbb{R}}.$$

Faces Q of the polytope Δ containing the vertex v correspond bijectively to faces Q_v of the cone C_v via the mutually inverse maps

$$Q \mapsto Q_v = \mathrm{Cone}\,(Q \cap M - v)$$
$$Q_v \mapsto Q = (Q_v + v) \cap \Delta.$$

So all facets of C_v come from facets of Δ that contain v, and thus

$$C_v = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq 0 \text{ for all facets } F \text{ containing } v\}.$$

It follows that the dual cone is given by $\sigma_v = \mathrm{Cone}\,(u_F \mid F \text{ contains } v)$. This can be generalised to arbitrary faces $Q \preceq \Delta$ by setting

$$\sigma_Q = \mathrm{Cone}\,(u_F \mid F \text{ contains } Q).$$

We then define the normal fan to the polytope Δ as $\Sigma_\Delta = \{\sigma_Q \mid Q \preceq \Delta\}$. By Theorem 2.3.2 in [15], this is indeed a fan in the sense of Definition 2.1.8.

The second way to associate a fan to a polytope Δ, used in f-duality (see §2.3.3 of this thesis), is a face fan. If $0 \in \Delta$, we define the face fan of Δ as follows. For every facet $\Phi \prec \Delta$ such that $0 \notin \mathrm{Relint}\,\Phi$ (where $\mathrm{Relint}\,\Phi$ is the relative interior of Φ, i.e. the interior of Φ in its span), we consider the cone

$$\sigma(\Phi) := \{r \cdot m \mid m \in \Phi,\ r \in \mathbb{R}_{\geq 0}\} \subseteq M_{\mathbb{R}}.$$

The face fan of Δ is defined as the collection of cones τ such that there is a face Φ of Δ with $\tau \in \sigma(\Phi)$. In other words, the face fan is the collection of cones $\{\tau \mid \exists \Phi \prec \Delta : \tau \in \sigma(\Phi)\}$.

Example 2.1.31. A way to think about this geometrically is to draw the lattice polytope and then connect all the vertices to 0. Consider for example the triangle in Figure 2.2. We can think of the fan being obtained by connecting the origin to the 3 vertices $(-1,-1)$, $(1,0)$ and $(0,1)$ and considering the cones over the faces we see. In this case, the face fan gives the fan for $\mathbb{P}^2$ that we saw in example 2.1.10.

We now know that every lattice polytope has a fan associated to it, but they are also useful when working with divisors. In fact, to every torus invariant divisor D we can assign a polyhedron. Consider a toric variety X_Σ, associated to a fan Σ. For $D = \sum_\rho a_\rho D_\rho$, we

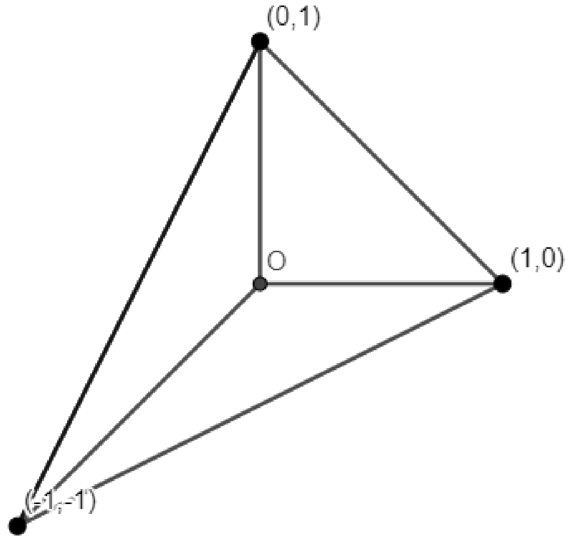

Figure 2.2: An example of a face fan

define the polyhedron

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

We obtain the following interesting proposition.

Proposition 2.1.32 (Proposition 4.3.3 in [15]). *If D is a T_N-invariant Weil divisor on X_Σ, then*

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

For each ray $\rho \in \Sigma(1)$, introduce a variable x_ρ. Then the toric variety X_Σ has the total coordinate ring $\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ of X_Σ. For additional information on the grading of the variables and more we refer the reader to §5 in [15].

As a particular special case of Proposition 2.1.32, we note the following. Take $D = 0$, so as to obtain the trivial sheaf on X_Σ. Then the polyhedron P_D is simply $|\Sigma|^\vee$. Hence, a point m in $|\Sigma|^\vee$ corresponds to a global section $x^m \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma})$, i.e. to a global function, via

$$x^m = \prod_{\rho \in \Sigma(1)} x_\rho^{\langle m, u_\rho \rangle},$$

where u_ρ is the primitive generator of the ray $\rho \in \Sigma(1)$ and x_ρ is the variable associated to it.

2.1.4 The Cox construction

Since associating a variety to a fan by gluing explicitly becomes unwieldy with an increasing number of cones, we would like a more efficient way to do this. One idea is to express toric varieties as quotients of complex space by a group acting on it. A motivating example is $\mathbb{P}^n = (\mathbb{C}^n \setminus \{0\}) / \mathbb{C}^*$. Building on that, we would like to express the toric variety associated to a fan as (almost geometric) quotient $(\mathbb{C}^r \setminus Z) /\!\!/ G$ for some exceptional set Z and reductive group G. This can be achieved by means of the Cox construction (see §5 of [15]). First, let us define what we mean by almost geometric quotient.

Definition 2.1.33. Let X, Y be varieties, G a group acting on X, and $\pi : X \to Y$ a morphism that is constant on G-orbits. Then π is a good categorical quotient if:

- If $U \subseteq Y$ is open, then the natural map $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ induces an isomorphism
$$\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\pi^{-1}(U))^G.$$
- If $W \subseteq X$ is closed and G-invariant, then $\pi(W) \subseteq Y$ is closed.
- If W_1, W_2 are closed, disjoint, and G-invariant in X, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint in Y.

Write a good categorical quotient as $\pi : X \to X /\!\!/ G$. A particularly nice form of good categorical quotient is a geometric quotient. A geometric quotient is a good categorical quotient which satisfies the conditions of the following proposition.

Proposition 2.1.34. *Let $\pi : X \to X /\!\!/ G$ be a good categorical quotient. Then the following are equivalent:*

- *All G-orbits are closed in X.*
- *Given points $x, y \in X$, we have*

$$\pi(x) = \pi(y) \Leftrightarrow x \textit{ and } y \textit{ lie in the same } G - \textit{orbit.}$$

- *π induces a bijection*

$$\{G - \textit{orbits in } X\} \simeq X /\!\!/ G.$$

- *The image of the morphism $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is the fiber product $X \times_{X /\!\!/ G} X$.*

Since the points of $X /\!\!/ G$ are in bijection with G-orbits in X, we write a geometric quotient as $\pi : X \to X/G$. Not all good categorical quotients are geometric, but some are very close to it. These are called almost geometric quotients and are characterised by the equivalent properties of the following proposition.

Proposition 2.1.35. *Let $\pi : X \to X /\!\!/ G$ be a good categorical quotient. Then the following are equivalent:*

- *X has a G-invariant Zariski dense open subset U_0 such that $G \cdot x$ is closed in X for all $x \in U_0$.*
- *$X /\!\!/ G$ has a Zariski dense open subset U such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient.*

Before we describe how to describe toric varieties as almost geometric quotients, we first define the kind of groups G that we will work with.

Definition 2.1.36. An affine algebraic group G is called reductive if its maximal connected solvable subgroup is a torus.

Example 2.1.37. Finite groups and tori are reductive groups and will be the reductive groups that we treat in this thesis.

Reductive groups have some key properties which will become vital in the following.

Proposition 2.1.38. *Let G be a reductive group acting algebraically on an affine variety $X = \operatorname{Spec}(R)$. Then:*

- *R^G is a finitely generated $\mathbb{C}$-algebra.*
- *The morphism $\pi : X \to \operatorname{Spec}(R^G)$ induced by $R^G \subseteq R$ is a good categorical quotient.*

We can now proceed with the construction of toric varieties as almost geometric quotients. Recall the exact sequence (2.5) (from Theorem 2.1.19, with slightly adjusted notation):

$$0 \to M \xrightarrow{\iota} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}\, D_\rho \to \operatorname{coker} \iota \to 0, \tag{2.6}$$

where $\iota(m) := \operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$.

We will write $\mathbb{Z}^{\Sigma(1)} := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}\, D_\rho$. Since $\mathbb{C}^*$ is a divisible group and hence an injective module, the functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ is exact, so applying it to (2.6) yields the exact sequence:

$$1 \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \iota, \mathbb{C}^*) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \to \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \to 1. \tag{2.7}$$

Define

$$G_\Sigma := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker} \iota, \mathbb{C}^*). \tag{2.8}$$

Note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\Sigma(1)}$ and $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \simeq T_N$, where T_N is the torus of the variety. Hence we may rewrite (2.7) as

$$1 \to G_\Sigma \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1. \tag{2.9}$$

When describing G_Σ explicitly, the following lemma is useful.

Lemma 2.1.39 (Lemma 5.1.1(c) in [15]). *Let $G_\Sigma \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ be as in* (2.9). *Given a basis $e_1, \ldots, e_n$ of M, we have*

$$G_\Sigma = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_\rho t_\rho^{\langle e_i, u_\rho \rangle} = 1 \textit{ for } 1 \leq i \leq n\}.$$

We now have both an affine space $\mathbb{C}^{\Sigma(1)}$ and a group G_Σ, which can be shown to be reductive and thus only further require an exceptional set Z in order to construct the toric variety X_Σ as a geometric quotient. For each ray $\rho \in \Sigma(1)$, introduce a variable x_ρ and consider the total coordinate ring of X_Σ,

$$S := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)].$$

For each cone $\sigma \in \Sigma$, let $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. We define the irrelevant ideal

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle \subseteq \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)].$$

If $\tau \preceq \sigma$, then $x^{\hat{\sigma}}$ is a factor of $x^{\hat{\tau}}$. Hence we only need to consider maximal cones to generate the irrelevant ideal. Define $Z(\Sigma) = Z(B(\Sigma)) \subseteq \mathbb{C}^{\Sigma(1)}$. We then have:

Theorem 2.1.40 (Theorem 5.1.11 in [15]). *Let X_Σ be a toric variety without torus factors, associated to a fan Σ. Then*

$$X_\Sigma \simeq (\mathbb{C}^{|\Sigma(1)|} \backslash Z(\Sigma)) /\!\!/ G.$$

Here, a toric variety is said to have a torus factor if it is equivariantly isomorphic to the product of a nontrivial torus and a toric variety of smaller dimension. By Proposition 3.3.9 in [15], a toric variety of a fan Σ has a torus factor if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ does not span $N_\mathbb{R}$.

Most of the following discussion happens on the level of *toric stacks* instead of the level of toric varieties. The section § 2.1.5, gives a gentle introduction to stacks in general.

In this thesis, we will use the following working definition of a toric stack.

Definition 2.1.41. Let Σ be a fan. Define the Cox fan $\mathrm{Cox}(\Sigma) \subseteq \mathbb{R}^{|\Sigma(1)|}$ to be

$$\mathrm{Cox}(\Sigma) := \left\{ \mathrm{Cone}(e_\rho \mid \rho \in \sigma) \middle| \sigma \in \Sigma \right\}.$$

Denote by n the number of rays in the fan Σ. Then the Cox fan of Σ is a subfan of the standard fan corresponding to the toric variety $\mathbb{A}^n$. Thus, $U_\Sigma := X_{\mathrm{Cox}(\Sigma)}$ is an open subset of $\mathbb{A}^n$. Recall G_Σ as in Equation (2.8). Then we define the following stack:

Definition 2.1.42. We call U_Σ the Cox open set associated to Σ and define the Cox stack associated to Σ to be the quotient stack

$$\mathcal{X}_\Sigma := [U_\Sigma / G_\Sigma].$$

In the smooth and orbifold case, we have the following result relating $\mathcal{X}_\Sigma$ to X_Σ.

Theorem 2.1.43 ([17])**.** *If Σ is simplicial, then $\mathcal{X}_\Sigma$ is a smooth Deligne-Mumford stack with coarse moduli space X_Σ. When Σ is smooth (or equivalently X_Σ is smooth) $\mathcal{X}_\Sigma \cong X_\Sigma$.*

Example 2.1.44. We illustrate the Cox construction on an example. Let us reuse the example of $\mathbb{P}^2$ given previously in Example 2.1.10. So consider the standard fan Σ for $\mathbb{P}^2$, as depicted in figure 2.1. In this case, the exact sequence (2.7) takes the form

$$0 \to \mathbb{Z}^2 \xrightarrow{\iota} \mathbb{Z}^3 \to \mathrm{coker}\,\iota \to 0,$$

where ι can be represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$.

We first note that $|\Sigma(1)| = 3$. This has Smith normal form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, and therefore the cokernel has no torsion, and hence $G_\Sigma = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{coker}\,\iota, \mathbb{C}^*) \cong \mathbb{C}^*$.

Using the Lemma 2.1.39, we compute G_Σ explicitly. Elements of G_Σ are of the form (t_1, t_2, t_3), subject to

$$t_1 t_3^{-1} = 1,$$
$$t_2 t_3^{-1} = 1.$$

In other words, $t_1 = t_3 = t_2$. Thus, $G_\Sigma = \{(t, t, t) \mid t \in \mathbb{C}^*\}$.

The exceptional set $Z(\Sigma)$ is found by considering the maximal cones $\sigma_{0,1}, \sigma_{1,2}$ and $\sigma_{2,0}$. We have

$$x^{\widehat{\sigma_{0,1}}} = x_2,$$
$$x^{\widehat{\sigma_{1,2}}} = x_0,$$
$$x^{\widehat{\sigma_{2,0}}} = x_1.$$

Thus, $B(\Sigma) = \langle x_0, x_1, x_2 \rangle$ and hence $Z(\Sigma) = Z(B(\Sigma)) = Z(\langle x_0, x_1, x_2 \rangle) = \{0\}$.

Theorem 2.1.40 then gives $X_\Sigma = (\mathbb{C}^3 \setminus \{0\}) /\!\!/ \mathbb{C}^*$, with $\mathbb{C}^*$ acting by $(t, t, t) \sim (1, 1, 1)$. This is the quotient description of $\mathbb{P}^2$, showing that the Cox Construction also obtains $\mathbb{P}^2$ as the variety obtained for the fan Σ.

2.1.5 A note on stacks

The aim of this section is to give a general idea of what a stack is. Roughly speaking, a stack over a scheme S is a category $\mathcal{M}$ equipped with a functor $p : \mathcal{M} \to \mathbf{Sch}/S$ fulfilling some lifting and gluing properties. Important to note here is that the underlying topology we work with is a Grothendieck topology, usually étale.

Another way to introduce stacks is more differential geometric in flavour than algebraic, in particular it uses the usual topology on manifolds.

Start by recalling Yoneda's lemma applied to the category of manifolds, which states that any manifold/space M is determined by the functor $Map(-, M) : \mathbf{Manifolds} \to \mathbf{Sets}$.

The idea to take away from this is that sometimes, instead of directly describing an object, we can equivalently describe it via a functor associated to it.

This works for stacks as well, so instead of defining a stack $\mathcal{M}$ algebraically as a category with some properties, we give a definition of stacks via 2-functors.

Definition 2.1.45 ([29], Definition 1.1)**.** A stack $\mathcal{M}$ is a 2-functor

$$\mathcal{M} : \mathbf{Manifolds} \to \mathbf{Groupoids} \subset Cat,$$

fulfilling some gluing properties[1]

A. We can glue objects: Given an open covering U_i of X, objects $P_i \in \mathcal{M}(U_i)$ and isomorphisms $\varphi_{ij} : P_i|_{U_i \cap U_j} \to P_j|_{U_i \cap U_j}$ satisfying the cocycle condition on threefold intersections, there is an object $P \in \mathcal{M}(X)$ together with isomorphisms $\varphi_i : P|_{U_i} \to P_i$ such that $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$.

B. We can glue morphisms: Given $P, P' \in \mathrm{Ob}(\mathcal{M}(X))$, an open covering U_i of X and isomorphisms $\varphi_i : P|_{U_i} \to P'|_{U_i}$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there is a unique $\varphi : P \to P'$ such that $\varphi_i = \varphi|_{U_i}$.

The toric stacks briefly defined above belong to a certain class of stacks, known as *quotient stacks*.

Definition 2.1.46 ([29], Example 1.5)**.** Let G be a Lie group acting on a manifold X on the left. We define the quotient stack $[X/G]$ as

$$[X/G](Y) := \langle (P \xrightarrow{\mathrm{p}} Y, P \xrightarrow{\mathrm{f}} X) | P \to Y \text{ a G-bundle, f is G-equivariant} \rangle.$$

2.1.6 Toric morphisms and vector bundles

Toric varieties admit morphisms between them that respect the toric structure. We call these toric morphisms.

[1] Adopting the notation $|_U$ instead of $j^* : \mathcal{M}(X) \to \mathcal{M}(U)$ for open embeddings $j : U \hookrightarrow X$.

Definition 2.1.47. Let $X_{\Sigma_1}, X_{\Sigma_2}$ be normal toric varieties, with Σ_i a fan in $(N_i)_\mathbb{R}$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is toric if ϕ maps the torus $T_{N_1} \subseteq X_{\Sigma_1}$ into the torus $T_{N_2} \subseteq X_{\Sigma_2}$, i.e. if $\phi(T_{N_1}) \subseteq T_{N_2}$, and $\phi|_{T_{N_1}}$ is a group homomorphism.

Toric morphisms correspond to maps of the underlying cocharacter lattices N_1, N_2 that are compatible with the fans of the varieties.

Definition 2.1.48. Let N_1, N_2 be two lattices with a fan Σ_1 in $(N_1)_\mathbb{R}$ and a fan Σ_2 in $(N_2)_\mathbb{R}$. A $\mathbb{Z}$-linear mapping $\overline{\phi} : N_1 \to N_2$ is compatible with the fans Σ_1, Σ_2 if for every cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ such that $\overline{\phi}_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.

Theorem 2.1.49 (Theorem 3.3.4 in [15])**.** *Let N_1, N_2 be lattices and let Σ_i be a fan in $(N_i)_\mathbb{R}$, $i = 1, 2$.*

A. If $\overline{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1 and Σ_2, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\phi|_{T_{N_1}}$ is the map

$$\overline{\phi} \otimes 1 : N_1 \otimes_\mathbb{Z} \mathbb{C}^* \to N_2 \otimes_\mathbb{Z} \mathbb{C}^* .$$

B. Conversely, if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$ that is compatible with the fans Σ_1 and Σ_2.

A Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on a toric variety X_Σ gives the line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$, which is the sheaf of sections of a rank 1 vector bundle $\pi : V_\mathcal{L} \to X_\Sigma$. The variety $V_\mathcal{L}$ is toric and π is a toric morphism. This is shown by directly constructing the fan of $V_\mathcal{L}$ in terms of Σ and D, which we will do now.

Define the fan $\Sigma \times D$ in $N_\mathbb{R}$ as follows. Given a cone $\sigma \in \Sigma$, set

$$\tilde{\sigma} = \operatorname{Cone}\left((0, 1), (u_\rho, -a_\rho) \mid \rho \in \sigma(1)\right).$$

Then $\tilde{\sigma}$ is a strongly convex rational polyhedral cone in $N_\mathbb{R} \times \mathbb{R}$ for all cones $\sigma \in \Sigma$. We then let $\Sigma \times D$ be the collection of cones $\tilde{\sigma}$ for $\sigma \in \Sigma$ and their faces. This is a fan in

$N_{\mathbb{R}} \times \mathbb{R}$ and the projection $\overline{\pi} : N \times \mathbb{Z} \to N$ is compatible with $\Sigma \times D$ and Σ, thus inducing a toric morphism

$$\pi : X_{\Sigma \times D} \to X_{\Sigma}.$$

Proposition 2.1.50 (Proposition 7.3.1 in [15])**.** *$\pi : X_{\Sigma \times D} \to X_{\Sigma}$ is a rank 1 vector bundle whose sheaf of sections is $\mathcal{O}_{X_{\Sigma}}(D)$.*

The variety $X_{\Sigma \times D}$ will sometimes also be denoted by $X_{\Sigma, D}$.

For decomposable vector bundles of rank higher than 1, we can repeatedly apply Proposition 2.1.50 to construct the total space of the vector bundle, following [19]. Taking r torus-invariant Weil divisors $D_i = \sum_{\rho \in \Sigma} a_{i\rho} D_{\rho}$, we define

$$\sigma_{D_1, \ldots, D_r} := \operatorname{Cone}\left(\{u_{\rho} - a_{1\rho} e_1 - \cdots - a_{r\rho} e_r \mid \rho \in \sigma(1)\} \cup \{e_i | i \in \{1, \ldots, r\}\}\right) \subset N_{\mathbb{R}} \oplus \mathbb{R}^r .$$

Let $\Sigma_{D_1, \ldots, D_r}$ be the fan generated by the cones $\sigma_{D_1, \ldots, D_r}$ and their proper faces, and call $\mathcal{X}_{\Sigma, D_1, \ldots, D_r}$ the associated stack. We obtain the following result.

Proposition 2.1.51 (Proposition 4.13 in [19])**.** *Let $D_1, \ldots, D_r$ be divisors on X_{Σ}. There is an isomorphism of stacks*

$$\mathcal{X}_{\Sigma, D_1, \ldots, D_r} \cong \operatorname{tot}\left(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathcal{X}_{\Sigma}}(D_i)\right).$$

Example 2.1.52. As an example, let us construct a fan for $\operatorname{tot} \mathcal{O}_{P^2}(-3)$. We use the same fan as before, depicted in Figure 2.1. Firstly, we note that $\operatorname{Cl}(\mathbb{P}^2) = \mathbb{Z}$ and $\mathcal{O}(D_{\rho_0}) \simeq \mathcal{O}(D_{\rho_1}) \simeq \mathcal{O}(D_{\rho_2}) \simeq \mathcal{O}(1)$. Thus we represent the anticanonical divisor $-K_{\mathbb{P}^2}$ as $-(D_{\rho_0} + D_{\rho_1} + D_{\rho_2})$. The fan $\Sigma \times -K_{\mathbb{P}^2}$ thus has the four rays $r_0 = (-1, -1, 1)$, $r_1 = (1, 0, 1)$, $r_2 = (0, 1, 1)$ and $r_3 = (0, 0, 1)$. The higher dimensional cones are, listed via their ray generators, $\operatorname{Cone}(r_0, r_1, r_3), \operatorname{Cone}(r_0, r_2, r_3)$ and $\operatorname{Cone}(r_1, r_2, r_3)$ as well as their faces.

2.1.7 GIT quotients

Geometric invariant theory (GIT), developed by Mumford, is a powerful tool in modern algebraic geometry. We will here discuss the toric version of it, following §14 of [15].

Roughly speaking, GIT deals with ways to take almost geometric quotients of spaces by some reductive groups acting on them. As a model for this, recall the Cox construction in §2.1.4. It gives a toric variety as almost geometric quotient $X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)) /\!\!/ G_\Sigma$. Fundamentally, we start with $\mathbb{C}^{\Sigma(1)}$ and remove a special Zariski closed subset in order to obtain an almost geometric quotient. GIT provides the machinery to do so, but the way to apply it is often not unique. The subsets that are removed depend on a choice of stability condition, parameterised by a choice of line bundle. The different choices can give different, birational quotients.

In GIT, deciding which points are removed is done via a lifting of the G-action on $\mathbb{C}^r$ to the rank 1 trivial vector bundle $\mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^r$. Define the character group of G to be

$$\widehat{G} = \{\chi : G \to \mathbb{C}^* \,|\chi \text{ is a homomorphism of algebraic groups}\}.$$

A character $\chi \in \widehat{G}$ then gives the action of G on $\mathbb{C}^r \times \mathbb{C}$ defined by

$$g \cdot (p,t) = (g \cdot p, \chi(g)t), \quad g \in G, \ (p,t) \in \mathbb{C}^r \times \mathbb{C}\,.$$

This lifts the G-action on $\mathbb{C}^r$. Furthermore, all possible liftings arise this way.

Let $\mathcal{L}_\chi$ or $\mathcal{O}(\chi)$ denote the sheaf of sections of $\mathbb{C}^r \times \mathbb{C}$ with this G-action. It is called the linearised line bundle with character χ. For $d \in \mathbb{Z}$, the tensor product $\mathcal{O}(\chi)^{\otimes d}$ is the linearised line bundle with character χ^d. As a line bundle on $\mathbb{C}^r$, $\mathcal{O}(\chi) \simeq \mathcal{O}_{\mathbb{C}^r}$, forgetting the G-action. Thus, a global section $s \in \Gamma(\mathbb{C}^r, \mathcal{O}(\chi))$ can be written as

$$s : \mathbb{C}^r \to \mathbb{C}^r \times \mathbb{C}$$

$$p \mapsto (p, F(p)),$$

for some unique $F_s \in \mathbb{C}[x_1, \ldots, x_r]$.

Definition 2.1.53. Fix $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, with linearised line bundle $\mathcal{O}(\chi)$. Given a global section s of $\mathcal{O}(\chi)$, we denote

$$(\mathbb{C}^r)_s := \{p \in \mathbb{C}^r \mid s(p) \neq 0\}.$$

This is an affine open subset of $\mathbb{C}^r$, as $s(p) \neq 0$ means $F_s(p) \neq 0$. Furthermore, G acts on $(\mathbb{C}^r)_s$ when s is G-invariant. We define:

A. $p \in \mathbb{C}^r$ is semistable with respect to χ if there exist $d > 0$ and $s \in \Gamma(\mathbb{C}^r, \mathcal{O}(\chi^d))^G$ such that $p \in (\mathbb{C}^r)_s$.

B. $p \in \mathbb{C}^r$ is stable with respect to χ if there exist $d > 0$ and $s \in \Gamma(\mathbb{C}^r, \mathcal{O}(\chi^d))^G$ such that $p \in (\mathbb{C}^r)_s$, the isotropy subgroup G_p is finite, and all G-orbits in $(\mathbb{C}^r)_s$ are closed in $(\mathbb{C}^r)_s$.

C. The set of all semistable (resp. stable) points with respect to χ is denoted $(\mathbb{C}^r)^{ss}_\chi$ (resp. $(\mathbb{C}^r)^s_\chi$).

Given a group $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, we next need to define the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$. Consider the graded ring $R_\chi = \bigoplus_{d=0}^\infty \Gamma(\mathbb{C}^r, \mathcal{O}(\chi^d))^G$.

Definition 2.1.54. For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is

$$\mathbb{C}^r /\!\!/_\chi G = \operatorname{Proj}(R_\chi).$$

An important property of GIT quotients is that in principle, this is the same as taking the quotient of $(\mathbb{C}^r)^{ss}_\chi$ under the action of G.

Proposition 2.1.55 (Proposition 14.1.12.c) in [15]). *For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is a good categorical quotient of $(\mathbb{C}^r)^{ss}_\chi$ under the action of G, i.e. $\mathbb{C}^r /\!\!/_\chi G \simeq (\mathbb{C}^r)^{ss}_\chi /\!\!/ G$.*

Theorem 14.2.13 of [15] shows, using a polyhedron associated to the character χ, that the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is a toric variety.

2.1.8 GKZ Fans

Let $G \subseteq (\mathbb{C}^*)^r$. Studying the GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ as χ varies gives rise to the GKZ fan, or also secondary fan, of a toric variety, which has the structure of a generalised fan.

Definition 2.1.56. A generalised fan Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:

A. Every $\sigma \in \Sigma$ is a rational polyhedral cone.

B. For all $\sigma \in \Sigma$, each face of σ is also in Σ.

C. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

This agrees with the Definition 2.1.8 of a fan, with the exception that cones are not necessarily strongly convex. Consider the cone $\sigma_0 = \bigcap_{\sigma \in \Sigma} \sigma$. It has no proper faces and is thus a subspace of $N_\mathbb{R}$. We consider the lattice $\overline{N} = N/(\sigma_0 \cap N)$. To associate a toric variety for the generalised fan Σ, one constructs the fan $\overline{\Sigma}$ where each cone comes from a cone of Σ quotiented by σ_0. This is a fan in the sense of Definition 2.1.8, and hence we can associate a toric variety to it as usual. Then $X_\Sigma := X_{\overline{\Sigma}}$.

We will now discuss the notion of a GKZ fan, following both [15] and [18]. Consider a toric variety X. It can be written as a GIT quotient $(\mathbb{C}^r \setminus Z) /\!\!/_\chi G$. Recall the character group $\widehat{G}$ of G. Each choice of character $\chi \in \widehat{G}$ determines an open subset $U_\chi := (\mathbb{C}^r)^{ss}_\chi$, the semi-stable locus of X with respect to χ. Several different characters can give the same semi-stable locus. Thinking of the vector space $\mathrm{Hom}(\widehat{G}, T_N) \otimes_\mathbb{Z} \mathbb{Q}$ as parameter space for linearisations, we investigate where the semi-stable locus U_ψ is the same as U_χ for a given character χ. It turns out that dividing the vector space into chambers where U_χ remains the same gives the space a natural fan structure. This fan-structure Σ_{GKZ} is called the GKZ fan. Maximal cones are called chambers and codimension one cones are called walls.

Consider an arbitrary fan Σ, we can construct the GKZ fan as follows. Take the group $G = G_\Sigma \subseteq (\mathbb{C}^*)^r$ acting on X_Σ to be the group in Equation (2.8). There is a well-known bijection between chambers of GKZ fans and regular triangulations of a certain set of points, constructed as follows. In the general setting, apply $\mathrm{Hom}(-, \mathbb{C}^*)$ to the sequence

$$0 \to G \xrightarrow{i_G} (\mathbb{C}^*)^r \xrightarrow{proj} \mathrm{coker}(i_G) \to 0$$

to obtain the sequence

$$\mathrm{Hom}(\mathrm{coker}(i_G), \mathbb{C}^*) \xrightarrow{\widehat{proj}} \mathbb{Z}^r \xrightarrow{\widehat{i_G}} \mathrm{Hom}(G, \mathbb{C}^*) \to 0.$$

Let $\nu_i(G)$ be the element of $\mathrm{Hom}(\mathrm{coker}(i_G), \mathbb{C}^*)^\vee$ given by composing $\widehat{proj}$ with the projection of $\mathbb{Z}^r$ onto its i^{th} factor. Compare this sequence with the sequence (2.6). We in fact reversed the process of obtaining (2.9) from (2.6). Starting with the correct group acting on the space, we thus recover the map corresponding to ι as $\widehat{proj}$. Hence, the $\nu(G)$ correspond to the primitive generators u_ρ of the rays of Σ. Then the set we will triangulate is the convex hull of the set $\nu(G) = \{\nu_1(G), \ldots, \nu_r(G)\}$.

Recall here what regular triangulations are.

Definition 2.1.57. Let ν be a collection of points in $N_\mathbb{R}$. A triangulation $\mathcal{T}$ of ν is a collection of simplices satisfying:

- Each simplex in $\mathcal{T}$ has codimension 1 in $N_\mathbb{R}$ with vertices in ν.
- The intersection of any two simplices in $\mathcal{T}$ is a face of each.
- The union of the simplices in $\mathcal{T}$ is $\mathrm{Conv}(\nu)$.

Definition 2.1.58. Let $\nu = \{\nu_1, \ldots, \nu_r\}$. Given nonnegative weights $\omega = (w_1, \ldots, w_r) \in \mathbb{R}_{\geq 0}$, we define the cone

$$C_{\nu,\omega} = \mathrm{Cone}((\nu_1, w_1), \ldots, (\nu_r, w_r)) \subseteq N_\mathbb{R} \times \mathbb{R}\,.$$

The lower hull of $C_{\nu,\omega}$ consists of all facets of $C_{\nu,\omega}$ whose inner normal has a positive last coordinate. Projecting the facets in the lower hull and their faces gives a fan Σ_ω in $N_\mathbb{R}$ such that $|\Sigma_\omega| = \mathrm{Cone}(\nu)$ and $\Sigma_\omega(1) \subseteq \{\mathrm{Cone}(\nu_i) | 1 \leq i \leq r\}$. A triangulation $\mathcal{T}$ of ν is regular if there are weights ω such that Σ_ω is simplicial and $\mathcal{T} = \Sigma_\omega \cap \mathrm{Conv}(\nu)$.

We can now formulate the following result, which enables us to efficiently study GKZ fans.

Theorem 2.1.59 (Proposition 15.2.9 in [15])**.** *There is a bijection between chambers of the GKZ fan for the action of G on $\mathbb{C}^r$ and regular triangulations of the set $\nu(G)$. In particular, there are only finitely many chambers of the GKZ fan.*

Thus we can enumerate the chambers of the GKZ fan, say by $\sigma_1, \ldots, \sigma_k$. For any of those chambers, we can choose a character in its interior and consider the semi-stable locus with respect to it. As this locus does not depend on the choice of character, but solely on the choice of chamber, denote the open affine associated to chamber σ_p by U_p. By the above theorem, it will also correspond to a specific triangulation $\mathcal{T}_p$ of $\{\nu_1(G), \ldots, \nu_r(G)\}$.

Example 2.1.60. In this example, we illustrate hands-on methods from §15 of [15] on how to construct secondary fans. We consider $\mathrm{tot}\, \mathcal{O}_{\mathbb{P}^2}(-3)$, with the fan constructed in Example 2.1.52. Consider the exact sequence (2.7) for this fan, being

$$0 \to \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^3 \xrightarrow{B} \mathrm{coker} \to 0.$$

The rows of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ give the points $\nu_i(G)$ we want to triangulate.

Since all these points lie in the hyperplane $z = 1$, we can intersect with that hyperplane and are left to triangulate the 4 points $(1, 0), (0, 1), (-1, -1), (0, 0)$. There are two ways to do this, illustrated in Figure 2.3.

Figure 2.3: Possible triangulations

Figure 2.4: Secondary fan of $\operatorname{tot}\mathcal{O}_{\mathbb{P}^2}(-3)$.

The map B is represented by the matrix $(1, 1, 1, -3)$. Each of the columns gives a ray of the secondary fan (in this case, one of the rays is counted thrice). The secondary fan is thus the one in Figure 2.4.

2.2 Derived categories and Homological Mirror Symmetry

In this section, we will state Homological Mirror Symmetry. Kontsevich [33] conjectured that mirror symmetry between two Calabi-Yau manifolds $X, X^\vee$ is expressed as an equivalence between a certain pair of categories one can define on the manifolds $X, X^\vee$. In the following, we will be taking a closer look at these categories in the statement of Homological Mirror Symmetry.

2.2.1 Derived Categories

We begin with the categorical setup, introducing derived categories, largely following [12]. Derived categories stem from the idea that while complexes are good and carry a lot of information, reducing to their homology is bad and loses some of that information. An example of this are simplicial complexes in algebraic topology. There exist non-homotopy equivalent simplicial complexes X, Y with isomorphic homology. An example of this are homology spheres, which are n-manifolds of the same homology as n-spheres. The concept of them goes back to Poincaré (1904); a treatment of the case $n = 1$ is given in [16]. Further examples of this phenomenon include knot complements, since all knot complements have the same homology but for distinct knots, the complements may differ in homotopy.

An upside to this is Whitehead's theorem, which loosely states that two complexes X and Y are homotopy equivalent if and only if there is a third simplicial complex Z with quasi-isomorphisms $f_* : C_\bullet(Z) \to C_\bullet(X)$, $g_* : C_\bullet(Z) \to C_\bullet(Y)$ (where $C_\bullet(-)$ is the chain complex associated to a simplicial complex).

The aim now is to construct a category which carries the data of a complex. Start with an abelian category $\mathcal{A}$ and construct a new category whose objects are complexes of objects of $\mathcal{A}$, and where the morphisms are chain maps. An object of that category will thus have the form $A^\bullet = \dots \overset{d_A^{i-2}}{\to} A^{i-1} \overset{d_A^{i-1}}{\to} A^i \overset{d_A^i}{\to} A^{i+1} \overset{d_A^{i+1}}{\to} \dots$, with $A^i \in \mathrm{Ob}(\mathcal{A})$ and the d_A^j are differential maps, so $d_A^{i+1} \circ d_A^i = 0$. Note the fact that the notation is cohomological (so upper indices and increasing degree).

We say that two morphisms $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are homotopic if and only if there is a chain map $h^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^i - g^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i$ for all i. The $[-1]$ here refers to the shift functor, and means that the component maps of $h^\bullet$ are of the form $h^i : A^i \to B^{i-1}$.

Definition 2.2.1 (Definition 1.1 in [12])**.** Let $\mathcal{A}$ be an abelian category. We define the homotopy category of $\mathcal{A}$, $\mathbf{K}(\mathcal{A})$, to be the category whose objects are complexes of objects of $\mathcal{A}$ and morphisms between complexes are chain maps modulo the homotopy equivalence relation.

Remark 2.2.2. Some authors use $H^0\mathcal{A}$ for the homotopy category, see for instance [28]. We shall adapt that notation later on when introducing dg-categories and A_∞-categories. In the more abstract concept of derived categories in general, we decided to keep the notation of [12]. The homotopy category of a dg-category $\mathcal{C}$ is also denoted by $[\mathcal{C}]$, which is a very convenient notation used later in §2.2.5, 2.2.7 and 4.

Definition 2.2.3 (Definition 1.3 in [12])**.** A chain map of complexes $f^\bullet : A^\bullet \to B^\bullet$ is called a quasi-isomorphism if the induced maps $H^i(f^\bullet) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all i.

Definition 2.2.4. The derived category $\mathbf{D}(\mathcal{A})$ of the abelian category $\mathcal{A}$ is obtained by considering quasi-isomorphisms to be isomorphisms in $\mathbf{K}(\mathcal{A})$.

This is called *localisation*, analogously to rings. Morphisms in $\mathbf{D}(\mathcal{A})$ will be *roofs*, which are diagrams of the following form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A^\bullet \arrow[rr, dotted] & & B^\bullet, \end{tikzcd}$$

where f, g are morphisms in $\mathbf{K}(\mathcal{A})$ and where f is a quasi-isomorphism. The roof in the diagram represents $g \circ f^{-1}$, even though f is not necessarily an actual isomorphism, hence not necessarily invertible.[1]

Example 2.2.5. An important example (see [3]) of derived categories is the bounded derived category of coherent sheaves. Recall some facts from algebraic geometry. Let X be an algebraic variety with sheaf of functions $\mathcal{O}_X$. We will restrict ourselves to quasi-projective (and for the most part actually smooth projective) varieties over a field k, usually $\mathbb{C}$.

A sheaf $\mathcal{F}$ on $\mathcal{O}_X$ is called quasi-coherent if locally $\mathcal{F}$ is the cokernel of a morphism of free $\mathcal{O}_X$-modules. If those $\mathcal{O}_X$-modules can be chosen to be of finite rank, then the sheaf $\mathcal{F}$ is called coherent.

[1]The way to think about it analogously to localisations of rings is that $2/3 = 2 \cdot 3^{-1} \in \mathbb{Q}$, even though 3^{-1} does not exist in $\mathbb{Z}$.

We note that the category of coherent sheaves on X, $\mathrm{Coh}(X)$, is abelian, and so is the category of quasi-coherent sheaves $\mathrm{Qcoh}(X)$. Furthermore, $\mathrm{Qcoh}(X)$ possesses enough injective objects (i.e. every object admits a monomorphism into an injective object). This allows to resolve any quasi-coherent sheaf by some bounded below complex of injective sheaves.

Consider the subcategory of all injectives with quasi-coherent cohomology, meaning that all chain homology groups are quasi-coherent sheaves. Taking its homotopy category, we have now formed the derived category of $\mathrm{QCoh}(X)$.

Similarly, consider the subcategory of $\mathrm{Coh}(X)$ of all bounded below complexes of injectives with bounded coherent cohomology. Taking the homotopy category, we obtain the bounded derived category of $\mathrm{Coh}(X)$, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.

2.2.2 Triangulated Categories

When passing to the homotopy category, we lost the notion of exactness. To get some similar notion in the homotopy and derived categories, Verdier introduced exact triangles. For this, we introduce the concept of triangulated categories.

Definition 2.2.6 (Definition 1.1 in [37]). Let $\mathcal{C}$ be an additive category and $T : \mathcal{C} \to \mathcal{C}$ be an additive auto-equivalence. A triangle in $\mathcal{C}$ with respect to T is a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA.$$

A morphism of triangles is a commutative diagram of the form given in Figure 2.5:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle Tu} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA' \end{array}$$

Figure 2.5: A morphism of triangles.

Definition 2.2.7 (Definition 1.2 in [37]). A triangulated category is a triple $(\mathcal{C}, T, \mathcal{D})$ where $\mathcal{C}$ is a category, $T : \mathcal{C} \to \mathcal{C}$ is an additive auto-equivalence and $\mathcal{D}$ is a class of distinguished triangles, satisfying the following axioms:

(TR0) The class of distinguished triangles is closed under isomorphisms. Moreover, the triangle

$$A \xrightarrow{\mathrm{id}_A} A \to 0 \to TA$$

is distinguished.

(TR1) For any morphism $f : A \to B$ in $\mathcal{C}$ there exists a distinguished triangle of the form

$$A \xrightarrow{f} B \to C \to TA.$$

(TR2) Consider the two triangles

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA \tag{2.10}$$

and

$$B \xrightarrow{g} C \xrightarrow{h} TA \xrightarrow{-Tf} TB. \tag{2.11}$$

Then (2.10) is a distinguished triangle if and only if (2.11) is so.

(TR3) For any commutative solid diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r, "h"] \arrow[d, dotted, "w"] & TA \arrow[d, "Tu"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & TA' \end{tikzcd}$$

There exists a dotted arrow making the diagram commutative.

(TR4) Assume we are given morphisms $f : A \to B$, $g : B \to C$ fitting into distinguished triangles:

$$A \xrightarrow{f} B \to C' \to TA$$
$$B \xrightarrow{g} C \to A' \to TB$$
$$A \xrightarrow{g\circ f} C \to B' \to TA.$$

Then, there exists a distinguished triangle

$$C' \to B' \to A' \to TC'$$

making the following diagram commutative:

$$\begin{tikzcd}
A \arrow[rr, bend left, "g\circ f"] \arrow[dr, "f"] & & C \arrow[rr, bend left] \arrow[dr] & & A' \arrow[rr, dotted, bend left] \arrow[dr] & & TC' \\
& B \arrow[ur, "g"] \arrow[dr] & & B' \arrow[ur, dotted] \arrow[dr] & & TB \arrow[ur] & \\
& & C' \arrow[ur, dotted] \arrow[rr, bend right] & & TA \arrow[ur] & &
\end{tikzcd}$$

Figure 2.6: Octahedron axiom of triangulated categories

Example 2.2.8. The derived category is a triangulated category, with the shift functor $T : A^\bullet \to A^\bullet, A^i \mapsto A[1]^i = A^{i+1}$ being the additive auto-equivalence. Hence, we tend to simply talk about shifts when meaning the auto-equivalence, which is otherwise also known as translation. We will use the notation $A[1]$ to mean TA.

The distinguished triangles in this category are called exact triangles. We obtain the basic set of exact triangles via the cone construction.

Given a chain map $f^\bullet : A^\bullet \to B^\bullet$, let $\mathrm{Cone}(f^\bullet)$ be the complex with $\mathrm{Cone}(f^\bullet)^i = A^{i+1} \oplus B^i$ and differential $d^i_C = \begin{pmatrix} d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$. From the definition of the cone, it is also clear how to define maps $g^\bullet : B^\bullet \to \mathrm{Cone}(f^\bullet)$, $h^\bullet : \mathrm{Cone}(f^\bullet) \to A^\bullet[1]$ to build an exact triangle. The full collection of exact triangles consists of those which by the axioms above need to be exact as well.

Given a triangulated category $\mathcal{T}$ with a full subcategory $\mathcal{N}$, we can form the Verdier quotient of $\mathcal{T}$ by $\mathcal{N}$ (see [39, 40]). Denote by $\Sigma(\mathcal{N})$ a class of morphisms in $\mathcal{T}$ fitting into an exact triangle

$$X \xrightarrow{s} Y \to N \to X[1],$$

with $N \in \mathcal{N}$. $\Sigma(\mathcal{N})$ is a multiplicative system and thus we can localise. The quotient $\mathcal{T}/\mathcal{N}$ is defined as the localisation $\mathcal{T}[\Sigma(\mathcal{N})^{-1}]$ and is a triangulated category. Indeed, the translation functor is induced from the translation on $\mathcal{T}$ and exact triangles in the quotient $\mathcal{T}/\mathcal{N}$ are those triangles that are isomorpic to the image of exact triangles in $\mathcal{T}$.

An important tool in studying derived categories is the concept of semiorthogonal decompositions, discussed in great detail, for example, by Orlov and Bondal [41, 11].

Definition 2.2.9. Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{A}$ be a full subcategory of $\mathcal{T}$. The right orthogonal to $\mathcal{A}$ is the full subcategory $\mathcal{A}^{\perp} \subseteq \mathcal{T}$ consisting of objects B such that $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{A}$. The left orthogonal ${}^{\perp}\mathcal{A}$ is defined analogously. Both ${}^{\perp}\mathcal{A}$ and $\mathcal{A}^{\perp}$ are also triangulated.

Definition 2.2.10. Let $\mathcal{T}$ be a triangulated category. A sequence of full triangulated subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is called a semiorthogonal collection if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$. If a semiorthogonal collection $\mathcal{A}_1, \ldots, \mathcal{A}_n$ generates $\mathcal{T}$ as a triangulated category, we call it a semiorthogonal decomposition and denote this as follows:

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle.$$

Let us dissect here what it means for the semiorthogonal collection to generate the triangulated category. This is the case if for every object $T \in \mathcal{T}$, there exists a chain of morphisms $0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of the morphism $T_k \to T_{k-1}$ is contained in $\mathcal{A}_k$ for each $k = 1, 2, \ldots, n$. We can think of this as a filtration of the object T with factors in the $\mathcal{A}_i$. Semiorthogonality then implies that this filtration is both unique and functorial.

Semiorthogonal decompositions are linked to admissible subcategories.

Definition 2.2.11. A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is called right admissible if for the inclusion functor $i : \mathcal{A} \to \mathcal{T}$ there is a right adjoint $i^! : \mathcal{T} \to \mathcal{A}$, and left admissible if there is a left adjoint $i^* : \mathcal{T} \to \mathcal{A}$. The subcategory $\mathcal{A}$ is called admissible if it is both right and left admissible.

An example of the relation between admissible subcategories of a triangulated category $\mathcal{T}$ and semiorthogonal decompositions is the following Lemma.

Lemma 2.2.12 (Lemma 2.3 in [34]). *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is a semiorthogonal sequence in $\mathcal{T}$ such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are left admissible and $\mathcal{A}_{k+1}, \ldots, \mathcal{A}_n$ are right admissible, then*

$$\langle \mathcal{A}_1, \ldots, \mathcal{A}_k, {}^{\perp}\langle \mathcal{A}_1, \ldots, \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n \rangle^{\perp}, \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n \rangle$$

is a semiorthogonal decomposition.

Examples of admissible subcategories are the ones generated by a exceptional objects.

Definition 2.2.13. An object E is exceptional if $\mathrm{Hom}(E, E) = k$ and $\mathrm{Hom}(E, E[t]) = 0$ for $t \neq 0$. An exceptional collection is a collection of exceptional objects $E_1, E_2, \ldots, E_m$ such that $\mathrm{Hom}(E_i, E_j[t]) = 0$ for all $i > j$ and all $t \in \mathbb{Z}$.

An exceptional collection in $\mathcal{T}$ gives rise to a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}, E_1, \ldots, E_m \rangle,$$

where $\mathcal{A} = \langle E_1, \ldots, E_m \rangle^{\perp}$. If $\mathcal{A}$ is zero, then the exceptional collection is called full.

Example 2.2.14. A result due to Beilinson states that there is a full exceptional collection $\mathrm{D^b}(\mathrm{coh}\,\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$, known as Beilinson's collection. Beilinson's collection gives one of the simplest examples of semiorthogonal decompositions.

Remark 2.2.15. Given a variety X with an exceptional collection $E_1, \dots, E_m \in \mathrm{D^b}(\mathrm{coh}\, X)$, we observed that the right orthogonal complement $\langle E_1, \dots, E_m \rangle^\perp$ gives rise to a semiorthogonal decomposition

$$\langle \langle E_1, \dots, E_m \rangle^\perp, E_1, \dots, E_m \rangle.$$

We refer to $\langle E_1, \dots, E_m \rangle^\perp$ as the Kuznetsov component of X, denoted by $\mathcal{K}u(X)$. It is an active area of research to deduce as much geometric information of X as possible from an appropriate Kuznetsov component. This includes a study of categorical resolutions, which we will talk about later in this thesis.

2.2.3 dg-categories

The reason why triangulated categories are not quite "enough" is that extending diagrams in the axiom (TR3) need not happen in a unique way. As such, derived categories as defined are not the best way to proceed. In most constructions of derived categories, a preferred map is given that is supposed to be used when filling the diagram 2.2.7, but the definition of a derived category itself does not specify this. That is why we need more complicated objects, in this case dg-categories and A_∞-categories, to advance in the theory. We will follow the exposition by Harder [28] to introduce those.

Definition 2.2.16. A category $\mathcal{C}$ is called a differential graded (dg) category if for each $a, b \in \mathrm{Ob}(\mathcal{C})$, there is a vector space $\mathrm{Hom}_\mathcal{C}(a, b)$ over a field k satisfying the following:

- It is a $\mathbb{Z}$-graded vectorspace, the graded piece of weight i being denoted by $\mathrm{Hom}^i_\mathcal{C}(a, b)$.
- It has a chosen differential $d_\mathcal{C} : \mathrm{Hom}_\mathcal{C}(a, b) \to \mathrm{Hom}_\mathcal{C}(a, b)$ increasing the degree by 1.
- If f, g are in $\mathrm{Hom}^i_\mathcal{C}(a, b), \mathrm{Hom}^j_\mathcal{C}(b, c)$ respectively, then

$$d_\mathcal{C}(g \cdot f) = (d_\mathcal{C} g) \cdot f + (-1)^{i+j} g \cdot (d_\mathcal{C} f) \in \mathrm{Hom}^{i+j+1}(a, c).$$

- For each $a \in \mathrm{Ob}(\mathcal{C})$, there is some $i_a \in \mathrm{Hom}^0_\mathcal{C}(a, a)$ so that $i_a \cdot f = f$ for any

$f \in \mathrm{Hom}^j_{\mathcal{C}}(b,a)$ and $g \cdot i_a = g = i_b \cdot g$ for all $g \in \mathrm{Hom}^j_{\mathcal{C}}(a,b)$.

In particular, the category of chain complexes over an abelian category $\mathcal{A}$ is a dg-category. To any dg-category $\mathcal{C}$, one can associate its homotopy category $H^0\mathcal{C}$ (again, note that we switch notations here).

2.2.4 A_∞-categories

There is a second class of objects that comes into play when discussing Homological Mirror Symmetry, slightly more general than *dg*-categories. They are called A_∞-categories. These are not categories in the classical sense, since composition of morphisms is not always associative. The way associativity fails is, however, controlled.

Definition 2.2.17 (Definition 2.3 [28]). An A_∞-category $\mathcal{A}$ is a collection of objects $\mathrm{Ob}(\mathcal{A})$ along with a $\mathbb{Z}$-graded vector space $\mathrm{Hom}_{\mathcal{A}}(a,b)$ for any pair $a, b \in \mathrm{Ob}(\mathcal{A})$ such that

- For all $n > 0$ and every set of objects $a_0, \dots, a_n \in \mathrm{Ob}(\mathcal{A})$, there are maps

$$m_n^{\mathcal{A}}(a_0, \dots, a_n) : \mathrm{Hom}_{\mathcal{A}}(a_{n-1}, a_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(a_0, a_1) \to \mathrm{Hom}_{\mathcal{A}}(a_0, a_n)[2-n].$$

- These maps satisfy quadratic A_∞-associativity relations,

$$\sum_{m,n} (-1)^{\tau_n} m^{\mathcal{A}}_{d-m+1}(f_d, \dots, f_{m+n+1}, m^{\mathcal{A}}_n(f_{m+n}, \dots, f_{n+1}), f_n, \dots, f_1) = 0,$$

  where $f_i \in \mathrm{Hom}_{\mathcal{A}}(a_{i-1}, a_i)$ and $\tau_n = -n + \sum_i |a_i|$.

To digest this, it is worth looking at the first few of the quadratic relations. The first one gives $m_1^{\mathcal{A}}(m_1^{\mathcal{A}}(f)) = 0$, so we have chain complexes of objects of $\mathcal{A}$ with differential m_1. The second relation gives more information about the notion of composition:

$$\begin{aligned} m_2^{\mathcal{A}}(f_1, m_2^{\mathcal{A}}(f_2, f_3)) - m_2^{\mathcal{A}}(m_2^{\mathcal{A}}(f_1, f_2), f_3) = \; & m_1^{\mathcal{A}}(m_3^{\mathcal{A}}(f_1, f_2, f_3)) + m_3^{\mathcal{A}}(m_1^{\mathcal{A}}(f_1), f_2, f_3) \\ & + m_3^{\mathcal{A}}(f_1, m_1^{\mathcal{A}}(f_2), f_3) + m_3^{\mathcal{A}}(f_1, f_2, m_1^{\mathcal{A}}(f_3)). \end{aligned}$$

This relation shows that we can think of $m_2^{\mathcal{A}}$ as composition, up to some factor involving $m_3^{\mathcal{A}}$. In other words, the way associativity fails is measured by $m_3^{\mathcal{A}}$.

As for *dg*-categories, one can construct the homotopy category $H^0\mathcal{A}$ to an A_∞-category $\mathcal{A}$. The objects are those of $\mathcal{A}$ and homomorphisms are the 0-th cohomology of the morphism complexes of $\mathcal{A}$ with respect to $m_1^{\mathcal{A}}$.

If m_i vanishes for $i > 2$, A_∞-categories are *dg*-categories, possibly without units. Therefore the category of *dg*-categories embeds into the category of A_∞-categories.

An A_∞-functor between two A_∞-categories is a map on objects and homomorphisms in the usual way which also satisfies some additional conditions with respect to the $m_i^{\mathcal{A}}$. In particular, we call an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ a quasi-equivalence if the induced functor on the homotopy categories $H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence of categories in the usual sense.

Given any *dg* or A_∞-category $\mathcal{C}$, one would like to find some triangular category containing it. One way to do this is by introducing twisted complexes over $\mathcal{C}$, denoted by $\mathrm{Tw}\,\mathcal{C}$.

Definition 2.2.18 (Definition 2.6 in [28])**.** Let $\mathbb{Z}\mathcal{C}$ be the category whose objects are formal pairs (a, n) with $a \in \mathrm{Ob}(\mathcal{C})$ and $n \in \mathbb{Z}$. We define

$$\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}((a, n), (b, m)) = \mathrm{Hom}_{\mathcal{C}}(a, b)[m - n].$$

The A_∞-structure is the same as on $\mathcal{C}$.

This category has a natural notion of shift, sending an object $\bigoplus_i a_i[i]$ to $\bigoplus_i a_i[i+1]$. We note that $\mathcal{C}$ is the full subcategory of objects in the form $a[0]$. To make this a triangulated category, we need to add formal mapping cones.

Definition 2.2.19 (Definition 2.7 in [28])**.** Let us take the category $\mathrm{Tw}\,\mathcal{C}$ so that $\mathrm{Ob}(\mathrm{Tw}\,\mathcal{C})$ is made up of pairs (A, δ) for $A = (a_1, \ldots, a_n) \in \mathrm{Ob}(\mathbb{Z}\mathcal{C})^n$ and δ is a strictly upper triangular matrix of morphisms. Furthermore, we require the Maurer-Cartan equation to hold, which states

$$\sum_{i=1}^{\infty} m_i \mathbb{Z}\mathcal{C}(\delta, \ldots, \delta) = 0.$$

Here the extension of $m_i^{\mathbb{Z}\mathcal{C}}$ to matrices is straightforward. Since δ is triangular, the sum in the Maurer-Cartan equation is finite. The space of morphisms between $(A,\delta),(B,\tau)$ is $\bigoplus_{i,j}\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(a_i,b_j)$ equipped with a twisted set of composition maps. Taking (A_i,δ_i) for $i=1,2,\dots,n$ and $f_i\in\mathrm{Hom}_{\mathrm{Tw}\,\mathcal{C}}((A_{i-1},\delta_{i-1}),(A_i,\delta_i))$, then

$$m_d^{\mathrm{Tw}\,\mathcal{C}}(f_d,\dots,f_1)=\sum_{j_0,\dots,j_d\geq 0} m_i^{\mathbb{Z}\mathcal{C}}(\underbrace{\delta_d,\dots,\delta_d}_{j_d},a_d,\underbrace{\delta_{d-1},\dots,\delta_{d-1}}_{j_{d-1}},a_{d-1},\dots).$$

An A_∞-category is called triangulated if the natural embedding $\mathcal{C}\hookrightarrow\mathrm{Tw}\,\mathcal{C}$ is a quasi-equivalence of A_∞-categories.

At this point, one might hope that the groundwork we set out is enough to formulate Homological Mirror Symmetry, saying that $\mathrm{Tw}\,\mathrm{Fuk}(X)$ should be equivalent to some A_∞-category with homotopy category $\mathrm{D^b}(\mathrm{coh}\,X)$. This is not quite true, as $\mathrm{Tw}\,\mathrm{Fuk}(X)$ is not necessarily Karoubi complete. This is a property of triangulated categories we will not elaborate on here, but we note that every triangulated A_∞-category $\mathcal{C}$ has a Karoubi completion. That is, a functor into another A_∞-category $\mathcal{S}$ such that the induced functor on homotopy categories is full, faithful and gives a Karoubi complete structure on $H^0\mathcal{S}$.

Seidel in [48] (see also Proposition 2.10 in [28]) proves that any pair of Karoubi completions of a given A_∞-category $\mathcal{C}$ are quasi-equivalent. There is an explicit construction due to Seidel, and we denote this Karoubi completion of $\mathcal{C}$ by $\Pi\mathcal{C}$.

The last thing we need to discuss before talking about Homological Mirror Symmetry are *dg*-enhancements of the category $\mathrm{D^b}(\mathrm{coh}\,X)$.

Definition 2.2.20 (Definition 2.11 in [28])**.** A *dg*- (respectively A_∞-) enhancement of a triangulated category $\mathcal{T}$ is a *dg*- (respectively A_∞-) category $\mathcal{C}$ whose homotopy category is equivalent to $\mathcal{T}$.

With Example 2.2.5, we have a good candidate for a category that we would like to find a *dg*-enhancement for.

Example 2.2.21. Let X be a smooth projective variety over a field $\mathbb{C}$. We revisit here the construction of $\mathrm{D^b}(\mathrm{coh}\,X)$ in light of the new kinds of categories we have introduced.

Recall the category of quasi-coherent sheaves on X, $\mathrm{QCoh}(X)$. One takes the category of complexes over it, $\mathbf{K}(\mathrm{QCoh}(X))$. This is naturally a *dg*-category, with homomorphisms being $\mathrm{Hom}^l((A^\bullet, d_A), (B^\bullet, d_B)) = \prod_i \mathrm{Hom}_{\mathrm{QCoh}(X)}(A^i, B^{l+i})$ and differential $df = d_B f + (-1)^l f d_A$. A complex $I^\bullet$ is said to be h-injective if for every complex $J^\bullet$ isomorphic to 0 in $\mathbf{D}(\mathrm{QCoh}(X))$, we have that $\mathrm{Hom}_{\mathbf{K}(\mathrm{QCoh}(X))}(J^\bullet, I^\bullet)$ is quasi-isomorphic to 0. The full subcategory $\mathcal{I}(X)$ of h-injective complexes of quasi-coherent sheaves has homotopy category equivalent to the derived category of quasi-coherent sheaves, $\mathbf{D}(\mathrm{QCoh}(X))$. Then $\mathrm{D^b}(\mathrm{coh}\, X)$, was introduced as a full subcategory of $\mathbf{D}(\mathrm{QCoh}(X))$ made up of bounded complexes whose cohomological sheaves are coherent. Thus, we have a full subcategory of $\mathbf{D}(\mathrm{QCoh}(X))$ equivalent to $\mathrm{D^b}(\mathrm{coh}\, X)$. $\mathcal{I}(X)$ has, up to equivalence in the homotopy category, the same objects as $\mathbf{D}(\mathrm{QCoh}(X))$, so we can define $\mathrm{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))$ to be the full subcategory of $\mathcal{I}(X)$ made up of objects equivalent to objects in $\mathrm{D^b}(\mathrm{coh}\, X) \subseteq \mathbf{D}(\mathrm{QCoh}(X))$.

$\mathrm{D}^b_{\mathrm{dg}}(\mathrm{coh}(X))$ has homotopy category equivalent to $\mathrm{D^b}(\mathrm{coh}\, X)$, and hence is a *dg*-enhancement of $\mathrm{D^b}(\mathrm{coh}\, X)$.

In Homological Mirror Symmetry, the Fukaya category is the counterpart to the derived category of coherent sheaves defined in § 2.2.1. In this thesis, we focus on the complex algebraic side of mirror symmetry, specifically relationships between derived categories. As such, the symplectic side (which studies the Fukaya category) lies outside the scope of this thesis. Hence we will not properly define what the Fukaya category of a symplectic manifold is, but we will note that it is an A_∞-category whose objects are Lagrangian submanifolds and whose morphisms are Floer chain groups. A good introduction to Fukaya categories can be found in [1].

2.2.5 Factorisation categories

In the following, we will introduce Landau-Ginzburg models (or LG models for short) and categories of factorisations. LG models naturally appear in mirror symmetry as proposed mirrors to Fano varieties. They give an effective way to keep track of data and in this thesis give the level of abstraction necessary to compare derived categories of different mirror constructions. To do the latter, we will pass from the derived category of coherent sheaves to categories associated to factorisations. The introduction here is based on work by Hirano, Ballard-Favero-Katzarkov and Favero-Kelly [2, 31, 19].

Let X be a separated scheme of finite type over an algebraically-closed field k of characteristic zero. Let G be an affine algebraic group over k acting on X. Denote by $m : G \times G \to G$ the group action, by $\sigma : G \times X \to X$ the G-action and by $\pi : G \times X \to X$ the projection onto X.

Definition 2.2.22. A quasi-coherent (resp. coherent) G-equivariant sheaf is a pair $(\mathcal{F}, \theta)$ of a quasi-coherent (resp. coherent) sheaf $\mathcal{F}$ and an isomorphism $\theta : \pi^*\mathcal{F} \xrightarrow{\simeq} \sigma^*\mathcal{F}$ such that

$$((1_G \times \sigma) \circ (\tau \times 1_X))^*\theta \circ (1_G \times \pi)^*\theta = (m \times 1_X)^*\theta,$$

where $\tau : G \times G \times X \to G \times G \times X$ switches the two factors of G, $s : X \to G \times X$ is induced by the identity and

$$s^*\theta = 1_{\mathcal{F}}.$$

A G-invariant morphism $\varphi : (\mathcal{F}_1, \theta_1) \to (\mathcal{F}_2, \theta_2)$ of equivariant sheaves is a morphism of sheaves $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ which commutes with the θ_i, i.e. $\sigma^*\varphi \circ \theta_1 = \theta_2 \circ \pi^*\varphi$.

We denote by $\mathrm{Qcoh}_G(X)$ (resp. $\mathrm{coh}_G(X)$) the category of quasi-coherent (resp. coherent) G-equivariant sheaves on X whose morphisms are G-invariant morphisms.

Let k be an algebraically closed field of characteristic zero. In practice, we work over $\mathbb{C}$ in this thesis. Let X be a smooth variety over k and G an affine algebraic group acting on it. Let w be a G-invariant section of an invertible G-equivariant sheaf, $\mathcal{L}$, i.e. $w \in \Gamma(X, \mathcal{L})^G$.

Definition 2.2.23. We call the data $(X, G, w, \mathcal{L})$ a gauged Landau-Ginzburg model. If the choice of $\mathcal{L}$ is clear, we abbreviate the notation to (X, G, w).

Definition 2.2.24. A factorisation is the data $\mathcal{E} = (\mathcal{E}_{-1}, \mathcal{E}_0, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0)$ where $\mathcal{E}_{-1}, \mathcal{E}_0$ are G-equivariant quasi-coherent sheaves and

$$\mathcal{E}_{-1} \xrightarrow{\phi^{\mathcal{E}}_0} \mathcal{E}_0 \xrightarrow{\phi^{\mathcal{E}}_{-1}} \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}$$

are morphisms such that

$$\begin{aligned} &\phi^{\mathcal{E}}_{-1} \circ \phi^{\mathcal{E}}_0 = w, \\ &(\phi^{\mathcal{E}}_0 \otimes \mathcal{L}) \circ \phi^{\mathcal{E}}_{-1} = w. \end{aligned}$$

A morphism between two factorisations of even degree $f : \mathcal{E} \to \mathcal{F}[2k]$ is a pair $f = (f_0, f_1)$ defined by

$$\mathrm{Hom}^{2k}_{\mathtt{Fact}(X,G,W)}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathrm{Qcoh}_G X}(\mathcal{E}_{-1}, \mathcal{F}_{-1}, \otimes_{\mathcal{O}_X} \mathcal{L}^k) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_G X}(\mathcal{E}_0, \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^k)$$

and, similarly, a morphism of odd degree $f : \mathcal{E} \to \mathcal{F}[2k+1]$ is a pair $f = (f_0, f_1)$ defined by

$$\mathrm{Hom}^{2k+1}_{\mathtt{Fact}(X,G,W)}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathrm{Qcoh}_G X}(\mathcal{E}_0, \mathcal{F}_{-1}, \otimes_{\mathcal{O}_X} \mathcal{L}^{k+1}) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_G X}(\mathcal{E}_{-1}, \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{k+1}).$$

These Hom sets can be equipped with a differential, yielding a dg-category $\mathtt{Fact}(X, G, w)$. Restricting to factorisations with only coherent components gives a full dg-subcategory $\mathtt{fact}(X, G, w)$. Another subcategory of $\mathtt{Fact}(X, G, w)$ we consider is the category with the same objects as $\mathtt{Fact}(X, G, w)$ but where the morphisms are restricted to closed degree zero morphisms. This category, denoted by $Z^0\mathtt{Fact}(X, G, w)$, is abelian and we can therefore form complexes of objects in $Z^0\mathtt{Fact}(X, G, w)$. Consider such a complex

$$\cdots \to \mathcal{E}^b \xrightarrow{f^b} \mathcal{E}^{b+1} \xrightarrow{f^{b+1}} \ldots$$

We consider a special factorisation $\mathcal{T} \in \texttt{Fact}(X, G, w)$ given by the data:

$$
\begin{aligned}
\mathcal{T}_{-1} &:= \bigoplus_{i=2k} \mathcal{E}^i_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-k} \oplus \bigoplus_{i=2k-1} \mathcal{E}^i_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{-k}, \\
\mathcal{T}_{\prime} &:= \bigoplus_{i=2k} \mathcal{E}^i_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{-k} \oplus \bigoplus_{i=2k+1} \mathcal{E}^i_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-k}, \\
\phi^{\mathcal{T}}_0 &:= \bigoplus_{i=2k} f^i_0 \otimes \mathcal{L}^{-k} \oplus \bigoplus_{i=2k-1} f^i_{-1} \otimes \mathcal{L}^{-k}, \\
\phi^{\mathcal{T}}_{-1} &:= \bigoplus_{i=2k} f^i_{-1} \otimes \mathcal{L}^{-k} \oplus \bigoplus_{i=2k-1} f^i_0 \otimes \mathcal{L}^{-k}.
\end{aligned}
$$

This factorisation $\mathcal{T}$ is called the totalisation of the complex. Let $\texttt{Acyc}(X, G, w)$ be the full subcategory of objects of $\texttt{Fact}(X, G, w)$ consisting of totalisations of bounded exact complexes of $Z^0\texttt{Fact}(X, G, w)$. Also, let $\texttt{acyc}(X, G, w) := \texttt{Acyc}(X, G, w) \cap \texttt{fact}(X, G, w)$.

Definition 2.2.25. The absolute derived category $\mathrm{D}^{\mathrm{abs}}[\texttt{Fact}(X, G, w)]$ of $[\texttt{Fact}(X, G, w)])$ is the Verdier quotient of $[\texttt{Fact}(X, G, w)]$ by $[\texttt{Acyc}(X, G, w)]$.

For our purposes, we focus on objects which are coherent sheaves. As such, we introduce the following, abbreviated version of this notation.

Definition 2.2.26. The absolute derived category $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ of $[\texttt{fact}(X, G, w)]$ is the idempotent completion of the Verdier quotient of $[\texttt{fact}(X, G, w)]$ by $[\texttt{acyc}(X, G, w)]$. Note that this is the full subcategory of $\mathrm{D}^{\mathrm{abs}}[\texttt{Fact}(X, G, w)]$ split-generated by objects in $\texttt{fact}(X, G, w)$.

Remark 2.2.27. As noted by the authors in [19], the category $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ can be thought of as the derived category of the Landau-Ginzburg model (X, G, w). It is triangulated with shift functor

$$\mathcal{E}[1] := (\mathcal{E}_0, \mathcal{E}_{-1} \otimes \mathcal{L}, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0 \otimes \mathcal{L}).$$

In particular, the double shift corresponds to tensoring with $\mathcal{L}$, $[2] = - \otimes \mathcal{L}$.

2.2.6 Homological Mirror Symmetry

Now that we know the types of categories involved, we can finally state Homological Mirror Symmetry. Mirror symmetry in general states that to every Calabi-Yau manifold X with complex structure and symplectic structure, there is a dual manifold $X^\vee$ so that the properties of X associated to the complex structure (i.e. periods, bounded derived category of coherent sheaves, etc.) reproduce properties of $X^\vee$ associated to its symplectic structure (e.g. counts of pseudo-holomorphic curves and disks) [28]. Homological Mirror Symmetry (HMS) takes these observations and puts them into a more categorical context, with important early works by Kontsevich [33], de la Ossa, Katz and Candelas [14] and Batyrev-Borisov [6, 7]. The name of HMS is due to the observation that for dual Calabi-Yau manifolds $X, X^\vee$ of dimension n, we get a flip in the Hodge diamond, i.e.

$$\dim H^p(X, \Omega^q) = \dim H^{n-p}(X^\vee, \Omega^q).$$

The two sides of HMS are known as the A-side (the symplectic structure, best approached by symplectic geometry) and the B-side (the complex structure, best approached by algebraic geometry). We will focus on the B-side in this thesis. In HMS, the A-side is concerned with Lagrangian submanifolds whereas the B-side works with complexes of coherent sheaves on X. The equivalence between these two sides can be interpreted as an equivalence between the bounded derived category of coherent sheaves on X and the Fukaya category of $X^\vee$. In other words, the HMS conjecture (Kontsevich, [33]) is that for a mirror pair of Calabi-Yau manifolds $X, X^\vee$, we have

$$\mathrm{Fuk}(X) \cong \mathrm{D^b}(\mathrm{coh}\, X^\vee) \text{ and } \mathrm{Fuk}(X^\vee) \cong \mathrm{D^b}(\mathrm{coh}\, X).$$

Since Kontsevich stated the HMS conjecture, more work has been done on the subject and we would like to state what HMS means for Calabi-Yau manifolds using the categories introduced earlier, making the notion of being a mirror more precise.

Definition 2.2.28 (Definition 2.12 in [28])**.** Let X and $X^\vee$ be a pair of Calabi-Yau varieties. If there is a quasi-equivalence of A_∞-categories

$$\Pi\,\mathrm{Tw}\,\mathrm{Fuk}(X) \cong \mathrm{D}^b_{\mathrm{dg}}(\mathrm{coh}(X^\vee))$$

and vice versa, then we say that X and $X^\vee$ are homologically mirror to one another.

The Homological Mirror Symmetry conjecture then states that for a given Calabi-Yau variety X, there is a homologically mirror Calabi-Yau variety $X^\vee$.

2.2.7 Categorical resolutions

In this section we are going to explore the concept of categorical resolutions, as introduced by Kuznetsov [34]. Our exposition will mainly follow [34, 19, 11].

A strong motivation behind introducing categorical resolutions as below lies in the Minimal Model Program (MMP). The goal of the MMP is to classify algebraic varieties up to birational transformations. Starting with a complex algebraic variety, the goal of the MMP is to find the "simplest" possible birationally equivalent variety. Often, this requires the study of varieties with terminal singularities. In [11], the authors provide evidence that a generalised flip from X to X^+ has the categorical meaning of breaking off semiorthogonal summands from $\mathrm{D^b}(\mathrm{coh}\, X)$. As such, a more modern point of view on the MMP interprets it as a process that minimises the derived category of coherent sheaves in a given birational class of an algebraic variety X. Using this approach to the MMP, searching for a minimal model keeps the triangulated categories well-behaved, using the notion of a minimal categorical resolution of singularities.

Definition 2.2.29. Let Y be an algebraic variety. An object $F \in \mathcal{D}(Y)$ in the derived category of coherent sheaves on Y is said to be a perfect complex if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. The full subcategory of perfect objects is denoted by $\mathcal{D}^{perf}(Y)$ or by $\mathrm{Perf}(Y)$.

Remark 2.2.30. Morally, we think of perfect complexes as being the smooth parts of our derived category. It is the complexes which are not perfect which we can think of as being the categorical equivalent of singularities.

Let Y be a singular variety with a resolution of singularities $\pi : \tilde{Y} \to Y$. The derived categories of coherent sheaves on $\tilde{Y}$ and Y can then be related by the derived pushforward and derived pullback functors:

$$\pi_* : \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \tilde{Y}) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y), \quad \text{and} \quad \pi^* : \mathcal{D}^{perf}(Y) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \tilde{Y}).$$

The functors π^*, π_* are mutually adjoint (π^* being the left adjoint to π_*). Moreover, if the singularities of Y are rational, then the composition $\pi_* \circ \pi^*$ is isomorphic to the identity functor. If furthermore π is a crepant resolution (which means that the relative canonical class is trivial), then π^* is isomorphic to the right adjoint functor of π_*.

This structure is what motivates Definition 2.2.33 below. To give a good definition of categorical resolution we cannot rely on underlying varieties and their resolution of singularities in the more conventional sense. Instead, we need to find a category that takes the place of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \tilde{Y})$, and therefore a notion of "smooth" category. In [34], the definition of a regular triangulated category is given, but that definition was always meant to be provisional. In [19], the authors give a different notion which we define here.

Definition 2.2.31. A *dg*-category $\mathcal{A}$ is called homologically smooth if $\mathcal{A}$ is a compact object of $\mathcal{D}(\mathcal{A} \otimes \mathcal{A}^{op} - Mod)$, i.e. $A \in \mathcal{D}^{perf}(\mathcal{A} \otimes \mathcal{A}^{op})$.

Apart from homological smoothness, we also require the *dg*-category to be proper. Recall the following terminology from [10]. Let $\mathcal{A}$ be a triangulated category, and E an object of $\mathcal{A}$. We denote by $\langle E \rangle_1$ the full subcategory of $\mathcal{A}$ consisting of objects of $\mathcal{A}$ isomorphic to direct summands of finite direct sums of shifts of E

$$\bigoplus_{i=1,\dots,r} E[n_i].$$

For $n > 1$, we denote by $\langle E\rangle_n$ the full subcategory of $\mathcal{A}$ isomorphic to direct summands of objects C which fit into a distinguished triangle

$$A \to C \to B \to A[1]$$

where $A \in \mathrm{Ob}(\langle E\rangle_1$ and $B \in \mathrm{Ob}(\langle E\rangle_{n-1})$. We call E a strong generator of $\mathcal{A}$ if $\mathcal{A} = \langle E\rangle_n$ for some $n \geq 1$.

Definition 2.2.32. A dg-category $\mathcal{A}$ is called proper if there exists a strong generator E of the homotopy category of $\mathcal{A}$ such that

$$\bigoplus_r H^r(\mathrm{Hom}_{\mathcal{A}}(E, E))$$

is finite dimensional.

Let Z be a variety with a G-action and $\mathcal{D}$ be an admissible subcategory of $\mathrm{D^b}(\mathrm{coh}[Z/G])$. Analogously to Definition 2.2.29, we denote by $\mathcal{D}^{perf}$ the full subcategory of $\mathcal{D}$ consisting of G-equivariant perfect complexes on Z.

Definition 2.2.33. Let $\tilde{\mathcal{D}}$ be the homotopy category of a homologically smooth and proper pretriangulated dg-category. A pair of exact functors

$$F : \tilde{\mathcal{D}} \to \mathcal{D}$$
$$G : \mathcal{D}^{perf} \to \tilde{\mathcal{D}}$$

is a categorical resolution of singularities if G is left adjoint to F and the natural morphism of functors $\mathrm{Id}_{\mathcal{D}^{perf}} \to FG$ is an isomorphism. We say that the categorical resolution of singularities is crepant if G is also right adjoint to F.

Example 2.2.34. For a resolution of singularities $\pi : \tilde{Y} \to Y$ of an algebraic variety with rational singularities Y, the category $\tilde{\mathcal{D}} = \mathrm{D^b}(\mathrm{coh}\,\tilde{Y})$ with the pushforward functor $\pi_* : \mathrm{D^b}(\mathrm{coh}\,\tilde{Y}) \to \mathrm{D^b}(\mathrm{coh}\,Y)$ and the pullback functor $\pi^* : \mathcal{D}^{perf}(Y) \to \mathrm{D^b}(\mathrm{coh}\,\tilde{Y})$ form a categorical resolution of $\mathrm{D^b}(\mathrm{coh}\, Y)$. This fits the situation we gave above that motivates the definition of categorical resolutions. We note that the pullback on the whole derived category $\mathrm{D^b}(\mathrm{coh}\, Y)$ takes values in $\mathcal{D}^-(\tilde{Y})$ (which is the unbounded from below derived category). Because of that, we restrict the functor G in Definition 2.2.33 to be defined only on the subcategory $\mathcal{D}^{perf}$.

Remark 2.2.35. This definition of categorical resolution extends the notion of noncommutative resolution of singularities introduced by van den Bergh [52, 8]. Indeed, if Y is a singular projective algebraic variety and $\mathcal{A}$ is a sheaf of noncommutative algebras on Y giving a noncommutative resolution of singularities of Y, then one can show that $\tilde{\mathcal{D}} = \mathrm{D^b}(\mathrm{coh}\, \mathcal{A} - mod)$ is a categorical resolution of $\mathrm{D^b}(\mathrm{coh}\, Y)$. If the noncommutative resolution of singularities was crepant, then so is this categorical resolution.

Remark 2.2.36. In [19], the authors provide an interpretation of crepant categorical resolutions in terms of Landau-Ginzburg models. Such resolutions get a geometric interpretation as partial compactifications of Landau-Ginzburg models. More details of this can be found in §4.1.

2.3 Some Mirror Constructions

In §3 of this thesis, we relate a construction by Libgober and Teitelbaum to the more well-studied mirror construction by Batyrev and Borisov. In §3 we then generalise the construction to complete intersections of two degree n polynomials in $\mathbb{P}^{2n-1}$, inspired by a construction of Rossi's. In this section, we introduce the relevant constructions for the later discussions.

2.3.1 The Batyrev-Borisov construction

Batyrev's original construction gives a way to find a mirror to a section of the anticanonical divisor of a toric variety, and was later generalised in [7] to complete intersections in toric varieties such that the divisors corresponding to the hypersurfaces sum up to the anticanonical divisor. We start by introducing the main tool used in this generalisation, nef partitions.

Definition 2.3.1. Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive lattice polytope. A nef partition of length r of Δ is a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ where $\Delta_1, \ldots, \Delta_r$ are lattice polytopes with $0 \in \Delta_i$.

Consider a reflexive polytope $\Delta \subseteq M_{\mathbb{R}}$ with nef partition $\Delta = \Delta_1 + \cdots + \Delta_r$. Then, for $1 \leq j \leq r$, we define

$$\nabla_j := \{n \in N_{\mathbb{R}} \mid \langle m, n\rangle \geq -\delta_{ij} \text{ for all } m \in \Delta_i, \text{ for } 1 \leq i \leq r\}.$$

We note that these polytopes are all lattice polytopes, and define the polytope ∇ as their Minkowski sum $\nabla := \nabla_1 + \cdots + \nabla_r$. We call $\nabla_1, \ldots, \nabla_r$ the dual nef partition to $\Delta_1, \ldots, \Delta_r$. Note that by Theorem 4.10 in [7], $\nabla^\vee = \operatorname{Conv}\{\Delta_1, \ldots, \Delta_r\}$.

To understand the statement of Batyrev-Borisov duality, we note that a lattice polytope Δ corresponds to a d-dimensional Gorenstein Fano toric variety X_Δ. Each of the polytopes Δ_i corresponds to a divisor D_i on X_Δ. The nef partition $\Delta = \Delta_1 + \cdots + \Delta_r$ decomposes the anticanonical sheaf $\mathcal{O}(-K_{X_\Delta})$ as tensor product $\bigotimes_{i=1}^r \mathcal{O}_{X_\Delta}(D_i)$. Now the lattice points inside the Δ_i correspond to global sections of these line bundles. Taking the zero-sets of such sections, we can associate to each polytope a family of hypersurfaces. By intersecting these, a nef partition corresponds to a family of $(d-r)$-dimensional Calabi-Yau complete intersections in X_Δ. Similarly, the dual nef partition $\nabla = \nabla_1 + \cdots + \nabla_r$ gives a family of $(d-r)$-dimensional Calabi-Yau complete intersections in X_∇.

Remark 2.3.2. The generic complete intersection in the family associated to the dual nef partition $\nabla_1, \ldots, \nabla_r$ may be singular.

In [5], Batyrev formulates the original construction in a way that fixes this problem. In this case, one uses a maximal projective crepant partial desingularization (MPCP-desingularization), which reduces to a combinatorial manipulation of the normal fan to

∇.

For every maximal cone of the normal fan, we choose a regular triangulation of it. Therefore, all maximal cones should contain exactly the minimal number of rays dictated by the dimension, since a triangulation uses simplices. Doing this for all maximal cones gives exactly a maximal projective triangulation. When speaking of X_∇ we will thus think of a MPCP-desingularization of the variety associated to the normal fan of ∇, obtained in this way.

Batyrev and Borisov prove the following result, showing that their construction produces complete mirror duality for $(1, q)$-Hodge numbers.

Theorem 2.3.3 (Theorem 9.6 in [7]). *Let V be a Calabi-Yau complete intersection of r hypersurfaces in $\mathbb{P}^d$ corresponding to a nef-partition $\Delta_1, \ldots, \Delta_r$ with $d - r \geq 3$. Suppose that $\widehat{W}$ is a MPCP-desingularization of the dual Calabi-Yau complete intersection $W \subseteq X_\nabla$. Then*

$$h^q(\Omega^1_{\widehat{W}}) = h^{d-r-q}(\Omega^1_V) \text{ for } 0 \leq q \leq d - r.$$

2.3.2 A construction by Libgober and Teitelbaum

We now recall the family $W_{LT,\lambda}$ that Libgober and Teitelbaum give as mirror to the generic complete intersection of two cubics in $\mathbb{P}^5$. To start, define $V_\lambda \subseteq \mathbb{P}^5$ to be the vanishing set of the following two polynomials:

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \qquad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2. \tag{2.12}$$

For generic λ, this gives a smooth complete intersection in $\mathbb{P}^5$ which is a Calabi-Yau threefold.

Let ζ_n denote a primitive n-th root of unity. Let $\alpha, \beta, \delta, \epsilon \in \mathbb{Z} \pmod 3$ and $\mu \in \mathbb{Z} \pmod 9$ with $3\mu = \alpha + \beta = \delta + \epsilon$. Define the diagonal matrix

$$g_{\alpha,\beta,\delta,\epsilon,\mu} := \operatorname{diag}\left(\zeta_3^\alpha \zeta_9^\mu,\ \zeta_3^\beta \zeta_9^\mu,\ \zeta_9^\mu,\ \zeta_3^{-\delta} \zeta_9^{-\mu},\ \zeta_3^{-\epsilon} \zeta_9^{-\mu},\ \zeta_9^{-\mu}\right)$$

and let $G_{81} \subset PGL(5, \mathbb{C})$ denote the order 81 group generated by the $g_{\alpha,\beta,\delta,\epsilon,\mu}$. Note that G_{81} acts on $\mathbb{P}^5$ by restricting the natural action of $PGL(5, \mathbb{C})$ on $\mathbb{P}^5$. The polynomials $Q_{1,\lambda}, Q_{2,\lambda}$ are invariant with respect to the action of G_{81}, hence G_{81} acts on V_λ.

Note that G_{81} is of isomorphism type $(\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/9\mathbb{Z})$ and can be generated by $(\zeta_3, \zeta_3^{-1}, 1, 1, 1, 1)$, $(1, 1, 1, \zeta_3^{-1}, \zeta_3, 1)$ and $(\zeta_9, \zeta_9^4, \zeta_9, \zeta_9^{-1}, \zeta_9^{-4}, \zeta_9^{-1})$.

Let $V_{LT,\lambda}$ be the quotient of V_λ by the action of G_{81} and let $W_{LT,\lambda}$ be a minimal resolution of singularities of $V_{LT,\lambda}$ which is a Calabi-Yau manifold.

2.3.3 f-duality

In this section we discuss the notion of f-duality, as introduced by Rossi in [44] and [45]. The motivation behind f-duality is to extend the polar duality of Fano toric varieties, creating a more versatile method of producing mirror partners to hypersurfaces and complete intersections in toric varieties. In particular, it is a direct generalisation of Batyrev-Borisov duality and relates to many other mirror constructions, as briefly described in §4 of [44].

We will mainly focus on complete intersections in toric varieties, referring the reader to Rossi's papers for a more complete treatment of the subject. However, to get a better grasp on the case of complete intersections, we will start with the general notion of f-duality first.

Definition 2.3.4. A framed toric variety(ftv) is a pair (X, D) where:

- X is a complete toric variety, with $\dim(X) = n$ and $\mathrm{rk}(\mathrm{Pic}(X)) = r$.
- Let Σ be a fan for X. Denote by $D_1, \ldots, D_r$ the $m = n + r$ divisors associated to the rays $\rho \in \Sigma(1)$. Then $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho = \sum_{i=1}^m a_i D_i \in \mathrm{Div}_{T_N}(X)$, is a strictly effective, torus invariant Weil divisor, called a framing of X.

A morphism of framed toric varieties $f : (X, D) \to (X', D')$ is a morphism of underlying toric varieties $f : X \to X'$ inducing a well defined pull-back morphism on torus invariant Weil divisors $f^* : \mathrm{Div}_{T_N}(X') \to \mathrm{Div}_{T_N}(X)$ such that $f^*D' = D$. If additionally f is an isomorphism of toric varieties, then it gives an isomorphism of framed toric varieties $f : (X, D) \cong (X', D')$. The category **ftv** of framed toric varieties is well defined.

Since all torus invariant divisors can be written as $D_{\mathbf{a}} = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ for some vector $\mathbf{a} = a_\rho \in \mathbb{Z}^{|\Sigma(1)|}$, giving a toric variety X and a vector $\mathbf{a}$ is sufficient to determine an associated ftv. Thus we will write

$$(X, \mathbf{a}) := \left(X(\Sigma), D_{\mathbf{a}} = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \right).$$

Given such a ftv, we consider the polytope associated to $D_{\mathbf{a}}$, $\Delta_{\mathbf{a}} = \Delta_{D_{\mathbf{a}}}$. Since the divisor $D_{\mathbf{a}}$ is strictly effective, we have that $0 \in \mathrm{Int}(\Delta_{\mathbf{a}})$, and hence 0 is also an interior point of $k\Delta_{\mathbf{a}}$ for any positive integer k. We define the integer part of a polytope $\Delta \subseteq M_{\mathbb{R}}$ as

$$[\Delta] := \mathrm{Conv}\left(\{m \in M \cap \Delta\}\right).$$

Definition 2.3.5. The framing polytope (f-polytope) of a ftv $(X, \mathbf{a})$ is the lattice polytope $\Delta(X, \mathbf{a}) \subseteq M_{\mathbb{R}}$ defined by

$$\Delta(X, \mathbf{a}) := [k_0 \Delta_{\mathbf{a}}], \quad k_0 := \min\{k \in \mathbb{N} \mid 0 \in \mathrm{Int}[k\Delta_{\mathbf{a}}]\}.$$

An important case of this is $\mathbf{a} = \mathbf{1}$. Then $D_{\mathbf{a}} = -K_X$. If $\Delta_{\mathbf{1}}$ is reflexive, we get $\Delta(X, \mathbf{a}) = \Delta_{\mathbf{a}} = \Delta_{\mathbf{1}} = \Delta_{-K_X}$.

We associate to the lattice polytope $\Delta(X, \mathbf{a})$ the complete toric variety $\mathbb{X}_{\mathbf{a}} := \mathbb{X}_{\Sigma_{\mathbf{a}}}$, where $\Sigma_{\mathbf{a}}$ is the face fan over the polytope $\Delta(X, \mathbf{a})$. Define $\Lambda_{\mathbf{a}} \in M_{n \times m'}(\mathbb{Z})$ to be the fan matrix of $\mathbb{X}_{\mathbf{a}}$ (see Definition 2.1.14). Given a fan matrix $V \in M_{n \times m}(\mathbb{Z})$ of X, we define

$$M_{\mathbf{a}} := V^T \cdot \Lambda_{\mathbf{a}} \in M_{m \times m'}(\mathbb{Z}).$$

Furthermore, we define $\mathbf{b} = (b_j)_{j=1}^{m'}$ to be the minimum strictly positive column vector such that

$$M_{\mathbf{a}}^T + B \geq \mathbf{0}, \text{ where } B := (\underbrace{\mathbf{b}, \ldots, \mathbf{b}}_{m \text{ times}}) \in M_{m' \times m}(\mathbb{Z}_{>0}).$$

Definition 2.3.6. Define $D'_{\mathbf{b}} := \sum_{j=1}^{m'} b_j D'_j$, where D'_i are the torus invariant prime divisors generating $\mathrm{Div}_{T_M}(\mathbb{X}_{\mathbf{a}})$. Then $(\mathbb{X}_{\mathbf{a}}, \mathbf{b}) := (\mathbb{X}_{\mathbf{a}}, D'_{\mathbf{b}})$ is a ftv, called the framed dual (f-dual) of $(X, \mathbf{a})$.

Thinking back to the case of $\mathbf{a} = 1$, f-duality reduces to Batyrev's duality between Fano toric varieties.

Remark 2.3.7. We can apply f-duality to $(\mathbb{X}_{\mathbf{a}}, \mathbf{b})$, giving rise to a ftv $(\mathbb{X}_{\mathbf{b}}, \mathbf{c}) := (\mathbb{X}_{\mathbf{b}}, D''_{\mathbf{c}})$. The double application of f-duality is called f-process. The question how a ftv is related to the result of an f-process is an interesting one considered in [44]. A notion of calibrated f-processes is defined. Roughly speaking, an f-process is calibrated if one can take simplicial refinements of the two fans at either end of the process, and find $\mathbb{Q}$-factorial resolutions that are isomorphic as toric varieties. Furthermore, one requires the push-forwards of the framing divisors to be equal under the isomorphism.

In particular, calibrated f-processes are key ingredients in introducing an involutive duality between ftvs, extending the Batyrev duality. In the remainder of the chapter, we will focus on how to use f-duality.

To apply f-duality to a hypersurface Y in a complete toric variety X, we make two assumptions. So assume that:

A. There is a divisor $D_{\mathbf{a}} \in \mathrm{Div}_{T_N}(X)$ such that Y is a generic element in the linear system $|D_{\mathbf{a}}| := \iota^{-1}([D_{\mathbf{a}}])$, with ι the morphism in (2.6).

B. $(X, D_{\mathbf{a}})$ is a ftv for which the f-process is calibrated.

Definition 2.3.8. A generic element $Y^\vee \in |D_{\mathbf{b}}| := \iota^{-1}([D_{\mathbf{b}}])$ is called a f-mirror partner of $Y \in |D_{\mathbf{a}}|$.

In his paper, Rossi exhibits an explicit way to describe the defining polynomials of Y and $Y^\vee$ in the Cox rings of $X, \mathbb{X}_{\mathbf{a}}$ respectively. While we omit this, we refer the reader to §4 in [44].

Remark 2.3.9. We notice that the divisor $D_{\mathbf{a}}$ satisfying the first condition is not necessarily unique. If there are two distinct divisors $D_{\mathbf{a}_1} \sim D_{\mathbf{a}_2}$ with $Y \in |D_{\mathbf{a}_1}| = |D_{\mathbf{a}_2}|$ and $(X, D_{\mathbf{a}_i})$ being a ftv for both, then f-duality may assign distinct mirror partners $Y_i^\vee$ who may not even be isomorphic. This problem does not occur in the Calabi-Yau/Fano case, as there is a unique choice of strictly effective divisor in the anticanonical class of X, which is $D_{\mathbf{1}} \in [-K_X]$. In the general case however, Rossi introduces the notion of a Mirror web of toric hypersurfaces, a notion generalising mirror symmetry. The mirror web will connect hypersurfaces by calibrated f-processes and contains different sub-webs depending on whether the f-process yielded a topological or Hodge mirror pair. For Calabi-Yau hypersurfaces all these sub-webs will be equal to the complete web.

f-duality for complete intersections in Toric Varieties.

We will now explain how to extend f-duality to families of complete intersections in a fixed variety X, generalising the Batyrev-Borisov duality, following §6 of [44].

Definition 2.3.10. Let $(X, D_{\mathbf{a}})$ be a ftv and V be a fan matrix of X, where $m = n + r$. A partition of the framing $D_{\mathbf{a}}$ is the datum of a partition

$$\exists l \in \mathbb{Z}_{\geq 0}: \quad I_1 \cup \cdots \cup I_l = \{1, \dots, m\}, \quad I_i \neq \emptyset \text{ for all } i \text{ such that if } i \neq j, \quad I_i \cap I_j = \emptyset$$

and divisors $D_{\mathbf{a}_1}, \dots, D_{\mathbf{a}_l}$ such that

$$D_{\mathbf{a}_k} := \sum_{i \in I_k} a_i D_i \text{ for } k = 1, \dots, l.$$

Since $\mathbf{a} = \sum_{k=1}^{l} \mathbf{a}_k$, we have $D_{\mathbf{a}} = \sum_{k=1}^{l} D_{\mathbf{a}_k}$. The ftv $(X, D_{\mathbf{a}})$ with framing partition $\mathbf{a} = \sum_{k=1}^{l} \mathbf{a}_k$ is called a partitioned ftv and we denote it by $(X, \mathbf{a} = \sum_{k=1}^{l} \mathbf{a}_k)$. Given such a partitioned ftv, we now describe an algorithm for the f-process on it. This will correspond to a mirror construction for complete intersections.

1. Let $\Delta_{\mathbf{a}}$ and $\Delta_{\mathbf{a}_1}, \ldots, \Delta_{\mathbf{a}_l}$ be the polytopes associated with divisors $D_{\mathbf{a}}$ and $D_{\mathbf{a}_1}, \ldots, D_{\mathbf{a}_l}$ respectively. Then the intersection of all $\Delta_{\mathbf{a}_i}$ turns out to be the origin, and their Minkowski sum is $\Delta_{\mathbf{a}}$.

2. Define the polytope
$$\widehat{\Delta_{\mathbf{a}}} := \mathrm{Conv}(\Delta_{\mathbf{a}_1}, \ldots, \Delta_{\mathbf{a}_l}) \subset M_{\mathbb{R}}.$$
We have $0 \in \mathrm{Int}(\widehat{\Delta_{\mathbf{a}}})$. We further have
$$\bigcap_{k=1}^{l} [k_0 \Delta_{\mathbf{a}_k}] = \{0\} \text{ and } 0 \in \mathrm{Int}(\Delta(X, \mathbf{a})),$$
noting that $\Delta(X, \mathbf{a}) = \left[\sum_{k=1}^{l} k_0 \Delta_{\mathbf{a}_k}\right]$. Then $0 \in \mathrm{Int}(\widehat{\Delta}(X, \mathbf{a}))$, defined as $\widehat{\Delta}(X, \mathbf{a}) := [k_0 \widehat{\Delta_{\mathbf{a}}}]$.

3. Set $\widehat{\mathbb{X}_{\mathbf{a}}}$ to be the variety associated to the face fan $\widehat{\Sigma_{\mathbf{a}}}$ of $\widehat{\Delta}(X, \mathbf{a})$, calling the fan matrix $\widehat{\Lambda}_{\mathbf{a}} \in M_{n \times \widehat{m}}(\mathbb{Z})$, with $\widehat{m} = |\widehat{\Sigma_{\mathbf{a}}}(1)|$.

4. For $k = 1, \ldots, l$, set $m_k := |I_k|$ and consider the matrix
$$\widehat{M_{\mathbf{a}_k}} := (V_{I_k})^T \cdot \widehat{\Lambda_{\mathbf{a}}} \in M_{m_k \times \widehat{m}}(\mathbb{Z}).$$
Here, V_{I_k} is the matrix obtained from V by only using the columns of V with index corresponding to an element of I_k.

Further, set $\mathbf{b}_k = (b_{j_k})_{j=1}^{\widehat{m}}$ to be the minimum non-negative column vector with
$$\widehat{M_{\mathbf{a}_k}}^T + B_k \geq \mathbf{0}, \text{ where } B_k = (\underbrace{\mathbf{b}_k \ldots \mathbf{b}_k}_{m_k \text{ times}}) \in M_{\widehat{m} \times m_k}(\mathbb{Z}_{\geq 0}).$$
Define $\widehat{\mathbf{b}} := \sum_{k=1}^{l} \mathbf{b}_k$. The group of torus invariant Weil divisors of $\widehat{\mathbb{X}_{\mathbf{a}}}$ is generated by the divisors corresponding to the $\widehat{m}$ rays of $\widehat{\Sigma}_{\mathbf{a}}$, call them $\widehat{D}_1, \ldots, \widehat{D}_{\widehat{m}}$. There

is now a unique partition $J_1 \cup \cdots \cup J_l = \{1, \ldots, \widehat{m}\}$ induced by $\mathbf{b}_k$. This partition gives a partitioned ftv, $(\widehat{\mathbb{X}_{\mathbf{a}}}, \widehat{D}_{\widehat{\mathbf{b}}} = \sum_{k=1}^{l} \widehat{D}_{\mathbf{b}_k})$, where $\widehat{D}_{\mathbf{b}_k} := \sum_{j \in J_k} b_{j_k} \widehat{D}_j$.

Details regarding this algorithm are worked out in the paper [45].

Definition 2.3.11. Following the above algorithm produces the paritioned ftv $(\widehat{\mathbb{X}_{\mathbf{a}}}, \widehat{\mathbf{b}} = \sum_{k=1}^{l} \mathbf{b}_k)$, called the partitioned f-dual of $(X, \mathbf{a} = \sum_{k=1}^{l} \mathbf{a}_k)$.

Applying these 4 steps again defines a partitioned f-process, analogously to the non-partitioned version, resulting in a partitioned ftv $(\widehat{\mathbb{X}}_{\widehat{\mathbf{b}}}, \widehat{\mathbf{c}} := \sum_{k=1}^{l} \mathbf{c}_k)$. A partitioned f-process is called calibrated under similar conditions as for the non-partitioned f-process. This is basically the case if and only if the fan matrices on either end of the process are the same, up to a permutation of columns and $\mathbf{c}_k = \mathbf{a}_k$ for all k (see Proposition 6.3 in [44]).

Definition 2.3.12. Given the partitioned ftv $(X, \mathbf{a} = \sum_{k=1}^{l} \mathbf{a}_k)$, assume that the associated partitioned f-process is calibrated. Consider the complete intersection subvariety

$$Y := \bigcap_{k=1}^{l} Y_k \subset X \text{ with } Y_k \in |D_{\mathbf{a}_k}|.$$

The generic complete intersection subvariety

$$Y^{\vee} := \bigcap_{k=1}^{l} Y_k^{\vee} \subset \widehat{\mathbb{X}}_{\mathbf{a}} \text{ with } Y_k \in |\widehat{D}_{\mathbf{b}_k}|$$

is called a f-mirror partner of Y.

Remark 2.3.13. As in the non-partitioned case, Rossi describes a way to explicitly define the polynomials of both Y and $Y^{\vee}$ in the respective Cox rings of X and $\widehat{\mathbb{X}}_{\mathbf{a}}$.

In the case of $\mathbf{a} = \mathbf{1}$ on a toric Fano, partitions correspond to nef-partitions and f-duality reduces to Batyrev-Borisov mirror symmetry between Calabi-Yau complete intersections.

In the paper [45], Rossi computes (stringy) Hodge numbers for framed toric varieties, working towards the topological mirror test for f-dual mirror families.

CHAPTER 3

DERIVED EQUIVALENCES BETWEEN MIRROR CONSTRUCTIONS VIA VGIT

In this chapter (which corresponds to the paper [36]), we will exhibit how variations of geometric invariant theory (VGIT) can be used to establish equivalences between the derived categories of coherent sheaves associated to complete intersections in toric varieties. In particular, we will use VGIT to prove the following Theorem, which is the main result of this chapter.

Recall the mirror constructions by Batyrev-Borisov and Libgober-Teitelbaum from §2.3.1, 2.3.2.

Theorem 3.0.1 (=Theorem 1.0.1)**.** *Let $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 0, 1$. Consider the two polynomials*

$$p_{1,\lambda} = x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8,$$

$$p_{2,\lambda} = x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11}.$$

Let $\mathcal{Z}_\lambda = Z(p_{1,\lambda}, p_{2,\lambda}) \subseteq \mathcal{X}_\nabla$ and $\mathcal{V}_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq [\mathbb{P}^5 / G_{81}]$. Then

$$\mathrm{D^b}(\mathrm{coh}\, \mathcal{V}_{LT,\lambda}) \simeq \mathrm{D^b}(\mathrm{coh}\, \mathcal{Z}_\lambda).$$

3.1 Categories of singularities and some results on the equivalence of derived categories.

In this section, we introduce the categories of singularities (as outlined in [41]) and their equivalences to derived categories through VGIT, reviewing §4 of [18].

Let X be a variety and G an algebraic group acting on X (on the left).

Definition 3.1.1. Recall that an object of $\mathrm{D^b}(\mathrm{coh}[X/G])$ is called perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles. We denote the full subcategory of perfect objects by $\mathrm{Perf}([X/G])$. The Verdier quotient of $\mathrm{D^b}(\mathrm{coh}[X/G])$ by $\mathrm{Perf}([X/G])$ is called the category of singularities and denoted by:

$$\mathrm{D_{sg}}([X/G]) := \mathrm{D^b}(\mathrm{coh}[X/G]) / \mathrm{Perf}([X/G]).$$

By the following observation of Orlov's, the category can be viewed as studying the geometry of the singular locus.

Proposition 3.1.2 (Orlov, [41])**.** *Assume that* $\mathrm{coh}[X/G]$ *has enough locally free sheaves. Let* $i : U \to X$ *be a* G*-equivariant open immersion such that the singular locus of* X *is contained in* $i(U)$*. Then the restriction,*

$$i^* : \mathrm{D_{sg}}([X/G]) \to \mathrm{D_{sg}}([U/G]),$$

is an equivalence of categories.

Next, consider a G-equivariant vector bundle $\mathcal{E}$ on X. Denote by Z the zero locus of a G-invariant section $s \in H^0(X, \mathcal{E})$. Then $\langle -, s \rangle$ induces a global function on $\mathrm{tot}\, \mathcal{E}^\vee$. Let Y be the zero-section of this pairing and consider the fibrewise dilation action on the torus $\mathbb{G}_m$ (which for us in the following will be $\mathbb{C}^*$). Then we have the following result.

Theorem 3.1.3 (Isik [32], Shipman [50], Hirano [30])**.** *Suppose the Koszul complex on s* *is exact. Then there is an equivalence of categories*

$$\mathrm{D_{sg}}([Y/(G \times \mathbb{G}_m)]) \cong \mathrm{D^b}(\mathrm{coh}[Z/G]).$$

Combining the previous two results gives the following result.

Corollary 3.1.4 (Corollary 3.4 in [18])**.** *Let V be an algebraic variety with a $G \times \mathbb{G}_m$ action. Suppose there is an open subset $U \subseteq V$ such that U is $G \times \mathbb{G}_m$ equivariantly isomorphic to Y as above and that U contains the singular locus of X. Then*

$$\mathrm{D_{sg}}([V/(G \times \mathbb{G}_m)]) \cong \mathrm{D^b}(\mathrm{coh}[Z/G]).$$

We will now work on making these results applicable to the contents treated in this thesis, adapting [18].

Consider an affine space $X := \mathbb{A}^{n+t}$ with coordinates x_i, u_j for $1 \leq i \leq n, 1 \leq j \leq t$. Let T denote the standard open torus $\mathbb{G}_m^{n+t}$ and consider a subgroup $S \subseteq T$, with $\tilde{S}$ the connected component that contains the identity. We now study the possible GIT quotients for the action of $\tilde{S}$ on X.

Recall from § 2.1.8 the notion of GKZ fans. We adjust the notation so that S above corresponds to the group G from § 2.1.8, and the treatment here is a bit more general, so we use $\mathbb{A}$ with torus $\mathbb{G}_m$ as opposed to $\mathbb{C}, \mathbb{C}^*$ considered before.

We will now explain how to construct varieties corresponding to the chambers of the GKZ fan, and the goal of this setup is to apply Corollary 3.1.4 and VGIT to provide equivalences between derived categories.

First, we will need to introduce some additional data.

Definition 3.1.5. Let G be a group acting on a space X and f a global function on X. f is said to be semi-invariant with respect to a character χ if, for any $g \in G$, $f(g \cdot x) = \chi(g) f(x)$.

Remark 3.1.6. We note that a global function f is semi-invariant if and only if it is a section of the equivariant line bundle $\mathcal{O}_X(\chi)$ on the global quotient stack $[X/G]$, i.e.

$f \in \Gamma(X, \mathcal{O}_X(\chi))^G$.

To apply Corollary 3.1.4, we will add a $\mathbb{G}_m$-action which is S-invariant and $\mathbb{G}_m$-semi-invariant, acting with weight 0 on the x_i and 1 on the u_j. We refer to this action as R-charge. Consider the action of S on the scheme $\operatorname{Spec} \mathbb{C}[u_j]$. It corresponds to a character γ_j of S. Let $f_1, \ldots, f_t$ be a collection of S-semi-invariant functions in the x_i with respect to γ_j^{-1}. Then define a function, called superpotential, by

$$w := \sum_{j=1}^{t} u_j f_j.$$

The superpotential w is S-invariant and χ-semi-invariant with respect to the projection character $\chi : S \times \mathbb{G}_m \to \mathbb{G}_m$, hence w is homogeneous of degree 0 with respect to the S-action and of degree 1 with respect to the R-charge. Let $Z(w) \subseteq X$ be its zero-locus and define $Y_p := Z(w) \cap U_p$. Then we have the following result.

Theorem 3.1.7 (Herbst-Walcher, Theorem 4.5 in [18])**.** *If S is quasi-Calabi-Yau, there is an equivalence of categories*

$$\mathrm{D}_{\mathrm{sg}}([Y_p/S \times \mathbb{G}_m]) \cong \mathrm{D}_{\mathrm{sg}}([Y_q/S \times \mathbb{G}_m])$$

for all $1 \leq p, q \leq r$.

We will use this result to show a useful equivalence of derived categories. We start by explicitly describing the open sets U_p corresponding to a chamber σ_p of the GKZ fan, defined in § 2.1.8. For $1 \leq p \leq k$ we associate an irrelevant ideal $\mathcal{I}_p$ to σ_p by considering the (regular) triangulation $\mathcal{T}_p$ that the chamber corresponds to. So, let

$$\mathcal{I}_p := \left\langle \prod_{i \notin I} x_i \prod_{j \notin J} u_j \;\middle|\; \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j \in J} \nu_{n+j}(S) \text{ is the set of vertices of a simplex in } \mathcal{T}_p \right\rangle.$$

Then $U_p = X \setminus Z(\mathcal{I}_p)$. Another ideal we will need is a subideal of $\mathcal{I}_p$, given similarly to

$\mathcal{I}_p$ by requiring J to be the full set $\{1, \dots, t\}$, i.e.,

$$\mathcal{J}_p := \left\langle \prod_{i \notin I} x_i \;\middle|\; \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j=1}^{t} \nu_{n+j}(S) \text{ is the set of vertices of a simplex in } \mathcal{T}_p \right\rangle.$$

This ideal is therefore generated by those simplices whose sets of vertices contain all ν_{n+j} for $1 \leq j \leq t$. Using this subideal, we get a new open set $V_p := X \setminus Z(\mathcal{J}_p) \subseteq U_p$. Since $\mathcal{J}_p$ has no u_j in its generators, we can see it as ideal $\mathcal{J}_p^x$ in $\mathbb{C}[x_1, \dots, x_n]$, giving an open subset of $\mathbb{A}^n$ by $V_p^x := \mathbb{A}^n \setminus Z(\mathcal{J}_p^x)$. This set gives us a toric stack $X_p := [V_p^x/S]$. Now suppose $\mathcal{J}_p$ is non-zero. Then the last two quantities defined are nonempty, and one can show $[V_p/S]$ is a vector bundle over X_p, with the inclusion of rings $\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_t]$ restricting to a S-equivariant morphism

$$[V_p/S] \to [V_p^x/S] = X_p.$$

This morphism gives the following proposition.

Proposition 3.1.8 (Proposition 4.6 in [18]). *Suppose $\mathcal{J}_p$ is non-zero. The morphism $[V_p/S] \to X_p$ realises $[V_p/S]$ as the total space of a vector bundle*

$$[V_p/S] \cong \operatorname{tot} \bigoplus_{j=1}^{t} \mathcal{O}(\gamma_j).$$

Furthermore, the R-charge action of $\mathbb{G}_m$ is the dilation action along fibers. Finally, for each j, the function f_j gives a section of $\mathcal{O}(\gamma_j^{-1})$ and the superpotential $w = \sum u_j f_j$ restricts to the pairing with the section $\oplus f_j$.

In particular, from this we can view the function $\oplus f_j$ as a section of V_p which defines, for all p, a complete intersection $Z_p := Z(\oplus f_j) \subseteq X_p$. Finally, we introduce the Jacobian ideal ∂w, generated by the partial derivatives of w with respect to the coordinates x_i, u_j.

Proposition 3.1.9 (Proposition 4.7 in [18]). *Suppose $\mathcal{J}_p$ is non-zero. If $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$, then*

$$\mathrm{D}_{\mathrm{sg}}([Y_p/S \times \mathbb{G}_m]) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z_p).$$

This finally leads us to the following result, allowing us to establish equivalences of derived categories of coherent sheaves.

Corollary 3.1.10. *Assume S satisfies the quasi-Calabi-Yau condition and that $\mathcal{J}_p$ and $\mathcal{J}_q$ are non-zero. If $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$ and $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$ for some $1 \leq p, q \leq r$, then*

$$\mathrm{D^b}(\mathrm{coh}\, Z_p) \cong \mathrm{D^b}(\mathrm{coh}\, Z_q).$$

3.2 The Batyrev-Borisov mirror construction in $\mathbb{P}^5$

We now construct a Batyrev-Borisov mirror to a complete intersection of two cubics in $\mathbb{P}^5$. We will do this by giving a nef partition of the anticanonical polytope of $\mathbb{P}^5$ which corresponds to a complete intersection. Then we will apply the Batyrev-Borisov construction to that nef partition, obtaining a polytope ∇ corresponding to the mirror. Fix the lattice $M \cong \mathbb{Z}^5$ and its dual lattice N.

Remark 3.2.1. Due to the way we will derive certain fans in this section via methods inspired by mirror symmetry (see § 3.3), our first fan lives in $M_{\mathbb{R}}$ and not in the conventional $N_{\mathbb{R}}$.

Define the rays $\overline{\rho_0}, \ldots, \overline{\rho_{11}}$ in $M_{\mathbb{R}} \oplus \mathbb{R}^2$ with primitive generators

$$\begin{aligned}
u_{\overline{\rho_0}} &= (3,0,0,-1,-1,0,1), & u_{\overline{\rho_6}} &= (2,-1,-1,0,0,1,0),\\
u_{\overline{\rho_1}} &= (0,3,0,-1,-1,0,1), & u_{\overline{\rho_7}} &= (-1,2,-1,0,0,1,0),\\
u_{\overline{\rho_2}} &= (0,0,3,-1,-1,0,1), & u_{\overline{\rho_8}} &= (-1,-1,2,0,0,1,0),\\
u_{\overline{\rho_3}} &= (-1,-1,-1,3,0,1,0), & u_{\overline{\rho_9}} &= (0,0,0,2,-1,0,1),\\
u_{\overline{\rho_4}} &= (-1,-1,-1,0,3,1,0), & u_{\overline{\rho_{10}}} &= (0,0,0,-1,2,0,1),\\
u_{\overline{\rho_5}} &= (-1,-1,-1,0,0,1,0), & u_{\overline{\rho_{11}}} &= (0,0,0,-1,-1,0,1),\\
u_{\tau_1} &= (0,0,0,0,0,1,0), & u_{\tau_2} &= (0,0,0,0,0,0,1).
\end{aligned}$$

Notation 3.2.2. For $0 \leq j \leq 11$, we denote by u_{ρ_j} the lattice point in M obtained from $u_{\overline{\rho_j}}$ by projecting onto the first 5 coordinates. Denote by ρ_j the ray generated by u_{ρ_j} in $M_{\mathbb{R}}$.

Proposition 3.2.3. *Consider the fan Σ_∇ with rays $\rho_0, \dots, \rho_{11}$ defined above and maximal cones listed in Table* 3.1 *(page 72). Then a general complete intersection in the toric variety X_∇ corresponding to the fan Σ_∇ is a Batyrev-Borisov mirror to a complete intersection of two cubics in $\mathbb{P}^5$.*

Proof. The anticanonical sheaf of $\mathbb{P}^5$ is $\mathcal{O}_{\mathbb{P}^5}(6)$, corresponding to the divisor class

$$-K_{\mathbb{P}^5} = T_0 + \cdots + T_5 = (T_0 + T_1 + T_2) + (T_3 + T_4 + T_5).$$

The anticanonical polytope for $\mathbb{P}^5$ is given by

$$\Delta_{-K_{\mathbb{P}^5}} = \{m \in M_{\mathbb{R}} | \langle m, u_\rho \rangle \geq -1 \text{ for } \rho \in \Sigma_{\mathbb{P}^5}(1)\} \subseteq M_{\mathbb{R}},$$

which is the convex hull of the six points

$$\begin{gathered}(5,-1,-1,-1,-1), \quad (-1,5,-1,-1,-1), \quad (-1,-1,5,-1,-1),\\ (-1,-1,-1,5,-1), \quad (-1,-1,-1,-1,5), \quad (-1,-1,-1,-1,-1).\end{gathered}$$

A nef partition with respect to the origin of the polytope $\Delta_{-K_{\mathbb{P}^5}}$ is given by the polytopes Δ_1, Δ_2 associated to the divisors $T_0 + T_1 + T_2$ and $T_3 + T_4 + T_5$, since the Minkowski sum $\Delta_1 + \Delta_2$ is equal to $\Delta_{-K_{\mathbb{P}^5}}$. These polytopes are

$$\begin{aligned}\Delta_1 = \ & \mathrm{Conv}((2,-1,-1,0,0),(-1,2,-1,0,0),(-1,-1,2,0,0),\\ & (-1,-1,-1,3,0),(-1,-1,-1,0,3),(-1,-1,-1,0,0)),\\ \Delta_2 = \ & \mathrm{Conv}((0,0,0,-1,2),(0,0,0,2,-1),(0,0,3,-1,-1),\\ & (0,3,0,-1,-1),(3,0,0,-1,-1),(0,0,0,-1,-1)).\end{aligned}$$

Next, we shall compute the dual nef partition, as defined in § 2.3.1. We have:

$$\begin{aligned}\nabla_1 = &\quad \mathrm{Conv}((1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,0,0))\\ \nabla_2 = &\quad \mathrm{Conv}((0,0,0,0,1),(0,0,0,1,0),(0,0,0,0,0),(-1,-1,-1,-1,-1)).\end{aligned}$$

Their Minkowski sum $\nabla \subseteq N_{\mathbb{R}}$ is then the convex hull of the 15 points

$$\begin{array}{llll}(1,0,0,0,0), & (0,1,0,0,0), & (0,0,1,0,0), & (0,0,0,1,0),\\ (0,0,0,0,1), & (1,0,0,1,0), & (1,0,0,0,1), & (0,1,0,1,0),\\ (0,1,0,0,1), & (0,0,1,1,0), & (0,0,1,0,1), & (-1,-1,-1,-1,-1),\\ (0,-1,-1,-1,-1), & (-1,0,-1,-1,-1), & (-1,-1,0,-1,-1). & \end{array}$$

A SAGE computation shows the normal fan of ∇, $\Sigma'_{\nabla} \subseteq M_{\mathbb{R}}$, has rays $\rho_0, \ldots, \rho_{11}$ from Notation 3.2.2. The maximal-dimensional cones are the following 15 cones:

$$\begin{array}{llll}\rho_0\rho_1\rho_2\rho_9\rho_{10}, & \rho_0\rho_1\rho_3\rho_4\rho_6\rho_7\rho_9\rho_{10}, & \rho_0\rho_2\rho_3\rho_4\rho_6\rho_8\rho_9\rho_{10}, & \rho_1\rho_2\rho_3\rho_4\rho_7\rho_8\rho_9\rho_{10},\\ \rho_1\rho_2\rho_4\rho_5\rho_7\rho_8\rho_{10}\rho_{11}, & \rho_0\rho_1\rho_2\rho_9\rho_{11}, & \rho_0\rho_1\rho_2\rho_{10}\rho_{11}, & \rho_3\rho_4\rho_5\rho_6\rho_7,\\ \rho_0\rho_1\rho_3\rho_5\rho_6\rho_7\rho_9\rho_{11}, & \rho_0\rho_1\rho_2\rho_5\rho_6\rho_7\rho_{10}\rho_{11}, & \rho_3\rho_4\rho_5\rho_6\rho_8, & \rho_0\rho_2\rho_3\rho_5\rho_6\rho_8\rho_9\rho_{11},\\ \rho_0\rho_2\rho_4\rho_5\rho_6\rho_8\rho_{10}\rho_{11}, & \rho_3\rho_4\rho_5\rho_7\rho_8, & \rho_1\rho_2\rho_3\rho_5\rho_7\rho_8\rho_9\rho_{10}\rho_{11}. & \end{array}$$

We listed the cones by giving the rays generating them. For instance, $\rho_0\rho_1\rho_2\rho_9\rho_{10}$ stands for the cone $\mathrm{Cone}\,(\rho_0, \rho_1, \rho_2, \rho_9, \rho_{10})$. Note here that some of these maximal cones contain more rays than the others. So, as described in Remark 2.3.2, we want a MPCP-resolution of the variety associated to the above fan. To do this, we subdivide each of the maximal cones which has more than 5 rays. This procedure involves choice, as each cone can be subdivided in 24 ways (being a total of 24^9 possible choices!). However, all these choices are related by GIT, so any choice gives us a mirror family, all of which are birational. Following this procedure, the Table 3.1 (see below) gives the 42 maximal cones in the fan corresponding to a MPCP-resolution of the variety associated to the fan Σ'_{∇}. Define the fan Σ_{∇} to be the fan consisting of those 42 5-dimensional cones and all of their faces. Determining the variety X_{∇} explicitly is not straightforward, but also not necessary for our purposes, so long as we have the fan Σ_{∇}. □

$\rho_0\rho_1\rho_2\rho_9\rho_{10}$	$\rho_0\rho_1\rho_6\rho_9\rho_{10}$	$\rho_3\rho_4\rho_6\rho_7\rho_9$	$\rho_1\rho_6\rho_7\rho_9\rho_{10}$	$\rho_4\rho_6\rho_7\rho_9\rho_{10}$	$\rho_0\rho_2\rho_6\rho_9\rho_{10}$
$\rho_2\rho_6\rho_8\rho_9\rho_{10}$	$\rho_3\rho_4\rho_6\rho_8\rho_9$	$\rho_4\rho_6\rho_8\rho_9\rho_{10}$	$\rho_1\rho_2\rho_7\rho_9\rho_{10}$	$\rho_2\rho_7\rho_8\rho_9\rho_{10}$	$\rho_3\rho_4\rho_7\rho_8\rho_{10}$
$\rho_5\rho_7\rho_8\rho_9\rho_{10}$	$\rho_1\rho_2\rho_7\rho_{10}\rho_{11}$	$\rho_2\rho_7\rho_8\rho_{10}\rho_{11}$	$\rho_4\rho_5\rho_7\rho_8\rho_{10}$	$\rho_5\rho_7\rho_8\rho_{10}\rho_{11}$	$\rho_0\rho_1\rho_2\rho_9\rho_{11}$
$\rho_0\rho_1\rho_2\rho_{10}\rho_{11}$	$\rho_3\rho_4\rho_5\rho_6\rho_7$	$\rho_0\rho_1\rho_6\rho_9\rho_{11}$	$\rho_1\rho_6\rho_7\rho_9\rho_{11}$	$\rho_3\rho_5\rho_6\rho_7\rho_9$	$\rho_5\rho_6\rho_7\rho_9\rho_{11}$
$\rho_0\rho_1\rho_6\rho_{10}\rho_{11}$	$\rho_1\rho_6\rho_7\rho_{10}\rho_{11}$	$\rho_4\rho_5\rho_6\rho_7\rho_{10}$	$\rho_5\rho_6\rho_7\rho_{10}\rho_{11}$	$\rho_3\rho_4\rho_5\rho_6\rho_8$	$\rho_0\rho_2\rho_6\rho_9\rho_{11}$
$\rho_2\rho_6\rho_8\rho_9\rho_{11}$	$\rho_3\rho_5\rho_6\rho_8\rho_9$	$\rho_5\rho_6\rho_8\rho_9\rho_{11}$	$\rho_0\rho_2\rho_6\rho_{10}\rho_{11}$	$\rho_2\rho_6\rho_8\rho_{10}\rho_{11}$	$\rho_4\rho_5\rho_6\rho_8\rho_{10}$
$\rho_5\rho_6\rho_8\rho_{10}\rho_{11}$	$\rho_3\rho_4\rho_5\rho_7\rho_8$	$\rho_1\rho_2\rho_7\rho_9\rho_{11}$	$\rho_2\rho_7\rho_8\rho_9\rho_{11}$	$\rho_3\rho_5\rho_7\rho_8\rho_9$	$\rho_5\rho_7\rho_8\rho_9\rho_{11}$

Table 3.1: Maximal cones of X_{∇}

For $i = 0, \ldots, 11$, call D'_i the torus-invariant divisor on X_{∇} corresponding to the ray ρ_i of Σ_{Δ}. Let $D'_a = D'_0 + D'_1 + D'_2 + D'_9 + D'_{10} + D'_{11}$ and $D'_b = D'_3 + D'_4 + D'_5 + D'_6 + D'_7 + D'_8$.

Corollary 3.2.4. *Let $\Sigma_{\nabla, D'_a, D'_b}$ be the fan with rays $\overline{\rho_0}, \ldots, \overline{\rho_{11}}, \tau_1, \tau_2$, and cones over those rays inherited from Σ_{∇}. Then $\Sigma_{\nabla, D'_a, D'_b}$ is a fan corresponding to* $\mathrm{tot}(\mathcal{O}_{X_{\nabla}}(-D'_b) \oplus \mathcal{O}_{X_{\nabla}}(-D'_a))$.

Proof. Apply Proposition 2.1.50 twice to get the result (recalling that we can do this by Proposition 2.1.51). □

3.3 Torically expressing the Libgober-Teitelbaum construction

Recall the Libgober-Teitelbaum mirror construction from §2.3.2. In the following, we aim to give a toric description of $V_{LT,\lambda}$. First we give a fan for the toric variety $X_{LT} := \mathbb{P}^5 / G_{81}$ and then employ methods of §7.3 of [15] to construct a vector bundle over X_{LT} that has the global section $Q_{1,\lambda} \oplus Q_{2,\lambda}$.

Proposition 3.3.1. *Consider the 1-dimensional cones $\rho_0, \ldots, \rho_5$ with corresponding primitive generators*

$$
\begin{aligned}
u_{\rho_0} &= (3,0,0,-1,-1), & u_{\rho_1} &= (0,3,0,-1,-1), & u_{\rho_2} &= (0,0,3,-1,-1),\\
u_{\rho_3} &= (-1,-1,-1,3,0), & u_{\rho_4} &= (-1,-1,-1,0,3), & u_{\rho_5} &= (-1,-1,-1,0,0).
\end{aligned}
$$

Consider the collection $\mathcal{C}$ of sets of the form

$$\{\rho_i \mid i \in I, I \subseteq \{0,\dots,5\}, |I| = 5\}.$$

Let $\Sigma_{LT} \subseteq M_{\mathbb{R}}$ be the fan consisting of maximal cones

$$\{\mathrm{Cone}(C) | C \in \mathcal{C}\}$$

and all their faces.

Then the toric stack associated to Σ_{LT} is the stack corresponding to the Libgober-Teitelbaum construction, $\mathcal{X}_{LT} = [\mathbb{C}^6 \setminus \{0\} / (\mathbb{C}^* \times G_{81})]$, with $\mathbb{C}^*$ acting by $(\lambda x_0, \dots, \lambda x_5) \sim (x_0, \dots, x_5)$ and G_{81} acting as described above in § 2.3.2.

Proof. We use the Cox construction described in §2.1.4. By Lemma 2.1.39, we obtain the following system of equations characterising elements of $G := G_\Sigma$

$$t_3t_4t_5 = t_0^3 \tag{3.1}$$

$$t_3t_4t_5 = t_1^3 \tag{3.2}$$

$$t_3t_4t_5 = t_2^3 \tag{3.3}$$

$$t_0t_1t_2 = t_3^3 \tag{3.4}$$

$$t_0t_1t_2 = t_4^3 \tag{3.5}$$

First we note that we have a copy of $\mathbb{C}^*$ in G, given by $\{t \cdot (1,1,1,1,1,1) \mid t \in \mathbb{C}^*\}$, so to compute G we consider the group H of cosets of $\mathbb{C}^*$. We will explicitly describe H and subsequently use the direct product theorem to compute G. Consider an element

$(t_0, \dots, t_5) \in G$. By an appropriate choice of coset representative of $(t_0, t_1, t_2, t_3, t_4, t_5) \cdot \mathbb{C}^*$, we may assume $\prod_{i=0}^5 t_i = 1$.

Using equations (3.1), (3.2) and (3.3), we have $t_0^3 = t_1^3 = t_2^3$, and thus $t_0 = \zeta_3^\alpha t_2, t_1 = \zeta_3^\beta t_2$ for some $\alpha, \beta \in \mathbb{Z}_3$. Using equations (3.1) – (3.5), we have that $t_3^3 t_4^3 t_5^3 = t_0^3 t_1^3 t_2^3 = t_3^6 t_4^3 = t_3^3 t_4^6$, which implies

$$t_5^3 = t_3^3 = t_4^3. \tag{3.6}$$

Hence, similarly to above, we obtain $t_3 = \zeta_3^{-\delta} t_5, t_4 = \zeta_3^{-\varepsilon} t_5$ for some $\delta, \varepsilon \in \mathbb{Z}_3$.
By combining (3.3), (3.4) and (3.6) we obtain

$$t_2^3 t_5^3 = t_0 t_1 t_2 t_3 t_4 t_5 = 1. \tag{3.7}$$

Equation (3.7) implies $t_5^3 = (t_2^{-1})^3$, thus $t_5 = \zeta_3^\nu \cdot t_2^{-1}$ for some $\nu \in \mathbb{Z}_3$. Using $t_3 = \zeta_3^{-\delta} t_5$ and $t_4 = \zeta_3^{-\varepsilon} t_5$ and equation (3.3), we obtain

$$t_2^3 = t_3 t_4 t_5 = t_5^3 \zeta_3^{-(\delta+\varepsilon)} = t_2^{-3} \zeta_3^{-(\delta+\varepsilon)}.$$

Hence $t_2^{18} = 1$. So we can write $t_2 = \zeta_{18}^l$ for some $l \in \mathbb{Z}_{18}$.

We now claim that t_2 can be assumed to be a ninth root of unity and t_5 to be its inverse, i.e. $t_2 = \zeta_9^\mu, t_5 = \zeta_9^{-\mu}$ for some $\mu \in \mathbb{Z}_9$. Indeed, note that $(\zeta_6, \dots, \zeta_6) \in (1,1,1,1,1,1) \cdot \mathbb{C}^* \subseteq G$, so we can scale an element $(t_0, \dots, t_5) \in G$ by sixth roots of unity, leaving the product $\prod_{i=1}^6 t_i$ invariant. The claim follows by multiplication with an appropriate sixth root of unity.

Expressing all the t_i in terms of t_2, the assumption $1 = \prod_{i=1}^6 t_i$ implies $1 = \zeta_3^{\alpha+\beta-\delta-\varepsilon}$, or, equivalently,

$$\alpha + \beta = \delta + \varepsilon \pmod 3.$$

Finally, using (3.3) gives $\zeta_9^{3\mu} = \zeta_3^{-\delta+\varepsilon} \zeta_9^{-3\mu}$ and therefore $\zeta_9^{3\mu} = \zeta_3^{\delta+\varepsilon}$. Thus H, the group of cosets of $\mathbb{C}^*$, is isomorphic to G_{81}, where G_{81} is the same group described in § 2.3.2. In particular, all elements of G are of the form $g \cdot \lambda$ with $g \in G_{81},\ \lambda \in (1,1,1,1,1,1) \cdot \mathbb{C}^*$

and $G_{81} \cap \{(1,1,1,1,1,1) \cdot \lambda | \lambda \in \mathbb{C}^*\} = \{(1,1,1,1,1,1)\}$. Hence, by the direct product theorem, $G \cong \mathbb{C}^* \times G_{81}$.

The Cox fan of Σ_{LT} can be described as follows. It has six rays $e_{\rho_0}, \ldots, e_{\rho_5}$. It is straightforward to see that the maximal cones are all 5-dimensional cones generated by any 5 of the rays above. Therefore, we obtain $U_{\Sigma_{LT}} = \mathbb{A}^6 \setminus \{0\}$. Thus, the Cox stack associated to Σ_{LT} is

$$\mathcal{X}_{LT} = [U_{\Sigma_{LT}}/G] = [\mathbb{C}^6 \setminus \{0\} / (\mathbb{C}^* \times G_{81})],$$

with the prescribed action, as required. $\square$

Remark 3.3.2. We note that by Theorem 2.1.43 the coarse moduli space of the stack $\mathcal{X}_{LT}$ is X_{LT}, since Σ_{LT} is simplicial.

Starting with the fan Σ_{LT} of X_{LT}, we apply Proposition 2.1.50 twice to construct a vector bundle. Let D_i be the Weil divisor corresponding to the ray ρ_i in Σ_{LT}. Let $D_a = D_0 + D_1 + D_2$ and $D_b = D_3 + D_4 + D_5$.

Corollary 3.3.3. *Denote by the rays $\overline{\rho_0}, \ldots, \overline{\rho_5}$, τ_1 and τ_2 the rays*[1] *generated by the primitive generators:*

$$\begin{aligned} &u_{\overline{\rho_0}} = (3,0,0,-1,-1,0,1), && u_{\overline{\rho_1}} = (0,3,0,-1,-1,0,1), && u_{\overline{\rho_2}} = (0,0,3,-1,-1,0,1),\\ &u_{\overline{\rho_3}} = (-1,-1,-1,3,0,1,0), && u_{\overline{\rho_4}} = (-1,-1,-1,0,3,1,0), && u_{\overline{\rho_5}} = (-1,-1,-1,0,0,1,0),\\ &u_{\tau_1} = (0,0,0,0,0,1,0), && u_{\tau_2} = (0,0,0,0,0,0,1). \end{aligned}$$

Consider the collection $\mathcal{S}$ of sets of the form

$$\{\overline{\rho_i} \mid i \in I, I \subseteq \{0,\ldots,5\}, |I| = 5\} \cup \{\tau_1, \tau_2\}.$$

[1]These are the same as on page 69.

Let Σ_{LT,D_a,D_b} be the fan in $M_{\mathbb{R}} \oplus \mathbb{R}^2$ consisting of the maximal cones

$$\{\mathrm{Cone}(S) | S \in \mathcal{S}\}$$

and all their faces. Then:

(a) Σ_{LT,D_a,D_b} is a fan corresponding to $\mathrm{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$;

(b) The vector bundle $\mathcal{O}_{X_{LT}}(D_b) \oplus \mathcal{O}_{X_{LT}}(D_a)$ has the global section $Q_{1,\lambda} \oplus Q_{2,\lambda}$.

Proof. Applying Proposition 2.1.50 twice yields (a).

We now turn to (b) and show that $Q_{1,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$ and $Q_{2,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$. We start by noting that on X_{LT} we have $\mathrm{div}(x_i^3) = 3D_i$, so $\mathrm{div}(x_i^3) - 3D_i \geq 0$, i.e. $x_i^3 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(3D_i))$. Similarly, $x_0x_1x_2 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$ and $x_3x_4x_5 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$.

To show the linear equivalence of two divisors, it suffices to consider their difference and show it is principal. We recall that $\mathrm{div}(\chi^n) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, n \rangle D_\rho$, corresponding to the map ι in the exact sequence (2.6). So, for instance $3D_1 - 3D_0 = \mathrm{div}(x_0^{-3}x_1^3)$ which is the character associated to the lattice point $(-1, 1, 0, 0, 0)$. Hence $3D_1 - 3D_0 = 0$ in $\mathrm{Cl}(X_{LT})$, i.e. $3D_0 \sim 3D_1$. Similarly $3D_1 \sim 3D_2$ and $3D_3 \sim 3D_4 \sim 3D_5$. Using the lattice points $(-1, 0, 0, 0, 0)$ and $(0, 0, 0, -1, 0)$ respectively, we also see that $3D_0 \sim D_b$ and $3D_3 \sim D_a$.

Thus

$$\mathcal{O}_{X_{LT}}(3D_0) \simeq \mathcal{O}_{X_{LT}}(3D_1) \simeq \mathcal{O}_{X_{LT}}(3D_2) \simeq \mathcal{O}_{X_{LT}}(D_b)$$

and

$$\mathcal{O}_{X_{LT}}(3D_3) \simeq \mathcal{O}_{X_{LT}}(3D_4) \simeq \mathcal{O}_{X_{LT}}(3D_5) \simeq \mathcal{O}_{X_{LT}}(D_a),$$

implying $Q_{2,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$ and $Q_{1,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$, as required. □

Intuition for constructing X_{LT} torically

We now explain how we found an explicit description for the fan Σ_{LT}. We start by considering the standard fan $\Sigma_{\mathbb{P}^5} \subseteq N_{\mathbb{R}}$ for $\mathbb{P}^5$ in the standard basis. It is the fan consisting of the cones generated by any proper subset of the six rays $\nu_0, \ldots, \nu_5$ with primitive generators

$$\begin{aligned}
u_{\nu_0} &= (1,0,0,0,0), \quad & u_{\nu_1} &= (0,1,0,0,0), \quad & u_{\nu_2} &= (0,0,1,0,0), \\
u_{\nu_3} &= (0,0,0,1,0), \quad & u_{\nu_4} &= (0,0,0,0,1), \quad & u_{\nu_5} &= (-1,-1,-1,-1,-1).
\end{aligned}$$

Denote by $T_0, \ldots, T_5$ the six primitive Weil divisors corresponding to the rays $u_{\nu_0}, \ldots, u_{\nu_5}$ respectively. Then

$$\mathcal{O}(-\underbrace{(T_0+T_1+T_2)}_{:=T_a}) = \mathcal{O}(-\underbrace{(T_3+T_4+T_5)}_{:=T_b}) = \mathcal{O}(-3),$$

and we can use the methods of §7.3 of [15] again to construct a fan of $\mathrm{tot}(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus \mathcal{O}_{\mathbb{P}^5}(-3))$. This yields the fan $\Sigma_{\mathbb{P}^5,T_a,T_b}$ in $N_{\mathbb{R}} \oplus \mathbb{R}^2$ with the 8 rays $\overline{\nu_0}, \ldots, \overline{\nu_5}, \tau_1$ and τ_2 having primitive ray generators

$$\begin{aligned}
u_{\overline{\nu_0}} &= (1,0,0,0,0,1,0), \quad & u_{\overline{\nu_4}} &= (0,0,0,0,1,0,1), \\
u_{\overline{\nu_1}} &= (0,1,0,0,0,1,0), \quad & u_{\overline{\nu_5}} &= (-1,-1,-1,-1,-1,0,1), \\
u_{\overline{\nu_2}} &= (0,0,1,0,0,1,0), \quad & u_{\tau_1} &= (0,0,0,0,0,1,0), \\
u_{\overline{\nu_3}} &= (0,0,0,1,0,0,1), \quad & u_{\tau_2} &= (0,0,0,0,0,0,1).
\end{aligned} \tag{3.8}$$

The fan $\Sigma_{\mathbb{P}^5,T_a,T_b}$ is the star subdivision of $\mathrm{Cone}(u_{\overline{\nu_0}}, \ldots, u_{\overline{\nu_5}}, u_{\tau_1}, u_{\tau_2})$ along u_{τ_1} and u_{τ_2} (noting the abuse of notation by which u_{τ_i} represent the same vector in both lattices M, N). The dual cone to $\Sigma_{\mathbb{P}^5,T_a,T_b}$ in $M_{\mathbb{R}} \oplus \mathbb{R}^2$ is spanned by the 12 rays $\overline{\rho_0}, \ldots, \overline{\rho_{11}}$ defined in § 2.3.1 (page 55).

We recall that each lattice point in the interior of the dual cone corresponds to a global function of $X_{\Sigma_{\mathbb{P}^5,T_a,T_b}}$ by associating m to the monomial

$$x^m := \prod_{\rho \in \Sigma_{\mathbb{P}^5,T_1,T_2}(1)} x_\rho^{\langle m, u_\rho \rangle}.$$

Now a section $s_1 \oplus s_2 \in \Gamma(\mathbb{P}^5, \mathcal{O}(3) \oplus \mathcal{O}(3))$ will correspond to a global function on tot $(\mathcal{O}(-3) \oplus \mathcal{O}(-3))$ of the form $u_1 s_1 + u_2 s_2$, where u_i is the variable corresponding to u_{τ_i}. Recalling the polynomials Q_i from (2.12) in § 2.3.2, we would like to express the global function $F := u_2 Q_{1,\lambda} + u_1 Q_{2,\lambda}$ as a linear combination of global functions of the form x^m. We do this by finding the lattice points in the dual cone corresponding to each monomial in F.

By splitting it up into its monomials, $u_2 Q_{1,\lambda}$ corresponds to the 4 points $(3, 0, 0, -1, -1, 0, 1)$, $(0, 3, 0, -1, -1, 0, 1)$, $(0, 0, 3, -1, -1, 0, 1)$ and $(0, 0, 0, 0, 0, 0, 1)$.
Similarly, $u_1 Q_{2,\lambda}$ corresponds to the points $(-1, -1, -1, 3, 0, 1, 0)$, $(-1, -1, -1, 0, 3, 1, 0)$, $(-1, -1, -1, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 0, 1, 0)$.

We find that these 8 points are the primitive generators for the rays of Σ_{LT,D_a,D_b} (see Corollary 3.3.3).

Quotienting $M_{\mathbb{R}} \oplus \mathbb{R}^2$ by the rays associated to the bundle coordinates (i.e. the lattice points that are the elements of the dual basis dual to u_{τ_1} and u_{τ_2}) corresponds to a toric morphism $X_{\Sigma_{LT,D_a,D_b}} \to X_{\Sigma_{LT}}$. We emphasize that the dual cone to $\mathrm{Cone}(\Sigma_{\mathbb{P}^5,T_a,T_b}(1))$ is given by $\mathrm{Conv}(u_{\overline{\rho_0}}, \ldots, u_{\overline{\rho_{11}}})$. Here, we take a subcone generated by a subset of $\{u_{\overline{\rho_0}}, \ldots, u_{\overline{\rho_{11}}}\}$.

Expressing the zerolocus of $Q_{1,\lambda}, Q_{2,\lambda}$

We remark that the cone $|\Sigma_{LT,D_a,D_b}|$ is not a reflexive Gorenstein cone, hence the Batyrev-Borisov construction does not apply to it.

The variety $V_{LT,\lambda} \subseteq X_{LT}$ is the zero-locus of the polynomials $Q_{1,\lambda}, Q_{2,\lambda}$, where $Q_{1,\lambda} \oplus Q_{2,\lambda}$ is a section of the vector bundle constructed above in Corollary 3.3.3. Proceeding in the same way as in §3.3, we consider lattice points on the cone $|\Sigma_{LT,D_a,D_b}|^\vee \subseteq N_{\mathbb{R}} \oplus \mathbb{R}^2$ to get global functions of $\mathcal{X}_{\Sigma_{LT,D_a,D_b}}$. The cone $|\Sigma_{LT,D_a,D_b}|^\vee$ is the cone over the convex hull of the following 12 points:

$$\begin{array}{lll}
(1,0,0,0,0,1,0), & (0,1,0,0,0,1,0), & (0,0,1,0,0,1,0), \\
(0,0,0,1,0,0,1), & (0,0,0,0,1,0,1) & (2,-1,-1,0,0,0,3), \\
(-1,2,-1,0,0,0,3), & (-1,-1,2,0,0,0,3), & (1,1,1,3,0,3,0), \\
(1,1,1,0,3,3,0), & (-1,-1,-1,-1,-1,0,1), & (-2,-2,-2,-3,-3,3,0).
\end{array}$$

The points corresponding to the monomials in $u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$, and hence to the section $Q_{1,\lambda} \oplus Q_{2,\lambda}$, are the lattice points $u_{\overline{\nu_i}}$ and u_{τ_i} in (3.8). Later on, describing V_{LT} by these 8 points will allow us to work with $\mathrm{D^b}(\operatorname{coh} V_{LT})$, using results in [18].

Remark 3.3.4. In their recent work [44, 45], Rossi proposes a generalisation of the Batyrev-Borisov mirror construction, called framed duality (f-duality). f-duality gives an algorithm to obtain mirror candidates of hypersurfaces and complete intersections in toric varieties. Applying f-duality to $V_{LT} \subset \mathbb{P}^5 / G_{81}$ produces $V_\lambda \subset \mathbb{P}^5$, which in turn gives the same mirror as the Batyrev-Borisov construction when applying f-duality to it. Theorem 1.0.1 suggests that different mirror candidates obtained via f-duality may be derived equivalent and prompts the question under what conditions this is the case.

3.4 A derived equivalence between Batyrev-Borisov and Libgober-Teitelbaum

Here we will prove the main result, Theorem 1.0.1.

3.4.1 Picking a partial compactification

Looking at the dual of the fan Σ_{LT,D_a,D_b} as in Corollary 3.3.3, we recall from §3.3 that the global function $u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$ corresponds to the points

$$(1,0,0,0,0,1,0),\quad (0,1,0,0,0,1,0),\quad (0,0,1,0,0,1,0),$$
$$(0,0,0,1,0,0,1),\quad (0,0,0,0,1,0,1),\quad (-1,-1,-1,-1,-1,0,1),$$
$$(0,0,0,0,0,0,1),\quad (0,0,0,0,0,1,0).$$

Consider the GKZ fan of $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$. We note that the chambers of this GKZ fan correspond to regular triangulations of the polytope $\mathfrak{P} = \mathrm{Conv}(\mathfrak{C})$, where $\mathfrak{C}$ is the collection of the following 14 points:

$$\begin{aligned}
P_0 &= (3,0,0,-1,-1,0,1), & P_6 &= (2,-1,-1,0,0,1,0),\\
P_1 &= (0,3,0,-1,-1,0,1), & P_7 &= (-1,2,-1,0,0,1,0),\\
P_2 &= (0,0,3,-1,-1,0,1), & P_8 &= (-1,-1,2,0,0,1,0),\\
P_3 &= (-1,-1,-1,3,0,1,0), & P_9 &= (0,0,0,2,-1,0,1),\\
P_4 &= (-1,-1,-1,0,3,1,0), & P_{10} &= (0,0,0,-1,2,0,1),\\
P_5 &= (-1,-1,-1,0,0,1,0), & P_{11} &= (0,0,0,-1,-1,0,1),\\
S_1 &= (0,0,0,0,0,1,0), & S_2 &= (0,0,0,0,0,0,1).
\end{aligned}$$

In the (regular) triangulations of $\mathfrak{P}$, we look for a subtriangulation corresponding to Σ_{LT,D_a,D_b}, as then we obtain a partial compactification of $\mathrm{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ from Corollary 3.3.3.

Proposition 3.4.1. *There exists a chamber σ_{LT} in the GKZ fan of* $\mathrm{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$ *(from Corollary 3.2.4) so that the triangulation $\mathcal{T}$ corresponding to the chamber σ_{LT} (in the sense of 2.1.59) has the following properties:*

- *$\mathcal{T}$ contains the following set of simplices, listed via their vertices:*

$$\mathcal{T}_0 := \{\{P_i, S_1, S_2 \mid i \in I\} \mid I \subset \{0,2,\ldots,5\}, |I| = 5\}.$$

- *Any simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills either of the two following conditions:*

  A. $S_1, P_6, P_7, P_8 \notin T$ and $\exists\ 3 \le j \le 6$ such that $P_j, P_{6+j} \notin T$.

B. $S_2, P_9, P_{10}, P_{11} \notin T$ *and* $\exists\ 0 \leq j \leq 2$ *such that* $P_j, P_{6+j} \notin T$.

Moreover, the toric variety X_Σ *corresponding to the chamber* σ_{LT} *is a partial compactification of the variety* $\mathrm{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ *from Corollary 3.3.3.*

The first property of $\mathcal{T}$ means that the associated variety X_Σ is a partial compactification of X_{LT}, so proving the existence of the triangulation $\mathcal{T}$ is sufficient to prove the Proposition. The second property of $\mathcal{T}$ is not a natural one to consider, but will become necessary to apply results from § 3.1.

The proposition can be checked via a simple SAGE program [47] using the TOPCOM package [42]; however, we include an explicit proof on how such a triangulation can be constructed.

To prove the proposition, we break the statement up into 3 steps.

Step 1: We start by defining an explicit regular polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$ containing $\mathcal{T}_0$.

Step 2: We prove that the polyhedral subdivision $\mathcal{S}$ can be refined to a regular triangulation $\mathcal{T}$ of $\mathfrak{P}$ containing $\mathcal{T}_0$.

Step 3: We show that any regular triangulation obtained this way fulfills the conditions outlined in the Proposition.

Step 1:

We note that $\mathcal{T}_0$ is a regular triangulation of the set of points $P_0, \ldots, P_5, S_1, S_2$. It is in fact a star subdivision with respect to S_1, S_2 of the convex hull $\mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$. Indeed, an example of an explicit weight function w giving the triangulation $\mathcal{T}_0$ is $w(S_1) = w(S_2) = 1, w(P_i) = 2$ for $0 \leq i \leq 5$. To complete Step 1, we extend this weight function to all 14 points of $\mathfrak{C}$.

Consider the weight function $w(P_i) = 2$ for $0 \leq i \leq 5$, $w(S_1) = w(S_2) = 1$ and $w(P_j) = 5$ for $6 \leq j \leq 11$. The convex hull of the points

$$Z_i = (P_i, w(P_i)), \quad R_j = (S_j, w(S_j)), \quad (0 \leq i \leq 11,\ j = 1, 2)$$

then forms a polyhedron $\mathcal{Q}$ in $\mathbb{R}^8$. To obtain the regular subdivision of $\mathfrak{P}$ corresponding to the weight function w, we need to project the lower facets of the polyhedron $\mathcal{Q}$ down to $\mathbb{R}^7$ along the last coordinate. A lower facet is defined to be a facet of $\mathcal{Q}$ where the inward pointing normal has a positive last coordinate.

We claim there are exactly 12 lower facets of $\mathcal{Q}$. We write each lower facet F_i in the form $u_i \cdot x + a_i = 0$ where u_i is the inward pointing normal of the i^{th} facet. Take H_i to be the halfspace corresponding to the lower facet F_i, i.e. the halfspace given by $u_i \cdot x + a \geq 0$. The normals and additive constants are:

- $H_0 : (5, -1, -1, 0, 0, 0, 0, 3)x - 3 \geq 0$
- $H_1 : (-1, 5, -1, 0, 0, 0, 0, 3)x - 3 \geq 0$
- $H_2 : (-1, -1, 5, 0, 0, 0, 0, 3)x - 3 \geq 0$
- $H_3 : (1, 1, 1, 6, 0, 0, 0, 3)x - 3 \geq 0$
- $H_4 : (1, 1, 1, 0, 6, 0, 0, 3)x - 3 \geq 0$
- $H_5 : (-5, -5, -5, -6, -6, 0, 0, 3)x - 3 \geq 0$
- $H_6 : (3, -1, -1, 0, 0, 0, 2, 1)x - 1 \geq 0$
- $H_7 : (-1, 3, -1, 0, 0, 0, 2, 1)x - 1 \geq 0$
- $H_8 : (-1, -1, 3, 0, 0, 0, 2, 1)x - 1 \geq 0$
- $H_9 : (1, 1, 1, 4, 0, 0, -2, 1)x + 1 \geq 0$
- $H_{10} : (1, 1, 1, 0, 4, 0, -2, 1)x + 1 \geq 0$
- $H_{11} : (-3, -3, -3, -4, -4, 0, -2, 1)x + 1 \geq 0.$

An easy computation shows that all 14 points lie in the intersection of the relevant half-spaces. This is a direct consequence of the fact that $\mathcal{Q} \subseteq H_i$ for $i = 0, \ldots, 11$. Table 3.2 shows which points lie on each lower facet.

Facet	contains
F_0	$Z_1, \dots, Z_5, R_1, R_2$
$\vdots$	$\vdots$
F_5	$Z_0, \dots, Z_4, R_1, R_2$
F_6	$Z_1, \dots, Z_5, R_1, Z_7, Z_8$
F_7	$Z_0, Z_2, \dots, Z_5, R_1, Z_6, Z_8$
F_8	$Z_0, Z_1, Z_3, Z_4, Z_5, R_1, Z_6, Z_7$
F_9	$Z_0, Z_1, Z_2, Z_4, Z_5, R_2, Z_{10}, Z_{11}$
F_{10}	$Z_0, Z_1, Z_2, Z_3, Z_5, R_2, Z_9, Z_{11}$
F_{11}	$Z_0, \dots, Z_4, R_2, Z_9, Z_{10}.$

Table 3.2: Dictionary of points contained in each lower facet of $\mathcal{Q}$.

To obtain the polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$ corresponding to the weight function w, we now project these facets down to $\mathbb{R}^7$ along the last coordinate. Denoting by $\widehat{F}_i$ the polyhedron obtained by projecting the facet F_i, we obtain the set of 12 polyhedra given in Table 3.3. We note here that when projecting, all points that lied on the facet F_i lie in the polyhedron $\widehat{F}_i$, by convexity of the polyhedron $\mathcal{Q}$ in $\mathbb{R}^8$.

$\widehat{F}_0 =$	$\mathrm{Conv}(P_1, \dots, P_5, S_1, S_2)$
$\vdots$	$\vdots$
$\widehat{F}_5 =$	$\mathrm{Conv}(P_0, \dots, P_4, S_1, S_2)$
$\widehat{F}_6 =$	$\mathrm{Conv}(P_1, \dots, P_5, S_1, P_7, P_8)$
$\widehat{F}_7 =$	$\mathrm{Conv}(P_0, P_2, \dots, P_5, S_1, P_6, P_8)$
$\widehat{F}_8 =$	$\mathrm{Conv}(P_0, P_1, P_3, P_4, P_5, S_1, P_6, P_7)$
$\widehat{F}_9 =$	$\mathrm{Conv}(P_0, P_1, P_2, P_4, P_5, S_2, P_{10}, P_{11})$
$\widehat{F}_{10} =$	$\mathrm{Conv}(P_0, \dots, P_3, P_5, S_2, P_9, P_{11})$
$\widehat{F}_{11} =$	$\mathrm{Conv}(P_0, \dots, P_4, S_2, P_9, P_{10}).$

Table 3.3: Polyhedra in the regular subdivision.

It remains to show that the above collection F_i contains all the lower facets of $\mathcal{Q}$. Showing that there is no other lower facet of $\mathcal{Q}$ apart from $F_0, \dots, F_{11}$ is equivalent to showing that $\bigcup F_i + \langle(0, \dots, 0, 1)\rangle_{\mathbb{R}_{\geq 0}}$ contains the entire polyhedron $\mathcal{Q}$. Since all vertices of $\mathcal{Q}$ lie inside each half-space H_i, it suffices to show that the union of the projections $\widehat{F}_i$ contains the convex hull of $P_0, \dots, P_{11}, S_1, S_2$, i.e. contains $\mathfrak{P}$. This is equivalent to saying that they give a polyhedral subdivision (regularity is given by construction).

So we aim to prove the following claim.

Lemma 3.4.2. *For $\widehat{F}_i$ and $\mathfrak{P}$ as above, we have $\bigcup_{i=0}^{11} \widehat{F}_i = \mathfrak{P}$.*

To prove Lemma 3.4.2, we will need the following result.

Lemma 3.4.3. *Suppose we are given a set of m inequalities $L_j \leq R_j$ with $\sum_{j=1}^{m} L_j \leq C \leq \sum_{j=1}^{m} R_j$, then there exists an m-tuple of real numbers a_j such that $L_j \leq a_j \leq R_j$ and $\sum_{j=1}^{m} a_j = C$.*

Proof. To show that the claim holds, we define $a_j(x) = L_j + x(R_j - L_j)$. This is a linear function such that, for all $x \in [0,1]$, $L_j \leq a_j(x) \leq R_j$. Define $f(x) = \sum a_j(x)$. f is itself linear and thus continuous in x, with $f(0) = \sum_{j=1}^{m} L_j \leq C \leq \sum_{j=1}^{m} R_j = f(1)$. By the intermediate value theorem, there is an $x_C \in [0,1]$ such that $f(x) = \sum_{j=1}^{m} a_j(x_C) = C$. Setting $a_j = a_j(x_C)$ gives the m-tuple, proving the claim. $\square$

Proof of Lemma 3.4.2. The first thing to note is that

$$\mathfrak{P} = \mathrm{Conv}(P_0, \ldots, P_{11}, S_1, S_2) = \mathrm{Conv}(P_0, \ldots, P_{11}).$$

So we will show that $\bigcup_{i=0}^{11} \widehat{F}_i = \mathrm{Conv}(P_0, \ldots, P_{11})$.

We start by showing that $\bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$, which is equivalent to saying that $\widehat{F}_0, \ldots, \widehat{F}_5$ form a polyhedral subdivision of $\mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$.

The inclusion $\subseteq$ is immediate from Table 3.3, so it remains to check the opposite inclusion. Any point $X \in \mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$ can be written as $X = \sum_{i=0}^{5} \lambda_i P_i + \mu_1 S_1 + \mu_2 S_2$ for some $\lambda_i, \mu_j \in \mathbb{R}_{\geq 0}$ with $\sum \lambda_i + \mu_1 + \mu_2 = 1$. Note also that $\sum_{i=0}^{5} P_i = 3(S_1 + S_2)$. Now define j such that $\lambda_j = \min_{0 \leq i \leq 5}\{\lambda_i\}$. Then

$$X = \sum_{i=0}^{5} (\lambda_i - \lambda_j) P_i + (3\lambda_j + \mu_1) S_1 + (3\lambda_j + \mu_2) S_2 = \sum_{\substack{0 \leq i \leq 5 \\ i \neq j}} (\lambda_i - \lambda_j) P_i + (3\lambda_j + \mu_1) S_1 + (3\lambda_j + \mu_2) S_2.$$

Since $\lambda_j = \min_{0 \leq i \leq 5}\{\lambda_i\} \leq \lambda_i$ for $0 \leq i \leq 5$, we have that $(\lambda_i - \lambda_j) \geq 0$ for $0 \leq i \leq 5$. As

$\lambda_i, \mu_1, \mu_2 \geq 0$, we also have $3\lambda_j + \mu_1, 3\lambda_j + \mu_2 \geq 0$. Also,

$$\sum_{\substack{0 \leq i \leq 5 \\ i \neq j}} (\lambda_i - \lambda_j) + (3\lambda_j + \mu_1) + (3\lambda_j + \mu_2) = \sum_{i=0}^{5} \lambda_i + \mu_1 + \mu_2 = 1,$$

and thus $X \in \widehat{F}_j$. This shows $\bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$.

To show $\bigcup_{i=0}^{11} \widehat{F}_i = \mathfrak{P}$, we note again that the inclusion $\subseteq$ is immediate. For the opposite inclusion $\supseteq$, take a general point X in $\mathfrak{P}$. Then X can be written as $X = \sum_{i=0}^{11} \lambda_i P_i$ with $\lambda_i \geq 0$ for $0 \leq i \leq 11$ and $\sum_{i=0}^{11} \lambda_i = 1$.

Without loss of generality, assume that $(\lambda_6 + \lambda_7 + \lambda_8) \geq (\lambda_9 + \lambda_{10} + \lambda_{11})$ (the case where the inequality is reversed is analogous). We will now show that if $X \notin \bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$, then $X \in \bigcup_{i=6}^{8} \widehat{F}_i$ (if the inequality had been reversed, then X would be in $\bigcup_{i=9}^{11} \widehat{F}_i$).

Let

$$\begin{aligned}
\nu_i = &\quad \lambda_i + \lambda_{6+i} - \tfrac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right) && \text{for } 0 \leq i \leq 2, \\
\nu_i = &\quad \lambda_i + \lambda_{6+i} && \text{for } 3 \leq i \leq 5, \\
\mu_1 = &\quad \left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right), \\
\mu_2 = &\quad 0.
\end{aligned}$$

Then

$$\sum_{i=0}^{5} \nu_i P_i + \mu_1 S_1 + \mu_2 S_2 = \sum_{i=0}^{11} \lambda_i P_i$$

and

$$\sum_{i=0}^{5} \nu_i + \mu_1 + \mu_2 = \sum_{i=0}^{11} \lambda_i = 1.$$

Note $\mu_1 \geq 0$ by assumption and $\mu_2 = 0$. Thus, if $\nu_i \geq 0$ for $0 \leq i \leq 5$, X is expressed as an element of $\mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2) = \bigcup_{i=0}^{5} \widehat{F}_i$ using the above equations. Otherwise, we will claim that $X \in \bigcup_{i=6}^{8} \widehat{F_i}$. For $3 \leq i \leq 5$, we have $\nu_i \geq 0$ as both λ_i and λ_{6+i} are ≥ 0. We turn our attention to the ν_i for $i = 0, 1, 2$.

For $0 \le i \le 2$, $\nu_i \ge 0$ is equivalent to

$$\frac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right) \le \lambda_i + \lambda_{6+i},$$

so the condition that all ν_i are non-negative is equivalent to

$$\frac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right) \le \min_{0 \le i \le 2}\{\lambda_i + \lambda_{6+i}\}.$$

Therefore, $X \in \bigcup_{i=0}^{5} \widehat{F}_i = \mathrm{Conv}(P_0, \ldots, P_5, S_1, S_2)$ if $\frac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right) \le \min_{0 \le i \le 2}\{\lambda_i + \lambda_{6+i}\}$. Suppose this condition does not hold, i.e.

$$\min_{0 \le i \le 2}\{\lambda_i + \lambda_{6+i}\} < \frac{1}{3}\left((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})\right). \tag{3.9}$$

Without loss of generality, we may assume that $\lambda_0 + \lambda_6 = \min_{0 \le i \le 2}\{\lambda_i + \lambda_{6+i}\}$ (by symmetry, the other cases are analogous). We will show that $X \in \widehat{F}_6$. Any point Y in $\widehat{F}_6 = \mathrm{Conv}\left(P_1, \ldots, P_5, S_1, P_7, P_8\right)$ can be written as

$$Y = \sum_{i=1}^{5} \nu_i P_i + \sum_{i=7}^{8} \nu_i P_i + \mu_1 S_1.$$

If we find ν_i, μ_1 such that this sum is equal to $\sum_{i=0}^{11} \lambda_i P_i = X$, we are done as we will have expressed X as an element of $\widehat{F}_6$.

Given a choice of real numbers α_1, α_2 with $\alpha_1 + \alpha_2 = 1$, define

$$\begin{aligned}
\nu_i ={}& \lambda_i + \alpha_i(3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})) - (\lambda_0 + \lambda_6) && \text{for } 1 \le i \le 2,\\
\nu_i ={}& \lambda_i + \lambda_{6+i} && \text{for } 3 \le i \le 5,\\
\mu_1 ={}& 3\lambda_0 + 3\lambda_6, &&\\
\nu_{6+i} ={}& \lambda_{6+i} + \alpha_i(-3\lambda_0 - 2\lambda_6 - (\lambda_9 + \lambda_{10} + \lambda_{11})) && \text{for } 1 \le i \le 2.
\end{aligned}$$

Substituting these values into the expression for Y gives

$$Y = \sum_{i=1}^{5} \nu_i P_i + \sum_{i=7}^{8} \nu_i P_i + \mu_1 S_1 = \sum_{i=0}^{11} \lambda_i P_i = X,$$

as well as

$$\sum \nu_i + \mu_1 = \sum_{i=0}^{11} \lambda_i = 1.$$

For this choice of ν_i's and μ_1 to define an element $Y \in \widehat{F}_6$, we require $\nu_i \geq 0$ for all i and $\mu_1 \geq 0$. We note that, as $\lambda_0, \lambda_6 \geq 0$, we have $\mu_1 \geq 0$.

Therefore, what remains to prove is that there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$ such that $\nu_i \geq 0$ for $i \in \{1, \ldots, 5, 7, 8\}$. For $i = 1, 2$, we can arrange the inequalities $\nu_i \geq 0$ and $\nu_{6+i} \geq 0$ to give

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq \alpha_i \leq \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})}. \tag{3.10}$$

This works provided $3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11}) \neq 0$ but if that term was zero, then by non-negativity of the λ_i we would have $\lambda_0 = \lambda_6 = \lambda_9 = \cdots = \lambda_{11} = 0$ and thus $X \in \widehat{F}_6$. So if there exists a pair (α_1, α_2) with (3.10) holding for $i = 1, 2$ and $\alpha_1 + \alpha_2 = 1$, then $X \in \widehat{F}_6$.

Note that for all i,

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})},$$

as $\lambda_0 + \lambda_6 = \min_{0 \leq i \leq 2}\{\lambda_i + \lambda_{6+i}\}$.

Furthermore,

$$\begin{aligned}
& & 0 &\leq \lambda_0 + \lambda_1 + \lambda_2 + \lambda_9 + \lambda_{10} + \lambda_{11} \\
&\Leftrightarrow & 2\lambda_0 + 2\lambda_6 &\leq 3\lambda_0 + \lambda_1 + \lambda_2 + 2\lambda_6 + \lambda_9 + \lambda_{10} + \lambda_{11} \\
&\Leftrightarrow & \textstyle\sum_{i=1}^{2} \lambda_0 + \lambda_6 - \lambda_i &\leq 3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11}) \\
&\Leftrightarrow & \textstyle\sum_{i=1}^{2} \frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} &\leq 1.
\end{aligned}$$

Lastly, we are given that $\lambda_0+\lambda_6 = \min_{0\le i\le 2}\{\lambda_i+\lambda_{6+i}\} \le \frac{1}{3}\left((\lambda_6+\lambda_7+\lambda_8)-(\lambda_9+\lambda_{10}+\lambda_{11})\right)$. This leads to the following sequence of implications:

$$\begin{aligned} && \lambda_0+\lambda_6 &\le \tfrac{1}{3}\left((\lambda_6+\lambda_7+\lambda_8)-(\lambda_9+\lambda_{10}+\lambda_{11})\right) \\ \Leftrightarrow && 3\lambda_0+2\lambda_6+\lambda_9+\lambda_{10}+\lambda_{11} &\le \lambda_7+\lambda_8 \\ \Leftrightarrow && 1 &\le \textstyle\sum_{i=1}^{2} \frac{\lambda_{6+i}}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})}. \end{aligned}$$

In summary, we have shown that for $i=1,2$, we have

$$\frac{\lambda_0+\lambda_6-\lambda_i}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})} \le \frac{\lambda_{6+i}}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})},$$

and that

$$\sum_{i=1}^{2}\frac{\lambda_0+\lambda_6-\lambda_i}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})} \le 1 \le \sum_{i=1}^{2}\frac{\lambda_{6+i}}{3\lambda_0+2\lambda_6+(\lambda_9+\lambda_{10}+\lambda_{11})}.$$

Applying Lemma 3.4.3 gives us the existence of a pair α_1, α_2 as required, concluding the proof to Lemma 3.4.2. $\square$

The Lemma 3.4.2 shows that we have indeed found all lower facets of the polyhedron $\mathcal{Q}$, meaning that the collection $\widehat{F}_1, \ldots, \widehat{F}_{11}$ gives a regular polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$, thus concluding Step 1.

Step 2:

This is true by general convex geometry (using the poset of refinements and the secondary polytope). By Theorem 2.4 in Chapter 7 of [22], the poset of (non-empty) faces of the secondary polytope $\Sigma(\mathfrak{P})$ is isomorphic to the poset of all regular subdivisions of $\mathfrak{P}$, partially ordered by refinement (see also Theorem 16.4.1 in [23]). The vertices of $\Sigma(\mathfrak{P})$ correspond to regular triangulations. Thus, our regular subdivision obtained by projection must correspond to some face of $\Sigma(\mathfrak{P})$ and any vertex of that face will correspond to a regular triangulation refining it.

Step 3:

Consider a regular triangulation $\mathcal{T}$ obtained by refining $\mathcal{S}$. By definition, it is a regular triangulation of $\mathfrak{P}$. Recall Table 3.3. Denote by C_i the collection of points used to define the polyhedron $\widehat{F}_i$ in the table. Note that $\widehat{F}_0, \dots, \widehat{F}_5$ are the simplices in $\mathcal{T}_0$, and therefore any simplices in $\mathcal{T} \setminus \mathcal{T}_0$ do not originate from refining any of $\widehat{F}_0, \dots, \widehat{F}_5$.

Thus the last step of the proof reduces to showing that none of the polyhedra $\widehat{F}_i,\ 0 \leq i \leq 11$, contain any of the points we did not define it by, i.e. $\widehat{F}_i \cap \mathfrak{C} = C_i$. Indeed, in that case we note that, by consulting Table 3.3, the polyhedra $\widehat{F}_i$ each fulfill at least one of the conditions A or B in the proposition. If $\widehat{F}_i \cap \mathfrak{C} = C_i$, then all simplices in a refinement of $\widehat{F}_i$ are defined as the convex hull of a subset of C_i (as there is no interior point to refine upon), thus inheriting the properties A or B from $\widehat{F}_i$.

Showing that $\widehat{F}_i \cap \mathfrak{C} = C_i$ for $6 \leq i \leq 11$ reduces to a simple computation. We shall do the computation for $\widehat{F}_6$, as the remaining cases are analogous by symmetry.

We need to show that $P_0, P_6, P_9, P_{10}, P_{11}, S_1 \notin \widehat{F}_6$. Any point X in $\widehat{F}_6$ can be written as

$$\begin{aligned}&\lambda_1 P_1 + \cdots + \lambda_5 P_5 + \mu_1 S_1 + \lambda_7 P_7 + \lambda_8 P_8 = (-\lambda_3 - \lambda_4 - \lambda_5 - \lambda_7 - \lambda_8,\\ &3\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_7 - \lambda_8, 3\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_7 + 2\lambda_8, -\lambda_1 - \lambda_2 + 3\lambda_3,\\ &\lambda_1 + \lambda_2 + 3\lambda_4, \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8, \lambda_1 + \lambda_2),\end{aligned} \tag{3.11}$$

with $\lambda_i, \mu_1 \geq 0$ and $\sum \lambda_i + \mu_1 = 1$. We note that the last two coordinates of X are $\lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8$ and $\lambda_1 + \lambda_2$ respectively. Assume $P_0 \in \widehat{F}_6$ and had an expression as in Equation (3.11). Then, as $\lambda_i, \mu_1 \geq 0$, we can see by looking at the last two coordinates that $\lambda_3 = \lambda_4 = \lambda_5 = \mu_1 = \lambda_7 = \lambda_8 = 0$ and $\lambda_1 + \lambda_2 = 1$. But then the first coordinate is $\lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0 \neq 2$, hence we get a contradiction and $P_0 \notin \widehat{F}_6$. By an analogous reasoning, for S_2, P_9, P_{10}, P_{11} we obtain that all but λ_1, λ_2 would need to be 0 again and the sum of these two would need to be 1, which means that not both the second and third coordinate (being $3\lambda_1, 3\lambda_2$) can be 0. Hence $S_2, P_9, P_{10}, P_{11} \notin \widehat{F}_6$.

Finally, we need to show $P_6 \notin \widehat{F}_6$. Assume we had an expression for P_6 as in Equation (3.11). Since $\lambda_i \geq 0$, considering the last two coordinates gives $\lambda_1 = \lambda_2 = 0$ (since $\lambda_i \geq 0$) and $\lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8 = 1$. But then the first coordinate is $-(\lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8) \leq 0 < 2$, a contradiction. Thus $P_6 \notin \widehat{F}_6$, and thus $\widehat{F}_6 \cap \mathfrak{C} = C_6$ as claimed.

The other cases are analogous by symmetry. Thus we finished Step 3, hence proving Proposition 3.4.1.

3.4.2 The ideals associated to the partial compactification

Recalling the notation from §3.1, we denote by x_i the variable in $\mathbb{C}[x_0, \ldots, x_{11}, u_1, u_2]$ corresponding to the point P_i and by u_j the variable corresponding to S_j. These 14 variables correspond to the rays of the fan $\Sigma_{\nabla, D'_a, D'_a}$ from Corollary 3.2.4.

Lemma 3.4.4. *There exists a global function on X_{∇, D'_a, D'_b} that has the form*

$$\begin{aligned} w = u_1(c_0x_0^3x_6^3 + c_1x_1^3x_7^3 + c_2x_2^3x_8^3 - 3\lambda_1x_3x_4x_5x_6x_7x_8) \\ + u_2(c_3x_3^3x_9^3 + c_4x_4^3x_{10}^3 + c_5x_5^3x_{11}^3 - 3\lambda_2x_0x_1x_2x_9x_{10}x_{11}), \end{aligned} \tag{3.12}$$

for some $c_i, \lambda_j \in \mathbb{C}$.

Proof. Consider the hyperplane

$$H := \{(m, t_1, t_2) \in M_{\mathbb{R}} \oplus \mathbb{R}^2 \mid t_1 + t_2 = 1\} \tag{3.13}$$

in $M_{\mathbb{R}} \oplus \mathbb{R}^2$. The cone $|\Sigma_{\nabla, D'_a, D'_b}|^{\vee}$ is given by the cone over the convex hull of the following 8 points on H:

$$(0,0,0,0,1,0,1), \quad (-1,-1,-1,-1,-1,0,1), \quad (0,0,0,1,0,0,1), \quad (0,0,1,0,0,1,0),$$
$$(0,1,0,0,0,1,0), \quad (0,0,0,0,0,0,1), \quad (1,0,0,0,0,1,0), \quad (0,0,0,0,0,1,0).$$

Recall that there is a correspondence between points in the dual cone $|\Sigma_{LT,D_a,D_b}|^\vee$ and global functions on X_{LT,D_a,D_b}. Note that, when one constructs a superpotential for X_{LT,D_a,D_b} using this correspondence with the 8 points above, one obtains the superpotential $w = u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$. To see the global function on X_{∇,D'_a,D'_b}, it suffices to compute what monomials these 8 points correspond to on X_{∇,D'_a,D'_b}. This gives the 8 monomials in (3.12). □

For our purposes of comparing the Batyrev-Borisov construction with the one by Libgober-Teitelbaum, we choose $c_i = 1$ and $\lambda_1 = \lambda_2 =: \lambda$.

We fix a triangulation $\mathcal{T}$ fulfilling the properties of Proposition 3.4.1. Let $X = \mathbb{C}^{14}$ and consider the group G_Σ corresponding to the fan $\Sigma_{\nabla,D'_a,D'_b}$ with its action on X. From the triangulation $\mathcal{T}$ we obtain the ideals:

$$\mathcal{I} := \left\langle \prod_{i \notin I} x_i \prod_{j \notin J} u_j \;\middle|\; \bigcup_{i \in I} u_{\overline{\rho}_i} \cup \bigcup_{j \in J} u_{\tau_j} \text{ give the set of vertices of a simplex in } \mathcal{T} \right\rangle,$$
$$\mathcal{J} := \left\langle \prod_{i \notin I} x_i \;\middle|\; \bigcup_{i \in I} u_{\overline{\rho}_i} \cup \bigcup_{j=1}^{2} u_{\tau_j} \text{ give the set of vertices of a simplex in } \mathcal{T} \right\rangle.$$

Before we can apply Proposition 3.1.9 and Corollary 3.1.10, we need to ensure the condition $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$ holds.

Lemma 3.4.5. *For any triangulation $\mathcal{T}$ as in Proposition 3.4.1, defining $\mathcal{I}, \mathcal{J}$ and w as above with $\lambda^6 \neq 0, 1$, we have $\mathcal{I} \subseteq \sqrt{\mathcal{J}, \partial w}$. Therefore, this choice of superpotential fulfills the condition of Proposition* 3.1.9.

Proof. To show the containment $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$, we prove that all the generators of $\mathcal{I}$ are in $\sqrt{\partial w, \mathcal{J}}$. The ideal $\mathcal{I}$ is, by definition, generated by the monomials which correspond to the simplices in the triangulation $\mathcal{T}$. For a simplex $T \in \mathcal{T}_0$, both S_1 and S_2 are vertices. Thus, by definition, the monomial associated to T is in $\mathcal{J}$ and hence in $\sqrt{\partial w, \mathcal{J}}$.

For any simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$, either condition (A) or (B) of Proposition 3.4.1 holds. We claim that the monomial associated to a simplex T fulfilling either of those two conditions is an element of $\sqrt{\partial w}$ and therefore an element of $\sqrt{\partial w, \mathcal{J}}$.

We note that if $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills condition (A), i.e. does not contain any of the points $S_1.P_6, P_7, P_8$ and there is a pair of points of the form P_j, P_{6+j} with $3 \leq j \leq 5$ also not contained, then by definition $u_1 x_j x_{6+j} x_6 x_7 x_8$ divides the monomial associated to T. Similarly, if T fulfilled condition (B) instead, $u_2 x_j x_{6+j} x_9 x_{10} x_{11}$ (for some $0 \leq j \leq 2$) would divide the monomial generator of $\mathcal{I}$ associated to T.

To show that any monomial associated to a simplex in $\mathcal{T} \setminus \mathcal{T}_0$ is in $\sqrt{\partial w} \subseteq \sqrt{\partial w, \mathcal{J}}$, it is thus sufficient to prove that the six monomials $u_2 x_0 x_6 x_9 x_{10} x_{11}$, $u_2 x_1 x_7 x_9 x_{10} x_{11}$, $u_2 x_2 x_8 x_9 x_{10} x_{11}$, $u_1 x_3 x_9 x_6 x_7 x_8$, $u_1 x_4 x_{10} x_6 x_7 x_8$ and $u_1 x_5 x_{11} x_6 x_7 x_8$ are elements of $\sqrt{\partial w}$.

By symmetry of the x_i in w, we note that it is sufficient to show that $u_2 x_0 x_6 x_9 x_{10} x_{11} \in \sqrt{\partial w}$. Start by explicitly writing down the ideal $\langle \partial w \rangle$, i.e. the ideal generated by the partial derivatives of w.

$$\begin{aligned}
\langle \partial w \rangle = \; & \langle 3u_1 x_0^2 x_6^3 - 3\lambda u_2 x_1 x_2 x_9 x_{10} x_{11}, 3u_1 x_1^2 x_7^3 - 3\lambda u_2 x_0 x_2 x_9 x_{10} x_{11}, \\
& 3u_1 x_2^2 x_8^3 - 3\lambda u_2 x_0 x_1 x_9 x_{10} x_{11}, 3u_2 x_3^2 x_9^3 - 3\lambda u_1 x_4 x_5 x_6 x_7 x_8, \\
& 3u_2 x_4^2 x_{10}^3 - 3\lambda u_1 x_3 x_5 x_6 x_7 x_8, 3u_2 x_5^2 x_{11}^3 - 3\lambda u_1 x_3 x_4 x_6 x_7 x_8, \\
& 3u_1 x_0^3 x_6^2 - 3\lambda u_1 x_3 x_4 x_5 x_7 x_8, 3u_1 x_1^3 x_7^2 - 3\lambda u_1 x_3 x_4 x_5 x_6 x_8, \\
& 3u_1 x_2^3 x_8^2 - 3\lambda u_1 x_3 x_4 x_5 x_6 x_7, 3u_2 x_3^3 x_9^2 - 3\lambda u_2 x_0 x_1 x_2 x_{10} x_{11}, \\
& 3u_2 x_4^3 x_{10}^2 - 3\lambda u_2 x_0 x_1 x_2 x_9 x_{11}, 3u_2 x_5^3 x_{11}^2 - 3\lambda u_2 x_0 x_1 x_2 x_9 x_{10}, \\
& x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8, x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11} \rangle
\end{aligned}$$

We see that $3u_1 x_i^2 x_{6+i}^3 - 3\lambda u_2 \frac{x_0 x_1 x_2 x_9 x_{10} x_{11}}{x_i} \in \langle \partial w \rangle$ for $0 \leq i \leq 2$. Notice that since $ac - bd = c(a-b) + b(c-d)$, if $a-b, c-d$ are elements in an ideal, then so is $ac - bd$.

Hence by iterating this we obtain that

$$27u_1^3x_0^2x_1^2x_2^2x_6^3x_7^3x_8^3 - 27\lambda^3u_2^3x_0^2x_1^2x_2^2x_9^3x_{10}^3x_{11}^3 \in \langle \partial w \rangle.$$

Similarly,

$$27u_2^3x_3^2x_4^2x_5^2x_9^3x_{10}^3x_{11}^3 - 27\lambda^3u_1^3x_3^2x_4^2x_5^2x_6^3x_7^3x_8^3 \in \langle \partial w \rangle.$$

Therefore,

$$\begin{aligned}
&(27)^2u_1^3u_2^3x_0^2\dots x_5^2x_6^3\dots x_{11}^3 - (27)^2\lambda^6u_1^3u_2^3x_0^2\dots x_5^2x_6^3\dots x_{11}^3 \in \langle \partial w \rangle \\
&\Rightarrow 27^2(1-\lambda^6)u_1^3u_2^3x_0^2\dots x_5^2x_6^3\dots x_{11}^3 \in \langle \partial w \rangle \\
&\Rightarrow u_1^3u_2^3x_0^2\dots x_5^2x_6^3\dots x_{11}^3 \in \langle \partial w \rangle \\
&\Rightarrow (u_1u_2x_0\dots x_{11})^3 \in \langle \partial w \rangle. \qquad (3.14)
\end{aligned}$$

Consider $\frac{\partial w}{\partial u_1}$, giving

$$x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8 \in \langle \partial w \rangle. \qquad (3.15)$$

Furthermore, we note that $\sum_{i=0}^{2} x_i \frac{\partial w}{\partial x_i} \in \langle \partial w \rangle$, and thus

$$\sum_{i=0}^{2}(3u_1x_i^3x_{6+i}^3 - 3\lambda u_2x_0x_1x_2x_9x_{10}x_{11}) \in \langle \partial w \rangle. \qquad (3.16)$$

By (3.15) we have that $x_3x_4x_5x_6x_7x_8 + \langle \partial w \rangle = \frac{1}{3\lambda}(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3) + \langle \partial w \rangle$. We use this to substitute into (3.14) to obtain that

$$\frac{1}{27\lambda^3}u_2^3x_0^3x_1^3x_2^3x_9^3x_{10}^3x_{11}^3(u_1(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3))^3 \in \langle \partial w \rangle.$$

Performing the same style of substitution with (3.16), we obtain

$$u_2^3x_0^3x_1^3x_2^3x_9^3x_{10}^3x_{11}^3u_2^3x_0^3x_1^3x_2^3x_9^3x_{10}^3x_{11}^3 = (u_2x_0x_1x_2x_9x_{10}x_{11})^6 \in \langle \partial w \rangle.$$

Thus $u_2x_0x_1x_2x_9x_{10}x_{11} \in \sqrt{\partial w}$.

By comparing the elements $x_2\frac{\partial w}{\partial x_2}, u_2x_0x_1x_2x_9x_{10}x_{11} \in \sqrt{\partial w}$, we obtain that $u_1x_2^3x_8^3 \in \sqrt{\partial w}$, implying $u_1x_2x_8 \in \sqrt{\partial w}$. This, in turn, implies that $u_2x_0x_1x_9x_{10}x_{11} \in \sqrt{\partial w}$, by inspection of $\frac{\partial w}{\partial x_2} \in \langle \partial w\rangle \subseteq \sqrt{\partial w}$. Similarly, $u_2x_0x_2x_9x_{10}x_{11} \in \sqrt{\partial w}$. We also have $u_2x_5x_{11} \in \sqrt{\partial w}$ by an analogous computation. Finally, $\frac{\partial w}{\partial u_1} = x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8 \in \sqrt{\partial w}$.

Therefore, one can intuit and then compute that

$$\begin{aligned}(u_2x_0x_6x_9x_{10}x_{11})^4 = &-u_2^3x_1^2x_6x_7^3x_9^3x_{10}^3x_{11}^3 \cdot (u_2x_0x_1x_9x_{10}x_{11})\\ &- u_2^3x_2^2x_6x_8^3x_9^3x_{10}^3x_{11}^3 \cdot (u_2x_0x_2x_9x_{10}x_{11})\\ &+ 3\lambda u_2^3x_0x_3x_4x_6^2x_7x_8x_9^4x_{10}^4x_{11}^4 \cdot (u_2x_5x_{11})\\ &+ u_2^4x_0x_6x_9^4x_{10}^4x_{11}^4 \cdot (x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8) \in \sqrt{\partial w}.\end{aligned} \tag{3.17}$$

Thus, we have shown that $u_2x_0x_6x_9x_{10}x_{11} \in \sqrt{\partial w}$. By symmetry, any simplex fulfilling properties (A) or (B) corresponds to a monomial in $\sqrt{\partial w, \mathcal{J}}$. Hence any monomial associated to a simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$ is an element of $\sqrt{\partial w, \mathcal{J}}$, concluding the proof that $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$. □

Corollary 3.4.6. *Consider the GKZ fan of* $\operatorname{tot}(\mathcal{O}_{X_\nabla}(-D_b') \oplus \mathcal{O}_{X_\nabla}(-D_a'))$ *and the group* G_Σ *from above. There is a chamber* σ_p *with affine open* U_p *such that:*

(i) $[U_p/G_\Sigma]$ *is a partial compactification of* $\operatorname{tot}(\mathcal{O}_{\mathcal{X}_{LT}}(-D_b) \oplus \mathcal{O}_{\mathcal{X}_{LT}}(-D_a))$.

(ii) There is a superpotential corresponding to the eight points in $|\Sigma_{\nabla,D_a',D_b'}|^\vee \cap H$ *taking the form* $w = u_1(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8) + u_2(x_3^3x_9^3 + x_4^3x_{10}^3 + x_5^3x_{11}^3 - 3\lambda x_0x_1x_2x_9x_{10}x_{11})$.

(iii) With $\mathcal{I}_p, \mathcal{J}_p$ *as defined in* §3.1, *we have* $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$.

Proof. Proposition 3.4.1 proves (i), Lemma 3.4.4 proves (ii) and finally Lemma 3.4.5 shows (iii). □

3.4.3 Relating X_∇ and X_{LT}

Recall that the partial compactification of the total space $\operatorname{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ in Corollary 3.4.6 corresponds to a chamber σ_p of the GKZ fan of $\operatorname{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$. We then know that it is birationally equivalent to $\operatorname{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$.

Thus we want to now explicitly find a triangulation of $\mathfrak{P}$ corresponding to the Batyrev-Borisov mirror family. There, the superpotential will take the form

$$w = u_1(x_0^3x_6^3+x_1^3x_7^3+x_2^3x_8^3-3\lambda x_3x_4x_5x_6x_7x_8)+u_2(x_3^3x_9^3+x_4^3x_{10}^3+x_5^3x_{11}^3-3\lambda x_0x_1x_2x_9x_{10}x_{11}).$$

Note that, by Lemma 3.4.4, this is the form the superpotential should take in the Batyrev-Borisov mirror. In other words, we need a chamber σ_q in the GKZ fan corresponding to $\operatorname{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$, where a general section of $\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a)$ will yield a complete intersection in X_∇, and thus a Batyrev-Borisov mirror.

Lemma 3.4.7. *Consider the GKZ fan of* $\operatorname{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$ *and recall the group G_Σ from above. There is a chamber σ_q with affine open U_q such that:*

(i) $[U_q/G_\Sigma] = \operatorname{tot}(\mathcal{O}_{\mathcal{X}_\nabla}(-D'_b) \oplus \mathcal{O}_{\mathcal{X}_\nabla}(-D'_a))$.

(ii) A superpotential corresponding to the eight lattice points of $|\Sigma_{\nabla,D'_a,D'_b}|^\vee \cap H$ is of the form $w = u_1(x_0^3x_6^3 + x_1^3x_7^3 + x_2^3x_8^3 - 3\lambda x_3x_4x_5x_6x_7x_8) + u_2(x_3^3x_9^3 + x_4^3x_{10}^3 + x_5^3x_{11}^3 - 3\lambda x_0x_1x_2x_9x_{10}x_{11})$.

(iii) For $\mathcal{I}_q, \mathcal{J}_q$ as defined in §3.1, $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$.

Proof. This proof will construct the triangulation $\mathcal{T}_q$ corresponding to the chamber σ_q. We consider the 42 maximal cones from Table 3.1. For each of those cones $\sigma_i, 1 \leq i \leq 42$, we associate a simplex given as convex hull of the 5 vertices corresponding to the 5 rays of σ_i plus the two vertices corresponding to the bundle coordinates, i.e. $(0,0,0,0,0,1,0)$ and $(0,0,0,0,0,0,1)$. So for example the first cone, with rays $\rho_0, \rho_1, \rho_2, \rho_9, \rho_{10}$, will correspond to the simplex with vertices $(3,0,0,-1,-1,1,0)$, $(0,3,0,-1,-1,1,0)$, $(0,0,3,-1,-1,1,0)$,

$(0,0,0,0,0,1,0)$, $(0,0,0,0,0,0,1)$, $(0,0,0,2,-1,1,0)$, $(0,0,0,-1,2,1,0)$. Another way to formulate this is that we take the star subdivision of the cones from Table 3.1 on the two bundle points S_1, S_2.

Regularity of this triangulation of the 14 points is an easy consequence of its construction as a star subdivision, hence it corresponds to some chamber σ_q in the GKZ fan. Indeed, the star subdivision can be obtained by giving the points S_1, S_2 a weight of 1 and giving all other points the same weight of $w = 2$ and then refining the resulting regular polyhedral subdivision into a triangulation. Alternatively, one can check the regularity of this triangulation by using `SAGE`.

The third item follows from the fact that we do not partially compactify, hence $\mathcal{J}_q = \mathcal{I}_q$ and therefore $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$, as required. $\square$

We now have all the necessary tools to prove the main result of this chapter, Theorem 3.0.1.

Proof of Theorem 1.0.1. Recall the chambers σ_p and σ_q in the GKZ fan of the toric variety $\operatorname{tot}(\mathcal{O}_{X_\nabla}(-D'_b) \oplus \mathcal{O}_{X_\nabla}(-D'_a))$ given in Corollary 3.4.6 and Lemma 3.4.7. By applying Corollary 3.1.10, we have $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{Z}_\lambda) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{V}_{LT,\lambda})$, as required. $\square$

CHAPTER 4

CATEGORICAL RESOLUTIONS VIA VGIT

In the previous chapter we saw how methods of partial compactifications and VGIT can be used to provide equivalences of derived categories. The key result used to relate the category of singularities of the partial compactification to the derived category of coherent sheaves, Proposition 3.1.9, comes with an algebraic condition, $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$. This condition can be thought of, intuitively, as a way to ensure singularities are manageable.

In the case of a singular complete intersection, the Proposition may thus fail to apply. In an attempt to generalise the Libgober-Teitelbaum construction (see § 4.2), we encountered a family of singular complete intersections where computer-based searches failed to verify the algebraic condition. Fortunately, the methods of partial compactifications and VGIT apply in further generality, and [19] elaborates on how to use them to obtain categorical resolutions in the sense of Definition 2.2.33.

4.1 Categorical resolutions via partial compactifications and VGIT

In this section, we provide an interpretation of crepant categorical resolutions in terms of Landau-Ginzburg models. Given a Landau-Ginzburg model (X, G, w) with superpotential w, we consider its singular locus. If the singular locus is not proper, we can make it so by means of partial compactification. This process is what gives us a crepant categorical resolution.

We follow the notation and conventions of Favero and Kelly [19], who base their work on the paper [2] by Ballard, Favero and Katzarkov.

Let X be a smooth variety over $\mathbb{C}$, G an affine algebraic group acting on X and χ a character of G. Then consider a semi-invariant function with respect to χ, $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$. We will study the absolute derived category $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ of the Landau-Ginzburg model (X, G, w), which (see Remark 2.2.27) should be thought of as the equivalent to the derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\circ)$ for Landau-Ginzburg models.

To obtain a crepant categorical resolution, we will examine a *dg*-enhancement of $\mathrm{D}^{\mathrm{abs}}[X, G, W]$ that is both homologically smooth and proper. Consider $\mathtt{Inj}_{coh}(X, G, w)$, which is the full subcategory of $\mathtt{Fact}(X, G, w)$ consisting of objects with injective components which are isomorphic in $\mathrm{D}^{\mathrm{abs}}[\mathtt{Fact}(X, G, w)]$ to objects with coherent components. The following results respectively show that $\mathtt{Inj}_{coh}(X, G, w)$ is a *dg*-enhancement of $\mathrm{D}^{\mathrm{abs}}[X, G, w]$ and that it is homologically smooth and proper, as desired.

Proposition 4.1.1 (Proposition 5.11 in [2])**.** *The dg-category* $\mathtt{Inj}_{coh}(X, G, w)$ *is a dg-enhancement of* $\mathrm{D}^{\mathrm{abs}}[X, G, w]$*.*

Lemma 4.1.2 (Lemma 2.11 and 2.14 in [19])**.** *Let X be a smooth algebraic variety and G a linearly reductive group acting on X. Let $\chi : G \to \mathbb{C}^*$ be a non-trivial character and $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$ a semi-invariant function. Denote by ∂w the critical locus. Assume that $[X/\, Ker\, \chi]$ has finite diagonal, is proper over* $\operatorname{Spec}\mathbb{C}$ *and that $\partial w \subseteq Z(w)$. Then the dg-category* $\mathtt{Inj}_{coh}(X, G, w)$ *is homologically smooth and proper.*

Remark 4.1.3. Note that any separated Deligne-Mumford stack has finite diagonal (see Remark 2.12 in [19]). Furthermore, to show that a variety Y is proper over $\operatorname{Spec}\mathbb{C}$, it is sufficient to show that Y is compact with respect to the usual complex analytic topology. Indeed, Proposition 6 of Serre's famous GAGA paper [49] asserts that an algebraic variety Y is compact if and only if it is complete. If a variety Y is complete, then it is proper over $\operatorname{Spec}\mathbb{C}$.

The two results above are important for us to establish categorical resolutions, but we first need to frame them in the right context. Let U be a variety with the action of a linearly reductive group G, χ a character of G, and w a section of $O_U(\chi)$. Consider a G-equivariant open immersion

$$i : V \to U.$$

Then i induces an adjoint pair of functors

$$i_* : \mathrm{D}^{\mathrm{abs}}[\mathtt{Fact}(V, G, w)] \to \mathrm{D}^{\mathrm{abs}}[\mathtt{Fact}(U, G, w)]$$
$$i^* : \mathrm{D}^{\mathrm{abs}}[\mathtt{Fact}(U, G, w)] \to \mathrm{D}^{\mathrm{abs}}[\mathtt{Fact}(V, G, w)].$$

We will use the existence of an adjoint pair of functors in this situation, together with the results above and a Theorem of Hirano's, to obtain a crepant categorical resolution. Consider a smooth quasi-projective variety Y with a G-action. Suppose s is a regular section of a G-equivariant vector bundle $\mathcal{E}$ on Y with $Z := Z(s)$. Consider the fibrewise dilation action of $\mathbb{C}^*$ on $\operatorname{tot}\mathcal{E}^\vee$ and view the pairing $w = \langle s, - \rangle$ as section of the line bundle $\mathcal{O}_{\operatorname{tot}\mathcal{E}^\vee}(\chi_p)$ associated to the projection character χ_p.

The gauged Landau-Ginzburg model associated to the complete intersection Z is defined to be the data $(\operatorname{tot}\mathcal{E}^\vee, G \times \mathbb{C}^*, w)$. The next result is another formulation of Theorem 3.1.3, due to Hirano [30].

Theorem 4.1.4. *Assume that w is a regular section of $\mathcal{E}$. There is an equivalence of categories* $\Omega : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G]) \to \mathrm{D}^{\mathrm{abs}}[\operatorname{tot}\mathcal{E}^\vee, G \times \mathbb{C}^*, \langle w, - \rangle]$.

Combining Proposition 4.1.1, Lemma 4.1.2 and Theorem 4.1.4, Favero and Kelly prove the following key result. The proof is included due to its simplicity and elegance.

Theorem 4.1.5. *With the setup as above, assume that Y admits a G-ample line bundle. Let U be a $G \times \mathbb{C}^*$-equivariant partial compactification of* $\operatorname{tot}\mathcal{E}^\vee$. *Assume that*

- *w extends to U as a section of $\mathcal{O}(\chi)$,*
- *$[U/G]$ has finite diagonal, and*

- *$[\partial w/G] \subseteq [U/G]$ is proper over $\operatorname{Spec}\mathbb{C}$ and that $\partial w \subseteq Z(w)$ in U.*

Then the functors

$$i_* \circ \Omega : \operatorname{Perf}([Z/G]) \to \mathrm{D}^{\mathrm{abs}}[U, G, w]$$
$$\Omega^{-1} \circ i^* : \mathrm{D}^{\mathrm{abs}}[U, G, w] \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh}[Z/G])$$

form a crepant categorical resolution.

Proof. Lemma 4.1.2 and Proposition 4.1.1 together with Proposition 5.11 in [2] assure $\mathrm{D}^{\mathrm{abs}}[U, G, w]$ is the homotopy category of a homologically smooth and proper dg-category. i is an open immersion, so the functors i_*, i^* are both left and right adjoint. Thus also $i_* \circ \Omega$ and $\Omega^{-1} \circ i^*$ are left and right adjoint. By definition, they hence form a crepant categorical resolution, as required. □

It is important to note that an extension of w need not exist in general. However, using variations of geometric invariant theory, we can generate some crepant categorical resolutions.

We set up the notation in parallel to § 3.1. Consider the affine space

$$X := \mathbb{C}^{n+r} = \operatorname{Spec}\mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_r].$$

Consider a subgroup of the open dense torus, $S \subset (\mathbb{C}^*)^{n+r}$. For $1 \leq i \leq n + r$, let χ_i be the character obtained by composition of the inclusion and projection onto the i^{th} summand. Recall that S satisfies the Calabi-Yau condition when $\sum_{i=1}^{n+r} \chi_i$ is torsion. Recall that a superpotential is a semi-invariant function w with respect to the character χ_R obtained by projecting the action $S \times \mathbb{C}^*$ on $\mathbb{C}^{n+r}$ induced by the R-charge onto the second factor. A choice of such a subgroup S with corresponding R-charge and projection character χ_R gives the data of a gauged Landau-Ginzburg model

$$(\mathbb{C}^{n+r}, S \times \mathbb{C}^*, w, \chi_R).$$

As before, VGIT gives invariant open subsets of $\mathbb{C}^{n+r}$ that yield new gauged Landau-Ginzburg models. These subsets are the open sets U_p associated to the various chambers of the GKZ fan. Hence, for each chamber σ_p of the GKZ fan, we have a Landau-Ginzburg model $(U_p, S \times \mathbb{C}^*, w)$ and, by previous discussion, an absolute derived category associated to it, $\mathrm{D}^{\mathrm{abs}}[U_p, S\times\mathbb{C}^*, w]$. The following result relates these categories associated to different chambers of the GKZ fan. The result in this form is Theorem 4.18 in [19], but is based on work by Herbst-Walcher, Ballard-Favero-Katzarkov, Halpern-Leistner, Cox-Little-Schenck and Segal.

Theorem 4.1.6. *For any two chambers σ_p and σ_q of a GKZ fan as above, if S satisfies the quasi-Calabi-Yau condition, then there is an equivalence of categories:*

$$\mathrm{D}^{\mathrm{abs}}[U_p, S \times \mathbb{C}^*, w] \simeq \mathrm{D}^{\mathrm{abs}}[U_q, S \times \mathbb{C}^*, w].$$

4.1.1 Specialising to toric varieties

From here on, we will focus on toric varieties. Fix two dual lattices M and N, both of dimension m. Consider a collection of points $\nu = \{v_1, \dots, v_n\}$.

Definition 4.1.7. We say that the collection ν is geometric if each v_i is non-zero and generates a distinct ray in $N_{\mathbb{Q}}$.

Given a collection ν of points, we define a group S_ν in an analogous fashion to the way we defined the group G_Σ in (2.8) (this is part of the Cox construction, §2.1.4). There is a right exact sequence

$$M \xrightarrow{f_\nu} \mathbb{Z}^n \xrightarrow{\pi} \mathrm{coker}(f_\nu) \to 0, \tag{4.1}$$

where $f_\nu(m) = \sum_{i=1}^n \langle v_i, m\rangle e_i$. In the case where the points in ν correspond to the primitive generators of the rays $\Sigma(1)$ in a simplicial fan Σ, this sequence should be compared to (2.7). Like with the Cox construction, we apply the functor $\mathrm{Hom}(-, \mathbb{C}^*)$ to obtain a left exact sequence

$$0 \to \mathrm{Hom}(\mathrm{coker}(f_\nu), \mathbb{C}^*) \to (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m.$$

Define

$$S_\nu := \mathrm{Hom}(\mathrm{coker}(f_\nu), \mathbb{C}^*). \tag{4.2}$$

Let $\sigma \subseteq N_\mathbb{R}$ be a $\mathbb{Q}$-Gorenstein cone, Gorenstein with respect to the element $\mathfrak{m} \subseteq M$. Let $\nu \subseteq \sigma \cap N$ be a finite, geometric collection of lattice points containing the ray generators of σ. Partition ν into the two subsets

$$\nu_{=1} := \{v \in \nu \mid \langle \mathfrak{m}, v\rangle = 1\},$$
$$\nu_{\neq 1} := \{v \in \nu \mid \langle \mathfrak{m}, v\rangle \neq 1\}.$$

Since σ is $\mathbb{Q}$-Gorenstein, its ray generators are contained in $\nu_{=1}$ by definition. Choose a subset $R \subset \nu$. The set R determines an action of $\mathbb{C}^*$ on $\mathbb{C}^{|\nu|}$ by setting

$$\lambda_R \cdot x_i := \begin{cases} \lambda_R x_i & \text{if } v_i \in R \\ x_i & v_i \notin R, \end{cases}$$

and hence we obtain an action of $S_\nu \times \mathbb{C}^*$ on $\mathbb{C}^{|\nu|}$.

Now fix a simplicial fan $\Sigma \subseteq N_\mathbb{R}$ with $\Sigma(1) \subseteq \nu_{=1}$ such that X_Σ is semiprojective and $\mathrm{Cone}(\Sigma(1)) = \sigma$.

This data specifies an open subset

$$U_\Sigma \times (\mathbb{C}^*)^{\nu \setminus \Sigma(1)} \subseteq \mathbb{C}^{|\nu|},$$

and we can restrict the action of $S_\nu \times \mathbb{C}^*$ to it.

Remark 4.1.8. In the later parts of this thesis, we consider toric vector bundles $X_{\Sigma_{D_1,\ldots,D_r}}$ and set $\nu = \Sigma_{-D_1,\ldots,-D_r}(1)$.

Let χ be the projection character of the action $S_\nu \times \mathbb{C}^* \to \mathbb{C}^*$ onto the second factor and let $w \in \Gamma(U_\Sigma \times (\mathbb{C}^*)^{\nu\setminus\Sigma(1)}, \mathcal{O}(\chi))^{S_\nu \times \mathbb{C}^*}$. We recall from Remark 3.1.6 that this defines a global function on $U_\Sigma \times (\mathbb{C}^*)^{\nu\setminus\Sigma(1)}$ which is semi-invariant with respect to χ. For each

$v_i \in \nu$, there is a character χ_{D_i} obtained as the following composition:

$$S_\nu \hookrightarrow (\mathbb{C}^*)^{|\nu|} \to \mathbb{C}^* .$$

Here, the second map is the projection onto the i^{th} factor. For $\nu_j \in \nu \setminus \Sigma(1)$, the character χ_{D_j} we obtain is non-trivial, hence defining a surjective map

$$\begin{aligned} S_\nu \times \mathbb{C}^* &\xrightarrow{p} (\mathbb{C}^*)^{\nu\setminus\Sigma(1)} \\ (g,\lambda) &\mapsto \prod_{\nu_j \in \nu\setminus\Sigma(1)} \chi_{D_j}(g)\cdot \lambda_{R\cap(\nu\setminus\Sigma(1))}. \end{aligned} \tag{4.3}$$

Definition 4.1.9. Define $H_{\Sigma,R} \subseteq S_\nu \times \mathbb{C}^*$ to be the kernel of the map (4.3). When clear in the context, we suppress Σ and R from the notation and simply write H.

We define $\bar{w}$ on $\mathbb{C}^{|\Sigma(1)|}$ by setting all variables associated to points in $\nu \setminus \Sigma(1)$ to one. This corresponds to the restriction of w to $\mathbb{C}^{|\Sigma(1)|}$. In this setup, Favero and Kelly study wall-crossings and use them to prove the following Theorem.

Theorem 4.1.10. *Let $\tilde{\Sigma}$ be any simplicial fan such that $\tilde{\Sigma}(1) = \nu$ and $X_{\tilde{\Sigma}}$ is semi-projective. Similarly, let Σ be any simplicial fan such that $\Sigma(1) \subseteq \nu_{=1}$, X_Σ is semi-projective, and* $\mathrm{Cone}(\Sigma(1)) = \sigma$. *We have the following:*

(1) If $\langle \mathfrak{m}, a\rangle > 1$ for all $a \in \nu_{\neq 1}$, then there is a fully faithful functor,

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \to \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{C}^*, w].$$

(2) If $\langle \mathfrak{m}, a\rangle < 1$ for all $a \in \nu_{\neq 1}$, then there is a fully faithful functor,

$$\mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{C}^*, w] \to \mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}].$$

(3) If $\nu_{\neq 1} = \emptyset$, then there is an equivalence,

$$\mathrm{D}^{\mathrm{abs}}[U_\Sigma, H, \bar{w}] \simeq \mathrm{D}^{\mathrm{abs}}[U_{\tilde{\Sigma}}, S_\nu \times \mathbb{C}^*, w].$$

Theorem 4.1.10 uses VGIT to compare the absolute derived categories associated to different chambers of a GKZ fan. The result we are after is a categorical resolution of the derived category of coherent sheaves on a complete intersection. To obtain it, we need a way to relate the derived category of coherent sheaves of a complete intersection to an absolute derived category associated to an appropriate Landau-Ginzburg model. We will set up the situation we consider and show how to use the Theorem 4.1.10.

Let $\Psi \subseteq N_{\mathbb{R}}$ be a complete fan such that X_Ψ is projective and $D_1, \ldots, D_t$ are nef-divisors. Assume $|\Psi_{-D_1,\ldots,-D_t}|$ is a $\mathbb{Q}$-Gorenstein cone. Denote by e_i the standard basis of the sublattice $\mathbb{Z}^t$ in $N \times \mathbb{Z}^t$, and set

$$\mathfrak{n} := \sum_{i=1}^{t} e_i.$$

Restrict to the case where $\mathrm{Cone}(\nu) = |\Psi_{-D_1,\ldots,-D_t}|$ and $\{u_\rho \mid \rho \in \Psi_{-D_1,\ldots,-D_t}(1)\} \subseteq \nu$. Set R to be the subset $\{e_i \mid 1 \leq i \leq t\}$. Now, suppose Σ is any fan such that

- X_Σ is semi-projective,
- $|\Sigma| = |\Psi_{-D_1,\ldots,-D_t}|$, and
- for any $\delta \in \Sigma(1)$, we have $u_\delta \in \nu$ and $\langle \mathfrak{m}, u_\delta \rangle = 1$.

The last condition ensures that $\Sigma(1) \subseteq \nu_{=1}$. We consider the set Ξ of lattice points in $|\Psi_{-D_1,\ldots,-D_t}|^\vee$, defined as

$$\Xi := \{m \in |\Psi_{-D_1,\ldots,D_t}|^\vee \cap (M \times \mathbb{Z}^t) \mid \langle m, \mathfrak{n} \rangle = 1\}.$$

Thus, $\mathrm{Conv}(\Xi)$ is the height 1 slice of the dual cone to $|\Psi_{-D_1,\ldots,-D_t}|$. Elements in Ξ are points in the dual to $|\Psi_{-D_1,\ldots,D_t}|$ and thus correspond to monomial functions (see Proposition 2.1.32). The case we work with is the one where the superpotential can be constructed using such monomials, i.e. where

$$w = \sum_{m \in \Xi} c_m x^m,$$

with $c_m \in \mathbb{C}$. Here, the t rays $e_1, \ldots, e_t$ generated by the elements of R are associated to variables $u_1, \ldots, u_t$ and the remaining $n-t$ points in $\nu \setminus R$ are associated to variables $x_1, \ldots, x_{n-t}$. Denote by ρ_j the ray with associated variable x_j.

This allows us to rewrite the superpotential w as

$$w = \sum_{i=1}^{t} u_i f_i,$$

where f_j is a global section of D_j. This reformulation allows us to establish the relation between derived categories of complete intersections and absolute derived categories of Landau-Ginzburg models we sought.

Proposition 4.1.11 (Proposition 5.11 in [19])**.** *Assume that $f_1, \ldots, f_t$ define a complete intersection $\mathcal{Z} = Z(f_1, \ldots, f_t) \subseteq \mathcal{X}_\Psi$. Then there is an equivalence of categories,*

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{Z}) \simeq \mathrm{D}^{\mathrm{abs}}[U_\psi, S_\nu \times \mathbb{C}^*, w].$$

In the remainder of the chapter, we will use these results to construct various categorical resolutions. The fundamental strategy in those constructions is the following. For a complete intersection Z inside a toric variety, we construct the toric vector bundle associated to it. We then partially compactify the toric vector bundle and construct the associated GKZ fan to the partial compactification. In this GKZ fan, we find an appropriate chamber that corresponds to a toric vector bundle itself. We then prove that the absolute derived category is homologically smooth and *dg*-proper and apply Proposition 4.1.11 and Theorem 4.1.10 to obtain a categorical resolution of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,Z)$.

4.2 A generalisation of the Libgober-Teitelbaum mirror construction

In this section we discuss a generalisation of the Libgober-Teitelbaum mirror [35] to the complete intersection of two cubics in $\mathbb{P}^5$. In particular, we show that the proposed generalisation, while singular, is categorically resolved by a Batyrev-Borisov mirror.

For an integer $n \geq 2$, denote by ζ_n a primitive n-th root of unit. Let $\alpha_i, \beta_i \in \mathbb{Z} \pmod{n}$ for $1 \leq i \leq n$ and $\delta \in \mathbb{Z} \pmod{n^2}$ subject to the conditions

$$\begin{cases} \zeta_{n^2}^{\delta} = \zeta_n^{\beta_1+\cdots+\beta_{n-1}} = \zeta_n^{-(\alpha_1+\cdots+\alpha_{n-1})} & \text{if } n \text{ odd,} \\ \zeta_{n^2}^{\delta} = \zeta_n^{\beta_1+\cdots+\beta_{n-1}} = \zeta_n^{-(\alpha_1+\cdots+\alpha_{n-1})} & \text{if } n \text{ even.} \end{cases} \tag{4.4}$$

Let $G_n \subset PGL(2n-1, \mathbb{C})$ be the group given by automorphisms of the following form:

$$g_{\underline{\alpha},\underline{\beta},\delta} = \begin{cases} \operatorname{diag}\left(\zeta_{n^2}^{\delta}, \zeta_{n^2}^{\delta}\zeta_n^{\alpha_1}, \ldots, \zeta_{n^2}^{\delta}\zeta_n^{\alpha_{n-1}}, \zeta_{n^2}^{-\delta}\zeta_n^{\beta_1}, \ldots, \zeta_{n^2}^{-\delta}\zeta_n^{\beta_{n-1}}, \zeta_{n^2}^{-\delta}\right) & \text{if } n \text{ odd,} \\ \operatorname{diag}\left(\zeta_{2n^2}^{\delta}, \zeta_{2n^2}^{\delta}\zeta_n^{\alpha_1}, \ldots, \zeta_{2n^2}^{\delta}\zeta_n^{\alpha_{n-1}}, \zeta_{2n^2}^{-\delta}\zeta_n^{\beta_1}, \ldots, \zeta_{2n^2}^{-\delta}\zeta_n^{\beta_{n-1}}, \zeta_{2n^2}^{-\delta}\right) & \text{if } n \text{ even.} \end{cases}$$

Given n, we fix the lattice $M_n \cong \mathbb{Z}^{2n-1}$ and its dual lattice N_n.

Theorem 4.2.1. *Let $n \geq 2$, and $\lambda^{2n} \neq 0, n^{2n}$. Consider the complete intersection in $\mathbb{P}^{2n-1}$ given by the vanishing set of the two polynomials*

$$Q_{1,n,\lambda} = x_1^n + x_2^n + \cdots + x_n^n - \lambda x_{n+1}x_{n+2}\ldots x_{2n},$$

$$Q_{2,n,\lambda} = x_{n+1}^n + x_{n+2}^n + \cdots + x_{2n}^n - \lambda x_1 x_2 \ldots x_n.$$

Then $G_n \cong (\mathbb{Z}/n\,\mathbb{Z})^{2(n-2)} \times (\mathbb{Z}/n^2\,\mathbb{Z})$ and the hypersurfaces $Q_1 = 0$ and $Q_2 = 0$ are preserved under the action of G_n on $\mathbb{P}^{2n-1}$. Let $Z_n = Z(Q_{1,n,\lambda}, Q_{2,n,\lambda}) \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/(\mathbb{C}^* \times G_n)\right]$, and let Y_n be a Batyrev-Borisov mirror to $Z(Q_{1,n,\lambda}, Q_{2,n,\lambda}) \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/\,\mathbb{C}^*\right]$.

Then there is a categorical resolution $\mathrm{D^b}(\operatorname{coh} Y_n) \to \mathrm{D^b}(\operatorname{coh} Z_n)$.

The invariance of the hypersurfaces under the action of G_n is easy to see, as is the isomorphism $G_n \cong (\mathbb{Z}/n\,\mathbb{Z})^{2(n-2)} \times (\mathbb{Z}/n^2\,\mathbb{Z})$.

Note that the case $n = 3$ corresponds to the Libgober-Teitelbaum construction, and the categorical resolution there turns out to be a derived equivalence by Theorem 3.0.1. For $n = 2$, we also obtain an equivalence by a similar computation to the one in Chapter §3.

Theorem 4.2.2 (Theorem 4.8 in [36]). *Let* $Q_1 = x_1^2 + x_2^2 - x_3x_4,\ Q_2 = x_3^2 + x_4^2 - x_1x_2$ *and let* $p_1 = x_1^2x_5^2 + x_2^2x_6^2 - x_3x_4x_5x_6,\ p_2 = x_3^2x_7^2 + x_4^2x_8^2 - x_1x_2x_7x_8$. *We define the group* $G_4 \subseteq PGL(3, \mathbb{C})$ *given by the four automorphisms*

$$\operatorname{diag}(1,1,1,1),\ \operatorname{diag}(\zeta_8, -\zeta_8, -\zeta_8^{-1}, \zeta_8^{-1}),\ \operatorname{diag}(\zeta_4, \zeta_4, \zeta_4^{-1}, \zeta_4^{-1}),\ \operatorname{diag}(\zeta_8^3, -\zeta_8^3, -\zeta_8^{-3}, \zeta_8^{-3}),$$

where ζ_k *is a primitive* k^{th} *root of unity.*

The Batyrev-Borisov mirror to $Z(Q_1, Q_2) \subseteq \mathbb{P}^3$ *can be computed to be a complete intersection* $\mathcal{Z}_2$ *in a 3-dimensional toric stack* $\mathcal{X}_{BB}$ *given as the zero locus* $Z_2 = Z(p_1, p_2) \subseteq \mathcal{X}_{BB}$. *Take the stacky complete intersection* $\mathcal{V}_2 := Z(Q_1, Q_2) \subseteq [(\mathbb{C}^4 \setminus \{0\})/(\mathbb{C}^* \times G_4)]$. *Then*

$$\mathrm{D^b}(\operatorname{coh} \mathcal{V}_2) \cong \mathrm{D^b}(\operatorname{coh} \mathcal{Z}_2).$$

Remark 4.2.3. This Theorem 4.2.1 is a special, explicit application of Theorem 4.3.7 and Corollary 4.3.10 (see Remark 4.3.13). We can imitate the proof of Theorem 4.2.1 to obtain categorical resolutions of sufficiently "well-behaved" hypersurfaces and complete intersections in complete, projective toric varieties. The meaning of "well-behaved" is made more precise in § 4.3, leading to Theorem 4.3.7. As for the case of Theorem 4.2.2, Theorem 4.3.7 will yield derived equivalences should the derived categories be homologically smooth and *dg*-proper already. It remains a question under what conditions on the complete intersections involved this becomes the case.

This is a general theme. If we do the process of this construction for a smooth complete intersection, then a categorical resolution of it is simply a derived equivalence by definition of categorical resolutions.

4.2.1 A fan for the generalisation of Libgober and Teitelbaums construction

The proof to Theorem 4.2.1 is split into several steps. We start by giving a fan for the variety $\mathbb{P}^{2n-1}/G_n$.

Consider the rays $\rho_{n,1}, \ldots, \rho_{n,2n}$ with primitive generators:

$$\begin{aligned}
u_{\rho_{n,1}} &= (\underbrace{n, 0, \ldots, 0}_{n}, \underbrace{-1, \ldots, -1}_{n-1}), & u_{\rho_{n,n+1}} &= (\underbrace{-1, \ldots, -1}_{n}, \underbrace{n, 0, \ldots, 0}_{n-1}), \\
&\vdots \qquad\qquad \vdots & &\vdots \qquad\qquad \vdots \\
u_{\rho_{n,n-1}} &= (0, \ldots, n, 0, -1, \ldots, -1), & u_{\rho_{n,2n-1}} &= (-1, \ldots, -1, 0, \ldots, 0, n), \\
u_{\rho_{n,n}} &= (0, \ldots, 0, n, -1, \ldots, -1), & u_{\rho_{n,2n}} &= (\underbrace{-1, \ldots, -1}_{n}, \underbrace{0, \ldots, 0, 0}_{n-1}).
\end{aligned}$$

We define the fan Σ_n as the fan with rays $\rho_{n,i}$ and $2n$ maximal cones, which are the ones spanned by $2n-1$ of the $2n$ rays respectively, together with their faces. In other words, the collection of maximal cones of Σ_n is

$$\mathcal{M}_n = \left\{ \operatorname{Cone}(\rho_{n,j} \mid j \in I) \;\middle|\; I \subset \{1, 2, \ldots, 2n\}, |I| = 2n-1 \right\}.$$

We should at this point check that this gives a well-defined fan. Since there are $2n$ rays in $2n-1$ dimensional space, we really only need to show that Σ_n is complete, as this means that not all $2n$ rays are contained in some strictly convex cone. We note that $\sum_i^{2n} \frac{1}{2n} u_{\rho_{n,i}} = 0$, implying that 0 is in the interior of the support $|\Sigma_n|$. Furthermore, $|\Sigma_n|$ is full dimensional, and therefore Σ_n is indeed complete.

Lemma 4.2.4. *The toric stack associated to Σ_n is $\mathcal{X}_n := \left[\mathbb{C}^{2n} \setminus \{0\} / (\mathbb{C}^* \times G_n)\right]$.*

Proof. To prove this, we will proceed analogously to the $n = 3$ case of Libgober-Teitelbaum and use the Cox construction. The exact sequence (2.7) takes the form

$$0 \to N_n \xrightarrow{\iota} \bigoplus_{i=1}^{2n} D_{\rho_{n,i}} \to \operatorname{coker} \iota \to 0.$$

Now consider the group $G = \operatorname{Hom}(\operatorname{coker} \iota, \mathbb{C}^*)$. By Lemma 2.1.39, we have $G = \{(t_i) \in$

$(\mathbb{C}^*)^{2n}\}$, subject to the following system of conditions:

$$
\begin{aligned}
t_1^n &= t_{n+1}t_{n+2}\dots t_{2n} && (4.5)\\
\vdots \quad & \quad \vdots\\
t_n^n &= t_{n+1}t_{n+2}\dots t_{2n} && (4.6)\\
t_{n+1}^n &= t_1t_2\dots t_n && (4.7)\\
\vdots \quad & \quad \vdots\\
t_{2n-1}^n &= t_1t_2\dots t_n. && (4.8)
\end{aligned}
$$

We note that the group $H := \{t\cdot(1,1,\dots,1)|t \in \mathbb{C}^*\}$ is a subgroup of G. As we did before, we compute G by computing the group of cosets of H. The equations (4.5)-(4.6) imply that

$$t_1^n = t_2^n = \dots = t_n^n \Rightarrow t_2 = \zeta_n^{\alpha_1}t_1, \quad \dots \quad , t_n = \zeta_n^{\alpha_{n-1}}t_1. \tag{4.9}$$

Similarly, the equations (4.7)-(4.8) imply

$$t_{n+1} = \zeta_n^{\beta_1}t_{2n}, \quad t_{n+2} = \zeta_n^{\beta_2}t_{2n}, \quad \dots \quad , t_{2n-1} = \zeta_n^{\beta_{n-1}}t_{2n}. \tag{4.10}$$

Substituting (4.10) into (4.5) and (4.9) into (4.8) gives

$$\left.\begin{aligned} t_1^n &= \zeta_n^{\beta_1+\dots+\beta_{n-1}}t_{2n}^n \\ t_{2n}^n &= \zeta_n^{\alpha_1+\dots+\alpha_{n-1}}t_1^n \end{aligned}\right\} \beta_1+\dots+\beta_{n-1} \equiv -(\alpha_1+\dots+\alpha_{n-1}) \pmod n. \tag{4.11}$$

Picking an appropriate representative of the coset of $(t_1,\dots,t_{2n})\cdot H$, we can assume $\prod_{i=1}^{2n} t_i = 1$. We then have

$$1 = \prod_{i=1}^{2n} t_i \Rightarrow 1 = \zeta_n^{\alpha_1+\dots+\alpha_{n-1}+\beta_1+\dots+\beta_{n-1}}t_1^n t_{2n}^n = t_1^n t_{2n}^n \Rightarrow t_{2n} = \zeta_n^{\gamma}t_1^{-1}, \tag{4.12}$$

for some $\gamma \in (\mathbb{Z}/n\mathbb{Z})$. Substituting what we have into equation (4.5) now gives

$$t_1^n = \zeta_n^{\beta_1+\cdots+\beta_{n-1}} t_1^{-n} \Rightarrow t_1^{2n^2} = 1.$$

Therefore, we know that for some $\delta \in (\mathbb{Z}/2n^2\mathbb{Z})$,

$$t_1 = \zeta_{2n^2}^{\delta}. \tag{4.13}$$

For the rest of the proof, we will split into the two cases n even and n odd.

A. <u>n is odd:</u> In this case, we note that $(\zeta_{2n}^k, \zeta_{2n}^k, \ldots, \zeta_{2n}^k) \in H$ for all $k \in \mathbb{Z}$, and also that $\zeta_{2n}^{k \cdot 2n} = 1$, so multiplying each t_i of a given coset $(t_1, \ldots, t_{2n})$ by ζ_{2n}^k changes neither the coset of H, nor does it change that $\prod_{i=1}^{2n} t_i = 1$.

   Since n is odd, we can therefore pick a representative of the coset with $t_1 = \zeta_{n^2}^{\delta_2}$ for some $\delta_2 \in (\mathbb{Z}/n^2\mathbb{Z})$, and such that all the above equations still hold. By abuse of notation, we will actually denote said δ_2 by δ. In particular, by (4.12), $t_{2n} = \zeta_{n^2}^{-\delta}\zeta_n^{\gamma}$.

   As n is odd, there exists a k_n such that $2k_n \equiv -1 \pmod{n}$. Multiplying each t_i by $\zeta_{2n}^{2k_n\gamma} = \zeta_n^{k_n\gamma}$ then gives a representation of the coset $(t_1, \ldots, t_{2n}) \cdot H$ as $(t_1', \ldots, t_{2n}') \cdot H$ with

$$\begin{aligned}(t_1', \ldots, t_{2n}') = \quad & (\zeta_{n^2}^{\delta}\zeta_n^{k_n\gamma}, \zeta_{n^2}^{\delta}\zeta_n^{\alpha_1}\zeta_n^{k_n\gamma}, \ldots, \zeta_{n^2}^{\delta}\zeta_n^{\alpha_{n-1}}\zeta_n^{k_n\gamma}, \\ & \zeta_{n^2}^{-\delta}\zeta_n^{\gamma}\zeta_n^{\beta_1}\zeta_n^{k_n\gamma}, \ldots, \zeta_{n^2}^{-\delta}\zeta_n^{\gamma}\zeta_n^{\beta_{n-1}}\zeta_n^{k_n\gamma}, \zeta_{n^2}^{-\delta}\zeta_n^{\gamma}\zeta_n^{k_n\gamma}).\end{aligned}$$

   Note that, using the definition of k_n,

$$t_1' t_{2n}' = \zeta_{n^2}^{\delta}\zeta_n^{k_n\gamma}\zeta_{n^2}^{-\delta}\zeta_n^{\gamma}\zeta_n^{k_n\gamma} = \zeta_n^{\gamma+2k_n\gamma} = 1.$$

   Hence, by picking appropriate representatives of $(t_1, \ldots, t_{2n})$, we can assume $t_{2n} = t_1^{-1}$, and so the group of cosets of H has elements of the form

$$\left(\zeta_{n^2}^{\delta}, \zeta_{n^2}^{\delta}\zeta_n^{\alpha_1}, \ldots, \zeta_{n^2}^{\delta}\zeta_n^{\alpha_{n-1}}, \zeta_{n^2}^{-\delta}\zeta_n^{\beta_1}, \ldots, \zeta_{n^2}^{-\delta}\zeta_n^{\beta_{n-1}}, \zeta_{n^2}^{-\delta}\right) \cdot H,$$

for $\alpha_i, \beta_j \in (\mathbb{Z}/n\,\mathbb{Z})$ and $\delta \in (\mathbb{Z}/n^2\,\mathbb{Z})$ with

$$\zeta_{n^2}^{\delta} = \zeta_n^{\beta_1+\cdots+\beta_{n-1}} = \zeta_n^{-(\alpha_1+\cdots+\alpha_{n-1})}.$$

By the direct product theorem (as in the Libgober-Teitelbaum case of $n = 3$), we thus have that $G \cong \mathbb{C}^* \times G_n$.

B. n is even: Like for the case of n odd, we can multiply each of the t_i by ζ_{2n}^k without affecting their product or which coset they represent. So consider $(t'_1, \ldots, t'_{2n})$ with $t'_i = \zeta_{2n}^{-\gamma}$. Then

$$t_1 t_{2n} = \zeta_{2n^2}^{\delta}\zeta_{2n}^{-\gamma}\zeta_{2n^2}^{-\delta}\zeta_n^{\gamma}\zeta_{2n}^{-\gamma} = \zeta_n^{\gamma}\zeta_{2n}^{-2\gamma} = 1.$$

Thus we have $t'_{2n} = t_1'^{-1}$. Note that $(-1, \ldots, -1) \in H$, so we may assume that $t'_1 = \zeta_{2n^2}^{\delta_2}$ for some $\delta_2 \in (\mathbb{Z}/n^2\,\mathbb{Z})$. By abuse of notation, we denote that δ_2 simply by δ. This means that the group of cosets of H is given by

$$\left(\zeta_{2n^2}^{\delta}, \zeta_{2n^2}^{\delta}\zeta_n^{\alpha_1}, \ldots, \zeta_{2n^2}^{\delta}\zeta_n^{\alpha_{n-1}}, \zeta_{2n^2}^{-\delta}\zeta_n^{\beta_1}, \ldots, \zeta_{2n^2}^{-\delta}\zeta_n^{\beta_{n-1}}, \zeta_{2n^2}^{-\delta}\right),$$

subject to the condition

$$\zeta_{n^2}^{\delta} = \zeta_n^{\beta_1+\cdots+\beta_{n-1}} = \zeta_n^{-(\alpha_1+\cdots+\alpha_{n-1})}.$$

By the direct product theorem, we thus have $G \cong \mathbb{C}^* \times G_n$.

The Cox fan of Σ_{LT} is the fan over the $2n$ rays $e_{\rho_{n,1}}, \ldots, e_{\rho_{n,2n}}$ with maximal cones being those spanned by $2n-1$ of the $2n$ rays. Therefore, we obtain $U_{\Sigma_n} = \mathbb{A}^{2n} \setminus \{0\}$. Thus, the toric stack associated to Σ_n is, as claimed,

$$\mathcal{X}_n = [U_{\Sigma_n}/G] = [\mathbb{C}^{2n} \setminus \{0\}/\left(\mathbb{C}^* \times G_n\right)].$$

$\square$

Analogously to § 3, we will need a vector bundle over $X_n = \mathbb{P}^{2n-1}/G_n$ with section $Q_2 \oplus Q_1$. To each ray ρ_j of Σ_n we can associate a torus-invariant Weil divisor D_{ρ_i}. Let $D_a = D_{\rho_1} + D_{\rho_2} + \cdots + D_{\rho_n}$ and $D_b = D_{\rho_{n+1}} + \cdots + D_{\rho_{2n}}$. Applying Proposition 2.1.50 twice gives a fan Σ_{n,D_a,D_b} for $\operatorname{tot}(\mathcal{O}_{X_n}(-D_a) \oplus \mathcal{O}_{X_n}(-D_b))$. It has rays $\overline{\rho_{n,1}}, \ldots, \overline{\rho_{n,2n}}, \tau_{n,1}, \tau_{n,2}$ with primitive generators

$$\begin{aligned}
u_{\overline{\rho_{n,1}}} &= (\underbrace{n, 0, \ldots, 0}_{n}, \underbrace{-1, \ldots, -1}_{n-1}, 0, 1), & u_{\overline{\rho_{n,n+1}}} &= (\underbrace{-1, \ldots, -1}_{n}, \underbrace{n, 0, \ldots, 0}_{n-1}, 1, 0),\\
&\vdots \qquad\qquad \vdots & &\vdots \qquad\qquad \vdots\\
u_{\overline{\rho_{n,n-1}}} &= (0, \ldots, n, 0, -1, \ldots, -1, 0, 1), & u_{\overline{\rho_{n,2n-1}}} &= (-1, \ldots, -1, 0, \ldots, 0, n, 0, 1),\\
u_{\overline{\rho_{n,n}}} &= (0, \ldots, 0, n, -1, \ldots, -1, 1, 0), & u_{\overline{\rho_{n,2n}}} &= (\underbrace{-1, \ldots, -1}_{n}, \underbrace{0, \ldots, 0, 0}_{n-1}, 1, 0),\\
u_{\tau_{n,1}} &= (0, \ldots, 0, 1, 0), & u_{\tau_{n,2}} &= (0, \ldots, 0, 0, 1).
\end{aligned}$$

The maximal cones of Σ_{n,D_a,D_b} are those cones generated by $2n+1$ of the $2n+2$ rays, always including $\tau_{n,1}$ and $\tau_{n,2}$. So the maximal cones are of the form $\operatorname{Cone}(\{\tau_{n,1}, \tau_{n,2}, u_{\overline{\rho_{n,i}}} \mid i \in I\})$ where $I \subseteq \{1, \ldots, 2n\}$ with $|I| = 2n - 1$.

Lemma 4.2.5. *The vector bundle $\mathcal{O}(D_a) \oplus \mathcal{O}(D_b)$ has global section $Q_2 \oplus Q_1$.*

Proof. We note that $x_j^n \in \Gamma(X_n, \mathcal{O}(nD_{\rho_{n,j}}))$, as $\operatorname{div}(x_j^n) = nD_{\rho_{n,j}}$, and this for all $1 \leq j \leq 2n$. By similar reasoning, $\prod_{i=1}^n x_i \in \Gamma(X_n, \mathcal{O}(D_{\rho_{n,1}} + \cdots + D_{\rho_{n,n}}))$ and $\prod_{j=n+1}^{2n} x_j \in \Gamma(X_n, \mathcal{O}(D_{\rho_{n,n+1}} + \cdots + D_{\rho_{n,2n}}))$. So it remains to show

$$\begin{aligned}
&nD_{\rho_{n,1}} \simeq nD_{\rho_{n,2}} \simeq \cdots \simeq nD_{\rho_{n,n}} \simeq D_b,\\
&nD_{\rho_{n,n+1}} \simeq nD_{\rho_{n,n+2}} \simeq \cdots \simeq nD_{\rho_{n,2n}} \simeq D_a.
\end{aligned}$$

Proceeding as in Corollary 3.3.3, we show that the monomials differ by $\operatorname{div}(\chi^m)$ for some $m \in M_n$, hence by a principal divisor. And indeed, $m = (-1, 0, \ldots, 0)$ gives $\operatorname{div}(\chi^m) = x_1^{-n} x_{n+1} \ldots x_{2n}$ and thus $nD_{\rho_{n,1}} \simeq D_b$. Similarly, $m = (\underbrace{0, \ldots, 0}_{n}, -1, 0, \ldots, 0)$ has $\operatorname{div}(\chi^m) = x_1 \ldots x_n x_{n+1}^{-n}$ and thus $nD_{\rho_{n,n+1}} \simeq D_a$.

For $2 \leq j \leq n$, we have $nD_{\rho_{n,1}} \simeq nD_{\rho_{n,j}}$ by looking at $\operatorname{div}(\chi^{m_j}) = x_1^{-n}x_j^n$ with $m_j = (-1, 0, \dots, 0, \underbrace{-1}_{j^{th} \text{ position}}, 0, \dots, 0)$. Showing $nD_{\rho_{n,n+1}} \simeq nD_{\rho_{n,n+2}} \simeq \dots \simeq nD_{\rho_{n,2n}}$ is analogous.

Therefore, $Q_2 \in \Gamma(X_n, \mathcal{O}_{X_n}(D_b))$ and $Q_1 \in \Gamma(X_n, \mathcal{O}_{X_n}(D_a))$, as required. $\square$

4.2.2 Proof of Theorem 4.2.1

To start the proof, we find two chambers σ_p, σ_q in the same GKZ fan Σ_{GKZ} which correspond to the two complete intersections Y_n, Z_n in the statement of the Theorem. We can then employ methods of VGIT in the form of the results of § 4.1 on these two chambers. Denote by U_p, U_q the affines corresponding to the chambers σ_p, σ_q. We lay out the strategy of the proof.

Step 0: Show the existence of the chambers σ_p, σ_q.

Step 1: Show $\mathrm{D}^{\mathrm{abs}}([U_p, G, w]) \cong \mathrm{D}^{\mathrm{abs}}([U_q, G, w])$.

Step 2: Show $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_n)$ is homologically smooth and proper.

Step 3: Show $\mathrm{D}^{\mathrm{abs}}([U_q, G, w]) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_n)$.

Step 4: Conclude that there is a categorical resolution of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z_n)$ by $\mathrm{D}^{\mathrm{abs}}([U_p, G, w])$.

This strategy can also be summarised by the following diagram, where the double arrow on the left represents a categorical resolution and the bracketed numbers label the steps of the proof.

$$\begin{tikzcd}
 & \mathrm{D}^{\mathrm{abs}}([U_p, G, w]) \arrow[r, leftrightarrow, "\cong", "(1)"'] \arrow[dl, shift left, "(4)"] & \mathrm{D}^{\mathrm{abs}}([U_q, G, w]) \arrow[dr, leftrightarrow, "\cong", "(3)"'] & \\
\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z_n) \arrow[ur, shift left] & & & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_n) \quad (2)
\end{tikzcd}$$

Step 0: There is a GKZ fan with chambers σ_p, σ_q corresponding to the mirror construction generalising LT and the BB construction.

Consider the polytope $\mathfrak{P}$ that is the convex hull of the following $4n+2$ points in $\mathbb{R}^{2n-1} \times \mathbb{R}^2 = \mathbb{R}^{2n+1}$:

$$
\begin{array}{llll}
P_1 = & (\underbrace{n, 0, \dots, 0}_{n}, \underbrace{-1, \dots, -1}_{n-1}, 0, 1), & P_{2n+1} = & (\underbrace{n-1, -1, \dots, -1}_{n}, \underbrace{0, \dots, 0}_{n-1}, 1, 0), \\
\vdots & \vdots & \vdots & \vdots \\
P_n = & (0, \dots, 0, n, -1, \dots, -1, 0, 1), & P_{3n} = & (-1, \dots, -1, n-1, 0, \dots, 0, 1, 0), \\
P_{n+1} = & (\underbrace{-1, \dots, -1}_{n}, \underbrace{n, 0, \dots, 0}_{n-1}, 1, 0), & P_{3n+1} = & (\underbrace{0, \dots, 0}_{n}, \underbrace{n-1, -1, \dots, -1}_{n-1}, 0, 1), \\
\vdots & \vdots & \vdots & \vdots \\
P_{2n-1} = & (-1, \dots, -1, 0, \dots, 0, n, 1, 0), & P_{4n-1} = & (0, \dots, 0, -1, \dots, -1, n-1, 0, 1), \\
P_{2n} = & (\underbrace{-1, \dots, -1}_{n}, \underbrace{0, \dots, 0}_{n-1}, 1, 0), & P_{4n} = & (\underbrace{0, \dots, 0}_{n}, \underbrace{-1, \dots, -1}_{n-1}, 0, 1), \\
S_1 = & (0, \dots, 0, 1, 0), & S_2 = & (0, \dots, 0, 0, 1).
\end{array}
$$

We note that this is the complete dual polytope to $|\operatorname{tot}(\mathcal{O}_{\mathbb{P}^{2n}}(-n) \oplus \mathcal{O}_{\mathbb{P}^{2n}}(-n))|$, with the lattice points corresponding to global functions on $\operatorname{tot}(\mathcal{O}_{\mathbb{P}^{2n}}(-n) \oplus \mathcal{O}_{\mathbb{P}^{2n}}(-n))$ in the usual way. Thus, the lattice points correspond to sections of the line bundle $\mathcal{O}_{\mathbb{P}^{2n}}(n) \oplus \mathcal{O}_{\mathbb{P}^{2n}}(n)$, and all sections of this line bundle can be generated by the lattice points of $\mathfrak{P}$.

Consider the $4n+2$ points of $\mathfrak{P}$ as primitive generators $u_{\overline{\rho_{n,1}}}, \dots, u_{\overline{\rho_{n,4n}}}, u_{\tau_{n,1}}, u_{\tau_{n,2}}$ of $4n+2$ rays in $\mathbb{R}^{4n+1}$. Let $\overline{\rho_{n,i}}$ be the ray generated by $u_{\overline{\rho_{n,i}}}$ and $\tau_{n,j}$ be the ray generated by $u_{\tau_{n,j}}$. For $1 \le i \le 4n$, denote by $u_{\rho_{n,i}}$ the primitive generator of the ray $\rho_{n,i}$ obtained by projecting $u_{\overline{\rho_{n,i}}}$ down to the first $\mathbb{R}^{2n-1}$ coordinates.

Since $\mathfrak{P}$ is the dual polytope to $|\operatorname{tot}(\mathcal{O}_{\mathbb{P}^{2n}}(-n) \oplus \mathcal{O}_{\mathbb{P}^{2n}}(-n))|$, the fans of the varieties in the Batyrev-Borisov mirror family to $\mathcal{O}_{\mathbb{P}^{2n}}(-n) \oplus \mathcal{O}_{\mathbb{P}^{2n}}(-n)$ have 1-skeletons consisting of the rays $u_{\rho_{n,1}}, \dots, u_{\rho_{n,4n}}$. In other words, if $X_{BB,n}$ is the toric variety that the Batyrev-Borisov mirror Y_n lives in, with fan $\Sigma_{BB,n}$ for $X_{BB,n}$, then $\Sigma_{BB,n}(1) = \{u_{\rho_{n,1}}, \dots, u_{\rho_{n,4n}}\}$.

Denote by $D'_{n,i}$ the torus-invariant Weil divisor of $X_{BB,n}$ corresponding to the ray $\rho_{n,i}$. Then let $D'_{n,a}$ be the divisor $D'_{n,a} = D'_{n,1} + \cdots + D_{n,n} + D'_{n,3n+1} + \cdots + D'_{n,4n}$

and let $D'_{n,b}$ be the divisor $D'_{n,b} = D'_{n,n+1} + \cdots + D'_{n,3n}$. We consider the vector bundle $\mathcal{O}_{X_{BB,n}}(D'_a) \oplus \mathcal{O}_{X_{BB,n}(D'_b)}$, which we construct torically. This gives us a fan Σ_{BB,n,D'_a,D'_b} with $\Sigma_{BB,n,D'_a,D'_b}(1) = \{u_{\overline{\rho_{n,1}}}, \ldots, u_{\overline{\rho_{n,4n}}}, u_{\tau_{n,1}}, u_{\tau_{n,2}}\}$.

The secondary fan Σ_{GKZ} of Σ_{BB,n,D'_a,D'_b} is the fan on which we will perform VGIT. To show the existence of two chambers σ_p, σ_q corresponding to Y_n, Z_n is equivalent to the existence of two regular triangulations of the $4n+2$ points of $\mathfrak{P}$ that can be associated to the two constructions. We start by showing that there is a triangulation partially compactifying the generalisation of the Libgober-Teitelbaum construction.

Definition 4.2.6. A triangulation $\mathcal{T}$ is called n-viable if the following properties hold.

- $\mathcal{T}$ is a regular triangulation of the $4n+2$ points of $\mathfrak{P}$.
- $\mathcal{T}$ contains the following set of simplices, listed via their vertices:

$$\mathcal{T}_0 := \{\{P_i, S_1, S_2 \mid i \in I\} \mid I \subset \{1, 2, \ldots, 2n\}, |I| = 2n-1\}.$$

Theorem 4.2.7. *For all $n \geq 2$, an n-viable triangulation exists.*

Proof. The proof can be divided into two parts.

Part 1: We start by defining an explicit regular polyhedral subdivision $\mathcal{S}$ of P containing $\mathcal{T}_0$.

Part 2: We prove that the polyhedral subdivision $\mathcal{S}$ can be refined to a regular triangulation $\mathcal{T}$ of $\mathfrak{P}$ containing $\mathcal{T}_0$.

Part 1: We note that $\mathcal{T}_0$ is a regular subdivision of the points $P_1, \ldots, P_{2n}, S_1, S_2$. It is in fact a star subdivision upon S_1, S_2 of the convex hull $\mathrm{Conv}(P_1, \ldots, P_{2n}, S_1, S_2)$. Indeed, an example of an explicit weight function w giving the triangulation $\mathcal{T}_0$ is $w(S_1) = w(S_2) = 1, w(P_i) = 2$ for $1 \leq i \leq 2n$. To complete Step 1, we extend this weight function to all $4n+2$ points of $\mathfrak{P}$.

Consider the weight function $w(P_i) = 2$ for $1 \leq i \leq 2n$, $w(S_1) = w(S_2) = 1$ $(j = 1, 2)$ and $w(P_j) = n+2$ for $2n+1 \leq j \leq 4n$. The points $(P_i, w(P_i)), (S_j, w(S_j))$ $(1 \leq i \leq$

$4n, j = 1,2)$ then form a polyhedron $\mathcal{Q}$ in $\mathbb{R}^{2n+2}$. To obtain the regular subdivision of $\mathfrak{P}$ corresponding to this weight function, we need to project the lower facets of this polyhedron down onto $\mathbb{R}^{2n+1}$ along the last coordinate (a lower facet being one where the inward pointing normal has a positive last coordinate).

By using SAGE for low dimensional cases, we found the form of $4n$ lower facets, written in the form $u_i \cdot x + a \geq 0$ were u_i is the inward pointing normal of the i^{th} facet. The normals and additive constants are (in this order):

- $F_1 : (2n-1, -1, \ldots, -1, 0, \ldots, 0, 0, 0, n)x - n \geq 0$
- $F_2 : (-1, 2n-1, -1, \ldots, -1, 0, \ldots, 0, 0, 0, n)x - n \geq 0$

  $\vdots$
- $F_n : (-1, \ldots, -1, 2n-1, 0, \ldots, 0, 0, 0, n)x - n \geq 0$
- $F_{n+1} : (1, \ldots, 1, 2n, 0, \ldots, 0, 0, 0, n)x - n \geq 0$
- $F_{n+2} : (1, \ldots, 1, 0, 2n, 0, \ldots, 0, 0, 0, n)x - n \geq 0$

  $\vdots$
- $F_{2n-1} : (1, \ldots, 1, 0, \ldots, 0, 2n, 0, 0, n)x - n \geq 0$
- $F_{2n} : (-(2n-1), \ldots, -(2n-1), -2n, \ldots, -2n, 0, 0, n)x - n \geq 0$
- $F_{2n+1} : (n, -1, \ldots, -1, 0, \ldots, 0, 0, n-1, 1)x - 1 \geq 0$
- $F_{2n+2} : (-1, n, -1, \ldots, -1, 0, \ldots, 0, 0, n-1, 1)x - 1 \geq 0$

  $\vdots$
- $F_{3n} : (-1, \ldots, -1, n, 0, \ldots, 0, 0, n-1, 1)x - 1 \geq 0$
- $F_{3n+1} : (1, \ldots, 1, n+1, 0, \ldots, 0, 0, -(n-1), 1)x + (n-2) \geq 0$
- $F_{3n+2} : (1, \ldots, 1, 0, n+1, 0, \ldots, 0, 0, -(n-1), 1)x + (n-2) \geq 0$

  $\vdots$

- $F_{4n-1} : (1, \dots, 1, 0, \dots, 0, n+1, 0, -(n-1), 1)x + (n-2) \geq 0$
- $F_{4n} : (-n, \dots, -n, -(n+1), \dots, -(n+1), 0, -(n-1), 1)x + (n-2) \geq 0.$

An easy computation shows that, indeed, we have all $4n+2$ points in the intersection of the relevant half-spaces. Furthermore, we have the following points lying on the facets:

Facet	contains
F_1	$P_2, \dots, P_{2n}, S_1, S_2$
$\vdots$	$\vdots$
F_{2n}	$P_1, \dots, P_{2n-1}, S_1, S_2$
F_{2n+1}	$P_2, \dots, P_{2n}, S_1, P_{2n+2}, \dots, P_{3n}$
$\vdots$	$\vdots$
F_{3n}	$P_1, \dots, P_{n-1}, P_{n+1}, \dots, P_{2n}, S_1, P_{2n+1}, \dots, P_{3n-1}$
F_{3n+1}	$P_1, \dots, P_n, P_{n+2}, \dots, P_{2n}, S_2, P_{3n+2}, \dots, P_{4n}$
$\vdots$	$\vdots$
F_{4n}	$P_1, \dots, P_{2n-1}, S_2, P_{3n+1}, \dots, P_{4n-1}.$

To obtain the polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$ corresponding to the weight function w, we now project these facets down to $\mathbb{R}^{2n+1}$ along the last coordinate. Denoting by $\widehat{F}_i$ the polyhedron obtained by projecting the facet F_i, we obtain the following set of $4n$ polyhedra:

$\widehat{F}_1 =$	$\mathrm{Conv}(P_2, \dots, P_{2n}, S_1, S_2)$
$\vdots$	$\vdots$
$\widehat{F}_{2n} =$	$\mathrm{Conv}(P_1, \dots, P_{2n-1}, S_1, S_2)$
$\widehat{F}_{2n+1} =$	$\mathrm{Conv}(P_2, \dots, P_{2n}, S_1, P_{2n+2}, \dots, P_{3n})$
$\vdots$	$\vdots$
$\widehat{F}_{3n} =$	$\mathrm{Conv}(P_1, \dots, P_{n-1}, P_{n+1}, \dots, P_{2n}, S_1, P_{2n+1}, \dots, P_{3n-1})$
$\widehat{F}_{3n+1} =$	$\mathrm{Conv}(P_1, \dots, P_n, P_{n+2}, \dots, P_{2n}, S_2, P_{3n+2}, \dots, P_{4n})$
$\vdots$	$\vdots$
$\widehat{F}_{4n} =$	$\mathrm{Conv}(P_1, \dots, P_{2n-1}, S_2, P_{3n+1}, \dots, P_{4n-1}).$

Table 4.1: Polyhedra in the regular subdivision.

We note here that when projecting, all points that lied on the facet F_i lie in the polyhedron $\widehat{F}_i$ by convexity of the polyhedron $\mathcal{Q}$ in $\mathbb{R}^{2n+2}$.

It remains to show that we have found all the lower facets of $\mathcal{Q}$. Showing that $F_1, \dots, F_{4n}$ is the complete set of lower facets is the same as showing that $\bigcup F_i + \langle(0, \dots, 0, 1)\rangle_{\mathbb{R}^+}$ contains the entire polyhedron $\mathcal{Q}$. Since we know all points lie on the inside of those polyhedra, it suffices to show $\mathcal{Q} \subset \bigcup F_i + \langle(0, \dots, 0, 1)\rangle$. Thus, what remains to prove is that the union of the projections $\widehat{F}_i$ contains the convex hull of $P_1, \dots, P_{4n}, S_1, S_2$, i.e. contains $\mathfrak{P}$. This is equivalent to saying that they give a polyhedral subdivision (regularity is given by construction, we reduced the statement to showing it is one).

So we aim to prove the following claim.

Lemma 4.2.8. *For $\widehat{F}_i$ and $\mathfrak{P}$ as above, we have $\bigcup_{i=1}^{4n} \widehat{F}_i = \mathfrak{P}$.*

Proof. The first thing to note is that $\mathfrak{P} = \operatorname{Conv}(P_1, \dots, P_{4n}, S_1, S_2) = \operatorname{Conv}(P_1, \dots, P_{4n})$. So we will show that $\bigcup_{i=1}^{4n} \widehat{F}_i = \operatorname{Conv}(P_1, \dots, P_{4n})$. We start by showing that $\bigcup_{i=1}^{2n} \widehat{F}_i = \operatorname{Conv}(P_1, \dots, P_{2n}, S_1, S_2)$, which is equivalent to saying that $\widehat{F}_1, \dots, \widehat{F}_{2n}$ form a polyhedral subdivision of $\operatorname{Conv}(P_1, \dots, P_{2n}, S_1, S_2)$.
The inclusion $\subseteq$ is immediate, so it remains to show the opposite inclusion. Any point $X \in \operatorname{Conv}(P_1, \dots, P_{2n}, S_1, S_2)$ can be written as $X = \sum_{i=1}^{2n} \lambda_i P_i + \mu_1 S_1 + \mu_2 S_2$ for some $\lambda_i, \mu_j \in \mathbb{R}_{\geq 0}$ with $\sum \lambda_i + \mu_1 + \mu_2 = 1$. Note also that $\sum_{i=1}^{2n} P_i = n(S_1 + S_2)$. Now define j such that $\lambda_j = \min_{1 \leq i \leq 2n}\{\lambda_i\}$. Then

$$X = \sum_{i=1}^{2n}(\lambda_i - \lambda_j)P_i + (n\lambda_j + \mu_1)S_1 + (n\lambda_j + \mu_2)S_2 = \sum_{\substack{1 \leq i \leq 2n \\ i \neq j}} (\lambda_i - \lambda_j)P_i + (n\lambda_j + \mu_1)S_1 + (n\lambda_j + \mu_2)S_2.$$

Since $\lambda_j = \min_{1 \leq i \leq 2n}\{\lambda_i\} \leq \lambda_i$ for $1 \leq i \leq 2n$, we have that $(\lambda_i - \lambda_j) \geq 0$ for $1 \leq i \leq 2n$, and as $\lambda_i, \mu_1, \mu_2 \geq 0$ we also have $n\lambda_j + \mu_1, n\lambda_j + \mu_2 \geq 0$. Also,

$$\sum_{\substack{1 \leq i \leq 2n \\ i \neq j}} (\lambda_i - \lambda_j) + (n\lambda_j + \mu_1) + (n\lambda_j + \mu_2) = \sum_{i=1}^{2n} \lambda_i + \mu_1 + \mu_2 = 1,$$

and thus $X \in \widehat{F}_j$. This shows $\bigcup_{i=1}^{2n} \widehat{F}_i = \mathrm{Conv}(P_1, \dots, P_{2n}, S_1, S_2)$.

To show $\bigcup_{i=1}^{4n} \widehat{F}_i = \mathfrak{P}$, we note again that the inclusion $\subseteq$ is immediate. To show the opposite inclusion $\supseteq$, take a general point X in $\mathfrak{P}$. Then X can be written as $X = \sum_{i=1}^{4n} \lambda_i P_i$ with $\lambda_i \geq 0$ for $1 \leq i \leq 4n$ and $\sum_{i=1}^{4n} \lambda_i = 1$.

Without loss of generality, assume that $(\lambda_{2n+1} + \cdots + \lambda_{3n}) \geq (\lambda_{3n+1} + \cdots + \lambda_{4n})$ (the case where the inequality is reversed is analogous). We will now show that if $X \notin \bigcup_{i=1}^{2n} \widehat{F}_i = \mathrm{Conv}(P_1, \dots, P_{2n}, S_1, S_2)$, then $X \in \bigcup_{i=2n+1}^{3n} \widehat{F}_i$ (if the inequality was reversed, then X would be in $\bigcup_{i=3n+1}^{4n} \widehat{F}_i$).

Let

$$\begin{aligned} \nu_i &= \lambda_i + \lambda_{2n+i} - \tfrac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) && \text{for } 1 \leq i \leq n, \\ \nu_i &= \lambda_i + \lambda_{2n+i} && \text{for } n+1 \leq i \leq 2n, \\ \mu_1 &= \left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right), \\ \mu_2 &= 0. \end{aligned}$$

Then

$$\sum_{i=1}^{2n} \nu_i P_i + \mu_1 S_1 + \mu_2 S_2 = \sum_{i=1}^{4n} \lambda_i P_i,$$

and

$$\sum_{i=1}^{2n} \nu_i + \mu_1 + \mu_2 = \sum_{i=1}^{4n} \lambda_i = 1.$$

Since $\mu_1 \geq 0$ by assumption and $\mu_2 = 0 \geq 0$, we now require all ν_i to be ≥ 0 to have found an expression of X as element of $\mathrm{Conv}(P_1, \dots, P_{2n}, S_1, S_2) = \bigcup_{i=1}^{2n} \widehat{F}_i$. For $n+1 \leq i \leq 2n$, $\nu_i \geq 0$ as both λ_i and λ_{2n+i} are.

For $1 \leq i \leq n$, $\nu_i \geq 0$ is equivalent to

$$\frac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) \leq \lambda_i + \lambda_{2n+i},$$

so the condition that all ν_i are non-negative is equivalent to

$$\frac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) \leq \min_{1 \leq i \leq n} \{\lambda_i + \lambda_{2n+i}\}.$$

Therefore, $X \in \bigcup_{i=1}^{2n} \widehat{F}_i = \mathrm{Conv}(P_1, \ldots, P_{2n}, S_1, S_2)$ if

$$\frac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) \leq \min_{1 \leq i \leq n} \{\lambda_i + \lambda_{2n+i}\}.$$

Suppose this condition does not hold, i.e.

$$\min_{1 \leq i \leq n} \{\lambda_i + \lambda_{2n+i}\} < \frac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right). \tag{4.14}$$

Without loss of generality, we may assume that $\lambda_1 + \lambda_{2n+1} = \min_{1 \leq i \leq n}\{\lambda_i + \lambda_{2n+i}\}$ (by symmetry, the other cases are analogous). We will show that $X \in \widehat{F}_{2n+1}$. Any point Y in $\widehat{F}_{2n+1} = \mathrm{Conv}\,(P_2, \ldots, P_{2n}, S_1, P_{2n+2}, \ldots, P_{3n})$ can be written as

$$Y = \sum_{i=2}^{2n} \nu_i P_i + \mu_1 S_1 + \sum_{i=2n+2}^{3n} \nu_i P_i.$$

If we find ν_i, μ_1 such that this sum is equal to $\sum_{i=1}^{4n} \lambda_i P_i = X$, we are done as we will have expressed X as an element of $\widehat{F}_{2n+1}$. Given a choice of real numbers $\alpha_2, \ldots, \alpha_n$ with $\alpha_2 + \cdots + \alpha_n = 1$, define

$$\begin{aligned}
\nu_i &= \alpha_i(n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})) - (\lambda_1 + \lambda_{2n+1}) && \text{for } 2 \leq i \leq n,\\
\nu_i &= \lambda_i + \lambda_{2n+i} && \text{for } n+1 \leq i \leq 2n,\\
\mu_1 &= n\lambda_1 + n\lambda_{2n+1}, &&\\
\nu_{2n+i} &= \lambda_{2n+i} + \alpha_i(-n\lambda_1 - (n-1)\lambda_{2n+1} - (\lambda_{3n+1} + \cdots + \lambda_{4n})) && \text{for } 2 \leq i \leq n.
\end{aligned}$$

Substituting these values into the expression for Y gives

$$Y = \sum_{i=2}^{2n} \nu_i P_i + \mu_1 S_1 + \sum_{i=2n+2}^{3n} \nu_i P_i = \sum_{i=1}^{4n} \lambda_i P_i = X,$$

as well as

$$\sum \nu_i + \mu_1 = \sum_{i=1}^{4n} \lambda_i = 1.$$

For this choice of ν_i's and μ_1 to define an element $Y \in \widehat{F}_{2n+1}$, we require $\nu_i \geq 0$ for all i and $\mu_1 \geq 0$. We note that, as $\lambda_1, \lambda_{2n+1} \geq 0$, we have $\mu_1 \geq 0$. Therefore, what remains to prove is that there exist $\alpha_2, \ldots, \alpha_n \in \mathbb{R}$ with $\alpha_2 + \cdots + \alpha_n = 1$ such that $\nu_i \geq 0$ for $i \in \{2, \ldots, 2n, 2n+2, \ldots, 3n\}$.

For each $2 \leq i \leq n$, we can arrange the inequalities $\nu_i \geq 0$ and $\nu_{2n+i} \geq 0$ to give

$$\frac{\lambda_1 + \lambda_{2n+1} - \lambda_i}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})} \leq \alpha_i \leq \frac{\lambda_{2n+i}}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})}. \tag{4.15}$$

This works provided $n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n}) \neq 0$ but if that term is zero, then by non-negativity of the λ_i we have that $\lambda_1 = \lambda_{2n+1} = \lambda_{3n+1} = \cdots = \lambda_{4n} = 0$ and thus $X \in \widehat{F}_{2n+1}$. So if there exists a tuple $(\alpha_2, \ldots, \alpha_n)$ with (3.10) holding for $2 \leq i \leq n$ and $\alpha_2 + \cdots + \alpha_n = 1$, then $X \in \widehat{F}_{2n+1}$. We note that for all i,

$$\frac{\lambda_1 + \lambda_{2n+1} - \lambda_i}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})} \leq \frac{\lambda_{2n+i}}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})},$$

as $\lambda_1 + \lambda_{2n+1} = \min_{1 \leq i \leq n}\{\lambda_i + \lambda_{2n+i}\}$.

Furthermore,

$$\begin{aligned}
&0 \leq \lambda_1 + \cdots + \lambda_n + \lambda_{3n+1} + \cdots + \lambda_{4n} \\
&\Leftrightarrow (n-1)\lambda_1 + (n-1)\lambda_{2n+1} \leq \lambda_2 + \cdots + \lambda_n + n\lambda_1 + (n-1)\lambda_{2n+1} + \lambda_{3n+1} + \cdots + \lambda_{4n} \\
&\Leftrightarrow \sum_{i=2}^{n} \lambda_1 + \lambda_{2n+1} - \lambda_i \leq n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n}) \\
&\Leftrightarrow \sum_{i=2}^{n} \frac{\lambda_1 + \lambda_{2n+1} - \lambda_i}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})} \leq 1.
\end{aligned}$$

Lastly, we are given that

$$\lambda_1 + \lambda_{2n+1} = \min_{1 \leq i \leq n}\{\lambda_i + \lambda 2n + i\} \leq \frac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right).$$

This leads to the following sequence of implications:

$$\lambda_1 + \lambda_{2n+1} \leq \frac{1}{n}\left((\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right)$$

$$\Leftrightarrow n(\lambda_1 + \lambda_{2n+1}) \leq (\lambda_{2n+1} + \cdots + \lambda_{3n}) - (\lambda_{3n+1} + \cdots + \lambda_{4n})$$

$$\Leftrightarrow n\lambda_1 + (n-1)\lambda_{2n+1} + \lambda_{3n+1} + \cdots + \lambda_{4n} \leq \lambda_{2n+2} + \cdots + \lambda_{3n}$$

$$\Leftrightarrow 1 \leq \sum_{i=2}^{n} \frac{\lambda_{2n+i}}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})}.$$

In summary, we have shown that for $2 \leq i \leq n$, we have

$$\frac{\lambda_1 + \lambda_{2n+1} - \lambda_i}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})} \leq \frac{\lambda_{2n+i}}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})},$$

and that

$$\sum_{i=2}^{n} \frac{\lambda_1 + \lambda_{2n+1} - \lambda_i}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})} \leq 1 \leq \sum_{i=2}^{n} \frac{\lambda_{2n+i}}{n\lambda_1 + (n-1)\lambda_{2n+1} + (\lambda_{3n+1} + \cdots + \lambda_{4n})}.$$

We apply Lemma 3.4.3 by setting $m = n - 1$, shifting the index by 1, and setting $L_j = \frac{\lambda_1+\lambda_{2n+1}-\lambda_j}{n\lambda_1+(n-1)\lambda_{2n+1}+(\lambda_{3n+1}+\cdots+\lambda_{4n})}$, $C = 1$ and $R_j = \frac{\lambda_{2n+j}}{n\lambda_1+(n-1)\lambda_{2n+1}+(\lambda_{3n+1}+\cdots+\lambda_{4n})}$. This gives us the existence of an $(n-1)$-tuple $\alpha_2, \ldots, \alpha_n$ as required, concluding the proof of Lemma 3.4.2. □

The Lemma 3.4.2 shows that we have indeed found all lower facets of the polyhedron $\mathcal{Q}$, meaning that the collection $\widehat{F}_1, \ldots, \widehat{F}_{4n}$ gives a regular polyhedral subdivision $\mathcal{S}$ of $\mathfrak{P}$, finishing Step 1 of the proof of Theorem 4.2.7.

Part 2: This is true by general convex geometry (using the poset of refinements and the secondary polytope). By Theorem 16.4.1 in [23], the poset of (non-empty) faces of the secondary polytope $\Sigma(\mathfrak{P})$ is isomorphic to the poset of all regular subdivisions of $\mathfrak{P}$, partially ordered by refinement. The vertices of $\Sigma(\mathfrak{P})$ correspond to regular triangulations. Thus, our regular subdivision obtained by projection must correspond to some face of

$\Sigma(\mathfrak{P})$ and any vertex of that face will correspond to a regular triangulation refining it.

By definition 4.2.6, we have, for arbitrary $n \geq 2$, constructed an n-viable triangulation.

$\square$

Remark 4.2.9. We can compare this to Proposition 3.4.1, where we showed the existence of a suitable chamber σ_{LT} of the GKZ fan considered in Chapter § 3. The triangulation we prove to exist in Proposition 3.4.1 is n-viable for $n = 3$. In fact, a stronger result than Theorem 4.2.7 holds. Given $n \geq 2$, the n-viable triangulation $\mathcal{T}$ constructed in the proof of Theorem 4.2.7 has the additional property that any simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills either of the two following conditions:

A. T does not contain either of the points S_1, $P_{2n+1}, \ldots, P_{3n}$ and does not contain one pair of points of the form P_j, P_{2n+j} with $n+1 \leq j \leq 2n$.

B. T does not contain either of the points S_2, $P_{3n+1}, \ldots, P_{4n}$ and does not contain one pair of points of the form P_j, P_{2n+j} with $1 \leq j \leq n$.

This is the same additional property we find in Proposition 3.4.1 and is proved in a similar manner. For $n = 2, 3$, this additional property ensured a containment of ideals $\mathcal{I} \subseteq \sqrt{\partial w, \mathcal{J}}$. For $n \geq 4$, this is no longer clear (we have not been able to verify such a containment) and this property is likely not enough to give such a containment. If it was true, then the categorical resolution would turn out to be a derived equivalence, but due to the singularities in the varieties for $n \geq 4$, we do not think this is likely to be the case.

For the sake of completeness, we prove the claim made above about the additional property holding. Consider a regular triangulation $\mathcal{T}$ of $\mathfrak{P}$ as constructed in the proof of Theorem 4.2.7. Recall Table 3.3. Denote by C_i the collection of points used to define the polyhedron $\widehat{F}_i$ in the table. We aim to show that none of the polyhedra $\widehat{F}_i, 1 \leq i \leq 4n$ contains any of the points we did not define it by, i.e. $\widehat{F}_i \cap \mathfrak{P} = C_i$. Indeed, $\widehat{F}_1, \ldots, \widehat{F}_{2n}$ are the simplices in $\mathcal{T}_0$ and do not get refined any further, as none of these simplices contains an interior point. It thus remains to show that all the simplices obtained by refining $\widehat{F}_{2n+1}, \ldots, \widehat{F}_{4n}$ fulfill one of the two conditions A or B from 4.2.6.

We note that the polyhedra $\widehat{F}_i$, $2n+1 \leq i \leq 4n$, each fulfill one of the two conditions A or B. So suppose we have a simplex $T \in \mathcal{T}$ obtained by refining one of the polyhedra, say $\widehat{F}_k$. We claim that T fulfills the same condition as $\widehat{F}_k$. T being obtained by refining $\widehat{F}_k$ implies that the vertex set of T is a subset of $\widehat{F}_k \cap \mathfrak{P}$. If $\widehat{F}_k \cap \mathfrak{P} = C_k$, then we are done as T inherits the condition A or B from T. Showing that $\widehat{F}_i \cap \mathfrak{P} = C_i$ for $1 \leq i \leq 4n$ reduces to a simple computation. □

Unpacking what an n-viable triangulation is, we obtain the following Corollary.

Corollary 4.2.10. *In the GKZ fan Σ_{GKZ}, there is a chamber σ_p which is associated to a partial compactification of $\mathcal{X}_n$ (from Lemma 4.2.4).*

Indeed, by being a regular triangulation of $\mathfrak{P}$, $\mathcal{T}$ corresponds to a chamber in the GKZ fan Σ_{GKZ} via the usual bijection of regular triangulations and chambers. Containing $\mathcal{T}_0$ means that we have a partial compactification of $\mathcal{X}_n$.

Showing a chamber exists in Σ_{GKZ} that corresponds to the Batyrev-Borisov mirror is more straightforward, since the GKZ fan is in fact the secondary fan of Σ_{BB,n,D'_a,D'_b}. As such, the chamber σ_q associated to the Batyrev-Borisov mirror Y_n will simply correspond to the regular triangulation obtained by performing a star subdivision of the points $P_1, \ldots, P_{4n}, S_1, S_2$ on S_1, S_2.

Theorem 4.2.11. *In the GKZ fan Σ_{GKZ}, there is a chamber σ_q which is associated to the Batyrev-Borisov mirror.*

Proof. As done previously, we will explicitly construct a regular subdivision of the polyope $\mathfrak{P}$ that will refine to a triangulation corresponding to the Batyrev-Borisov mirror. We start by defining the weight function w by $w(P_i) = 2$ for $1 \leq i \leq 4n$ and $w(S_j) = 1$ for $j = 1, 2$.

Let $\mathcal{Q}$ be the polyhedron given by the convex hull of the $4n+2$ points $R_i := (P_i, w(P_i)), Z_j := (S_j, w(S_j))$, $1 \leq i \leq 4n$ and $j = 1, 2$. We claim that there are exactly the following $n^2 + 2n$ lower facets, written in the form $u_i \cdot x + a \geq 0$, where u_i is the inward pointing normal of the i^{th} facet.

- $F_{i,j} : (\underbrace{0,\dots,0,1,0,\dots,0}_{i^{th}\text{ position}},\underbrace{0,\dots,0,1,0,\dots,0}_{j^{th}\text{ position}},0,0,1)x - 1 \geq 0$ for $1 \leq i \leq n, n+1 \leq j \leq 2n-1$,
- $F_{i,2n} : (\underbrace{-1,-1,\dots,-1,0,-1,\dots,-1}_{i^{th}\text{ position}},0,0,1)x - 1 \geq 0$ for $1 \leq i \leq n$,
- $F_k : (\underbrace{0,\dots,0,1,0,\dots,0}_{k^{th}\text{ position}},0,0,1)x - 1 \geq 0$ for $1 \leq k \leq 2n-1$,
- $F_{2n} : (-1,\dots,-1,0,0,1)x - 1 \geq 0$.

We note also that the polyhedron $\mathcal{Q}$ is contained in the hyperplane $(0,\dots,0,1,1,0)x-1 = 0$.

A straightforward computation shows that we have all $4n+2$ points in the intersection of the relevant half-spaces. Furthermore, we get the following inclusion of points on the facets

Facet	Points contained	range
$F_{i,j}$	$\{R_1,\dots,R_{4n},Z_1,Z_2\} \setminus \{R_i,R_j,R_{2n+i},R_{2n+j}\}$	$1 \leq i \leq n, n+1 \leq j \leq 2n$
F_i	$\{R_{n+1},\dots,R_{3n},Z_1,Z_2\} \setminus \{R_{2n+i}\}$	$1 \leq i \leq n$
F_j	$\{R_1,\dots,R_n,R_{3n+1},\dots,R_{4n},Z_1,Z_2\} \setminus \{R_{2n+j}\}$	$n+1 \leq j \leq 2n$

Project these facets down to $\mathbb{R}^{2n+1}$ by projecting down the last coordinate which corresponds to the weight. The facets will then give a subdivision of $\mathfrak{P} \subseteq Z(x_{2n}+x_{2n+1}-1) \subseteq \mathbb{R}^{2n+1}$. Denote by $\widehat{F}_{i,j}$ and $\widehat{F}_k$ the polyhedron obtained by projecting the facet $F_{i,j}, F_k$ respectively. Note that when projecting, all points that lied on $F_{i,j}, F_k$ now lie in the polyhedron $\widehat{F}_{i,j}, \widehat{F}_k$ by convexity. This gives the following n^2+2n polyhedra:

$\widehat{F}_{i,j} =$	$\operatorname{Conv}(\{P_1,\dots,P_{4n},S_1,S_2\} \setminus \{P_i,P_j,P_{2n+i},P_{2n+j}\})$	$1 \leq i \leq n, n+1 \leq j \leq 2n$
$\widehat{F}_i =$	$\operatorname{Conv}(\{P_{n+1},\dots,P_{3n},S_1,S_2\} \setminus \{P_{2n+i}\})$	$1 \leq i \leq n$
$\widehat{F}_j =$	$\operatorname{Conv}(\{P_1,\dots,P_n,P_{3n+1},\dots,P_{4n},S_1,S_2\} \setminus \{P_{2n+j}\})$	$n+1 \leq j \leq 2n$

To prove Theorem 4.2.11, we need to show that the n^2+2n polyhedra form a polyhedral subdivision of $\mathfrak{P}$ (regularity is given by construction) and that they refine to a triangulation corresponding to Batyrev-Borisov. The latter is a combination of the general fact that regular subdivisions can be refined to triangulations and the observation that the weight function we chose corresponds to taking a star subdivision of the $4n+2$ points along S_1 and S_2. In particular, this corresponds to taking the face fan of the polytope $\mathfrak{P}'$ obtained by projecting the polytope $\mathfrak{P}$ along the last two coordinates. So it remains to prove that we have a polyhedral subdivision, which is equivalent to show that the union of the polyhedra $\widehat{F}_{i,j}, \widehat{F}_k$ form a convex set. Specifically, we will show $\bigcup \widehat{F}_{i,j} \cup \bigcup \widehat{F}_k = \mathrm{Conv}\{P_1, \ldots, P_{4n}, S_1, S_2\}$.

First note that $\mathrm{Conv}\{P_1, \ldots, P_{4n}, S_1, S_2\} = \mathrm{Conv}\{P_1, \ldots, P_{4n}\}$. The inclusion $\bigcup \widehat{F}_{i,j} \cup \bigcup \widehat{F}_k \subseteq \mathrm{Conv}\{P_1, \ldots, P_{4n}, S_1, S_2\}$ is immediate, so we need to prove the converse.

So consider a point $X \in \mathrm{Conv}\{P_1, \ldots, P_{4n}, S_1, S_2\}$. Then we can write X as $X = \sum \lambda_i P_i$. We will prove the following claim:

A. Suppose

$$\min_{n+1 \leq j \leq 2n} (\lambda_j + \lambda_{2n+j}) \leq \frac{1}{n} \left((\lambda_1 + \cdots + \lambda_n) + (\lambda_{3n+1} + \cdots + \lambda_{4n}) \right)$$

and

$$\min_{1 \leq i \leq n} (\lambda_i + \lambda_{2n+i}) \leq \frac{1}{n} \left((\lambda_{n+1} + \cdots + \lambda_{2n}) + (\lambda_{2n+1} + \cdots + \lambda_{3n}) \right).$$

Further, suppose $\lambda_k + \lambda_{2n+k} = \min_{1 \leq i \leq n} (\lambda_i + \lambda_{2n+i})$ and $\lambda_r + \lambda_{2n+r} = \min_{n+1 \leq j \leq 2n} (\lambda_j + \lambda_{2n+j})$. Then $X \in \widehat{F}_{k,r}$.

B. Suppose

$$\min_{1 \leq i \leq n} (\lambda_i + \lambda_{2n+i}) \geq \frac{1}{n} \left((\lambda_1 + \cdots + \lambda_n) + (\lambda_{3n+1} + \cdots + \lambda_{4n}) \right).$$

If $(\lambda_k + \lambda_{2n+k}) = \min_{1 \leq i \leq n} (\lambda_i + \lambda_{2n+i})$, then $X \in \widehat{F_k}$.

C. Suppose

$$\min_{n+1 \leq j \leq 2n} (\lambda_j + \lambda_{2n+j}) \geq \frac{1}{n} \left((\lambda_{n+1} + \cdots + \lambda_{2n}) + (\lambda_{2n+1} + \cdots + \lambda_{3n}) \right).$$

If $(\lambda_r + \lambda_{2n+r}) = \min_{n+1 \leq j \leq 2n} (\lambda_j + \lambda_{2n+j})$, then $X \in \widehat{F_r}$.

This classification covers all cases and therefore, we have $X \in \bigcup \widehat{F}_{i,j} \cup \bigcup \widehat{F}_k$.

In the first case, let α_i, β_j be a collection of real numbers such that

$$\frac{\lambda_k + \lambda_{2n+k} - \lambda_{2n+i}}{\lambda_k + \lambda_{2n+k}} \leq \alpha_i \leq \frac{\lambda_i}{\lambda_k + \lambda_{2n+k}} \text{ for } 2 \leq i \neq k \leq n, \tag{4.16}$$

$$\frac{\lambda_r + \lambda_{2n+r} - \lambda_{2n+j}}{\lambda_r + \lambda_{2n+r}} \leq \beta_j \leq \frac{\lambda_j}{\lambda_r + \lambda_{2n+r}} \text{ for } n+1 \leq j \neq r \leq 2n. \tag{4.17}$$

We further require $\sum \alpha_i(\lambda_k + \lambda_{2n+k}) - \sum \beta_j(\lambda_r + \lambda_{2n+r}) = \lambda_r - \lambda_k$. Such α_i, β_j exist by Lemma 3.4.3. Checking the conditions of the Lemma, we have

$$\frac{\lambda_k + \lambda_{2n+k} - \lambda_{2n+i}}{\lambda_k + \lambda_{2n+k}} \leq \frac{\lambda_i}{\lambda_k + \lambda_{2n+k}} \text{ for } 2 \leq i \neq k \leq n \text{ by minimality of } \lambda_k + \lambda_{2n+k};$$

$$\frac{\lambda_r + \lambda_{2n+r} - \lambda_{2n+j}}{\lambda_r + \lambda_{2n+r}} \leq \frac{\lambda_j}{\lambda_r + \lambda_{2n+r}} \text{ for } n+1 \leq j \neq r \leq 2n \text{ by minimality of } \lambda_r + \lambda_{2n+r};$$

$$\sum_{1 \leq i \neq k \leq n} \frac{\lambda_k + \lambda_{2n+k} - \lambda_{2n+i}}{\lambda_k + \lambda_{2n+k}}(\lambda_k + \lambda_{2n+k}) - \sum_{n+1 \leq j \neq r \leq 2n} \frac{\lambda_j}{\lambda_r + \lambda_{2n+r}}(\lambda_r + \lambda_{2n+r}) \leq \lambda_r - \lambda_k$$

$$\Leftrightarrow (n-1)(\lambda_k + \lambda_{2n+k}) - \sum_{1 \leq i \neq k \leq n} \lambda_{2n+i} - \sum_{n+1 \leq j \neq r \leq 2n} \lambda_j \leq \lambda_r - \lambda_k$$

$$\Leftrightarrow \lambda_k + \lambda_{2n+k} \leq \frac{1}{n}\left((\lambda_{n+1} + \cdots + \lambda_{2n}) + (\lambda_{2n+1} + \cdots + \lambda_{3n})\right) \text{ which is true by assumption;}$$

$$\lambda_r - \lambda_k \leq \sum_{1 \leq i \neq k \leq n} \lambda_i - \sum_{n+1 \leq j \neq r \leq 2n} \lambda_r + \lambda_{2n+r} - \lambda_{2n+j}$$

$$\Leftrightarrow \lambda_r + \lambda_{2n+r} \leq \frac{1}{n}\left((\lambda_1 + \cdots + \lambda_n) + (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) \text{ which is true by assumption.}$$

Hence we can apply the Lemma 3.4.3, obtaining a collection $(\alpha_i)_{1 \leq i \neq k \leq n}, (\beta_j)_{n+1 \leq j \neq r \leq 2n}$ as required. Now define

- $\nu_i = \lambda_i - \alpha_i(\lambda_k + \lambda_{2n+k})$ and $\nu_{2n+i} = \lambda_{2n+i} - (1 - \alpha_i)(\lambda_k + \lambda_{2n+k})$ for $1 \leq i \neq k \leq n$,
- $\nu_j = \lambda_j - \beta_j(\lambda_r + \lambda_{2n+r})$ and $\nu_{2n+j} = \lambda_{2n+j} - (1 - \beta_j)(\lambda_r + \lambda_{2n+r})$ for $n+1 \leq j \neq r \leq 2n$,
- $\kappa_1 = \lambda_r + n\lambda_{2n+k} + (n-1)\lambda_k + \sum \beta_j(\lambda_r + \lambda_{2n+r}) - \sum \alpha_i(\lambda_k + \lambda_{2n+k})$,
- $\kappa_2 = \lambda_k + n\lambda_{2n+r} + (n-1)\lambda_r + \sum \alpha_i(\lambda_k + \lambda_{2n+k}) - \sum \beta_j(\lambda_r + \lambda_{2n+r})$.

Then

$$X = \sum \nu_i P_i + \kappa_1 S_1 + \kappa_2 S_2.$$

It remains to show that all $\nu_i, \kappa_1, \kappa_2$ are non-negative to obtain $X \in \widehat{F}_{k,r}$. This is clear by the inequalities (4.16) and (4.17). Indeed, for $1 \leq i \neq k \leq 2n$, we have

$$\begin{aligned}
&\nu_i = \lambda_i - \alpha_i(\lambda_k + \lambda_{2n+k}) \geq 0 \\
&\Leftrightarrow \lambda_i \geq \alpha_i(\lambda_k + \lambda_{2n+k}) \\
&\Leftrightarrow \lambda_i = \frac{\lambda_i}{\lambda_k + \lambda_{2n+k}}(\lambda_k + \lambda_{2n+k}) \geq \alpha_i(\lambda_k + \lambda_{2n+k}) \text{ by (4.16);} \\
&\nu_{2n+i} = \lambda_{2n+i} - (1-\alpha_i)(\lambda_k + \lambda_{2n+k}) \geq 0 \\
&(\lambda_k + \lambda_{2n+k}) - \lambda_{2n+i} \leq \frac{(\lambda_k + \lambda_{2n+k}) - \lambda_{2n+i}}{\lambda_k + \lambda_{2n+k}} \leq \alpha_i(\lambda_k + \lambda_{2n+k}) \text{ by (4.16).}
\end{aligned}$$

An analogous computation shows that $\nu_j, \nu_{2n+j} \geq 0$ for $n+1 \leq j \neq r \leq 2n$. Also,

$$\begin{aligned}
\kappa_1 &= \lambda_r + n\lambda_{2n+k} + (n-1)\lambda_k + \sum \beta_j(\lambda_r + \lambda_{2n+r}) - \sum \alpha_i(\lambda_k + \lambda_{2n+k}) \\
&= \lambda_r + n\lambda_{2n+k} + (n-1)\lambda_k + \lambda_k - \lambda_r \\
&= n(\lambda_k + \lambda_{2n+k}) \geq 0,
\end{aligned}$$

and similarly $\kappa_2 \geq 0$. Finally, we also have

$$\sum \nu_i + \kappa_1 + \kappa_2 = \sum_{s=1}^{4n} \lambda_s = 1, \text{ as required.}$$

This concludes the first part of the claim.

In the second case, we have

$$\min_{1 \leq i \leq n}(\lambda_i + \lambda_{2n+i}) \geq \frac{1}{n}\left((\lambda_1 + \cdots + \lambda_n) + (\lambda_{3n+1} + \cdots + \lambda_{4n})\right).$$

The claim then is, if $(\lambda_k + \lambda_{2n+k}) = \min_{1 \leq i \leq n}(\lambda_i + \lambda_{2n+i})$, then $X \in \widehat{F_k}$.

$\widehat{F_k} = \text{Conv}\{P_{n+1}, \ldots, P_{3n} \setminus \{P_{2n+k}\}\}$, so we need to express $X = \sum_s^{4n} \lambda_s P_s$ as

$$\sum_{n+1 \leq s \neq 2n+k \leq 3n} \nu_s P_s + \kappa_1 S_1 + \kappa_2 S_2$$

with $\nu_s, \kappa_i \geq 0$ and $\sum \nu_s + \kappa_1 + \kappa_2 = \sum_{s=1}^{4n} \lambda_s = 1$.

So define

- $\nu_j = \lambda_j + \lambda_{2n+j} - \frac{1}{n}\left((\lambda_1 + \cdots + \lambda_n) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right)$ for $n+1 \leq j \leq 2n$.
- $\nu_{2n+i} = \lambda_i + \lambda_{2n+i} - (\lambda_k + \lambda_{2n+k})$ for $1 \leq i \leq n$.
- $\kappa_1 = n(\lambda_k + \lambda_{2n+k})$.
- $\kappa_2 = (\lambda_1 + \cdots + \lambda_n) + (\lambda_{3n+1} + \cdots + \lambda_{4n})$.

We indeed have

$$X = \sum_{n+1 \leq s \neq 2n+k \leq 3n} \nu_s P_s + \kappa_1 S_1 + \kappa_2 S_2,$$

so we need to check that $\nu_s, \kappa_i \geq 0$ and $\sum \nu_s + \kappa_1 + \kappa_2 = \sum_{s=1}^{4n} \lambda_s = 1$.

$$\nu_j = \lambda_j + \lambda_{2n+j} - \frac{1}{n}\left((\lambda_1 + \cdots + \lambda_n) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) \geq 0$$
$$\Leftrightarrow \lambda_j + \lambda_{2n+j} \geq \frac{1}{n}\left((\lambda_1 + \cdots + \lambda_n) - (\lambda_{3n+1} + \cdots + \lambda_{4n})\right) \text{ which is true by assumption;}$$
$$\nu_{2n+i} = \lambda_i + \lambda_{2n+i} - (\lambda_k + \lambda_{2n+k}) \geq 0$$
$$\Leftrightarrow \lambda_i + \lambda_{2n+i} \geq \lambda_k + \lambda_{2n+k} \text{ which is true by minimality;}$$
$$\kappa_1 = n(\lambda_k + \lambda_{2n+k}) \geq 0 \text{ which is true by non-negativity of } \lambda_k, \lambda_{2n+k};$$
$$\kappa_2 = (\lambda_1 + \cdots + \lambda_n) + (\lambda_{3n+1} + \cdots + \lambda_{4n}) \geq 0 \text{ which is true by non-negativity of the } \lambda_s.$$

Finally $\sum \nu_s + \kappa_1 + \kappa_2 = \sum_{s=1}^{4n} \lambda_s = 1$, as required, thus showing $X \in \widehat{F_k}$ and proving the second part of the claim.

The third case in the claim is analogous to the second one. Hence, we have shown the claim, which proves that $\bigcup \widehat{F}_{i,j} \cup \bigcup \widehat{F}_k$ is a convex set and thus that we have found all lower facets of the polytope $\mathcal{Q}$. Therefore, the triangulation obtained by projecting $\mathcal{Q}$ indeed gives a regular triangulation of the points $P_1, \dots, P_{4n}, S_1, S_2$ which hence gives a chamber σ_q in the GKZ fan. It corresponds to the Batyrev-Borisov construction, proving Theorem 4.2.11. $\square$

Before we continue with the next steps of the proof of Theorem 4.2.1, we note that the triangulation constructed to correspond to the chamber σ_q is in fact the regular triangulation corresponding to $\operatorname{tot}(\mathcal{O}_{X_{\Sigma_{BB,n}}}(-D'_a) \oplus \mathcal{O}_{X_{\Sigma_{BB,n}}}(-D'_b))$.

Step 1: $\mathrm{D}^{\mathrm{abs}}([U_p, G, w]) \cong \mathrm{D}^{\mathrm{abs}}([U_q, G, w])$**.**

This is true by Theorem 4.1.10. The cone $\mathrm{Cone}(\nu)$ is Gorenstein with respect to the element $\mathfrak{m} = (0, 0, \dots, 0, 1, 1)$ and hence $\nu_{\neq 1} = \emptyset$. By the third item of Theorem 4.1.10, we thus have $\mathrm{D}^{\mathrm{abs}}([U_p, G, w]) \cong \mathrm{D}^{\mathrm{abs}}([U_q, G, w])$.

Step 2: $\mathrm{D}^{\mathrm{abs}}([U_q, G, w])$ **is homologically smooth and** dg**-proper.**

Theorem 4.2.12. $\mathrm{D}^{\mathrm{abs}}([U_q, G, w])$ *is homologically smooth and dg-proper.*

Proof. We use the same arguments as in the proof of Theorem 4.1.5, namely we need to show that $[\partial w/G]$ is proper over $\operatorname{Spec} \mathbb{C}$ and that $\partial w \subseteq Z(w)$ in the affine open U_q associated to the chamber σ_q. We first consider $[\partial w/G]$ as subset of $X_{\mathcal{S}}$, the variety associated to the polyhedral subdivision $\mathcal{S}$ exhibited in Theorem 4.2.11. We note that $X_{\mathcal{S}}$ is by construction a rank 2 split vector bundle over some variety Y_{∇}. The ideal $\mathcal{I}$ associated to the subdivision $\mathcal{S}$ is

$$\mathcal{I} = \underbrace{\langle x_1 \dots x_n x_{3n+1} \dots x_{4n} \rangle \langle x_{2n+1}, \dots, x_{3n} \rangle}_{\mathcal{I}_1} + \underbrace{\langle x_{n+1} \dots x_{3n} \rangle \langle x_{3n+1}, \dots, x_{4n} \rangle}_{\mathcal{I}_2}$$

$$+ \underbrace{\langle x_1 x_{2n+1}, \dots, x_n x_{3n} \rangle \langle x_{n+1} x_{3n+1}, \dots, x_{2n} x_{4n}}_{\mathcal{I}_3} .$$

We note $Z(\mathcal{I}) = Z(\mathcal{I}_1) \cap Z(\mathcal{I}_2) \cap Z(\mathcal{I}_3)$. This means we can express $Z(\mathcal{I})$ via the eight components in the following table:

Component name	Component description
C_1	$x_1 \dots x_n x_{3n+1} \dots x_{4n} = 0$, $x_{n+1} \dots x_{3n} = 0$ and $x_i x_{2n+i} = 0 \ \forall 1 \leq i \leq n$
C_2	$x_1 \dots x_n x_{3n+1} \dots x_{4n} = 0$, $x_{n+1} \dots x_{3n} = 0$ and $x_j x_{2n+j} = 0 \ \forall n+1 \leq j \leq 2n$
C_3	$x_{2n+1}, \dots, x_{3n} = 0$
C_4	$x_{2n+1}, \dots, x_{3n} = 0$ and $x_j x_{2n+j} = 0 \ \forall n+1 \leq j \leq 2n$
C_5	$x_{3n+1}, \dots, x_{4n} = 0$, $x_j x_{2n+j} = 0 \ \forall n+1 \leq j \leq 2n$
C_6	$x_{3n+1}, \dots, x_{4n} = 0$
C_7	$x_{2n+1}, \dots, x_{4n} = 0$
C_8	$x_{2n+1}, \dots, x_{4n} = 0$

We note that the following chains of containments exist $C_4 \subseteq C_3 \supseteq C_7 = C_8 \subseteq C_6 \supseteq C_5$, so the components we need to consider are C_1, C_2, C_3 and C_6. The space ∂w is the zero locus of the partial derivatives of w, and thus $\partial w \subseteq Z(\frac{\partial w}{\partial u_1}, \frac{\partial w}{\partial u_2}) = Z(f_1, f_2) = Z(w) \subseteq U_q$. Let us examine the points in ∂w by distinguishing 3 cases:

- $u_1, u_2 \neq 0$,
- One of u_1, u_2 is 0,
- $u_1, u_2 = 0$.

In the first case, consider the product $\prod_{i=2n+1}^{3n} \frac{\frac{\partial w}{\partial x_i}}{u_1}$. We obtain

$$n^n x_1^n \dots x_n^n x_{2n+1}^{n-1} \dots x_{3n}^{n-1} = \lambda^n x_{n+1}^n \dots x_{2n}^n x_{2n+1}^{n-1} \dot{x}_{3n}^{n-1}. \tag{4.18}$$

If $x_{2n+1} \dots x_{3n} \neq 0$, then this yields

$$n^n (x_1 \dots x_n)^n = \lambda^n (x_{n+1} \dots x_{2n})^n. \tag{4.19}$$

Similarly, if $x_{3n+1} \dots x_{4n} \neq 0$, we obtain

$$n^n (x_{n+1} \dots x_{2n})^n = \lambda^n (x_1 \dots x_n)^n. \tag{4.20}$$

These two equations combine to give $x_1 \dots x_n = x_{n+1} \dots x_{2n} = 0$ or $n^{2n} = \lambda^{2n}$, which we excluded in the statement of the Theorem 4.2.1. This gives us three cases to distinguish

- $\underline{x_{2n+1} \dots x_{3n} = 0.}$ By considering $\frac{\partial w}{\partial x_{n+1}}, \dots, \frac{\partial w}{\partial x_{2n}}$ and $nu_2 \neq 0$, we obtain $x_{n+1}^{n-1} x_{3n+1}^n = \dots = x_{2n}^{n-1} x_{4n}^n = 0$. That is equivalent to

  $$x_{n+1} x_{3n+1} = \dots = x_{2n} x_{4n} = 0. \tag{4.21}$$

  Similarly, using $\frac{\partial w}{\partial x_{2n+1}}, \dots, \frac{\partial w}{\partial x_{3n}}$, we obtain

  $$x_1 x_{2n+1} = \dots = x_n x_{3n} = 0. \tag{4.22}$$

  But then by $\frac{\partial w}{\partial u_2}$, we have $\lambda x_1 \dots x_n x_{3n+1} \dots x_{4n} = 0$. Since $\lambda \neq 0$, we have $x_1 \dots x_n x_{3n+1} \dots x_{4n} = 0$. Also, as $x_{2n+1} \dots x_{3n} = 0$ by assumption in this case, we have $x_{n+1} \dots x_{3n} = 0$. These two conditions with (4.22) however imply that any such point lies in C_1.

- $\underline{x_{3n+1} \dots x_{4n} \neq 0.}$ This case is similar to the case above.

- $\underline{x_1 \dots x_n = x_{n+1} \dots x_{2n} = 0,\ x_{2n+1} \dots x_{3n} \neq 0,\ x_{3n+1} \dots x_{4n} \neq 0.}$ Considering the partial derivatives $\frac{\partial w}{\partial x_i}$ for $2n+1 \leq i \leq 4n$ gives $x_1 x_{2n+1} = \dots = x_n x_{3n} = 0$ and $x_{n+1} x_{3n+1} = \dots = x_{2n} x_{4n} = 0$. Together with the fact that $x_1 \dots x_n x_{3n+1} \dots x_{4n} = 0$ and $x_{n+1} \dots x_{3n} = 0$ (obtained by conditions of this case), we conclude that any point fitting this case lies in C_1 or C_2.

This concludes the discussion of the case where $u_1, u_2 \neq 0$, as all such points would be contained in the exceptional locus $Z(\mathcal{I})$ and hence not in the variety $X_{\mathcal{S}}$ associated to the subdivision $\mathcal{S}$.

Next we discuss the points in ∂w for which exactly one of u_1, u_2 is 0. Without loss of generality, $u_1 = 0, u_2 \neq 0$.

By $\frac{\partial w}{\partial x_i}$ for $1 \leq i \leq n$, we obtain

$$0 = -\lambda x_2 \ldots x_n x_{3n+1} \ldots x_{4n} = \cdots = -\lambda x_1 \ldots x_{n-1} x_{3n+1} \ldots x_{4n}, \tag{4.23}$$

and by $\frac{\partial w}{\partial x_j}$ for $n+1 \leq j \leq 2n$, we obtain

$$0 = x_{n+1} x_{3n+1} = \cdots = x_{2n} x_{4n}. \tag{4.24}$$

Furthermore, considering $\frac{\partial w}{\partial x_{2n+j}}$ for $n+1 \leq j \leq 2n$, we obtain

$$0 = -\lambda x_1 \ldots x_n x_{3n+2} \ldots x_{4n} = \cdots = -\lambda x_1 \ldots x_n x_{3n+1} \ldots x_{4n-1}. \tag{4.25}$$

This means the point lies in C_2 unless $x_{3n+1} = \cdots = x_{4n} = 0$ and $x_{n+1}, \ldots, x_{2n} \neq 0$. Indeed, otherwise $x_1 \ldots x_n x_{3n+1} \ldots x_{4n} = 0$ by (4.23), so to not be in C_2, we would require $x_{n+1} \ldots x_{3n} \neq 0$ (since (4.23), (4.24) and $x_{n+1} \ldots x_{3n} = 0$ is the description of C_2). Then (4.24) and $x_{n+1} \ldots x_{3n} \neq 0$ imply $x_{3n+1} = \cdots = x_{4n} = 0$ and $x_{n+1}, \ldots, x_{2n} \neq 0$. But then the point is contained in C_6.

Hence all points in ∂w with $u_1 = 0, u_2 \neq 0$ are in the exceptional locus $Z(\mathcal{I})$, and hence not in the variety $X_{\mathcal{S}}$ associated to $\mathcal{S}$. The case where $u_1 \neq 0, u_2 = 0$ is analogous.

This leaves us with the points of ∂w which have $u_1 = u_2 = 0$. Thus, $\partial w \cap X_{\mathcal{S}} \subseteq Z(u_1, u_2)$. In particular, $[\partial w] \subseteq [Z(f_1, f_2, u_1, u_2)] \subseteq [Y_\nabla \times \{0\}^2] \subseteq [Y_\nabla \times \mathbb{C}^2]$. Note that ∂w is closed in the Euclidian topology as it is a Zariski closed set. After refining $\mathcal{S}$ to a regular triangulation corresponding to a toric variety $\mathbb{P}_\nabla / G_n$, we obtain $[\partial w] \subseteq [Z(f_1, f_2, u_1, u_2)] \subseteq [\mathbb{P}_\nabla / G_n \times \{0\}^2] \subseteq [\mathbb{P}_\nabla / G_n \times \mathbb{C}^2]$. We note that this is the total space of a vector bundle over a toric variety with fan $\Sigma_{BB,n}$. This is a complete fan, and thus $\mathbb{P}_\nabla / G_n$ is compact in the Euclidian topology. Hence $[\partial w]$ is a closed subset of $\mathbb{P}_\nabla / G_n \times \{0\}^2$, which is compact. Thus, $[\partial w]$ is compact, which implies properness over $\operatorname{Spec} \mathbb{C}$.

As in the proof of Theorem 4.1.5, this shows that $\mathrm{D}^{\mathrm{abs}}([U_q, G, w])$ is homologically smooth and dg-proper, concluding the proof of Theorem 4.2.12. $\square$

Step 3: $\mathrm{D}^{\mathrm{abs}}([U_q, G, w]) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_n)$

We will apply Proposition 4.1.11 to obtain $\mathrm{D}^{\mathrm{abs}}([U_q, G, w]) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_n)$. Choosing $\Psi = \Sigma_{BB,n}$, we note that D'_a, D'_b are nef. We also note that Σ_{BB,n,D'_a,D'_b} is Gorenstein with respect to the element $\mathfrak{m} = (0, \ldots, 0, 1, 1)$. We need to make sure the Proposition is applicable, i.e. we need to check the conditions which are set up in § 4.1 before the Proposition 4.1.11.

- $X_{\Sigma_{BB,n,D'_a,D'_b}}$ is semi-projective;
- $|\Sigma_{BB,n,D'_a,D'_b}| = |\Psi_{-D'_a,\ldots,-D'_b}|$;
- For any $\delta \in \Sigma_{BB,n,D'_a,D'_b}(1)$, we have $u_\delta \in \nu$ and $\langle \mathfrak{m}, u_\delta \rangle = 1$;

These three conditions are all immediate by construction. Further, we need to check that functions $f_{1,n,\lambda}, f_{2,n,\lambda}$ extracted from the superpotential define a complete intersection in the toric variety associated to the chamber σ_q. For that purpose, it is sufficient to check the Jacobian has full rank, noting that in the chamber σ_q the functions take the form $f_{1,n,\lambda} = x_1^n x_{2n+1}^n + \cdots + x_n^n x_{3n}^n - \lambda x_{n+1} \cdot \cdots \cdot x_{3n}$ and $f_{2,n,\lambda} = x_{n+1}^n x_{3n+1}^n + \cdots + x_{2n}^n x_{4n}^n - \lambda x_1 \cdot \cdots \cdot x_n \cdot x_{3n+1} \cdot \cdots \cdot x_{4n}$.

Hence $\frac{\partial f_{1,n,\lambda}}{\partial x_j} = 0$ for $3n+1 \leq j \leq 4n$ and $\frac{\partial f_{2,n,\lambda}}{\partial x_k} = 0$ for $2n+1 \leq k \leq 3n$. For the Jacobian to not have full rank, there needs to be a linear dependence between the two row vectors $\left(\frac{\partial f_{i,n,\lambda}}{\partial x_j}\right)_{j=1}^{4n}$, $i = 1, 2$. Because of the above partial derivatives being zero, we obtain the following equations:

$$
\begin{aligned}
&nx_1^n x_{2n+1}^{n-1} - \lambda x_{n+1} \dots x_{2n} x_{2n+2} \dots x_{3n} = 0, \\
&\qquad\vdots \\
&nx_n^n x_{3n}^{n-1} - \lambda x_{n+1} \dots x_{3n-1} = 0, \\
&nx_{n+1}^n x_{3n+1}^{n-1} - \lambda x_1 \dots x_n x_{3n+2} \dots x_{4n} = 0, \\
&\qquad\vdots \\
&nx_{2n}^n x_{4n}^{n-1} - \lambda x_1 \dots x_n x_{3n+1} \dots x_{4n-1} = 0.
\end{aligned}
\tag{4.26}
$$

We thus have $nx_1^n x_{2n+1}^n = \cdots = nx_n^n x_{3n}^n = \lambda x_{n+1} \dots x_{3n}$. Indeed, we get

$$n^n x_1^n \dots x_n^n x_{2n+1}^n \dots x_{3n}^n = \lambda^n x_{n+1}^n \dots x_{3n}^n.$$

If $x_{2n+1} \dots x_{3n} \neq 0$, then $n^n(x_1 \dots x_n)^n = \lambda^n (x_{n+1} \dots x_{2n})^n$. Similarly, if $x_{3n+1} \dots x_{4n} \neq 0$, we also have $\lambda^n (x_1 \dots x_n)^n = n^n (x_{n+1} \dots x_{2n})^n$, which implies $n^{2n} = \lambda^{2n}$. But this is excluded by the conditions of Theorem 4.2.1.

Thus we require $x_{2n+1} \cdot \dots \cdot x_{3n} = 0$ or $x_{3n+1} \cdot \dots \cdot x_{4n} = 0$ for the Jacobian to not have full rank. By the same arguments as in Step 2, this is not possible (the zero loci of $x_{2n+1} \dots x_{3n} = 0$ and $x_{3n+1} \dots x_{4n} = 0$ are both in the exceptional locus of the chamber).

Step 4: There is a categorical resolution of $\mathrm{D^b}(\mathrm{coh}\, Z_n)$ by $\mathrm{D^{abs}}([U_p, G, w])$

We have, by the previous steps, that $\mathrm{D^{abs}}([U_p, G, w]) \cong \mathrm{D^{abs}}([U_q, G, w]) \cong \mathrm{D^b}(\mathrm{coh}\, Y_n)$. In particular $\mathrm{D^{abs}}([U_p, G, w])$ is homologically smooth and dg-proper by Step 2. Thus, we can use the proof of Theorem 4.1.5 to obtain a categorical resolution of $\mathrm{D^b}(\mathrm{coh}\, Z_n)$ by $\mathrm{D^{abs}}([U_p, G, w])$, concluding the proof to Theorem 4.2.1. $\square$

Remark 4.2.13. The above proof quite straightforwardly generalises to weighted projective space (i.e. taking the appropriate 2 polynomials in $W\,\mathbb{P}(q_1, \dots, q_n, q_1, \dots, q_n)$).

4.3 A result on categorical resolutions of toric complete intersections

In this section, we will give a more general result on the existence of crepant categorical resolutions for complete intersections in toric varieties fulfilling a certain condition. The condition that needs to be fulfilled will be further examined, and Theorem 4.2.1 will be framed as a special case.

Definition 4.3.1. We say a cone $\sigma \subseteq M_{\mathbb{R}} \times \mathbb{R}^r$ has property $(\mathbf{r} - \mathbf{P})$ if there exists a simplicial fan Σ' in a $(\dim M_{\mathbb{R}})$-dimensional sublattice of $N_{\mathbb{R}} \times \mathbb{R}^r$ and torus-invariant, effective, nef basepoint free divisors $D_1, \dots, D_r$ on $X_{\Sigma'}$ so that there is a fan Ξ where $X_\Xi = \operatorname{tot}(\oplus_{i=1}^r \mathcal{O}_{X_{\Sigma'}}(-D_i))$ and the following properties hold:

- $\sigma^\vee = |\Xi|$
- X'_Σ is Gorenstein Fano;
- for all $\rho \in \Sigma'_{-D_1,\dots,-D_r}(1)$ we have that $u_\rho \in H_{\deg^\vee}(1)$;
- $\sum_{i=1}^r D_i = -K_{X_{\Sigma'}}$.

This property seems unwieldy at first, but in fact we can relate it to different properties of cones that are more well-studied in the context of mirror symmetry and which have been introduced in §2.

Before we relate the property $(\mathbf{r} - \mathbf{P})$ to other properties, we first state the following two results, due to Batyrev and Nill [4].

Lemma 4.3.2 (Corollary 3.7 in [4]). *A reflexive Gorenstein cone σ is associated to a nef-partition if and only if both cones σ and $\sigma^\vee$ are completely split. A Gorenstein polytope $\tilde{\Delta}$ of index r is the Cayley polytope of a nef-partition if and only if $\tilde{\Delta}$ and $\tilde{\Delta}^\vee$ are Cayley polytopes of length r.*

Lemma 4.3.3. *Let $\tilde{\Delta}$ be a Gorenstein polytope such that both $\tilde{\Delta}$ and $\tilde{\Delta}^\vee$ are integrally closed. Then $\tilde{\Delta}$ (and also $\tilde{\Delta}^\vee$) is a Cayley polytope associated to a nef partition.*

We now formulate the following Proposition, giving us an explicit class of cones with the property $(\mathbf{r}-\mathbf{P})$.

Proposition 4.3.4. *If σ and $\sigma^\vee$ are both reflexive completely split Gorenstein cones of index r, then σ has the property $(\mathbf{r}-\mathbf{P})$.*

Proof. As σ is completely split, it is a Cayley cone of r lattice polytopes $\Delta_1,\dots,\Delta_r$. By Lemma 4.3.2, these lattice polytopes form a nef-partition if and only if $\sigma^\vee$ is also completely split. Since this was the assumption of the Proposition, we obtain that the Δ_i form a nef-partition. By the same result, $\sigma^\vee$ is associated to a nef-partition $\nabla_1+\cdots+\nabla_r=\nabla$, i.e. $\sigma^\vee=|\Sigma_{\nabla,-D_1,\dots,-D_r}|$ where D_i is the divisor on X_∇ defined by ∇_i and Σ_∇ is the normal fan to ∇. As ∇_i is a nef-partition, ∇ is reflexive and thus the associated variety is Gorenstein Fano. By the structure of σ as Cayley polytope $\Delta_1*\cdots*\Delta_r$ for a nef-partition Δ_i, we note that the unique interior point of the support $\sigma_{(r)}$ is $e_1+\cdots+e_r$ and thus for all $\rho\in\Sigma_{\nabla,-D_1,\dots,-D_r}(1)$, we indeed have $u_\rho\in H_{\deg^\vee}(1)$. This proves that σ has the property $(\mathbf{r}-\mathbf{P})$, as required. $\square$

Conjecture 4.3.5. *If σ is a reflexive completely split Gorenstein cone of index r, then σ has property $(\mathbf{r}-\mathbf{P})$.*

In our treatment of categorical resolutions, we will rely on the following result.

Proposition 4.3.6. *Let X_Σ be a projective toric variety. Suppose there is a collection of divisors D_i on X_Σ such that $\Xi:=\Sigma_{-D_1,\dots,-D_r}$ is a simplicial fan and there are functions $f_i\in\Gamma(X_\Sigma,\mathcal{O}(D_i))$ so that $Z(f_1,\dots,f_r)$ is a smooth complete intersection. Write $X_\Xi=[U_\Xi/G_\Xi]$. Then consider the G_Ξ-function*

$$w:=u_if_i:U_\Xi\to\mathbb{C}\,.$$

Then $[\partial w/G_\Xi]\subseteq[U_\Xi/G_\Xi]$ is proper over $\operatorname{Spec}\mathbb{C}$ and $\partial w\subseteq Z(w)$ in U_Ξ.

Proof. Note that $X_\Xi=\mathrm{tot}(\bigoplus_{i=1}^r\mathcal{O}_{X_\Sigma}(-D_i))$, which is locally isomorphic to $X_\Sigma\times\mathbb{C}^r$ (since it is the total space of a vector bundle). Consider a point $\mathbf{p}\in\partial w$. Then there is an open patch U_σ corresponding to some maximal cone σ of Ξ such that $\mathbf{p} \in U_\sigma$. Let $\mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_r]$ be the corresponding coordinate ring of U_σ, where the u_j are the coordinates of the $\mathbb{C}^r$ component.

We will first prove that $\mathbf{p} \in Z(w) \cap U_\sigma$, and hence that $\partial w \cap U_\sigma \subseteq Z(w) \cap U_\sigma$.

$$\partial w = Z\left(\{\frac{\partial w}{\partial x_i} \mid 1 \leq i \leq n\} \cup \{\frac{\partial w}{\partial u_j} | 1 \leq j \leq r\}\right) \subseteq Z\left(\{\frac{\partial w}{\partial u_j} | 1 \leq j \leq r\}\right) = Z(f_1, \dots, f_r) \subseteq Z(w).$$

The last step follows as $f_i(p) = 0$ for some point p, $1 \leq i \leq n$, then $\sum u_i f_i(p) = 0$ and so $p \in Z(w)$.

Next, we will prove that $[\partial w / G_\Xi] \subseteq [U_\Xi / G_\Xi]$ is proper over $\operatorname{Spec} \mathbb{C}$. By Remark 4.1.3, it is sufficient that $[\partial w / G_\Xi]$ is compact with respect to the usual topology. Express $\mathbf{p}$ in local coordinates as $\mathbf{p} = (p_1, \dots, p_n, y_1, \dots, y_r) \in \partial w \cap U_\sigma$. Noting that the f_k are functions of the x_i only, we have

$$\begin{aligned}
0 &= \frac{\partial w}{\partial x_i}(\mathbf{p}) = \sum_{k=1}^{r} u_k \frac{\partial f_k}{\partial x_i}(\mathbf{p}) \\
\Leftrightarrow 0 &= \sum_{k=1}^{r} y_k \frac{\partial f_k}{\partial x_i}((p_1, \dots, p_n)), \qquad (4.27) \\
0 &= \frac{\partial w}{\partial u_j}(\mathbf{p}) \\
\Leftrightarrow 0 &= f_j((p_1, \dots, p_n)). \qquad (4.28)
\end{aligned}$$

For $1 \leq i \leq r$, denote by $\mathbf{v}_k$ the vector

$$\mathbf{v}_k = \begin{pmatrix} \frac{\partial f_k}{\partial x_1} \\ \vdots \\ \frac{\partial f_k}{\partial x_n} \end{pmatrix}.$$

This vector $\mathbf{v}_k$ should be viewed as a function of $x_1, \dots, x_n$. Note that the Jacobian J of

$Z(f_1,\dots,f_r) \subseteq X_\Sigma$ is given by the $n\times r$ matrix $(\mathbf{v}_1 \quad \dots \quad \mathbf{v}_r)$. Now we have

$$\begin{aligned}
&\sum_{k=1}^{r} y_k \mathbf{v}_i((p_1,\dots,p_n)) \\
&= \sum_{k=1}^{r} y_k \begin{pmatrix} \frac{\partial f_k}{\partial x_1}((p_1,\dots,p_n)) \\ \vdots \\ \frac{\partial f_k}{\partial x_n}((p_1,\dots,p_n)) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=1}^{r} y_k \frac{\partial f_k}{\partial x_1}((p_1,\dots,p_n)) \\ \vdots \\ \sum_{k=1}^{r} y_k \frac{\partial f_k}{\partial x_n}((p_1,\dots,p_n)) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\end{aligned} \tag{4.29}$$

Here, the last step follows by applying (4.27). By (4.28), $(p_1,\dots,p_n)\in Z(f_1,\dots,f_r)$. But then (4.29) gives a linear dependence between the vectors $\mathbf{v}_k((p_1,\dots,p_n))$, unless $y_k = 0$ for all k. Hence, unless $y_k = 0$ for $1\le k\le r$, we have that the Jacobian J does not have full rank at $(p_1,\dots,p_n)\in Z(f_1,\dots,f_r)$. But $Z(f_1,\dots,f_r)$ is a complete intersection, so of codimension $n-r$ in X_Σ. Its Jacobian not having full rank at a point $(p_1,\dots,p_n)$ is thus equivalent to the point $(p_1,\dots,p_n)$ being a singular point of $Z(f_1,\dots,f_r)$. Since we assumed $Z(f_1,\dots,f_r)$ to be smooth, this would be a contradiction. Therefore, if $(p_1,\dots,p_n,y_1,\dots,y_r)\in \partial w\cap U_\sigma$, then $y_k = 0$ for $1\le k\le r$.

We have therefore shown that $\partial w \cap U_\sigma \subseteq Z(f_1,\dots,f_r)\times\{0\}^r \subseteq \mathbb{C}^n\times\mathbb{C}^r$. Thus, $U_\sigma\cap \partial w/G_\Xi \subseteq X_\Sigma\times\{0\}^r\subseteq U_\sigma/G_\Xi$. Since X_Σ is projective, it is in particular compact. Therefore, $[U_\sigma\cap\partial w/G_\Xi]$ is a closed subset of a compact variety $[X_\Sigma\times\{0\}^r]$ and hence compact itself. Thus $\partial w = \bigcup_{\sigma\in\Xi(max)} \partial w\cap U_\sigma$ is a finite union of compact varieties, hence compact itself. By Remark 4.1.3, we have that $[\partial w/G_\Xi]\subseteq[U_\Xi/G_\Xi]$ is proper over $\operatorname{Spec}\mathbb{C}$. This completes the proof of Proposition 4.3.6.

□

Let X_Σ be a projective toric variety with vector bundle $\mathcal{V} = \oplus_{i=1}^r \mathcal{O}(D_i)$ so that $D_1 + \cdots + D_r = -K_{X_\Sigma}$ such that the divisors D_i partition the torus-invariant Weil divisors associated to the rays of Σ. $\mathcal{V}$ corresponds to a fan $\Sigma_\mathcal{V}$ in $N_\mathbb{R} \times \mathbb{R}^r$. Let $\sigma_\mathcal{V} = |\Sigma_\mathcal{V}|$. Let e_i be the generators of the $\mathbb{R}^r$ component in $M_\mathbb{R} \times \mathbb{R}^r$ (and by abuse of notation also for $N_\mathbb{R} \times \mathbb{R}^r$). Let the elements of the total coordinate ring of $X_{\Sigma_\mathcal{V}}$ be x_ρ for $\rho \in \Sigma_\mathcal{V}(1)$ apart from the variables associated to the e_i, which we will call u_i.

We recall that points m' in $\sigma_\mathcal{V}^\vee$ correspond to global functions, and hence to sections of the dual, via $x^{m'} = \prod_{\rho \in \Sigma_\mathcal{V}} x_\rho^{\langle m', u_\rho \rangle}$.

For each $1 \leq i \leq r$, we consider the intersection $H_i = \sigma_\mathcal{V}^\vee \cap \{e_i = 1, e_j = 0 | j \neq i\} \subseteq M_\mathbb{R} \times \mathbb{R}^r$. Any lattice point m is H_i is of the form $m' + e_i$ where $m' \in M$. It corresponds to a family of global functions of the form $c_m x^m$. We note that, by construction, $u_i \mid x^m, u_i^2 \nmid x^m$ and $u_j \nmid x^m$ for $j \neq i$. Thus, a subset of lattice points $A_i \subseteq H_i \cap M \times \mathbb{Z}^r$ can be associated to a family of global functions of the form $u_i s_i = \sum_{m \in A_i} c_m x^m, c_m \neq 0$. We note that $u_j \nmid s_i$ for $1 \leq j \leq r$ and that $s_i \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_i))$, by construction.

Denote by A the union of the sets A_i. Thus $A = \{m' \in M_\mathbb{R} \times \mathbb{R}^r \mid x^{m'}$ is nontrivial summand of som
Now each $s_i = \sum_{m \in A_i} c_m x^m$ defines a family of sections of $\mathcal{O}_{X_\Sigma}(D_i)$ for $1 \leq i \leq r$.

By construction, m' is of the form $m + e_i, m \in M_\mathbb{R}$ for some i. In particular, since there is no summands of the form $u_i u_j s$, we have

$$\langle m', e_1 + \cdots + e_r \rangle = 1. \tag{4.30}$$

Here, by abuse of notation, the e_i also denote the generators of the $\mathbb{R}^r$ component of $\mathbb{N}_\mathbb{R} \times \mathbb{R}^r$. Denote by $\overline{A}$ the set

$$\overline{A} = \{m \in M_\mathbb{R} \mid m + e_i \in A \text{ for some } i\}.$$

Further, for $i = 1, \ldots, r$, define

$$\overline{A}_i = \{m \in M_\mathbb{R} \mid m + e_i \in A\}.$$

We define $S = A \cup \{e_i \mid i = 1, \dots, r\}$ and let

$$\sigma_S := \mathrm{Cone}(\mathrm{Conv}(S)) \subseteq \sigma_{\mathcal{V}}^{\vee}.$$

If the cone σ_S has the property $(\mathbf{r}-\mathbf{P})$, then we note that we have a toric vector bundle $X_\Xi = \mathrm{tot}(\bigoplus_{i=1}^r \mathcal{O}_{X_{\Sigma'}}(-T_i))$ for a projective fan Σ' with divisors T_i such that the conditions in Definition 4.3.1 hold. In particular, $\sigma_S^\vee = |\Xi|$ and so $f_i = \sum_{m \in \overline{A}_i} c_m x^m$ gives a family of sections of $\Gamma(X_{\Sigma'}, \mathcal{O}_{X_{\Sigma'}}(T_i))$. Consider a choice of coefficients $c_m \in \mathbb{C}^*$ such that the section f_i is generic for $1 \le i \le r$. As the divisors T_i are basepoint free, we can apply Bertini's theorem to obtain a smooth complete intersection $Z_{res} := Z(f_1, \dots, f_r)$ in $X_{\Sigma'}$. We consider the complete intersection Z in X_Σ given by $Z := Z(s_1, \dots, s_r)$ for the choice of coefficients c_m above and obtain the following result.

Theorem 4.3.7. *Let $X_\Sigma, \mathcal{V}$, and σ_S be as above. If σ_S has the property $(\mathbf{r}-\mathbf{P})$, consider Z and Z_{res} as above. Then there is a categorical resolution of $\mathrm{D^b}(\mathrm{coh}\, Z)$ by the derived category $\mathrm{D^b}(\mathrm{coh}\, Z_{res})$ associated to the complete intersection Z_{res} in the toric stack $[X_{\Sigma'}]$.*

Proof. Recall that $\sigma_S \subseteq \sigma_{\mathcal{V}}^\vee$, and so

$$|\Xi| = \sigma_S^\vee \supseteq ((\sigma_{\mathcal{V}})^\vee)^\vee = \sigma_{\mathcal{V}} = |\Sigma_{\mathcal{V}}|.$$

Consider the primitive generators of rays in $\Sigma'(1)$. Each primitive generator can be interpreted as a point, and we denote these points by $\nu_1, \dots, \nu_k$. Further, consider the primitive generators of rays in $\Sigma_{\mathcal{V}}(1)$, and denote the corresponding points by $\nu_1', \dots, \nu_m'$. Let L be the set $\{\nu_i\} \cup \{\nu_j'\}$. Then regular triangulations of $\mathrm{Conv}(L)$ correspond to chambers in a GKZ fan Σ_{GKZ} which is the secondary fan associated to the variety X_Ω, where Ω is a common refinement of Ξ and $\Sigma_{\mathcal{V}}$.

Then there is a chamber σ_{res} corresponding to X_Ξ in Σ_{GKZ}. Indeed, there is a regular triangulation $\mathcal{T}_{res}$ of $\mathrm{Conv}(\{\nu_i\}_{i=1}^k)$ which corresponds to X_Ξ in the secondary fan of X_Ξ simply by performing a star subdivision on the r "central" rays $(0, \dots, 0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 0, \dots, 0, 1)$. Now, as $|\Xi| \supseteq |\Sigma_{\mathcal{V}}|$, we have $\mathrm{Conv}(\{\nu_i\}_{i=1}^k) = \mathrm{Conv}(L)$, so the triangulation $\mathcal{T}_{res}$ giving X_Ξ in its own secondary fan is also a triangulation of $\mathrm{Conv}(L)$. The chamber in Σ_{GKZ} associated to this triangulation $\mathcal{T}_{res}$ thus still corresponds to X_Ξ and we denote it by σ_{res}. Its associated open affine is called U_{res}.

We claim that there is a chamber in Σ_{GKZ} which corresponds to a partial compactification of $\mathcal{V}$. Firsty, note that there is a regular triangulation $\mathcal{T}_0$ of $\{\nu'_j\}_{j=1}^m$ which corresponds to the vector bundle $\mathcal{V}$ in its own GKZ fan. Therefore, there is a weight function w_0 defined on $\{\nu'_j\}$ giving that triangulation when projecting the lower facets of $\mathrm{Conv}((\nu'_j, w_0(\nu'_j))_{j=1}^m)$. We now extend this weight function w_0 on $\mathrm{Conv}(\{\nu'_j\}_{j=1}^m)$ to a weight function w_1 on L by setting $w_1(\nu'_j) := w_0(\nu'_j)$ for $j = 1, \ldots, m$ and $w_1(v)$ to be big enough for $v \in L \setminus \{\nu'_j\}_{j=1}^m$. Projecting the lower facets of the polytope $\mathrm{Conv}((v, w_1(v)|v \in L)$ gives a regular subdivision $\mathcal{S}$ of L which contains the triangulation $\mathcal{T}_0$ of $\{\nu'_j\}_{j=1}^m$. Refining to a regular triangulation $\mathcal{T}_1$ can then be seen as an extension of $\mathcal{T}_0$. We denote by σ_{pc} the chamber of Σ_{GKZ} corresponding to $\mathcal{T}_1$, with associated open affine U_{pc}. By construction, the variety associated to σ_{pc} is a partial compactification of $\mathcal{V}$.

Note that the superpotential w takes the form $w = \sum_{i=1}^r u_i f_i$. As the cone $|\Xi|$ is Gorenstein with respect to the element $(0, \ldots, 0, 1, \ldots, 1)$, we can apply Theorem 4.1.10 and obtain that $\mathrm{D}^{\mathrm{abs}}([U_{res}, G, w]) \simeq \mathrm{D}^{\mathrm{abs}}([U_{pc}, G, w])$.

Next, we claim that $\mathrm{D}^{\mathrm{abs}}([U_{\sigma_{res}}, G, w]) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z_{res})$. To prove this claim, we want to apply Proposition 4.1.11. For this, we check the conditions outlined in § 4.1.1 (the conditions are listed before the statement of Proposition 4.1.11). Firstly X_Ξ is semiprojective by construction, and $|\Xi| = |\Xi|$ is tautological. $\langle \mathfrak{m}, u_\delta \rangle = 1$ is also true for all rays δ, as the cone is Gorenstein with respect to $\mathfrak{m}$. Finally, the f_i defining a complete intersection follows from the smoothness of the chosen section via Bertini. By Proposition 4.1.11, we obtain that $\mathrm{D}^{\mathrm{abs}}([U_{\sigma_{res}}, G, w]) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z_{res})$.

Now, by Proposition 4.3.6 and the proof of Theorem 4.1.5, $\mathrm{D}^{\mathrm{abs}}([U_{res}, G, w]) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z_{res})$ is homologically smooth and dg-proper, as $[\partial w/G]$ is proper and $\partial w \subseteq Z(w)$.

We now have

$$\mathrm{D}^{\mathrm{abs}}([U_{pc}, G, w]) \simeq \mathrm{D}^{\mathrm{abs}}([U_{res}, G, w]) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z_{res}),$$

and thus $\mathrm{D}^{\mathrm{abs}}([U_{pc}, G, w])$ is homologically smooth and dg-proper. By the proof of Theorem 3.7 in [19], $\mathrm{D}^{\mathrm{abs}}([U_{pc}, G, w]) \longleftrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z)$ is thus a categorical resolution and we have a categorical resolution

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z_{res}) \longleftrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z),$$

as required. □

Remark 4.3.8. If σ_S is the cone over the fan of an appropriate toric vector bundle itself, the construction in the proof corresponds to a Batyrev-Borisov mirror.

Next, we examine when the cone σ_S has the required property $(\mathbf{r} - \mathbf{P})$. This will allow us to make the Theorem applicable. In particular, we will see that existence of a crepant categorical resolution in Theorem 4.2.1 is a consequence of Theorem 4.3.7. We start by assuming that Conjecture 4.3.5 holds. Then we obtain the following result.

Conjecture 4.3.9. *If* $\operatorname{Conv}(S)$ *is a Gorenstein polytope of index* r*, then* σ_S *has property* $(\mathbf{r} - \mathbf{P})$.

Sketch of a proof. Using Proposition 4.3.5, it is sufficient to show that σ_S is reflexive completely split Gorenstein.

By (4.30), σ_S is Gorenstein with Gorenstein element $n_{\sigma_S} = e_1 + \cdots + e_r$. Applying Proposition 2.1.27, we find that σ_S being reflexive of index k is equivalent to $(\sigma_S)_{(1)}$ being a Gorenstein polytope of index k. But $(\sigma_S)_{(1)} = \operatorname{Conv}(S)$, which by assumption is a Gorenstein polytope of index r. Thus σ_S is a reflexive Gorenstein cone of index r.

We note that all elements of S are of the form $m + e_i$ for some $m \in \overline{A}_i \cup \{0\}$. Thus, $(e_1, \operatorname{Conv}(\overline{A}_i \cup \{0\})), \ldots, (e_r, \operatorname{Conv}(\overline{A}_r \cup \{0\}))$ generate σ_S. Letting $\Delta_i = \operatorname{Conv}(\overline{A}_i \cup \{0\})$, we have $\sigma_S = \operatorname{Cone}(\Delta_1 * \cdots * \Delta_r)$. Since the index of σ_S is r, the cone σ_S is therefore completely split. Therefore, σ_S is a reflexive completely split Gorenstein cone and thus, by Conjecture 4.3.5, has property $(\mathbf{r}-\mathbf{P})$. $\square$

Combining Conjecture 4.3.9 and Theorem 4.3.7 gives the following Corollary.

Conjecture 4.3.10. *If* $\mathrm{Conv}(S)$ *is a Gorenstein polytope of index* r*, then there is a categorical resolution of* $\mathrm{D^b}(\mathrm{coh}\, Z)$ *given by a Batyrev-Borisov mirror.*

Should Conjecture 4.3.5 not hold, we still obtain the following result.

Lemma 4.3.11. *If* $\mathrm{Conv}(S)$ *is a Gorenstein polytope of index* r *such that* $\mathrm{Conv}(S)$ *and* $\mathrm{Conv}(S)^\vee$ *are integrally closed, then* σ_S *has property* $(\mathbf{r}-\mathbf{P})$*.*

Proof. Using Proposition 4.3.4, it is sufficient to show that σ_S and $\sigma_S^\vee$ reflexive completely split Gorenstein.

By (4.30), σ_S is Gorenstein with Gorenstein element $n_{\sigma_S} = e_1 + \cdots + e_r$. Applying Proposition 2.1.27, we find that σ_S being reflexive of index k is equivalent to $(\sigma_S)_{(1)}$ being a Gorenstein polytope of index k. But $(\sigma_S)_{(1)} = \mathrm{Conv}(S)$, which by assumption is a Gorenstein polytope of index r. Thus σ_S is a reflexive Gorenstein cone of index r, and so is $\sigma_S^\vee$ by definition of reflexivity.

We note that all elements of S are of the form $m + e_i$ for some $m \in \overline{A}_i \cup \{0\}$. Thus, $(e_1, \mathrm{Conv}(\overline{A}_i \cup \{0\})), \ldots, (e_r, \mathrm{Conv}(\overline{A}_r \cup \{0\}))$ generate σ_S. Letting $\Delta_i = \mathrm{Conv}(\overline{A}_i \cup \{0\})$, we have $\sigma_S = \mathrm{Cone}(\Delta_1 * \cdots * \Delta_r)$. Since the index of σ_S is r, the cone σ_S is therefore completely split.

Since $\mathrm{Conv}(S)$ and $\mathrm{Conv}(S)^\vee$ are both integrally closed and $\mathrm{Conv}(S)$ is a Gorenstein polytope of index r, Lemma 4.3.3 implies that $\mathrm{Conv}(S)$ and $\mathrm{Conv}(S)^\vee$ are Cayley polytopes associated to nef-partitions. Since $\mathrm{Conv}(S)$ is associated to a nef-partition, $\sigma_S^\vee$ is completely split by Proposition 2.12 in [20]. Therefore, σ_S and $\sigma_s^\vee$ are reflexive completely split Gorenstein cones and thus, by Proposition 4.3.4, σ_S has property $(\mathbf{r}-\mathbf{P})$. $\square$

Combining this with Theorem 4.3.7, we obtain the following Corollary.

Corollary 4.3.12. *If* $\mathrm{Conv}(S)$ *is a Gorenstein polytope of index* r *such that* $\mathrm{Conv}(S)$ *and* $\mathrm{Conv}(S)^\vee$ *are integrally closed, then there is a categorical resolution of* $\mathrm{D^b}(\mathrm{coh}\, Z)$ *given by a Batyrev-Borisov mirror.*

Remark 4.3.13. We can put $Z = Z(Q_{1,n,\lambda}, Q_{2,n,\lambda}) \subseteq \mathbb{P}^{2n-1}/G_n$ from Section § 4.2 into the setup above. We note that the set S associated to $Q_{2,n,\lambda} \oplus Q_{1,n,\lambda}$ is

$$\{e_i+e_{2n} \mid 1 \leq i \leq n\} \cup \{e_i+e_{2n+1} \mid n+1 \leq i \leq 2n-1\} \cup \{-e_1+\cdots-e_{2n-1}+e_{2n+1}\} \cup \{e_{2n}, e_{2n+1}\},$$

where $\{e_i \mid 1 \leq i \leq 2n+1\}$ is the standard basis for $\mathbb{Z}^{2n-1} \oplus \mathbb{Z}^2$. $\mathrm{Conv}(S)$ is a Gorenstein polytope of index 2, and thus Conjecture 4.3.10 would apply.

Furthermore, one can construct the cones σ_S and $\sigma_S^\vee$ and check that they are indeed both reflexive Gorenstein and completely split, so we can also apply Proposition 4.3.4 directly to obtain the property $(\mathbf{r} - \mathbf{P})$, which by Theorem 4.3.7 gives the existence of a categorical resolution.

CHAPTER 5

FURTHER DIRECTIONS OF RESEARCH

In this chapter, we will talk about further directions of research that the author has considered during their PhD, extending the work exhibited in the previous chapters of this thesis. These directions of research are based on f-duality, see § 2.3.3.

5.1 f-duality from a VGIT perspective

In this section we elaborate on the Remark 3.3.4 and show that the mirror construction by Libgober and Teitelbaum naturally fits into the context of f-duality. By comparing this to the computations in § 3, we will see that applying f-duality to the Libgober-Teitelbaum corresponds to constructing a toric vector bundle and considering it under partial compactifications and VGIT.

To fit the Libgober-Teitelbaum construction into the context of f-duality, we recall its toric representation from § 3. In particular, recall Proposition 3.3.1, giving a fan for $\mathbb{P}^5/G_{81}$, as well as the divisors D_a, D_b appearing in Corollary 3.3.3. We shall represent the Libgober-Teitelbaum variety as framed toric variety via the pair $(X_{LT}, -K_{X_{LT}} = D_a + D_b)$.

Here, writing the anticanonical divisor as sum $D_a + D_b$ gives a partition and we can apply the f-duality algorithm for complete intersections, see § 2.3.3 (page 61). The computations were done using SAGE.

The polytopes associated to the partitioned ftv $(X_{LT}, -K_{X_{LT}} = D_a + D_b)$ are

$$\begin{aligned}
\Delta_{-K_{X_{LT}}} =& \mathrm{Conv}(\left(\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3}, 0, 0\right), \left(-\frac{1}{3}, \frac{5}{3}, -\frac{1}{3}, 0, 0\right), \left(-\frac{1}{3}, -\frac{1}{3}, \frac{5}{3}, 0, 0\right),\\
&\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 2, 0\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 2\right), \left(-\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3}, -2, -2\right))\\
\Delta_a =& \mathrm{Conv}(\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0, 0\right), \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, 0, 0\right), (-1, -1, -1, -1, -1),\\
&(0, 0, 0, 0, 1), (0, 0, 0, 1, 0), \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, 0, 0\right))\\
\Delta_b =& \mathrm{Conv}((1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, 0\right),\\
&\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 1\right), \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, -1, -1\right)).
\end{aligned}$$

We define the polytope

$$\widehat{\Delta}_{-K_{X_{LT}}} = \mathrm{Conv}(\Delta_a, \Delta_b).$$

In this case, we obtain the framing polytope

$$\begin{aligned}
[\Delta_{-K_{X_{LT}}}] =& \mathrm{Conv}((-1, -1, -1, -1, -1), (0, -1, -1, -1, -1), (-1, 0, -1, -1, -1),\\
&(-1, -1, 0, -1, -1), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0),\\
&(0, 0, 0, 0, 1), (1, 0, 0, 1, 0), (1, 0, 0, 0, 1), (0, 1, 0, 1, 0), (0, 1, 0, 0, 1),\\
&(0, 0, 1, 1, 0), (0, 0, 1, 0, 1)).
\end{aligned}$$

Next, note that

$$\begin{aligned}
[\Delta_a] =& \mathrm{Conv}((-1, -1, -1, -1, -1), (0, 0, 0, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)),\\
[\Delta_b] =& \mathrm{Conv}((0, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)).
\end{aligned}$$

As described in the f-duality algorithm, we indeed have $[\Delta_a] \cap [\Delta_b] = \{0\}$ and $0 \in \mathrm{Int}(\Delta(X_{LT}, -K_{X_{LT}}))$, where $\Delta(X_{LT}, -K_{X_{LT}}) = [\Delta_a + \Delta_b] = [\Delta_{-K_{X_{LT}}}]$ is the framing polytope.

Hence, we obtain the polytope

$$\widehat{\Delta}(X_{LT}, -K_{X_{LT}}) = [\widehat{\Delta}_{-K_{X_{LT}}}] = \mathrm{Conv}((1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),$$
$$(0,0,0,1,0),(0,0,0,0,1),(-1,-1,-1,-1,-1)).$$

The face fan of $\Delta(X_{LT}, -K_{X_{LT}})$ is the standard fan for $\mathbb{P}^5$, with fan matrix

$$\widehat{\Lambda}_{-K_{X_{LT}}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We compute the two matrices

$$\widehat{M}_a = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 \end{pmatrix} \cdot \widehat{\Lambda}_{-K_{X_{LT}}} = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & -1 \end{pmatrix},$$
$$\widehat{M}_b = \begin{pmatrix} -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix} \cdot \widehat{\Lambda}_{-K_{X_{LT}}} = \begin{pmatrix} -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 \end{pmatrix}.$$

By inspection, the minimum non-negative column vectors in the next step of the algorithm are

$$\mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So the f-dual to $(X_{LT}, D_a + D_b)$ is $(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$. Thus, following Definition 2.3.12, we obtain that $Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5$ is a f-mirror partner of $V_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq X_{LT}$ (recalling $Q_{1,\lambda}, Q_{2,\lambda}$ from the Libgober-Teitelbaum construction § 2.3.2).

Since $\mathbb{P}^5$ is Fano and the two divisors that make up the framing sum to the anticanonical, §3.1 of [44] claims that the f-dual of $(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$ should correspond to the Batyrev-Borisov mirror of $Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5$. Further, this should lead to a calibrated f-process and thus to an involutive relationship between $(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$ and the Batyrev-Borisov mirror. In the following, we shall verify this.

The polytopes associated to the framing divisors are

$$\begin{aligned}
\Delta_{\mathbf{b}_1} =& \operatorname{Conv}((3,0,0,-1,-1),(0,3,0,-1,-1),(0,0,3,-1,-1),\\
&(0,0,0,2,-1),(0,0,0-1,2),(0,0,0,-1,-1)),\\
\Delta_{\mathbf{b}_2} =& \operatorname{Conv}((2,-1,-1,0,0),(-1,2,-1,0,0),(-1,-1,2,0,0),\\
&(-1,-1,-1,3,0),(-1,-1,-1,0,3),(-1,-1,-1,0,0)),\\
\widehat{\Delta}_{\mathbf{b}} =& \operatorname{Conv}(\Delta_{\mathbf{b}_1}, \Delta_{\mathbf{b}_2}).
\end{aligned}$$

Checking the conditions, we see that

$$\bigcap_{k=1}^{2} [\Delta_{b_k}] = \{0\} \text{ and } 0 \in \operatorname{Int}(\Delta(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)),$$

where $\Delta(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2) = [\Delta_{\mathbf{b}_1} + \Delta_{\mathbf{b}_2}]$. Hence, we obtain $\widehat{\Delta}(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2) = [\widehat{\Delta}_{\mathbf{b}}]$ and we have $0 \in \widehat{\Delta}(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$.

The face fan $\Sigma_{\hat{\mathbf{b}}}$ of $\widehat{\Delta}(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$ has the same rays as X_{∇} in Proposition 3.2.3. The fan matrix is

$$\widehat{\Lambda}_{\hat{\mathfrak{b}}} = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{pmatrix}.$$

In fact, the face fan $\Sigma_{\hat{\mathfrak{b}}}$ is precisely the normal fan to ∇ considered in the proof to Proposition 3.2.3 in § 3.2. Computing the framing, we obtain the following two matrices:

$$\widehat{M}_{a,b,1} = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{M}_{a,b,2} = \begin{pmatrix} -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

This gives the non-negative column vectors

$$\mathbf{c}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{c}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus the f-dual to $(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$ is $(X_\nabla, \mathbf{c}_1 + \mathbf{c}_2)$.

Finally, we will check that the f-dual to this partitioned ftv $(X_\nabla, \mathbf{c}_1 + \mathbf{c}_2)$ is $(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$, i.e. that f-duality is involutive in this case.

Note that the polytopes associated to the framing are

$$\begin{aligned}
\Delta_{\mathbf{c}_1} &= \mathrm{Conv}((1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,0,0)) = [\Delta_b], \\
\Delta_{\mathbf{c}_2} &= \mathrm{Conv}((0,0,0,0,0),(-1,-1,-1,-1,-1),(0,0,0,1,0),(0,0,0,0,1)) = [\Delta_a]\ .
\end{aligned}$$

Therefore, following through with the f-duality algorithm will lead to the same result as when we considered the partitioned ftv $(X_{LT}, -K_{X_{LT}} = D_a + D_b)$. Since this result is $(\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2)$, we have shown that we are now indeed in an involutive loop. Summarising, applying f-duality gives the following sequence:

$$V_{LT,\lambda} \xrightarrow{\quad f-dual \quad} Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5 \xleftrightarrow{f-dual} Z_\lambda \subseteq X_\nabla$$

$$(X_{LT}, -K_{X_{LT}} = D_a + D_b) \xrightarrow{f-dual} (\mathbb{P}^5, \mathbf{b}_1 + \mathbf{b}_2) \xleftrightarrow{f-dual} (X_\nabla, \mathbf{c}_1 + \mathbf{c}_2).$$

The above computations relate to the ones in § 3, when we constructed X_{LT} torically. Applying f-duality to ($\mathbb{P}^5$ $\mathbf{b}_1$+$\mathbf{b}_2$) corresponds to applying the Batyrev-Borisov construction (as noted by Rossi in [44]). However, we note that this is equivalent to considering the cone σ over the toric vector bundle $\mathrm{tot}(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus \mathcal{O}_{\mathbb{P}^5}(-3))$ and constructing its dual cone $\sigma^\vee$. Attach the vectors $\mathbf{c}_1^T, \mathbf{c}_2^T$ as rows to the fan matrix $\widehat{\Lambda}_{\hat{\mathbf{b}}}$, and consider its columns as primitive generators for 12 rays. These rays are the generators of the cone $\sigma^\vee$.

An interesting question is to consider under what circumstances in general applying f-duality corresponds to taking the dual of a cone over a toric vector bundle. This would provide a natural setting to apply methods of VGIT to the underlying vector bundles, allowing to obtain relations between associated derived categories as in § 3,4 of this thesis. Obtaining relations between derived categories would be helpful in establishing under what conditions applying f-duality can be expected to yield mirror partners.

Remark 5.1.1. In their recent paper [46], Rossi has been building on the paper [36] by the author of this thesis, examining when f-duality can be used to obtain equivalences between derived categories. This has been related to the Bondal-Orlov-Kawamata conjecture, giving empirical evidence of it in a number of examples. It would be a worthwhile project to continue this work and use the results of this thesis to strengthen the notion of f-duality, which can potentially lead to some bigger unification results in the study of mirror symmetry for toric varieties.

5.2 f-duality and the Gross-Siebert program

Another formulation of mirror symmetry is due to Gross and Siebert, first introduced in the paper [27], and is known as the Gross-Siebert program. The underlying motivation for the Gross-Siebert program was to produce a method of constructing mirror pairs which combines the Strominger-Yau-Zaslow (SYZ) approach with the Batyrev-Borisov approach. The SYZ approach to mirror symmetry has a differential geometric flavour, whereas the Batyrev-Borisov approach (as we illustrated in § 2.3.1) is algebro-geometric. We focus on the earlier parts of the Gross-Siebert program, which treat toric degenerations. To a toric degeneration of Calabi-Yau varieties $\mathcal{X} \to \mathcal{S}$, one can associate an integral affine manifold with singularities B, known as the dual intersection complex. If additionally, there is a polarization of $\mathcal{X} \to \mathcal{S}$ by a relatively ample divisor, then we obtain a convex, piecewise linear multi-valued function φ. Gross and Siebert defined a notion of discrete Legendre transform, which led to a new affine manifold with singularities $\check{B}$ and a new function $\check{\varphi}$. This $\check{B}$ is the dual intersection complex of a new degeneration $\check{\mathcal{X}} \to \mathcal{S}$. This new degeneration is studied and its special fibre $\check{\mathcal{X}}_0$ is supposed to give us an element of the mirror family by smoothing it.

In [25], Gross proves that the Batyrev-Borisov mirror construction is a case of a discrete Legendre transform, giving rise to toric degenerations. Given a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ with a dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$, Gross defines integral affine manifold with singularities B_Δ, B_∇ together with polyhedral decompositions $\mathcal{P}_\Delta, \mathcal{P}_\nabla$. He then equips B_Δ, B_∇ with piecewise linear functions $\check{\varphi}_\Delta, \varphi_\nabla$ and proves that $(B_\Delta, \mathcal{P}_\Delta, \check{\varphi}_\Delta)$ is the discrete Legendre transform of $(B_\nabla, \mathcal{P}_\nabla, \varphi_\nabla)$ in the sense of [27], further defining a toric degeneration associated to $\Delta = \Delta_1 + \cdots + \Delta_r$ with dual intersection complex $B_\nabla, \mathcal{P}_\nabla$.

With f-duality being an extension to the polar duality which is at the core of the Batyrev-Borisov construction, the natural question to ask is whether the methods of Gross can be modified to apply to f-duality.

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