# On the regularity of Fourier transforms and maximal functions

by

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# Abstract

In the first part of this thesis, we construct a function that lies in $L^p(\mathbb{R}^d)$ for every $p \in (1, \infty]$ and whose Fourier transform has no Lebesgue points in a Cantor set of full Hausdorff dimension. We apply Kovač's maximal restriction principle to show that the same full-dimensional set is avoided by any Borel measure satisfying a nontrivial Fourier restriction theorem. As a consequence of a near-optimal fractal restriction theorem of Łaba and Wang, we hence prove that no previously unknown relations hold between the Hausdorff dimension of a set and the range of valid Fourier restriction exponents for measures supported in the set.

In the second part, we prove sharp local and global variation bounds for the centred Hardy–Littlewood maximal functions of indicator functions in one dimension, establishing that they are variation diminishing. We characterise maximisers, treat both the continuous and discrete settings and extend our results to a larger class of functions. This is partial progress towards proving a conjecture of Kurka and Bober, Carneiro, Hughes and Pierce.

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# Contents

# Notation

By $|X|$ we denote the *Lebesgue measure* of a set in Euclidean space and by $\#S$ we denote the *cardinality* of a finite set. We use the *average integral* notation

$$\fint_X f(x)\,dx = \frac{1}{|X|}\int_X f(x)\,dx$$

and the *average sum* notation

$$\underset{m=n-r}{\overset{n+r}{\fint\!\!\sum}} f(m) = \frac{1}{2r+1}\sum_{m=n-r}^{n+r} f(m).$$

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of *Schwartz functions*. The *Lebesgue space* $L^p(\mathbb{R}^d)$ is associated to Lebesgue measure, while $L^p(\mu)$ is associated to another Borel measure on $\mathbb{R}^d$. We write $A \lesssim B$ if $A \leq CB$ for some finite constant $C$ independent of all parameters.

# Chapter 1

# Introduction

This thesis comprises two parts, each concerned with a different regularity issue in harmonic analysis.

In the first part, we study Lebesgue points of Fourier transforms, a subject that arose from the recent introduction of maximal functions as an object of study in Fourier restriction theory. We explain how our results clarify the relationship between the size of a set and its restriction properties. This discussion reveals interesting open questions, which we highlight. We introduce our point of view and our results in Section 1.1 and the proofs can be found in Chapter 2. This material is based on the published paper [Bil22].

In the second part, we study the regularity of maximal functions themselves. This area of research continues to attract plenty of interest, particularly in questions of boundedness and continuity in Sobolev spaces. However, progress on establishing sharp versions of known inequalities has been slow. We establish the first sharp variation bounds for the centred Hardy–Littlewood maximal function. Our results are discussed in Section 1.2 and the proofs are contained in Chapter 3. This material is based on the joint work [BW21] with Julian Weigt, submitted for publication.

## 1.1 Lebesgue points of Fourier transforms

In this section, we are concerned with the regularity of the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\, dx$$

of a function $f \in L^p(\mathbb{R}^d)$. Two fundamental consequences of Hölder's inequality and the dominated convergence theorem are that if $p = 1$, then the above integral converges for any $\xi \in \mathbb{R}^d$ and $\hat{f}$ is continuous. Neither fact remains true when $p \in (1, 2]$, in which case the Fourier transform is defined by the standard procedure involving dense continuation and interpolation.

Does the Fourier transform retain a weaker kind of continuity for $p$ close to 1? Our goal is to show that this is not the case if we measure this weaker kind of continuity by the prevalence of Lebesgue points.

**Definition 1.1.1** (Lebesgue point)**.** Let $g\colon \mathbb{R}^d \to \mathbb{C}$ be a Borel measurable function. We say that a point $\xi \in \mathbb{R}^d$ is a *Lebesgue point* of $g$ if there exists a number $c \in \mathbb{C}$ such that

$$\lim_{r\to 0} \fint_{\{|\eta|<r\}} |g(\xi - \eta) - c|\, d\eta \to 0.$$

In this case, we call $c$ the *regularised* value of $g$ at $\xi$. The *non-Lebesgue set* of $g$ is the set of its non-Lebesgue points.

By continuity, any point is a Lebesgue point of $\hat{f}$ if $p = 1$. We complement this with the following main result.

**Theorem 1.1.2.** *There exists a function in $\bigcap_{p\in(1,\infty]} L^p(\mathbb{R}^d)$ whose Fourier transform has no Lebesgue points in some compact set of Hausdorff dimension $d$.*

The Lebesgue differentiation theorem states that non-Lebesgue sets of locally integrable functions have Lebesgue measure zero. Theorem 1.1.2 shows that this cannot be sharpened in terms of Hausdorff dimension for the class of Fourier transforms of $L^p(\mathbb{R}^d)$ functions when $p > 1$.

### 1.1.1 Restriction inequalities

Our interest in this problem stems from its connection to *restriction inequalities*

$$\|\hat{f}\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mathbb{R}^d)}, \quad f \in \mathcal{S}(\mathbb{R}^d) \tag{1.1.1}$$

where $p \in [1, 2]$, $q \in [1, \infty]$, $\mu$ is a Borel measure on $\mathbb{R}^d$ and $C$ is a constant that is independent of $f$, but may depend on $p$, $q$, $d$ and $\mu$. The first such inequality was proved by Stein in 1967, see [Ste93, p. 374]. Restriction inequalities are therefore a relatively late discovery in the field of harmonic analysis, which has roots 200 years ago in the work of Fourier [Fou22].

The main focus of the now extensive literature on restriction inequalities lies on natural measures $\mu$ supported in hypersurfaces such as the sphere, the paraboloid and the cone, as well as lower-dimensional submanifolds and curves. The range of valid restriction inequalities strongly depends on the curvature properties of the underlying submanifold. This line of research has important connections to other areas of mathematics, including PDE, number theory and geometric measure theory. Despite significant progress, Stein's restriction conjecture [Ste79] remains unresolved in most cases. We refer the reader to the surveys [Sto19; Tao04] and the references therein.

The Stein–Tomas argument [Tom75] is a useful tool in the case that $q = 2$, where it can be used to prove (1.1.1) in the best possible range of exponents $p$ on compact hypersurfaces of positive Gaussian curvature. It is also robust enough to yield interesting restriction inequalities for fractals, as Mockenhaupt [Moc00] and Mitsis [Mit02] have observed, see Section 1.1.3.

### 1.1.2 Maximal and intrinsic restriction

The connection between Theorem 1.1.2 and the restriction inequality (1.1.1) arises from the pointwise perspective recently introduced by Müller, Ricci and Wright [MRW19]. If (1.1.1) holds, then the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ extends to a bounded *restriction operator* $\mathcal{R}_\mu \colon L^p(\mathbb{R}^d) \to L^q(\mu)$. In the case of a singular measure $\mu$, this operator can be regarded as a natural way of assigning values $\mu$-almost everywhere to the Fourier transform of an $L^p(\mathbb{R}^d)$ function. Indeed, (1.1.1) readily implies that for any $f \in L^p(\mathbb{R})$ there exists a sequence of radii $r_n \to 0$ such that

$$\lim_{n\to\infty} \fint_{\{|\eta|<r_n\}} \hat{f}(\xi - \eta)\, d\eta = \mathcal{R}_\mu f(\xi) \qquad \mu\text{-a.e.}$$

Of course this does not mean that a limit exists for any sequence of radii $r_n \to 0$ or that all such limits are equal. Therefore it is unclear at first how the $\mu$-almost everywhere defined $\mathcal{R}_\mu f$ can be recovered from the only Lebesgue-almost everywhere defined $\hat{f}$.

Müller, Ricci and Wright addressed this issue in the case of planar curves by strengthening the mode of convergence as follows:

$$\lim_{r\to 0} \fint_{\{|\eta|<r\}} |\hat{f}(\xi - \eta) - \mathcal{R}_\mu f(\xi)|\, d\eta = 0 \qquad \mu\text{-a.e.} \tag{1.1.2}$$

when $\mu$ is the affine arc length measure on a smooth planar curve and $f \in L^p(\mathbb{R}^2)$, $1 \le p < 8/7$. This means that $\mu$-almost every $\xi \in \mathbb{R}^2$ is a Lebesgue point of $\hat{f}$ with regularised value $\mathcal{R}_\mu f(\xi)$. The proof of (1.1.2) relies on the *maximal* restriction inequality

$$\left\| \sup_{r\in(0,1)} \fint_{\{|\eta|<r\}} |\hat{f}(\xi-\eta)|\, d\eta \right\|_{L^q_\xi(\mu)} \le C\|f\|_{L^p(\mathbb{R}^d)} \tag{1.1.3}$$

in the same way that the proof of the Lebesgue differentiation theorem relies on the Hardy–Littlewood maximal inequality. Later, Vitturi [Vit22] obtained a similar result for the surface measure of the sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ and Kovač and Oliveira e Silva [KO21] proved a stronger *variational* restriction inequality for spheres, where the supremum in (1.1.3) is replaced by a variation norm.

Then, Kovač proved a general variational restriction principle that for instance implies the following result.

**Theorem 1.1.3** (see [Kov19, Remark 3])**.** *If the restriction inequality* (1.1.1) *holds for some finite Borel measure $\mu$ on $\mathbb{R}^d$ and some exponents $p \in [1,2]$ and $q \in (p,\infty)$, then the maximal restriction inequality* (1.1.3) *holds with the exponents $p_1 = \frac{2p}{p+1}$ and $q_1 = 2q$. Hence* (1.1.2) *holds for any $f \in L^{2p/(p+1)}(\mathbb{R}^d)$.*

Kovač's methods also apply to more singular averaging kernels in place of the ball averages in (1.1.2), see also the work of Ramos [Ram22].

Theorem 1.1.3 shows that restriction inequalities imply an *intrinsic* restriction property that can conceivably be studied independently. The following question seems natural, but nothing appears to be known.

**Question 1.1.4.** Does intrinsic restriction imply the presence of an underlying restriction inequality, i.e. does the second part of Theorem 1.1.3 have a converse?

The proof of Theorem 1.1.3 relies on the Christ–Kiselev lemma and this is why $q > p$ is assumed. Based on a *multi-parameter* Christ–Kiselev lemma, Bulj and Kovač [BK22] proved a multi-parameter maximal restriction principle that implies a version of Theorem 1.1.3 for anisotropic averages. It is unknown whether general maximal restriction principles can be proved without assuming that $q > p$.

Similarly, it is unknown whether the conclusion of Theorem 1.1.3 can in general be strengthened by replacing $\frac{2p}{p+1}$ with $p$ and $2q$ with $q$. Here, the seeming inefficiency is due to a reflection argument that is needed in order to obtain the *positive* (also called *strong*) maximal inequality (1.1.3) from an oscillatory version with the modulus pulled outside of the integral. Ramos [Ram20] used a linearisation method to circumvent this issue for strictly convex $C^2$ curves. The $C^2$ regularity assumption was later removed by Fraccaroli [Fra21]. The above-mentioned reflection argument involves a maximal $L^2$-average operator and Ramos [Ram22] and Jesurum [Jes22] proved versions of (1.1.3) for maximal $L^r$-averages.

The non-Lebesgue sets of $L^q(\mathbb{R}^d)$ functions, $q \in [1, \infty]$, were characterised by D'yachkov [Dya93] as the $G_{\delta\sigma}$ sets of zero Lebesgue measure, a class that does not depend on $q$. This is in contrast with the smaller class of non-Lebesgue sets of Fourier transforms of $L^p(\mathbb{R}^d)$ functions. No nontrivial examples of such sets appear to be known beyond what is provided by Theorem 1.1.2. A positive answer to Question 1.1.4 would provide a sufficient condition.

However, $p$-dependent necessary conditions follow from maximal restriction theorems such as the above. The second part of Theorem 1.1.3 states that, under fairly general assumptions, non-Lebesgue sets of Fourier transforms are avoided by measures satisfying restriction inequalities. Since many such measures are known, this is a strong structural condition on such sets. Our Theorem 1.1.2 shows that these sets can nevertheless be large in a metric sense.

By combining Theorems 1.1.2 and 1.1.3 we prove the existence of a set of full Hausdorff dimension and without nontrivial restriction theorems.

**Corollary 1.1.5.** *There exists a compact subset $E$ of $\mathbb{R}^d$ such that $E$ has Hausdorff dimension $d$ and for any Borel measure $\mu$ on $\mathbb{R}^d$ with $\mu(E) > 0$ and for any $p \in (1, 2]$ and any $q \in [1, \infty]$ it holds that*

$$\sup_{f \in \mathcal{S}(\mathbb{R}^d)} \frac{\|\hat{f}\|_{L^q(\mu)}}{\|f\|_{L^p(\mathbb{R}^d)}} = \infty.$$

We further strengthen this result by showing that no relations except for a well-known energy-theoretic inequality hold between the Hausdorff dimension of a set and the supremum of the range of exponents $p$ for which $L^p(\mathbb{R}^d)$-based restriction inequalities hold on that set, see Corollary 1.1.9 in the next subsection.

### 1.1.3 Restriction to fractals

It was first observed by Mockenhaupt [Moc00] and Mitsis [Mit02] that the Stein–Tomas argument mentioned in Section 1.1.1 can be used to prove interesting restriction inequalities on fractals. In its endpoint form due to Bak and Seeger [BS11], this idea implies the following: if $\mu$ is a finite Borel measure on $\mathbb{R}^d$ with Fourier decay

$$\sup_{\xi \in \mathbb{R}^d} |\xi|^{\beta/2} |\hat{\mu}(\xi)| < \infty \tag{1.1.4}$$

for some $\beta \in [0, d)$, then it satisfies the restriction inequality (1.1.1) for any $p \in [1, 4d/(4d - \beta)]$ and $q = 2$.

The full Mockenhaupt–Mitsis–Bak–Seeger theorem contains an additional dimensionality condition that leads to a larger range of restriction exponents $p$ in most cases. The exponent $4d/(4d - \beta)$ that we give above corresponds to the dimensionality that is implied by (1.1.4), see e.g. [Mit02, Corollary 3.1].

The sharpness of the Mockenhaupt–Mitsis–Bak–Seeger theorem in one dimension was established by Chen [Che16] following work of Hambrook and Łaba [HŁ13]. In higher dimensions, sharpness in the case that $\beta > d - 1$ was established by Hambrook and Łaba [HŁ16].

Fourier decay gives rise to the following notion of dimension.

**Definition 1.1.6** (Fourier dimension)**.** The supremum of the set of rates $\beta \in [0, d)$ for which (1.1.4) holds is the *Fourier dimension* of the measure $\mu$. The *Fourier dimension* of a subset of $\mathbb{R}^d$ is the supremum of the set of Fourier dimensions of all finite nonzero Borel measures that are compactly supported in that set.

Hence, the Mockenhaupt–Mitsis–Bak–Seeger theorem yields a nontrivial restriction theorem for any Borel measure or set of strictly positive Fourier dimension. This makes it widely applicable not only to curved submanifolds, but also to pseudorandom fractals. In order to demonstrate this, we mention a few interrelated classes of deterministic and random fractal sets that have positive Fourier dimension, with selected references:

- images of stochastic processes [Kah85] and random diffeomorphisms [Eks16],
- certain sets arising from Diophantine approximation [Kau81; JS16],
- limit sets of Fuchsian groups [BD17] and
- various constructions based on Cantor sets [Sal51; ŁW18].

On the other hand, fractal measures can have Fourier dimension zero. For instance, this is the case for any measure supported in a hyperplane. Erdős [Erd39] showed that the natural measure on a Cantor set of dissection ratio $1/\alpha$ has Fourier dimension zero if $\alpha$ is a Pisot number, e.g. if $\alpha$ is 3 or the golden ratio. The set that we use to prove Corollary 1.1.5 is also a Cantor set and by the Mockenhaupt–Mitsis–Bak–Seeger theorem it necessarily has Fourier dimension zero.

We refer the reader to Łaba's survey [Łab14] for further information on the role of Fourier dimension in harmonic analysis.

The Mockenhaupt–Mitsis–Bak–Seeger theorem is not the only way of proving fractal restriction inequalities. Chen [Che14] showed that any finite Borel measure with a convolution power in $L^r(\mathbb{R}^d)$ satisfies a nontrivial restriction inequality. This theorem, too, is applicable to a large class of fractal measures: Körner [Kör08] showed that, in a Baire category sense, many Borel probability measures with support of Hausdorff dimension 1/2 on the real line have a continuous self-convolution. For such a measure, Chen's result implies the restriction inequality (1.1.1) with $1 \leq p \leq 4/3$ and $q = 2$. This range of exponents $p$ is optimal in the following sense.

**Definition 1.1.7** (Endpoint restriction exponent)**.** Given a subset $E$ of $\mathbb{R}^d$, we denote by $p_{\text{res}}(E)$ the supremum of the range of exponents $p \in [1, 2]$ for which there exists a Borel measure $\mu$ with $\mu(E) > 0$ such that the restriction inequality (1.1.1) holds for some exponent $q \in [1, \infty]$.

The universal $L^1(\mathbb{R}^d) \to L^\infty(\mu)$ bound implies that $p_{\text{res}}(E) \geq 1$. If $E$ has positive Lebesgue measure, then by the Plancherel theorem we have $p_{\text{res}}(E) = 2$. An energy integral argument, see e.g. [Moc00, Section 2], shows that $p_{\text{res}}(E)$ cannot be too large depending on the Hausdorff dimension $\dim_H(E)$ and the ambient dimension $d$. Namely, it holds that

$$p_{\text{res}}(E) \le \frac{2d}{2d - \dim_H(E)}. \tag{1.1.5}$$

The combination of the results of Chen and Körner described above gives a restriction inequality with $d = 1$ and $\dim_H(E) = 1/2$ that is optimal with respect to (1.1.5). Their methods were extended by Chen and Seeger [CS17] to the case that $d \ge 1$ and $\dim_H(E) = d/n$ for some integer $n$. Related results were obtained by Shmerkin and Suomala [SS18] using a different approach involving so called spatially independent martingales.

Łaba and Wang used yet another method to show that equality in (1.1.5) can be attained even in the general case.

**Theorem 1.1.8** ([ŁW18]). *Let $d \ge 1$ and $\alpha \in (0, d)$. There exists a Borel probability measure $\mu$ with compact support of Hausdorff dimension $\alpha$ such that the restriction inequality* (1.1.1) *holds for any $1 \le p < 2d/(2d - \alpha)$ and $q = 2$.*

Their proof is based on a decoupling argument and the existence of $\Lambda(p)$ sets of asymptotically largest possible cardinality.

We combine Corollary 1.1.5 and Theorem 1.1.8 to show that (1.1.5) is the only relation that holds between the endpoint restriction exponent, the Hausdorff dimension and the ambient dimension.

**Corollary 1.1.9.** *Let $\alpha \in [0, d]$ and $p \in [1, 2d/(2d - \alpha)]$. There exists a compact set $E \subseteq \mathbb{R}^d$ of Hausdorff dimension $\alpha$ such that $p_{\text{res}}(E) = p$.*

At least in the cases covered by the above-mentioned results of [Che14; CS17; SS18], the set $E$ can be chosen so that the endpoint exponent $p = 2d/(2d-\alpha)$ itself satisfies the restriction inequality (1.1.1) for some $\mu$ and $q$. It is unknown whether this can be achieved in general. Theorem 1.1.8 and hence Corollary 1.1.9 do not address this question.

Corollary 1.1.9 may be compared to the well-known result that for any $\alpha \in [0, d]$ and any $\beta \in [0, \alpha]$ there exists a set of Hausdorff dimension $\alpha$ and Fourier dimension $\beta$. Körner [Kör11] proved a stronger version of that statement where the set is further guaranteed to be precisely the support of a measure of Fourier dimension $\beta$. It would be interesting to show a similarly strengthened version of Corollary 1.1.9. Our proof strategy is unsuitable for this problem.

Körner's result in the last paragraph is perhaps unsurprising given that Hausdorff dimension is a metric property of a set and Fourier dimension is not, see e.g. [Eks16]. Similarly, Corollary 1.1.9 is perhaps unsurprising if restriction estimates rely on some pseudorandomness of the underlying measure, manifesting itself, for instance, in a positive Fourier dimension, see e.g. [Łab14]. However, a converse of, for instance, the Mockenhaupt–Mitsis–Bak–Seeger theorem is not known, except in the case of hypersurfaces due to a result of Iosevich and Lu [IL00]. Therefore, we do not know whether Körner's result can be used to prove Corollary 1.1.9. In this connection, an answer to the following question would shed further light on the nature of fractal restriction.

**Question 1.1.10.** Except for the Mockenhaupt–Mitsis–Bak–Seeger theorem, what other relations, if any, hold between the Fourier dimension of a measure and its range of valid restriction inequalities?

## 1.2 Sharp regularity of maximal functions

It is a common theme in analysis that averaging operations can improve the regularity of a function. An instance of this is the following consequence of Young's convolution inequality: For any fixed radius $r > 0$ and any function in the Sobolev space $W^{1,p}(\mathbb{R})$, $p \in [1, \infty]$, it holds that

$$\left\| \frac{d}{dx} \fint_{x-r}^{x+r} |f(y)|\, dy \right\|_{L^p_x(\mathbb{R})} \leq \left\| \frac{df}{dx} \right\|_{L^p(\mathbb{R})}.$$

Since the right-hand side does not depend on $r$, it is natural to ask what happens if we replace the average on the left-hand side by a maximal average. This was studied by Kinnunen [Kin97], who proved that for $p > 1$,

$$\left\| \frac{d}{dx} \sup_{r>0} \fint_{x-r}^{x+r} |f(y)|\, dy \right\|_{L^p_x(\mathbb{R})} \leq C(p) \left\| \frac{df}{dx} \right\|_{L^p(\mathbb{R})}, \tag{1.2.1}$$

where $C(p)$ is the constant in the Hardy–Littlewood maximal inequality. The case $p = 1$ was resolved by Kurka [Kur15], who in fact proved a more general statement for functions of bounded variation. But his methods do not provide the optimal constant $C(1)$ in the above inequality.

We determine optimal constants and characterise maximisers of Kurka's inequality when it is restricted to special classes of functions. Section 1.2.1 discusses bounds on the real line and Section 1.2.2 discusses bounds on the integers.

### 1.2.1 Continuous setting

On the real line $\mathbb{R}$, the *Hardy–Littlewood maximal function* is defined by

$$Mf(x) = \sup_{r>0} \fint_{x-r}^{x+r} |f(y)| \, \mathrm{d}y.$$

The *variation* of a function $f\colon \mathbb{R} \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ is

$$\mathrm{var}_I(f) = \sup_{\phi\colon \mathbb{Z} \to I \text{ monotone}} \sum_{i \in \mathbb{Z}} |f(\phi(i)) - f(\phi(i+1))|.$$

We write $\mathrm{var}(f) = \mathrm{var}_{\mathbb{R}}(f)$ and say that $f$ is of *bounded variation* if $\mathrm{var}(f) < \infty$.

Kurka [Kur15] proved that for any such function it holds that

$$\mathrm{var}(Mf) \le C \, \mathrm{var}(f) \tag{1.2.2}$$

for some large constant $C$ independent of $f$. It is an open conjecture that the optimal constant in this inequality is $C = 1$, see e.g. [BCHP12; Kur15]. The following main result establishes this in the case of indicator functions.

**Theorem 1.2.1.** *Let $f\colon \mathbb{R} \to \{0,1\}$ be a function of bounded variation. Then* (1.2.2) *holds with $C = 1$. Equality is attained if and only if $f$ is constant or the set $\{x \in \mathbb{R} \mid f(x) = 1\}$ is a bounded interval of positive length.*

An indicator function is of bounded variation precisely if it has at most finitely many jumps. This immediately implies that $f(x) = 0$ or $f(x) = Mf(x)$ for Lebesgue-almost every $x \in \mathbb{R}$. Our methods only require this weaker assumption, allowing us to prove the following more general result for nonnegative functions.

**Theorem 1.2.2.** *Let $f\colon \mathbb{R} \to [0,\infty)$ be a function of bounded variation such that for almost every $x \in \mathbb{R}$ we have that $f(x) = 0$ or $f(x) = Mf(x)$. Then (1.2.2) holds with $C = 1$. Equality is attained if and only if $f$ is constant or the set $\{x \in \mathbb{R} \mid f(x) > 0\}$ is a bounded interval of positive length and for any $x \in \mathbb{R}$,*

$$\liminf_{y\to x} f(y) \le f(x) \le \limsup_{y\to x} f(y).$$

Another common notion of variation is given by the total variation $|\mathrm{D}f|(\mathbb{R}^d)$ of the distributional derivative $\mathrm{D}f$, i.e. the measure satisfying the integration by parts rule

$$\int_{\mathbb{R}^d} f\varphi' \,\mathrm{d}x = -\int_{\mathbb{R}^d} \varphi \,\mathrm{d}(\mathrm{D}f)$$

for all functions $\varphi \in C_c^\infty(\mathbb{R}^d)$. The variation of a function on $\mathbb{R}^d$ with $d > 1$ is usually defined in this way. For any function $f\colon \mathbb{R} \to \mathbb{R}$ of bounded variation it holds that $|\mathrm{D}f|(\mathbb{R}) \le \operatorname{var}(f)$. Conversely, if $|\mathrm{D}f|(\mathbb{R}) < \infty$, then there exists a function $\bar{f}$ equal to $f$ almost everywhere such that $\operatorname{var}(\bar{f}) = |\mathrm{D}f|(\mathbb{R})$, see e.g. [Leo09, Theorem 7.2]. If $f$ satisfies the hypotheses of Theorem 1.2.2, then it follows that

$$|\mathrm{D}Mf|(\mathbb{R}) \le \operatorname{var}(Mf) = \operatorname{var}(M\bar{f}) \le \operatorname{var}(\bar{f}) = |\mathrm{D}f|(\mathbb{R}).$$

Hence Theorem 1.2.2 remains true for this definition of the variation.

The regularity of maximal functions was first studied by Kinnunen [Kin97], who proved that the $d$-dimensional centred Hardy–Littlewood maximal operator is bounded on the Sobolev space $W^{1,p}(\mathbb{R}^d)$ when $1 < p \le \infty$ and $d \ge 1$. The one-dimensional case of this statement already appeared in (1.2.1). Hajłasz and

Onninen [HO04] later asked whether the endpoint inequality

$$\|\nabla Mf\|_{L^1(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^1(\mathbb{R}^d)} \tag{1.2.3}$$

also holds and Kurka's inequality (1.2.2) provides a positive answer to this question in one dimension. The higher-dimensional case remains completely open.

In comparison to the one-dimensional *centred* Hardy–Littlewood maximal function, its *uncentred* counterpart

$$\widetilde{M}f(x) = \sup_{x_0<x<x_1} \fint_{x_0}^{x_1} |f(y)| \, \mathrm{d}y$$

allows averages over a larger class of intervals and hence may be expected to be smoother. Indeed, Tanaka [Tan02] gave a short proof of the uncentred version of (1.2.2) with $C = 2$ and later Aldaz and Pérez Lázaro [AP07] showed that the optimal constant is $C = 1$. Ramos [Ram19] studied the sharp version of (1.2.2) for a family of *nontangential* maximal functions interpolating between the centred and uncentred Hardy–Littlewood maximal functions.

Similarly, higher-dimensional partial results are available for the uncentred maximal function where the corresponding results are not known in the centred case. The first such result is due to Aldaz and Pérez Lázaro [AP09] who proved the uncentred version of (1.2.3) for so-called block decreasing functions. Later, Luiro [Lui18] proved the same for radial functions and Weigt [Wei22] proved an analogous inequality for indicator functions.

### 1.2.2 Discrete setting

Our methods also imply discrete analogues of Theorems 1.2.1 and 1.2.2. The *centred Hardy–Littlewood maximal function* $Mf\colon \mathbb{Z} \to \mathbb{R}$ of a bounded function $f\colon \mathbb{Z} \to \mathbb{R}$ is defined by

$$Mf(n) = \sup_{r\in\mathbb{Z}_{\geq 0}} \sum_{m=n-r}^{n+r} |f(m)|.$$

For a *discrete interval* $I \subseteq \mathbb{Z}$, i.e. the intersection of $\mathbb{Z}$ and a real interval, the *variation* of $f$ on $I$ is

$$\mathrm{var}_I(f) = \sum_{n,n+1\in I} |f(n) - f(n+1)|.$$

We say that $f$ is of *bounded variation* if $\mathrm{var}_{\mathbb{Z}}(f) < \infty$.

Bober, Carneiro, Hughes and Pierce [BCHP12] proved that

$$\mathrm{var}_{\mathbb{Z}}(Mf) \leq C \sum_{n\in\mathbb{Z}} |f(n)|$$

for $C = 2 + \frac{146}{315}$. They asked whether the optimal constant in this inequality is $C = 2$ and whether the stronger inequality

$$\mathrm{var}_{\mathbb{Z}}(Mf) \leq C\,\mathrm{var}_{\mathbb{Z}}(f) \tag{1.2.4}$$

analogous to (1.2.2) holds. Madrid [Mad17] affirmatively answered the first question and Temur [Tem13] adapted Kurka's method to prove (1.2.4) with a large constant. We improve a special case of Temur's result by establishing the optimal constant $C = 1$ in the case of indicator functions.

**Theorem 1.2.3.** *Let $f\colon \mathbb{Z} \to \{0,1\}$ be a function of bounded variation. Then* (1.2.4) *holds with $C = 1$. Equality is attained if and only if $f$ is constant or the set $\{n \in \mathbb{Z} \mid f(n) = 1\}$ is a bounded nonempty discrete interval.*

In fact this result quickly follows from the continuous Theorem 1.2.1 and an embedding argument. When combined with a complementary approximation argument, this embedding argument also implies the following relationship between the optimal constants in the continuous and discrete variation bounds for general functions of bounded variation.

**Proposition 1.2.4.** *The optimal constants in* (1.2.2) *and* (1.2.4) *are the same, i.e. if* (1.2.2) *holds for all functions of bounded variation, then the same is true for* (1.2.4) *with the same constant $C$ and vice versa.*

However, we do not know whether an embedding argument can be used to prove the following discrete analogue of the stronger Theorem 1.2.2. This is because of the additional assumptions in these theorems. Instead, we adapt the proof of Theorem 1.2.2 to the discrete setting for the following result.

**Theorem 1.2.5.** *Let $f\colon \mathbb{Z} \to [0,\infty)$ be a function of bounded variation such that for any $n \in \mathbb{Z}$ we have $f(n) = 0$ or $f(n) = Mf(n)$. Then* (1.2.4) *holds with $C = 1$. Equality is attained if and only if $f$ is constant or the set $\{n \in \mathbb{Z} \mid f(n) > 0\}$ is a bounded nonempty discrete interval.*

Although the proofs of Theorems 1.2.2 and 1.2.5 are quite similar, different technical difficulties arise in each case. In the continuous setting, we have to deal with compactness issues and exceptional sets of measure zero. In the discrete setting, one inconvenience is that not every integer interval has an integer midpoint.

# Chapter 2

# Large sets without Fourier restriction theorems

In this chapter, we prove the results contained in Section 1.1. The proof of Theorem 1.1.2 is made up of Sections 2.2 to 2.4 and the derivations of Corollaries 1.1.5 and 1.1.9 are contained in Section 2.5.

## 2.1 Proof strategy

Let us first comment on the proof strategy for Theorem 1.1.2, which involves a somewhat delicate construction based on a Cantor set. In Section 2.2, we introduce a family of Cantor sets parameterised by their dissection ratios $\theta_j \in (0, 1/2)$, $j \geq 0$, at different scales. We establish conditions under which such a Cantor set $E$ is the non-Lebesgue set of a certain natural function $g$ and we calculate the inverse Fourier transform $\check{g}$. The proof of Theorem 1.1.2 then comes down to choosing the dissection ratios in such a way that $E$ has full Hausdorff dimension while $\check{g}$ is $p$-integrable for any $p > 1$. The key terms in the $p$-integral of $\check{g}$ are products of cosines resembling

$$\prod_{j=0}^{k-1} \cos(\tfrac{1}{2}\theta_0\theta_1 \cdots \theta_{j-1}\xi). \tag{2.1.1}$$

In Section 2.3, motivated by Euler's formula for the sinc function, we bound $p$-integrals of products of cosines with dyadic phases of the form

$$\prod_{j\in J}\cos(2^{-j}\xi) \tag{2.1.2}$$

where $J$ is a set of positive integers. Our estimate involves some loss depending on the number of components of $J$.

The dissection ratios $\theta_j$ that we fix in Section 2.4 to complete the proof of Theorem 1.1.2 have the following essential properties:

(i) *The dissection ratios are very close to* $1/2$ *in an average sense.* This ensures that the Cantor set has full Hausdorff dimension and it enables us to approximate the products (2.1.1) appearing in the $p$-norm of $\check{g}$ by products of cosines with dyadic phases (2.1.2), for which we have bounds.

(ii) *Infinitely many consecutive pairs of dissection ratios are bounded away from* $0$ *and* $1/2$*.* Under this condition, the Cantor set is the non-Lebesgue set of the associated function $g$. However, the boundedness away from $1/2$ could break the comparability of the products (2.1.1) and (2.1.2). Therefore, we are led to the following condition.

(iii) *The dissection ratios that are not close to* $1/2$ *are powers of* $1/2$*.* Then, in the analysis of (2.1.1), these small dissection ratios conveniently translate into gaps in (2.1.2), so that the set $J$ has multiple components.

(iv) *We raise* $1/2$ *to exponents that are large on average.* This ameliorates the aforementioned loss in our integral estimate of (2.1.2) that depends on the number of components of $J$. However, a subsequence of the exponents must remain bounded because of Item (ii).

## 2.2 Cantor sets as non-Lebesgue sets

Let $\theta_j \in (0, 1/2)$, $j \geq 0$, and let $S$ be an infinite set of nonnegative integers. Write

$$\Theta_k = \theta_0 \theta_1 \cdots \theta_{k-1}.$$

Using a Cantor set with dissection ratios $\theta_j$ we prove the following Proposition 2.2.1 which serves as the starting point for the proof of Theorem 1.1.2.

In that proof, it will be important to ensure that the dissection ratios have a certain asymptotic behaviour. The set $S$ will be the set of indices for which the dissection ratios are not close to $1/2$ and this set will have natural density zero.

**Proposition 2.2.1.** *Let $\theta_j$, $\Theta_k$ and $S$ be as above. Assume that*

$$\lim_{k\to\infty} \Theta_k^{1/k} = \tfrac{1}{2} \tag{2.2.1}$$

*and that there exists an $\epsilon > 0$ such that*

$$\theta_j, \theta_{j+1} \in (\epsilon, \tfrac{1}{2} - \epsilon) \quad \textit{for infinitely many } j \in S \textit{ with } j+1 \in S. \tag{2.2.2}$$

*Then there exists a Borel measurable function $g\colon \mathbb{R}^d \to \{-1, 0, 1\}$ such that*

*(i) the non-Lebesgue set of $g$ is compact and has Hausdorff dimension $d$ and*

*(ii) for any $p > 1$, the inverse Fourier transform $\check{g}$ lies in $\bigcap_{r\in[p,\infty]} L^r(\mathbb{R}^d)$ if*

$$\int_0^\infty \left( \sum_{k\in S} \frac{2^k \Theta_k}{1 + (1-2\theta_k)\Theta_k|\xi|} \prod_{j=0}^{k-1} |\cos((1-\theta_j)\Theta_j \pi\xi)| \right)^p d\xi < \infty.$$

We simultaneously construct the function $g\colon \mathbb{R}^d \to \{-1,0,1\}$ and the set $E \subseteq \mathbb{R}^d$ that we then show to be the non-Lebesgue set of $g$. Let $c(I)$ denote the *midpoint* of an interval. We define families $\mathcal{W}_k$ and $\mathcal{B}_k$ of *white* and *black intervals* of generations $k = 0, 1, \ldots$ by the recursion

$$\begin{aligned}
\mathcal{W}_0 &= \left\{ \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \right\}, \\
\mathcal{B}_k &= \{ B_k \text{ open interval} \mid c(B_k) = c(W_k) \text{ and } |B_k| = (1-2\theta_k)|W_k| \\
&\qquad\qquad \text{for some } W_k \in \mathcal{W}_k \}, \\
\mathcal{W}_{k+1} &= \text{set of connected components of} \bigcup_{W_k\in\mathcal{W}_k} W_k \setminus \bigcup_{B_k\in\mathcal{B}_k} B_k.
\end{aligned}$$

In each generation, one black interval is removed from the middle of each remaining white interval, leaving two white intervals of the next generation. The remaining white set $\bigcup_{W_k\in\mathcal{W}_k} W_k$ is decreasing in $k$. Its limit as $k \to \infty$ is a Cantor set $E^{(1)} \subseteq \mathbb{R}$ with dissection ratios $\theta_k$:

$$E^{(1)} = \bigcap_{k=0}^{\infty} \bigcup_{W_k\in\mathcal{W}_k} W_k = [-\tfrac{1}{2}, \tfrac{1}{2}] \setminus \bigcup_{k=0}^{\infty} \bigcup_{B_k\in\mathcal{B}_k} B_k.$$

We define an oscillating function $g^{(1)}\colon \mathbb{R} \to \{-1,0,1\}$ associated to the black intervals by

$$g^{(1)} = \sum_{k\in S} (-1)^k \sum_{B_k\in\mathcal{B}_k} \chi_{B_k}. \tag{2.2.3}$$

Here, $\chi_{B_k} \colon \mathbb{R} \to \{0,1\}$ is the characteristic function of $B_k$. The series (2.2.3) converges pointwise and in $L^1(\mathbb{R})$ and takes values in $\{-1,0,1\}$ since the black intervals $B_k$ are pairwise disjoint.

From $E^{(1)}$ and $g^{(1)}$ we construct the corresponding $d$-dimensional objects $E \subseteq \mathbb{R}^d$ and $g \colon \mathbb{R}^d \to \{-1,0,1\}$ by taking a tensor product:

$$E = \Big\{(x_1,\dots,x_d) \in [-\tfrac{1}{2},\tfrac{1}{2}]^d \mid x_i \in E^{(1)} \text{ for some } i\Big\},$$
$$g(x_1,\dots,x_d) = g^{(1)}(x_1)\cdots g^{(1)}(x_d).$$

We next show that $E$ has full Hausdorff dimension precisely when (2.2.1) holds.

**Lemma 2.2.2.** *The Hausdorff dimension of $E$ is $d$ if and only if $\Theta_k^{1/k} \to 1/2$ as $k \to \infty$.*

The proof is based on the following version of Frostman's lemma, see e.g. [Mat15, Theorem 2.7].

**Theorem 2.2.3.** *A compact set $E$ in $\mathbb{R}^d$ has Hausdorff dimension at least $\alpha \in [0,d]$ if and only if for any $\beta \in [0,\alpha)$ there exists a Borel measure $\mu$ on $\mathbb{R}^d$ such that $\mu(E) > 0$ and $\mu(U) \le \operatorname{diam}(U)^\beta$ for any Borel set $U$.*

*Proof of Lemma 2.2.2.* From the above recursive construction one can verify that the lengths of any $W_k \in \mathcal{W}_k$ and $B_k \in \mathcal{B}_k$ are

$$|W_k| = \Theta_k \quad \text{and} \quad |B_k| = (1-2\theta_k)\Theta_k. \tag{2.2.4}$$

The families $\mathcal{W}_k$ and $\mathcal{B}_k$ each contain $2^k$ intervals.

First assume that $E$ has Hausdorff dimension $d$. Let $\beta \in [0, d)$ and let the measure $\mu$ be as in Theorem 2.2.3. Since $E$ is covered by $d$ isometric copies of $\bigcup_{W_k \in \mathcal{W}_k} W_k \times [-1/2, 1/2]^{d-1}$ and $\operatorname{diam}(W_k) = \Theta_k$, we can cover $E$ by at most $2^k \Theta_k^{-d+1} d$ boxes of diameter comparable to $\Theta_k$ and hence

$$0 < \mu(E) \lesssim 2^k \Theta_k^{-d+1} d \cdot \Theta_k^{\beta}.$$

Taking $k$th roots, followed by $\liminf_{k\to\infty}$ and then letting $\beta \to d$, this implies

$$\liminf_{k\to\infty} \Theta_k^{1/k} \geq \tfrac{1}{2}.$$

Since $\Theta_k = \theta_0 \cdots \theta_{k-1} < 2^{-k}$, we in fact have $\Theta_k^{1/k} \to 1/2$ as $k \to \infty$.

Now we assume that $\Theta_k^{1/k} \to 1/2$ as $k \to \infty$. Let $\mu$ be the Borel probability measure supported in $E$ given by

$$\mu(W_k \times T) = 2^{-k} |T|$$

for any $W_k \in \mathcal{W}_k$, $k \geq 0$, and any Borel set $T \subseteq [-1/2, 1/2]^{d-1}$.

Fix a $\beta < d$ and choose a large integer $N = N(\beta) \geq 2$ such that

$$\Theta_k^{1/k} \geq 2^{-1/(1-d+\beta)} \quad \text{for any } k \geq N.$$

Let $U \subseteq \mathbb{R}^d$ be a Borel set of diameter $\operatorname{diam}(U) < \Theta_N$. Since $\Theta_k \to 0$ as $k \to \infty$, there is a $k \geq N$ such that

$$\Theta_{k+1} \leq \operatorname{diam}(U) < \Theta_k.$$

Then $U$ intersects $W_k \times [-1/2, 1/2]^{d-1}$ for at most two intervals $W_k \in \mathcal{W}_k$. Hence,

$$\tfrac{1}{4}\mu(U) \leq 2^{-k-1} \operatorname{diam}(U)^{d-1} \leq \Theta_{k+1}^{1-d+\beta} \operatorname{diam}(U)^{d-1} \leq \operatorname{diam}(U)^{\beta}.$$

Since $\Theta_N < 1/4$ and $\mu(\mathbb{R}^d) = 1$, it follows that the renormalised measure $\Theta_N^{\beta}\mu$ satisfies the inequality in Theorem 2.2.3 for any Borel set $U$. Letting $\beta \to d$, it follows that $E$ has Hausdorff dimension $d$. □

Note that $\check{g}$ is bounded since $g$ is integrable. Hence, we can prove Proposition 2.2.1(ii) by showing the following result.

**Lemma 2.2.4.** *Let $p > 1$. It holds that*

$$\|\check{g}\|_{L^p(\mathbb{R}^d)}^{p/d} \lesssim \int_0^{\infty} \left( \sum_{k \in S} \frac{2^k \Theta_k}{1 + (1 - 2\theta_k)\Theta_k |\xi|} \prod_{j=0}^{k-1} |\cos((1-\theta_j)\Theta_j \pi \xi)| \right)^p d\xi.$$

*Proof.* By Fubini's theorem we have $\check{g}(\xi_1, \dots, \xi_d) = \check{g}^{(1)}(\xi_1) \cdots \check{g}^{(1)}(\xi_d)$ and therefore

$$\|\check{g}\|_{L^p(\mathbb{R}^d)}^{p/d} = \int_{-\infty}^{\infty} |\check{g}^{(1)}(\xi)|^p \, d\xi. \tag{2.2.5}$$

Since the series (2.2.3) converges in $L^1(\mathbb{R})$, we have

$$\check{g}^{(1)}(\xi) = \sum_{k \in S} (-1)^k \sum_{B_k \in \mathcal{B}_k} \frac{\sin(|B_k| \pi \xi)}{\pi \xi} e^{2\pi i c(B_k)\xi} \tag{2.2.6}$$

and the outer sum converges uniformly. The midpoints of the black (and the white) intervals are given by

$$\{c(B_k) \mid B_k \in \mathcal{B}_k\} = \left\{ \sum_{j=0}^{k-1} \sigma_j \frac{1-\theta_j}{2} \Theta_j \;\middle|\; \sigma_j \in \{-1, 1\} \right\}.$$

We use this to rewrite the inner sum in (2.2.6) and then we apply the identity $e^{-i\alpha} + e^{i\alpha} = 2\cos(\alpha)$ to get

$$\sum_{B_k \in \mathcal{B}_k} e^{2\pi i c(B_k)\xi} = \prod_{j=0}^{k-1} \Big( e^{-i(1-\theta_j)\Theta_j \pi \xi} + e^{i(1-\theta_j)\Theta_j \pi \xi} \Big) = 2^k \prod_{j=0}^{k-1} \cos((1-\theta_j)\Theta_j \pi \xi).$$

Combining this with (2.2.6), (2.2.4) and the elementary estimate $|\sin(\eta)/\eta| \lesssim (1+|\eta|)^{-1}$ gives the pointwise bound

$$|\check{g}^{(1)}(-\xi)| = |\check{g}^{(1)}(\xi)| \lesssim \sum_{k\in S} \frac{2^k \Theta_k}{1+(1-2\theta_k)\Theta_k|\xi|} \prod_{j=0}^{k-1} |\cos((1-\theta_j)\Theta_j \pi \xi)|.$$

In view of (2.2.5), this completes the proof of the lemma. □

We now complete the proof of Proposition 2.2.1(i) by showing the following result.

**Lemma 2.2.5.** *If there exists an $\epsilon > 0$ such that (2.2.2) holds, then $E$ is the non-Lebesgue set of $g$.*

Note that (2.2.2) implies $\liminf_{k\to\infty} \theta_k < 1/2$ and hence by (2.2.4):

$$|E| \leq d \cdot \lim_{k\to\infty} \sum_{W_k \in \mathcal{W}_k} |W_k| \cdot |[-\tfrac{1}{2}, \tfrac{1}{2}]|^{d-1} = d \cdot \lim_{k\to\infty} 2^k \Theta_k = 0.$$

By the Lebesgue differentiation theorem, this is a necessary condition for $E$ to be the non-Lebesgue set of a locally integrable function.

The key step in the proof of Lemma 2.2.5 is the following lower bound on the oscillation of averages of the one-dimensional function $g^{(1)}$.

**Lemma 2.2.6.** *For any $k \geq 0$ and any $W_k, W_k' \in \mathcal{W}_k$ it holds that*

$$\fint_{W_k} g^{(1)}\,dx = \fint_{W_k'} g^{(1)}\,dx$$

*and for any $k, k+1 \in S$ and any $W_k \in \mathcal{W}_k$ and $W_{k+1} \in \mathcal{W}_{k+1}$ it holds that*

$$\left|\fint_{W_k} g^{(1)}\,dx - \fint_{W_{k+1}} g^{(1)}\,dx\right| \geq 2(1-2\theta_k)(1-2\theta_{k+1}).$$

Before proving this lemma, we show how it can be used to prove Lemma 2.2.5.

*Proof of Lemma 2.2.5.* The function $g$ is constant in each of the connected components of the open set $\mathbb{R}^d \setminus E$. Hence, every point of that set is a Lebesgue point.

It remains to show that there are no Lebesgue points in $E$. By (2.2.2) and Lemma 2.2.6, we can find sequences of indices $k(a), k'(a) \in S$, $a \geq 0$, with $k'(a) = k(a) \pm 1$ and $k(a) \to \infty$ as $a \to \infty$ such that

$$\theta_{k(a)}, \theta_{k'(a)} > \epsilon, \quad \left|\fint_{W_{k(a)}} g^{(1)}\,dx - \fint_{W_{k'(a)}} g^{(1)}\,dx\right| > 8\epsilon^2, \quad \left|\fint_{W_{k(a)}} g^{(1)}\,dx\right| > 4\epsilon^2 \tag{2.2.7}$$

for any $W_{k(a)} \in \mathcal{W}_{k(a)}$ and $W_{k'(a)} \in \mathcal{W}_{k'(a)}$.

Fix a point $(x_1, \ldots, x_d) \in E$ and let $a \geq 0$ be so large that $g^{(1)}$ is identically equal to 1 or identically equal to $-1$ in a $\Theta_{k(a)}$-neighborhood of any $x_i \notin E^{(1)}$, $i = 1, \ldots, d$. For $x_i \notin E^{(1)}$, let $W^i_{k(a)}$ be any interval, not necessarily in $\mathcal{W}_{k(a)}$, of length $\Theta_{k(a)}$ containing $x_i$. For $x_i \in E^{(1)}$, choose intervals $W^i_{k(a)} \in \mathcal{W}_{k(a)}$ and $W^i_{k'(a)} \in \mathcal{W}_{k'(a)}$ that contain $x_i$. In both cases, we have by $g = \pm 1$ and the third inequality in (2.2.7), respectively:

$$\left| \fint_{W^i_{k(a)}} g^{(1)}\, dx \right| > 4\epsilon^2. \tag{2.2.8}$$

Fix an index $j$ for which $x_j \in E$ and consider the Cartesian products

$$Q_{2a} = \prod_{i=1}^{d} W^i_{k(a)}, \qquad Q_{2a+1} = \prod_{i=1}^{j-1} W^i_{k(a)} \times W^j_{k'(a)} \times \prod_{i=j+1}^{d} W^i_{k(a)}.$$

Making use of the tensor product structure of $g$, we can use the second inequality in (2.2.7) and (2.2.8) as follows:

$$\begin{aligned} \left| \fint_{Q_{2a}} g - \fint_{Q_{2a+1}} g \right| &= \left| \fint_{W^j_{k(a)}} g^{(1)} - \fint_{W^j_{k'(a)}} g^{(1)} \right| \cdot \prod_{i \neq j} \left| \fint_{W^i_{k(a)}} g^{(1)} \right| \\ &> 2^{2d+1}\epsilon^{2d} > 0, \end{aligned}$$

which holds for any large enough $a$, i.e. the averages of $g$ over the boxes $Q_b$ do not converge as $b \to \infty$. On the other hand, by the first inequality in (2.2.7) these boxes have bounded eccentricity. This shows that $(x_1, \dots, x_d) \in \bigcap_{b \geq 0} Q_b$ is not a Lebesgue point of $g$. □

To complete the proof of Lemma 2.2.5 and hence of Proposition 2.2.1, we need to perform the calculations leading to Lemma 2.2.6.

*Proof of Lemma 2.2.6.* By construction, it holds that $B_j \cap W_k = \emptyset$ when $B_j \in \mathcal{B}_j$, $W_k \in \mathcal{W}_k$ and $j < k$. For $j \geq k$ we either have $B_j \cap W_k = \emptyset$ or $B_j \subseteq W_k$. Hence,

$$\fint_{W_k} g^{(1)}\, dx = \fint_{W_k} \sum_{j \in S;\, j \geq k} (-1)^j \sum_{B_j \in \mathcal{B}_j;\, B_j \subseteq W_k} \chi_{B_j}. \tag{2.2.9}$$

For a fixed $W_k \in \mathcal{W}_k$, there are precisely $2^{j-k}$ intervals $B_j \in \mathcal{B}_j$ for which $B_j \subseteq W_k$. Furthermore for fixed $k$ and $j$, the average $\fint_{W_k} \chi_{B_j}\, dx$ does by (2.2.4) not depend on the choice of $W_k \in \mathcal{W}_k$ and $B_j \in \mathcal{B}_j$ as long as $B_j \subseteq W_k$. This shows the first claim of the lemma.

It remains to prove the inequality in the second claim. Let $k, k+1 \in S$ and let $W_k \in \mathcal{W}_k$ and $W_{k+1} \in \mathcal{W}_{k+1}$. As $W_k$ is the disjoint union of two intervals in $\mathcal{W}_{k+1}$ and one interval in $\mathcal{B}_k$, we have by (2.2.4):

$$\fint_{W_k} g^{(1)}\, dx = 2\theta_k \fint_{W_{k+1}} g^{(1)}\, dx + (-1)^k(1 - 2\theta_k).$$

Since $|g^{(1)}(x)| \leq 1$ for any $x \in \mathbb{R}$, it follows from ignoring all but the first term of the outer sum on the right-hand side of (2.2.9) that

$$(-1)^{k+1} \fint_{W_{k+1}} g^{(1)}\, dx \geq -1 + 2(1 - 2\theta_{k+1}).$$

Together with the last equation this implies that

$$(-1)^k \left( \fint_{W_k} g^{(1)}\, dx - \fint_{W_{k+1}} g^{(1)}\, dx \right) \geq 2(1 - 2\theta_k)(1 - 2\theta_{k+1}).$$

Because the right-hand side is positive, this completes the proof of Lemma 2.2.6. □

We have now proved Proposition 2.2.1.

*Remark* 2.2.7. In the definition (2.2.3) of $g^{(1)}$, the oscillating coefficients $\pm 1$ may be replaced by 0 and 1, respectively, yielding $\{0, 1\}$-valued functions $g^{(1)}$ and $g$ instead of $\{-1, 0, 1\}$-valued ones. The thus modified function $g$ still satisfies all properties that are asserted in Proposition 2.2.1. However, if $d \geq 2$, then the non-

Lebesgue set of $g$, while still of full Hausdorff dimension, would be a proper subset of $E$ since (2.2.8) would fail for some $x_i \notin E^{(1)}$.

## 2.3 Incomplete cosine expansions of $\sin(x)/x$

In the proof of Theorem 1.1.2 we need to verify the inequality in Proposition 2.2.1(ii). In this section, we provide a tool for this task by bounding $p$-integrals of products of cosines with dyadic phases. Our motivation is Euler's product expansion of the sinc function:

$$\prod_{j=1}^{\infty} \cos(2^{-j}\xi) = \frac{\sin(\xi)}{\xi}. \tag{2.3.1}$$

A quick proof of this identity can be obtained by iterating the double-angle formula $\sin(\xi) = 2\sin(\xi/2)\cos(\xi/2)$ and using that $2^n \sin(\xi/2^n) \to \xi$ as $n \to \infty$. Other proofs are possible, see e.g. the probabilistic proof in [Kac59].

The function in (2.3.1) lies in $L^p_\xi(\mathbb{R})$ for any $p > 1$. We are interested in the stability of this property under omission of factors from the product of cosines. First, it follows from (2.3.1) that the product can be truncated after logarithmically in $|\xi|$ many steps without a loss in the decay rate. More precisely,

$$\left|\frac{\sin(\xi)}{\xi}\right| \leq \prod_{j=1}^{n} |\cos(2^{-j}\xi)| \leq \frac{\pi}{2} \cdot \left|\frac{\sin(\xi)}{\xi}\right| \qquad \text{if } |\xi| \leq 2^{n-1}\pi. \tag{2.3.2}$$

However, if any further factor is omitted from this finite product, then the pointwise upper bound fails dramatically for some $|\xi| \leq 2^{n-1}\pi$. We therefore focus on integral estimates. Given a finite set $J$ of integers, we define its *number of components* $b(J)$ as follows:

$$b(J) = \#\{j \in J \mid j - 1 \notin J\}.$$

**Lemma 2.3.1.** *Let* $n \geq 1$ *be an integer and let* $J \subseteq \{1, 2, \ldots, n\}$*. Then it holds for every* $p \in (1, \infty)$ *that*

$$\int_0^{2^{n-1}\pi} \prod_{j \in J} |\cos(2^{-j}\xi)|^p \, d\xi \leq 2^{n-|J|-1} \pi C_p^{b(J)}, \tag{2.3.3}$$

*where* $C_p$ *is a finite constant that depends only on* $p$*.*

*Proof.* We prove the lemma with the (non-optimal) constant

$$C_p = 2\pi^p \sum_{s=1}^{\infty} \frac{1}{s^p} < \infty.$$

We proceed by induction on the number of components $b(J)$. If $b(J) = 0$, then $J$ is empty and (2.3.3) is immediate.

Now fix a nonnegative integer $b$ and assume that (2.3.3) holds whenever $b(J) = b$. Let $n$ be a positive integer and let $J_1$ be a subset of $\{1, \ldots, n\}$ such that $b(J_1) = b+1$. We can decompose this set as $J_1 = J_0 \cup \{\ell, \ell+1, \ldots, m\}$ where $b(J_0) = b$ and $\sup(J_0) + 2 \leq \ell \leq m \leq n$. Write $n_0 = \max(J_0 \cup \{0\})$.

We need to show (2.3.3) for $J_1$. To this end, we cover the domain of integration $[0, 2^{n-1}\pi]$ by the essentially disjoint intervals of equal length

$$A(q, r) = [(2^{m-1}q + 2^{n_0-1}r)\pi, (2^{m-1}q + 2^{n_0-1}(r+1))\pi]$$

for any integers $q$ and $r$ with $0 \leq q < 2^{n-m}$ and $0 \leq r < 2^{m-n_0}$. If $j \in J_0$, then the function $\xi \mapsto |\cos(2^{-j}\xi)|$ is even and $2^{n_0}\pi$-periodic. We use this and the induction hypothesis to obtain

$$\begin{aligned}\int_{A(q,r)} \prod_{j\in J_0} |\cos(2^{-j}\xi)|^p \, d\xi &= \int_0^{2^{n_0-1}\pi} \prod_{j\in J_0} |\cos(2^{-j}\xi)|^p \, d\xi \\ &\le 2^{n_0-|J_0|-1}\pi C_p^{b(J_0)} \\ &= 2^{n_0-|J|+m-\ell}\pi C_p^{b(J)-1}.\end{aligned}$$

Similarly for $j \le m$, the function $\xi \mapsto |\cos(2^{-j}\xi)|$ is even and $2^m\pi$-periodic. This gives

$$\begin{aligned}\sup_{\xi\in A(q,r)} \prod_{j=\ell}^{m} |\cos(2^{-j}\xi)| &= \sup_{\xi\in A(0,r)} \prod_{j=\ell}^{m} |\cos(2^{-j}\xi)| \\ &= \sup_{\xi\in A(0,r)} \prod_{j=1}^{m-\ell+1} |\cos(2^{-j}2^{1-\ell}\xi)| \\ &\le \frac{\pi}{1+2^{n_0-\ell}\pi r}.\end{aligned}$$

The last inequality follows from (2.3.2) and the definition of $A(0,r)$. We combine the last two estimates to get an estimate for the product over the full set of indices $J_1$:

$$\int_{A(q,r)} \prod_{j\in J_1} |\cos(2^{-j}\xi)|^p \, d\xi \le \frac{2^{n_0-|J|+m-\ell}\pi^{p+1}C_p^{b(J)-1}}{(1+2^{n_0-\ell}\pi r)^p}.$$

Note that the numerator does not depend on $q$ or $r$ and the denominator does not depend on $q$. Therefore, we can sum over $q$ and $r$ as follows:

$$\begin{aligned}\int_0^{2^{n-1}\pi} \prod_{j\in J_1} |\cos(2^{-j}\xi)|^p \, d\xi &= \sum_{q=0}^{2^{n-m}-1} \sum_{r=0}^{2^{m-n_0}-1} \int_{A(q,r)} \prod_{j\in J_1} |\cos(2^{-j}\xi)|^p \, d\xi \\ &\le 2^{n-m}2^{n_0-|J|+m-\ell}\pi^{p+1}C_p^{b(J)-1} \sum_{r=0}^{\infty} \frac{1}{(1+2^{n_0-\ell}\pi r)^p}\end{aligned}$$

$$\leq 2^{n-|J|}\pi^{p+1}C_p^{b(J)-1}\sum_{s=1}^{\infty}\frac{1}{s^p}.$$

For the last inequality, we replaced $\pi r$ by the largest multiple of $2^{\ell-n_0}$ not exceeding $r$. By our choice of $C_p$, this shows (2.3.3) for $J_1$ and hence closes the induction. □

In the proof of Theorem 1.1.2 we need the following perturbed version of the previous result.

**Lemma 2.3.2.** *Let $n \geq 1$ be an integer, let $J \subseteq \{1, 2, \ldots, n\}$, let $1 < p \leq p_0 < \infty$ and let $\epsilon > 0$. There exists a number $\delta = \delta(n, p_0, \epsilon) > 0$ that does not depend on $J$ or $p$ such that if*

$$\phi_j \in (2^{-j}(1-\delta), 2^{-j}(1+\delta)) \tag{2.3.4}$$

*for all $j \in J$, then the following inequality holds:*

$$\int_0^{2^{n-1}\pi}\prod_{j\in J}|\cos(\phi_j\xi)|^p\,d\xi \leq (1+\epsilon)2^{n-|J|-1}\pi C_p^{b(J)}.$$

*Proof.* First, let $J \subseteq \{1, \ldots, n\}$ be fixed. Consider the integrals

$$I^p(\{\phi_j\}_{j\in J}) = \int_0^{2^{n-1}\pi}\prod_{j\in J}|\cos(\phi_j\xi)|^p\,d\xi \in (0,\infty).$$

By the dominated convergence theorem, $I^p(\{\phi_j\}_{j\in J})$ is continuous in $p \in [1,\infty)$ and $\phi_j \in \mathbb{R}$. Hence it is uniformly continuous once $p$ and the $\phi_j$ are confined to a compact domain. We have by compactness that

$$\int_0^{2^{n-1}\pi}\prod_{j\in J}|\cos(2^{-j}\xi)|^p\,d\xi > 0$$

uniformly in $p \in [1, p_0]$. Now this together with uniform continuity allows us to find $\delta = \delta(n, J, p_0, \epsilon) > 0$ such that

$$\int_0^{2^{n-1}\pi} \prod_{j\in J} |\cos(\phi_j \xi)|^p \, d\xi \leq (1+\epsilon) \int_0^{2^{n-1}\pi} \prod_{j\in J} |\cos(2^{-j}\xi)|^p \, d\xi$$

whenever $1 \leq p \leq p_0$ and (2.3.4) holds. As there are only finitely many subsets of $\{1, \ldots, n\}$, the number $\delta$ can in fact be chosen independently of $J$. An application of Lemma 2.3.1 completes the proof. □

## 2.4 Proof of Theorem 1.1.2

In this section, we use the criteria of Proposition 2.2.1 and the analytical Lemma 2.3.2 to prove Theorem 1.1.2 in five steps.

### 2.4.1 Choice of parameters

We fix a set $S$ of nonnegative integers and a weight function $w : S \to \{2, 3, 4, \ldots\}$ and define

$$w^+(k) = \sum_{j\in S;\, j<k} w(j) \qquad \text{and} \qquad \chi^+(k) = \#(S \cap [0, k-1])$$

such that there is a finite constant $M \geq 2$ and there are infinitely many $k \in S$ for which $k+1 \in S$ and $w(k), w(k+1) \leq M$ and such that we have the following asymptotics:

$$\lim_{\substack{k\to\infty \\ k\in S}} \frac{w(k)}{w^+(k)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\chi^+(k)}{w^+(k)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{w^+(k)}{k} = 0. \tag{2.4.1}$$

For example, we may choose $S = \{n^3 \mid n \geq 2\} \cup \{n^6 + 1 \mid n \geq 2\}$ and $w(n^3) = n$ if $n$ is not a square and $w(n^6) = w(n^6 + 1) = 2$ for any $n \geq 2$.

Next we fix a function $r : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ satisfying

$$\lim_{s\to\infty} r(s) = \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{r(s)}{w^+(s)} = 0. \tag{2.4.2}$$

Let $\delta(n, p_0, \epsilon)$ be the numbers from Lemma 2.3.2 and write $\delta(n) = \delta(n, 2, 1)$. We may assume that $0 < \delta(n+1) < \delta(n) < 1/2$ for any $n$. Finally we choose dissection ratios $\theta_j \in (0, 1/2)$ as follows:

$$\theta_j = \begin{cases} 2^{-w(j)} & \text{if } j \in S, \\ \frac{1}{2}(1 - \alpha_j) & \text{if } j \notin S, \end{cases}$$

with error terms $\alpha_j$ satisfying $0 < 2\alpha_{j+1} \leq \alpha_j \leq 1/4$ and

$$2\alpha_j \leq \inf\{\delta(2s) \mid s \geq 0, r(s) \leq j\} \tag{2.4.3}$$

for any $j \geq 0$. Note that the infimum above is positive since $r(s) \to \infty$ as $s \to \infty$.

### 2.4.2 Asymptotics of products of dissection ratios

It follows that

$$\prod_{j=k}^{\infty}(1 - \alpha_j) \geq 1 - \sum_{j=k}^{\infty} \alpha_j \geq 1 - 2\alpha_k.$$

This bound is significant because of the following expansion of $\Theta_k$ based on our choice of dissection ratios:

$$\begin{aligned}\Theta_k &= \prod_{j\in S;\, j<k} 2^{-w(j)} \cdot \prod_{j\notin S;\, j<k} \tfrac{1}{2}(1-\alpha_j)\\ &= 2^{-k-w^+(k)+\chi^+(k)} \cdot \prod_{j\notin S;\, j<k} (1-\alpha_j).\end{aligned}$$

For convenience, we define corresponding to any index $i$ a larger index by

$$i^* = i + w^+(i) - \chi^+(i).$$

Then, we obtain the following inequalities for any $k \geq 0$ and $j \geq r(s)$:

$$2^{-k^*-1} \leq \Theta_k \leq 2^{-k^*}, \tag{2.4.4}$$

$$(1-\delta(2s))2^{-j^*+r(s)^*} \leq \Theta_{r(s)}^{-1}\Theta_j \leq 2^{-j^*+r(s)^*}. \tag{2.4.5}$$

Hence, $\Theta_k$ is close to a particular power of $1/2$ and this relation is even tighter for the partial product $\Theta_{r(s)}^{-1}\Theta_j$. Using the last two limits in (2.4.1), it follows from (2.4.4) that $\Theta_k^{1/k} \to 1/2$ as $k \to \infty$. This verifies assumption (2.2.1) of Proposition 2.2.1. Notice that assumption (2.2.2) is satisfied for any $\epsilon \in (0, 2^{-M})$ by our choices of $S$ and $\theta_j$.

### 2.4.3 Integral decomposition

In order to prove Theorem 1.1.2, it remains to verify the inequality in Proposition 2.2.1(ii) for any $p \in (1,2]$. Since $1-2\theta_k \geq 1/2$ for $k \in S$ we may omit this term from the left-hand side of that inequality. Hence, in order to prove Theorem 1.1.2

it now suffices to show that

$$H(\xi) = \sum_{k\in S} \frac{2^k\Theta_k}{1+\Theta_k|\xi|} \prod_{j=0}^{k-1} |\cos((1-\theta_j)\Theta_j\pi\xi)|$$

lies in $L^p_\xi([0,\infty))$ for any $p \in (1,2]$. Consider the integrals at scales $s \geq k$:

$$I^p_{k,s} = \int_0^{\Theta_s^{-1}} \prod_{j=0}^{k-1} |\cos((1-\theta_j)\Theta_j\pi\xi)|^p \, d\xi. \tag{2.4.6}$$

We use Minkowski's inequality in $L^p([0,\infty))$ and for every $k \in S$ we decompose $[0,\infty)$ into the subintervals $[0,\Theta_k^{-1})$ and $[\Theta_s^{-1},\Theta_{s+1}^{-1})$ for $s \geq k$ to obtain the bound

$$\|H\|_{L^p([0,\infty))} \lesssim \sum_{k\in S} 2^k\Theta_k \left( I^p_{k,k} + \sum_{s=k}^{\infty} \frac{\Theta_s^p}{\Theta_k^p} I^p_{k,s+1} \right)^{1/p} =: \sum_{k\in S} A^p_k. \tag{2.4.7}$$

We let go of some factors in the product of cosines in (2.4.6), perform a linear change of variables and then slightly enlarge the domain of integration using (2.4.4) to obtain

$$\begin{aligned} I^p_{k,s} &\leq \pi^{-1}\Theta^{-1}_{r(s)} \int_0^{\Theta_{r(s)}\Theta_s^{-1}\pi} \prod_{\substack{j\notin S\\ r(s)\leq j\leq k-1}} |\cos((1-\theta_j)\Theta^{-1}_{r(s)}\Theta_j\xi)|^p \, d\xi \\ &\lesssim \Theta^{-1}_{r(s)} \int_0^{2^{s^*-r(s)^*+1}\pi} \prod_{\substack{j\notin S\\ r(s)\leq j\leq k-1}} |\cos((1-\theta_j)\Theta^{-1}_{r(s)}\Theta_j\xi)|^p \, d\xi. \end{aligned} \tag{2.4.8}$$

### 2.4.4 Application of Lemma 2.3.2

We next analyze the phases of the cosines in (2.4.8). If $j \notin S$ and $r(s) \leq j \leq k-1 < s$, then (2.4.5), (2.4.3) and the inequality $1-\theta_j > 1/2$ imply

$$|2^{-(j^*-r(s)^*+1)} - (1-\theta_j)\Theta_{r(s)}^{-1}\Theta_j| \leq 2^{-(j^*-r(s)^*+1)}\delta(2s).$$

If $k$ and hence $s$ are larger than some sufficiently large constant $K = K_{S,\theta_j}$, then we have by (2.4.1) that $s^* \leq 2s-2$ and $j^* - r(s)^* + 1 \leq 2s$. Therefore, we verified the assumption (2.3.4) in Lemma 2.3.2 in the case when $k \geq K$ with the near-dyadic phases

$$\phi_{j^*-r(s)^*+1} = \phi_{j^*-r(s)^*+1,s} = (1-\theta_j)\Theta_{r(s)}^{-1}\Theta_j$$

and the following set of dyadic exponents:

$$J = J_{k,s} = \{j^* - r(s)^* + 1 \mid j \notin S \text{ and } r(s) \leq j \leq k-1\}.$$

Since the map $j \mapsto j^*$ is strictly increasing and therefore injective we have

$$\#J_{k,s} \geq k - \chi^+(k) - r(s) = k^* - w^+(k) - r(s).$$

We have $(j+1)^* = j^* + 1$ if and only if $j \notin S$. Hence for any $j \notin S$ with $r(s) < j \leq k-1$ the condition $j^* - r(s)^* \notin J_{k,s}$ is equivalent to $j-1 \in S$. This allows us to bound the number of components of $J_{k,s}$:

$$b(J_{k,s}) \leq \#(S \cap [r(s), k-2]) + 1 \leq \chi^+(k) + 1.$$

Furthermore, the set $J_{k,s}$ is bounded from above:

$$\sup(J_{k,s}) \leq (k-1)^* - r(s)^* + 1 \leq s^* - r(s)^*.$$

Compare this to the upper bound of integration in (2.4.8) to see that Lemma 2.3.2 is applicable to the integral in (2.4.8). We obtain that

$$I^p_{k,s} \lesssim \Theta^{-1}_{r(s)} 2^{s^* - r(s)^* - \# J_{k,s}} C_p^{b(J_{k,s})}, \quad \text{if } k \geq K.$$

We use (2.4.4) and the above bounds on $\# J_{k,s}$ and $b(J_{k,s})$ to bring this estimate into a more convenient form:

$$I^p_{k,s} \lesssim \Theta^{-1}_s 2^{-\# J_{k,s}} C_p^{b(J_{k,s})} \lesssim \Theta^{-1}_s \Theta_k 2^{w^+(k)+r(s)} C_p^{\chi^+(k)+1}, \quad \text{if } k \geq K.$$

### 2.4.5 Conclusion

We use the previous inequality and (2.4.4) to estimate the terms $A^p_k$, $k \in S$, of the sum in (2.4.7):

$$A^p_k \lesssim 2^{-(1-1/p)w^+(k)+\chi^+(k)} C_p^{(\chi^+(k)+1)/p} \left( 2^{r(k)} + \sum_{s=k}^{\infty} \frac{\Theta_s^{p-1}}{\Theta_k^{p-1}} \theta_s^{-1} 2^{r(s+1)} \right)^{1/p}.$$

Fix a positive number $\epsilon$ such that $2\epsilon < 1 - 1/p$ and $\epsilon < p - 1$. After possibly increasing $K$, we obtain from the limits (2.4.1) and (2.4.2) that

$$\theta_s^{-1} 2^{r(s+1)} \leq 2^{\epsilon w^+(s)} = 2^{\epsilon w^+(k)} 2^{\epsilon(w^+(s)-w^+(k))} \leq 2^{\epsilon w^+(k)} \Theta_s^{-\epsilon} \Theta_k^{\epsilon}, \quad \text{if } s \geq K$$

and further for $k \in S$ with $k \geq K$:

$$A_k^p \lesssim 2^{-(1-1/p-2\epsilon)w^+(k)} \left(1 + \sum_{s=k}^{\infty} \frac{\Theta_s^{p-1-\epsilon}}{\Theta_k^{p-1-\epsilon}}\right)^{1/p} \leq 2^{-(1-1/p-2\epsilon)w^+(k)} S_{p,\epsilon},$$

where $S_{p,\epsilon}$ is a constant depending on $p$ and $\epsilon$. We split the sum in (2.4.7) as follows:

$$\|H\|_{L^p([0,\infty))} \lesssim \sum_{k\in S;\, k<K} A_k^p + \sum_{k\in S;\, k\geq K} A_k^p.$$

The sum over $k \geq K$ is dominated by a convergent geometric series due to the last bound on $A_k^p$ and since $1 - 1/p - 2\epsilon > 0$ and $w^+(k_1) \leq w^+(k_2) - 2$ for any $k_1, k_2 \in S$ with $k_1 < k_2$. The sum over $k < K$ above is finite since it is the sum of finitely many terms $A_k^p$, each of which is finite. Hence, $H$ lies in $L^p([0,\infty))$. This completes the proof of Theorem 1.1.2.

## 2.5 Proofs of Corollaries 1.1.5 and 1.1.9

We first prove Corollary 1.1.5 using Theorems 1.1.2 and 1.1.3 and then we prove Corollary 1.1.9 using Corollary 1.1.5 and Theorem 1.1.8.

*Proof of Corollary 1.1.5.* By Theorem 1.1.2, there is a function $f \in \bigcap_{p\in(1,\infty]} L^p(\mathbb{R}^d)$ such that the non-Lebesgue set $E$ of $\hat{f}$ is compact and has full Hausdorff dimension $d$. We now show that $E$ satisfies the remaining assertion of Corollary 1.1.5.

To this end, let $\mu$ be a nonzero Borel measure such that the restriction inequality (1.1.1) holds for some exponents $p \in (1,2]$ and $q \in [1,\infty]$. It suffices to show that $\mu(E) = 0$. A scaling argument shows that since $p > 1$, we necessarily have $q < \infty$. Therefore, it follows from (1.1.1) that $\mu$ is $\sigma$-finite and an interpolation with the trivial $L^1(\mathbb{R}^d) \to L^\infty(\mu)$ bound gives a $L^{p_1}(\mathbb{R}^d) \to L^{q_1}(\mu)$ restriction estimate with $1 < p_1 < q_1 < \infty$. Hence, by Theorem 1.1.3 and since $f$ lies in $L^{2p_1/(p_1+1)}(\mathbb{R}^d)$, $\mu$-almost every point is a Lebesgue point of $\hat{f}$. But $\hat{f}$ has no Lebesgue points in $E$ and therefore $\mu(E) = 0$. This completes the proof. □

We need the following theorem of Besicovitch [Bes52], see also [Dav52].

**Theorem 2.5.1.** *Any closed set in $\mathbb{R}^d$ has subsets of any smaller Hausdorff dimension.*

*Proof of Corollary 1.1.9.* If $p = 2$, then $\alpha = d$ and any compact set of positive Lebesgue measure proves the corollary. We may now assume that $p < 2$.

Let $E$ be the set from Corollary 1.1.5. Let $E_1$ be a compact subset of $E$ of Hausdorff dimension equal to $\alpha$. Such a subset can be obtained by appropriately reducing the Cantor set of Section 2.2 or by applying Theorem 2.5.1.

If $p = 1$, then we are finished since $p_{\text{res}}(E) = 1$ and hence $p_{\text{res}}(E_1) = 1$. We may now assume that $1 < p \le 2d/(2d - \alpha)$. Hence, there is an $\alpha_0 \in (0, \alpha]$ such that $p = 2d/(2d - \alpha_0)$. Using the previous reduction to the case $p < 2$, we see that $\alpha_0 < d$. By [ŁW18, Theorem 2], there is a compact set $E_2$ of Hausdorff dimension $\alpha_0$ such that $p_{\text{res}}(E_2) = p$. Now we have for the Hausdorff dimensions,

$$\dim_H(E_1 \cup E_2) = \max(\dim_H(E_1), \dim_H(E_2)) = \max(\alpha, \alpha_0) = \alpha$$

and similarly for the endpoint restriction exponents,

$$p_{\text{res}}(E_1 \cup E_2) = \max(p_{\text{res}}(E_1), p_{\text{res}}(E_2)) = \max(1, p) = p.$$

Hence, the compact set $E_1 \cup E_2$ has the claimed properties. □

# Chapter 3

# The one-dimensional centred maximal function diminishes the variation of indicator functions

In this chapter, we prove the results contained in Section 1.2. In Section 3.1, we introduce a local variation bound in the continuous setting which we then prove in Section 3.2. This bound is the key to the proof of Theorem 1.2.2, and hence Theorem 1.2.1, in Section 3.3.

Our proofs in the discrete setting are contained in Section 3.4. In Section 3.4.3 we prove an analogous discrete local variation bound which we then apply in Sections 3.4.4 and 3.4.5 to show the discrete Theorem 1.2.5. These proofs can be read mostly independently from Sections 3.2 and 3.3. Section 3.4.1 contains the embedding argument leading to one of the inequalities in Proposition 1.2.4, as well as to the derivation of Theorem 1.2.3 from Theorem 1.2.1. In Section 3.4.2, we use an approximation argument to establish the remaining inequality in Proposition 1.2.4.

## 3.1 Proof strategy

Let us explain our ideas in the continuous setting since they are largely the same in the discrete setting. Our main observation is that, for a function $f\colon \mathbb{R} \to [0, \infty)$

satisfying the assumptions of Theorem 1.2.2, the local variation bound

$$\operatorname{var}_{[a,b]}(Mf) \leq \operatorname{var}_{[a,b]}(f) \tag{3.1.1}$$

holds for any real numbers $a < b$ such that $f(a) = Mf(a)$ and $f(b) = Mf(b)$, i.e. such that $Mf$ is *attached* to $f$ at $a$ and $b$. Our proof of Theorem 1.2.2 heavily relies on this property. The following example shows a typical situation. Denote

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a < x < b, \\ 1/2 & \text{if } x = a \text{ or } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 3.1.1.** Let $c \in (1,3)$ and $f = \chi_{[-c,-1]} + \chi_{[1,c]}$. Then $Mf$ is attached to $f$ at any point $x$ with $1 \leq |x| \leq c$ and $\operatorname{var}_{[-1,1]}(Mf) = c^{-1} < 1 = \operatorname{var}_{[-1,1]}(f)$. The maximal function $Mf$ has a strict local maximum of value $(c-1)/c$ at 0 and two strict local minima of value $(3c-3)/(4c)$ at $\pm c/3$, see Fig. 3.1.

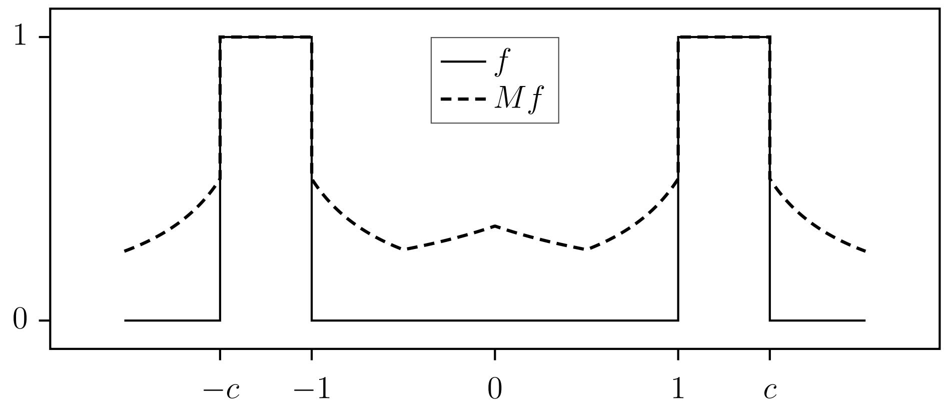


Figure 3.1: The functions $f$ and $Mf$ in Example 3.1.1 with $c = 3/2$.

By taking $c \to 1$, we obtain a sequence of functions with $\operatorname{var}(Mf)/\operatorname{var}(f) \to 1$ that does not converge pointwise modulo symmetries to a nonzero maximiser of Theorem 1.2.2, even though maximisers exist. This lack of compactness presents a difficulty that any proof of a sharp version of (1.2.2) has to overcome.

The calculations leading to Example 3.1.1 and Fig. 3.1, as well as to Example 3.1.3 and Figs. 3.2 and 3.3 below, are straightforward because for step functions it holds that

$$Mf(x) = \sup_{y \neq x \text{ is a jump of } f} \fint_{x-|x-y|}^{x+|x-y|} |f(z)| \,\mathrm{d}z.$$

The local variation bound (3.1.1) will follow from part (i) of the following result. An analogue for unbounded intervals is contained in part (ii).

**Proposition 3.1.2.** *Let $f\colon \mathbb{R} \to [0, \infty)$ be a bounded Borel measurable function and let $I \subseteq \mathbb{R}$ be an interval such that $f(x) = 0$ for almost every $x \in I$. Then the following holds:*

*(i) If $I = [a, b]$ for some real numbers $a < b$, then $\operatorname{var}_{[a,(a+b)/2]}(Mf) \leq Mf(a)$ and $\operatorname{var}_{[(a+b)/2,b]}(Mf) \leq Mf(b)$. Both of these inequalities are strict unless $f$ vanishes almost everywhere on $\mathbb{R}$.*

*(ii) If $I = (-\infty, a]$ or $I = [a, \infty)$ for some real $a$, then $Mf$ is monotone on $I$ and $\operatorname{var}_I(Mf) = Mf(a) - \inf_{x\in I} Mf(x)$.*

Our approach may be compared to the strategy of Aldaz and Pérez Lázaro [AP07] for the uncentred Hardy–Littlewood maximal function $\widetilde{M}f$. They show that if $f\colon \mathbb{R} \to \mathbb{R}$ is of bounded variation and satisfies $f(x) = \limsup_{y\to x} f(y)$ for any $x \in \mathbb{R}$, then $\widetilde{M}f \geq f$ and $\widetilde{M}f$ is attached to $f$ at any strict local maximum point of $\widetilde{M}f$. This can be used to show (3.1.1) when $Mf$ is replaced by $\widetilde{M}f$ and $a$ and $b$ are neighbouring strict local maximum points of $\widetilde{M}f$.

However, in the centred case, $Mf$ is not necessarily attached to $f$ at strict local maxima of $Mf$, see Example 3.1.1 above. We overcome this by making use of a gradient bound for $Mf$ in the proof of Proposition 3.1.2. On the other hand, this bound becomes less useful for our purposes if a function fails to satisfy the assumptions of Theorem 1.2.2. In fact, for general functions of bounded variation, the local variation bound (3.1.1) often fails between points of attachment. This is what prevents us from generalising our results to a substantially larger class of functions than in Theorem 1.2.2.

**Example 3.1.3.** Let $h = 2/5$ and $f = \chi_{[-3/2,-1]} + h \cdot \chi_{[-1/2,1/2]} + \chi_{[1,3/2]}$. Then $f$ is constant in $(-1/2, 1/2)$ and $Mf$ is attached to $f$ at any point $x$ with $2 \leq 8|x| \leq 3$, but $Mf$ has a strict local maximum of value $7/15 > h$ at 0. In particular, (3.1.1) fails between the points of attachment $a = -1/3$ and $b = 1/3$, see Fig. 3.2.

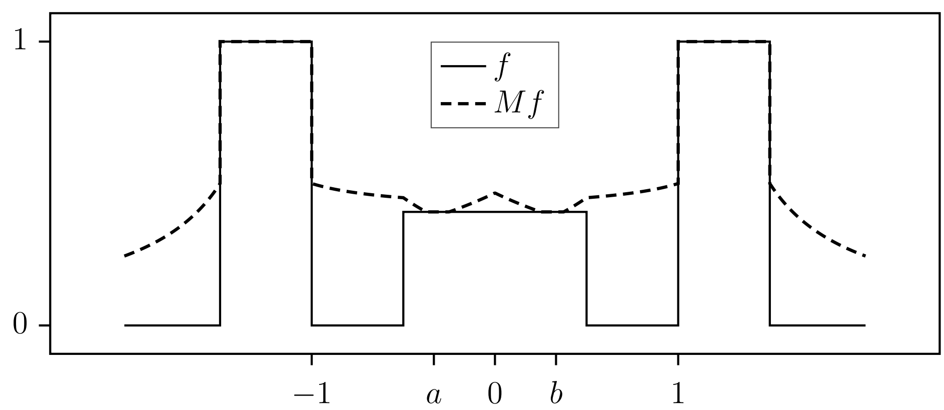


Figure 3.2: The functions $f$ and $Mf$ in Example 3.1.3.

## 3.2 Proof of Proposition 3.1.2

Throughout this section, let $f\colon \mathbb{R} \to [0,\infty)$ be a bounded Borel measurable function. The following result proves the unbounded case in Proposition 3.1.2(ii). By symmetry, it suffices to take $I = [a,\infty)$.

**Lemma 3.2.1.** *Let $a \in \mathbb{R}$ be such that $f(x) = 0$ for almost every $x \geq a$. Then $Mf$ is nonincreasing on $[a,\infty)$ and hence*

$$\mathrm{var}_{[a,\infty)}(Mf) = Mf(a) - \inf_{x\in[a,\infty)} Mf(x).$$

*Proof.* Let $a \leq x \leq y$. By the definition of $Mf$ and the assumptions on $f$,

$$Mf(x) = \sup_{r>x-a} \fint_{x-r}^{x+r} f(z)\,\mathrm{d}z \geq \sup_{r>x-a} \fint_{x-r}^{x+r+2(y-x)} f(z)\,\mathrm{d}z = Mf(y).$$

This completes the proof. □

The rest of this section is devoted to the proof of Proposition 3.1.2(i), i.e. the case that $I = [a,b]$ for some real numbers $a < b$. It suffices to consider the special case that $a = -1$ and $b = 1$ and to prove the strict inequality

$$\mathrm{var}_{[0,1]}(Mf) < Mf(1) \tag{3.2.1}$$

under the assumption that $f(x) = 0$ for almost every $x \in [-1,1]$ and that $f$ does not vanish almost everywhere on $\mathbb{R}$. The general case follows from this because for any nonconstant affine map $\phi\colon \mathbb{R} \to \mathbb{R}$ we have that $M(f\circ\phi)(1) = Mf(\phi(1))$ and

$$\mathrm{var}_{\phi([0,1])}(Mf) = \mathrm{var}_{[0,1]}((Mf)\circ\phi) = \mathrm{var}_{[0,1]}(M(f\circ\phi)).$$

For the proof of (3.2.1) we first note that $Mf$ restricted to $[0,\infty)$ is the pointwise maximum of the auxiliary maximal functions $M_0 f, M_1 f\colon [0,\infty) \to [0,\infty)$ defined by

$$M_0 f(x) = \sup_{r \le 1+x} \fint_{x-r}^{x+r} f(y)\,\mathrm{d}y, \qquad M_1 f(x) = \sup_{r \ge 1+x} \fint_{x-r}^{x+r} f(y)\,\mathrm{d}y,$$

see Fig. 3.3 for an example. Of these, $M_1 f$ only permits averages over large radii. Based on this, our first lemma bounds the difference quotients of $M_1 f$.

**Lemma 3.2.2.** *Let $x, y \ge 0$ be distinct and let $r \ge 1 + x$ be such that*

$$M_1 f(x) = \sup_{s \ge r} \fint_{x-s}^{x+s} f(z)\,\mathrm{d}z.$$

*Then,*

$$\frac{M_1 f(x) - M_1 f(y)}{|x-y|} \le \frac{M_1 f(x)}{r + |x-y|} \le \frac{M_1 f(y)}{r}.$$

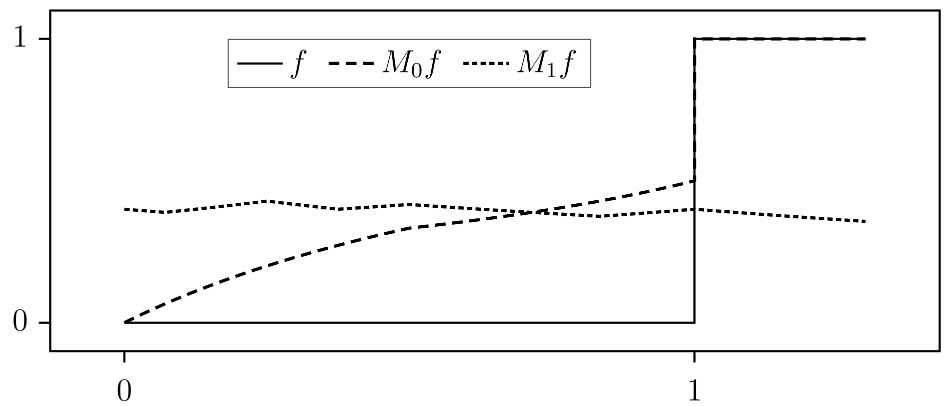


Figure 3.3: The auxiliary maximal functions $M_0 f$ and $M_1 f$ on $[0, 1.3]$ for the function $f = \chi_{[-5/2,-2]} + \chi_{[-3/2,-1]} + \chi_{[1,2]} + \chi_{[3,7/2]}$.

Note that by the definition of $M_1 f$, we can always take $r$ to be at least $1 + x$. The lemma also holds for $Mf$ instead of $M_1 f$, but then we are not guaranteed a good lower bound on $r$.

*Proof.* We have that $M_1 f(x) < \infty$ since $f$ is bounded. Hence, for any $\epsilon > 0$ there exists an $s \geq r$ such that $(1-\epsilon) M_1 f(x) \leq \fint_{x-s}^{x+s} f(z)\,\mathrm{d}z$ and therefore,

$$\begin{aligned}
(1-\epsilon) M_1 f(x) - M_1 f(y) &\leq \fint_{x-s}^{x+s} f(z)\,\mathrm{d}z - \fint_{y-s-|x-y|}^{y+s+|x-y|} f(z)\,\mathrm{d}z \\
&\leq \left( \frac{1}{2s} - \frac{1}{2s + 2|x-y|} \right) \int_{x-s}^{x+s} f(z)\,\mathrm{d}z \\
&= \frac{|x-y|}{s + |x-y|} \fint_{x-s}^{x+s} f(z)\,\mathrm{d}z \\
&\leq \frac{|x-y|}{r + |x-y|} M_1 f(x).
\end{aligned}$$

The first inequality uses the definition of $M_1 f(y)$ together with the fact that $s + |x-y| \geq 1 + y$. In the second inequality, we use the nonnegativity of $f$ to reduce the domain of integration of the second integral. The last two relations follow from definitions. Now the first inequality in the lemma follows by letting $\epsilon \to 0$. The second inequality follows after rearranging terms. □

Bounds similar to Lemma 3.2.2 have frequently appeared in the literature, including in higher dimensions. The related inequality $|\nabla M_\alpha f(x)| \leq C M_{\alpha-1} f(x)$ for the fractional maximal function $M_\alpha f$ with $1 \leq \alpha \leq d$ was proved by Kinnunen and Saksman [KS03]. A generalisation to the range $0 \leq \alpha \leq d$ is due to Beltran, González-Riquelme, Madrid Padilla and Weigt [BGMW21, Section 2.5].

We now employ the previous result to prove a local variation bound for $M_1 f$. The strictness of this inequality will be crucial to our characterisation of maximisers.

**Lemma 3.2.3.** *It holds that* $\mathrm{var}_{[0,1]}(M_1 f) \le M_1 f(1)$ *and this inequality is strict if* $f(x) = 0$ *for almost every* $x \in [-1, 1]$ *and* $f(x) > 0$ *for any* $x$ *in some set of positive measure.*

*Proof.* First assume that $f(x) = 0$ for almost every $x \in [-1, 1]$ and $f(x) > 0$ for any $x$ in some set of positive measure. Then $M_1 f(x) > 0$ for any $x \ge 0$. By Lemma 3.2.2, $M_1 f$ is continuous. Since the map $(x, s) \mapsto \fint_{x-s}^{x+s} f(y)\,\mathrm{d}y$ is continuous at $(x, s) = (0, 1)$ and $f(y) = 0$ for almost every $y \in [-1, 1]$, this implies the existence of a $\delta \in (0, 1)$ such that for any $x \in [0, \delta)$,

$$M_1 f(x) = \sup_{s \ge 1+x+\delta} \fint_{x-s}^{x+s} f(y)\,\mathrm{d}y$$

and hence $f$ and $x$ satisfy the hypotheses of Lemma 3.2.2 with $r = 1 + x + \delta$. Without the additional assumptions that $f(x) = 0$ for almost every $x \in [-1, 1]$ and that $f(x) > 0$ for any $x$ in some set of positive measure, this remains true for $\delta = 0$.

In order to estimate the variation of $M_1 f$ on $[0, 1]$, we let $k \ge 1$ and

$$0 = x_0 < x_1 < \ldots < x_k = 1$$

We write $\delta_i = \delta$ if $x_i < \delta$ and $\delta_i = 0$ otherwise. By the two inequalities in Lemma 3.2.2 and the monotonicity of the sequences $x_i$ and $\delta_i$,

$$\begin{aligned}
&\sum_{i=0}^{k-1} |M_1 f(x_i) - M_1 f(x_{i+1})| \\
&\qquad \le \sum_{i=0}^{k-1} \max\Big(\frac{M_1 f(x_i)}{1 + x_{i+1} + \delta_i}, \frac{M_1 f(x_{i+1})}{1 + 2x_{i+1} - x_i + \delta_{i+1}}\Big)(x_{i+1} - x_i)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{k-1} \max(M_1 f(x_i), M_1 f(x_{i+1})) \frac{x_{i+1} - x_i}{1 + x_i + \delta_{i+1}} \\
&\leq \sum_{i=0}^{k-1} \max\left( \frac{2 + \delta_i}{1 + x_i + \delta_i}, \frac{2 + \delta_{i+1}}{1 + x_{i+1} + \delta_{i+1}} \right) \frac{x_{i+1} - x_i}{1 + x_i + \delta_{i+1}} M_1 f(1) \\
&\leq \sum_{i=0}^{k-1} \frac{(2 + \delta_i)(x_{i+1} - x_i)}{(1 + x_i + \delta_{i+1})^2} M_1 f(1).
\end{aligned}
$$

Except for the summand corresponding to the case $x_i < \delta \leq x_{i+1}$, this is a Riemann sum and therefore

$$
\begin{aligned}
\operatorname{var}_{[0,1]}(M_1 f) &\leq \left( \int_0^\delta \frac{2 + \delta}{(1 + x + \delta)^2} \, \mathrm{d}x + \int_\delta^1 \frac{2}{(1 + x)^2} \, \mathrm{d}x \right) M_1 f(1) \\
&\leq \int_0^1 \frac{2}{(1 + x)^2} \, \mathrm{d}x \cdot M_1 f(1) \\
&= M_1 f(1)
\end{aligned}
$$

and the second inequality is strict if $\delta > 0$. This completes the proof. □

*Remark* 3.2.4. Let us sketch a shorter but less elementary version of the second part of the above proof. By Lemma 3.2.2, the auxiliary maximal function $M_1 f$ is Lipschitz continuous. Hence it is differentiable almost everywhere and

$$
\operatorname{var}_{[0,1]}(M_1 f) = \int_0^1 |(M_1 f)'(x)| \, \mathrm{d}x.
$$

At any point of differentiability $x \in (0, 1)$ we have by Lemma 3.2.2 that

$$
|(M_1 f)'(x)| \leq \frac{M_1 f(x)}{1 + x} \leq \frac{2 M_1 f(1)}{(1 + x)^2}
$$

and the first inequality is strict in some neighbourhood of 0. Plugging this into the above formula for $\operatorname{var}_{[0,1]}(M_1 f)$ yields Lemma 3.2.3.

The next lemma concerns the other auxiliary maximal function $M_0 f$.

**Lemma 3.2.5.** *Let $f(x) = 0$ for almost every $x \in [-1, 1]$. Then $M_0 f$ is nondecreasing on $[0, 1]$.*

*Proof.* This is similar to the proof of Lemma 3.2.1. Let $0 < x \leq y \leq 1$. Then,

$$M_0 f(x) = \sup_{1-x<r\leq 1+x} \fint_{x-r}^{x+r} f(z)\,\mathrm{d}z \leq \sup_{1-x<r\leq 1+x} \fint_{x-r+2(y-x)}^{x+r} f(z)\,\mathrm{d}z \leq M_0 f(y).$$

Since $M_0 f(0) = 0$ and $M_0 f$ is nonnegative, this completes the proof. □

We have established the monotonicity of $M_0 f$ in Lemma 3.2.5 and a variation bound for $M_1 f$ in Lemma 3.2.3. The next result will allow us to deduce a variation bound for the pointwise maximum $Mf = \max(M_0 f, M_1 f)$.

**Lemma 3.2.6.** *Let $g, h\colon [0, 1] \to \mathbb{R}$ be functions such that $g(1) \leq h(1)$ and let $g$ be nondecreasing. Then $\operatorname{var}_{[0,1]}(\max(g, h)) \leq \operatorname{var}_{[0,1]}(h)$.*

*Proof.* Write $u = \max(g, h)$. We need to show that for any $k \geq 1$ and any

$$0 = x_0 < x_1 < \ldots < x_k = 1$$

it holds that

$$\sum_{i=0}^{k-1} |u(x_i) - u(x_{i+1})| \leq \operatorname{var}_{[0,1]}(h).$$

Write $x_{k+1} = 1$ and consider the set

$$P = \{-1\} \cup \{i \in \{0, \ldots, k\} \mid h(x_i) \geq u(x_{i+1})\}.$$

Since $x_k = x_{k+1} = 1$ and by assumption, $h(x_k) = u(x_{k+1})$ and hence $k \in P$. Let $\ell = \#P - 2$ and let $p(-1) < p(0) < \ldots < p(\ell)$ be the elements of $P$. Clearly, $p(-1) = -1$ and $p(\ell) = k$. If $i \in \{0, \ldots, k\} \setminus P$, then $h(x_i) < u(x_{i+1})$. Since $g$ is nondecreasing, we also have that $g(x_i) \leq g(x_{i+1})$ and hence $u(x_i) \leq u(x_{i+1})$. On the other hand, if $i \in P \setminus \{-1\}$, then

$$h(x_i) \geq u(x_{i+1}) \geq g(x_{i+1}) \geq g(x_i)$$

and hence $h(x_i) = u(x_i)$ and $u(x_i) \geq u(x_{i+1})$. This shows that for any $0 \leq j \leq \ell$,

$$u(x_{p(j-1)+1}) \leq u(x_{p(j-1)+2}) \leq \ldots \leq u(x_{p(j)}) = h(x_{p(j)})$$

and $h(x_{p(j)}) \geq u(x_{p(j)+1})$. We conclude that

$$\begin{aligned}
\sum_{i=0}^{k-1} |u(x_i) - u(x_{i+1})| &= \sum_{j=0}^{\ell} \sum_{i=p(j-1)+1}^{p(j)} |u(x_i) - u(x_{i+1})| \\
&= \sum_{j=0}^{\ell} 2h(x_{p(j)}) - u(x_{p(j-1)+1}) - u(x_{p(j)+1}) \\
&\leq \sum_{j=0}^{\ell} 2h(x_{p(j)}) - h(x_{p(j-1)+1}) - h(x_{p(j)+1}) \\
&\leq \operatorname{var}_{[0,1]}(h).
\end{aligned}$$

This completes the proof. □

We are now ready to prove (3.2.1). Let $f(x) = 0$ for almost every $x \in [-1, 1]$ and let $h\colon [0, 1] \to [0, \infty)$ be the function defined by $h(x) = M_1 f(x)$ for $0 \leq x < 1$ and $h(1) = Mf(1)$. Then $M_0 f(1) \leq h(1)$ and $Mf$ restricted to $[0, 1]$ is the pointwise maximum of $M_0 f$ and $h$. Hence by an application of Lemmas 3.2.5 and 3.2.6 and then Lemma 3.2.3,

$$\operatorname{var}_{[0,1]}(Mf) \leq \operatorname{var}_{[0,1]}(h) \leq \operatorname{var}_{[0,1]}(M_1 f) + Mf(1) - M_1 f(1) \leq Mf(1).$$

The last inequality is strict if $f$ does not vanish almost everywhere on $\mathbb{R}$. This shows (3.2.1) and hence completes the proof of Proposition 3.1.2.

*Remark* 3.2.7. One can show that $M_0 f$ is also Lipschitz continuous on $[0,1]$ and that $M_0 f$ and $M_1 f$ do not coincide at more than one point in $[0,1]$ if $f$ does not vanish almost everywhere on $\mathbb{R}$. Let us only sketch a proof of the fact that if $y \in [0,1]$ is such that $M_0 f(y) \geq M_1 f(y)$, then $M_0 f(x) > M_1 f(x)$ for any $x \in (y,1]$. By Lemma 3.2.2,

$$\frac{M_1 f(x) - M_1 f(y)}{x - y} \leq \frac{M_1 f(y)}{1 + x}.$$

Similarly as in the proofs of Lemmas 3.2.2 and 3.2.5, one can show that

$$\frac{M_0 f(x) - M_0 f(y)}{x - y} \geq \frac{M_0 f(y)}{1 - x + 2y}.$$

Since $M_0 f(y) \geq M_1 f(y) > 0$ and $y < x$ it follows that $M_0 f(x) > M_1 f(x)$.

## 3.3 Proof of Theorem 1.2.2

Throughout this section, let $f\colon \mathbb{R} \to [0,\infty)$ be a function of bounded variation such that for almost every $x \in \mathbb{R}$ we have that $f(x) = 0$ or $f(x) = Mf(x)$. In order to prove Theorem 1.2.2, we need to show the inequality

$$\operatorname{var}(Mf) \leq \operatorname{var}(f) \tag{3.3.1}$$

and determine its cases of equality. We will accomplish this by using a certain canonical representative $\bar{f}$ whose properties facilitate the application of Proposition 3.1.2. In Section 3.3.1, we define $\bar{f}$, show that $f$ and $\bar{f}$ agree almost everywhere and that

$$\operatorname{var}(\bar{f}) \leq \operatorname{var}(f). \tag{3.3.2}$$

There, we also establish some further properties of $\bar{f}$. In Section 3.3.2, we apply Proposition 3.1.2 to show (3.3.1) for $\bar{f}$, i.e. we show that

$$\operatorname{var}(Mf) \leq \operatorname{var}(\bar{f}). \tag{3.3.3}$$

Inequalities (3.3.2) and (3.3.3) together imply (3.3.1). In Section 3.3.3, we characterise the cases of equality in (3.3.1) by characterising the cases of equality in (3.3.3) and then characterising the cases of equality in (3.3.2) under the assumption that equality holds in (3.3.3).

### 3.3.1 Canonical representative

Let us define a function $\bar{f}\colon \mathbb{R} \to [0,\infty)$ as follows. If $x \in \mathbb{R}$ is such that

$$\limsup_{r \searrow 0} ⨍_{x-r}^{x+r} f(y)\,\mathrm{d}y = 0, \tag{3.3.4}$$

then we let $\bar{f}(x) = 0$ and otherwise we let $\bar{f}(x) = Mf(x)$. This *canonical representative* is related to but distinct from the homonymous object in [AP07]. By the Lebesgue differentiation theorem and the assumption on $f$, we have that $f(x) = \bar{f}(x)$ for almost every $x \in \mathbb{R}$ and hence $Mf(x) = M\bar{f}(x)$ for any $x \in \mathbb{R}$. Since $f$ is of bounded variation, its one-sided limits exist at any point. It follows that (3.3.4) can be rewritten without the use of an integral, but we will not need this.

The following lemma will be used multiple times throughout this section.

**Lemma 3.3.1.** *The maximal function $Mf$ is lower semi-continuous, i.e. for any $x \in \mathbb{R}$ it holds that* $\liminf_{y\to x} Mf(y) \geq Mf(x)$.

*Proof.* By definition, $Mf$ is the pointwise supremum of the continuous functions

$$\mathbb{R} \ni x \mapsto \fint_{x-r}^{x+r} f(y)\,\mathrm{d}y, \quad r > 0.$$

The lemma follows from this. □

We now show that the canonical representative does not increase the variation.

**Lemma 3.3.2.** *Inequality* (3.3.2) *holds.*

*Proof.* We first claim that it suffices to show that for any $x \in \mathbb{R}$ and $\epsilon > 0$ there exist $y_1, y_2 \in (x - \epsilon, x + \epsilon)$ such that $f(y_1) - \epsilon \leq \bar{f}(x) \leq f(y_2) + \epsilon$. Let $k \geq 1$ and let

$$-\infty < x_0 < x_1 < \ldots < x_k < \infty.$$

By iteratively removing any points $x_i$ with $1 \leq i \leq k-1$ for which $\bar{f}(x_i)$ lies in the convex hull of $\{\bar{f}(x_{i-1}), \bar{f}(x_{i+1})\}$, we obtain a subsequence $x'_0 < \ldots < x'_\ell$ such that

$$\sum_{i=0}^{k-1} |\bar{f}(x_i) - \bar{f}(x_{i+1})| = \sigma \sum_{i=0}^{\ell-1} (-1)^i \bar{f}(x'_i) + (-1)^{i+1} \bar{f}(x'_{i+1})$$

for some $\sigma \in \{-1, 1\}$. Let $\epsilon > 0$. By the claim made at the beginning of the proof, there exist points $y_i \in (x'_i - \epsilon, x'_i + \epsilon)$ such that

$$\sigma(-1)^i \bar{f}(x'_i) \leq \sigma(-1)^i f(y_i) + \epsilon$$

for any $0 \leq i \leq \ell$. If $\epsilon$ is small enough, then $y_i$ is increasing in $i$ and hence

$$\sum_{i=0}^{k-1} |\bar{f}(x_i) - \bar{f}(x_{i+1})| - 2\ell\epsilon \leq \sigma \sum_{i=0}^{\ell-1} (-1)^i f(y_i) + (-1)^{i+1} f(y_{i+1}) \leq \operatorname{var}(f).$$

Let $\epsilon \to 0$ and then take the supremum over all $k$ and $x_i$ as above to show (3.3.2).

It remains to show that for any $x \in \mathbb{R}$ and $\epsilon > 0$ there exist points $y_1$ and $y_2$ as above. Let $r \in (0, \epsilon)$. We start with the existence of $y_1$. By the definitions of $\bar{f}(x)$ and $Mf(x)$,

$$\bar{f}(x) \geq \limsup_{r \searrow 0} \fint_{x-r}^{x+r} f(y)\,\mathrm{d}y.$$

Hence if $r$ is sufficiently small, then the integral on the right-hand side is at most $\bar{f}(x) + \epsilon$ and so there exists a $y_1 \in (x - r, x + r)$ with $f(y_1) - \epsilon \leq \bar{f}(x)$, as required.

We complete the proof by showing the existence of $y_2$. If $\bar{f}(x) = 0$, then we may simply choose $y_2 = x$ because $f$ is nonnegative. So we assume that $\bar{f}(x) = Mf(x) > 0$. Since (3.3.4) fails, $f(y) > 0$ for any $y$ in some subset of positive measure of $(x - r, x + r)$. As $f$ and $\bar{f}$ are equal almost everywhere, it follows that

$f(y_2) = \bar{f}(y_2) = Mf(y_2)$ for some $y_2 \in (x - r, x + r)$. Hence if $r$ is small enough, then Lemma 3.3.1 implies that $f(y_2) + \epsilon \geq \bar{f}(x)$, as required. $\square$

In particular, (3.3.2) shows that $\bar{f}$ is of bounded variation. Together with the definition of $\bar{f}$, this implies some topological properties of the vanishing set

$$V = \{x \in \mathbb{R} \mid \bar{f}(x) = 0\}.$$

**Lemma 3.3.3.** *The set $V$ is open and its boundary has no limit points in $\mathbb{R}$.*

This can be stated equivalently as follows: There exists a finite or countably infinite nondecreasing sequence of points $a_i \in \mathbb{R} \cup \{\pm\infty\}$ without accumulation points in $\mathbb{R}$ such that $V = \bigcup_i (a_{2i}, a_{2i+1})$.

*Proof.* If $f$ vanishes almost everywhere, then $V = \mathbb{R}$ and the lemma follows. Since $f$ is nonnegative, we may therefore assume that $f$ is positive in a set of positive measure. Let $x \in \mathbb{R}$. Then $Mf(x) > 0$ and by Lemma 3.3.1 there exists an $\epsilon > 0$ such that $Mf(y) > \epsilon$ for any $y \in (x - \epsilon, x + \epsilon)$.

We first show that $x$ is not a limit point of the boundary of $V$. Let $k \geq 0$ and let

$$x - \epsilon < x_0 < x_1 < \ldots < x_{2k+1} < x + \epsilon$$

be a sequence of points with $\bar{f}(x_{2i}) = 0$ and $\bar{f}(x_{2i+1}) = M\bar{f}(x_{2i+1})$ for any $0 \leq i \leq k$. It suffices to show that $k$ is bounded by a constant that only depends on $\bar{f}$ and $\epsilon$. Such a bound holds because

$$(2k + 1)\epsilon < \sum_{i=0}^{2k} |\bar{f}(x_{i+1}) - \bar{f}(x_i)| \leq \operatorname{var}(\bar{f}) < \infty.$$

The first inequality above holds by the properties of $\epsilon$ and $x_k$. The second inequality holds by definition. Hence $x$ is not a limit point of the boundary of $V$.

It remains to show that $V$ is open. To this end, let $x$ be a boundary point of $V$. We need to show that $\bar{f}(x) > 0$. By the first part of the proof, $f(y) > \epsilon$ for any $y$ in some one-sided neighbourhood of $x$, i.e. for any $y$ in $(x - r, x)$ or $(x, x + r)$ for some $r > 0$. Since $f$ is nonnegative, we see that (3.3.4) fails and hence $\bar{f}(x) = Mf(x) > 0$. This completes the proof. □

### 3.3.2 Global variation bound

In Section 3.3.1, we proved (3.3.2). Together with the following result, this implies (3.3.1), proving the first part of Theorem 1.2.2.

**Proposition 3.3.4.** *Inequality* (3.3.3) *holds.*

*Proof.* By Lemma 3.3.3 and a subdivision of $\mathbb{R}$ we see that (3.3.3) holds if

$$\mathrm{var}_I(Mf) \leq \mathrm{var}_I(\bar{f}) \tag{3.3.5}$$

whenever $I$ is a connected component of $\mathbb{R} \setminus V$ or the closure of a connected component of $V$. If $I$ is a connected component of $\mathbb{R} \setminus V$, then $\bar{f}$ and $Mf$ agree on $I$, so that (3.3.5) holds with equality. Now let $I$ be the closure of a connected component of $V$. If $I = \mathbb{R}$, then both sides of (3.3.5) are zero. On the other hand, if $I \neq \mathbb{R}$, then by Lemma 3.3.3, $\bar{f}$ and $Mf$ agree on the boundary of $I$ and therefore (3.3.5) follows from either (i) or (ii) in Proposition 3.1.2. This completes the proof. □

### 3.3.3 Cases of equality

It remains to characterise the cases of equality in (3.3.1). We first establish certain regularity properties of $\bar{f}$.

**Lemma 3.3.5.** *Any connected component of $V$ or $\mathbb{R} \setminus V$ has positive length.*

*Proof.* Let $x \in \mathbb{R}$. If $\bar{f}(y) > 0$ for any $y \neq x$ in some compact neighbourhood of $x$, then by Lemma 3.3.1 there exists an $\epsilon > 0$ such that $\bar{f}(y) = Mf(y) > \epsilon$ for any such $y$. Hence (3.3.4) fails and $\bar{f}(x) = Mf(x) > \epsilon$. This shows that $\{x\}$ is not a connected component of $V$.

On the other hand, if $\bar{f}(y) = 0$ for any $y \neq x$ in some neighbourhood of $x$, then (3.3.4) holds and hence $\bar{f}(x) = 0$. This shows that $\{x\}$ is not a connected component of $\mathbb{R} \setminus V$. Since $x \in \mathbb{R}$ was arbitrary, it follows that any connected component of $V$ or $\mathbb{R} \setminus V$ has positive length. □

Now we investigate the behaviour of the canonical representative $\bar{f}$ on connected components of its support

$$\mathbb{R} \setminus V = \{x \in \mathbb{R} \mid \bar{f}(x) > 0\}.$$

This set is closed by Lemma 3.3.3. Our next result will only be applied in the case of an unbounded connected component, but its proof is identical in the bounded and unbounded cases.

**Lemma 3.3.6.** *The function $\bar{f}$ is concave on any connected component of $\mathbb{R} \setminus V$.*

*Proof.* Suppose for a contradiction that $\bar{f}$ is not concave on some connected component $I$ of $\mathbb{R} \setminus V$. Then there exist points $x_0 < x_1 < x_2$ in $I$ such that

$\bar{f}(x_1) < L(x_1)$ where $L\colon \mathbb{R} \to \mathbb{R}$ is the affine linear function defined by $L(x_0) = \bar{f}(x_0)$ and $L(x_2) = \bar{f}(x_2)$. Hence for $g = \bar{f} - L$ we have $g(x_1) < 0$ and $g(x_0) = g(x_2) = 0$.

Since $\bar{f}$ and $Mf$ are equal in $I$, Lemma 3.3.1 and the continuity of $L$ imply that there exists a smallest $x_1' \in [x_0, x_2]$ such that

$$g(x_1') = \inf_{x_0 \le y \le x_2} g(y) < 0.$$

Since $g(x_0) = 0$, there exists an $r > 0$ such that $[x_1' - r, x_1' + r] \subseteq [x_0, x_2]$. We have that $g(y) \ge g(x_1')$ for any $y \in [x_0, x_2]$ and the inequality is strict if $y < x_1'$. Hence by the mean value property for $L$,

$$Mf(x_1') \ge \fint_{x_1'-r}^{x_1'+r} \bar{f}(y)\,\mathrm{d}y = \fint_{x_1'-r}^{x_1'+r} g(y)\,\mathrm{d}y + L(x_1') > g(x_1') + L(x_1') = \bar{f}(x_1').$$

This is a contradiction to the fact that $x_1' \in \mathbb{R} \setminus V$. Therefore $\bar{f}$ is concave on $I$. □

The following result is a consequence of Lemma 3.3.6.

**Lemma 3.3.7.** *Let $I$ be an unbounded connected component of $\mathbb{R} \setminus V$. Then,*

$$\lim_{|x|\to\infty;\, x \in I} \bar{f}(x) > 0.$$

*Furthermore, if $I = \mathbb{R}$, then $\bar{f}$ is constant.*

*Proof.* Suppose for a contradiction that one of the conclusions of the lemma is false. Let $x_0 \in I$, meaning that $\bar{f}(x_0) > 0$. Then by symmetry, we may assume that $[x_0, \infty) \subseteq I$ and that there exists a $x_1 > x_0$ such that $\bar{f}(x_1) < \bar{f}(x_0)$. By Lemma 3.3.6, it follows that $\bar{f}(x_2) \le L(x_2)$ for any $x_2 \ge x_1$ where $L\colon \mathbb{R} \to \mathbb{R}$ is the affine linear function defined by $L(x_0) = \bar{f}(x_0)$ and $L(x_1) = \bar{f}(x_1)$. Notice that $L$ is strictly decreasing and hence $\bar{f}(x_2) < 0$ if $x_2$ is large enough. This is a contradiction to the nonnegativity of $\bar{f}$. □

We can now characterise the cases of equality in the intermediate inequality (3.3.3).

**Proposition 3.3.8.** *Equality holds in* (3.3.3) *if and only if* $\bar{f}$ *is constant or* $\mathbb{R} \setminus V$ *is a compact interval of positive length.*

*Proof.* It suffices to consider the case that $\bar{f}$ is not constant since otherwise both sides of (3.3.3) are zero. Then $\mathbb{R} \setminus V$ is nonempty. By the second part of Lemma 3.3.7, we also have that $V$ is nonempty.

By the proof of Proposition 3.3.4, equality holds in (3.3.3) if and only if (3.3.5) holds with equality whenever $I$ is the closure of some connected component of $V$. Any such $I$ has positive length by Lemma 3.3.5. By the strictness in Proposition 3.1.2(i), this means that (3.3.3) can only hold with equality if all connected components of $V$ are unbounded, i.e. if $\mathbb{R} \setminus V$ is a nonempty interval. This interval is closed by Lemma 3.3.3 and has positive length by Lemma 3.3.5.

Now let $I \neq \mathbb{R}$ be an unbounded connected component of $V$. Since the function $\bar{f}$ is of bounded variation, its limits at $\pm\infty$ exist and for any $x \in \mathbb{R}$,

$$Mf(x) \geq \lim_{r\to\infty} \fint_{x-r}^{x+r} \bar{f}(y)\,\mathrm{d}y = \lim_{y\to\infty} \frac{\bar{f}(y) + \bar{f}(-y)}{2}.$$

Furthermore, if the right-hand side is zero, then $\lim_{|x|\to\infty} Mf(x) = 0$. By Proposition 3.1.2(ii), it follows that (3.3.5) holds with equality if and only if $\lim_{|x|\to\infty} \bar{f}(x) = 0$. By Lemma 3.3.7, this is the case precisely when $\mathbb{R} \setminus V$ has no unbounded components. We conclude that if $\bar{f}$ is not constant, then (3.3.3) holds with equality if and only if $\mathbb{R} \setminus V$ is a compact interval of positive length. This completes the proof. □

We can now characterise the cases of equality in (3.3.1). We first assume that equality holds and show that $f$ is constant or the set $\{x \in \mathbb{R} \mid f(x) > 0\}$ is a bounded interval of positive length and for any $x \in \mathbb{R}$,

$$\liminf_{y \to x} f(y) \leq f(x) \leq \limsup_{y \to x} f(y). \tag{3.3.6}$$

If $\bar{f}$ is constant, then $Mf$ is constant and equality in (3.3.1) implies that $f$ is constant. Now consider the case that $\bar{f}$ is not constant. By Lemma 3.3.2 and Proposition 3.3.4, equality in (3.3.1) implies equality in (3.3.2) and (3.3.3). Hence by Proposition 3.3.8, it follows that $\mathbb{R} \setminus V = [a, b]$ for some real numbers $a < b$. By Lemma 3.3.6, the canonical representative $\bar{f}$ is concave on $[a, b]$ and hence continuous on $(a, b)$ with

$$0 = \lim_{y \nearrow a} \bar{f}(y) \leq \bar{f}(a) \leq \lim_{y \searrow a} \bar{f}(y) \quad \text{and} \quad 0 = \lim_{y \searrow b} \bar{f}(y) \leq \bar{f}(b) \leq \lim_{y \nearrow b} \bar{f}(y). \tag{3.3.7}$$

Since $f$ and $\bar{f}$ are equal almost everywhere and $\bar{f}$ is continuous in $\mathbb{R} \setminus \{a, b\}$, equality in (3.3.2) now implies that $f(x) = \bar{f}(x)$ for any $x \in \mathbb{R} \setminus \{a, b\}$. It follows that $\{x \in \mathbb{R} \mid f(x) > 0\}$ contains $(a, b)$ and is contained in $[a, b]$, verifying that this set is a bounded interval of positive length. Furthermore, (3.3.6) holds if $x \notin \{a, b\}$. At $x = a$, using (3.3.7) together with the equality in (3.3.2) we obtain that

$$0 = \lim_{y \nearrow a} \bar{f}(y) \leq f(a) \leq \lim_{y \searrow a} \bar{f}(y)$$

and a similar statement holds at $x = b$. Since we already established that $f(y) = \bar{f}(y)$ for $y \in \mathbb{R} \setminus \{a, b\}$, we can replace $\bar{f}(y)$ by $f(y)$ in the above limits. Hence, (3.3.6) is true also for $x \in \{a, b\}$, and we conclude that $f$ has the properties stated in the previous paragraph.

Conversely, assume that $f$ is constant or the set $\{x \in \mathbb{R} \mid f(x) > 0\}$ is a bounded interval of positive length and (3.3.6) holds for any $x \in \mathbb{R}$. We need to show equality in (3.3.1). This is immediate if $f$ is constant, so it remains to consider the case that there exist real numbers $a < b$ such that $\{x \in \mathbb{R} \mid f(x) > 0\}$ contains $(a, b)$ and is contained in $[a, b]$. Using the Lebesgue differentiation theorem and Lemma 3.3.3 it follows that $\mathbb{R} \setminus V = [a, b]$. Hence equality holds in (3.3.3) by Proposition 3.3.8 and $\bar{f}$ is continuous in $(a, b)$ by Lemma 3.3.6. It remains to show equality in (3.3.2). Since $f$ and $\bar{f}$ agree almost everywhere and are of bounded variation, we have that

$$\liminf_{y \to x} f(y) = \liminf_{y \to x} \bar{f}(y) = \bar{f}(x)$$

for every $x \in (a, b)$ and the same holds with $\liminf$ replaced by $\limsup$. Now $f(x) = \bar{f}(x)$ holds for any $x \in (a, b)$ by (3.3.6) and for any $x \in \mathbb{R} \setminus [a, b]$ by assumption on $f$, i.e. $f(x)$ and $\bar{f}(x)$ may only disagree if $x \in \{a, b\}$. However, Lemma 3.3.6 implies (3.3.7) as before and (3.3.7) continues to hold if $\bar{f}$ is replaced by $f$, which is a consequence of (3.3.6). We can conclude equality in (3.3.2), which implies equality in (3.3.1) because we already showed equality in (3.3.3). This completes the proof of Theorem 1.2.2.

## 3.4 Discrete setting

In this section, we first use an embedding argument and a complementary approximation argument to prove the conditional result Proposition 1.2.4 and to derive the discrete Theorem 1.2.3 from the continuous Theorem 1.2.1. Afterwards, we adapt the arguments in Sections 3.2 and 3.3 to show the general discrete Theorem 1.2.5.

### 3.4.1 Embedding

Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation and let $Mf\colon \mathbb{Z} \to \mathbb{R}$ be the discrete maximal function as defined in Section 1.2.2. We define an associated step function $f_{\mathrm{c}}\colon \mathbb{R} \to \mathbb{R}$ by setting $f_{\mathrm{c}}(x) = f(n)$ for any integer $n$ and any $x \in [n-1/2, n+1/2)$. Let $Mf_{\mathrm{c}}\colon \mathbb{R} \to \mathbb{R}$ be the continuous maximal function as defined in Section 1.2.1.

For any monotone map $\phi\colon \mathbb{Z} \to \mathbb{Z}$ there exists a monotone map $\psi\colon \mathbb{Z} \to \mathbb{R}$ such that $f \circ \phi = f_{\mathrm{c}} \circ \psi$ and vice versa. Hence $\mathrm{var}_{\mathbb{Z}}(f) = \mathrm{var}(f_{\mathrm{c}})$. Our next claim is that $\mathrm{var}_{\mathbb{Z}}(Mf) \leq \mathrm{var}(Mf_{\mathrm{c}})$. This is an immediate consequence of the following result.

**Lemma 3.4.1.** *We have that $Mf(n) = Mf_{\mathrm{c}}(n)$ for any integer $n$.*

*Proof.* For any nonnegative integer $m$, the step function $f_{\mathrm{c}}$ is constant on the intervals $[n-m-1/2, n-m+1/2)$ and $[n+m-1/2, n+m+1/2)$. Thus for any positive radius $r$ with $|r-m| \leq 1/2$ we have that

$$\fint_{n-r}^{n+r} |f_{\mathrm{c}}(y)|\,\mathrm{d}y = \frac{1}{2r}\int_{n-m}^{n+m} |f_{\mathrm{c}}(y)|\,\mathrm{d}y + \frac{r-m}{2r}(|f_{\mathrm{c}}(n-m)| + |f_{\mathrm{c}}(n+m)|).$$

The right-hand side is of the form $A + B/r$ for some constants $A$ and $B$ independent of $r$, where $B = 0$ if $m = 0$. It follows that the map $r \mapsto \fint_{n-r}^{n+r} f_{\mathrm{c}}(y)\,\mathrm{d}y$ is constant on $(0,1/2]$ and monotone on $[m-1/2, m+1/2]$ for any positive integer $m$. Hence,

$$Mf_{\mathrm{c}}(n) = \sup_{r\in\mathbb{Z}_{\geq 0}} \fint_{n-r-1/2}^{n+r+1/2} |f_{\mathrm{c}}(y)| \,\mathrm{d}y = \sup_{r\in\mathbb{Z}_{\geq 0}} \sum_{m=n-r}^{n+r} |f(m)| = Mf(n).$$

This completes the proof. □

If $f_{\mathrm{c}}$ satisfies (1.2.2) for some constant $C$, then it follows from the above that

$$\operatorname{var}_{\mathbb{Z}}(Mf) \leq \operatorname{var}(Mf_{\mathrm{c}}) \leq C\operatorname{var}(f_{\mathrm{c}}) = C\operatorname{var}_{\mathbb{Z}}(f) \tag{3.4.1}$$

and hence $f$ satisfies (1.2.4) with the same constant. This proves a part of Proposition 1.2.4, namely that the optimal constant in the continuous inequality (1.2.2) is not strictly smaller than the optimal constant in the discrete inequality (1.2.4). Similarly, (3.4.1) enables us to derive Theorem 1.2.3 from Theorem 1.2.1.

*Proof of Theorem 1.2.3.* By assumption, $f$ is $\{0,1\}$-valued and of bounded variation and so the same is true for $f_{\mathrm{c}}$. Hence by (3.4.1) and Theorem 1.2.1, we see that $f$ satisfies (1.2.4) with $C = 1$. Equality can only hold if equality holds in Theorem 1.2.1. For a nonconstant $f$, this implies that the set $\{x \in \mathbb{R} \mid f_{\mathrm{c}}(x) = 1\}$ is a bounded interval of positive length and hence the set $\{n \in \mathbb{Z} \mid f(n) = 1\}$ is a bounded nonempty discrete interval. On the other hand, if $f$ is of this form, then equality is attained because for any integer $n$ with $f(n) = 1$ we have that

$$\operatorname{var}_{\mathbb{Z}}(Mf) \geq 2Mf(n) - \lim_{m\to\infty} Mf(m) + Mf(-m) = 2 - 0 = \operatorname{var}_{\mathbb{Z}}(f).$$

This completes the proof. □

### 3.4.2 Approximation

In order to complete the proof of Proposition 1.2.4, it remains to show that the optimal constant in the continuous inequality (1.2.2) is not strictly larger than the optimal constant in the discrete inequality (1.2.4).

Assume that (1.2.4) holds for some constant $C$. By the following result, it suffices to show that $\operatorname{var}(Mf_{\mathrm{c}}) \leq C \operatorname{var}(f_{\mathrm{c}})$ for any nonnegative function $f_{\mathrm{c}} \colon \mathbb{R} \to [0, \infty)$ of bounded variation.

**Lemma 3.4.2.** *For any function $g \colon \mathbb{R} \to \mathbb{R}$ of bounded variation, $Mg = M|g|$ and $\operatorname{var}(|g|) \leq \operatorname{var}(g)$.*

*Proof.* The equality is immediate from the definition of the maximal function and the inequality follows from the reverse triangle inequality:

$$\big||g(x)| - |g(y)|\big| \leq |g(x) - g(y)|$$

for any real numbers $x$ and $y$. □

Let $a$ be a nonnegative integer and define a function $f^a \colon \mathbb{Z} \to [0, \infty)$ by setting

$$f^a(n) = \fint_{2^{-a}(n-1/2)}^{2^{-a}(n+1/2)} f_{\mathrm{c}}(x)\,\mathrm{d}x, \quad n \in \mathbb{Z}.$$

**Lemma 3.4.3.** *It holds that $\operatorname{var}_{\mathbb{Z}}(f^a) \leq \operatorname{var}(f_{\mathrm{c}})$.*

*Proof.* Similarly as in the proof of Lemma 3.3.2, it suffices to show that for any $n \in \mathbb{Z}$ there exist points $y_1, y_2 \in 2^{-a}(n - 1/2, n + 1/2)$ such that $f_{\mathrm{c}}(y_1) \leq f^a(n) \leq f_{\mathrm{c}}(y_2)$. This follows from the definition of $f^a(n)$ as an average. □

By the last result and (1.2.4) it suffices to show that

$$\operatorname{var}(Mf_{\mathrm{c}}) \le \sup_{a\ge 0} \operatorname{var}_{\mathbb{Z}}(Mf^a).$$

Our proof of this inequality consists of two parts: Lemma 3.4.4 relates $Mf_{\mathrm{c}}$ to $Mf^a$ on the set

$$\mathbb{Z}[\tfrac{1}{2}] = \{2^{-a}n \mid a, n \in \mathbb{Z}\}$$

whose elememts are known as the *dyadic rationals*. Then, Lemma 3.4.6 shows that the variation of $Mf_{\mathrm{c}}$ is exhausted by dyadic rationals.

**Lemma 3.4.4.** *If $x$ is a dyadic rational, then $\lim_{a\to\infty} Mf^a(2^a x) = Mf_{\mathrm{c}}(x)$.*

*Proof.* If $a$ is sufficiently large, then $Mf^a(2^a x)$ is defined and we have

$$Mf^a(2^a x) = \sup_{r\in\mathbb{Z}_{\ge 0}} \sum_{n=2^a x-r}^{2^a x+r} f^a(n) = \sup_{r\in 2^{-a}(\mathbb{Z}_{\ge 0}+1/2)} \fint_{x-r}^{x+r} f_{\mathrm{c}}(y)\,\mathrm{d}y.$$

If $U$ is an nonempty open subset of $[0,\infty]$, then the set of radii $2^{-a}(\mathbb{Z}_{\ge 0} + 1/2)$ intersects $U$ for any sufficiently large $a$. Since $f_{\mathrm{c}}$ is of bounded variation, the map

$$[0,\infty] \ni r \mapsto \lim_{\substack{s\to r\\ s\in(0,\infty)}} \fint_{x-s}^{x+s} f_{\mathrm{c}}(y)\,\mathrm{d}y$$

is continuous and hence $\lim_{a\to\infty} Mf^a(2^a x) = \sup_{r>0} \fint_{x-r}^{x+r} f_{\mathrm{c}}(y)\,\mathrm{d}y$, as required. □

Now we need to strengthen the lower semicontinuity established in Lemma 3.3.1. Note that Lemma 3.3.1 applies to $f_{\mathrm{c}}$ since its proof does not use the additional assumptions of Section 3.3.

**Lemma 3.4.5.** *Let $x_i$ be a sequence of real numbers converging to some real limit $x$. If $\lim_{i\to\infty} f_c(x_i) = \liminf_{y\to x} f_c(y)$, then $\lim_{i\to\infty} Mf_c(x_i) = Mf_c(x)$. In particular, $Mf_c$ is continuous at points where $f_c$ is continuous.*

*Proof.* By Lemma 3.3.1 it suffices to show that $\limsup_{i\to\infty} Mf_c(x_i) \le Mf_c(x)$. Let $r_i \in [0,\infty]$ be such that

$$Mf_c(x_i) = \lim_{\substack{s\to r_i \\ s\in(0,\infty)}} \fint_{x_i-s}^{x_i+s} f_c(y)\,\mathrm{d}y.$$

After extracting a subsequence we may assume that $r_i$ converges to some $r^* \in [0,\infty]$. Note that the map

$$\mathbb{R}\times(0,\infty] \ni (z,r) \mapsto \lim_{\substack{s\to r \\ s\in(0,\infty)}} \fint_{z-s}^{z+s} f_c(y)\,\mathrm{d}y$$

is continuous. Therefore, if $r^* > 0$, then

$$\lim_{i\to\infty} Mf_c(x_i) = \lim_{\substack{s\to r^* \\ s\in(0,\infty)}} \fint_{x-s}^{x+s} f_c(y)\,\mathrm{d}y \le Mf_c(x),$$

as required. We now assume that $r^* = 0$. Since $f_c$ is of bounded variation, it has one-sided limits at $x$. By our assumption that $\lim_{i\to\infty} f_c(x_i) = \liminf_{y\to x} f_c(y)$, it follows that one of the two limits

$$\lim_{i\to\infty} \fint_{x_i-r_i}^{x_i} f_c(y)\,\mathrm{d}y \quad \text{and} \quad \lim_{i\to\infty} \fint_{x_i}^{x_i+r_i} f_c(y)\,\mathrm{d}y$$

is equal to $\liminf_{y\to x} f_{\mathrm{c}}(y)$ and the other is at most $\limsup_{y\to x} f_{\mathrm{c}}(y)$. Hence,

$$\lim_{i\to\infty} Mf_{\mathrm{c}}(x_i) \le \frac{1}{2}\Big(\liminf_{y\to x} f_{\mathrm{c}}(y) + \limsup_{y\to x} f_{\mathrm{c}}(y)\Big) = \lim_{s\searrow 0} \fint_{x-s}^{x+s} f_{\mathrm{c}}(y)\,\mathrm{d}y \le Mf_{\mathrm{c}}(x),$$

as required. □

This allows us to prove that the variation of $Mf_{\mathrm{c}}$ is exhausted by dyadic rationals. We write

$$\operatorname{var}_{\mathbb{Z}[1/2]}(Mf_{\mathrm{c}}) = \sup_{\phi\colon \mathbb{Z}\to\mathbb{Z}[1/2] \text{ monotone}} \sum_{i\in\mathbb{Z}} |Mf_{\mathrm{c}}(\phi(i)) - Mf_{\mathrm{c}}(\phi(i+1))|.$$

**Lemma 3.4.6.** *It holds that* $\operatorname{var}(Mf_{\mathrm{c}}) \le \operatorname{var}_{\mathbb{Z}[1/2]}(Mf_{\mathrm{c}})$.

*Proof.* Similarly as in the proof of Lemma 3.3.2, it suffices to show that for any $x \in \mathbb{R}$ and $\epsilon > 0$ there exists a dyadic rational $y \in (x-\epsilon, x+\epsilon)$ such that $Mf_{\mathrm{c}}(y) - \epsilon \le Mf_{\mathrm{c}}(x) \le Mf_{\mathrm{c}}(y) + \epsilon$. This follows from Lemma 3.4.5 since the dyadic rationals are dense in $\mathbb{R}$ and since the one-sided limits of the function of bounded variation $f_{\mathrm{c}}$ exist. □

By Lemmas 3.4.4 and 3.4.6 we have

$$\operatorname{var}(Mf_{\mathrm{c}}) \le \operatorname{var}_{\mathbb{Z}[1/2]}(Mf_{\mathrm{c}}) \le \sup_{a\ge 0} \operatorname{var}_{\mathbb{Z}}(Mf^a)$$

and this completes the proof of Proposition 1.2.4.

### 3.4.3 Discrete local variation bound

The following result is the discrete analogue of Proposition 3.1.2. We will use it to derive Theorem 1.2.5 similarly as Theorem 1.1.2 in the continuous setting, but without any of the technical difficulties related to compactness issues or exceptional sets of measure zero.

**Proposition 3.4.7.** *Let* $f\colon \mathbb{Z} \to [0,\infty)$ *be a bounded function and let* $I \subseteq \mathbb{R}$ *be an interval such that* $f(n) = 0$ *for any integer* $n$ *in the interior of* $I$*. Then the following holds:*

*(i) If* $I = [a,b]$ *for some integers* $a < b$*, then* $\operatorname{var}_{I\cap\mathbb{Z}}(Mf) \leq Mf(a) + Mf(b)$*. The inequality is strict unless* $f$ *vanishes everywhere on* $\mathbb{Z}$*.*

*(ii) If* $I = (-\infty, a]$ *or* $I = [a,\infty)$ *for some integer* $a$*, then* $Mf$ *is monotone on* $I \cap \mathbb{Z}$ *and* $\operatorname{var}_{I\cap\mathbb{Z}}(Mf) = Mf(a) - \inf_{n\in I\cap\mathbb{Z}} Mf(n)$*.*

The proof of this result goes along similar lines of the proof of Proposition 3.1.2, although some details differ. In particular we have to work around the fact that not all integer intervals have integer midpoints.

We first prove the unbounded case in Proposition 3.4.7(ii). By symmetry, it suffices to take $I = [a,\infty)$.

**Lemma 3.4.8.** *Let* $f\colon \mathbb{Z} \to [0,\infty)$ *be a bounded function and let* $a \in \mathbb{Z}$ *be such that* $f(n) = 0$ *for every integer* $n > a$*. Then* $Mf$ *is nonincreasing on* $[a,\infty)\cap\mathbb{Z}$ *and hence*

$$\operatorname{var}_{[a,\infty)\cap\mathbb{Z}}(Mf) = Mf(a) - \inf_{n\in[a,\infty)\cap\mathbb{Z}} Mf(n).$$

*Proof.* This is similar to the proof of Lemma 3.2.1. Let $n, m \in \mathbb{Z}$ be such that $a \leq n \leq m$. Then,

$$Mf(n) = \sup_{r\geq n-a} \sum_{k=n-r}^{n+r} f(k) \geq \sup_{r\geq n-a} \sum_{k=n-r}^{n+r+2(m-n)} f(k) = Mf(m). \qquad \square$$

The rest of this section is devoted to the proof of Proposition 3.4.7(i), i.e. the case that $I = [a, b]$ for some integers $a < b$. We start with a reduction using translation invariance. We also insert a midpoint in the case that $a + b$ is odd. For this, let $f$ be as in Proposition 3.4.7. Set

$$S = \begin{cases} \mathbb{Z} & \text{if } a+b \text{ is even,} \\ \mathbb{Z} + \frac{1}{2} = \left\{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\right\} & \text{if } a+b \text{ is odd} \end{cases}$$

and write $S_0 = S \cup \{0\}$. We define a translated function $\tilde{f} \colon S \to [0, \infty)$ by

$$\tilde{f}(n) = f\Big(n + \frac{a+b}{2}\Big)$$

and we define its centred maximal function $M\tilde{f} \colon S_0 \to [0, \infty)$ by

$$M\tilde{f}(n) = \sup_{v \in S;\, v \leq n} \sum_{m=v}^{2n-v} \tilde{f}(m).$$

Given a domain $T \in \{S, S_0\}$, a function $g : T \to [0, \infty)$ and an interval $I \subseteq \mathbb{R}$ we define the variation of $g$ on the *discrete interval* $I \cap T$ by

$$\operatorname{var}_{I \cap T}(g) = \sup_{\phi \colon \mathbb{Z} \to I \cap T \text{ monotone}} \sum_{i \in \mathbb{Z}} |g(\phi(i)) - g(\phi(i+1))|.$$

If $S = \mathbb{Z}$, then these definitions agree with those in Section 1.2.2. Note that

$$\operatorname{var}_{[a,b] \cap \mathbb{Z}}(Mf) = \operatorname{var}_{[-(b-a)/2,(b-a)/2] \cap S}(M\tilde{f}) \leq \operatorname{var}_{[-(b-a)/2,(b-a)/2] \cap S_0}(M\tilde{f})$$

and

$$M\tilde{f}\Big(-\frac{b-a}{2}\Big) = Mf(a), \qquad M\tilde{f}\Big(\frac{b-a}{2}\Big) = Mf(b).$$

From now on and for the rest of the proof of Proposition 3.4.7(i), let $f\colon S \to [0,\infty)$ be a bounded nonzero function. By the above relations and by symmetry, it is enough to show the strict inequality

$$\operatorname{var}_{[0,a]\cap S_0}(Mf) < Mf(a) \tag{3.4.2}$$

for any positive $a \in S$ such that $f(n) = 0$ for all $n \in S$ with $-a < n < a$. This is analogous to (3.2.1).

Similarly as in the continuous setting, $Mf$ restricted to $[0,a]\cap S_0$ is the pointwise maximum of the auxiliary maximal functions $M_0 f, M_1 f\colon [0,a]\cap S_0 \to [0,\infty)$ defined by

$$M_0 f(n) = \max_{v\in S;\, -a<v\leq n} \sum_{m=v}^{2n-v} f(m), \qquad M_1 f(n) = \sup_{v\in S;\, v\leq -a} \sum_{m=v}^{2n-v} f(m).$$

The following gradient bound for $M_1 f$ is analogous to the continuous Lemma 3.2.2. Since admissible radii in the above discrete setting are separated by a distance of 1, an additional term 1/2 appears in this bound. Because of this, we also dispense with the additional lower bound on the radii in Lemma 3.2.2. Except for these differences, the proof is similar to the continuous case.

**Lemma 3.4.9.** *Let $n, m \in [0,\infty)\cap S_0$ be distinct. Then,*

$$\frac{M_1 f(n) - M_1 f(m)}{|n-m|} \leq \frac{M_1 f(n)}{n+a+1/2+|n-m|} \leq \frac{M_1 f(m)}{n+a+1/2}.$$

*Proof.* We have $M_1 f(n) < \infty$ since $f$ is bounded. Hence for any $\epsilon > 0$ there exists a $v \in S$ with $v \leq -a$ such that $(1-\epsilon) M_1 f(n) \leq \sum_{k=v}^{2n-v} f(k)$. Let $w$ be such that

$$m - w = n - v + |n - m|.$$

Then $w \in S$ because $v - w$ is an integer. Since $w \leq v < 2n - v \leq 2m - w$,

$$\begin{aligned}
(1-\epsilon) M_1 f(n) - M_1 f(m) &\leq \sum_{k=v}^{2n-v} f(k) - \sum_{k=w}^{2m-w} f(k) \\
&\leq \left( \frac{1}{2(n-v)+1} - \frac{1}{2(m-w)+1} \right) \sum_{k=v}^{2n-v} f(k) \\
&= \frac{2|n-m|}{2(m-w)+1} \sum_{k=v}^{2n-v} f(k) \\
&\leq \frac{|n-m|}{n+a+1/2+|n-m|} M_1 f(n).
\end{aligned}$$

The first, third and fourth relations follow from definitions and the fact that $w \leq -a$. In the second line, we use that $f$ is nonnegative to reduce the range of summation of the second sum. Now the first inequality in the lemma follows by letting $\epsilon \to 0$. The second inequality follows after rearranging terms. $\square$

Our next result is a local variation bound for $M_1 f$ analogous to the continuous Lemma 3.2.3. Here the proof is somewhat simplified due to a telescoping argument. Furthermore, due to the additional term $1/2$ in Lemma 3.4.9 above, we are able to show a slightly stronger inequality than in the continuous setting. This artefact already allows us to obtain a strict inequality, whereas in the continuous setting we have to work a little harder to get the strict inequality in Lemma 3.2.3.

**Lemma 3.4.10.** *Let $a \in S$ be nonnegative. Then,*

$$\operatorname{var}_{[0,a]\cap S_0}(M_1 f) \leq \frac{2a}{2a+1} M_1 f(a).$$

*Proof.* Let $n < m$ be elements of $[0,a] \cap S_0$. By the two inequalities in Lemma 3.4.9,

$$\begin{aligned}
|M_1 f(n) - M_1 f(m)| &\leq \frac{m-n}{m+a+1/2} \max(M_1 f(n), M_1 f(m)) \\
&\leq \frac{(m-n)(2a+1/2)}{(n+a+1/2)(m+a+1/2)} M_1 f(a) \\
&= \Big( \frac{2a+1/2}{n+a+1/2} - \frac{2a+1/2}{m+a+1/2} \Big) M_1 f(a).
\end{aligned}$$

Now let $0 = n_0 < n_1 < \ldots < n_k = a$ be an enumeration of $[0,a] \cap S_0$. We use the above estimate and evaluate the resulting telescoping sum to obtain that

$$\operatorname{var}_{[0,a]\cap S_0}(M_1 f) = \sum_{i=0}^{k-1} |M_1 f(n_i) - M_1 f(n_{i+1})| \leq \Big( \frac{2a+1/2}{a+1/2} - 1 \Big) M_1 f(a).$$

This completes the proof. □

Regarding the other auxiliary maximal function $M_0 f$, the following result similar to Lemmas 3.2.5 and 3.4.8 holds.

**Lemma 3.4.11.** *Let $a \in S$ be nonnegative and let $f(n) = 0$ for any $n \in S$ with $-a < n < a$. Then $M_0 f$ is nondecreasing on $[0,a] \cap S_0$.*

*Proof.* Let $n, m \in S$ be such that $0 < n \leq m \leq a$. Then,

$$M_0 f(n) = \max_{v \in S;\, -a < v \leq 2n-a} \sum_{k=v}^{2n-v} f(k) \leq \max_{v \in S;\, -a < v \leq 2n-a} \sum_{k=v+2(m-n)}^{2n-v} f(k) \leq M_0 f(m).$$

Since $M_0 f(0) = 0$ and $M_0 f$ is nonnegative, this completes the proof. □

Having established the monotonicity of $M_0 f$ and a variation bound for $M_1 f$ similarly as in the continuous setting, the next step is to combine these results using the following analogue of Lemma 3.2.6. We omit the proof because it is the same.

**Lemma 3.4.12.** *Let $a \in S$ be nonnegative. Let $g, h\colon [0,a] \cap S_0 \to \mathbb{R}$ be functions such that $g(a) \leq h(a)$ and let $g$ be nondecreasing. Then,*

$$\operatorname{var}_{[0,a]\cap S_0}(\max(g,h)) \leq \operatorname{var}_{[0,a]\cap S_0}(h).$$

We are now ready to prove (3.4.2). Let $a \in S$ be positive such that $f(n) = 0$ for any $n \in S$ with $-a < n < a$ and let $h\colon [0,a] \cap S_0 \to [0,\infty)$ be the function defined by $h(n) = M_1 f(n)$ for $n < a$ and $h(a) = Mf(a)$. Then $M_0 f(a) \leq h(a)$ and $Mf$ restricted to $[0,a] \cap S_0$ is the pointwise maximum of $M_0 f$ and $h$. Hence we can apply Lemmas 3.4.11 and 3.4.12 and then Lemma 3.4.10 to obtain that

$$\operatorname{var}_{[0,a]\cap S_0}(Mf) \leq \operatorname{var}_{[0,a]\cap S_0}(h) \leq \operatorname{var}_{[0,a]\cap S_0}(M_1 f) + Mf(a) - M_1 f(a) < Mf(a).$$

This proves (3.4.2) and thus completes the proof of Proposition 3.4.7.

### 3.4.4 Discrete global variation bound

We now prove the inequality in Theorem 1.2.5. Throughout this subsection and the next subsection, let $f\colon \mathbb{Z} \to [0,\infty)$ be a function of bounded variation such that for any $n \in \mathbb{Z}$ we have $f(n) = 0$ or $f(n) = Mf(n)$. For possibly infinite endpoints

$a \leq b$ we write

$$[a,b] \cap \mathbb{Z} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}.$$

There exists a possibly unbounded discrete interval $\mathcal{I} \subseteq \mathbb{Z}$ and a nondecreasing sequence $(a_i)_{i\in\mathcal{I}}$ of points in $\mathbb{Z} \cup \{\pm\infty\}$ such that

$$\{n \in \mathbb{Z} \mid f(n) > 0\} = \bigcup_{i,i+1\in\mathcal{I};\, i \text{ odd}} [a_i, a_{i+1}] \cap \mathbb{Z} = \mathbb{Z} \setminus \bigcup_{i,i+1\in\mathcal{I};\, i \text{ even}} (a_i, a_{i+1})$$

and $a_i + 2 \leq a_{i+1}$ for any even $i \in \mathcal{I}$ such that $i+1 \in \mathcal{I}$. We may further assume that the points $\pm\infty$ each occur at most once in the sequence $(a_i)_{i\in\mathcal{I}}$.

Let $i \in \mathcal{I}$ be such that $i+1 \in \mathcal{I}$. If $i$ is even, then by Proposition 3.4.7,

$$\operatorname{var}_{[a_i,a_{i+1}]\cap\mathbb{Z}}(Mf) \leq \operatorname{var}_{[a_i,a_{i+1}]\cap\mathbb{Z}}(f). \tag{3.4.3}$$

On the other hand, if $i$ is odd, then by assumption it holds that $f(n) = Mf(n)$ for all $n \in [a_i, a_{i+1}] \cap \mathbb{Z}$ and thus (3.4.3) holds with equality. We can conclude that

$$\operatorname{var}_{\mathbb{Z}}(Mf) = \sum_{i,i+1\in\mathcal{I}} \operatorname{var}_{[a_i,a_{i+1}]\cap\mathbb{Z}}(Mf) \leq \sum_{i,i+1\in\mathcal{I}} \operatorname{var}_{[a_i,a_{i+1}]\cap\mathbb{Z}}(f) = \operatorname{var}_{\mathbb{Z}}(f). \tag{3.4.4}$$

This proves the inequality in Theorem 1.2.5.

### 3.4.5 Cases of equality

For the characterisation of the cases of equality in Theorem 1.2.5, we may assume that $f$ is not constant since otherwise both sides of (3.4.4) are zero. By the last subsection, equality holds in (3.4.4) if and only if for every even $i \in \mathcal{I}$ with $i+1 \in \mathcal{I}$ we have equality in (3.4.3). We need the following concavity result whose proof we omit because it is similar to the proof of Lemma 3.3.7. The conclusion of this result slightly differs from Lemma 3.3.7 because here we already assume $f$ to be nonconstant.

**Lemma 3.4.13.** *Let $i \in \mathcal{I}$ be odd and such that $i+1 \in \mathcal{I}$ and $a_{i+1} = \infty$. Then $\lim_{n\to\infty} f(n) > 0$ and $a_i > -\infty$.*

By definition, $\mathcal{I}$ has at least two elements. Since $f$ is not constant, it is not the zero function. Hence $\mathcal{I}$ is not of the form $\{i, i+1\}$ for any even $i$. By Lemma 3.4.13, it is also not of this form for any odd $i$. Hence $\mathcal{I}$ has at least three elements. If there exists an even $i \in \mathcal{I}$ with $i+1 \in \mathcal{I}$ and $a_i, a_{i+1} \in \mathbb{Z}$, then (3.4.3) is a strict inequality by Proposition 3.4.7 and hence (3.4.4) is strict. It remains to consider the case that no such $i$ exists. After re-indexing and up to symmetry, this means that $\mathcal{I}$ is either $\{0, 1, 2, 3\}$ or $\{0, 1, 2\}$.

In the first case, $f$ is finitely supported and hence, by Proposition 3.4.7(ii), equality holds in (3.4.3) for the even indices $i = 0$ and $i = 2$. Thus (3.4.4) holds with equality. In the second case, by Lemma 3.4.13,

$$Mf(n) \geq \lim_{r\to\infty} \sum_{m=n-r}^{n+r} f(m) = \lim_{m\to\infty} \frac{f(m)}{2} > 0.$$

for any integer $n$. By Proposition 3.4.7(ii), this means that (3.4.3) is strict for $i = 0$ and hence (3.4.4) is strict. We conclude that equality holds in (3.4.4) if and only if $f$ is constant or $\{n \in \mathbb{Z} \mid f(n) > 0\} = [a, b] \cap \mathbb{Z}$ for some integers $a \leq b$. This completes the proof of Theorem 1.2.5.

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