PATH AND CYCLE DECOMPOSITIONS OF GRAPHS AND DIGRAPHS

by

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Abstract

In this thesis, we make progress on five long standing conjectures on path and cycle decompositions of graphs and digraphs. Firstly, we confirm a conjecture of Jackson from 1981 by showing that the edges of any sufficiently large regular bipartite tournament can be decomposed into Hamilton cycles. Along the way, we also prove several further results, including a conjecture of Liebenau and Pehova on Hamilton decompositions of dense bipartite digraphs.

Secondly, we determine the minimum number of paths required to decompose the edges of any sufficiently large tournament of even order, thus resolving a conjecture of Alspach, Mason, and Pullman from 1976. We also prove an asymptotically optimal result for tournaments of odd order.

Finally, we give asymptotically best possible upper bounds on the minimum number of paths, cycles, and cycles and edges required to decompose the edges of any sufficiently large dense graph. This makes progress on three famous conjectures from the 1960s: Gallai's conjecture, Hajós' conjecture, and the Erdős–Gallai conjecture, respectively.

This includes joint work with António Girão [40, 41], Daniela Kühn [40, 41], Allan Lo [40], and Deryk Osthus [40, 41].

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PART I

DISCUSSION OF RESULTS

CHAPTER 1

INTRODUCTION

Given a (di)graph G, a *decomposition* of G is a set of edge-disjoint sub(di)graphs $F_1, \dots, F_\ell$ of G which altogether cover all the edges of G. The study of decompositions has a long history and encompasses a wide range of structures $F_1, \dots, F_\ell$. For example, triangle decompositions of complete graphs were studied by Kirkman [67] in 1847, while Walecki [88] showed in 1883 that the complete graph on n vertices can be decomposed into Hamilton cycles if n is odd and into Hamilton paths if n is even. In 1963, Ringel [101] conjectured that the complete graph on $2n+1$ vertices can be decomposed into edge-disjoint copies of any tree with n edges. (This was recently verified by Montgomery, Pokrovskiy, and Sudakov [90] and Keevash and Staden [63] for large complete graphs.) Decompositions of general graphs into complete graphs where considered in 1966 by Erdős, Goodman, and Pósa [28], who showed that any graph on n vertices with no isolated vertex can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ complete graphs. In 1968, Lovász [87] studied decompositions into paths and cycles.

Theorem 1.1 ([87])**.** *Any graph on n vertices can be decomposed into at most $\lfloor \frac{n}{2} \rfloor$ paths and cycles.*

Observe that Theorem 1.1 is tight. Indeed, let G be graph on n vertices and let $\mathcal{D}$ be a decomposition of G into paths and cycles. Then, observe that any vertex of odd degree in G must be the endpoint of a path in $\mathcal{D}$, so $|\mathcal{D}| \geq \frac{\mathrm{odd}(G)}{2}$ (where $\mathrm{odd}(G)$ denotes the number of odd-degree vertices of G).

1.1 Hamilton decompositions

One of the most natural and most extensively studied type of decomposition is a *Hamilton decomposition*, that is, a decomposition into Hamilton cycles. (Here and throughout this thesis, a Hamilton cycle in a digraph is always assumed to have all its edges consistently oriented.)

Note that in a graph G, any set of edge-disjoint Hamilton cycles of G induce an even-regular subgraph of G, so G must be regular of even degree to admit a Hamilton decomposition. However, even-regularity is not a sufficient condition for the existence of a Hamilton decomposition. For example, the graph in Figure 1.1 is 4-regular but cannot be decomposed into Hamilton cycles since any Hamilton cycle must contain the two dashed edges.

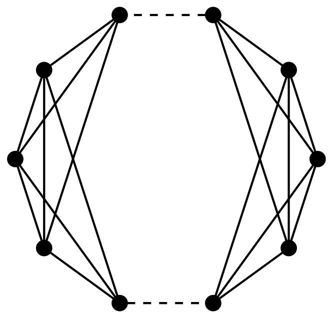


Figure 1.1: Example of an even-regular graph which cannot be decomposed into Hamilton cycles.

Similarly, a digraph D which can be decomposed into Hamilton cycles must be regular. (Here, a digraph D is *regular* if there exists r such that every vertex $v \in V(D)$ has both its indegree $d^-_D(v)$ and its outdegree $d^+_D(v)$ equal to r.) As for the undirected case, regularity does not guarantee the existence of a Hamilton decomposition of a digraph. For example, it was verified by Bermond and Faber [8] that the complete digraphs on 4 and 6 vertices cannot be decomposed into Hamilton cycles.

Extensive research has been done into classifying the (di)graphs which admit a Hamilton decomposition.

Problem 1.2. *Given a (di)graph G, can G be decomposed into Hamilton cycles?*

A well-known result of Karp, Lawler, and Tarjan [61] states that the problem of determining whether a (di)graph G has a Hamilton cycle is NP-complete. Péroche [97] showed that Problem 1.2 is also NP-complete and so, since it is expected that P $\neq$ NP, a general solution to Problem 1.2 cannot be expected.

1.1.1 Hamilton decompositions of dense graphs and digraphs

As mentioned at the start of this chapter, constructions of Hamilton decompositions of complete graphs of odd order date back to the 19$^{\text{th}}$ century [88] (see also [4] for an English description of Walecki's construction). More than 130 years later, Walecki's result was extended by Csaba, Kühn, Lo, Osthus, and Treglown [20], who gave an exact minimum degree threshold for the existence of a Hamilton decomposition in sufficiently large graphs.

Theorem 1.3 ([20])**.** *There exists $n_0 \in \mathbb{N}$ for which the following holds. Let G be a graph on $n \geq n_0$ vertices and suppose that G is r-regular for some $r \geq \lfloor \frac{n}{2} \rfloor$. Then, G can be decomposed into edge-disjoint Hamilton cycles and at most one perfect matching.*

Analogous results can be obtained for directed graphs. Bermond and Faber [8] observed that Walecki's result [88] implies that complete digraphs of odd order have a Hamilton decomposition. In 1980, Tillson [107] showed that complete digraphs on $2n \geq 8$ vertices can be decomposed into Hamilton cycles. (Bermond and Faber [8] verified that such decompositions do not exist for $2n \in \{4, 6\}$.) The analogous question for regular tournaments was posed by Kelly in 1968 (see [91]).

Conjecture 1.4 (Kelly)**.** *Any regular tournament can be decomposed into Hamilton cycles.*

Kelly's conjecture was first proved approximately for large tournaments by Kühn, Osthus, and Treglown [78] and later resolved for such tournaments by Kühn and Osthus [76] in 2013. In fact, the methods in [76] are more general and apply to "robust outexpanders". Roughly speaking, a robust outexpander is a digraph D such that for any $S \subseteq V(D)$ which is neither too small nor too large, there are significantly many more than $|S|$ vertices of D which have a linear number of inneighbours in S. Robust outexpanders were first introduced in [79] by Kühn, Osthus, and Treglown and have since then been used to prove a wide range of results (see e.g. [48, 74]).

More precisely, given a digraph D on n vertices and $S \subseteq V(D)$, the *ν-robust outneighbourhood of S*, denoted by $RN^+_{\nu,D}(S)$, consists of all the vertices of D which have at least νn inneighbours in S. A digraph D on n vertices is called a *robust (ν,τ)-outexpander* if $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$ for every $S \subseteq V(D)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$.

Theorem 1.5 ([76])**.** *For any $\delta > 0$, there exists $\tau > 0$ such that, for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let D be a robust (ν,τ)-outexpander on $n \geq n_0$ vertices and suppose that D is r-regular for some $r \geq \delta n$. Then, D has a Hamilton decomposition.*

One can show that sufficiently dense digraphs and oriented graphs are robust outexpanders. (Throughout this thesis, a *digraph* refers to a directed graph which contains at most one edge of each direction between any two distinct vertices, while an *oriented graph* refers to a directed graph which contains at most one edge between any two distinct vertices.) Thus, Theorem 1.5 can be used to obtain an approximate analogue of Theorem 1.3 for digraphs and oriented graphs. In particular, this implies that Kelly's conjecture holds for sufficiently large tournaments.

Theorem 1.6 ([76])**.** *For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ for which the following hold.*

(i) *Let D be a digraph on n vertices and suppose that D is r-regular for some $r \geq \left(\frac{1}{2}+\varepsilon\right)n$. Then, D has a Hamilton decomposition.*

(ii) *Let D be an oriented graph on n vertices and suppose that D is r-regular for some $r \geq \left(\frac{3}{8}+\varepsilon\right)n$. Then, D has a Hamilton decomposition.*

Although Theorem 1.6(i) is asymptotically best possible (there exist disconnected $(\lfloor \frac{n}{2} \rfloor - 1)$-regular digraphs on n vertices), it is not clear whether Theorem 1.6(ii) is (asymptotically) best possible. On the one hand, there exist oriented graphs on n vertices which are very close to being $\frac{3n}{8}$-regular but are not Hamiltonian (see e.g. [76]). On the other hand, Jackson [56] conjectured that any r-regular oriented graph on n vertices with $r > 2$ and $r \geq \frac{n-1}{4}$ is Hamiltonian. This conjecture was recently confirmed approximately for large oriented graphs by Lo, Patel, and Yıldız [86]. See the survey [75] for a further discussion.

Beyond Kelly's conjecture, Theorem 1.5 has had a wide range of applications (see e.g. [20, 35, 42, 77], which are discussed in more detail below). In particular, all the results in this thesis either use Theorem 1.5 or the main tool of [76] which was used to prove it.

1.1.2 Hamilton decompositions of partite graphs and digraphs

Hamilton decompositions of partite (di)graphs also have a long history. Walecki [88] used their construction for complete graphs to show that a complete bipartite graph on vertex classes of size $2n$ can also be decomposed into Hamilton cycles. This was extended in 1972 by Dirac [25], who showed that a complete bipartite graph on vertex classes of size $2n + 1$ can be decomposed into n Hamilton cycles and one perfect matching. More generally, Hetyei [53] and Laskar and Auerbach [82] independently showed in the 1970's that complete r-partite graphs on vertex classes of size n can be decomposed into $\lfloor \frac{n(r-1)}{2} \rfloor$ Hamilton cycles and at most one perfect matching (depending on the parity of $n(r - 1)$). In 1997, Ng [93] showed that complete r-partite digraphs on vertex classes of size n have a Hamilton decomposition if and only if $(r, n) \notin \{(4, 1), (6, 1)\}$.

An *r-partite tournament* is a digraph which is obtained by orienting the edges of a complete r-partite graph. For $r \geq 4$, Kühn and Osthus [77] showed that sufficiently large regular r-partite tournaments are in fact robust outexpanders and so, by Theorem 1.5, can be decomposed into Hamilton cycles. They also conjectured that regular tripartite tournaments have a Hamilton decomposition [77]. However, we observe that this conjecture is false. Indeed, we will see in Section 6.1 that a regular tripartite tournament obtained by flipping the orientation of precisely one triangle in a consistently oriented blow-up of a triangle cannot be decomposed into Hamilton cycles.

Proposition 1.7. *For any integer $n \geq 2$, there exists a regular tripartite tournament on vertex classes of size n which does not have a Hamilton decomposition.*

In 1981, Jackson [56] showed than any regular bipartite tournament is Hamiltonian and conjectured that such digraphs have a Hamilton decomposition.

Conjecture 1.8 (Jackson). *Any regular bipartite tournament can be decomposed into Hamilton cycles.*

Some progress on this conjecture was made by Liebenau and Pehova [84], who showed that any sufficiently large regular bipartite digraph of sufficiently large degree has an approximate Hamilton decomposition.

Theorem 1.9 ([84]). *For any $\delta > \frac{1}{2}$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that any δn-regular bipartite digraph on $2n \geq n_0$ vertices contains at least $(1 - \varepsilon)\delta n$ edge-disjoint Hamilton cycles.*

Note however that Theorem 1.9 does not actually apply to regular bipartite tournaments, i.e. even the existence of an approximate Hamilton decomposition of regular bipartite tournaments was not known so far. The main result of this thesis consists of a proof of Conjecture 1.8 for sufficiently large bipartite tournaments. (Note that the number of vertices in Theorem 1.10 is a multiple of 4 since any regular bipartite tournament must necessarily have vertex classes of the same even size.)

Theorem 1.10. *There exists $n_0 \in \mathbb{N}$ such that any regular bipartite tournament T on $4n \geq n_0$ vertices has a Hamilton decomposition.*

Our proof of Theorem 1.10 is split into two cases: T is a "bipartite robust outexpander" and T is "close to the complete blow-up C_4". (This is discussed more thoroughly in Chapter 4.) Along the way, we also prove a bipartite analogue of Theorem 1.5. In particular, this allows us to extend Theorem 1.9 and fully decompose sufficiently large dense bipartite digraphs into edge-disjoint Hamilton cycles. This resolves a conjecture of Liebenau and Pehova [84].

Theorem 1.11. *For any $\delta > \frac{1}{2}$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let D be a bipartite digraph on vertex classes of size $n \geq n_0$ and suppose that D is r-regular for some $r \geq \delta n$. Then, D has a Hamilton decomposition.*

1.1.3 Some related problems

One may consider the following generalisation of Problem 1.2.

Problem 1.12. *Given a (di)graph G, how many edge-disjoint Hamilton cycles does G contain?*

Given a (di)graph G, a *packing of Hamilton cycles* in G is a set of edge-disjoint Hamilton cycles of G. A packing is *optimal* if it has maximum size. Given a graph G, denote by $\mathrm{reg}_{\mathrm{even}}(G)$ the maximum degree of an even-regular spanning subgraph of G. Given a digraph D, denote by $\mathrm{reg}(D)$ the maximum degree of a regular spanning subdigraph of D. Clearly, $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$ and $\mathrm{reg}(D)$ provide natural upper bounds on the size of an optimal packing of Hamilton cycles in a graph G and a digraph D, respectively. Kühn, Lapinskas, and Osthus [73] conjectured that a graph G of sufficiently large minimum degree contains $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$ edge-disjoint Hamilton cycles.

Conjecture 1.13 (Kühn, Lapinskas, and Osthus)**.** *Let G be a graph on n vertices. If $\delta(G) \geq \frac{n}{2}$, then G contains $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$ edge-disjoint Hamilton cycles.*

Partial results towards Conjecture 1.13 were obtained by Kühn, Lapinskas, and Osthus [73] and Csaba, Kühn, Lo, Osthus, and Treglown [20]. Moreover, Kühn and Osthus [77] verified this conjecture when n is sufficiently large and the minimum degree is at least $(2-\sqrt{2})n+o(n)$, while Ferber, Krivelevich, and Sudakov [34] proved an approximate version of Conjecture 1.13. Joos, Kühn, and Schülke [60] recently extended this approximate result to k-uniform hypergraphs.

Given a digraph D, denote by $\delta^0(D) := \min\{d_D^+(v), d_D^-(v) \mid v \in V(D)\}$ the *minimum semidegree* of D. Based on Theorem 1.5, Kühn and Osthus [77] confirmed a conjecture of Erdős (see [106]) which states that a random tournament T contains $\mathrm{reg}(T) = \delta^0(T)$

edge-disjoint Hamilton cycles with high probability. Together, results of Knox, Kühn, and Osthus [69], Krivelevich and Samotij [72], and Kühn and Osthus [77] imply that an analogous result holds for the binomial random graph $G_{n,p}$ for arbitrary p, i.e. $G_{n,p}$ contains $\frac{\mathrm{reg}_{\mathrm{even}}(G_{n,p})}{2} = \left\lfloor \frac{\delta(G_{n,p})}{2} \right\rfloor$ Hamilton cycles with high probability. (Here and throughout this thesis, $\delta(G)$ denotes the *minimum degree* of a graph G.) This resolves a conjecture of Frieze and Krivelevich [37]. For Hamilton decompositions of random regular graphs, see e.g. [65]. Note that [69] introduced the method of "iterative absorption", which was used and further developed to prove Theorem 1.5. For further uses of this proof technique, see e.g. [44, 80, 98].

Progress on Problem 1.12 was also obtained by Frieze and Krivelevich [36] for ε-regular graphs of linear minimum degree. A bipartite graph G on vertex classes A and B of size n is *ε-regular* if for any $A' \subseteq A$ and $B' \subseteq B$ of size at least εn, we have

$$\left| \frac{e_G(A,B)}{|A||B|} - \frac{e_G(A',B')}{|A'||B'|} \right| < \varepsilon.$$

Similarly, a bipartite digraph D on vertex classes A and B of size n is *ε-regular* if for any $A' \subseteq A$ and $B' \subseteq B$ of size at least εn, we have both

$$\left| \frac{e_D(A,B)}{|A||B|} - \frac{e_D(A',B')}{|A'||B'|} \right| < \varepsilon \qquad \text{and} \qquad \left| \frac{e_D(B,A)}{|A||B|} - \frac{e_D(B',A')}{|A'||B'|} \right| < \varepsilon.$$

(Here and throughout this thesis, given a digraph D and disjoint $A, B \subseteq V(D)$, $e_D(A,B)$ denotes the number of edges which start in A and end in B.)

Theorem 1.14 ([36])**.** *For any $0 < \delta < 1$, there exist $0 < \varepsilon < \delta$ and $n_0 \in \mathbb{N}$ for which the following hold. Let G be an ε-regular balanced bipartite graph on vertex classes of size $n \geq n_0$ with minimum degree $\delta(G) \geq \delta n$. Then, G contains $\frac{\delta(G)}{2} - O(\varepsilon)n \geq \frac{\mathrm{reg}_{\mathrm{even}}(G)}{2} - O(\varepsilon)n$ edge-disjoint Hamilton cycles.*

Frieze and Krivelevich [36] also proved a non-bipartite analogue of Theorem 1.14 for graphs and digraphs. Using our bipartite analogue of Theorem 1.5 (see Chapter 4 for details), we improve Theorem 1.14 to best possible bounds and prove a directed analogue. We also deduce that the binomial bipartite graph $G_{n,n,p}$ and the binomial bipartite digraph $D_{n,n,p}$ contain, with high probability, respectively $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$ and $\mathrm{reg}(D)$ edge-disjoint Hamilton cycles. An analogous result can be obtained for random bipartite tournaments.

Corollary 1.15. *For any $0 < p \leq 1$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ for which the following hold.*

(i) *Let G be a bipartite ε-regular graph on vertex classes of size n and suppose that $\delta(G) \geq pn$. Then, G contains $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$ edge-disjoint Hamilton cycles.*

(ii) *Let D be a bipartite ε-regular digraph on vertex classes of size n and suppose that $\delta^0(D) \geq pn$. Then, D contains $\mathrm{reg}(D)$ edge-disjoint Hamilton cycles.*

(iii) *With high probability, $G_{n,n,p}$ contains $\frac{\mathrm{reg}_{\mathrm{even}}(G_{n,n,p})}{2}$ edge-disjoint Hamilton cycles.*

(iv) *With high probability, $D_{n,n,p}$ contains $\mathrm{reg}(D_{n,n,p})$ edge-disjoint Hamilton cycles.*

(v) *Let T be chosen uniformly at random among the bipartite tournaments on vertex classes of size n. Then, T contains $\mathrm{reg}(T)$ edge-disjoint Hamilton cycles with high probability.*

Another related line of research has been to count Hamilton decompositions. For example, Glebov, Luria, and Sudakov [42] showed that, for any $c > \frac{1}{2}$ and any sufficiently large even $r = cn$, any r-regular graph on n vertices contains $r^{\frac{(1+o(n))nr}{2}}$ distinct Hamilton cycles. Similarly, Ferber, Long, and Sudakov [35] showed that, for any $c > \frac{3}{8}$, any sufficiently large cn-regular oriented graph on n vertices contains $n^{(1-o(1))cn^2}$ distinct Hamilton decompositions. Both proofs are based on Theorem 1.5.

1.2 Cycle decompositions

More generally, one can consider *cycle decompositions*, that is, decompositions into (not necessarily Hamilton) cycles. (Here and throughout this thesis, a cycle in a digraph is always assumed to have all its edges consistently oriented.) Unlike Hamilton decompositions, the class of (di)graphs which admit a cycle decomposition has a simple characterisation: a (di)graph can be decomposed into cycles if and only if it is Eulerian. (Here and throughout this thesis, a digraph D is called *Eulerian* if each vertex $v \in V(D)$ satisfies $d_D^+(v) = d_D^-(v)$.) So instead, the main line of research focuses on decomposing Eulerian (di)graphs into as few cycles as possible.

Problem 1.16. *Given an Eulerian (di)graph G, what is the minimum size of a cycle decomposition of G?*

The answer to Problem 1.16 is called the *cycle number* of G and denoted by $\mathrm{cn}(G)$. Note that a Hamilton decomposition of an n-vertex (di)graph G is a cycle decomposition of size $\frac{e(G)}{n}$ and so since Problem 1.2 is already NP-complete, determining the cycle number of a (di)graph is also an NP-complete problem. Thus, a general solution to Problem 1.16 cannot be expected.

1.2.1 Cycle decompositions of graphs

Let G be an Eulerian graph on n vertices. A natural lower bound on the cycle number of G can be expressed in terms of the maximum degree $\Delta(G)$ of G. That is,

$$\mathrm{cn}(G) \geq \frac{\Delta(G)}{2}. \tag{1.1}$$

In general, $\frac{\Delta(G)}{2}$ cycles may not suffice to decompose G. For example, it is easy to see that the graph in Figure 1.2 has maximum degree 4, but has a unique cycle decomposition of size 3.

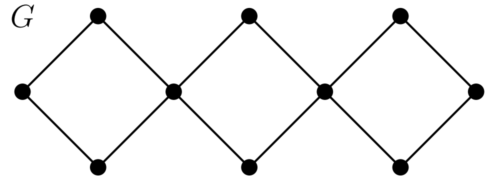


Figure 1.2: Example of a graph G which satisfies $\mathrm{cn}(G) > \frac{\Delta(G)}{2}$.

By considering the maximum possible value for $\Delta(G)$ in (1.1), one can deduce that there exist n-vertex graphs which cannot be decomposed into fewer than $\lfloor \frac{n-1}{2} \rfloor$ cycles. Hajós (see [87]) conjectured that $\lfloor \frac{n-1}{2} \rfloor$ cycles are actually sufficient.

Conjecture 1.17 (Hajós)**.** *Any Eulerian graph on n vertices satisfies* $\mathrm{cn}(G) \leq \lfloor \frac{n-1}{2} \rfloor$.

(Note that Hajós originally asked for a decomposition of Eulerian n-vertex graphs into at most $\lfloor \frac{n}{2} \rfloor$ cycles, but Dean [22] observed that this is equivalent to Conjecture 1.17.)

Hajós' conjecture is still open but has been verified for some specific classes of graphs. First, note that a Hamilton decomposition of a (di)graph G on n vertices is a cycle decomposition of size $\frac{e(G)}{n} \leq \lfloor \frac{n-1}{2} \rfloor$, so Conjecture 1.17 holds for all the classes of (di)graphs which have a Hamilton decomposition (see Section 1.1 for some examples). Moreover, Hajós' conjecture has been verified for graphs on at most 12 vertices by Heinrich, Natale, and Streicher [52], as well as for planar graphs by Jiang [58] and Seyffarth [102]. Favaron and Kouider [32] and Granville and Moisiadis [47] independently proved Conjecture 1.17 for graphs of maximum degree at most 4. Favaron and Kouider [32] also verified Hajós' conjecture for minimally 2-connected and minimally 2-edge-connected graphs. Other lines of investigation on Hajós' conjecture include K_6^--minor free graphs by Fan and Xu [31], graphs of treewidth at most 3 by Botler, Sambinelli, Coelho, and Lee [16], graphs of pathwidth at most 6 by Fuchs, Gellert, and Heinrich [38], projective graphs by Fan and Xu [31], and (quasi)random graphs by Glock, Kühn, and Osthus [45].

Together with Girão, Kühn, and Osthus [41], we prove an approximate version of Conjecture 1.17 for sufficiently large graphs of linear minimum degree.

Theorem 1.18 ([41])**.** *For any $\alpha, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Let G be an Eulerian graph on $n \geq n_0$ vertices of minimum degree $\delta(G) \geq \alpha n$. Then,*

$\mathrm{cn}(G) \leq \frac{n}{2} + \delta n$.

In fact, our methods allow us to show that the bound in (1.1) is asymptotically the correct value when G also a satisfies weak quasirandom property. (This is discussed more thoroughly in Chapter 2.) In particular, these refined bounds settle Conjecture 1.17 for sufficiently large graphs of sufficiently large minimum degree and sufficiently small maximum degree.

Theorem 1.19 ([41])**.** *For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Let G be an Eulerian graph on $n \geq n_0$ vertices satisfying $\left(\frac{1}{2} + \varepsilon\right) n \leq \delta(G) \leq \Delta(G) \leq (1-\varepsilon)n$. Then, $\mathrm{cn}(G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.*

1.2.2 Cycle and edge decompositions of graphs

As discussed above, Hajós' conjecture has been verified for some specific classes of graphs, but very little progress has been made for general graphs. In fact, the related problem of decomposing Eulerian graphs into $O(n)$ cycles is still open. This is equivalent to a problem posed in [28], which is known as the Erdős-Gallai conjecture (see [27]).

Conjecture 1.20 (Erdős-Gallai)**.** *Any graph on n vertices can be decomposed into $O(n)$ cycles and edges.*

To see why the Erdős-Gallai conjecture is equivalent to decomposing Eulerian graphs into a linear number of cycles, let G be an Eulerian graph on n vertices and suppose that $\mathcal{D}$ is a decomposition of G into $O(n)$ cycles and edges. Denote by $\mathcal{C}$ the set of cycles in $\mathcal{D}$ and by $\mathcal{E}$ the set of edges in $\mathcal{D}$. Since G is Eulerian and all the cycles in $\mathcal{C}$ are edge-disjoint, $\mathcal{E}$ induces an Eulerian subgraph G' of G. Thus, G' can be greedily decomposed into a set $\mathcal{C}'$ of at most $|\mathcal{E}|$ edge-disjoint cycles. Then, $\mathcal{C} \cup \mathcal{C}'$ is a cycle decomposition of G of size at most $|\mathcal{D}| = O(n)$, as desired. Conversely, suppose that G is a graph on n vertices. By greedily removing cycles from G, partition the edges of G into an Eulerian subgraph G' and a forest F. Note that F contains at most $n - 1 = O(n)$ edges. Thus, a decomposition of G' into $O(n)$ cycles would induce a decomposition of G into $O(n)$ cycles and edges.

Some progress toward Conjecture 1.20 was made by Conlon, Fox, and Sudakov [19], who showed that $O(n \log \log n)$ cycles and edges are sufficient to decompose any graph on n vertices. Moreover, they verified Conjecture 1.20 for graphs of linear minimum degree.

Theorem 1.21 ([19]). *For any $\alpha > 0$, the following holds. Let G be a graph on n vertices of minimum degree $\delta(G) \geq \alpha n$. Then, G can be decomposed into $O(\alpha^{-12}n)$ cycles and edges.*

Together with Girão, Kühn, and Osthus [41], we observe that Theorem 1.18 can be used to improve the bounds in Theorem 1.21.

Theorem 1.22 ([41]). *For any $\alpha, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices of minimum degree $\delta(G) \geq \alpha n$. Then, G can be decomposed into at most $\frac{3n}{2} + \delta n$ cycles and edges.*

There exist graphs on n vertices which cannot be decomposed into fewer than $\frac{3n}{2} - o(n)$ cycles and edges, so Theorem 1.22 gives the asymptotically best possible constant. Indeed, let $\varepsilon > 0$ and fix a natural number $n \geq \frac{1-\varepsilon}{\varepsilon^2}$ such that εn is odd. Let G be the complete bipartite graph on vertex classes A and B of size εn and $(1-\varepsilon)n$, respectively. We show that G cannot be decomposed into fewer than $\left(\frac{3}{2} - 2\varepsilon\right) n$ cycles and edges. Let $\mathcal{D}$ be a decomposition of G into cycles and edges. By assumption, every vertex of B has odd degree in G and so must be adjacent to at least one edge in $\mathcal{D}$. Thus, there exists $\mathcal{E} \subseteq \mathcal{D}$ which consists of $(1-\varepsilon)n$ edges. Then, there exists $v \in A$ which is adjacent to at most $\frac{1-\varepsilon}{\varepsilon}$ edges in $\mathcal{E}$ and so $|\mathcal{D}| \geq |\mathcal{E}| + \frac{1}{2}\left(d_G(v) - \frac{1-\varepsilon}{\varepsilon}\right) \geq \left(\frac{3}{2} - 2\varepsilon\right) n$, as desired.

1.2.3 Cycle decompositions of digraphs

Jackson [55] conjectured that the analogue of Hajós' conjecture holds for Eulerian oriented graphs, that is, any Eulerian oriented graph on n vertices can be decomposed into at most $\lfloor \frac{n}{2} \rfloor$ cycles. However, Dean [22] observed that this conjecture is false and proposed the following alternative.

Conjecture 1.23 (Dean)**.** *Any Eulerian oriented graph D on n vertices satisfies* $\mathrm{cn}(D) \leq \lfloor \frac{2n}{3} \rfloor$.

More generally, Bienia and Meyniel [9] conjectured that Eulerian digraphs can be decomposed into linearly many cycles.

Conjecture 1.24 (Bienia and Meyniel)**.** *There exists $\alpha > 0$ such that any Eulerian digraph D on n vertices satisfies* $\mathrm{cn}(D) \leq \alpha n$*.*

As mentioned in [9, 22], unions of complete digraphs on 4 vertices which are all sharing a common vertex show that, if Conjecture 1.24 is true, then $\alpha \geq \frac{4}{3}$. On the other hand, Dean [22] conjectured that $\alpha \leq \frac{8}{3}$.

Conjecture 1.25 (Dean)**.** *Any Eulerian digraph D on $n > 1$ vertices satisfies* $\mathrm{cn}(D) \leq \frac{8n}{3} - 3$.

These conjectures are still open, but some progress was recently made by Knierim, Larcher, Martinsson, and Noever [68].

Theorem 1.26 ([68])**.** *Any Eulerian digraph D on n vertices with maximum degree Δ satisfies* $\mathrm{cn}(D) = O(n \log \Delta)$*.*

1.2.4 Some related problems

A related line of research consists in finding decompositions into cycles of prescribed lengths. For example, we already discussed Hamilton decompositions in Section 1.1. Asymptotically best possible minimum degree thresholds for the existence of $C_{2\ell}$-decompositions were obtained by Barber, Kühn, Lo, and Osthus [7]. The study of triangle decompositions has also attracted a lot of attention. Kirkman [67] showed that the complete graph on n vertices can be decomposed into triangles if and only if $n \equiv 1$ or $3 \pmod 6$. More generally, Nash-Williams [92] conjectured that any sufficiently large Eulerian graph G on n vertices which satisfies $e(G) \equiv 0 \pmod 3$ and $\delta(G) \geq \frac{3n}{4}$ can be decomposed into triangles. Delcourt and Postle [24] showed that the minimum degree threshold for a fractional triangle decomposition is at most $\left(\frac{7+\sqrt{21}}{14}\right) n + o(n)$. Together with previous results of Barber, Kühn, Lo, and Osthus [7], this implies that the currently best known minimum degree threshold for triangle decompositions is equal to $\left(\frac{7+\sqrt{21}}{14}\right) n + o(n)$. For further details on the history of Nash-Williams' conjecture, see e.g. the survey [46].

Another line of research consists of decompositions into cycle factors. The Oberwolfach problem, due to Ringel (see e.g. [83]), asks whether the complete graph on an odd number of vertices can be decomposed into edge-disjoint copies of a given cycle factor F. This was answered positively for large complete graphs by Glock, Joos, Kim, Kühn, and Osthus [43]. Subsequently, another proof was given by Keevash and Staden [64]. In fact, both results are more general and cover analogues of the Oberwolfach problem for almost complete graphs [43], dense quasirandom graphs [64], and digraphs [64]. They also allow for decompositions into a prescribed set of cycles factor rather than copies of the same cycle factor F. In particular, this resolves several variants of the Oberwolfach problem. For further details, we direct the readers to the introductions of [43, 64].

1.3 Path decompositions

Finally, we consider *path decompositions*, that is, decompositions which consist of paths. (Here and throughout this thesis, a path in a digraph is always assumed to have all its edges consistently oriented.) For any (di)graph G, it is easy to see that the set $E(G)$ of edges of G forms a path decomposition of G. Thus, as for cycle decompositions, we are interested in decomposing (di)graphs into as few paths as possible.

Problem 1.27. *Given a (di)graph G, what is the minimum size of a path decomposition of G?*

The answer to Problem 1.27 is called the *path number* of G and denoted by $\mathrm{pn}(G)$. Péroche [97] showed that Problem 1.27 is an NP-complete problem, so a general solution cannot be expected.

1.3.1 Path decompositions of graphs

Let G be a graph on n vertices. As for cycle decompositions (recall (1.1)), $\frac{\Delta(G)}{2}$ provides a natural lower bound on $\mathrm{pn}(G)$. However, this lower bound can be refined by considering vertices of odd degree. Indeed, note that any vertex of odd degree in G must be the endpoint of a path in a path decomposition of G. Thus,

$$\mathrm{pn}(G) \geq \max\left\{\frac{\Delta(G)}{2}, \frac{\mathrm{odd}(G)}{2}\right\}. \tag{1.2}$$

(Recall that $\mathrm{odd}(G)$ denotes the number of odd-degree vertices in G.) In general, $\max\left\{\frac{\Delta(G)}{2}, \frac{\mathrm{odd}(G)}{2}\right\}$ paths may not suffice to decompose G. For example, a cycle is a 2-regular graph but can only be decomposed into at least two paths. (Further examples are discussed in Chapter 2, see Propositions 2.4 and 2.5.)

On the other hand, a linear upper bound on the path number can be obtained from Lovász' result on path and cycle decompositions. Indeed, let G be a graph on n vertices and recall that Theorem 1.1 guarantees a decomposition $\mathcal{D}$ of G into $\lfloor \frac{n}{2} \rfloor$ paths and cycles. Thus, by splitting each cycle in $\mathcal{D}$ into two paths, we obtain a path decomposition of G of size at most n, that is, $\mathrm{pn}(G) \leq n$. In fact, Lovász' main result [87] is slightly more general than Theorem 1.1 and implies that any graph G on n vertices satisfies $\mathrm{pn}(G) \leq n - 1$. This was later improved by Donald [26], who showed that $\lfloor \frac{3n}{4} \rfloor$ paths suffice to decompose a graph of order n. Subsequently, Dean and Kouider [23] and Yan [109] independently refined this bound.

Theorem 1.28 ([23, 109])**.** *Any graph G on n vertices satisfies* $\mathrm{pn}(G) \leq \lfloor \frac{2n}{3} \rfloor$.

By considering a disjoint union of triangles (and at most one isolated vertex or edge if $n \equiv 1$ or $2 \pmod 3$), one can see that the bound in Theorem 1.28 is best possible for general graphs. However, Gallai (see [87]) conjectured that it can be improved for connected graphs.

Conjecture 1.29 (Gallai)**.** *Any connected graph G on n vertices satisfies* $\mathrm{pn}(G) \leq \lceil \frac{n}{2} \rceil$.

By considering complete graphs, one can show that Gallai's conjecture would be best possible. Similarly to Hajós' conjecture, Conjecture 1.29 is still open but has been verified for some specific classes of graphs. For example, Stanton, Cowan, and James [103] and Harary and Schwenk [50] proved Gallai's conjecture for complete graphs, complete bipartite graphs, trees, and 3-regular graphs. Gallai's conjecture has been verified for graphs of maximum degree at most 4 by Favaron and Kouider [32] and at most 5 by Bonamy and Perrett [11]. Partial results for graphs of maximum degree 6 were obtained by Chu, Fan, and Liu [18]. Gallai's conjecture for planar graphs has also attracted a lot of attention (see e.g. [14, 16, 39]) and was recently resolved by Blanché, Bonamy, and Bonichon [10]. Much work has been done on families of graphs which impose conditions on the vertices of even degree (see e.g. [15, 30, 32, 87, 100]) or on the girth (see e.g. [13, 16, 51]). Other lines of investigation on Gallai's conjecture include series-parallel graphs by Kindermann, Schlipf, and Schulz [66], families of sparse triangle-free graphs by Jiménez and Wakabayashi [59], graphs of treewidth at most 3 by Botler, Sambinelli, Coelho, and Lee [16], and (quasi)random graphs by Glock, Kühn, and Osthus [45].

Together with Girão, Kühn, and Osthus [41], we prove an approximate version of Conjecture 1.29 for sufficiently large graphs of linear minimum degree.

Theorem 1.30 ([41])**.** *For any $\alpha, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices of minimum degree $\delta(G) \geq \alpha n$. Then,* $\mathrm{pn}(G) \leq \frac{n}{2} + \delta n$.

In fact, our methods allow us to show that the bound in (1.2) is asymptotically the correct value when G also satisfies a weak quasirandom property. (This is discussed more thoroughly in Chapter 2.)

1.3.2 Path decompositions of tournaments

While undirected graphs have a linear or sublinear path number (recall Theorem 1.28), the path number of a digraph may be quadratic on the number of vertices. Indeed, Alspach and Pullman [6] proved that an oriented graph D on n vertices satisfies $\mathrm{pn}(D) \leq \frac{n^2}{4}$, with equality holding for transitive tournaments. This was then generalised to digraphs on at least 4 vertices by O'Brien [95].

Theorem 1.31 ([95])**.** *Let D be a digraph on $n \geq 4$ vertices. Then,* $\mathrm{pn}(D) \leq \frac{n^2}{4}$.

Let D be a digraph. To derive a lower bound on $\mathrm{pn}(D)$, we need to introduce some notation. Let $v \in V(D)$ and define the *excess* at v as $\mathrm{ex}_D(v) \coloneqq d_D^+(v) - d_D^-(v)$. Let $\mathrm{ex}_D^+(v) \coloneqq \max\{0, \mathrm{ex}_D(v)\}$ and $\mathrm{ex}_D^-(v) \coloneqq \max\{0, -\mathrm{ex}_D(v)\}$ be the *positive excess* and *negative excess* at v, respectively. Then, as observed in [6], if $d_D^+(v) > d_D^-(v)$, then a path decomposition of D contains at most $d_D^-(v)$ paths which have v as an internal vertex, and thus at least $d_D^+(v) - d_D^-(v) = \mathrm{ex}_D^+(v)$ paths starting at v. Similarly, a path decomposition of D will contain at least $\mathrm{ex}_D^-(v)$ paths ending at v. Thus, the *excess* of D, defined as

$$\mathrm{ex}(D) \coloneqq \sum_{v \in V(D)} \mathrm{ex}_D^+(v) = \sum_{v \in V(D)} \mathrm{ex}_D^-(v) = \frac{1}{2} \sum_{v \in V(D)} |\,\mathrm{ex}_D(v)|, \tag{1.3}$$

provides a natural lower bound for the path number of D. That is, any digraph D satisfies

$$\mathrm{pn}(D) \geq \mathrm{ex}(D). \tag{1.4}$$

It was shown in [6] that equality is satisfied for acyclic digraphs. A digraph satisfying equality in (1.4) is called *consistent*. Clearly, not all digraphs are consistent (e.g. regular digraphs have excess 0). However, Alspach, Mason, and Pullman [5] conjectured in 1976 that tournaments of even order are consistent.

Conjecture 1.32 (Alspach, Mason, and Pullman)**.** *Any tournament T of even order satisfies* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

By Theorem 1.31 and (1.4), Conjecture 1.32 holds for tournaments of excess $\frac{n^2}{4}$. Moreover, Lo, Patel, Skokan, and Talbot [85] observed that Conjecture 1.32 for tournaments of excess $\frac{n}{2}$ is equivalent to Kelly's conjecture on Hamilton decompositions of regular tournaments (see Conjecture 1.4). Indeed, suppose that T is a regular tournament on n vertices and let $v \in V(T)$. Then, one can verify that n is odd and $T - \{v\}$ is a tournament of excess $\frac{n-1}{2}$. Thus, Conjecture 1.32 would imply that $T - \{v\}$ has a path decomposition $\mathcal{D}$ of size $\frac{n-1}{2}$. This decomposition $\mathcal{D}$ would thus have to consist of Hamilton paths and every vertex of $T - \{v\}$ would have to be the starting point of precisely one path in $\mathcal{D}$ and the ending point of precisely one path in $\mathcal{D}$. A Hamilton decomposition of T could thus be obtained by incorporating v in each of the Hamilton paths in $\mathcal{D}$. Conversely, suppose that T is a tournament of excess $\frac{n}{2}$. Then, one can verify that there exists a partition $U^+ \cup U^-$ of $V(T)$ such that U^+ consists of $\frac{n}{2}$ vertices satisfying $\mathrm{ex}_T^+(v) = d_T^+(v) - d_T^-(v) = 1$ and U^- consists of $\frac{n}{2}$ vertices satisfying $\mathrm{ex}_T^-(v) = d_T^-(v) - d_T^+(v) = 1$. Let T' be the tournament obtained from T by adding a new vertex w with outneighbourhood U^+ and inneighbourhood U^-. Then, T' is regular and so Kelly's conjecture would imply that T' has a decomposition $\mathcal{D}$ which consists of $\frac{n}{2}$ Hamilton cycles. Removing w from each Hamilton cycle in $\mathcal{D}$ gives a decomposition of T into $\frac{n}{2} = \mathrm{ex}(T)$ (Hamilton) paths.

Recently, Lo, Patel, Skokan, and Talbot [85] verified Conjecture 1.32 for sufficiently large tournaments of sufficiently large excess. Moreover, they extended this result to tournaments of odd order n whose excess is at least $n^{2-\frac{1}{18}}$.

Theorem 1.33 ([85])**.** *The following hold.*

(i) *There exists $C \in \mathbb{N}$ for which the following holds. Let T be a tournament of even order n and suppose that $\mathrm{ex}(T) \geq Cn$. Then, $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

(ii) *There exists $n_0 \in \mathbb{N}$ for which the following holds. Let T be a tournament on $n \geq n_0$ vertices satisfying $\mathrm{ex}(T) \geq n^{2-\frac{1}{18}}$. Then, $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

With Girão, Kühn, Lo, and Osthus [40], we build on the results and methods of [76, 85] to prove Conjecture 1.32 for large tournaments.

Theorem 1.34 ([40])**.** *There exists $n_0 \in \mathbb{N}$ such that any tournament T of even order $n \geq n_0$ satisfies $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

In fact, our methods are more general and allow us to study tournaments of odd order. In particular, we obtain asymptotically best possible bounds on the path number of sufficiently large tournaments of odd order, as well as exact bounds for almost all such tournaments. (This is discussed more thoroughly in Chapter 3.) We also determine the path number of regular tournaments, which resolves a conjecture of Alspach, Mason, and Pullman [5].

Theorem 1.35 ([40]). *There exists $n_0 \in \mathbb{N}$ such that any regular tournament T on $n \geq n_0$ vertices satisfies* $\mathrm{pn}(T) = \frac{n+1}{2}$.

More generally, one may consider the path number of general digraphs. The arguments of Péroche [97] can be adapted to show that determining whether a digraph is consistent or not is an NP-complete problem (see e.g. [21]), so one cannot expect to fully characterise consistent digraphs. Nevertheless, this line of study was recently initiated by Espuny Díaz, Patel, and Stroh [29], who proved that, for a wide range of densities p, the random binomial digraph $D_{n,p}$ is consistent with high probability.

1.3.3 Some related problems

Instead of decomposing graphs into paths, one may consider the related problem of finding a decomposition into *linear forests*, that is, forests whose components are all paths. The linear arboricity conjecture, due to Akiyama, Exoo, and Harary [1], states that any graph G can be decomposed into at most $\left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ linear forests. An approximate version of this conjecture was verified by Alon [2] for graphs of sufficiently large maximum degree. The best current bound, due to Lang and Postle [81], states that $\frac{\Delta(G)}{2} + 3\sqrt{\Delta(G)}\log^4(\Delta(G))$ linear forests suffice provided that $\Delta(G)$ is sufficiently large. For further details on the history of the linear arboricity conjecture, see e.g. [33].

Instead of considering general linear forests, with components of any length, one may consider matching decompositions, that is, edge-colourings. Vizing's theorem states that the chromatic index $\chi'(G)$ of any graph G satisfies $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$. An active line of research consists in classifying graphs in terms of their chromatic index. Given a graph G on n vertices and a subgraph H of G on m vertices, we say that H is *overfull* if H

has more than $\Delta(G)\lfloor \frac{m}{2} \rfloor$ edges. The overfull subgraph conjecture, due to Chetwynd and Hilton [17], states that a graph G on n vertices with $\Delta(G) > \frac{n}{3}$ satisfies $\chi'(G) = \Delta(G) + 1$ if and only if G contains an overfull subgraph. Niessen and Volkmann [94], Plantholt [99], and Bongard, Hoffmann, and Volkmann [12] considered minimum degree conditions for which the overfull subgraph conjecture holds. Some results on the overfull conjecture when the maximum degree is large are surveyed in [54]. Moreover, Glock, Kühn, and Osthus [45] verified this conjecture for sufficiently large almost regular quasirandom graphs with linear minimum degree.

One can show that the overfull subgraph conjecture is a generalisation of the 1-factorisation conjecture, which states that any sufficiently dense regular graph on an even number of vertices can be decomposed into perfect matchings. This latter conjecture was verified for sufficiently large graphs by Csaba, Kühn, Lo, Osthus, and Treglown [20]. For further discussions on the overfull and 1-factorisation conjectures, see e.g. [104].

1.4 Organisation

This thesis is organised as follows. First, we give a detailed overview of our main results: in Chapter 2, we discuss our results on path and cycle decompositions of dense graphs; in Chapter 3, we discuss our results on path decompositions of tournaments; and in Chapter 4, we discuss our results on Hamilton decompositions of bipartite tournaments. Due to space constraints, we will only provide a proof of Jackson's conjecture (and some easy applications). This is achieved in Chapters 5–18. We give a more detailed overview of these chapters at the start of Chapter 5.

CHAPTER 2

PATH AND CYCLE DECOMPOSITIONS OF DENSE GRAPHS

This chapter summarises the results from [41], which are joint work with António Girão, Daniela Kühn, and Deryk Osthus.

In this chapter, we discuss our results on path and cycle decompositions of dense graphs more thoroughly. Recall that given a graph G, we denote by $\mathrm{pn}(G)$ the path number of G (that is, the minimum number of paths in a path decomposition of G) and, if G is Eulerian, we denote by $\mathrm{cn}(G)$ the cycle number of G (that is, the minimum number of cycles in a cycle decomposition of G). As mentioned in the introduction, we obtain asymptotically best possible upper bounds on the path number (Theorem 1.30) and cycle number (Theorem 1.18) of dense graphs, as well as improve previously known bounds on the number of cycles and edges required to decompose such graphs (Theorem 1.22). For convenience, we restate these theorems here.

Theorem 2.1. *For any $\alpha, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that if G is a graph on $n \geq n_0$ vertices of minimum degree $\delta(G) \geq \alpha n$, then the following hold.*

(i) $\mathrm{pn}(G) \leq \frac{n}{2} + \delta n$.

(ii) *If G is Eulerian, then* $\mathrm{cn}(G) \leq \frac{n}{2} + \delta n$.

(iii) *G can be decomposed into at most $\frac{3n}{2} + \delta n$ cycles and edges.*

Due to space constraints we will omit the proof of Theorem 2.1. However, a proof overview can be found in Section 2.2.

2.1 Weak quasirandomness

As briefly mentioned in the introduction, we can improve the bounds in Theorem 2.1(i) and (ii) when G also satisfies a weak quasirandom property.

More precisely, we say that an n-vertex graph G is *weakly-(ε, p)-quasirandom* if for any partition $A \cup B$ of $V(G)$ with $|A|, |B| \geq \varepsilon n$ we have $e_G(A, B) \geq p|A||B|$. This notion of weak quasirandomness implies that the reduced graph obtained after applying the regularity lemma to a dense graph is connected. This is the only property required to obtain the bounds in the following theorem.

Theorem 2.2. *For any $\alpha, \delta, p > 0$, there exists $n_0 \in \mathbb{N}$ such that if G is a weakly-$(\frac{\alpha}{2}, p)$-quasirandom graph on $n \geq n_0$ vertices with $\delta(G) \geq \alpha n$, then the following hold.*

(i) $\mathrm{pn}(G) \leq \max\left\{\frac{\mathrm{odd}(G)}{2}, \frac{\Delta(G)}{2}\right\} + \delta n$.

(ii) *If G is Eulerian, then* $\mathrm{cn}(G) \leq \frac{\Delta(G)}{2} + \delta n$.

Note that n-vertex graphs with minimum degree at least $\frac{n}{2} + o(n)$ are easily seen to be weakly quasirandom, so the following holds.

Corollary 2.3. *For any $\delta, \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \frac{n}{2} + \varepsilon n$, then the following hold.*

(i) $\mathrm{pn}(G) \leq \max\left\{\frac{\mathrm{odd}(G)}{2}, \frac{\Delta(G)}{2}\right\} + \delta n$.

(ii) *If G is Eulerian, then* $\mathrm{cn}(G) \leq \frac{\Delta(G)}{2} + \delta n$.

Note that, if in addition G is regular, then the error terms of εn and δn can be removed in Corollary 2.3(ii) (see Theorem 1.3). Moreover, Corollary 2.3(ii) automatically implies that Hajós' conjecture (Conjecture 1.17) holds for very dense n-vertex graphs whose maximum degree is bounded away from n (see Theorem 1.19). In fact, Theorem 2.2(ii) implies that Theorem 1.19 holds more generally for sufficiently large weakly-quasirandom graphs on n vertices with maximum degree bounded away from n.

For Theorem 2.1, the linear minimum degree condition is likely to be an artefact of our proof. On the other hand, the next results show that neither the linear minimum degree condition (or even the stronger assumption of linear connectivity), nor the weakly-$(\frac{\alpha}{2}, p)$-quasirandom property is sufficient on its own to obtain the bounds in Theorem 2.2.

Proposition 2.4. *For any odd integer $n \geq 20$, there exists an $\lfloor \frac{n}{10} \rfloor$-connected Eulerian graph G on $2n$ vertices such that the following hold.*

(i) $\mathrm{pn}(G) \geq \max\left\{\frac{\mathrm{odd}(G)}{2}, \frac{\Delta(G)}{2}\right\} + \frac{n}{10}$.

(ii) $\mathrm{cn}(G) \geq \frac{\Delta(G)}{2} + \frac{n}{10}$.

Proposition 2.5. *For any $0 < \alpha \leq 1$ and $n_0 \in \mathbb{N}$, the following hold.*

(i) *There exists a weakly-$\left(\frac{\alpha}{2}, \frac{\alpha^2}{100}\right)$-quasirandom graph G on $n \geq n_0$ vertices such that* $\mathrm{pn}(G) \geq \max\left\{\frac{\mathrm{odd}(G)}{2}, \frac{\Delta(G)}{2}\right\} + \frac{\alpha n}{10}$.

(ii) *There exists an Eulerian weakly-$\left(\frac{\alpha}{2}, \frac{\alpha^2}{100}\right)$-quasirandom graph G on $n \geq n_0$ vertices such that* $\mathrm{cn}(G) \geq \frac{\Delta(G)}{2} + \frac{\alpha n}{10}$.

On the other hand, the next result shows that one can drop the linear minimum degree condition in Theorem 2.2(i) if the quasirandomness covers a larger range of partition class sizes.

Theorem 2.6. *For any $\delta, p > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that any weakly-(ε, p)-quasirandom graph G on $n \geq n_0$ vertices satisfies* $\mathrm{pn}(G) \leq \max\left\{\frac{\mathrm{odd}(G)}{2}, \frac{\Delta(G)}{2}\right\} + \delta n$.

Surprisingly, it turns out that the Erdős-Gallai conjecture (Conjecture 1.20) is equivalent to the following analogue of Theorem 2.6 for cycle decompositions of Eulerian graphs.

Conjecture 2.7. *For any $\delta, p > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that any Eulerian weakly-(ε, p)-quasirandom graph G on $n \geq n_0$ vertices satisfies* $\mathrm{cn}(G) \leq \frac{\Delta(G)}{2} + \delta n$.

We can prove Conjecture 2.7 if weak-(ε, p)-quasirandomness is replaced by weak-$(\frac{\varepsilon}{\log\log n}, p)$-quasirandomness.

Proposition 2.8. *For any $\delta, p > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that any Eulerian weakly-$(\frac{\varepsilon}{\log\log n}, p)$-quasirandom graph G on $n \geq n_0$ vertices satisfies $\mathrm{cn}(G) \leq \frac{\Delta(G)}{2} + \delta n$.*

2.2 Proof overview

The proofs of Theorems 2.2 and 2.6 follow a similar strategy as those of Theorem 2.1(i) and (ii), and so, for simplicity, we only sketch the proof of Theorem 2.1.

2.2.1 Cycle and edge decompositions: proof overview of Theorem 2.1(iii)

First, we observe that Theorem 2.1(iii) can be derived from Theorem 2.1(ii) as follows. Fix an additional constant n_0 such that $0 < \frac{1}{n_0} \ll \alpha, \delta \leq 1$. Let G be a graph on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq \alpha n$. Using the minimum degree condition of G, we construct a set $\mathscr{P}$ which consists of $\frac{\mathrm{odd}(G)}{2}$ edge-disjoint short paths of G such that every vertex of odd degree in G is an endpoint of precisely one path in $\mathscr{P}$. We do so in such a way that $\mathscr{P}$ covers at most $n + \frac{\delta n}{2}$ edges of G and at most $\frac{\alpha n}{2}$ edges incident to each vertex of G. Thus, $G \setminus E(\mathscr{P})$ is an Eulerian graph of minimum degree at least $\frac{\alpha n}{2}$ and so we can use Theorem 2.1(ii) (with $\frac{\alpha}{2}$ and $\frac{\delta}{2}$ playing the roles of α and δ) to decompose it into at most $\frac{n}{2} + \frac{\delta n}{2}$ cycles. Altogether, we obtain a decomposition of G into at most $\frac{3n}{2} + \delta n$ cycles and edges, as desired.

2.2.2 Cycle decompositions: proof overview of Theorem 2.1(ii)

Fix additional constants $\varepsilon, \zeta, \beta$, and n_0 such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \zeta \ll \beta \ll \alpha, \delta \leq 1$. Let G be an Eulerian graph on $n \geq n_0$ vertices with $\delta(G) \geq \alpha n$. We decompose G by repeatedly constructing cycles. For simplicity, whenever edges are used to form a cycle, they are implicitly deleted from the graph (so all the cycles constructed below are edge-disjoint, as desired). We obtain the bulk of our cycles in Step 3, all other cycles will contribute to the error term. In Step 3, we need to be very efficient (i.e. the average length of the cycles needs to be large), while there is room to spare in the other steps.

Step 1: Applying Szemerédi's regularity lemma and setting aside some random subgraphs Γ and Γ'. We start by applying Szemerédi's regularity lemma and a cleaning procedure similar to the one used to prove the degree form of the regularity lemma. From this application, we obtain a subgraph $H \subseteq G$ of small maximum degree and a partition of $V(G)$ into k clusters $V_1, \ldots, V_k$ and an exceptional set V_0. Moreover, in each non-empty pair of clusters of $G \setminus H$, almost all vertices have degree close to the density of the pair, while the few other vertices are isolated. Moreover, in each pair, the vertices of positive degree span an ε-regular bipartite graph.

We also set aside two sparse edge-disjoint random spanning subgraphs $\Gamma, \Gamma' \subseteq G \setminus H$ such that, in Γ, each non-empty pair of clusters has density close to β, while in Γ' each such pair has density close to ζ. By Theorem 1.1 and by splitting clusters if necessary, we may assume that the reduced graph R' of Γ can be decomposed into at most $\frac{|R'|}{2} = \frac{k}{2}$ cycles of even length (this will be needed in Step 5). Let $G^* \coloneqq G \setminus (H \cup \Gamma \cup \Gamma')$. Denote by G^*_{ij} the ε-regular (almost spanning) subgraph of the pair $G^*[V_i, V_j]$, and define Γ_{ij} similarly. The random subgraphs Γ and Γ' will be used to tie together given sets of paths of G^* into cycles.

Step 2: Covering the edges of $G[V_0]$. Apply Theorem 1.1 to $G[V_0]$. The paths obtained are extended to paths with endpoints in $V(G) \setminus V_0$ and then closed into cycles using edges of Γ. Since V_0 is small, this results in only a few cycles and we can use edges of Γ sparingly so that its properties are not destroyed.

Step 3: Covering most of G^* with at most roughly $\frac{n}{2}$ cycles. The idea is to decompose the edges of G^* into paths and then link some of these paths together using the edges in $\Gamma \cup \Gamma'$ to form cycles. The bipartite graph $G^*[V_0, V(G) \setminus V_0]$ is decomposed into paths of length 2 with midpoints in V_0, called *exceptional paths*, while ε-regular pairs G^*_{ij} are approximately decomposed into long but not spanning paths, so that a few vertices are set aside for tying up paths. We then use edges of $\Gamma \cup \Gamma'$ to link these paths into cycles. More precisely, we proceed as follows. Suppose first that the reduced graph R of G is connected. We construct an auxiliary reduced graph $\widehat{R}$ such that the multiplicity of the edges between V_i and V_j in $\widehat{R}$ is proportional to the density of corresponding pair G^*_{ij} of G^*. We optimally decompose $\widehat{R}$ into matchings. Given a matching M of $\widehat{R}$, we form sets $\mathcal{P}$ of paths consisting of exactly one path of G^*_{ij} for each $V_iV_j \in M$, and of exceptional paths which cover vertices of V_0 with highest degree. Since M is a matching of clusters and our non-exceptional paths do not span entire clusters, we can ensure that each set $\mathcal{P}$ of paths obtained in this way consists of vertex-disjoint paths and does not span entire clusters. Thus, after this step, we still have some uncovered vertices, called *reservoir vertices*, which can be used to link the paths in each set $\mathcal{P}$ into a cycle using edges of $\Gamma \cup \Gamma'$.

Since the edge multiplicity between two clusters in $\widehat{R}$ is proportional to the density of the corresponding pair of G^* and at each stage we cover exceptional vertices of highest degree, we obtain an upper bound of roughly $\frac{\Delta(G^*)}{2}$ cycles in total. In general, R may be disconnected and, by construction, $\Gamma \cup \Gamma'$ contains no edges between the different components of R. Thus, we cannot tie together paths from different components and we need to apply the above argument to each component of R separately. But, if a component of R contains n' vertices of G^* (say), then the subgraph of G^* induced by this component has maximum degree at most n' and we obtain at most roughly $\frac{n'}{2}$ cycles from that component. Thus, we get an upper bound of roughly $\frac{n}{2}$ cycles in total.

By alternating which vertices are used as reservoir vertices, we ensure that the leftover graph H' has small maximum degree. Moreover, we use edges of Γ sparingly so that the properties of Γ are maintained. Since the density ζ of Γ' is small, we can add the remaining edges of Γ' to H' without significantly increasing the maximum degree of H'.

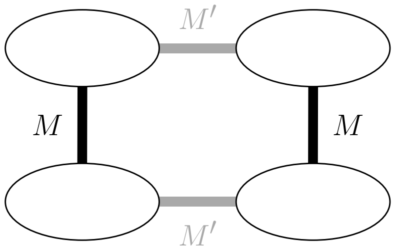
We remark that in Step 2 it was possible to tie together paths using only Γ because we had some room to spare (in the sense that the number of cycles produced might be fairly large compared to the number of edges covered). But in Step 3, we need to use edges of both Γ and Γ' in order to be efficient and obtain the desired number of cycles. (The reason that using $\Gamma \cup \Gamma'$ is more efficient is that the reduced graph of $\Gamma \cup \Gamma'$ equals that of G^*. We cannot guarantee this property for Γ alone since for Step 4 the non-empty pairs Γ_{ij} of Γ need to be fairly dense.)

Step 4: Covering the leftovers. By construction, $H \cup H'$ has small maximum degree and so can be decomposed into few small matchings. We tie the edges of each matching into a cycle using edges of Γ. Once again, we make sure that the relevant properties of Γ are preserved.

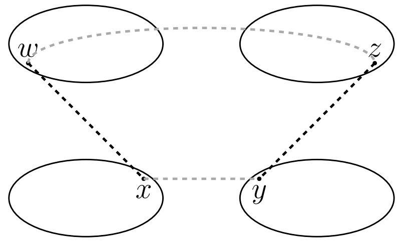
Step 5: Fully decomposing Γ. It only remains to decompose (the remainder of) Γ. The idea is to initially decompose the reduced graph of Γ into $\frac{k}{2}$ cycles of even length (as discussed in Step 1). For each such cycle C, the subgraph Γ_C of Γ corresponding to the blow-up of C is first approximately decomposed into Hamilton cycles of Γ_C that "wind around" C. The leftover is then decomposed using the main technical result of [76] as follows.

The cycle C is initially decomposed into a pair (M, M') of matchings. For each $V_iV_j \in M \cup M'$, we first set aside a small set $\mathcal{E}_{ij}$ of edges of Γ_{ij} and then decompose the remaining edges into a set $\mathcal{H}_{ij}$ of Hamilton paths. We make sure the set of endpoints of the paths in $\bigcup_{V_iV_j \in M} \mathcal{H}_{ij}$ equals the set of endpoints of the edges in $\bigcup_{V_iV_j \in M'} \mathcal{E}_{ij}$, and similarly for M and M' exchanged. Thus we can tie together a path of $\mathcal{H}_{ij}$ for each $V_iV_j \in M$ using exactly one edge of $\mathcal{E}_{i'j'}$ for each $V_{i'}V_{j'} \in M'$. We proceed similarly to tie paths of $\bigcup_{V_iV_j \in M'} \mathcal{H}_{ij}$ into cycles. We thus obtain a Hamilton decomposition of Γ_C.

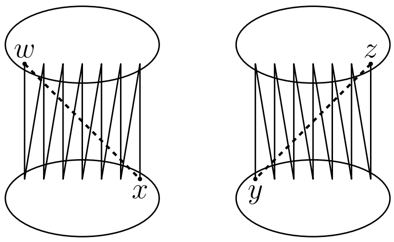
In order to prescribe the endpoints of the Hamilton paths, we add some suitable edges to Γ_C, called *fictive edges*, and then actually find a Hamilton decomposition of each pair $\Gamma_{ij} \setminus \mathcal{E}_{ij}$ such that each cycle in the decomposition contains exactly one fictive edge (see Figure 2.1). Such decompositions are guaranteed by the "robust decomposition lemma" of [76]. Since by construction all pairs of Γ have density close to β, we obtain, in total,



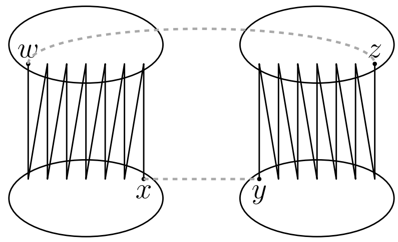
(A) Pair of matchings (M, M') in the reduced graph.



(B) We set aside an edge from each pair in M' (dashed grey) and replace them by a fictive edge in each pair of M (dashed black).



(C) We find a Hamilton cycle of each pair of M containing a single fictive edge (dashed black).



(D) We remove the fictive edges from the cycles of pairs of M and insert back the edges set aside from pairs of M' (dashed grey).

Figure 2.1: Construction of a cycle of Γ.

about $\frac{\beta n}{2} \ll \delta n$ cycles.

2.2.3 Path decompositions: proof overview of Theorem 2.1(i)

We will find the required path decomposition by applying the previous arguments to a suitable auxiliary graph. More precisely, fix an additional constant n_0 such that $0 < \frac{1}{n_0} \ll \alpha, \delta \leq 1$. Let G be a graph on $n \geq n_0$ vertices of minimum degree $\delta(G) \geq \alpha n$. Our strategy consists in adding to G a (multi)set E_{fict} of at most roughly $\frac{n}{2}$ fictive edges in such a way that the (multi)graph $G \cup E_{\text{fict}}$ is Eulerian. Then, we apply the arguments of Theorem 2.1(ii) to find a cycle decomposition $\mathcal{D}$ of $G \cup E_{\text{fict}}$ of size at most $\frac{n}{2} + \frac{\delta n}{2}$. We do this in such a way that $\mathcal{D}$ has a partition $\mathcal{D}_1 \cup \mathcal{D}_2$ where each cycle in $\mathcal{D}_1$ contains precisely one fictive edge from E_{fict} and $\mathcal{D}_2$ is a set of at most $\frac{\delta n}{2}$ cycles which do not contain any fictive edge. (Roughly speaking, the cycles in $\mathcal{D}_1$ are those constructed in

Step 3 of the above proof overview, while $\mathcal{D}_2$ consists of all the other cycles.) Observe that $\mathcal{D}_1$ induces a set $\mathcal{D}_1'$ of edge-disjoint paths of G. Let $\mathcal{D}_2'$ be obtained from $\mathcal{D}_2$ by splitting each cycle into two paths. Then, $\mathcal{D}_1' \cup \mathcal{D}_2'$ is a path decomposition of G of size at most $|\mathcal{D}| + \frac{\delta n}{2} \leq \frac{n}{2} + \delta n$, as desired.

CHAPTER 3

PATH DECOMPOSITIONS OF TOURNAMENTS

This chapter summarises the results from [40], which are joint work with António Girão, Daniela Kühn, Allan Lo, and Deryk Osthus.

In this chapter, we discuss our results on path decompositions of tournaments. Recall from (1.4), that the excess $\mathrm{ex}(D) = \frac{1}{2}\sum_{v\in V(D)} |d_D^+(v) - d_D^-(v)|$ of a digraph D provides a natural lower bound on the path number $\mathrm{pn}(D)$. Our main contribution (Theorem 1.34) consists of a proof of Alspach, Mason, and Pullman's conjecture (Conjecture 1.32), which states any tournament T of even order satisfies $\mathrm{pn}(T) = \mathrm{ex}(T)$. For simplicity, we restate this theorem here.

Theorem 3.1. *There exists $n_0 \in \mathbb{N}$ such that any tournament T of even order $n \geq n_0$ satisfies $\mathrm{pn}(T) = \mathrm{ex}(T)$.*

Due to space constraints, we will omit the proof of Theorem 3.1. However, a proof overview can be found in Section 3.2.

3.1 Tournaments of odd order

As briefly mentioned in the introduction, our methods also apply to tournaments of odd order. We now discuss this more thoroughly.

Let D be a digraph. Let $\Delta^0(D)$ denote the *maximum semidegree* of D, that is $\Delta^0(D) \coloneqq \max\{d^+(v), d^-(v) \mid v \in V(D)\}$. Note that $\Delta^0(D)$ is a natural lower bound for pn(D) as every vertex $v \in V(D)$ must be in at least $\max\{d^+(v), d^-(v)\}$ paths. This leads to the notion of the *modified excess* of a digraph D, which is defined as

$$\widetilde{\mathrm{ex}}(D) \coloneqq \max\{\mathrm{ex}(D), \Delta^0(D)\}.$$

This provides a natural lower bound for the path number of any digraph D. That is, any digraph D satisfies

$$\mathrm{pn}(D) \geq \widetilde{\mathrm{ex}}(D). \tag{3.1}$$

(Note that one can easily verify that any tournament T of even order satisfies $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, so (3.1) is consistent with Conjecture 1.32.)

Observe that, by Theorem 1.33(ii), equality holds in (3.1) for large tournaments of excess at least $n^{2-\frac{1}{18}}$. However, note that equality does not hold for regular digraphs. Indeed, by considering the number of edges, one can show that any path decomposition of an r-regular digraph will contain at least $r+1$ paths. Thus, any r-regular digraph satisfies

$$\mathrm{pn}(D) \geq r + 1 = \widetilde{\mathrm{ex}}(D) + 1.$$

Alspach, Mason, and Pullman [5] conjectured that equality holds in this inequality whenever D is a regular tournament. We verify this conjecture for sufficiently large tournaments (see Theorem 1.35). In fact, our argument also applies to regular oriented graphs of large enough degree.

Theorem 3.2. *For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if D is an r-regular oriented graph on $n \geq n_0$ vertices satisfying $r \geq \left(\frac{3}{8} + \varepsilon\right) n$, then* $\mathrm{pn}(D) = r + 1 = \widetilde{\mathrm{ex}}(D) + 1$.

Recall from Section 1.1.1 that a robust (ν, τ)-outexpander is a digraph D on n vertices such that $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$ for every $S \subseteq V(D)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$ (where $RN^+_{\nu,D}(S)$ denotes the set of vertices of D which have at least νn inneighbours in S). Theorem 3.2 can be extended to regular digraphs of linear degree which are robust outexpanders.

Theorem 3.3. *For any $\delta > 0$, there exists $\tau > 0$ such that, for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let D be a robust (ν, τ)-outexpander on $n \geq n_0$ vertices and suppose that D is r-regular for some $r \geq \delta n$. Then,* $\mathrm{pn}(D) = \widetilde{\mathrm{ex}}(D) + 1 = r + 1$.

There also exist non-regular tournaments for which equality does not hold in (3.1). Indeed, let $\mathcal{T}_{\mathrm{apex}}$ be the set of tournaments T on $n \geq 5$ vertices for which there exists a partition $V(T) = V_0 \cup \{v_+\} \cup \{v_-\}$ such that $T[V_0]$ is a regular tournament on $n - 2$ vertices (and so n is odd), $N_T^+(v_+) = V_0 = N_T^-(v_-)$, $N_T^-(v_+) = \{v_-\}$, and $N_T^+(v_-) = \{v_+\}$. The tournaments in $\mathcal{T}_{\mathrm{apex}}$ are called *apex tournaments*.

Theorem 3.4. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \in \mathcal{T}_{\mathrm{apex}}$ satisfies* $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T) + 1$.

Denote by $\mathcal{T}_{\mathrm{reg}}$ the class of regular tournaments and let $\mathcal{T}_{\mathrm{excep}} := \mathcal{T}_{\mathrm{apex}} \cup \mathcal{T}_{\mathrm{reg}}$. The tournaments in $\mathcal{T}_{\mathrm{excep}}$ are called *exceptional*. We conjecture that the tournaments in $\mathcal{T}_{\mathrm{excep}}$ are the only ones which do not satisfy equality in (3.1).

Conjecture 3.5. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \notin \mathcal{T}_{\mathrm{excep}}$ on $n \geq n_0$ vertices satisfies* $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$.

We prove an approximate version of this conjecture (see Corollary 3.7). Moreover, in Theorem 3.6, we prove Conjecture 3.5 exactly unless n is odd and T is extremely close to being a regular tournament.

Theorem 3.6. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. If T is a tournament on $n \geq n_0$ vertices such that $T \notin \mathcal{T}_{\mathrm{excep}}$ and*

(i) $\widetilde{\mathrm{ex}}(T) \geq \frac{n}{2} + \beta n$, *or*

(ii) $N^+(T), N^-(T) \geq \beta n$, *where* $N^+(T) := |\{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ *and* $N^-(T) := |\{v \in V(T) \mid \mathrm{ex}_T^-(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$,

then $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$.

One can verify that a tournament T of even order satisfies $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$ and so Theorem 3.1 (i.e. the exact solution when n is even) can be derived from Theorem 3.6. We also derive an approximate version of Conjecture 3.5 from Theorem 3.6.

Corollary 3.7. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any tournament T on $n \geq n_0$ vertices, $\mathrm{pn}(T) \leq \widetilde{\mathrm{ex}}(T) + \beta n$.*

Note that Theorem 3.6(ii) corresponds to the case where linearly many different vertices can be used as endpoints of paths in a path decomposition of size $\widetilde{\mathrm{ex}}(T)$. Indeed, let T be a tournament and $\mathcal{P}$ be a path decomposition of T. Then, as mentioned above, each $v \in V(T)$ must be the starting point of at least $\mathrm{ex}_T^+(v)$ paths in $\mathcal{P}$. Thus, for any tournament T, $N^+(T)$ is the maximum number of distinct vertices which can be a starting point of a path in a path decomposition of T of size $\widetilde{\mathrm{ex}}(T)$ and similarly for $N^-(T)$ and the ending points of paths.

One can show that almost all large tournaments satisfy $\mathrm{ex}(T) = n^{\frac{3}{2}+o(1)}$. Indeed, consider a tournament T on n vertices, where the orientation of each edge is chosen uniformly at random, independently of all other orientations. For the upper bound on $\mathrm{ex}(T)$, one can simply apply a Chernoff bound to show that for a given vertex v and $\varepsilon > 0$, we have $\mathrm{ex}_T^+(v) \leq n^{\frac{1}{2}+\varepsilon}$ with probability $1 - o\left(\frac{1}{n}\right)$. The result follows by a union bound over all vertices. For the lower bound, let X denote the number of vertices v with $d_T^-(v) \in \left[\frac{n}{2} - 2\sqrt{n}, \frac{n}{2} - \sqrt{n}\right]$. Then it is easy to see that, for large enough n, we have $\mathbb{E}[X] \geq \frac{n}{10^4}$, say. Moreover, Chebyshev's inequality can be used to show that, with probability $1 - o(1)$, we have $X \geq \frac{n}{2 \cdot 10^4}$, again with room to spare. Thus, Theorem 3.6 implies the following. (Note that the case when n is even already follows from Theorem 1.33(i).)

Corollary 3.8. *As $n \to \infty$, the proportion of tournaments T on n vertices satisfying $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$ tends to 1.*

Finally, we observe that our methods give a short proof of (a stronger version of) a result of Osthus and Staden [96], which guarantees an approximate decomposition of regular robust outexpanders of linear degree into Hamilton cycles and which was used as a tool in the proof of Kelly's conjecture [76]. This new approximate decomposition result will play an important role in our proof of Jackson's conjecture (see Chapter 4 for more details).

Theorem 3.9. *For all $\delta > 0$, there exists $\tau > 0$ such that, for all $\nu, \eta > 0$, there exists $\varepsilon > 0$ such that there exists $n_0 \in \mathbb{N}$ for which the following holds. Let $\ell \leq (\delta - \eta)n$. Let D be a robust (ν, τ)-outexpander on $n \geq n_0$ vertices and suppose that every $v \in V(D)$ satisfies $(\delta - \varepsilon)n \leq d_D^+(v), d_D^-(v) \leq (\delta + \varepsilon)n$. Suppose that $F_1, \dots, F_\ell$ are linear forests on $V(D)$ satisfying the following properties.*

(i) *For each $i \in [\ell]$, $e(F_i) \leq \varepsilon n$.*

(ii) *For each $v \in V(D)$, there exist at most εn indices $i \in [\ell]$ such that $v \in V(F_i)$.*

Define a multidigraph $\mathcal{F}$ by $\mathcal{F} := \bigcup_{i \in [\ell]} F_i$. Then, the multidigraph $D \cup \mathcal{F}$ contains edge-disjoint Hamilton cycles $C_1, \dots, C_\ell$ such that $F_i \subseteq C_i$ for each $i \in [\ell]$.

3.2 Proof overview

We now give a brief proof overview of (a simplified case of) Theorem 3.6. (Recall that Theorem 3.1 follows from Theorem 3.6.)

3.2.1 Robust outexpanders

Recall from Theorem 1.5 that any regular robust outexpander of linear degree has a Hamilton decomposition. We can apply this to obtain an optimal path decomposition in the following setting. Let D be a digraph on n vertices, $0 < \eta < 1$, and suppose that $X^+ \cup X^- \cup X^0$ is a partition of $V(D)$ such that $|X^+| = |X^-| = \eta n$ and the following hold.

$$\begin{aligned} &\textit{Each } v \in X^0 \textit{ satisfies } d_D^+(v) = \eta n = d_D^-(v).\\ &\textit{Each } v \in X^+ \textit{ satisfies } d_D^+(v) = \eta n \textit{ and } d_D^-(v) = \eta n - 1.\\ &\textit{Each } v \in X^- \textit{ satisfies } d_D^+(v) = \eta n - 1 \textit{ and } d_D^-(v) = \eta n.\end{aligned} \tag{†}$$

Then, the digraph D' obtained from D by adding a new vertex v with $N^+_{D'}(v) = X^+$ and $N^-_{D'}(v) = X^-$ is ηn-regular. Thus, if D is a robust outexpander, then so is D' and there exists a decomposition of D' into Hamilton cycles. This induces a decomposition $\mathcal{P}$ of D into ηn Hamilton paths, where each vertex in X^+ is the starting point of exactly one path in $\mathcal{P}$ and each vertex in X^- is the ending point of exactly one path in $\mathcal{P}$. (A similar observation was already made and used in [85].) Our main strategy will be to reduce our tournament to a digraph of the above form. This will be achieved as follows.

3.2.2 Simplified approach for well-behaved tournaments

Let $\beta > 0$ and fix additional constants such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \gamma \ll \eta \ll \beta$. Let T be a tournament on $n \geq n_0$ vertices. For simplicity, we assume that each $v \in V(T)$ satisfies $|\operatorname{ex}_T(v)| \leq \varepsilon n$ (i.e. T is almost regular), $\widetilde{\operatorname{ex}}(T) = \operatorname{ex}(T)$, and both $|\{v \in V(T) \mid \operatorname{ex}^+_T(v) > 0\}|, |\{v \in V(T) \mid \operatorname{ex}^-_T(v) > 0\}| \geq \eta n$. (Further details on how the argument can be generalised if any of these conditions is not satisfied can be found in [40].)

Since T is almost regular, it is a robust outexpander. Let Γ be obtained by including each edge of T with probability γ. Using Chernoff bounds, we may assume that Γ is a robust outexpander of density almost γ and $D := T \setminus \Gamma$ is almost regular. The digraph Γ will serve two purposes. Firstly, its robust outexpansion properties will be used to construct an approximate path decomposition of T. Secondly, provided few edges of Γ are used throughout this approximate decomposition, it will guarantee that the leftover (consisting of all of those edges of T not covered by the approximate path decomposition) is still a robust outexpander, as required to complete our decomposition of T in the way described in Section 3.2.1.

Fix $X^+ \subseteq \{v \in V(T) \mid \operatorname{ex}^+_T(v) > 0\}$ and $X^- \subseteq \{v \in V(T) \mid \operatorname{ex}^-_T(v) > 0\}$, both of size ηn and denote $X^0 := V(T) \setminus (X^+ \cup X^-)$. Our goal is then to find an approximate path decomposition $\mathcal{P}$ of T such that $|\mathcal{P}| = \widetilde{\operatorname{ex}}(T) - \eta n$ and such that the leftover $T \setminus E(\mathcal{P})$ satisfies the degree conditions in (†). Thus, it suffices to show that $\mathcal{P}$ satisfies the following.

(i) Each $v \in X^+$ is the starting point of exactly $\operatorname{ex}^+_T(v) - 1$ paths in $\mathcal{P}$, while each

$v \in V(T) \setminus X^+$ is the starting point of exactly $\mathrm{ex}_T^+(v)$ paths in $\mathcal{P}$. Similarly, each $v \in X^-$ is the ending point of exactly $\mathrm{ex}_T^-(v) - 1$ paths in $\mathcal{P}$, while each $v \in V(T) \setminus X^-$ is the ending point of exactly $\mathrm{ex}_T^-(v)$ paths in $\mathcal{P}$.

(ii) Each $v \in V(T)\setminus(X^+ \cup X^-)$ is the internal vertex of exactly $\frac{(n-1)-|\mathrm{ex}_T(v)|}{2} - \eta n$ paths in $\mathcal{P}$, while each $v \in X^+ \cup X^-$ is the internal vertex of exactly $\frac{(n-1)-|\mathrm{ex}_T(v)|}{2} - \eta n + 1$ paths in $\mathcal{P}$.

Indeed, (i) ensures that $|\mathcal{P}| = \mathrm{ex}(T) - \eta n$ and each vertex has the desired excess in $T \setminus E(\mathcal{P})$, namely $\mathrm{ex}_{T\setminus E(\mathcal{P})}(v) = +1$ if $v \in X^+$, $\mathrm{ex}_{T\setminus E(\mathcal{P})}(v) = -1$ if $v \in X^-$, and $\mathrm{ex}_{T\setminus E(\mathcal{P})}(v) = 0$ otherwise. In addition, (ii) ensures that the degrees in $T \setminus E(\mathcal{P})$ satisfy (†).

Recall that, by assumption, T is almost regular. Thus, in a nutshell, (i) and (ii) state that we need to construct edge-disjoint paths with specific endpoints and such that each vertex is covered by about $(\frac{1}{2} - \eta)n$ paths. To ensure the latter, we will in fact approximately decompose T into about $(\frac{1}{2} - \eta)n$ spanning sets of internally vertex-disjoint paths. To ensure the former, we will start by constructing $(\frac{1}{2} - \eta)n$ auxiliary digraphs on $V(T)$ such that, for each $v \in V(T)$, the total number of edges starting (and ending) at v is the number of paths that we want to start (and end, respectively) at v. These auxiliary digraphs will be called *layouts*. Then, it will be enough to construct, for each layout L, a spanning set $\mathcal{P}_L$ of paths, called a *spanning configuration of shape* L, such that each path $P \in \mathcal{P}_L$ corresponds to some edge $e \in E(L)$ and such that the starting and ending points of P equal those of e. Roughly speaking, a spanning configuration $\mathcal{P}_L$ is a set of internally vertex-disjoint paths and L indicates the starting and ending points of these paths.

These spanning configurations will be constructed one by one as follows. (See also Figure 3.1.) At each stage, given a layout L, fix an edge $yz \in E(L)$. Then, using the robust outexpansion properties of (the remainder of) Γ, find short internally vertex-disjoint paths with endpoints corresponding to the endpoints of the edges in $L \setminus \{yz\}$. Denote by $\mathcal{P}_L'$ the set containing these paths. Then, it only remains to construct a path from y to z spanning $V(T) \setminus V(\mathcal{P}_L')$. We achieve this as follows.

Let D' and Γ' be obtained from (the remainders of) $D - V(\mathcal{P}_L')$ and $\Gamma - V(\mathcal{P}_L')$

by merging the vertices y and z into a new vertex v_{yz} whose outneighbourhood is the outneighbourhood of y and whose inneighbourhood is the inneighbourhood of z. Then, observe that a Hamilton cycle of $D' \cup \Gamma'$ corresponds to a path from y to z of T which spans $V(T) \setminus V(\mathcal{P}'_L)$. To construct such a Hamilton cycle of $D' \cup \Gamma'$, one can simply use the fact that Γ' is a robust outexpander to find a Hamilton cycle. However, if we proceed in this way, then the robust outexpansion property of Γ' might be destroyed before constructing all the desired spanning configurations.

So instead we construct a Hamilton cycle of $D' \cup \Gamma'$ with only few edges in Γ' as follows. As a preparatory step in advance of choosing the spanning configurations, we consider a random partition of $V(T)$ into $A_1, \dots, A_a$ each of size $\frac{n}{a}$. We choose one A_i for the current layout. We restrict ourselves to use Γ' inside A_i only. Note that $\Gamma'[A_i]$ is a robust outexpander and $D' - A_i$ is a dense almost regular digraph. The latter means that we can find a spanning linear forest F in $D' - A_i$ which has few components. Since F has few components, we can then greedily extend the components of F to obtain a linear forest $F' \subseteq D'$ which covers all the vertices in $V(D') \setminus A_i$ and whose endpoints are all in A_i. Finally, we use the robust outexpansion properties of $\Gamma'[A_i]$ to close F' in to a Hamilton cycle of $D' \cup \Gamma'$. None of the A_i will be used too often when constructing the spanning configurations, which will mean that $\Gamma'[A_i]$ is always a robust outexpander. When the desired spanning configuration is a Hamilton cycle, this approach of finding many edge-disjoint spanning configurations by first finding a suitable linear forest F, and then tying F together together via some small set A_i (with varying A_i in order to avoid over-using a particular set of vertices) has been used successfully in several earlier papers (e.g. [35, 76]).

We illustrate this argument with the following example. Suppose that L is a layout consisting of three edges uv, wx, and yz (Figure 3.1(A)). We want to construct a spanning configuration of shape L, that is, a set of paths which consists of a path from u to v, a path from w to x, and a path from y to z such that these three paths are vertex-disjoint and altogether cover all the vertices of T. First, we use robust outexpansion to construct

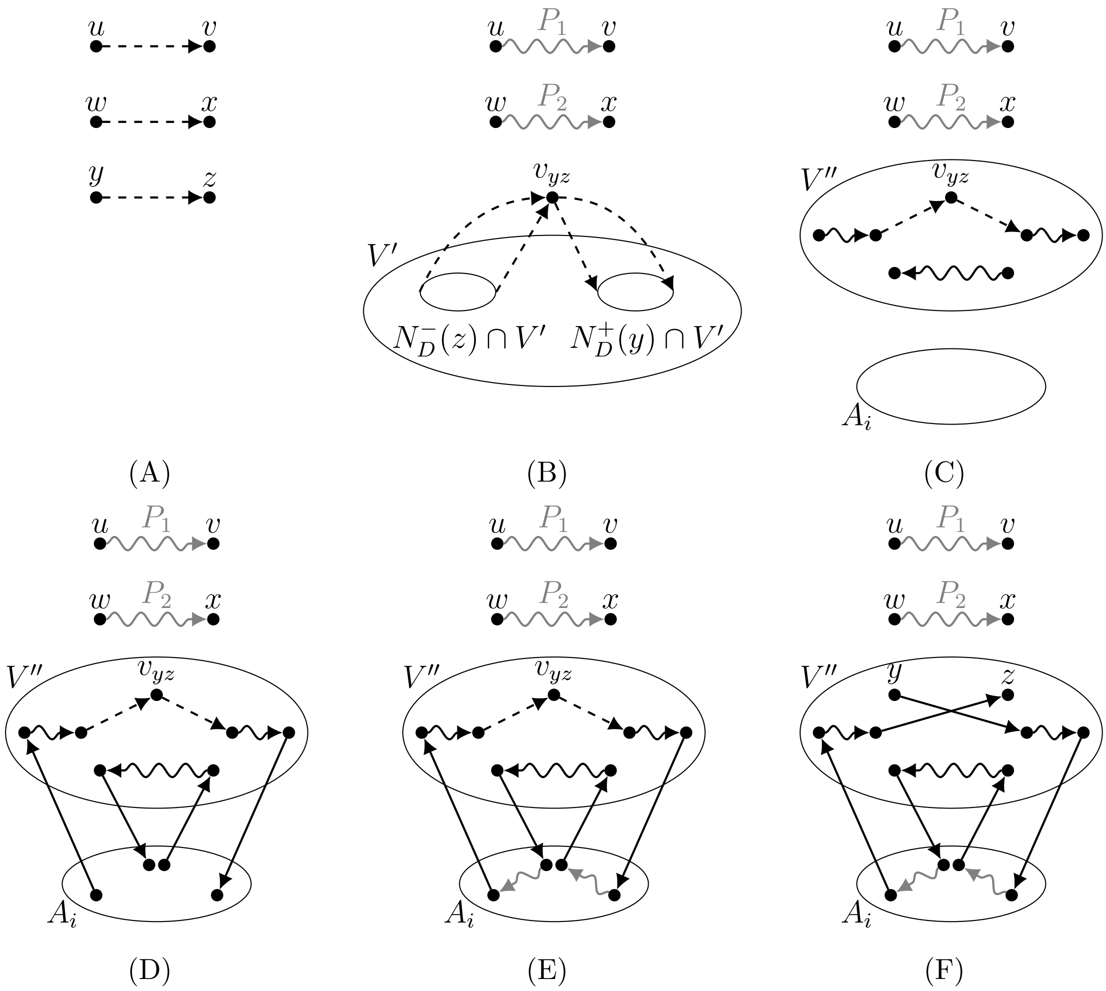


Figure 3.1: Constructing a spanning set of vertex-disjoint paths in $D \cup \Gamma$ with prescribed endpoints and few edges of Γ. Dashed edges represent auxiliary edges, full black edges represent edges of D, and grey edges represent edges of Γ. Wavy black edges represent paths in D and wavy grey edges represent paths in Γ.

a short path P_1 from u to v and a short path P_2 from w to x in Γ (Figure 3.1(B)). Denote $V' \coloneqq V(T) \setminus (V(P_1) \cup V(P_2) \cup \{y, z\})$. The goal is now to construct a path from y to z which covers all the vertices in V'. To do so, we replace y and z by an auxiliary vertex v_{yz} whose outneighbourhood is $N^+(v_{yz}) \coloneqq N_D^+(y) \cap V'$ and whose inneighbourhood is $N^-(v_{yz}) \coloneqq N_D^-(z) \cap V'$ (Figure 3.1(B)) and we consider a small preselected random set of vertices $A_i \subseteq V'$. It is then enough to find a cycle on $V' \cup \{v_{yz}\}$ which uses Γ inside A_i only. Denote $V'' \coloneqq (V' \cup \{v_{yz}\}) \setminus A_i$. Firstly, we use almost regularity of D to find a spanning linear forest on V'' which consists of few components (Figure 3.1(C)). Secondly, we use the large degree of D to extend the endpoints of the linear forest to A_i (Figure 3.1(D)).

Finally, we use the robust outexpansion of Γ to close the linear forest into a cycle which covers all the vertices in A_i (Figure 3.1(E)). This gives a cycle on $V' \cup \{v_{yz}\}$. Replacing the auxiliary vertex v_{yz} by the original vertices y and z, we obtain a path from y to z which covers all the vertices in V', as desired (Figure 3.1(F)).

CHAPTER 4

HAMILTON DECOMPOSITIONS OF REGULAR BIPARTITE TOURNAMENTS

In this chapter, we discuss our results on Hamilton decompositions of regular bipartite tournaments. Our main contribution consists of a proof of Jackson's conjecture (Conjecture 1.8), which states any regular bipartite tournament can be decomposed into Hamilton cycles.

Theorem 1.10. *There exists $n_0 \in \mathbb{N}$ such that any regular bipartite tournament T on $4n \geq n_0$ vertices has a Hamilton decomposition.*

Along the way, we also prove a conjecture of Liebenau and Pehova [84] on Hamilton decompositions of dense regular bipartite digraphs.

Theorem 1.11. *For any $\delta > \frac{1}{2}$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let D be a bipartite digraph on vertex classes of size $n \geq n_0$ and suppose that D is r-regular for some $r \geq \delta n$. Then, D has a Hamilton decomposition.*

Recall from Section 1.1.2 that the analogue of Theorem 1.10 when T is an r-partite tournament for some $r \geq 4$ was already proven in [77]. In Section 6.1, we will construct a family of regular tripartite tournaments which cannot be decomposed into Hamilton cycles.

Proposition 1.7. *For any integer $n \geq 2$, there exists a regular tripartite tournament on vertex classes of size n which does not have a Hamilton decomposition.*

Thus, Theorem 1.10 completes the study of Hamilton decompositions of partite tournaments. As briefly mentioned in the introduction, our proof will be split into two cases: T is a "bipartite robust outexpander" and T is "close to the complete blow-up C_4". We now discuss this more thoroughly.

4.1 Bipartite robust outexpanders

Recall from Section 1.1.1 that a robust (ν, τ)-outexpander is a digraph D on n vertices such that $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$ for every $S \subseteq V(D)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$ (where $RN^+_{\nu,D}(S)$ denotes the set of vertices of D which have at least νn inneighbours in S). One can check that bipartite digraphs are not robust outexpanders (the largest vertex class does not expand). However, we can easily define a bipartite analogue of robust outexpansion as follows. (Note that an undirected version of bipartite robust outexpansion was introduced in [74] by Kühn, Lo, Osthus, and Staden.) A balanced bipartite digraph D on vertex classes A and B of size n is called a *bipartite robust* (ν, τ)*-outexpander with bipartition* (A, B) if the following holds. Let $S \subseteq V(D)$ satisfy $\tau n \leq |S| \leq (1-\tau)n$. If $S \subseteq A$ or $S \subseteq B$, then $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$.

Recall that in Theorem 3.9 we approximately decomposed almost regular robust outexpanders into Hamilton cycles. We will see that these arguments can easily be adapted to the bipartite case. Then, the leftovers can be covered using tools of [76] to obtain the following bipartite version of Theorem 1.5.

Theorem 4.1. *For any $\delta > 0$, there exists $\tau > 0$ such that, for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let D be a balanced bipartite digraph on vertex classes A and B of size $n \geq n_0$. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B) and that D is r-regular for some $r \geq \delta n$. Then, D has a Hamilton decomposition.*

Theorem 4.1 can be used to prove an analogous result for undirected graphs. Given a graph G on n vertices and $S \subseteq V(G)$, the *ν-robust neighbourhood of S*, denoted by

$RN_{\nu,G}(S)$, consists of all the vertices of G which have at least νn neighbours in S. A balanced bipartite graph G on vertex classes A and B of size n is called a *bipartite robust* (ν, τ)*-expander with bipartition* (A, B) if, for any $S \subseteq A$ which satisfies $\tau|A| \leq |S| \leq (1-\tau)|A|$, we have $|RN_{\nu,G}(S)| \geq |S| + \nu n$. (Note that the ordering of A and B matters here.)

Corollary 4.2. *For any $\delta > 0$, there exists $\tau > 0$ such that, for all $\nu > 0$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let G be a bipartite graph on vertex classes A and B of size $n \geq n_0$. Suppose that G is a bipartite robust (ν, τ)-expander with bipartition (A, B), as well as with bipartition (B, A), and that G is r-regular for some even $r \geq \delta n$. Then, G has a Hamilton decomposition.*

Moreover, it turns out that regular digraphs of sufficiently large degree are bipartite robust outexpanders, so Theorem 1.11 is an immediate corollary of Theorem 4.1. One can also derive Corollary 1.15 from Theorem 4.1 and Corollary 4.2. (See Chapter 6 for more details.)

4.2 The complete blow-up C_4 case

The *complete blow-up C_4 with vertex classes of size n* is the n-fold blow-up of the directed C_4. We say that a regular bipartite tournament is *ε-close to the complete blow-up C_4 on vertex classes on size n* if it can be obtained from the complete blow-up C_4 with vertex classes of size n by flipping the direction of at most $4\varepsilon n^2$ edges.

As discussed in the introduction, a regular tournament is a robust outexpander and so Theorem 1.5 directly implies Kelly's conjecture on Hamilton decompositions of regular tournaments (Conjecture 1.4). However, regular bipartite tournaments are not necessarily bipartite robust outexpanders: for example, the vertex classes of the complete blow-up C_4 do not expand. It is thus much more difficult to prove the existence of a Hamilton decomposition in the bipartite case. However, from the definition of a bipartite robust outexpander, one can easily verify that a regular bipartite tournament is either a bipartite robust outexpander or close to the complete blow-up C_4.

Lemma 4.3. *For any $\tau > 0$, there exists $\nu > 0$ such that, for all $0 < \nu' \leq \nu$, there exists $n_0 \in \mathbb{N}$ for which the following holds. Let T be a regular bipartite tournament on vertex classes A and B of size $2n \geq n_0$. Then, one of the following holds.*

(i) *T is a bipartite robust (ν', τ)-outexpander with bipartition (A, B).*

(ii) *T is $\sqrt{\nu'}$-close to the complete blow-up C_4 on vertex classes of size n.*

Thus, Theorem 1.10 follows from Theorem 4.1, Lemma 4.3, and the following.

Theorem 4.4. *There exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ for which the following holds. Let T be a regular bipartite tournament on vertex classes of size $2n \geq n_0$. Suppose that T is ε-close to the complete blow-up C_4 on vertex classes of size n. Then, T has a Hamilton decomposition.*

Proof of Theorem 1.10. Define $\delta := \frac{1}{2}$. Let $\tau > 0$ be the constant obtained by applying Theorem 4.1, let $\nu > 0$ be the constant obtained by applying Lemma 4.3, and let $\varepsilon > 0$ be the constant obtained by applying Theorem 4.4. Define $\nu' := \min\{\nu, \varepsilon^2\}$. Let n_0' the largest of the constants obtained by applying Theorems 4.1 and 4.4 and Lemma 4.3. Define $n_0 := 2n_0'$. Let T be a regular bipartite tournament on $4n \geq n_0$ vertices. Denote by A and B the vertex classes of T. By definition of a regular bipartite tournament, we have $|A| = |B| = 2n \geq n_0'$ and T is n-regular. If T is a bipartite robust (ν, τ)-outexpander with bipartition (A, B), then Theorem 4.1 (applied with T and $2n$ playing the roles of D and n) implies that T has a Hamilton decomposition, as desired. Otherwise, Lemma 4.3 implies that T is ε-close to the complete blow-up C_4 on vertex classes of size n and so Theorem 4.4 implies that T also has a Hamilton decomposition. □

Most of this thesis will be devoted to the proof of Theorem 4.4. The core of the proof will be to decompose and incorporate the few edges with reversed direction. The approximate decomposition will be constructed using the bipartite analogue of Theorem 3.9. To decompose the leftovers, we will use the "robust decomposition lemma" of [76]. Roughly speaking, this tool states that a robust outexpander D contains an absorber $D^{\text{rob}} \subseteq D$

which can decompose any sparse regular leftover of D. In this thesis, we derive an analogue of this lemma for blow-up cycles.

4.3 Proof overview

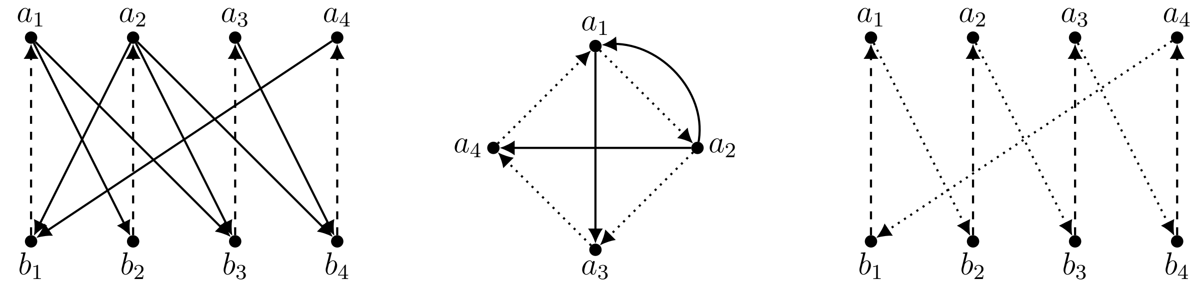
First, we give a proof overview of our main theorems, that is, Theorems 4.1 and 4.4. Given a bipartite digraph D on vertex classes A and B, we denote by $E_D(A, B)$ the set of edges of D which are oriented from A to B and by $D[A, B]$ the bipartite (undirected) graph induced by $E_D(A, B)$.

4.3.1 Constructing a Hamilton cycle in a bipartite digraph

Most of our Hamilton cycles will be constructed using the following procedure. (See also Figure 4.1.) Let D be a balanced bipartite digraph on vertex classes A and B. First, we find a perfect matching M of D whose edges are all oriented from B to A. (For example, if D is regular, then we can simply obtain M by applying Hall's theorem in $D[B, A]$.) Then, we restrict ourselves to constructing a Hamilton cycle which contains M. That is, we need to find a perfect matching M' of D whose edges are all oriented from A to B and such that $M \cup M'$ forms a Hamilton cycle. To do so, we construct an auxiliary digraph D_M on A whose edge set is obtained from $E_D(A, B)$ by identifying the vertices which are matched in M. (This digraph will be called the *M-contraction of* $D[A, B]$, see Definition 7.25(i) for a formal definition.) Then, we find a Hamilton cycle C in D_M. Finally, we observe that C corresponds to a perfect matching M' of D whose edges are all oriented from A to B and such that $M \cup M'$ forms a Hamilton cycle, as desired.

4.3.2 The bipartite robust expander case: proof overview of Theorem 4.1

Let T be a regular bipartite tournament on vertex classes A and B of size n and suppose that T is a bipartite robust outexpander with bipartition (A, B). (The same arguments



(A) A bipartite digraph D on vertex classes $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ which contains a perfect matching M (dashed edges) whose edges are all oriented from B to A.

(B) The digraph D_M on $A = \{a_1, a_2, a_3, a_4\}$ whose edge set is obtained from $E_D(A, B)$ by identifying the vertices which are matched in M. The dotted edges form a Hamilton cycle C of D_M.

(C) The Hamilton cycle C of D_M induces a perfect matching M' (dotted edges) of D whose edges are all oriented from A to B and such that $M \cup M'$ forms a Hamilton cycle of D.

Figure 4.1: Constructing a Hamilton cycle in a bipartite digraph.

hold for a regular bipartite robust outexpander of linear degree.)

Step 1: Constructing an absorber. First, we apply Szemerédi's regularity lemma to exhibit the structure required to apply (the bipartite version of) the robust decomposition lemma of [76]. This then guarantees a sparse regular absorber $D^{\rm abs} \subseteq T$ which satisfies the following property: for any sparse regular leftover $H \subseteq T \setminus D^{\rm abs}$, the digraph $H \cup D^{\rm abs}$ has a Hamilton decomposition.

Step 2: Approximate decomposition. Denote $D \coloneqq T \setminus D^{\rm abs}$ and note that since $D^{\rm abs}$ is regular and sparse, D is still a very dense regular bipartite robust outexpander. We approximately decompose D into Hamilton cycles using the procedure described in Section 4.3.1. More precisely, we can construct a Hamilton cycle of D as follows. Since D is regular, we can obtain a perfect matching M of D whose edges are all oriented from B to A simply by applying Hall's theorem in $D[B, A]$. Denote by D_M the auxiliary digraph as defined in Section 4.3.1. Since D is a regular bipartite robust outexpander, one can verify that D_M is a regular robust outexpander. Then, we use arguments of [40] to construct a Hamilton cycle C of D_M. Let M' be the perfect matching of D induced by C. As explained in Section 4.3.1, $M \cup M'$ is Hamilton cycle of D.

Of course, removing the edges of $M \cup M'$ from D affects the bipartite robust outexpansion and, in general, we would not be be able to repeat this argument sufficiently many times to obtain an approximate decomposition. However, the arguments of [40] allow us to preserve bipartite robust outexpansion in a sufficiently strong way that we can repeat the above arguments to construct many edge-disjoint Hamilton cycles $C_1, \dots, C_{\frac{(1-\varepsilon)n}{2}}$ of D. (For more details, see the second half of Section 3.2.2, which describes the arguments used in [40] to repeatedly construct spanning linear forests in a robust outexpander without affecting the robust outexpansion too much. One can proceed analogously to approximately decompose a robust outexpander into Hamilton cycles.)

Step 3: Decomposing the leftovers. Let $H \coloneqq D \setminus \bigcup_{i \in [\frac{(1-\varepsilon)n}{2}]} E(C_i)$. Note that H is sparse and regular. Thus, the absorbing property described in Step 1 implies that $H \cup D^{\mathrm{abs}}$ can be decomposed into edge-disjoint Hamilton cycles. Together with $C_1, \dots, C_{\frac{(1-\varepsilon)n}{2}}$, this gives us a Hamilton decomposition of T, as desired.

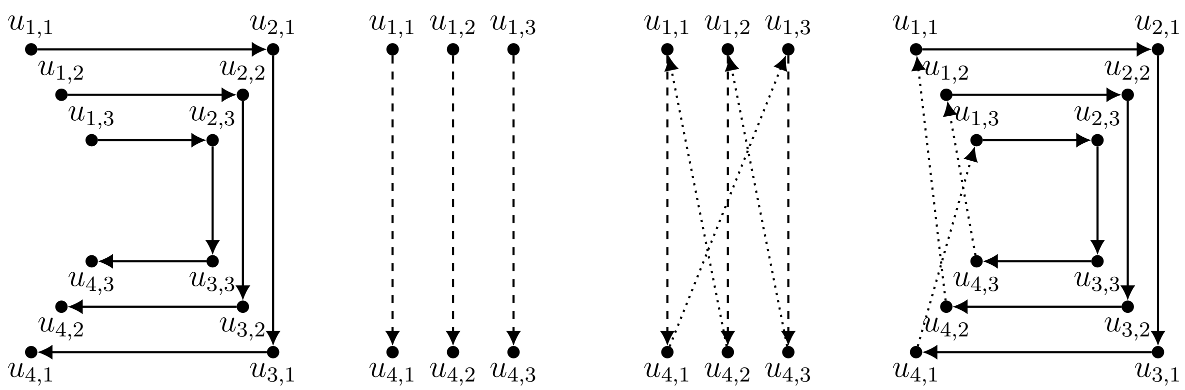
Note that Theorem 4.1 is proved in Chapter 9. The tools for constructing the absorber are introduced in Sections 8.2 and 8.3. The approximate decomposition step is discussed more thoroughly in Section 8.1.

4.3.3 The complete blow-up C_4 case: proof overview of a special case of Theorem 4.4

Let T be the complete blow-up C_4 with vertex classes of size n. That is, there is a partition of $V(T)$ into vertex classes $U_1, \dots, U_4$ of size n such that $E(T)$ consists of all the edges which start in U_i and end in U_{i+1} for some $i \in [4]$ (where $U_5 \coloneqq U_1$).

Note that the vertex classes $U_1, \dots, U_4$ do not expand, so T is not a bipartite robust outexpander and we cannot apply the above arguments. (Recall that robust outexpansion was key to construct the approximate decomposition. It is also needed to apply the robust decomposition lemma.) However, we can (roughly) reduce the decomposition of T to the bipartite robust outexpander case as follows.

First, we discuss how to construct a single Hamilton cycle. (See also Figure 4.2.) For each $i \in [3]$, observe that $T[U_i, U_{i+1}]$ is a complete balanced bipartite graph and so Hall's theorem implies that it contains a perfect matching M_i. Then, $M_1 \cup M_2 \cup M_3$ induces a set $\mathscr{P}$ of n vertex-disjoint paths of T, each starting in U_1 and ending in U_4. Moreover, $\mathscr{P}$ covers all of the vertices of T. We restrict ourselves to constructing a Hamilton cycle of T which contains $E(\mathscr{P})$. Let M be the auxiliary perfect matching from U_1 to U_4 obtained by replacing each path in $\mathscr{P}$ by an edge from its starting point to its ending point. Then, it suffices to find a perfect matching $M' \subseteq E_T(U_4, U_1)$ such that $M \cup M'$ forms a Hamilton cycle on $U_4 \cup U_1$. This can be done using the arguments of Section 4.3.1 (with $A = U_4$, $B = U_1$, and $E(D) = M \cup E_T(U_4, U_1)$).



(A) A spanning set $\mathscr{P}$ of n vertex-disjoint paths which start in U_1 and end in U_4.

(B) The auxiliary perfect matching M from U_1 to U_4 obtained by replacing each path in $\mathscr{P}$ by an edge from its starting to its ending point.

(C) A Hamilton cycle on $U_1 \cup U_4$ which consists of M (dashed edges) and a perfect matching M' (dotted edges) whose edges are all oriented from U_4 to U_1.

(D) The edges of $\mathscr{P}$ and M' induce a Hamilton cycle on $U_1 \cup U_2 \cup U_3 \cup U_4$.

Figure 4.2: Constructing a Hamilton cycle in the complete blow-up C_4 on vertex classes of size $n = 3$, where $U_i = \{u_{i,1}, u_{i,2}, u_{i,3}\}$ for each $i \in [4]$.

In fact, the above argument can be repeated to obtain an approximate decomposition of T into Hamilton cycles. Indeed, for each $i \in [3]$, $T[U_i, U_{i+1}]$ is a complete bipartite graph and so Hall's theorem can be applied repeatedly to (approximately) decompose it into perfect matchings. Moreover, $T[U_4, U_1]$ is also a complete bipartite graph and so one can verify that it is a bipartite robust expander. Thus, it can be approximately decomposed into suitable perfect matchings using the same arguments as in the bipartite robust outexpander case.

To obtain a full Hamilton decomposition of T, one needs to find an absorber D^{abs} which will decompose the edges which are leftover after the approximate decomposition. Unfortunately, we cannot directly apply the tools of [76] in T since it is not a robust outexpander. However, we will derive an analogue of the robust decomposition lemma which can be applied in a blow-up C_4. This argument is discussed in Chapter 11 (a detailed proof overview is given in Section 11.1).

4.3.4 The ε-close to the complete blow-up C_4 case: proof overview of Theorem 4.4

Let T be a regular bipartite tournament and suppose that T is ε-close to the complete blow-up C_4 on vertex classes of size n. That is, there is a partition of $V(T)$ into vertex classes $U_1, \dots, U_4$ of size n such that $E(T)$ satisfies the following properties (where $U_5 \coloneqq U_1$).

- For each $i \in [4]$, $u \in U_i$, and $v \in U_{i+1}$, $E(T)$ contains either the edge uv from u to v or the edge vu from v to u (but not both).
- $E(T)$ does not contain any other edges.
- $\sum_{i\in[4]} |E_T(U_{i+1}, U_i)| \leq 4\varepsilon n^2$.

Note that if $E_T(U_{i+1}, U_i) = \emptyset$ for each $i \in [4]$, then T is in fact the complete blow-up C_4 on vertex classes of size n and so it can be decomposed using the arguments presented in Section 4.3.3. In general, the sets $E_T(U_{i+1}, U_i)$ will be non-empty and the main difficulty will be to incorporate these edges, which we call *backward edges*.

Our overall strategy is the following. First, we decompose all the backward edges into n small digraphs $\mathcal{F}_1, \dots, \mathcal{F}_n$. Then, we restrict ourselves to constructing a Hamilton decomposition of T where each Hamilton cycle contains precisely one of the $\mathcal{F}_i$'s.

For this to be possible, $\mathcal{F}_1, \dots, \mathcal{F}_n$ will need to have a very specific structure. First, each $\mathcal{F}_i$ will have to be a linear forest (any proper subdigraph of a Hamilton cycle is a linear forest). Moreover, each $\mathcal{F}_i$ will need to contain a "balanced" number of backward edges. To see this, suppose that C is cycle of T such that C contains a backward edge, say from U_1 to U_4, and all other edges of C are from U_i to U_{i+1} for some $i \in [4]$. Then, one can verify that C covers one more vertex from each of U_1 and U_4 than from each of U_2 and U_3. Thus, C cannot be a Hamilton cycle of T (recall that $U_1, \dots, U_4$ are equal sized vertex classes which partition $V(T)$). This example shows no $\mathcal{F}_i$ can consist of a single backward edge.

More generally, we will have restrictions on the number of backward edges contained in each $\mathcal{F}_i$. (Formally, each $\mathcal{F}_i$ will have to be a *feasible system*, as defined in Section 13.1.) To illustrate this further, consider the simple example where T is obtained from the complete blow-up C_4 on vertex classes of size n by flipping the orientation of precisely one C_4. Then, T contains precisely four backward edges and since these form a small cycle, they cannot all be included into a common Hamilton cycle. As discussed above, they also cannot be spread across four different Hamilton cycles. Thus, they will be incorporated two by two as follows:

- $\mathcal{F}_1$ will consist of the backward edge from U_2 to U_1 and the backward edge from U_4 to U_3,
- $\mathcal{F}_2$ will consist of the backward edge from U_1 to U_4 and the backward edge from U_3 to U_2, and
- $\mathcal{F}_3, \dots, \mathcal{F}_n$ will be empty.

We will see that this decomposition of backward edges will ensure that the vertex classes $U_1, \dots, U_4$ can be covered in a "balanced" way (as opposed to the previous example where C covered more vertices from U_1 and U_4 than from U_2 and U_3).

Decomposing the backward edges of T into suitable digraphs $\mathcal{F}_1, \dots, \mathcal{F}_n$ will be the core of the proof and the sole focus of Chapters 13 and 15–18. We defer further discussions about how to decompose backward edges to these chapters, which contain further intuition and motivation.

Once we have constructed suitable digraphs $\mathcal{F}_1, \dots, \mathcal{F}_n$, the Hamilton decomposition will be constructed using the arguments presented in Section 4.3.3. However, the backward edges will introduce additional problems. In particular, recall that in Section 4.3.3 we decomposed $T[U_1, U_2]$, $T[U_2, U_3]$, and $T[U_3, U_4]$ into perfect matchings by applying Hall's theorem. But, this is no longer possible since $T[U_1, U_2]$, $T[U_2, U_3]$, and $T[U_3, U_4]$ may no longer be regular bipartite graphs. Moreover, the matchings will now have to incorporate some of the backward edges, as prescribed by $\mathcal{F}_1, \dots, \mathcal{F}_n$. Thus, $T[U_1, U_2]$, $T[U_2, U_3]$, and $T[U_3, U_4]$ will now need to be decomposed building on methods from [40]. (For more detail on how construct an approximate decomposition which incorporates given $\mathcal{F}_i$'s, see Section 14.1.)

As mentioned in Section 4.3.3, the absorber required to decompose the leftovers will be constructed using an analogue of the robust decomposition lemma for blow-up cycles (see Chapter 11 for more detail). The decomposition properties of this absorber will be robust enough to allow us to prescribe the backward edges of the $\mathcal{F}_i$'s left over by the approximate decomposition.

PART II

PROOF OF JACKSON'S CONJECTURE

CHAPTER 5

ORGANISATION AND NOTATION

In this chapter, we give a brief overview of the remainder of this thesis and introduce some key notation and concepts that will be used throughout the next chapters.

5.1 Organisation

The rest of this thesis is organised as follows. In Chapter 6, we prove Proposition 1.7, Theorem 1.11, and Corollaries 1.15 and 4.2. All other chapters are dedicated to the proofs of Theorems 4.1 and 4.4 and Lemma 4.3. More precisely, useful tools and preliminary results are collected in Chapter 7, while our main tools for constructing approximate decompositions and leftovers are collected in Chapter 8. In Chapter 9, we prove Theorem 4.1, while Chapters 10–18 are devoted to proving Lemma 4.3 and Theorem 4.4 (which are derived in Chapters 10 and 14, respectively).

5.2 Notation

For simplicity, we collect the key notation and concepts that will be used throughout the rest of this thesis. The core definitions will be defined when first needed and are indexed in the glossary at the end of this thesis. Given $n \in \mathbb{N}$, we define $[n] \coloneqq \{1, \dots, n\}$. Given $a, b, c \in \mathbb{R}$, we write $a = b \pm c$ to mean that $b - c \leq a \leq b + c$.

5.2.1 Graphs and digraphs

In this thesis, all (directed) graphs are without loops and, unless otherwise specified, without multiple edges. A *digraph* is a directed graph which contains at most two edges between any pair of distinct vertices and at most one in each direction. An *oriented graph* is a digraph which contains at most one edge between any pair of distinct vertices. Given a (di)graph G, a *sub(di)graph* of G is a (di)graph whose vertex and edge sets are subsets of those of G. Let G be an undirected graph. An *orientation* of G is an oriented graph which can be obtained by orienting the edges of G. Given an orientation D of G, we say that G is the undirected graph *underlying* D. A directed edge from a vertex u to a vertex v is denoted by uv. If e is the directed edge uv, we say that u and v are the starting and ending points of e, respectively.

A *multigraph* is an undirected graph without loops which may contain multiple edges between the same pair of distinct vertices. Similarly, a *multidigraph* is a directed graph without loops which may contain multiple edges of the same direction between the same pair of distinct vertices. Given a multi(di)graph G, a *submulti(di)graph* of G is a multi(di)graph whose vertex set is a subset of the vertex set of G and whose edge multiset is a submultiset of the edge multiset of G. The edges of a multi(di)graph are always considered to be distinct. More precisely, given a multi(di)graph G and distinct vertices u and v, denote by $\mu_G(uv)$ the *multiplicity* of the edge uv in G (that is, $\mu_G(uv)$ is the number of edges between u and v if G is undirected and the number of edges from u to v if G is directed). Then, given a multi(di)graph G and submulti(di)graphs G_1 and G_2 of G, we say that G_1 and G_2 are *edge-disjoint* if $\mu_{G_1}(uv) + \mu_{G_2}(uv) \leq \mu_G(uv)$ for any distinct vertices u and v of G.

5.2.2 Edge sets

Let G be a (di)graph. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. The *order* of G is $|V(G)|$ and we define the *size* of G as $e(G) := |E(G)|$. We say that G is *empty* if $E(G) = \emptyset$.

Let G be a (di)graph and let $A, B \subseteq V(G)$ be disjoint. If G is undirected, we denote by $E_G(A, B)$ the set of undirected edges of G which have an endpoint in A and an endpoint in B. If G is directed, we denote by $E_G(A, B)$ the set of directed edges of G which start in A and end in B. Define $e_G(A, B) := |E_G(A, B)|$. Given any disjoint vertex sets A' and B' which are not necessarily contained in $V(G)$, we sometimes abuse the above notation and write $E_G(A', B') := E_G(A' \cap V(G), B' \cap V(G))$ and $e_G(A', B') := |E_G(A', B')|$.

Let G be a (di)graph and let A and B be any disjoint vertex sets. We denote by $G[A, B]$ the undirected bipartite graph on vertex classes A and B induced by $E_G(A, B)$ and, if G is directed, we denote by $G(A, B)$ the directed bipartite graph on vertex classes A and B induced by $E_G(A, B)$. (Thus, if G is directed, then $G[A, B]$ is the undirected graph underlying $G(A, B)$.)

All the above definitions from this subsection extend naturally to multi(di)graphs. That is, if G is a multi(di)graph and A and B are disjoint vertex sets, then $E(G)$ and $E_G(A, B)$ are now multisets of edges, while $G[A, B]$, as well as $G(A, B)$ if G is directed, are now multi(di)graphs. The vertex set $V(G)$ of a multi(di)graph is still a set rather than a multiset.

We sometime abuse notation and consider a set of (directed) edges as a (di)graph. In particular, given a set of edges E, we write $V(E)$ for the set of vertices which are incident to an edge in E.

5.2.3 Subgraphs

Let G and H be (di)graphs and let F be a sub(di)graph of G. We write $F \subseteq G$, and, if $V(F) = V(G)$, we say that F is *spanning*. Given $S \subseteq V(G)$, we write $G[S]$ for the sub(di)graph of G *induced* by S and define $G - S := G[V(G) \setminus S]$. We denote by $G \setminus H$ the (di)graph obtained from G by deleting all the edges in $E(G) \cap E(H)$, we denote by $G \cup H$ the (di)graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and we denote by $G \cap H$ the (di)graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

Given a set of (directed) edges E, we sometimes abuse the above notation and write $G \setminus E$, $G \cup E$, and $G \cap E$ for the (di)graphs obtained as above when E is viewed as a (di)graph.

All the above definitions from this subsection extend naturally to multi(di)graphs, with unions and differences now considered as multiset unions and differences. More precisely, let G and H be multi(di)graphs. We denote by $G \setminus H$ the multi(di)graph with vertex set $V(G)$ where $\mu_{G\setminus H}(uv) := \max\{\mu_G(uv) - \mu_H(uv), 0\}$ for any distinct $u, v \in V(G)$, we denote by $G \cup H$ the multi(di)graph with vertex set $V(G) \cup V(H)$ where $\mu_{G\cup H}(uv) := \mu_G(uv) + \mu_H(uv)$ for any distinct $u, v \in V(G) \cup V(H)$, and we denote by $G \cap H$ the multi(di)graph with vertex set $V(G) \cap V(H)$ where $\mu_{G\cap H}(uv) := \min\{\mu_G(uv), \mu_H(uv)\}$ for any distinct $u, v \in V(G) \cap V(H)$.

5.2.4 Neighbourhoods and degrees

We use standard notation for neighbourhoods and degrees. More precisely, let G be an undirected graph. Given $v \in V(G)$, we denote by $N_G(v)$ the *neighbourhood* of v in G and by $d_G(v) := |N_G(v)|$ the *degree* of v in G. The *maximum degree* of G is $\Delta(G) := \max\{d_G(v) \mid v \in V(G)\}$ and the *minimum degree* of G is $\delta(G) := \min\{d_G(v) \mid v \in V(G)\}$.

Similarly, let D be a digraph. Given $v \in V(D)$, we denote by $N_D^+(v)$ and $N_D^-(v)$ the *outneighbourhood* and *inneighbourhood* of v in D, respectively, and by $d_D^+(v) := |N_D^+(v)|$ and $d_D^-(v) := |N_D^-(v)|$ the *outdegree* and *indegree* of v in D, respectively. The *neighbourhood* of a vertex $v \in V(D)$ is the set $N_D(v) := N_D^+(v) \cup N_D^-(v)$ and the *degree* of a vertex $v \in V(D)$ is $d_D(v) := d_D^+(v) + d_D^-(v)$. The *maximum* and *minimum outdegree* of D are $\Delta^+(D) := \max\{d_D^+(v) \mid v \in V(G)\}$ and $\delta^+(D) := \min\{d_D^+(v) \mid v \in V(G)\}$, respectively. The *maximum/minimum indegree* and *maximum/minimum degree* of D are defined analogously and denoted by $\Delta^-(D)$, $\delta^-(D)$, $\Delta(D)$, and $\delta(D)$, respectively. We denote by $\Delta^0(D) := \max\{\Delta^+(D), \Delta^-(D)\}$ the *maximum semidegree* of D and by $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$ the *minimum semidegree* of D.

Let G be a (di)graph and $S \subseteq V(G)$. The *neighbourhood* of S in G is the set $N_G(S) := \bigcup_{v\in S} N_G(v)$. If G is directed, the *outneighbourhood* $N_G^+(S)$ and *inneighbourhood*

$N^-_G(S)$ of S in G are defined analogously.

5.2.5 Regularity

An undirected graph G is *r-regular* if $d_G(v) = r$ for all $v \in V(G)$ and a digraph D is *r-regular* if $d^+_D(v) = r = d^-_D(v)$ for all $v \in V(D)$. A (di)graph is *regular* if it is r-regular for some $r \in \mathbb{N}$. An undirected graph G on n vertices is *(δ, ε)-almost regular* if $d_G(v) = (\delta \pm \varepsilon)n$ for all $v \in V(G)$ and a digraph D is *(δ, ε)-almost regular* if both $d^+_D(v), d^-_D(v) = (\delta \pm \varepsilon)n$ for all $v \in V(D)$.

5.2.6 Matchings

A *matching* is a set of pairwise non-adjacent edges. Given a vertex set V, a matching M is called *perfect* if $V(M) = V$.

Let M be a directed matching. We say that M is a *matching from A to B* if $V(M) \subseteq A \cup B$ and all the edges of M are directed from A to B. We say that M is a *perfect matching from A to B* if M is a matching from A to B satisfying $V(M) = A \cup B$.

5.2.7 Blow-ups

Let D be a digraph and $r \in \mathbb{N}$. The *r-fold blow-up of D* is the digraph D' defined as follows. The vertex set $V(D')$ consists of r copies of v for each $v \in V(D)$. Let $u, v \in V(D)$ and $u', v' \in V(D')$. Suppose that u' is a copy of u and v' is a copy of v. Then, $u'v' \in E(D')$ if and only if $uv \in E(D)$. For each $v \in V(D)$, the set of r copies of v in $V(D')$ is called a *vertex class* of D'.

5.2.8 Paths and cycles

Throughout this thesis, all paths and cycles are directed, with consistently oriented edges. The number of edges contained in a path/cycle P is called the *length* of P and denoted by $e(P)$. A path P is *trivial* if $e(P) = 0$. Given a vertex set V, a *Hamilton cycle* is a cycle C

satisfying $V(C) = V$.

Let $P = v_1 \dots v_\ell$ be a path. The *starting point* of P is v_1, the *ending point* of P is v_ℓ, the *endpoints* of P are v_1 and v_ℓ, and the *internal vertices* of P are $v_2, \dots, v_{\ell-1}$. We denote $V^+(P) \coloneqq \{v_1\}$, $V^-(P) \coloneqq \{v_\ell\}$, and $V^0(P) \coloneqq \{v_2, \dots, v_\ell\}$. A (u, v)-*path* is a path which starts at u and ends at v. Given $1 \leq i \leq j \leq \ell$, denote by $v_i P v_j \coloneqq v_i v_{i+1} \dots v_j$ the (v_i, v_j)-path induced by P.

A set of vertex-disjoint paths is sometimes called a *linear forest*. Given a set $\mathscr{P}$ of (not necessarily disjoint) paths, we denote by $V^+(\mathscr{P})$ the set $\bigcup_{P\in\mathscr{P}} V^+(P)$ of vertices which are the starting point of a path in $\mathscr{P}$. Define $V^-(\mathscr{P})$ and $V^0(\mathscr{P})$ analogously. Note that $V^+(\mathscr{P}), V^-(\mathscr{P})$, and $V^0(\mathscr{P})$ are always sets rather than multisets.

Let $\mathscr{P}$ be a set of (not necessarily disjoint) paths. We sometimes abuse notation and view $\mathscr{P}$ as a multidigraph. In particular, we denote by $V(\mathscr{P})$ the set $\bigcup_{P\in\mathscr{P}} V(P)$ and by $E(\mathscr{P})$ the multiset $\bigcup_{P\in\mathscr{P}} E(P)$. For any vertex $v \in V(\mathscr{P})$, we denote $d_{\mathscr{P}}(v) = \sum_{P\in\mathscr{P}} d_P(v)$, and define the out- and indegrees $d^+_{\mathscr{P}}(v)$ and $d^-_{\mathscr{P}}(v)$ of v in $\mathscr{P}$ analogously. Given a digraph D, we write $D \setminus \mathscr{P} \coloneqq D \setminus E(\mathscr{P})$.

5.2.9 Decompositions

Given a (di)graph G, a *decomposition* of G is a set of edge-disjoint sub(di)graphs of G which altogether cover all the edges of G. A *Hamilton decomposition* is a decomposition into Hamilton cycles.

Recall that the edges of a multi(di)graph are considered to be distinct. Thus, a *decomposition* of a multi(di)graph G is a set $\{H_1, \dots, H_\ell\}$ of submulti(di)graphs of G such that $\mu_G(uv) = \sum_{i\in[\ell]} \mu_{H_i}(uv)$ for any distinct $u, v \in V(G)$.

5.2.10 Hierarchies

In a statement, the hierarchy $0 < a \ll b \ll c \leq 1$ means that there exist non-decreasing functions $f\colon (0, 1] \longrightarrow (0, 1]$ and $g\colon (0, 1] \longrightarrow (0, 1]$ for which the statement holds for all $0 < a, b, c \leq 1$ satisfying $b \leq f(c)$ and $a \leq g(b)$. Hierarchies with more constants are defined analogously and should always be read from right to left. Whenever a constant appears in the form $\frac{1}{a}$ in a hierarchy, we implicit assume that $a \in \mathbb{N}$.

5.2.11 $\pm$-notation

To avoid repetitions, we sometime write statements of the form $\mathcal{C}^{\pm}$ to mean that the statements $\mathcal{C}^{+}$ and $\mathcal{C}^{-}$ both hold. In particular, if $\mathcal{C}^{\pm}$ is a statement of the form "$\mathcal{A}^{\pm}$ implies $\mathcal{B}^{\pm}$", then we mean that "$\mathcal{A}^{+}$ implies $\mathcal{B}^{+}$" and "$\mathcal{A}^{-}$ implies $\mathcal{B}^{-}$". Similarly, a statement of the form "$\mathcal{A}^{\pm}$ implies $\mathcal{B}^{\mp}$" means that "$\mathcal{A}^{+}$ implies $\mathcal{B}^{-}$" and "$\mathcal{A}^{-}$ implies $\mathcal{B}^{+}$".

CHAPTER 6

TRIPARTITE TOURNAMENTS AND SOME APPLICATIONS OF THEOREM 4.1

In this section, we construct a family of regular tripartite tournaments which cannot be decomposed into Hamilton cycles and derive several consequences of Theorem 4.1. In particular, we prove Proposition 1.7, Theorem 1.11, and Corollaries 1.15 and 4.2.

6.1 Tripartite tournaments: proof of Proposition 1.7

Let $n \geq 2$. We show that if T is obtained from the n-fold blow-up of the directed C_3 by flipping the orientation of precisely one triangle, then T does not have a Hamilton decomposition.

Proof of Proposition 1.7. Let U_1, U_2, and U_3 be disjoint vertex sets of size $n \geq 2$. For each $i \in [3]$, let $u_i \in U_i$. Denote $E \coloneqq \{u_1u_3, u_3u_2, u_2u_1\}$ and let T be the digraph on $U_1 \cup U_2 \cup U_3$ defined by

$$E(T) \coloneqq E \cup (\{uv \mid i \in [3], u \in U_i, v \in U_{i+1}\} \setminus \{u_1u_2, u_2u_3, u_3u_1\})$$

(where $U_4 \coloneqq U_1$). Note that $E(T[U_i]) = \emptyset$ for each $i \in [3]$.

Suppose for a contradiction that $\mathscr{C}$ is a Hamilton decomposition of T. Since $u_1u_3u_2$ is a non-spanning cycle of T, the edges u_1u_3, u_3u_2, and u_2u_1 do not all lie on a common Hamilton cycle in $\mathscr{C}$. By the pigeon-hole principle, there exists $i \in [3]$ and $C \in \mathscr{C}$ such that $E(C) \cap E = \{u_i u_{i-1}\}$ (where $u_0 \coloneqq u_3$). Denote $C = u_i u_{i-1} v_1 \dots v_{3n-2}$. Then, for each $j \in [3n-2]$, we have $v_j \in U_{i-1+j}$ (where the index $i-1+j$ is taken modulo 3). In particular, $v_{3n-2} \in U_i$. But $u_i \in U_i$ and so $v_{3n-2}u_i \notin E(T)$, a contradiction. $\square$

Note that the above arguments can easily be extended to show that none of the edges in E lie on a Hamilton cycle.

Moreover, the same arguments can be used to show that, if T is obtained from the complete blow-up C_4 on vertex classes of size n by flipping the orientation of a set E' of edges, then no Hamilton cycle of T contains a single edge from E'. This illustrates the fact that the edges of T with reversed direction in Theorem 4.4 will have to be decomposed in a "balanced" way.

6.2 Bipartite robust expanders: proof of Corollary 4.2

The arguments of [77, Lemma 3.6] can be easily adapted to the bipartite case to show that the edges of a regular bipartite robust expander can be oriented to form a regular bipartite robust outexpander.

Lemma 6.1. *Let $0 < \frac{1}{n} \ll \nu' \ll \nu \leq \tau \ll \delta \leq 1$ and let $r \geq \delta n$ be even. Let G be an r-regular bipartite graph on vertex classes A and B of size n. Suppose that G is a bipartite robust (ν, τ)-expander with bipartition (A, B), as well as with bipartition (B, A). Then, there exists an $\frac{r}{2}$-regular orientation D of G such that D is a bipartite robust (ν', τ)-outexpander with bipartition (A, B).*

This can be proved by considering a random orientation of the edges of G and then adjusting the orientations of a small proportion of edges to ensure that D is $\frac{r}{2}$-regular.

Proof of Corollary 4.2. Let $\delta > 0$ and let τ' be the constant obtained by applying Theorem 4.1. We may assume without loss of generality that $\delta \ll 1$. Fix additional constants such that $0 < \frac{1}{n_0} \ll \tau \ll \tau', \delta$ and $\frac{1}{n_0} \ll \nu$. Let $n \geq n_0$ and $r \geq \delta n$. Suppose that r is even. Let G be an r-regular balanced bipartite graph on vertex classes A and B of size n.

Suppose that G is a bipartite robust (ν, τ)-expander with bipartition (A, B), as well as with bipartition (B, A). By definition of a bipartite robust expander, we have $\nu \leq \tau$. Fix an additional constant such that $\frac{1}{n} \ll \nu' \ll \nu$.

By Lemma 6.1, there exists an $\frac{r}{2}$-regular orientation D of G such that D is a bipartite robust (ν', τ)-outexpander with bipartition (A, B). By definition, D is also a bipartite robust (ν', τ')-outexpander with bipartition (A, B). Apply Theorem 4.1 (with ν' and τ' playing the roles of ν and τ) to obtain a Hamilton decomposition $\mathscr{C}$ of D. Let $\mathscr{C}'$ be obtained from $\mathscr{C}$ by replacing each directed edge $uv \in E(\mathscr{C})$ by an undirected edge between u and v. By construction, $\mathscr{C}'$ is a Hamilton decomposition of G. □

6.3 Dense bipartite digraphs: proof of Theorem 1.11

We show that any bipartite digraph of sufficiently large minimum semidegree is a bipartite robust outexpander.

Lemma 6.2. *Let $0 < \nu \leq \tau \ll \varepsilon < 1$. Let D be a bipartite digraph on vertex classes A and B of size n. Suppose that $\delta^0(D) \geq (\frac{1}{2} + \varepsilon)n$. Then, D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B).*

Proof. Let $S \subseteq A$ satisfy $\tau n \leq |S| \leq (1 - \tau)n$ and denote $T := RN^+_{\nu,D}(S)$. We show that $|T| \geq |S| + \nu n$. If $|S| \geq \frac{n}{2}$, then each $v \in B$ satisfies $|N^-_D(v) \cap S| \geq \varepsilon n$ and so $T = B$. We may therefore assume that $|S| \leq \frac{n}{2}$. Then,

$$\left(\frac{1}{2} + \varepsilon\right) n|S| \leq e_D(S, B) \leq \nu n^2 + |S||T| \leq \frac{\nu n}{\tau}|S| + |S||T|$$

and so $|T| \geq (\frac{1}{2} + \varepsilon - \frac{\nu}{\tau})n \geq |S| + \nu n$.

Similarly, if $S \subseteq B$ satisfies $\tau n \leq |S| \leq (1 - \tau)n$, then $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$. Thus, D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B). □

Proof of Theorem 1.11. Let $\delta > \frac{1}{2}$ and let $\tau > 0$ be the constant obtained by applying Theorem 4.1. Let $0 < \nu \ll \tau, \delta$ and let $n_0 \in \mathbb{N}$ be the constant obtained by applying

Theorem 4.1. Let D be a bipartite digraph on vertex classes A and B of size $n \geq n_0$. Suppose that D is r-regular for some $r \geq \delta n$. Denote $\varepsilon := \frac{r}{n} - \frac{1}{2}$. By assumption, $0 < \delta - \frac{1}{2} \leq \varepsilon \leq \frac{n-1}{n} - \frac{1}{2} \leq 1$. Fix an additional constant τ' such that $0 < \nu \leq \tau' \ll \varepsilon, \tau$. By Lemma 6.2, D is a bipartite robust (ν, τ')-outexpander with bipartition (A, B). By definition of a bipartite robust outexpander, D is also a bipartite robust (ν, τ)-outexpander with bipartition (A, B) and so Theorem 4.1 implies that D has a Hamilton decomposition. □

6.4 Optimal packings of Hamilton cycles: proof of Corollary 1.15

The proof of Corollary 1.15 is standard, so we only give a brief proof overview. (Full details can be found in Appendix A.) Let G, D, and T be defined as in Corollary 1.15. Observe that by Theorem 4.1 and Corollary 4.2, it is enough to show that each of G, D, $G_{n,n,p}$, $D_{n,n,p}$, and T contain, with high probability, a spanning regular sub(di)graph of degree $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$, $\mathrm{reg}(D)$, $\frac{\mathrm{reg}_{\mathrm{even}}(G_{n,n,p})}{2}$, $\mathrm{reg}(D_{n,n,p})$, and $\mathrm{reg}(T)$, respectively, which is a bipartite robust (out)expander.

Arguments of [36] imply that $\mathrm{reg}_{\mathrm{even}}(G) \geq (p-2\varepsilon)n$. Thus, one can use basic properties of ε-regular bipartite graphs to show that any $\mathrm{reg}_{\mathrm{even}}(G)$-regular spanning subgraph of G is still ε-regular. It is also easy to see that any ε-regular bipartite graph is also a bipartite robust expander. Thus, Corollary 1.15(i) holds. Similar arguments hold for the directed case and so Corollary 1.15(ii) is satisfied.

A simple Chernoff bound can be used to show that $G_{n,n,p}$ is an ε-regular bipartite graph of minimum degree at least $(p-\varepsilon)n$ with high probability. Thus, Corollary 1.15(iii) follows from Corollary 1.15(i). Similarly, Corollary 1.15(iv) and (v) follow from Corollary 1.15(ii).

CHAPTER 7

PRELIMINARIES

We now introduce some preliminary tools and results which will be used throughout this thesis.

7.1 (Bipartite) robust (out)expanders

In Sections 1.1.1 and 4.1, we introduced the concept of (bipartite) robust (out)expansion. We start that by recalling and expanding on these definitions.

7.1.1 Definitions

Let D be a digraph on n vertices. Recall that for any $S \subseteq V(D)$, we denote by $RN^+_{\nu,D}(S)$ the set of vertices $v \in V(D)$ which satisfy $|N^-_D(v) \cap S| \geq \nu n$. Then, we say that D is a *robust* (ν, τ)*-outexpander* if, for any $S \subseteq V(D)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$, we have $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$.

Let G be a graph on n vertices. Recall that for any $S \subseteq V(G)$, we denote by $RN_{\nu,G}(S)$ the set of vertices $v \in V(G)$ which satisfy $|N_D(v) \cap S| \geq \nu n$. Then, we say that G is a *robust* (ν, τ)*-expander* if, for any $S \subseteq V(G)$ satisfying $\tau n \leq |S| \leq (1-\tau)n$, we have $|RN_{\nu,G}(S)| \geq |S| + \nu n$. Let G be a bipartite graph on vertex classes A and B of size n. We say that G is a *bipartite robust* (ν, τ)*-expander with bipartition* (A, B) if, for any $S \subseteq A$ satisfying $\tau n \leq |S| \leq (1-\tau)n$, we have $|RN_{\nu,G}(S)| \geq |S| + \nu n$. Note that the order of A

and B matters.

In Section 4.1, we defined an analogue of bipartite robust expanders for digraphs. Let D be a bipartite digraph on vertex classes A and B of size n. We say that D is a *bipartite robust (ν, τ)-outexpander with bipartition (A, B)* if

- for any $S \subseteq A$ such that $\tau n \leq |S| \leq (1-\tau)n$, we have $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$; and
- for any $S \subseteq B$ such that $\tau n \leq |S| \leq (1-\tau)n$, we have $|RN^+_{\nu,D}(S)| \geq |S| + \nu n$.

Note that, here, the order of A and B does not matter.

7.1.2 Basic properties of (bipartite) robust (out)expanders

The following facts hold by definition.

Fact 7.1. *A digraph D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B) if and only if $D[A, B]$ is a bipartite robust (ν, τ)-expander with bipartition (A, B) and $D[B, A]$ is a robust (ν, τ)-expander with bipartition (B, A).*

Fact 7.2. *Suppose that G is a bipartite robust (ν, τ)-expander with bipartition (A, B). Then, for any $\nu' \leq \nu$ and $\tau' \geq \tau$, G is a bipartite robust (ν', τ')-expander with bipartition (A, B).*

By definition, bipartite robust outexpansion is preserved when only a few edges are removed at each vertex.

Lemma 7.3. *Let $0 < \frac{1}{n} \ll \varepsilon \leq \nu \leq 1$. Let D be a bipartite digraph on vertex classes A and B of size n. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B). If D' is obtained from D by removing at most εn inedges and εn outedges at each vertex, then D' is a bipartite robust $(\nu - \varepsilon, \tau)$-expander with bipartition (A, B).*

In [62, 79], Keevash, Kühn, Osthus, and Treglown showed that a robust outexpander of linear minimum degree is Hamiltonian.

Theorem 7.4 ([79, Theorem 16])**.** *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$. Let D be a robust (ν, τ)-outexpander on n vertices with $\delta^0(D) \geq \delta n$. Then, D is Hamiltonian.*

The analogue of Theorem 7.4 holds for bipartite robust outexpanders. This can be derived from Theorem 7.4 using the procedure presented in Section 4.3.1. The formal proof is deferred to the end of Section 7.5, where we introduce the required definitions.

Corollary 7.5. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \delta \leq 1$. Let D be a balanced bipartite digraph on vertex classes A and B of size n. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B) and that $\delta^0(D) \geq \delta n$. Then, D is Hamiltonian.*

Almost complete bipartite graphs are bipartite robust expanders.

Proposition 7.6. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll 1$. Let G be a bipartite graph on vertex classes A and B of size n. If $\delta(G) \geq (1-\varepsilon)n$, then G is a bipartite robust (ν, τ)-expander with bipartition (A, B).*

Proof. Let $S \subseteq A$ satisfy $\tau n \leq |S| \leq (1-\tau)n$. Each $v \in B$ satisfies $|N_G(v) \cap S| \geq (1-\varepsilon)n - |A \setminus S| \geq (\tau - \varepsilon)n \geq \nu n$. Thus, $|RN_{\nu,G}(S)| = |B| \geq |S| + \tau n \geq |S| + \nu n$. Similarly, if $S' \subseteq B$ satisfies $\tau n \leq |S'| \leq (1-\tau)n$, then $|RN_{\nu,G}(S')| \geq |S'| + \nu n$. □

Recall the definition of an r-fold blow-up from Section 5.2.7. The next lemma states that bipartite robust outexpansion is preserved when taking r-fold blow-ups. The proof is very similar to that of its non-bipartite analogue (see [76, Lemma 5.3]), so we omit the details.

Lemma 7.7. *Let $0 < 3\nu \leq \tau < 1$ and $r \geq 3$. Let D be a balanced bipartite robust (ν, τ)-outexpander with bipartition (A, B). Let D' be the r-fold blow-up of D. Let A' be the set of vertices in $V(D')$ which are a copy of a vertex in A. Let $B' \coloneqq V(D') \setminus A'$. Then, D' is a bipartite robust $(\nu^3, 2\tau)$-outexpander with bipartition (A', B').*

7.2 (Super)regularity

In Section 1.1.3, we introduced the concept of ε-regular (di)graphs. We start by recalling and expanding on these definitions.

7.2.1 Definitions

Suppose that G is an undirected bipartite graph on vertex classes A and B. The *density* of G is defined as

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

Let $\varepsilon > 0$. We say that G is *ε-regular* if, for any $A' \subseteq A$ and $B' \subseteq B$ satisfying $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$, we have $|d_G(A, B) - d_G(A', B')| < \varepsilon$. Let $0 \leq d \leq 1$. We say that G is *(ε, d)-regular* if G is ε-regular and has density $d_G(A, B) = d$. We say that G is *$(\varepsilon, \geq d)$-regular* if there exists $d' \geq d$ such that G is (ε, d')-regular. We say that G is *$[\varepsilon, d]$-superregular* if G is ε-regular, each $a \in A$ satisfies $d_G(a) = (d \pm \varepsilon)|B|$, and each $b \in B$ satisfies $d_G(b) = (d \pm \varepsilon)|A|$. We say that G is *$[\varepsilon, \geq d]$-superregular* if there exists $d' \geq d$ such that G is $[\varepsilon, d']$-superregular.

7.2.2 Basic properties of (super)regular pairs

The next proposition states that (super)regularity is preserved when few vertices and edges are removed and/or added to a bipartite graph. This follows easily from the definitions and a similar observation was already made (and proved) in [76, Proposition 4.3], so we omit its proof here.

Proposition 7.8. *Let $0 < \frac{1}{m} \ll \varepsilon \leq \varepsilon' \leq d \leq 1$. Let G be a bipartite graph on vertex classes A and B of size at least m. Let A' and B' be disjoint vertex sets satisfying $|A \triangle A'| \leq \varepsilon|A'|$ and $|B \triangle B'| \leq \varepsilon|B'|$. Let G' be a bipartite graph on vertex classes A' and B' and suppose that $G'[A' \cap A, B' \cap B]$ is obtained from $G[A' \cap A, B' \cap B]$ by removing and adding at most $\varepsilon'|B'|$ edges incident to each vertex in $A' \cap A$ and at most $\varepsilon'|A'|$ edges incident to each vertex in $B' \cap B$.*

(i) *If G is $(\varepsilon, \geq d)$-regular, then G' is $(3\sqrt{\varepsilon'}, \geq d - 3\sqrt{\varepsilon'})$-regular.*

(ii) *Suppose that $A' \subseteq A$ and $B' \subseteq B$. If G is $[\varepsilon, d]$-superregular, then G' is $[3\sqrt{\varepsilon'}, d]$-superregular.*

Lemma 7.9 ([76, Proposition 4.14])**.** *Let $0 < \frac{1}{m} \ll \varepsilon \ll \delta \leq 1$. Let G be a balanced bipartite graph on vertex classes of size m. Suppose that G is ε-regular and $\delta(G) \geq \delta m$. Then, G contains a perfect matching.*

One can easily verify from the definition of superregularity that bipartite graphs of very high minimum degree are superregular.

Proposition 7.10. *Let $0 < \frac{1}{m} \ll \varepsilon \ll \varepsilon' \ll 1$. Let G be a bipartite graph on vertex classes A and B of size at least m. Suppose that each $a \in A$ satisfies $d_G(a) \geq (1-\varepsilon)|B|$ and each $b \in B$ satisfies $d_G(b) \geq (1-\varepsilon)|A|$. Then, G is $[\varepsilon', \geq 1-\varepsilon']$-superregular.*

Lemma 7.11 ([76, Corollary 4.15])**.** *Let $0 < \frac{1}{m} \ll \varepsilon \ll d \leq 1$ and $k \geq 4$. Let D be a digraph and $V_1 \cup \dots \cup V_k$ be a partition of $V(D)$ into k clusters of size m. Suppose that $D[V_i, V_{i+1}]$ is $[\varepsilon, \geq d]$-superregular for each $i \in [k-1]$. Let $u_1, \dots, u_m$ and $v_1, \dots, v_m$ be enumerations of V_1 and V_k, respectively. Then, D contains a spanning set $\mathscr{P}$ of m vertex-disjoint paths, one (u_i, v_i)-path for each $i \in [m]$.*

If the pair $D[V_k, V_1]$ is also superregular, one can find a matching in $D(V_k, V_1)$ to tie the paths obtained with Lemma 7.11 into a Hamilton path.

Corollary 7.12. *Let $0 < \frac{1}{m} \ll \varepsilon \ll d \leq 1$ and $k \geq 4$. Let D be a digraph and $V_1 \cup \dots \cup V_k$ be a partition of $V(D)$ into k clusters of size m. Suppose that $D[V_i, V_{i+1}]$ is $[\varepsilon, \geq d]$-superregular for each $i \in [k]$ (where $V_{k+1} \coloneqq V_1$). Let $u \in V_1$ and $v \in V_k$. Then, D contains a Hamilton (u, v)-path.*

Proof. By Proposition 7.8, $D[V_k \setminus \{v\}, V_1 \setminus \{u\}]$ is still $[3\sqrt{\varepsilon}, \geq d]$-superregular and so Lemma 7.9 implies that there exists a perfect matching $M \subseteq E_D(V_k \setminus \{v\}, V_1 \setminus \{u\})$. Let $v_1u_1, \dots, v_{m-1}u_{m-1}$ be an enumeration of M. Denote $u_0 \coloneqq u$ and $v_m \coloneqq v$. Let $\mathscr{P}$ be the spanning set of vertex-disjoint paths obtained by applying Lemma 7.11 with $u, u_1, \dots, u_{m-1}$ and $v_1, \dots, v_{m-1}, v$ playing the roles of $u_1, \dots, u_m$ and $v_1, \dots, v_m$. For each $i \in [m]$, let P_i denote the (u_{i-1}, v_i)-path contained in $\mathscr{P}$. Then, $uP_1v_1u_1P_2v_2 \dots u_{m-1}P_mv$ is a Hamilton (u, v)-path of D. □

Let D be a digraph and suppose that $V(D)$ is partitioned into clusters which form superregular pairs. Then, one can adjust this partition in such a way that superregularity is preserved and all the vertices of a small given set S are concentrated into few of the clusters.

Lemma 7.13. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \frac{1}{k} \ll \varepsilon' \ll \varepsilon'' \ll 1$. Let $U_1, \dots, U_4$ be disjoint vertex sets of size n. Let D be a digraph on $U_1 \cup \dots \cup U_4$. For each $i \in [4]$, let $\mathcal{P}_i$ be a partition of U_i into k clusters of size $\frac{n}{k}$. Suppose that for each $i \in [4]$, $D[V, W]$ is $[\varepsilon', \geq 1-\varepsilon']$-superregular whenever $V \subseteq U_i$ and $W \subseteq U_{i+1}$ are unions of clusters in $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$, respectively (where $U_5 := U_1$ and $\mathcal{P}_5 := \mathcal{P}_1$). Let $S \subseteq V(D)$ satisfy $|S| \leq \varepsilon n$. Define $\mathcal{P}'_3 := \mathcal{P}_3$ and $\mathcal{P}'_4 := \mathcal{P}_4$. Then, there exists, for each $i \in [2]$, a partition $\mathcal{P}'_i$ of U_i into k clusters of size $\frac{n}{k}$ such that the following hold.*

(i) *For each $i \in [4]$, $D[V, W]$ is $[\varepsilon'', \geq 1-\varepsilon'']$-superregular whenever $V \subseteq U_i$ and $W \subseteq U_{i+1}$ are unions of clusters in $\mathcal{P}'_i$ and $\mathcal{P}'_{i+1}$, respectively (where $\mathcal{P}'_5 := \mathcal{P}'_1$).*

(ii) *For each $i \in [2]$, there exists a cluster $V \in \mathcal{P}'_i$ for which $S \cap U_i \subseteq V$.*

Proof. For each $i \in [2]$, denote by $V_{i,1}, \dots, V_{i,k}$ the clusters in $\mathcal{P}_i$ and observe that since $|S \cap U_i| \leq |V_{i,k}|$, one can greedily swap each vertex in $S \cap (U_i \setminus V_{i,k})$ with a distinct vertex in $V_{i,k} \setminus S$ to obtain a partition $\mathcal{P}'_i$ of U_i into k clusters $V'_{i,1}, \dots, V'_{i,k}$ such that $S \cap U_i \subseteq V'_{i,k}$ and

$$|V_{i,j} \triangle V'_{i,j}| \leq |S \cap U_i| \leq \frac{\varepsilon' n}{k}$$

for each $j \in [k]$. Then, (ii) holds. Moreover, (i) follows easily from Proposition 7.8. □

7.2.3 The regularity lemma

We now state a degree form of Szemerédi's regularity lemma for balanced bipartite digraphs. In [3], Alon and Shapira proved a regularity lemma for digraphs. A degree form can be derived using similar arguments as the undirected version (see e.g. [105]). The bipartite version stated below can easily be obtained by adjusting the partition obtained with the degree form regularity lemma for digraphs.

Lemma 7.14 (Degree form regularity lemma for balanced bipartite digraphs)**.** *For all $\varepsilon > 0$ and $M' \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ such that, if D is a balanced bipartite digraph on vertex classes A and B of size $n \geq n_0$ and $d \in [0, 1]$, then there exist a spanning subdigraph $D' \subseteq D$ and a partition of $V(D)$ into an* exceptional set *V_0 and $2k$* clusters *$V_1, \dots, V_{2k}$ such that the following hold.*

(i) *$M' \leq 2k \leq M$.*

(ii) *$|V_0 \cap A| = |V_0 \cap B| \leq \varepsilon n$.*

(iii) *For each $i \in [2k]$, either $V_i \subseteq A$ or $V_i \subseteq B$.*

(iv) *$|V_1| = \cdots = |V_{2k}| =: m$. In particular, there are precisely k indices $i \in [2k]$ such that $V_i \subseteq A$ and precisely k indices $i \in [2k]$ such that $V_i \subseteq B$.*

(v) *For each $v \in V(D)$, $d^{\pm}_{D'}(v) > d^{\pm}_{D}(v) - (d + \varepsilon)n$.*

(vi) *For each $i \in [2k]$, $D'[V_i]$ is empty.*

(vii) *Let $i, j \in [2k]$ be distinct. Then, $D'[V_i, V_j]$ is either empty or $(\varepsilon, \geq d)$-regular. Moreover, if $D'[V_i, V_j]$ is non-empty, then $D'[V_i, V_j] = D[V_i, V_j]$.*

Let $\varepsilon > 0$, $M' \in \mathbb{N}$, and $d \in [0, 1]$. Let D be a balanced bipartite digraph. The *bipartite pure digraph of D with parameters ε, d, and M'* is the digraph $D' \subseteq D$ obtained by applying Lemma 7.14 with these parameters. The *bipartite reduced digraph of D with parameters ε, d, and M'* is the digraph R defined as follows. Let $V_0, V_1, \dots, V_{2k}$ be the partition of $V(D)$ obtained by applying Lemma 7.14 with parameters ε, d, and M'. Denote by D' the bipartite pure digraph of D with parameters ε, d, and M'. Then, $V(R) := \{V_i \mid i \in [2k]\}$ and, for any distinct $U, V \in V(R)$, $UV \in E(R)$ if and only if $D'[U, V]$ is non-empty. Note that Lemma 7.14(vii) implies that $D'[U, V] = D[U, V]$ is $(\varepsilon, \geq d)$-regular for any $UV \in E(R)$ and Lemma 7.14(iii) implies that R is a bipartite digraph on vertex classes $\{V \in V(R) \mid V \subseteq A\}$ and $\{V \in V(R) \mid V \subseteq B\}$.

The following lemma states that if a balanced bipartite digraph D is a robust outexpander, then so is its corresponding bipartite reduced digraph. The proof is very similar to that of its non-bipartite analogue (see [79, Lemma 14]) and is therefore omitted.

Lemma 7.15. *Let $0 < \frac{1}{n} \ll \varepsilon \ll d \ll \nu, \tau, \delta \leq 1$ and $\frac{M'}{n} \ll 1$. Let D be a balanced bipartite digraph on vertex classes A and B of size n. Suppose that D is a bipartite robust (ν, τ)-outexpander and that $\delta^0(D) \geq \delta n$. Let R be the bipartite reduced digraph of D with parameters ε, d, and M'. Then, $\delta^0(R) \geq \frac{\delta |R|}{4}$ and R is a bipartite robust $(\frac{\nu}{2}, 2\tau)$-outexpander with bipartition $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A} \coloneqq \{V \in V(R) \mid V \subseteq A\}$ and $\mathcal{B} \coloneqq \{V \in V(R) \mid V \subseteq B\}$.*

7.3 Probabilistic estimates

Let X be a random variable. We write $X \sim \mathrm{Bin}(n, p)$ if X follows a binomial distribution with parameters n and p. Let $N, n, m \in \mathbb{N}$ be such that $\max\{n, m\} \leq N$. Let Γ be a set of size N and $\Gamma' \subseteq \Gamma$ be of size m. Recall that X has a *hypergeometric distribution with parameters $N, n,$ and m* if $X = |\Gamma_n \cap \Gamma'|$, where Γ_n is a random subset of Γ with $|\Gamma_n| = n$ (i.e. Γ_n is obtained by drawing n elements of Γ without replacement). We will denote this by $X \sim \mathrm{Hyp}(N, n, m)$.

7.3.1 Chernoff's bound

First, we will need Chernoff's bound.

Lemma 7.16 (Chernoff's bound, see e.g. [57, Theorems 2.1 and 2.10])**.** *Assume $X \sim \mathrm{Bin}(n, p)$ or $X \sim \mathrm{Hyp}(N, n, m)$. Then, for any $0 < \varepsilon \leq 1$, the following hold.*

(i) $\mathbb{P}\left[X \leq (1-\varepsilon)\mathbb{E}[X]\right] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$.

(ii) $\mathbb{P}\left[X \geq (1+\varepsilon)\mathbb{E}[X]\right] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$.

One can use Lemma 7.16 to show that (super)regularity is preserved with high probability when taking a random edge-slice, i.e. when selecting a random spanning subgraph by including each edge independently with some fixed probability p. This was already observed in (the proof of) [76, Lemma 4.10(iv)] and so we omit the details here.

Lemma 7.17. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \varepsilon' \ll d \leq 1$ and let $\varepsilon \ll p \leq 1$. Let G be a bipartite graph on vertex classes of size n and let G' be obtained from G by selecting each edge independently with probability p.*

(i) *If G is $(\varepsilon, \geq d)$-regular, then G' is $(\varepsilon', \geq pd - \varepsilon)$-regular with high probability.*

(ii) *If G is $[\varepsilon, d]$-superregular, then G' is $[\varepsilon', pd]$-superregular with high probability.*

Corollary 7.18. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \varepsilon' \ll \frac{1}{k} \ll d \ll 1$. Let $U_1, \dots, U_4$ be disjoint vertex sets of size n. Let D be a digraph on $U_1 \cup \dots \cup U_4$. For each $i \in [4]$, let $\mathcal{P}$ be a partition of U_i into k clusters of size $\frac{n}{k}$. Suppose that for each $i \in [4]$, $D[V, W]$ is $[\varepsilon, \geq 1 - \varepsilon]$-superregular whenever V and W are unions of clusters in $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$, respectively (where $\mathcal{P}_5 \coloneqq \mathcal{P}_1$). Let D_1 be obtained by selecting each edge of D independently with probability $1 - 2d$. Let $D_2 \coloneqq D \setminus D_1$. Then, the following holds with high probability. For each $i \in [4]$, $D_1[V, W]$ is $[\varepsilon', \geq 1 - 3d]$-superregular and $D_2[V, W]$ is $[\varepsilon', \geq d + \varepsilon']$-superregular whenever V and W are unions of clusters in $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$, respectively.*

One can also use Lemma 7.16 to show that bipartite robust outexpansion is preserved with high probability when taking random edge-slices. The arguments are similar to those used in the proof of [77, Lemma 3.2(ii)] and are therefore omitted.

Lemma 7.19. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq 1$. Let D be a balanced bipartite digraph on vertex classes A and B of size n. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B). Let D' be obtained from D by taking each edge independently with probability $\frac{1}{2}$. Then, with high probability, both D' and $D \setminus D'$ are bipartite robust $(\frac{\nu}{4}, \tau)$-outexpanders with bipartition (A, B).*

7.3.2 McDiarmid's inequality

We will also need McDiarmid's inequality.

Lemma 7.20 (McDiarmid's inequality [89])**.** *Let $X_1, \dots, X_n$ be independent random variables, each taking values in $\{0, 1\}$. Let $c_1, \dots, c_n \in \mathbb{R}$ and let $f \colon \{0,1\}^n \longrightarrow \mathbb{R}$ be a* *measurable function. Suppose that for any $i \in [n]$ and $x_1, \dots, x_n, x'_i \in \{0,1\}$, we have*

$$|f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq c_i.$$

Then, for any $t > 0$,

$$\mathbb{P}[|f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)]| > t] \leq 2\exp\left(-\frac{2t^2}{\sum_{i\in[n]} c_i^2}\right).$$

Lemma 7.21. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \frac{1}{k} \ll 1$. Let G be a bipartite graph on vertex classes A and B of size n. Suppose that $\Delta(G) \leq \varepsilon n$ and $e(G) \geq \frac{n}{2}$. Let $A_1 \cup \dots \cup A_k$ be a random partition of A such that, for each $i \in [k]$ and $v \in A$, $v \in A_i$ with probability $\frac{1}{k}$ independently of all other vertices. Similarly, let $B_1 \cup \dots \cup B_k$ be a random partition of B such that, for each $i \in [k]$ and $v \in B$, $v \in B_i$ with probability $\frac{1}{k}$ independently of all other vertices. Then, with probability at least $\frac{4}{5}$, we have $e_G(A_i, B_j) \geq \frac{e(G)}{2k^2}$ for all $i, j \in [k]$.*

Proof. Denote $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. Let $i, j \in [k]$. For each $\ell \in [n]$, let

$$X_\ell \coloneqq \begin{cases} 1 & \text{if } a_\ell \in A_i; \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad X_{2n+1-\ell} \coloneqq \begin{cases} 1 & \text{if } b_\ell \in B_j; \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(X_1, \dots, X_{2n}) \coloneqq e_G(A_i, B_j)$. Then, $\mathbb{E}[f(X_1, \dots, X_{2n})] = \frac{e(G)}{k^2}$. Observe that, for each $\ell \in [n]$, we have

$$f(X_1, \dots, X_{\ell-1}, 1, X_{\ell+1}, \dots, X_{2n}) - f(X_1, \dots, X_{\ell-1}, 0, X_{\ell+1}, \dots, X_{2n}) \leq d_G(a_\ell)$$

and

$$f(X_1, \dots, X_{2n-\ell}, 1, X_{2n-\ell+2}, \dots, X_{2n}) - f(X_1, \dots, X_{2n-\ell}, 0, X_{2n-\ell+2}, \dots, X_{2n}) \leq d_G(b_\ell).$$

Moreover, $\sum_{\ell\in[n]}((d_G(a_\ell))^2 + (d_G(b_\ell))^2) \leq 2\frac{e(G)}{\Delta(G)}(\Delta(G))^2 \leq 2e(G)\varepsilon n$. Thus, Lemma 7.20

implies that

$$\mathbb{P}\left[e_G(A_i, B_j) < \frac{e(G)}{2k^2}\right] \leq \mathbb{P}\left[|f(X_1, \ldots, X_{2n}) - \mathbb{E}[f(X_1, \ldots, X_{2n})]| > \frac{e(G)}{2k^2}\right]$$
$$\leq 2\exp\left(-\frac{e(G)}{4k^4\varepsilon n}\right) \leq 2\exp\left(-\frac{1}{8\varepsilon k^4}\right).$$

Therefore, a union bound implies that, with probability at least $1 - 2k^2 \exp\left(-\frac{1}{8\varepsilon k^4}\right) \geq \frac{4}{5}$, we have $e_G(A_i, B_j) \geq \frac{e(G)}{2k^2}$ for all $i, j \in [k]$. □

7.4 Matchings

In this section, we collect tools for constructing and working with matchings. First, we need the following two propositions, which follow from König's theorem [70] (see also [71] for a German translation).

Proposition 7.22. *Let G be a bipartite graph with maximum degree at most Δ. Then, G contains a matching of size $\frac{e(G)}{\Delta}$.*

Proposition 7.23 (see e.g. [108, Exercise 7.1.33])**.** *Let G be a bipartite graph with maximum degree at most Δ. Then, G can be decomposed into edge-disjoint matchings $M_1, \ldots, M_\Delta$ such that, for any $i, j \in [\Delta]$, $||M_i| - |M_j|| \leq 1$.*

We will also need the following corollary of Hall's theorem [49].

Proposition 7.24. *Let G be a bipartite graph on vertex classes A and B with $|A| \leq |B|$. Suppose that, for each $a \in A$, $d_G(a) \geq \frac{|B|}{2}$ and, for each $b \in B$, $d_G(b) \geq |A| - \frac{|B|}{2}$. Then, G contains a matching covering A.*

7.5 Matching contractions

Note that the concepts introduced in this section will not be used formally until Chapter 11. However, we introduce them here as they will help us to explain the approximate decomposition strategy presented in Section 8.1.

As discussed in the proof overview, most of our Hamilton cycles will be formed by first constructing a perfect matching, which is then extended to a Hamilton cycle by constructing a Hamilton cycle in an auxiliary digraph which is, roughly speaking, obtained by contracting the edges of M. In this section, we give a formal definition of this auxiliary digraph and state its main properties.

Definition 7.25 (Matching contraction and matching expansion)**.** Let A and B be disjoint vertex sets of equal size. Let M be an auxiliary directed perfect matching from B to A.

(i) Let G be a bipartite graph on vertex classes A and B. The *M-contraction of G* is the digraph G_M on vertex set A defined as follows. Let $a, a' \in A$ be distinct and denote by b the (unique) neighbour of a in M. Then, $a'a \in E(G_M)$ if and only if $a'b \in E(G)$.

(ii) Let D be a digraph on vertex set A. The *M-expansion of D* is the bipartite graph D_M on vertex classes A and B defined as follows. Let $b \in B$ and let a be the (unique) neighbour of b in M. Then, for any $a' \in A$, $a'b \in E(D_M)$ if and only if $a'a \in E(D)$.

(Recall that Definition 7.25 and all other main definitions are indexed in the glossary at the end of this thesis.)

The condition that a and a' have to be distinct in Definition 7.25(i) ensures that the resulting digraph G_M does not contain any loop. However, this implies that the edges lying along M are lost in the process of contraction and expansion. (Of course, one could slightly change Definition 7.25(i) to allow M-contractions to have loops. In this way, the M-expansion would the exact reverse operation of the M-contraction. But working with loops is impractical for our purposes.)

Fact 7.26. *Let A and B be disjoint vertex sets of equal size. Let M be a directed perfect matching from B to A. Let G be a bipartite graph on vertex classes A and B. Denote by D the M-contraction of G and by G' the M-expansion of D. Then, $e(D) = e(G \setminus M[B, A])$ and $G' = G \setminus M[B, A]$.*

Let A, B, M, G, and D be as in Fact 7.26. By Definition 7.25(i), the outneighbourhood of a vertex $a \in A$ in D corresponds to the neighbourhood of a in G, while the inneighbourhood of a in D corresponds to the neighbourhood of $N_M(a)$ in G. We also observe for later use that there is a one-to-one correspondence between the connected components of $G \cup M[B, A]$ and D.

Fact 7.27. *Let A, B, M, G, and D be as in Fact 7.26. Then, the following hold.*

(i) *Each $a \in A$ satisfies $N_D^+(a) = N_M(N_G(a)) \setminus \{a\}$ and $N_D^-(a) = N_G(N_M(a)) \setminus \{a\}$.*

(ii) *Any $a, a' \in A$ belong to a common connected component of D if and only if they belong to a common connected component of $G \cup M[B, A]$.*

Let A, B, and M be as in Fact 7.26. Let D be a digraph on A and denote by G the M-expansion of D. By Definition 7.25(ii), the neighbourhood of a vertex $a \in A$ in G corresponds to the outneighbourhood of a in D, while the neighbourhood of $N_M(a)$ in G corresponds to the inneighbourhood of a in D.

Fact 7.28. *Let A, B, and M be as in Fact 7.26. Let D be a digraph on A and denote by G the M-expansion of D. Then, the following hold.*

(i) *Each $a \in A$ satisfies $N_G(a) = N_M(N_D^+(a))$.*

(ii) *Each $b \in B$ satisfies $N_G(b) = N_D^-(N_M(b))$.*

We now state our key property of matching contractions and matching expansions: finding a Hamilton cycle in a bipartite digraph D on vertex classes A and B is equivalent to finding a perfect matching M from B to A in D and then finding a Hamilton cycle in the M-contraction of $D[A, B]$.

Fact 7.29. *Let A and B be disjoint vertex sets of equal size. Let M be a directed perfect matching from B to A. Let H be a directed Hamilton cycle on A. Let G be obtained by orienting from A to B all the edges in the M-expansion of H. Then, G is a directed perfect matching from A to B and $G \cup M$ is a directed Hamilton cycle on $A \cup B$.*

To construct Hamilton cycles in matching contractions, we will use the following proposition, which states that almost regularity, superregularity, and robust outexpansion are preserved in contracted digraphs. Its proof follows easily from definitions and is therefore omitted. (Similar observations were also made and proved in [74].)

Proposition 7.30. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \delta, d \leq 1$. Let G be a bipartite graph on vertex classes A and B of size n and M be a directed perfect matching from B to A. Then, the M-contraction D of G satisfies the following properties.*

(i) *If G is (δ, ε)-almost regular, then D is $(2\delta, 2\varepsilon)$-almost regular.*

(ii) *Let $A_1, A_2 \subseteq A$ be disjoint. If $G[A_1, N_M(A_2)]$ is $[\varepsilon, d]$-superregular, then $D[A_1, A_2]$ is $[\varepsilon, d]$-superregular.*

(iii) *If G is a bipartite robust (ν, τ)-expander with bipartition (A, B), then D is a robust $(\frac{\nu}{2}, \tau)$-outexpander.*

Note that Corollary 7.5 follows from Theorem 7.4 and Proposition 7.30(iii).

Proof of Corollary 7.5. First, we find a perfect matching from A to B as follows. Let M be an arbitrary perfect matching from B to A. Let D_M be the M-contraction of $D[A, B]$. By Proposition 7.30(iii), D_M is a robust $(\frac{\nu}{2}, \tau)$-outexpander. Moreover, Fact 7.27(i) implies that $\delta^0(D_M) \geq \frac{\delta n}{2}$. Thus, Theorem 7.4 implies that D_M contains a Hamilton cycle H. Let M' be obtained by orienting from A to B all the edges in the M-expansion of H. Then, Facts 7.27(i), 7.28, and 7.29 imply that M' is a perfect matching of D from A to B.

We close M' into a Hamilton cycle as follows. Let $D_{M'}$ be the M'-contraction of $D[B, A]$. By the same arguments as above, $D_{M'}$ contains a Hamilton cycle H'. Let M'' be obtained by orienting from B to A all the edges in the M'-expansion of H'. By the same arguments as above, M'' is a perfect matching of D from B to A. Moreover, Fact 7.29 implies that $M' \cup M''$ is a Hamilton cycle. □

Finally, observe that contracting a linear forest F gives a linear forest with endpoints corresponding to those of F. This follows easily from Fact 7.27 and so we omit the details.

Proposition 7.31. *Let F be a balanced bipartite directed linear forest on vertex classes A and B. Suppose that $F[B, A]$ is a perfect matching and let $M \coloneqq E_F(B, A)$. Denote by D the M-contraction of $F[A, B]$. Then, D is a linear forest satisfying*

$$V^+(D) = N_M(V^+(F)), \quad V^-(D) = V^-(F), \quad \textit{and} \quad V^0(D) = (V^0(F) \cap A) \setminus N_M(V^+(F)).$$

CHAPTER 8

MAIN TOOLS

We now introduce our main tools for constructing approximate decompositions and decomposing leftovers. These will be used in the proofs of Theorems 4.1 and 4.4.

8.1 Approximate decomposition tools

As mentioned in Section 4.3, we adapt arguments of [40] using the concept of matching contraction. We now discuss this in more detail. In [40], we showed that any dense almost regular robust outexpander D can be approximately decomposed into Hamilton cycles (see Theorem 3.9). (Moreover, one can ensure that each Hamilton cycle contains a small set of prescribed edges.) The key idea behind the proof is to reserve a sparse random edge-slice $\Gamma \subseteq D$ and then construct, one by one, edge-disjoint Hamilton cycles which use very few edges of Γ. This ensures that robust outexpansion is preserved throughout the approximate decomposition.

Let D be a bipartite digraph on vertex classes A and B and suppose that $D[A, B]$ is an almost regular bipartite robust expander. Let $M_1, \dots, M_\ell$ be edge-disjoint perfect matchings whose edges are all oriented from B to A. Then, we can extend $M_1, \dots, M_\ell$ into edge-disjoint Hamilton cycles as follows. For each $i \in [\ell]$, denote by D_i the M_i-contraction of D and note that, by Proposition 7.30, D_i is an almost regular robust outexpander. By Fact 7.29, it is enough to find, for each $i \in [\ell]$, a Hamilton cycle of D_i. Since the D_i's are distinct, we cannot apply Theorem 3.9 directly. However, we can adapt the strategy discussed above as follows. We initially reserve a randomly chosen edge-slice $\Gamma \subseteq D[A, B]$ and denote, for each $i \in [\ell]$, by Γ_i the corresponding random edge-slice of D_i. At each stage $i \in [\ell]$, we use the arguments of Theorem 3.9 to construct a Hamilton cycle of D_i which uses very few edges of Γ_i. This ensures that, overall, very few edges of Γ are used and so $D[A, B]$ remains a bipartite robust expander throughout the approximate decomposition. By Proposition 7.30, this implies that, at each stage $i \in [\ell]$, D_i is still a robust outexpander and so the approximate decomposition can be completed. (See Appendix B for details.)

Recall from Section 5.2.5 that a balanced bipartite digraph D on vertex classes of size n is (δ, ε)-regular if all its vertices have in- and outdegree both roughly equal to $\delta|V(D)| = 2\delta n$. This justifies the factor of 2 in the upper bound on ℓ in Theorem 8.1. Moreover, recall from Section 5.2.1 that the parallel edges of a multi(di)graph are considered to be distinct. Thus, we do not require the linear forests $F_1, \dots, F_\ell$ in Theorem 8.1 to be edge-disjoint and the theorem states that each edge of D is covered by at most one of the resulting Hamilton cycles $C_1, \dots, C_\ell$ (while each linear forest F_i is fully incorporated into its corresponding cycle C_i).

Theorem 8.1 (Extending an approximate perfect matching decomposition into an approximate Hamilton decomposition)**.** *Let $0 < \frac{1}{n} \ll \tau \ll \delta \leq 1$ and $0 < \frac{1}{n} \ll \varepsilon \ll \eta, \nu \leq 1$. Let $\ell \leq 2(\delta - \eta)n$. Let D be a balanced bipartite digraph on vertex classes A and B of size n. Suppose that $D[A, B]$ is a (δ, ε)-almost regular bipartite robust (ν, τ)-expander with bipartition (A, B). Suppose that $F_1, \dots, F_\ell$ are bipartite directed linear forests on vertex classes A and B satisfying the following properties.*

(i) *For each $i \in [\ell]$, $e_{F_i}(B, A) = n$.*

(ii) *For each $i \in [\ell]$, $e_{F_i}(A, B) \leq \varepsilon n$.*

(iii) *For each $v \in V(D)$, there exist at most εn indices $i \in [\ell]$ such that $d_{F_i[A,B]}(v) = 1$.*

Define a multidigraph $\mathcal{F}$ by $\mathcal{F} := \bigcup_{i \in [\ell]} F_i$. Then, the multidigraph $D \cup \mathcal{F}$ contains edge-disjoint Hamilton cycles $C_1, \dots, C_\ell$ such that $F_i \subseteq C_i$ for each $i \in [\ell]$. Moreover, $D[A, B] \setminus \bigcup_{i \in [\ell]} C_i$ is still a bipartite robust $(\frac{\nu}{2}, \tau)$-expander with bipartition (A, B).

If D is a bipartite robust outexpander (i.e. if both $D[A, B]$ and $D[B, A]$ are bipartite robust expanders (recall Fact 7.1)), then we can apply Theorem 8.1 twice in a row to construct an approximate Hamilton decomposition of D: first, we apply Theorem 8.1 with arbitrary perfect matchings from B to A to approximately decompose the edges of D from A to B into edge-disjoint perfect matchings, and then we apply Theorem 8.1 a second time to extend these perfect matchings into edge-disjoint Hamilton cycles of D.

Corollary 8.2 (Approximate Hamilton decomposition)**.** *Let $0 < \frac{1}{n} \ll \tau \ll \delta \leq 1$ and $0 < \frac{1}{n} \ll \varepsilon \ll \eta, \nu \leq 1$. Let $\ell \leq 2(\delta - \eta)n$. Let D be a balanced bipartite digraph on vertex classes A and B of size n. Suppose that D is a (δ, ε)-almost regular bipartite robust (ν, τ)-outexpander with bipartition (A, B). Suppose that $F_1, \dots, F_\ell$ are bipartite directed linear forests on vertex classes A and B satisfying the following properties.*

(i) *For each $i \in [\ell]$, $e(F_i) \leq \varepsilon n$.*

(ii) *For each $v \in V(D)$, there exist at most εn indices $i \in [\ell]$ such that $v \in V(F_i)$.*

Define a multidigraph $\mathcal{F}$ by $\mathcal{F} := \bigcup_{i \in [\ell]} F_i$. Then, the multidigraph $D \cup \mathcal{F}$ contains edge-disjoint Hamilton cycles $C_1, \dots, C_\ell$ such that $F_i \subseteq C_i$ for each $i \in [\ell]$. Moreover, $D \setminus \bigcup_{i \in [\ell]} C_i$ is still a bipartite robust $(\frac{\nu}{2}, \tau)$-outexpander with bipartition (A, B).

Proof. First, we extend $F_1, \dots, F_\ell$ to auxiliary linear forests which satisfy Theorem 8.1(i)–(iii).

Claim 1. *For each $i \in [\ell]$, there exists a bipartite linear forest F_i' on vertex classes A and B such that $F_i'[A, B] = F_i[A, B]$ and $F_i'[B, A]$ is a perfect matching containing $F_i[B, A]$.*

Proof of Claim. Let $i \in [\ell]$.

- Denote by $a_1, \dots, a_q$ the vertices $v \in A$ satisfying $d_{F_i}^-(v) = 0$ and $d_{F_i}^+(v) = 1$.
- Denote by $a_{q+1}, \dots, a_{q+r}$ the vertices $v \in A$ satisfying $d_{F_i}^-(v) = 0$ and $d_{F_i}^+(v) = 0$.
- Denote by $b_1, \dots, b_s$ the vertices $v \in B$ satisfying $d_{F_i}^-(v) = 1$ and $d_{F_i}^+(v) = 0$.
- Denote by $b_{s+1}, \dots, b_{s+t}$ the vertices $v \in B$ satisfying $d_{F_i}^-(v) = 0$ and $d_{F_i}^+(v) = 0$.

Observe that there exist exactly r vertices in A which have degree 0 in F_i. Therefore,

$$r \geq |A| - e(F_i) \overset{\text{(i)}}{\geq} (1-\varepsilon)n > 0.$$

Note that $a_1, \dots, a_q$ is an enumeration of $V^+(F_i) \cap A$ and $b_1, \dots, b_s$ is an enumeration of $V^-(F_i) \cap B$. Since F_i is a linear forest, we may therefore assume without loss of generality that, if F_i contains an (a_j, b_k)-path for some $j \in [q]$ and $k \in [s]$, then $j = k$. Note that $e(F_i[B, A]) = |A| - (q + r) = |B| - (s + t)$ and so $q + r = s + t$. Let $F_i' := F_i \cup \{b_i a_{i+1} \mid i \in [q + r]\}$ (where $a_{q+r+1} := a_1$). Then, F_i is a bipartite digraph on vertex classes A and B such that $F_i'[A, B] = F_i[A, B]$ and $F_i'[B, A]$ is a perfect matching which contains $F_i[B, A]$. It is easy to check that F_i is a linear forest. $\diamond$

Let $F_1', \dots, F_\ell'$ be the linear forests obtained by applying Claim 1. Observe that Theorem 8.1(i)–(iii) are satisfied with $F_1', \dots, F_\ell'$ playing the roles of $F_1, \dots, F_\ell$. Define a multidigraph $\mathcal{F}'$ by $\mathcal{F}' := \bigcup_{i \in [\ell]} F_i'$. By Theorem 8.1 (applied with $F_1', \dots, F_\ell'$ playing the roles of $F_1, \dots, F_\ell$), the multidigraph $D \cup \mathcal{F}'$ contains edge-disjoint Hamilton cycles $C_1', \dots, C_\ell'$ such that $F_i' \subseteq C_i'$ for each $i \in [\ell]$. For each $i \in [\ell]$, let $F_i'' := C_i'[A, B] \cup F_i[B, A] \subsetneq C_i'$ and note that $F_i \subseteq F_i'' \subseteq \mathcal{F} \cup D[A, B]$. Moreover, Theorem 8.1(i)–(iii) are satisfied with B, A, and $F_1'', \dots, F_\ell''$ playing the roles of A, B, and $F_1, \dots, F_\ell$. Define a multidigraph $\mathcal{F}''$ by $\mathcal{F}'' := \bigcup_{i \in [\ell]} F_i''$. Let $D' := D \setminus \mathcal{F}''$. Note that $D'[B, A] = D[B, A]$ and, by the "moreover part" of Theorem 8.1, $D'[A, B]$ is a bipartite robust $(\frac{\nu}{2}, \tau)$-expander with bipartition (A, B).

By Theorem 8.1 (applied with D', B, A, and $F_1'', \dots, F_\ell''$ playing the roles of D, A, B, and $F_1, \dots, F_\ell$), the multidigraph $D' \cup \mathcal{F}'' = D \cup \mathcal{F}$ contains edge-disjoint Hamilton cycles $C_1, \dots, C_\ell$ such that $F_i \subseteq F_i'' \subseteq C_i$ for each $i \in [\ell]$. Let $D'' := D \setminus \bigcup_{i \in [\ell]} C_i$. By the "moreover part" of Theorem 8.1, $D''[B, A]$ is a bipartite robust $(\frac{\nu}{2}, \tau)$-expander with bipartition (B, A). By construction, $D''[A, B] = D'[A, B]$ and so $D''[A, B]$ is also a bipartite robust $(\frac{\nu}{2}, \tau)$-expander with bipartition (A, B). Therefore, Fact 7.1 implies that D'' is a bipartite robust $(\frac{\nu}{2}, \tau)$-outexpander with bipartition (A, B). $\square$

Note that the "moreover part" of Corollary 8.2 implies that we can prescribe some edges to most of the Hamilton cycles in the decomposition given by Theorem 4.1.

Similarly, one can apply Theorem 8.1 with auxiliary perfect matchings from B to A to obtain an approximate decomposition of a bipartite robust expander into perfect matchings which extend given small matchings.

Corollary 8.3 (Approximate perfect matching decomposition). *Let $0 < \frac{1}{n} \ll \tau \ll \delta \leq 1$ and $0 < \frac{1}{n} \ll \varepsilon \ll \eta, \nu \leq 1$. Let $\ell \leq 2(\delta - \eta)n$. Let G be a balanced bipartite graph on vertex classes A and B of size n. Suppose that G is a (δ, ε)-almost regular bipartite robust (ν, τ)-expander with bipartition (A, B). Suppose that $F_1, \dots, F_\ell$ are bipartite matchings on vertex classes A and B satisfying the following properties.*

(i) *For each $i \in [\ell]$, $e(F_i) \leq \varepsilon n$.*

(ii) *For each $v \in V(G)$, there exist at most εn indices $i \in [\ell]$ such that $v \in V(F_i)$.*

Define a multigraph $\mathcal{F}$ by $\mathcal{F} := \bigcup_{i\in[\ell]} F_i$. Then, the multigraph $G \cup \mathcal{F}$ contains edge-disjoint perfect matchings $M_1, \dots, M_\ell$ such that $F_i \subseteq M_i$ for each $i \in [\ell]$. Moreover, $G \setminus \bigcup_{i\in[\ell]} M_i$ is still a bipartite robust $(\frac{\nu}{2}, \tau)$-expander with bipartition (A, B).

8.2 The robust decomposition lemma

In this section, we state (a modified version of) the robust decomposition lemma of [76]. Roughly speaking, this result guarantees the existence of a sparse absorber $D^{\rm rob}$ which can decompose any sparse leftover H into Hamilton cycles. To state this lemma, we need some definitions. These are needed in order to describe the structure within which the sparse absorber $D^{\rm rob}$ can be found. Roughly speaking, the structure consists of a "quasirandom" blow-up of a graph consisting of a cycle and a suitable set of chords on this cycle.

8.2.1 Equivalent linear forests

We start with a simple concept which will enable us to simplify some arguments.

Definition 8.4 (Equivalent linear forests)**.** Two linear forests F and F' are *equivalent* if $V(F) = V(F')$ and there exist enumerations $P_1, \dots, P_\ell$ and $P'_1, \dots, P'_\ell$ of the components of F and F' such for each $i \in [\ell]$, P_i and P'_i have the same starting and ending points.

Fact 8.5. *Let V be a vertex set and D be a digraph with $V(D) \subseteq V$. Let F and F' be two equivalent linear forests. Then, $D \cup F$ is a Hamilton cycle on V if and only if $D \cup F'$ is a Hamilton cycle on V.*

Roughly speaking, Fact 8.5 states that, if F is a linear forest that we want to extend into a Hamilton cycle, then the internal structure of F is irrelevant. This simple observation will enable us to simplify the statement and application of the robust decomposition lemma of [76].

8.2.2 Refinements

Let D be a digraph and $\mathcal{P}$ be a partition of $V(D)$ into an exceptional set V_0 and k clusters $V_1, \dots, V_k$ of size m. Let $\mathcal{P}'$ be a partition of $V(D)$. We say that $\mathcal{P}'$ is an *ℓ-refinement of $\mathcal{P}$* if $\mathcal{P}'$ is obtained by splitting each cluster in $\mathcal{P}$ into ℓ subclusters of size $\frac{m}{\ell}$. (Thus, $\mathcal{P}'$ consists of the exceptional set V_0 and ℓk clusters.)

Definition 8.6 (Uniform refinement)**.** Let D be a digraph and $\mathcal{P}$ be a partition of $V(D)$ into an exceptional set V_0 and k clusters $V_1, \dots, V_k$ of size m. An ℓ-refinement $\mathcal{P}'$ of $\mathcal{P}$ is *ε-uniform (with respect to D)* if the following condition holds, where for each $i \in [k]$, $V_{i,1} \cup \dots \cup V_{i,\ell}$ denotes the partition of V_i induced by $\mathcal{P}'$.

(URef) Let $v \in V(D)$, $i \in [k]$, $j \in [\ell]$, and $\diamond \in \{+, -\}$. If $|N_D^\diamond(v) \cap V_i| \geq \varepsilon m$, then $|N_D^\diamond(v) \cap V_{i,j}| = (1 \pm \varepsilon)\frac{|N_D^\diamond(v) \cap V_i|}{\ell}$.

Given a partition $\mathcal{P}$ and a random ℓ-refinement $\mathcal{P}'$ of $\mathcal{P}$, one can use Lemma 7.16 to show that $\mathcal{P}'$ is ε-uniform with high probability.

Lemma 8.7 ([76, Lemma 4.7])**.** *Let $0 < \frac{1}{m} \ll \frac{1}{k}, \varepsilon \ll \frac{1}{\ell} \leq 1$ and suppose that $\frac{m}{\ell} \in \mathbb{N}$. Let D be a digraph on $n \leq 2km$ vertices and let $\mathcal{P}$ be a partition of $V(D)$ into an exceptional set V_0 and k clusters of size m. Then, there exists an ε-uniform ℓ-refinement of $\mathcal{P}$.*

Using the definition of ε-regularity, one can easily verify that (super)regularity is preserved under taking uniform refinements.

Lemma 8.8 ([76, Lemma 4.7]). *Let $0 < \frac{1}{m} \ll \frac{1}{k}, \varepsilon \ll d, \frac{1}{\ell} \leq 1$ and $\varepsilon \ll \varepsilon' \leq 1$. Suppose that $\frac{m}{\ell} \in \mathbb{N}$. Let D be a digraph on $n \leq 2km$ vertices and let $\mathcal{P}$ be a partition of $V(D)$ into an exceptional set V_0 and k clusters of size m. Let $\mathcal{P}'$ is an ε-uniform ℓ-refinement of $\mathcal{P}$ and let $V, W \in \mathcal{P}$ and $V', W' \in \mathcal{P}'$ be distinct clusters satisfying $V' \subseteq V$ and $W' \subseteq W$.*

(i) *If $D[V, W]$ is $(\varepsilon, \geq d)$-regular, then $D[V', W']$ is $(\varepsilon', \geq d - \varepsilon)$-regular.*

(ii) *If $D[V, W]$ is $[\varepsilon, \geq d]$-superregular, then $D[V', W']$ is $[\varepsilon', \geq d]$-superregular.*

Using Lemma 7.16, one can easily verify that the uniformity of a refinement is preserved with high probability when considering edge-slices.

Lemma 8.9. *Let $0 < \frac{1}{m} \ll \frac{1}{k}, \varepsilon \ll \frac{1}{\ell}, p \leq 1$. Let D be a digraph on $n \leq 2km$ vertices and let $\mathcal{P}$ be a partition of $V(D)$ into an exceptional set V_0 and k clusters of size m. Let $\mathcal{P}'$ be an ε-uniform ℓ-refinement of $\mathcal{P}$ with respect to D. Let D' be obtained from D by selecting each edge independently with probability p. Then, with high probability, $\mathcal{P}'$ is 2ε-uniform with respect to both D' and $D \setminus D'$.*

Finally, observe that refinements are always uniform in digraphs of very high minimum degree.

Lemma 8.10. *Let $0 < \frac{1}{m} \ll \varepsilon \ll \frac{1}{k}, \frac{1}{\ell} \leq 1$ and suppose that $\frac{m}{\ell} \in \mathbb{N}$. Let D be a digraph on $n \leq 2km$ vertices and suppose that $\delta^0(D) \geq (1-\varepsilon)n$. Let $\mathcal{P}$ be a partition of $V(D)$ into an exceptional set V_0 and k clusters of size m. Then, any ℓ-refinement of $\mathcal{P}$ is $\sqrt{\varepsilon}$-uniform with respect to D.*

Proof. Let $\mathcal{P}'$ be an ℓ-refinement of $\mathcal{P}$. Let $v \in V(D)$ and fix clusters $V \in \mathcal{P}$ and $W \in \mathcal{P}'$ satisfying $W \subseteq V$. By assumption, both

$$|N^{\pm}(v) \cap W| \leq \frac{m}{\ell} \leq \frac{|N^{\pm}(v) \cap V| + \varepsilon n}{\ell} \leq (1 + \sqrt{\varepsilon}) \frac{|N^{\pm}(v) \cap V|}{\ell}$$

and

$$|N^{\pm}(v) \cap W| \geq \frac{m}{\ell} - \varepsilon n \geq \frac{|N^{\pm}(v) \cap V|}{\ell} - \frac{\varepsilon \ell n}{\ell} \geq (1 - \sqrt{\varepsilon}) \frac{|N^{\pm}(v) \cap V|}{\ell}.$$

Thus, (URef) holds and we are done. □

8.2.3 (Bi)-universal walks

Let R be a digraph whose vertices are $V_1, \dots, V_k$ and suppose that $C = V_1 \dots V_k$ is a Hamilton cycle of R. Let $i, j \in [k]$. A *chord sequence $CS(V_i, V_j)$ from V_i to V_j in R* is an ordered sequence of edges of the form

$$CS(V_i, V_j) = (V_{i_1 - 1} V_{i_2}, V_{i_2 - 1} V_{i_3}, \dots, V_{i_t - 1} V_{i_{t+1}}),$$

where $V_{i_1} \coloneqq V_i$, $V_{i_{t+1}} \coloneqq V_j$ and, for each $s \in [t]$, $V_{i_s - 1} V_{i_{s+1}} \in E(R)$. Thus, the simplest example of a chord sequence $CS(V_i, V_j)$ is simply $(V_{i-1} V_j)$. Chord sequences are used in the proof of the robust decomposition lemma in [76] to extend arbitrary edges into cycles which meet each cluster V_i the same number of times.

Definition 8.11 (Universal walk)**.** Suppose that R is a digraph whose vertices are k clusters $V_1, \dots, V_k$ and that $C \coloneqq V_1 \dots V_k$ is a Hamilton cycle of R. A closed walk U in R is a *universal walk for C with parameter ℓ'* if the following conditions hold.

(U1) For every $i \in [k]$, U contains a chord sequence $CS(V_i, V_{i+1})$ from V_i to V_{i+1} (where $V_{k+1} \coloneqq V_1$) such that (U2), (U3), and the following hold. All the remaining edges of U lie on C.

(U2) For each $i \in [k]$, $CS(V_i, V_{i+1})$ consists of at most $\frac{\sqrt{\ell'}}{2}$ edges.

(U3) For each $i \in [k]$, both $d_U^{\pm}(V_i) = \ell'$.

(Recall that Definition 8.11, as well as all the core definitions and their main properties are indexed in the glossary at the end of this thesis.)

Lemma 8.12 ([20, Lemma 2.9.1]). *Let R be a complete digraph and C be a Hamilton cycle of R. For any $\ell' \geq 4$, R contains a universal walk for C with parameter ℓ'.*

We will also need the bipartite analogue of a universal walk.

Definition 8.13 (Bi-universal walk). Suppose that R is a digraph whose vertices are k clusters $V_1, \dots, V_k$, where k is even, and that $C \coloneqq V_1 \dots V_k$ is a Hamilton cycle of R. A closed walk U in R is a *bi-universal walk for C with parameter ℓ'* if the following conditions hold.

(BU1) The edge set of U has a partition into U_{odd} and U_{even} and, for every $i \in [k]$, U contains a chord sequence $CS(V_i, V_{i+2})$ from V_i to V_{i+2} (where $V_{k+1} \coloneqq V_1$ and $V_{k+2} \coloneqq V_2$) such that (BU2), (BU3), and the following hold. All of the edges in the multiset $\bigcup\{CS(V_i, V_{i+2}) \mid i \in [k] \text{ is odd}\}$ are contained in U_{odd}, all of the edges in the multiset $\bigcup\{CS(V_i, V_{i+2}) \mid i \in [k] \text{ is even}\}$ are contained in U_{even}, and all the remaining edges of U lie on C.

(BU2) For each $i \in [k]$, $CS(V_i, V_{i+2})$ consists of at most $\frac{\sqrt{\ell'}}{2}$ edges.

(BU3) For each $i \in [k]$, both $d^{\pm}_{U_{\text{odd}}}(V_i) = \frac{\ell'}{2}$ and both $d^{\pm}_{U_{\text{even}}}(V_i) = \frac{\ell'}{2}$

8.2.4 (Bi)-setups

We introduce the key structures required to construct the absorber in the robust decomposition lemma.

Definition 8.14 (Setup). We say that $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \ell^*, k, m, \varepsilon, d)$-*setup* if the following properties are satisfied.

(ST1) D is a digraph. $\mathcal{P}$ is a partition of $V(D)$ into an exceptional set V_0 of size $|V_0| \leq \varepsilon |V(D)|$ and k clusters $V_1, \dots, V_k$ of size m.

(ST2) R is a digraph on the clusters in $\mathcal{P}$, that is, $V(R) = \{V_i \mid i \in [k]\}$. For each $VW \in E(R)$, the corresponding pair $D[V, W]$ is $(\varepsilon, \geq d)$-regular.

(ST3) C is a Hamilton cycle of R and, for each $VW \in E(C)$, the corresponding pair $D[V, W]$ is $[\varepsilon, \geq d]$-superregular.

(ST4) U is a universal walk for C in R with parameter ℓ'.

(ST5) $\mathcal{P}'$ is an ε-uniform ℓ'-refinement of $\mathcal{P}$.

(ST6) For each $i \in [k]$, let $V_{i,1}, \ldots, V_{i,\ell'}$ denote the subclusters of V_i contained in $\mathcal{P}'$. Then, U' is a closed walk on the clusters in $\mathcal{P}'$ which is obtained from U as follows. For each $i \in [k]$ and $j \in [\ell']$, when U visits V_i for the j^{th} time, U' visits the subcluster $V_{i,j}$.

(ST7) For each $VW \in E(U')$, the corresponding pair $D[V, W]$ is $[\varepsilon, \geq d]$-superregular.

(ST8) $\mathcal{P}^*$ is an ε-uniform ℓ^*-refinement of $\mathcal{P}$.

We will also need the bipartite analogue of a setup.

Definition 8.15 (Bi-setup)**.** We say that $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \ell^*, 2k, m, \varepsilon, d)$-*bi-setup* if $k \in \mathbb{N}$ and the following properties are satisfied.

(BST1) D is a balanced bipartite digraph on vertex classes A and B. $\mathcal{P}$ is a partition of $V(D)$ into an exceptional set V_0 which satisfies $|V_0 \cap A| = |V_0 \cap B| \leq \varepsilon |A| = \varepsilon |B|$, and $2k$ clusters $V_1, \ldots, V_{2k}$ of size m. Let $\mathcal{A}$ be the set of clusters $V \in \mathcal{P}$ such that $V \subseteq A$. Define $\mathcal{B}$ analogously. Then, $A \setminus V_0 = \bigcup \mathcal{A}$ and $B \setminus V_0 = \bigcup \mathcal{B}$. (In particular, each cluster $V \in \mathcal{P}$ satisfies $V \subseteq A$ or $V \subseteq B$.)

(BST2) R is a balanced bipartite digraph on vertex classes $\mathcal{A}$ and $\mathcal{B}$. For each $VW \in E(R)$, the corresponding pair $D[V, W]$ is $(\varepsilon, \geq d)$-regular.

(BST3) C is a Hamilton cycle of R and for each $VW \in E(C)$ the corresponding pair $D[V, W]$ is $[\varepsilon, \geq d]$-superregular.

(BST4) U is a bi-universal walk for C in R with parameter ℓ'.

(BST5) $\mathcal{P}'$ is an ε-uniform ℓ'-refinement of $\mathcal{P}$.

(BST6) For each $i \in [2k]$, let $V_{i,1}, \dots, V_{i,\ell'}$ denote the subclusters of V_i contained in $\mathcal{P}'$. Then, U' is a closed walk on the clusters in $\mathcal{P}'$ which is obtained from U as follows. For each $i \in [2k]$ and $j \in [\ell']$, when U visits V_i for the j^{th} time, U' visits the subcluster $V_{i,j}$.

(BST7) For each $VW \in E(U')$, the corresponding pair $D[V, W]$ is $[\varepsilon, \geq d]$-superregular.

(BST8) $\mathcal{P}^*$ is an ε-uniform ℓ^*-refinement of $\mathcal{P}$.

Note that these definitions of a setup and a bi-setup are slightly different to that of [76]. The original definitions required the exceptional set V_0 to form an independent set in D. Here, we only need the definition of a (bi)-setup within the setting of the robust decomposition lemma (Lemma 8.23 below), where V_0 is empty. The independent set condition is therefore redundant and we omit it. In [76], the refinement $\mathcal{P}^*$ is added in the statement of the robust decomposition lemma directly. For convenience, we incorporate $\mathcal{P}^*$ into the definition of a (bi)-setup and, for technical reasons, we also require that $\mathcal{P}^*$ is ε-uniform. Finally, the definition of a bi-setup in [76] did not require D to be a bipartite digraph. We add this constraint here for convenience. For clarity, we also specify that the clusters in $\mathcal{P}$ must be a subset of one of the vertex classes of D (this actually follows from (BST3) and the fact that D is bipartite).

By Proposition 7.8, a (bi)-setup remains a (bi)-setup (with slightly worse parameters) if only a few edges are removed and added at each vertex. A similar observation was already made (and proved) in [76, Lemma 9.2], so we omit the details here.

Proposition 8.16. *Let $0 < \frac{1}{m} \ll \frac{1}{k}, \varepsilon \leq \varepsilon' \ll d \ll \frac{1}{\ell'} \ll 1$ and $\varepsilon' \ll \frac{1}{\ell^*}$. Let D be a digraph and suppose that D' is obtained from D by removing and adding at most $\varepsilon' m$ inedges and at most $\varepsilon' m$ outedges incident to each vertex. If $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \ell^*, k, m, \varepsilon, d)$-(bi)-setup, then $(D', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \ell^*, k, m, (\varepsilon')^{\frac{1}{3}}, \frac{d}{2})$-(bi)-setup.*

Note that any partition is an ε-uniform 1-refinement of itself.

Fact 8.17. *Suppose that* $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *is an* $(\ell', \ell^*, 2k, m, \varepsilon, d)$*-(bi)-setup. Then,* $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}, R, C, U, U')$ *is an* $(\ell', 1, 2k, m, \varepsilon, d)$*-(bi)-setup.*

By definition, one can delete the exceptional vertices of a (bi)-setup.

Fact 8.18. *Let* $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *be an* $(\ell', \ell^*, 2k, m, \varepsilon, d)$*-(bi)-setup. Denote by* V_0 *the exceptional set contained in* $\mathcal{P}$*. Let* $\mathcal{P}_\emptyset, \mathcal{P}'_\emptyset$*, and* $\mathcal{P}^*_\emptyset$ *be obtained from* $\mathcal{P}, \mathcal{P}'$*, and* $\mathcal{P}^*$ *by replacing the exceptional set* V_0 *by the empty set. Then,* $(D - V_0, \mathcal{P}_\emptyset, \mathcal{P}'_\emptyset, \mathcal{P}^*_\emptyset, R, C, U, U')$ *is an* $(\ell', \ell^*, 2k, m, \varepsilon, d)$*-(bi)-setup.*

Finally, observe that if D forms a (bi)-setup, then the edges of D can be randomly partitioned to obtain, with high probability, two edge-disjoint digraphs which both form a (bi)-setup. Indeed, properties (ST1), (ST4), and (ST6) of a setup and properties (BST1), (BST4), and (BST6) of a bi-setup are automatically preserved. Moreover, Lemma 8.9 implies that properties (ST5) and (ST8) of a setup and properties (BST5) and (BST8) of a bi-setup hold with high probability. Finally, Lemma 7.17 implies that (super)regularity is preserved with high probability, as desired for properties (ST2), (ST3), and (ST7) of a setup and properties (BST2), (BST3), and (BST7) of a bi-setup.

Lemma 8.19. *Let* $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll \varepsilon' \ll d \ll \frac{1}{\ell'} \ll 1$ *and* $\varepsilon \ll \frac{1}{\ell^*}$*. Let* D *be a digraph and suppose that* D' *is obtained from* D *by selecting each edge independently with probability* $\frac{1}{2}$*. If* $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *is an* $(\ell', \ell^*, k, m, \varepsilon, d)$*-(bi)-setup, then, with high probability, both* $(D', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *and* $(D \setminus D', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *are* $(\ell', \ell^*, k, m, \varepsilon', \frac{d}{2})$*-(bi)-setups.*

8.2.5 Special path systems and special factors

Roughly speaking, special path systems can be viewed as blocks of prescribed edges for our Hamilton cycles; in the robust decomposition lemma (Lemma 8.23 below), each special path system will be extended to a distinct Hamilton cycle. Special path systems are then organised into special factors to provide a convenient way of finding them and incorporating them into Hamilton cycles in a balanced way.

Definition 8.20 (Canonical interval partition)**.** Let V be a vertex set and $\mathcal{P}$ be a partition of V into an exceptional set and k clusters. Suppose that $C = V_1 \dots V_k$ is a Hamilton cycle on the clusters in $\mathcal{P}$. Suppose that $\frac{k}{f} \in \mathbb{N}$. The *canonical interval partition of C into f intervals* is $\mathcal{I} = \{I_1, \dots, I_f\}$, where

$$I_i = V_{(i-1)\frac{k}{f}+1} V_{(i-1)\frac{k}{f}+2} \dots V_{i\frac{k}{f}+1}$$

for each $i \in [f]$. For each $i \in [f]$, the clusters $V_{(i-1)\frac{k}{f}+2}, V_{(i-1)\frac{k}{f}+3} \dots, V_{i\frac{k}{f}}$ are called the *internal clusters* of the interval I_i.

Definition 8.21 (Special path system)**.** Let V be a vertex set and $\mathcal{P}$ be a partition of V into an exceptional set V_0 and k clusters of size m. Suppose that $C = V_1 \dots V_k$ is a Hamilton cycle on the clusters in $\mathcal{P}$. Suppose that $\frac{k}{f} \in \mathbb{N}$. Suppose that $\mathcal{P}^*$ is an ℓ^*-refinement of $\mathcal{P}$. For each $i \in [k]$, let $V_{i,1}, \dots, V_{i,\ell^*}$ be an enumeration of the subclusters of V_i contained in $\mathcal{P}^*$. For any $(h, j) \in [\ell^*] \times [f]$, an *$(\ell^*, f, h, j)$-special path system SPS with respect to $\mathcal{P}^*$ and C* is a set of $\frac{m}{\ell^*}$ vertex-disjoint paths satisfying the following conditions.

(SPS1) $V^+(SPS) = V_{(j-1)\frac{k}{f}+1,h}$ and $V^-(SPS) = V_{j\frac{k}{f}+1,h}$.

(SPS2) $V^0(SPS) = V_{(j-1)\frac{k}{f}+2,h} \cup \dots \cup V_{j\frac{k}{f},h}$.

Roughly speaking, an (ℓ^*, f, h, j)-special path system with respect to $\mathcal{P}^*$ and C is a set of vertex-disjoint paths which lies along the "h^{th} refinement" of the j^{th} interval in the canonical interval partition of C into f intervals (see also Figure 8.1).

Definition 8.22 (Special factor)**.** Let $V, \mathcal{P}^*, C$, and V_0 be as in Definition 8.21. An *(ℓ^*, f)-special factor SF with respect to $\mathcal{P}^*$ and C* is a 1-regular digraph on $V \setminus V_0$ which has a decomposition $\{SPS_{h,j} \mid (h,j) \in [\ell^*] \times [f]\}$ where, for each $(h,j) \in [\ell^*] \times [f]$, $SPS_{h,j}$ induces an (ℓ^*, f, h, j)-special path system in D.

Observe that in the original definition of a special path system in [76], most of the edges belonged to a host digraph D and the other edges, called "fictive edges", had to satisfy some additional properties. Thanks to Fact 8.5, we can omit these conditions.

Indeed, suppose that we want to construct a Hamilton cycle which contains a special path system SPS, but SPS does not satisfy all the desired internal conditions. Then, we can temporarily consider a suitable equivalent special path system SPS' and construct a Hamilton cycle H containing SPS' instead of SPS since, by Fact 8.5, H induces a Hamilton cycle containing SPS, as desired.

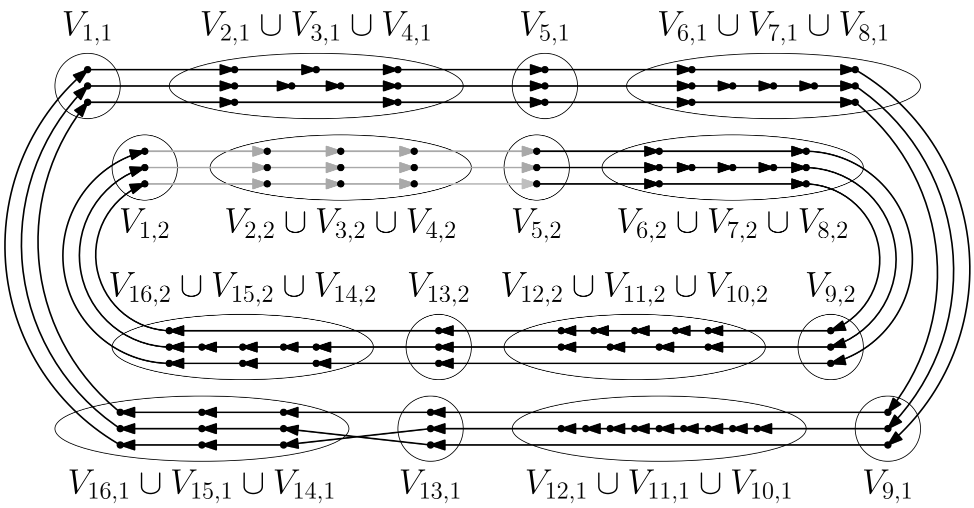


Figure 8.1: A $(2,4)$-special factor with respect to $\mathcal{P}^* = \{V_0, V_{1,1}, V_{1,2}, V_{2,1}, \dots, V_{16,2}\}$ and $C = V_1 \dots V_{16}$. The grey edges form a $(2,4,2,1)$-special path system with respect to $\mathcal{P}^*$ and C.

8.2.6 Statement of the robust decomposition lemma

We are now ready to state a modified version of the robust decomposition lemma of [76]. We discuss the differences from the original version after the statement. (A formal derivation of Lemma 8.23 is available in Appendix C.)

Observe that $\mathcal{SF}$ and $\mathcal{SF}'$ may have edges with common starting and ending points in Lemma 8.23. Indeed, recall our convention that the edges of a multidigraph are all distinct (see Section 5.2.1) and that, in particular, a decomposition of a multidigraph covers each edge according to its multiplicity (see Section 5.2.9). Thus, Lemma 8.23 simply states that each occurrence of an edge in each of $H, D^{\mathrm{rob}}, \mathcal{SF}$, and $\mathcal{SF}'$ is covered by precisely one of the Hamilton cycles in $\mathscr{C}$.

Lemma 8.23 (Modified robust decomposition lemma [76]). *Let* $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{q} \ll$

$\frac{1}{f} \ll \frac{r_1}{m} \ll d \ll \frac{1}{\ell'}, \frac{1}{g} \ll 1$ *and suppose that* $rk^2 \leq m$. *Let*

$$r_2 := 96\ell' g^2 kr, \quad r_3 := \frac{rfk}{q}, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s' := rfk + 7r^\diamond,$$

and suppose that $\frac{k}{14}, \frac{k}{f}, \frac{k}{g}, \frac{q}{f}, \frac{m}{4\ell'}, \frac{fm}{q}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$. *Suppose that* $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *is an* $(\ell', \frac{q}{f}, k, m, \varepsilon, d)$*-setup with empty exceptional set* V_0. *Let* $\mathcal{SF}$ *be a multidigraph which consists of the union of* r_3 $(\frac{q}{f}, f)$*-special factors with respect to* $\mathcal{P}^*$ *and* C *and let* $\mathcal{SF}'$ *be a multidigraph which consists of the union of* $r^\diamond$ $(1,7)$*-special factors with respect to* $\mathcal{P}$ *and* C.

Then, D *contains an* $(r_1+r_2+5r^\diamond)$*-regular spanning subdigraph* $D^{\rm rob}$ *for which the holds. For any* r*-regular digraph* H *on* $V(D)$ *which is edge-disjoint from* $D^{\rm rob}$*, the multidigraph* $H \cup D^{\rm rob} \cup \mathcal{SF} \cup \mathcal{SF}'$ *has a decomposition* $\mathscr{C}$ *into* s' *Hamilton cycles such that each cycle in* $\mathscr{C}$ *contains precisely one of the special path systems in the multidigraph* $\mathcal{SF} \cup \mathcal{SF}'$.

The analogue holds if $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *is an* $(\ell', \frac{q}{f}, k, m, \varepsilon, d)$*-bi-setup and* H *is an* r*-regular bipartite digraph on the same vertex classes as* D.

Lemma 8.23 differs from [76, Lemma 12.1] in four minor points. (i) As discussed in Section 8.2.5, our definition of special factor is more general: there is no restriction on how many edges can lie outside D. (ii) $\mathcal{SF}$ and $\mathcal{SF}'$ are multidigraphs rather than digraphs. (iii) The robustly decomposable digraph $D^{\rm rob}$ is now constructed in only one stage. In [76, Lemma 12.1], we first input a set of r_3 (edge-disjoint) $(\frac{q}{f}, f)$-special factors to obtain a subdigraph $CA^\diamond(r) \subseteq D$ and then, in a second stage, we input a set of $r^\diamond$ (edge-disjoint) $(1,7)$-special factors to obtain a second subdigraph $PCA^\diamond(r) \subseteq D$. In Lemma 8.23, these two stages are condensed into one: $D^{\rm rob}$ from Lemma 8.23 corresponds to $CA^\diamond(r) \cup PCA^\diamond(r)$ from [76, Lemma 12.1]. (iv) H only needs to be edge-disjoint from $D^{\rm rob}$.

Using the concept of equivalent sets of vertex-disjoint paths, modifications (i)–(iii) can be derived immediately from [76, Lemma 12.1]. Indeed, since each special path system goes into a different Hamilton cycle, Fact 8.5 implies that it is enough to apply [76, Lemma

12.1] with special path systems which are equivalent to those in the multidigraph $\mathcal{SF} \cup \mathcal{SF}'$. In both stages of the application of [76, Lemma 12.1], one can use the superregular pairs in D (which exist by (ST3) an (BST3)) to find suitable special path systems in D which are equivalent to those contained in the multidigraph $\mathcal{SF} \cup \mathcal{SF}'$ and edge-disjoint from each other (as well as from $CA^\diamond(r)$ in the second stage).

Modification (iv) cannot be derived immediately from the statement of [76, Lemma 12.1] but follows easily from its proof and Fact 8.5. More precisely, using the equivalent special path system approach described above, [76, Lemma 12.1] requires that H is edge-disjoint from D^{rob} as well as the auxiliary special path systems we used to apply [76, Lemma 12.1]. But, the proof of [76, Lemma 12.1] implies that the relevant absorbing properties come from $CA^\diamond(r)$ and $PCA^\diamond(r)$. Thus, if H is not edge-disjoint from the auxiliary special path systems used to apply [76, Lemma 12.1], then Fact 8.5 implies that we can simply replace these special path systems by equivalent ones which are edge-disjoint from H. (More details on how to obtain these modifications can be found in Appendix C.)

8.2.7 Incorporating the exceptional vertices

Recall that (Lemma 8.23) can only be applied with an empty exceptional set V_0. In general, V_0 will be non-empty and so we will have to apply Lemma 8.23 with $D - V_0$ playing the role of D. As a result, the cycles obtained via Lemma 8.23 will not be Hamilton cycles on $V(D)$, they will only span $V(D) \setminus V_0$. We will incorporate the exceptional vertices into these almost spanning cycles using the special path systems as follows. (Note that a special cover as defined below is a generalisation of an exceptional cover as defined in [76], while a complete special sequence as defined below is the analogue of a complete exceptional sequence as defined in [76].)

Definition 8.24 (Special cover)**.** Let D be a digraph and $V_0 \subseteq V(D)$ be an exceptional set. A *special cover in D (with respect to V_0)* is a linear forest $SC \subseteq D$ such that $V^0(SC) = V_0$.

Definition 8.25 (Complete special sequence)**.** Let D be a digraph and $V_0 \subseteq V(D)$ be an exceptional set. Suppose that SC is a special cover in D. Let $P_1, \dots, P_\ell$ be an enumeration of the components of SC which are not isolated vertices. For each $i \in [\ell]$, denote by u_i and v_i the starting and ending points of P_i. The *complete special sequence associated to* SC is the directed matching $M_{SC} \coloneqq \{u_i v_i \mid i \in [\ell]\}$.

Then, observe that the following holds.

Fact 8.26. *Let D be a digraph and $V_0 \subseteq V(D)$ be an exceptional set. Let SC be a special cover in D with respect to V_0 and denote by M_{SC} the complete special sequence associated to SC. Suppose that C is a spanning cycle on $V(D) \setminus V_0$ satisfying $M_{SC} \subseteq C \subseteq D \cup M_{SC}$. Then, $(C \setminus M_{SC}) \cup SC$ is a Hamilton cycle of D.*

Our strategy for incorporating the exceptional vertices into the cycles obtained via the robust decomposition lemma will thus be as follows. Before applying the robust decomposition lemma, we will reserve the edges from s' special covers in D with respect to V_0. Then, we will construct the special factors for Lemma 8.23 in such a way that each of the s' special path systems contains the complete special sequence associated to one of the reserved special covers. Since each cycle obtained via the robust decomposition lemma will contain precisely one of these special path systems, these cycles will contain precisely one of these complete special sequences. Using Fact 8.26 and the reserved special covers, we will thus be able to transform the cycles from the robust decomposition lemma into Hamilton cycles on $V(D)$. We discuss how to find these special sequences in Section 9.1.

8.3 The preprocessing step

Recall that in the robust decomposition lemma (Lemma 8.23), the exceptional set must be empty. In the proof of Theorem 4.1, the partition $\mathcal{P}$ in the bi-setup will be obtained via the regularity lemma (Lemma 7.14) and so we will have a non-empty exceptional set V_0. Thus, we will only be able to apply the (bipartite) robust decomposition lemma in $D - V_0$ rather than D. This means that the absorber $D^{\rm rob}$ will only be able to decompose leftovers on $V(D) \setminus V_0$ and so we need an additional absorber to cover the leftover edges incident to V_0. This absorber will be constructed by adapting the preprocessing step of [76] to the bipartite case. Roughly speaking, the preprocessing lemma (Lemma 8.30 below) says that given a bipartite digraph D and an exceptional set V_0, one can find a sparse absorber $PG \subseteq D$ (called a preprocessing graph in [76]) such that for any very sparse leftover H which is edge-disjoint from PG, the digraph $H \cup PG$ contains edge-disjoint Hamilton cycles which cover all the edges incident to V_0.

8.3.1 Consistent bi-systems

First, we need a bipartite analogue of a consistent system defined in [76]. This is the key structure required to construct the absorber in the preprocessing lemma. It is similar to a bi-setup.

Definition 8.27 (Consistent bi-system)**.** We say that $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a *consistent $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$-bi-system* if the following properties are satisfied.

(CBSys1) D, R_0, and R are balanced bipartite digraphs on vertex classes A and B, $\mathcal{A}_0$ and $\mathcal{B}_0$, and $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover, D, R_0, and R are bipartite robust (ν, τ)-outexpanders with bipartitions (A, B), $(\mathcal{A}_0, \mathcal{B}_0)$, and $(\mathcal{A}, \mathcal{B})$, respectively. Furthermore, $\delta^0(D) \geq \delta|D|$, $\delta^0(R_0) \geq \delta|R_0|$, and $\delta^0(R) \geq \delta|R|$.

(CBSys2) $\mathcal{P}_0$ is a partition of $V(D)$ into an exceptional set V_0 which satisfies $|V_0 \cap A| = |V_0 \cap B| \leq \varepsilon|A| = \varepsilon|B|$, and $\frac{k}{\ell^*}$ clusters of size $m\ell^*$. The vertex set of R_0 consists of these clusters. (Thus, $|R_0| = \frac{k}{\ell^*}$.)

(CBSys3) $\mathcal{P}$ is an ℓ^*-refinement of $\mathcal{P}_0$ (and so the clusters in $\mathcal{P}$ have size m). The vertex set of R consists of the clusters in $\mathcal{P}$. (Thus, $|R| = k$.)

(CBSys4) For each $VW \in E(R)$, the corresponding pair $D[V, W]$ is $(\varepsilon, \geq d)$-regular.

(CBSys5) C_0 is a Hamilton cycle in R_0 and C is a Hamilton cycle in R. For each $VW \in E(C)$, the corresponding pair $D[V, W]$ is $[\varepsilon, \geq d]$-superregular.

(CBSys6) Suppose that $W, W' \in V(R_0)$ and $V, V' \in E(R)$ satisfy $V \subseteq W$ and $V' \subseteq W'$. Then, $WW' \in E(R_0)$ if and only if $VV' \in E(R)$.

(CBSys7) C can be viewed as obtained by winding ℓ^* times around C_0, i.e. for every edge $WW' \in E(C_0)$, there are precisely ℓ^* edges $VV' \in E(C)$ such that $V \subseteq W$ and $V' \subseteq W'$.

(CBSys8) Let V be a cluster in $\mathcal{P}_0$ and $W \subseteq V$ be a cluster in $\mathcal{P}$. Let $v \in V(D)$ and $\diamond \in \{+,-\}$. If $|N_D^\diamond(v) \cap V| \geq \tau|V|$, then $|N_D^\diamond(v) \cap W| \geq \frac{\theta|N_D^\diamond(v)\cap V|}{\ell^*}$.

Observe that the original definition of a consistent bi-system in [76] also required the exceptional set V_0 to be an independent set. However, we will see that this condition is not necessary for our purposes and so we omit it here for convenience.

By Lemma 7.3 and Proposition 7.8, a consistent bi-system remains a consistent bi-system (with slightly worse parameters) if only a few edges are removed at each vertex. An analogous observation was made (and proved) in [76, Lemma 7.1], so we omit the details here.

Proposition 8.28. *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \leq \varepsilon' \ll d \ll \nu \ll \tau \ll \delta, \theta \leq 1$. Let D be a digraph and suppose that D' is obtained from D by removing at most $\varepsilon' m$ inedges and at most $\varepsilon' m$ outedges incident to each vertex. If $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a consistent $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$-bi-system, then $(D', \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is still a consistent $(\ell^*, k, m, 3\sqrt{\varepsilon'}, \frac{d}{2}, \frac{\nu}{2}, \tau, \frac{\delta}{2}, \frac{\theta}{2})$-bi-system.*

Finally, we observe that if D forms a consistent bi-system, then the edges of D can be randomly partitioned to obtain, with high probability, two edge-disjoint digraphs which both form a consistent bi-system. Indeed, (CBSys2), (CBSys3), (CBSys6), and (CBSys7) are automatically preserved and, by Lemma 7.19, bipartite robust outexpansion is preserved with high probability, as desired for (CBSys1). Moreover, a simple application of Lemma 7.16 can guarantee suitable minimum degree conditions for (CBSys1) and (CBSys8). Finally, Lemma 7.17 implies that (super)regularity is preserved with high probability, as desired for (CBSys4) and (CBSys5).

Lemma 8.29. *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll \varepsilon' \ll d \ll \nu \ll \tau \ll \delta, \theta \leq 1$. Suppose that $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a consistent $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$-bi-system. Let D' be* *obtained from D by selecting each edge independently with probability $\frac{1}{2}$. Then, with high probability, both $(D', \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ and $(D \setminus D', \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ are consistent $(\ell^*, k, m, \varepsilon', \frac{d}{2}, \frac{\nu}{4}, \tau, \frac{\delta}{3}, \frac{\theta}{3})$-bi-systems.*

8.3.2 Statement of the preprocessing lemma for bipartite digraphs

The following lemma is a bipartite analogue of [76, Corollary 8.5 and Lemma 8.6]. Since it can be proved using very similar arguments as those of [76], we omit its proof here. (A detailed explanation on how to derive Lemma 8.30 can be found in Appendix D.) As mentioned at the beginning of Section 8.3, Lemma 8.30 states that a consistent bi-system contains a sparse absorber PG which can cover all the exceptional edges of a very sparse leftover H with edge-disjoint Hamilton cycles.

Lemma 8.30 (Preprocessing lemma for bipartite digraphs [76]). *Let $0 < \frac{1}{m} \ll \frac{r}{m} \ll \frac{r'}{m} \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{\ell^*} \ll d \ll \nu \ll \tau \ll \delta, \theta \leq 1$. Denote $s \coloneqq \frac{10^7}{\nu^2}$ and suppose that $\frac{m}{50}, \frac{50\ell^*}{s-1} \in \mathbb{N}$. Let $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ be a consistent $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$-bi-system. Then, there exists a spanning subdigraph $PG \subseteq D$ such that the following hold.*

(i) *Each $v \in V_0$ satisfies $d^{\pm}_{PG}(v) = r(s-1)$ and each $w \in V(D) \setminus V_0$ satisfies $d^{\pm}_{PG}(w) = r'$.*

(ii) *Let H be an r-regular bipartite digraph on the same vertex classes as D. If H is edge-disjoint from PG and satisfies $e(H[V_0]) = 0$, then $H \cup PG$ contains rs edge-disjoint Hamilton cycles $C_1, \dots, C_{rs}$ such that the following hold.*

   (a) *$H \subseteq \bigcup_{i \in [rs]} C_i \subseteq H \cup PG$.*

   (b) *Let $PG' \coloneqq PG \setminus \bigcup_{i \in [rs]} C_i$. Then, each $v \in V_0$ satisfies $d^{\pm}_{PG'}(v) = 0$ and each $w \in V(D) \setminus V_0$ satisfies $d^{\pm}_{PG'}(w) = r' - r(s-1)$.*

Note that in [76, Corollary 8.5], H must be a spanning subdigraph of the host digraph D. However, this condition is not necessary since, as H is very sparse, its edges could be added to the host digraph without affecting the parameters of the consistent system significantly. This is why, in our bipartite version of the preprocessing lemma (Lemma 8.30), we may omit the condition that H is a spanning subdigraph of D. Moreover, [76, Corollary 8.5] requires the exceptional set V_0 to form an independent set in D. But, Proposition 8.28 implies that $D \setminus D[V_0]$ also contains a consistent bi-system (with slightly worse parameters). Thus, this condition can be omitted in Lemma 8.30.

Since PG' is a regular digraph on $V(D) \setminus V_0$, we can use $D^{\rm rob}$ from the robust decomposition lemma (Lemma 8.23) to decompose it into Hamilton cycles on $V(D) \setminus V_0$. (The vertices in V_0 will later be incorporated into these cycles via the special path systems as discussed in Section 8.2.7.) Thus, we can combine PG and $D^{\rm rob}$ to form an absorber $D^{\rm abs}$ which can decompose a sparse leftover digraph H which have edges incident to V_0. This is made precise in the following corollary.

Corollary 8.31. *Let* $0 < \frac{1}{m} \ll \frac{r}{m} \ll \frac{r'}{m} \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{\ell^*}, \frac{1}{f} \ll \frac{r_1}{m} \ll d \ll \nu, \frac{1}{\ell'}, \frac{1}{g} \ll \tau \ll \delta, \theta \leq 1$. *Let*

$$s := \frac{10^7}{\nu^2}, \quad r^* := r' - (s-1)r,$$

$$r_2 := 96\ell' g^2 k r^*, \quad r_3 := \frac{r^* f k}{q}, \quad r^\diamond := r_1 + r_2 + r^* - (q-1)r_3, \quad s' := r^* f k + 7r^\diamond.$$

Suppose that $\frac{k}{14}, \frac{k}{f}, \frac{k}{g}, \frac{q}{f}, \frac{m}{50}, \frac{m}{4\ell'}, \frac{fm}{q}, \frac{2fk}{3g(g-1)}, \frac{50\ell^*}{s-1} \in \mathbb{N}$. *Suppose that* $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ *is a consistent* $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$*-bi-system and suppose that* $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *is an* $(\ell', \frac{q}{f}, k, m, \varepsilon, d)$*-bi-setup. Suppose that the exceptional set* V_0 *forms an independent set in* D*. Let* $\mathcal{SF}$ *be a multidigraph which consists of the union of* r_3 $(\frac{q}{f}, f)$*-special factors with respect to* $\mathcal{P}^*$ *and* C *and let* $\mathcal{SF}'$ *be a multidigraph which consists of the union of* $r^\diamond$ $(1,7)$*-special factors with respect to* $\mathcal{P}$ *and* C*. Then,* D *contains a spanning subdigraph* $D^{\rm abs}$ *for which the following hold.*

(i) *Each* $v \in V_0$ *satisfies* $d^\pm_{D^{\rm abs}}(v) = r(s-1)$ *and each* $w \in V(D) \setminus V_0$ *satisfies* $d^\pm_{D^{\rm abs}}(w) = r' + r_1 + r_2 + 5r^\diamond$.

(ii) *Let* H *be an* r*-regular bipartite digraph on the same vertex classes as* D*. If*

H is edge-disjoint from D^{abs} and satisfies $e(H[V_0]) = 0$, then the multidigraph $H \cup D^{\mathrm{abs}} \cup \mathcal{SF} \cup \mathcal{SF}'$ has a decomposition $\mathscr{C} \cup \mathscr{C}'$ into $rs + s'$ edge-disjoint cycles satisfying the following properties.

(a) *$\mathscr{C} \subseteq H \cup D^{\mathrm{abs}}$ and consists of rs edge-disjoint Hamilton cycles on $V(D)$.*

(b) *$\mathscr{C}'$ consist of s' edge-disjoint Hamilton cycles on $V(D) \setminus V_0$ such that each cycle in $\mathscr{C}'$ contains precisely one of the special path systems in the multidigraph $\mathcal{SF} \cup \mathcal{SF}'$.*

Proof. First, let PG be the spanning subdigraph of D obtained by applying Lemma 8.30. Define $D' := D \setminus PG$ and let $\mathcal{P}_\emptyset, \mathcal{P}'_\emptyset$, and $\mathcal{P}^*_\emptyset$ be obtained by replacing the exceptional set V_0 by the empty set in $\mathcal{P}, \mathcal{P}'$, and $\mathcal{P}^*$, respectively. By Lemma 8.30(i), Proposition 8.16, and Fact 8.18, $(D' - V_0, \mathcal{P}_\emptyset, \mathcal{P}'_\emptyset, \mathcal{P}^*_\emptyset, R, C, U, U')$ is an $(\ell', \frac{q}{f}, k, m, \varepsilon^{\frac{1}{3}}, \frac{d}{2})$-bi-setup with empty exceptional set. Let D^{rob} be the spanning subdigraph of $D' - V_0$ obtained by applying Lemma 8.23 with $D' - V_0, \mathcal{P}_\emptyset, \mathcal{P}'_\emptyset, \mathcal{P}^*_\emptyset, \varepsilon^{\frac{1}{3}}, \frac{d}{2}$, and r^* playing the roles of $D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \varepsilon, d$, and r.

Define $D^{\mathrm{abs}} := PG \cup D^{\mathrm{rob}}$. Then, (i) follows from Lemma 8.30(i) and Lemma 8.23. For (ii), let H be an r-regular bipartite digraph on the same vertex classes as D. Suppose that H is edge-disjoint from D^{abs} and satisfies $e(H[V_0]) = 0$. Let $\mathscr{C}$ be the set of rs Hamilton cycles obtained by applying Lemma 8.30(ii). In particular, (ii.a) holds and Lemma 8.30(ii.a) implies that $E(H) \subseteq E(\mathscr{C}) \subseteq E(H) \cup E(PG)$. Let $PG' := PG \setminus \mathscr{C}$. Then, Lemma 8.30(ii.b) implies that $E(PG') = E(PG' - V_0)$ and $PG' - V_0$ is r^*-regular. Let $\mathscr{C}'$ be the set Hamilton cycles on $V(D) \setminus V_0$ obtained by applying Lemma 8.23 with PG' and r^* playing the roles of H and r. Then, (ii.b) holds and we are done. $\square$

CHAPTER 9

THE BIPARTITE ROBUST OUTEXPANDER CASE: PROOF OF THEOREM 4.1

In this section, we combine the approximate decomposition (Corollary 8.2) and the absorption step (Corollary 8.31) to derive Theorem 4.1.

9.1 Applying the robust decomposition lemma in a bipartite robust outexpander

In order to apply Corollary 8.31, we will need to find a consistent bi-system, a bi-setup, and special factors. In this section, we discuss how these can be found in a bipartite robust outexpander.

First, we explain how to form the special factors required for Corollary 8.31. As discussed in Section 8.2.7, their role is to incorporate the exceptional vertices into the cycles obtained via the robust decomposition lemma. By Fact 8.26, we would like each special path system to consist of a complete special sequence (recall Definition 8.25) and edges of D. (Note that such special path systems were called complete exceptional path systems in [76].) One can achieve this by adapting the arguments of [76, Lemma 7.6] to the bipartite case (see Appendix D for more details).

Lemma 9.1 (Constructing special covers and special factors)**.** *Let* $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll d \ll \nu \ll \tau \ll \delta, \theta \leq 1$ *and* $\varepsilon \ll \frac{1}{\ell'}, \frac{1}{f}$ *and* $\frac{\ell' r}{m} \ll d$*. Suppose that* $\frac{\ell^*}{f}, \frac{m}{\ell'} \in \mathbb{N}$ *and* $\frac{f}{\ell^*} \ll 1$*. Let*

$(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ *be a consistent* $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$*-bi-system with exceptional set* V_0*. Let* $\mathcal{P}'$ *be an* ε*-uniform* ℓ'*-refinement of* $\mathcal{P}$*. Then, there exist*

(i) *a set* $\mathcal{SC} = \{SC_{i,h,j} \mid (i, h, j) \in [r] \times [\ell'] \times [f]\}$ *of* $r\ell' f$ *edge-disjoint special covers in* D *with respect to* V_0 *and*

(ii) r (ℓ', f)*-special factors* $SF_1, \ldots, SF_r$ *with respect to* $\mathcal{P}'$ *and* C

such that the following hold, where for each $(i, h, j) \in [r] \times [\ell'] \times [f]$*,* $M_{i,h,j}$ *denotes the complete special sequence associated to* $SC_{i,h,j}$ *and* $SPS_{i,h,j}$ *denotes the* (ℓ', f, h, j)*-special path system contained in* SF_i*.*

(iii) *For each* $(i, h, j) \in [r] \times [\ell'] \times [f]$*, we have* $M_{i,h,j} \subseteq SPS_{i,h,j} \subseteq (D \setminus \mathcal{SC}) \cup M_{i,h,j}$*.*

(iv) *Let* $(i, h, j), (i', h', j') \in [r] \times [\ell'] \times [f]$ *be distinct. Then, we have* $(SPS_{i,h,j} \setminus M_{i,h,j}) \cap (SPS_{i',h',j'} \setminus M_{i',h',j'}) = \emptyset$*.*

Roughly speaking, Lemma 9.1(iii) means that each complete special sequence is incorporated into a distinct special path system, while Lemma 9.1(iv) states that each edge of $D \setminus \mathcal{SC}$ is incorporated into at most one of the special path systems. Note that in [76, Lemma 7.6], the exceptional set V_0 must form an independent set in D. But, Proposition 8.28 implies that $D \setminus D[V_0]$ also contains a consistent bi-system (with slightly worse parameters). Thus, this condition can be omitted in Lemma 9.1.

Consistent bi-systems and bi-setups can be constructed from Szemerédi's regularity lemma (Lemma 7.14) as follows. Recall that, by Lemma 7.15, the reduced digraph R_0 obtained by applying the regularity lemma (Lemma 7.14) to a bipartite robust outexpander D is also a bipartite robust outexpander. By Lemma 7.7, this property is also preserved when taking refinements. Thus, one can obtain a consistent bi-system by applying the regularity lemma to obtain a partition $\mathcal{P}_0$ of $V(D)$, then selecting a uniform refinement $\mathcal{P}$ of $\mathcal{P}_0$ using Lemma 8.7, and finally using Corollary 7.5 to find the desired Hamilton cycles C_0 and C.

The additional refinements $\mathcal{P}'$ and $\mathcal{P}^*$ of $\mathcal{P}$ required to form a bi-setup can also be obtained by applying Lemma 8.7 (the superregular pairs required for (BST7) can be obtained using Lemma 8.8). The next lemma gives the bi-universal walk required for (BST4). (Lemma 9.2 is easily proved by adapting the arguments of [76, Lemma 9.1] to the bipartite case, so we defer its proof to Appendix D.)

Lemma 9.2. *Let $0 < \frac{1}{k} \ll \nu \ll \tau \ll \delta < 1$ and let ℓ' be an even integer satisfying $\ell' \geq 36\nu^{-2}$. Suppose that R a balanced bipartite robust (ν, τ)-outexpander with bipartition $(\mathcal{A}, \mathcal{B})$, where $|\mathcal{A}| = |\mathcal{B}| = k$. Suppose that $\delta^0(R) \geq \delta k$. Let C be a Hamilton cycle in R. Then, there exists a bi-universal walk U for C with parameter ℓ'.*

Altogether, we obtain a consistent bi-system and a bi-setup (as defined in Definitions 8.15 and 8.27). Then, we apply Lemmas 8.19 and 8.29 to partition the edges of D into two edge-disjoint subdigraphs D_1 and D_2 which each form a consistent bi-system and a bi-setup. We will use D_1 to construct the special factors (Lemma 9.1) and D_2 to construct the absorber (Corollary 8.31). This is necessary because, after constructing all the special factors, the clusters in $\mathcal{P}$ and $\mathcal{P}'$ may no longer form suitable (super)regular pairs (that is, properties (CBSys4) and (CBSys5) of a consistent bi-system and properties (BST2), (BST3), and (BST7) of a bi-setup might not hold anymore), so we cannot apply Corollary 8.31 directly after Lemma 9.1.

Lemma 9.3. *Let $0 < \frac{1}{M'} \ll \varepsilon$. Then, there exist $M'', n_0 \in \mathbb{N}$ such that the following holds. Suppose that $\varepsilon \ll \frac{1}{q} \ll \frac{1}{f}, \frac{1}{\ell^*} \ll d \ll \nu \ll \tau \ll \delta, \theta \ll 1$ and $d \ll \frac{1}{g} \ll 1$. Moreover, let $\ell' \geq 324\nu^{-2}$ be even. Let D be a balanced bipartite digraph on vertex classes A and B of size $n \geq n_0$. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B) and that $\delta^0(D) \geq \delta n$. Then, there exist $m, k \in \mathbb{N}$ and edge-disjoint $D_1, D_2 \subseteq D$ such that the following conditions are satisfied.*

(i) *$M' \leq k \leq M''$ and $\frac{k}{7}, \frac{k}{f}, \frac{k}{g}, \frac{m}{50}, \frac{m}{4\ell'}, \frac{fm}{q}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$.*

(ii) *There exist $\mathcal{P}_0, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R_0, R, C_0, C, U$, and U' which satisfy the following conditions for each $i \in [2]$.*

- *$(D_i, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a consistent $(\ell^*, 2k, m, \varepsilon, d, \nu^4, 8\tau, \frac{\delta}{9}, \theta)$-bi-system.*

– $(D_i, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ *is an* $(\ell', \frac{q}{f}, 2k, m, \varepsilon, d)$*-bi-setup.*

Note that in order to apply Corollary 8.31, we need a constant g which divides k. We also require that this constant g is sufficiently large but sufficiently small compared to k and f. We would not be able to fix such a parameter if, for instance, f is a prime number and $k = 7f$. It is therefore necessary to introduce g in Lemma 9.3 even though it does not explicitly appear in Lemma 9.3(ii).

For a formal proof of Lemma 9.3, see Appendix E.

9.2 Proof of Theorem 4.1

We are now ready to derive Theorem 4.1. Our strategy is as follows. In Step 1, we construct consistent bi-systems and bi-setups using Lemma 9.3. In Step 2, we construct an absorber $D^{\rm abs}$ using Corollary 8.31 (the required special factors are constructed with Lemma 9.1). In Step 3, we approximately decompose D using Corollary 8.2. In Step 4, we decompose the leftover using $D^{\rm abs}$.

Proof of Theorem 4.1. We may assume without loss of generality that $\delta \ll 1$. Let $0 < \frac{1}{n_0} \ll \tau \ll \delta$ and $\frac{1}{n_0} \ll \nu$. By Fact 7.2, we may assume that $\nu \ll \tau$. Let $n \geq n_0$ and $r \geq \delta n$. Let D be an r-regular bipartite digraph on vertex classes A and B of size n. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B). Let $s := 10^7\nu^{-8}$. Fix additional parameters such that

$$0 < \frac{1}{n} \ll \frac{1}{M''} \ll \frac{1}{M'} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll d \ll \nu, \frac{1}{g} \ll \tau \ll \delta, \theta \ll 1$$

and $\frac{q}{f}, \frac{f^2}{7}, \frac{50f^2}{s-1} \in \mathbb{N}$. Let ℓ' be the smallest even integer satisfying $\ell' \geq 324\nu^{-2}$. Denote $\ell^* := f^2$ and observe that $\frac{\ell^*}{7}, \frac{50\ell^*}{s-1} \in \mathbb{N}$ and $\frac{f}{\ell^*} = \frac{1}{f} \ll 1$.

Step 1: Constructing consistent bi-systems and bi-setups. Apply Lemma 9.3 to obtain $m, k, D_1, D_2, \mathcal{P}_0, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R_0, R, C_0, C, U$, and U' such that the following hold for each $i \in [2]$.

(i) $M' \leq k \leq M''$ and $\frac{k}{7}, \frac{k}{f}, \frac{k}{g}, \frac{m}{50}, \frac{m}{4\ell'}, \frac{fm}{q}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$.

(ii) $(D_i, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a consistent $(\ell^*, 2k, m, \varepsilon, d, \nu^4, 8\tau, \frac{\delta}{9}, \theta)$-bi-system.

(iii) $(D_i, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \frac{q}{f}, 2k, m, \varepsilon, d)$-bi-setup.

Fix additional constants ε', r_0, r_0', and r_1 such that $\frac{1}{n} \ll \varepsilon' \ll \frac{r_0}{m} \ll \frac{r_0'}{m} \ll \frac{1}{M''}$ and $\frac{1}{f} \ll \frac{r_1}{m} \ll d$. Let

$$r^* := r_0' - (s-1)r_0, \quad r_2 := 192\ell' g^2 k r^*, \quad r_3 := \frac{2r^* fk}{q},$$

$$r^\diamond := r_1 + r_2 + r^* - (q-1)r_3, \quad s' := 2r^* fk + 7r^\diamond.$$

Let V_0 denote the exceptional set in $\mathcal{P}$.

Step 2: Constructing the absorber. We will use Corollary 8.31. First, we construct the required special factors in D_1.

Note that (iii) and (BST8) imply that $\mathcal{P}^*$ is an ε-uniform $\frac{q}{f}$-refinement of $\mathcal{P}$ with respect to D_1. Apply Lemma 9.1 with $D_1, r_3, 2k, \nu^4, 8\tau, \frac{\delta}{9}, \mathcal{P}^*$, and $\frac{q}{f}$ playing the roles of $D, r, k, \nu, \tau, \delta, \mathcal{P}'$, and ℓ' to obtain

(a) a set $\mathcal{SC} = \{SC_{i,h,j} \mid (i,h,j) \in [r_3] \times [\frac{q}{f}] \times [f]\}$ of qr_3 edge-disjoint special covers in $D_1 \subseteq D$ with respect to V_0 and

(b) r_3 $(\frac{q}{f}, f)$-special factors $SF_1, \dots, SF_{r_3}$ with respect to $\mathcal{P}^*$ and C

for which the following hold, where for each $(i,h,j) \in [r_3] \times [\frac{q}{f}] \times [f]$, $M_{i,h,j}$ denotes the complete special sequence associated to $SC_{i,h,j}$ and $SPS_{i,h,j}$ denotes the $(\frac{q}{f}, f, h, j)$-special path system contained in SF_i.

(c) For each $(i,h,j) \in [r_3] \times [\frac{q}{f}] \times [f]$, we have $M_{i,h,j} \subseteq SPS_{i,h,j} \subseteq (D_1 \setminus \mathcal{SC}) \cup M_{i,h,j}$.

(d) Let $(i,h,j), (i',h',j') \in [r_3] \times [\frac{q}{f}] \times [f]$ be distinct. Then, we have $(SPS_{i,h,j} \setminus M_{i,h,j}) \cap (SPS_{i',h',j'} \setminus M_{i',h',j'}) = \emptyset$.

Define a multiset $\mathcal{M}$ by $\mathcal{M} := \{M_{i,h,j} \mid (i,h,j) \in [r_3] \times [\frac{q}{f}] \times [f]\}$ and a multidigraph $\mathcal{SF}$ by $\mathcal{SF} := SF_1 \cup \cdots \cup SF_{r_3}$. Let $D_1' := D_1 \setminus (\mathcal{SC} \cup \mathcal{SF})$. Observe that $\mathcal{SC}$ consists of $r_3 q$ linear forests and $\mathcal{SF}$ consists of r_3 digraphs of maximum semidegree 1. Thus, Proposition 8.28 and (ii) imply that $(D_1', \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a consistent $(\ell^*, 2k, m, 3\sqrt{\varepsilon}, \frac{d}{2}, \frac{\nu^4}{2}, 8\tau, \frac{\delta}{18}, \frac{\theta}{2})$-bi-system. Note that $\mathcal{P}$ is a $3\sqrt{\varepsilon}$-uniform refinement of itself with respect to D_1'. Apply Lemma 9.1 with $D_1', r^\diamond, 2k, 3\sqrt{\varepsilon}, \frac{d}{2}, \frac{\nu^4}{2}, 8\tau, \frac{\delta}{18}, \frac{\theta}{2}, \mathcal{P}, 1$, and 7 playing the roles of $D, r, k, \varepsilon, d, \nu, \tau, \delta, \theta, \mathcal{P}', \ell'$, and f to obtain

(a′) a set $\mathcal{SC}' = \{SC'_{i,h,j} \mid (i,h,j) \in [r^\diamond] \times [1] \times [7]\}$ of $7r^\diamond$ edge-disjoint special covers in $D_1' \subseteq D$ with respect to V_0 and

(b′) $r^\diamond$ $(1,7)$-special factors $SF_1', \dots, SF_{r^\diamond}'$ with respect to $\mathcal{P}$ and C

for which the following hold, where for each $(i,h,j) \in [r^\diamond] \times [1] \times [7]$, $M'_{i,h,j}$ denotes the complete special sequence associated to $SC'_{i,h,j}$ and $SPS'_{i,h,j}$ denotes the $(1,7,h,j)$-special path system contained in SF_i'.

(c′) For each $(i,h,j) \in [r^\diamond] \times [1] \times [7]$, we have $M'_{i,h,j} \subseteq SPS'_{i,h,j} \subseteq (D_1' \setminus \mathcal{SC}') \cup M'_{i,h,j}$.

(d′) Let $(i,h,j), (i',h',j') \in [r^\diamond] \times [1] \times [7]$ be distinct. Then, we have $(SPS'_{i,h,j} \setminus M'_{i,h,j}) \cap (SPS'_{i',h',j'} \setminus M'_{i',h',j'}) = \emptyset$.

Define a multiset $\mathcal{M}'$ by $\mathcal{M}' := \{M'_{i,h,j} \mid (i,h,j) \in [r^\diamond] \times [1] \times [7]\}$ and a multidigraph $\mathcal{SF}'$ by $\mathcal{SF}' := SF_1' \cup \cdots \cup SF_{r^\diamond}'$.

By (ii) and (iii), we can let D^{abs} be the absorber obtained by applying Corollary 8.31 with $D_2, r_0, r_0', 2k, \nu^4, 8\tau$, and $\frac{\delta}{9}$ playing the roles of D, r, r', k, ν, τ, and δ.

Step 3: Approximate decomposition. In this step, we approximately decompose $D' := D \setminus (\mathcal{SC} \cup \mathcal{SC}' \cup \mathcal{SF} \cup \mathcal{SF}' \cup D^{\mathrm{abs}})$ into Hamilton cycles, with a sparse leftover. Let $r' := r - r_0(s-1) - s'$.

Claim 1. *D' is an r'-regular bipartite robust $(\frac{\nu}{2}, \tau)$-outexpander with bipartition (A, B).*

Proof of Claim. By Lemma 7.3, it is enough to show that D' is r'-regular. By Corollary 8.31(i), each $v \in V(D)$ satisfies

$$d^{\pm}_{D^{\mathrm{abs}}}(v) = \begin{cases} r_0(s-1) & \text{if } v \in V_0; \\ r_0' + r_1 + r_2 + 5r^{\diamond} & \text{otherwise.} \end{cases}$$

By (a) and (a′), $\mathcal{SC}$ and $\mathcal{SC}'$ are edge-disjoint sets of edge-disjoint special covers in D with respect to V_0 and so Definitions 8.24 and 8.25 imply that each $v \in V(D)$ satisfies

$$d^{\pm}_{\mathcal{SC}\cup\mathcal{SC}'}(v) = d^{\pm}_{\mathcal{SC}}(v) + d^{\pm}_{\mathcal{SC}'}(v) = \begin{cases} qr_3 + 7r^{\diamond} & \text{if } v \in V_0; \\ d^{\pm}_{\mathcal{M}}(v) + d^{\pm}_{\mathcal{M}'}(v) & \text{otherwise.} \end{cases}$$

By Definition 8.22, (b), and (b′), each $v \in V(D)$ satisfies

$$d^{\pm}_{\mathcal{SF}}(v) + d^{\pm}_{\mathcal{SF}'}(v) = \begin{cases} 0 & \text{if } v \in V_0; \\ r_3 + r^{\diamond} & \text{otherwise.} \end{cases}$$

By construction, D^{abs} and $\mathcal{SC}\cup\mathcal{SC}'$ are edge-disjoint. By (c), (d), (c′), and (d′), $\mathcal{SF}\cap D$ and $\mathcal{SF}'\cap D$ are edge-disjoint digraphs (rather than multidigraphs) and are both subdigraphs of $D \setminus (\mathcal{SC} \cup \mathcal{SC}' \cup D^{\mathrm{abs}})$. Thus, each $v \in V_0$ satisfies

$$\begin{aligned} d^{\pm}_{D'}(v) &= d^{\pm}_{D}(v) - d^{\pm}_{D^{\mathrm{abs}}}(v) - d^{\pm}_{\mathcal{SC}\cup\mathcal{SC}'}(v) - d^{\pm}_{\mathcal{SF}\cap D}(v) - d^{\pm}_{\mathcal{SF}'\cap D}(v) \\ &= r - r_0(s-1) - (qr_3 + 7r^{\diamond}) - 0 - 0 = r'. \end{aligned}$$

Moreover, (c) and (c′) imply that each $v \in V(D) \setminus V_0$ satisfies

$$\begin{aligned} d^{\pm}_{D'}(v) &= d^{\pm}_{D}(v) - d^{\pm}_{D^{\mathrm{abs}}}(v) - d^{\pm}_{\mathcal{SC}\cup\mathcal{SC}'}(v) - d^{\pm}_{\mathcal{SF}\cap D}(v) - d^{\pm}_{\mathcal{SF}'\cap D}(v) \\ &= r - d^{\pm}_{D^{\mathrm{abs}}}(v) - d^{\pm}_{\mathcal{SC}\cup\mathcal{SC}'}(v) - (d^{\pm}_{\mathcal{SF}}(v) - d^{\pm}_{\mathcal{M}}(v)) - (d^{\pm}_{\mathcal{SF}'}(v) - d^{\pm}_{\mathcal{M}'}(v)) \\ &= r - (r_0' + r_1 + r_2 + 5r^{\diamond}) - (r_3 + r^{\diamond}) = r'. \end{aligned}$$

Thus, D' is r'-regular, as desired. $\diamond$

Note that by Corollary 8.31(ii), D^{abs} cannot absorb any edges which entirely lie in V_0. Thus, we start by covering the edges of $D'[V_0]$ with a small number Hamilton cycles as follows. By (iii) and (BST1),

$$\ell_0 := |V_0 \cap A| = |V_0 \cap B| \leq \varepsilon n \leq r' - 2\nu n.$$

Apply König's theorem to decompose $D'[V_0]$ into ℓ_0 edge-disjoint matchings $M_1, \dots, M_{\ell_0}$ and observe that Claim 1 and Lemma 7.3 imply that $D' \setminus \bigcup_{i \in [\ell_0]} M_i$ is an $(\frac{r'}{2n}, \varepsilon)$-almost regular bipartite robust $(\frac{\nu}{3}, \tau)$-outexpander with bipartition (A, B). Note that Corollary 8.2(i) and (ii) hold with $M_1, \dots, M_{\ell_0}$ playing the roles of $F_1, \dots, F_\ell$. Apply Corollary 8.2 with $D' \setminus \bigcup_{i \in [\ell_0]} M_i, \ell_0, \frac{r'}{2n}, \nu, \frac{\nu}{3}$, and $M_1, \dots, M_{\ell_0}$ playing the roles of $D, \ell, \delta, \eta, \nu$, and $F_1, \dots, F_\ell$ to obtain a set $\mathscr{C}_0$ of ℓ_0 edge-disjoint Hamilton cycles of D' such that $D'[V_0] \subseteq E(\mathscr{C}_0)$ and $D'' := D' \setminus E(\mathscr{C}_0)$ is a bipartite robust $(\frac{\nu}{6}, \tau)$-outexpander with bipartition (A, B). Note that D'' is regular of degree $r'' := r' - \ell_0$.

We now approximately decompose D'' as follows. For each $i \in [r'' - r_0]$, let F_i be the empty digraph (so Corollary 8.2(i) and (ii) are satisfied). Apply Corollary 8.2 with $D'', r'' - r_0, \frac{r''}{2n}, \varepsilon', \frac{r_0}{2n}$, and $\frac{\nu}{6}$ playing the roles of $D, \ell, \delta, \varepsilon, \eta$, and ν to obtain a set $\mathscr{C}_{\mathrm{approx}}$ of $r'' - r_0$ edge-disjoint Hamilton cycles of D''.

Step 4: Absorbing the leftovers. Finally, we decompose $H := D'' \setminus \mathscr{C}_{\mathrm{approx}} = D' \setminus (\mathscr{C}_0 \cup \mathscr{C}_{\mathrm{approx}})$ using the absorber D^{abs} constructed in Step 2. Note that H is an r_0-regular subdigraph of $D' \setminus D'[V_0]$ (by Claim 1, H is obtained from the r'-regular digraph D' by removing the edges of $r' - r_0$ edge-disjoint Hamilton cycles of D'). Define a multidigraph D''' by $D''' := H \cup D^{\mathrm{abs}} \cup \mathcal{SF} \cup \mathcal{SF}'$.

Claim 2. *$D''' \setminus (\mathcal{M} \cup \mathcal{M}')$ is a digraph (rather than a multidigraph) and satisfies $D''' \setminus (\mathcal{M} \cup \mathcal{M}') = D \setminus (\mathcal{SC} \cup \mathcal{SC}' \cup \mathscr{C}_0 \cup \mathscr{C}_{\mathrm{approx}})$.*

Proof of Claim. By (c), (d), (c′), and (d′), $\mathcal{SF} \setminus \mathcal{M}$ and $\mathcal{SF}' \setminus \mathcal{M}'$ are digraphs rather than multidigraphs and are edge-disjoint subdigraphs of $D \setminus (\mathcal{SC} \cup \mathcal{SC}')$. By Step 2, $D^{\text{abs}} \subseteq D \setminus (\mathcal{SC} \cup \mathcal{SF} \cup \mathcal{SC}' \cup \mathcal{SF}')$ and, by definition,

$$H = D'' \setminus \mathscr{C}_{\text{approx}} = D \setminus (\mathcal{SC} \cup \mathcal{SC}' \cup \mathcal{SF} \cup \mathcal{SF}' \cup D^{\text{abs}} \cup \mathscr{C}_0 \cup \mathscr{C}_{\text{approx}}). \tag{9.1}$$

Thus, $\mathcal{SF} \setminus \mathcal{M}$, $\mathcal{SF}' \setminus \mathcal{M}'$, D^{abs}, and H are all pairwise edge-disjoint subdigraphs of D. Therefore, $D''' \setminus (\mathcal{M} \cup \mathcal{M}')$ is a digraph. Moreover, recall from Step 3 that $\mathscr{C}_0 \cup \mathscr{C}_{\text{approx}} \subseteq D' \subseteq D \setminus (\mathcal{SF} \cup \mathcal{SF}' \cup D^{\text{abs}})$. Thus, (9.1) implies that $D''' \setminus (\mathcal{M} \cup \mathcal{M}') = D \setminus (\mathcal{SC} \cup \mathcal{SC}' \cup \mathscr{C}_0 \cup \mathscr{C}_{\text{approx}})$, as desired. $\diamond$

Let $\mathscr{C} \cup \mathscr{C}'$ be the decomposition of D''' obtained by applying Corollary 8.31(ii) (with r_0 and r_0' playing the roles of r and r'). By Corollary 8.31(ii.b), $\mathscr{C}'$ is a set of s' Hamilton cycles on $V(D) \setminus V_0$, each containing precisely one of the special path systems in the multidigraph $\mathcal{SF} \cup \mathcal{SF}'$. That is, there exists an enumeration

$$\{C_{i,h,j} \mid (i,h,j) \in [r_3] \times [\tfrac{q}{f}] \times [f]\} \cup \{C'_{i,h,j} \mid (i,h,j) \in [r^\diamond] \times [1] \times [7]\}$$

of $\mathscr{C}'$ such that $C_{i,h,j} \cap (\mathcal{SF} \cup \mathcal{SF}') = SPS_{i,h,j}$ for each $(i,h,j) \in [r_3] \times [\frac{q}{f}] \times [f]$ and $C'_{i',h',j'} \cap (\mathcal{SF} \cup \mathcal{SF}') = SPS'_{i',h',j'}$ for each $(i',h',j') \in [r^\diamond] \times [1] \times [7]$.

For each $(i,h,j) \in [r_3] \times [\frac{q}{f}] \times [f]$, define $C^*_{i,h,j} \coloneqq (C_{i,h,j} \setminus M_{i,h,j}) \cup SC_{i,h,j}$ and note that (c) and Fact 8.26 imply that $C^*_{i,h,j}$ is a Hamilton cycle of D. For each $(i,h,j) \in [r^\diamond] \times [1] \times [7]$, define $C''_{i,h,j} \coloneqq (C'_{i,h,j} \setminus M'_{i,h,j}) \cup SC'_{i,h,j}$ and note that (c′) and Fact 8.26 imply that $C''_{i,h,j}$ is a Hamilton cycle of D. Let

$$\mathscr{C}'' \coloneqq \{C^*_{i,h,j} \mid (i,h,j) \in [r_3] \times [\tfrac{q}{f}] \times [f]\} \cup \{C''_{i,h,j} \mid (i,h,j) \in [r^\diamond] \times [1] \times [7]\}.$$

By (a), (a′), and Step 3, $\mathcal{SC}$, $\mathcal{SC}'$, and $\mathscr{C}_0 \cup \mathscr{C}_{\text{approx}}$ are all pairwise edge-disjoint. Thus, Claim 2 implies that $\mathscr{C}''$ is a Hamilton decomposition of

$$(D''' \setminus (\mathcal{M} \cup \mathcal{M}' \cup \mathscr{C})) \cup (\mathcal{SC} \cup \mathcal{SC}') = D \setminus (\mathscr{C}_0 \cup \mathscr{C}_{\text{approx}} \cup \mathscr{C}).$$

By Step 3, $\mathscr{C}_0 \cup \mathscr{C}_{\rm approx}$ is a set of edge-disjoint Hamilton cycles of D and, by Corollary 8.31(ii.a) and Claim 2, $\mathscr{C}$ is a set of edge-disjoint Hamilton cycles of $H \cup D^{\rm abs} \subseteq D \setminus (\mathscr{C}_0 \cup \mathscr{C}_{\rm approx})$. Thus, $\mathscr{C}_0 \cup \mathscr{C}_{\rm approx} \cup \mathscr{C} \cup \mathscr{C}''$ is a Hamilton decomposition of D. This completes the proof of Theorem 4.1. $\square$

CHAPTER 10

BLOW-UP CYCLES: DEFINITIONS AND PROOF OF LEMMA 4.3

The remainder of this thesis is devoted to the proofs of Lemma 4.3 and Theorem 4.4. In this section, we recall and expand on the definitions of the complete blow-up C_4 and a digraph which is ε-close to the complete blow-up C_4. We also state a few properties of blow-up cycles and prove Lemma 4.3.

10.1 Blow-up cycles

We now generalise the concept of complete blow-up C_4 introduced in Section 4.2. Let $K \geq 3$. The *complete blow-up C_K on vertex classes of size n* is the n-fold blow-up of the consistently directed C_K. Any spanning subdigraph of the n-fold blow-up of the directed C_K is called a *blow-up C_K on vertex classes of size n*. Recall from Section 4.2 that a digraph is *ε-close to the complete blow-up C_4 on vertex classes of size n* if it can be obtained from the complete blow-up C_4 on vertex classes of size n by flipping the direction of at most $4\varepsilon^2 n$ edges.

Let $K \geq 3$. It will sometimes be convenient to label the vertex classes of the (complete) blow-up C_K. This motivates the following variants of the above definitions. Let $U_1, \dots, U_K$ be disjoint vertex sets of size n and let $\mathcal{U} \coloneqq (U_1, \dots, U_K)$. The *complete blow-up C_K with vertex partition $\mathcal{U}$* is the digraph D on $\bigcup \mathcal{U} = \bigcup_{i \in [K]} U_i$ defined by $E(D) \coloneqq \{uv \mid u \in U_i, v \in U_{i+1}, i \in [K]\}$ (where $U_{K+1} \coloneqq U_1$). Note that the ordering of $U_1, \dots, U_K$

matters. The vertex sets $U_1, \dots, U_K$ are the *vertex classes* of D. In informal discussions, we sometime refer to $(U_1, U_2), \dots, (U_K, U_1)$ as the *pairs of the blow-up* C_K. (In other words, D is the n-fold blow-up of the directed C_K whose vertex classes are denoted by $U_1, \dots, U_K$ and ordered according to the ordering of the vertices in the directed C_K.) Let D' be a digraph on $\bigcup \mathcal{U}$. We say that D' is a *blow-up* C_K *with vertex partition* $\mathcal{U}$ if D' is a spanning subdigraph of D. If $K = 4$, we say that D' is ε*-close to the complete blow-up* C_4 *with vertex partition* $\mathcal{U}$ if D' can be obtained from D by flipping the direction of at most $4\varepsilon n^2$ edges of D.

Definition 10.1 (($\varepsilon, 4$)-partition)**.** Let $U_1, \dots, U_4$ be disjoint vertex sets of size n and denote $\mathcal{U} \coloneqq (U_1, \dots, U_4)$. Let D be a digraph on $\bigcup \mathcal{U}$. We say that $\mathcal{U}$ is an $(\varepsilon, 4)$*-partition for* D if D is ε-close to the complete blow-up C_4 with vertex partition $\mathcal{U}$.

Note that while the ordering of $U_1, \dots, U_K$ in the above definitions matters, this ordering can be shifted without effect. We also emphasise that, in the above definitions, we assume that the vertex classes $U_1, \dots, U_K$ are equally sized.

Fact 10.2. *Let D be a digraph on $4n$ vertices and suppose that $\mathcal{U} = (U_1, \dots, U_4)$ is an $(\varepsilon, 4)$-partition for D. Then, the following hold.*

(i) $|U_1| = \dots = |U_4| = n$.

(ii) *For each $i \in [4]$, $(U_i, \dots, U_{i+3})$ is an $(\varepsilon, 4)$-partition for D.*

Throughout this thesis, when we work with a vertex partition $\mathcal{U} = (U_1, \dots, U_K)$, the subscripts in $U_1, \dots, U_K$ are always taken modulo K (so $U_{K+1} \coloneqq U_1$ for example).

10.2 Forward and backward edges

Let $U_1, \dots, U_K$ be disjoint vertex sets (not necessarily of the same size) and denote $\mathcal{U} \coloneqq (U_1, \dots, U_K)$. Let $u, v \in \bigcup \mathcal{U}$ be distinct. We say that uv is a *forward edge (with respect to* $\mathcal{U}$*)* if there exists $i \in [K]$ such that $u \in U_i$ and $v \in U_{i+1}$. We say that uv is a *backward edge (with respect to* $\mathcal{U}$*)* if there exists $i \in [K]$ such that $u \in U_{i+1}$ and $v \in U_i$. Let

D be a digraph on $\bigcup\mathcal{U}$. We denote by $\overrightarrow{D}_{\mathcal{U}}$ the subdigraph of D induced by the forward edges of D with respect to $\mathcal{U}$ and by $\overleftarrow{D}_{\mathcal{U}}$ the subdigraph of D induced by the backward edges of D with respect to $\mathcal{U}$. Let $v \in V(D)$. The *forward (in/out)degree of v in D (with respect to $\mathcal{U}$)* is the (in/out)degree of v in $\overrightarrow{D}_{\mathcal{U}}$ and the *backward (in/out)degree of v in D (with respect to $\mathcal{U}$)* is the (in/out)degree of v in $\overleftarrow{D}_{\mathcal{U}}$. These are denoted by

$$\overrightarrow{d}_{D,\mathcal{U}}(v) \coloneqq d_{\overrightarrow{D}_{\mathcal{U}}}(v), \quad \overrightarrow{d}^{\pm}_{D,\mathcal{U}}(v) \coloneqq d^{\pm}_{\overrightarrow{D}_{\mathcal{U}}}(v), \quad \overleftarrow{d}_{D,\mathcal{U}}(v) \coloneqq d_{\overleftarrow{D}_{\mathcal{U}}}(v), \quad \overleftarrow{d}^{\pm}_{D,\mathcal{U}}(v) \coloneqq d^{\pm}_{\overleftarrow{D}_{\mathcal{U}}}(v).$$

Similarly, the *forward (in/out)neighbourhood of v in D (with respect to $\mathcal{U}$)* is the (in/out)-neighbourhood of v in $\overrightarrow{D}_{\mathcal{U}}$ and the *backward (in/out)neighbourhood of v in D (with respect to $\mathcal{U}$)* is the (in/out)neighbourhood of v in $\overleftarrow{D}_{\mathcal{U}}$. These are denoted by

$$\overrightarrow{N}_{D,\mathcal{U}}(v) \coloneqq N_{\overrightarrow{D}_{\mathcal{U}}}(v), \ \overrightarrow{N}^{\pm}_{D,\mathcal{U}}(v) \coloneqq N^{\pm}_{\overrightarrow{D}_{\mathcal{U}}}(v), \ \overleftarrow{N}_{D,\mathcal{U}}(v) \coloneqq N_{\overleftarrow{D}_{\mathcal{U}}}(v), \ \overleftarrow{N}^{\pm}_{D,\mathcal{U}}(v) \coloneqq N^{\pm}_{\overleftarrow{D}_{\mathcal{U}}}(v).$$

10.3 Regular bipartite tournaments

We now make a few observations about regular bipartite tournaments. Let $U_1, \dots, U_4$ be disjoint vertex sets of size n and denote $\mathcal{U} \coloneqq (U_1, \dots, U_4)$. Let T be a bipartite tournament and suppose that $\mathcal{U}$ is an $(\varepsilon, 4)$-partition for T. Then, it is easy to see that T is a bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Moreover, note that the complete blow-up C_4 with vertex partition $\mathcal{U}$ is a regular digraph. Thus, T must be obtained by changing, for each $v \in \bigcup\mathcal{U}$, the direction of the same number of in- and outedges incident to v.

Fact 10.3. *Let $U_1, \dots, U_4$ be disjoint vertex sets and $\mathcal{U} \coloneqq (U_1, \dots, U_4)$. Let T be a regular bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Then, each $v \in V(T)$ satisfies*

$$\overrightarrow{d}^{+}_{T,\mathcal{U}}(v) = \overrightarrow{d}^{-}_{T,\mathcal{U}}(v) \quad \textit{and} \quad \overleftarrow{d}^{+}_{T,\mathcal{U}}(v) = \overleftarrow{d}^{-}_{T,\mathcal{U}}(v).$$

In particular, this implies that the number of forward/backward edges is the same in each pair of the blow-up C_4.

Fact 10.4. *Let $U_1, \dots, U_4$ be disjoint vertex sets and denote $\mathcal{U} := (U_1, \dots, U_4)$. Let T be a regular bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Then,*

$$e_T(U_1, U_2) = e_T(U_2, U_3) = e_T(U_3, U_4) = e_T(U_4, U_1)$$

and

$$e_T(U_1, U_4) = e_T(U_4, U_3) = e_T(U_3, U_2) = e_T(U_2, U_1).$$

Thus, we may use the following alternative definition of an $(\varepsilon, 4)$-partition.

Fact 10.5. *Let $U_1, \dots, U_4$ be disjoint vertex sets and $\mathcal{U} := (U_1, \dots, U_4)$. Let T be a regular bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Then, $\mathcal{U}$ is an $(\varepsilon, 4)$-partition for T if and only if $e_T(U_i, U_{i-1}) \leq \varepsilon n^2$ for each $i \in [4]$.*

10.4 Proof of Lemma 4.3

We are now ready to prove Lemma 4.3, which states that if a regular bipartite tournament T is not a bipartite robust outexpander, then T is close to the complete blow-up C_4.

Proof of Lemma 4.3. Let $0 < \frac{1}{n_0} \ll \nu' \leq \nu \ll \tau$ and let T be a regular bipartite tournament on vertex classes A and B of size $2n \geq n_0$. Note that T is n-regular. Suppose that T is not a bipartite robust (ν', τ)-outexpander with bipartition (A, B). We show that T is $\sqrt{\nu'}$-close to the complete blow-up C_4 on vertex classes of size n.

We may assume without loss of generality that there exists $A' \subseteq A$ satisfying $2\tau n \leq |A'| \leq 2(1-\tau)n$ for which

$$|RN^+_{\nu',T}(A')| < |A'| + 2\nu' n. \tag{10.1}$$

Denote $B' := RN^+_{\nu',T}(A')$. By definition of a bipartite robust outexpander, we have

$$e_T(A', B \setminus B') < 2\nu' n |B \setminus B'| \leq 4\nu' n^2. \tag{10.2}$$

Thus,

$$|A'||B'| \geq e_T(A',B') = e_T(A',B) - e_T(A',B\setminus B') \overset{(10.2)}{\geq} n|A'| - 4\nu' n^2 \geq \left(1 - \frac{2\nu'}{\tau}\right) n|A'|.$$

Therefore,

$$|B'| \geq \left(1 - \frac{2\nu'}{\tau}\right) n \tag{10.3}$$

and so

$$|A'| \overset{(10.1),(10.3)}{\geq} \left(1 - \frac{3\sqrt{\nu'}}{4}\right) n. \tag{10.4}$$

Moreover,

$$n|B\setminus B'| \geq e_T(B\setminus B', A') = |B\setminus B'||A'| - e_T(A', B\setminus B') \overset{(10.2)}{\geq} |B\setminus B'|(|A'| - 2\nu' n).$$

Therefore,

$$|A'| \leq (1 + 2\nu')\, n \tag{10.5}$$

and so

$$|B'| \overset{(10.1),(10.5)}{\leq} (1 + 4\nu')\, n. \tag{10.6}$$

Let $U_1 \cup U_3$ be a partition of A such that $|U_1| = n = |U_3|$ and $|U_1 \triangle A'| = |n - |A'||$. Similarly, let $U_2 \cup U_4$ be a partition of B such that $|U_2| = n = |U_4|$ and $|U_2 \triangle B'| = |n - |B'||$. Note that

$$|U_1 \triangle A'| \overset{(10.4),(10.5)}{\leq} \frac{3\sqrt{\nu'} n}{4} \quad \text{and} \quad |U_2 \triangle B'| \overset{(10.3),(10.6)}{\leq} \frac{3\sqrt{\nu'} n}{4}. \tag{10.7}$$

By Facts 10.4 and 10.5, it is enough to show that $e_T(U_1, U_4) \leq 4\sqrt{\nu'} n^2$. We have

$$\begin{aligned} e_T(U_1,U_4) \quad &\leq \quad e_T(A'\cap U_1, U_4 \setminus B') + e_T(U_1 \setminus A', U_4) + e_T(U_1, B'\cap U_4) \\ &\overset{(10.2),(10.7)}{\leq} \quad 4\nu' n^2 + \frac{6\sqrt{\nu'} n^2}{4} + \frac{6\sqrt{\nu'} n^2}{4} \leq 4\sqrt{\nu'} n^2, \end{aligned}$$

as desired. $\square$

CHAPTER 11

A ROBUST DECOMPOSITION LEMMA FOR BLOW-UP CYCLES

In this section, we introduce a robust decomposition lemma for blow-up cycles. This result will be used in the proof of Theorem 4.4 to decompose the edges leftover after the approximate decomposition. (See Section 4.3 for a proof overview of Theorem 4.4.)

First, observe that the standard robust decomposition lemma (Lemma 8.23) cannot be directly applied when T is (ε-close to) the complete blow-up C_4 because we cannot find a setup or a bi-setup: since (almost) all the edges lie along a blow-up cycle, we cannot find the necessary chord edges to form a universal or bi-universal walk. Thus, we will need to derive an analogue of Lemma 8.23 which holds for blow-up C_4's.

11.1 Aim and strategy

Let D be a blow-up C_4 with vertex partition $\mathcal{U} = (U_1, \dots, U_4)$. (In the proof of Theorem 4.4, D will be (a subdigraph of) $\overrightarrow{T}_{\mathcal{U}}$, that is, D will consist of (some of) the forward edges of T.) We want to find an absorber $D^{\rm rob} \subseteq D$ such that for any sparse regular leftover $H \subseteq D \setminus D^{\rm rob}$, the digraph $H \cup D^{\rm rob}$ has a Hamilton decomposition.

Roughly speaking, the overall strategy is as follows. Recall the notion of matching contractions from Section 7.5. For each $i \in [4]$ in turn, we apply the standard robust decomposition lemma in a suitable auxiliary "contracted" digraph corresponding to the pair (U_i, U_{i+1}) of the blow-up C_4. This enables us to decompose small leftovers into suitable auxiliary "contracted" Hamilton cycles spanning U_i. These are then "expanded" into full Hamilton cycles of D. (See also Section 4.3.3 for an informal discussion about how to construct a Hamilton cycle in a blow-up C_4.)

We now explain this strategy in more detail. Note that it is enough to consider each pair (U_i, U_{i+1}) of the blow-up C_4 in turn. Indeed, suppose that for each $i \in [4]$, we have constructed an absorber $D_i^{\rm rob} \subseteq D$ such that for any sparse leftover $H_i \subseteq D(U_i, U_{i+1}) \setminus D_i^{\rm rob}$, the digraph $H_i \cup D_i^{\rm rob}$ has a decomposition into Hamilton cycles of D. Let $D^{\rm rob} \coloneqq \bigcup_{i\in[4]} D_i^{\rm rob}$. Then, for any sparse leftover $H \subseteq D \setminus D^{\rm rob}$, we can use each $D_i^{\rm rob}$ in turn to decompose the edges of $H(U_i, U_{i+1})$. Altogether, this induces a Hamilton decomposition of $H \cup D^{\rm rob}$ (recall that D, and so H, only contains edges which lie in one of the pairs (U_i, U_{i+1})).

Let $i \in [4]$. We now explain our strategy for constructing the absorber $D_i^{\rm rob}$. First, as mentioned above, observe that the problem of constructing Hamilton cycles of D can be reduced to constructing Hamilton cycles on $U_i \cup U_{i+1}$. Then, the following holds.

Fact 11.1. *Fix an auxiliary perfect matching M_i from U_{i+1} to U_i. Let M be a perfect matching from U_i to U_{i+1} and suppose that $M \cup M_i$ forms a Hamilton cycle on $U_i \cup U_{i+1}$. Let $\mathscr{P}$ be a spanning set of vertex-disjoint paths on $\bigcup \mathcal{U}$ which consists of a (u,v)-path for each $uv \in M_i$. Then, $M \cup \mathscr{P}$ forms a Hamilton cycle on $\bigcup \mathcal{U}$.*

In our robust decomposition lemma for blow-up cycles, we will input such spanning sets of vertex-disjoint paths (these will be incorporated into the special factors). Thus, we have reduced the original problem to that of finding an absorber $D_i^{\rm rob}$ such that the following holds: for any sparse leftover $H_i \subseteq D(U_i, U_{i+1}) \setminus D_i^{\rm rob}$, the digraph $H_i \cup D_i^{\rm rob}$ has a decomposition into perfect matchings from U_i to U_{i+1}, each of which forms a Hamilton cycle on $U_i \cup U_{i+1}$ with a fixed auxiliary matching M_i.

We now discuss the construction of $D_i^{\rm rob}$. We have already discussed (e.g. in Sections 4.3.1, 7.5, and 8.1) that one can construct Hamilton cycles which contain a prescribed perfect matching by considering contracted digraphs. More precisely, fix an auxiliary perfect matching M_i from U_{i+1} to U_i and let $\widetilde{D}_i$ be the M_i-contraction of $D[U_i, U_{i+1}]$

(recall Definition 7.25(i)). Then, as seen in Fact 7.29, a Hamilton cycle in $\widetilde{D}_i$ induces a perfect matching from U_i to U_{i+1} in D which forms a Hamilton cycle on $U_i \cup U_{i+1}$ with M_i. Thus, we can let $D_i^{\rm rob}$ be the M_i-expansion of the absorber $\widetilde{D}_i^{\rm rob}$ obtained by applying the robust decomposition lemma in $\widetilde{D}_i$. Indeed, suppose that $H_i \subseteq D(U_i, U_{i+1}) \setminus D_i^{\rm rob}$ is a sparse leftover. Denote by $\widetilde{H}_i$ the M_i-contraction of H_i. Then, Lemma 8.23 implies that $\widetilde{H}_i \cup \widetilde{D}_i^{\rm rob}$ has a decomposition into Hamilton cycles on U_i. By Fact 7.29, this induces a decomposition of $H_i \cup D_i^{\rm rob}$ into perfect matchings from U_i to U_{i+1} which form Hamilton cycles on $U_i \cup U_{i+1}$ with M_i, as desired.

Note that since we consider each pair (U_i, U_{i+1}) of the blow-up C_4 in turn, our methods hold for more general blow-up cycles of any length. Thus, we write the rest of this section for general blow-up C_K's for possible future use. In this thesis, we will only need the case $K = 4$ (to prove Theorem 4.4).

11.2 Definitions

First, we introduce the cycle analogues of setups, special path systems, and special factors (which were defined in Sections 8.2.4 and 8.2.5).

11.2.1 Cycle-setups

Let D be a blow-up C_K with vertex partition $\mathcal{U} = (U_1, \dots, U_K)$. For each $i \in [K]$, let M_i be an auxiliary perfect matching from U_{i+1} to U_i and let $\widetilde{D}_i$ be the M_i-contraction of $D[U_i, U_{i+1}]$. As discussed in Section 11.1, we aim to apply the standard robust decomposition lemma (Lemma 8.23) in each $\widetilde{D}_i$ in turn and so we will need a setup in each $\widetilde{D}_i$. This motivates the next definition: roughly speaking, a cycle-setup consists of K setups, one in each $\widetilde{D}_i$.

Definition 11.2 (Cycle-setup)**.** $(D, \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ is a $(K, \ell', \ell^*, k, m, \varepsilon, d)$-*cycle-setup* if the following properties are satisfied.

(CST1) D is a blow-up C_K with vertex partition $\mathcal{U} = (U_1, \dots, U_K)$.

(CST2) $\mathcal{M} = (M_1, \dots, M_K)$ where, for each $i \in [K]$, M_i is an auxiliary directed perfect matching from U_{i+1} to U_i.

(CST3) $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_K)$, $\mathcal{P}' = (\mathcal{P}'_1, \dots, \mathcal{P}'_K)$, $\mathcal{P}^* = (\mathcal{P}^*_1, \dots, \mathcal{P}^*_K)$, $\mathcal{C} = (C^1, \dots, C^K)$, $\mathcal{R} = (R_1, \dots, R_K)$, $\mathscr{U} = (U^1, \dots, U^K)$, and $\mathscr{U}' = (U'^1, \dots, U'^K)$ are such that the following holds for each $i \in [K]$. Let $\widetilde{D}_i$ be the M_i-contraction of $D[U_i, U_{i+1}]$. Then, $(\widetilde{D}_i, \mathcal{P}_i, \mathcal{P}'_i, \mathcal{P}^*_i, R_i, C^i, U^i, U'^i)$ is an $(\ell', \ell^*, k, m, \varepsilon, d)$-setup with an empty exceptional set.

Whenever $(D,\mathcal{U},\mathcal{P},\mathcal{P}',\mathcal{P}^*,\mathcal{R},\mathcal{C},\mathscr{U},\mathscr{U}',\mathcal{M})$ is a $(K, \ell', \ell^*, k, m, \varepsilon, d)$-cycle-setup, we implicitly use the notation $\mathcal{U} = (U_1, \dots, U_K)$, $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_K)$, $\mathcal{P}' = (\mathcal{P}'_1, \dots, \mathcal{P}'_K)$, $\mathcal{P}^* = (\mathcal{P}^*_1, \dots, \mathcal{P}^*_K)$, $\mathcal{R} = (R_1, \dots, R_K)$, $\mathcal{C} = (C^1, \dots, C^K)$, $\mathscr{U} = (U^1, \dots, U^K)$, $\mathscr{U}' = (U'^1, \dots, U'^K)$, and $\mathcal{M} = (M_1, \dots, M_K)$.

Definition 11.3 (Cycle-framework)**.** To avoid repetitions, we say that $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is a (K, ℓ^*, k, n)-*cycle-framework* if $\mathcal{U} = (U_1, \dots, U_K)$, $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_K)$, $\mathcal{P}^* = (\mathcal{P}^*_1, \dots, \mathcal{P}^*_K)$, $\mathcal{C} = (C^1, \dots, C^K)$, and $\mathcal{M} = (M_1, \dots, M_K)$ satisfy the following properties for each $i \in [K]$.

(CF1) U_i is a vertex set of size n which is disjoint from the other sets in $\mathcal{U}$.

(CF2) $\mathcal{P}_i$ is a partition of U_i into an empty exceptional set and k clusters of size $\frac{n}{k}$.

(CF3) $\mathcal{P}^*_i$ is an ℓ^*-refinement of $\mathcal{P}_i$.

(CF4) C^i is a Hamilton cycle on the clusters in $\mathcal{P}_i$.

(CF5) M_i is an auxiliary perfect matching from U_{i+1} to U_i.

Whenever we say that $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is a (K, ℓ^*, k, n)-cycle-framework, we implicitly use the notation $\mathcal{U} = (U_1, \dots, U_K)$, $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_K)$, $\mathcal{P}^* = (\mathcal{P}^*_1, \dots, \mathcal{P}^*_K)$, $\mathcal{C} = (C^1, \dots, C^K)$, and $\mathcal{M} = (M_1, \dots, M_K)$.

One can easily verify that a cycle-setup induces a cycle-framework.

Fact 11.4. *Let* $(D, \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ *be a* $(K, \ell', \ell^*, k, m, \varepsilon, d)$*-cycle-setup. Then,* $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ *is a* (K, ℓ^*, k, n)*-cycle-framework where* $n \coloneqq |U_1|$*.*

Any partition is a 1-refinement of itself, so the following holds.

Fact 11.5. *Let* $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ *be a* (K, ℓ^*, k, n)*-cycle-framework. Then,* $(\mathcal{U}, \mathcal{P}, \mathcal{P}, \mathcal{C}, \mathcal{M})$ *is a* $(K, 1, k, n)$*-cycle-framework.*

Recall from Proposition 8.16 that a setup remains a setup (with slightly worse parameters) after removing a few edges incident to each vertex. Using similar arguments, one can show that the analogue holds for a cycle-setup.

Proposition 11.6. *Let* $0 < \frac{1}{m} \ll \frac{1}{k}, \varepsilon \leq \varepsilon' \ll d \ll \frac{1}{\ell'} \ll 1$ *and* $\varepsilon' \ll \frac{1}{\ell^*}$*. Let* D *be a digraph and let* D' *be obtained from* D *by removing at most* $\varepsilon' m$ *inedges and* $\varepsilon' m$ *outedges incident to each vertex. If* $(D, \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ *is a* $(K, \ell', \ell^*, k, m, \varepsilon, d)$*-cycle-setup, then* $(D', \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ *is a* $(K, \ell', \ell^*, k, m, (\varepsilon')^{\frac{1}{3}}, \frac{d}{2})$*-cycle-setup.*

11.2.2 Extended special path systems and extended special factors

We will now introduce the concept of extended special path systems. Roughly speaking, these can be viewed as the analogues of the special path systems for blow-up C_K's. As discussed in Section 8.2.5, special path systems can viewed as building blocks for Hamilton cycles; in Lemma 8.23, each special path system that we input gives rise to a distinct Hamilton cycle. Analogously, extended special path systems (defined formally below) will be building blocks for constructing Hamilton cycles in a blow-up cycle; in the blow-up cycle version of the robust decomposition lemma (Lemma 11.10 below), each extended special path system that we input will give rise to a distinct Hamilton cycle.

The structure of an extended special path system follows naturally from the proof idea described in Section 11.1. More precisely, let D be a blow-up C_K with vertex partition $\mathcal{U} = (U_1, \dots, U_K)$. For each $i \in [K]$, let M_i be an auxiliary perfect matching from U_{i+1}

to U_i and let $\widetilde{D}_i$ be the M_i-contraction of $D[U_i, U_{i+1}]$. As discussed in Section 11.1, the leftovers in each of the pairs (U_i, U_{i+1}) will be decomposed in two steps. First, we use the robust decomposition lemma in $\widetilde{D}_i$ to decompose the leftovers into Hamilton cycles in the contracted pair (U_i, U_{i+1}). Then, we expand each of these contracted Hamilton cycles using a spanning set of vertex-disjoint paths whose endpoints are prescribed by M_i (see Fact 11.1). Thus, an extended special path system will consist of two parts: a special path system SPS in the contracted pair (U_i, U_{i+1}) (which will be used to apply Lemma 8.23 in $\widetilde{D}_i$) and a spanning set of paths with prescribed endpoints (which will be used to expand the contracted Hamilton cycle containing SPS). (Recall that special path systems were introduced in Definition 8.21.)

Definition 11.7 (Friendly extended special path system)**.** Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a (K, ℓ^*, k, n)-cycle-framework and suppose that $\frac{k}{f} \in \mathbb{N}$. For any $(h, i, j) \in [\ell^*] \times [K] \times [f]$, a *friendly* (ℓ^*, K, f, h, i, j)*-extended special path system with respect to* $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$*, and* $\mathcal{M}$ is a linear forest $FESPS$ for which the following hold.

(FESPS1) The digraph obtained by deleting all the isolated vertices in the M_i-contraction of $FESPS[U_i, U_{i+1}]$ is an (ℓ^*, f, h, j)-special path system with respect to $\mathcal{P}_i^*$ and C^i.

(FESPS2) $FESPS \setminus E_{FESPS}(U_i, U_{i+1})$ is a spanning linear forest on $\bigcup \mathcal{U}$ which consists of n components, one (u, v)-path for each $uv \in M_i$.

Recall that the main purpose of special path systems is to prescribe edges in our Hamilton decompositions. In particular, in the ε-close to the blow-up C_4 case (Theorem 4.4), we will need to incorporate prescribed sets of backward edges. It turns out that the concept of friendly extended special path systems is very inconvenient for doing so. However, as discussed in Section 8.2.1, if we want to incorporate a linear forest F into a Hamilton cycle, then the internal structure of F is not important; we can always consider an equivalent linear forest instead (recall Definition 8.4). Thus, we can generalise the concept of friendly extended special path systems as follows.

Definition 11.8 (Extended special path system)**.** Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a (K, ℓ^*, k, n)-cycle-framework and suppose that $\frac{k}{f} \in \mathbb{N}$. For any $(h, i, j) \in [\ell^*] \times [K] \times [f]$, a linear forest is an (ℓ^*, K, f, h, i, j)*-extended special path system with respect to* $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$*, and* $\mathcal{M}$ if it is equivalent to a friendly (ℓ^*, K, f, h, i, j)-extended special path system with respect to $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$.

Note that since a linear forest is equivalent to itself, a friendly extended special path system is indeed an extended special path system.

Definition 11.9 (Extended special factor)**.** Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a (K, ℓ^*, k, n)-cycle-framework and suppose that $\frac{k}{f} \in \mathbb{N}$. An (ℓ^*, K, f)*-extended special factor with respect to* $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$*, and* $\mathcal{M}$ is a multidigraph which has a decomposition $\{ESPS_{h,i,j} \mid (h, i, j) \in [\ell^*] \times [K] \times [f]\}$ where, for each $(h, i, j) \in [\ell^*] \times [K] \times [f]$, $ESPS_{h,i,j}$ is an (ℓ^*, K, f, h, i, j)-extended special path system with respect to $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$.

11.3 Statement of the robust decomposition lemma for blow-up cycles

We are now ready to state a blow-up cycle version of the robust decomposition lemma.

Lemma 11.10 (Robust decomposition lemma for blow-up cycles)**.** *Let* $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll \frac{r_1}{m} \ll d \ll \frac{1}{\ell'}, \frac{1}{g} \ll 1$ *and suppose that* $rk^2 \leq m$*. Let*

$$r_2 := 96\ell' g^2 kr, \quad r_3 := \frac{rfk}{q}, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s' := rfk + 7r^\diamond,$$

and suppose that $\frac{k}{14}, \frac{k}{f}, \frac{k}{g}, \frac{q}{f}, \frac{m}{4\ell'}, \frac{fm}{q}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$*. Let* $(D, \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ *be a* $(K, \ell', \frac{q}{f}, k, m, \varepsilon, d)$*-cycle-setup. Let* $\mathcal{ESF}$ *be a multidigraph which consists of the union of* r_3 $(\frac{q}{f}, K, f)$*-extended special factors with respect to* $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$*, and* $\mathcal{M}$ *and let* $\mathcal{ESF}'$ *be a multidigraph which consists of the union of* $r^\diamond$ $(1, K, 7)$*-extended special factors with respect to* $\mathcal{U}, \mathcal{P}, \mathcal{C}$*, and* $\mathcal{M}$*. Then,* D *contains an* $(r_1 + r_2 + 5r^\diamond)$*-regular spanning subdigraph* D^{rob}

for which the following holds. Let H be an r-regular blow-up C_K with vertex partition $\mathcal{U}$. Suppose that H is edge-disjoint from D^{rob} and that $E(H) \cap \{uv \mid vu \in \bigcup \mathcal{M}\} = \emptyset$. Then, the multidigraph $H \cup D^{\text{rob}} \cup \mathcal{ESF} \cup \mathcal{ESF}'$ has a decomposition $\mathscr{C}$ into Ks' edge-disjoint Hamilton cycles such that each cycle in $\mathscr{C}$ contains precisely one of the extended special path systems in the multidigraph $\mathcal{ESF} \cup \mathcal{ESF}'$.

By Fact 8.5, we may assume without loss of generality that all extended special path systems contained in $\mathcal{ESF} \cup \mathcal{ESF}'$ are friendly. Thus, as discussed in Section 11.1, Lemma 11.10 can be obtained by applying the original robust decomposition lemma (Lemma 8.23) to each contracted pair (U_i, U_{i+1}) of the blow-up cycle in turn. (A formal derivation is available in Appendix C.)

CHAPTER 12

APPLYING THE ROBUST DECOMPOSITION LEMMA IN A VERY DENSE BLOW-UP C_4

In this section, we discuss how to apply Lemma 11.10 in the context of Theorem 4.4. Let T be a bipartite tournament which is ε-close to a blow-up C_4 with vertex partition $\mathcal{U}$. Then, observe that $\overrightarrow{T}_{\mathcal{U}}$ (that is, the subdigraph of T induced by the forward edges of T (see Section 10.2)) is a very dense blow-up C_4 with vertex partition $\mathcal{U}$ (only at most an ε proportion of the edges are missing for $\overrightarrow{T}_{\mathcal{U}}$ to be the complete blow-up C_4). We will find our absorber $D^{\rm rob}$ by applying the robust decomposition lemma for blow-up cycles (Lemma 11.10) to (a subdigraph of) $\overrightarrow{T}_{\mathcal{U}}$. In this section, we show how to construct, in a very dense blow-up C_4, the extended special factors and the cycle-setup required to apply Lemma 11.10.

12.1 An alternative description of extended special path systems

To construct extended special path systems, it will be convenient to consider the following alternative description of extended special path systems.

Proposition 12.1. *Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a (K, ℓ^*, k, n)-cycle-framework and suppose that $\frac{k}{f} \in \mathbb{N}$. Let $(h, i, j) \in [\ell^*] \times [K] \times [f]$ and denote $k' \coloneqq \frac{k}{f} + 1$. Denote by $I = W_1 \dots W_{k'}$ the $j^{\rm th}$ interval in the canonical interval partition of C^i into f intervals. Let $W_{1,h}, \dots, W_{k',h}$*

denote the h^{th} subclusters of $W_1, \dots, W_{k'}$ contained in $\mathcal{P}_i^*$, respectively. Then, a linear forest $ESPS$ is an (ℓ^*, K, f, h, i, j)-extended special path system if and only if the following properties are satisfied.

(i) $V(ESPS) = \bigcup \mathcal{U}$.

(ii) $V^+(ESPS) = U_{i+1} \setminus N_{M_i}(W_{2,h} \cup \dots \cup W_{k',h})$.

(iii) $V^-(ESPS) = U_i \setminus (W_{1,h} \cup \dots \cup W_{k'-1,h})$.

(iv) *If $uv \in M_i - (W_{1,h} \cup \dots \cup W_{k',h})$, then $ESPS$ has a component which is a (u, v)-path.*

We now give a brief overview of the idea behind Proposition 12.1. Recall from Definition 8.21 that a special path system is a linear forest which covers a given interval of C^i. By Fact 7.28, the M_i-expansion of a special path system is thus a matching which covers a given interval. Thus, Definition 11.7 implies that a friendly extended special path system is a spanning linear forest whose components have endpoints which avoid a given interval and which are matched according to the auxiliary matching M_i. By Definition 8.4, these properties are shared by any linear forest which is equivalent to a friendly extended special path system, that is, by any extended special path system (recall Definition 11.8). Thus, an extended special path system is simply a spanning linear forest with suitably prescribed endpoints.

Proof of Proposition 12.1. ($\Rightarrow$) Firstly, assume that $ESPS$ is an (ℓ^*, K, f, h, i, j)-extended special path system. We need to show that (i)–(iv) are satisfied. By Definition 8.4, we may assume without loss of generality that $ESPS$ is friendly. Denote $D_1 \coloneqq ESPS \setminus ESPS(U_i, U_{i+1})$ and $D_2 \coloneqq ESPS \setminus D_1$ (i.e. $E(D_2) = E_{ESPS}(U_i, U_{i+1})$). Then, (FESPS2) implies that each $v \in \bigcup \mathcal{U}$ satisfies

$$d_{D_1}^+(v) = \begin{cases} 1 & \text{if } v \in \bigcup \mathcal{U} \setminus U_i; \\ 0 & \text{if } v \in U_i; \end{cases} \quad \text{and} \quad d_{D_1}^-(v) = \begin{cases} 1 & \text{if } v \in \bigcup \mathcal{U} \setminus U_{i+1}; \\ 0 & \text{if } v \in U_{i+1}. \end{cases}$$

In particular, (i) holds. Let SPS be the M_i-contraction of $ESPS[U_i, U_{i+1}]$. By (FESPS1) and Definition 8.21, each $v \in U_i \cup U_{i+1}$ satisfies

$$d^+_{SPS}(v) = \begin{cases} 1 & \text{if } v \in \bigcup_{j' \in [k'-1]} W_{j',h}; \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad d^-_{SPS}(v) = \begin{cases} 1 & \text{if } v \in \bigcup_{j' \in [k'-1]} W_{j'+1,h}; \\ 0 & \text{otherwise.} \end{cases}$$

By (FESPS2), $ESPS$ contains a (u, v)-path for each $uv \in M_i$. Since $ESPS$ is a linear forest, this implies that $E(ESPS) \cap \{uv \mid vu \in M_i\} = \emptyset$. Therefore, Fact 7.26 implies that $ESPS[U_i, U_{i+1}]$ is the M_i-expansion of SPS. Thus, Fact 7.28 implies that each $v \in \mathcal{U}$ satisfies

$$d^+_{D_2}(v) = \begin{cases} 1 & \text{if } v \in \bigcup_{j' \in [k'-1]} W_{j',h}; \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad d^-_{D_2}(v) = \begin{cases} 1 & \text{if } v \in \bigcup_{j' \in [k'-1]} N_{M_i}(W_{j'+1,h}); \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (ii) and (iii) are satisfied. For (iv), suppose that $uv \in M_i - (W_{1,h} \cup \cdots \cup W_{k',h})$. By (FESPS2), D_1 has a component P_{uv} which is a (u, v)-path. Moreover, $d_{D_2}(u) = 0 = d_{D_2}(v)$. Thus, P_{uv} is also a component of $ESPS$ and so (iv) holds.

($\Leftarrow$) Secondly, suppose that $ESPS$ is a linear forest which satisfies (i)–(iv). We need to show that $ESPS$ is an (ℓ^*, K, f, h, i, j)-extended special path system. By Definition 11.8, it is enough to construct a friendly (ℓ^*, K, f, h, i, j)-extended special path system $FESPS$ which is equivalent to $ESPS$.

In order to satisfy (FESPS2), our friendly extended special path system will need to contain a spanning set of vertex-disjoint paths whose endpoints "correspond" to the edges of M_i. We construct this set of paths as follows. For each $i' \in [K]$, let $u_{i',1}, \dots, u_{i',n}$ be an enumeration of $U_{i'}$. Suppose without loss of generality that $M_i = \{u_{i+1,j'}u_{i,j'} \mid j' \in [n]\}$. Let $\mathscr{P} := \{u_{i+1,j'}u_{i+2,j'} \dots u_{i+K-1,j'}u_{i,j'} \mid j' \in [n]\}$. Note that (FESPS2) holds with $\mathscr{P}$ playing the role of $FESPS \setminus E_{FESPS}(U_i, U_{i+1})$.

We now list the components of $ESPS$ and specify their endpoints. (This will enable to us to construct a friendly extended special path system which is equivalent to $ESPS$.)

For each $uv \in M_i - (W_{1,h} \cup \cdots \cup W_{k',h})$, denote by P_{uv} the component of $ESPS$ which is a (u,v)-path (P_{uv} exists by (iv) and is unique since $ESPS$ is a linear forest). Let $\mathscr{P}_1 := \{P_{uv} \mid uv \in M_i - (W_{1,h} \cup \cdots \cup W_{k',h})\}$. Let $\mathscr{P}_2$ be the set of components of $ESPS \setminus \mathscr{P}_1$. By (ii) and (iii), $\mathscr{P}_2$ consists of $m' := |W_{1,h}|$ paths, each of which starts in $N_{M_i}(W_{1,h})$ and ends in $W_{k',h}$.

We are now ready to select the edges from U_i to U_{i+1}. For each $j' \in [k']$, let $v_{j',1}, \dots, v_{j',m'}$ be an enumeration of $W_{j',h}$ and denote by $w_{j',1}, \dots, w_{j',m'}$ the (unique) neighbours of $v_{j',1}, \dots, v_{j',m'}$ in M_i, respectively. Suppose without loss of generality that $\mathscr{P}_2$ consists of a $(w_{1,j'}, v_{k',j'})$-path $P_{j'}$ for each $j' \in [m']$. For each $j' \in [k'-1]$, let $M'_{j'} := \{v_{j',1}w_{j'+1,1}, \dots, v_{j',m'}w_{j'+1,m'}\}$. Let $M' := M'_1 \cup \cdots \cup M'_{k'-1}$.

Let $FESPS$ be the digraph on $\bigcup \mathcal{U}$ defined by $E(FESPS) := E(\mathscr{P}) \cup M'$. Observe that $FESPS \setminus FESPS(U_i, U_{i+1}) = \mathscr{P}$ and so (FESPS2) holds. Thus, it remains to prove that $FESPS$ is a spanning linear forest which is equivalent to $ESPS$ and that (FESPS1) holds.

Claim 1. *$FESPS$ is a spanning linear forest satisfying both $V^\pm(FESPS) = V^\pm(ESPS)$.*

Proof of Claim. By construction of $\mathscr{P}$, each $v \in \bigcup \mathcal{U}$ satisfies

$$d^+_{\mathscr{P}}(v) = \begin{cases} 1 & \text{if } v \in \bigcup \mathcal{U} \setminus U_i; \\ 0 & \text{if } v \in U_i; \end{cases} \quad \text{and} \quad d^-_{\mathscr{P}}(v) = \begin{cases} 1 & \text{if } v \in \bigcup \mathcal{U} \setminus U_{i+1}; \\ 0 & \text{if } v \in U_{i+1}. \end{cases}$$

By definition of M', each $v \in \bigcup \mathcal{U}$ satisfies

$$d^+_{M'}(v) = \begin{cases} 1 & \text{if } v \in \bigcup_{j' \in [k'-1]} W_{j',h}; \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad d^-_{M'}(v) = \begin{cases} 1 & \text{if } v \in \bigcup_{j' \in [k'-1]} N_{M_i}(W_{j'+1,h}); \\ 0 & \text{otherwise}. \end{cases}$$

Thus, $FESPS$ is spanning and $\Delta^0(FESPS) = 1$. Moreover, (ii) and (iii) imply that both $V^\pm(FESPS) = V^\pm(ESPS)$.

Suppose for a contradiction that $FESPS$ contains a cycle C. Since $\mathscr{P}$ is a linear forest, $E(C) \cap M' \neq \emptyset$. Let $j' \in [k'-1]$ be the largest index such that $E(C) \cap M'_{j'} \neq \emptyset$ and let $vw \in E(C) \cap M'_{j'}$. By construction of $M'_{j'}$, we have $w \in N_{M_i}(W_{j'+1,h})$. Let w' be the (unique) neighbour of w in M_i. Note that $w' \in W_{j'+1,h} \subseteq U_i$. By definition of $\mathscr{P}$, we have $w' \in V(C)$ and $d^+_{\mathscr{P}}(w') = 0$. Therefore, there exists $e \in E(C) \cap M'$ which starts at w'. By construction of M', we have $j' < k'-1$ and $e \in M'_{j'+1}$. But this contradicts the maximality of j' and so $FESPS$ does not contain a cycle. $\diamond$

Claim 2. *$ESPS$ and $FESPS$ are equivalent.*

Proof of Claim. Recall Definition 8.4. By Claim 1 and (i), we have $V(ESPS) = \bigcup \mathcal{U} = V(FESPS)$. Thus, it remains to find a bijection ϕ from the components of $ESPS$ to the components of $FESPS$ such that for each component P of $ESPS$, the paths P and $\phi(P)$ have the same starting and ending points.

Recall that $\mathscr{P}_1 \cup \mathscr{P}_2$ denotes the set of components of $ESPS$, where $\mathscr{P}_1$ consists of a (u,v)-path P_{uv} for each $uv \in M_i - (W_{1,h} \cup \dots \cup W_{k',h})$ and $\mathscr{P}_2$ consists of a $(w_{1,j'}, v_{k',j'})$-path $P_{j'}$ for each $j' \in [m']$.

Let $uv \in M_i - (W_{1,h} \cup \dots \cup W_{k',h})$. Let P'_{uv} be the (u,v)-path contained in $\mathscr{P}$. By construction of M', both $u, v \notin V(M')$. Thus, P'_{uv} is a component of $FESPS$ and so we can let $\phi(P_{uv}) \coloneqq P'_{uv}$.

Let $j' \in [m']$. By definition of M', we have both $w_{1,j'}, v_{k',j'} \notin V(M')$. Moreover,

$$v_{1,j'}w_{2,j'}, v_{2,j'}w_{3,j'}, \dots, v_{k'-1,j'}w_{k',j'} \in M'.$$

For each $i' \in [k']$, let $Q_{i'}$ be the $(w_{i',j'}, v_{i',j'})$-path contained in $\mathscr{P}$. Then,

$$P'_{j'} \coloneqq w_{1,j'}Q_1v_{1,j'}w_{2,j'}Q_2v_{2,j'}w_{3,j'} \dots w_{k'-1,j'}Q_{k'-1}v_{k'-1,j'}w_{k',j'}$$

is a component of $FESPS$ and so we can let $\phi(P_{j'}) \coloneqq P'_{j'}$.

By construction, ϕ is an injection from the components of $ESPS$ to the components of $FESPS$ such that for each component P of $ESPS$, the paths P and $\phi(P)$ have the same starting and ending points. Since $FESPS$ is a linear forest satisfying both $V^{\pm}(FESPS) = V^{\pm}(ESPS)$, ϕ is also a surjection. $\diamond$

Claim 3. (FESPS1) *is satisfied.*

Proof of Claim. Let D be the M_i-contraction of $FESPS[U_i, U_{i+1}] = M'[U_i, U_{i+1}]$ and let SPS be obtained from D by deleting all the isolated vertices. We need to show that SPS is an (ℓ^*, f, h, j)-special path system with respect to $\mathcal{P}_i^*$ and C^i. By (FESPS2), $F := M' \cup M_i$ is obtained from $FESPS$ by contracting each path in $\mathscr{P}$ into an edge from its starting point to its ending point. Together with Claim 1, (ii), and (iii), this implies that F is a linear forest satisfying the following properties.

- $V^+(F) = V^+(FESPS) = V^+(ESPS) = U_{i+1} \setminus N_{M_i}(W_{2,h} \cup \cdots \cup W_{k',h})$.
- $V^-(F) = V^-(FESPS) = V^-(ESPS) = U_i \setminus (W_{1,h} \cup \cdots \cup W_{k'-1,h})$.
- $V^0(F) \cap U_i = U_i \setminus (V^+(F) \cup V^-(F)) = W_{1,h} \cup \cdots \cup W_{k'-1,h}$.

Thus, Proposition 7.31 (applied with U_i, U_{i+1}, and M_i playing the roles of A, B, and M) implies that D is a linear forest satisfying the following properties.

- $V^+(D) = N_{M_i}(V^+(F)) = U_i \setminus (W_{2,h} \cup \cdots \cup W_{k',h})$.
- $V^-(D) = V^-(F) = U_i \setminus (W_{1,h} \cup \cdots \cup W_{k'-1,h})$.
- $V^0(D) = (V^0(F) \cap U_i) \setminus N_{M_i}(V^+(F)) = W_{2,h} \cup \cdots \cup W_{k'-1,h}$.

In particular, SPS is a linear forest satisfying $V^0(SPS) = V^0(D) = W_{2,h} \cup \cdots \cup W_{k'-1,h}$ and so (SPS2) holds. Note that the set of isolated vertices in D is precisely $V^+(D) \cap V^-(D)$. Thus, $V^+(SPS) = V^+(D) \setminus V^-(D) = W_{1,h}$ and $V^-(SPS) = V^-(D) \setminus V^+(D) = W_{k',h}$, so (SPS1) holds. Therefore, SPS is an (ℓ^*, f, h, j)-special path system and so (FESPS1) is satisfied. $\diamond$

This concludes the proof of Proposition 12.1. $\square$

Corollary 12.2. *Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a (K, ℓ^*, k, n)-cycle-framework and suppose that $\frac{k}{f} \in \mathbb{N}$. An (ℓ^*, K, f)-extended special factor ESF is a $(1+\ell^*(K-1)f)$-regular multidigraph.*

Proof. Let $\{ESPS_{h,i,j} \mid (h,i,j) \in [\ell^*] \times [K] \times [f]\}$ be the decomposition of ESF which witnesses that ESF is an (ℓ^*, K, f)-extended special factor. Let $i \in [K]$ and $v \in U_i$. By Proposition 12.1, there is a pair $(h,j) \in [\ell^*] \times [f]$ such that $v \notin V^-(ESPS_{h,i,j})$ but $v \in V^-(ESPS_{h',i,j'})$ for all $(h',j') \in ([\ell^*] \times [f]) \setminus \{(h,j)\}$. Moreover, $v \notin V^-(ESPS_{h',i',j'})$ for each $(h',i',j') \in [\ell^*] \times ([K] \setminus \{i\}) \times [f]$. By Proposition 12.1(i), $ESPS$ is spanning linear forest on $\bigcup \mathcal{U}$ and so

$$
\begin{aligned}
d^+_{ESF}(v) = d^+_{ESPS_{h,i,j}}(v) + &\sum_{(h',j') \in ([\ell^*] \times [f]) \setminus \{(h,j)\}} d^+_{ESPS_{h',i,j'}}(v) \\
&+ \sum_{(h',i',j') \in [\ell^*] \times ([K] \setminus \{i\}) \times [f]} d^+_{ESPS_{h',i',j'}}(v) \\
&= 1 + (\ell^* f - 1) \cdot 0 + \ell^*(K-1)f \cdot 1 = 1 + \ell^*(K-1)f.
\end{aligned}
$$

Since M_{i-1} is a perfect matching from U_i to U_{i-1}, one can apply similar arguments to show that there are precisely $\ell^* f - 1$ tuples $(h, i', j) \in [\ell^*] \times [K] \times [f]$ for which $v \in V^+(ESF)$ and so $d^-_{ESF}(v) = 1 + \ell^*(K-1)f$. □

12.2 Constructing extended special factors

Recall that the blow-up cycle version of the robust decomposition lemma (Lemma 11.10) can only be applied when there are no exceptional vertices (see (CST3)). In general, we will have a non-empty exceptional set $U^* \subseteq V(D)$ and so we will apply Lemma 11.10 with $D - U^*$ playing the role of D. As a result, the cycles obtained via Lemma 11.10 will not be Hamilton cycles on $V(D)$, but will only span $V(D) \setminus U^*$. We will incorporate the exceptional vertices into these cycles using the strategy described in Section 8.2.7: we will initially reserve $4s'$ special covers in D (see Definition 8.24) and then construct the extended special factors for Lemma 11.10 in such a way that each extended special path system contains the complete special sequence (see Definition 8.25) associated to one of the reserved special covers.

However, as described in Proposition 12.1, an extended special path system is a linear forest whose components have prescribed endpoints. Thus, our special covers will need to satisfy certain constraints. More precisely, let SC be a special cover and denote by M_{SC} the associated complete special sequence. Suppose that we want to construct an extended special path system which contains M_{SC}. Let P be a component of SC which is not an isolated vertex. By definition, M_{SC} contains an edge from the starting point u of P to the ending point v of P and so for any linear forest $F \supseteq M_{SC}$, we have $u \notin V^-(F)$ and $v \notin V^+(F)$. Thus, Proposition 12.1(ii) and (iii) imply that we require $u \notin U_i \setminus (W_{1,h} \cup \dots \cup W_{k'-1,h})$ and $v \notin U_{i+1} \setminus N_{M_i}(W_{2,h} \cup \dots \cup W_{k',h})$. Moreover, u and v will lie in a common connected component of F, so Proposition 12.1(iv) implies that we cannot have u and v lying in different edges of $M_i - (W_{1,h} \cup \dots \cup W_{k',h})$. For convenience, we will require that u and v completely avoid the vertices of $M_i - (W_{1,h} \cup \dots \cup W_{k',h})$. Altogether, this motivates the following definition.

Definition 12.3 (Localised special cover)**.** Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a $(4, \ell^*, k, n)$-cycle-framework. Let D be a digraph with $V(D) \supseteq \bigcup \mathcal{U}$ and denote by $U^* \coloneqq V(D) \setminus \bigcup \mathcal{U}$ the exceptional set of D. Suppose that $\frac{k}{f} \in \mathbb{N}$ and denote $k' \coloneqq \frac{k}{f}+1$. Let $(h, i, j) \in [\ell^*] \times [4] \times [f]$ and let $W_{1,h}, \dots, W_{k',h}$ be defined as in Proposition 12.1. A special cover SC in D with respect to U^* is $(\ell^*, 4, f, h, i, j)$*-localised (with respect to* $\mathcal{P}^*, \mathcal{C}$*, and* $\mathcal{M}$*)* if the following holds.

$$(V^+(SC) \cup V^-(SC)) \cap (U_i \cup U_{i+1}) \subseteq (W_{1,h} \cup \dots \cup W_{k'-1,h}) \cup N_{M_i}(W_{2,h} \cup \dots \cup W_{k',h}).$$

The next fact follows immediately from Definition 8.25.

Fact 12.4. *Let* $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ *be a* $(4, \ell^*, k, n)$*-cycle-framework. Let* D *be a digraph with* $V(D) \supseteq \bigcup \mathcal{U}$ *and denote by* $U^* \coloneqq V(D) \setminus \bigcup \mathcal{U}$ *the exceptional set of* D*. Suppose that* $\frac{k}{f} \in \mathbb{N}$ *and denote* $k' \coloneqq \frac{k}{f} + 1$*. Let* $(h, i, j) \in [\ell^*] \times [4] \times [f]$ *and let* $W_{1,h}, \dots, W_{k',h}$ *be defined as in Proposition* 12.1*. Suppose that* SC *is an* $(\ell^*, 4, f, h, i, j)$*-localised special cover in* D *with respect to* U^**. Then, the complete special sequence* M_{SC} *associated to* SC

satisfies

$$V(M_{SC}) \cap (U_i \cup U_{i+1}) \subseteq (W_{1,h} \cup \cdots \cup W_{k'-1,h}) \cup N_{M_i}(W_{2,h} \cup \cdots \cup W_{k',h}).$$

Our strategy for incorporating the complete special sequence M_{SC} into an extended special path system will be to extend each edge of M_{SC} into a longer path by "winding around" $\mathcal{U}$. This can be done greedily, with room to spare, so it will be possible to ensure that these paths all start and end in given small sets of vertices X and Y and avoid the vertices of a small set Z.

Lemma 12.5. *Let $0 < \frac{1}{n} \ll \varepsilon \ll 1$. Let D be a digraph and $U_1 \cup \cdots \cup U_4$ be a partition of $V(D)$ into vertex classes of size n. Suppose that $\delta(D[U_i, U_{i+1}]) \geq (1-\varepsilon)n$ for each $i \in [4]$ (where $U_5 \coloneqq U_1$). Let $X \subseteq U_1$ and $Y \subseteq U_4$. Let $Z \subseteq V(D) \setminus (X \cup Y)$ satisfy $|Z| \leq \varepsilon n$. Let M be a matching on $V(D) \setminus (X \cup Y \cup Z)$. Suppose that $|M| \leq |X| = |Y| \leq \varepsilon n$. For each $i \in [4]$, let n_i^+ and n_i^- be the number of edges in M which start and end in U_i, respectively. Suppose that $n_i^+ = n_{i+1}^-$ for each $i \in [4]$ (where $n_5^- \coloneqq n_1^-$). Then, there exists a set $\mathscr{P}$ of $|M|$ vertex-disjoint paths for which the following hold.*

(i) $M \subseteq \mathscr{P} \subseteq D \cup M$.

(ii) $V^+(\mathscr{P}) \subseteq X$, $V^-(\mathscr{P}) \subseteq Y$, *and* $V^0(\mathscr{P}) \subseteq V(D) \setminus (X \cup Y \cup Z)$.

(iii) $|V(\mathscr{P}) \cap U_1| = \cdots = |V(\mathscr{P}) \cap U_4| \leq 4|M|$.

Proof. Let $u_1v_1, \dots, u_mv_m$ be an enumeration of M. Let $x_1, \dots, x_m \in X$ and $y_1, \dots, y_m \in Y$ be distinct. For each $j \in [m]$, we will construct a path $x_jP'_ju_jv_jQ'_jy_j$ such that P'_j and Q'_j are paths of length between 4 and 8 which "wind around" $\mathcal{U}$.

For each $j \in [m]$, let $j^+, j^- \in [4]$ be such that $u_j \in U_{j^+}$ and $v_j \in U_{j^-}$. For each $i \in [4]$, let $U'_i \coloneqq U_i \setminus (X \cup Y \cup Z \cup V(M))$. Note that each $i \in [4]$ and $v \in U_i$ satisfy

$$|N_D^-(v) \cap U'_{i-1}| \geq (1-\varepsilon)n - |X| - |Y| - |Z| - 2|M| \geq (1-6\varepsilon)n > \frac{n}{2} + 4m \qquad (12.1)$$

and, similarly,

$$|N_D^+(v) \cap U'_{i+1}| > \frac{n}{2} + 4m. \tag{12.2}$$

Thus, one can greedily construct vertex-disjoint paths $P_1, \dots, P_m, Q_1, \dots, Q_m \subseteq D$ such that for each $j \in [m]$, $P_j = u_{j,3}u_{j,4}u'_{j,1} \dots u'_{j,j^+-1}u_j$ for some $u_{j,3} \in U'_3$, $u_{j,4} \in U'_4$, $u'_{j,1} \in U'_1, \dots, u'_{j,j^+-1} \in U'_{j^+-1}$ and, similarly, $Q_j = v_jv'_{j,j^-+1} \dots v'_{j,4}v_{j,1}v_{j,2}$ for some $v'_{j,j^-+1} \in U'_{j^-+1}, \dots, v'_{j,4} \in U'_4$, $v_{j,1} \in U'_1$, $v_{j,2} \in U'_2$. Then, (12.1) and (12.2) imply that there exist distinct $u_{1,2}, \dots, u_{m,2} \in U'_2 \setminus \bigcup_{j \in [m]} V(P_m \cup Q_m)$ and $v_{1,3}, \dots, v_{m,3} \in U'_3 \setminus \bigcup_{j \in [m]} V(P_m \cup Q_m)$ such that, for each $j \in [m]$, $u_{j,2} \in N_D^+(x_j) \cap N_D^-(u_{j,3})$ and $v_{j,3} \in N_D^+(v_{j,2}) \cap N_D^-(y_j)$. For each $j \in [m]$, denote $P'_j := x_ju_{j,2}u_{j,3}P_ju_j$ and $Q'_j := v_jQ_jv_{j,2}v_{j,3}y_j$. Let $\mathscr{P} := \{x_jP'_ju_jv_jQ'_jy_j \mid j \in [m]\}$. By construction, $\mathscr{P}$ is a set of vertex-disjoint paths satisfying (i) and (ii). It remains to verify (iii). For each $i \in [4]$ and $j \in [m]$, we have

$$|V(P'_j) \cap U_i| = \begin{cases} 2 & \text{if } i \leq j^+; \\ 1 & \text{otherwise;} \end{cases} \quad \text{and} \quad |V(Q'_j) \cap U_i| = \begin{cases} 2 & \text{if } i \geq j^-; \\ 1 & \text{otherwise.} \end{cases} \tag{12.3}$$

Recall that for each $i \in [4]$, n_i^+ denotes the number of indices $j \in [m]$ for which $j^+ = i$ and n_i^- denotes the number of indices $j \in [m]$ for which $j^- = i$. Therefore, each $i \in [4]$ satisfies

$$\begin{aligned} |V(\mathscr{P}) \cap U_i| &= \sum_{j \in [m]} |V(P'_j) \cap U_i| + \sum_{j \in [m]} |V(Q'_j) \cap U_i| \\ &\overset{(12.3)}{=} (m + n_i^+ + \dots + n_4^+) + (m + n_1^- + \dots + n_i^-) \\ &= 2m + (n_{i+1}^- + \dots + n_5^-) + (n_1^- + \dots + n_i^-) = 3m + n_1^-. \end{aligned}$$

Thus, (iii) holds. □

Note that in the proof of Lemma 12.5 the conditions on the number of paths starting and ending in each vertex class was necessary to obtain a set $\mathscr{P}$ of vertex-disjoint paths which covers the same number of vertices from each vertex class (see Lemma 12.5(iii)). Eventually, we will want to extend $\mathscr{P}$ to a full extended special path system. The number of vertices covered by $\mathscr{P}$ will thus be of particular importance since, by Proposition 12.1(i), an extended special path system needs to span all the vertex classes $U_1, \dots, U_4$, which are all of the same size (see (CF1)). This motivates the following definition.

Definition 12.6 (Balanced special cover)**.** Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a $(4, \ell^*, k, n)$-cycle-framework. Let D be a digraph with $V(D) \supseteq \bigcup \mathcal{U}$ and denote by $U^* \coloneqq V(D) \setminus \bigcup \mathcal{U}$ the exceptional set of D. Let SC be a special cover in D with respect to U^*. For each $i \in [4]$, let n_i^+ and n_i^- be the number of components of SC which are not isolated vertices and start and end in U_i, respectively. We say that SC is *$\mathcal{U}$-balanced* if $n_i^+ = n_{i+1}^-$ for each $i \in [4]$ (where $n_5^- \coloneqq n_1^-$).

The next lemma states that given small special covers which are localised and balanced, one can incorporate the associated complete special sequences into extended special path systems. (Note that Lemma 12.7 below is the analogue of Lemma 9.1 from the bipartite robust outexpander case. The only difference is that, in Lemma 9.1, we also constructed the special covers at the same time. In the context of Theorem 4.4, constructing the special covers is much more difficult because of the backward edges and so this will be done separately at a later stage.)

Lemma 12.7 (Constructing extended special path systems and factors from special covers)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \frac{1}{k} \ll \varepsilon' \ll 1$ and $\frac{1}{k} \ll \frac{1}{f}, \frac{1}{\ell^*} \leq 1$. Let r be an integer satisfying $\frac{f(\ell^*)^2 rk}{n} \ll 1$. Suppose that $\ell^* f \geq 2$ and $\frac{k}{f} \in \mathbb{N}$. Let $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ be a $(4, \ell^*, k, n)$-cycle-framework. Let D be a digraph with $V(D) \supseteq \bigcup \mathcal{U}$ and denote by $U^* \coloneqq V(D) \setminus \bigcup \mathcal{U}$ the exceptional set of D. Suppose that the following hold for each $i \in [4]$.*

(i) *For any cluster $V \in \mathcal{P}_i^*$, the set $N_{M_i}(V)$ is a cluster in $\mathcal{P}_{i+1}^*$ (where $\mathcal{P}_5^* \coloneqq \mathcal{P}_1^*$).*

(ii) *$D[V, W]$ is $[\varepsilon', \geq 1 - \varepsilon']$-superregular whenever $V \subseteq U_i$ and $W \subseteq U_{i+1}$ are unions of clusters in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$, respectively.*

Let $\mathcal{SC} = \{SC_{\ell,h,i,j} \mid (\ell, h, i, j) \in [r] \times [\ell^*] \times [4] \times [f]\}$ be a set of edge-disjoint special covers in D with respect to U^* such that the following hold for each $(\ell, h, i, j) \in [r] \times [\ell^*] \times [4] \times [f]$.

(iii) $|SC_{\ell,h,i,j}| \leq \varepsilon n$. *In particular, Definition 8.25 implies that the complete special sequence* $M_{\ell,h,i,j}$ *associated to* $SC_{\ell,h,i,j}$ *satisfies* $|M_{\ell,h,i,j}| \leq \varepsilon n$.

(iv) $SC_{\ell,h,i,j}$ *is* $(\ell^*, 4, f, h, i, j)$*-localised.*

(v) $SC_{\ell,h,i,j}$ *is* $\mathcal{U}$*-balanced.*

Then, there exist r $(\ell^*, 4, f)$*-extended special factors* $ESF_1, \dots, ESF_r$ *with respect to* $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$*, and* $\mathcal{M}$ *such that the following hold, where for each* $(\ell, h, i, j) \in [r] \times [\ell^*] \times [4] \times [f]$*,* $ESPS_{\ell,h,i,j}$ *denotes the* $(\ell^*, 4, f, h, i, j)$*-special path system contained in* ESF_ℓ*.*

(a) *For each* $(\ell, h, i, j) \in [r] \times [\ell^*] \times [4] \times [f]$*, we have* $M_{\ell,h,i,j} \subseteq ESPS_{\ell,h,i,j} \subseteq (D \setminus \mathcal{SC}) \cup M_{\ell,h,i,j}$.

(b) *Let* $(\ell, h, i, j), (\ell', h', i', j') \in [r] \times [\ell^*] \times [4] \times [f]$ *be distinct. Then, we have* $(ESPS_{\ell,h,i,j} \setminus M_{\ell,h,i,j}) \cap (ESPS_{\ell',h',i',j'} \setminus M_{\ell',h',i',j'}) = \emptyset$.

Roughly speaking, Lemma 12.7(a) means that each complete special sequence is incorporated into a distinct extended special path system, while Lemma 12.7(b) states that each edge of $D \setminus \mathcal{SC}$ is incorporated into at most one of the extended special path systems.

Proof of Lemma 12.7. First, recall from (CF2) and (CF3) that for each $i \in [4]$, U_i is the union of the clusters in $\mathcal{P}_i^*$. Thus, (ii) implies that each $i \in [4]$ satisfies

$$\delta(D[U_i, U_{i+1}]) \geq (1 - 2\varepsilon')n. \tag{12.4}$$

Fix additional constants ε_1 and ε_2 such that $\frac{f(\ell^*)^2 rk}{n}, \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll 1$. Suppose inductively that for some $0 \leq t \leq 4r\ell^* f$ we have constructed a set $T_t \subseteq [r] \times [\ell^*] \times [4] \times [f]$ of size t and a set $\mathcal{ESPS}_t = \{ESPS_{\ell,h,i,j} \mid (\ell, h, i, j) \in T_t\}$ such that the following properties hold.

(a′) For each $(\ell, h, i, j) \in T_t$, we have $M_{\ell,h,i,j} \subseteq ESPS_{\ell,h,i,j} \subseteq (D \setminus \mathcal{SC}) \cup M_{\ell,h,i,j}$.

(b′) Let $(\ell, h, i, j), (\ell', h', i', j') \in T_t$ be distinct. Then, we have $(ESPS_{\ell,h,i,j} \setminus M_{\ell,h,i,j}) \cap (ESPS_{\ell',h',i',j'} \setminus M_{\ell',h',i',j'}) = \emptyset$.

(c′) For each $(\ell, h, i, j) \in T_t$, $ESPS_{\ell,h,i,j}$ is an $(\ell^*, 4, f, h, i, j)$-extended special path system with respect to $\mathcal{U}, \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$.

First, suppose that $t = 4r\ell^* f$. Then, define $ESF_\ell \coloneqq \bigcup_{(h,i,j)\in[\ell^*]\times[4]\times[f]} ESPS_{\ell,h,i,j}$ for each $\ell \in [r]$. By (c′), $ESF_1, \dots, ESF_r$ are $(\ell^*, 4, f)$-extended special factors. Moreover, (a) and (b) follow from (a′) and (b′).

We may therefore assume that $t < 4r\ell^* f$. Let $(\ell, h, i, j) \in ([r] \times [\ell^*] \times [4] \times [f]) \setminus T_t$ and define $T_{t+1} \coloneqq T_t \cup \{(\ell, h, i, j)\}$. We will now construct $ESPS_{\ell,h,i,j}$ as follows. Let $D' \coloneqq D \setminus (\mathcal{SC} \cup \mathcal{ESPS}_t)$. Since $\mathcal{SC}$ consists of $4r\ell^* f$ linear forests and $\mathcal{ESPS}_t$ consists of t linear forests, we have

$$\Delta^0(D \setminus D') \leq 4r\ell^* f + t \leq \frac{\varepsilon_1 n}{k\ell^*}. \tag{12.5}$$

By (iii) and Lemma 7.13 (applied with $2\varepsilon, U_{i-2}, U_{i-1}, U_i, U_{i+1}, \mathcal{P}^*_{i-2}, \dots, \mathcal{P}^*_{i+1}$, and $V(M_{\ell,h,i,j})$ playing the roles of $\varepsilon, U_1, \dots, U_4, \mathcal{P}_1, \dots, \mathcal{P}_4$, and S), we may assume without loss of generality that there exists, for each $i' \in [4] \setminus \{i, i+1\}$, a cluster $V_{i'} \in \mathcal{P}^*_{i'}$ which satisfies

$$V(M_{\ell,h,i,j}) \cap U_{i'} \subseteq V_{i'}. \tag{12.6}$$

(Otherwise, we may simply apply the arguments below with the partitions $\mathcal{P}'_{i-2}, \dots, \mathcal{P}'_{i+1}$ guaranteed by Lemma 7.13 playing the roles of $\mathcal{P}^*_{i-2}, \dots, \mathcal{P}^*_{i+1}$. This is possible since these satisfy (ii) up to a slightly worse ε'-parameter.)

To ensure that (c′) is satisfied, we will use Proposition 12.1 and construct a linear forest which satisfies Proposition 12.1(i)–(iv). Let $W_{1,h}, \dots, W_{k',h}$ be defined as in Proposition 12.1 and denote $M'_i \coloneqq M_i - (W_{1,h} \cup \dots \cup W_{k',h})$. Let

$$X_i \coloneqq W_{k',h} \quad \text{and} \quad X_{i+1} \coloneqq N_{M_i}(W_{1,h}). \tag{12.7}$$

By (iv) and Fact 12.4, we have

$$V(M_{\ell,h,i,j}) \cap (X_i \cup X_{i+1}) = \emptyset. \tag{12.8}$$

For each $i' \in [4] \setminus \{i, i+1\}$, fix a cluster

$$X_{i'} \in \mathcal{P}^*_{i'} \setminus \{V_{i'}\}. \tag{12.9}$$

By (12.6), we have

$$V(M_{\ell,h,i,j}) \cap X_{i'} = \emptyset. \tag{12.10}$$

By (ii), $D[X_{i'}, X_{i'+1}]$ is $[\varepsilon', \geq 1-\varepsilon']$-superregular for each $i' \in [4]$ (where $X_5 \coloneqq X_1$). We will reserve these superregular pairs to finish off the construction of $ESPS_{\ell,h,i,j}$.

Step 1: Constructing the components for Proposition 12.1(iv). In this step, we will use Lemma 7.11 to construct a set $\mathscr{P}_1$ of vertex-disjoint paths which consists of one (u,v)-path for each $uv \in M'_i$. The paths in $\mathscr{P}_1$ will eventually be incorporated as components of $ESPS_{\ell,h,i,j}$ to ensure that Proposition 12.1(iv) is satisfied.

Let

$$U'_i \coloneqq U_i \cap V(M'_i) \quad \text{and} \quad U'_{i+1} \coloneqq U_{i+1} \cap V(M'_i). \tag{12.11}$$

By (CF2), (CF3), (CF5), and (i), U'_i and U'_{i+1} are unions of $k\ell^* - k'$ clusters in $\mathcal{P}^*_i$ and $\mathcal{P}^*_{i+1}$, respectively. Moreover, note for later that (iv) and Fact 12.4 imply that

$$V(M_{\ell,h,i,j}) \cap (U'_i \cup U'_{i+1}) = \emptyset. \tag{12.12}$$

For each $i' \in [4] \setminus \{i, i+1\}$, let $U'_{i'} \subseteq U_i$ be the union of $k\ell^* - k'$ clusters in $\mathcal{P}^*_{i'} \setminus \{V_{i'}, X_{i'}\}$ and note for later that, by (12.6), we have

$$V(M_{\ell,h,i,j}) \cap U'_{i'} = \emptyset. \tag{12.13}$$

By (CF1)–(CF3), we have

$$n' \coloneqq |U'_1| = \cdots = |U'_4| = n - \frac{k'n}{k\ell^*}. \tag{12.14}$$

Denote $\mathcal{U}' \coloneqq (U'_1, \dots, U'_4)$. By (ii), (12.5), and Proposition 7.8(ii), $D'[U'_{i'}, U'_{i'+1}]$ is $[\varepsilon_2, \geq 1-\varepsilon_2]$-superregular for each $i' \in [4]$. Let $\mathscr{P}_1$ be the set of vertex-disjoint paths obtained by applying Lemma 7.11 with $D'[\bigcup \mathcal{U}'], U'_{i+1}, \dots, U'_i, 4, n', \varepsilon_2, 1-\varepsilon_2$, and M'_i playing the roles of $D, V_1, \dots, V_k, k, m, \varepsilon, d$, and $\{u_1v_1, \dots, u_mv_m\}$. Then, $V(\mathscr{P}_1) = \bigcup \mathcal{U}'$ and $\mathscr{P}_1$ consists of a (u,v)-path for each $uv \in M'_i$, as desired.

Step 2: Incorporating $M_{\ell,h,i,j}$. In order to satisfy (a′), we will now use Lemma 12.5 to construct a set $\mathscr{P}_2$ of vertex-disjoint paths which cover all the edges in $M_{\ell,h,i,j}$.

Recall from Step 1, (12.7), and (12.9) that, for each $i' \in [4]$, $U'_{i'}$ is the union of $k\ell^* - k'$ clusters in $\mathcal{P}^*_{i'}$ and $X_{i'} \subseteq U_{i'} \setminus U'_{i'}$ is a cluster in $\mathcal{P}^*_{i'}$. In particular, (CF2) and (CF3) imply that for each $i' \in [4]$, $U_{i'} \setminus U'_{i'}$ is the union of k' clusters in $\mathcal{P}^*_{i'}$, so

$$|U_1 \setminus U'_1| = \cdots = |U_4 \setminus U'_4| = \frac{k'n}{k\ell^*} \tag{12.15}$$

and $Z \coloneqq \bigcup_{i' \in [4] \setminus \{i,i+1\}} X_{i'}$ satisfies

$$\frac{|M_{\ell,h,i,j}|}{\varepsilon_2} \overset{\text{(iii)}}{\leq} \frac{n}{k\ell^*} = |X_1| = \cdots = |X_4| \leq |Z| \leq \frac{\varepsilon_1 k'n}{k\ell^*}. \tag{12.16}$$

By (ii), (12.5), and (12.15), each $i' \in [4]$ satisfies

$$\delta(D'[U_{i'} \setminus U'_{i'}, U_{i'+1} \setminus U'_{i'+1}]) \geq (1-2\varepsilon')\frac{k'n}{k\ell^*} - \frac{\varepsilon_1 n}{k\ell^*} \geq (1-\varepsilon_1)\frac{k'n}{k\ell^*}.$$

Moreover, (12.8), (12.10), (12.12), and (12.13) imply that

$$V(M_{\ell,h,i,j}) \subseteq \bigcup \mathcal{U} \setminus \left(\bigcup_{i' \in [4]} X_{i'} \cup \bigcup \mathcal{U}' \right).$$

For each $i' \in [4]$, let $n_{i'}^+$ and $n_{i'}^-$ be the number of edges of $M_{\ell,h,i,j}$ which start and end in $U_{i'}$, respectively. Since by (v) $SC_{\ell,h,i,j}$ is $\mathcal{U}$-balanced, Definitions 8.25 and 12.6 imply that $n_{i'}^+ = n_{i'+1}^-$ for each $i' \in [4]$. Thus, we can let $\mathscr{P}_2$ be the set of vertex-disjoint paths obtained by applying Lemma 12.5 with $D' - \bigcup \mathcal{U}', U_{i+1} \setminus U'_{i+1}, U_{i+2} \setminus U'_{i+2}, \dots, U_{i-1} \setminus U'_{i-1}, U_i \setminus U'_i, X_{i+1}, X_i, \varepsilon_1, \frac{k'n}{k\ell^*}$, and $M_{\ell,h,i,j}$ playing the roles of $D, U_1, \dots, U_4, X, Y, \varepsilon, n$, and M.

Step 3: Covering the remaining vertices. In order to satisfy Proposition 12.1(i), we will now use Lemma 7.11 and Corollary 7.12 to construct a set $\mathscr{P}_3$ of vertex-disjoint paths which cover all the vertices in $\bigcup \mathcal{U} \setminus V(\mathscr{P}_1 \cup \mathscr{P}_2)$.

By (12.16) and Lemma 12.5(ii), there exist $x \in X_{i+1} \setminus V(\mathscr{P}_2)$ and $y \in X_i \setminus V(\mathscr{P}_2)$. Denote $U''_{i+1} \coloneqq X_{i+1} \setminus (V(\mathscr{P}_2) \cup \{x\})$ and $U''_i \coloneqq X_i \setminus (V(\mathscr{P}_2) \cup \{y\})$. Let $U^*_{i+1} \coloneqq U_{i+1} \setminus (V(\mathscr{P}_1 \cup \mathscr{P}_2) \cup U''_{i+1})$ and $U^*_i \coloneqq U_i \setminus (V(\mathscr{P}_1 \cup \mathscr{P}_2) \cup U''_i)$. By Step 1 and Lemma 12.5(ii) and (iii), we have

$$n'' \coloneqq |U''_{i+1}| = |U''_i| = |X_i| - |M_{\ell,h,i,j}| - 1 \overset{(12.16),(iii)}{\geq} \frac{n}{k\ell^*} - \varepsilon n - 1 \geq (1 - \varepsilon_1)\frac{n}{k\ell^*} \tag{12.17}$$

and

$$n^* \coloneqq |U^*_{i+1}| = |U^*_i| = n - n' - |X_i| + 1 \overset{(12.14),(12.16)}{\geq} (1 - \varepsilon_1)\frac{k'n}{k\ell^*}. \tag{12.18}$$

For each $i' \in [4] \setminus \{i, i+1\}$, Step 1 and Lemma 12.5(ii) imply that $X_{i'} \cap V(\mathscr{P}_1 \cup \mathscr{P}_2) = \emptyset$ and so we can let $U''_{i'} \subseteq X_{i'}$ satisfy $|U''_{i'}| = n''$. For each $i' \in [4] \setminus \{i, i+1\}$, let $U^*_{i'} \coloneqq U_{i'} \setminus V(\mathscr{P}_1 \cup \mathscr{P}_2 \cup U''_{i'})$ and observe that, by Step 1 and Lemma 12.5(iii), we have $|U^*_{i'}| = n^*$. Denote $\mathcal{U}'' \coloneqq (U''_1, \dots, U''_4)$ and $\mathcal{U}^* \coloneqq (U^*_1, \dots, U^*_4)$.

Claim 1. *For each $i' \in [4]$, $D'[U''_{i'}, U''_{i'+1}]$ and $D'[U^*_{i'}, U^*_{i'+1}]$ are both $[\varepsilon_2, \geq 1 - \varepsilon_2]$-superregular.*

Proof of Claim. Let $i' \in [4]$. Recall that $X_{i'}$ and $X_{i'+1}$ are clusters of size $\frac{n}{k\ell^*}$ in $\mathcal{P}^*_{i'}$ and $\mathcal{P}^*_{i'+1}$, respectively. Thus, (ii) implies that $D[X_{i'}, X_{i'+1}]$ is $[\varepsilon', \geq 1 - \varepsilon']$-superregular. By (12.17), $U''_{i'}$ and $U''_{i'+1}$ are obtained from $X_{i'}$ and $X_{i'+1}$ by deleting at most $\varepsilon_1|X_{i'}| = \varepsilon_1|X_{i'+1}|$

vertices. Thus, (12.5) and Proposition 7.8(ii) imply that $D'[U''_{i'}, U''_{i'+1}]$ is still $[\varepsilon_2, \geq 1-\varepsilon_2]$-superregular.

Similarly, recall from Step 1 that $U_{i'} \setminus V(\mathscr{P}_1)$ and $U_{i'+1} \setminus V(\mathscr{P}_1)$ are the unions of k' clusters of size $\frac{n}{k\ell^*}$ in $\mathcal{P}^*_{i'}$ and $\mathcal{P}^*_{i'+1}$, respectively. Thus, (ii) implies that $D[U_{i'} \setminus V(\mathscr{P}_1), U_{i'+1} \setminus V(\mathscr{P}_1)]$ is $[\varepsilon', \geq 1-\varepsilon']$-superregular. By (12.18), $U^*_{i'}$ and $U^*_{i'+1}$ are obtained from $U_{i'} \setminus V(\mathscr{P}_1)$ and $U_{i'+1} \setminus V(\mathscr{P}_1)$ by deleting at most $\varepsilon_1|U_{i'} \setminus V(\mathscr{P}_1)| = \varepsilon_1|U_{i'+1} \setminus V(\mathscr{P}_1)|$ vertices. Thus, (12.5) and Proposition 7.8(ii) imply that $D'[U^*_{i'}, U^*_{i'+1}]$ is still $[\varepsilon_2, \geq 1-\varepsilon_2]$-superregular. $\diamond$

Let $u_1, \dots, u_{n''}$ and $v_1, \dots, v_{n''}$ be enumerations of U''_{i+1} and U''_i. Let $\mathscr{P}'_3$ be the set of vertex-disjoint paths obtained by applying Lemma 7.11 with $D'[\bigcup \mathcal{U}''], U''_{i+1}, \dots, U''_i, 4, n''$, ε_2, and $1-\varepsilon_2$ playing the roles of $D, V_1, \dots, V_k, k, m, \varepsilon$, and d. Apply Corollary 7.12 with $D'[\bigcup \mathcal{U}^*], U^*_{i+1}, \dots, U^*_i, 4, n^*, \varepsilon_2, 1-\varepsilon_2, x$, and y playing the roles of $D, V_1, \dots, V_k, k, m, \varepsilon, d$, u, and v to obtain a Hamilton (x,y)-path P of $D'[\bigcup \mathcal{U}^*]$. Denote $\mathscr{P}_3 \coloneqq \mathscr{P}'_3 \cup \{P\}$. By Lemma 7.11 and Corollary 7.12, $\mathscr{P}_3$ is a set of vertex-disjoint paths satisfying $V(\mathscr{P}_3) = \bigcup \mathcal{U} \setminus V(\mathscr{P}_1 \cup \mathscr{P}_2)$, $V^+(\mathscr{P}_3) = X_{i+1} \setminus V(\mathscr{P}_2)$ and $V^-(\mathscr{P}_3) = X_i \setminus V(\mathscr{P}_2)$.

Let $ESPS_{\ell,h,i,j} \coloneqq \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$ and denote $\mathcal{ESPS}_{t+1} \coloneqq \mathcal{ESPS}_t \cup \{ESPS_{\ell,h,i,j}\}$. Then, (a$'$) holds by Lemma 12.5(i) and definition of D', while (b$'$) holds by definition of D'. It remains to show that (c$'$) holds. By construction, $ESPS_{\ell,h,i,j}$ is a linear forest and so Proposition 12.1 implies that it is enough to verify that $ESPS_{\ell,h,i,j}$ satisfies Proposition 12.1(i)–(iv). Note that Proposition 12.1(i) follows from Step 3 and Proposition 12.1(iv) follows from Step 1. By construction, (12.7), and (12.11), we have $V^+(ESPS_{\ell,h,i,j}) = U'_{i+1} \cup X_{i+1} = U_{i+1} \setminus N_{M_i}(W_{2,h} \cup \dots \cup W_{k',h})$ and $V^-(ESPS_{\ell,h,i,j}) = U'_i \cup X_i = U_i \setminus (W_{1,h} \cup \dots \cup W_{k'-1,h})$. Thus, Proposition 12.1(ii) and (iii) hold and we are done. $\square$

12.3 Constructing a cycle-setup

Finally, we build the cycle-setup required for the robust decomposition lemma for blow-up cycles (Lemma 11.10). Given a very dense blow-up C_4, say D, one could of course construct a cycle-setup by first randomly partitioning the vertices of D and then use Lemma 7.16 to exhibit the desired (super)regular pairs for (CST3). However, our cycle-setup will need to satisfy additional properties.

- To construct the extended special factors, $\mathcal{P}^*$ and $\mathcal{M}$ need to satisfy Lemma 12.7(i) and (ii). (This motivates Lemma 12.10(i) and (ii) below.)
- After constructing the extended special factors, the clusters in $\bigcup \mathcal{P}$ and $\bigcup \mathcal{P}'$ may no longer form suitable (super)regular pairs. To solve this problem, we randomly partition the edges of D into a dense digraph D_1 and a sparse digraph D_2. We will only use D_1 to construct the extended special factors and reserve D_2 for the application of Lemma 11.10. (This motivates Lemma 12.10(iii) below.) (Recall that a similar strategy was used in the robust outexpander case, see Chapter 9.)
- To be able to construct the localised and balanced special covers required for Lemma 12.7, we will need the backward edges of our bipartite tournament T to be well distributed across the clusters in $\mathcal{P}$ and $\mathcal{P}^*$. Since there may be relatively few backward edges, this cannot be guaranteed via a simple application of Lemma 7.16. This explains why, in Lemma 12.10, we construct a cycle-setup with respect to given sets of partitions $\mathcal{P}$ and $\mathcal{P}^*$. (To help the reader gain intuition for this step, we will only detail the construction of $\mathcal{P}$ and $\mathcal{P}^*$ after we have discussed our strategy for decomposing backward edges.)

Note that in Lemma 12.10, we assume that the minimum semidegree of D is very large. While $\overrightarrow{T}_{\mathcal{U}}$ (that is, the digraph which consists of all the forward edges of T (see Section 10.2)) is very dense, its minimum semidegree may be low (the backward edges may be concentrated on a few vertices). To solve this problem, we will assign the few vertices of large backward degree into the exceptional set U^* and then apply Lemma 12.10 with $\overrightarrow{T}_{\mathcal{U}} - U^*$ playing the role of D. This will ensure that the minimum semidegree condition in Lemma 12.10 is satisfied.

For technical reasons, we will need the matchings in $\mathcal{M}$ to satisfy a stronger property than Lemma 12.7(i). Roughly speaking, Lemma 12.7(i) states that, for each $i \in [4]$, M_i matches the clusters in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$. It will be convenient that each M_i matches "corresponding" clusters together.

Definition 12.8 (Consistent cycle-framework)**.** We say that a $(4, \ell^*, k, n)$-cycle-framework $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is *consistent* if the following holds for each $(h, i, j) \in [\ell^*] \times [4] \times [k]$. Let $V_{i,j}$ and $V_{i+1,j}$ denote the j^{th} clusters in $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$, respectively. Let $V_{i,j,h}$ and $V_{i+1,j,h}$ denote the h^{th} subclusters of $V_{i,j}$ and $V_{i+1,j}$ contained in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$, respectively. Then, $N_{M_i}(V_{i,j,h}) = V_{i+1,j,h}$.

Recall from Fact 11.5 that a cycle-framework remains a cycle-framework when $\mathcal{P}^*$ is replaced by $\mathcal{P}$. Observe that consistency is also preserved.

Fact 12.9. *Suppose that $(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is a consistent $(4, \ell^*, k, n)$-cycle-framework. Then, $(\mathcal{U}, \mathcal{P}, \mathcal{P}, \mathcal{C}, \mathcal{M})$ is also a consistent $(4, 1, k, n)$-cycle-framework.*

We are now ready to construct our cycle-setup.

Lemma 12.10. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \varepsilon' \ll \frac{1}{k} \ll \frac{1}{\ell'}, \frac{1}{\ell^*}, d \ll 1$ and denote $m := \frac{n}{k}$. Suppose that $\frac{m}{\ell'}, \frac{m}{\ell^*} \in \mathbb{N}$. Let $U_1, \dots, U_4$ be disjoint vertex sets of size n and denote $\mathcal{U} := (U_1, \dots, U_4)$. Let D be a blow-up C_4 with vertex partition $\mathcal{U}$. Suppose that $\delta^0(D) \geq (1-\varepsilon)n$. For each $i \in [4]$, let $\mathcal{P}_i$ be a partition of U_i into an empty exceptional set and k clusters of size m, let $\mathcal{P}_i^*$ be an ℓ^*-refinement of $\mathcal{P}_i$, and let C^i be a Hamilton cycle on the clusters in $\mathcal{P}_i$. Denote $\mathcal{P} := (\mathcal{P}_1, \dots, \mathcal{P}_4)$, $\mathcal{P}^* := (\mathcal{P}_1^*, \dots, \mathcal{P}_4^*)$, and $\mathcal{C} := (C^1, \dots, C^4)$. Then, there exist $D_1, \mathcal{P}', \mathcal{R}, \mathscr{U}, \mathscr{U}'$, and $\mathcal{M}$ for which the following hold, where $D_2 := D \setminus D_1$.*

(i) *$(\mathcal{U}, \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is a consistent $(4, \ell^*, k, n)$-cycle-framework. In particular, the following hold. For any $i \in [4]$ and any cluster $V \in \mathcal{P}_i$, the set $N_{M_i}(V)$ is a cluster in $\mathcal{P}_{i+1}$ (where $\mathcal{P}_5 := \mathcal{P}_1$). The analogue holds for the partitions in $\mathcal{P}^*$.*

(ii) *For each $i \in [4]$, $D_1[V, W]$ is $[\varepsilon', \geq 1-3d]$-superregular whenever $V \subseteq U_i$ and $W \subseteq U_{i+1}$ are unions of clusters in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$, respectively. In particular, since $\mathcal{P}_i^*$ is a refinement of $\mathcal{P}_i$ for each $i \in [4]$, the analogue holds for the partitions in $\mathcal{P}$.*

(iii) *$(D_2, \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ is a $(4, \ell', \ell^*, k, m, \varepsilon', d)$-cycle-setup.*

Proof. Fix additional constants $\varepsilon_1, \varepsilon_2$, and ε_3 satisfying $\varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon'$. First, we construct the matchings in $\mathcal{M}$. For each $i \in [4]$, let $V_{i,1}, \dots, V_{i,k}$ be an enumeration of the clusters in $\mathcal{P}_i$ and, for each $(h, j) \in [\ell^*] \times [k]$, denote by $V_{i,j,h}$ the h^{th} subcluster of $V_{i,j}$ contained in $\mathcal{P}_i^*$. For each $(h, i, j) \in [\frac{q}{f}] \times [4] \times [k]$, let $M_{h,i,j}$ be an auxiliary perfect matching from $V_{i+1,j,h}$ to $V_{i,j,h}$. For each $i \in [4]$, define $M_i \coloneqq \bigcup_{(h,j) \in [\frac{q}{f}] \times [k]} M_{h,i,j}$. Let $\mathcal{M} \coloneqq (M_1, \dots, M_4)$ and observe that (i) and (CST2) hold.

Let $i \in [4]$. Denote by $\widetilde{D}_i$ the M_i-contraction of $D[U_i, U_{i+1}]$. By Fact 7.27(i), $\delta^0(\widetilde{D}_i) \geq (1-2\varepsilon)n$ and so Lemma 8.10 implies that $\mathcal{P}_i^*$ is an $\sqrt{2\varepsilon}$-uniform refinement of $\mathcal{P}_i$ with respect to $\widetilde{D}_i$.

Let $i \in [4]$ and let $V \subseteq U_i$ and $W \subseteq U_{i+1}$ be unions of clusters in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$, respectively. Then, observe that each $v \in V$ satisfies $|N_D^+(v) \cap W| \geq |W| - \varepsilon n \geq (1-\varepsilon_1)|W|$ and, similarly, each $w \in W$ satisfies $|N_D^-(v) \cap V| \geq |V| - \varepsilon n \geq (1-\varepsilon_1)|V|$. Thus, Proposition 7.10 implies that $D[V, W]$ is $[\varepsilon_2, \geq 1-\varepsilon_2]$-superregular.

Let D_1 be obtained from D by selecting each edge independently with probability $1-2d$. Denote $D_2 \coloneqq D \setminus D_1$. For each $i \in [4]$, denote by $\widetilde{D}_i'$ the M_i-contraction of $D_2[U_i, U_{i+1}]$ and observe that, by definition, $\widetilde{D}_i'$ is obtained from $\widetilde{D}_i$ by selecting each edge independently with probability $2d$. Thus, Corollary 7.18 and Lemma 8.9 imply that we may assume that the following hold.

(ii′) For each $i \in [4]$, $D_1[V, W]$ is $[\varepsilon_3, \geq 1-3d]$-superregular and $D_2[V, W]$ is $[\varepsilon_3, \geq d+\varepsilon_3]$-superregular whenever $V \subseteq U_i$ and $W \subseteq U_{i+1}$ are unions of clusters in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$, respectively.

(iii′) For each $i \in [4]$, $\mathcal{P}_i^*$ is an ε'-uniform ℓ^*-refinement of $\mathcal{P}_i$ with respect to $\widetilde{D}_i'$.

In particular, (ii) is satisfied.

We now construct the cycle-setup. First, observe that since D is a blow-up C_4 with vertex partition $\mathcal{U}$, D_2 satisfies (CST1). Let $\mathcal{P}_i'$ be the ε-uniform ℓ'-refinement of $\mathcal{P}_i$ obtained by applying Lemma 8.7 with $\widetilde{D}_i', \mathcal{P}_i$, and ℓ' playing the roles of $D, \mathcal{P}$, and ℓ. Let R_i be the complete digraph on the clusters in $\mathcal{P}_i$ and note that C^i is a Hamilton cycle of R_i. Let U^i be the universal walk for C^i with parameter ℓ' obtained by applying Lemma 8.12 with R_i and C^i playing the roles of R and C. Let U'^i be the closed walk on the clusters in $\mathcal{P}_i'$ obtained from U^i as described in (ST6).

Denote $\mathcal{P}' := (\mathcal{P}_1', \dots, \mathcal{P}_4')$, $\mathcal{R} := (R_1, \dots, R_4)$, $\mathscr{U} := (U^1, \dots, U^4)$, and $\mathscr{U}' := (U'^1, \dots, U'^4)$. Let $i \in [4]$. We need to show that $(\widetilde{D}_i', \mathcal{P}_i, \mathcal{P}_i', \mathcal{P}_i^*, R_i, C^i, U^i, U'^i)$ is an $(\ell', \ell^*, k, m, \varepsilon', d)$-setup. By construction and (iii′), properties (ST1), (ST4)–(ST6), and (ST8) are satisfied. Moreover, (ii′) and Proposition 7.30(ii) imply that $\widetilde{D}_i'[V, W]$ is $[\varepsilon_3, \geq d + \varepsilon_3]$-superregular for any distinct clusters $V, W \in \mathcal{P}_i$. Thus, (ST2) and (ST3) are satisfied. Moreover, Lemma 8.8(ii) implies that $\widetilde{D}_i'[V, W]$ is $[\varepsilon', \geq d]$-superregular for any distinct clusters $V, W \in \mathcal{P}_i'$. Therefore, (ST7) holds and so $(\widetilde{D}_i', \mathcal{P}_i, \mathcal{P}_i', \mathcal{P}_i^*, R_i, C^i, U^i, U'^i)$ is an $(\ell', \ell^*, k, m, \varepsilon', d)$-setup. By construction, the exceptional set in $\mathcal{P}_i, \mathcal{P}_i'$, and $\mathcal{P}_i^*$ is empty and so (CST3) holds. Thus, $(D_2, \mathcal{U}, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ is a $(4, \ell', \ell^*, k, m, \varepsilon', d)$-cycle-setup and so (iii) is satisfied. $\square$

CHAPTER 13

DECOMPOSING BACKWARD EDGES

As briefly mentioned in the proof overview, the backward edges of T will be decomposed separately at the beginning of the proof of Theorem 4.4. We now discuss this in more detail.

13.1 Feasible systems

The strategy for Theorem 4.4 will be to first decompose all the backward edges into n edge-disjoint subdigraphs $\mathcal{F}_1, \dots, \mathcal{F}_n$ and then incorporate each $\mathcal{F}_i$ into a distinct Hamilton cycle of the decomposition of T. Each $\mathcal{F}_i$ will need to have a very specific structure, which will be called a *feasible system*; otherwise we would not be able to incorporate it into a Hamilton cycle. (See Definition 13.2 below for a formal definition.) To gain intuition, we start by giving some informal motivation.

First, each $\mathcal{F}_i$ will have to be linear forest (since any proper subdigraph of a Hamilton cycle is a linear forest). This is property (F3) below. Moreover, we will show that for any Hamilton cycle of T, the set of backward edges satisfy the following "balance property". (To gain intuition behind Proposition 13.1, observe that in the proof of Proposition 1.7, we in fact showed that, in a tripartite tournament, one cannot construct a Hamilton cycle which contains a single backward edge. The analogue holds in the blow-up C_4 case: a cycle which does not contain a "balanced" number of backward edges will not cover all the vertex classes equitably.) Recall Definition 10.1.

Proposition 13.1. *Let T be a bipartite tournament on $4n$ vertices and suppose that $\mathcal{U} = (U_1, \dots, U_4)$ is an $(\varepsilon, 4)$-partition for T. Then, any Hamilton cycle C of T satisfies $e_C(U_1, U_4) = e_C(U_3, U_2)$ and $e_C(U_4, U_3) = e_C(U_2, U_1)$.*

Thus, we will need to make sure that each $\mathcal{F}_i$ contains the same number of backward edges in non-adjacent pairs of the blow-up C_4. This is property (F1) below. For convenience, we will also allow each $\mathcal{F}_i$ to contain a few forward edges to ensure that all the exceptional vertices are covered. This is property (F2) below. Roughly speaking, this means that we will decompose all the exceptional edges at the same time as the backward edges, which will enable us to "ignore" the exceptional vertices when constructing our Hamilton cycles.

Altogether, this motivates the next definition.

Definition 13.2 (Feasible system)**.** Let $U_1, \dots, U_4$ be disjoint vertex sets and denote $\mathcal{U} \coloneqq (U_1, \dots, U_4)$. Let $U^* \subseteq \bigcup_{i \in [4]} U_i$ be an exceptional set. We say that $\mathcal{F}$ is a *feasible system (with respect to $\mathcal{U}$ and U^*)* if the following hold.

(F1) $e_{\mathcal{F}}(U_1, U_4) = e_{\mathcal{F}}(U_3, U_2)$ and $e_{\mathcal{F}}(U_4, U_3) = e_{\mathcal{F}}(U_2, U_1)$.

(F2) For each $v \in U^*$, $d_{\mathcal{F}}^+(v) = 1 = d_{\mathcal{F}}^-(v)$.

(F3) $\mathcal{F}$ is a linear forest.

Note that Proposition 13.1 follows immediately from Fact 10.2(i) and the next result (which will also be used in the proof of Lemma 13.6 below).

Proposition 13.3. *Let $U_1, \dots, U_4$ be disjoint vertex sets (not necessarily of the same size) and let C be a bipartite cycle on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Then, there exists $\ell \in \mathbb{Z}$ such that, for each $i \in [4]$, $|U_i| = \ell + e_C(U_{i+1}, U_i) + e_C(U_i, U_{i-1})$.*

Proof. We proceed by induction on $|C|$. For the base case, suppose that $|C| = 4$. If $|U_i| = 1$ and $e_C(U_{i+1}, U_i) = 0$ for each $i \in [4]$, then we can let $\ell \coloneqq 1$ and we are done. If $|U_i| = 1$ and $e_C(U_{i+1}, U_i) = 1$ for each $i \in [4]$, then we can let $\ell \coloneqq -1$ and we are done. Suppose that there exists $i \in [4]$ such that $C = v_1v_2v_3v_4$ for some $v_1 \in U_i$, $v_2, v_4 \in U_{i+1}$, and $v_3 \in U_{i+2}$. Then, each $j \in [4]$ satisfies

$$|U_j| = \begin{cases} 2 & \text{if } j = i+1; \\ 1 & \text{if } j \in \{i, i+2\}; \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad e_C(U_{j+1}, U_j) = \begin{cases} 1 & \text{if } j \in \{i, i+1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can let $\ell \coloneqq 0$ and we are done. We may therefore assume that there exists $i \in [4]$ such that $C = v_1v_2v_3v_4$ for some $v_1, v_3 \in U_i$ and $v_2, v_4 \in U_{i+1}$. Then, each $j \in [4]$ satisfies

$$|U_j| = \begin{cases} 2 & \text{if } j \in \{i, i+1\}; \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad e_C(U_{j+1}, U_j) = \begin{cases} 2 & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can let $\ell \coloneqq 0$ and we are done.

For the induction step, let $k > 2$ and suppose that the proposition holds for any cycle of length $2(k-1)$. Assume that $|C| = 2k$ and denote $C = v_1v_2 \dots v_{2k}$. Suppose without loss of generality that $v_1 \in U_1$. Then, observe that $v_{2k-2}, v_{2k} \in U_2 \cup U_4$ and $v_{2k-1} \in U_1 \cup U_3$. For each $i \in [4]$, let $U_i' \coloneqq U_i \setminus \{v_{2k-1}, v_{2k}\}$. Define a cycle $C' \coloneqq v_1 \dots v_{2k-2}$. Then, $|C'| = 2(k-1)$ and C' is a bipartite cycle on vertex classes $U_1' \cup U_3'$ and $U_2' \cup U_4'$. Thus, by the induction hypothesis, there exists $\ell' \in \mathbb{Z}$ such that $|U_i'| = \ell' + e_{C'}(U_{i+1}', U_i') + e_{C'}(U_i', U_{i-1}')$ for each $i \in [4]$. If $v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}$, and $v_{2k}v_1$ are all forward edges with respect to $\mathcal{U} \coloneqq (U_1, \dots, U_4)$, then let $\ell \coloneqq \ell' + 1$. If $v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}$, and $v_{2k}v_1$ are all backward edges with respect to $\mathcal{U}$, then let $\ell \coloneqq \ell' - 1$. Otherwise, let $\ell \coloneqq \ell'$.

We now verify that $|U_i| = \ell + e_C(U_{i+1}, U_i) + e_C(U_i, U_{i-1})$ for each $i \in [4]$. We consider the case where $v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}$, and $v_{2k}v_1$ are all forward edges with respect to $\mathcal{U}$ (the other cases can be verified with similar arguments). Note that $v_{2k} \in U_4$, $v_{2k-1} \in U_3$, and $v_{2k-2} \in U_2$. Then,

$$|U_i'| = \begin{cases} |U_i| & \text{if } i \in [2]; \\ |U_i| - 1 & \text{otherwise}; \end{cases} \quad \text{and} \quad e_{C'}(U_{i+1}', U_i') = \begin{cases} e_C(U_{i+1}, U_i) + 1 & \text{if } i = 1; \\ e_C(U_{i+1}, U_i) & \text{otherwise.} \end{cases}$$

Then, for each $i \in [2]$,

$$\begin{aligned}|U_i| = |U_i'| &= \ell' + e_{C'}(U_{i+1}', U_i') + e_{C'}(U_i', U_{i-1}') \\ &= \ell - 1 + e_{C'}(U_{i+1}', U_i') + e_{C'}(U_i', U_{i-1}') = \ell + e_C(U_{i+1}, U_i) + e_C(U_i, U_{i-1}).\end{aligned}$$

Moreover, for each $i \in [4] \setminus [2]$,

$$\begin{aligned}|U_i| = |U_i'| + 1 &= \ell' + 1 + e_{C'}(U_{i+1}', U_i') + e_{C'}(U_i', U_{i-1}') \\ &= \ell + e_C(U_{i+1}, U_i) + e_C(U_i, U_{i-1}),\end{aligned}$$

so we are done. □

We now state a few useful properties of feasible systems. Observe that forward edges are only required to cover the exceptional set U^*, so any forward edge which is not incident to U^* may be deleted or added from a feasible system.

Fact 13.4. *Let $U_1, \dots, U_4$ be disjoint vertex sets. Denote $\mathcal{U} := (U_1, \dots, U_4)$ and let $U^* \subseteq \bigcup_{i \in [4]} U_i$. Let $\mathcal{F}$ be a feasible system with respect to $\mathcal{U}$ and U^*. Let e be a forward edge with respect to $\mathcal{U}$ which satisfies $V(e) \cap U^* = \emptyset$. Then, $\mathcal{F} \setminus \{e\}$ is a feasible system and, if $\mathcal{F} \cup \{e\}$ is a linear forest, then $\mathcal{F} \cup \{e\}$ is also a feasible system.*

Note that isolated vertices play no role in a feasible system and so may be deleted.

Fact 13.5. *Let $\mathcal{U}, U^*$, and $\mathcal{F}$ be as in Fact 13.4. Let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by deleting all isolated vertices. Then, $\mathcal{F}'$ is also a feasible system with respect to $\mathcal{U}$ and U^*.*

As discussed above, we will decompose the backward and exceptional edges into n feasible systems and then restrict ourselves to construct a Hamilton decomposition where each Hamilton cycle contains precisely one of the feasible systems. The incorporation of feasible systems into the approximate decomposition is discussed in Section 14.1. To decompose the leftovers, recall that we will be using the robust decomposition lemma for blow-up cycles (Lemma 11.10) and, as discussed in Section 12.2, all the cycles obtained via Lemma 11.10 will be turned into Hamilton cycles of T by incorporating a special cover. Thus, we will require some of our feasible systems to form special covers. For Lemma 12.7, these will also need to be balanced (recall Definition 12.6). In the next lemma, we verify that feasible systems can induce balanced special covers.

Lemma 13.6. *Let D be a digraph and $U_1, \dots, U_4$ be a partition of $V(D)$. Denote $\mathcal{U} := (U_1, \dots, U_4)$ and let $U^* \subseteq \bigcup_{i\in[4]} U_i$ be an exceptional set satisfying $|U^* \cap U_1| = \dots = |U^* \cap U_4|$. Let $\mathcal{F} \subseteq D$ be a feasible system with respect to $\mathcal{U}$ and U^*. If $V^0(\mathcal{F}) = U^*$, then $\mathcal{F}$ is a $\mathcal{U}$-balanced special cover in D with respect to U^*. In particular, $\mathcal{F}$ is $\mathcal{U}'$-balanced, where $\mathcal{U}' := (U_1 \setminus U^*, \dots, U_4 \setminus U^*)$.*

Proof. Clearly, $\mathcal{F}$ is a special cover in D with respect to U^* and the "in particular part" holds since all the endpoints of the components of $\mathcal{F}$ lie in $\bigcup \mathcal{U}'$. We show that $\mathcal{F}$ is $\mathcal{U}$-balanced. By Definition 12.6 and Fact 13.5, we may assume without loss of generality that $\mathcal{F}$ does not contain any isolated vertex. For each $i \in [4]$, let $n_i^\pm := |V^\pm(\mathcal{F}) \cap U_i|$. By symmetry, it is enough to show that $n_1^+ = n_2^-$. Using new vertices and edges, extend each component of $\mathcal{F}$ to obtain a linear forest $\mathcal{F}' \supseteq \mathcal{F}$ and vertex sets $U_1' \supseteq U_1, \dots, U_4' \supseteq U_4$ such that the following hold, where $\mathcal{U}'' := (U_1', \dots, U_4')$.

(a) $\mathcal{F}'$ is a bipartite linear forest on vertex classes $U_1' \cup U_3'$ and $U_2' \cup U_4'$.

(b) $E(\overleftarrow{\mathcal{F}'}_{\mathcal{U}''}) = E(\overleftarrow{\mathcal{F}}_{\mathcal{U}})$.

(c) Each component of $\mathcal{F}'$ is a path which starts in U_1' and ends in U_4'.

(d) Each component of $\mathcal{F}' \setminus \mathcal{F}$ is a path of length at most 3.

Let $\mathcal{F}''$ be obtained from $\mathcal{F}'$ by adding an edge from the ending point to the starting point of each component of $\mathcal{F}'$. By (a)–(c), the following hold.

(a′) $\mathcal{F}''$ is a bipartite 1-factor on vertex classes $U_1' \cup U_3'$ and $U_2' \cup U_4'$.

(b′) $E(\overleftarrow{\mathcal{F}''}_{\mathcal{U}''}) = E(\overleftarrow{\mathcal{F}}_{\mathcal{U}})$.

Let $\mathscr{C}$ be the set of components of $\mathcal{F}''$. For each $C \in \mathscr{C}$, let $\ell_C \in \mathbb{Z}$ be the constant obtained by applying Proposition 13.3 with $V(C) \cap U'_1, \dots, V(C) \cap U'_4$ playing the roles of $U_1, \dots, U_4$. Then,

$$\begin{aligned}
|U'_1| &= \sum_{C \in \mathscr{C}} |V(C) \cap U_1| \overset{\text{Proposition 13.3}}{=} \sum_{C \in \mathscr{C}} (\ell_C + e_C(U'_2, U'_1) + e_C(U'_1, U'_4)) \\
&= \sum_{C \in \mathscr{C}} \ell_C + e_{\mathcal{F}''}(U'_2, U'_1) + e_{\mathcal{F}''}(U'_1, U'_4) \overset{\text{(b')}}{=} \sum_{C \in \mathscr{C}} \ell_C + e_{\mathcal{F}''}(U'_2, U'_1) + e_{\mathcal{F}}(U_1, U_4) \\
&\overset{\text{(F1)}}{=} \sum_{C \in \mathscr{C}} \ell_C + e_{\mathcal{F}''}(U'_2, U'_1) + e_{\mathcal{F}}(U_3, U_2) \overset{\text{(b')}}{=} \sum_{C \in \mathscr{C}} \ell_C + e_{\mathcal{F}''}(U'_2, U'_1) + e_{\mathcal{F}''}(U'_3, U'_2) \\
&= \sum_{C \in \mathscr{C}} (\ell_C + e_C(U'_2, U'_1) + e_C(U'_3, U'_2)) \overset{\text{Proposition 13.3}}{=} \sum_{C \in \mathscr{C}} |V(C) \cap U'_2| \\
&= |U'_2|. \qquad (13.1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(|V^0(\mathcal{F}) \cap U_1| + n_1^+ + n_1^-) + (n_2^+ + n_3^+ + n_4^+) \\
&\qquad \overset{\text{(b)–(d)}}{=} |U_1| + |U'_1 \setminus U_1| = |U'_1| \overset{(13.1)}{=} |U'_2| = |U_2| + |U'_2 \setminus U_2| \\
&\qquad \overset{\text{(b)–(d)}}{=} (|V^0(\mathcal{F}) \cap U_2| + n_2^+ + n_2^-) + (n_1^- + n_3^+ + n_4^+).
\end{aligned}$$

Thus, $n_1^+ = n_2^-$, as desired. □

13.2 Optimal partitions

We now introduce our main tool for decomposing the backward and exceptional edges into feasible systems. Suppose that T is ε-close to the complete blow-up C_4 with vertex partition $\mathcal{U} = (U_1, \dots, U_4)$. As observed in Section 10.3, T is a bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Thus, the partition $\mathcal{U}$ is not fixed (one may swap some vertices between U_1 and U_3, as well as some vertices between U_2 and U_4). We will consider a partition $\mathcal{U}$ which minimises the number of backward edges.

Definition 13.7 (Optimal partition)**.** Let T be a bipartite tournament. An $(\varepsilon, 4)$-partition

$\mathcal{U}$ for T is *optimal* if it minimises the number of backward edges in T, that is, if

$$|E(\overleftarrow{T}_{\mathcal{U}})| = \min\{|E(\overleftarrow{T}_{\mathcal{U}'})| \mid \mathcal{U}' \text{ is an } (\varepsilon, 4)\text{-partition for } T\}.$$

Roughly speaking, an optimal $(\varepsilon, 4)$-exceptional partition of T will guarantee the existence of a subdigraph $H \subseteq T$ of small maximum degree which contains many backward edges. This will enable us to apply König's theorem (Proposition 7.22) to find large matchings of backward edges. This is Lemma 13.8 below. To state and prove this lemma, we need some additional notation.

Let $U_1, \dots, U_4$ be disjoint vertex sets of size n and denote $\mathcal{U} := (U_1, \dots, U_4)$. Let T be a regular bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. Recall from Fact 10.3 that $\overleftarrow{d}^{+}_{T,\mathcal{U}}(v) = \overleftarrow{d}^{-}_{T,\mathcal{U}}(v)$ for each $v \in V(T)$. For each $i \in [4]$ and $0 \le \gamma < 1$, denote by

$$U_i^{\gamma,\mathcal{U}}(T) := \left\{ v \in U_i \mid \overleftarrow{d}^{+}_{T,\mathcal{U}}(v) = \overleftarrow{d}^{-}_{T,\mathcal{U}}(v) > \gamma n \right\} \tag{13.2}$$

the set of vertices in U_i whose backward out- and indegree is greater that γn. Define $U^{\gamma,\mathcal{U}}(T) := \bigcup_{i \in [4]} U_i^{\gamma,\mathcal{U}}(T)$. In practice, $\mathcal{U}$ will always be clear from the context and so we will omit the second superscript. That is, we will write $U_i^{\gamma}(T)$ and $U^{\gamma}(T)$ instead of $U_i^{\gamma,\mathcal{U}}(T)$ and $U^{\gamma,\mathcal{U}}(T)$. Throughout the rest of this thesis, the subscript i in the above notation will always be taken modulo 4, so $U_5^{\gamma}(T) := U_1^{\gamma}(T)$ for example.

Lemma 13.8. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \gamma \le \frac{1}{2}$. Let T be a regular bipartite tournament on $4n$ vertices and let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T. Then, there exists $H \subseteq \overleftarrow{T}_{\mathcal{U}}$ satisfying the following properties.*

(i) $\Delta^0(H) \le \gamma n$.

(ii) *For each $v \in U^{1-\gamma}(T)$, $d_H(v) = 0$.*

(iii) *For each $i \in [4]$, $e_{H - U^{1-\gamma}(T)}(U_i, U_{i-1}) \ge (1 - 2\gamma) n |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$.*

Note that Lemma 13.8(ii) implies that $H - U^{1-\gamma}(T)$ is simply H in Lemma 13.8(iii). However, we emphasise for later applications that none the edges are incident to $U^{1-\gamma}(T)$.

Indeed, H will be used in conjunction with other sets of edges incident to $U^{1-\gamma}(T)$ to construct feasible systems and the role of H will be to balance out the number of backward edges chosen incident to $U^{1-\gamma}(T)$ (recall property (F1) of a feasible system). At this stage, it will be crucial that there are many edges which are not incident to $U^{1-\gamma}(T)$.

To prove Lemma 13.8, we will need the next two results.

Fact 13.9. *Let $U_1, \dots, U_4$ be disjoint vertex sets of size n and denote $\mathcal{U} \coloneqq (U_1, \dots, U_4)$. Let T be a regular bipartite tournament on vertex classes $U_1 \cup U_3$ and $U_2 \cup U_4$. For any $0 \leq \gamma \leq \gamma' < 1$, $U^{\gamma'}(T) \subseteq U^{\gamma}(T)$.*

Lemma 13.10. *Let $0 < \varepsilon \leq 1$ and $0 < \gamma \leq \frac{1}{2}$. Let T be a bipartite tournament and let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T. Then, for each $i \in [2]$, there exists $j_i \in \{i, i+2\}$ such that*

$$U_i^{1-\gamma}(T) = \emptyset = U_{i+2}^{1-\gamma}(T) \quad \text{or} \quad U_{j_i}^{\gamma}(T) = \emptyset.$$

In particular,

$$U_{j_1}^{1-\gamma}(T) = \emptyset = U_{j_2}^{1-\gamma}(T).$$

Proof. The "in particular part" follows immediately from Fact 13.9. By symmetry, it suffices to show that the lemma holds for $i = 1$. Let $j_1 \in \{1, 3\}$ minimise $\max_{v \in U_{j_1}} \overleftarrow{d}_{T,\mathcal{U}}(v)$. Suppose for a contradiction that there exist $u \in U_{j_1+2}^{1-\gamma}(T)$ and $v \in U_{j_1}^{\gamma}(T)$. We claim that swapping u and v decreases the number of backward edges and so $\mathcal{U}$ is not optimal. Indeed, let $U_2' \coloneqq U_2$, $U_4' \coloneqq U_4$, $U_{j_1}' \coloneqq (U_{j_1} \setminus \{v\}) \cup \{u\}$, and $U_{j_1+2}' \coloneqq (U_{j_1+2} \setminus \{u\}) \cup \{v\}$. Define $\mathcal{U}' \coloneqq (U_1', \dots, U_4')$. Then,

$$\begin{aligned} e_T(U_{j_1}', U_{j_1-1}') &= \overleftarrow{d}^{+}_{T,\mathcal{U}'}(u) + \sum_{w \in U_{j_1} \setminus \{v\}} \overleftarrow{d}^{+}_{T,\mathcal{U}'}(w) = \overrightarrow{d}^{+}_{T,\mathcal{U}}(u) + \sum_{w \in U_{j_1} \setminus \{v\}} \overleftarrow{d}^{+}_{T,\mathcal{U}}(w) \\ &< \gamma n + \sum_{w \in U_{j_1} \setminus \{v\}} \overleftarrow{d}^{+}_{T,\mathcal{U}}(w) < \overleftarrow{d}^{+}_{T,\mathcal{U}}(v) + \sum_{w \in U_{j_1} \setminus \{v\}} \overleftarrow{d}^{+}_{T,\mathcal{U}}(w) = e_T(U_{j_1}, U_{j_1-1}). \end{aligned}$$

Thus, Facts 10.4 and 10.5 imply that $\mathcal{U}'$ is an $(\varepsilon, 4)$-partition for T which satisfies

$|E(\overleftarrow{T}_{\mathcal{U}'})| < |E(\overleftarrow{T}_{\mathcal{U}})|$. This contradicts the fact that $\mathcal{U}$ is optimal and so

$$U^{1-\gamma}_{j_1+2}(T) = \emptyset \quad \text{or} \quad U^{\gamma}_{j_1}(T) = \emptyset.$$

Thus, it suffices to prove that if $U^{1-\gamma}_{j_1+2}(T) = \emptyset$, then $U^{1-\gamma}_{j_1}(T) = \emptyset$. Suppose not. Then, $\max_{w \in U_{j_1+2}} \overleftarrow{d}_{T,\mathcal{U}}(w) \leq (1-\gamma)n < \max_{w \in U_{j_1}} \overleftarrow{d}_{T,\mathcal{U}}(w)$, a contradiction to the definition of j_1. $\square$

Proof of Lemma 13.8. By Fact 10.2(ii) and Lemma 13.10, we may assume without loss of generality that, for each $i \in [2]$,

$$U^{1-\gamma}_{i}(T) = \emptyset = U^{1-\gamma}_{i+2}(T) \quad \text{or} \quad U^{\gamma}_{i+2}(T) = \emptyset. \tag{13.3}$$

In particular, we have

$$U^{1-\gamma}_{3}(T) = \emptyset \quad \text{and} \quad U^{1-\gamma}_{4}(T) = \emptyset. \tag{13.4}$$

For each $i \in \{1,4\}$ and $v \in U_i \setminus U^{1-\gamma}(T)$, let $E_v \subseteq E_T(\{v\}, U_{i-1}) \subseteq E(\overleftarrow{T}_{\mathcal{U}})$ satisfy $|E_v| = |N^-_T(v) \cap U^{1-\gamma}_{i+1}(T)|$ (this is possible by Fact 10.3). Similarly, for each $i \in \{2,3\}$ and $v \in U_i \setminus U^{1-\gamma}(T)$, let $E_v \subseteq E_T(U_{i+1}, \{v\}) \subseteq E(\overleftarrow{T}_{\mathcal{U}})$ satisfy $|E_v| = |N^+_T(v) \cap U^{1-\gamma}_{i-1}(T)|$. Let $E_{14} \coloneqq \bigcup_{v \in U_1 \setminus U^{1-\gamma}(T)} E_v$ and $E_{32} \coloneqq \bigcup_{v \in U_2 \setminus U^{1-\gamma}(T)} E_v$. Define

$$E_{43} \coloneqq \begin{cases} \emptyset & \text{if } U^{1-\gamma}_1(T) = \emptyset = U^{1-\gamma}_2(T); \\ E_T(U_4, U_3) & \text{if } U^{1-\gamma}_1(T) \neq \emptyset \neq U^{1-\gamma}_2(T); \\ \bigcup_{v \in U_4 \setminus U^{1-\gamma}(T)} E_v & \text{if } U^{1-\gamma}_1 \neq \emptyset = U^{1-\gamma}_2(T); \\ \bigcup_{v \in U_3 \setminus U^{1-\gamma}(T)} E_v & \text{if } U^{1-\gamma}_1 = \emptyset \neq U^{1-\gamma}_2(T). \end{cases}$$

Let H be the digraph on $V(T)$ defined by $E(H) \coloneqq E_{14} \cup E_{32} \cup E_{43}$.

By definition, $H \subseteq \overleftarrow{T}_{\mathcal{U}}$. We verify that (i)–(iii) are satisfied. One can easily verify that, for each $v \in V(T) \setminus U^{1-\gamma}(T)$ and $e \in E_v$, we have $V(e) \setminus \{v\} \subseteq U_3 \cup U_4$. Together with (13.4) and the fact that $E(H) \subseteq E_T(U_4, U_3) \cup \bigcup_{v \in V(T) \setminus U^{1-\gamma}(T)} E_v$, this implies that

(ii) holds.

For (i), it only remains to check that $d_H^{\pm}(v) \leq \gamma n$ for each $v \in V(T) \setminus U^{1-\gamma}(T)$. By Fact 10.5, the following holds for each $i \in [4]$.

$$|U_i^{1-\gamma}(T)| \leq \frac{\varepsilon n^2}{(1-\gamma)n} \leq \gamma n. \tag{13.5}$$

Thus, for each $v \in U_1 \setminus U^{1-\gamma}(T)$, we have

$$d_H(v) = d_{E_{14}}(v) = |E_v| \leq |U_2^{1-\gamma}(T)| \overset{(13.5)}{\leq} \gamma n,$$

as desired. Similarly, each $v \in U_2 \setminus U^{1-\gamma}(T)$ satisfies $d_H(v) \leq \gamma n$. It remains to verify that $d_H^{\pm}(v) \leq \gamma n$ for each $v \in (U_3 \cup U_4) \setminus U^{1-\gamma}(T)$. If $U_3^{\gamma}(T) = \emptyset = U_4^{\gamma}(T)$, then we have $d_H^{\pm}(v) \leq \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) \leq \gamma n$ for each $v \in U_3 \cup U_4$, as desired. Moreover, (13.3) implies that if $U_3^{\gamma}(T) \neq \emptyset \neq U_4^{\gamma}(T)$, then $U_i^{1-\gamma}(T) = \emptyset$ for each $i \in [4]$ and so $E(H) = \emptyset$. Thus, by symmetry, we may assume that $U_3^{\gamma}(T) = \emptyset \neq U_4^{\gamma}(T)$. Then, each $v \in U_3$ satisfies $d_H^{\pm}(v) \leq \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) \leq \gamma n$, as desired. By (13.3), we have $U_2^{1-\gamma}(T) = \emptyset$. Therefore, $E_{14} = \emptyset$ and so each $v \in U_4 \setminus U^{1-\gamma}(T)$ satisfies $d_H^-(v) = d_{E_{14}}(v) = 0$. Moreover, each $v \in U_4 \setminus U^{1-\gamma}(T)$ satisfies

$$d_H^+(v) = d_{E_{43}}(v) \leq |E_v| \leq |U_1^{1-\gamma}(T)| \overset{(13.5)}{\leq} \gamma n.$$

Thus, (i) is satisfied.

Finally, we verify (iii). By (13.4), (iii) holds for $i = 2$. Moreover,

$$\begin{aligned}
e_{H-U^{1-\gamma}(T)}(U_1, U_4) &= |E_{14}| = \sum_{v \in U_1 \setminus U^{1-\gamma}(T)} |N_T^-(v) \cap U_2^{1-\gamma}(T)| \\
&= e_T(U_2^{1-\gamma}(T), U_1 \setminus U^{1-\gamma}(T)) \geq \sum_{v \in U_2^{1-\gamma}(T)} (\overleftarrow{d}^{+}_T(v) - |U_1^{1-\gamma}(T)|) \\
&\geq |U_2^{1-\gamma}(T)|((1-\gamma)n - |U_1^{1-\gamma}(T)|) \overset{(13.5)}{\geq} (1-2\gamma)n|U_2^{1-\gamma}(T)| \\
&\overset{(13.4)}{=} (1-2\gamma)n(|U_2^{1-\gamma}(T)| + |U_3^{1-\gamma}(T)|).
\end{aligned}$$

Thus, (iii) holds for $i = 1$. Similarly, (iii) holds for $i = 3$. If $U_1^{1-\gamma}(T) = \emptyset = U_2^{1-\gamma}(T)$, then (iii) is clearly satisfied for $i = 4$. If $U_1^{1-\gamma}(T) \neq \emptyset \neq U_2^{1-\gamma}(T)$, then

$$\begin{aligned} e_{H-U^{1-\gamma}(T)}(U_4, U_3) &= |E_{43}| = e_T(U_4, U_3) \overset{\text{Fact 10.4}}{=} e_T(U_2, U_1) \\ &\geq (|U_1^{1-\gamma}(T)| + |U_2^{1-\gamma}(T)|)(1-\gamma)n - |U_1^{1-\gamma}(T)||U_2^{1-\gamma}(T)| \\ &\overset{(13.5)}{\geq} (1-2\gamma)n(|U_1^{1-\gamma}(T)| + |U_2^{1-\gamma}(T)|) \end{aligned}$$

and so (iii) holds for $i = 4$. If $U_1^{1-\gamma}(T) \neq \emptyset = U_2^{1-\gamma}(T)$, then

$$\begin{aligned} e_{H-U^{1-\gamma}(T)}(U_4, U_3) = |E_{43}| = \sum_{v \in U_4 \setminus U^{1-\gamma}(T)} |N_T^-(v) \cap U_1^{1-\gamma}(T)| \overset{(13.4)}{=} e_T(U_1^{1-\gamma}(T), U_4) \\ \geq (1-\gamma)n|U_1^{1-\gamma}(T)| = (1-\gamma)n(|U_1^{1-\gamma}(T)| + |U_2^{1-\gamma}(T)|) \end{aligned}$$

and so (iii) holds for $i = 4$. Similarly, (iii) holds for $i = 4$ if $U_1^{1-\gamma}(T) = \emptyset \neq U_2^{1-\gamma}(T)$. Therefore, (iii) is satisfied. □

13.3 Decomposing backward and exceptional edges into feasible systems

Finally, we state and motivate our main decomposition lemma. First, we need an additional definition. As discussed in Section 12.3, the exceptional set U^* will have to contain all the vertices of high backward degree (otherwise we would not be able to construct the cycle-setup required for the robust decomposition lemma). This motivates (ES1) below. To facilitate the incorporation of the exceptional vertices into the Hamilton decomposition, we also require that U^* is small and contain the same number of vertices from each vertex class. This is (ES2) below.

Definition 13.11 ((ε, $\mathcal{U}$)-exceptional set)**.** Let T be a regular bipartite tournament on $4n$ vertices. Suppose that $\mathcal{U} = (U_1, \ldots, U_4)$ is an $(\varepsilon, 4)$-partition for T. We say that U^* is an

$(\varepsilon',\mathcal{U})$*-exceptional set for* T if the following hold, where $U_i^* := U_i \cap U^*$ for each $i \in [4]$.

(ES1) $U^{\varepsilon'}(T) \subseteq U^* \subseteq V(T)$.

(ES2) $|U_1^*| = \dots = |U_4^*| \leq \varepsilon' n$.

Let T be a bipartite tournament on $4n$ vertices. Let $\mathcal{U}$ be an optimal $(\varepsilon, 4)$-partition for T and let U^* be an $(\varepsilon',\mathcal{U})$-exceptional set for T. Then, Lemma 13.12 states that T contains n edge-disjoint feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_n$ which contain all the backward edges of T (see Lemma 13.12(a)). By Lemma 13.12(b), all these feasible systems are small, which will enable us to incorporate them into our Hamilton cycles. The first t feasible systems will be those which will be incorporated into the Hamilton cycles given by the robust decomposition lemma (Lemma 11.10) and so we will require those to form special covers which are localised and balanced (see Lemma 12.7). Together with Lemma 13.6, Lemma 13.12(c) will imply that $\mathcal{F}_1, \dots, \mathcal{F}_t$ are balanced special covers, as required for Lemma 12.7. Additionally, Lemma 13.12(d) ensures that $\mathcal{F}_1, \dots, \mathcal{F}_t$ are constructed out of prescribed sets $H_1, \dots, H_s$ of edges of T. These edges will be chosen in such a way that $\mathcal{F}_1, \dots, \mathcal{F}_t$ form localised special covers, as desired for Lemma 12.7. The last $n-t$ feasible systems will be incorporated into the approximate decomposition. For simplicity, we require that all of the components of these feasible systems are paths which start in U_1 and end in U_4 (see Lemma 13.12(e)). Finally, Lemma 13.12(a) will allow us to incorporate a small prescribed set E of forward edges of T into the feasible systems. In practice, E will consists of all the edges of T which cannot be decomposed via the robust decomposition lemma. (Recall from Lemma 11.10 that the robustly decomposable digraph $D^{\rm rob}$ cannot decompose the edges which are lying along the auxiliary matchings in $\mathcal{M}$.) This will ensure that they are not left over at the end of the approximate decomposition.

Roughly speaking, Lemma 13.12(i)–(iv) ensure that $H_1, \dots, H_s$ contain many well distributed backward and exceptional edges. This is necessary, for otherwise we may not be able to construct the feasible systems satisfying Lemma 13.12(d). More precisely, Lemma 13.12(i) and (ii) ensure that each exceptional vertex in U^* has many in- and outneighbours in each H_i (recall from (F2) that a feasible system has to cover U^*). Additionally, Lemma 13.12(iii) and (iv) ensure that there are many backward edges and that these are evenly distributed across the non-exceptional vertices. This will enable us to use König's theorem to find large matchings of backward edges, which will be convenient for adjusting the number of backward edges in each feasible system (recall from (F1) that a feasible system must contain a balanced number of backward edges).

Note that $H_1, \dots, H_s$ will be constructed using Lemma 13.8. (Compare the bounds in Lemma 13.12(iii) and (iv) to those in Lemma 13.8(i) and (iii).) This is a point where we make crucial use of the concept of optimal partitions.

Finally, observe that $H_1, \dots, H_s$ need not be edge-disjoint in Lemma 13.12. The upper bound on t will be sufficient to ensure that there are, overall, sufficiently many edges in $H_1, \dots, H_s$ to construct the edge-disjoint feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_t$, each within its prescribed H_i.

Lemma 13.12 (Decomposing the backward and exceptional edges into feasible systems)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \varepsilon' \ll \eta \ll \gamma \ll 1$ and $s \in \mathbb{N}$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that, for each $j \in [s]$, $H_j \subseteq T$ satisfies the following.*

(i) *For each $v \in U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{H_j,\mathcal{U}}(v) \geq 3\gamma n$.*

(ii) *For each $v \in U^* \setminus U^{1-\gamma}(T)$, $\overrightarrow{d}^{\pm}_{H_j,\mathcal{U}}(v) \geq \gamma^2 n$.*

(iii) *For each $v \in V(T) \setminus U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{H_j,\mathcal{U}}(v) \leq 2\gamma n$.*

(iv) *For each $i \in [4]$, $e_{H_j - U^{1-\gamma}(T)}(U_i, U_{i-1}) \geq 110\gamma n |U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)|$.*

For each $i \in [s]$, let $s_i \in \mathbb{N}$ and $t_i \coloneqq \sum_{j \in [i-1]} s_j$. Let $t \coloneqq \sum_{i \in [s]} s_i$ and suppose that $t \leq \eta n$. Let $E \subseteq E(T)$ be such that the following hold.

(v) *$E \subseteq E(\overrightarrow{T}_{\mathcal{U}} - U^*)$.*

(vi) *For each $v \in V(T) \setminus U^*$, $d^{\pm}_E(v) \leq 1$.*

Then, there exist edge-disjoint feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_n$ such that the following hold.

(a) $E(\overleftarrow{T}_{\mathcal{U}}) \cup E \subseteq \bigcup_{i\in[n]} E(\mathcal{F}_i) \subseteq E(T)$.

(b) *For each* $i \in [n]$, $e(\mathcal{F}_i) \leq \varepsilon' n$.

(c) *For each* $i \in [t]$, $V^0(\mathcal{F}_i) = U^*$.

(d) *For each* $i \in [s]$ *and* $j \in [s_i]$, $\mathcal{F}_{t_i+j} \subseteq H_i \setminus E$.

(e) *For each* $i \in [n-t]$, *we have* $V^+(\mathcal{F}_{t+i}) \subseteq U_1$ *and* $V^-(\mathcal{F}_{t+i}) \subseteq U_4$.

To provide intuition into its formulation, we will first assume that Lemma 13.12 holds and derive Theorem 4.4. The proof of Lemma 13.12 is spread over Chapters 15–18. These chapters also include a detailed proof overview of Lemma 13.12.

CHAPTER 14

THE ε-CLOSE TO THE COMPLETE BLOW-UP C_4 CASE: PROOF OF THEOREM 4.4

We will now prove Theorem 4.4. First, we use our tools from Section 8.1 to incorporate feasible systems into an approximate Hamilton decomposition (see Lemma 14.1 below). Then, we derive Theorem 4.4 from the robust decomposition lemma for blow-up cycles (Lemma 11.10), the decomposition lemma for backward and exceptional edges (Lemma 13.12), and the approximate decomposition lemma (Lemma 14.1).

14.1 Approximate decomposition

Let T be a regular bipartite tournament on $4n$ vertices and suppose that T is ε-close to the complete blow-up C_4 with vertex partition $\mathcal{U} = (U_1, \dots, U_4)$. Our strategy for approximately decomposing T is the following (see also Figure 4.2). First, we use Corollary 8.3 to approximately decompose $T[U_1, U_2]$, $T[U_2, U_3]$, and $T[U_3, U_4]$ into perfect matchings. Combining a matching from each pair, we obtain an approximate decomposition of $T[U_1, U_2] \cup T[U_2, U_3] \cup T[U_3, U_4]$ into linear forests, each consisting of n components which start in U_1 and end in U_4. Finally, using Theorem 8.1, we close each of these linear forests into a Hamilton cycle by approximately decomposing $T[U_4, U_1]$ into "suitable" perfect matchings.

Recall that in Theorem 8.1 and Corollary 8.3, there is the flexibility of prescribing a few edges. This enables us to construct an approximate decomposition of T which incorporates given feasible systems.

Lemma 14.1 (Incorporating feasible systems into an approximate Hamilton decomposition)**.** *Let $0 < \frac{1}{n} \ll \tau \ll \delta \leq 1$ and $0 < \frac{1}{n} \ll \varepsilon \ll \eta, \nu \leq 1$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Let $\ell \leq 2(\delta - \eta)\left(n - \frac{|U^*|}{4}\right)$. Let $\mathcal{F}_1, \dots, \mathcal{F}_\ell \subseteq T$ be edge-disjoint feasible systems and $D \subseteq T \setminus \bigcup_{i \in [\ell]} \mathcal{F}_i$. Suppose that the following hold.*

(i) *For each $i \in [4]$, $D[U_i \setminus U^*, U_{i+1} \setminus U^*]$ is (δ, ε)-almost regular.*

(ii) *For each $i \in [4]$, $D[U_i \setminus U^*, U_{i+1} \setminus U^*]$ is a bipartite robust (ν, τ)-expander with bipartition $(U_i \setminus U^*, U_{i+1} \setminus U^*)$.*

(iii) *For each $i \in [\ell]$, $e(\mathcal{F}_i) \leq \varepsilon n$.*

(iv) *For each $v \in V(T) \setminus U^*$, there exist at most εn indices $i \in [\ell]$ such that $v \in V(\mathcal{F}_i)$.*

(v) *For each $i \in [\ell]$, $V^+(\mathcal{F}_i) \subseteq U_1$ and $V^-(\mathcal{F}_i) \subseteq U_4$.*

Then, there exist edge-disjoint Hamilton cycles $C_1, \dots, C_\ell$ of T such that $\mathcal{F}_i \subseteq C_i \subseteq D \cup \mathcal{F}_i$ for each $i \in [\ell]$.

Given two digraphs D and D', we say that D' is a *subdivision* of D if D' can be obtained from D by replacing some edges by internally vertex-disjoint paths.

Proof of Lemma 14.1. For each $i \in [4]$, define $U_i^* := U^* \cap U_i$. Let $A := U_4 \setminus U^*$ and $B := U_1 \setminus U^*$. By Fact 10.2(i) and (ES2), we have

$$|A| = |B| \geq (1 - \varepsilon)n. \tag{14.1}$$

Step 1: Approximately decomposing $D[U_1, U_2] \cup D[U_2, U_3] \cup D[U_3, U_4]$. For each $i \in [3]$ in turn, we will use Corollary 8.3 to approximately decompose $D[U_i, U_{i+1}]$ into matchings. We will then combine a matching from each pair to get an approximate decomposition of $D[U_1, U_2] \cup D[U_2, U_3] \cup D[U_3, U_4]$ into spanning linear forests.

Let $i \in [3]$ and $j \in [\ell]$. Denote by $S^+_{i,j}$ the set of vertices $v \in U_i \setminus U^*$ such that $d^+_{\mathcal{F}_j}(v) = 1$ and let $S^-_{i+1,j}$ be the set of vertices $v \in U_{i+1} \setminus U^*$ such that $d^-_{\mathcal{F}_j}(v) = 1$. (Thus, $S^+_{i,j} \cup S^-_{i+1,j}$ is the set of vertices which are already covered by $\mathcal{F}_j$ and so need to be avoided by the j^{th} matching of $D[U_i, U_{i+1}]$.) Note that

$$\begin{aligned} |S^+_{i,j}| &\overset{\text{(F2),(F3)}}{=} e_{\mathcal{F}_j}(U_i, U_{i+1}) + e_{\mathcal{F}_j}(U_i, U_{i-1}) - |U^*_i| \\ &\overset{\text{(F1),(ES2)}}{=} e_{\mathcal{F}_j}(U_i, U_{i+1}) + e_{\mathcal{F}_j}(U_{i+2}, U_{i+1}) - |U^*_{i+1}| \overset{\text{(F2),(F3)}}{=} |S^-_{i+1,j}|. \end{aligned} \tag{14.2}$$

Let $F_{i,j}$ be an auxiliary perfect matching between $S^+_{i,j}$ and $S^-_{i+1,j}$. Then,

$$e(F_{i,j}) \overset{(14.2)}{\leq} e(\mathcal{F}_j) \overset{\text{(iii)}}{\leq} \varepsilon n \overset{\text{(ES2)}}{\leq} 2\varepsilon(n - |U^*_i|)$$

and so Corollary 8.3(i) holds with $n - |U^*_i|, F_{i,j}$, and 2ε playing the roles of n, F_i, and ε.

Let $i \in [3]$. For each $j \in [\ell]$, we have $V(F_{i,j}) \subseteq V(\mathcal{F}_j)$. Thus, (iv) implies that Corollary 8.3(ii) holds with $D[U_i \setminus U^*, U_{i+1} \setminus U^*]$ and $F_{i,1}, \dots, F_{i,\ell}$ playing the roles of G and $F_1, \dots, F_\ell$. Let $M_{i,1}, \dots, M_{i,\ell}$ be the matchings obtained by applying Corollary 8.3 with $D[U_i \setminus U^*, U_{i+1} \setminus U^*], n - |U^*_i|, 2\varepsilon$, and $F_{i,1}, \dots, F_{i,\ell}$ playing the roles of G, n, ε, and $F_1, \dots, F_\ell$. For each $j \in [\ell]$, let $F'_{i,j}$ be obtained from $M_{i,j} \setminus F_{i,j}$ by orienting all the edges from U_i to U_{i+1} and observe that $F'_{i,j} \subseteq D(U_i \setminus U^*, U_{i+1} \setminus U^*)$.

For each $j \in [\ell]$, let $\mathcal{F}'_j \coloneqq \mathcal{F}_j \cup \bigcup_{i \in [3]} F'_{i,j}$. We claim that $\mathcal{F}'_1, \dots, \mathcal{F}'_\ell$ are edge-disjoint spanning linear forests whose components are paths which start in $B = U_1 \setminus U^*$ and end in $A = U_4 \setminus U^*$.

Claim 1. *$\mathcal{F}'_1, \dots, \mathcal{F}'_\ell$ are edge-disjoint linear forests such that the following hold for each $i \in [\ell]$.*

(a) $E(\mathcal{F}'_i) \cap E_D(A, B) = \emptyset$.

(b) $V(\mathcal{F}'_i) = V(T)$.

(c) $|V^0(\mathcal{F}'_i) \cap (A \cup B)| \leq 3\varepsilon|A|$.

(d) $V^+(\mathcal{F}_i') \subseteq B$ *and* $V^-(\mathcal{F}_i') \subseteq A$.

Proof of Claim. By assumption, $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ are edge-disjoint. For each $i \in [3]$, Corollary 8.3 implies that $F_{i,1}', \dots, F_{i,\ell}'$ are edge-disjoint matchings in $E_D(U_i, U_{i+1})$. Therefore, $\mathcal{F}_1', \dots, \mathcal{F}_\ell'$ are edge-disjoint, as desired.

Let $j \in [\ell]$. Suppose for a contradiction that $\mathcal{F}_j'$ is not a linear forest. By (F3) and construction, each $v \in V(T)$ satisfies both $d^{\pm}_{\mathcal{F}_j'}(v) \leq 1$. Thus, $\mathcal{F}_j'$ contains a cycle C. Clearly, $\bigcup_{i \in [3]} F_{i,j}'$ is a linear forest. Thus, there exists $e \in E(\mathcal{F}_j) \cap E(C)$. Let v be the starting point of the component of $\mathcal{F}_j$ which contains e. Then, $v \in V(C)$. Let u be the inneighbour of v in C. By assumption, $uv \notin E(\mathcal{F}_j)$ (otherwise v would not be the starting point of one of the components of $\mathcal{F}_j$). Thus, $uv \in \bigcup_{i \in [3]} E(F_{i,j}')$ and so $v \in U_2 \cup U_3 \cup U_4$, which contradicts (v). Therefore, $\mathcal{F}_j'$ is a linear forest, as desired.

Let $j \in [\ell]$. We show that (a)–(d) are satisfied. By construction and since $E(\mathcal{F}_j) \cap E(D) = \emptyset$, (a) holds. By (F2), $U^* \subseteq V(\mathcal{F}_j) \subseteq V(\mathcal{F}_j')$ and, by construction, $(U_i \cup U_{i+1}) \setminus (V(\mathcal{F}_j) \cup U^*) \subseteq V(F_{i,j}') \subseteq V(\mathcal{F}_j')$ for each $i \in [3]$. Thus, (b) holds. By construction, $\bigcup_{i \in [3]} F_{i,j}$ does not contain any edge which starts in $A = U_4 \setminus U^*$ or ends in $B = U_1 \setminus U^*$. Thus,

$$V^0(\mathcal{F}_j') \cap A \subseteq (V^0(\mathcal{F}_j) \cap A) \cup (V^+(\mathcal{F}_j) \cap A)$$

and

$$V^0(\mathcal{F}_j') \cap B \subseteq (V^0(\mathcal{F}_j) \cap B) \cup (V^-(\mathcal{F}_j) \cap B).$$

Therefore,

$$|V^0(\mathcal{F}_j') \cap (A \cup B)| \leq |V(\mathcal{F}_j)| \overset{\text{(iii)}}{\leq} 2\varepsilon n \overset{(14.1)}{\leq} 3\varepsilon |A|$$

and so (c) holds. Finally, we verify that $V^+(\mathcal{F}_j') \subseteq B$ and $V^-(\mathcal{F}_j') \subseteq A$. By (F2), each $v \in U^*$ satisfies $d^+_{\mathcal{F}_j'}(v) = 1 = d^-_{\mathcal{F}_j'}(v)$ and so $V^+(\mathcal{F}_j') \cap U^* = \emptyset = V^-(\mathcal{F}_j') \cap U^*$. Let $i \in [3]$. Suppose that $v \in U_i \setminus U^*$. If $v \in S^+_{i,j}$, then $d^+_{\mathcal{F}_j}(v) = 1$; otherwise, $v \in V(F_{i,j}')$. Thus, $d^+_{\mathcal{F}_j'}(v) = 1$ and so $v \notin V^-(\mathcal{F}_j')$. Therefore, $V^-(\mathcal{F}_j') \subseteq U_4 \setminus U^* = A$. Similarly, if $w \in U_{i+1} \setminus U^*$, then either $d^-_{\mathcal{F}_j}(w) = 1$ or $w \in V(F'_{i,j})$. Thus, $d^-_{\mathcal{F}'_j}(w) = 1$ for each $w \in U_{i+1} \setminus U^*$ and so $V^+(\mathcal{F}'_j) \subseteq U_1 \setminus U^* = B$. Therefore, (d) is satisfied. $\diamond$

Step 2: Approximately decomposing $D[U_4, U_1]$. In this step, we use Theorem 8.1 to approximately decompose $D[U_4, U_1]$ into matchings which close $\mathcal{F}'_1, \dots, \mathcal{F}'_\ell$ into Hamilton cycles of T.

To apply Theorem 8.1, we first need to contract $\mathcal{F}'_1, \dots, \mathcal{F}'_\ell$ into auxiliary linear forests on $A \cup B$. For each $j \in [\ell]$, let $\widetilde{\mathcal{F}}_j$ be the digraph on $A \cup B$ defined as follows. For any distinct $u, v \in A \cup B$, we let $uv \in E(\widetilde{\mathcal{F}}_j)$ if and only if $\mathcal{F}'_j$ contains a (u, v)-subpath P which satisfies $V^0(P) \subseteq (U_2 \cup U_3 \cup U^*)$.

Claim 2. *Let $i \in [\ell]$. Then, $\mathcal{F}'_i$ is a subdivision of $\widetilde{\mathcal{F}}_i$. In particular, $\widetilde{\mathcal{F}}_i$ is a linear forest satisfying the following properties.*

(α) $V^0(\widetilde{\mathcal{F}}_i) = V^0(\mathcal{F}'_i) \cap (A \cup B) = (A \setminus V^-(\mathcal{F}'_i)) \cup (B \setminus V^+(\mathcal{F}'_i))$.

(β) $V^+(\widetilde{\mathcal{F}}_i) = V^+(\mathcal{F}'_i) \subseteq B$ *and* $V^-(\widetilde{\mathcal{F}}_i) = V^-(\mathcal{F}'_i) \subseteq A$.

Proof of Claim. Let $i \in [\ell]$. Using Claim 1, it is easy to check that $\mathcal{F}'_i$ is a subdivision of $\widetilde{\mathcal{F}}_i$. Then, each $v \in V(\widetilde{\mathcal{F}}_i)$ satisfies both $d^\pm_{\widetilde{\mathcal{F}}_i}(v) = d^\pm_{\mathcal{F}'_i}(v)$. Moreover, each cycle in $\widetilde{\mathcal{F}}_i$ would induce a cycle in $\mathcal{F}'_i$. Recall from Claim 1 that $\mathcal{F}'_i$ is a linear forest. Thus, $\widetilde{\mathcal{F}}_i$ is also a linear forest and (α) and (β) follow from (b) and (d). $\diamond$

Since $\widetilde{\mathcal{F}}_1, \dots, \widetilde{\mathcal{F}}_\ell$ may not be bipartite on vertex classes A and B, we cannot apply Theorem 8.1 directly and need to consider equivalent linear forests (recall Definition 8.4).

Claim 3. *There exist bipartite linear forests $\widetilde{\mathcal{F}}'_1, \dots, \widetilde{\mathcal{F}}'_\ell$ on vertex classes A and B such that $\widetilde{\mathcal{F}}_i$ and $\widetilde{\mathcal{F}}'_i$ are equivalent for each $i \in [\ell]$. In particular, the following hold for each $i \in [\ell]$.*

(α') $V^0(\widetilde{\mathcal{F}}'_i) = V^0(\widetilde{\mathcal{F}}_i) = V^0(\mathcal{F}'_i) \cap (A \cup B) = (A \setminus V^-(\mathcal{F}'_i)) \cup (B \setminus V^+(\mathcal{F}'_i))$.

(β') $V^+(\widetilde{\mathcal{F}}'_i) = V^+(\widetilde{\mathcal{F}}_i) = V^+(\mathcal{F}'_i) \subseteq B$ *and* $V^-(\widetilde{\mathcal{F}}'_i) = V^-(\widetilde{\mathcal{F}}_i) = V^-(\mathcal{F}'_i) \subseteq A$.

Proof of Claim. Let $i \in [\ell]$. By Claim 2, $\widetilde{\mathcal{F}}_i$ is a linear forest which spans $A \cup B$ and whose components are all paths which start in B and end in A. Recall from (14.1) that $|A| = |B|$. Thus, one can easily construct an auxiliary bipartite linear forest $\widetilde{\mathcal{F}}_i'$ which is equivalent to $\widetilde{\mathcal{F}}_i$. Then, ($\alpha'$) and ($\beta'$) follow from ($\alpha$) and ($\beta$). $\diamond$

Let $\widetilde{\mathcal{F}}_1', \dots, \widetilde{\mathcal{F}}_\ell'$ be the linear forests obtained by applying Claim 3. We now verify that Theorem 8.1(i)–(iii) hold for $\widetilde{\mathcal{F}}_1', \dots, \widetilde{\mathcal{F}}_\ell'$. Let $i \in [\ell]$. Since $\widetilde{\mathcal{F}}_i'$ is a spanning bipartite linear forest on vertex classes A and B, we have

$$e(\widetilde{\mathcal{F}}_i'[B, A]) = \sum_{v \in B} d^+_{\widetilde{\mathcal{F}}_i'}(v) = |B \setminus V^-(\widetilde{\mathcal{F}}_i')| \overset{(\beta')}{=} |B| \overset{(14.1)}{=} |A|$$

and

$$e(\widetilde{\mathcal{F}}_i'[A, B]) = \sum_{v \in A} d^+_{\widetilde{\mathcal{F}}_i'}(v) = |A \setminus V^-(\widetilde{\mathcal{F}}_i')| \overset{(\beta')}{=} |V^0(\widetilde{\mathcal{F}}_i) \cap A| \overset{\text{(c)},(\alpha')}{\leq} 3\varepsilon|A|.$$

Thus, Theorem 8.1(i) and (ii) hold with $\widetilde{\mathcal{F}}_i', |A|$, and 3ε playing the roles of F_i, n, and ε. Moreover, each $v \in A$ satisfies

$$d_{\widetilde{\mathcal{F}}_i'[A,B]}(v) = d^+_{\widetilde{\mathcal{F}}_i'}(v) \overset{(\alpha'),(\beta')}{=} d^+_{\mathcal{F}_i'}(v) \overset{\text{(a)}}{=} d^+_{\mathcal{F}_i}(v) = d_{\mathcal{F}_i[A,B]}(v)$$

and, similarly, each $w \in B$ satisfies $d_{\widetilde{\mathcal{F}}_i'[A,B]}(w) = d_{\mathcal{F}_i[A,B]}(w)$. Thus, (iv) and (14.1) imply that Theorem 8.1(iii) holds with $\widetilde{\mathcal{F}}_i', |A|$, and 3ε playing the roles of F_i, n, and ε.

Apply Theorem 8.1 with $D[A \cup B], |A|, 3\varepsilon$, and $\widetilde{\mathcal{F}}_1', \dots, \widetilde{\mathcal{F}}_\ell'$ playing the roles of D, n, ε, and $F_1, \dots, F_\ell$ to obtain edge-disjoint cycles $\widetilde{C}_1, \dots, \widetilde{C}_\ell$ such that, for each $i \in [\ell]$, $V(\widetilde{C}_i) = A \cup B$ and $\widetilde{\mathcal{F}}_i' \subseteq \widetilde{C}_i \subseteq D(A, B) \cup \widetilde{\mathcal{F}}_i'$. For each $i \in [\ell]$, let $\widetilde{C}_i' := (\widetilde{C}_i \setminus \widetilde{\mathcal{F}}_i') \cup \widetilde{\mathcal{F}}_i$ and $C_i := (\widetilde{C}_i' \setminus \widetilde{\mathcal{F}}_i) \cup \mathcal{F}_i' = (\widetilde{C}_i \setminus \widetilde{\mathcal{F}}_i') \cup \mathcal{F}_i'$.

Step 3: Verifying the conclusions of the lemma. We now verify that $C_1, \dots, C_\ell$ are edge-disjoint Hamilton cycles of T such that, for each $i \in [\ell]$, $\mathcal{F}_i \subseteq C_i \subseteq D \cup \mathcal{F}_i$.

Recall from Claim 1 that $\mathcal{F}_1', \dots, \mathcal{F}_\ell'$ are edge-disjoint. Thus, (a) and Theorem 8.1

imply that $C_1, \dots, C_\ell$ are edge-disjoint. Let $i \in [\ell]$. By construction, $\mathcal{F}_i' \subseteq D \cup \mathcal{F}_i$ and $\widetilde{C}_i \setminus \widetilde{\mathcal{F}}_i' \subseteq D$. Thus, $C_i = (\widetilde{C}_i \setminus \widetilde{\mathcal{F}}_i') \cup \mathcal{F}_i' \subseteq D \cup \mathcal{F}_i$. Moreover, $\mathcal{F}_i \subseteq \mathcal{F}_i' \subseteq C_i$, as desired.

Let $i \in [\ell]$ and recall from Claim 3 that $\widetilde{\mathcal{F}}_i$ and $\widetilde{\mathcal{F}}_i'$ are equivalent. Thus, Fact 8.5 implies that $\widetilde{C}_i'$ is also a Hamilton cycle on $A \cup B$. By Claim 2, $\mathcal{F}_i'$ is a subdivision of $\widetilde{\mathcal{F}}_i$ and so C_i is a subdivision of $\widetilde{C}_i'$. In particular, C_i is a cycle satisfying

$$V(T) \supseteq V(C_i) = V(\widetilde{C}_i') \cup V(\mathcal{F}_i') \overset{\text{(b)}}{\supseteq} V(T).$$

That is, C_i is a Hamilton cycle of T. $\square$

14.2 Proof of Theorem 4.4

We are now ready to derive Theorem 4.4. Our strategy is as follows. In Step 1, we use Lemma 12.10 to construct a cycle-setup for the robust decomposition lemma (Lemma 11.10). In Step 2, we decompose the backward edges and exceptional edges into feasible systems using Lemma 13.12. In Step 3, we apply the robust decomposition lemma (Lemma 11.10) to obtain an absorber D^{rob} (the required extended special factors are constructed using Lemma 12.7). In Step 4, we construct an approximate Hamilton decomposition using Lemma 14.1. In Step 5, we decompose the leftovers using D^{rob}.

Proof of Theorem 4.4. Fix additional constants such that

$$0 < \frac{1}{n_0} \ll \varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll \eta \ll \frac{1}{k} \ll \varepsilon_3 \ll \gamma \ll \frac{1}{q} \ll \frac{1}{f} \ll d \ll \frac{1}{\ell'}, \frac{1}{g}, \nu \ll \tau \ll 1$$

and $\frac{k}{14}, \frac{k}{f}, \frac{k}{g}, \frac{q}{f}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$. Let $m_0 \in \mathbb{N}$ be such that $\varepsilon_1^2 n \leq m_0 \leq \varepsilon_1 n$ and $m := \frac{n-m_0}{k}, \frac{m}{4\ell'}, \frac{fm}{q} \in \mathbb{N}$. Fix additional constants such that $\frac{1}{f} \ll \frac{r_1}{m} \ll d$ and $\eta \ll \frac{r}{m} \ll \frac{1}{k}$. Define

$$r_2 := 96\ell' g^2 kr, \quad r_3 := \frac{rfk}{q}, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s' := rfk + 7r^\diamond. \tag{14.3}$$

For simplicity, we denote

$$Q \coloneqq [r_3] \times [\tfrac{q}{f}] \times [4] \times [f] \quad \text{and} \quad Q' \coloneqq [r^\diamond] \times [1] \times [4] \times [7]. \tag{14.4}$$

Let T be a regular bipartite tournament which is ε-close to the complete blow-up C_4 on vertex classes of size $n \geq n_0$. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T and denote by $H \subseteq \overleftarrow{T}_{\mathcal{U}}$ the digraph obtained by applying Lemma 13.8.

Step 1: Constructing a cycle-setup. We will use Lemma 12.10. We first construct the partitions in $\mathcal{P}^*$ randomly, to ensure that the edges of H are well distributed across the clusters. More precisely, for each $(h, i, j) \in [\frac{q}{f}] \times [4] \times [k]$, let $V_{i,j,h} \subseteq U_i$ be obtained by including each $v \in U_i$ with probability $\frac{f}{kq}$ independently of all other vertices. For each $(i, j) \in [4] \times [k]$, denote $V_{i,j} \coloneqq \bigcup_{h \in [\frac{q}{f}]} V_{i,j,h}$. For each $i \in [4]$, let $\widetilde{C}^i \coloneqq V_{i,1} \dots V_{i,k}$ and let $\mathcal{I}_i \coloneqq \{I_1, \dots, I_f\}$ denote the canonical interval partition of $\widetilde{C}^i$ into f interval, that is,

$$I_j \coloneqq V_{i,(j-1)\frac{k}{f}+1} V_{i,(j-1)\frac{k}{f}+2} \dots V_{i,j\frac{k}{f}+1}$$

for each $j \in [f]$ (see Definition 8.20). For each $(h, i, j) \in [\frac{q}{f}] \times [4] \times [f]$, define

$$S_{h,i,j} \coloneqq V_{i,(j-1)\frac{k}{f}+2,h} \cup V_{i,(j-1)\frac{k}{f}+3,h} \cup \dots \cup V_{i,j\frac{k}{f},h}, \tag{14.5}$$

that is, $S_{h,i,j}$ is the union of the h^{th} subclusters of the internal clusters in the j^{th} interval in the canonical interval partition of $\widetilde{C}^i$ into f intervals.

Claim 1. *With positive probability, all of the following hold.*

(i) *For each $(h, i, j) \in [\frac{q}{f}] \times [4] \times [k]$, we have $|V_{i,j,h}| \geq \frac{(1-\varepsilon)fn}{kq}$.*

(ii) *For each $(h, i, j) \in [\frac{q}{f}] \times [4] \times [k]$ and $v \in V(T)$, we have $|N_T^{\pm}(v) \cap V_{i,j,h}| \geq \frac{f|N_T^{\pm}(v) \cap U_i|}{kq} - \varepsilon n$.*

(iii) *For each* $h, h' \in [\frac{q}{f}]$, $i \in [4]$, *and* $j, j' \in [f]$, *we have*

$$e_{H-U^{1-\gamma}(T)}(S_{h,i,j}, S_{h',i-1,j'}) \geq 110\gamma n|U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)|.$$

Proof of Claim. By Lemma 7.16 and a union bound, (i) and (ii) hold with probability at least $1 - \frac{1}{n}$.

Let $i \in [4]$. If $|U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)| = 0$, then (iii) holds for i with probability 1. Suppose that $|U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)| \geq 1$. Denote $G := H[U_i, U_{i-1}]$. By Lemma 13.8(i) and (iii), $\Delta(G) \leq \gamma n$ and $e(G) \geq \frac{n}{2}$. Also observe that, for each $i' \in [4]$, the q sets $S_{1,i',1}, \dots, S_{\frac{q}{f},i',1}, S_{1,i',2}, \dots, S_{\frac{q}{f},i',f}$ randomly partition $U_{i'}$. Thus, Lemma 7.21 (applied with $q, \gamma, U_i, U_{i-1}, S_{1,i,1}, \dots, S_{\frac{q}{f},i,f}$, and $S_{1,i-1,1}, \dots, S_{\frac{q}{f},i-1,f}$ playing the roles of k, ε, A, B, $A_1, \dots, A_k$, and $B_1, \dots, B_k$) implies that, with probability at least $\frac{4}{5}$, all $h, h' \in [\frac{q}{f}]$ and $j, j' \in [f]$ satisfy

$$e_H(S_{h,i,j}, S_{h',i-1,j'}) \geq \frac{e_H(U_i, U_{i-1})}{2q^2} \overset{\text{Lemma 13.8(iii)}}{\geq} 110\gamma n|U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)|.$$

Thus, Lemma 13.8(ii) implies that (iii) holds for i with probability at least $\frac{4}{5}$.

Therefore, a union bound over all $i \in [4]$ implies that (iii) holds with probability at least $\frac{1}{5}$. Then, a union bound implies that, with positive probability, (i)–(iii) are all satisfied. $\diamond$

We may therefore assume that (i)–(iii) are all satisfied. We now equalise the partition classes to achieve (i′) below, without affecting the bounds in (ii) and (iii) too much. For this, note that

$$|U^{\varepsilon_1}_i(T)| \overset{(13.2)}{\leq} \frac{e_T(U_i, U_{i-1})}{\varepsilon_1 n} \overset{\text{Fact 10.5}}{\leq} \frac{\varepsilon n^2}{\varepsilon_1 n} \leq \varepsilon_1^3 n$$

for each $i \in [4]$. Moreover, (i) implies that

$$|V_{i,j,h}| \geq \frac{fm}{q} + \frac{f(m_0 - \varepsilon n)}{kq} \geq \frac{fm}{q} + \frac{(\varepsilon_1^2 - \varepsilon)fn}{kq} \geq \frac{fm}{q} + |U^{\varepsilon_1}_i(T)|$$

for each $(h,i,j) \in [\frac{q}{f}] \times [4] \times [k]$. Thus, for each $i \in [4]$, we can let U_i^* be obtained from $U_i^{\varepsilon_1}(T)$ by adding, for each $(h,j) \in [\frac{q}{f}] \times [k]$, precisely $|V_{i,j,h} \setminus U_i^{\varepsilon_1}(T)| - \frac{fm}{q}$ vertices of $V_{i,j,h} \setminus U_i^{\varepsilon_1}(T)$. For each $(h,i,j) \in [\frac{q}{f}] \times [4] \times [k]$, let $V'_{i,j,h} := V_{i,j,h} \setminus U_i^*$ and $V'_{i,j} := V_{i,j} \setminus U_i^*$. For each $i \in [4]$, define $C^i := V'_{i,1} \dots V'_{i,k}$. For each $(h,i,j) \in [\frac{q}{f}] \times [4] \times [f]$, let $S'_{h,i,j} := S_{h,i,j} \setminus U_i^*$ and observe that $S'_{h,i,j}$ is the union of the h^{th} subclusters of the internal clusters in the j^{th} interval in the canonical interval partition of C^i into f intervals. Then, (i)–(iii) imply that the following properties are satisfied.

(i′) For each $(h,i,j) \in [\frac{q}{f}] \times [4] \times [k]$, we have $|V'_{i,j,h}| = \frac{fm}{q}$ and $|U_i^*| = m_0$.

(ii′) For each $(h,i,j) \in [\frac{q}{f}] \times [4] \times [f]$ and $v \in V(T)$, we have

$$\begin{aligned} \left|N_T^{\pm}(v) \cap \left(U_i^* \cup S'_{h,i,j}\right)\right| &\geq \left(\frac{k}{f} - 1\right) \cdot \left(\frac{f|N_T^{\pm}(v) \cap U_i|}{kq} - \varepsilon n\right) \\ &\geq \frac{|N_T^{\pm}(v) \cap U_i|}{2q} - \varepsilon_1 n. \end{aligned}$$

(iii′) For each $h, h' \in [\frac{q}{f}]$, $i \in [4]$, and $j, j' \in [f]$, we have

$$e_{H - U^{1-\gamma}(T)}\left(U_i^* \cup S'_{h,i,j}, U_{i-1}^* \cup S'_{h',i-1,j'}\right) \geq 110\gamma n |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|.$$

By (i′) and construction, $U^* := \bigcup_{i \in [4]} U_i^*$ is an $(\varepsilon_1, \mathcal{U})$-exceptional set for T (see Definition 13.11). Thus, Fact 10.2(i) and (ES2) imply that

$$n' := |U_1 \setminus U^*| = \cdots = |U_4 \setminus U^*| = n - \frac{|U^*|}{4} \geq (1 - \varepsilon_1)n. \tag{14.6}$$

Therefore, each $i \in [4]$ satisfies

$$\delta(T[U_i \setminus U^*, U_{i+1} \setminus U^*]) \geq n' - \varepsilon_1 n \geq (1 - \sqrt{\varepsilon_1})n'.$$

Define

$$\mathcal{U}' := (U_1 \setminus U^*, \dots, U_4 \setminus U^*).$$

For each $i \in [4]$, let $\mathcal{P}_i$ be the partition of $U_i \setminus U^*$ into an empty exceptional set and the k clusters $V'_{i,1}, \dots, V'_{i,k}$ and let $\mathcal{P}_i^*$ be the partition of $U_i \setminus U^*$ into an empty exceptional set and the $\frac{kq}{f}$ clusters $V'_{i,1,1}, \dots, V'_{i,1,\frac{q}{f}}, V'_{i,2,1}, \dots, V'_{i,k,\frac{q}{f}}$. Denote $\mathcal{P} := (\mathcal{P}_1, \dots, \mathcal{P}_4)$, $\mathcal{P}^* := (\mathcal{P}_1^*, \dots, \mathcal{P}_4^*)$, and $\mathcal{C} := (C^1, \dots, C^4)$. For each $i \in [4]$, note that $\mathcal{P}_i^*$ is a $\frac{q}{f}$-refinement of $\mathcal{P}_i$ and C^i is a Hamilton cycle on the clusters in $\mathcal{P}_i$. Let $D_1, D_2, \mathcal{P}', \mathcal{R}, \mathscr{U}, \mathscr{U}'$, and $\mathcal{M}$ be obtained by applying Lemma 12.10 with $\overrightarrow{T}_{\mathcal{U}} - U^*, \mathcal{U}', n', \frac{q}{f}, \sqrt{\varepsilon_1}$, and ε_2 playing the roles of $D, \mathcal{U}, n, \ell^*, \varepsilon$, and ε'. Let $D_1' := T \setminus D_2$ and observe that D_1' is obtained from D_1 by adding backward and exceptional edges only. Thus, Lemma 12.10(i)–(iii) are still satisfied with D_1' playing the role of D_1. That is, the following hold.

(iv) $(\mathcal{U}', \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is a consistent $(4, \ell^*, k, n')$-cycle-framework. In particular, the following hold.

- By Fact 12.9, $(\mathcal{U}', \mathcal{P}, \mathcal{P}, \mathcal{C}, \mathcal{M})$ is a consistent $(4, \ell^*, k, n')$-cycle-framework.
- For any $i \in [4]$ and any cluster $V \in \mathcal{P}_i$, the set $N_{M_i}(V)$ is a cluster in $\mathcal{P}_{i+1}$ (where $\mathcal{P}_5 := \mathcal{P}_1$).
- The analogue holds for the partitions in $\mathcal{P}^*$.

(v) For any $i \in [4]$, $D_1'[V, W]$ is $[\varepsilon_2, \geq 1 - 3d]$-superregular whenever $V \subseteq U_i \setminus U^*$ and $W \subseteq U_{i+1} \setminus U^*$ are unions of clusters in $\mathcal{P}_i^*$ and $\mathcal{P}_{i+1}^*$, respectively. In particular, since $\mathcal{P}_i^*$ is a refinement of $\mathcal{P}_i$ for each $i \in [4]$, the analogue holds for the partitions in $\mathcal{P}$.

(vi) $(D_2, \mathcal{U}', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ is a $(4, \ell', \frac{q}{f}, k, m, \varepsilon_2, d)$-cycle-setup. In particular, Facts 11.4 and 11.5 imply that $(\mathcal{U}', \mathcal{P}, \mathcal{P}^*, \mathcal{C}, \mathcal{M})$ is a $(4, \frac{q}{f}, k, n')$-cycle-framework and $(\mathcal{U}', \mathcal{P}, \mathcal{P}, \mathcal{C}, \mathcal{M})$ is a $(4, 1, k, n')$-cycle-framework.

As discussed in Section 12.3, we will use D_1' to construct the required extended special factors (via Lemma 12.7), while D_2 will be reserved for the application of the robust decomposition lemma for blow-up cycles (Lemma 11.10).

Step 2: Decomposing the backward and exceptional edges. We will use

Lemma 13.12, so we start by building digraphs $H_1, \dots, H_s$ which satisfy Lemma 13.12(i)–(iv).

First, observe that if we apply Lemma 13.8, then H satisfies Lemma 13.12(iii) and (iv), but not Lemma 13.12(i) and (ii). To achieve the latter, we add edges as follows. Let $\widetilde{H}$ be the spanning subdigraph of $\overleftarrow{T}_{\mathcal{U}}$ which consists of all the edges incident to $U^{1-\gamma}(T)$. Let $\widehat{H}$ be the spanning subdigraph of $\overrightarrow{T}_{\mathcal{U}}$ which consists of all the edges incident to $U^* \setminus U^{1-\gamma}(T)$. By (13.2), we see that $H \cup \widetilde{H} \cup \widehat{H}$ now satisfies all the bounds in Lemma 13.12(i)–(iv).

However, as discussed in Section 13.3, the feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_t$ constructed within $H_1, \dots, H_s$ will form the special covers required for Lemma 12.7. By Lemmas 13.6 and 13.12(c), these feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_t$ will automatically form balanced special covers. Additionally, Lemma 12.7 requires $\mathcal{F}_1, \dots, \mathcal{F}_t$ to be localised (recall Definition 12.3). To achieve this, each H_ℓ will be associated to one set of "locality parameters" (that is, a choice of (h, i, j) in Definition 12.3) and then obtained from $H \cup \widetilde{H} \cup \widehat{H}$ by removing all the edges which are forbidden with respect to this set of parameters. In Claim 2 below, we will verify that not too many edges are removed and so $H_1, \dots, H_s$ still satisfy Lemma 13.12(i)–(iv). Moreover, each of the $\mathcal{F}_i \subseteq H_\ell$ will automatically be localised.

Recall from Lemma 11.10 that we need two types of extended special factors: some $(\frac{q}{f}, 4, f)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$ and some $(1, 4, 7)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}, \mathcal{C}$, and $\mathcal{M}$. These will be constructed separately by applying Lemma 12.7 successively. First, we construct the H_ℓ's for the first application of Lemma 12.7, that is, for the construction of the $(\frac{q}{f}, 4, f)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$.

More precisely, let $(h, i, j) \in [\frac{q}{f}] \times [4] \times [f]$. Let $k' := \frac{k}{f} + 1$ and denote by $W_1 \dots W_{k'}$ the j^{th} interval in the canonical interval partition of C^i into f intervals. Denote by $W_{1,h}, \dots, W_{k',h}$ the h^{th} subclusters of $W_1, \dots, W_{k'}$ contained in $\mathcal{P}_i^*$. Let $E_{h,i,j}$ be the set of edges $e \in E(T)$ such that

$$V(e) \cap (U_i \cup U_{i+1}) \nsubseteq U^* \cup (W_{1,h} \cup \dots \cup W_{k'-1,h}) \cup N_{M_i}(W_{2,h} \cup \dots \cup W_{k',h}). \qquad (14.7)$$

(Roughly speaking, the set $E_{h,i,j}$ consists of all the edges of T which cannot be included in a $(\frac{q}{f}, 4, f, h, i, j)$-localised special cover. See the proof of Claim 2(a) below for details.) Let $H_{h,i,j} \coloneqq (H \cup \widetilde{H} \cup \widehat{H}) \setminus E_{h,i,j}$. We now claim that any special cover in $H_{h,i,j}$ is $(\frac{q}{f}, 4, f, h, i, j)$-localised with respect to $\mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$, and that $H_{h,i,j}$ satisfies Lemma 13.12(i)–(iv).

Claim 2. *For each $(h, i, j) \in [\frac{q}{f}] \times [4] \times [f]$, the following properties are satisfied.*

(a) *Let SC be a special cover in T with respect to U^*. If $SC \subseteq H_{h,i,j}$, then SC is $(\frac{q}{f}, 4, f, h, i, j)$-localised with respect to $\mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$.*

(b) *For each $v \in U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{H_{h,i,j},\mathcal{U}}(v) \geq 3\gamma n$.*

(c) *For each $v \in U^* \setminus U^{1-\gamma}(T)$, $\overrightarrow{d}^{\pm}_{H_{h,i,j},\mathcal{U}}(v) \geq \gamma^2 n$.*

(d) *For each $v \in V(T) \setminus U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{H_{h,i,j},\mathcal{U}}(v) \leq 2\gamma n$.*

(e) *For each $i' \in [4]$, $e_{H_{h,i,j} - U^{1-\gamma}(T)}(U_{i'}, U_{i'-1}) \geq 110\gamma n |U^{1-\gamma}_{i'-2}(T) \cup U^{1-\gamma}_{i'-3}(T)|$.*

Proof of Claim. Let $(h, i, j) \in [\frac{q}{f}] \times [4] \times [f]$. Denote by $W_1 \dots W_{k'}$ the j^{th} interval in the canonical interval partition of C^i into f intervals. Denote by $W_{1,h}, \dots, W_{k',h}$ the h^{th} subclusters of $W_1, \dots, W_{k'}$ contained in $\mathcal{P}^*_i$. Note that $S_i \coloneqq S'_{h,i,j} = W_{2,h} \cup \dots \cup W_{k'-1,h}$ and, by (iv), $S_{i+1} \coloneqq S'_{h,i+1,j} = N_{M_i}(W_{2,h} \cup \dots \cup W_{k'-1,h})$. For each $i' \in [4] \setminus \{i, i+1\}$, let $S_{i'} \coloneqq S'_{1,i',1}$.

Let $SC \subseteq H_{h,i,j}$ be a special cover in T with respect to U^*. By Definition 8.24, $V^+(SC) \cup V^-(SC) \subseteq V(T) \setminus U^*$ and so (14.7) implies that

$$(V^+(SC) \cup V^-(SC)) \cap (U_i \cup U_{i+1}) \subseteq (W_{1,h} \cup \dots \cup W_{k'-1,h}) \cup N_{M_i}(W_{2,h} \cup \dots \cup W_{k',h}).$$

Thus, SC is $(\frac{q}{f}, 4, f, h, i, j)$-localised with respect to $\mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$, and so (a) holds.

Let $i' \in [4]$ and $v \in U^{1-\gamma}_{i'}(T)$. By Fact 13.9 and (ES1), $v \in U^*$ and so (14.7) implies that $E_{h,i,j}$ does not contain any edge from v to $U^*_{i'-1} \cup S_{i'-1}$. Thus,

$$\overleftarrow{d}^{+}_{H_{h,i,j},\mathcal{U}}(v) \geq d^{+}_{\widetilde{H} \setminus E_{h,i,j}}(v) \geq |N^+_T(v) \cap (U^*_{i'-1} \cup S_{i'-1})| \overset{\text{(ii')}}{\geq} \frac{(1-\gamma)n}{2q} - \varepsilon_1 n \geq 3\gamma n.$$

Similarly, $\overleftarrow{d}^{-}_{H_{h,i,j},\mathcal{U}}(v) \geq 3\gamma n$ and so (b) is satisfied.

Let $i' \in [4]$ and $v \in U^*_{i'} \setminus U^{1-\gamma}(T)$. Since $v \in U^*$, (14.7) implies that $E_{h,i,j}$ does not contain any edge from $U^*_{i'-1} \cup S_{i'-1}$ to v. Thus,

$$\overrightarrow{d}^{-}_{H_{h,i,j},\mathcal{U}}(v) \geq d^{-}_{\widehat{H}\setminus E_{h,i,j}}(v) \geq |N^{-}_{T}(v) \cap (U^*_{i'-1} \cup S_{i'-1})| \overset{\text{(ii')}}{\geq} \frac{\gamma n}{2q} - \varepsilon_1 n \geq \gamma^2 n.$$

Similarly, $\overrightarrow{d}^{+}_{H_{h,i,j},\mathcal{U}}(v) \geq \gamma^2 n$ and so (c) holds.

For any $v \in V(T) \setminus U^{1-\gamma}(T)$, we have

$$\begin{aligned}\overleftarrow{d}^{\pm}_{H_{h,i,j},\mathcal{U}}(v) &\leq d^{\pm}_{H}(v) + d^{\pm}_{\widetilde{H}}(v) \overset{\text{Lemma 13.8(i)}}{\leq} \gamma n + |U^{1-\gamma}(T)| \\ &\overset{\text{Fact 13.9,Definition 13.11}}{\leq} \gamma n + 4\varepsilon_1 n \leq 2\gamma n\end{aligned}$$

and so (d) holds.

Let $i' \in [4]$. By (14.7), $E_{h,i,j}$ does not contain any edge from $U^*_{i'} \cup S_{i'}$ to $U^*_{i'-1} \cup S_{i'-1}$. Thus,

$$\begin{aligned} e_{H_{h,i,j}-U^{1-\gamma}(T)}(U_{i'}, U_{i'-1}) &\geq e_{(H\setminus E_{h,i,j})-U^{1-\gamma}(T)}(U_{i'}, U_{i'-1}) \\ &\geq e_{H-U^{1-\gamma}(T)}(U^*_{i'} \cup S_{i'}, U^*_{i'-1} \cup S_{i'-1}) \\ &\overset{\text{(iii')}}{\geq} 110\gamma n |U^{1-\gamma}_{i'-2}(T) \cup U^{1-\gamma}_{i'-3}(T)| \end{aligned}$$

and so (e) is satisfied. $\diamond$

The H_ℓ's for the second application of Lemma 12.7, that is, for the construction of the $(1,4,7)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}, \mathcal{C}$, and $\mathcal{M}$, can be constructed analogously. More precisely, let $(h,i,j) \in [1] \times [4] \times [7]$. Let $k'' := \frac{k}{7} + 1$ and denote by $W'_1 \dots W'_{k''}$ the j^{th} interval in the canonical interval partition of C^i into 7 intervals. Let $E'_{h,i,j}$ be the set of edges $e \in E(T)$ such that

$$V(e) \cap (U_i \cup U_{i+1}) \not\subseteq U^* \cup (W'_1 \cup \dots \cup W'_{k''-1}) \cup N_{M_i}(W'_2 \cup \dots \cup W'_{k''}).$$

Let $H'_{h,i,j} := (H \cup \widetilde{H} \cup \widehat{H}) \setminus E'_{h,i,j}$. Since $f > 14$, note that every interval in the canonical interval partition of C^i into 7 intervals contains, as a subinterval, an interval from the canonical interval partition of C^i into f intervals. That is, there exists $(h', j') \in [\frac{q}{f}] \times [f]$ such that $S'_{h',i,j'} \subseteq W'_2 \cup \cdots \cup W'_{k''-1}$. Thus, we can apply similar arguments as in Claim 2, to show that the following hold.

(a′) Let SC be a special cover in T with respect to U^*. If $SC \subseteq H'_{h,i,j}$, then SC is $(1, 4, 7, h, i, j)$-localised with respect to $\mathcal{P}, \mathcal{C}$, and $\mathcal{M}$.

(b′) For each $v \in U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{H'_{h,i,j},\mathcal{U}}(v) \geq 3\gamma n$.

(c′) For each $v \in U^* \setminus U^{1-\gamma}(T)$, $\overrightarrow{d}^{\pm}_{H'_{h,i,j},\mathcal{U}}(v) \geq \gamma^2 n$.

(d′) For each $v \in V(T) \setminus U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{H'_{h,i,j},\mathcal{U}}(v) \leq 2\gamma n$.

(e′) For each $i' \in [4]$, $e_{H'_{h,i,j}-U^{1-\gamma}(T)}(U_{i'}, U_{i'-1}) \geq 110\gamma n|U^{1-\gamma}_{i'-2}(T) \cup U^{1-\gamma}_{i'-3}(T)|$.

Denote $\mathcal{H} := \{H_{h,i,j} \mid (h, i, j) \in [\frac{q}{f}] \times [4] \times [f]\}$ and $\mathcal{H}' := \{H'_{h,i,j} \mid (h, i, j) \in [1] \times [4] \times [7]\}$. Let $s := 4q + 28$ and let $H_1, \ldots, H_s$ be an enumeration of $\mathcal{H} \cup \mathcal{H}'$. Recall from Lemma 11.10 that we need to construct r_3 $(\frac{q}{f}, 4, f)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$, as well as $r^\diamond$ $(1, 4, 7)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}, \mathcal{C}$, and $\mathcal{M}$. Also recall that in Lemma 13.12, $s_1, \ldots, s_s$ denote the number of feasible systems constructed within $H_1, \ldots, H_s$, respectively. Thus, for each $i \in [s]$, define

$$s_i := \begin{cases} r_3 & \text{if } H_i \in \mathcal{H}; \\ r^\diamond & \text{if } H_i \in \mathcal{H}'. \end{cases}$$

Note that $\sum_{i \in [s]} s_i = 4qr_3 + 28r^\diamond = 4s' \leq \varepsilon_3 n$ (see (14.3) for the definition of s'). By (b)–(e) and (b′)–(e′), Lemma 13.12(i)–(iv) hold.

Let $E := \{uv \in E(T) \mid vu \in \bigcup \mathcal{M}\}$ (this is precisely the set of edges which cannot be decomposed via Lemma 11.10). By (vi) and (CF5), Lemma 13.12(v) and (vi) hold. Recall the notation introduced in (14.4). By Lemma 13.12 (applied with $\varepsilon_1, \varepsilon_2$, and ε_3 playing the roles of $\varepsilon, \varepsilon'$, and η), there exist disjoint sets

$$S = \{\mathcal{F}_{\ell,h,i,j} \mid (\ell,h,i,j) \in Q\}, \ \ S' = \{\mathcal{F}'_{\ell,h,i,j} \mid (\ell,h,i,j) \in Q'\}, \ \ S'' = \{\mathcal{F}_i \mid i \in [n-4s']\}$$

such that $S^* := S \cup S' \cup S''$ is a set of n edge-disjoint feasible systems which satisfy the following properties.

(α) $E(\overleftarrow{T}_{\mathcal{U}}) \cup E \subseteq E(S^*) \subseteq E(T)$.

(β) For each $\mathcal{F} \in S^*$, $e(\mathcal{F}) \leq \varepsilon_2 n$.

(γ) For each $\mathcal{F} \in S \cup S'$, $V^0(\mathcal{F}) = U^*$.

(δ) For each $(\ell,h,i,j) \in Q$, we have $\mathcal{F}_{\ell,h,i,j} \subseteq H_{h,i,j} \setminus E$.

(ε) For each $(\ell,h,i,j) \in Q'$, we have $\mathcal{F}'_{\ell,h,i,j} \subseteq H'_{h,i,j} \setminus E$.

(ζ) For each $\mathcal{F} \in S''$, we have $V^+(\mathcal{F}) \subseteq U_1$ and $V^-(\mathcal{F}) \subseteq U_4$.

By Fact 13.4, we may assume without loss of generality that all the forward edges in $E(S^*)$ are either edges of E or incident to U^*. Thus, Lemma 13.12(vi) implies that the following holds.

(η) For each $v \in V(T) \setminus U^*$, we have $|\overrightarrow{N}^{\pm}_{S^*}(v) \setminus U^*| \leq 1$.

We next observe that the feasible systems in $S \cup S'$ are localised and balanced special covers.

Claim 3. *The following hold.*

(γ') *Each $\mathcal{F} \in S \cup S'$ is a $\mathcal{U}'$-balanced special cover in D'_1 with respect to U^*.*

(δ') *For each $(\ell,h,i,j) \in Q$, $\mathcal{F}_{\ell,h,i,j}$ is $(\frac{q}{f}, 4, f, h, i, j)$-localised with respect to $\mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$.*

(ε') *For each $(\ell,h,i,j) \in Q'$, $\mathcal{F}'_{\ell,h,i,j}$ is $(1, 4, 7, h, i, j)$-localised with respect to $\mathcal{P}, \mathcal{C}$, and $\mathcal{M}$.*

Proof of Claim. By assumption, (δ), and (ε), all the forward edges in $E(S \cup S')$ are incident to U^*. Since by Step 1 $D_2 = T \setminus D'_1 \subseteq \overrightarrow{T}_{\mathcal{U}} - U^*$, this implies that $E(S \cup S') \subseteq E(D'_1)$. Moreover, (ES2) implies that $|U^*_1| = \cdots = |U^*_4|$ and so Lemma 13.6 and (γ) imply that each $\mathcal{F} \in S \cup S'$ is a $\mathcal{U}'$-balanced special cover in D'_1 with respect to U^*. In particular, (γ') holds. Then, (δ') follows from (a) and (δ) and, similarly, (ε') holds by (a$'$) and (ε). $\diamond$

Step 3: Applying the robust decomposition lemma. In this step, we will apply the robust decomposition lemma (Lemma 11.10) to obtain a robustly decomposable digraph $D^{\rm rob}$ which will enable us to decompose the leftovers after the approximate decomposition. First, we use Lemma 12.7 to construct the required extended special factors.

Recall from Claim 3 that $S \cup S'$ consists of special covers in D'_1 with respect to U^*. For each $(\ell, h, i, j) \in Q$, let $M_{\ell,h,i,j}$ denote the complete special sequence associated to $\mathcal{F}_{\ell,h,i,j} \in S$ (see Definition 8.25). Define a multiset $\mathscr{M}$ by $\mathscr{M} := \{M_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q\}$. For each $(\ell, h, i, j) \in Q'$, let $M'_{\ell,h,i,j}$ denote the complete special sequence associated to $\mathcal{F}'_{\ell,h,i,j} \in S'$. Define a multiset $\mathscr{M}'$ by $\mathscr{M}' := \{M_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q'\}$.

First, we construct the $(\frac{q}{f}, 4, f)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$ required for Lemma 11.10. By (β), (γ'), and (δ'), Lemma 12.7(iii)–(v) hold with $\mathcal{F}_{\ell,h,i,j}, \frac{q}{f}, \varepsilon_2$, and $\mathcal{U}'$ playing the roles of $SC_{\ell,h,i,j}, \ell^*, \varepsilon$, and $\mathcal{U}$. Moreover, Lemma 12.7(i) follows immediately from (iv). Let $D''_1 := D'_1 \setminus (S' \cup S'')$. By Proposition 7.8(ii), (v), and (η), Lemma 12.7(ii) holds with D''_1 and $3d$ playing the roles of D and ε'. Apply Lemma 12.7 with $D''_1, \mathcal{U}', r_3, \frac{q}{f}, n', \varepsilon_2, 3d$, and S playing the roles of $D, \mathcal{U}, r, \ell^*, n, \varepsilon, \varepsilon'$, and SC to obtain r_3 $(\frac{q}{f}, 4, f)$-extended special factors $ESF_1, \dots, ESF_{r_3}$ with respect to $\mathcal{U}', \mathcal{P}^*, \mathcal{C}$, and $\mathcal{M}$ which satisfy the following properties, where for each $(\ell, h, i, j) \in Q$, $ESPS_{\ell,h,i,j}$ denotes the $(\frac{q}{f}, 4, f, h, i, j)$-extended special path system contained in ESF_ℓ.

(I) For each $(\ell, h, i, j) \in Q$, we have $M_{\ell,h,i,j} \subseteq ESPS_{\ell,h,i,j} \subseteq (D''_1 \setminus S) \cup M_{\ell,h,i,j}$.

(II) Let $(\ell, h, i, j), (\ell', h', i', j') \in Q$ be distinct. Then, we have $(ESPS_{\ell,h,i,j} \setminus M_{\ell,h,i,j}) \cap (ESPS_{\ell',h',i',j'} \setminus M_{\ell',h',i',j'}) = \emptyset$.

Define a multidigraph $\mathcal{ESF}$ by $\mathcal{ESF} := ESF_1 \cup \cdots \cup ESF_{r_3}$.

Next, we construct the $(1, 4, 7)$-extended special factors with respect to $\mathcal{U}', \mathcal{P}, \mathcal{C}$, and

$\mathcal{M}$ required for Lemma 11.10. By (β), (γ'), and (ε'), Lemma 12.7(iii)–(v) hold with $\mathcal{F}'_{\ell,h,i,j}, 1, 7, \varepsilon_2$, and $\mathcal{U}'$ playing the roles of $SC_{\ell,h,i,j}, \ell^*, f, \varepsilon$, and $\mathcal{U}$. By (iv), Lemma 12.7(i) holds with $\mathcal{P}$ playing the role of $\mathcal{P}^*$. Let $D_1''' \coloneqq D_1' \setminus (S \cup S'' \cup \mathcal{ESF})$. By Proposition 7.8(ii), (v), (η), and Corollary 12.2, Lemma 12.7(ii) holds with $D_1''', \mathcal{P}$, and $3d$ playing the roles of $D, \mathcal{P}^*$, and ε'. Apply Lemma 12.7 with $D_1''', \mathcal{U}', \mathcal{P}, r^\diamond, 1, 7, n', \varepsilon_2, 3d$, and S' playing the roles of $D, \mathcal{U}, \mathcal{P}^*, r, \ell^*, f, n, \varepsilon, \varepsilon'$, and SC to obtain $r^\diamond$ $(1,4,7)$-extended special factors $ESF_1', \dots, ESF_{r^\diamond}'$ with respect to $\mathcal{U}', \mathcal{P}, \mathcal{C}$, and $\mathcal{M}$ which satisfy the following properties, where for each $(\ell, h, i, j) \in Q'$, $ESPS'_{\ell,h,i,j}$ denotes the $(1,4,7,h,i,j)$-extended special path system contained in ESF'_ℓ.

(I′) For each $(\ell, h, i, j) \in Q'$, we have $M'_{\ell,h,i,j} \subseteq ESPS'_{\ell,h,i,j} \subseteq (D_1''' \setminus S') \cup M'_{\ell,h,i,j}$.

(II′) Let $(\ell, h, i, j), (\ell', h', i', j') \in Q'$ be distinct. Then, we have $(ESPS'_{\ell,h,i,j} \setminus M'_{\ell,h,i,j}) \cap (ESPS'_{\ell',h',i',j'} \setminus M'_{\ell',h',i',j'}) = \emptyset$.

Define a multidigraph $\mathcal{ESF}'$ by $\mathcal{ESF}' \coloneqq ESF_1' \cup \dots \cup ESF_{r^\diamond}'$.

We are now ready to apply the robust decomposition lemma. Let $D_2' \coloneqq D_2 \setminus S^*$ and recall from Step 1 that $D_2 \subseteq \overrightarrow{T}_{\mathcal{U}} - U^*$. By (vi), (η), and Proposition 11.6, $(D_2', \mathcal{U}', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, \mathcal{R}, \mathcal{C}, \mathscr{U}, \mathscr{U}', \mathcal{M})$ is a $(4, \ell', \frac{q}{f}, k, m, \varepsilon_3, \frac{d}{2})$-cycle-setup. Let $D^{\rm rob}$ be the robustly decomposable digraph obtained by applying Lemma 11.10 with $D_2', \mathcal{U}', 4, \varepsilon_3$, and $\frac{d}{2}$ playing the roles of $D, \mathcal{U}, K, \varepsilon$, and d.

Step 4: Approximate decomposition. Let $D \coloneqq T \setminus (S^* \cup \mathcal{ESF} \cup \mathcal{ESF}' \cup D^{\rm rob})$. In this step, we will approximately decompose $D \cup S''$ using Lemma 14.1. By (β) and (ζ), Lemma 14.1(iii) and (v) hold with $S'', n - t$, and ε_2 playing the roles of $\{\mathcal{F}_1, \dots, \mathcal{F}_\ell\}, \ell$, and ε. Moreover, (η) implies that each $v \in V(T) \setminus U^*$ satisfies

$$d_{S''}(v) \leq \overleftarrow{d}_{T,\mathcal{U}}(v) + |U^*| + 2 \overset{\text{Definition 13.11}}{\leq} 2\varepsilon_1 n + 4\varepsilon_1 n + 2 \leq \varepsilon_2 n \tag{14.8}$$

and so Lemma 14.1(iv) holds with $S'', n - t$, and ε_2 playing the roles of $\{\mathcal{F}_1, \dots, \mathcal{F}_\ell\}, \ell$, and ε. It remains to show that $D[U_i \setminus U^*, U_{i+1} \setminus U^*]$ is an almost regular bipartite robust expander for each $i \in [4]$.

Claim 4. *Each $v \in V(T)$ satisfies*

$$d^{\pm}_{D}(v) = \begin{cases} 0 & \textit{if } v \in U^*; \\ n - (4s' - r) - d^{\pm}_{S''}(v) & \textit{otherwise.} \end{cases} \tag{14.9}$$

Proof of Claim. By (F2) and since the feasible systems in S^* are edge-disjoint, each $v \in U^*$ satisfies $d^{\pm}_{S^*}(v) = n$ and so $d^{\pm}_{D}(v) = 0$, as desired. Let $v \in V(T) \setminus U^*$. First, note that

$$|N^{\pm}_{\mathcal{ESF}}(v) \setminus N^{\pm}_{T}(v)| \overset{\text{(I)}}{=} d^{\pm}_{\mathscr{M}}(v) \overset{\text{Definitions 8.24 and 8.25}}{=} d^{\pm}_{S}(v) \tag{14.10}$$

and, similarly,

$$|N^{\pm}_{\mathcal{ESF}'}(v) \setminus N^{\pm}_{T}(v)| \overset{\text{(I')}}{=} d^{\pm}_{\mathscr{M}'}(v) \overset{\text{Definitions 8.24 and 8.25}}{=} d^{\pm}_{S'}(v). \tag{14.11}$$

Moreover, Corollary 12.2 implies that

$$d^{\pm}_{\mathcal{ESF}}(v) = (1+3q)r_3 \quad \text{and} \quad d^{\pm}_{\mathcal{ESF}'}(v) = 22r^{\diamond}. \tag{14.12}$$

By (I), (II), (I'), and (II'), $\mathcal{ESF} \setminus \mathscr{M}$ and $\mathcal{ESF}' \setminus \mathscr{M}'$ are edge-disjoint subdigraphs of $D_1' \setminus S^* \subseteq T \setminus S^*$. Moreover, Step 3 and Lemma 11.10 imply that $D^{\text{rob}} \subseteq D_2' \subseteq T \setminus (S^* \cup \mathcal{ESF} \cup \mathcal{ESF}')$. Therefore, Lemma 11.10 implies that

$$\begin{aligned} d^{\pm}_{D}(v) &= d^{\pm}_{T}(v) - d^{\pm}_{D^{\text{rob}}}(v) - d^{\pm}_{\mathcal{ESF}\setminus\mathscr{M}}(v) - d^{\pm}_{\mathcal{ESF}'\setminus\mathscr{M}'}(v) - d^{\pm}_{S^*}(v) \\ &\overset{(14.10),(14.11)}{=} n - d^{\pm}_{D^{\text{rob}}}(v) - d^{\pm}_{\mathcal{ESF}}(v) - d^{\pm}_{\mathcal{ESF}'}(v) - d^{\pm}_{S''}(v) \\ &\overset{(14.12)}{=} n - (r_1 + r_2 + 5r^{\diamond}) - (1+3q)r_3 - 22r^{\diamond} - d^{\pm}_{S''}(v) \\ &\overset{(14.3)}{=} n - (4s' - r) - d^{\pm}_{S''}(v), \end{aligned}$$

as desired. ◇

Let $\delta \coloneqq \frac{1}{2}\left(1 - \frac{4s'-r}{n}\right)$. By Claim 4, (14.6), and (14.8), each $v \in V(T) \setminus U^*$ satisfies $d_D^\pm(v) = 2\delta n \pm \varepsilon_2 n = 2(\delta \pm \varepsilon_2)n'$. Since by ($\alpha$) D only contains forward edges, this implies that $D[U_i \setminus U^*, U_{i+1} \setminus U^*]$ is (δ, ε_2)-almost regular for each $i \in [4]$. Thus, Lemma 14.1(i) holds with ε_2 playing the role of ε. Moreover, Proposition 7.6 (applied with $\frac{4s'-r}{n} + \varepsilon_2$ playing the role of ε) implies that Lemma 14.1(ii) is satisfied. By (14.6), we have

$$n - 4s' = 2\delta n - r \leq 2\delta n' + 2\varepsilon_1 n' - r \leq 2(\delta - \eta)n'.$$

Finally, recall that U^* is an $(\varepsilon_1, \mathcal{U})$-exceptional set for T. Thus, Fact 13.9 implies that U^* is also an $(\varepsilon_2, \mathcal{U})$-exceptional set for T. Let $\mathscr{C}_{\text{approx}}$ be the set of $n - 4s'$ Hamilton cycles of T obtained by applying Lemma 14.1 with $n - 4s', \varepsilon_2$, and S'' playing the roles of ℓ, ε, and $\{\mathcal{F}_1, \dots, \mathcal{F}_\ell\}$.

Step 5: Absorbing the leftovers. Finally, we decompose $H \coloneqq D \setminus \mathscr{C}_{\text{approx}}$ using the robustly decomposable digraph D^{rob}. Define a multidigraph D' by $D' \coloneqq (H - U^*) \cup D^{\text{rob}} \cup \mathcal{ESF} \cup \mathcal{ESF}'$.

Claim 5. *$D' \setminus (\mathscr{M} \cup \mathscr{M}')$ is a digraph (rather than a multidigraph) and satisfies $E(D' \setminus (\mathscr{M} \cup \mathscr{M}')) = E(T \setminus (S \cup S' \cup \mathscr{C}_{\text{approx}}))$.*

Proof of Claim. By (I), (II), (I′), and (II′), $\mathcal{ESF} \setminus \mathscr{M}$ and $\mathcal{ESF}' \setminus \mathscr{M}'$ are digraphs (rather than multidigraphs) and are edge-disjoint subdigraphs of $D_1' \setminus S^* \subseteq D_1' \setminus (S \cup S')$. By Step 3, $D^{\text{rob}} \subseteq D_2 \setminus S^* \subseteq T \setminus (D_1' \cup S \cup S')$. By definition,

$$H = D \setminus \mathscr{C}_{\text{approx}} = T \setminus (S^* \cup \mathcal{ESF} \cup \mathcal{ESF}' \cup D^{\text{rob}} \cup \mathscr{C}_{\text{approx}}). \tag{14.13}$$

Thus, $\mathcal{ESF} \setminus \mathscr{M}$, $\mathcal{ESF}' \setminus \mathscr{M}'$, D^{rob}, and $H - U^*$ are pairwise edge-disjoint subdigraphs of T. Therefore, $D' \setminus (\mathcal{M} \cup \mathcal{M}')$ is a digraph. Moreover, Lemma 14.1 implies that $S'' \subseteq \mathscr{C}_{\text{approx}} \subseteq D \cup S'' \subseteq T \setminus (S \cup S' \cup \mathcal{ESF} \cup \mathcal{ESF}' \cup D^{\text{rob}})$. Thus, (14.13) implies that $D' \setminus (\mathscr{M} \cup \mathscr{M}') = T \setminus (S \cup S' \cup \mathscr{C}_{\text{approx}})$, as desired. $\diamond$

We are now ready to decompose D'. By Lemma 14.1, $\mathscr{C}_{\text{approx}}$ is a set of $n - 4s'$

edge-disjoint Hamilton cycles of $D \cup S''$ which altogether cover S'' and so Claim 4 implies that each $v \in V(T)$ satisfies

$$d_H^{\pm}(v) = \begin{cases} 0 & \text{if } v \in U^*; \\ r & \text{otherwise.} \end{cases}$$

In particular, $H - U^*$ is r-regular. Moreover, (α) implies that H only consists of forward edges and avoids the edges in $E = \{uv \in E(T) \mid vu \in \bigcup \mathcal{M}\}$. In particular, $H - U^*$ is a blow-up C_4 with vertex partition $\mathcal{U}'$. By definition, $H \subseteq D \subseteq T \setminus D^{\mathrm{rob}}$. Thus, Lemma 11.10 implies that the multidigraph D' has a decomposition $\mathscr{C}_{\mathrm{rob}}$ into $4s'$ edge-disjoint Hamilton cycles on $V(T) \setminus U^*$ such that each cycle in $\mathscr{C}_{\mathrm{rob}}$ contains precisely one of the extended special path systems in the multidigraph $\mathcal{ESF} \cup \mathcal{ESF}'$. That is, there is an enumeration

$$\{C_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q\} \cup \{C'_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q'\}$$

of $\mathscr{C}_{\mathrm{rob}}$ such that $C_{\ell,h,i,j} \cap (\mathcal{ESF} \cup \mathcal{ESF}') = ESPS_{\ell,h,i,j}$ for each $(\ell, h, i, j) \in Q$ and $C'_{\ell',h',i',j'} \cap (\mathcal{ESF} \cup \mathcal{ESF}') = ESPS'_{\ell',h',i',j'}$ for each $(\ell', h', i', j') \in Q'$.

Recall from (γ') that $S = \{\mathcal{F}_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q\}$ and $S' = \{\mathcal{F}'_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q'\}$ are edge-disjoint sets of edge-disjoint special covers in D_1' with respect to U^*. By Step 3, $\mathcal{M}$ and $\mathcal{M}'$ are multisets which consist of the complete special sequences associated to the special covers in S and S', respectively.

For each $(\ell, h, i, j) \in Q$, define $C^*_{\ell,h,i,j} \coloneqq (C_{\ell,h,i,j} \setminus M_{\ell,h,i,j}) \cup \mathcal{F}_{\ell,h,i,j}$ and observe that, by (I) and Fact 8.26, $C^*_{\ell,h,i,j}$ is a Hamilton cycle of T. For each $(\ell, h, i, j) \in Q'$, define $C''_{\ell,h,i,j} \coloneqq (C'_{\ell,h,i,j} \setminus M'_{\ell,h,i,j}) \cup \mathcal{F}'_{\ell,h,i,j}$ and observe that, by (I′) and Fact 8.26, $C''_{\ell,h,i,j}$ is a Hamilton cycle of T. Let

$$\mathscr{C}'_{\mathrm{rob}} \coloneqq \{C^*_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q\} \cup \{C''_{\ell,h,i,j} \mid (\ell, h, i, j) \in Q'\}.$$

By Steps 2 and 4, S, S', and $\mathcal{C}_{\mathrm{approx}}$ are pairwise edge-disjoint. Thus, Claim 5 implies that

$\mathscr{C}'_{\rm rob}$ is a Hamilton decomposition of $(D' \setminus (\mathscr{M} \cup \mathscr{M}')) \cup (S \cup S') = T \setminus \mathscr{C}_{\rm approx}$. Recall from Step 4 that $\mathscr{C}_{\rm approx}$ is a set of edge-disjoint Hamilton cycles of T. Therefore, $\mathscr{C}_{\rm approx} \cup \mathscr{C}'_{\rm rob}$ is a Hamilton decomposition of T. This completes the proof of Theorem 4.4. $\square$

CHAPTER 15

PSEUDO-FEASIBLE SYSTEMS

It remains to prove Lemma 13.12. First, we observe that one can initially decompose the backward and exceptional edges into structures (called pseudo-feasible systems) which are slightly more general than feasible systems.

15.1 Definitions

We now define the concept of a *placeholder* (defined formally below). Roughly speaking, an edge e with precisely one endpoint $v \in U^*$ is called a *placeholder* if T contains many edges of the same type: if e is a forward in/outedge at v, then e is a placeholder if T has many forward in/outedges at v; similarly, if e is a backward in/outedge at v, then e is a placeholder if T has many backward in/outedges at v. A placeholder will be used to hold the place for an edge e' of the same type as e. A suitable e' will exist since, by definition of a placeholder, there exist many edges which are of the same type as e.

Definition 15.1 (Placeholder)**.** Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and let U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. For each $i \in [4]$, denote $U_i^* := U_i \cap U^*$. Let $uv \in E(T)$ and denote by $i, j \in [4]$ the unique indices such that $u \in U_i$ and $v \in U_j$. We say that uv is a (γ, T)-*placeholder (with respect to* $\mathcal{U}$ *and* U^**)* if one of the following holds.

- $u \in U_i^*$, $v \in U_j \setminus U_j^*$, and $|N_T^+(u) \cap U_j| > \gamma n$.

– $u \in U_i \setminus U_i^*$, $v \in U_j^*$, and $|N_T^-(v) \cap U_i| > \gamma n$.

Fact 15.2. *Let $0 \leq \varepsilon \ll \gamma \leq 1$. Let T be a regular bipartite tournament. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $e \in E(\overrightarrow{T})$ is a (γ, T)-placeholder. Then, $V(e) \cap U^{1-\gamma}(T) = \emptyset$.*

Recall from Definition 13.2 that a feasible system is a linear forest which contains an appropriate number of backward and exceptional edges. Roughly speaking, we say that $\mathcal{F}$ is a *pseudo-feasible system* (defined formally below) if the only obstructions to $\mathcal{F}$ being a feasible system are caused by placeholders. More precisely, a pseudo-feasible system $\mathcal{F}$ may not form a linear forest (that is, $\mathcal{F}$ may not satisfy property (F3) of a feasible system), but all of the cycles in $\mathcal{F}$ contain a placeholder (see (F4′) below) and all of the excess degree can be accounted for by placeholders (see (F3′) below). Additionally, a pseudo-feasible system $\mathcal{F}$ may not cover all of the vertices in U^* (that is, $\mathcal{F}$ may not satisfy property (F2)), but the uncovered vertices in U^* have high forward degree and so the missing edges at U^* are forward placeholders (see (F2′) below).

Definition 15.3 (Pseudo-feasible system)**.** We say that $\mathcal{F}$ is a (γ, T)-*pseudo-feasible system (with respect to $\mathcal{U}$ and U^*)* if $\mathcal{F} \subseteq T$, (F1) is satisfied, and the following hold.

(F2′) For each $v \in U^*$, both $d_{\mathcal{F}}^{\pm}(v) \leq 1$ and, if $v \in U^{1-\gamma}(T)$, then both $d_{\mathcal{F}}^{\pm}(v) = 1$.

(F3′) Let $v \in V(T) \setminus U^*$. Then, $\mathcal{F}$ contains at most one edge which starts at v and is not a (γ, T)-placeholder. Similarly, $\mathcal{F}$ contains at most one edge which ends at v and is not a (γ, T)-placeholder.

(F4′) Each cycle in $\mathcal{F}$ contains a (γ, T)-placeholder.

As for feasible systems (recall Fact 13.4), forward edges which are not incident to U^* play no role in a pseudo-feasible system. Additionally, Fact 15.2 implies that all of the forward placeholders can be deleted.

Fact 15.4. *Let $0 \leq \varepsilon \ll \gamma \leq 1$. Let T be a regular bipartite tournament. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Let*

$\mathcal{F}$ be a (γ, T)-pseudo-feasible system and $e \in E(\overrightarrow{\mathcal{F}}_{\mathcal{U}})$. If $V(e) \cap U^* = \emptyset$ or e is a (γ, T)-placeholder, then $\mathcal{F} \setminus \{e\}$ is a (γ, T)-pseudo-feasible system.

15.2 Transforming pseudo-feasible systems into feasible systems: proof overview

The next lemma states that pseudo-feasible systems can be transformed into feasible systems. The idea behind the proof of Lemma 15.5 is to replace the placeholders with edges of the same type to form linear forests and then add forward edges to cover $U^* \setminus U^{1-\gamma}(T)$.

More precisely, let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U}$ be an $(\varepsilon, 4)$-partition for T and let U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $\mathcal{F}$ is a (γ, T)-pseudo-feasible system. We can transform $\mathcal{F}$ into a feasible system as follows. Suppose that $\mathcal{F}$ contains a cycle C. By (F4′), C contains a (γ, T)-placeholder e, say e is a backward edge from $u \in U^*$ to $v \notin U^*$ for instance. Then, by definition of a placeholder, T contains many backward edges which start at u and end in $V(T) \setminus U^*$. Therefore, we can find a backward edge $e' = uv'$ with $v' \in V(T) \setminus (U^* \cup V(\mathcal{F}))$. Then, replacing e by e' in $\mathcal{F}$ breaks the cycle C without affecting (F1) and (F2′)–(F4′). Repeating this argument, we can eventually remove all cycles in $\mathcal{F}$. By (F2′) and (F3′), we can use similar arguments to ensure that $\Delta^0(\mathcal{F}) \leq 1$. Then, $\mathcal{F}$ satisfies (F3). Finally, we add forward edges to ensure that (F2) holds as follows. Suppose that $x \in U^*$ satisfies $d^+_{\mathcal{F}}(x) = 0$. Then, (F2′) implies that $x \notin U^{1-\gamma}(T)$ and so T contains many forward outedges at x. Thus, we can find a forward edge $e'' = xy$ with $y \in V(T) \setminus V(\mathcal{F})$. Then, adding e'' to $\mathcal{F}$ does not affect (F1) and (F3). Repeating this argument, $\mathcal{F}$ eventually satisfies (F2) and so $\mathcal{F}$ becomes a feasible system.

Additionally, we will add forward edges to incorporate a given suitable set of edges E (see Lemma 13.12(a)) and to ensure that all the components of the feasible systems start in U_1 and end in U_4 (see Lemma 13.12(e)).

Lemma 15.5 (Transforming pseudo-feasible systems into feasible ones)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $(1-\eta)n \leq r \leq n$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $D \subseteq T$ satisfies $\delta^0(D) \geq r$. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be edge-disjoint (γ, T)-pseudo-feasible systems satisfying the following properties.*

(i) $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i) \subseteq E(D)$.

(ii) *For each $i \in [r]$, $e(\mathcal{F}_i) \leq \varepsilon n$.*

Let $E \subseteq E(D)$ be such that the following hold.

(iii) $E \subseteq E(\overrightarrow{D}_{\mathcal{U}} - U^*)$.

(iv) *For each $v \in V(T) \setminus U^*$, $d_E^\pm(v) \leq 1$.*

Then, there exist edge-disjoint feasible systems $\mathcal{F}_1', \dots, \mathcal{F}_r'$ such that the following hold.

(a) $E(\overleftarrow{D}_{\mathcal{U}}) \cup E \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i') \subseteq E(D)$.

(b) *For each $i \in [r]$, $e(\mathcal{F}_i') \leq \varepsilon^{\frac{1}{3}} n$.*

(c) *For each $v \in V(T) \setminus U^*$, there exist at most $\varepsilon^{\frac{1}{3}} n$ indices $i \in [r]$ such that $v \in V(\mathcal{F}_i')$.*

(d) *For each $i \in [r]$, $V^+(\mathcal{F}_i') \subseteq U_1$ and $V^-(\mathcal{F}_i') \subseteq U_4$.*

Note that $E(D[U^*])$ appears in Lemma 15.5(i) for technical reasons (this ensures that all the edges available for transforming the pseudo-feasible systems into feasible ones do not entirely lie in the exceptional set U^*). On the other hand, $E(D[U^*])$ does not need to explicitly appear in Lemma 15.5(a) since these edges will automatically be covered by definition of a feasible system. Indeed, recall from Lemma 13.12 that we aim to construct n edge-disjoint feasible systems. But property (F2) of a feasible system states that each exceptional vertex is covered by both an in- and an outedge, so any set of n edge-disjoint feasible systems automatically covers all the edges incident to U^*.

In practice, the above argument needs to be carried out to all of the pseudo-feasible systems in parallel. To gain intuition, we first derive Lemma 13.12 and defer the proof of

Lemma 15.5 to Chapter 16.

15.3 Proof of Lemma 13.12

By Lemma 15.5, it is enough to decompose the backward edges into pseudo-feasible systems (rather than feasible ones). However, recall from Lemma 13.12(d) that we require a few of the feasible systems to be constructed out of prescribed sets $H_1, \dots, H_s$ of edges of T. It is therefore more convenient to construct these feasible systems straight away. Thus, it is most convenient to prove the following pseudo-feasible system analogue of Lemma 13.12. (The proof of Lemma 15.6, as well as a detailed proof overview, can be found in Chapter 17.)

Lemma 15.6 (Decomposing the backward and exceptional edges into pseudo-feasible systems)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $s \in \mathbb{N}$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that, for each $i \in [s]$, $H_i \subseteq T$ satisfies Lemma 13.12(i)–(iv). For each $i \in [s]$, let $s_i \in \mathbb{N}$ and $t_i \coloneqq \sum_{j \in [i-1]} s_j$. Let $t \coloneqq \sum_{i \in [s]} s_i$ and suppose that $t \leq \eta n$. Then, there exist edge-disjoint (γ, T)-pseudo-feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_n$ for which the following hold.*

(a) $E(\overleftarrow{T}_{\mathcal{U}}) \cup E(T[U^*]) \subseteq \bigcup_{i \in [n]} E(\mathcal{F}_i) \subseteq E(T)$.

(b) *For each $i \in [n]$, $e(\mathcal{F}_i) \leq \sqrt{\varepsilon} n$.*

(c) *For each $i \in [t]$, $\mathcal{F}_i$ is a feasible system with $V^0(\mathcal{F}_i) = U^*$.*

(d) *For each $i \in [s]$ and $j \in [s_i]$, $\mathcal{F}_{t_i + j} \subseteq H_i$.*

Proof of Lemma 13.12. Let $\mathcal{F}_1^*, \dots, \mathcal{F}_n^*$ be the (γ, T)-pseudo-feasible systems obtained by applying Lemma 15.6. For each $i \in [t]$, let $\mathcal{F}_i \coloneqq \mathcal{F}_i^* \setminus E$. By (v), Fact 13.4, and Lemma 15.6(c) and (d), $\mathcal{F}_1, \dots, \mathcal{F}_t$ are feasible systems which satisfy (c) and (d).

We transform $\mathcal{F}_{t+1}^*, \dots, \mathcal{F}_n^*$ into feasible systems using Lemma 15.5 as follows. Let $r \coloneqq n - t$ and $D \coloneqq T \setminus \bigcup_{i \in [t]} \mathcal{F}_i$. By (F3), $\delta^0(D) \geq r$ and, by Lemma 15.6(a) and (b), Lemma 15.5(i) and (ii) hold with $\mathcal{F}^*_{t+1}, \dots, \mathcal{F}^*_n$, and $\sqrt{\varepsilon}$ playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$, and ε. By construction, $E \subseteq E(D)$ and so Lemma 15.5(iii) and (iv) follow from (v) and (vi). Let $\mathcal{F}_{t+1}, \dots, \mathcal{F}_n$ be the feasible systems obtained by applying Lemma 15.5 with $\mathcal{F}^*_{t+1}, \dots, \mathcal{F}^*_n$, and $\sqrt{\varepsilon}$ playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$, and ε. Then, (a) follows from Lemma 15.5(a) and Lemma 15.6(a), while (b) follows from Lemma 15.5(b) and Lemma 15.6(b). Finally, (e) holds by Lemma 15.5(d). $\square$

CHAPTER 16

TRANSFORMING PSEUDO-FEASIBLE SYSTEMS INTO FEASIBLE SYSTEMS: PROOF OF LEMMA 15.5

We proceed as described in Section 15.2. First, we redistribute all the placeholders contained in $\mathcal{F}_1, \dots, \mathcal{F}_r$ to break all the cycles and reduce the maximum semidegree to 1 (Lemma 16.2). Then, we add some forward edges to cover U^* and thus form feasible systems (Lemma 16.3). Next, we incorporate the set E of prescribed edges (Lemma 16.4). Finally, we add some additional forward edges to ensure that each component of the feasible systems have its endpoints in the desired vertex classes (Lemma 16.6).

16.1 Extending linear forests

As mentioned above, the feasible systems will be constructed in stages. At each stage, we will consider linear forests and need to extend them in a prescribed way (e.g. in Lemma 16.3 we will need to cover precisely the uncovered vertices in U^*). Most of the time, this will be done using the next lemma.

Roughly speaking, Lemma 16.1 states that a sufficiently dense bipartite digraph D on vertex classes A and B contains edge-disjoint linear forests $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$, where each $\mathcal{Q}_i$ covers a prescribed set $S_i^+ \subseteq A$ with outedges, covers a prescribed set $S_i^- \subseteq A$ with inedges, and avoids a prescribed set $T_i \subseteq B$ (see Lemma 16.1(a) below). Moreover, these linear forests can be constructed in such a way that every vertex of B is not covered by too many of the linear forests (see Lemma 16.1(b)) and is adjacent to at most one edge in each linear forest (see Lemma 16.1(c)).

In general, given linear forests $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ that we want to extend, we will apply Lemma 16.1 with $T_i = V(E(\mathcal{F}_i)) \cap B$ for each $i \in [\ell]$. This will ensure that the linear forests $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ guaranteed by Lemma 16.1 can be incorporated into $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ to form larger linear forests. For each $i \in [\ell]$, the sets S_i^+ and S_i^- will correspond to the sets of vertices that need to be covered in $\mathcal{F}_i$ with out- and inedges (e.g. in the proof of Lemma 16.3 we will apply this to the vertices of U^* which are not yet covered with out- and inedges by $\mathcal{F}_i$).

Note that Lemma 16.1(c) will not be used until Chapter 18.

Lemma 16.1. *Let D be a bipartite digraph on vertex classes A and B. For each $i \in [\ell]$, let $S_i^+, S_i^- \subseteq A$ and $T_i \subseteq B$. For each $v \in B$, let n_v denote the number of indices $i \in [\ell]$ such that $v \in T_i$. Let $1 \leq N \leq 2|A|$. For each $i \in [\ell]$, $\diamond \in \{+, -\}$, and $v \in S_i^\diamond$, denote*

$$c_{i,\diamond,v} := \max\{|\{i' \in [\ell] \mid v \in S_{i'}^\diamond\}|, 2(|S_i^+| + |S_i^-| + |T_i|), 2(\max_{w \in B} n_w + N)\}$$

and suppose that

$$d_D^\diamond(v) \geq \begin{cases} c_{i,\diamond,v} & \text{if } N = 2|A|; \\ \frac{1}{\lfloor N \rfloor} \sum_{i' \in [\ell]} (|S_{i'}^+| + |S_{i'}^-|) + c_{i,\diamond,v} & \text{if } N < 2|A|. \end{cases} \tag{16.1}$$

Then, D contains edge-disjoint linear forests $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ such that the following hold.

(a) *For each $i \in [\ell]$, $\mathcal{Q}_i$ consists of a matching of $D(B \setminus T_i, S_i^-)$ of size $|S_i^-|$ and a matching of $D(S_i^+, B \setminus T_i)$ of size $|S_i^+|$.*

(b) *For each $v \in B$, there exists at most N indices $i \in [\ell]$ such that $v \in V(\mathcal{Q}_i)$.*

(c) *For each $i \in [\ell]$ and $v \in B$, we have $d_{\mathcal{Q}_i}(v) \leq 1$ (i.e. the two matchings in (a) do not intersect in B).*

In particular, if $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ are linear forests which are edge-disjoint from each other and from D such that both

$$S_i^\pm \cap V(\mathcal{F}_i) \subseteq V^\mp(\mathcal{F}_i) \quad \textit{and} \quad V(E(\mathcal{F}_i)) \cap B \subseteq T_i$$

for each $i \in [\ell]$, then $\mathcal{F}_1 \cup \mathcal{Q}_1, \dots, \mathcal{F}_\ell \cup \mathcal{Q}_\ell$ are edge-disjoint linear forests.

Proof. Note that the "in particular part" follows immediately from (a) and (c). Thus, it suffices to construct edge-disjoint linear forests $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ which satisfy (a)–(c). Let $S^\pm := \bigcup_{i\in[\ell]} S_i^\pm$ and denote $S := \{(+, v) \mid v \in S^+\} \cup \{(-, v) \mid v \in S^-\}$. We will consider each tuple $(\diamond, v) \in S$ in turn and, at each stage, choose all the edges corresponding to the current tuple $(\diamond, v) \in S$ (that is, all the outedges at v if $\diamond = +$ and all the inedges at v if $\diamond = -$).

Suppose inductively that, for some $0 \leq k \leq |S|$, there exist $S^k \subseteq S$ of size k and edge-disjoint linear forests $\mathcal{Q}_1^k, \dots, \mathcal{Q}_\ell^k$ such that the following hold, where $S^{\pm,k} := \{v \mid (\pm, v) \in S^k\}$.

- (α) For each $i \in [\ell]$, $\mathcal{Q}_i^k$ consists of a matching of $D(B \setminus T_i, S_i^- \cap S^{-,k})$ of size $|S_i^- \cap S^{-,k}|$ and a matching of $D(S_i^+ \cap S^{+,k}, B \setminus T_i)$ of size $|S_i^+ \cap S^{+,k}|$.
- (β) For each $v \in B$, there exist at most N indices $i \in [\ell]$ such that $v \in V(\mathcal{Q}_i^k)$.
- (γ) For each $i \in [\ell]$ and $v \in B$, we have $d_{\mathcal{Q}_i^k}(v) \leq 1$ (i.e. the two matchings in (α) do not intersect in B).

First, suppose that $k = |S|$. Let $\mathcal{Q}_i := \mathcal{Q}_i^k$ for each $i \in [\ell]$. Then, (a)–(c) follow from (α)–(γ).

We may therefore assume that $k < |S|$. Let $(\diamond, v) \in S \setminus S^k$ and define $S^{k+1} := S^k \cup \{(\diamond, v)\}$. Let X be the set of vertices $w \in B$ such that there exist $\lfloor N \rfloor$ indices $i \in [\ell]$ such that $w \in V(\mathcal{Q}_i^k)$ (so X is the set of vertices of B that cannot be used anymore). Let $Y := \{i \in [\ell] \mid v \in S_i^\diamond\}$ (so Y lists the $\mathcal{Q}_i^k$'s to which we need to add an edge incident to v in this step).

Claim 1. *For each $i \in Y$, the following hold.*

(I) $d_D^\diamond(v) \geq |X| + |Y|$.

(II) $d_D^\diamond(v) \geq |X| + 2(|S_i^+| + |S_i^-| + |T_i|)$.

(III) $d_D^\diamond(v) \geq |X| + 2(\max_{w \in B} n_w + N)$.

Proof of Claim. If $N = 2|A|$, then $N \geq |S| > k$ and so $X = \emptyset$. Thus, (I)–(III) follow immediately from (16.1). We may therefore assume that $N < 2|A|$. Since D is a bipartite graph on vertex classes A and B, we have

$$|X| \leq \frac{\sum_{i \in [\ell]} e(\mathcal{Q}_i^k)}{\lfloor N \rfloor} \overset{(\alpha)}{\leq} \frac{1}{\lfloor N \rfloor} \sum_{i' \in [\ell]} (|S_{i'}^+| + |S_{i'}^-|).$$

Therefore, (I)–(III) follow from (16.1). $\diamond$

If $\diamond = +$, then let $Z \coloneqq \{vw \in E(D) \mid w \notin X\}$; otherwise, let $Z \coloneqq \{wv \in E(D) \mid w \notin X\}$ (so Z consists of the edges of D that we may use to extend $\mathcal{Q}_1^k, \dots, \mathcal{Q}_\ell^k$ in this step). Let G be the auxiliary bipartite graph on vertex classes Y and Z defined as follows. For each $i \in Y$ and $e \in Z$, $ie \in E(G)$ if and only if $V(e) \cap B \cap (V(\mathcal{Q}_i^k) \cup T_i) = \emptyset$. Note that

$$\begin{aligned} |Z| &\geq d_D^\diamond(v) - |X| \\ &\overset{\text{(I)}}{\geq} |Y|. \end{aligned} \tag{16.2}$$

Then, each $i \in Y$ satisfies

$$\begin{aligned} d_G(i) &\geq |Z| - |V(\mathcal{Q}_i^k) \cap B| - |T_i| \overset{(\alpha)}{\geq} |Z| - (|S_i^+| + |S_i^-|) - |T_i| \\ &\overset{\text{(II)}}{\geq} |Z| - \frac{d_D^\diamond(v) - |X|}{2} \overset{(16.2)}{\geq} \frac{|Z|}{2}. \end{aligned}$$

Let $e \in Z$ and denote by w the (unique) vertex $w \in V(e) \cap B$. Let n_w' be the number of indices $i \in [\ell]$ such that $w \in V(\mathcal{Q}_i^k) \cup T_i$. Then,

$$d_G(e) \geq |Y| - n'_w \overset{(\beta)}{\geq} |Y| - (n_w + N) \overset{\text{(III)}}{\geq} |Y| - \frac{d_D^\diamond(v) - |X|}{2} \overset{(16.2)}{\geq} |Y| - \frac{|Z|}{2}.$$

Apply Proposition 7.24 with Y and Z playing the roles of A and B to obtain a matching M of G which covers Y. For each $i \in Y$, let e_i denote the (unique) neighbour of i in M and let $\mathcal{Q}_i^{k+1} := \mathcal{Q}_i^k \cup \{e_i\}$. For each $i \in [\ell] \setminus Y$, let $\mathcal{Q}_i^{k+1} := \mathcal{Q}_i^k$.

Since M is a matching, $\mathcal{Q}_1^{k+1}, \dots, \mathcal{Q}_\ell^{k+1}$ are pairwise edge-disjoint. Moreover, the definition of G and the induction hypothesis imply that (γ) holds with $k+1$ playing the role of k and $\mathcal{Q}_i^{k+1}$ is a linear forest for each $i \in [\ell]$. By definition of Y and G and the induction hypothesis, (α) holds with $k+1$ playing the role of k. Finally, (β) holds with $k+1$ playing the role of k by definition of X and Z and the induction hypothesis. □

16.2 Proof of Lemma 15.5

We are now ready to prove Lemma 15.5, which states that edge-disjoint pseudo-feasible systems can be transformed into edge-disjoint feasible systems. As discussed at the start of Chapter 16, we spilt the proof into several lemmas. First, we redistribute placeholders to break all the cycles and reduce the maximum semidegree to 1.

Lemma 16.2 (Redistributing placeholders)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $(1-\eta)n \leq r \leq n$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $D \subseteq T$ satisfies $\delta^0(D) \geq r$. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be edge-disjoint (γ, T)-pseudo-feasible systems which satisfy the following properties.*

(i) *$E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i) \subseteq E(D)$.*

(ii) *For each $i \in [r]$, $e(\mathcal{F}_i) \leq \varepsilon n$.*

(iii) *Suppose that $e \in \bigcup_{i \in [r]} E(\mathcal{F}_i)$ is a forward edge. Then, $V(e) \cap U^* \neq \emptyset$ and e is not a (γ, T)-placeholder.*

Then, there exist edge-disjoint (γ, T)-pseudo-feasible systems $\mathcal{F}_1', \dots, \mathcal{F}_r'$ such that the following hold.

(a) $\bigcup_{i\in[r]} E(\mathcal{F}_i') = \bigcup_{i\in[r]} E(\mathcal{F}_i)$. *In particular,* $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i\in[r]} E(\mathcal{F}_i') \subseteq E(D)$.

(b) *For each* $i \in [r]$, $e(\mathcal{F}_i') = e(\mathcal{F}_i) \leq \varepsilon n$.

(c) *For each* $i \in [r]$, $\mathcal{F}_i'$ *is a linear forest.*

Proof. Let E be the set of (γ, T)-placeholders contained in $\bigcup_{i\in[\ell]} E(\mathcal{F}_i)$. For each $i \in [r]$, let $\widetilde{\mathcal{F}}_i$ be obtained from $\mathcal{F}_i \setminus E$ by removing all isolated vertices and denote $E_i := E \cap E(\mathcal{F}_i)$. Note that, by (F2′)–(F4′), $\widetilde{\mathcal{F}}_1, \dots, \widetilde{\mathcal{F}}_r$ are linear forests. Thus, we may assume without loss of generality that $E \neq \emptyset$ and so, by Definition 15.1, $U^* \neq \emptyset$.

We will redistribute the placeholders in E into the linear forests $\widetilde{F}_1, \dots, \widetilde{F}_r$ using Lemma 16.1. More precisely, we will add, for each $v \in U^*$ and $i \in [r]$, an in/outedge at v from E to $\widetilde{F}_i$ if and only if $\mathcal{F}_i$ contains a placeholder which is an in/outedge at v.

Let $A := U^*$ and $B := V(T) \setminus U^*$. Let D' be the digraph on $V(T)$ defined by $E(D') := E$. By Definition 15.1, D' is a bipartite digraph on vertex classes A and B. For each $i \in [r]$, let $S_i^+, S_i^- \subseteq U^*$ be the sets of vertices which are incident to an out/inedge in E_i, respectively (so S_i^+ and S_i^- list the vertices in U^* which we need to cover with an out/inedge from E) and define $T_i := V(E(\widetilde{\mathcal{F}}_i)) \cap B$. Note for later that (F2′) implies that both

$$S_i^{\pm} \cap V(\widetilde{\mathcal{F}}_i) \subseteq V^{\mp}(\widetilde{\mathcal{F}}_i) \quad \text{and} \quad V(E(\widetilde{\mathcal{F}}_i)) \cap B \subseteq T_i \tag{16.3}$$

for each $i \in [r]$. Define $N := 2|U^*|$. For each $v \in B$, let n_v denote the number of indices $i \in [r]$ such that $v \in T_i$.

We verify that (16.1) holds with D' and r playing the roles of D and ℓ. By Definition 15.1, each $i \in [r]$ satisfies

$$\begin{aligned} |S_i^+| + |S_i^-| &= |E_i| && (16.4) \\ &\stackrel{\text{(F2')}}{\leq} 2|U^*| \stackrel{\text{(ES2)}}{\leq} 8\varepsilon n && (16.5) \end{aligned}$$

and

$$|T_i| \leq |V(E(\mathcal{F}_i))| \stackrel{\text{(ii)}}{\leq} 2\varepsilon n. \tag{16.6}$$

Moreover, each $v \in B = V(T) \setminus U^*$ satisfies

$$n_v \leq \sum_{i \in [r]} d_{\mathcal{F}_i}(v) \stackrel{\text{(i),(iii)}}{\leq} \overleftarrow{d}_{D,\mathcal{U}}(v) + |\overrightarrow{N}_{D,\mathcal{U}}(v) \cap U^*| \stackrel{\text{(ES1)}}{\leq} 2\varepsilon n + |U^*| \stackrel{\text{(ES2)}}{\leq} 6\varepsilon n. \tag{16.7}$$

Therefore, each $v \in U^\gamma(T) \subseteq U^*$ satisfies

$$\begin{aligned} d^{\pm}_{D'}(v) &= d^{\pm}_E(v) = |\{i \in [r] \mid v \in S_i^{\pm}\}| && (16.8) \\ &\stackrel{\text{Definition 15.1,(i)}}{=} \overleftarrow{d}^{\pm}_{D,\mathcal{U}}(v) - |\overleftarrow{N}^{\pm}_{D,\mathcal{U}}(v) \cap U^*| \\ &\geq \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) - \Delta^0(T \setminus D) - |\overleftarrow{N}^{\pm}_{T,\mathcal{U}}(v) \cap U^*| \geq \gamma n - (n-r) - |U^*| \\ &\stackrel{\text{(ES2),(16.5)–(16.7)}}{\geq} 2\max_{i \in [r]}(|S_i^+| + |S_i^-| + |T_i|) + 2(\max_{w \in B} n_w + N). && (16.9) \end{aligned}$$

By (iii), all the edges in E are backward edges and so Definition 15.1 implies that each edge in E is incident to a vertex in $U^\gamma(T) \subseteq U^*$. Thus, $S_i^+ \cup S_i^- \subseteq U^\gamma(T)$ for each $i \in [r]$ and so (16.1) follows from (16.8) and (16.9).

Let $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ be the edge-disjoint linear forests obtained by applying Lemma 16.1 with D' and r playing the roles of D and ℓ. For each $i \in [r]$, denote $\mathcal{F}'_i \coloneqq \widetilde{\mathcal{F}}_i \cup \mathcal{Q}_i$. We claim that $\mathcal{F}'_1, \dots, \mathcal{F}'_r$ are edge-disjoint (γ, T)-pseudo-feasible systems which satisfy (a)–(c). By construction, $\widetilde{\mathcal{F}}_1, \dots, \widetilde{\mathcal{F}}_r$ are edge-disjoint from each other and from D'. Thus, (16.3) and the "in particular part" of Lemma 16.1 imply that $\mathcal{F}'_1, \dots, \mathcal{F}'_r$ are edge-disjoint linear forests. In particular, (c), (F3′), and (F4′) are satisfied.

By Lemma 16.1(a), $\bigcup_{i\in[r]} E(\mathcal{F}'_i) \subseteq \bigcup_{i\in[r]} E(\widetilde{\mathcal{F}}_i) \cup E = \bigcup_{i\in[r]} E(\mathcal{F}_i)$. Moreover,

$$\sum_{i\in[r]} e(\mathcal{F}'_i \setminus \widetilde{\mathcal{F}}_i) \overset{\text{Lemma 16.1(a)}}{=} \sum_{i\in[r]} (|S_i^+| + |S_i^-|) \overset{(16.4)}{=} \sum_{i\in[r]} e(\mathcal{F}_i \setminus \widetilde{\mathcal{F}}_i).$$

Thus, (a) is satisfied. For each $i \in [r]$,

$$e(\mathcal{F}'_i) \overset{\text{Lemma 16.1(a)}}{=} e(\widetilde{\mathcal{F}}_i) + |S_i^+| + |S_i^-| \overset{(16.4)}{=} e(\mathcal{F}_i) \overset{\text{(ii)}}{\leq} \varepsilon n,$$

so (b) is satisfied. Let $j \in [r]$. By Lemma 16.1(a) and definition of $S_j^\pm$, each $v \in U^*$ satisfies

$$\overleftarrow{d}^{\pm}_{\mathcal{F}'_j,\mathcal{U}}(v) = \overleftarrow{d}^{\pm}_{\mathcal{F}_j,\mathcal{U}}(v). \tag{16.10}$$

Thus, (F2′) follows from the fact that $\mathcal{F}_j$ is a (γ, T)-pseudo-feasible system. Recall that $E(\mathcal{F}'_j \setminus \widetilde{\mathcal{F}}_j) \cup E(\mathcal{F}_j \setminus \widetilde{\mathcal{F}}_j) \subseteq E$ and so, by Definition 15.1 and (iii), $E(\mathcal{F}'_j \setminus \widetilde{\mathcal{F}}_j) \cup E(\mathcal{F}_j \setminus \widetilde{\mathcal{F}}_j)$ is a set of backward edges which have one endpoint in $U^\gamma(T) \subseteq U^*$ and one endpoint in $V(T) \setminus U^*$. Thus, the following holds for each $i \in [4]$.

$$\begin{aligned}
e_{\mathcal{F}'_j}(U_i, U_{i-1}) &= e_{\widetilde{\mathcal{F}}_j}(U_i, U_{i-1}) + \sum_{v\in U_i^\gamma(T)} \left(\overleftarrow{d}^{+}_{\mathcal{F}'_j,\mathcal{U}}(v) - \overleftarrow{d}^{+}_{\widetilde{\mathcal{F}}_j,\mathcal{U}}(v)\right) \\
&\quad + \sum_{v\in U_{i-1}^\gamma(T)} \left(\overleftarrow{d}^{-}_{\mathcal{F}'_j,\mathcal{U}}(v) - \overleftarrow{d}^{-}_{\widetilde{\mathcal{F}}_j,\mathcal{U}}(v)\right) \\
&\overset{(16.10)}{=} e_{\widetilde{\mathcal{F}}_j}(U_i, U_{i-1}) + \sum_{v\in U_i^\gamma(T)} \left(\overleftarrow{d}^{+}_{\mathcal{F}_j,\mathcal{U}}(v) - \overleftarrow{d}^{+}_{\widetilde{\mathcal{F}}_j,\mathcal{U}}(v)\right) \\
&\quad + \sum_{v\in U_{i-1}^\gamma(T)} \left(\overleftarrow{d}^{-}_{\mathcal{F}_j,\mathcal{U}}(v) - \overleftarrow{d}^{-}_{\widetilde{\mathcal{F}}_j,\mathcal{U}}(v)\right) \\
&= e_{\mathcal{F}_j}(U_i, U_{i-1}).
\end{aligned}$$

Thus, (F1) follows from the fact that $\mathcal{F}_j$ is a (γ, T)-pseudo-feasible system. Therefore, $\mathcal{F}'_j$ is a (γ, T)-pseudo-feasible system, as desired. □

Lemma 16.3 (Covering U^*). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $(1-\eta)n \leq r \leq n$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \ldots, U_4)$ be an $(\varepsilon, 4)$-partition* *for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $D \subseteq T$ satisfies $\delta^0(D) \geq r$. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be edge-disjoint (γ, T)-pseudo-feasible systems which satisfy the following.*

(i) $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i\in[r]} E(\mathcal{F}_i) \subseteq E(D)$.

(ii) *For each $i \in [r]$, $e(\mathcal{F}_i) \leq \varepsilon n$.*

(iii) *Suppose that $e \in \bigcup_{i\in[r]} E(\mathcal{F}_i)$ is a forward edge. Then, $V(e) \cap U^* \neq \emptyset$ and e is not a (γ, T)-placeholder.*

(iv) *For each $i \in [r]$, $\mathcal{F}_i$ is a linear forest.*

Then, there exist edge-disjoint feasible systems $\mathcal{F}_1', \dots, \mathcal{F}_r'$ such that the following hold.

(a) $\bigcup_{i\in[r]} E(\mathcal{F}_i) \subseteq \bigcup_{i\in[r]} E(\mathcal{F}_i') \subseteq E(D)$. *In particular,* $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i\in[r]} E(\mathcal{F}_i') \subseteq E(D)$.

(b) *For each $i \in [r]$, $e(\mathcal{F}_i') \leq 9\varepsilon n$.*

(c) *For each $v \in V(T) \setminus U^*$, there exist at most $6\varepsilon n$ indices $i \in [r]$ such that $v \in V(\mathcal{F}_i')$.*

Proof. First, note that we may assume without loss of generality that $U^* \neq \emptyset$. Indeed, if $U^* = \emptyset$, then (F2) holds automatically and so $\mathcal{F}_1, \dots, \mathcal{F}_r$ are already feasible systems.

We extend the linear forests $\mathcal{F}_1, \dots, \mathcal{F}_r$ into larger linear forests which cover U^* (and so satisfy (F2)) using Lemma 16.1. Let $A := U^*$ and $B := V(T) \setminus U^*$. Let D' be the bipartite digraph on vertex classes A and B induced by $\overrightarrow{D}_{\mathcal{U}} - U^{1-\gamma}(T)$. Note for later that $E(D')$ is a set of (γ, T)-placeholders, so (iii) implies that

$$E(D') \cap \bigcup_{i\in[r]} E(\mathcal{F}_i) = \emptyset. \tag{16.11}$$

For each $i \in [r]$, let $S_i^\pm$ be the set of vertices $v \in U^*$ which satisfy $d_{\mathcal{F}_i}^\pm(v) = 0$ (so S_i^+ and S_i^- list the vertices in U^* which are not yet covered with an out/inedge in $\mathcal{F}_i$) and define $T_i := V(E(\mathcal{F}_i)) \cap B$. Note for later that both

$$S_i^\pm \cap V(\mathcal{F}_i) \subseteq V^\mp(\mathcal{F}_i) \quad \text{and} \quad V(E(\mathcal{F}_i)) \cap B \subseteq T_i \tag{16.12}$$

for each $i \in [r]$. Define $N := 2|U^*|$. For each $v \in B$, let n_v denote the number of indices $i \in [\ell]$ such that $v \in T_i$.

We verify that (16.1) holds with D' and r playing the roles of D and ℓ. For each $i \in [r]$, we have

$$|S_i^+| + |S_i^-| \leq 2|U^*| \overset{\text{(ES2)}}{\leq} 8\varepsilon n \tag{16.13}$$

and

$$|T_i| \leq |V(E(\mathcal{F}_i))| \overset{\text{(ii)}}{\leq} 2\varepsilon n. \tag{16.14}$$

Moreover, each $v \in B = V(T) \setminus U^*$ satisfies

$$n_v \leq \sum_{i \in [r]} d_{\mathcal{F}_i}(v) \overset{\text{(i),(iii)}}{\leq} \overleftarrow{d}_{D,\mathcal{U}}(v) + |\overrightarrow{N}_{D,\mathcal{U}}(v) \cap U^*| \overset{\text{(ES1)}}{\leq} 2\varepsilon n + |U^*| \overset{\text{(ES2)}}{\leq} 6\varepsilon n. \tag{16.15}$$

Therefore, each $v \in U^* \setminus U^{1-\gamma}(T)$ satisfies

$$\begin{aligned} d_{D'}^{\pm}(v) &= |\overrightarrow{N}_{D,\mathcal{U}}^{\pm}(v) \setminus U^*| \geq \gamma n - |U^*| \\ &\overset{\text{(ES2),(16.13)–(16.15)}}{\geq} 2\max_{i \in [r]}(|S_i^+| + |S_i^-| + |T_i|) + 2(\max_{w \in B} n_w + N) \end{aligned} \tag{16.16}$$

and

$$\begin{aligned} d_{D'}^{\pm}(v) &= |\overrightarrow{N}_{D,\mathcal{U}}^{\pm}(v) \setminus U^*| \overset{\delta^0(D) \geq r}{\geq} r - \overleftarrow{d}_{D,\mathcal{U}}^{\pm}(v) - |\overrightarrow{N}_{D,\mathcal{U}}^{\pm}(v) \cap U^*| \overset{\text{(i)}}{\geq} r - \sum_{i \in [r]} d_{\mathcal{F}_i}^{\pm}(v) \\ &\overset{\text{(iv)}}{=} |\{i \in [r] \mid v \in S_i^{\pm}\}|. \end{aligned} \tag{16.17}$$

By (F2′), the vertices in $U^{1-\gamma}(T)$ are already covered with both an in- and outedge in each of the pseudo-feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_r$, so $S_i^+ \cup S_i^- \subseteq U^* \setminus U^{1-\gamma}(T)$ for each $i \in [r]$. Thus, (16.1) follows from (16.16) and (16.17).

Let $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ be the edge-disjoint linear forests obtained by applying Lemma 16.1 with D' and r playing the roles of D and ℓ. For each $i \in [r]$, denote $\mathcal{F}_i' := \mathcal{F}_i \cup \mathcal{Q}_i$. We claim that $\mathcal{F}_1', \dots, \mathcal{F}_r'$ are edge-disjoint feasible systems which satisfy (a)–(c). By assumption, $\mathcal{F}_1, \dots, \mathcal{F}_r$ are edge-disjoint. Thus, (16.11), (16.12), and the "in particular part" of Lemma 16.1 imply that $\mathcal{F}_1', \dots, \mathcal{F}_r'$ are edge-disjoint linear forests. In particular, (F3) is satisfied. By construction and Lemma 16.1(a), each $i \in [r]$ satisfies

$$E(\mathcal{F}_i) \subseteq E(\mathcal{F}_i') \subseteq E(\mathcal{F}_i) \cup \{e \in E(\overrightarrow{D}_{\mathcal{U}}) \mid V(e) \cap U^* \neq \emptyset\}. \tag{16.18}$$

In particular, (a) follows from (i), while (b) follows from (ii), (16.13), and Lemma 16.1(a). For each $v \in V(T) \setminus U^*$, we have

$$\sum_{i \in [r]} d_{\mathcal{F}_i'}(v) \overset{\text{(i),(iii),(16.18)}}{\leq} \overleftarrow{d}_{D,\mathcal{U}}(v) + |\overrightarrow{N}_{D,\mathcal{U}}(v) \cap U^*| \overset{\text{(ES1)}}{\leq} 2\varepsilon n + |U^*| \overset{\text{(ES2)}}{\leq} 6\varepsilon n.$$

Thus, (c) holds. Let $i \in [r]$. By (16.18), $\mathcal{F}_i'$ is obtained from $\mathcal{F}_i$ by adding forward edges, so (F1) follows from the fact that $\mathcal{F}_i$ is a (γ, T)-pseudo-feasible system. By definition of S_i^+ and S_i^-, Lemma 16.1(a) implies that (F2) is satisfied. Therefore, $\mathcal{F}_i'$ is a feasible system, as desired. □

Lemma 16.4 (Incorporating E)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $(1-\eta)n \leq r \leq n$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $D \subseteq T$ satisfies $\delta^0(D) \geq r$. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be edge-disjoint feasible systems which satisfy the following.*

(i) *$E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i) \subseteq E(D)$.*

(ii) *For each $i \in [r]$, $e(\mathcal{F}_i) \leq \varepsilon n$.*

(iii) *For each $v \in V(T) \setminus U^*$, there exist at most εn indices $i \in [r]$ such that $v \in V(\mathcal{F}_i)$.*

Let $E \subseteq E(D)$ satisfy the following properties.

(iv) *$E \subseteq E(\overrightarrow{D}_{\mathcal{U}} - U^*)$.*

(v) *For each $v \in V(T) \setminus U^*$, $d_E^{\pm}(v) \leq 1$.*

Then, there exist edge-disjoint feasible systems $\mathcal{F}_1', \dots, \mathcal{F}_r'$ such that the following hold.

(a) $\bigcup_{i\in[r]} E(\mathcal{F}_i') = \bigcup_{i\in[r]} E(\mathcal{F}_i) \cup E$. *In particular,* $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \cup E \subseteq \bigcup_{i\in[r]} E(\mathcal{F}_i') \subseteq E(D)$.

(b) *For each* $i \in [r]$, $e(\mathcal{F}_i') \leq e(\mathcal{F}_i) + 5 \leq 2\varepsilon n$.

Proof. To ensure that we do not create any cycle when adding the edges in E, we will separate the feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_r$ into four groups. For each $i \in [4]$, the edges of E from U_i to U_{i+1} will be distributed among the feasible systems from the i^{th} group. In this way, each $\mathcal{F}_j'$ will be obtained from $\mathcal{F}_j$ by adding a matching of forward edges. The edges of E will be distributed using Hall's theorem (Proposition 7.24).

For each $i \in [4]$, let $A_i := E_E(U_i, U_{i+1}) \setminus \bigcup_{j\in[r]} E(\mathcal{F}_j)$. Let $B_1 \cup \dots \cup B_4$ be a partition of $[r]$ such that $|B_i| \geq \lfloor \frac{r}{4} \rfloor =: r'$ for each $i \in [4]$. For each $i \in [4]$, let B_i' be the multiset which consists of 5 copies of each $j \in B_i$ and let G_i be the auxiliary bipartite graph on vertex classes A_i and B_i' defined as follows. For each $e \in A_i$ and each (copy of) $j \in B_i'$, $ej \in E(G)$ if and only if $V(e) \cap V(E(\mathcal{F}_j)) = \emptyset$.

Let $i \in [4]$. By (v) and Fact 10.2(i), $|A_i| \leq n \leq 5r' \leq |B_i'|$. For each $e \in A_i$, we have

$$d_G(e) \overset{\text{(iii)}}{\geq} 5(r' - 2\varepsilon n) \geq \frac{|B_i'|}{2}.$$

Moreover, each (copy of) $j \in B_i'$ satisfies

$$d_{G_i}(j) \overset{\text{(v)}}{\geq} |A_i| - |V(E(\mathcal{F}_j))| \geq |A_i| - 2e(\mathcal{F}_j) \overset{\text{(ii)}}{\geq} |A_i| - \frac{|B_i'|}{2}.$$

Apply Proposition 7.24 to obtain a matching M_i of G_i which covers A_i.

Denote $A := \bigcup_{i\in[4]} A_i$ and $M := \bigcup_{i\in[4]} M_i$. For each $j \in [r]$, let $\mathcal{F}_j'$ be obtained from $\mathcal{F}_j$ by adding all the edges $e \in A$ such that e is adjacent to a copy of j in M. We now verify that $\mathcal{F}_1', \dots, \mathcal{F}_r'$ are edge-disjoint feasible systems for which (a) and (b) are satisfied. By construction, M is a matching covering A and (iv) implies that $A = E \setminus \bigcup_{j\in[r]} E(\mathcal{F}_j)$. Therefore, $\mathcal{F}_1', \dots, \mathcal{F}_r'$ are edge-disjoint and (a) holds. Moreover, (b) holds by (ii) and definition of $B_1', \dots, B_4'$.

Let $j \in [r]$. Recall that $\mathcal{F}_j$ is a feasible system. In particular, (F3) implies that $\mathcal{F}_j$ is a linear forest. By definition of $G_1, \dots, G_4$, we have $V(E(\mathcal{F}'_j \setminus \mathcal{F}_j)) \cap V(E(\mathcal{F}_j)) = \emptyset$. Moreover, (v) implies that $A_1, \dots, A_4$ are all matchings, so, by construction, $E(\mathcal{F}'_j \setminus \mathcal{F}_j)$ is a matching. Thus, $\mathcal{F}'_j$ is also a linear forest and so Fact 13.4 and (iv) imply that $\mathcal{F}'_j$ is also a feasible system. $\square$

In the following lemma, we add forward edges to ensure that all the components of each feasible system have their ending point in U_4.

Lemma 16.5 (Extending the ending points of feasible systems into U_4)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $(1-\eta)n \leq r \leq n$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that $D \subseteq T$ satisfies $\delta^0(D) \geq r$. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be edge-disjoint feasible systems which satisfy the following.*

(i) $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i) \subseteq E(D)$.

(ii) *For each $i \in [r]$, $e(\mathcal{F}_i) \leq \varepsilon n$.*

(iii) *For each $v \in V(T) \setminus U^*$, there exist at most εn indices $i \in [r]$ such that $v \in V(\mathcal{F}_i)$.*

Then, there exist edge-disjoint feasible systems $\mathcal{F}'_1, \dots, \mathcal{F}'_r$ such that the following hold.

(a) $\bigcup_{i \in [r]} E(\mathcal{F}_i) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}'_i) \subseteq E(D)$. *In particular,* $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}'_i) \subseteq E(D)$.

(b) *For each $i \in [r]$, $e(\mathcal{F}'_i) \leq 4e(\mathcal{F}_i) \leq 4\varepsilon n$.*

(c) *For each $v \in V(T) \setminus U^*$, there exist at most $4\sqrt{\varepsilon}n$ indices $i \in [r]$ such that $v \in V(\mathcal{F}'_i)$.*

(d) *For each $i \in [r]$, $V^+(\mathcal{F}'_i) = V^+(\mathcal{F}_i)$ and $V^-(\mathcal{F}'_i) \subseteq U_4$.*

Proof. We extend the components of the feasible systems in three stages as follows. At each stage $i \in [3]$, we use edges of $D(U_i, U_{i+1})$ to extend the components of the feasible systems which currently end in U_i into components which end in U_{i+1}. This is achieved via Lemma 16.1.

By Fact 13.5, we may assume without loss of generality that $\mathcal{F}_1, \dots, \mathcal{F}_r$ do not contain any isolated vertices. For each $i \in [r]$, let $\mathcal{F}_i^0 := \mathcal{F}_i$. Suppose inductively that, for some $0 \le i \le 3$, we have constructed edge-disjoint feasible systems $\mathcal{F}_1^i, \dots, \mathcal{F}_r^i$ such that the following hold.

(a′) For each $j \in [r]$, $E(\mathcal{F}_j) \subseteq E(\mathcal{F}_j^i) \subseteq E(D)$.

(b′) For each $j \in [r]$, $e(\mathcal{F}_j^i) \le (i+1)e(\mathcal{F}_j) \le (i+1)\varepsilon n$.

(c′) For each $v \in V(T) \setminus U^*$, there exist at most $(i+1)\sqrt{\varepsilon} n$ indices $j \in [r]$ such that $v \in V(\mathcal{F}_j^i)$.

(d′) For each $j \in [r]$, $V^+(\mathcal{F}_j^i) = V^+(\mathcal{F}_j)$.

(e′) For each $j \in [r]$, $V^-(\mathcal{F}_j^i) \subseteq V(T) \setminus \bigcup_{i' \in [i]} U_{i'}$.

(f′) For each $j \in [r]$, $\mathcal{F}_j^i$ does not contain any isolated vertex.

First, assume that $i = 3$. For each $j \in [r]$, let $\mathcal{F}_j' := \mathcal{F}_j^i$. Then, (a)–(d) follow from (a′)–(e′).

We may therefore assume that $i < 3$. We construct $\mathcal{F}_1^{i+1}, \dots, \mathcal{F}_r^{i+1}$ using Lemma 16.1 as follows. Let $A := U_{i+1} \setminus U^*$ and $B := U_{i+2} \setminus U^*$. Let D' be the bipartite digraph on vertex classes A and B which is induced by $(\overrightarrow{D}_{\mathcal{U}} \setminus \bigcup_{j \in [r]} \mathcal{F}_j^i) - U^*$. For each $j \in [r]$, let $S_j^- := \emptyset$, let $S_j^+ := V^-(\mathcal{F}_j^i) \cap A$ (so S_j^+ lists the ending points of the components that currently end in $U_{i+1} \setminus U^*$ and which we want to extend in this step), and define $T_j := V(E(\mathcal{F}_j^i)) \cap B$. Note for later that both

$$S_j^{\pm} \cap V(\mathcal{F}_j^i) \subseteq V^{\mp}(\mathcal{F}_j^i) \quad \text{and} \quad V(E(\mathcal{F}_j^i)) \cap B \subseteq T_j \tag{16.19}$$

for each $j \in [r]$. Define $N := \sqrt{\varepsilon} n$. For each $v \in A \cup B$, let n_v denote the number of indices $j \in [r]$ such that $v \in V(\mathcal{F}_j^i)$ and observe that

$$|\{j \in [r] \mid v \in S_j^+\}| \le n_v \overset{\text{(c')}}{\le} 4\sqrt{\varepsilon} n. \tag{16.20}$$

We verify that (16.1) holds with D' and r playing the roles of D and ℓ. Recall that $\mathcal{F}_1, \dots, \mathcal{F}_r$ do not contain any isolated vertices. Thus, each $j \in [r]$ satisfies

$$\begin{aligned} |S_j^+| + |S_j^-| &\le |V^-(\mathcal{F}_j^i)| = |V^+(\mathcal{F}_j^i)| \overset{\text{(d')}}{=} |V^+(\mathcal{F}_j)| \le e(\mathcal{F}_j) && (16.21)\\ &\overset{\text{(ii)}}{\le} \varepsilon n && (16.22) \end{aligned}$$

and

$$|T_j| \le |V(E(\mathcal{F}_j^i))| \overset{\text{(b')}}{\le} 8\varepsilon n. \tag{16.23}$$

Then,

$$\frac{1}{\lfloor N \rfloor} \sum_{j \in [r]} (|S_j^+| + |S_j^-|) \overset{(16.22)}{\le} \frac{\varepsilon n r}{\lfloor \sqrt{\varepsilon} n \rfloor} \le 2\sqrt{\varepsilon} n. \tag{16.24}$$

For each $v \in U_{i+1} \setminus U^* = A$, we have

$$\begin{aligned} d_{D'}^+(v) &\ge |N_D^+(v) \cap U_{i+2}| - |N_D^+(v) \cap U^*| - \sum_{j \in [r]} d_{\mathcal{F}_j^i}^+(v)\\ &\overset{\text{(F3)}}{\ge} \overrightarrow{d}^{\,+}_{D,\mathcal{U}}(v) - |U^*| - n_v\\ &\ge \left(\overrightarrow{d}^{\,+}_{T,\mathcal{U}}(v) - (n-r)\right) - |U^*| - n_v\\ &\overset{\text{Definition 13.11,(16.20)}}{\ge} (1-2\eta)n - 4\varepsilon n - 4\sqrt{\varepsilon} n\\ &\overset{(16.20),(16.22)\text{–}(16.24)}{\ge} \frac{1}{\lfloor N \rfloor} \sum_{j \in [r]} (|S_j^+| + |S_j^-|) + |\{j \in [r] \mid v \in S_j^+\}|\\ &\qquad + 2\max_{j \in [r]}(|S_j^+| + |S_j^-| + |T_j|) + 2(\max_{w \in B} n_w + N). \end{aligned}$$

Therefore, (16.1) is satisfied.

Let $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ be the edge-disjoint linear forests obtained by applying Lemma 16.1 with D' and r playing the roles of D and ℓ. For each $j \in [r]$, denote $\mathcal{F}_j^{i+1} \coloneqq \mathcal{F}_j^i \cup \mathcal{Q}_j$. We claim that $\mathcal{F}_1^{i+1}, \dots, \mathcal{F}_r^{i+1}$ are edge-disjoint feasible systems such that (a′)–(f′) are satisfied with $i+1$ playing the role of i. By assumption and definition of D', $\mathcal{F}_1^i, \dots, \mathcal{F}_r^i$ are edge-disjoint from each other and from D'. Thus, (16.19) and the "in particular part" of Lemma 16.1 imply that $\mathcal{F}_1^{i+1}, \dots, \mathcal{F}_r^{i+1}$ are edge-disjoint linear forests. Moreover, (f′) follows from Lemma 16.1(a) and the induction hypothesis, while (c′) holds by Lemma 16.1(b) and the induction hypothesis. Furthermore, (b′) follows from (ii), (16.21), Lemma 16.1(a), and the induction hypothesis. By Lemma 16.1(a), each $j \in [r]$ satisfies

$$E(\mathcal{F}_j^i) \subseteq E(\mathcal{F}_j^{i+1}) \subseteq E(\mathcal{F}_j^i) \cup E(\overrightarrow{D}_{\mathcal{U}} - U^*).$$

Therefore, (a′) follows from the induction hypothesis and, by Fact 13.4, $\mathcal{F}_j^{i+1}$ is still a feasible system for each $j \in [r]$. For each $j \in [r]$, the definition of S_j^+ and S_j^- and Lemma 16.1(a) imply that all the edges of $\mathcal{F}_j^{i+1} \setminus \mathcal{F}_j^i = \mathcal{Q}_j$ start at a vertex in $V^-(\mathcal{F}_j^i)$. Thus, (d′) holds. By Lemma 16.1(a), each $j \in [r]$ satisfies

$$\begin{aligned} V^-(\mathcal{F}_j^{i+1}) &\subseteq (V^-(\mathcal{F}_j^i) \setminus A) \cup B \subseteq (V^-(\mathcal{F}_j^i) \setminus U_{i+1}) \cup (V^-(\mathcal{F}_j^i) \cap U^*) \cup U_{i+2} \\ &\overset{\text{(F2)}}{=} (V^-(\mathcal{F}_j^i) \setminus U_{i+1}) \cup U_{i+2} \overset{\text{(e′)}}{=} \left(V(T) \setminus \left(\bigcup_{i' \in [i]} U_{i'} \cup U_{i+1} \right) \right) \cup U_{i+2} \\ &= V(T) \setminus \bigcup_{i' \in [i+1]} U_{i'}. \end{aligned}$$

Therefore, (e′) holds. $\square$

By symmetry, we can proceed analogously to ensure that the starting point of each component of the feasible systems also lies in the correct vertex class.

Lemma 16.6 (Extending the starting points of feasible systems into U_1)**.** *Under the conditions of Lemma 16.5, there exist edge-disjoint feasible systems $\mathcal{F}_1', \dots, \mathcal{F}_r'$ such that the following hold.*

(a) $\bigcup_{i \in [r]} E(\mathcal{F}_i) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i') \subseteq E(D)$. *In particular, $E(\overleftarrow{D}_{\mathcal{U}}) \cup E(D[U^*]) \subseteq \bigcup_{i \in [r]} E(\mathcal{F}_i') \subseteq E(D)$.*

(b) *For each* $i \in [r]$, $e(\mathcal{F}_i') \leq 7e(\mathcal{F}_i) \leq 7\varepsilon n$.

(c) *For each* $v \in V(T) \setminus U^*$, *there exist at most* $7\sqrt{\varepsilon}n$ *indices* $i \in [r]$ *such that* $v \in V(\mathcal{F}_i')$.

(d) *For each* $i \in [r]$, $V^+(\mathcal{F}_i') \subseteq U_1$ *and* $V^-(\mathcal{F}_i') \subseteq U_4$.

We are now ready to derive Lemma 15.5.

Proof of Lemma 15.5. Let $i \in [r]$. Suppose that $e \in E(\mathcal{F}_i)$ is a forward edge such that $V(e) \cap U^* = \emptyset$ or e is a (γ, T)-placeholder. Then, (i)–(iv) are still satisfied if we replace $\mathcal{F}_i$ by $\mathcal{F}_i \setminus \{e\}$. Moreover, Fact 15.4 implies that $\mathcal{F}_i \setminus \{e\}$ is still a (γ, T)-pseudo-feasible system. Thus, by deleting edges if necessary, we may assume that Lemma 16.2(iii) is satisfied. Moreover, Lemma 16.2(i) and (ii) follow from (i) and (ii). Apply Lemma 16.2 to obtain edge-disjoint (γ, T)-pseudo-feasible systems $\mathcal{F}_1^1, \dots, \mathcal{F}_r^1$ satisfying Lemma 16.2(a)–(c).

By Lemma 16.2(a)–(c), Lemma 16.3(i)–(iv) are satisfied with $\mathcal{F}_1^1, \dots, \mathcal{F}_r^1$ playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$. Apply Lemma 16.3 with $\mathcal{F}_1^1, \dots, \mathcal{F}_r^1$ playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$ to obtain edge-disjoint feasible systems $\mathcal{F}_1^2, \dots, \mathcal{F}_r^2$ satisfying Lemma 16.3(a)–(c).

By Lemma 16.3(a)–(c), Lemma 16.4(i)–(iii) are satisfied with $\mathcal{F}_1^2, \dots, \mathcal{F}_r^2$, and 9ε playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$, and ε. By (iii) and (iv), Lemma 16.4(iv) and (v) are satisfied. Apply Lemma 16.4 with $\mathcal{F}_1^2, \dots, \mathcal{F}_r^2$, and 9ε playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$, and ε to obtain edge-disjoint feasible systems $\mathcal{F}_1^3, \dots, \mathcal{F}_r^3$ for which Lemma 16.4(a) and (b) are satisfied (with 9ε playing the role of ε).

By (iv), Lemma 16.3(c), and Lemma 16.4(a) and (b), Lemma 16.5(i)–(iii) are satisfied with $\mathcal{F}_1^3, \dots, \mathcal{F}_r^3$, and 18ε playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$, and ε. Apply Lemma 16.6 with $\mathcal{F}_1^3, \dots, \mathcal{F}_r^3$, and 18ε playing the roles of $\mathcal{F}_1, \dots, \mathcal{F}_r$, and ε to obtain edge-disjoint feasible systems $\mathcal{F}_1', \dots, \mathcal{F}_r'$ satisfying Lemma 16.6(a)–(d) (with 18ε playing the role of ε). Then, (a) follows from Lemma 16.4(a) and Lemma 16.6(a). Moreover, (b)–(d) follow from Lemma 16.6(b)–(d), respectively. □

CHAPTER 17

CONSTRUCTING PSEUDO-FEASIBLE SYSTEMS: PROOF OF LEMMA 15.6

In this section, we prove Lemma 15.6, which states that the backward and exceptional edges of a regular bipartite tournament can be decomposed into pseudo-feasible systems.

17.1 Proof overview

Let T be a bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon,\mathcal{U})$-exceptional set for T. Suppose that we want to decompose the backward edges of T into n pseudo-feasible systems. The main difficulty is to construct linear forests (up to placeholders) which have a balanced number of backward edges (recall (F1)).

17.1.1 Simplified argument

For simplicity, first assume that $\Delta^0(\overleftarrow{T}_{\mathcal{U}}) \leq \frac{n}{2}$. The idea is to decompose each pair $E_T(U_i, U_{i-1})$ into $\frac{n}{2}$ matchings (which is possible by König's theorem) and then construct pseudo-feasible systems from these matchings as follows. First, we pair each of the $\frac{n}{2}$ matchings from $E_T(U_1, U_4)$ with a distinct matching from $E_T(U_3, U_2)$. Overall, we obtain a decomposition of $E_D(U_1, U_4) \cup E_D(U_3, U_2)$ into $\frac{n}{2}$ matchings $\mathcal{F}_1, \dots, \mathcal{F}_{\frac{n}{2}}$. Similarly, we pair each of the $\frac{n}{2}$ matchings from $E_T(U_4, U_3)$ with a distinct matching from $E_T(U_2, U_1)$

to obtain a decomposition of $E_T(U_4, U_3) \cup E_T(U_2, U_1)$ into $\frac{n}{2}$ matchings $\mathcal{F}_{\frac{n}{2}+1}, \dots, \mathcal{F}_n$. In particular, $\mathcal{F}_1, \dots, \mathcal{F}_n$ are linear forests and so satisfy (F3′) and (F4′). By assumption, T does not contain any vertex of very high backward degree and so (F2′) also holds. Thus, for $\mathcal{F}_1, \dots, \mathcal{F}_n$ to be pseudo-feasible systems, we only need them to contain a balanced number of backward edges (see (F1)). More precisely, each of $\mathcal{F}_1, \dots, \mathcal{F}_{\frac{n}{2}}$ must contain the same number of edges from $E_T(U_1, U_4)$ and $E_T(U_3, U_2)$ and each of $\mathcal{F}_{\frac{n}{2}+1}, \dots, \mathcal{F}_n$ must contain the same number of edges from $E_T(U_4, U_3)$ and $E_T(U_2, U_1)$. Since T contains the same number of backward edges in each pair of the blow-up C_4 (recall Fact 10.4), this can be easily achieved by using Proposition 7.23 to initially decompose each $E_T(U_i, U_{i-1})$ into $\frac{n}{2}$ matchings of (almost) the same size.

17.1.2 General argument

In general, $\Delta^0(\overleftarrow{T}_{\mathcal{U}})$ may be larger than $\frac{n}{2}$ and so the above strategy does not work (we cannot decompose each pair into $\frac{n}{2}$ matchings of backward edges). However, we adapt the above argument by constructing $\frac{n}{2}$ pseudo-feasible systems which mostly consist of edges from $E_T(U_1, U_4) \cup E_T(U_3, U_2)$ and $\frac{n}{2}$ pseudo-feasible systems which mostly consist of edges from $E_T(U_4, U_3) \cup E_T(U_2, U_1)$.

To do so, we consider an auxiliary digraph D obtained from $\overleftarrow{T}_{\mathcal{U}}$ as follows. For each $v \in U^{\frac{1}{2}}(T)$ (that is, for each $v \in V(T)$ satisfying $\overleftarrow{d}^{+}_{T,\mathcal{U}}(v) = \overleftarrow{d}^{-}_{T,\mathcal{U}}(v) > \frac{n}{2}$ (recall Fact 10.3 and (13.2))), we replace v by two copies v and replace all the edges incident to v by an edge incident to one of the copies of v. By splitting neighbourhoods evenly between the two copies of each vertex in $U^{\frac{1}{2}}(T)$, we can ensure that $\Delta^0(D) \leq \frac{n}{2}$. Then, we can proceed as above to partition $E_D(U_1, U_4) \cup E_D(U_3, U_2)$ into $\frac{n}{2}$ pseudo-feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_{\frac{n}{2}}$ and partition $E_D(U_4, U_3) \cup E_D(U_2, U_1)$ into $\frac{n}{2}$ pseudo-feasible systems $\mathcal{F}_{\frac{n}{2}+1}, \dots, \mathcal{F}_n$.

Denote by $\mathcal{F}_1', \dots, \mathcal{F}_n'$ the decomposition of $\overleftarrow{T}_{\mathcal{U}}$ induced by $\mathcal{F}_1, \dots, \mathcal{F}_n$. Then, $\mathcal{F}_1', \dots, \mathcal{F}_n'$ each contain a balanced number of backward edges but may contain up to two edges (of the same direction) incident to the vertices in $U^{\frac{1}{2}}(T)$ (all the other vertices have degree at most one). We solve this problem as follows. For each $i \in [n]$, we move an edge of

$\mathcal{F}'_i$ at each vertex of degree two to $\mathcal{F}'_{\frac{n}{2}+i}$ (where the index $\frac{n}{2}+i$ is taken modulo n). By construction, $\mathcal{F}'_i$ and $\mathcal{F}'_{\frac{n}{2}+i}$ are constructed from different pairs of the blow-up C_4. Thus, we do not create additional vertices of semidegree at least two and so we now have $\Delta^0(\mathcal{F}'_i) = 1$ for each $i \in [n]$. In particular, (F3′) is now satisfied. Of course, some cycles may be created in the process. However, each cycle will contain one of the moved edges and so will contain a backward edge incident to vertex of very high backward degree, that is, a placeholder. Thus, (F4′) is also satisfied. We may have unbalanced the number of backward edges in each $\mathcal{F}'_i$ in the process but, by moving a few additional edges, we will be able to satisfy (F1) without affecting (F3′) and (F4′). This latter step will be achieved using Lemma 17.4 below. The overall argument corresponds to Lemma 17.3 below.

17.1.3 Limitations

To sum up, we have so far decomposed the edges of $\overleftarrow{T}_{\mathcal{U}}$ into digraphs $\mathcal{F}'_1, \dots, \mathcal{F}'_n$ satisfying (F1), (F3′), and (F4′). Moreover, $\Delta^0(\mathcal{F}'_i) = 1$ for each $i \in [n]$. Thus, $\mathcal{F}'_1, \dots, \mathcal{F}'_n$ are almost pseudo-feasible systems and it only remains to cover $U^{1-\gamma}(T)$ to ensure that (F2′) is satisfied. Unfortunately, this may not be possible. Indeed, suppose that $v \in U^{1-\gamma}(T)$ satisfies $\overleftarrow{d}^{+}_{T,\mathcal{U}}(v) = n-1$. Then, there is precisely one $\mathcal{F}'_i$ which does not contain an outedge at v and so, to turn this $\mathcal{F}'_i$ into a pseudo-feasible system, we would have to add the unique forward outedge at v in T to $\mathcal{F}'_i$ (so that we satisfy (F2′)). However, this edge is not a placeholder and so we may break (F3′) and/or (F4′) in the process. More generally, we may not be able to cover the vertices in $U^{1-\gamma}(T)$ which are incident to at least one forward edge. (The vertices $v \in U^{1-\gamma}(T)$ with no forward edges are not a problem because the n backward in- and outedges at v are already entirely covering v in each $\mathcal{F}'_i$.)

This why, in Lemma 17.3, we will assume that none of the vertices in $U^{1-\gamma}(T)$ are adjacent to a forward edge (see Lemma 17.3(ii) and (iii)). Before applying Lemma 17.3, we will thus have to construct a few pseudo-feasible systems which cover all the forward edges incident to $U^{1-\gamma}(T)$. This is achieved in Lemma 17.2 below.

In Lemma 17.2, we also cover all the forward edges which entirely lie in U^*. Recall from Lemma 15.6(a) that we will have to incorporate these edges into our pseudo-feasible systems. But, it may not be possible to incorporate them into the pseudo-feasible systems constructed with the above arguments. Indeed, the edges which lie entirely in U^* are not placeholders and so incorporating them may break (F3′) and/or (F4′). Thus, we will cover them separately with a few pseudo-feasible systems in Lemma 17.2.

Finally, observe that, with the above arguments, we are only constructing pseudo-feasible systems (rather than feasible systems) and we have no control on which edges are used in each of the pseudo-feasible system. Thus, we will have to construct the t feasible systems satisfying Lemma 15.6(c) and (d) separately. This is achieved in Lemma 17.1 below.

17.2 Proof of Lemma 15.6

First, we build the t feasible systems which satisfy Lemma 15.6(c) and (d).

Lemma 17.1 (Constructing the few feasible systems). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $s \in \mathbb{N}$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Suppose that, for each $i \in [s]$, $H_i \subseteq T$ satisfies Lemma 13.12(i)–(iv). For each $i \in [s]$, let $s_i \in \mathbb{N}$ and $t_i := \sum_{j \in [i-1]} s_j$. Let $t := \sum_{i \in [s]} s_i$ and suppose that $t \leq \eta n$. Then, there exist edge-disjoint feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_t \subseteq T$ for which the following hold, where $\mathcal{F} := \bigcup_{i \in [t]} \mathcal{F}_i$.*

(a) *For each $i \in [4]$, we have $e_{\mathcal{F} - U^{1-\gamma}(T)}(U_i, U_{i-1}) = t|U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$.*

(b) *For each $i \in [t]$, we have $e(\mathcal{F}_i) \leq \sqrt{\varepsilon} n$.*

(c) *For each $i \in [t]$, we have $V^0(\mathcal{F}_i) = U^*$.*

(d) *For each $i \in [s]$ and $j \in [s_i]$, we have $\mathcal{F}_{t_i + j} \subseteq H_i$.*

Note that Lemma 17.1(c) and (d) will automatically imply Lemma 15.6(c) and (d); while Lemma 17.1(b) corresponds to Lemma 15.6(b).

Then, we construct a few pseudo-feasible systems which cover all the forward edges in U^* and all the forward edges incident to $U^{1-\gamma}(T)$.

Lemma 17.2 (Covering the forward edges in U^* and incident to $U^{1-\gamma}(T)$)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \gamma \ll 1$. Let $t \leq \eta n$ and $t' \in \{\lfloor \gamma n \rfloor, \lfloor \gamma n \rfloor + 1\}$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U}$ be an optimal $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Let $D \subseteq T$ and suppose that the following hold.*

(i) $\Delta^0(T \setminus D) \leq t$.

(ii) *For each $i \in [4]$, we have $e_{(T \setminus D) - U^{1-\gamma}(T)}(U_i, U_{i-1}) \leq t|U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)|$.*

Then, there exist edge-disjoint (γ, T)-pseudo-feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_{t'}$ such that the following hold, where $\mathcal{F} \coloneqq \bigcup_{i \in [t']} \mathcal{F}_i$.

(a) $E(\overrightarrow{D}_{\mathcal{U}}[U^*]) \subseteq E(\mathcal{F}) \subseteq E(D)$.

(b) *For each $i \in [t']$, we have $e(\mathcal{F}_i) \leq \sqrt{\varepsilon} n$.*

(c) *For each $v \in U^{1-\gamma}(T)$, we have $\overrightarrow{d}^{\pm}_{\mathcal{F}, \mathcal{U}}(v) = \overrightarrow{d}^{\pm}_{D, \mathcal{U}}(v)$.*

(d) *For each $v \in U^{1-2\gamma}(T) \setminus U^{1-\gamma}(T)$, we have $\overleftarrow{d}^{\pm}_{\mathcal{F}, \mathcal{U}}(v) \geq t' - 4\varepsilon n$.*

Finally, we use the arguments outlined in Section 17.1.2 to construct the remaining pseudo-feasible systems using all the leftover backward edges of T.

Lemma 17.3 (Decomposing the backward edges)**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \gamma \ll 1$ and $\gamma n < r \leq \frac{n}{2}$. Let T be a regular bipartite tournament on $4n$ vertices. Let $\mathcal{U}$ be an $(\varepsilon, 4)$-partition for T and U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Let $D \subseteq \overleftarrow{T}_{\mathcal{U}}$ satisfy the following.*

(i) $e_D(U_1, U_4) = e_D(U_3, U_2)$ *and* $e_D(U_4, U_3) = e_D(U_2, U_1)$.

(ii) $\Delta^0(D) \leq 2r$.

(iii) *For each $v \in U^{1-\gamma}(T)$, $d^{\pm}_D(v) = 2r$.*

Then, there exist edge-disjoint (γ, T)-pseudo-feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_{2r}$ such that $D = \bigcup_{i\in[2r]} \mathcal{F}_i$ and $e(\mathcal{F}_i) \le \sqrt{\varepsilon} n$ for each $i \in [2r]$.

We first assume that Lemmas 17.1–17.3 hold and derive Lemma 15.6 as follows. Lemmas 17.1 and 17.2 will be proved in Chapter 18.

Proof of Lemma 15.6. Apply Lemma 17.1 to obtain edge-disjoint feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_t \subseteq T$ which satisfy Lemma 17.1(a)–(d). In particular, (c) and (d) hold.

Let $D \coloneqq T \setminus \bigcup_{i\in[t]} \mathcal{F}_i$. Then, Lemma 17.2(i) follows from (F3) and Lemma 17.2(ii) follows from Lemma 17.1(a). Let $t' \in \{\lfloor \gamma n \rfloor, \lfloor \gamma n \rfloor + 1\}$ be such that $n - t - t'$ is even. Let $\mathcal{F}_{t+1}, \dots, \mathcal{F}_{t+t'}$ be the edge-disjoint (γ, T)-pseudo-feasible systems obtained by applying Lemma 17.2.

Let $D' \coloneqq D \setminus \bigcup_{i\in[t']} \mathcal{F}_{t+i} = T \setminus \bigcup_{i\in[t+t']} \mathcal{F}_i$. Let $r \coloneqq \frac{n-t-t'}{2}$ and note that $r \in \mathbb{N}$. We claim that Lemma 17.3(i)–(iii) are satisfied with $\overleftarrow{D'}_{\mathcal{U}}$ playing the role of D. Indeed, Lemma 17.3(i) holds by Fact 10.4 and (F1). By (13.2), each $v \in V(T) \setminus U^{1-2\gamma}(T)$ satisfies $\overleftarrow{d}^{\pm}_{D',\mathcal{U}}(v) \le \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) \le (1-2\gamma)n \le 2r$. Moreover, each $v \in U^{1-2\gamma}(T) \setminus U^{1-\gamma}(T)$ satisfies

$$\overleftarrow{d}^{\pm}_{D',\mathcal{U}}(v) = \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) - \sum_{i\in[t+t']} \overleftarrow{d}^{\pm}_{\mathcal{F}_i,\mathcal{U}}(v) \overset{\text{Lemma 17.2(d)}}{\le} (1-\gamma)n - (t' - 4\varepsilon n) \le 2r.$$

Similarly, each $v \in U^{1-\gamma}(T)$ satisfies

$$\overleftarrow{d}^{\pm}_{D',\mathcal{U}}(v) = \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) - \sum_{i\in[t+t']} \overleftarrow{d}^{\pm}_{\mathcal{F}_i,\mathcal{U}}(v) \overset{\text{(F2),(F2'),Lemma 17.2(c)}}{=} \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) - (t+t' - \overrightarrow{d}^{\pm}_{T,\mathcal{U}}(v)) = 2r.$$

Thus, Lemma 17.3(ii) and (iii) are satisfied. Apply Lemma 17.3 with $\overleftarrow{D'}_{\mathcal{U}}$ playing the role of D to obtain $2r$ edge-disjoint (γ, T)-pseudo-feasible systems $\mathcal{F}_{t+t'+1}, \dots, \mathcal{F}_n$ such that $\overleftarrow{D'}_{\mathcal{U}} = \bigcup_{i\in[2r]} \mathcal{F}_{t+t'+i}$ and $e(\mathcal{F}_{t+t'+i}) \le \sqrt{\varepsilon} n$ for each $i \in [2r]$.

Then, (a) follows from Lemma 17.1, Lemma 17.2(a), and Lemma 17.3. Moreover, (b) holds by Lemma 17.1(b), Lemma 17.2(b), and Lemma 17.3. Finally, (c) and (d) follow from Lemma 17.1(c) and (d). $\square$

17.3 Proof of Lemma 17.3

We use the arguments presented in Section 17.1.2. To keep the number of backward edges balanced in each pseudo-feasible system, we will use the following lemma. Roughly speaking, Lemma 17.4 states that if we have two equal sized matchings M_1 and M_2 in an auxiliary digraph where a set W of vertices is replaced by two copies W^1 and W^2 of W, then there exist equal sized $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$ (see Lemma 17.4(a)) such that each of $M_1', M_1 \setminus M_1', M_2'$, and $M_2 \setminus M_2'$ cover at most one copy of each vertex in W (see Lemma 17.4(c)) and all edges in M_1' or M_2' are incident to W^2 (see Lemma 17.4(b)). In the proof of Lemma 17.3, we will move the matchings M_1' and M_2' from $\mathcal{F}_i'$ to $\mathcal{F}_{\frac{n}{2}+i}'$ (as discussed in Section 17.1.2) and so Lemma 17.4(c) will ensure that the maximum semidegree is reduced to 1. Moreover, we will construct our auxiliary digraph (see Section 17.1.2) in such a way that all the edges incident to the second copy W^2 of $W := U^{\frac{1}{2}}(T)$ correspond to placeholders. Thus, Lemma 17.4(b) will ensure that no problematic cycle is created (i.e. every cycle will contain a placeholder). Since $|M_1'| = |M_2'|$ and $|M_1 \setminus M_1'| = |M_2 \setminus M_2'|$, the number of backward edges will remain balanced.

Lemma 17.4. *Let W and V' be disjoint vertex sets and suppose that W^1 and W^2 are two copies of W. Let M_1 and M_2 be undirected matchings satisfying the following properties.*

(i) $|M_1| = |M_2|$.

(ii) $V(M_1 \cup M_2) \subseteq V' \cup W^1 \cup W^2$.

(iii) $e_{M_1 \cup M_2}(W^2, W^1 \cup W^2) = 0$.

Then, there exist $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$ such that the following hold.

(a) $|M_1'| = |M_2'|$.

(b) *There exists $i \in [2]$ such that all the edges in M_i' are incident to W^2.*

(c) *For each $i \in [2]$ and $w \in W$, both $V(M_i')$ and $V(M_i \setminus M_i')$ contain at most one copy of w.*

Proof. By induction on $m \coloneqq |M_1| = |M_2|$. If $m \in \{0,1\}$, then we let $M_1' \coloneqq \emptyset =: M_2'$ and we are done. For the induction step, suppose that $m \geq 2$ and that the lemma holds for any matchings of size less than m which satisfy (i)–(iii).

For any $w \in W$, denote by $w^1 \in W^1$ and $w^2 \in W^2$ the copies of w. For each $i \in [2]$, denote by $X_i^1 \coloneqq \{w^1 \in W^1 \mid w^1, w^2 \in V(M_i)\}$ the set of vertices $w^1 \in W^1$ such that M_i covers both w^1 and its corresponding vertex $w^2 \in W^2$. Note that if $X_1^1 = \emptyset = X_2^1$, then we may simply set $M_1' = \emptyset = M_2'$ and we are done. Thus, we may view X_1^1 and X_2^1 as the set of (first copies of the) problematic vertices in M_1 and M_2. For each $i \in [2]$, let $Y_i^1 \coloneqq \{w^1 \in X_i^1 \mid N_{M_i}(w^1) \subseteq X_i^1\}$, define $Z_i^1 \coloneqq X_i^1 \setminus Y_i^1$, and note that $M_i[Y_i^1]$ is a matching of size $\frac{|Y_i^1|}{2}$.

Case 1: $\min\{|Y_1^1|, |Y_2^1|\} \neq 0$. Then, there exist $v_1^1 w_1^1 \in M_1[Y_1^1]$ and $v_2^1 w_2^1 \in M_2[Y_2^1]$. For each $i \in [2]$, denote by e_i and e_i' the edges of M_i which are incident to v_i^2 and w_i^2, respectively (e_i and e_i' exist by definition of $X_i^1 \supseteq Y_i^1$). Note that (iii) implies that $e_i \neq e_i'$ for each $i \in [2]$. Then, observe that $\widetilde{M}_1 \coloneqq M_1 \setminus \{v_1^1 w_1^1, e_1, e_1'\}$ and $M_2 \setminus \{v_2^1 w_2^1, e_2, e_2'\}$ are matchings of size $m-3$ which still satisfy (i)–(iii). Thus, the induction hypothesis implies that there exist $\widetilde{M}_1' \subseteq \widetilde{M}_1$ and $\widetilde{M}_2' \subseteq \widetilde{M}_2$ such that (a)–(c) hold with $\widetilde{M}_1, \widetilde{M}_1', \widetilde{M}_2$, and $\widetilde{M}_2'$ playing the roles of M_1, M_1', M_2, and M_2'. Let $M_1' \coloneqq \widetilde{M}_1' \cup \{e_1, e_1'\}$ and $M_2' \coloneqq \widetilde{M}_2' \cup \{e_2, e_2'\}$. Since (a) and (b) hold with $\widetilde{M}_1, \widetilde{M}_1', \widetilde{M}_2$, and $\widetilde{M}_2'$ playing the roles of M_1, M_1', M_2, and M_2' and since e_1, e_1', e_2, and e_2' are all incident to W^2, (a) and (b) hold. For (c), let $i \in [2]$. By construction, (c) holds for v_i and w_i. Let $w \in W \setminus \{v_i, w_i\}$. By definition, $v_i^1 w_i^1$ does not cover a copy of w and, by (iii), neither e_i nor e_i' covers a copy of w. Therefore, since (c) holds with $\widetilde{M}_i$ and $\widetilde{M}_i'$ playing the roles of M_i and M_i', we have

$$|V(M_i \setminus M_i') \cap \{w^1, w^2\}| = |V(\widetilde{M}_i \setminus \widetilde{M}_i') \cap \{w^1, w^2\}| \leq 1$$

and

$$|V(M_i') \cap \{w^1, w^2\}| = |V(\widetilde{M}_i') \cap \{w^1, w^2\}| \leq 1.$$

Thus, (c) holds and we are done.

Case 2: $\min\{|Z_1^1|, |Z_2^1|\} \neq 0$. Then, there exist $w_1^1 \in Z_1^1$ and $w_2^1 \in Z_2^1$. For each $i \in [2]$, denote by e_i^1 and e_i^2 the edges of M_i which are incident to w_i^1 and w_i^2, respectively (e_i^1 and e_i^2 exist by definition of $X_i^1 \supseteq Z_i^1$). Note that, by (iii), e_i^1 and e_i^2 are distinct. Moreover, $\widehat{M_1} := M_1 \setminus \{e_1^1, e_1^2\}$ and $\widehat{M_2} := M_2 \setminus \{e_2^1, e_2^2\}$ are matchings of size $m-2$ and still satisfy (i)–(iii). Thus, the induction hypothesis implies that there exist $\widehat{M_1'} \subseteq \widehat{M_1}$ and $\widehat{M_2'} \subseteq \widehat{M_2}$ such that (a)–(c) hold with $\widehat{M_1}, \widehat{M_1'}, \widehat{M_2}$, and $\widehat{M_2'}$ playing the roles of M_1, M_1', M_2, and M_2'. Let $M_1' := \widehat{M_1'} \cup \{e_1^2\}$ and $M_2' := \widehat{M_2'} \cup \{e_2^2\}$. Since (a) and (b) hold with $\widehat{M_1}, \widehat{M_1'}, \widehat{M_2}$, and $\widehat{M_2'}$ playing the roles of M_1, M_1', M_2, and M_2' and since e_1^2 and e_2^2 are both incident to W^2, (a) and (b) hold. For (c), let $i \in [2]$. By construction, (c) holds for w_i. Suppose that $w \in W \setminus \{w_i\}$. If $w^1 \notin X_i^1$, then $V(M_i)$ contains at most one copy of w and so (c) holds for w. We may therefore assume that $w^1 \in X_i^1$. By (iii), e_i^2 does not cover a copy of w and, by definition of Z_i^1, neither does e_i^1. Therefore, since (c) holds with $\widehat{M_i}$ and $\widehat{M_i'}$ playing the roles of M_i and M_i', we have

$$|V(M_i \setminus M_i') \cap \{w^1, w^2\}| = |V(\widehat{M_i} \setminus \widehat{M_i'}) \cap \{w^1, w^2\}| \leq 1$$

and

$$|V(M_i') \cap \{w^1, w^2\}| = |V(\widehat{M_i'}) \cap \{w^1, w^2\}| \leq 1.$$

Thus, (c) holds and we are done.

Case 3: $\min\{|Y_1^1|, |Y_2^1|\} = 0 = \min\{|Z_1^1|, |Z_2^1|\}$. Define $y := \max\{|Y_1^1|, |Y_2^1|\}$ and $z := \max\{|Z_1^1|, |Z_2^1|\}$. For each $i \in [2]$, denote $X_i^2 := \{w^2 \mid w^1 \in X_i^1\}$ and recall that $M_i[Y_i^1]$ is a matching of size $\frac{|Y_i^1|}{2}$. Thus, (iii) implies that each $i \in [2]$ and $S^2 \subseteq X_i^2$ satisfies

$$|\{e \in M_i \mid V(e) \cap S^2 \neq \emptyset\}| = |S^2| \tag{17.1}$$

and

$$\begin{aligned}|\{e \in M_i \mid V(e) \cap (X_i^1 \cup X_i^2) \neq \emptyset\}| &= |\{e \in M_i \mid V(e) \cap Y_i^1 \neq \emptyset\}| \\ &\quad + |\{e \in M_i \mid V(e) \cap Z_i^1 \neq \emptyset\}| \\ &\quad + |\{e \in M_i \mid V(e) \cap X_i^2 \neq \emptyset\}| \\ &\overset{(17.1)}{=} \frac{|Y_i^1|}{2} + |Z_i^1| + |X_i^2| = \frac{3|Y_i^1|}{2} + 2|Z_i^1|. \end{aligned} \tag{17.2}$$

We proceed as follows.

Case 3.1: There exists $i \in [2]$ such that $|Y_i^1| = 0 = |Z_i^1|$. Suppose without loss of generality that $|Y_1^1| = 0 = |Z_1^1|$, $|Y_2^1| = y$, and $|Z_2^1| = z$. Then,

$$|M_1| \overset{(i)}{=} |M_2| \overset{(17.2)}{\geq} \frac{3y}{2} + 2z \geq y + z.$$

Let $M_1' \subseteq M_1$ satisfy $|M_1'| = y + z$ and let M_2' consists of all the edges of M_2 which are incident to X_2^2. Then, all the edges of M_2' are incident to W^2 and so (b) holds. Moreover,

$$|M_2'| \overset{(17.1)}{=} y + z = |M_1'|.$$

Thus, (a) holds. Recall that $X_1^1 = \emptyset = X_1^2$. By construction, $X_2^2 \subseteq V(M_2')$ and, by (iii), $V(M_2') \cap X_2^1 = \emptyset$. Therefore, (c) holds and we are done.

Case 3.2: $z \geq y$. By Case 3.1, we may assume without loss of generality that $|Y_1^1| = y$, $|Z_1^1| = 0 = |Y_2^1|$, and $|Z_2^1| = z$. Then,

$$|M_1 - (X_1^1 \cup X_1^2)| \overset{(17.2)}{=} |M_1| - \frac{3y}{2} \overset{(i)}{=} |M_2| - \frac{3y}{2} \overset{(17.2)}{\geq} 2z - \frac{3y}{2} \geq z - y.$$

Let M_1' consist of all the edges of M_1 which are incident to X_1^2 plus $z - y$ edges of $M_1 - (X_1^1 \cup X_1^2)$. Let M_2' consist of all the edges of M_2 which are incident to X_2^2. Then, all the edges in M_2' are incident to W^2 and so (b) holds. Moreover,

$$|M_1'| \overset{(17.1)}{=} y + (z - y) = z \overset{(17.1)}{=} |M_2'|.$$

Thus, (a) holds. Let $i \in [2]$. By construction, $X_i^2 \subseteq V(M_i')$ and, by (iii), $V(M_i') \cap X_i^1 = \emptyset$. Thus, (c) holds and we are done.

Case 3.3: $y \geq 2z$**.** By Case 3.1, we may assume without loss of generality that $|Y_1^1| = y$, $|Z_1^1| = 0 = |Y_2^1|$, and $|Z_2^1| = z$. Then,

$$|M_2 - (X_2^1 \cup X_2^2)| \overset{(17.2)}{=} |M_2| - 2z \overset{(i)}{=} |M_1| - 2z \overset{(17.2)}{\geq} \frac{3y}{2} - 2z \geq y - z.$$

Let M_1' consist of all the edges of M_1 which are incident to X_1^2. Let M_2' consist of all the edges of M_2 which are incident to X_2^2 plus $y - z$ edges of $M_2 - (X_2^1 \cup X_2^2)$. Then, all the edges in M_1' are incident to W^2 and so (b) holds. Moreover,

$$|M_1'| \overset{(17.1)}{=} y = z + (y - z) \overset{(17.1)}{=} |M_2'|.$$

Thus, (a) holds. Let $i \in [2]$. By construction, $X_i^2 \subseteq V(M_i')$ and, by (iii), $V(M_i') \cap X_i^1 = \emptyset$. Thus, (c) holds and we are done.

Case 3.4: $2z \geq y \geq z$**.** By Case 3.1, we may assume without loss of generality that $|Y_1^1| = y$, $|Z_1^1| = 0 = |Y_2^1|$, and $|Z_2^1| = z$. Then, $M_1[Y_1^1]$ is a matching of size $\frac{y}{2} \geq y - z$. Let $S^1 \subseteq M_1[Y_1^1]$ satisfy $|S^1| = y - z$ and define $S^2 := \{w^2 \mid w^1 \in V(S^1)\}$. Let M_1' be obtained from S^1 by adding all the edges M_1 which are incident to $X_1^2 \setminus S^2$. Let M_2' consist of all the edges of M_2 which are incident to X_2^2. Then, all the edges in M_2' are incident to W^2 and so (b) holds. Note that

$$|M_1'| \overset{(17.1)}{=} |S^1| + (y - 2|S^1|) = z \overset{(17.1)}{=} |M_2'|.$$

Thus, (a) holds. By construction, $X_2^2 \subseteq V(M_2')$ and, by (iii), $V(M_2') \cap X_2^1 = \emptyset$. Moreover,

(iii) implies that $V(M_1') \cap X_1^2 = X_1^2 \setminus S^2$ and $V(M_1') \cap X_1^1 = V(S^1)$. Thus, (c) holds and we are done. □

Proof of Lemma 17.3. We proceed as follows.

Step 1: Constructing auxiliary graphs. For each $i \in [4]$, we construct an auxiliary (undirected) graph H_i as follows. Let $i \in [4]$. Let W_i be the set of vertices $w \in U_i \cup U_{i-1}$ such that $d_{D[U_i,U_{i-1}]}(w) \geq r$. Observe that

$$W_i \subseteq U^\gamma(T) \overset{\text{(ES1),Fact 13.9}}{\subseteq} U^*. \tag{17.3}$$

Let W_i^1 and W_i^2 be two copies of W_i. For each $w \in W_i$, we denote by $w^1 \in W_i^1$ and $w^2 \in W_i^2$ the copies of w. For each $w \in W_i$, let $N_i^1(w) \cup N_i^2(w)$ be a partition of $N_{D[U_i,U_{i-1}]}(w)$ satisfying $|N_i^j(w)| \leq r$ for each $j \in [2]$ (this is possible by (ii)). By (ES2), $|U^*| \leq r$ and so we may assume that

$$U^* \cap N_i^2(w) = \emptyset \tag{17.4}$$

for each $w \in W_i$.

For each $i \in [4]$, let H_i be the (undirected) graph on $((U_i \cup U_{i-1}) \setminus W_i) \cup (W_i^1 \cup W_i^2)$ which contains all of the following edges, and no other edges.

- If $uv \in E(D[U_i \setminus W_i, U_{i-1} \setminus W_i])$, then $uv \in E(H_i)$.
- If $uv \in E(D[U_i \cap W_i, U_{i-1} \cap W_i])$, then $u^1v^1 \in E(H_i)$.
- Suppose that $uv \in E(D[U_i \cap W_i, U_{i-1} \setminus W_i])$. If $v \in N_i^1(u)$, then $u^1v \in E(H_i)$. Otherwise, $v \in N_i^2(u)$ and $u^2v \in E(H_i)$.
- Suppose that $uv \in E(D[U_i \setminus W_i, U_{i-1} \cap W_i])$. If $u \in N_i^1(v)$, then $uv^1 \in E(H_i)$. Otherwise, $u \in N_i^2(v)$ and $uv^2 \in E(H_i)$.

Claim 1. *For each $i \in [4]$, H_i is a bipartite graph which satisfies the following properties.*

(a) $V(H_i) = ((U_i \cup U_{i-1}) \setminus W_i) \cup (W_i^1 \cup W_i^2)$.

(b) $e(H_i) = e_D(U_i, U_{i-1})$.

(c) $\Delta(H_i) \leq r$.

(d) $e_{H_i}(W_i^2, W_i^1 \cup W_i^2) = 0$.

Proof of Claim. Let $i \in [4]$. One can easily verify that H_i is a bipartite graph on vertex classes $(U_i \setminus W_i) \cup \{w^j \mid j \in [2], w \in W_i \cap U_i\}$ and $(U_{i-1} \setminus W_i) \cup \{w^j \mid j \in [2], w \in W_i \cap U_{i-1}\}$. In particular, (a) holds. Moreover, there is a one-to-one correspondence between the edges of $D[U_i, U_{i-1}]$ and H_i and so (b) holds. For (c), observe that each $v \in (U_i \cup U_{i-1}) \setminus W_i$ satisfies $d_{H_i}(v) = d_{D[U_i,U_{i-1}]}(v) \leq r$, while each $w^1 \in W_i^1$ satisfies

$$d_{H_i}(w^1) = |N_{D[U_i,U_{i-1}]}(w) \cap W_i| + |N_i^1(w) \setminus W_i| \overset{(17.3),(17.4)}{=} |N_i^1(w)| \leq r.$$

Moreover, each $w^2 \in W_i^2$ satisfies $N_{H_i}(w^2) = N_i^2(w) \setminus W_i$ and so $d_{H_i}(w^2) \leq r$. Thus, (c) and (d) hold. $\diamond$

Step 2: Decomposing the auxiliary graphs. For each $i \in [4]$, apply Proposition 7.23 to decompose H_i into r edge-disjoint (undirected) matchings $M_{i,1}, \dots, M_{i,r}$ such that, for any $j, j' \in [r]$, $||M_{i,j}| - |M_{i,j'}|| \leq 1$ (this is possible by (c)). By (i) and (b), we may assume without loss of generality that, for each $i \in [r]$, we have

$$|M_{1,i}| = |M_{3,i}| \quad \text{and} \quad |M_{4,i}| = |M_{2,i}|. \tag{17.5}$$

Moreover, each $i \in [4]$ and $j \in [r]$ satisfy

$$|M_{i,j}| \overset{\text{Fact 10.5}}{\leq} \left\lceil \frac{\varepsilon n^2}{r} \right\rceil \leq \frac{\sqrt{\varepsilon} n}{4}. \tag{17.6}$$

Step 3: Decomposing D. Let $i \in [2]$ and $j \in [r]$. Define $W := W_i \cup W_{i+2}$, $W^1 := W_i^1 \cup W_{i+2}^1$, and $W^2 := W_i^2 \cup W_{i+2}^2$. Let $V' := V(T) \setminus W$. Note that Lemma 17.4(i)–(iii) are satisfied with $M_{i,j}$ and $M_{i+2,j}$ playing the roles of M_1 and M_2. Indeed, Lemma 17.4(i) follows from (17.5), while Lemma 17.4(ii) holds by (a). Moreover, Lemma 17.4(iii) follows from (a) and (d). Apply Lemma 17.4 with $M_{i,j}$ and $M_{i+2,j}$ playing the roles of M_1 and M_2 to obtain $M'_{i,j} \subseteq M_{i,j}$ and $M'_{i+2,j} \subseteq M_{i+2,j}$ satisfying Lemma 17.4(a)–(c). For each $i' \in \{i, i+2\}$, let $\widetilde{M}_{i',j}$ and $\widetilde{M}'_{i',j}$ be obtained from $M_{i',j}$ and $M'_{i',j}$ by replacing, for each $j \in [2]$, each $w^j \in W^j_{i'}$ by w, and then orienting all the edges from $U_{i'}$ to $U_{i'-1}$. By definition of H_i and H_{i+2}, we have $\widetilde{M}'_{i,j} \subseteq \widetilde{M}_{i,j} \subseteq E_D(U_i, U_{i-1})$ and $\widetilde{M}'_{i+2,j} \subseteq \widetilde{M}_{i+2,j} \subseteq E_D(U_{i+2}, U_{i+1})$.

For each $i \in [r]$, let

- $\mathcal{F}_i \coloneqq (\widetilde{M}_{1,i} \setminus \widetilde{M}'_{1,i}) \cup \widetilde{M}'_{2,i} \cup (\widetilde{M}_{3,i} \setminus \widetilde{M}'_{3,i}) \cup \widetilde{M}'_{4,i}$ and
- $\mathcal{F}_{r+i} \coloneqq \widetilde{M}'_{1,i} \cup (\widetilde{M}_{2,i} \setminus \widetilde{M}'_{2,i}) \cup \widetilde{M}'_{3,i} \cup (\widetilde{M}_{4,i} \setminus \widetilde{M}'_{4,i})$.

By definition, there is a one-to-one correspondence between the edges of $H_1 \cup \cdots \cup H_4$ and D. Thus, $\mathcal{F}_1, \dots, \mathcal{F}_{2r}$ are edge-disjoint and $D = \bigcup_{i \in [2r]} \mathcal{F}_i$. By (17.6), we have $e(\mathcal{F}_i) \leq \sqrt{\varepsilon} n$ for each $i \in [2r]$.

Step 4: Verifying (F1) and (F2′)–(F4′). Finally, we verify that $\mathcal{F}_1, \dots, \mathcal{F}_{2r}$ are (γ, T)-pseudo-feasible systems. Let $j \in [r]$. First, observe that

$$e_{\mathcal{F}_j}(U_1, U_4) = |\widetilde{M}_{1,j}| - |\widetilde{M}'_{1,j}| \overset{(17.5),\text{Lemma 17.4(a)}}{=} |\widetilde{M}_{3,j}| - |\widetilde{M}'_{3,j}| = e_{\mathcal{F}_j}(U_3, U_2)$$

and

$$e_{\mathcal{F}_j}(U_4, U_3) = |\widetilde{M}'_{4,j}| \overset{\text{Lemma 17.4(a)}}{=} |\widetilde{M}'_{2,j}| = e_{\mathcal{F}_j}(U_2, U_1).$$

Thus, (F1) holds. By Lemma 17.4(c), $\widetilde{M}'_{i,j'}$ and $\widetilde{M}_{i,j'} \setminus \widetilde{M}'_{i,j'}$ are matchings for each $i \in [4]$ and $j' \in [r]$. Therefore, $\Delta^0(\mathcal{F}_{j'}) \leq 1$ for each $j' \in [2r]$. In particular, (F3′) holds for $\mathcal{F}_j$. Moreover, (iii) implies that each $v \in U^{1-\gamma}(T)$ satisfies $d^+_{\mathcal{F}_{j'}}(v) = 1 = d^-_{\mathcal{F}_{j'}}(v)$ for each $j' \in [2r]$. In particular, (F2′) holds for $\mathcal{F}_j$. For (F4′), suppose that C is a cycle in $\mathcal{F}_j$. By Lemma 17.4(b), there exists $i \in \{2, 4\}$ such that all the edges in $M'_{i,j}$ are incident to W^2_i. Since $C \subseteq D \subseteq \overleftarrow{T}_{\mathcal{U}}$, there exists $e \in E_C(U_i, U_{i-1}) \subseteq \widetilde{M}'_{i,j}$. Denote by e' the edge of $M'_{i,j}$ which witnesses that $e \in \widetilde{M}'_{i,j}$. By assumption, e' is incident to W^2_i, say $e' = w^2 v$ for some

$w^2 \in W_i^2$ (similar arguments hold if the ending point of e' is in W_i^2). By (d), we have $v \in U_{i-1} \setminus W_i$ and so, by construction, $e = wv$ with $w \in W_i \cap U_i$ and $v \in N_i^2(w)$. Therefore, (17.3) and (17.4) imply that e is a backward edge from $U^\gamma(T) \cap U^*$ to $V(T) \setminus U^*$. Thus, e is a (γ, T)-placeholder and so (F4′) holds. Therefore, $\mathcal{F}_j$ is a (γ, T)-pseudo-feasible system and, by similar arguments, $\mathcal{F}_{r+j}$ is also a (γ, T)-pseudo-feasible system. $\square$

CHAPTER 18

CONSTRUCTING A FEW SPECIAL (PSEUDO)-FEASIBLE SYSTEMS: PROOFS OF LEMMAS 17.1 AND 17.2

Finally, we show how to construct a few (pseudo)-feasible systems which satisfy some special properties: in Lemma 17.1, we construct a few feasible systems out of prescribed sets of edges and, in Lemma 17.2, we construct a few pseudo-feasible systems which incorporate a given set of exceptional edges.

18.1 Proof overview

Each (pseudo)-feasible system $\mathcal{F}$ in Lemmas 17.1 and 17.2 will be constructed using the following approach. Let T be a regular bipartite tournament. Let $\mathcal{U} = (U_1, \ldots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T and let U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T.

Step 1: Selecting the forward edges which are incident to $U^{1-\gamma}(T)$. First, we fix the forward edges incident to $U^{1-\gamma}(T)$ that we want $\mathcal{F}$ to cover. We make sure that these edges form a linear forest $\mathcal{F}^1$. (Note that this step will be void in the proof of Lemma 17.1, all these forward edges will be covered in Lemma 17.2.)

Step 2: Covering $U^{1-\gamma}(T)$. Then, to ensure that (F2′) is satisfied, we add backward edges to $\mathcal{F}^1$ to cover all the uncovered vertices in $U^{1-\gamma}(T)$. Since the vertices in $U^{1-\gamma}(T)$ have, by definition, very high backward degree, we can do so greedily and in such a way that we still have a linear forest $\mathcal{F}^2$.

Step 3: Balancing the number of backward edges. Next, we construct a linear forest $\mathcal{F}^3$ by adding to $\mathcal{F}^2$ precisely $m^{i\downarrow} := e_{\mathcal{F}^2}(U_{i-2}, U_{i-3})$ additional backward edges of $T(U_i, U_{i-1})$ for each $i \in [4]$ (where the superscript i is taken modulo 4). Then, $e_{\mathcal{F}^3}(U_i, U_{i-1}) = m^{(i+2)\downarrow} + m^{i\downarrow}$ for each $i \in [4]$ and so (F1) holds.

These new edges will be selected using König's theorem as follows. First, we find a subdigraph $H \subseteq \overleftarrow{T}_{\mathcal{U}} - U^{1-\gamma}(T)$ which contains many edges but has small maximum degree. (In practice, H is already given in Lemma 17.1 (see Lemma 13.12(iii) and (iv)). For Lemma 17.2, H will be constructed using Lemma 13.8.) Then, we can use Proposition 7.22 to find, for each $i \in [4]$, a matching of size $m^{i\downarrow}$ in $H(U_i, U_{i-1})$ which avoids all the vertices in $\mathcal{F}^2$.

Step 4: Adding forward edges incident to $U^* \setminus U^{1-\gamma}(T)$. Finally, we add to $\mathcal{F}^3$ a few extra forward edges incident to $U^* \setminus U^{1-\gamma}(T)$. We do this in such a way that the resulting digraph $\mathcal{F}$ is still a linear forest.

For Lemma 17.1, this is necessary because we want $\mathcal{F}$ to be a feasible system and so U^* needs to be entirely covered by in- and outedges (see (F2)). This can be done greedily since the vertices in $U^* \setminus U^{1-\gamma}(T)$ have high forward degree (see Lemma 13.12(ii)).

For Lemma 17.2, we only need a pseudo-feasible system and so U^* need not be entirely covered (see (F2′)). However, recall that we will need to incorporate all the forward edges of T which lie inside U^* (see Lemma 17.2(a)). We will thus add a few of these to $\mathcal{F}^3$ at this stage (and distribute the remaining such edges to the other pseudo-feasible systems).

In practice, all the (pseudo)-feasible systems in Lemmas 17.1 and 17.2 will be constructed in parallel. In particular, for Lemma 17.2, we will decompose all the forward edges of T which are incident to $U^{1-\gamma}(T)$ at the start of the proof (this corresponds to Step 1 above) and the other exceptional forward edges of T will be distributed greedily at the end of the proof (this corresponds to Step 4).

18.2 Selecting backward edges

Steps 2 and 3 from Section 18.1 are combined into the following lemma. Let T be a regular bipartite tournament. Let $\mathcal{U} = (U_1, \dots, U_4)$ be an optimal $(\varepsilon, 4)$-partition for T and let U^* be an $(\varepsilon, \mathcal{U})$-exceptional set for T. Let $H \subseteq \overleftarrow{T}_{\mathcal{U}}$ contain many well distributed backward edges. Roughly speaking, Lemma 18.1 states that H contains edge-disjoint linear forests $\mathcal{F}_1, \dots, \mathcal{F}_\ell$, where each $\mathcal{F}_j(U_i, U_{i-1})$ covers all vertices of $U_i^{1-\gamma}(T) \cup U_{i-1}^{1-\gamma}(T)$ apart from those in a given prescribed set $S_j^{i\downarrow}$ of vertices to avoid and contains a prescribed number $m_j^{i\downarrow}$ of additional edges (see Lemma 18.1(a)). Moreover, these linear forests can be constructed in such a way that every vertex of $V(T) \setminus U^{1-\gamma}(T)$ is not covered by too many of the linear forests (see Lemma 18.1(b)) and is adjacent to at most one edge in each linear forest (see Lemma 18.1(c)).

In our applications, the sets $S_j^{i\downarrow}$ of vertices to avoid will consist of the vertices which are already covered at the end of Step 1 from Section 18.1 and the constants $m_j^{i\downarrow}$ will be chosen as described in Step 3 from Section 18.1 in order to balance the number of backward edges in each pair of the blow-up C_4.

As explained in Section 18.1, the backward edges will be chosen in two stages, depending on whether they are incident to $U^{1-\gamma}(T)$ or not. However, we will swap the order of Steps 2 and 3. That is, we will select the backward edges incident to $U^{1-\gamma}(T)$ only after the other backward edges have been selected. This is because it is much easier to select backward edges incident to $U^{1-\gamma}(T)$ (the vertices in $U^{1-\gamma}(T)$ have high backward degree and so can be covered greedily).

Lemma 18.1 (Selecting backward edges). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \gamma \ll 1$ and $\ell \leq \gamma n$. Let T be a regular bipartite tournament on $4n$ vertices and $\mathcal{U} = (U_1, \dots, U_4)$ be an $(\varepsilon, 4)$-partition for T. Suppose that $H \subseteq \overleftarrow{T}$ satisfies the following.*

(i) *For each $v \in U^{1-\gamma}(T)$, $d_H^{\pm}(v) \geq 2\gamma n$.*

(ii) *For each $v \in V(T) \setminus U^{1-\gamma}(T)$, $d_H^{\pm}(v) \leq 2\gamma n$.*

(iii) *For each* $i \in [4]$, $e_{H-U^{1-\gamma}(T)}(U_i, U_{i-1}) \geq 109\gamma n|U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$.

For each $i \in [4]$ *and* $j \in [\ell]$, *let* $S_j^{i\downarrow} \subseteq U_i \cup U_{i-1}$ *and* $m_j^{i\downarrow} \in \mathbb{N}$ *satisfy the following.*

(iv) $|S_j^{i\downarrow} \setminus U^{1-\gamma}(T)| \leq |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$.

(v) $m_j^{i\downarrow} \leq |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$.

Then, H *contains edge-disjoint linear forests* $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ *such that the following hold, where* $\mathcal{F} \coloneqq \bigcup_{i \in [\ell]} \mathcal{F}_i$.

(a) *For each* $j \in [\ell]$, $\mathcal{F}_j$ *consists of*

(α) *a matching of* $H_j^{i\downarrow} \coloneqq H(U_i \setminus (U^{1-\gamma}(T) \cup S_j^{i\downarrow}), U_{i-1} \setminus (U^{1-\gamma}(T) \cup S_j^{i\downarrow}))$ *of size* $m_j^{i\downarrow}$ *for each* $i \in [4]$;

(β) *a matching of* $H(U_i^{1-\gamma}(T) \setminus S_j^{i\downarrow}, U_{i-1} \setminus (U^{1-\gamma}(T) \cup S_j^{i\downarrow}))$ *of size* $|U_i^{1-\gamma}(T) \setminus S_j^{i\downarrow}|$ *for each* $i \in [4]$; *and*

(γ) *a matching of* $H(U_i \setminus (U^{1-\gamma}(T) \cup S_j^{i\downarrow}), U_{i-1}^{1-\gamma}(T) \setminus S_j^{i\downarrow})$ *of size* $|U_{i-1}^{1-\gamma}(T) \setminus S_j^{i\downarrow}|$ *for each* $i \in [4]$.

(b) *For each* $v \in V(T) \setminus U^{1-\gamma}(T)$, *we have* $d_{\mathcal{F}}(v) \leq \frac{\gamma n}{6}$.

(c) *For each* $i \in [\ell]$ *and* $v \in V(T) \setminus U^{1-\gamma}(T)$, *we have* $d_{\mathcal{F}_i}(v) \leq 1$.

Proof. We will first select the edges which are not incident to $U^{1-\gamma}(T)$ (i.e. those in (a.α)) as follows.

Claim 1. H *contains edge-disjoint linear forests* $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ *such that the following hold, where* $\mathcal{Q} \coloneqq \bigcup_{i \in [\ell]} \mathcal{Q}_i$.

(a′) *For each* $j \in [\ell]$, $\mathcal{Q}_j$ *consists of a matching of* $H_j^{i\downarrow}$ *of size* $m_j^{i\downarrow}$ *for each* $i \in [4]$.

(b′) *For each* $v \in V(T) \setminus U^{1-\gamma}(T)$, *we have* $d_{\mathcal{Q}}(v) \leq \frac{\gamma n}{6}$.

(c′) *For each* $i \in [\ell]$ *and* $v \in V(T) \setminus U^{1-\gamma}(T)$, *we have* $d_{\mathcal{Q}_i}(v) \leq 1$.

First, we assume that Claim 1 holds and select the edges incident to $U^{1-\gamma}(T)$ (i.e. those in (a.β) and (a.γ)) using Lemma 16.1 as follows. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ be the edge-disjoint linear forests obtained by applying Claim 1 and denote $\mathcal{Q} := \bigcup_{i\in[\ell]} \mathcal{Q}_i$. If $U^{1-\gamma}(T) = \emptyset$, then let $\mathcal{F}_i := \mathcal{Q}_i$ for each $i \in [\ell]$ and observe that (a)–(c) follow from (a′)–(c′). We may therefore assume that $U^{1-\gamma}(T) \neq \emptyset$.

First, note that each $i \in [\ell]$ satisfies

$$|\mathcal{Q}_i| \leq 2e(\mathcal{Q}_i) \overset{\text{(v),(a')}}{\leq} 4|U^{1-\gamma}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} 16\varepsilon n. \tag{18.1}$$

Let X be the set of vertices $v \in V(T) \setminus U^{1-\gamma}(T)$ such that $d_{\mathcal{Q}}(v) \geq \frac{\gamma n}{7}$ (so X is the set of vertices which are already covered by many of the linear forests $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$). Note that

$$|X| \overset{(18.1),(c')}{\leq} \frac{7 \cdot 16\varepsilon n\ell}{\gamma n} \leq 112\varepsilon n. \tag{18.2}$$

For each $j \in [\ell]$, denote by

$$S_j := \bigcup_{i\in[4]} (S_j^{i\downarrow} \setminus U^{1-\gamma}(T))$$

the set of vertices outside $U^{1-\gamma}(T)$ that need to be avoided by $\mathcal{F}_j$ and observe that

$$|S_j| \leq \sum_{i\in[4]} |S_j^{i\downarrow} \setminus U^{1-\gamma}(T)| \overset{\text{(iv)}}{\leq} 2|U^{1-\gamma}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} 8\varepsilon n. \tag{18.3}$$

Let Y be the set of vertices $v \in V(T) \setminus U^{1-\gamma}(T)$ for which there exist at least $\frac{\gamma n}{7}$ indices $i \in [\ell]$ such that $v \in S_i$ (so Y is the set vertices which need to be avoided by many of the linear forests $\mathcal{F}_1, \dots, \mathcal{F}_\ell$). Note that

$$|Y| \overset{(18.3)}{\leq} \frac{7 \cdot 8\varepsilon n\ell}{\gamma n} \leq 56\varepsilon n. \tag{18.4}$$

Let $A := U^{1-\gamma}(T)$ and $B := V(T) \setminus (A \cup X \cup Y)$. Let D be the bipartite digraph on vertex classes A and B induced by H. For each $j \in [\ell]$, let $S_j' := V(E(\mathcal{Q}_j)) \cup S_j$ and define

$$T_j^+ := \bigcup_{i\in[4]} U_i^{1-\gamma}(T) \setminus S_j^{i\downarrow} \quad \text{and} \quad T_j^- := \bigcup_{i\in[4]} U_{i-1}^{1-\gamma}(T) \setminus S_j^{i\downarrow}.$$

By (a′), $V(\mathcal{Q}_i) \cap U^{1-\gamma}(T) = \emptyset$ for each $i \in [\ell]$. Thus, note for later that both

$$T_i^{\pm} \cap V(\mathcal{Q}_i) = \emptyset \quad \text{and} \quad V(E(\mathcal{Q}_i)) \cap B \subseteq S_i' \tag{18.5}$$

for each $i \in [\ell]$. Define $N := 2|U^{1-\gamma}(T)| = 2|A|$. For each $v \in B$, let n_v denote the number of indices $i \in [\ell]$ such that $v \in S_i'$.

We verify that (16.1) holds with $T_1^+, \dots, T_\ell^+, T_1^-, \dots, T_\ell^-$, and $S_1', \dots, S_\ell'$ playing the roles of $S_1^+, \dots, S_\ell^+, S_1^-, \dots, S_\ell^-$, and $T_1, \dots, T_\ell$. For each $i \in [\ell]$, we have

$$|T_i^+| + |T_i^-| \leq 2|U^{1-\gamma}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} 8\varepsilon n \tag{18.6}$$

and

$$|S_i'| \leq |\mathcal{Q}_i| + |S_i| \overset{(18.1),(18.3)}{\leq} 24\varepsilon n. \tag{18.7}$$

By definition of X and Y, each $v \in B \subseteq V(T) \setminus (U^{1-\gamma}(T) \cup X \cup Y)$ satisfies

$$n_v \leq d_{\mathcal{Q}}(v) + |\{i \in [\ell] \mid v \in S_i\}| \leq \frac{\gamma n}{7} + \frac{\gamma n}{7} = \frac{2\gamma n}{7}. \tag{18.8}$$

Thus, (18.2), (18.4), and (18.6)–(18.8) imply that each $v \in U^{1-\gamma}(T)$ satisfies

$$\begin{aligned} d_D^{\pm}(v) &\overset{\text{(i)}}{\geq} 2\gamma n - |U^{1-\gamma}(T)| - |X| - |Y| \\ &\overset{\text{Fact 13.9,Definition 13.11}}{\geq} \ell + 2\max_{j \in [\ell]}(|T_j^+| + |T_j^-| + |S_j'|) + 2(\max_{w \in B} n_w + N). \end{aligned}$$

Thus, (16.1) holds with $T_1^+, \dots, T_\ell^+, T_1^-, \dots, T_\ell^-$, and $S_1', \dots, S_\ell'$ playing the roles of $S_1^+, \dots, S_\ell^+, S_1^-, \dots, S_\ell^-$, and $T_1, \dots, T_\ell$.

Let $\mathcal{Q}_1', \dots, \mathcal{Q}_\ell'$ be the edge-disjoint linear forests obtained by applying Lemma 16.1 with $T_1^+, \dots, T_\ell^+, T_1^-, \dots, T_\ell^-$, and $S_1', \dots, S_\ell'$ playing the roles of $S_1^+, \dots, S_\ell^+, S_1^-, \dots, S_\ell^-$, and $T_1, \dots, T_\ell$. For each $i \in [\ell]$, denote $\mathcal{F}_i := \mathcal{Q}_i \cup \mathcal{Q}_i'$. Let $\mathcal{F} := \bigcup_{i \in [\ell]} \mathcal{F}_i$. We claim that $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ are edge-disjoint linear forests which satisfy (a)–(c). By Claim 1, $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ are edge-disjoint linear forests. By (a′), their edges are not adjacent to $U^{1-\gamma}(T)$ and so

$\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$ are edge-disjoint from D. Therefore, (18.5) and the "in particular part" of Lemma 16.1 implies that $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ are edge-disjoint linear forests. Moreover, (a) holds by (a′) and Lemma 16.1(a). For (b), let $v \in V(T) \setminus U^{1-\gamma}(T)$. If $v \in X$, then recall that $X \cap V(D) = \emptyset$ and so (b′) implies that $d_{\mathcal{F}}(v) = d_{\mathcal{Q}}(v) \leq \frac{\gamma n}{6}$, as desired. Otherwise, the definition of X implies that

$$d_{\mathcal{F}}(v) \leq d_{\mathcal{Q}}(v) + d_D(v) \leq \frac{\gamma n}{7} + 2|U^{1-\gamma}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} \frac{\gamma n}{7} + 8\varepsilon n \leq \frac{\gamma n}{6},$$

so (b) holds. For (c), let $i \in [\ell]$ and $v \in V(T) \setminus U^{1-\gamma}(T)$. If $d_{\mathcal{Q}_i}(v) = 0$, then Lemma 16.1(c) implies that $d_{\mathcal{F}_i}(v) \leq 1$. Otherwise, Lemma 16.1(a) and the definition of S_i' imply that $d_{\mathcal{Q}_i'}(v) = 0$ and so (c′) implies that $d_{\mathcal{F}_i}(v) \leq 1$. Thus, (c) holds.

Proof of Claim 1. We will construct, for each $i \in [4]$ and $j \in [\ell]$, a matching $M_j^{i\downarrow} \subseteq H_j^{i\downarrow}$ of size $m_j^{i\downarrow}$. Then, we will let each $\mathcal{Q}_j$ consist of the union $\bigcup_{i \in [4]} M_j^{i\downarrow}$. In this way, (a′) will be automatically satisfied.

These matchings will be constructed one by one using Proposition 7.22 as follows. Suppose that we want to construct $M_j^{i\downarrow}$ for some $i \in [4]$ and $j \in [\ell]$, and suppose furthermore that we have already constructed $M_j^{i'\downarrow}$ for some $i' \in [4] \setminus \{i\}$. Then, in order to satisfy (c′), we need to avoid the vertices in $V(M_j^{i'\downarrow})$. Thus, in order to minimise the number of vertices we have to avoid at each stage (and thus maximise the number of available edges for each matching), we will construct the matchings in ascending size order.

Let $\emptyset =: X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{4\ell} := [4] \times [\ell]$ be such that, for each $k \in [4\ell]$, $(i,j) \in X_k \setminus X_{k-1}$, and $(i',j') \in X_{k-1}$, we have $m_{j'}^{i'\downarrow} \leq m_j^{i\downarrow}$. Note that $|X_k \setminus X_{k-1}| = 1$ for each $k \in [4\ell]$.

Suppose inductively that, for some $0 \leq k \leq 4\ell$, we have constructed a set $\mathcal{M}_k = \{M_j^{i\downarrow} \mid (i,j) \in X_k\}$ of edge-disjoint matchings such that the following hold.

(a″) For each $(i,j) \in X_k$, $M_j^{i\downarrow}$ is a matching of $H_j^{i\downarrow}$ of size $m_j^{i\downarrow}$.

(b″) For each $v \in V(T)$, we have $\sum_{(i,j) \in X_k} d_{M_j^{i\downarrow}}(v) \leq \frac{\gamma n}{6}$.

(c″) For each $j \in [\ell]$ and $v \in V(T) \setminus U^{1-\gamma}(T)$, we have $\sum_{i\colon (i,j)\in X_k} d_{M_j^{i\downarrow}}(v) \leq 1$.

First, suppose that $k = 4\ell$. For each $j \in [\ell]$, let $\mathcal{Q}_j := \bigcup_{i\in[4]} M_j^{i\downarrow}$. Then, (a′)–(c′) follow from (a″)–(c″).

We may therefore assume that $k < 4\ell$. Let $(i,j) \in X_{k+1} \setminus X_k$ and let H' be obtained from $H(U_i \setminus U^{1-\gamma}(T), U_{i-1} \setminus U^{1-\gamma}(T))$ by deleting all the edges in $\bigcup \mathcal{M}_k$. The matching $M_j^{i\downarrow}$ will be constructed in H' using Proposition 7.22. By definition of $X_0, \dots, X_{4\ell}$, we have

$$\begin{aligned}
e(H') &\overset{(a'')}{=} e_{H-U^{1-\gamma}(T)}(U_i, U_{i-1}) - \sum_{j'\colon (i,j')\in X_k} |M_{j'}^{i\downarrow}| \\
&\overset{(a'')}{=} e_{H-U^{1-\gamma}(T)}(U_i, U_{i-1}) - \sum_{j'\colon (i,j')\in X_k} m_{j'}^{i\downarrow} \geq e_{H-U^{1-\gamma}(T)}(U_i, U_{i-1}) - \ell m_j^{i\downarrow} \\
&\overset{(iii),(v)}{\geq} 108\gamma n |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|. \qquad (18.9)
\end{aligned}$$

We will now list and count all the vertices that $M_j^{i\downarrow}$ needs to avoid. For (a″), we will need to avoid the vertices in $S_j^{i\downarrow} \setminus U^{1-\gamma}(T)$, where

$$|S_j^{i\downarrow} \setminus U^{1-\gamma}(T)| \overset{(iv)}{\leq} |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|. \qquad (18.10)$$

(The vertices in $S_j^{i\downarrow} \cap U^{1-\gamma}(T)$ will automatically be avoided since $V(H') \cap U^{1-\gamma}(T) = \emptyset$.) For (b″), we will need to avoid the set of vertices $Y_j^{i\downarrow}$ of vertices $v \in V(H')$ for which $\sum_{(i',j')\in X_k} d_{M_{j'}^{i'\downarrow}}(v) = \lfloor \frac{\gamma n}{6} \rfloor$ where, by definition of $X_0, \dots, X_{4\ell}$,

$$\begin{aligned}
|Y_j^{i\downarrow}| &\leq \frac{\sum_{(i',j')\in X_k} |V(M_{j'}^{i'\downarrow})|}{\lfloor \frac{\gamma n}{6} \rfloor} \overset{(a'')}{=} \frac{2\sum_{(i',j')\in X_k} m_{j'}^{i'\downarrow}}{\lfloor \frac{\gamma n}{6} \rfloor} \leq \frac{2 \cdot 4\ell \cdot m_j^{i\downarrow}}{\lfloor \frac{\gamma n}{6} \rfloor} \\
&\overset{(v)}{\leq} 49 |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|. \qquad (18.11)
\end{aligned}$$

Finally, for (c″), we will need to avoid all the vertices in $Z_j^{i\downarrow} := \bigcup_{i'\colon (i',j)\in X_k} (V(M_j^{i'\downarrow}) \cap V(H'))$

where, by definition of $X_0, \dots, X_{4\ell}$,

$$\begin{aligned}|Z_j^{i\downarrow}| &\leq \sum_{i' \colon (i',j) \in X_k} \left| V(M_j^{i'\downarrow}) \cap (U_i \cup U_{i-1}) \right| \overset{\text{(a'')}}{\leq} \sum_{i' \colon (i',j) \in X_k} m_j^{i'\downarrow} \leq 3m_j^{i\downarrow} \\ &\overset{\text{(v)}}{\leq} 3|U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|. \end{aligned} \tag{18.12}$$

Let $\widetilde{S}_j^{i\downarrow}$ consist of all the above mentioned vertices, that is, let

$$\widetilde{S}_j^{i\downarrow} := (S_j^{i\downarrow} \setminus U^{1-\gamma}(T)) \cup Y_j^{i\downarrow} \cup Z_j^{i\downarrow}.$$

Let $\widetilde{H}_j^{i\downarrow} := H' - \widetilde{S}_j^{i\downarrow} \subseteq H_j^{i\downarrow}$. By (18.10)–(18.12), we have $|\widetilde{S}_j^{i\downarrow}| \leq 53|U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$ and so

$$e(\widetilde{H}_j^{i\downarrow}) \geq e(H') - \Delta(H')|\widetilde{S}_j^{i\downarrow}| \overset{\text{(ii),(18.9)}}{\geq} 2\gamma n|U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)| \overset{\text{(ii),(v)}}{\geq} \Delta(\widetilde{H}_j^{i\downarrow})m_j^{i\downarrow}.$$

Thus, Proposition 7.22 (applied with the undirected graph underlying $\widetilde{H}_j^{i\downarrow}$ playing the role of G) implies that $\widetilde{H}_j^{i\downarrow}$ contains a matching $M_j^{i\downarrow}$ of size $m_j^{i\downarrow}$. One can easily verify that (a'')–(c'') hold with $k+1$ playing the role of k. ◇

This completes the proof of Lemma 18.1. □

18.3 Proofs of Lemmas 17.1 and 17.2

We are now ready to prove Lemmas 17.1 and 17.2, using the arguments described in Section 18.1.

Proof of Lemma 17.1. Denote $t_0 := 0 =: s_0$. Suppose inductively that, for some $0 \leq k \leq s$, we have constructed edge-disjoint feasible systems $\mathcal{F}_1, \dots, \mathcal{F}_{t_k+s_k}$ such that the following hold.

(α) For each $i \in [4]$ and $j \in [t_k + s_k]$, we have $e_{\mathcal{F}_j - U^{1-\gamma}(T)}(U_i, U_{i-1}) = |U_{i-2}^{1-\gamma}(T) \cup U_{i-3}^{1-\gamma}(T)|$.

(β) For each $i \in [t_k + s_k]$, $e(\mathcal{F}_i) \leq \sqrt{\varepsilon} n$.

(γ) For each $i \in [t_k + s_k]$ and $v \in V(T) \setminus U^*$, $d_{\mathcal{F}_i}(v) \leq 1$.

(δ) For each $i \in [k]$ and $j \in [s_i]$, $\mathcal{F}_{t_i+j} \subseteq H_i$.

First, suppose that $k = s$. Then, (a), (b), and (d) follow from (α), (β), and (δ), respectively. Let $i \in [t]$. By (F2), $U^* \subseteq V^0(\mathcal{F}_i)$ and, by (γ), $V^0(\mathcal{F}_i) \subseteq U^*$. Therefore, (c) holds.

We may therefore assume that $k < s$. Note that $t_k + s_k = t_{k+1}$. We will now construct $\mathcal{F}_{t_{k+1}+1}, \dots, \mathcal{F}_{t_{k+1}+s_{k+1}}$. Let $H \coloneqq H_{k+1} \setminus (H_{k+1}[U^{1-\gamma}(T)] \cup \bigcup_{i \in [t_{k+1}]} \mathcal{F}_i)$.

Step 1: Selecting backward edges. First, we show that Lemma 18.1(i)–(iii) hold with $\overleftarrow{H}_{\mathcal{U}}$ playing the role of H. Note that Lemma 18.1(ii) follows immediately from Lemma 13.12(iii). For each $v \in U^{1-\gamma}(T)$, we have

$$\begin{aligned} \overleftarrow{d}^{\pm}_{H,\mathcal{U}}(v) &\overset{\text{(F3)}}{\geq} \overleftarrow{d}^{\pm}_{H_{k+1},\mathcal{U}}(v) - |\overleftarrow{N}^{\pm}_{H_{k+1},\mathcal{U}}(v) \cap U^{1-\gamma}(T)| - t_{k+1} \\ &\overset{\text{Lemma 13.12(i)}}{\geq} 3\gamma n - |U^{1-\gamma}(T)| - \eta n \overset{\text{Fact 13.9,Definition 13.11}}{\geq} 2\gamma n. \end{aligned}$$

Thus, Lemma 18.1(i) holds. For each $i \in [4]$, we have

$$\begin{aligned} e_{H-U^{1-\gamma}(T)}(U_i, U_{i-1}) &= e_{H_{k+1}-U^{1-\gamma}(T)}(U_i, U_{i-1}) \\ &\qquad - \sum_{j \in [t_{k+1}]} e_{\mathcal{F}_j - U^{1-\gamma}(T)}(U_i, U_{i-1}) \\ &\overset{(\alpha),\text{Lemma 13.12(iv)}}{\geq} 109\gamma n |U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)|. \end{aligned}$$

Thus, Lemma 18.1(iii) is satisfied.

For each $i \in [4]$ and $j \in [s_{k+1}]$, let $S_j^{i\downarrow} \coloneqq \emptyset$ and

$$m_j^{i\downarrow} \coloneqq |U^{1-\gamma}_{i-2}(T) \cup U^{1-\gamma}_{i-3}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} 2\varepsilon n. \tag{18.13}$$

Then, Lemma 18.1(iv) and (v) hold with s_{k+1} playing the role of ℓ. Let $\mathcal{F}'_{t_{k+1}+1}, \dots, \mathcal{F}'_{t_{k+1}+s_{k+1}}$ be the edge-disjoint linear forests obtained by applying Lemma 18.1 with $\overleftarrow{H}_{\mathcal{U}}$

and s_{k+1} playing the roles of H and ℓ.

Step 2: Covering U^*. We now add forward edges to $\mathcal{F}'_{t_{k+1}+1}, \dots, \mathcal{F}'_{t_{k+1}+s_{k+1}}$ to ensure that (F2) is satisfied. We will use Lemma 16.1 as follows. Let $A \coloneqq U^*$ and $B \coloneqq V(T) \setminus U^*$. Let D be the bipartite digraph on vertex classes A and B induced by $\overrightarrow{H}_{\mathcal{U}}$. For each $i \in [s_{k+1}]$, let $S_i^\pm$ be the set of vertices $v \in U^*$ which satisfy $d^\pm_{\mathcal{F}'_{t_{k+1}+i}}(v) = 0$ (so S_i^+ and S_i^- list the vertices in U^* which are not yet covered with an out/inedge in $\mathcal{F}'_{t_{k+1}+i}$) and define $T_i \coloneqq V(E(\mathcal{F}'_{t_{k+1}+i})) \cap B$. Note for later that both

$$S_i^\pm \cap V(\mathcal{F}'_{t_{k+1}+i}) \subseteq V^\mp(\mathcal{F}'_{t_{k+1}+i}) \quad \text{and} \quad V(E(\mathcal{F}'_{t_{k+1}+i}) \cap B \subseteq T_i \tag{18.14}$$

for each $i \in [s_{k+1}]$. Define $N \coloneqq 2|U^*| = 2|A|$. For each $v \in B$, let n_v denote the number of indices $i \in [s_{k+1}]$ such that $v \in T_i$.

We verify that (16.1) holds with s_{k+1} playing the role of ℓ. For each $i \in [s_{k+1}]$, we have

$$|S_i^+| + |S_i^-| \leq 2|U^*| \overset{\text{(ES2)}}{\leq} 8\varepsilon n \tag{18.15}$$

and

$$|T_i| \leq |V(E(\mathcal{F}'_{t_{k+1}+i}))| \overset{\text{Lemma 18.1(a),(18.13)}}{\leq} 4|U^{1-\gamma}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} 16\varepsilon n. \tag{18.16}$$

Moreover, each $v \in B = V(T) \setminus U^*$ satisfies

$$n_v \leq \sum_{i \in [s_{k+1}]} d_{\mathcal{F}'_{t_{k+1}+i}}(v) \overset{\text{Lemma 18.1(a)}}{\leq} \overleftarrow{d}_{H,\mathcal{U}}(v) \overset{\text{(ES1)}}{\leq} 2\varepsilon n. \tag{18.17}$$

Therefore, each $v \in U^* \setminus U^{1-\gamma}(T)$ satisfies

$$\begin{aligned}
d_D^{\pm}(v) &= \overrightarrow{d}^{\pm}_{H_{k+1},\mathcal{U}}(v) - |\overrightarrow{N}^{\pm}_{H_{k+1},\mathcal{U}}(v) \cap U^*| - \sum_{j\in[t_{k+1}]} \overrightarrow{d}^{\pm}_{\mathcal{F}_j}(v) \\
&\overset{\text{Lemma 13.12(ii),(F3)}}{\geq} \gamma^2 n - |U^*| - t_{k+1} \overset{\text{(ES2)}}{\geq} \gamma^3 n \\
&\overset{\text{(ES2),(18.15)–(18.17)}}{\geq} s_{k+1} + 2 \max_{i\in[s_{k+1}]} (|S_i^+| + |S_i^-| + |T_i|) \\
&\qquad\qquad + 2(\max_{w\in B} n_w + N). \qquad (18.18)
\end{aligned}$$

By Lemma 18.1(a.β) and (a.γ), each vertex in $U^{1-\gamma}(T)$ is already covered with both an in- and an outedge in each of the linear forests $\mathcal{F}'_{t_{k+1}+1}, \dots, \mathcal{F}'_{t_{k+1}+s_{k+1}}$, so $S_i^+ \cup S_i^- \subseteq U^* \setminus U^{1-\gamma}(T)$ for each $i \in [s_{k+1}]$. Thus, (16.1) follows from (18.18).

Let $\mathcal{Q}_1, \dots, \mathcal{Q}_{s_{k+1}}$ be the edge-disjoint linear forests obtained by applying Lemma 16.1 with s_{k+1} playing the role of ℓ. For each $i \in [s_{k+1}]$, denote $\mathcal{F}_{t_{k+1}+i} \coloneqq \mathcal{F}'_{t_{k+1}+i} \cup \mathcal{Q}_i$.

Step 3: Verifying (α)–(δ). We now check that $\mathcal{F}_{t_{k+1}+1}, \dots, \mathcal{F}_{t_{k+1}+s_{k+1}}$ are edge-disjoint feasible systems and that (α)–(δ) hold with $k+1$ playing the role of k. By Lemma 18.1, $\mathcal{F}'_{t_{k+1}+1}, \dots, \mathcal{F}'_{t_{k+1}+s_{k+1}}$ are edge-disjoint linear forests and, by Lemma 18.1(a), $\mathcal{F}'_{t_{k+1}+1}, \dots, \mathcal{F}'_{t_{k+1}+s_{k+1}}$ consist of backward edges and so are edge-disjoint from D. Thus, (18.14) and the "in particular part" of Lemma 16.1 imply that $\mathcal{F}_{t_{k+1}+1}, \dots, \mathcal{F}_{t_{k+1}+s_{k+1}}$ are edge-disjoint linear forests. Moreover, (α) holds by Lemma 18.1(a), (18.13), and the fact that Step 2 involves only forward edges. Note that each $j \in [s_{k+1}]$ satisfies

$$\begin{aligned}
e(\mathcal{F}_{t_{k+1}+j}) &\leq e(\mathcal{F}'_{t_{k+1}+j}) + e(\mathcal{Q}_j) \\
&\overset{\text{Lemma 18.1(a),Lemma 16.1(a)}}{\leq} \left(\sum_{i\in[4]} m_j^{i\downarrow} + 2|U^{1-\gamma}(T)|\right) + (|S_j^+| + |S_j^-|) \\
&\overset{\text{(18.13),(18.15)}}{\leq} 4|U^{1-\gamma}(T)| + 2|U^*| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} \sqrt{\varepsilon} n,
\end{aligned}$$

so (β) holds. Recall from Fact 13.9 and Definition 13.11 that $U^{1-\gamma}(T) \subseteq U^*$. Thus, Lemmas 18.1(c) and 16.1(c) imply that each $v \in V(T) \setminus U^*$ and $i \in [s_{k+1}]$ satisfy both $d_{\mathcal{F}_{t_{k+1}+i}}(v) \leq 1$ and $d_{\mathcal{Q}_i}(v) \leq 1$. Therefore, Lemma 16.1(a) and the definition of $T_1, \dots, T_{s_{k+1}}$ imply that (γ) holds. Moreover, (δ) follows from Lemma 18.1(a) and Lemma 16.1(a).

Let $j \in [s_{k+1}]$. We now check that $\mathcal{F}_{t_{k+1}+j}$ is a feasible system. We have already verified that (F3) holds. By Lemma 18.1(a) and since Step 2 only involves forward edges, each $i \in [4]$ satisfies

$$e_{\mathcal{F}_{t_{k+1}+j}}(U_i, U_{i-1}) = e_{\mathcal{F}'_{t_{k+1}+j}}(U_i, U_{i-1}) = m_j^{i\downarrow} + |U_i^{1-\gamma}(T)| + |U_{i-1}^{1-\gamma}(T)| \overset{(18.13)}{=} |U^{1-\gamma}(T)|.$$

Thus, (F1) is satisfied. By Lemma 16.1(a), (F2) is satisfied. Therefore, $\mathcal{F}_{t_{k+1}+j}$ is a feasible system, as desired. □

Let D be a digraph. For each $e \in E(D)$, let $L(e)$ be a list of colours. A *proper list edge-colouring of D* is a colouring of the edges of D such that each edge $e \in E(D)$ is coloured with one of the colours in its list $L(e)$ and no two adjacent edges receive the same colour. The next proposition states that if the lists are all sufficiently large, then a proper list edge-colouring exists. Its proof follows from a simple greedy colouring argument and is therefore omitted.

Proposition 18.2. *Let D be a digraph. For each $e = uv \in E(D)$, let $L(e)$ be a list of colours satisfying $|L(e)| \geq d_D(u) + d_D(v) + 1$. Then, D has a proper list edge-colouring.*

Proof of Lemma 17.2. For each $i \in [4]$, denote $U_i^* := U^* \cap U_i$. By Fact 10.2(ii) and the "in particular part" of Lemma 13.10 (applied with 2γ playing the role of γ), we may assume without loss of generality that

$$U_3^{1-2\gamma}(T) = \emptyset = U_4^{1-2\gamma}(T). \tag{18.19}$$

Step 1: Decomposing the forward edges of D which are incident to $U^{1-\gamma}(T)$ and the forward edges of $D[U^*]$ which are incident to $U^{1-2\gamma}(T)$. By (13.2), each

$v \in U^{1-\gamma}(T)$ satisfies

$$\overrightarrow{d}^{\pm}_{D,\mathcal{U}}(v) \leq \overrightarrow{d}^{\pm}_{T,\mathcal{U}}(v) \leq \lfloor \gamma n \rfloor \leq t'.$$

Moreover, each $v \in U^{1-2\gamma}(T)$ satisfies

$$|\overrightarrow{N}^{\pm}_{D,\mathcal{U}}(v) \cap U^*| \leq |U^*| \overset{\text{(ES2)}}{\leq} t'.$$

Also recall from Fact 13.9 and Definition 13.11 that $|U^{1-\gamma}(T)| \leq |U^{1-2\gamma}(T)| \leq t'$. For each $i \in [4]$, Proposition 7.23 (applied with the corresponding underlying undirected graph playing the role of G) implies that the digraph

$$D(U_i^{1-\gamma}(T), U_{i+1}) \cup D(U_i, U_{i+1}^{1-\gamma}(T)) \cup D(U_i^{1-2\gamma}(T), U_{i+1}^*) \cup D(U_i^*, U_{i+1}^{1-2\gamma}(T))$$

can be decomposed into t' edge-disjoint matchings $M_1^{i\uparrow}, \dots, M_{t'}^{i\uparrow}$. Observe that the following hold.

(α) For each $i \in [4]$ and $j \in [t']$, $M_j^{i\uparrow} \subseteq E_D(U_i, U_{i+1})$.

(β) $\bigcup_{(i,j)\in[4]\times[t']} M_j^{i\uparrow} = \{e \in E(\overrightarrow{D}_{\mathcal{U}}) \mid V(e) \cap U^{1-\gamma}(T) \neq \emptyset\} \cup \{e \in E(\overrightarrow{D}_{\mathcal{U}}[U^*]) \mid V(e) \cap U^{1-2\gamma}(T) \neq \emptyset\}$. In particular, Fact 13.9 implies that each edge in $\bigcup_{(i,j)\in[4]\times[t']} M_j^{i\uparrow}$ is incident to $U^{1-2\gamma}(T)$.

Step 2: Selecting backward edges. In this step, we use backward edges to ensure that each vertex in $U^{1-2\gamma}(T)$ is covered by both an in- and an outedge in each of the feasible system. Covering $U^{1-\gamma}(T)$ is necessary for (F2′) to be satisfied, while covering $U^{1-2\gamma}(T) \setminus U^{1-\gamma}(T)$ will ensure that (d) is satisfied. We will also balance the number of backward edges using edges which are not incident to $U^{1-2\gamma}(T)$. This corresponds to Steps 2 and 3 of the proof overview presented in Section 18.1 and will be carried out using Lemma 18.1.

Let $H \subseteq \overleftarrow{T}_{\mathcal{U}}$ be the digraph obtained by applying Lemma 13.8 with 2γ playing the role of γ. Note that the following hold.

(I) $\Delta^0(H) \leq 2\gamma n$.

(II) For each $v \in U^{1-2\gamma}(T)$, $d_H(v) = 0$.

(III) For each $i \in [4]$, $e_{H-U^{1-2\gamma}(T)}(U_i, U_{i-1}) \geq (1-4\gamma)n|U^{1-2\gamma}_{i-2}(T) \cup U^{1-2\gamma}_{i-3}(T)|$.

Let H' be obtained from $H \cap D$ by adding all the edges of $\overleftarrow{D}_{\mathcal{U}}$ which have precisely one endpoint in $U^{1-2\gamma}(T)$ and precisely one endpoint in $V(T) \setminus U^*$. Note that

$$e_{H'}(U^{1-2\gamma}(T), U^*) + e_{H'}(U^*, U^{1-2\gamma}(T)) \overset{\text{(II)}}{=} 0. \tag{18.20}$$

We now verify that Lemma 18.1(i)–(iii) hold with H' and 2γ playing the roles of H and γ. For each $v \in U^{1-2\gamma}(T)$,

$$\begin{aligned} d^{\pm}_{H'}(v) &\overset{\text{(II)}}{=} |\overleftarrow{N}^{\pm}_{D,\mathcal{U}}(v) \setminus U^*| \geq \overleftarrow{d}^{\pm}_{T,\mathcal{U}}(v) - \Delta^0(T \setminus D) - |U^*| \\ &\overset{\text{(i)}}{\geq} (1-2\gamma)n - t - |U^*| \overset{\text{(ES2)}}{\geq} (1-3\gamma)n. \end{aligned}$$

Therefore, Lemma 18.1(i) is satisfied, with room to spare. For each $v \in V(T) \setminus U^{1-2\gamma}(T)$,

$$d^{\pm}_{H'}(v) \leq d^{\pm}_H(v) + |U^{1-2\gamma}(T)| \overset{\text{(I),Fact 13.9,Definition 13.11}}{\leq} 3\gamma n.$$

Thus, Lemma 18.1(ii) is satisfied, with room to spare. For each $i \in [4]$,

$$e_{H'-U^{1-2\gamma}(T)}(U_i, U_{i-1}) \overset{\text{(ii),(III)}}{\geq} (1-5\gamma)n|U^{1-2\gamma}_{i-2}(T) \cup U^{1-2\gamma}_{i-3}(T)|.$$

Therefore, Lemma 18.1(iii) holds, with room to spare.

Let $i \in [4]$ and $j \in [t']$. Recall from Lemma 18.1(a) that, when applying Lemma 18.1, $S^{i\downarrow}_j$ denotes the set of vertices that need to be avoided by the edges from U_i to U_{i-1}. Here, we need to avoid the vertices in U_i which are already covered with an outedge and the vertices in U_{i-1} which are already covered with an inedge. By (α), these are precisely the vertices in

$$S^{i\downarrow}_j := (V(M^{i\uparrow}_j) \cup V(M^{(i-2)\uparrow}_j)) \cap (U_i \cup U_{i-1}). \tag{18.21}$$

Then,

$$\begin{aligned}
|S_j^{i\downarrow} \setminus U^{1-2\gamma}(T)| &\stackrel{(\alpha)}{=} |V(M_j^{i\uparrow}) \cap (U_i \setminus U^{1-2\gamma}(T))| + |V(M_j^{(i-2)\uparrow}) \cap (U_{i-1} \setminus U^{1-2\gamma}(T))| \\
&\stackrel{(\alpha),(\beta)}{\leq} |V(M_j^{i\uparrow}) \cap U_{i+1}^{1-2\gamma}(T)| + |V(M_j^{(i-2)\uparrow}) \cap U_{i-2}^{1-2\gamma}(T)| \\
&\leq |U_{i+1}^{1-2\gamma}(T)| + |U_{i-2}^{1-2\gamma}(T)|.
\end{aligned}$$

Therefore, Lemma 18.1(iv) is satisfied with 2γ playing the role of γ. Let

$$m_j^{i\downarrow} := \left|\left(U_{i-2}^{1-2\gamma}(T) \cup U_{i-3}^{1-2\gamma}(T)\right) \setminus S_j^{(i-2)\downarrow}\right|. \tag{18.22}$$

Then, Lemma 18.1(v) holds with 2γ playing the role of γ.

Let $\mathcal{F}_1', \dots, \mathcal{F}_{t'}'$ be the edge-disjoint linear forests obtained by applying Lemma 18.1 with H', t', and 2γ playing the roles of H, ℓ, and γ. For each $j \in [t']$, note that

$$\mathcal{F}_j' \subseteq \overleftarrow{D}_{\mathcal{U}} \subseteq \overleftarrow{T}_{\mathcal{U}} \tag{18.23}$$

and let $\mathcal{F}_j'' := \bigcup_{i \in [4]} M_j^{i\uparrow} \cup \mathcal{F}_j'$. Denote $\mathcal{F}'' := \bigcup_{j \in [t']} \mathcal{F}_j''$.

Claim 1. *$\mathcal{F}_1'', \dots, \mathcal{F}_{t'}''$ are edge-disjoint and satisfy the following properties.*

(A) *For each $j \in [t']$, $\mathcal{F}_j''$ is a (γ, T)-pseudo-feasible system.*

(B) *For each $j \in [t']$, $e(\mathcal{F}_j'') \leq 24\varepsilon n$.*

(C) *For each $v \in U^{1-\gamma}(T)$, $\overrightarrow{d}^{\pm}_{\mathcal{F}'',\mathcal{U}}(v) = \overrightarrow{d}^{\pm}_{D,\mathcal{U}}(v)$.*

(D) *For each $v \in U^{1-2\gamma}(T) \setminus U^{1-\gamma}(T)$, $\overleftarrow{d}^{\pm}_{\mathcal{F}'',\mathcal{U}}(v) \geq t' - 4\varepsilon n$.*

(E) *For each $v \in V(T) \setminus U^{1-2\gamma}(T)$, $d_{\mathcal{F}''}(v) \leq \frac{2\gamma n}{5}$.*

(F) *$E(\overrightarrow{\mathcal{F}''}_{\mathcal{U}}) = \{e \in E(\overrightarrow{D}_{\mathcal{U}}) \mid V(e) \cap U^{1-\gamma}(T) \neq \emptyset\} \cup \{e \in E(\overrightarrow{D}_{\mathcal{U}}[U^*]) \mid V(e) \cap U^{1-2\gamma}(T) \neq \emptyset\}$. In particular, $E(\overrightarrow{\mathcal{F}''}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)]) = \emptyset$.*

Proof of Claim. By (α) and (18.23), $E(\mathcal{F}_j') \cap M_{j'}^{i\uparrow} = \emptyset$ for any $i \in [4]$ and $j, j' \in [t']$. Thus, $\mathcal{F}_1'', \dots, \mathcal{F}_{t'}''$ are edge-disjoint. Moreover, (18.23) implies that $\overrightarrow{\mathcal{F}''}_{\mathcal{U}} = \bigcup_{(i,j) \in [4] \times [t']} M_j^{i\uparrow}$. Thus,

(C) and (F) follow from (β). Moreover, note for later that each $v \in U^{1-2\gamma}(T) \setminus U^{1-\gamma}(T)$ satisfies

$$\overrightarrow{d}^{\pm}_{\mathcal{F}'',\mathcal{U}}(v) \overset{(\beta)}{=} |\overrightarrow{N}^{\pm}_{D,\mathcal{U}}(v) \cap U^*| \overset{\text{(ES2)}}{\leq} 4\varepsilon n. \tag{18.24}$$

For each $j \in [t']$,

$$\begin{aligned}
e(\mathcal{F}''_j) &= \sum_{i\in[4]} |M_j^{i\uparrow}| + e(\mathcal{F}'_j) \\
&\overset{(\alpha),(\beta),\text{Lemma 18.1(a)}}{\leq} 2|U^{1-2\gamma}(T)| + \left(\sum_{i\in[4]} m_j^{i\downarrow} + 2|U^{1-2\gamma}(T)|\right) \\
&\overset{(18.22)}{\leq} 6|U^{1-2\gamma}(T)| \overset{\text{Fact 13.9,Definition 13.11}}{\leq} 24\varepsilon n,
\end{aligned}$$

so (B) holds. For each $i \in [4]$ and $v \in U_i \setminus U^{1-2\gamma}(T)$, we have

$$\begin{aligned}
d_{\mathcal{F}''}(v) &\overset{(\alpha)}{=} \sum_{j\in[t']} \left(\left(d_{M_j^{i\uparrow}}(v) + d_{M_j^{(i-1)\uparrow}}(v)\right) + d_{\mathcal{F}'_j}(v)\right) \\
&\overset{(\beta),\text{Lemma 18.1(b)}}{\leq} 2|U^{1-2\gamma}(T)| + \frac{\gamma n}{3} \overset{\text{Fact 13.9,Definition 13.11}}{\leq} \frac{2\gamma n}{5},
\end{aligned}$$

so (E) holds. Let $j \in [t']$. We show that $\mathcal{F}''_j$ is a (γ, T)-pseudo-feasible system. First, note that

$$\begin{aligned}
e_{\mathcal{F}''_j}(U_1, U_4) &\overset{(\alpha)}{=} e_{\mathcal{F}'_j}(U_1, U_4) \overset{\text{Lemma 18.1(a)}}{=} m_j^{1\downarrow} + |(U_1^{1-2\gamma}(T) \cup U_4^{1-2\gamma}(T)) \setminus S_j^{1\downarrow}| \\
&\overset{(18.22)}{=} |(U_3^{1-2\gamma}(T) \cup U_2^{1-2\gamma}(T)) \setminus S_j^{3\downarrow}| + m_j^{3\downarrow} \\
&\overset{\text{Lemma 18.1(a)}}{=} e_{\mathcal{F}'_j}(U_3, U_2) \overset{(\alpha)}{=} e_{\mathcal{F}''_j}(U_3, U_2).
\end{aligned}$$

Similarly, $e_{\mathcal{F}''_j}(U_4, U_3) = e_{\mathcal{F}''_j}(U_2, U_1)$ and so (F1) is satisfied. For each $i \in [4]$ and $v \in U_i$,

$$d^+_{\mathcal{F}''_j}(v) \overset{(\alpha),(18.23)}{=} d_{M_j^{i\uparrow}}(v) + d_{\mathcal{F}'_j(U_i,U_{i-1})}(v) \overset{(18.21),\text{Lemma 18.1(a)}}{\leq} 1.$$

Similarly, each $v \in V(T)$ satisfies $d^-_{\mathcal{F}''_j}(v) \leq 1$. Thus, (F3′) holds. Moreover, each $i \in [4]$

and $v \in U_i^{1-2\gamma}(T)$ satisfy

$$d^+_{\mathcal{F}''_j}(v) \overset{(\alpha),(18.23)}{=} d_{M_j^{i\uparrow}}(v) + d_{\mathcal{F}'_j(U_i,U_{i-1})}(v) \overset{(18.21),\text{Lemma 18.1(a)}}{=} 1.$$

Similarly, each $v \in U^{1-2\gamma}(T)$ satisfies $d^-_{\mathcal{F}''_j}(v) = 1$. Thus, (F2′) holds and (D) follows from (18.24).

Finally, to verify (F4′), suppose that C is a cycle in $\mathcal{F}''_j$. We show that C contains a (γ, T)-placeholder. By (18.19), (β), and (18.23),

$$e_{\mathcal{F}''_j}(U_3, U_4) + e_{\mathcal{F}''_j}(U_4^{1-2\gamma}(T), U_3) + e_{\mathcal{F}''_j}(U_4, U_3^{1-2\gamma}(T)) = 0.$$

Therefore, Lemma 18.1(c) implies that each edge in $E_{\mathcal{F}''_j}(U_4, U_3)$ forms a component in $\mathcal{F}''_j$. Altogether, we have

$$e_C(U_3, U_4) + e_C(U_4, U_3) = 0. \tag{18.25}$$

Suppose that there exists $e \in E_C(U_2, U_1) \subseteq E_{\mathcal{F}''_j}(U_2, U_1)$. By ($\alpha$), $e \in E(\mathcal{F}'_j)$ and, by (18.19) and (18.22), $m_j^{2\downarrow} = 0$. Thus, Lemma 18.1(a) implies that $V(e) \cap U^{1-2\gamma}(T) \neq \emptyset$. By Lemma 18.1(a), (18.20), Fact 13.9, and (ES1), e is a backward edge with precisely one endpoint in $U^{1-2\gamma}(T) \subseteq U^*$ and one endpoint in $V(T) \setminus U^*$. Thus, e is a (γ, T)-placeholder, as desired.

We may therefore assume that $e_C(U_2, U_1) = 0$. Then, (α), (18.23), and (18.25) imply that either $V(C) \subseteq U_4 \cup U_1$ or $V(C) \subseteq U_3 \cup U_2$. Suppose the former (similar arguments hold in the other case). By Lemma 18.1, $\mathcal{F}'_j$ is a linear forest and so (α) implies that there exists $uv \in M_j^{4\uparrow} \cap E(C)$. By (18.19), ($\alpha$), and ($\beta$), $v \in U_1^{1-2\gamma}(T)$. Let w denote the outneighbour of v on C. By assumption, (α), and (18.23), $w \in U_4$. In particular, vw is a backward edge and so (α) and Lemma 18.1(a) imply that $vw \in E(H')$. Thus, (18.20) implies that $w \in V(T) \setminus U^*$ and so vw is a (γ, T)-placeholder, as desired. Therefore, (F4′) holds and so $\mathcal{F}''_j$ is a (γ, T)-pseudo-feasible system. Thus, (A) holds. $\diamond$

Step 3: Covering the edges of $\overrightarrow{D}[U^* \setminus U^{1-2\gamma}(T)]$. We will use Proposition 18.2 as follows. For each $e \in E(\overrightarrow{D}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)])$, let $L(e)$ be the set of colours $i \in [t']$ such that $V(e) \cap V(E(\mathcal{F}_i'')) = \emptyset$. For each $e = uv \in E(\overrightarrow{D}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)])$, we have

$$|L(e)| \geq t' - d_{\mathcal{F}''}(u) - d_{\mathcal{F}''}(v) \overset{\text{(E)}}{\geq} \frac{\gamma n}{6} \overset{\text{(ES2)}}{\geq} |\overrightarrow{N}_{D,\mathcal{U}}(u) \cap U^*| + |\overrightarrow{N}_{D,\mathcal{U}}(v) \cap U^*| + 1.$$

Thus, Proposition 18.2 implies that $\overrightarrow{D}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)]$ has a proper list edge-colouring $\phi : E(\overrightarrow{D}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)]) \longrightarrow [t']$. For each $i \in [t']$, let $\mathcal{F}_i \coloneqq \mathcal{F}_i'' \cup \phi^{-1}(i)$. Since ϕ is an edge-colouring of $\overrightarrow{D}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)]$, any distinct $i, i' \in [t']$ satisfy

$$E(\mathcal{F}_i \setminus \mathcal{F}_i'') \cap E(\mathcal{F}_{i'} \setminus \mathcal{F}_{i'}'') = \emptyset. \tag{18.26}$$

Denote $\mathcal{F} \coloneqq \bigcup_{i \in [t']} \mathcal{F}_i$ and note that

$$E(\mathcal{F} \setminus \mathcal{F}'') \overset{\text{(F)}}{=} E(\overrightarrow{D}_{\mathcal{U}}[U^* \setminus U^{1-2\gamma}(T)]). \tag{18.27}$$

Step 4: Verifying (a)–(d). We claim that $\mathcal{F}_1, \dots, \mathcal{F}_{t'}$ are edge-disjoint (γ, T)-pseudo-feasible systems satisfying (a)–(d). Recall from Claim 1 that $E(\mathcal{F}_i'') \cap E(\mathcal{F}_{i'}'') = \emptyset$ for any distinct $i, i' \in [t']$. Thus, (18.26), (18.27), and the "in particular part" of (F) imply that $\mathcal{F}_1, \dots, \mathcal{F}_{t'}$ are edge-disjoint.

Let $i \in [t']$. We now show that $\mathcal{F}_i$ is a (γ, T)-pseudo-feasible system. By construction, $E(\mathcal{F}_i) \setminus E(\mathcal{F}_i'') \subseteq E(\overrightarrow{T}_{\mathcal{U}})$. Thus, (F1) follows from (A). By construction of the lists of colours, we have $V(E(\mathcal{F}_i \setminus \mathcal{F}_i'')) \cap V(E(\mathcal{F}_i'')) = \emptyset$ and, since ϕ is proper, $\phi^{-1}(i)$ is a matching for each $i \in [t']$. Thus, (F2′) and (F3′) follow from (A). Moreover, each cycle in $\mathcal{F}_i$ is a cycle in $\mathcal{F}_i''$. Thus, (F4′) also follows from (A). Therefore, $\mathcal{F}_i$ is a (γ, T)-pseudo-feasible system, as desired.

Moreover, (a) follows from (F) and (18.27), (c) follows from (C) and (18.27), and (d) follows from (D). Finally, the fact that $\phi^{-1}(i)$ is a matching implies that

$$e(\mathcal{F}_i) = e(\mathcal{F}_i'') + |\phi^{-1}(i)| \overset{\text{(B)}}{\leq} 24\varepsilon n + |U^* \setminus U^{1-2\gamma}(T)| \overset{\text{(ES2)}}{\leq} \sqrt{\varepsilon} n$$

and so (b) holds. □

APPENDICES

APPENDIX A

OPTIMAL PACKINGS OF HAMILTON CYCLES: PROOF OF COROLLARY 1.15

In this appendix, we prove Corollary 1.15. First, we will need the following properties of (ε, d)-regular bipartite graphs. (Recall that those were defined in Section 7.2.) The next two lemmas hold by definition (and so their proofs are omitted).

Lemma A.1. *Let $0 < \frac{1}{m} \ll \varepsilon \ll d < 1$ and $\varepsilon \leq \eta \ll 1$. Let G be a bipartite graph on vertex classes A and B of size m. Suppose that G is (ε, d)-regular. Let $A' \subseteq A$ and $B' \subseteq B$ satisfy $|A'|, |B'| \geq \eta m$. Then, $G[A', B']$ is $(\frac{\varepsilon}{\eta}, \geq d - \varepsilon)$-regular.*

Lemma A.2 ([76, Proposition 4.2])**.** *Let $0 < \varepsilon \leq d \leq 1$. Let G be an (ε, d)-regular bipartite graph on vertex classes A and B. Then, fewer than $\varepsilon|A|$ vertices $a \in A$ satisfy $d_G(a) \geq (d + \varepsilon)|B|$ and fewer than $\varepsilon|A|$ vertices $a \in A$ satisfy $d_G(a) \leq (d - \varepsilon)|B|$.*

One can easily deduce that ε-regular bipartite graphs of linear minimum degree are also bipartite robust expanders.

Lemma A.3. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \delta \leq 1$. Let G be a bipartite graph on vertex classes A and B of size n. Suppose that G is ε-regular and $\delta(G) \geq \delta n$. Then, G is a bipartite robust (ν, τ)-expander with bipartition (A, B).*

Proof. Note that G is $(\varepsilon, \geq \delta)$-regular. Let $S \subseteq A$ satisfy $\tau n \leq |S| \leq (1 - \tau)n$. If suffices to show that $|RN_{\nu,G}(S)| \geq |S| + \nu n$. By Lemma A.1, $G[S, B]$ is $(\sqrt{\varepsilon}, \geq \delta - \varepsilon)$-regular and so Lemma A.2 implies that all but at most $\sqrt{\varepsilon} n$ vertices $v \in B$ satisfy

$|N_G(v) \cap S| \geq (\delta - \varepsilon - \sqrt{\varepsilon})|S| \geq \frac{\delta \tau n}{2} \geq \nu n$. Thus, $|RN_{\nu,G}(S)| \geq (1-\sqrt{\varepsilon})n \geq |S| + \nu n$, as desired. □

Using the max-flow min-cut theorem, Frieze and Krivelevich [36] showed that ε-regular bipartite graphs of linear minimum degree contain a dense regular spanning subgraph.

Lemma A.4 ([36]). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \delta \leq 1$. Let G be an ε-regular bipartite graph on vertex classes of size n. Suppose that $\delta(G) \geq \delta n$. Then, $\mathrm{reg}_{\mathrm{even}}(G) \geq (\delta - 2\varepsilon)n$.*

By Proposition 7.10, the complete bipartite graph $K_{n,n}$ is $[\varepsilon, 1]$-superregular. Therefore, Lemma 7.17 implies that $G_{n,n,p}$ is also superregular with high probability.

Corollary A.5. *Let $0 < \frac{1}{n} \ll \varepsilon \ll p \leq 1$. With high probability, $G_{n,n,p}$ is ε-regular and $\delta(G_{n,n,p}) \geq (p - \varepsilon)n$.*

We are now ready to prove Corollary 1.15.

Proof of Corollary 1.15. Let $0 < p \leq 1$. Fix additional constants such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \nu \ll \tau \ll p$. By Corollary A.5, (iii) follows immediately from (i) (with $p - \varepsilon$ playing the role of p).

For (iv), denote by A and B the vertex classes of $D_{n,n,p}$. Observe that $D_{n,n,p}[A, B] \sim G_{n,n,p}$ and $D_{n,n,p}[B, A] \sim G_{n,n,p}$. Thus, Corollary A.5 implies that $D_{n,n,p}$ is ε-regular of minimum semidegree $\delta^0(D_{n,n,p}) \geq (p - \varepsilon)n$ with high probability and so (iv) follows from (ii) (with $p - \varepsilon$ playing the role of p).

For (v), let T be chosen uniformly at random among the bipartite tournaments on vertex classes A and B of size n. Observe that $T[A, B] \sim G_{n,n,\frac{1}{2}}$ and $T[B, A] \sim G_{n,n,\frac{1}{2}}$. Thus, Corollary A.5 implies that T is ε-regular of minimum semidegree $\delta^0(T) \geq (\frac{1}{2} - \varepsilon)n$ with high probability and so (v) also follows from (ii) (with $\frac{1}{2} - \varepsilon$ playing the role of p).

For (i), let G be an ε-regular bipartite graph on vertex classes A and B of size n and suppose that $\delta(G) \geq pn$. Note that G is (ε, d)-regular for some $d \geq p$. Let S be the set of vertices $v \in V(G)$ which satisfy $d_G(v) \geq (d + \varepsilon)n$. By Lemma A.2, $|S| \leq \varepsilon n$ and so Proposition 7.8(i) implies that $G - S$ is still ε_1-regular. Let G' be a spanning

$\mathrm{reg}_{\mathrm{even}}(G)$-regular subgraph of G. By Lemma A.4, $G' - S$ is obtained from $G - S$ by deleting at most $2\varepsilon n$ edges incident to each vertex and so Proposition 7.8(i) implies that $G' - S$ is ε_2-regular. Another application of Proposition 7.8(i) implies that G' is ε_3-regular. Thus, Lemma A.3 implies that G' is a bipartite robust (ν, τ)-expander with bipartition (A, B), as well as with bipartition (B, A). Apply Corollary 4.2 to decompose G' into $\frac{\mathrm{reg}_{\mathrm{even}}(G)}{2}$ edge-disjoint Hamilton cycles.

For (ii), let D be an ε-regular bipartite digraph on vertex classes A and B of size n and suppose that $\delta^0(D) \geq pn$. Let D' be a $\mathrm{reg}(D)$-regular spanning subdigraph of D. Observe that Lemma A.4 implies that $\mathrm{reg}(D) \geq (p - 2\varepsilon)n$ and so, by similar arguments as above, D' is a bipartite robust (ν, τ)-outexpander with bipartition (A, B). Apply Theorem 4.1 to decompose D' into $\mathrm{reg}(D)$ edge-disjoint Hamilton cycles. $\square$

APPENDIX B

APPROXIMATE DECOMPOSITION: PROOF OF THEOREM 8.1

In this appendix, we adapt the arguments of [40] to prove Theorem 8.1.

B.1 Preliminaries

We will need some additional preliminary results.

B.1.1 Regularity

Recall the definition of an (ε, d)-regular bipartite graph from Section 7.2. We need the (non-bipartite version of the) degree form regularity lemma for digraphs.

Lemma B.1 (Degree form regularity lemma for digraphs)**.** *For all $\varepsilon > 0$ and $M' \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ such that if D is a digraph on $n \geq n_0$ vertices and $d \in [0,1]$, then there exist a spanning subdigraph $D' \subseteq D$ and a partition of $V(D)$ into an* exceptional set *V_0 and k* clusters *$V_1, \ldots, V_k$ such that the following hold.*

(i) $M' \leq k \leq M$.

(ii) $|V_0| \leq \varepsilon n$.

(iii) $|V_1| = \cdots = |V_k| =: m$.

(iv) *For each $v \in V(D)$, both $d^{\pm}_{D'}(v) > d^{\pm}_{D}(v) - (d + \varepsilon)n$.*

(v) *For each $i \in [k]$, $D'[V_i]$ is empty.*

(vi) *Let $i, j \in [k]$ be distinct. Then, $D'[V_i, V_j]$ is either empty or $(\varepsilon, \geq d)$-regular. Moreover, if $D'[V_i, V_j]$ is non-empty, then $D'[V_i, V_j] = D[V_i, V_j]$.*

Let $\varepsilon > 0$, $M' \in \mathbb{N}$, and $d \in [0, 1]$. Let D be a digraph. The *pure digraph of D with parameters ε, d, and M'* is the digraph $D' \subseteq D$ obtained by applying Lemma B.1 with these parameters. The *reduced digraph of D with parameters ε, d, and M'* is the digraph R defined as follows. Let $V_0, V_1, \dots, V_k$ be the partition obtained by applying Lemma B.1 with parameters ε, d, and M'. Denote by D' the pure digraph of D with parameters ε, d, and M'. Then, $V(R) \coloneqq \{V_i \mid i \in [k]\}$ and, for any distinct $U, V \in V(R)$, $UV \in E(R)$ if and only if $D'[U, V]$ is non-empty. Note that Lemma B.1(vi) implies that $D'[U, V] = D[U, V]$ is $(\varepsilon, \geq d)$-regular for any $UV \in E(R)$.

The following result states that robust outexpansion is inherited by the reduced digraph.

Lemma B.2 ([79, Lemma 14])**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll d \ll \nu, \tau, \delta \leq 1$ and $\frac{M'}{n} \ll 1$. Let D be a robust (ν, τ)-outexpander on n vertices. Suppose that $\delta^0(D) \geq \delta n$. Let R be the reduced digraph of D with parameters ε, d, and M'. Then, $\delta^0(R) \geq \frac{\delta|R|}{2}$ and R is a robust $(\frac{\nu}{2}, 2\tau)$-outexpander.*

B.1.2 Robust outexpanders

By definition of a bipartite robust outexpander, the τ-parameter can be made arbitrarily small. An analogous observation for the non-bipartite setting was made (and proved) in [40, Lemma 4.3], so we omit the details here.

Lemma B.3. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leq \frac{\delta}{2} \leq 1$. Let D be a bipartite digraph on vertex classes A and B of size n. Suppose that D is a bipartite robust $(\nu, \frac{\delta}{2})$-outexpander with bipartition (A, B). Suppose furthermore that $\delta^0(D) \geq \delta n$. Then, D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B).*

Lemma B.4 ([40, Lemma 4.2])**.** *Let $0 < \varepsilon \leq \nu \ll \tau \leq 1$. Let D be a robust (ν, τ)-outexpander on n vertices.*

(i) *If D' is obtained from D by removing at most εn inedges and at most εn outedges at each vertex, then D' is a robust $(\nu - \varepsilon, \tau)$-outexpander.*

(ii) *If D' is obtained from D by adding or removing at most εn vertices, then D' is a robust $(\nu - \varepsilon, 2\tau)$-outexpander.*

We will need [40, Lemma 14.3], which states that robust outexpansion is inherited by random vertex subsets with high probability. Note that we only gave a brief proof overview of this lemma in [40] as it was only used to sketch a new shorter proof of the main result of [96]. Thus, for completeness, we include its proof here.

Lemma B.5 ([40, Lemma 14.3])**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu' \ll \delta, \nu, \tau \ll 1$. Fix a positive integer $n' \geq \varepsilon n$. Suppose that D is a robust (ν, τ)-outexpander on n vertices satisfying $\delta^0(D) \geq \delta n$. Suppose that V is chosen uniformly at random among the subsets of $V(D)$ of size n'. Then, $D[V]$ is a robust $(\nu', 4\tau)$-outexpander with probability at least $1 - n^{-2}$.*

Proof. Fix additional constants such that $\frac{1}{n} \ll \varepsilon' \ll \varepsilon \ll \nu' \ll d \ll \nu, \tau$ and $\frac{M'}{n} \ll 1$. Let $V_0, V_1, \dots, V_k$ be the partition of $V(D)$ obtained by applying Lemma B.1 with ε' playing the role of ε and define $m := |V_1| = \cdots = |V_k|$. Denote by R the reduced digraph of D with parameters ε', d, and M'. By Lemma B.2, R is a robust $(\frac{\nu}{2}, 2\tau)$-outexpander. Let $n' \geq \varepsilon n$ and suppose that V is chosen uniformly at random among the subsets of $V(D)$ of size n'. We show that $D' := D[V]$ is a robust $(\nu', 4\tau)$-outexpander with probability at least $1 - n^{-2}$.

For each $i \in [k]$, denote $V_i' := V_i \cap V(D')$. Let $i \in [k]$. Then, $\mathbb{E}[|V_i'|] = \frac{n'm}{n} =: m' \geq \varepsilon m$. Then, Lemma 7.16 implies that

$$\mathbb{P}[|V_i'| \neq (1 \pm \varepsilon)m'] \leq 2\exp\left(-\frac{\varepsilon^3 m}{3}\right).$$

Thus, a union bound implies that $|V_i'| = (1 \pm \varepsilon)m'$ for each $i \in [k]$ with probability at least $1 - n^{-2}$. Therefore, we assume that $|V_i'| = (1 \pm \varepsilon)m'$ for each $i \in [k]$ and show that D' is a robust $(\nu', 4\tau)$-outexpander.

Note that Lemma A.1 (with ε' and ε^2 playing the roles of ε and η) implies that, for each $V_iV_j \in E(R)$ and $S \subseteq V_i'$ satisfying $|S| \geq \varepsilon m' \geq \varepsilon^2 m$, the pair $D'[S, V_j']$ is still $(\varepsilon, \geq d - \varepsilon')$-regular. Let $S \subseteq V(D')$ satisfy $4\tau n' \leq |S| \leq (1-4\tau)n'$. We need to show that $|RN^+_{\nu', D'}(S)| \geq |S| + \nu' n'$. Let $S' \coloneqq \{V_i \mid i \in [k], |S \cap V_i| = |S \cap V_i'| \geq dm'\}$. Then,

$$\begin{aligned} |S'| &\geq \frac{|S| - dm'k}{m'} \\ &\geq \frac{4\tau n'}{m'} - dk = \frac{4\tau n}{m} - dk \geq 2\tau k. \end{aligned} \tag{B.1}$$

If $|S'| \leq (1-2\tau)k$, then let $S'' \coloneqq S'$; otherwise, choose $S'' \subseteq S'$ of size $(1-2\tau)k$. Then, $|RN^+_{\frac{\nu}{2}, R}(S'')| \geq |S''| + \frac{\nu k}{2}$.

Let $V_i \in S''$ and $S_i \coloneqq V_i' \cap S = V_i \cap S$. By definition of S', we have $|S_i| \geq dm'$ and so $D'[S_i, V_j']$ is $(\varepsilon, \geq d - \varepsilon')$-regular for each $V_iV_j \in E(R)$. Then, Lemma A.2 implies that, for each $V_iV_j \in E(R)$, all but at most $\varepsilon(1+\varepsilon)m' \leq 2\varepsilon m'$ vertices $v \in V_j'$ satisfy $d_{D'[S_i, V_j']}(v) \geq (d - \varepsilon' - \varepsilon)|S_i| \geq (d - 2\varepsilon)dm'$.

Thus, for each $V_i \in RN^+_{\frac{\nu}{2}, R}(S'')$, all but at most $\frac{2\varepsilon m' \cdot k}{2\sqrt{\varepsilon} k} = \sqrt{\varepsilon} m'$ vertices $v \in V_i'$ satisfy

$$|N^-_{D'}(v) \cap S| \geq \left(\frac{\nu}{2} - 2\sqrt{\varepsilon}\right) k \cdot (d - 2\varepsilon)dm' \geq \left(\frac{\nu d^2}{2} - 2\sqrt{\varepsilon} d^2 - \nu\varepsilon d\right) km' \overset{\text{Lemma B.1(ii)}}{\geq} \nu' n'.$$

Therefore,

$$\begin{aligned} |RN^+_{\nu', D'}(S)| &\geq |RN^+_{\frac{\nu}{2}, R}(S'')| \left(1 - \varepsilon - \sqrt{\varepsilon}\right) m' \geq \left(|S''| + \frac{\nu k}{2}\right)(1 - 2\sqrt{\varepsilon})m' \\ &\overset{\text{Lemma B.1(ii)}}{\geq} |S''|m' + \frac{\nu n'}{3}. \end{aligned}$$

If $|S''| = (1-2\tau)k$, then Lemma B.1(ii) implies that $|S''|m' \geq (1-4\tau)n' \geq |S|$ and so $|RN^+_{\nu', D'}(S)| \geq |S| + \nu' n'$, as desired. We may therefore assume that $S'' = S'$. Then, (B.1) implies that $|S''|m' \geq |S| - dn'$ and so $|RN^+_{\nu', D'}(S)| \geq |S| + \nu' n'$, as desired. $\square$

B.1.3 Probabilistic estimates

The following lemmas are easy consequences of Lemma 7.16.

Lemma B.6. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \delta \leq 1$. Let G be a balanced bipartite graph on vertex classes A and B of size n. Suppose that G is a (δ, ε)-almost regular bipartite robust (ν, τ)-expander with bipartition (A, B). Let Γ be obtained from G by taking each edge independently with probability $\frac{\gamma}{\delta}$. Then, with positive probability, all of the following hold.*

(i) *$G \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular.*

(ii) *Γ is (γ, ε)-almost regular.*

(iii) *Γ is a bipartite robust $(\frac{\gamma\nu}{2\delta}, \tau)$-expander with bipartition (A, B).*

Lemma B.7. *Let $0 < \frac{1}{n} \ll \varepsilon \leq 1$ and fix positive integers $k, \ell \geq \varepsilon n$. Let A and B be disjoint vertex sets of size n. Suppose that $M_1, \dots, M_\ell$ are bipartite perfect matchings on vertex classes A and B. Suppose that $A_1, \dots, A_\ell$ are chosen independently and uniformly at random among subsets of A of size k. Then, with probability at least $1 - n^{-1}$, both of the following hold.*

(i) *For each $v \in A$, there exist at most $\frac{(1+\varepsilon)\ell k}{n}$ indices $i \in [\ell]$ such that $v \in A_i$.*

(ii) *For each $v \in B$, there exist at most $\frac{(1+\varepsilon)\ell k}{n}$ indices $i \in [\ell]$ such that $v \in N_{M_i}(A_i)$.*

Lemma B.8. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \delta \leq 1$ and fix a positive integer $k \geq \varepsilon n$. Let D be a (δ, ε)-almost regular digraph on n vertices. Let A be chosen uniformly at random among subsets of $V(D)$ of size k. Then, the following hold with probability at least $1 - n^{-2}$.*

(i) *$D[A]$ and $D - A$ are $(\delta, 3\varepsilon)$-almost regular.*

(ii) *Each $v \in V(D) \setminus A$ satisfies $|N_D^\pm(v) \cap A| \geq (\delta - 3\varepsilon)|A|$.*

B.1.4 Matching contractions

Let G be a bipartite graph on vertex classes A and B and let D be a digraph on A. Let M be an auxiliary perfect matching from B to A. Recall Definition 7.25 and note that there is a one-to-one correspondence between the edges of $G \setminus M[B, A]$ and the M-contraction of G, as well as between the edges of D and the M-expansion of D. Thus, edge-disjointness and sub(di)graph relations are preserved when considering M-contractions and M-expansions.

Fact B.9. *Let A and B be disjoint vertex sets of equal size. Let M be a directed perfect matching from B to A. Let G and G' be bipartite graphs on vertex classes A and B and denote by G_M and G'_M the M-contractions of G and G', respectively. Let D and D' be digraphs on A and let D_M and D'_M be the M-expansions of D and D', respectively. Then, the following hold.*

(i) *If $G' \subseteq G$, then $G'_M \subseteq G_M$.*

(ii) *If G and G' are edge-disjoint, then G_M and G'_M are edge-disjoint.*

(iii) *If $D' \subseteq D$, then $D'_M \subseteq D_M$.*

(iv) *If D and D' are edge-disjoint, then D_M and D'_M are edge-disjoint.*

B.2 Proof of Theorem 8.1

We need a (simplified) bipartite analogue of [40, Lemma 7.3].

Lemma B.10. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu' \ll \nu \ll \tau \ll \gamma \ll \eta, \delta \leq 1$ and $\ell \leq 2(\delta - \eta)n$. Let D and Γ be edge-disjoint balanced bipartite digraphs on common vertex classes A and B of size n. Suppose that $D[A, B]$ is (δ, ε)-almost regular and $\Gamma[A, B]$ is (γ, ε)-almost regular. Suppose that $\Gamma[A, B]$ is a bipartite robust (ν, τ)-expander with bipartition (A, B). Suppose that, for each $i \in [\ell]$, F_i is a bipartite directed linear forest on vertex classes A and B such that the following hold.*

(i) *For each $i \in [\ell]$, $e_{F_i}(B, A) = n$.*

(ii) *For each* $i \in [\ell]$, $e_{F_i}(A, B) \leq \varepsilon^4 n$.

(iii) *For each* $v \in V(D)$, *there exist at most* $\varepsilon^3 n$ *indices* $i \in [\ell]$ *such that* $d_{F_i[A,B]}(v) > 0$.

Define a multidigraph $\mathcal{F}$ *by* $\mathcal{F} := \bigcup_{i\in[\ell]} F_i$. *Then, the multidigraph* $D \cup \Gamma \cup \mathcal{F}$ *contains edge-disjoint Hamilton cycles* $C_1, \dots, C_\ell$ *such that* $F_i \subseteq C_i$ *for each* $i \in [\ell]$ *and the following hold, where* $D' := D \setminus \bigcup_{i\in[\ell]} C_i$ *and* $\Gamma' := \Gamma \setminus \bigcup_{i\in[\ell]} C_i$.

(a) *If* $\ell \leq \varepsilon^2 n$, *then* $\Gamma'[A, B]$ *is obtained from* $\Gamma[A, B]$ *by removing at most* $3\varepsilon^3(\nu')^{-4}n$ *edges incident to each vertex.*

(b) *If* $\ell \leq (\nu')^5 n$, *then* $D'[A, B]$ *is* $(\delta - \frac{\ell}{2n}, 2\varepsilon)$*-almost regular and* $\Gamma'[A, B]$ *is* $(\gamma, 2\varepsilon)$*-almost regular. Moreover,* $\Gamma'[A, B]$ *is a bipartite robust* $(\nu - \varepsilon, \tau)$*-expander with bipartition* (A, B).

(c) $D'[A, B] \cup \Gamma'[A, B]$ *is a bipartite robust* $(\frac{\nu}{2}, \tau)$*-expander with bipartition* (A, B).

We first derive Theorem 8.1 from Lemma B.10(c).

Proof of Theorem 8.1. By Fact 7.2 and Lemma B.3, we may assume without loss of generality that $\varepsilon \ll \nu \ll \tau \ll \eta, \delta$. Define additional constants such that $\varepsilon \ll \nu' \ll \nu$ and $\tau \ll \gamma \ll \eta, \delta$. By Lemma B.6, there exists $\Gamma \subseteq D$ such that $(D \setminus \Gamma)[A, B]$ is $(\delta - \gamma, \varepsilon)$-almost regular and $\Gamma[A, B]$ is a (γ, ε)-almost regular bipartite robust $(\frac{\gamma\nu}{2\delta}, \tau)$-expander with bipartition (A, B). Apply Lemma B.10(c) with $D \setminus \Gamma, \delta - \gamma, \frac{\gamma\nu}{2\delta}$, and $\varepsilon^{\frac{1}{4}}$ playing the roles of D, δ, ν, and ε. This completes the proof of Theorem 8.1. □

We now prove Lemma B.10. First, Lemma B.10(b) follows by repeated applications of Lemma B.10(a) and Lemma B.10(c) follows by repeated applications of Lemma B.10(b). The arguments are the same as in [40], so we omit these proofs here. It remains to prove Lemma B.10(a).

Proof of Lemma B.10(a). Let $i \in [\ell]$. Define $M_i := E_{F_i}(B, A)$ and note that (i) implies that M_i is a perfect matching from B to A. Let $\widetilde{D}_i$, $\widetilde{\Gamma}_i$, and $\widetilde{F}_i$ be the M_i-contractions of $D[A, B]$, $\Gamma[A, B]$, and $F_i[A, B]$, respectively. (In the rest of the proof, tildes will be used to denote "contracted" digraphs on vertex set A.) By Proposition 7.30(i) and (iii), $\widetilde{D}_i$ is $(2\delta, 2\varepsilon)$-regular and $\widetilde{\Gamma}_i$ is a $(2\gamma, 2\varepsilon)$-regular robust $(\frac{\nu}{2}, \tau)$-outexpander. By Proposition 7.31, $\widetilde{F}_i$ is a linear forest. Let $\widetilde{P}_{i,1}, \dots, \widetilde{P}_{i,k_i}$ be an enumeration of the non-trivial components of $\widetilde{F}_i$. For each $j \in [k_i]$, denote by $x^+_{i,j}$ and $x^-_{i,j}$ the starting and ending points of $\widetilde{P}_{i,j}$. Let S_i be the set of vertices $v \in A$ which are not isolated in $\widetilde{F}_i$. Note that

$$k_i \leq e(\widetilde{F}_i) \overset{\text{Fact 7.26}}{\leq} e(F_i[A,B]) \overset{\text{(ii)}}{\leq} \varepsilon^4 n \tag{B.2}$$

and

$$|S_i| \leq 2e(\widetilde{F}_i) \overset{\text{Fact 7.26}}{\leq} 2e(F_i[A,B]) \overset{\text{(ii)}}{\leq} 2\varepsilon^4 n. \tag{B.3}$$

By Lemmas B.5, B.7, and B.8, there exist $A_1, \dots, A_\ell \subseteq A$ such that the following hold.

(α) For each $v \in A$, there exist at most $2\varepsilon^3(\nu')^{-4}n$ indices $i \in [\ell]$ such that $v \in A_i$.

(β) For each $v \in B$, there exist at most $2\varepsilon^3(\nu')^{-4}n$ indices $i \in [\ell]$ such that $v \in N_{F_i[B,A]}(A_i)$.

(γ) For each $i \in [\ell]$, $|A_i| = \lfloor \varepsilon(\nu')^{-4}n \rfloor$.

(δ) For each $i \in [\ell]$, $\widetilde{D}_i - A_i$ is $(2\delta, 6\varepsilon)$-almost regular.

(ε) For each $i \in [\ell]$ and $v \in A \setminus A_i$, $|N^{\pm}_{\widetilde{D}_i}(v) \cap A_i| \geq \frac{\varepsilon\delta n}{(\nu')^4}$.

(ζ) For each $i \in [\ell]$, $\widetilde{\Gamma}_i[A_i]$ and $\widetilde{\Gamma}_i - A_i$ are $(2\gamma, 6\varepsilon)$-almost regular.

(η) For each $i \in [\ell]$, $\widetilde{\Gamma}_i[A_i]$ is a robust $(\nu', 4\tau)$-outexpander.

For each $i \in [\ell]$, let $A'_i \coloneqq A_i \setminus S_i$. By (B.3), we have $|A'_i| \geq |A_i| - 2\varepsilon^4 n$. Therefore, Lemma B.4(ii) and (α)–(η) imply that the following hold for each $i \in [\ell]$.

(α') For each $v \in A$, there exist at most $2\varepsilon^3(\nu')^{-4}n$ indices $i \in [\ell]$ such that $v \in A'_i$.

(β') For each $v \in B$, there exist at most $2\varepsilon^3(\nu')^{-4}n$ indices $i \in [\ell]$ such that $v \in N_{F_i[B,A]}(A'_i)$.

(γ') $|A'_i| = \varepsilon((\nu')^{-4} \pm 1)n$.

(δ') $\widetilde{D}_i - A'_i$ is $(2\delta, 7\varepsilon)$-almost regular.

(ε') For each $v \in A \setminus A'_i$, $|N^{\pm}_{\widetilde{D}_i}(v) \cap A'_i| \geq \frac{\varepsilon\delta n}{2(\nu')^4}$.

(ζ') $\widetilde{\Gamma}_i[A'_i]$ and $\widetilde{\Gamma}_i - A'_i$ are both $(2\gamma, 7\varepsilon)$-almost regular.

(η') $\widetilde{\Gamma}_i[A'_i]$ and $\widetilde{\Gamma}_i - A'_i$ are both robust $(\frac{\nu'}{2}, 8\tau)$-outexpanders.

Assume inductively that for some $0 \leq m \leq \ell$ we have constructed, for each $i \in [m]$, a set $\widetilde{\mathcal{Q}}_i = \{\widetilde{Q}_{i,j} \mid j \in [k_i]\}$ of paths in $\widetilde{D}_i \cup \widetilde{\Gamma}_i$ such that the following hold, where, for each $i \in [m]$, $\mathcal{Q}_i$ is obtained from the M_i-expansion of $\widetilde{\mathcal{Q}}_i$ by orienting all the edges from A to B.

(A) $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ are edge-disjoint.

(B) Let $i \in [m]$. For each $j \in [k_i]$, $\widetilde{Q}_{i,j}$ is an $(x^-_{i,j}, x^+_{i,j+1})$-path (where $x^+_{i,k_i+1} \coloneqq x^+_{i,1}$). Moreover, the paths in $\widetilde{\mathcal{Q}}_i \cup \{\widetilde{P}_{i,1}, \dots, \widetilde{P}_{i,k_i}\}$ are internally vertex-disjoint and span A. In particular, $\widetilde{C}_i \coloneqq \widetilde{\mathcal{Q}}_i \cup \widetilde{F}_i$ is a Hamilton cycle on A.

(C) For each $i \in [m]$ and $j \in [k_i - 1]$, $V(\widetilde{Q}_{i,j}) \cap A'_i = \emptyset$ and $e(\widetilde{Q}_{i,j}) \leq 9(\nu')^{-1}$. Moreover, for each $v \in A$, there exist at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V(E(\widetilde{\mathcal{Q}}_i \setminus \{\widetilde{Q}_{i,k_i}\}) \cap E(\widetilde{\Gamma}_i))$ and for each $v \in B$, there exist at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in N_{F_i[B,A]}(V(E(\widetilde{\mathcal{Q}}_i \setminus \{\widetilde{Q}_{i,k_i}\}) \cap E(\widetilde{\Gamma}_i)))$.

(D) For each $i \in [m]$, $E(\widetilde{Q}_{i,k_i}) \cap E(\widetilde{\Gamma}_i) \subseteq E(\widetilde{\Gamma}_i[A'_i])$.

Denote $D_{m+1} \coloneqq D \setminus \bigcup_{i\in[m]} \mathcal{Q}_i$ and $\Gamma_{m+1} \coloneqq \Gamma \setminus \bigcup_{i\in[m]} \mathcal{Q}_i$.

Claim 1. *$\Gamma_{m+1}[A, B]$ is obtained from $\Gamma[A, B]$ by removing at most $3\varepsilon^3(\nu')^{-4}n$ edges incident to each vertex and $D_{m+1}[A, B]$ is obtained from $D[A, B]$ by removing at most m and at least $m - 4\varepsilon^3(\nu')^{-4}n$ edges incident to each vertex.*

Proof of Claim. Let $v \in A$. By (B), $d^+_{\widetilde{\mathcal{Q}}_i}(v) \le 1$ for each $i \in [m]$ and so

$$\begin{aligned} |N_{D[A,B]}(v) \setminus N_{D_{m+1}[A,B]}(v)| &= \sum_{i\in[m]} |N_{D[A,B]}(v) \cap N_{\mathcal{Q}_i[A,B]}(v)| \\ &\overset{\text{Fact 7.28(i)}}{=} \sum_{i\in[m]} |N^+_{\widetilde{D}_i}(v) \cap N^+_{\widetilde{\mathcal{Q}}_i}(v)| \\ &\le m. \end{aligned} \tag{B.4}$$

Moreover, (C) implies that there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V(E(\widetilde{\mathcal{Q}}_i \setminus \{\widetilde{Q}_{i,k_i}\}) \cap E(\widetilde{\Gamma}_i))$ and, by (α') and (D), there are at most $2\varepsilon^3(\nu')^{-4}n$ indices $i \in [m]$ such that $v \in V(E(\widetilde{Q}_{i,k_i}) \cap E(\widetilde{\Gamma}_i))$. Thus,

$$\begin{aligned} |N_{\Gamma[A,B]}(v) \setminus N_{\Gamma_{m+1}[A,B]}(v)| &= \sum_{i\in[m]} |N_{\Gamma[A,B]}(v) \cap N_{\mathcal{Q}_i[A,B]}(v)| \\ &\overset{\text{Fact 7.28(i)}}{=} \sum_{i\in[m]} |N^+_{\widetilde{\Gamma}_i}(v) \cap N^+_{\widetilde{\mathcal{Q}}_i}(v)| \\ &\le \varepsilon^3 n + 2\varepsilon^3(\nu')^{-4}n \\ &\le 3\varepsilon^3(\nu')^{-4}n. \end{aligned} \tag{B.5}$$

Moreover, Proposition 7.31 implies that, for each $i \in [m]$, we have $d^+_{\widetilde{F}_i}(v) > 0$ if and only if $d^+_{F_i[A,B]}(v) > 0$. Thus, (iii) implies that there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $d^+_{\widetilde{F}_i}(v) > 0$ and so

$$\begin{aligned} |N_{D[A,B]}(v) \setminus N_{D_{m+1}[A,B]}(v)| &\overset{\text{(B.4)}}{=} \sum_{i\in[m]} |N^+_{\widetilde{D}_i}(v) \cap N^+_{\widetilde{\mathcal{Q}}_i}(v)| \\ &\overset{\text{(B)}}{=} m - \sum_{i\in[m]} |N^+_{\widetilde{\Gamma}_i}(v) \cap N^+_{\widetilde{\mathcal{Q}}_i}(v)| - \sum_{i\in[m]} |N^+_{\widetilde{F}_i}(v)| \\ &\overset{\text{(B.5)}}{\ge} m - 3\varepsilon^3(\nu')^{-4}n - \varepsilon^3 n \ge m - 4\varepsilon^3(\nu')^{-4}n. \end{aligned}$$

Similarly, let $v \in B$. For each $i \in [m]$, denote by v_i the unique vertex of A such that

$vv_i \in F_i[B,A]$ (v_i exists by (i)). By (B), $d^-_{\widetilde{\mathcal{Q}}_i}(v_i) \leq 1$ for each $i \in [m]$ and so

$$\begin{aligned} |N_{D[A,B]}(v) \setminus N_{D_{m+1}[A,B]}(v)| &= \sum_{i\in[m]} |N_{D[A,B]}(v) \cap N_{\mathcal{Q}_i[A,B]}(v)| \\ &\overset{\text{Fact 7.28(ii)}}{=} \sum_{i\in[m]} |N^-_{\widetilde{D}_i}(v_i) \cap N^-_{\widetilde{\mathcal{Q}}_i}(v_i)| \qquad \text{(B.6)} \\ &\leq m. \end{aligned}$$

Moreover, (C) implies that there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v_i \in V(E(\widetilde{\mathcal{Q}}_i \setminus \{\widetilde{Q}_{i,k_i}\}) \cap E(\widetilde{\Gamma}_i))$ and, by (β') and (D), there are at most $2\varepsilon^3(\nu')^{-4}n$ indices $i \in [m]$ such that $v_i \in V(E(\widetilde{Q}_{i,k_i}) \cap E(\widetilde{\Gamma}_i))$. Thus,

$$\begin{aligned} |N_{\Gamma[A,B]}(v) \setminus N_{\Gamma_{m+1}[A,B]}(v)| &= \sum_{i\in[m]} |N_{\Gamma[A,B]}(v) \cap N_{Q_i[A,B]}(v)| \\ &\overset{\text{Fact 7.28(ii)}}{=} \sum_{i\in[m]} |N^-_{\widetilde{\Gamma}_i}(v_i) \cap N^-_{\widetilde{\mathcal{Q}}_i}(v_i)| \\ &\leq \varepsilon^3 n + 2\varepsilon^3(\nu')^{-4}n \\ &\leq 3\varepsilon^3(\nu')^{-4}n. \qquad \text{(B.7)} \end{aligned}$$

Moreover, Proposition 7.31 implies that, for each $i \in [m]$, we have $d^-_{\widetilde{F}_i}(v_i) > 0$ if and only if $d^-_{F_i[A,B]}(v) > 0$. Thus, (iii) implies that there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $d^-_{\widetilde{F}_i}(v_i) > 0$ and so

$$\begin{aligned} |N_{D[A,B]}(v) \setminus N_{D_{m+1}[A,B]}(v)| &\overset{\text{(B.6)}}{=} \sum_{i\in[m]} |N^-_{\widetilde{D}_i}(v_i) \cap N^-_{\widetilde{\mathcal{Q}}_i}(v_i)| \\ &\overset{\text{(B)}}{=} m - \sum_{i\in[m]} |N^-_{\widetilde{\Gamma}_i}(v_i) \cap N^-_{\widetilde{\mathcal{Q}}_i}(v_i)| - \sum_{i\in[m]} |N^-_{\widetilde{F}_i}(v_i)| \\ &\overset{\text{(B.7)}}{\geq} m - 3\varepsilon^3(\nu')^{-4}n - \varepsilon^3 n \geq m - 4\varepsilon^3(\nu')^{-4}n. \end{aligned}$$

This completes the proof of Claim 1. $\diamond$

Suppose that $m = \ell$. For each $i \in [\ell]$, let $C_i := \mathcal{Q}_i \cup F_i$. By (A), $C_1, \ldots, C_\ell$ are edge-disjoint. Let $i \in [\ell]$. Note that Fact 7.26 implies that $F_i[A, B]$ is the M_i-expansion of $\widetilde{F}_i$. Thus, (B) implies that C_i is obtained from the M_i-expansion of $\widetilde{C}_i$ by orienting all the edges from A to B, and then adding the edges in $E_{F_i}(B, A) = M_i$. In particular, Fact 7.29 and (B) imply that C_i is a Hamilton cycle on $V(D)$. Moreover, (a) follows from Claim 1. It remains to show that $F_i \subseteq C_i \subseteq D \cup \Gamma \cup F_i$. By (B), $\widetilde{F}_i \subseteq \widetilde{C}_i \subseteq \widetilde{D}_i \cup \widetilde{\Gamma}_i \cup \widetilde{F}_i$ and, by Fact 7.26, the M_i-expansions of $\widetilde{D}_i$ and $\widetilde{\Gamma}_i$ are subgraphs of $D[A, B]$ and $\Gamma[A, B]$. Thus, Fact B.9(iii) implies that $F_i \subseteq C_i \subseteq D \cup \Gamma \cup F_i$ and so we are done.

Assume that $m < \ell$. Let $\widetilde{D}'_{m+1}$ and $\widetilde{\Gamma}'_{m+1}$ be the M_{m+1}-contractions of D_{m+1} and Γ_{m+1}, respectively. By Fact B.9(i), $\widetilde{D}'_{m+1} \subseteq \widetilde{D}_{m+1}$ and $\widetilde{\Gamma}'_{m+1} \subseteq \widetilde{\Gamma}_{m+1}$. By Fact 7.27(i) and Claim 1, $\widetilde{\Gamma}'_{m+1}$ is obtained from $\widetilde{\Gamma}_{m+1}$ by removing at most $3\varepsilon^3(\nu')^{-4}n$ inedges incident to each vertex and at most $3\varepsilon^3(\nu')^{-4}n$ outedges incident to each vertex. Similarly, $\widetilde{D}'_{m+1}$ is obtained from $\widetilde{D}_{m+1}$ by removing at most m and at least $m - 4\varepsilon^3(\nu')^{-4}n$ inedges incident to each vertex, as well as at most m and at least $m - 4\varepsilon^3(\nu')^{-4}n$ outedges incident to each vertex. Thus, using (γ')–(η') and Lemma B.4(i), it is easy to check that the following hold.

(I) $\widetilde{D}'_{m+1} - A'_{m+1}$ is $(2\delta - \frac{m}{n}, 8\varepsilon)$-almost regular.

(II) For each $v \in A \setminus A'_{m+1}$, $|N^{\pm}_{\widetilde{D}'_{m+1}}(v) \cap A'_{m+1}| \geq \frac{\varepsilon\delta n}{3(\nu')^4}$.

(III) $\widetilde{\Gamma}'_{m+1}[A'_{m+1}]$ and $\widetilde{\Gamma}'_{m+1} - A'_{m+1}$ are both $(2\gamma, 8\varepsilon)$-almost regular.

(IV) $\widetilde{\Gamma}'_{m+1}[A'_{m+1}]$ and $\widetilde{\Gamma}'_{m+1} - A'_{m+1}$ are robust $(\frac{\nu'}{4}, 8\tau)$-outexpanders.

Let S_A be the set of vertices $v \in A$ for which there exist $\lfloor \varepsilon^3 n \rfloor$ indices $i \in [m]$ such that $v \in V(E(\widetilde{\mathcal{Q}}_i \setminus \{\widetilde{Q}_{i,k_i}\}) \cap E(\widetilde{\Gamma}_i))$. Observe that, by (B.2), (γ'), and (C), $|S_A| \leq \frac{(9(\nu')^{-1}+1)\cdot\varepsilon^4 n\cdot m}{\lfloor \varepsilon^3 n \rfloor} \leq \varepsilon|A \setminus A'_{m+1}|$. Let S_B be the set of vertices $v \in B$ such that there exist $\lfloor \varepsilon^3 n \rfloor$ indices $i \in [m]$ such that $v \in N_{F_i[B,A]}(V(E(\widetilde{\mathcal{Q}}_i \setminus \{\widetilde{Q}_{i,k_i}\}) \cap E(\widetilde{\Gamma}_i)))$. Let $S'_B \coloneqq N_{F_{m+1}[B,A]}(S_B)$. Then, by similar arguments as above, $|S'_B| = |S_B| \leq \varepsilon|A \setminus A'_{m+1}|$. Recall that S_{m+1} denotes the set of vertices $v \in A$ which are not isolated in $\widetilde{F}_{m+1}$ (and so $\{x^+_{m+1,i}, x^-_{m+1,i} \mid i \in [k_{m+1}]\} \subseteq S_{m+1}$). Moreover, $S_{m+1} \cap A'_{m+1} = \emptyset$. By (B.3), (γ'), and the above, $|S_A \cup S'_B \cup S_{m+1}| \leq 3\varepsilon|A \setminus A'_{m+1}|$. By (B.2) and (γ'), $k_i \leq \varepsilon^4 n \leq \varepsilon^3|A \setminus A'_{m+1}|$. Thus, (I) implies that there exist distinct $y^-_{m+1,1}, \ldots, y^-_{m+1,k_{m+1}-1}, y^+_{m+1,2}, \ldots, y^+_{m+1,k_{m+1}} \in A \setminus$

$(A'_{m+1} \cup S_A \cup S'_B \cup S_{m+1})$ such that $y^-_{m+1,i} \in N^+_{\widetilde{D}'_{m+1}}(x^-_{m+1,i})$ and $y^+_{m+1,i+1} \in N^-_{\widetilde{D}'_{m+1}}(x^+_{m+1,i+1})$ for each $i \in [k_{m+1} - 1]$ and we can apply the arguments of [40] to construct vertex-disjoint paths $\widetilde{Q}'_{m+1,1}, \dots, \widetilde{Q}'_{m+1,k_{m+1}-1} \subseteq \widetilde{\Gamma}_{m+1} - A'_{m+1}$ such that the following hold for each $i \in [k_{m+1} - 1]$.

- $\widetilde{Q}'_{m+1,i}$ is a $(y^-_{m+1,i}, y^+_{m+1,i+1})$-path of length at most $8(\nu')^{-1}$.
- $V(\widetilde{Q}'_{m+1,i}) \subseteq A \setminus (A'_{m+1} \cup S_A \cup S'_B \cup S_{m+1})$.

(Roughly speaking, the paths $\widetilde{Q}'_{m+1,1}, \dots, \widetilde{Q}'_{m+1,k_{m+1}-1}$ are constructed greedily by applying the definition of robust outexpansion.) For each $i \in [k_{m+1} - 1]$, define

$$\widetilde{Q}_{m+1,i} \coloneqq x^-_{m+1,i} y^-_{m+1,i} \widetilde{Q}'_{m+1,i} y^+_{m+1,i+1} x^+_{m+1,i+1}.$$

Let $\widetilde{\mathcal{Q}}'_{m+1} \coloneqq \{\widetilde{Q}_{m+1,i} \mid i \in [k_{m+1} - 1]\}$ and proceed as in [40] to build an $(x^-_{m+1,k_{m+1}}, x^+_{m+1,1})$-path $\widetilde{Q}_{m+1,k_{m+1}}$ satisfying the following.

- $V^0(\widetilde{Q}_{m+1,k_{m+1}}) = A \setminus (V(\widetilde{\mathcal{Q}}'_{m+1}) \cup S_{m+1})$.
- $\widetilde{Q}_{m+1,k_{m+1}}[A'_{m+1}] \subseteq \widetilde{\Gamma}'_{m+1}$.
- $\widetilde{Q}_{m+1,k_{m+1}} \setminus \widetilde{Q}_{m+1,k_{m+1}}[A'_{m+1}] \subseteq \widetilde{D}'_{m+1}$.

(Roughly speaking, $\widetilde{Q}_{m+1,k_{m+1}}$ is constructed by applying Theorem 7.4 to find a Hamilton cycle in a suitable auxiliary digraph.) Let $\widetilde{\mathcal{Q}}_{m+1} \coloneqq \widetilde{\mathcal{Q}}'_{m+1} \cup \{\widetilde{Q}_{m+1,k_{m+1}}\}$. Then, (A)–(D) hold with $m + 1$ playing the role of m. $\square$

APPENDIX C

THE ROBUST DECOMPOSITION LEMMAS: PROOFS OF LEMMAS 8.23 AND 11.10

In this appendix, we derive the modified robust decomposition lemma (Lemma 8.23) and the robust decomposition lemma for blow-up cycles (Lemma 11.10).

C.1 Proof of Lemma 8.23

We need the following version of the robust decomposition lemma, which follows immediately from the proof of [76, Lemma 12.1] and the definition of a "chord absorber" in [76]. (Note that [76, Lemma 12.1] is not explicitly proven because its proof is identical to that of [76, Lemma 11.2]. See the paragraph before [76, Lemma 12.1] for more details.)

Lemma C.1 (Robust decomposition lemma [76])**.** *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll \frac{r_1}{m} \ll d \ll \frac{1}{\ell'}, \frac{1}{g} \ll 1$ and suppose that $rk^2 \leq m$. Let*

$$r_2 := 96\ell' g^2 kr, \quad r_3 := \frac{rfk}{q}, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3,$$

and suppose that $\frac{k}{14}, \frac{k}{f}, \frac{k}{g}, \frac{q}{f}, \frac{m}{4\ell'}, \frac{fm}{q}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$. Suppose that $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \frac{q}{f}, k, m, \varepsilon, d)$-setup with empty exceptional set V_0. Suppose that $\mathcal{SF} \subseteq D$ consists of r_3 edge-disjoint $(\frac{q}{f}, f)$-special factors with respect to $\mathcal{P}^*$ and C. Then, there exists a spanning subdigraph $CA^\diamond(r) \subseteq D$ for which the following hold.

(i) *$CA^\diamond(r)$ is an (r_1+r_2)-regular spanning subdigraph of D which is edge-disjoint from $\mathcal{SF}$.*

(ii) *Let H be an r-regular digraph on $V(D)$ which is edge-disjoint from $CA^\diamond(r)$. Suppose that $\mathcal{SF}^*$ consists of r_3 edge-disjoint $(\frac{q}{f}, f)$-special factors with respect to $\mathcal{P}^*$ and C which are edge-disjoint from $CA^\diamond(r) \cup H$. (Note that $\mathcal{SF}^*$ is not necessarily a subdigraph of D here.) Then, there exists a set $\mathscr{C}_1$ of rfk edge-disjoint Hamilton cycles such that $E(H) \cup E(\mathcal{SF}^*) \subseteq E(\mathscr{C}_1) \subseteq E(CA^\diamond(r)) \cup E(H) \cup E(\mathcal{SF}^*)$ and each cycle in $\mathscr{C}_1$ contains precisely one of the special path systems contained in $\mathcal{SF}^*$. Denote $H' \coloneqq CA^\diamond(r) \setminus E(\mathscr{C}_1)$.*

(iii) *Suppose that $\mathcal{SF}' \subseteq D \setminus (CA^\diamond(r) \cup \mathcal{SF})$ consists of $r^\diamond$ edge-disjoint $(1,7)$-special factors with respect to $\mathcal{P}$ and C. Then, there exists a spanning subdigraph $PCA^\diamond(r) \subseteq D$ for which the following hold.*

   (a) *$PCA^\diamond(r)$ is a $5r^\diamond$-regular spanning subdigraph of D which is edge-disjoint from $CA^\diamond(r) \cup \mathcal{SF} \cup \mathcal{SF}'$.*

   (b) *$H' \cup PCA^\diamond(r) \cup \mathcal{SF}'$ has a decomposition $\mathscr{C}_2$ into $7r^\diamond$ edge-disjoint Hamilton cycles such that each cycle in $\mathscr{C}_2$ contains precisely one of the special path systems in $\mathcal{SF}'$.*

The analogue holds if $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \frac{q}{f}, k, m, \varepsilon, d)$-bi-setup and H is an r-regular bipartite digraph on the same vertex classes as D.

Note that (i) and (iii.a) correspond to [76, Lemma 12.1(i) and (ii.a)]. To see (ii), observe that by the proof of [76, Lemma 12.1] (see the proof of [76, Lemma 11.2]), $CA^\diamond(r) \cup \mathcal{SF}$ is a "chord absorber". By definition, $CA^\diamond(r) \cup \mathcal{SF}^*$ is also a "chord absorber" in $D \cup \mathcal{SF}^*$. Moreover, since r_3 is very small, Proposition 8.16 implies that $(D \cup \mathcal{SF}^*, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ also forms a (bi)-setup (with slightly worse parameters). Thus, $\mathscr{C}_1$ in (ii) can be obtained by applying the arguments of [76, Lemma 11.2] with $D \cup \mathcal{SF}^*$ and $CA^\diamond(r) \cup \mathcal{SF}^*$ playing the role of G and $CA^\diamond(r) \cup \mathcal{SF}$. Then, $\mathscr{C}_2$ in (iii.b) is the set of Hamilton cycles obtained by applying the arguments of [76, Lemma 11.2] with H' playing the role of H_1.

Here, Lemma C.1 is stated in terms of our simplified definition of a special factors (see Section 8.2.5). The fact that $\mathcal{SF} \cup \mathcal{SF}' \subseteq D$ ensures that the special factors in $\mathcal{SF} \cup \mathcal{SF}'$ are indeed special factors in the stronger sense of [76] (the set of fictive edges is empty). As discussed above, the special factors in $\mathcal{SF}^*$ will be considered within the digraph $D \cup \mathcal{SF}^*$, so they also satisfy the original definition of special factors [76] with $D \cup \mathcal{SF}^*$ playing the role of G (once again, with an empty set of fictive edges).

As discussed in Section 8.2.6, the key idea for deriving Lemma 8.23 from Lemma C.1 is to consider equivalent special path systems. These will be constructed using the superregular pairs of the (bi)-setup.

Lemma C.2. *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll d \leq 1$ and $\frac{1}{k} \ll \frac{1}{f}, \frac{1}{\ell^*} \leq 1$ and $\frac{r\ell^*}{m} \ll d$. Let $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ be an $(\ell', \ell^*, k, m, \varepsilon, d)$-(bi)-setup and suppose that $\mathcal{P}^*$ is an ε-uniform ℓ^*-refinement of $\mathcal{P}$. Let $(h, j) \in [\ell^*] \times [f]$ and suppose that $SPS_1, \dots, SPS_r$ are (ℓ^*, f, h, j)-special path systems with respect to $\mathcal{P}^*$ and C. Then, there exist edge-disjoint (ℓ^*, f, h, j)-special path systems $SPS'_1, \dots, SPS'_r \subseteq D$ with respect to $\mathcal{P}^*$ and C such that SPS_i and SPS'_i are equivalent for each $i \in [r]$.*

Proof. Fix additional constants such that $\frac{r\ell^*}{m}, \varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll d$. Denote by $I = V_1 \dots V_{k'}$ the j^{th} interval in the canonical interval partition of C into f intervals. For each $i \in [k']$, denote by $V_{i,h}$ the h^{th} subcluster of V_i contained in $\mathcal{P}^*$ and observe that (ST1), (ST8), (BST1), and (BST8) imply that $|V_{i,h}| = \frac{m}{\ell^*}$. Suppose inductively that, for some $\ell \in [r]$, we have already constructed edge-disjoint (ℓ^*, f, h, j)-special path systems $SPS'_1, \dots, SPS'_{\ell-1} \subseteq D$ with respect to $\mathcal{P}^*$ and C such that SPS_i and SPS'_i are equivalent for each $i \in [\ell - 1]$. We construct SPS'_ℓ using Lemma 7.11 as follows.

Let $D' := D \setminus \bigcup_{i \in [\ell-1]} SPS_i$. We claim that $D'[V_{i,h}, V_{i+1,h}]$ is $[3\sqrt{\varepsilon_2}, \geq d]$-superregular for each $i \in [k' - 1]$. Indeed, (ST3), (BST3), and Lemma 8.8(ii) imply that $D[V_{i,h}, V_{i+1,h}]$ is $[\varepsilon_1, \geq d]$-superregular for each $i \in [k' - 1]$. Since D' is obtained from D by removing at most $r \leq \frac{\varepsilon_2 m}{\ell^*}$ in- and outedges incident to each vertex, Proposition 7.8(ii) implies that $D'[V_{i,h}, V_{i+1,h}]$ is $[3\sqrt{\varepsilon_2}, \geq d]$-superregular for each $i \in [k'-1]$. Let $u_1, \dots, u_{\frac{m}{\ell^*}}$ be an enumeration of $V^+(SPS_\ell)$. For each $i \in [\frac{m}{\ell^*}]$, let v_i denote the ending point of the path in SPS_ℓ which starts at u_i. By (SPS1), $u_1, \dots, u_{\frac{m}{\ell^*}}$ and $v_1, \dots, v_{\frac{m}{\ell^*}}$ are enumerations of $V_{1,h}$ and $V_{k',h}$, respectively. Let SPS'_ℓ be the set of paths obtained by applying Lemma 7.11 with $D'[\bigcup_{i \in [k']} V_{i,h}], \frac{m}{\ell^*}, k', 3\sqrt{\varepsilon_2}$, and $V_{1,h}, \dots, V_{k',h}$ playing the roles of D, m, k, ε, and $V_1, \dots, V_k$. Then, SPS'_ℓ is an (ℓ^*, f, h, j)-special path system with respect to $\mathcal{P}^*$ and C and, by construction, it is equivalent to SPS_ℓ, as desired. □

Proof of Lemma 8.23. We prove the setup and bi-setup versions of Lemma 8.23 in parallel. Fix additional constants such that $\varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll \frac{1}{q}$ and $\frac{r_1}{m} \ll \varepsilon_3 \ll d$. By Lemmas 8.9 and 8.19, there exist edge-disjoint $D_1, D_2 \subseteq D$ such that both $(D_1, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ and $(D_2, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ are $(\ell', \frac{q}{f}, k, m, \varepsilon_1, \frac{d}{2})$-(bi)-setups.

Since $\mathcal{SF}$ consists of special factors which are not necessarily edge-disjoint, we need to construct auxiliary special path systems which are edge-disjoint from each other and equivalent to those in $\mathcal{SF}$. For each $(h, j) \in [\frac{q}{f}] \times [f]$, denote by $SPS_{1,h,j}, \dots, SPS_{r_3,h,j}$ the $(\frac{q}{f}, f, h, j)$-special path systems contained in $\mathcal{SF}$.

Claim 1. *For each $(h, j) \in [\frac{q}{f}] \times [f]$, there exist edge-disjoint $(\frac{q}{f}, f, h, j)$-special path systems $SPS^\diamond_{1,h,j}, \dots, SPS^\diamond_{r_3,h,j}$ with respect to $\mathcal{P}^*$ and C such that $SPS_{i,h,j}$ and $SPS^\diamond_{i,h,j}$ are equivalent for each $i \in [r_3]$. In particular, $SF^\diamond_i := \bigcup_{(h,j) \in [\frac{q}{f}] \times [f]} SPS^\diamond_{i,h,j}$ is a $(\frac{q}{f}, f)$-special factor with respect to $\mathcal{P}^*$ and C for each $i \in [r_3]$.*

Proof of Claim. For the setup version of Lemma 8.23, let K be the complete graph on $V(D)$; for the bi-setup version, let K be the complete bipartite graph on the vertex classes of D. One can easily verify that $(K, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \frac{q}{f}, k, m, \varepsilon, d)$-(bi)-setup ((super)regular pairs exist by Proposition 7.10). For each $(h, j) \in [\frac{q}{f}] \times [f]$, let $SPS^\diamond_{1,h,j}, \dots, SPS^\diamond_{r_3,h,j}$ be the special path systems obtained by applying Lemma C.2 with $K, \frac{q}{f}, r_3$, and $SPS_{1,h,j}, \dots, SPS_{r_3,h,j}$ playing the roles of D, ℓ^*, r, and $SPS_1, \dots, SPS_r$. ◇

Let $\mathcal{SF}^\diamond$ be the union of the r_3 edge-disjoint $(\frac{q}{f}, f)$-special factors with respect to $\mathcal{P}^*$ and C obtained by applying Claim 1. Let $D'_1 := D_1 \cup \mathcal{SF}^\diamond$ and observe that D'_1 is obtained from D_1 by adding at most r_3 in- and outedges incident to each vertex (recall from Definition 8.22 that special factors are digraphs of maximum semidegree 1). Thus, Proposition 8.16 implies that $(D_1', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is still an $(\ell', \frac{q}{f}, k, m, \varepsilon_2, \frac{d}{4})$-(bi)-setup. Let $CA^\diamond(r)$ be the spanning subdigraph of D_1' obtained by applying Lemma C.1 with $D_1', \mathcal{SF}^\diamond, \varepsilon_2$, and $\frac{d}{4}$ playing the roles of $D, \mathcal{SF}, \varepsilon$, and d. Note that Lemma C.1(i) implies that $CA^\diamond(r) \subseteq D_1 \subseteq D$.

Since the special factors in $\mathcal{SF}'$ may not be edge-disjoint subdigraphs of $D_1'' := D_1' \setminus (CA^\diamond(r) \cup \mathcal{SF}^\diamond) = D_1 \setminus (CA^\diamond(r) \cup \mathcal{SF}^\diamond)$, we need to construct auxiliary special path systems which are equivalent to those in $\mathcal{SF}'$. For each $(h,j) \in [1] \times [7]$, let $SPS'_{1,h,j}, \dots, SPS'_{r^\diamond,h,j}$ denote the $(1,7,h,j)$-special path systems contained in $\mathcal{SF}'$.

Claim 2. *For each $(h,j) \in [1] \times [7]$, there exist edge-disjoint $(1,7,h,j)$-special path systems $SPS''_{1,h,j}, \dots, SPS''_{r^\diamond,h,j} \subseteq D_1''$ with respect to $\mathcal{P}$ and C such that $SPS'_{i,h,j}$ and $SPS''_{i,h,j}$ are equivalent for each $i \in [r^\diamond]$. In particular, $SF_i'' := \bigcup_{(h,j)\in[1]\times[7]} SPS_{i,h,j}$ is a $(1,7)$-special factor with respect to $\mathcal{P}$ and C for each $i \in [r^\diamond]$.*

Proof of Claim. By Fact 8.17, $(D_1, \mathcal{P}, \mathcal{P}', \mathcal{P}, R, C, U, U')$ is an $(\ell', 1, k, m, \varepsilon_1, \frac{d}{2})$-(bi)-setup. By Lemma C.1(i) and Definition 8.22, D_1'' is obtained from D_1 removing at most $r_1+r_2+r_3$ in- and outedges at each vertex, so Proposition 8.16 implies that $(D_1'', \mathcal{P}, \mathcal{P}', \mathcal{P}, R, C, U, U')$ is still an $(\ell', 1, k, m, \varepsilon_3, \frac{d}{4})$-(bi)-setup. For each $(h,j) \in [1] \times [7]$, let $SPS''_{1,h,j}, \dots, SPS''_{r^\diamond,h,j}$ be the special path systems obtained by applying Lemma C.2 with $D_1'', \mathcal{P}, \varepsilon_3, \frac{d}{4}, 1, 7, r^\diamond$, and $SPS'_{1,1,j}, \dots, SPS'_{r^\diamond,1,j}$ playing the roles of $D, \mathcal{P}^*, \varepsilon, d, \ell^*, f, r$, and $SPS_1, \dots, SPS_r$. $\diamond$

Let $\mathcal{SF}''$ be the union of the $r^\diamond$ edge-disjoint $(1,7)$-special factors with respect to $\mathcal{P}$ and C obtained by applying Claim 2. Let $PCA^\diamond(r)$ be the spanning subdigraph of D_1 obtained by applying Lemma C.1(iii) with $\mathcal{SF}''$ playing the role of $\mathcal{SF}'$. By Lemma C.1(iii.a), $PCA^\diamond(r) \subseteq D_1'' \setminus \mathcal{SF}'' \subseteq D_1 \subseteq D$. Define $D^{\mathrm{rob}} := CA^\diamond(r) \cup PCA^\diamond(r) \subseteq D$ and observe that Lemma C.1(i) and (iii.a) imply that D^{rob} is $(r_1 + r_2 + 5r^\diamond)$-regular, as desired.

Let H be an r-regular digraph on $V(D)$. For the bi-setup version of Lemma 8.23, suppose furthermore that H is a bipartite digraph on the same vertex classes as D. It remains to show that the multidigraph $H \cup D^{\rm rob} \cup \mathcal{SF} \cup \mathcal{SF}'$ has a decomposition $\mathscr{C}$ into s' edge-disjoint Hamilton cycles such that each Hamilton cycle in $\mathscr{C}$ contains precisely one of the special path systems in the multidigraph $\mathcal{SF} \cup \mathcal{SF}'$. We will use Lemma C.1(ii) and (iii.b).

Claim 3. *For each $(h, j) \in [\frac{q}{f}] \times [f]$, there exist edge-disjoint $(\frac{q}{f}, f, h, j)$-special path systems $SPS^*_{1,h,j}, \dots, SPS^*_{r_3,h,j}$ with respect to $\mathcal{P}^*$ and C which are edge-disjoint from $CA^\diamond(r) \cup H$ and such that $SPS_{i,h,j}$ and $SPS^*_{i,h,j}$ are equivalent for each $i \in [r_3]$. In particular, $SF^*_i := \bigcup_{(h,j)\in[\frac{q}{f}]\times[f]} SPS^*_{i,h,j}$ is a $(\frac{q}{f}, f)$-special factor with respect to $\mathcal{P}^*$ and C for each $i \in [r_3]$.*

Proof of Claim. Let $D_2' := D_2 \setminus H$. Since $CA^\diamond(r) \subseteq D_1$, note that D_2' is edge-disjoint from $CA^\diamond(r) \cup H$. By Proposition 8.16, $(D_2', \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \frac{q}{f}, k, m, \varepsilon_2, \frac{d}{4})$-(bi)-setup. For each $(h, j) \in [\frac{q}{f}] \times [f]$, let $SPS^*_{1,h,j}, \dots, SPS^*_{r_3,h,j}$ be the $(\frac{q}{f}, f)$-special path systems with respect to $\mathcal{P}^*$ and C obtained by applying Lemma C.2 with $D_2', \varepsilon_2, \frac{d}{4}, \frac{q}{f}$, and r_3 playing the roles of $D, \varepsilon, d, \ell^*$, and r. $\diamond$

Let $\mathcal{SF}^*$ be the union of the r_3 edge-disjoint $(\frac{q}{f}, f)$-special factors with respect to $\mathcal{P}^*$ and C obtained by applying Claim 3. By Lemma C.1(ii), there exists a set $\mathscr{C}_1$ of rfk edge-disjoint Hamilton cycles such that $E(H) \cup E(\mathcal{SF}^*) \subseteq E(\mathscr{C}_1) \subseteq E(CA^\diamond(r)) \cup E(H) \cup E(\mathcal{SF}^*)$ and each cycle in $\mathscr{C}_1$ contains precisely one of the special path systems contained in $\mathcal{SF}^*$. Denote $H' := CA^\diamond(r) \setminus E(\mathscr{C}_1)$. By Lemma C.1(iii.b), $H' \cup PCA^\diamond(r) \cup \mathcal{SF}''$ has a decomposition $\mathscr{C}_2$ into $7r^\diamond$ edge-disjoint Hamilton cycles such that each cycle in $\mathscr{C}_2$ contains precisely one of the special path systems in $\mathcal{SF}''$. Altogether, $\mathscr{C}_1 \cup \mathscr{C}_2$ forms a decomposition of the multidigraph $H \cup D^{\rm rob} \cup \mathcal{SF}^* \cup \mathcal{SF}''$ into s' Hamilton cycles such that each cycle in $\mathscr{C}_1 \cup \mathscr{C}_2$ contains precisely one of the special path systems in $\mathcal{SF}^* \cup \mathcal{SF}''$. By Claims 2 and 3 and Fact 8.5, $\mathscr{C}_1 \cup \mathscr{C}_2$ induces a decomposition $\mathscr{C}$ of the multidigraph $H \cup D^{\rm rob} \cup \mathcal{SF} \cup \mathcal{SF}'$ into s' edge-disjoint Hamilton cycles such that each Hamilton cycle in $\mathscr{C}$ contains precisely one of the special path systems contained in the multidigraph $\mathcal{SF} \cup \mathcal{SF}'$. $\square$

C.2 Proof of Lemma 11.10

We know derive the blow-up cycle version of the robust decomposition lemma using the strategy presented in Section 11.1.

Proof of Lemma 11.10. By Fact 8.5 and Definition 11.8, we may assume without loss of generality that all extended special path systems contained in the multidigraph $\mathcal{ESF}\cup\mathcal{ESF}'$ are friendly. For each $(h,i,j)\in[\frac{q}{f}]\times[K]\times[f]$, denote by $ESPS_{1,h,i,j},\dots,ESPS_{r_3,h,i,j}$ the r_3 friendly $(\frac{q}{f},K,f,h,i,j)$-extended special path systems contained in $\mathcal{ESF}$. For each $(h,i,j)\in[1]\times[K]\times[7]$, denote by $ESPS'_{1,h,i,j},\dots,ESPS'_{r^\diamond,h,i,j}$ the $r^\diamond$ friendly $(\frac{q}{f},K,f,h,i,j)$-extended special path systems contained in $\mathcal{ESF}'$. For each $i\in[K]$, define

$$\mathcal{ESF}_i \coloneqq \bigcup_{(\ell,h,j)\in[r_3]\times[\frac{q}{f}]\times[f]} ESPS_{\ell,h,i,j} \quad\text{and}\quad \mathcal{ESF}'_i \coloneqq \bigcup_{(\ell,h,j)\in[r^\diamond]\times[1]\times[7]} ESPS'_{\ell,h,i,j}.$$

Step 1: Applying the robust decomposition lemma in each contracted pair. Let $i\in[K]$. Denote by $\widetilde{D}_i$ the M_i-contraction of $D[U_i,U_{i+1}]$. By (CST3), $(\widetilde{D}_i,\mathcal{P}_i,\mathcal{P}'_i,\mathcal{P}^*_i,R_i,C^i,U^i,U'^i)$ is an $(\ell',\frac{q}{f},k,m,\varepsilon,d)$-setup with an empty exceptional set. We construct the required special factors for applying the robust decomposition lemma in $\widetilde{D}_i$ as follows. For each $(h,j)\in[\frac{q}{f}]\times[f]$, let $SPS_{1,h,i,j},\dots,SPS_{r_3,h,i,j}$ be obtained from the M_i-contractions of $ESPS_{1,h,i,j}[U_i,U_{i+1}],\dots,ESPS_{r_3,h,i,j}[U_i,U_{i+1}]$ by deleting all isolated vertices. For each $(h,j)\in[\frac{q}{f}]\times[f]$, (FESPS1) implies that $SPS_{1,h,i,j},\dots,SPS_{r_3,h,i,j}$ are $(\frac{q}{f},f,h,j)$-special path systems with respect to $\mathcal{P}^*_i$ and C^i. For each $\ell\in[r_3]$, let $SF_{\ell,i}\coloneqq\bigcup_{(h,j)\in[\frac{q}{f}]\times[f]}SPS_{\ell,h,i,j}$ and observe that $SF_{\ell,i}$ is a $(\frac{q}{f},f)$-special factor with respect to $\mathcal{P}^*_i$ and C^i. Define a multidigraph $\mathcal{SF}_i$ by $\mathcal{SF}_i\coloneqq\bigcup_{\ell\in[r_3]}SF_{\ell,i}$. Define $\mathcal{SF}'_i$ analogously. Let $\widetilde{D}_i^{\rm rob}$ be the spanning subdigraph of $\widetilde{D}_i$ obtained by applying Lemma 8.23 with $\widetilde{D}_i,\mathcal{P}_i,\mathcal{P}'_i,\mathcal{P}^*_i,R_i,C^i,U^i,U'^i,\mathcal{SF}_i$, and $\mathcal{SF}'_i$ playing the roles of $D,\mathcal{P},\mathcal{P}',\mathcal{P}^*,R,C,U,U'$, $\mathcal{SF}$, and $\mathcal{SF}'$.

Step 2: Constructing the robustly decomposable digraph. For each $i\in[K]$, let

D_i^{rob} be obtained from the M_i-expansion of $\widetilde{D}_i^{\mathrm{rob}}$ by orienting all the edges from U_i to U_{i+1}. Define $D^{\mathrm{rob}} \coloneqq \bigcup_{i\in[K]} D_i^{\mathrm{rob}}$. First, observe that D^{rob} is a regular spanning subdigraph of D.

Claim 1. *D^{rob} is an $(r_1 + r_2 + 5r^\diamond)$-regular spanning subdigraph of D.*

Proof of Claim. By definition, $V(D_i^{\mathrm{rob}}) = U_i \cup U_{i+1}$ for each $i \in [K]$ and so D^{rob} is spanning. For each $i \in [K]$, $\widetilde{D}_i^{\mathrm{rob}} \subseteq \widetilde{D}_i$ and Fact 7.26 implies that the M_i-expansion of $\widetilde{D}_i$ is a subdigraph of $D[U_i, U_{i+1}]$. Thus, Fact B.9(iii) implies that $D_i^{\mathrm{rob}}[U_i, U_{i+1}] \subseteq D[U_i, U_{i+1}]$ for each $i \in [K]$ and so $D^{\mathrm{rob}} \subseteq D$. Let $i \in [K]$ and $v \in U_i$. By construction,

$$d^+_{D^{\mathrm{rob}}}(v) = d^+_{D_i^{\mathrm{rob}}}(v) \overset{\text{Fact 7.28(i)}}{=} d^+_{\widetilde{D}_i^{\mathrm{rob}}}(v) \overset{\text{Lemma 8.23}}{=} r_1 + r_2 + 5r^\diamond.$$

Let v' be the unique neighbour of v in M_{i-1}. Then,

$$d^-_{D^{\mathrm{rob}}}(v) = d^-_{D_{i-1}^{\mathrm{rob}}}(v) \overset{\text{Fact 7.28(ii)}}{=} d^-_{\widetilde{D}_{i-1}^{\mathrm{rob}}}(v') \overset{\text{Lemma 8.23}}{=} r_1 + r_2 + 5r^\diamond.$$

Thus, D^{rob} is $(r_1 + r_2 + 5r^\diamond)$-regular. $\diamond$

Moreover, observe that each D_i^{rob} can decompose a sparse digraph in the pair (U_i, U_{i+1}) into perfect matchings.

Claim 2. *Let $i \in [K]$. Let H_i be a bipartite digraph on vertex classes U_i and U_{i+1} which is edge-disjoint from D_i^{rob} and such that $E(H_i) \cap \{uv \mid vu \in M_i\} = \emptyset$. Suppose that $H_i[U_i, U_{i+1}]$ is r-regular and $H_i[U_{i+1}, U_i]$ is empty. Denote by $\widetilde{H}_i$ the M_i-contraction of $H_i[U_i, U_{i+1}]$. Define a multidigraph $\mathcal{H}_i$ by $\mathcal{H}_i \coloneqq H_i \cup D_i^{\mathrm{rob}} \cup \mathcal{ESF}_i(U_i, U_{i+1}) \cup \mathcal{ESF}'_i(U_i, U_{i+1})$ and define a multidigraph $\widetilde{\mathcal{H}}_i$ by $\widetilde{\mathcal{H}}_i \coloneqq \widetilde{H}_i \cup \widetilde{D}_i^{\mathrm{rob}} \cup \mathcal{SF}_i \cup \mathcal{SF}'_i$. Then, the following hold.*

(i) *The multidigraph $\mathcal{H}_i$ can be obtained from the M_i-expansion of the multidigraph $\mathcal{H}'_i$ by orienting all the edges from U_i to U_{i+1}.*

(ii) *The multidigraph $\mathcal{H}_i$ has a decomposition $\mathscr{M}_i$ into s' perfect matchings from U_i to U_{i+1} such that the following hold for each $M \in \mathscr{M}_i$.*

  (a) *$M \cup M_i$ forms a Hamilton cycle on $U_i \cup U_{i+1}$.*

(b) *There exists an extended special path system $ESPS_M$ in the multidigraph $\mathcal{ESF}_i \cup \mathcal{ESF}'_i$ such that $M \cap (E(\mathcal{ESF}) \cup E(\mathcal{ESF}')) = E_{ESPS_M}(U_i, U_{i+1})$. (I.e. M contains precisely one of the "special path system parts" in $\mathcal{ESF} \cup \mathcal{ESF}'$.)*

Proof of Claim. First, we show (i). By Fact 7.26 and assumption, $H_i[U_i, U_{i+1}]$ is the M_i-expansion of $\widetilde{H}_i$. Moreover, recall that $H[U_{i+1}, U_i]$ is empty, so H_i can be obtained from the M_i-expansion of $\widetilde{H}_i$ by orienting all the edges from U_i to U_{i+1}. By definition, D_i^{rob} is obtained from the M_i-expansion of $\widetilde{D}_i^{\mathrm{rob}}$ by orienting all the edges from U_i to U_{i+1}. By Definition 11.7, each extended special path system $ESPS$ in the multidigraph $\mathcal{ESF}_i \cup \mathcal{ESF}'_i$ satisfies $ESPS[U_i, U_{i+1}] \cap M_i[U_{i+1}, U_i] = \emptyset$ (otherwise, (FESPS2) would imply that $ESPS$ is not a linear forest). Thus, Fact 7.26 implies that the M_i-expansions of $\mathcal{SF}_i$ and $\mathcal{SF}'_i$ are $\mathcal{ESF}_i[U_i, U_{i+1}]$ and $\mathcal{ESF}'_i[U_i, U_{i+1}]$. Altogether, this implies that the multidigraph $\mathcal{H}_i$ can indeed be obtained from the M_i-expansion of the multidigraph $\widetilde{\mathcal{H}}_i$ by orienting all the edges from U_i to U_{i+1}, as desired.

For (ii), we decompose the multidigraph $\mathcal{H}_i$ as follows. By Fact 7.27(i), $\widetilde{H}_i$ is an r-regular digraph on U_i and, by Fact B.9(ii), $\widetilde{H}_i$ is edge-disjoint from $\widetilde{D}_i^{\mathrm{rob}}$. Thus, Lemma 8.23 implies that the multidigraph $\widetilde{\mathcal{H}}_i$ has a decomposition $\widetilde{\mathscr{C}}_i$ into s' Hamilton cycles on U_i such that each cycle in $\widetilde{\mathscr{C}}_i$ contains precisely one of the special path systems contained in the multidigraph $\mathcal{SF}_i \cup \mathcal{SF}'_i$. Let $\mathscr{M}_i$ consist of the digraphs obtained by orienting all the edges from U_i to U_{i+1} in the M_i-expansions of the cycles in $\widetilde{\mathscr{C}}_i$.

Then, Fact 7.29 implies that (ii.a) holds and $\mathscr{M}_i$ is a set of s' perfect matchings from U_i to U_{i+1}. By Fact B.9(iv), the matchings in $\mathscr{M}_i$ are edge-disjoint. Thus, (i) implies that $\mathcal{M}_i$ is a decomposition of $\mathcal{H}_i$, as desired.

For (ii.b), let $M \in \mathscr{M}_i$ and denote by $C \in \widetilde{\mathscr{C}}_i$ its corresponding cycle. By Lemma 8.23, the multidigraph $\mathcal{SF}_i \cup \mathcal{SF}'_i$ contains a special path system SPS such that $E(\widetilde{\mathscr{C}}_i) \cap (E(\mathcal{SF}_i) \cup E(\mathcal{SF}'_i)) = E(SPS)$. Let SPS' be obtained from the M_i-expansion of SPS by orienting all the edges from U_i to U_{i+1}. By Fact B.9(iv), $M \cap (E(\mathcal{ESF}) \cup E(\mathcal{ESF}')) = E(SPS')$. By construction, there exists an extended special path system $ESPS$ in the multidigraph $\mathcal{ESF} \cup \mathcal{ESF}'$ such that SPS is obtained from the M_i-contraction of $ESPS[U_i, U_{i+1}]$ by deleting isolated vertices. By Definition 11.7, $ESPS[U_i, U_{i+1}] \cap M_i[U_{i+1}, U_i] = \emptyset$ and so Fact 7.26 implies that $E(SPS') = E_{ESPS}(U_i, U_{i+1})$ and so (ii.b) holds. $\diamond$

Step 3: Decomposing $H \cup D^{\mathrm{rob}} \cup \mathcal{ESF} \cup \mathcal{ESF}'$. For each $i \in [K]$, let H_i be digraph on $U_i \cup U_{i+1}$ defined by $E(H_i) := E_H(U_i, U_{i+1})$. Since H is a blow-up C_K with vertex partition $\mathcal{U}$, we have $H = \bigcup_{i \in [K]} H_i$ and so it is enough to show that, for each $i \in [K]$, the multidigraph $H_i \cup D_i^{\mathrm{rob}} \cup \mathcal{ESF}_i \cup \mathcal{ESF}_i'$ has a decomposition $\mathscr{C}_i$ into s' Hamilton cycles such that each cycle in $\mathscr{C}_i$ contains precisely one of the extended special path systems in the multidigraph $\mathcal{ESF}_i \cup \mathcal{ESF}_i'$.

Let $i \in [K]$. Denote by $\mathscr{M}_i$ the decomposition of the multidigraph $H_i \cup D_i^{\mathrm{rob}} \cup \mathcal{ESF}_i(U_i, U_{i+1}) \cup \mathcal{ESF}_i'(U_i, U_{i+1})$ obtained by applying Claim 2(ii). Let $\mathscr{C}_i$ be obtained from $\mathscr{M}_i$ by replacing each $M \in \mathscr{M}_i$ by the digraph $M \cup ESPS_M$ (recall that $ESPS_M$ was defined in Claim 2(ii.b)). Then, Fact 11.1, (FESPS2), and Claim 2(ii.a) imply that $\mathscr{C}_i$ is a Hamilton decomposition of the multidigraph $H_i \cup D_i^{\mathrm{rob}} \cup \mathcal{ESF}_i \cup \mathcal{ESF}_i'$. By Claim 2(ii.b), each cycle in $\mathscr{C}_i$ contains precisely one of the extended special path systems in the multidigraph $\mathcal{ESF}_i \cup \mathcal{ESF}_i'$. $\square$

APPENDIX D

THE PREPROCESSING STEP: PROOF OF LEMMA 8.30

In this appendix, we discuss how to derive the preprocessing lemma for bipartite digraphs (Lemma 8.30). First, note that Lemma 8.30 is a direct corollary of the bipartite versions of [76, Lemma 8.6] (which guarantees the existence of PG) and [76, Corollary 8.5] (which verifies the properties of PG). Figure D.1 illustrates the overall structure of the proofs of [76, Corollary 8.5 and Lemma 8.6]. In this appendix, we will discuss the bipartite versions of the dark grey lemmas from Figure D.1. The white lemmas from Figure D.1 can be used in their original versions. The statements of the light grey lemmas from Figure D.1 can be adapted simply by replacing a consistent system by a consistent bi-system whose exceptional set forms an independent set, while their proofs follow immediately from the white lemmas from Figure D.1 and the bipartite versions of the grey lemmas from Figure D.1.

Note that Lemma 9.1 corresponds to the bipartite version of [76, Lemma 7.6] and Lemma 7.7 is the bipartite version of [76, Lemma 5.3]. As already mentioned, Lemma 7.7 can be proven using the same arguments as in the proof of [76, Lemma 5.3], so we omit the details. Finally, note that we will use the bipartite version of [76, Lemma 7.2] (Lemma D.2 below) to derive Lemma 9.2 (that is, the bipartite version of [76, Lemma 9.1]) at the end of this appendix.

Lemma D.1 (Bipartite version of [76, Lemma 5.2])**.** *Let* $0 < \frac{1}{n} \ll \varepsilon \ll \nu \leq \tau \ll \delta < 1$

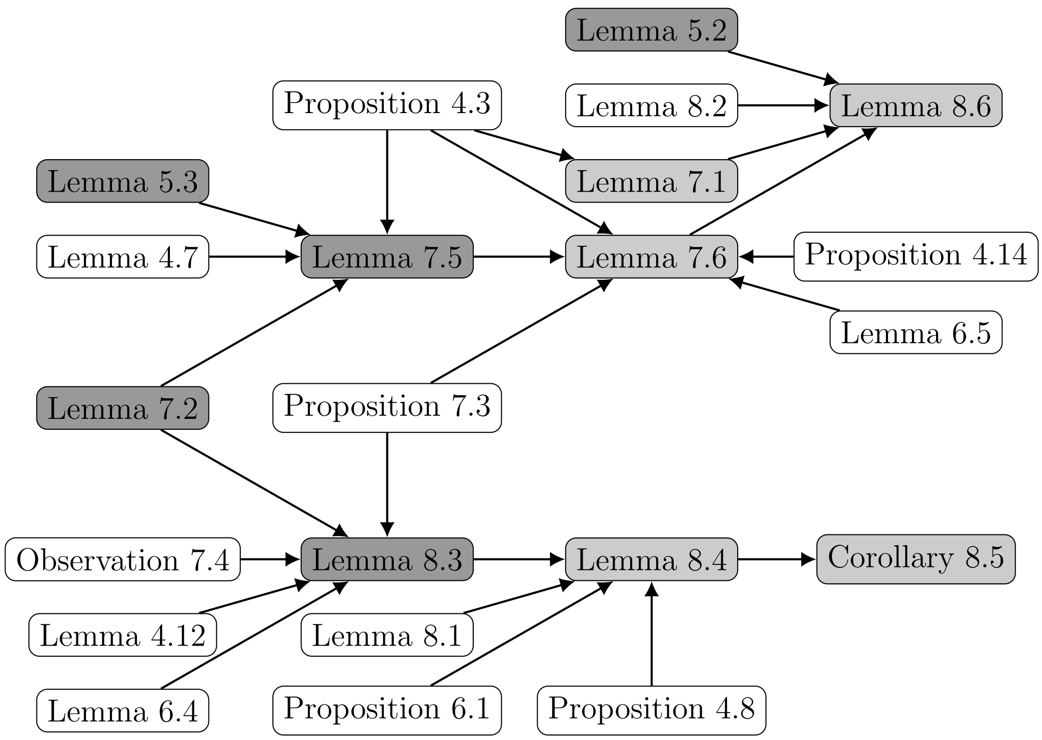


Figure D.1: The structure of the proofs of [76, Corollary 8.5 and Lemma 8.6].

and $\frac{1}{n} \ll \xi \leq \frac{\nu^2}{3}$. *Let* D *be a balanced bipartite digraph on vertex classes* A *and* B *of size* n. *Suppose that* $\delta^0(D) \geq \delta n$ *and that* D *is a bipartite robust* (ν, τ)*-outexpander with bipartition* (A, B). *For each* $v \in V(D)$, *let* $n_v^+, n_v^- \in \mathbb{N}$ *be such that* $(1-\varepsilon)\xi n \leq n_v^+, n_v^- \leq (1+\varepsilon)\xi n$ *and such that both* $\sum_{v\in A} n_v^\pm = \sum_{v \in B} n_v^\mp$. *Then,* D *contains a spanning subdigraph* D' *such that* $d_{D'}^+(v) = n_v^+$ *and* $d_{D'}^-(v) = n_v^-$ *for each* $v \in V(D)$.

Proof. By symmetry, it is enough to find a spanning subgraph $H \subseteq D[A, B]$ satisfying $d_D(a) = n_a^+$ for each $a \in A$ and $d_H(b) = n_b^-$ for each $b \in B$. Let N be the flow network obtained from $D(A, B)$ by giving each edge of $D(A, B)$ capacity 1, by adding a source s which is joined to every vertex $a \in A$ with an outedge of capacity n_a^+, and by adding a sink t which is joined to every vertex $b \in B$ with an inedge of capacity n_b^-. Let $r \coloneqq \sum_{v\in A} n_v^+ = \sum_{v\in B} n_v^-$. Using similar arguments as in [76, Lemma 5.2], one can show that any $s-t$ cut in N has capacity at least r. Thus, the max-flow min-cut theorem implies that N has an $s-t$ flow of value r. This flow corresponds to the desired spanning subgraph $H \subseteq D[A, B]$. $\square$

By definition, a bipartite digraph R can only contain a chord sequence between clusters which belong to a common vertex class of R. Such chord sequences can be constructed using the same arguments as in [76, Lemma 7.2], so we omit the details.

Lemma D.2 (Bipartite version of [76, Lemma 7.2]). *Let $0 < \frac{1}{k} \ll \nu \ll \tau \ll \delta < 1$. Let R be a balanced bipartite digraph on vertex classes $\mathcal{A}$ and $\mathcal{B}$ of size k. Suppose that R is a bipartite robust (ν, τ)-outexpander with bipartition (A, B) and that $\delta^0(R) \geq \delta k$. Let C be a Hamilton cycle in R. Let $\mathcal{V} \subseteq V(R)$ satisfy $|\mathcal{V}| \leq \frac{\nu k}{4}$. Suppose that $A_1, A_2 \in \mathcal{A}$. Then, there exists a chord sequence $CS(A_1, A_2) \subseteq E_R(\mathcal{B}, \mathcal{A})$ containing at most $3\nu^{-1}$ edges and such that $V(CS(A_1, A_2)) \cap \mathcal{V} \subseteq \{A_1^-, A_2\}$, where A_1^- denote the predecessor of A_1 on C.*

Since we can no longer construct chord sequences between any pair of clusters, we will need to be more careful and adapt the proofs of the lemmas which use [76, Lemma 7.2] (that is, [76, Lemmas 7.5 and 8.3]).

Define a *path system extender PE for C and R with parameters* $(\varepsilon, d, d', \zeta)$ as in [76]. (The precise definition is not relevant for our purposes and so we omit it here.)

Lemma D.3 (Bipartite version of [76, Lemma 8.3]). *Let $0 < \frac{1}{n} \ll d' \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{\ell^*} \ll d \ll \nu \ll \tau \ll \delta, \theta \leq 1$ and $d \ll \zeta \leq \frac{1}{2}$. Suppose that $\frac{m}{50} \in \mathbb{N}$. Let $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ be a consistent $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$-bi-system with $|D| = n$ and exceptional set V_0. Let $\mathcal{P}'$ be a $(d')^2$-uniform 50-refinement of $\mathcal{P}$. Let PE be a path system extender with parameters $(\varepsilon, d, d', \zeta)$ for C and R. Let $s := \frac{10^7}{\nu^2}$ and suppose that Q is a set of vertex-disjoint paths of D such that the following hold.*

(i) *Q and PE are edge-disjoint.*

(ii) *Q contains a special cover SC in D with respect to V_0 such that each component of SC is a path of length 2.*

(iii) *There exists a set $\mathcal{V}$ of five clusters of $\mathcal{P}'$ such that each $e \in E(Q)$ satisfies $V(e) \subseteq V_0 \cup \bigcup \mathcal{V}$.*

(iv) $|E(Q)| \leq \frac{1230n}{s}$.

Then, D contains a Hamilton cycle H such that $Q \subseteq H \subseteq PE \cup Q$.

Proof. Let M_{SC} be the complete special sequence associated to SC and denote by E the edge set of $(Q \setminus SC) \cup M_{SC}$. (Note that E is precisely $E(Q^{\text{basic}})$ with respect to the notation of [76]). For any cluster $V \in \mathcal{P}$, denote by V^- and V^+ the predecessor and successor of V on C. For any vertex $v \in V(D) \setminus V_0$, denote by V_v the cluster in $\mathcal{P}$ which contains v.

Claim 1. *There exist chord sequences $CS(W_1, \widetilde{W}_1^+), \dots, CS(W_{|E|}, \widetilde{W}_{|E|}^+)$ in R for which the following hold.*

(a) *There exists an enumeration $u_1, \dots, u_{|E|}$ of the starting points of the edges in E such that $\widetilde{W}_i = V_{u_i}$ for each $i \in [|E|]$.*

(b) *There exists an enumeration $v_1, \dots, v_{|E|}$ of the ending points of the edges in E such that $W_i = V_{u_i}$ for each $i \in [|E|]$.*

(c) *Altogether, $CS(W_1, \widetilde{W}_1^+), \dots, CS(W_{|E|}, \widetilde{W}_{|E|}^+)$ contain at most $\frac{21m}{100}$ edges incident to each cluster in $\mathcal{P}$ and at most $\frac{m}{50}$ occurrences of every edge of R.*

If Claim 1 holds, one can conclude the proof of Lemma D.3 using the arguments of [76, Lemma 8.3]. Thus, it suffices to prove Claim 1.

Proof of Claim 1. In the proof of [76, Lemma 8.3], [76, Lemma 7.2] is used to construct, for each edge $uv \in E$, a chord sequence $CS(V_v, V_u^+)$ in R. This is not possible here because (CBSys1) and (ii) imply that, for any $uv \in M_{SC} \subseteq E$, u and v belong to a common vertex class of D and so V_v and V_u^+ belong to distinct vertex classes of $C \subseteq R$.

We circumvent this problem as follows. Recall from (CBSys1) that D is a balanced bipartite digraph. Denote by A and B the vertex classes of D. Let $M_{SC,A} \coloneqq \{e \in V(M_{SC}) \mid V(e) \subseteq A\}$ and $M_{SC,B} \coloneqq \{e \in V(M_{SC}) \mid V(e) \subseteq B\}$. By (ii) and since D is bipartite, $M_{SC,A}$ and $M_{SC,B}$ partition M_{SC}. By (CBSys2), we have $|V_0 \cap A| = |V_0 \cap B|$ and so $|M_{SC,A}| = |M_{SC,B}|$. Fix a bijection $\phi\colon M_{SC,A} \longrightarrow M_{SC,B}$ and define $E' \coloneqq (Q \setminus SC) \cup \{ab', ba' \mid aa' \in M_{SC,A}, bb' = \phi(aa')\}$. By construction, each edge in E'

has precisely one endpoint in A and one endpoint in B. Using Lemma D.2 instead of its non-bipartite analogue [76, Lemma 7.2], one can apply the arguments of [76, Lemma 8.3] to construct, for each $uv \in E'$, a chord sequence $CS(V_v, V_u^+)$ in R such that, altogether, (c) holds. Then, (a) and (b) hold by definition of E'. $\diamond$

This completes the proof of Lemma D.3. $\square$

A similar problem arises when adapting [76, Lemma 7.5] to the bipartite case. Roughly speaking, [76, Lemma 7.5] guarantees the existence of a special cover SC whose components are paths of length 2, and chord sequences from V_v to V_u^+ for each edge uv of the complete special sequence M_{SC} associated to SC (where V_v denotes the cluster in $\mathcal{P}$ which contains v and V_u^+ denotes the successor on C of the cluster in $\mathcal{P}$ which contains u). As discussed in the proof of Lemma D.3, such chord sequences do not exist. We can only guarantee that the number of times a cluster is at the start/end of a chord sequence equals the number of edges in M_{SC} which start/end in that cluster. This is sufficient for proving the bipartite version of [76, Lemma 7.6] (that is, Lemma 9.1).

Lemma D.4 (Bipartite version of [76, Lemma 7.5]). *Let $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \ll \varepsilon' \ll d \ll \nu \ll \tau \ll \delta, \theta \leq 1$ and $\frac{f}{\ell^*} \ll 1$ and $\varepsilon \ll \frac{1}{\ell'}, \frac{1}{f}$. Suppose that $\frac{\ell^*}{f}, \frac{m}{\ell'} \in \mathbb{N}$. Let $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ be a consistent $(\ell^*, k, m, \varepsilon, d, \nu, \tau, \delta, \theta)$-bi-system with $|D| = n$ and exceptional set V_0. Suppose that D' is a spanning subdigraph of D and that $\mathcal{P}'$ is a partition of $V(D)$ such that the following conditions are satisfied.*

(i) *$\mathcal{P}'$ is an ε-uniform ℓ'-refinement of $\mathcal{P}$.*

(ii) *For any $v \in V_0$, we have $d_D^\pm(x) - d_{D'}^\pm(x) \leq \varepsilon n$.*

(iii) *For any $v \in V(D) \setminus V_0$, we have $d_D^\pm(v) - d_{D'}^\pm(v) \leq \frac{(\varepsilon')^3 m}{\ell'}$.*

For any cluster $V \in \mathcal{P}$, denote by V^- and V^+ the predecessor and successor of V on C. For any vertex $v \in V(D) \setminus V_0$, denote by V_v the cluster in $\mathcal{P}$ which contains v. Let $(h, j) \in [\ell'] \times [f]$ and let $I = W_1 \dots W_{k'}$ denote the j^{th} interval in the canonical interval partition of C into f intervals. For any cluster $V \in \mathcal{P}$, denote by V^h the h^{th} subcluster of

V in $\mathcal{P}'$. In particular, let $W_1^h, \ldots, W_{k'}^h$ denote the h^{th} subclusters of $W_1, \ldots, W_{k'}$ in $\mathcal{P}'$. Then, the following hold.

(a) *There exists a special cover SC in D' with respect to V_0 which satisfies the following properties.*

   – *Each component of SC is a path of length 2.*

   – *$V(SC) \subseteq V_0 \cup W_3^h \cup \ldots W_{k'-2}^h$.*

   – *Each cluster $V \in \mathcal{P}$ satisfies $|V \cap V(SC)| \leq \varepsilon^{\frac{1}{4}} m$.*

(b) *There exist chord sequences $CS(\widetilde{W}_1, \widehat{W}_1^+), \ldots, CS(\widetilde{W}_{|V_0|}, \widehat{W}_{|V_0|}^+)$ in R for which the following hold.*

   – *There exists an enumeration $u_1, \ldots, u_{|V_0|}$ of the starting points of the components of SC such that $\widehat{W}_i = V_{u_i}$ for each $i \in [|V_0|]$.*

   – *There exists an enumeration $v_1, \ldots, v_{|V_0|}$ of the ending points of the components of SC such that $\widetilde{W}_i = V_{v_i}$ for each $i \in [|V_0|]$.*

   – *For each $i \in [|V_0|]$, $CS(\widetilde{W}_i, \widehat{W}_i^+)$ contains at most $3\nu^{-3}$ edges and all its vertices lie in $W_2 \cup \cdots \cup W_{k'-1}$.*

   – *Altogether, $CS(\widetilde{W}_1, \widehat{W}_1^+), \ldots, CS(\widetilde{W}_{|V_0|}, \widehat{W}_{|V_0|}^+)$ contain at most $4\varepsilon^{\frac{1}{4}} m$ edges incident to each cluster in $\mathcal{P}$.*

(c) *D' contains a matching M which satisfies the following properties.*

   – *M can be obtained by replacing, for each $i \in [|V_0|]$, each edge UV of $CS(\widetilde{W}_i, \widehat{W}_i^+)$ by an edge of $D'(U^h, V^h)$.*

   – *$V(M) \cap V(SC) = \emptyset$.*

(d) *For each $UV \in E(C)$, the pair $D'[U^h, V^h]$ is $[\varepsilon', \geq d]$-superregular.*

Lemma D.4 can be proven using the same arguments as in [76, Lemma 7.5], with Lemma 7.7 and Lemma D.2 playing the roles of [76, Lemmas 5.3 and 7.2] and using the arguments of Claim 1 of the proof of Lemma D.3 to choose the endpoints of the chord sequences in Lemma D.4(b). Therefore, we omit the details here.

Finally, we use Lemma D.2 and adapt the arguments of [76, Lemma 9.1] to derive Lemma 9.2.

Proof of Lemma 9.2. Denote $C = V_1 \dots V_{2k}$. Since R is bipartite, we may assume without loss of generality that $\mathcal{A} = \{V_i \mid i \in [2k] \text{ is odd}\}$ and $\mathcal{B} = \{V_i \mid i \in [2k] \text{ is even}\}$. Denote $V_{2k+1} \coloneqq V_1$ and $V_{2k+2} \coloneqq V_2$. For simplicity, split (BU1) into two parts as follows.

(BU1a) The edge set of U has a partition into U_{odd} and U_{even} and, for every $i \in [2k]$, U contains a chord sequence $CS(V_i, V_{i+2})$ from V_i to V_{i+2} such that (BU2), (BU3), and the following hold. All of the edges in the multiset $\bigcup\{CS(V_i, V_{i+2}) \mid i \in [2k] \text{ is odd}\}$ are contained in U_{odd}, all of the edges in the multiset $\bigcup\{CS(V_i, V_{i+2}) \mid i \in [2k] \text{ is even}\}$ are contained in U_{even}, and

(BU1b) all the remaining edges of U lie on C.

Apply the arguments of [76, Lemma 9.1] with Lemma D.2 playing the role of [76, Lemma 7.2] to obtain chord sequences $CS(V_1, V_3), CS(V_2, V_4), \dots, CS(V_{2k}, V_{2k+2})$ which satisfy the following properties, where U'_{odd} denotes the multiset of edges defined by

$$U'_{\text{odd}} \coloneqq E(CS(V_1, V_3)) \cup E(CS(V_3, V_5)) \cup \dots \cup E(CS(V_{2k-1}, V_{2k+1}))$$

and U'_{even} denotes the multiset of edges defined by

$$U'_{\text{even}} \coloneqq E(CS(V_2, V_4)) \cup E(CS(V_4, V_6)) \cup \dots \cup E(CS(V_{2k}, V_{2k+2})).$$

(i) For each $i \in [2k]$, $CS(V_i, V_{i+2})$ contains at most $3\nu^{-1} \leq \frac{\sqrt{\ell'}}{2}$ edges.

(ii) For each $i \in [2k]$, we have $d_{U'_{\text{odd}}}(V_i) \leq \frac{2\ell'}{5}$ and $d_{U'_{\text{even}}}(V_i) \leq \frac{2\ell'}{5}$.

Let U' be the multidigraph on $V(R)$ whose multiset of edges is defined by $E(U') \coloneqq U'_{\text{odd}} \cup U'_{\text{even}} = \bigcup_{i \in [2k]} E(CS(V_i, V_{i+2}))$. By (i) and construction, U' satisfies (BU1a) and

(BU2).

For each $i \in [2k]$, let $n^{\pm}_{i,\mathrm{odd}} := d^{\pm}_{U'_{\mathrm{odd}}}(V_i)$ and $n^{\pm}_{i,\mathrm{even}} := d^{\pm}_{U'_{\mathrm{even}}}(V_i)$. By similar arguments as in the proof of [76, Lemma 9.1], we have $n^{-}_{i+1,\mathrm{odd}} = n^{+}_{i,\mathrm{odd}}$ and $n^{-}_{i+1,\mathrm{even}} = n^{+}_{i,\mathrm{even}}$ for each $i \in [2k]$ (where $n^{-}_{2k+1,\mathrm{odd}} := n^{-}_{1,\mathrm{odd}}$ and $n^{-}_{2k+1,\mathrm{even}} := n^{-}_{1,\mathrm{even}}$). For each $i \in [2k]$, let $\ell_{i,\mathrm{odd}} := \frac{\ell'}{2} - n^{-}_{i,\mathrm{odd}}$ and $\ell_{i,\mathrm{even}} := \frac{\ell'}{2} - n^{-}_{i,\mathrm{even}}$. Let U be obtained from U' by adding, for each $i \in [2k]$, exactly $\ell_{i,\mathrm{odd}} + \ell_{i,\mathrm{even}}$ copies of the edge $V_{i-1}V_i$. Let U_{odd} be obtained from U'_{odd} by adding exactly $\ell_{i,\mathrm{odd}}$ copies of the edge $V_{i-1}V_i$ and let U_{even} be obtained from U'_{even} by adding exactly $\ell_{i,\mathrm{even}}$ copies of the edge $V_{i-1}V_i$. Note that U_{odd} and U_{even} partition the edges of U. Since U' satisfies (BU1a) and (BU2), U also satisfies (BU1a) and (BU2). By construction, (BU1b) also holds.

For each $i \in [2k]$, we have $d^{-}_{U_{\mathrm{odd}}}(V_i) = n^{-}_{i,\mathrm{odd}} + \ell_{i,\mathrm{odd}} = \frac{\ell'}{2}$ and

$$d^{+}_{U_{\mathrm{odd}}}(V_i) = n^{+}_{i,\mathrm{odd}} + \ell_{i+1,\mathrm{odd}} = n^{-}_{i+1,\mathrm{odd}} + \ell_{i+1,\mathrm{odd}} = \frac{\ell'}{2}.$$

Similarly, both $d^{\pm}_{U_{\mathrm{even}}}(V_i) = \frac{\ell'}{2}$ for each $i \in [2k]$ and so (BU3) holds. One can show that U forms a closed walk in R using similar arguments as in [76, Lemma 9.1]. $\square$

APPENDIX E

APPLYING THE REGULARITY LEMMA: PROOF OF LEMMA 9.3

In this appendix, we will prove Lemma 9.3, which guarantees the existence of consistent bi-systems and bi-setups in a bipartite robust outexpander. We will need the bipartite analogue of Lemma B.4(ii). The proof follows easily from the definition of a bipartite robust outexpander and is therefore omitted.

Lemma E.1. *Let $0 < \frac{1}{n} \ll \varepsilon \leq \nu \ll \tau \leq 1$. Let D be a bipartite digraph on vertex classes A and B of size n and suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B). Let $A' \subseteq A$ and $B' \subseteq B$ satisfy $|A'| = |B'| \geq (1-\varepsilon)n$. Then, $D(A', B')$ is a bipartite robust $(\nu - \varepsilon, 2\tau)$-outexpander with bipartition (A', B').*

Proof of Lemma 9.3. Let $0 < \frac{1}{M'} \ll \varepsilon$. Fix additional constants such that $\frac{1}{M'} \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll \varepsilon$.

Step 1: Applying the regularity lemma. Let M and n_0 be the constants obtained by applying Lemma 7.14 with ε_1 playing the role of ε. By Lemma 7.14(i), we may assume without loss of generality that $0 < \frac{1}{n_0} \ll \frac{1}{M} \leq \frac{1}{M'} \ll \varepsilon_1$. Fix additional constants such that $\varepsilon \ll \frac{1}{q} \ll \frac{1}{f}, \frac{1}{\ell^*} \ll d \ll \nu \ll \tau \ll \delta, \theta \ll 1$ and $d \ll \frac{1}{g} \ll 1$. Moreover, let $\ell' \geq 324\nu^{-2}$ be even. Let D be a balanced bipartite on vertex classes A and B of size $n \geq n_0$. Suppose that D is a bipartite robust (ν, τ)-outexpander with bipartition (A, B) and that $\delta^0(D) \geq \delta n$.

Apply Lemma 7.14 with ε_1 and $4d$ playing the roles of ε and d to obtain a spanning subdigraph $D' \subseteq D$ and a partition $\widetilde{\mathcal{P}} = \{\widetilde{V}_0, \widetilde{V}_1, \dots, \widetilde{V}_{2\widetilde{k}}\}$ of $V(D)$ such that Lemma 7.14(i)–(vii) hold with $2\widetilde{k}, \widetilde{m}, \widetilde{V}_0, \widetilde{V}_1, \dots, \widetilde{V}_{2\widetilde{k}}$ playing the roles of $k, m, V_0, V_1, \dots, V_{2k}$. Denote $\widetilde{\mathcal{A}} \coloneqq \{\widetilde{V}_i \mid i \in [2\widetilde{k}], \widetilde{V}_i \subseteq A\}$ and $\widetilde{\mathcal{B}} \coloneqq \{\widetilde{V}_i \mid i \in [2\widetilde{k}], \widetilde{V}_i \subseteq B\}$. We may assume without loss of generality that $\widetilde{\mathcal{A}} \coloneqq \{V_{2i-1} \mid i \in [\widetilde{k}]\}$ and $\widetilde{\mathcal{B}} \coloneqq \{\widetilde{V}_{2i} \mid i \in [\widetilde{k}]\}$. Let $\widetilde{R}$ be the bipartite reduced digraph of D with parameters $\varepsilon_1, 4d$, and M'. By Lemma 7.15, $\delta^0(\widetilde{R}) \geq \frac{\delta \widetilde{k}}{2}$ and $\widetilde{R}$ is a bipartite robust $(\frac{\nu}{2}, 2\tau)$-outexpander with bipartition $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$. Observe that, by Lemma 7.14(vii), $D[U,V]$ is $(\varepsilon_1, \geq 4d)$-regular for each $UV \in E(\widetilde{R})$.

Step 2: Ensuring the desired divisibility conditions. Let $\widehat{k}$ be the largest integer satisfying $\widehat{k} \leq \widetilde{k}$ and $\frac{\widehat{k}}{21fg(g-1)} \in \mathbb{N}$. Let $\widehat{V}_0 \coloneqq \widetilde{V}_0 \cup \bigcup_{i \in [2\widetilde{k}-2\widehat{k}]} \widetilde{V}_{2\widehat{k}+i}$. By Lemma 7.14(i), (ii), and (iv),

$$|\widehat{V}_0 \cap A| = |\widehat{V}_0 \cap B| \leq \varepsilon_1 n + 21fg(g-1)\widetilde{m} \leq 2\varepsilon_1 n. \tag{E.1}$$

Let $\widehat{\mathcal{P}} \coloneqq \{\widehat{V}_0, \widetilde{V}_1, \dots, \widetilde{V}_{2\widehat{k}}\}$ and let $\widehat{V}_1, \dots, \widehat{V}_{2\widehat{k}}$ be a relabelling of $\widetilde{V}_1, \dots, \widetilde{V}_{2\widehat{k}}$. Let $\widehat{R} \coloneqq \widetilde{R} - \{\widetilde{V}_{2\widehat{k}+i} \mid i \in [2\widetilde{k} - 2\widehat{k}]\}$. Denote $\widehat{\mathcal{A}} \coloneqq \widetilde{\mathcal{A}} \setminus \{\widetilde{V}_{2\widehat{k}+i} \mid i \in [2\widetilde{k} - 2\widehat{k}]\}$ and $\widehat{\mathcal{B}} \coloneqq \widetilde{\mathcal{B}} \setminus \{\widetilde{V}_{2\widehat{k}+i} \mid i \in [2\widetilde{k} - 2\widehat{k}]\}$. Then, $\delta^0(\widehat{R}) \geq \frac{\delta \widetilde{k}}{2} - 21fg(g-1) \geq \frac{\delta \widehat{k}}{3}$ and, by Lemma E.1, $\widehat{R}$ is a bipartite robust $(\frac{\nu}{3}, 4\tau)$-outexpander with bipartition $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$.

Step 3: Finding a Hamilton cycle and a bi-universal walk in the reduced graph. Apply Corollary 7.5 with $\widehat{R}, \widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{k}, \frac{\nu}{3}, 4\tau$, and $\frac{\delta}{3}$ playing the roles of D, A, B, n, ν, τ, and δ to obtain a Hamilton cycle $\widehat{C}$ of $\widehat{R}$. We may assume without loss of generality that $\widehat{C} = \widehat{V}_1 \dots \widehat{V}_{2\widehat{k}}$. Apply Lemma 9.2 with $\widehat{R}, \widehat{C}, \widehat{k}, \frac{\nu}{3}, 4\tau$, and $\frac{\delta}{3}$ playing the roles of R, C, k, ν, τ, and δ to obtain a universal walk $\widehat{U}$ for $\widehat{C}$ in $\widehat{R}$ with parameter ℓ'. Denote $\widehat{U} = \widehat{V}_{i_1} \dots \widehat{V}_{i_{2\ell'\widehat{k}}}$.

Step 4: Forming superregular pairs. Let $E \coloneqq \{e \in E(\widehat{C}) \cup E(\widehat{U})\}$. We adjust the partition $\widehat{\mathcal{P}}$ to ensure that each edge in E corresponds to a superregular pair in D. By (BU3), each $V \in V(\widehat{R})$ satisfies $d_E^{\pm}(V) \leq \ell' + 1$. For each $e = UV \in E$, denote $d_e \coloneqq d_D(U,V)$ and observe that, by Step 1, $d_e \geq 4d$. Fix an integer m_0 such that $(1 - 2\sqrt{\varepsilon_1})\widetilde{m} \leq m_0 \leq (1 - \sqrt{\varepsilon_1})\widetilde{m}$ and $\frac{fm_0}{200q\ell'\ell^*} \in \mathbb{N}$. Let $i \in [2\widehat{k}]$. Let $\widehat{V}_i'$ consist of all the vertices $v \in \widehat{V}_i$ such that there exists $j \in [2\widehat{k}] \setminus \{i\}$ such that $e = \widehat{V}_i\widehat{V}_j \in E$ but $|N_D^+(v) \cap \widehat{V}_j| \neq (d_e \pm \varepsilon_1)\widetilde{m}$ or, $e' = \widehat{V}_j\widehat{V}_i \in E$ but $|N_D^-(v) \cap \widehat{V}_j| \neq (d_{e'} \pm \varepsilon_1)\widetilde{m}$. By Lemma A.2, $|\widehat{V}_i'| \leq 2\varepsilon_1\widetilde{m}d_E(\widehat{V}_i) \leq 4\varepsilon_1\widetilde{m}(\ell'+1) \leq \sqrt{\varepsilon_1}\widetilde{m}$. Let $\widehat{V}_i' \subseteq \widehat{V}_i'' \subseteq \widehat{V}_i$ satisfy $|\widehat{V}_i''| = \widetilde{m} - m_0$. Let $V_0 := \widehat{V}_0 \cup \bigcup_{i \in [2\widehat{k}]} \widehat{V}_i''$. By (E.1),

$$|V_0 \cap A| = |V_0 \cap B| \leq 2\varepsilon_1 n + 2\sqrt{\varepsilon_1}\widetilde{m} \cdot 2\widehat{k} \leq \varepsilon n.$$

For each $i \in [2\widehat{k}]$, let $V_i^0 := \widehat{V}_i \setminus \widehat{V}_i''$. By construction, $|V_1^0| = \cdots = |V_{2\widehat{k}}^0| = m_0$. Let $\mathcal{P}_0 := \{V_0, V_1^0, \ldots, V_{2\widehat{k}}^0\}$. Denote $\mathcal{A}_0 := \{V_i^0 \in \mathcal{P}_0 \mid \widehat{V}_i \in \widehat{\mathcal{A}}\}$ and $\mathcal{B}_0 := \{V_i^0 \in \mathcal{P}_0 \mid \widehat{V}_i \in \widehat{\mathcal{B}}\}$. Let R_0 be the digraph on $\mathcal{A}^0 \cup \mathcal{B}^0$ which is induced by $\widehat{R}$, i.e. defined as follows. For any $i, j \in [2\widehat{k}]$, $V_i^0V_j^0 \in E(R_0)$ if and only if $\widehat{V}_i\widehat{V}_j \in E(\widehat{R})$. By Lemma A.1, $D[U,V]$ is $(\varepsilon_2, \geq 3d)$-regular for each $UV \in E(R_0)$. Moreover, $\delta^0(R_0) \geq \frac{\delta\widehat{k}}{3}$ and R_0 is a bipartite robust $(\frac{\nu}{3}, 4\tau)$-outexpander with bipartition $(\mathcal{A}_0, \mathcal{B}_0)$. Let $C_0 := V_1^0 \ldots V_{2\widehat{k}}^0$ and $U_0 = V_{i_1}^0 \ldots V_{i_{2\ell'\widehat{k}}}^0$. By construction, C_0 is Hamilton cycle of R_0 and U_0 is a universal walk for C_0 in R_0 with parameter ℓ'. Moreover, $D[U,V]$ is $[\varepsilon_2, \geq 3d]$-superregular for each $UV \in E(C_0) \cup E(U_0)$.

Step 5: Finding the refinements. Apply Lemma 8.7 with $2n, m_0, 2\widehat{k}, \mathcal{P}_0, \varepsilon_2$, and ℓ^* playing the roles of $n, m, k, \mathcal{P}, \varepsilon$, and ℓ to obtain an ε_2-uniform ℓ^*-refinement $\mathcal{P}$ of $\mathcal{P}_0$. Let $\mathcal{A}$ be the set of clusters $V \in \mathcal{P}$ such that $V \subseteq W$ for some $W \in \mathcal{A}_0$. Let $\mathcal{B}$ be the set of clusters in $\mathcal{P} \setminus \mathcal{A}$. Let $k := \ell^*\widehat{k}$ and $m := \frac{m_0}{\ell^*}$. Let R be the ℓ^*-fold blow-up of R_0 induced by $\mathcal{P}$. Then, $\delta^0(R) = \ell^*\delta^0(R_0) \geq \frac{\delta k}{3}$. By Lemma 7.7, R is a bipartite robust $(4\nu^4, 8\tau)$-outexpander with bipartition $(\mathcal{A}, \mathcal{B})$. By Lemma 8.8(i) and Step 4, $D[U,V]$ is $(\varepsilon_3, \geq 2d)$-regular for each $UV \in E(R)$. For each $i \in [2\widehat{k}]$, denote by $V_{i,1}^0, \ldots, V_{i,\ell^*}^0$ the subclusters of V_i^0 contained in $\mathcal{P}$. Let $C := V_{1,1}^0V_{2,1}^0 \ldots V_{2\widehat{k},1}^0V_{1,2}^0 \ldots V_{2\widehat{k},\ell^*}^0$ and $U := V_{i_1,1}^0V_{i_2,1}^0 \ldots V_{i_{2\ell'\widehat{k}},1}^0V_{1,2}^0 \ldots V_{i_{2\ell'\widehat{k}},\ell^*}^0$. Then, C is a Hamilton cycle of R and U is a bi-universal walk for C in R with parameter ℓ'. Moreover, by Lemma 8.8(ii) and Step 4, $D[U,V]$ is $[\varepsilon_3, \geq 2d]$-superregular for each $UV \in E(C) \cup E(U)$.

Apply Lemma 8.7 with $2n, \varepsilon_3$, and ℓ' playing the roles of n, ε, and ℓ to obtain an ε_3-uniform ℓ'-refinement $\mathcal{P}'$ of $\mathcal{P}$. Let $V_1, \ldots, V_{2k}$ be a relabelling of the clusters in

$\mathcal{P}$ such that $C = V_1 \dots V_{2k}$ and let $i'_1, \dots, i'_{2\ell' k}$ be such that $U = V_{i'_1} \dots V_{i'_{2\ell' k}}$. For each $i \in [2k]$, denote by $V_{i,1}, \dots, V_{i,\ell'}$ the subclusters of V_i contained in $\mathcal{P}'$. Let $U' \coloneqq V_{i'_1,1} V_{i'_2,1} \dots V_{i'_{2\ell' k},1} V_{i'_1,2} \dots V_{i'_{2\ell' k},\ell'}$. By Lemma 8.8(ii), $D[U,V]$ is $[\varepsilon_4, \geq 2d]$-superregular for each $UV \in E(U')$.

Apply Lemma 8.7 with $2n, \varepsilon_3$, and $\frac{q}{f}$ playing the roles of n, ε, and ℓ to obtain an ε_3-uniform $\frac{q}{f}$-refinement $\mathcal{P}^*$ of $\mathcal{P}$.

Step 6: Verifying (i) and (ii). Let $M'' \coloneqq \ell^* M$. By our choice of $\widehat{k}$ in Step 2 and definition of k in Step 5, we have $\frac{\ell^* \widetilde{k}}{2} \leq \ell^* \widehat{k} = k \leq \ell^* \widetilde{k}$. Then, Lemma 7.14(i) (with $\widetilde{k}$ playing the role of k) implies that $M' \leq k \leq M''$. Moreover, Step 2 implies that $\frac{k}{7}, \frac{k}{f}, \frac{k}{g}, \frac{2fk}{3g(g-1)} \in \mathbb{N}$. By our choice of m_0 in Step 4 and definition of m in Step 5, we have $\frac{m}{50}, \frac{m}{4\ell'}, \frac{fm}{q} \in \mathbb{N}$. Thus, (i) is satisfied.

Let D_1 be obtained from D by taking each edge independently with probability $\frac{1}{2}$. Define $D_2 \coloneqq D \setminus D_1$. We need to show that (ii) holds with positive probability. By Lemmas 8.19 and 8.29, it suffices to show that the following properties are satisfied.

(a) $(D, \mathcal{P}_0, R_0, C_0, \mathcal{P}, R, C)$ is a consistent $(\ell^*, 2k, m, \varepsilon_4, 2d, 4\nu^4, 8\tau, \frac{\delta}{3}, 3\theta)$-bi-system.

(b) $(D, \mathcal{P}, \mathcal{P}', \mathcal{P}^*, R, C, U, U')$ is an $(\ell', \frac{q}{f}, 2k, m, \varepsilon_4, 2d)$-bi-setup.

By Lemma 7.14(iii) and Steps 4 and 5, (BST1) holds. Moreover, (BST2)–(BST8) follow from Step 5. Thus, (b) holds. By Step 4 and definition of k and m in Step 5, (CBSys2) holds. By Step 5, (CBSys3), (CBSys4), and (CBSys6) are satisfied and (CBSys8) follows from (URef). Moreover, (CBSys1), (CBSys5), and (CBSys7) follow from Steps 4 and 5. Therefore (a) holds. $\square$

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GLOSSARY

$(\varepsilon, 4)$-partition Definition 10.1.

$(\varepsilon,\mathcal{U})$-exceptional set Definition 13.11: (ES1) and (ES2).

balanced special cover Definition 12.6.

bi-setup Definition 8.15: (BST1)–(BST8).

bi-universal walk Definition 8.13: (BU1)–(BU3).

canonical interval partition Definition 8.20.

complete special sequence Definition 8.25.

consistent bi-system Definition 8.27: (CBSys1)–(CBSys8).

consistent cycle-framework Definition 12.8.

cycle-framework Definition 11.3: (CF1)–(CF5).

cycle-setup Definition 11.2: (CST1)–(CST3).

equivalent linear forests Definition 8.4.

extended special factor Definition 11.9.

extended special path system Definition 11.8.

feasible system Definition 13.2: (F1)–(F3).

friendly extended special path system Definition 11.7: (FESPS1) and (FESPS2).

localised special cover Definition 12.3.

matching contraction Definition 7.25(i).

matching expansion Definition 7.25(ii).

optimal partition Definition 13.7.

placeholder Definition 15.1.

pseudo-feasible system Definition 15.3: (F1) and (F2′)–(F4′).

setup Definition 8.14: (ST1)–(ST8).

special cover Definition 8.24.

special factor Definition 8.22.

special path system Definition 8.21: (SPS1) and (SPS2).

uniform refinement Definition 8.6: (URef).

universal walk Definition 8.11: (U1)–(U3).